Hence

$$M_2 < AF(T+1)^{1-A(\sigma-\frac{1}{2}-\delta)} \leqslant AF(T+1)(\log^{\frac{1}{2}} T)^{-A(\sigma-\frac{1}{2}-\delta)}.$$

This gives the required result if $\sigma \leqslant \frac{5}{4}$, and for $\frac{5}{4} \leqslant \sigma \leqslant 2$ it is trivial, if the A is small enough.

LEMMA γ. *For $\sigma > \frac{1}{2}$, $0 < \xi < \frac{1}{2}t$,*

$$\log \zeta(s) = i \int_{t-\xi}^{t+\xi} \frac{S(y)}{s-\frac{1}{2}-iy}\, dy + O\left\{\frac{\phi(2t)}{\xi}\right\} + O(1). \qquad (14.13.3)$$

We have

$$\int_{t+\xi}^{2t} \frac{S(y)}{s-\frac{1}{2}-iy}\, dy = \left[\frac{S_1(y)}{s-\frac{1}{2}-iy}\right]_{t+\xi}^{2t} - i\int_{t+\xi}^{2t} \frac{S_1(y)}{(s-\frac{1}{2}-iy)^2}\, dy$$

$$= O\left\{\frac{\phi(2t)}{\xi}\right\} + O\left\{\phi(2t)\int_{t+\xi}^{2t} \frac{dy}{(\sigma-\frac{1}{2})^2+(y-t)^2}\right\} = O\left\{\frac{\phi(2t)}{\xi}\right\},$$

and similarly for the integral over $(\frac{1}{2}t, t-\xi)$. The result therefore follows from (14.12.4).

Proof of Theorem 14.13. By Lemmas α and γ,

$$\log \zeta(s) = O\{\phi(4t)\log t\}^{\frac{1}{2}} \int_{t-\xi}^{t+\xi} \frac{dy}{\{(\sigma-\frac{1}{2})^2+(y-t)^2\}^{\frac{1}{2}}} + O\left\{\frac{\phi(2t)}{\xi}\right\} + O(1)$$

$$= O\left[\{\phi(4t)\log t\}^{\frac{1}{2}} \frac{\xi}{\sigma-\frac{1}{2}}\right] + O\left\{\frac{\phi(4t)}{\xi}\right\} + O(1)$$

for $\sigma-\frac{1}{2} \geqslant 1/\log\log T$, $4 \leqslant t \leqslant T$. Taking

$$\xi = A\left\{\frac{\phi(4t)}{\log t}\right\}^{\frac{1}{4}} \frac{1}{(\log\log T)^{\frac{1}{2}}},$$

we obtain $\quad \log \zeta(s) = O[(\log T)^{\frac{1}{4}}(\log\log T)^{\frac{1}{2}}\{\phi(4T)\}^{\frac{3}{4}}].$

Hence by Lemma β, for $\sigma-\frac{1}{2} \geqslant 1/\log\log T$,

$$\log \zeta(s) = O[(\log T)^{\frac{1}{4}}(\log\log T)^{\frac{1}{2}}\{\phi(4T+4)\}^{\frac{3}{4}} e^{-A(\sigma-\frac{1}{2})\log\log T}].$$

Hence

$$\int_{\frac{1}{2}+1/\log\log T}^{2} \log|\zeta(s)|\, d\sigma = O[(\log T)^{\frac{1}{4}}(\log\log T)^{-\frac{1}{2}}\{\phi(4T+4)\}^{\frac{3}{4}}]. \qquad (14.13.4)$$

Again, the real part of (14.13.3) may be written

$$\log|\zeta(s)| = \int_0^{\xi} \frac{x}{(\sigma-\frac{1}{2})^2+x^2}\{S(t-x)-S(t+x)\}\, dx + O\left\{\frac{\phi(2t)}{\xi}\right\} + O(1). \qquad (14.13.5)$$

Hence

$$\int_{\frac{1}{2}}^{\frac{1}{2}+\mu} \log|\zeta(s)|\, d\sigma = \int_0^{\xi} \arctan\frac{\mu}{x}\{S(t-x)-S(t+x)\}\, dx + $$

$$+ O\{\mu\phi(2t)/\xi\} + O(\mu)$$

$$= O[\xi\{\phi(4t)\log t\}^{\frac{1}{2}}] + O\{\mu\phi(2t)/\xi\} + O(\mu).$$

Taking $\mu = 1/\log\log T$, and ξ as before,

$$\int_{\frac{1}{2}}^{\frac{1}{2}+1/\log\log T} \log|\zeta(s)|\, d\sigma = O[(\log T)^{\frac{1}{4}}(\log\log T)^{-\frac{1}{2}}\{\phi(4T)\}^{\frac{3}{4}}]. \qquad (14.13.6)$$

