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Radon Transform
Second Edition

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Preface to the Second Edition

The first edition of this book has been out of print for some time and I have decided to follow the publisher's kind suggestion to prepare a new edition. Many examples of the explicit inversion formulas and range theorems have been added, and the group-theoretic viewpoint emphasized. For example, the integral geometric viewpoint of the Poisson integral for the disk leads to interesting analogies with the X-ray transform in Euclidean 3-space. To preserve the introductory flavor of the book the short and self-contained Chapter V on Schwartz' distributions has been added. Here §5 provides proofs of the needed results about the Riesz potentials while §§3–4 develop the tools from Fourier analysis following closely the account in Hörmander's books [1963] and [1983]. There is some overlap with my books [1984] and [1994b] which, however, rely heavily on Lie group theory. The present book is much more elementary.

I am indebted to Sine Jensen for a critical reading of parts of the manuscript and to Hilgert and Schlichtkrull for concrete contributions mentioned at specific places in the text. Finally I thank Jan Wetzel and Bonnie Friedman for their patient and skillful preparation of the manuscript.

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Preface to the First Edition

The title of this booklet refers to a topic in geometric analysis which has its origins in results of Funk [1916] and Radon [1917] determining, respectively, a symmetric function on the two-sphere $\mathbf{S}^2$ from its great circle integrals and a function of the plane $\mathbf{R}^2$ from its line integrals. (See references.) Recent developments, in particular applications to partial differential equations, X-ray technology, and radio astronomy, have widened interest in the subject.

These notes consist of a revision of lectures given at MIT in the Fall of 1966, based mosty on my papers during 1959–1965 on the Radon transform and its generalizations. (The term "Radon Transform" is adopted from John [1955].)

The viewpoint for these generalizations is as follows.

The set of points on $\mathbf{S}^2$ and the set of great circles on $\mathbf{S}^2$ are both homogeneous spaces of the orthoginal group $O(3)$. Similarly, the set of points in $\mathbf{R}^2$ and the set of lines in $\mathbf{R}^2$ are both homogeneous spaces of the group $\mathbf{M}(2)$ of rigid motions of $\mathbf{R}^2$. This motivates our general Radon transform definition from [1965a, 1966a] which forms the framwork of Chapter II: Given two homogeneous spaces G/K and G/H of the same group G, the Radon transform $u \to \widehat{u}$ maps functions u on the first space to functions $\widehat{u}$ on the second space. For $\xi \in G/H$, $\widehat{u}(\xi)$ is defined as the (natural) integral of u over the set of points $x \in G/K$ which are incident to ξ in the sense of Chern [1942]. The problem of inverting $u \to \widehat{u}$ is worked out in a few cases.

It happens when G/K is a Euclidean space, and more generally when G/K is a Riemannian symmetric space, that the natural differential operators A on G/K are transferred by $u \to \widehat{u}$ into much more manageable differential operators $\widehat{A}$ on G/H; the connection is $(Au)\widehat{} = \widehat{A}\widehat{u}$. Then the theory of the transform $u \to \widehat{u}$ has significant applications to the study of properties of A.

On the other hand, the applications of the original Radon transform on $\mathbf{R}^2$ to X-ray technology and radio astronomy are based on the fact that for an unknown density u, X-ray attenuation measurements give $\widehat{u}$ directly and therefore yield u via Radon's inversion formula. More precisely, let B be a convex body, $u(x)$ its density at the point x, and suppose a thin beam of X-rays is directed at B along a line ξ. Then the line integral $\widehat{u}(\xi)$ of u along ξ equals $\log(I_o/I)$ where I_o and I, respectively, are the intensities of the beam before hitting B and after leaving B. Thus while the function u is at first unknown, the function $\widehat{u}$ is determined by the X-ray data.

The lecture notes indicated above have been updated a bit by including a short account of some applications (Chapter I, §7), by adding a few cprollaries (Corollaries 2.8 and 2.12, Theorem 6.3 in Chapter I, Corollaries 2.8

and 4.1 in Chapter III), and by giving indications in the bibliographical notes of some recent developments.

An effort has been made to keep the exposition rather elementary. The distribution theory and the theory of Riesz potentials, occasionally needed in Chapter I, is reviewed in some detail in §8 (now Chapter V). Apart from the general homogeneous space framework in Chapter II, the treatment is restricted to Euclidean and isotropic spaces (spaces which are "the same in all directions"). For more general symmetric spaces the treatment is postposed (except for §4 in Chapter III) to another occasion since more machinery from the theorem of semisimple Lie groups is required.

I am indebted to R. Melrose and R. Seeley for helpful suggestions and to F. Gonzalez and J. Orloff for critical reading of parts of the manuscript.

Cambridge, MA 1980

CHAPTER I

THE RADON TRANSFORM ON $\mathbf{R}^N$

§1 Introduction

It was proved by J. Radon in 1917 that a differentiable function on $\mathbf{R}^3$ can be determined explicitly by means of its integrals over the planes in $\mathbf{R}^3$. Let $J(\omega, p)$ denote the integral of f over the hyperplane $\langle x, \omega \rangle = p$, ω denoting a unit vector and $\langle \ , \ \rangle$ the inner product. Then

$$f(x) = -\frac{1}{8\pi^2} L_x \left(\int_{\mathbf{S}^2} J(\omega, \langle \omega, x \rangle) \, d\omega \right) ,$$

where L is the Laplacian on $\mathbf{R}^3$ and $d\omega$ the area element on the sphere $\mathbf{S}^2$ (cf. Theorem 3.1).

We now observe that the formula above has built in a remarkable duality: first one integrates over the set of points in a hyperplane, then one integrates over the set of hyperplanes passing through a given point. This suggests considering the transforms $f \to \widehat{f}, \varphi \to \check{\varphi}$ defined below.

The formula has another interesting feature. For a fixed ω the integrand $x \to J(\omega, \langle \omega, x \rangle)$ is a *plane wave*, that is a function constant on each plane perpendicular to ω. Ignoring the Laplacian the formula gives a continuous decomposition of f into plane waves. Since a plane wave amounts to a function of just one variable (along the normal to the planes) this decomposition can sometimes reduce a problem for $\mathbf{R}^3$ to a similar problem for $\mathbf{R}$. This principle has been particularly useful in the theory of partial differential equations.

The analog of the formula above for the line integrals is of importance in radiography where the objective is the description of a density function by means of certain line integrals.

In this chapter we discuss relationships between a function on $\mathbf{R}^n$ and its integrals over k-dimensional planes in $\mathbf{R}^n$. The case $k = n - 1$ will be the one of primary interest. We shall occasionally use some facts about Fourier transforms and distributions. This material will be developed in sufficient detail in Chapter V so the treatment should be self-contained.

Following Schwartz [1966] we denote by $\mathcal{E}(\mathbf{R}^n)$ and $\mathcal{D}(\mathbf{R}^n)$, respectively, the space of complex-valued $\mathcal{C}^\infty$ functions (respectively $\mathcal{C}^\infty$ functions of compact support) on $\mathbf{R}^n$. The space $\mathcal{S}(\mathbf{R}^n)$ of rapidly decreasing functions on $\mathbf{R}^n$ is defined in connection with (6) below. $C^m(\mathbf{R}^n)$ denotes the space of m times continuously differentiable functions. We write $C(\mathbf{R}^n)$ for $C^0(\mathbf{R}^n)$, the space of continuous function on $\mathbf{R}^n$.

For a manifold M, $C^m(M)$ (and $C(M)$) is defined similarly and we write $\mathcal{D}(M)$ for $\mathcal{C}_c^\infty(M)$ and $\mathcal{E}(M)$ for $\mathcal{C}^\infty(M)$.

§2 The Radon Transform of the Spaces $\mathcal{D}(\mathbf{R}^n)$ and $\mathcal{S}(\mathbf{R}^n)$. The Support Theorem

Let f be a function on $\mathbf{R}^n$, integrable on each hyperplane in $\mathbf{R}^n$. Let $\mathbf{P}^n$ denote the space of all hyperplanes in $\mathbf{R}^n$, $\mathbf{P}^n$ being furnished with the obvious topology. The *Radon transform* of f is defined as the function $\widehat{f}$ on $\mathbf{P}^n$ given by

$$\widehat{f}(\xi) = \int_\xi f(x)dm(x)\,,$$

where dm is the Euclidean measure on the hyperplane ξ. Along with the transformation $f \to \widehat{f}$ we consider also the *dual transform* $\varphi \to \check{\varphi}$ which to a continuous function φ on $\mathbf{P}^n$ associates the function $\check{\varphi}$ on $\mathbf{R}^n$ given by

$$\check{\varphi}(x) = \int_{x\in\xi} \varphi(\xi)\, d\mu(\xi)$$

where $d\mu$ is the measure on the compact set $\{\xi \in \mathbf{P}^n : x \in \xi\}$ which is invariant under the group of rotations around x and for which the measure of the whole set is 1 (see Fig. I.1). We shall relate certain function spaces on $\mathbf{R}^n$ and on $\mathbf{P}^n$ by means of the transforms $f \to \widehat{f}, \varphi \to \check{\varphi}$; later we obtain explicit inversion formulas.

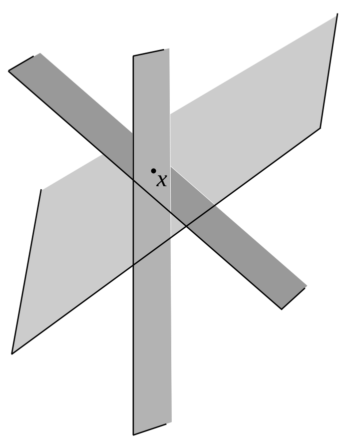

FIGURE I.1.

