Because the support of μ is not compact, there is a *finite interval* J', *disjoint from* J, and containing two disjoint open intervals I'_1 and I'_2 with

$$\int_{I'_1} |d\mu(t)| > 0, \quad \int_{I'_2} |d\mu(t)| > 0.$$

Repetition of the argument just made, with J' playing the rôle of J, shows now that μ, outside J' (and hence in particular *in* J!) is also supported on a countable set without finite limit point. Therefore the *whole support* of μ in E must be such a set, which is what we had to prove.

Remark. The support of μ must *really be infinite.* Otherwise it would be compact, and this, as we have seen, is impossible.

Now we are ready to establish the

Theorem (Louis de Branges). *Let $W(x) \geqslant 1$ be a weight having the properties stated at the beginning of this §, and let E be the associated closed set on which $W(x)$ is finite.*

Suppose that $\mathscr{E}_A$ is not $\| \quad \|_W$*-dense in* $\mathscr{C}_W(\mathbb{R})$, *and let μ be an* extreme point *of the set Σ of real signed measures ν on E such that*

$$\int_E |d\nu(t)| \leqslant 1$$

and

$$\int_E \frac{f(t)}{W(t)} d\nu(t) \;=\; 0 \quad \textit{for all } f \in \mathscr{E}_A.$$

Then μ is supported on an infinite sequence $\{x_n\}$ without finite limit point, lying in E. There is an entire function $S(z)$ of exponential type A having a simple zero *at each point x_n and* no other zeros, *with*

$$\boxed{\mu(\{x_n\}) \;=\; \frac{W(x_n)}{S'(x_n)}.}$$

Moreover,

$$\int_{-\infty}^{\infty} \frac{\log^+ |S(x)|}{1 + x^2} dx \;<\; \infty$$

and

$$\lim_{y \to \infty} \frac{\log |S(iy)|}{y} \;=\; \lim_{y \to -\infty} \frac{\log |S(iy)|}{|y|} \;=\; A.$$

Proof. Let us begin by first establishing an auxiliary proposition.

Given an extreme point μ of Σ, suppose that we have a sequence of functions $f_n \in \mathscr{E}_A$ such that

$$\int_{-\infty}^{\infty} |f_n(x) - f_m(x)| \frac{|\mathrm{d}\mu(x)|}{W(x)} \xrightarrow[n,m]{} 0.$$

We know from the third lemma of the preceding article that a subsequence of the f_n tends u.c.c. in $\mathbb{C}$ to some entire function F of exponential type $\leqslant A$, and here it is clear by Fatou's lemma that

$$(*) \qquad \int_{-\infty}^{\infty} |f_n(x) - F(x)| \frac{|\mathrm{d}\mu(x)|}{W(x)} \xrightarrow[n]{} 0,$$

whence surely

$$\int_{-\infty}^{\infty} \frac{F(x)}{W(x)} \mathrm{d}\mu(x) = 0$$

since $\int_{-\infty}^{\infty}(f_n(x)/W(x))\mathrm{d}\mu(x) = 0$ for all n.

Our auxiliary proposition says that the relations

$$\int_{-\infty}^{\infty} \frac{F(x) - F(a)}{x-a} \frac{\mathrm{d}\mu(x)}{W(x)} = 0, \qquad \int_{-\infty}^{\infty} (x-b)\frac{F(x) - F(a)}{x-a} \frac{\mathrm{d}\mu(x)}{W(x)} = 0$$

also hold for the limit function F; here, a and b are arbitrary complex numbers. Both of these formulas are proved in the same way, and it is enough to deal here only with the *second* one.

Wlog, $f_n(z) \xrightarrow[n]{} F(z)$ u.c.c., whence $f_n(a) \longrightarrow F(a)$ and thence, by $(*)$,

$$\int_{-\infty}^{\infty} \frac{|f_n(x) - f_n(a) - (F(x) - F(a))|}{W(x)} |\mathrm{d}\mu(x)| \xrightarrow[n]{} 0,$$

since $W(x) \geqslant 1$. From this is clear that

$$\int_{|x-a|\geqslant 1} \left| \frac{x-b}{x-a}\left(f_n(x) - f_n(a) - F(x) + F(a)\right)\right| \frac{|\mathrm{d}\mu(x)|}{W(x)} \xrightarrow[n]{} 0.$$

Also, the u.c.c. convergence of $f_n(z)$ to $F(z)$ makes

$$\frac{f_n(z) - f_n(a)}{z-a}(z-b) \xrightarrow[n]{} \frac{F(z) - F(a)}{z-a}(z-b)$$

u.c.c., by the elementary theory of analytic functions (Cauchy's formula!). Therefore we also have

$$\int_{|x-a|\leqslant 1} \left|(x-b)\left(\frac{f_n(x) - f_n(a)}{x-a} - \frac{F(x) - F(a)}{x-a}\right)\right| \frac{|\mathrm{d}\mu(x)|}{W(x)} \xrightarrow[n]{} 0.$$

and finally

$$\int_{-\infty}^{\infty}\left|\frac{f_n(x)-f_n(a)}{x-a}(x-b)-\frac{F(x)-F(a)}{x-a}(x-b)\right|\frac{|\mathrm{d}\mu(x)|}{W(x)} \xrightarrow[n]{} 0.$$

Since, however, the $f_n \in \mathscr{E}_A$, we have $(x-b)(f_n(x)-f_n(a))/(x-a) \in \mathscr{E}_A$, whence

$$\int_{-\infty}^{\infty}\frac{f_n(x)-f_n(a)}{x-a}(x-b)\frac{\mathrm{d}\mu(x)}{W(x)} = 0$$

for each n. Referring to the previous relation, we see that

$$\int_{-\infty}^{\infty}\frac{F(x)-F(a)}{x-a}(x-b)\frac{\mathrm{d}\mu(x)}{W(x)} = 0,$$

as we set out to show.

Now we turn to the theorem itself. According to the lemma at the beginning of this article, our measure μ is supported on a countable set $\{x_n\} \subseteq E$ without finite limit point, and, by the remark following that lemma, $\mu(\{x_n\}) \neq 0$ *for infinitely many* of the points x_n. There is thus no loss of generality in supposing that $\mu(\{x_n\}) \neq 0$ *for each* n.

Take any two points from among the x_n, say x_0 and x_1, and put

$$\varphi(x_0) = \frac{1}{\mu(\{x_0\})(x_0-x_1)},$$

$$\varphi(x_1) = \frac{1}{\mu(\{x_1\})(x_1-x_0)},$$

and $\varphi(x) = 0$ for $x \neq x_0$ or x_1. Then

$$\int_{-\infty}^{\infty}\varphi(x)\mathrm{d}\mu(x) = 0,$$

so, as in the proof of the preceding lemma, there is a sequence of $f_n \in \mathscr{E}_A$ with

$$\int_{-\infty}^{\infty}\left|\frac{f_n(x)}{W(x)}-\varphi(x)\right||\mathrm{d}\mu(x)| \xrightarrow[n]{} 0$$

and, wlog, $f_n(z) \xrightarrow[n]{} F(z)$ u.c.c., F being some entire function of exponential type $\leqslant A$. We see that

$$(\ast)\qquad \frac{F(x)}{W(x)} = \varphi(x) \text{ a.e. } (|\mathrm{d}\mu|),$$

whence

$$(\dagger) \qquad \int_{-\infty}^{\infty} \frac{|f_n(x) - F(x)|}{W(x)} |d\mu(x)| \xrightarrow[n]{} 0.$$

From (⁑) and the definition of φ, we have $F(x_0) \neq 0$, $F(x_1) \neq 0$, so $F(z) \not\equiv 0$. For the same reasons, however, $F(x)$ *vanishes at all the other points* x_n, $n \neq 0, 1$.

Put

$$S(z) = F(z)(z - x_0)(z - x_1).$$

Then S, like F, is an entire function of exponential type $\leqslant A$. $S(z)$ vanishes at each of the points x_n in the support of μ. Finally,

$$\int_{-\infty}^{\infty} \frac{\log^+ |S(x)|}{1 + x^2} dx \ < \ \infty,$$

since, by the third lemma of the preceding article, the function F has this property.

Let us compute the quantities $S'(x_n)$. We already know that

$$S'(x_0) \ = \ F(x_0)(x_0 - x_1) \ = \ W(x_0)\varphi(x_0)(x_0 - x_1) \ = \ \frac{W(x_0)}{\mu(\{x_0\})},$$

and similarly $S'(x_1) = W(x_1)/\mu(\{x_1\})$. Take any other point x_n, $n \neq 0, 1$, and form the function

$$\frac{S(x)}{(x - x_0)(x - x_n)} \ = \ \frac{F(x)}{x - x_n}(x - x_1).$$

Since $F(x_n) = 0$, ($\dagger$) implies, *by our auxiliary proposition*, that

$$\int_{-\infty}^{\infty} \frac{F(x)}{x - x_n}(x - x_1)\frac{d\mu(x)}{W(x)} \ = \ 0.$$

The function $S(x)/(x - x_0)(x - x_n)$ vanishes at all the x_k, save x_0 and x_n. The previous relation therefore reduces to

$$\frac{S'(x_0)\mu(\{x_0\})}{(x_0 - x_n)W(x_0)} + \frac{S'(x_n)\mu(\{x_n\})}{(x_n - x_0)W(x_n)} \ = \ 0,$$

i.e.,

$$\frac{S'(x_n)}{W(x_n)}\mu(\{x_n\}) \ = \ 1,$$

and finally $S'(x_n) = W(x_n)/\mu(\{x_n\})$.