Now (14.13.4), (14.13.6), and Theorem 9.9 give

$$S_1(t) = O[(\log T)^{\frac{1}{4}}(\log\log T)^{-\frac{1}{2}}\{\phi(5T)\}^{\frac{3}{4}}] \quad (4 \leqslant t \leqslant T). \quad (14.13.7)$$

Varying t and taking the maximum,

$$\phi(T) = O[(\log T)^{\frac{1}{4}}(\log\log T)^{-\frac{1}{2}}\{\phi(5T)\}^{\frac{3}{4}}].$$

Let $$\psi(T) = \max_{4 \leqslant t \leqslant T} \frac{(\log\log t)^2\phi(t)}{\log t},$$

so that $\psi(T)$ is non-decreasing and

$$\phi(T) \leqslant \frac{\log T}{(\log\log T)^2}\psi(T).$$

Then (14.13.7) gives

$$\phi(T) = O\left[\frac{\log T}{(\log\log T)^2}\{\psi(5T)\}^{\frac{3}{4}}\right]$$

or $$\frac{\phi(T)(\log\log T)^2}{\log T} = O[\{\psi(5T)\}^{\frac{3}{4}}] = O[\{\psi(5T_1)\}^{\frac{3}{4}}] \quad (T \leqslant T_1).$$

Varying T and taking the maximum,

$$\psi(T_1) = O[\{\psi(5T_1)\}^{\frac{3}{4}}].$$

But $\psi(5T_1) < 5\psi(T_1)$ for some arbitrarily large T_1; for otherwise

$$\psi(5^n t_0) \geqslant 5^n \psi(t_0),$$

i.e. $\psi(T) > AT$ for some arbitrarily large T, which is not so, since in fact $\phi(T) = O(\log T)$, $\psi(T) = O\{(\log\log T)^2\}$. Hence

$$\psi(T_1) < A\{\psi(T_1)\}^{\frac{3}{4}}, \qquad \psi(T_1) < A,$$

for some arbitrarily large T_1, and so for all T_1, since ϕ is non-decreasing.

Hence
$$\phi(T) = O\left\{\frac{\log T}{(\log\log T)^2}\right\}.$$

This proves (14.13.2), and (14.13.1) then follows from Lemma α.

The argument can be extended to show that, if $S_n(t)$ is the nth integral of $S(t)$, then
$$S_n(t) = O\left\{\frac{\log t}{(\log\log t)^{n+1}}\right\}. \tag{14.13.8}$$

14.14. Theorem 14.13 also enables us to prove inequalities for $\zeta(s)$ in the immediate neighbourhood of $\sigma = \frac{1}{2}$, a region not touched by previous arguments. We obtain first

THEOREM 14.14 (A).
$$\zeta(\tfrac{1}{2}+it) = O\left\{\exp\left(A\frac{\log t}{\log\log t}\right)\right\}. \tag{14.14.1}$$

We have
$$S(t+x)-S(t) = \{N(t+x)-N(t)\}-\{L(t+x)-L(t)\}-\{f(t+x)-f(t)\},$$
where $f(t)$ is the $O(1/t)$ of (9.3.2), and arises from the asymptotic formula for $\log\Gamma(s)$. Thus $f'(t) = O(1/t^2)$, and since $N(t+x) \geqslant N(t)$
$$S(t+x)-S(t) > -Ax\log t+O(x/t^2) > -Ax\log t.$$
Hence, by (14.13.5),
$$\log|\zeta(s)| < A\int_0^\xi \frac{x^2\log t}{(\sigma-\frac{1}{2})^2+x^2}\,dx+O\left\{\frac{\log t}{\xi(\log\log t)^2}\right\}+O(1)$$
$$< A\xi\log t+O\left\{\frac{\log t}{\xi(\log\log t)^2}\right\}+O(1)$$
uniformly for $\sigma > \frac{1}{2}$, and so by continuity for $\sigma = \frac{1}{2}$. Taking
$$\xi = 1/\log\log t$$
the result follows.