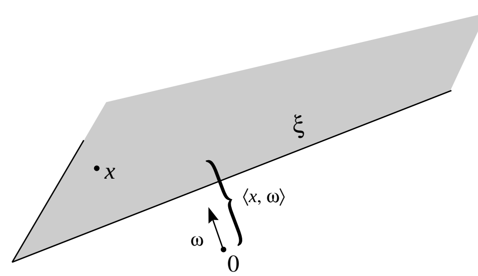

FIGURE I.2.

Each hyperplane $\xi \in \mathbf{P}^n$ can be written $\xi = \{x \in \mathbf{R}^n : \langle x, \omega\rangle = p\}$ where $\langle\ ,\ \rangle$ is the usual inner product, $\omega = (\omega_1, \ldots, \omega_n)$ a unit vector and $p \in \mathbf{R}$ (Fig. I.2). Note that the pairs (ω, p) and $(-\omega, -p)$ give the same ξ; the mapping $(\omega, p) \to \xi$ is a double covering of $\mathbf{S}^{n-1} \times \mathbf{R}$ onto $\mathbf{P}^n$. Thus $\mathbf{P}^n$ has a canonical manifold structure with respect to which this covering map is differentiable and regular. We thus identify continuous (differentiable) function on $\mathbf{P}^n$ with continuous (differentiable) functions φ on $\mathbf{S}^{n-1} \times \mathbf{R}$ satisfying the symmetry condition $\varphi(\omega, p) = \varphi(-\omega, -p)$. Writing $\widehat{f}(\omega, p)$ instead of $\widehat{f}(\xi)$ and f_t (with $t \in \mathbf{R}^n$) for the translated function $x \to f(t + x)$ we have

$$\widehat{f_t}(\omega, p) = \int_{\langle x,\omega\rangle = p} f(x+t)\, dm(x) = \int_{\langle y,\omega\rangle = p + \langle t,\omega\rangle} f(y)\, dm(y)$$

so

$$\widehat{f_t}(\omega, p) = \widehat{f}(\omega, p + \langle t, \omega\rangle)\,. \tag{1}$$

Taking limits we see that if $\partial_i = \partial/\partial x_i$

$$(\partial_i f)\widehat{\ }(\omega, p) = \omega_i \frac{\partial \widehat{f}}{\partial p}(\omega, p)\,. \tag{2}$$

Let L denote the Laplacian $\Sigma_i \partial_i^2$ on $\mathbf{R}^n$ and let $\square$ denote the operator

$$\varphi(\omega, p) \to \frac{\partial^2}{\partial p^2} \varphi(\omega, p)\,,$$

which is a well-defined operator on $\mathcal{E}(\mathbf{P}^n) = \mathcal{C}^\infty(\mathbf{P}^n)$. It can be proved that if $\mathbf{M}(n)$ is the group of isometries of $\mathbf{R}^n$, then L (respectively $\square$) generates the algebra of $\mathbf{M}(n)$-invariant differential operators on $\mathbf{R}^n$ (respectively $\mathbf{P}^n$).

Lemma 2.1. *The transforms $f \to \widehat{f}, \varphi \to \check{\varphi}$ intertwine L and $\square$, i.e.,*

$$(Lf)\widehat{\ } = \square(\widehat{f})\,, \quad (\square\varphi)^\vee = L\check{\varphi}\,.$$

Proof. The first relation follows from (2) by iteration. For the second we just note that for a certain constant c

$$\check{\varphi}(x) = c \int_{\mathbf{S}^{n-1}} \varphi(\omega, \langle x, \omega\rangle)\, d\omega\,, \tag{3}$$

where $d\omega$ is the usual measure on $\mathbf{S}^{n-1}$.

The Radon transform is closely connected with the Fourier transform

$$\widetilde{f}(u) = \int_{\mathbf{R}^n} f(x) e^{-i\langle x,\omega\rangle}\, dx \quad u \in \mathbf{R}^n.$$

In fact, if $s \in \mathbf{R}$, ω a unit vector,

$$\widetilde{f}(s\omega) = \int_{-\infty}^{\infty} dr \int_{\langle x,\omega\rangle = r} f(x) e^{-is\langle x,\omega\rangle}\, dm(x)$$

so

$$\widetilde{f}(s\omega) = \int_{-\infty}^{\infty} \widehat{f}(\omega, r) e^{-isr}\, dr\,. \tag{4}$$

This means that the n-dimensional Fourier transform is the 1-dimensional Fourier transform of the Radon transform. From (4), or directly, it follows that the Radon transform of the convolution

$$f(x) = \int_{\mathbf{R}^n} f_1(x-y) f_2(y)\, dy$$

is the convolution

$$\widehat{f}(\omega, p) = \int_{\mathbf{R}} \widehat{f_1}(\omega, p-q) \widehat{f_2}(\omega, q)\, dq\,. \tag{5}$$

We consider now the space $\mathcal{S}(\mathbf{R}^n)$ of complex-valued rapidly decreasing functions on $\mathbf{R}^n$. We recall that $f \in \mathcal{S}(\mathbf{R}^n)$ if and only if for each polynomial P and each integer $m \geq 0$,

$$\sup_x \big| |x|^m P(\partial_1, \ldots, \partial_n) f(x) \big| < \infty\,, \tag{6}$$

$|x|$ denoting the norm of x. We now formulate this in a more invariant fashion.

Lemma 2.2. *A function $f \in \mathcal{E}(\mathbf{R}^n)$ belongs to $\mathcal{S}(\mathbf{R}^n)$ if and only if for each pair $k, \ell \in \mathbb{Z}^+$*

$$\sup_{x \in \mathbf{R}^n} |(1+|x|)^k (L^\ell f)(x)| < \infty\,.$$

This is easily proved just by using the Fourier transforms.

In analogy with $\mathcal{S}(\mathbf{R}^n)$ we define $\mathcal{S}(\mathbf{S}^{n-1} \times \mathbf{R})$ as the space of $\mathcal{C}^\infty$ functions φ on $\mathbf{S}^{n-1} \times \mathbf{R}$ which for any integers $k, \ell \geq 0$ and any differential operator D on $\mathbf{S}^{n-1}$ satisfy

$$\sup_{\omega \in \mathbf{S}^{n-1}, r \in \mathbf{R}} \left| (1+|r|^k) \frac{d^\ell}{dr^\ell} (D\varphi)(\omega, r) \right| < \infty\,. \tag{7}$$

The space $\mathcal{S}(\mathbf{P}^n)$ is then defined as the set of $\varphi \in \mathcal{S}(\mathbf{S}^{n-1} \times \mathbf{R})$ satisfying $\varphi(\omega, p) = \varphi(-\omega, -p)$.

Lemma 2.3. *For each $f \in \mathcal{S}(\mathbf{R}^n)$ the Radon transform $\widehat{f}(\omega, p)$ satisfies the following condition: For $k \in \mathbb{Z}^+$ the integral*

$$\int_{\mathbf{R}} \widehat{f}(\omega, p) p^k\, dp$$

can be written as a k^{th} degree homogeneous polynomial in $\omega_1, \ldots, \omega_n$.

Proof. This is immediate from the relation

(8)
$$\int_{\mathbf{R}} \widehat{f}(\omega, p)p^k\,dp = \int_{\mathbf{R}} p^k\,dp \int_{\langle x,\omega\rangle = p} f(x)\,dm(x) = \int_{\mathbf{R}^n} f(x)\langle x,\omega\rangle^k\,dx\,.$$

In accordance with this lemma we define the space

$$\mathcal{S}_H(\mathbf{P}^n) = \left\{ F \in \mathcal{S}(\mathbf{P}^n) : \begin{array}{l} \text{For each } k \in \mathbb{Z}^+, \int_{\mathbf{R}} F(\omega,p)p^k\,dp \\ \text{is a homogeneous polynomial} \\ \text{in } \omega_1, \dots, \omega_n \text{ of degree } k \end{array} \right\}.$$

With the notation $\mathcal{D}(\mathbf{P}^n) = \mathcal{C}_c^\infty(\mathbf{P}^n)$ we write

$$\mathcal{D}_H(\mathbf{P}^n) = \mathcal{S}_H(\mathbf{P}^n) \cap \mathcal{D}(\mathbf{P}^n)\,.$$

According to Schwartz [1966], p. 249, the Fourier transform $f \to \widetilde{f}$ maps the space $\mathcal{S}(\mathbf{R}^n)$ onto itself. See Ch. V, Theorem 3.1. We shall now settle the analogous question for the Radon transform.

Theorem 2.4. *(The Schwartz theorem) The Radon transform $f \to \widehat{f}$ is a linear one-to-one mapping of $\mathcal{S}(\mathbf{R}^n)$ onto $\mathcal{S}_H(\mathbf{P}^n)$.*

Proof. Since
$$\frac{d}{ds}\widetilde{f}(s\omega) = \sum_{i=1}^n \omega_i(\partial_i \widetilde{f})$$
it is clear from (4) that for each fixed ω the function $r \to \widehat{f}(\omega, r)$ lies in $\mathcal{S}(\mathbf{R})$. For each $\omega_0 \in \mathbf{S}^{n-1}$ a subset of $(\omega_1, \dots, \omega_n)$ will serve as local coordinates on a neighborhood of ω_0 in $\mathbf{S}^{n-1}$. To see that $\widehat{f} \in \mathcal{S}(\mathbf{P}^n)$, it therefore suffices to verify (7) for $\varphi = \widehat{f}$ on an open subset $N \subset \mathbf{S}^{n-1}$ where ω_n is bounded away from 0 and $\omega_1, \dots, \omega_{n-1}$ serve as coordinates, in terms of which D is expressed. Since

(9) $$u_1 = s\omega_1, \dots, u_{n-1} = s\omega_{n-1}\,, \quad u_n = s(1 - \omega_1^2 - \cdots - \omega_{n-1}^2)^{1/2}$$

we have
$$\frac{\partial}{\partial\omega_i}(\widetilde{f}(s\omega)) = s\frac{\partial\widetilde{f}}{\partial u_i} - s\omega_i(1 - \omega_1^2 - \cdots - \omega_{n-1}^2)^{-1/2}\frac{\partial\widetilde{f}}{\partial u_n}\,.$$

It follows that if D is any differential operator on $\mathbf{S}^{n-1}$ and if $k, \ell \in \mathbb{Z}^+$ then

(10) $$\sup_{\omega \in N, s \in \mathbf{R}} \left|(1 + s^{2k})\frac{d^\ell}{ds^\ell}(D\widetilde{f})(\omega, s)\right| < \infty\,.$$

We can therefore apply D under the integral sign in the inversion formula to (4),
$$\widehat{f}(\omega, r) = \frac{1}{2\pi}\int_{\mathbf{R}} \widetilde{f}(s\omega)e^{isr}\,ds$$

and obtain

$$(1+r^{2k})\frac{d^\ell}{dr^\ell}\Big(D_\omega(\widehat{f}(\omega,r))\Big)=\frac{1}{2\pi}\int\left(1+(-1)^k\frac{d^{2k}}{ds^{2k}}\right)\Big((is)^\ell D_\omega(\widetilde{f}(s\omega))\Big)e^{isr}ds\,.$$

Now (10) shows that $\widehat{f}\in\mathcal{S}(\mathbf{P}^n)$ so by Lemma 2.3, $\widehat{f}\in\mathcal{S}_H(\mathbf{P}^n)$.