The function $S(z)$ *can have no zeros apart from the* x_n. Suppose, indeed, that $S(a) = 0$ with a different from all the x_n; then we would also have $F(a) = 0$, so, in the identity

$$\int_{-\infty}^{\infty} \frac{S(x)}{(x-x_0)(x-a)} \frac{d\mu(x)}{W(x)} = \int_{-\infty}^{\infty} \frac{F(x)}{x-a}(x-x_1)\frac{d\mu(x)}{W(x)},$$

the *right-hand integral would have to vanish* by our auxiliary proposition. The quantity $(F(x)/(x-a))(x-x_1)$ is, however, *different from* 0 *at only one of the points* x_n *in* μ's *support, namely, at* x_0, where

$$\frac{F(x_0)}{x_0-a}(x_0-x_1) = \frac{S'(x_0)}{x_0-a}.$$

We would thus get

$$\frac{S'(x_0)}{x_0-a}\cdot\frac{\mu(\{x_0\})}{W(x_0)} = 0,$$

i.e., in view of the computation made in the previous paragraph,

$$\frac{1}{x_0-a} = 0,$$

which is absurd.

The function $S(z)$ thus *vanishes once at each* x_n, and *only at those points.* As we have already seen, $\mu(\{x_n\}) = W(x_n)/S'(x_n)$, $S(z)$ is of exponential type $\leqslant A$, and

$$\int_{-\infty}^{\infty} \frac{\log^+|S(x)|}{1+x^2}dx < \infty.$$

To complete our proof, we have to show that $\log|S(iy)|/|y| \longrightarrow A$ for $y \to \pm\infty$.

In order to do this, let us first derive the partial fraction decomposition

$$\frac{1}{S(z)} = \sum_n \frac{1}{(z-x_n)S'(x_n)}.$$

Note that $\sum_n |1/S'(x_n)|$ is *surely convergent* because $\mu(\{x_n\}) = W(x_n)/S'(x_n)$, μ is a *finite* measure, and $W(x) \geqslant 1$. Take the function $G(t) = F(t)(t-x_1) = S(t)/(t-x_0)$, and, for *fixed* z, observe that

$$\frac{G(t)-G(z)}{t-z} = \frac{F(t)(t-x_1)-F(z)(z-x_1)}{t-z}$$

$$= \frac{F(t)-F(z)}{t-z}(t-x_1) + F(z).$$

By our auxiliary proposition,

$$\int_{-\infty}^{\infty} \frac{F(t)-F(z)}{t-z}(t-x_1)\frac{d\mu(t)}{W(t)} = 0,$$

and of course

$$\int_{-\infty}^{\infty} F(z)\cdot\frac{d\mu(t)}{W(t)} = 0$$

since $1 \in \mathscr{E}_A$. Therefore

$$\int_{-\infty}^{\infty} \frac{G(t)-G(z)}{t-z}\frac{d\mu(t)}{W(t)} = 0,$$

or, since $G(t)$ vanishes at *all the* x_n *save* x_0,

$$\frac{G(x_0)}{x_0-z}\frac{\mu(\{x_0\})}{W(x_0)} - G(z)\sum_n \frac{1}{(x_n-z)S'(x_n)} = 0.$$

This is the same as

$$\frac{S'(x_0)}{x_0-z}\cdot\frac{1}{S'(x_0)} = \frac{S(z)}{z-x_0}\sum_n \frac{1}{(x_n-z)S'(x_n)},$$

or

$$S(z)\sum_n \frac{1}{(z-x_n)S'(x_n)} = 1,$$

the desired relation.

From the result just found we derive a more general *interpolation formula.* Let $-A \leqslant \lambda \leqslant A$. Then $(e^{i\lambda t} - e^{i\lambda z})/(t-z)$ belongs, as a function of t, to $\mathscr{E}_A$, so

$$\int_{-\infty}^{\infty} \frac{e^{i\lambda t}-e^{i\lambda z}}{t-z}\frac{d\mu(t)}{W(t)} = 0.$$

In other words,

$$\sum_n \frac{e^{i\lambda x_n}}{(x_n-z)S'(x_n)} = e^{i\lambda z}\sum_n \frac{1}{(x_n-z)S'(x_n)}.$$

According to our previous result, the right-hand side is just $-e^{i\lambda z}/S(z)$. Therefore

$$\boxed{\frac{e^{i\lambda z}}{S(z)} = \sum_n \frac{e^{i\lambda x_n}}{(z-x_n)S'(x_n)} \quad \text{for} \quad -A \leqslant \lambda \leqslant A.}$$

(An analogous formula with $e^{i\lambda t}$ replaced by any $f(t) \in \mathscr{E}_A$ also holds, by the way – the proof is the same.)

In the boxed relation, put $\lambda = -A$ and take $z = iy$, $y > 0$. We get

$$\frac{e^{Ay}}{S(iy)} = \sum_n \frac{e^{-iAx_n}}{(iy - x_n)S'(x_n)}.$$

Since $\sum_n |1/S'(x_n)| < \infty$ and the x_n are real, the right side tends to 0 for $y \to \infty$. Thus,

$$\liminf_{y \to \infty} \frac{\log|S(iy)|}{y} \geqslant A.$$

But $\limsup_{y\to\infty} \log|S(iy)|/y \leqslant A$ since S is of exponential type $\leqslant A$. Therefore $\log|S(iy)|/y \to A$ for $y \to \infty$.

On taking $\lambda = A$ in the above boxed formula and making $y \to -\infty$, we see in like manner that

$$\frac{\log|S(iy)|}{|y|} \longrightarrow A \quad \text{for} \quad y \to -\infty.$$

De Branges' theorem is now completely proved. We are done.

3. Discussion of the theorem

De Branges' description of the extreme points of Σ is a most beautiful result; I still do not understand the full meaning of it.

Since $\int_{-\infty}^{\infty} (\log^+|S(x)|/(1 + x^2))\,dx < \infty$ and $S(z)$ is of exponential type, the set of zeros $\{x_n\}$ of S, on which the extremal measure μ corresponding to S is supported, has a distribution governed by *Levinson's theorem* (Chapter III; here the version in §H.2 suffices). Because

$$\frac{\log|S(iy)|}{|y|} \longrightarrow A \quad \text{for} \quad y \longrightarrow \pm\infty,$$

we see by that theorem that

$$\frac{\text{number of } x_n \text{ in } [0, t]}{t} \longrightarrow \frac{A}{\pi} \quad \text{as} \quad t \to \infty$$

and

$$\frac{\text{number of } x_n \text{ in } [-t, 0]}{t} \longrightarrow \frac{A}{\pi} \quad \text{as} \quad t \to \infty.$$

The zeros of $S(z)$ are distributed roughly (very *roughly*!) *like the points*

$$\frac{\pi}{A} n, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots.$$

(We shall see towards the end of Chapter IX that a certain refinement of this description is possible; we cannot, however obtain much more information about the actual *position* of the points x_n.)

De Branges' result is an *existence theorem.* It says that, if W is a weight of the kind considered in this § such that the $e^{i\lambda x}$, $-A \leqslant \lambda \leqslant A$, are *not* $\| \quad \|_W$-dense in $\mathscr{C}_W(\mathbb{R})$, *then* there *exists* an entire function $\Phi(z)$ of exponential type A with

$$\frac{\log|\Phi(iy)|}{|y|} \longrightarrow A \quad \text{for} \quad y \to \pm\infty, \qquad \int_{-\infty}^{\infty} \frac{\log^+|\Phi(x)|}{1+x^2}\,dx < \infty,$$

and $|\Phi(x_n)| \geqslant W(x_n)$ on a set of points x_n with $x_n \sim (\pi/A)n$ for $n \to \pm\infty$.

It suffices to take $\Phi(x) = S'(x)$ with one of the functions $S(z)$ furnished by the theorem. (There *will be* such a function S because here Σ is not reduced to $\{0\}$, and *will have* extreme points by the Krein–Millman theorem!) If $\{x_n\}$ is the set of zeros of S, we have

$$\sum_n \frac{W(x_n)}{|S'(x_n)|} \;=\; \int_{-\infty}^{\infty} |d\mu(x)| \;=\; 1,$$

so $|S'(x_n)| \geqslant W(x_n)$. Let us verify that

$$\int_{-\infty}^{\infty} \frac{\log^+|S'(x)|}{1+x^2}\,dx \;<\; \infty.$$

Our function $S(z)$ is of exponential type; *therefore, so is* $S'(z)$. The desired relation will hence follow in now familiar fashion via Fubini's theorem and §E of Chapter III from the inequality

$$\int_{-\infty}^{\infty} \frac{\log^+|S'(x+i)|}{1+x^2}\,dx \;<\; \infty,$$

which we proceed to establish (cf. the hall of mirrors argument at the end of §E.4).