THEOREM 14.14 (B). *We have*
$$-\frac{A\log t}{\log\log t}\log\left\{\frac{2}{(\sigma-\frac{1}{2})\log\log t}\right\} < \log|\zeta(s)| < \frac{A\log t}{\log\log t}$$
$$(\tfrac{1}{2} < \sigma \leqslant \tfrac{1}{2}+A/\log\log t), \tag{14.14.2}$$
$$\arg\zeta(s) = O\left(\frac{\log t}{\log\log t}\right) \quad (\tfrac{1}{2} \leqslant \sigma \leqslant \tfrac{1}{2}+A/\log\log t). \tag{14.14.3}$$

By (14.13.1) and (14.13.3),
$$\log\zeta(s) = O\left\{\frac{\log t}{\log\log t}\int_0^\xi \frac{dx}{\sqrt{\{(\sigma-\frac{1}{2})^2+x^2\}}}\right\}+O\left\{\frac{\log t}{\xi(\log\log t)^2}\right\}+O(1).$$

Now
$$\int_0^\xi \frac{du}{\sqrt{\{(\sigma-\frac{1}{2})^2+x^2\}}} = \int_0^{\xi/(\sigma-\frac{1}{2})} \frac{dx'}{\sqrt{(1+x'^2)}},$$
which is less than 1 if $\xi \leqslant \sigma-\frac{1}{2}$, and otherwise is less than
$$1+\int_1^{\xi/(\sigma-\frac{1}{2})}\frac{dx'}{x'} = 1+\log\frac{\xi}{\sigma-\frac{1}{2}}.$$

Taking $\xi = 1/\log\log t$, the lower bound in (14.14.2) follows. The upper bound follows from the argument of the previous section. Lastly, taking imaginary parts in (14.13.3),
$$\arg\zeta(s) = \int_0^\xi \frac{\sigma-\frac{1}{2}}{x^2+(\sigma-\frac{1}{2})^2}\{S(t+x)-S(t-x)\}\,dx+$$
$$+O\left\{\frac{\log t}{\xi(\log\log t)^2}\right\}+O(1)$$
$$= O\left\{\frac{\log t}{\log\log t}\int_0^\xi \frac{\sigma-\frac{1}{2}}{x^2+(\sigma-\frac{1}{2})^2}\,dx\right\}+O\left\{\frac{\log t}{\xi(\log\log t)^2}\right\}+O(1).$$

Now
$$\int_0^\xi \frac{\sigma-\frac{1}{2}}{x^2+(\sigma-\frac{1}{2})^2}\,dx < \int_0^\infty \frac{\sigma-\frac{1}{2}}{x^2+(\sigma-\frac{1}{2})^2}\,dx = \tfrac{1}{2}\pi.$$

Hence, taking $\xi = 1$, (14.14.3) follows uniformly for $\sigma > \frac{1}{2}$, and so by continuity for $\sigma = \frac{1}{2}$.

In particular
$$\log\zeta(s) = O\left(\frac{\log t}{\log\log t}\right) \quad \left(\sigma = \frac{1}{2}+\frac{A}{\log\log t}\right). \tag{14.14.4}$$

From (14.14.4), (14.5.2), and a Phragmén–Lindelöf argument it follows that
$$\log\zeta(s) = O\left\{\frac{(\log t)^{2-2\sigma}}{\log\log t}\right\} \tag{14.14.5}$$
uniformly for
$$\frac{1}{2}+\frac{A}{\log\log t} \leqslant \sigma \leqslant 1-\delta.$$

14.15. Another result in the same order of ideas is an approximate formula for $\log\zeta(s)$, which should be compared with Theorem 9.6 (B).

THEOREM 14.15. *For* $\frac{1}{2} \leqslant \sigma \leqslant 2$,
$$\log\zeta(s) = \sum_{|t-\gamma|<1/\log\log t}\log(s-\rho)+O\left(\frac{\log t\log\log\log t}{\log\log t}\right), \tag{14.15.1}$$
where $\rho = \frac{1}{2}+i\gamma$ *runs through zeros of* $\zeta(s)$.