Because of (4) and the fact that the Fourier transform is one-to-one it only remains to prove the surjectivity in Theorem 2.4. Let $\varphi\in\mathcal{S}_H(\mathbf{P}^n)$. In order to prove $\varphi=\widehat{f}$ for some $f\in\mathcal{S}(\mathbf{R}^n)$ we put

$$\psi(s,\omega)=\int_{-\infty}^{\infty}\varphi(\omega,r)e^{-irs}\,dr\,.$$

Then $\psi(s,\omega)=\psi(-s,-\omega)$ and $\psi(0,\omega)$ is a homogeneous polynomial of degree 0 in $\omega_1,\dots,\omega_n$, hence constant. Thus there exists a function F on $\mathbf{R}^n$ such that

$$F(s\omega)=\int_{\mathbf{R}}\varphi(\omega,r)e^{-irs}\,dr\,.$$

While F is clearly smooth away from the origin we shall now prove it to be smooth at the origin too; this is where the homogeneity condition in the definition of $\mathcal{S}_H(\mathbf{P}^n)$ enters decisively. Consider the coordinate neighborhood $N\subset\mathbf{S}^{n-1}$ above and if $h\in\mathcal{C}^\infty(\mathbf{R}^n-0)$ let $h^*(\omega_1,\dots,\omega_{n-1},s)$ be the function obtained from h by means of the substitution (9). Then

$$\frac{\partial h}{\partial u_i}=\sum_{j=1}^{n-1}\frac{\partial h^*}{\partial\omega_j}\frac{\partial\omega_j}{\partial u_i}+\frac{\partial h^*}{\partial s}\cdot\frac{\partial s}{\partial u_i}\quad(1\le i\le n)$$

and

$$\begin{aligned}\frac{\partial\omega_j}{\partial u_i}&=\frac{1}{s}(\delta_{ij}-\frac{u_iu_j}{s^2})\quad(1\le i\le n\,,\quad 1\le j\le n-1)\,,\\ \frac{\partial s}{\partial u_i}&=\omega_i\quad(1\le i\le n-1),\frac{\partial s}{\partial u_n}=(1-\omega_1^2-\cdots-\omega_{n-1}^2)^{1/2}\,.\end{aligned}$$

Hence

$$\begin{aligned}\frac{\partial h}{\partial u_i}&=\frac{1}{s}\frac{\partial h^*}{\partial\omega_i}+\omega_i\left(\frac{\partial h^*}{\partial s}-\frac{1}{s}\sum_{j=1}^{n-1}\omega_j\frac{\partial h^*}{\partial\omega_j}\right)\quad(1\le i\le n-1)\\ \frac{\partial h}{\partial u_n}&=(1-\omega_1^2-\cdots-\omega_{n-1}^2)^{1/2}\left(\frac{\partial h^*}{\partial s}-\frac{1}{s}\sum_{j=1}^{n-1}\omega_j\frac{\partial h^*}{\partial\omega_j}\right)\,.\end{aligned}$$

In order to use this for $h=F$ we write

$$F(s\omega)=\int_{-\infty}^{\infty}\varphi(\omega,r)\,dr+\int_{-\infty}^{\infty}\varphi(\omega,r)(e^{-irs}-1)\,dr\,.$$

By assumption the first integral is independent of ω. Thus using (7) we have for constant $K > 0$

$$\left|\frac{1}{s}\frac{\partial}{\partial\omega_i}(F(s\omega))\right| \leq K\int(1+r^4)^{-1}s^{-1}|e^{-isr}-1|\,dr \leq K\int\frac{|r|}{1+r^4}\,dr$$

and a similar estimate is obvious for $\partial F(s\omega)/\partial s$. The formulas above therefore imply that all the derivatives $\partial F/\partial u_i$ are bounded in a punctured ball $0 < |u| < \epsilon$ so F is certainly continuous at $u = 0$.

More generally, we prove by induction that

$$\frac{\partial^q h}{\partial u_{i_1}\dots\partial u_{i_q}} = \sum_{1\leq i+j\leq q, 1\leq k_1,\cdots,k_i\leq n-1} A_{j,k_1\dots k_i}(\omega,s)\frac{\partial^{i+j}h^*}{\partial\omega_{k_1}\dots\partial\omega_{k_i}\partial s^j} \tag{11}$$

where the coefficients A have the form

$$A_{j,k_1\dots k_i}(\omega,s) = a_{j,k_1\dots k_i}(\omega)s^{j-q}\,. \tag{12}$$

For $q = 1$ this is in fact proved above. Assuming (11) for q we calculate

$$\frac{\partial^{q+1}h}{\partial u_{i_1}\dots\partial u_{i_{q+1}}}$$

using the above formulas for $\partial/\partial u_i$. If $A_{j,k_1\dots k_i}(\omega,s)$ is differentiated with respect to $u_{i_{q+1}}$ we get a formula like (12) with q replaced by $q+1$. If on the other hand the $(i+j)^{\text{th}}$ derivative of h^* in (11) is differentiated with respect to $u_{i_{q+1}}$ we get a combination of terms

$$s^{-1}\frac{\partial^{i+j+1}h^*}{\partial\omega_{k_1}\dots\partial\omega_{k_{i+1}}\partial s^j}\ ,\ \frac{\partial^{i+j+1}h^*}{\partial\omega_{k_1}\dots\partial\omega_{k_i}\partial s^{j+1}}$$

and in both cases we get coefficients satisfying (12) with q replaced by $q+1$. This proves (11)–(12) in general. Now

$$F(s\omega) = \int_{-\infty}^{\infty}\varphi(\omega,r)\sum_0^{q-1}\frac{(-isr)^k}{k!}\,dr + \int_{-\infty}^{\infty}\varphi(\omega,r)e_q(-irs)\,dr\,, \tag{13}$$

where

$$e_q(t) = \frac{t^q}{q!} + \frac{t^{q+1}}{(q+1)!} + \cdots$$

Our assumption on φ implies that the first integral in (13) is a polynomial in $u_1,\dots,u_n$ of degree $\leq q-1$ and is therefore annihilated by the differential operator (11). If $0 \leq j \leq q$, we have

$$|s^{j-q}\frac{\partial^j}{\partial s^j}(e_q(-irs))| = |(-ir)^q(-irs)^{j-q}e_{q-j}(-irs)| \leq k_j r^q\,, \tag{14}$$

where k_j is a constant because the function $t \to (it)^{-p} e_p(it)$ is obviously bounded on $\mathbf{R}\,(p \geq 0)$. Since $\varphi \in \mathcal{S}(\mathbf{P}^n)$ it follows from (11)–(14) that each q^{th} order derivative of F with respect to $u_1, \dots, u_n$ is bounded in a punctured ball $0 < |u| < \epsilon$. Thus we have proved $F \in \mathcal{E}(\mathbf{R}^n)$. That F is rapidly decreasing is now clear from (7), Lemma 2.2 and (11). Finally, if f is the function in $\mathcal{S}(\mathbf{R}^n)$ whose Fourier transform is F then

$$\widetilde{f}(s\omega) = F(s\omega) = \int_{-\infty}^{\infty} \varphi(\omega, r) e^{-irs}\, dr\,;$$

hence by (4), $\widehat{f} = \varphi$ and the theorem is proved.