Since $S(z)$ is free of zeros in $\Im z > 0$, we have there, by §G.1 of Chapter III,

$$\begin{aligned}\log|S(z)| \;&=\; A\Im z + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log|S(t)|}{|z-t|^2}\,dt \\ &=\; A\Im z + \frac{1}{\pi}\int_{-\infty}^{\infty} \Im\left(\frac{1}{t-z}\right)\log|S(t)|\,dt.\end{aligned}$$

For the same reason one can define an analytic function $\log S(z)$ in $\Im z > 0$. Using the previous relation together with the Cauchy–Riemann equations we thus find that

$$\frac{S'(z)}{S(z)} = \frac{\mathrm{d}\log S(z)}{\mathrm{d}z} = \left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right)\log|S(z)|$$

$$= -\mathrm{i}A - \frac{\mathrm{i}}{\pi}\int_{-\infty}^{\infty}\frac{\log|S(t)|}{(z-t)^2}\mathrm{d}t, \quad \Im z > 0,$$

whence, taking $z = x + \mathrm{i}$,

$$\left|\frac{S'(x+\mathrm{i})}{S(x+\mathrm{i})}\right| \leqslant A + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|\log|S(t)||}{(x-t)^2+1}\mathrm{d}t.$$

Here, since

$$\int_{-\infty}^{\infty}\frac{\log^+|S(t)|}{t^2+1}\mathrm{d}t < \infty,$$

we of course have

$$\int_{-\infty}^{\infty}\frac{\log^-|S(t)|}{t^2+1}\mathrm{d}t < \infty,$$

(Chapter III, §G.2) so

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|\log|S(t)||}{1+t^2}\mathrm{d}t = \text{say } C,$$

a finite quantity. By §B.2 we also have

$$\left|\frac{t-\mathrm{i}}{t-\mathrm{i}-x}\right|^2 \leqslant (|x|+2)^2, \quad t\in\mathbb{R},$$

so

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|\log|S(t)||}{(t-x)^2+1}\mathrm{d}t \leqslant C(|x|+2)^2$$

and thence, by the previous relation,

$$\frac{|S'(x+\mathrm{i})|}{|S(x+\mathrm{i})|} \leqslant A + C(|x|+2)^2.$$

This means, however, that

$$\log|S'(x+\mathrm{i})| \leqslant \log(A + C(|x|+2)^2) + \log|S(x+\mathrm{i})|,$$

from which

$$\int_{-\infty}^{\infty}\frac{\log^+|S'(x+\mathrm{i})|}{x^2+1}\mathrm{d}t \leqslant \int_{-\infty}^{\infty}\frac{\log^+|S(x+\mathrm{i})|}{1+x^2}\mathrm{d}x$$

$$+ \int_{-\infty}^{\infty}\frac{\log^+(A + C(|x|+2)^2)}{x^2+1}\mathrm{d}x.$$

Both integrals on the right are *finite*, however, the *first* because

$$\int_{-\infty}^{\infty} \frac{\log^+ |S(x)|}{1+x^2} \mathrm{d}x \;<\; \infty,$$

and the *second* by inspection. Therefore

$$\int_{-\infty}^{\infty} \frac{\log^+ |S'(x+\mathrm{i})|}{1+x^2} \mathrm{d}x \;<\; \infty,$$

which is what we needed to show.

We still have to check that

$$\frac{\log |S'(\mathrm{i}y)|}{|y|} \longrightarrow A \quad \text{for} \quad y \longrightarrow \pm\infty.$$

There are several ways of doing this; one goes as follows. Since the limit relation in question is *true* for S, we have, for each $\varepsilon > 0$,

$$|S(z)| \;\leqslant\; M_\varepsilon \exp(A|\Im z| + \varepsilon|z|)$$

(see discussion at end of §B.2). Using Cauchy's formula (for the derivative) with circles of radius 1 centered on the imaginary axis, we see from this relation that

$$|S'(\mathrm{i}y)| \;\leqslant\; \text{const.}\mathrm{e}^{(A+\varepsilon)|y|},$$

so, since $\varepsilon > 0$ is arbitrary,

$$\limsup_{y\to\pm\infty} \frac{\log|S'(\mathrm{i}y)|}{|y|} \;\leqslant\; A.$$

However, $S(\mathrm{i}y) = S(0) + \mathrm{i}\int_0^y S'(\mathrm{i}\eta)\,\mathrm{d}\eta$. Therefore the above limit superior along either direction of the imaginary axis must *be* A, otherwise $\log|S(\mathrm{i}y)|/|y|$ could not tend to A as $y \to \pm\infty$. By a remark at the end of §G.1, Chapter III, it will follow from this fact that the ratio $\log|S'(\mathrm{i}y)|/|y|$ actually *tends* to A as $y \to \pm\infty$, if we can verify that $S'(z)$ *has only real zeros.*

To see this, write the Hadamard factorization (Chapter III, §A) for S:

$$S(z) \;=\; A\mathrm{e}^{cz} \prod_n \left(1 - \frac{z}{x_n}\right) \mathrm{e}^{z/x_n}.$$

(We are assuming that none of the zeros x_n of S is equal to 0; if one of them is, a slight modification in this formula is necessary.) Here, as we know, all the x_n are real, therefore

$$\left|\frac{S(\mathrm{i}y)}{S(-\mathrm{i}y)}\right| \;=\; \mathrm{e}^{-2y\Im c}.$$

Since $\log|S(\mathrm{i}y)|/y$ and $\log|S(-\mathrm{i}y)|/y$ both tend to the same limit, A, as $y \to \infty$, we must have $\Im c = 0$, i.e., c is *real*. Logarithmic differentiation of the above Hadamard product now yields

$$\frac{S'(z)}{S(z)} = c + \sum_n \left(\frac{1}{z - x_n} + \frac{1}{x_n} \right),$$

whence

$$\Im\left(\frac{S'(z)}{S(z)}\right) = -\sum_n \frac{\Im z}{|z - x_n|^2}.$$

The expression on the right is < 0 for $\Im z > 0$ and > 0 for $\Im z < 0$; $S'(z)$ can hence *not vanish* in *either of those half planes*. This argument (which goes back to Gauss, by the way), shows that *all the zeros of* $S'(z)$ *must be real*, as required.

We have now finished showing that the function $\Phi(z) = S'(z)$ has all the properties claimed for it. As an observation of general interest, let us just mention one more fact: *the zeros of* $S'(z)$ *are simple and lie between the zeros* x_n *of* $S(z)$. To see that, differentiate the above formula for $S'(z)/S(z)$ one more time, getting

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{S'(z)}{S(z)}\right) = -\sum_n \frac{1}{(z - x_n)^2}.$$

From this it is clear that $S'(x)/S(x)$ decreases strictly from ∞ to $-\infty$ on each open interval with endpoints at two successive points x_n, and hence *vanishes precisely once therein*. $S'(z)$ therefore has exactly *one* zero in each such interval, and, since all its zeros are *real*, *no others*.

This property implies that the (real) zeros of $S'(z)$ *have the same asymptotic distributions as the* x_n. From that it is easy to obtain another proof of the limit relation

$$\frac{\log|S'(\mathrm{i}y)|}{|y|} \longrightarrow A, \quad y \longrightarrow \pm\infty.$$

Just use the Hadamard factorization of $S'(z)$ to write $\log|S'(\mathrm{i}y)|$ as a *Stieltjes integral*, then perform an integration by parts in the latter. The desired result follows without difficulty (see a similar computation in §H.3, Chapter III).

Let us summarize. If, for a weight $W(x)$, the $\mathrm{e}^{\mathrm{i}\lambda x}$, $-A \leqslant \lambda \leqslant A$, are *not* $\|\ \|_W$-dense in $\mathscr{C}_W(\mathbb{R})$, Louis de Branges' theorem furnishes entire functions of *precise exponential type* A having *convergent* $\log^+$ *integrals* which are at the same time *large* ($\geqslant W$ in absolute value) *fairly often*, namely on a set of points x_n with $x_n \sim (\pi/A)n$ for $n \to \pm\infty$. These points

x_n are of course located in the set E where $W(x) < \infty$; the theorem, unfortunately, does not provide much more information about their position, even though some refinement in the description of their asymptotic distribution is possible (Chapter IX). One would like to know more about the location of the x_n.

4. Scholium. Krein's functions

Entire functions whose reciprocals have partial fraction decompositions like the one for $1/S(z)$ figuring in the proof of de Branges' theorem arise in the study of various questions. They were investigated by M.G. Krein, in connection, I believe, with the inverse Sturm–Liouville problem. We give some results about such functions here, limiting the discussion to those with *real zeros*. More material on Krein's work (he allowed complex zeros) can be found, together with references, in Levin's book.

Theorem. *Let $S(z)$ be entire, of exponential type, and have only the real simple zeros $\{x_n\}$. Suppose that $S(z) \to \infty$ as $z \to \infty$ along each of four rays*

$$\arg z = \alpha_k,$$

with

$$0 < \alpha_1 < \frac{\pi}{2} < \alpha_2 < \pi < \alpha_3 < \frac{3\pi}{2} < \alpha_4 < 2\pi,$$

and that also

$$\sum_n |1/S'(x_n)| < \infty.$$

Then

$$\frac{1}{S(z)} = \sum_n \frac{1}{(z-x_n)S'(x_n)}.$$

Proof. The function

$$L(z) = \sum_n \frac{S(z)}{(z-x_n)S'(x_n)}$$

is entire, since $S(x_n) = 0$ for each n and $\sum_n |1/S'(x_n)| < \infty$.

I claim that $L(z)$ *is of exponential type*. Clearly,

$$|L(z)| \leqslant \text{const.} \frac{|S(z)|}{|\Im z|},$$

so the growth of $L(z)$ is dominated by that of $S(z)$ outside the strip $|\Im z| \leqslant 1$. For $|\Im z| \leqslant 1$, one may use the following trick. The function $\sqrt{|L(z)|}$ is *subharmonic*, therefore

$$\sqrt{|L(z)|} \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}\sqrt{|L(z+2e^{i\theta})|}\,d\theta.$$

Substituting the preceding inequality into the integral on the right, we obtain for it a bound of the form $\text{const.}e^{K|z|/2}\int_{-\pi}^{\pi}d\theta/\sqrt{|\Im z + 2\sin\theta|}$, and this is clearly $\leqslant \text{const.}e^{K|z|/2}$ for $|\Im z| \leqslant 1$. We see in this way that $|L(z)| \leqslant Ce^{K|z|}$ for all z.

We have $L(x_k) = 1$ at each x_k. Therefore

$$\frac{L(z)-1}{S(z)} = \sum_n \frac{1}{(z-x_n)S'(x_n)} - \frac{1}{S(z)}$$

is *entire*; *as the ratio of two entire functions of exponential type it is also of exponential type by Lindelöf's theorem.* (See *third* theorem of §B, Chapter III.)