In Lemma α of § 3.9, let

$$f(s) = \zeta(s), \qquad s_0 = \frac{1}{2}+\frac{1}{\sqrt{3}}\delta+iT, \qquad r = \frac{4}{\sqrt{3}}\delta,$$

where $\delta = 1/\text{loglog}\, T$. By (14.14.4)

$$\left|\frac{1}{\zeta(s_0)}\right| \leqslant \exp\left(\frac{A\log T}{\text{loglog}\, T}\right).$$

The upper bound in (14.14.2) gives

$$|\zeta(s)| < \exp\left(\frac{A\log T}{\text{loglog}\, T}\right)$$

for $|s-s_0| \leqslant r$, $\sigma \geqslant \frac{1}{2}$; and for $|s-s_0| \leqslant r$, $\sigma < \frac{1}{2}$, the functional equation gives

$$|\zeta(s)| < At^{\frac{1}{2}-\sigma}|\zeta(1-s)| < At^{\sqrt{3}\delta}\exp\left(\frac{A\log T}{\text{loglog}\, T}\right) < \exp\left(\frac{A\log T}{\text{loglog}\, T}\right).$$

It therefore follows from (3.9.1) that

$$\log\zeta(s)-\log\zeta(s_0)-\sum_{|s_0-\rho|\leqslant 2\delta/\sqrt{3}}\log(s-\rho)+ \\ +\sum_{|s_0-\rho|\leqslant 2\delta/\sqrt{3}}\log(s_0-\rho) = O\left(\frac{\log T}{\text{loglog}\, T}\right)$$

for $|s-s_0| \leqslant \frac{3}{8}r$, and so in particular for $\frac{1}{2} \leqslant \sigma \leqslant \frac{1}{2}+\delta$, $t = T$.

Now $$\log\zeta(s_0) = O\left(\frac{\log T}{\text{loglog}\, T}\right).$$

Also $$s_0-\rho = \frac{1}{\sqrt{3}}\delta+i(T-\gamma).$$

Hence $$\frac{1}{\sqrt{3}}\delta \leqslant |s_0-\rho| < A,$$

and so, if the logarithm has its principal value,

$$\log(s_0-\rho) = O\left(\log\frac{1}{\delta}\right) = O(\text{logloglog}\, T).$$

Also the number of values of ρ in the above sums does not exceed

$$N\left(T+\frac{2\delta}{\sqrt{3}}\right)-N\left(T-\frac{2\delta}{\sqrt{3}}\right) = O(\delta\log T)+O\left(\frac{\log T}{\text{loglog}\, T}\right) = O\left(\frac{\log T}{\text{loglog}\, T}\right),$$

by Theorem 14.13. Hence

$$\sum_{|s_0-\rho|\leqslant 2\delta/\sqrt{3}}\log(s_0-\rho) = O\left(\frac{\log T\,\text{logloglog}\, T}{\text{loglog}\, T}\right).$$

Since $|T-\gamma| \leqslant \delta$ if $|s_0-\rho| \leqslant 2\delta/\sqrt{3}$, the result follows, with T for t and $\frac{1}{2} \leqslant \sigma \leqslant \frac{1}{2}+\delta$. It is also true for $\frac{1}{2}+\delta < \sigma \leqslant 2$, since in this region

$$\log\zeta(s) = O\left(\frac{\log T}{\text{loglog}\, T}\right),$$

and, as in the case of the other sum,

$$\sum_{|s_0-\rho|\leqslant 2\delta/\sqrt{3}}\log(s-\rho) = O\left(\frac{\log T\,\text{logloglog}\, T}{\text{loglog}\, T}\right).$$

This proves the theorem.

For $\zeta'(s)/\zeta(s)$ we obtain similarly from Lemma α of § 3.9

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{|t-\gamma|\leqslant 1/\text{loglog}\, t}\frac{1}{s-\rho}+O(\log t). \qquad (14.15.2)$$

14.16. THEOREM 14.16. *Each interval* $[T, T+1]$ *contains a value of* t *such that*

$$|\zeta(s)| > \exp\left(-A\frac{\log t}{\text{loglog}\, t}\right) \quad (\tfrac{1}{2} \leqslant \sigma \leqslant 2). \qquad (14.16.1)$$

Let $\delta = 1/\text{loglog}\, T$. Then the lower bound (14.16.1) holds automatically for $\sigma \geqslant \frac{1}{2}+\delta$, by (14.14.4). We therefore assume that $\frac{1}{2} \leqslant \sigma \leqslant \frac{1}{2}+\delta$. If $s = \sigma+it$ and $s_0 = \frac{1}{2}+\delta+it$ then, on integrating (14.15.2), we find

$$\log\frac{\zeta(s)}{\zeta(s_0)} = \sum_{|t-\gamma|\leqslant\delta}\log\left(\frac{s-\rho}{s_0-\rho}\right)+O\left(\frac{\log T}{\text{loglog}\, T}\right).$$