To make further progress we introduce some useful notation. Let $S_r(x)$ denote the sphere $\{y : |y - x| = r\}$ in $\mathbf{R}^n$ and $A(r)$ its area. Let $B_r(x)$ denote the open ball $\{y : |y - x| < r\}$. For a continuous function f on $S_r(x)$ let $(M^r f)(x)$ denote the mean value

$$(M^r f)(x) = \frac{1}{A(r)} \int_{S_r(x)} f(\omega)\, d\omega\,,$$

where $d\omega$ is the Euclidean measure. Let K denote the orthogonal group $\mathbf{O}(n)$, dk its Haar measure, normalized by $\int dk = 1$. If $y \in \mathbf{R}^n, r = |y|$ then

$$(15)\quad (M^r f)(x) = \int_K f(x + k \cdot y)\, dk\,.$$

(Fig. I.3) In fact, for x, y fixed both sides represent rotation-invariant functionals on $C(S_r(x))$, having the same value for the function $f \equiv 1$. The rotations being transitive on $S_r(x)$, (15) follows from the uniqueness of such invariant functionals. Formula (3) can similarly be written

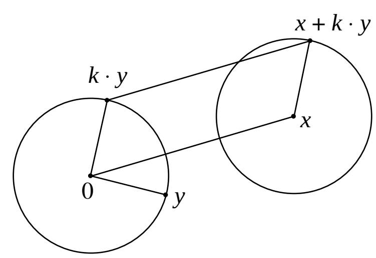

FIGURE I.3.

$$(16)\qquad \check{\varphi}(x) = \int_K \varphi(x + k \cdot \xi_0)\, dk$$

if ξ_0 is some fixed hyperplane through the origin. We see then that if $f(x) = 0(|x|^{-n}), \Omega_k$ the area of the unit sphere in $\mathbf{R}^k$, i.e., $\Omega_k = 2\frac{\pi^{k/2}}{\Gamma(k/2)}$,

$$\begin{aligned}(\widehat{f})^{\vee}(x) &= \int_K \widehat{f}(x + k \cdot \xi_0)\, dk = \int_K \left(\int_{\xi_0} f(x + k \cdot y)\, dm(y) \right)\, dk \\ &= \int_{\xi_0} (M^{|y|} f)(x)\, dm(y) = \Omega_{n-1} \int_0^{\infty} r^{n-2} \left(\frac{1}{\Omega_n} \int_{\mathcal{S}^{n-1}} f(x + r\omega)\, d\omega \right) dr\end{aligned}$$

so

$$(\widehat{f})^\vee(x) = \frac{\Omega_{n-1}}{\Omega_n} \int_{\mathbf{R}^n} |x-y|^{-1} f(y)\,dy\,. \tag{17}$$

We consider now the analog of Theorem 2.4 for the transform $\varphi \to \check{\varphi}$. But $\varphi \in \mathcal{S}_H(\mathbf{P}^n)$ does not imply $\check{\varphi} \in \mathcal{S}(\mathbf{R}^n)$. (If this were so and we by Theorem 2.4 write $\varphi = \widehat{f}, f \in \mathcal{S}(\mathbf{R}^n)$ then the inversion formula in Theorem 3.1 for $n = 3$ would imply $\int f(x)\,dx = 0$.) On a smaller space we shall obtain a more satisfactory result.

Let $\mathcal{S}^*(\mathbf{R}^n)$ denote the space of all functions $f \in \mathcal{S}(\mathbf{R}^n)$ which are orthogonal to all polynomials, i.e.,

$$\int_{\mathbf{R}^n} f(x)P(x)\,dx = 0 \quad \text{for all polynomials } P\,.$$

Similarly, let $\mathcal{S}^*(\mathbf{P}^n) \subset \mathcal{S}(\mathbf{P}^n)$ be the space of φ satisfying

$$\int_{\mathbf{R}} \varphi(\omega, r)p(r)\,dr = 0 \quad \text{for all polynomials } p\,.$$

Note that under the Fourier transform the space $\mathcal{S}^*(\mathbf{R}^n)$ corresponds to the subspace $\mathcal{S}_0(\mathbf{R}^n) \subset \mathcal{S}(\mathbf{R}^n)$ of functions all of whose derivatives vanish at 0.

Corollary 2.5. *The transforms $f \to \widehat{f}, \varphi \to \check{\varphi}$ are bijections of $\mathcal{S}^*(\mathbf{R}^n)$ onto $\mathcal{S}^*(\mathbf{P}^n)$ and of $\mathcal{S}^*(\mathbf{P}^n)$ onto $\mathcal{S}^*(\mathbf{R}^n)$, respectively.*

The first statement is clear from (8) if we take into account the elementary fact that the polynomials $x \to \langle x, \omega\rangle^k$ span the space of homogeneous polynomials of degree k. To see that $\varphi \to \check{\varphi}$ is a bijection of $\mathcal{S}^*(\mathbf{P}^n)$ onto $\mathcal{S}^*(\mathbf{R}^n)$ we use (17), knowing that $\varphi = \widehat{f}$ for some $f \in \mathcal{S}^*(\mathbf{R}^n)$. The right hand side of (17) is the convolution of f with the tempered distribution $|x|^{-1}$ whose Fourier transform is by Chapter V, §5 a constant multiple of $|u|^{1-n}$. (Here we leave out the trivial case $n = 1$.) By Chapter V, (12) this convolution is a tempered distribution whose Fourier transform is a constant multiple of $|u|^{1-n}\widetilde{f}(u)$. But, by Lemma 5.6, Chapter V this lies in the space $\mathcal{S}_0(\mathbf{R}^n)$ since $\widetilde{f}$ does. Now (17) implies that $\check{\varphi} = (\widehat{f})^\vee \in \mathcal{S}^*(\mathbf{R}^n)$ and that $\check{\varphi} \not\equiv 0$ if $\varphi \not\equiv 0$. Finally we see that the mapping $\varphi \to \check{\varphi}$ is surjective because the function

$$((\widehat{f})^\vee)\widetilde{\ }(u) = c|u|^{1-n}\widetilde{f}(u)$$

(where c is a constant) runs through $\mathcal{S}_0(\mathbf{R}^n)$ as f runs through $\mathcal{S}^*(\mathbf{R}^n)$.

We now turn to the space $\mathcal{D}(\mathbf{R}^n)$ and its image under the Radon transform. We begin with a preliminary result. (See Fig. I.4.)

Theorem 2.6. *(The support theorem.) Let $f \in C(\mathbf{R}^n)$ satisfy the following conditions:*

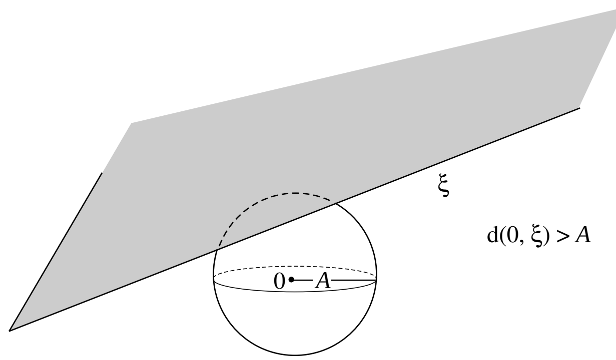

FIGURE I.4.

(i) *For each integer $k > 0, |x|^k f(x)$ is bounded.*

(ii) *There exists a constant $A > 0$ such that*

$$\widehat{f}(\xi) = 0 \ \textit{for} \ d(0, \xi) > A\,,$$

d denoting distance.

Then

$$f(x) = 0 \ \textit{for} \ |x| > A\,.$$

Proof. Replacing f by the convolution $\varphi * f$ where φ is a radial $\mathcal{C}^\infty$ function with support in a small ball $B_\epsilon(0)$ we see that it suffices to prove the theorem for $f \in \mathcal{E}(\mathbf{R}^n)$. In fact, $\varphi * f$ is smooth, it satisfies (i) and by (5) it satisfies (ii) with A replaced by $A+\epsilon$. Assuming the theorem for the smooth case we deduce that support $(\varphi * f) \subset B_{A+\epsilon}(0)$ so letting $\epsilon \to 0$ we obtain support $(f) \subset$ Closure $B_A(0)$.

To begin with we assume f is a radial function. Then $f(x) = F(|x|)$ where $F \in \mathcal{E}(\mathbf{R})$ and even. Then $\widehat{f}$ has the form $\widehat{f}(\xi) = \widehat{F}(d(0, \xi))$ where $\widehat{F}$ is given by

$$\widehat{F}(p) = \int_{\mathbf{R}^{n-1}} F((p^2 + |y|^2)^{1/2})\, dm(y)\,, \quad (p \geq 0)$$

because of the definition of the Radon transform. Using polar coordinates in $\mathbf{R}^{n-1}$ we obtain

$$\widehat{F}(p) = \Omega_{n-1} \int_0^\infty F((p^2 + t^2)^{1/2}) t^{n-2}\, dt\,. \tag{18}$$

Here we substitute $s = (p^2 + t^2)^{-1/2}$ and then put $u = p^{-1}$. Then (18) becomes

$$u^{n-3} \widehat{F}(u^{-1}) = \Omega_{n-1} \int_0^u (F(s^{-1}) s^{-n})(u^2 - s^2)^{(n-3)/2}\, ds\,.$$

We write this equation for simplicity

$$h(u) = \int_0^u g(s)(u^2 - s^2)^{(n-3)/2} \, ds \, . \tag{19}$$

This integral equation is very close to Abel's integral equation (Whittaker-Watson [1927], Ch. IX) and can be inverted as follows. Multiplying both sides by $u(t^2 - u^2)^{(n-3)/2}$ and integrating over $0 \le u \le t$ we obtain

$$\begin{aligned}
&\int_0^t h(u)(t^2 - u^2)^{(n-3)/2} u \, du \\
&\quad = \int_0^t \left[\int_0^u g(s)[(u^2 - s^2)(t^2 - u^2)]^{(n-3)/2} \, ds \right] u \, du \\
&\quad = \int_0^t g(s) \left[\int_{u=s}^t u[(t^2 - u^2)(u^2 - s^2)]^{(n-3)/2} \, du \right] ds \, .
\end{aligned}$$

The substitution $(t^2 - s^2)V = (t^2 + s^2) - 2u^2$ gives an explicit evaluation of the inner integral and we obtain

$$\int_0^t h(u)(t^2 - u^2)^{(n-3)/2} u \, du = C \int_0^t g(s)(t^2 - s^2)^{n-2} \, ds \, ,$$

where $C = 2^{1-n} \pi^{\frac{1}{2}} \Gamma((n-1)/2)/\Gamma(n/2)$. Here we apply the operator $\frac{d}{d(t^2)} = \frac{1}{2t} \frac{d}{dt}$ $(n-1)$ times whereby the right hand side gives a constant multiple of $t^{-1} g(t)$. Hence we obtain

$$F(t^{-1}) t^{-n} = ct \left[\frac{d}{d(t^2)} \right]^{n-1} \int_0^t (t^2 - u^2)^{(n-3)/2} u^{n-2} \widehat{F}(u^{-1}) \, du \tag{20}$$

where $c^{-1} = (n-2)! \Omega_n / 2^n$. By assumption (ii) we have $\widehat{F}(u^{-1}) = 0$ if $u^{-1} \ge A$, that is if $u \le A^{-1}$. But then (20) implies $F(t^{-1}) = 0$ if $t \le A^{-1}$, that is if $t^{-1} \ge A$. This proves the theorem for the case when f is radial.