Since $\sum_n |1/S'(x_n)| < \infty$,

$$\sum_n \frac{1}{(z-x_n)S'(x_n)} \longrightarrow 0$$

as $z \to \infty$ along each of the rays $\arg z = \alpha_k$, $k = 1, 2, 3$ and 4, and by hypothesis $1/S(z) \to 0$ for $z \to \infty$ along each of those rays. Therefore $(L(z)-1)/S(z)$ is certainly *bounded* on each of those rays, so, since it is *entire and of exponential type*, it is *bounded in each of the four sectors separated by them* (and having opening $< 180°$) according to the *second Phragmén–Lindelöf theorem* of §C, Chapter III. The entire function $(L(z)-1)/S(z)$ is thus *bounded in* $\mathbb{C}$, hence *equal to a constant*, by Liouville's theorem. Since, as we have seen, it tends to *zero* for z tending to ∞ along certain rays, *the constant must be zero.* Hence

$$\sum_n \frac{1}{(z-x_n)S'(x_n)} - \frac{1}{S(z)} \equiv 0, \qquad \text{Q.E.D.}$$

Remark. The hypothesis of the theorem just proved is very ungainly, and one would *like* to be able to affirm the following more general result:

Let $S(z)$, of exponential type, have only the real simple zeros x_n, let $\sum_n |1/S'(x_n)| < \infty$, and suppose that $S(iy) \to \infty$ for $y \to \pm\infty$. Then

$$\frac{1}{S(z)} = \sum_n \frac{1}{(z-x_n)S'(x_n)}.$$

One can waste much time attempting to prove this statement, all in vain, because it is false! In order to lay this ghost for good, here is a *counter example.*

Take

$$S(z) \;=\; \prod_1^\infty \left(1 - \frac{z}{2^n}\right) e^{z/2^n};$$

since $\sum_1^\infty 2^{-n} < \infty$, $S(z)$ is of exponential type. One readily computes $|S'(2^n)|$ by the method used in §C, and finds that

$$|S'(2^n)| \;\sim\; \frac{1}{e} 2^{(n(n-3)/2)} e^{2^n} (S(1))^2$$

for $n \to \infty$; we thus *certainly have*

$$\sum_1^\infty |1/S'(2^n)| < \infty.$$

It is also true that

$$|S(iy)|^2 \;=\; \prod_1^\infty \left(1 + \frac{y^2}{4^n}\right) \;\longrightarrow\; \infty$$

for $y \to \pm\infty$. However, $\prod_1^\infty (1 + |z|/2^n) \leqslant e^{o(|z|)}$ for $z \to \infty$, again by convergence of $\sum_1^\infty 2^{-n}$ (see calculations in §A, Chapter III!). So, for x *real and negative,*

$$S(x) \;=\; e^x \prod_1^\infty \left(1 + \frac{|x|}{2^n}\right) \;\leqslant\; e^{-|x| + o(|x|)},$$

and, for $x \to -\infty$, $1/S(x)$ tends to ∞ like an exponential (!). Therefore $1/S(x)$ certainly *cannot* equal

$$\sum_1^\infty \frac{1}{(x - 2^n) S'(2^n)}$$

which *tends to zero* as $x \to -\infty$.

Theorem (Krein). *Let an entire function $S(z)$ have only the real simple zeros x_n; suppose that $\sum_n |1/S'(x_n)| < \infty$ and that*

$$\frac{1}{S(z)} \;=\; \sum_n \frac{1}{(z - x_n) S'(x_n)}.$$

Then $S(z)$ is of exponential type, *and*

$$\int_{-\infty}^{\infty} \frac{\log^+ |S(x)|}{1 + x^2} dx \;<\; \infty.$$

Remark. In particular, for functions $S(z)$ satisfying the hypothesis of the *previous* theorem, we have

$$\int_{-\infty}^{\infty} \frac{\log^+ |S(x)|}{1+x^2} \mathrm{d}x \ < \ \infty .$$

The reader who wants only *this* result may skip *all but the last paragraph* of the following demonstration.

Proof of theorem. Without loss of generality, $\sum_n |1/S'(x_n)| = 1$, whence, by the assumed representation for $1/S(z)$,

$$\left|\frac{1}{S(z)}\right| \ \leqslant \ \frac{1}{|\Im z|}.$$

Given any $h > 0$, the reciprocal $1/S(z)$ is thus *bounded and non-zero* in each of the half-planes $\{\Im z \geqslant h\}$, $\{\Im z \leqslant -h\}$, as well as being *analytic* in slightly larger open half planes containing them. The representation of §G.1, Chapter III, therefore applies in each of those half planes, and we find that in fact

$$\int_{-\infty}^{\infty} \frac{\log^- |1/S(t+ih)|}{1+t^2} \mathrm{d}t \ < \ \infty,$$

and that

$$\log\left|\frac{1}{S(z)}\right| \ = \ -A_h(\Im z - h) + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(\Im z - h)\log|1/S(t+ih)|}{|z-t-ih|^2}\mathrm{d}t$$

for $\Im z > h$, while $\int_{-\infty}^{\infty}(\log^- |1/S(t-ih)|/(1+t^2))\,\mathrm{d}t < \infty$ and

$$\log\left|\frac{1}{S(z)}\right| \ = \ -B_h(|\Im z| - h) + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(|\Im z| - h)\log|1/S(t-ih)|}{|z-t+ih|^2}\mathrm{d}t$$

when $\Im z < -h$.

Here, A_h and B_h are constants which, *a priori*, depend on h. *In fact, they do not*, because, by the remark at the end of §G.1, Chapter III, $\lim_{y\to\infty}(1/y)\log|1/S(iy)|$ *exists* and equals $-A_h$, with a similar relation involving B_h for $y \to -\infty$. All the numbers $-A_h$ for $h > 0$ are thus equal to the limit just mentioned, say to $-A$, and all the B_h are similarly equal to some number B.

For each $h > 0$ we thus have, for $\Im z > h$,

$$\begin{aligned}\log|S(z)| \ &= \ A(\Im z - h) + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(\Im z - h)\log|S(t+ih)|}{|z-t-ih|^2}\mathrm{d}t \\ &\leqslant \ A(\Im z - h) + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(\Im z - h)\log^+|S(t+ih)|}{|z-t-ih|^2}\mathrm{d}t.\end{aligned}$$

Similar relations involving B, which we do not bother to write down, hold for $\Im z < -h$. Let us fix some value of h, say $h = 1$. We have

$$\int_{-\infty}^{\infty} \frac{\log^+ |S(t \pm \mathrm{i})|}{1+t^2} \mathrm{d}t \;=\; \int_{-\infty}^{\infty} \frac{\log^- |1/S(t \pm \mathrm{i})|}{1+t^2} \mathrm{d}t,$$

both of which are *finite*. Knowing this we can, by using the two inequalities for $\log|S(z)|$ involving integrals with $\log^+$, in $\{\Im z > 1\}$ and in $\{\Im z < -1\}$, verify immediately that $S(z)$ is *of exponential growth at most* in *each* of the two sectors $\delta < \arg z < \pi - \delta$, $\pi + \delta < \arg z < 2\pi - \delta$, $\delta > 0$ being arbitrary. This verification proceeds in the same way as the corresponding one made while proving Akhiezer's *second* theorem, §B.2.

It remains to show that $S(z)$ is of at most exponential growth in each of the two sectors $|\arg z| < \delta$, $|\arg z - \pi| < \delta$. This can be done by choosing $\delta < \pi/4$ and then following the Phragmén–Lindelöf procedure used at the end of the proof of Akhiezer's second theorem, *provided* that we know that $|S(z)| \leqslant \exp(\mathrm{O}(|z|^2))$ for large $|z|$ in *each* of those two sectors. This property we now proceed to establish.

The method followed here is like that used to discuss $L(z)$ in the proof of the previous theorem. For $h > 0$, we have $|S(t + \mathrm{i}h)| \geqslant h$, so

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log^- |S(t+\mathrm{i}h)|}{1+t^2} \mathrm{d}t \;\leqslant\; \log^+ \frac{1}{h}.$$

At the same time,

$$A + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log |S(t+\mathrm{i}h)|}{t^2+1} \mathrm{d}t \;=\; \log|S(\mathrm{i}+\mathrm{i}h)|,$$

and, since $\mathrm{i} + \mathrm{i}h$ lies on the positive imaginary axis, we *already know* that $\log|S(\mathrm{i}+\mathrm{i}h)| \leqslant C(h+1)$ for $h > 0$, for the positive imaginary axis lies in the sector $\delta < \arg z < \pi - \delta$ where (at most) exponential growth of $S(z)$ *is* clear. Because $\log^+ |S(t+\mathrm{i}h)| = \log|S(t+\mathrm{i}h)| + \log^- |S(t+\mathrm{i}h)|$, the above relations yield, for $h > 0$.

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log^+ |S(t+\mathrm{i}h)|}{t^2+1} \mathrm{d}t \;\leqslant\; -A + C(h+1) + \log^+ \frac{1}{h}.$$

In like manner,

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log^+ |S(t-\mathrm{i}h)|}{t^2+1} \mathrm{d}t \;\leqslant\; -B + C'(h+1) + \log^+ \frac{1}{h}$$

when $h > 0$.

From these two inequalities we now find, for large (!) R, that

$$\frac{1}{\pi}\int_{-R}^{R}\int_{-\infty}^{\infty} \frac{\log^+ |S(t+\mathrm{i}y)|}{1+t^2} \mathrm{d}t\,\mathrm{d}y \;\leqslant\; \text{const.} R^2$$

(note that $\int_{-R}^{R}\log^+(1/|y|)\,\mathrm{d}y \leqslant \text{const!}$). This inequality yields, in turn

$$\int_{-R}^{R}\int_{-R}^{R}\log^+|S(z)|\,\mathrm{d}x\,\mathrm{d}y \;\leqslant\; \text{const.}R^4$$

for large R.