Moreover $\log\zeta(s_0) = O\left(\dfrac{\log T}{\text{loglog}\, T}\right)$ by (14.14.4) so that, on taking real parts

$$\log|\zeta(s)| = \sum_{|t-\gamma|\leqslant\delta}\log\left|\frac{s-\rho}{s_0-\rho}\right|+O\left(\frac{\log T}{\text{loglog}\, T}\right)$$
$$\geqslant \sum_{|t-\gamma|\leqslant\delta}\log\frac{|t-\gamma|}{2\delta}+O\left(\frac{\log T}{\text{loglog}\, T}\right),$$

since $|s-\rho| \geqslant |t-\gamma|$ and $|s_0-\rho| \leqslant 2\delta$. We now observe that

$$\int_T^{T+1}\sum_{|t-\gamma|\leqslant\delta}\log\frac{|t-\gamma|}{2\delta}\,dt = \sum_{T-\delta\leqslant\gamma\leqslant T+1+\delta}\int_{\max(\gamma-\delta,\,T)}^{\min(\gamma+\delta,\,T+1)}\log\frac{|t-\gamma|}{2\delta}\,dt$$
$$\geqslant \sum_{T-\delta\leqslant\gamma\leqslant T+1+\delta}\int_{\gamma-\delta}^{\gamma+\delta}\log\frac{|t-\gamma|}{2\delta}\,dt$$
$$= \sum_{T-\delta\leqslant\gamma\leqslant T+1+\delta}(-2\delta-2\delta\log 2)$$
$$\geqslant -A\delta\log T,$$

as there are $O(\log T)$ terms in the sum. Hence there is a t for which

$$\sum_{|t-\gamma|\leqslant\delta}\log\frac{|t-\gamma|}{2\delta}\geqslant -A\delta\log T$$

and the result follows.

In particular, if ϵ is any positive number, each $(T, T+1)$ contains a t such that

$$\frac{1}{\zeta(s)}=O(t^{\epsilon})\quad(\tfrac{1}{2}\leqslant\sigma\leqslant 2). \tag{14.16.2}$$

14.17. Mean-value theorems† for $S(t)$ and $S_1(t)$. We consider first $S_1(t)$. We begin by proving

THEOREM 14.17. *For $\frac{1}{2}T\leqslant t\leqslant T$, $\delta = T^{-\frac{1}{2}}$,*

$$\pi S_1(t)=C-\sum_{n=2}^{\infty}\frac{\Lambda_1(n)\cos(t\log n)}{n^{\frac{1}{2}}\log n}e^{-\delta n}+O\left(\frac{1}{\log\log T}\right) \tag{14.17.1}$$

where

$$C=\int_{\frac{1}{2}}^{\infty}\log|\zeta(\sigma)|\,d\sigma.$$

Making $\beta\to\infty$ in (9.9.4), we have

$$\pi S_1(t)=C-\int_{\frac{1}{2}}^{\infty}\log|\zeta(\sigma+it)|\,d\sigma. \tag{14.17.2}$$

Now, integrating (14.4.1) from s to $\frac{9}{8}+it$,

$$\log\zeta(s)-\log\zeta(\tfrac{9}{8}+it)=\sum_{n=2}^{\infty}\frac{\Lambda_1(n)}{n^s}e^{-\delta n}-\sum_{n=2}^{\infty}\frac{\Lambda_1(n)}{n^{\frac{9}{8}+it}}e^{-\delta n}+$$
$$+\sum_{\rho}\int_{\sigma}^{\frac{9}{8}}\delta^{s_1-\rho}\Gamma(\rho-s_1)\,d\sigma_1+O(\delta^{\sigma-\frac{1}{4}}\log t)\quad(\tfrac{1}{2}\leqslant\sigma\leqslant\tfrac{9}{8}).$$

Also, if $\sigma\geqslant\frac{9}{8}$,

$$\log\zeta(\sigma+it)-\sum_{n=2}^{\infty}\frac{\Lambda_1(n)}{n^s}e^{-\delta n}=\sum_{n=2}^{\infty}\frac{\Lambda_1(n)}{n^s}(1-e^{-\delta n})$$
$$=O\left(\sum_{n=2}^{\infty}n^{-\sigma}(1-e^{-\delta n})\right)=O\left(\sum_{2\leqslant n\leqslant 1/\delta}n^{1-\sigma}\delta+\sum_{n>1/\delta}n^{-\sigma}\right)$$
$$=O\{(\delta^{\sigma-1}+2^{-\sigma}\delta)\log t\}. \tag{14.17.3}$$