We consider next the case of a general f. Fix $x \in \mathbf{R}^n$ and consider the function

$$g_x(y) = \int_K f(x + k \cdot y) \, dk$$

as in (15). Then g_x satisfies (i) and

$$\widehat{g}_x(\xi) = \int_K \widehat{f}(x + k \cdot \xi) \, dk \, , \tag{21}$$

$x + k \cdot \xi$ denoting the translate of the hyperplane $k \cdot \xi$ by x. The triangle inequality shows that

$$d(0, x + k \cdot \xi) \geq d(0, \xi) - |x|\,, \quad x \in \mathbf{R}^n, k \in K\,.$$

Hence we conclude from assumption (ii) and (21) that

$$\widehat{g}_x(\xi) = 0 \quad \text{if} \quad d(0,\xi) > A + |x|\,. \tag{22}$$

But g_x is a radial function so (22) implies by the first part of the proof that

$$\int_K f(x + k \cdot y)\, dk = 0 \quad \text{if} \quad |y| > A + |x|\,. \tag{23}$$

Geometrically, this formula reads: The surface integral of f over $S_{|y|}(x)$ is 0 if the ball $B_{|y|}(x)$ contains the ball $B_A(0)$. The theorem is therefore a consequence of the following lemma.

Lemma 2.7. *Let $f \in C(\mathbf{R}^n)$ be such that for each integer $k > 0$,*

$$\sup_{x \in \mathbf{R}^n} |x|^k |f(x)| < \infty\,.$$

Suppose f has surface integral 0 over every sphere S which encloses the unit ball. Then $f(x) \equiv 0$ for $|x| > 1$.

Proof. The idea is to perturb S in the relation

$$\int_S f(s)\, d\omega(s) = 0 \tag{24}$$

slightly, and differentiate with respect to the parameter of the perturbations, thereby obtaining additional relations. (See Fig. I.5.) Replacing, as above, f with a suitable convolution $\varphi * f$ we see that it suffices to prove the lemma for f in $\mathcal{E}(\mathbf{R}^n)$. Writing $S = S_R(x)$ and viewing the exterior of the ball $B_R(x)$ as a union of spheres with center x we have by the assumptions,

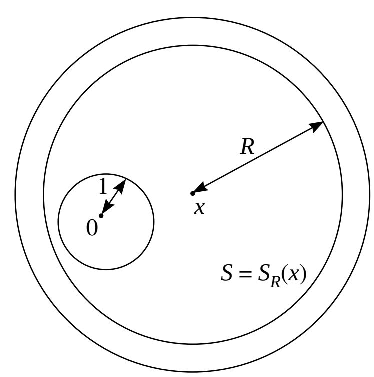

FIGURE I.5.

$$\int_{B_R(x)} f(y)\, dy = \int_{\mathbf{R}^n} f(y)\, dy\,,$$

which is a constant. Differentiating with respect to x_i we obtain

$$\int_{B_R(0)} (\partial_i f)(x+y)\,dy = 0\,. \tag{25}$$

We use now the divergence theorem

$$\int_{B_R(0)} (\mathrm{div} F)(y)\,dy = \int_{S_R(0)} \langle F, \mathbf{n}\rangle(s)\,d\omega(s) \tag{26}$$

for a vector field F on $\mathbf{R}^n$, $\mathbf{n}$ denoting the outgoing unit normal and $d\omega$ the surface element on $S_R(0)$. For the vector field $F(y) = f(x+y)\frac{\partial}{\partial y_i}$ we obtain from (25) and (26), since $\mathbf{n} = R^{-1}(s_1, \ldots, s_n)$,

$$\int_{S_R(0)} f(x+s)s_i\,d\omega(s) = 0\,. \tag{27}$$

But by (24)

$$\int_{S_R(0)} f(x+s)x_i\,d\omega(s) = 0$$

so by adding

$$\int_S f(s)s_i\,d\omega(s) = 0\,.$$

This means that the hypotheses of the lemma hold for $f(x)$ replaced by the function $x_i f(x)$. By iteration

$$\int_S f(s)P(s)\,d\omega(s) = 0$$

for any polynomial P, so $f \equiv 0$ on S. This proves the lemma as well as Theorem 2.6.

Corollary 2.8. *Let $f \in C(\mathbf{R}^n)$ satisfy (i) in Theorem 2.6 and assume*

$$\widehat{f}(\xi) = 0$$

for all hyperplanes ξ disjoint from a certain compact convex set C. Then

$$f(x) = 0 \quad \textit{for} \quad x \notin C\,. \tag{28}$$

In fact, if B is a closed ball containing C we have by Theorem 2.6, $f(x) = 0$ for $x \notin B$. But C is the intersection of such balls so (28) follows.

Remark 2.9. While condition (i) of rapid decrease entered in the proof of Lemma 2.7 (we used $|x|^k f(x) \in L^1(\mathbf{R}^n)$ for each $k > 0$) one may wonder whether it could not be weakened in Theorem 2.6 and perhaps even dropped in Lemma 2.7.

As an example, showing that the condition of rapid decrease can not be dropped in either result consider for $n = 2$ the function

$$f(x, y) = (x + iy)^{-5}$$

made smooth in $\mathbf{R}^2$ by changing it in a small disk around 0. Using Cauchy's theorem for a large semicircle we have $\int_\ell f(x)\, dm(x) = 0$ for every line ℓ outside the unit circle. Thus (ii) is satisfied in Theorem 2.6. Hence (i) cannot be dropped or weakened substantially.

This same example works for Lemma 2.7. In fact, let S be a circle $|z - z_0| = r$ enclosing the unit disk. Then $d\omega(s) = -ir\frac{dz}{z-z_0}$ so, by expanding the contour or by residue calculus,

$$\int_S z^{-5}(z - z_0)^{-1}\, dz = 0\,,$$

(the residue at $z = 0$ and $z = z_0$ cancel) so we have in fact

$$\int_S f(s)\, d\omega(s) = 0\,.$$

We recall now that $\mathcal{D}_H(\mathbf{P}^n)$ is the space of symmetric $\mathcal{C}^\infty$ functions $\varphi(\xi) = \varphi(\omega, p)$ on $\mathbf{P}^n$ of compact support such that for each $k \in \mathbb{Z}^+$, $\int_{\mathbf{R}} \varphi(\omega, p)p^k\, dp$ is a homogeneous kth degree polynomial in $\omega_1, \ldots, \omega_n$. Combining Theorems 2.4, 2.6 we obtain the following characterization of the Radon transform of the space $\mathcal{D}(\mathbf{R}^n)$. This can be regarded as the analog for the Radon transform of the Paley-Wiener theorem for the Fourier transform (see Chapter V).

Theorem 2.10. *(The Paley-Wiener theorem.) The Radon transform is a bijection of $\mathcal{D}(\mathbf{R}^n)$ onto $\mathcal{D}_H(\mathbf{P}^n)$.*

We conclude this section with a variation and a consequence of Theorem 2.6.

Lemma 2.11. *Let $f \in C_c(\mathbf{R}^n)$, $A > 0$, ω_0 a fixed unit vector and $N \subset S$ a neighborhood of ω_0 in the unit sphere $S \subset \mathbf{R}^n$. Assume*

$$\widehat{f}(\omega, p) = 0 \quad \textit{for} \quad \omega \in N,\, p > A\,.$$

Then

$$f(x) = 0 \textit{ in the half-space } \langle x, \omega_0 \rangle > A\,. \tag{29}$$

Proof. Let B be a closed ball around the origin containing the support of f. Let $\epsilon > 0$ and let H_ϵ be the union of the half spaces $\langle x, \omega \rangle > A + \epsilon$ as ω runs through N. Then by our assumption

$$\widehat{f}(\xi) = 0 \quad \text{if} \quad \xi \subset H_\epsilon\,. \tag{30}$$

Now choose a ball B_ϵ with a center on the ray from 0 through $-\omega_0$, with the point $(A+2\epsilon)\omega_0$ on the boundary, and with radius so large that any hyperplane ξ intersecting B but not B_ϵ must be in H_ϵ. Then by (30)

$$\widehat{f}(\xi) = 0 \quad \text{whenever} \quad \xi \in \mathbf{P}^n,\, \xi \cap B_\epsilon = \emptyset\,.$$

Hence by Theorem 2.6, $f(x) = 0$ for $x \notin B_\epsilon$. In particular, $f(x) = 0$ for $\langle x, \omega_0 \rangle > A + 2\epsilon$; since $\epsilon > 0$ is arbitrary, the lemma follows.

Corollary 2.12. *Let N be any open subset of the unit sphere $\mathbf{S}^{n-1}$. If $f \in C_c(\mathbf{R}^n)$ and*

$$\widehat{f}(\omega, p) = 0 \quad \textit{for } p \in \mathbf{R}, \omega \in N$$

then

$$f \equiv 0\,.$$

Since $\widehat{f}(-\omega, -p) = \widehat{f}(\omega, p)$ this is obvious from Lemma 2.11.

§3 The Inversion Formula

We shall now establish explicit inversion formulas for the Radon transform $f \to \widehat{f}$ and its dual $\varphi \to \check{\varphi}$.

Theorem 3.1. *The function f can be recovered from the Radon transform by means of the following inversion formula*

$$cf = (-L)^{(n-1)/2}((\widehat{f})^\vee) \quad f \in \mathcal{E}(\mathbf{R}^n)\,, \tag{31}$$

provided $f(x) = 0(|x|^{-N})$ for some $N > n$. Here c is the constant

$$c = (4\pi)^{(n-1)/2}\Gamma(n/2)/\Gamma(1/2)\,.$$

Proof. We use the connection between the powers of L and the Riesz potentials in Chapter V, §5. Using (17) we in fact have

$$(\widehat{f})^\vee = 2^{n-1}\pi^{\frac{n}{2}-1}\Gamma(n/2)I^{n-1}f\,. \tag{32}$$

By Chapter V, Proposition 5.7, we thus obtain the desired formula (31).