Let z_0 be given. Since $\log^+|S(z)|$ is *subharmonic*,

$$\log^+|S(z_0)| \;\leqslant\; \frac{1}{\pi|z_0|^2}\iint\limits_{|z-z_0|\leqslant|z_0|}\log^+|S(z)|\,\mathrm{d}x\,\mathrm{d}y$$

$$\leqslant\; \frac{4}{\pi R^2}\int_{-R}^{R}\int_{-R}^{R}\log^+|S(z)|\,\mathrm{d}x\,\mathrm{d}y,$$

where $R = 2|z_0|$. By what we have just seen, the expression on the *right* is $\leqslant (1/R^2)\mathrm{O}(R^4) = \mathrm{O}(|z_0|^2)$ for large values of $|z_0|$, i.e., $|S(z_0)| \leqslant \exp(\mathrm{O}(|z_0|^2))$ when $|z_0|$ is large. This is what we wanted to show; as explained above, it implies that $S(z)$ is actually of exponential growth in the two sectors $|\arg z| < \delta$, $|\arg z - \pi| < \delta$, and hence, finally, that the entire function $S(z)$ *is of exponential type.*

We still have to show that

$$\int_{-\infty}^{\infty}\frac{\log^+|S(x)|}{1+x^2}\,\mathrm{d}x \;<\; \infty.$$

That is, however, immediate. In the course of the argument just completed, we had (taking, for instance, $h = 1$) the relation

$$\int_{-\infty}^{\infty}\frac{\log^+|S(t+\mathrm{i})|}{1+t^2}\,\mathrm{d}t \;<\; \infty.$$

Because S is of *exponential type*, the desired inequality follows from this one by §E of Chapter III (applied in the half plane $\Im z < 1$) and Fubini's theorem, in the usual fashion (hall of mirrors). We are done.

Remark. We remind the reader that, since the functions $S(z)$ considered here have no zeros either in $\Im z > 0$ or in $\Im z < 0$, the representation of §G.1, Chapter III holds for them in each of those half planes. That is,

$$\log|S(z)| \;=\; A\Im z + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z\log|S(t)|}{|z-t|^2}\,\mathrm{d}t \quad \text{for} \quad \Im z > 0,$$

and

$$\log|S(z)| \;=\; B|\Im z| + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|\Im z|\log|S(t)|}{|z-t|^2}\,\mathrm{d}t \quad \text{for} \quad \Im z < 0.$$

Problem 9

Let $x_{-n} = -x_n$, let $\sum_1^\infty 1/x_n^2 < \infty$, and suppose that $\sum'^{\infty}_{-\infty} |1/S'(x_n)| < \infty$, where

$$S(z) = \prod_1^\infty \left(1 - \frac{z^2}{x_n^2}\right).$$

The x_n are assumed to be *real.*

Show that

$$\frac{1}{S(z)} = \sum_{-\infty}^{\infty}{}' \frac{1}{(z - x_n)S'(x_n)},$$

and hence that $S(z)$ is of *exponential type*, and that

$$\int_{-\infty}^{\infty} \frac{\log^+ |S(x)|}{1 + x^2} \mathrm{d}x < \infty.$$

(Hint: First put $S_R(z) = \prod_{0 < x_n \leqslant R}(1 - z^2/x_n^2)$, and show that one can make $R \to \infty$ in the Lagrange formula

$$1 = \sum_{|x_n| \leqslant R} \frac{S_R(z)}{(z - x_n)S_R'(x_n)}$$

so as to obtain

$$1 = \sum_{-\infty}^{\infty}{}' \frac{S(z)}{(z - x_n)S'(x_n)}.$$

At this point, one may either invoke Krein's theorem, or else look at the Poisson representation of the (negative) harmonic function $\log|1/S(z)|$ in a suitable half-plane $\{\Im z > H\}$, noting that here $|S(z)| \leqslant S(\mathrm{i}|z|)$.)

Problem 10

Let $S(z)$ be entire, of exponential type, and satisfy the rest of the hypothesis of the *first* theorem of this article. That is, S has only the real simple zeros x_n, $\sum_n |1/S'(x_n)| < \infty$, and $S(z) \to \infty$ for z tending to ∞ along *four rays, one* in the *interior* of *each* of the *four quadrants.* Suppose also that the *two limits* (which *exist* by the above discussion) of $\log|S(\mathrm{i}y)|/|y|$, for $y \to \infty$ and for $y \to -\infty$, *are equal,* say to $A > 0$. The purpose of this problem is to prove that

$$\sum_n \frac{\mathrm{e}^{\mathrm{i}\lambda x_n}}{S'(x_n)} = 0 \quad \text{for} \quad -A \leqslant \lambda \leqslant A.$$

(a) If

$$F(z) = \sum_n \frac{S(z)\mathrm{e}^{\mathrm{i}\lambda x_n}}{(z - x_n)S'(x_n)},$$

show that $F(z)$ is entire and of exponential type A, and that

$$\int_{-\infty}^{\infty} \frac{\log^+|F(x)|}{1+x^2}\,dx \;<\; \infty.$$

(Hint: Refer to the trick with $L(z)$, pp. 203–4.)

(b) Show that $Q(z) = (F(z) - e^{i\lambda z})/S(z)$ is *entire and of exponential type*, and that

$$\int_{-\infty}^{\infty} \frac{\log^+|Q(x)|}{1+x^2}\,dx \;<\; \infty.$$

(c) If $-A < \lambda < A$ and $Q(z)$ is the function constructed in (b), show that $Q(z) \equiv 0$. (Hint: First show that $Q(iy) \to 0$ for $y \to \pm\infty$ when $-A < \lambda < A$. Use this fact and the result proved in (b) to show that

$$\limsup_{R\to\infty} \frac{\log|Q(Re^{i\varphi})|}{R} \;\leqslant\; 0$$

if $\varphi \neq 0$ or π. Then use boundedness of Q on the imaginary axis and apply the Poisson representation for $\log|Q(z)|$ (or else Phragmén–Lindelöf) in the *right* and *left* half-planes.)

(d) $\sum_{-\infty}^{\infty}(e^{i\lambda x_n}/S'(x_n)) = 0$ for $-A \leqslant \lambda \leqslant A$. (Hint: Show this for $-A < \lambda < A$ and argue by continuity. Here, one may observe that

$$\sum_{-\infty}^{\infty} \frac{e^{i\lambda x_n}}{S'(x_n)} \;=\; \lim_{y\to\infty}\left(iy\cdot\sum_{-\infty}^{\infty}\frac{e^{i\lambda x_n}}{(iy-x_n)S'(x_n)}\right),$$

and use the result of (c).)

G. Weighted approximation with L_p norms

The results established in §§A–E apply to *uniform* weighted approximation, i.e., to approximation using the norm

$$\|\varphi\|_W \;=\; \sup_{t\in\mathbb{R}}\left|\frac{\varphi(t)}{W(t)}\right|.$$

One may ask what happens if, instead of *this* norm, we use a weighted L_p one, viz.

$$\|\varphi\|_{W,p} = \sqrt[p]{\int_{-\infty}^{\infty}\left|\frac{\varphi(x)}{W(x)}\right|^p dx};$$

here, p is some number $\geqslant 1$. The answer is that *all the results except for de Branges' theorem* (§F) *carry over with hardly any change, not even in the proofs.* Here, we of course have to assume that, for $x \to \pm\infty$, $W(x) \to \infty$ rapidly enough to make

$$\int_{-\infty}^{\infty}\left(\frac{1}{W(x)}\right)^p dx \;<\; \infty.$$

(some weakening of this restriction is possible; compare with the discussion in §E.4.)

It is enough to merely peruse the proofs of Mergelian's and Akhiezer's theorems, whether for approximation by polynomials or by functions in $\mathscr{E}_A$, to see that they are applicable *as is* with the norms $\| \quad \|_{W,p}$. Here the functions $\Omega(z)$, $\Omega_A(z)$, $W_*(z)$ and $W_A(z)$ have evidently to be defined using the appropriate norm $\| \quad \|_{W,p}$ instead of $\| \quad \|_W$. And it is no longer necessarily true that $W_*(x) \leqslant W(x)$.

Verification of all this is left to the reader. In general, in the *kind* of approximation problem considered here (that of the *density* of a certain simple class of functions in the whole space), *it makes very little difference which L_p norm is chosen.* If the proofs vary in difficulty, they are hardest for the L_1 norm or for the uniform one. Here, the *continuous functions* (with the uniform norm) play the rôle of '$\lim_{p\to\infty} L_p$', and *not* L_∞, which is not even separable.

H. Comparison of weighted approximation by polynomials and by functions in $\mathscr{E}_A$

We now turn to the examination of the *relations* between the $\| \quad \|_W$-closed subspaces of $\mathscr{C}_W(\mathbb{R})$ generated by the *polynomials* and by the *linear combinations of the* $e^{i\lambda x}$, $-A \leqslant \lambda \leqslant A$, for $A > 0$.

► *In order to consider the former subspace, it is of course necessary to assume that*

$$x^n/W(x) \longrightarrow 0 \quad for\ x \longrightarrow \pm\infty$$

when $n \geqslant 0$. This we do throughout the present §.

We also use systematically the following

Notation. $\mathscr{C}_W(0)$ is the $\| \quad \|_W$-closure of the set of polynomials in $\mathscr{C}_W(\mathbb{R})$. For $A > 0$, $\mathscr{C}_W(A)$ is the $\| \quad \|_W$-closure of the set of finite linear combinations of the $e^{i\lambda x}$; $-A \leqslant \lambda \leqslant A$. (*Equivalently, $\mathscr{C}_W(A)$ is the $\| \quad \|_W$-closure of $\mathscr{E}_A$*; see §E.1.)

It also turns out to be useful to introduce some intersections:

Definition. For $A \geqslant 0$ (*sic*!),

$$\mathscr{C}_W(A+) \;=\; \bigcap_{A' > A} \mathscr{C}_W(A').$$

In this §, we shall be especially interested in $\mathscr{C}_W(0+)$, the set of functions in $\mathscr{C}_W(\mathbb{R})$ *which can be* $\|\ \|_W$*-approximated by entire functions of arbitrarily small exponential type.*

We clearly have $\mathscr{C}_W(A) \subseteq \mathscr{C}_W(A+)$ for $A > 0$. But also:

Lemma. $\mathscr{C}_W(0) \subseteq \mathscr{C}_W(0+)$.

Proof. We have to show that $\mathscr{C}_W(0) \subseteq \mathscr{C}_W(A)$ for every $A > 0$. *Fix* any such A.

We have $x/W(x) \longrightarrow 0$ for $x \to \pm\infty$. Therefore, for the functions

$$f_h(x) = \frac{e^{i(\lambda+h)x} - e^{i\lambda x}}{h}, \quad h > 0,$$

we have $\|f_h\|_W \leqslant \text{const.}$, $h > 0$, and $f_h(x)/W(x) \longrightarrow 0$ *uniformly* for $h > 0$ as $x \to \pm\infty$. Since $f_h(x) \longrightarrow xe^{i\lambda x}$ u.c.c. in x for $h \to 0$, we thus have $\|f_h(x) - xe^{i\lambda x}\|_W \longrightarrow 0$ as $h \to 0$, and $xe^{i\lambda x} \in \mathscr{C}_W(A)$ if $-A < \lambda < A$.

By iterating this procedure, we find that $x^n e^{i\lambda x} \in \mathscr{C}_W(A)$ for $n = 0, 1, 2, 3, \ldots$ if $-A < \lambda < A$. In particular, then, all the *powers* x^n, $n = 0, 1, 2, \ldots$, belong to $\mathscr{C}_W(A)$, so $\mathscr{C}_W(0) \subseteq \mathscr{C}_W(A)$, as required.

Remark. This *justifies* the notation $\mathscr{C}_W(0)$ for the $\|\ \|_W$-closure of *polynomials* in $\mathscr{C}_W(\mathbb{R})$.

Once we know that $\mathscr{C}_W(0) \subseteq \mathscr{C}_W(0+)$, it is natural to ask whether $\mathscr{C}_W(0) = \mathscr{C}_W(0+)$ for the weights considered in this §, and, if the equality does not hold for *all* such weights, for *which ones* it *is* true. In other words, if a given function can be $\|\ \|_W$-approximated by *entire functions of arbitrarily small exponential type*, can it be $\|\ \|_W$-approximated by *polynomials*? This question, which interested some probabilists around 1960, was studied by Levinson and McKean who used the quadratic norm $\|\ \|_{W,2}$ (§G) instead of $\|\ \|_W$, and, simultaneously and independently, by me, in terms of the uniform norm $\|\ \|_W$. I learned later, around 1967, that *I.O. Khachatrian* had done some of the same work that I had a couple of years before me, in a somewhat different way. He has a paper in the Kharkov University Mathematics and Mechanics Faculty's *Uchonye Zapiski* for 1964, and a short note in the (more accessible) 1962 *Doklady* (vol. 145).

The remainder of this § is concerned with the question of equality of the subspaces $\mathscr{C}_W(0)$ and $\mathscr{C}_W(0+)$. It turns out that *in general* they are *not equal*, but that they *are* equal when the weight $W(x)$ enjoys a certain *regularity*.

1. Characterization of the functions in $\mathscr{C}_W(A+)$

Akhiezer's second theorem (§§B.2 and E.2) generally furnishes only a partial description of the functions in $\mathscr{C}_W(A)$ when that subspace does not coincide with $\mathscr{C}_W(\mathbb{R})$. One important reason for introducing the intersections $\mathscr{C}_W(A+)$ is that we can give a *complete* description of the functions belonging to any one of them which is properly contained in $\mathscr{C}_W(\mathbb{R})$.

Lemma. *Suppose that $f(z)$ is an entire function of exponential type with*

$$|f(z)| \leqslant C_\varepsilon \exp(A|\Im z| + \varepsilon|z|)$$

for each $\varepsilon > 0$. Then, if $\delta > 0$, the Fourier transform

$$F_\delta(\lambda) = \int_{-\infty}^{\infty} e^{-\delta|x|} e^{i\lambda x} f(x)\,dx$$

belongs to $L_1(\mathbb{R})$, and, if $A' > A$,

$$(*) \qquad \int_{|\lambda|>A'} |F_\delta(\lambda)|\,d\lambda \longrightarrow 0 \quad \textit{for} \quad \delta \to 0.$$

Proof. For each $\delta > 0$, $e^{-\delta|x|}f(x)$ is in $L_1(\mathbb{R})$ (choose $\varepsilon < \delta$ in the given condition on $f(x)$), so $F_\delta(\lambda)$ is *continuous* and *therefore* integrable on $[-A', A']$. The whole lemma will thus follow as soon as we prove $(*)$.

Fix $A' > A$, and suppose for the moment that $\delta > 0$ is also fixed. Take an $\varepsilon > 0$ *less than* both $\delta/2$ and $(A'-A)/2$. If $\lambda \geqslant A'$, we then have, for $y = \Im z \geqslant 0$,

$$|e^{-\delta z} e^{i\lambda z} f(z)| \leqslant C_\varepsilon e^{(A-A')y - \delta x + \varepsilon|z|},$$

and, for $x = \Re z \geqslant 0$, this is in turn $< C_\varepsilon e^{-\varepsilon x - \varepsilon y}$.

Let us now apply Cauchy's theorem using the following contour Γ_R:

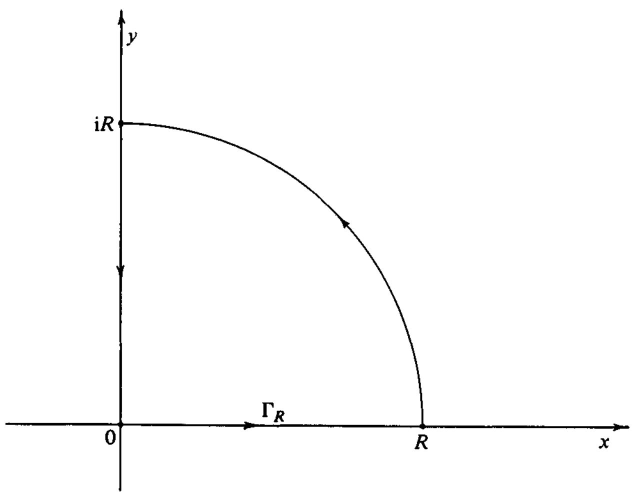

Figure 38

We have $\int_{\Gamma_R} e^{-\delta z}e^{i\lambda z}f(z)dx = 0$. For large R, $|e^{-\delta z}e^{i\lambda z}f(z)|$ is, by the preceding inequality, $< C_\varepsilon e^{-\varepsilon R/\sqrt{2}}$ on the *circular* part of Γ_R. Therefore the portion of our integral taken along this circular part tends to *zero* as $R \to \infty$, and we see that

$$\int_0^\infty e^{-\delta x}e^{i\lambda x}f(x)dx \;=\; i\int_0^\infty e^{-i\delta y}e^{-\lambda y}f(iy)dy.$$

This formula is valid whenever $\lambda \geqslant A' > A$ and $\delta > 0$. By integrating around the following contour

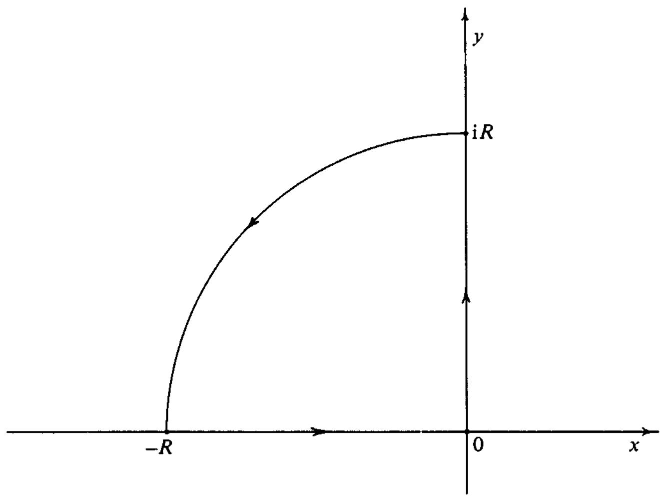

Figure 39

we see in like manner, on making $R \to \infty$, that

$$\int_{-\infty}^0 e^{\delta x}e^{i\lambda x}f(x)dx \;=\; -i\int_0^\infty e^{i\delta y}e^{-\lambda y}f(iy)dy,$$

whenever $\delta > 0$ and $\lambda \geqslant A' > A$. Combining this with the previous formula we get

$$F_\delta(\lambda) \;=\; \int_{-\infty}^\infty e^{-\delta|x|}e^{i\lambda x}f(x)dx \;=\; 2\int_0^\infty e^{-\lambda y}\sin\delta y f(iy)dy;$$

this holds whenever $\lambda \geqslant A' > A$ and $\delta > 0$.