For n odd one can give a more geometric proof of (31). We start with some general useful facts about the mean value operator M^r. It is a familiar fact that if $f \in C^2(\mathbf{R}^n)$ is a radial function, i.e., $f(x) = F(r), r = |x|$, then

$$(Lf)(x) = \frac{d^2F}{dr^2} + \frac{n-1}{r}\frac{dF}{dr}\,. \tag{33}$$

This is immediate from the relations

$$\frac{\partial^2 f}{\partial x_i^2} = \frac{\partial^2 f}{\partial r^2}\left(\frac{\partial r}{\partial x_i}\right)^2 + \frac{\partial f}{\partial r}\frac{\partial^2 r}{\partial x_i^2}\,.$$

Lemma 3.2. *(i)* $LM^r = M^r L$ *for each* $r > 0$.

(ii) For $f \in C^2(\mathbf{R}^n)$ *the mean value* $(M^r f)(x)$ *satisfies the "Darboux equation"*

$$L_x\big((M^r f)(x)\big) = \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right)(M^r f(x))\,,$$

that is, the function $F(x,y) = (M^{|y|}f)(x)$ *satisfies*

$$L_x(F(x,y)) = L_y(F(x,y))\,.$$

Proof. We prove this group theoretically, using expression (15) for the mean value. For $z \in \mathbf{R}^n, k \in K$ let T_z denote the translation $x \to x + z$ and R_k the rotation $x \to k \cdot x$. Since L is invariant under these transformations, we have if $r = |y|$,

$$\begin{aligned}(LM^r f)(x) = \int_K L_x(f(x+k\cdot y))\,dk &= \int_K (Lf)(x+k\cdot y)\,dk = (M^r Lf)(x) \\ &= \int_K [(Lf)\circ T_x \circ R_k](y)\,dk = \int_K [L(f\circ T_x\circ R_k)](y)\,dk \\ &= L_y\left(\int_K f(x+k\cdot y)\right)\,dk\end{aligned}$$

which proves the lemma.

Now suppose $f \in \mathcal{S}(\mathbf{R}^n)$. Fix a hyperplane ξ_0 through 0, and an isometry $g \in \mathbf{M}(n)$. As k runs through $\mathbf{O}(n)$, $gk \cdot \xi_0$ runs through the set of hyperplanes through $g \cdot 0$, and we have

$$\check{\varphi}(g\cdot 0) = \int_K \varphi(gk\cdot\xi_0)\,dk$$

and therefore

$$\begin{aligned}(\widehat{f})^\vee(g\cdot 0) &= \int_K \left(\int_{\xi_0} f(gk\cdot y)\,dm(y)\right)dk \\ &= \int_{\xi_0} dm(y)\int_K f(gk\cdot y)\,dk = \int_{\xi_0} (M^{|y|}f)(g\cdot 0)\,dm(y)\,.\end{aligned}$$

Hence

$$(34) \qquad ((\widehat{f}))^{\vee}(x) = \Omega_{n-1} \int_0^{\infty} (M^r f)(x) r^{n-2}\, dr\,,$$

where Ω_{n-1} is the area of the unit sphere in $\mathbf{R}^{n-1}$. Applying L to (34), using (33) and Lemma 3.2, we obtain

$$(35) \qquad L((\widehat{f})^{\vee}) = \Omega_{n-1} \int_0^{\infty} \left(\frac{d^2F}{dr^2} + \frac{n-1}{r}\frac{dF}{dr} \right) r^{n-2}\, dr$$

where $F(r) = (M^r f)(x)$. Integrating by parts and using

$$F(0) = f(x),\ \lim_{r\to\infty} r^k F(r) = 0\,,$$

we get

$$L((\widehat{f})^{\vee}) = \begin{cases} -\Omega_{n-1} f(x) & \text{if} \quad n = 3\,, \\ -\Omega_{n-1}(n-3) \int_0^{\infty} F(r) r^{n-4}\, dr & \quad (n > 3)\,. \end{cases}$$

More generally,

$$L_x \left(\int_0^{\infty} (M^r f)(x) r^k\, dr \right) = \begin{cases} -(n-2) f(x) & \text{if } k = 1\,, \\ -(n-1-k)(k-1) \int_0^{\infty} F(r) r^{k-2} dr, & (k > 1)\,. \end{cases}$$

If n is odd the formula in Theorem 3.1 follows by iteration. Although we assumed $f \in \mathcal{S}(\mathbf{R}^n)$ the proof is valid under much weaker assumptions.

Remark 3.3. The condition $f(x) = 0(|x|^{-N})$ for some $N > n$ cannot in general be dropped. In [1982] Zalcman has given an example of a smooth function f on $\mathbf{R}^2$ satisfying $f(x) = 0(|x|^{-2})$ on all lines with $\widehat{f}(\xi) = 0$ for all lines ξ and yet $f \neq 0$. The function is even $f(x) = 0(|x|^{-3})$ on each line which is not the x-axis. See also Armitage and Goldstein [1993].

Remark 3.4. It is interesting to observe that while the inversion formula requires $f(x) = 0(|x|^{-N})$ for *one* $N > n$ the support theorem requires $f(x) = 0(|x|^{-N})$ for *all* N as mentioned in Remark 2.9.

We shall now prove a similar inversion formula for the dual transform $\varphi \to \check{\varphi}$ on the subspace $\mathcal{S}^*(\mathbf{P}^n)$.

Theorem 3.5. *We have*

$$c\varphi = (-\Box)^{(n-1)/2} (\check{\varphi})\widehat{}\,, \quad \varphi \in \mathcal{S}^*(\mathbf{P}^n)\,,$$

where c is the constant $(4\pi)^{(n-1)/2}\Gamma(n/2)/\Gamma(1/2)$.

Here $\Box$ denotes as before the operator $\frac{d^2}{dp^2}$ and its fractional powers are again defined in terms of the Riesz' potentials on the 1-dimensional p-space.

If n is odd our inversion formula follows from the odd-dimensional case in Theorem 3.1 if we put $f = \check{\varphi}$ and take Lemma 2.1 and Corollary 2.5 into account. Suppose now n is even. We claim that

$$((-L)^{\frac{n-1}{2}} f)\widehat{\ } = (-\Box)^{\frac{n-1}{2}} \widehat{f} \qquad f \in \mathcal{S}^*(\mathbf{R}^n)\,. \tag{36}$$

By Lemma 5.6 in Chapter V, $(-L)^{(n-1)/2} f$ belongs to $\mathcal{S}^*(\mathbf{R}^n)$. Taking the 1-dimensional Fourier transform of $((-L)^{(n-1)/2} f)\widehat{\ }$ we obtain

$$\big((-L)^{(n-1)/2} f\big)\widetilde{\ }(s\omega) = |s|^{n-1} \widetilde{f}(s\omega)\,.$$

On the other hand, for a fixed ω, $p \to \widehat{f}(\omega, p)$ is in $\mathcal{S}^*(\mathbf{R})$. By the lemma quoted, the function $p \to ((-\Box)^{(n-1)/2} \widehat{f})(\omega, p)$ also belongs to $\mathcal{S}^*(\mathbf{R})$ and its Fourier transform equals $|s|^{n-1} \widetilde{f}(s\omega)$. This proves (36). Now Theorem 3.5 follows from (36) if we put in (36)

$$\varphi = \widehat{g}\,, \quad f = (\widehat{g})^\vee\,, \quad g \in \mathcal{S}^*(\mathbf{R}^n)\,,$$

because, by Corollary 2.5, $\widehat{g}$ belongs to $\mathcal{S}^*(\mathbf{P}^n)$.

Because of its theoretical importance we now prove the inversion theorem (3.1) in a different form. The proof is less geometric and involves just the one variable Fourier transform.

Let $\mathcal{H}$ denote the Hilbert transform

$$(\mathcal{H}F)(t) = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{F(p)}{t-p}\, dp \qquad F \in \mathcal{S}(\mathbf{R})$$

the integral being considered as the Cauchy principal value (see Lemma 3.7 below). For $\varphi \in \mathcal{S}(\mathbf{P}^n)$ let $\Lambda\varphi$ be defined by

$$(\Lambda\varphi)(\omega, p) = \begin{cases} \frac{d^{n-1}}{dp^{n-1}} \varphi(\omega, p) & n \text{ odd}, \\ \mathcal{H}_p \frac{d^{n-1}}{dp^{n-1}} \varphi(\omega, p) & n \text{ even}. \end{cases} \tag{37}$$

Note that in both cases $(\Lambda\varphi)(-\omega, -p) = (\Lambda\varphi)(\omega, p)$ so $\Lambda\varphi$ is a function on $\mathbf{P}^n$.