Take now any η, $0 < \eta < (A' - A)/2$, and *fix* it for the following computation. Since, for $y \geqslant 0$, $|f(iy)| \leqslant C_\eta e^{(A+\eta)y}$, the formula just derived yields, for $\lambda \geqslant A'$,

$$|F_\delta(\lambda)| \leqslant 2\int_0^\infty C_\eta e^{(A+\eta-\lambda)y}|\sin\delta y|\,dy.$$

By Schwarz' inequality, the right-hand side is in turn

$$\leqslant 2C_\eta\sqrt{\left(\int_0^\infty e^{-(\lambda-A-\eta)y}\,dy\int_0^\infty e^{-(\lambda-A-\eta)y}\sin^2\delta y\,dy\right)}$$

For the second integral under the radical we have

$$\int_0^\infty e^{-(\lambda-A-\eta)y}\sin^2\delta y\,dy = \tfrac{1}{2}\Re\int_0^\infty e^{-(\lambda-A-\eta)y}(1-e^{2i\delta y})\,dy$$

$$= \frac{2\delta^2}{(\lambda-A-\eta)|\lambda-A-\eta-2i\delta|^2}.$$

Therefore, for $\lambda \geqslant A'$,

$$|F_\delta(\lambda)| \leqslant \frac{2\sqrt{2}C_\eta\delta}{(\lambda-A-\eta)|\lambda-A-\eta-2i\delta|} \leqslant \frac{2\sqrt{2}C_\eta\delta}{(\lambda-A-\eta)^2}.$$

And

$$\int_{A'}^\infty |F_\delta(\lambda)|\,d\lambda \leqslant \frac{2\sqrt{2}C_\eta\delta}{A'-A-\eta} \leqslant \frac{2\sqrt{2}C_\eta\delta}{\eta}.$$

We see now that

$$\int_{A'}^\infty |F_\delta(\lambda)|\,d\lambda \longrightarrow 0 \quad \text{for} \quad \delta\to 0.$$

Working with contours in the *lower* half plane, we see in the same way that

$$\int_{-\infty}^{-A'} |F_\delta(\lambda)|\,d\lambda \longrightarrow 0 \quad \text{as} \quad \delta\to 0.$$

We have proved $(*)$, and are done.

Theorem (de Branges). *Let $f(z)$ be entire, and suppose that*

$$|f(z)| \leqslant C_\varepsilon e^{A|\Im z|+\varepsilon|z|}$$

for each $\varepsilon > 0$. Then,

$$\text{if} \quad f\in\mathscr{C}_W(\mathbb{R}), \quad f\in\mathscr{C}_W(A+).$$

Proof. We have to show that, if $f\in\mathscr{C}_W(\mathbb{R})$, then in fact $f\in\mathscr{C}_W(A')$ for each $A' > A$; this we do by duality.

Fix any $A' > A$. According to the Hahn–Banach theorem it is enough to show that if L is any bounded linear functional on functions of the form $\varphi(t)/W(t)$ with $\varphi\in\mathscr{C}_W(\mathbb{R})$, and if

$$L\left(\frac{e^{i\lambda t}}{W(t)}\right) = 0 \quad \text{for} \quad -A' \leqslant \lambda \leqslant A',$$

then

$$L\left(\frac{f(t)}{W(t)}\right) = 0.$$

To see this, observe in the first place that $\| f(t) - e^{-\delta|t|}f(t)\|_W \longrightarrow 0$ for $\delta \to 0$, so surely

$$L\left(\frac{f(t)}{W(t)}\right) = \lim_{\delta\to 0} L\left(\frac{e^{-\delta|t|}f(t)}{W(t)}\right).$$

Our task thus reduces to showing that the limit on the right is zero; this we do with the help of the above lemma.

Writing, as in the lemma,

$$F_\delta(\lambda) = \int_{-\infty}^{\infty} e^{-\delta|x|}e^{i\lambda x}f(x)\,dx,$$

we have $F_\delta\in L_1(\mathbb{R})$ as we have seen. Hence, by the Fourier inversion formula,

$$e^{-\delta|t|}f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda t}F_\delta(\lambda)\,d\lambda.$$

In order to bring the functional L into play, we approximate the integral on the right by *finite sums*.

Put

$$S_N(t) = \frac{1}{2\pi}\sum_{k=-N^2}^{N^2-1} e^{-i(k/N)t}\int_{k/N}^{(k+1)/N} F_\delta(\lambda)\,d\lambda;$$

since $F_\delta\in L_1(\mathbb{R})$,

$$S_N(t) \longrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda t}F_\delta(\lambda)\,d\lambda = e^{-\delta|t|}f(t)$$

u.c.c. in t as $N\to\infty$, and, at the same time, $|S_N(t)| \leqslant \|F_\delta\|_1$ on $\mathbb{R}$ for all N. Therefore, since $W(t)\to\infty$ for $t\to\pm\infty$,

$$\|e^{-\delta|t|}f(t) - S_N(t)\|_W \xrightarrow[N]{} 0,$$

so, by the *boundedness* of L,

$$L\left(\frac{e^{-\delta|t|}f(t)}{W(t)}\right) = \lim_{N\to\infty} L\left(\frac{S_N(t)}{W(t)}\right).$$

However, $\|e^{-i\lambda t} - e^{-i\lambda' t}\|_W \longrightarrow 0$ when $|\lambda - \lambda'| \to 0$, so $L(e^{-i\lambda t}/W(t))$ is a *continuous function* of λ on $\mathbb{R}$ as well as being *bounded* there (note that $|e^{i\lambda t}| = 1$!). Hence, since $F_\delta(\lambda) \in L_1(\mathbb{R})$, we have

$$L\left(\frac{S_N(t)}{W(t)}\right) = \frac{1}{2\pi}\sum_{k=-N^2}^{N^2-1} L\left(\frac{e^{-i(k/N)t}}{W(t)}\right)\int_{k/N}^{(k+1)/N} F_\delta(\lambda)\,d\lambda$$
$$\longrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} L\left(\frac{e^{-i\lambda t}}{W(t)}\right)F_\delta(\lambda)\,d\lambda$$

for $N \to \infty$. In view of the previous relation, we thus get

$$L\left(\frac{e^{-\delta|t|}f(t)}{W(t)}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} L\left(\frac{e^{-i\lambda t}}{W(t)}\right)F_\delta(\lambda)\,d\lambda.$$

We are assuming that

$$L\left(\frac{e^{-i\lambda t}}{W(t)}\right) = 0 \quad \text{for} \quad -A' \leqslant \lambda \leqslant A'.$$

The integral on the right thus reduces to

$$\frac{1}{2\pi}\int_{|\lambda|\geqslant A'} L\left(\frac{e^{-i\lambda t}}{W(t)}\right)F_\delta(\lambda)\,d\lambda.$$

Here, as already noted,

$$|L(e^{-i\lambda t}/W(t))| \leqslant \text{const.}, \quad \lambda \in \mathbb{R},$$

so the last integral is bounded in absolute value by

$$\text{const.}\int_{|\lambda|\geqslant A'} |F_\delta(\lambda)|\,d\lambda.$$

This, however, tends to 0 by the lemma as $\delta \to 0$. We see that

$$L\left(\frac{e^{-\delta|t|}f(t)}{W(t)}\right) \longrightarrow 0$$

for $\delta \to 0$, which is what was needed. The theorem is proved.

Remark. Since we are not supposing anything about *continuity* of $W(t)$, we are *not* in general permitted to write

$$L\left(\frac{f(t)}{W(t)}\right) \quad \text{as} \quad \int_{-\infty}^{\infty} \frac{f(t)}{W(t)}\,d\mu(t)$$

with a finite (complex-valued) Radon *measure* on $\mathbb{R}$. This makes the above proof *appear* a little more involved than in the case where use of such a measure is allowed. The difference is only in the *appearance*, however. The argument with a measure *is the same*, and only *looks* simpler.

Thanks to the above result, we can strengthen Akhiezer's second theorem so as to arrive at the following *characterization of the subspaces* $\mathscr{C}_W(A+)$. Recall the definition (§E.2):

$$W_A(z) = \sup\{|f(z)|: \ f \in \mathscr{E}_A \text{ and } \|f\|_W \leqslant 1\}.$$

Then we have the

Theorem. *Let* $A \geqslant 0$. *Either*

$$\int_{-\infty}^{\infty} \frac{\log W_{A'}(x)}{1+x^2}\,dx = \infty$$

for every $A' > A$, *in which case* $\mathscr{C}_W(A+)$ *is equal to* $\mathscr{C}_W(\mathbb{R})$, *or else* $\mathscr{C}_W(A+)$ *consists* precisely *of all the entire functions* f *such that* $f(x)/W(x) \longrightarrow 0$ *for* $x \to \pm\infty$ *and*

$$|f(z)| \leqslant C_\varepsilon e^{A|\Im z| + \varepsilon|z|}$$

for each $\varepsilon > 0$.

► ***Remark.*** *In the second case,* $\mathscr{C}_W(A+)$ *may* still *coincide with* $\mathscr{C}_W(\mathbb{R})$. *(If, for example, the set of points x where* $W(x) < \infty$ *is sufficiently sparse. See* §C *and end of* §E.2)

Proof. For the Mergelian function $\Omega_A(z)$ defined in §E.2, we have $\Omega_A(z) \geqslant W_A(z)$, so, if the *first alternative holds*,

$$\int_{-\infty}^{\infty} \frac{\log \Omega_{A'}(x)}{1+x^2}\,dx = \infty$$

for every $A' > A$. Then, by Mergelian's second theorem, $\mathscr{C}_W(A') = \mathscr{C}_W(\mathbb{R})$ for each $A' > A$, so $\mathscr{C}_W(A+) = \mathscr{C}_W(\mathbb{R})$.