Theorem 3.6. *Let Λ be as defined by (37). Then*

$$cf = (\Lambda \widehat{f})^\vee\,, \quad f \in \mathcal{S}(\mathbf{R}^n)\,,$$

where as before

$$c = (-4\pi)^{(n-1)/2} \Gamma(n/2)/\Gamma(1/2)\,.$$

Proof. By the inversion formula for the Fourier transform and by (4)

$$f(x) = (2\pi)^{-n} \int_{\mathbf{S}^{n-1}} d\omega \int_0^\infty \Big(\int_{-\infty}^{\infty} e^{-isp} \widehat{f}(\omega, p)\, dp \Big) e^{is\langle x, \omega\rangle} s^{n-1}\, ds$$

which we write as

$$f(x) = (2\pi)^{-n}\int_{\mathbf{S}^{n-1}} F(\omega, x)\, d\omega = (2\pi)^{-n}\int_{\mathbf{S}^{n-1}} \tfrac{1}{2}(F(\omega, x)+F(-\omega, x))\, d\omega\,.$$

Using $\widehat{f}(-\omega, p) = \widehat{f}(\omega, -p)$ this gives the formula

$$(38)\quad f(x) = \tfrac{1}{2}(2\pi)^{-n}\int_{\mathbf{S}^{n-1}} d\omega \int_{-\infty}^{\infty} |s|^{n-1} e^{is\langle x,\omega\rangle}\, ds \int_{-\infty}^{\infty} e^{-isp}\widehat{f}(\omega, p)\, dp\,.$$

If n is odd the absolute value on s can be dropped. The factor s^{n-1} can be removed by replacing $\widehat{f}(\omega, p)$ by $(-i)^{n-1}\frac{d^{n-1}}{dp^{n-1}}\widehat{f}(\omega, p)$. The inversion formula for the Fourier transform on $\mathbf{R}$ then gives

$$f(x) = \tfrac{1}{2}(2\pi)^{-n}(2\pi)^{+1}(-i)^{n-1}\int_{\mathbf{S}^{n-1}} \left\{\frac{d^{n-1}}{dp^{n-1}}\widehat{f}(\omega, p)\right\}_{p=\langle x,\omega\rangle} d\omega$$

as desired.

In order to deal with the case n even we recall some general facts.

Lemma 3.7. *Let S denote the Cauchy principal value*

$$S : \psi \to \lim_{\epsilon\to 0}\int_{|x|\geq\epsilon} \frac{\psi(x)}{x}\, dx\,.$$

Then S is a tempered distribution and $\widetilde{S}$ is the function

$$\widetilde{S}(s) = -\pi i\,\mathrm{sgn}(s) = \begin{cases} -\pi i & s \geq 0 \\ \pi i & s < 0 \end{cases}.$$

Proof. It is clear that S is tempered. Also $xS = 1$ so

$$2\pi\delta = \widetilde{1} = (xS)\widetilde{\ } = i(\widetilde{S})'\,.$$

But $\mathrm{sgn}' = 2\delta$ so $\widetilde{S} = -\pi i\,\mathrm{sgn} + C$. But $\widetilde{S}$ and sgn are odd so $C = 0$.

This implies

$$(39)\qquad (\mathcal{H}F)\widetilde{\ }(s) = \mathrm{sgn}(s)\widetilde{F}(s)\,.$$

For n even we write in (38), $|s|^{n-1} = \mathrm{sgn}(s)s^{n-1}$ and then (38) implies

$$(40)\qquad f(x) = c_0\int_{\mathbf{S}^{n-1}} d\omega \int_{\mathbf{R}} \mathrm{sgn}(s)e^{is\langle x,\omega\rangle}\, ds \int_{\mathbf{R}} \frac{d^{n-1}}{dp^{n-1}}\widehat{f}(\omega, p)e^{-isp}\, dp\,,$$

where $c_0 = \frac{1}{2}(-i)^{n-1}(2\pi)^{-n}$. Now we have for each $F \in \mathcal{S}(\mathbf{R})$ the identity

$$\int_{\mathbf{R}} \mathrm{sgn}(s)e^{ist}\left(\int_{\mathbf{R}} F(p)e^{-ips}\, dp\right) ds = 2\pi(\mathcal{H}F)(t)\,.$$

In fact, if we apply both sides to $\widetilde{\psi}$ with $\psi \in \mathcal{S}(\mathbf{R})$, the left hand side is by (39)

$$\begin{aligned}&\int_{\mathbf{R}}\Big(\int_{\mathbf{R}}\operatorname{sgn}(s)e^{ist}\widetilde{F}(s)\,ds\Big)\widetilde{\psi}(t)\,dt\\ &\quad=\int_{\mathbf{R}}\operatorname{sgn}(s)\widetilde{F}(s)2\pi\psi(s)\,ds=2\pi(\mathcal{H}F)\widetilde{\ }(\psi)=2\pi(\mathcal{H}F)(\widetilde{\psi})\,.\end{aligned}$$

Putting $F(p)=\frac{d^{n-1}}{dp^{n-1}}\widehat{f}(\omega,p)$ in (40) Theorem 3.6 follows also for n even.

For later use we add here a few remarks concerning $\mathcal{H}$. Let $F \in \mathcal{D}$ have support contained in $(-R,R)$. Then

$$-i\pi(\mathcal{H}F)(t)=\lim_{\epsilon\to 0}\int_{\epsilon<|t-p|}\frac{F(p)}{t-p}\,dp=\lim_{\epsilon\to 0}\int_I\frac{F(p)}{t-p}\,dp$$

where $I=\{p : |p|<R, \epsilon<|t-p|\}$. We decompose this last integral

$$\int_I\frac{F(p)}{t-p}\,dp=\int_I\frac{F(p)-F(t)}{t-p}\,dp+F(t)\int_I\frac{dp}{t-p}\,.$$

The last term vanishes for $|t|>R$ and all $\epsilon>0$. The first term on the right is majorized by

$$\int_{|p|<R}\Big|\frac{F(t)-F(p)}{t-p}\Big|\,dp\le 2R\sup|F'|\,.$$

Thus by the dominated convergence theorem

$$\lim_{|t|\to\infty}(\mathcal{H}F)(t)=0\,.$$

Also if $J\subset(-R,R)$ is a compact subset the mapping $F\to\mathcal{H}F$ is continuous from $\mathcal{D}_J$ into $\mathcal{E}(\mathbf{R})$ (with the topologies in Chapter V, §1).

§4 The Plancherel Formula

We recall that the functions on $\mathbf{P}^n$ have been identified with the functions φ on $\mathbf{S}^{n-1}\times\mathbf{R}$ which are even: $\varphi(-\omega,-p)=\varphi(\omega,p)$. The functional

$$\varphi\to\int_{\mathbf{S}^{n-1}}\int_{\mathbf{R}}\varphi(\omega,p)\,d\omega\,dp\qquad\varphi\in C_c(\mathbf{P}^n)\,,\tag{41}$$

is therefore a well defined measure on $\mathbf{P}^n$, denoted $d\omega\,dp$. The group $\mathbf{M}(n)$ of rigid motions of $\mathbf{R}^n$ acts transitively on $\mathbf{P}^n$: it also leaves the measure $d\omega\,dp$ invariant. It suffices to verify this latter statement for the translations

T in $\mathbf{M}(n)$ because $\mathbf{M}(n)$ is generated by them together with the rotations around 0, and these rotations clearly leave $d\omega\, dp$ invariant. But

$$(\varphi \circ T)(\omega, p) = \varphi(\omega, p + q(\omega, T))$$

where $q(\omega, T) \in \mathbf{R}$ is independent of p so

$$\iint (\varphi \circ T)(\omega, p)\, d\omega\, dp = \iint \varphi(\omega, p + q(\omega, T))\, d\omega\, dp = \iint \varphi(\omega, p)\, dp\, d\omega\,,$$

proving the invariance.

In accordance with (49)–(50) in Ch. V the fractional power $\Box^k$ is defined on $\mathcal{S}(\mathbf{P}^n)$ by

$$(42) \qquad (-\Box^k)\varphi(\omega, p) = \frac{1}{H_1(-2k)} \int_{\mathbf{R}} \varphi(\omega, q)|p-q|^{-2k-1}\, dq$$

and then the 1-dimensional Fourier transform satisfies

$$(43) \qquad ((-\Box)^k \varphi)\widetilde{\ }(\omega, s) = |s|^{2k}\widetilde{\varphi}(\omega, s)\,.$$

Now, if $f \in \mathcal{S}(\mathbf{R}^n)$ we have by (4)

$$\widehat{f}(\omega, p) = (2\pi)^{-1} \int \widetilde{f}(s\omega) e^{isp}\, ds$$

and

$$(44) \qquad (-\Box)^{\frac{n-1}{4}} \widehat{f}(\omega, p) = (2\pi)^{-1} \int_{\mathbf{R}} |s|^{\frac{n-1}{2}} \widetilde{f}(s\omega) e^{isp}\, ds\,.$$

Theorem 4.1. *The mapping* $f \to \Box^{\frac{n-1}{4}}\widehat{f}$ *extends to an isometry of* $L^2(\mathbf{R}^n)$ *onto the space* $L^2_e(\mathbf{S}^{n-1} \times \mathbf{R})$ *of even functions in* $L^2(\mathbf{S}^{n-1} \times \mathbf{R})$*, the measure on* $\mathbf{S}^{n-1} \times \mathbf{R}$ *being*

$$\tfrac{1}{2}(2\pi)^{1-n}\, d\omega\, dp\,.$$

Proof. By (44) we have from the Plancherel formula on $\mathbf{R}$

$$(2\pi) \int_{\mathbf{R}} |(-\Box)^{\frac{n-1}{4}} \widehat{f}(\omega, p)|^2\, dp = \int_{\mathbf{R}} |s|^{n-1} |\widetilde{f}(s\omega)|^2\, ds$$

so by integration over $\mathbf{S}^{n-1}$ and using the Plancherel formula for $f(x) \to \widetilde{f}(s\omega)$ we obtain

$$\int_{\mathbf{R}^n} |f(x)|^2\, dx = \tfrac{1}{2}(2\pi)^{1-n} \int_{\mathbf{S}^{n-1}\times\mathbf{R}} |\Box^{\frac{n-1}{4}} \widehat{f}(\omega, p)|^2\, d\omega\, dp\,.$$

It remains to prove that the mapping is surjective. For this it would suffice to prove that if $\varphi \in L^2(\mathbf{S}^{n-1} \times \mathbf{R})$ is even and satisfies

$$\int_{\mathbf{S}^{n-1}} \int_{\mathbf{R}} \varphi(\omega, p)(-\Box)^{\frac{n-1}{4}} \widehat{f}(\omega, p)\, d\omega\, dp = 0$$

for all $f \in \mathcal{S}(\mathbf{R}^n)$ then $\varphi = 0$. Taking Fourier transforms we must prove that if $\psi \in L^2(\mathbf{S}^{n-1} \times \mathbf{R})$ is even and satisfies

$$\int_{\mathbf{S}^{n-1}} \int_{\mathbf{R}} \psi(\omega, s)|s|^{\frac{n-1}{2}} \widetilde{f}(s\omega)\, ds\, d\omega = 0 \tag{45}$$

for all $f \in \mathcal{S}(\mathbf{R}^n)$ then $\psi = 0$. Using the condition $\psi(-\omega, -s) = \psi(\omega, s)$ we see that

$$\begin{aligned} &\int_{\mathbf{S}^{n-1}} \int_{-\infty}^{0} \psi(\omega, s)|s|^{\frac{1}{2}(n-1)} \widetilde{f}(s\omega)\, ds\, d\omega \\ = &\int_{\mathbf{S}^{n-1}} \int_{0}^{\infty} \psi(\omega, t)|t|^{\frac{1}{2}(n-1)} \widetilde{f}(t\omega)\, dt\, d\omega \end{aligned}$$

so (45) holds with $\mathbf{R}$ replaced with the positive axis $\mathbf{R}^+$. But then the function

$$\Psi(u) = \psi\left(\frac{u}{|u|}, |u|\right)|u|^{-\frac{1}{2}(n-1)}, \quad u \in \mathbf{R}^n - \{0\}$$

satisfies

$$\int_{\mathbf{R}^n} \Psi(u)\widetilde{f}(u)\, du = 0\,, \quad f \in \mathcal{S}(\mathbf{R}^n)$$

so $\Psi = 0$ almost everywhere, whence $\psi = 0$.

If we combine the inversion formula in Theorem 3.6 with (46) below we obtain the following version of the Plancherel formula

$$c \int_{\mathbf{R}^n} f(x)g(x)\, dx = \int_{\mathbf{P}^n} (\Lambda\widehat{f})(\xi)\widehat{g}(\xi)\, d\xi\,.$$

§5 Radon Transform of Distributions

It will be proved in a general context in Chapter II (Proposition 2.2) that

$$\int_{\mathbf{P}^n} \widehat{f}(\xi)\varphi(\xi)\, d\xi = \int_{\mathbf{R}^n} f(x)\check{\varphi}(x)\, dx \tag{46}$$

for $f \in C_c(\mathbf{R}^n), \varphi \in C(\mathbf{P}^n)$ if $d\xi$ is a suitable fixed $\mathbf{M}(n)$-invariant measure on $\mathbf{P}^n$. Thus $d\xi = \gamma\, d\omega\, dp$ where γ is a constant, independent of f and φ. With applications to distributions in mind we shall prove (46) in a somewhat stronger form.

Lemma 5.1. *Formula (46) holds (with $\widehat{f}$ and $\check{\varphi}$ existing almost anywhere) in the following two situations:*

(a) $f \in L^1(\mathbf{R}^n)$ *vanishing outside a compact set;* $\varphi \in C(\mathbf{P}^n)$.

(b) $f \in C_c(\mathbf{R}^n)$, φ *locally integrable.*

Also $d\xi = \Omega_n^{-1}\, d\omega\, dp$.

Proof. We shall use the Fubini theorem repeatedly both on the product $\mathbf{R}^n \times \mathbf{S}^{n-1}$ and on the product $\mathbf{R}^n = \mathbf{R} \times \mathbf{R}^{n-1}$. Since $f \in L^1(\mathbf{R}^n)$ we have for each $\omega \in \mathbf{S}^{n-1}$ that $\widehat{f}(\omega, p)$ exists for almost all p and

$$\int_{\mathbf{R}^n} f(x)\, dx = \int_{\mathbf{R}} \widehat{f}(\omega, p)\, dp\,.$$

We also conclude that $\widehat{f}(\omega, p)$ exists for almost all $(\omega, p) \in \mathbf{S}^{n-1} \times \mathbf{R}$. Next we consider the measurable function

$$(x, \omega) \to f(x)\varphi(\omega, \langle \omega, x\rangle) \;\text{ on }\; \mathbf{R}^n \times \mathbf{S}^{n-1}\,.$$

We have

$$\begin{aligned}
&\int_{\mathbf{S}^{n-1}\times\mathbf{R}^n} |f(x)\varphi(\omega, \langle\omega, x\rangle)|\, d\omega\, dx \\
&\quad = \int_{\mathbf{S}^{n-1}} \left(\int_{\mathbf{R}^n} |f(x)\varphi(\omega, \langle\omega, x\rangle)|\, dx \right) d\omega \\
&\quad = \int_{\mathbf{S}^{n-1}} \left(\int_{\mathbf{R}} |f|\widehat{\ }(\omega, p)|\varphi(\omega, p)|\, dp \right) d\omega\,,
\end{aligned}$$

which in both cases is finite. Thus $f(x) \cdot \varphi(\omega, \langle\omega, x\rangle)$ is integrable on $\mathbf{R}^n \times \mathbf{S}^{n-1}$ and its integral can be calculated by removing the absolute values above. This gives the left hand side of (46). Reversing the integrations we conclude that $\check{\varphi}(x)$ exists for almost all x and that the double integral reduces to the right hand side of (46).

The formula (46) dictates how to define the Radon transform and its dual for distributions (see Chapter V). In order to make the definitions formally consistent with those for functions we would require $\widehat{S}(\varphi) = S(\check{\varphi})$, $\check{\Sigma}(f) = \Sigma(\widehat{f})$ if S and Σ are distributions on $\mathbf{R}^n$ and $\mathbf{P}^n$, respectively. But while $f \in \mathcal{D}(\mathbf{R}^n)$ implies $\widehat{f} \in \mathcal{D}(\mathbf{P}^n)$ a similar implication does not hold for φ; we do not even have $\check{\varphi} \in \mathcal{S}(\mathbf{R}^n)$ for $\varphi \in \mathcal{D}(\mathbf{P}^n)$ so $\widehat{S}$ cannot be defined as above even if S is assumed to be tempered. Using the notation $\mathcal{E}$ (resp. $\mathcal{D}$) for the space of $\mathcal{C}^\infty$ functions (resp. of compact support) and $\mathcal{D}'$ (resp. $\mathcal{E}'$) for the space of distributions (resp. of compact support) we make the following definition.

Definition. For $S \in \mathcal{E}'(\mathbf{R}^n)$ we define the functional $\widehat{S}$ by

$$\widehat{S}(\varphi) = S(\check{\varphi}) \qquad \text{for } \varphi \in \mathcal{E}(\mathbf{P}^n)\,;$$

for $\Sigma \in \mathcal{D}'(\mathbf{P}^n)$ we define the functional $\check{\Sigma}$ by

$$\check{\Sigma}(f) = \Sigma(\widehat{f}) \qquad \text{for } f \in \mathcal{D}(\mathbf{R}^n)\,.$$

Lemma 5.2. *(i) For each* $\Sigma \in \mathcal{D}'(\mathbf{P}^n)$ *we have* $\check{\Sigma} \in \mathcal{D}'(\mathbf{R}^n)$.

(ii) For each $S \in \mathcal{E}'(\mathbf{R}^n)$ *we have* $\widehat{S} \in \mathcal{E}'(\mathbf{P}^n)$.

Proof. For $A > 0$ let $\mathcal{D}_A(\mathbf{R}^n)$ denote the set of functions $f \in \mathcal{D}(\mathbf{R}^n)$ with support in the closure of $B_A(0)$. Similarly let $\mathcal{D}_A(\mathbf{P}^n)$ denote the set of functions $\varphi \in \mathcal{D}(\mathbf{P}^n)$ with support in the closure of the "ball"

$$\beta_A(0) = \{\xi \in \mathbf{P}^n : d(0, \xi) < A\} .$$

The mapping of $f \to \widehat{f}$ from $\mathcal{D}_A(\mathbf{R}^n)$ to $\mathcal{D}_A(\mathbf{P}^n)$ being continuous (with the topologies defined in Chapter V, §1) the restriction of $\check{\Sigma}$ to each $\mathcal{D}_A(\mathbf{R}^n)$ is continuous so (i) follows. That $\widehat{S}$ is a distribution is clear from (3). Concerning its support select $R > 0$ such that S has support inside $B_R(0)$. Then if $\varphi(\omega, p) = 0$ for $|p| \leq R$ we have $\check{\varphi}(x) = 0$ for $|x| \leq R$ whence $\widehat{S}(\varphi) = S(\check{\varphi}) = 0$.

Lemma 5.3. *For* $S \in \mathcal{E}'(\mathbf{R}^n), \Sigma \in \mathcal{D}'(\mathbf{P}^n)$ *we have*

$$(LS)\widehat{\ } = \square \widehat{S} , \quad (\square \Sigma)^\vee = L\check{\Sigma} .$$

Proof. In fact by Lemma 2.1,

$$(LS)\widehat{\ }(\varphi) = (LS)(\check{\varphi}) = S(L\check{\varphi}) = S((\square\varphi)^\vee) = \widehat{S}(\square\varphi) = (\square\widehat{S})(\varphi) .$$

The other relation is proved in the same manner.

We shall now prove an analog of the support theorem (Theorem 2.6) for distributions. For $A > 0$ let $\beta_A(0)$ be defined as above and let supp denote support.

Theorem 5.4. *Let* $T \in \mathcal{E}'(\mathbf{R}^n)$ *satisfy the condition*

$$\operatorname{supp} \widehat{T} \subset C\ell(\beta_A(0)) , \quad (C\ell = \textit{ closure}) .$$

Then

$$\operatorname{supp}(T) \subset C\ell(B_A(0)) .$$

Proof. For $f \in \mathcal{D}(\mathbf{R}^n)$, $\varphi \in \mathcal{D}(\mathbf{P}^n)$ we can consider the "convolution"

$$(f \times \varphi)(\xi) = \int_{\mathbf{R}^n} f(y)\varphi(\xi - y)\, dy ,$$