The supremum $W_{A'}(z)$ is an *increasing* function of A' for each fixed z by virtue of the obvious inclusion of $\mathscr{E}_{A'}$ in $\mathscr{E}_{A''}$ when $A' \leqslant A''$. Therefore, if the *second alternative holds*, we have

$$(\dagger) \qquad \int_{-\infty}^{\infty} \frac{\log W_{A'}(x)}{1+x^2}\,dx < \infty$$

for each $A' \leqslant A_0$, some number *larger* than A.

Let $\varepsilon > 0$ be given, wlog $\varepsilon < A_0 - A$, and put $\delta = \varepsilon/2$. Then, if $f \in \mathscr{C}_W(A+)$, surely $f \in \mathscr{C}_W(A')$, where $A' = A + \delta$. For *this* A', (†) holds, so, by Akhiezer's *second theorem* (§E.2), we have

$$|f(z)| \leqslant K_\delta e^{A'|\Im z| + \delta|z|}.$$

Therefore

$$|f(z)| \leqslant K_{\varepsilon/2} e^{A|\Im z| + \varepsilon|z|}.$$

Saying that $f(x)/W(x) \longrightarrow 0$ for $x \to \pm\infty$ is simply another way of expressing the fact that $f \in \mathscr{C}_W(\mathbb{R})$. Thus, in the event of the *second alternative, all* the functions f in $\mathscr{C}_W(A+)$ *have* the two asserted properties.

However, *any* entire function f with those two properties *does* belong to $\mathscr{C}_W(A+)$. For such a function will be in $\mathscr{C}_W(\mathbb{R})$, and then *must* belong to $\mathscr{C}_W(A+)$ by the preceding theorem. The subspace $\mathscr{C}_W(A+)$ thus consists *precisely* of the functions having the two properties in question (and no others) when the second alternative holds. We are done.

Corollary. *For the intersections $\mathscr{C}_W(A+)$ the following alternative holds: Either $\mathscr{C}_W(A+) = \mathscr{C}_W(\mathbb{R})$, or, if $\mathscr{C}_W(A+) \neq \mathscr{C}_W(\mathbb{R})$, the former space consists* precisely *of the entire functions $f(z)$ belonging to $\mathscr{C}_W(\mathbb{R})$ with*

$$|f(z)| \leqslant C_\varepsilon e^{A|\Im z| + \varepsilon|z|}$$

for each $\varepsilon > 0$.

Remark. Even when $\mathscr{C}_W(A+) = \mathscr{C}_W(\mathbb{R})$, *all* the functions in $\mathscr{C}_W(A+)$ *may* have the form described in the *second* clause of this statement. That happens when $\mathscr{C}_W(\mathbb{R})$ consists entirely of the restrictions of such functions to the set of real x where $W(x) < \infty$. See remark following the statement of the preceding theorem.

2. Sufficient conditions for equality of $\mathscr{C}_W(0)$ and $\mathscr{C}_W(0+)$

Lemma. *Let $w(z) = c\prod_1^N(z - a_k)$, where the a_k are distinct, with $\Im a_k < 0$. Let $g(z)$ be an entire function of exponential type $\leqslant A$ with $|(x+\mathrm{i})g(x)|$ bounded for real x. Then, for $x \in \mathbb{R}$,*

$$\left| \frac{e^{-iAx}g(x)}{w(x)} - \sum_{k=1}^{N} \frac{g(a_k)e^{-iAa_k}}{w'(a_k)(x - a_k)} \right|$$
$$\leqslant \frac{e}{\pi}\int_{-\infty}^{\infty} \left| \frac{g(t + (\mathrm{i}/A))}{w(t + (\mathrm{i}/A))} \right| \left| \frac{\sin A(x - t)}{x - t - (\mathrm{i}/A)} \right| dt.$$

Proof. $(z + \mathrm{i})g(z)$ is of exponential type $\leqslant A$ and is bounded on $\mathbb{R}$, hence has modulus $\leqslant \mathrm{const.}e^{A|\Im z|}$ by the third Phragmén–Lindelöf theorem of §C, Chapter III. Hence

$$(*) \qquad |g(z)| \leqslant \text{const.}\frac{e^{A|\Im z|}}{|z+i|}.$$

We are going to use $(*)$ together with some contour integration.

Fix $b>0$, take any large R, and let Γ_R be the following contour:

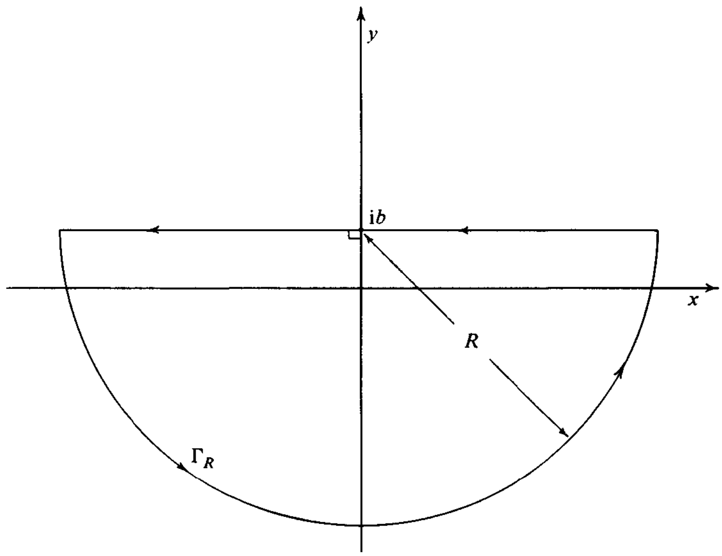

Figure 40

If R is large enough for Γ_R to encircle all the a_k and the real point x, the calculus of residues gives

$$\frac{1}{2\pi i}\int_{\Gamma_R}\frac{g(\zeta)e^{iA(x-\zeta)}}{W(\zeta)(x-\zeta)}d\zeta = \sum_{k=1}^{N}\frac{g(a_k)e^{iA(x-a_k)}}{W'(a_k)(x-a_k)} - \frac{g(x)}{W(x)}.$$

By $(*)$, $|g(\zeta)e^{-iA\zeta}|$ is $O(1/(|\zeta|-1)) = O(1/R)$ on the semi-circular part of Γ_R, so, as $R\to\infty$, the portion of the integral taken along that part of the contour tends to zero. Therefore

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{g(t+ib)e^{iA(x-t-ib)}}{W(t+ib)(x-t-ib)}dt = \frac{g(x)}{W(x)} - \sum_k\frac{g(a_k)e^{iA(x-a_k)}}{W'(a_k)(x-a_k)}.$$

We rewrite this relation as follows:

$$(\ast\!\ast) \qquad \frac{e^{Ab}}{2\pi i}\int_{-\infty}^{\infty}\frac{g(t+ib)e^{iA(x-t)}}{W(t+ib)(x-t-ib)}dt = \frac{g(x)}{W(x)} - \sum_k\frac{g(a_k)e^{iA(x-a_k)}}{W'(a_k)(x-a_k)}.$$

Let now Γ'_R be the contour obtained by reflecting Γ_R in the line $\Im z = b$:

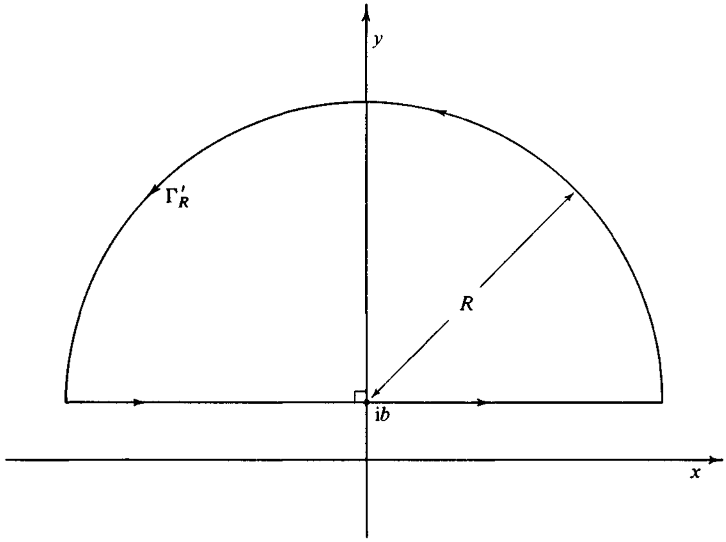

Figure 41

We have

$$\int_{\Gamma'_R} \frac{g(\zeta)e^{-iA(x-\zeta)}}{w(\zeta)(x-\zeta)} d\zeta \;=\; 0.$$

Here, $|g(\zeta)e^{iA\zeta}| = O(1/R)$ on the semi-circular part of Γ'_R, so, making $R \to \infty$, we get

$$\int_{-\infty}^{\infty} \frac{g(t+ib)e^{-iA(x-t-ib)}}{w(t+ib)(x-t-ib)} dt \;=\; 0,$$

that is,

$$\int_{-\infty}^{\infty} \frac{g(t+ib)e^{-iA(x-t)}}{w(t+ib)(x-t-ib)} dt \;=\; 0.$$

Multiplying the last relation by $e^{Ab}/2\pi i$ and subtracting the result from the left side of (*), we find

$$\frac{e^{Ab}}{\pi}\int_{-\infty}^{\infty} \frac{g(t+ib)\sin A(x-t)}{w(t+ib)(x-t-ib)} dt \;=\; \frac{g(x)}{w(x)} \;-\; \sum_k \frac{g(a_k)e^{iA(x-a_k)}}{w'(a_k)(x-a_k)}.$$

Now put $b = 1/A$ and multiply what has just been written by e^{-iAx}. After