Hence, for $\frac{1}{2}\leqslant\sigma\leqslant\frac{9}{8}$,

$$\log\zeta(s)=\sum_{n=2}^{\infty}\frac{\Lambda_1(n)}{n^s}e^{-\delta n}+\sum_{\rho}\int_{\sigma}^{\frac{9}{8}}\delta^{s_1-\rho}\Gamma(\rho-s_1)\,d\sigma_1+O(\delta^{\frac{1}{8}})+O(\delta^{\sigma-\frac{1}{4}}\log t),$$

† Littlewood (5), Titchmarsh (2).

and integrating over $\frac{1}{2}\leqslant\sigma\leqslant\frac{9}{8}$,

$$\int_{\frac{1}{2}}^{\frac{9}{8}}\log\zeta(s)\,d\sigma=\int_{\frac{1}{2}}^{\frac{9}{8}}\left(\sum_{n=2}^{\infty}\frac{\Lambda_1(n)}{n^s}e^{-\delta n}\right)d\sigma+\sum_{\rho}\int_{\frac{1}{2}}^{\frac{9}{8}}(\sigma_1-\tfrac{1}{2})\delta^{s_1-\rho}\Gamma(\rho-s_1)\,d\sigma_1+$$
$$+O(\delta^{\frac{1}{8}})+O(\delta^{\frac{1}{4}}\log t).$$

Also, by (14.17.3),

$$\int_{\frac{9}{8}}^{\infty}\log\zeta(s)\,d\sigma=\int_{\frac{9}{8}}^{\infty}\left(\sum_{n=2}^{\infty}\frac{\Lambda_1(n)}{n^s}e^{-\delta n}\right)d\sigma+O(\delta^{\frac{1}{8}}),$$

and

$$\int_{\frac{1}{2}}^{\infty}\sum_{n=2}^{\infty}\frac{\Lambda_1(n)}{n^s}e^{-\delta n}\,d\sigma=\sum_{n=2}^{\infty}\frac{\Lambda_1(n)}{n^{\frac{1}{2}+it}\log n}e^{-\delta n},$$

the inversion being justified by absolute convergence. Hence

$$\pi S_1(t)=C-\sum_{n=2}^{\infty}\frac{\Lambda_1(n)\cos(t\log n)}{n^{\frac{1}{2}}\log n}e^{-\delta n}+$$
$$+O\left\{\sum_{\rho}\int_{\frac{1}{2}}^{\frac{9}{8}}(\sigma_1-\tfrac{1}{2})\delta^{\sigma_1-\frac{1}{2}}|\Gamma(\rho-s_1)|\,d\sigma_1\right\}+O(\delta^{\frac{1}{8}})+O(\delta^{\frac{1}{4}}\log t).$$

Now

$$\Gamma(\rho-s_1)=O(e^{-A|\gamma-t|})\quad(|\gamma-t|\geqslant 1),$$

$$\Gamma(\rho-s_1)=O\left(\frac{1}{|\rho-s_1|}\right)=O\left\{\frac{1}{\{(\sigma_1-\frac{1}{2})^2+|\gamma-t|^2\}^{\frac{1}{2}}}\right\}\quad(|\gamma-t|<1).$$

Hence

$$\sum_{\rho}=\sum_{|\gamma-t|<1/\log\log t}+\sum_{1/\log\log t\leqslant|\gamma-t|<1}+\sum_{|\gamma-t|\geqslant 1}$$
$$=O\left(\sum_{|\gamma-t|<1/\log\log t}\int_{\frac{1}{2}}^{\frac{9}{8}}\delta^{\sigma_1-\frac{1}{2}}\,d\sigma_1\right)+$$
$$+O\left(\sum_{1/\log\log t\leqslant|\gamma-t|<1}\frac{1}{|\gamma-t|}\int_{\frac{1}{2}}^{\frac{9}{8}}(\sigma_1-\tfrac{1}{2})\delta^{\sigma_1-\frac{1}{2}}\,d\sigma_1\right)+$$
$$+O\left(\sum_{|\gamma-t|\geqslant 1}e^{-A|\gamma-t|}\int_{\frac{1}{2}}^{\frac{9}{8}}(\sigma_1-\tfrac{1}{2})\delta^{\sigma_1-\frac{1}{2}}\,d\sigma_1\right).$$

Now
$$\int_{\frac{1}{2}}^{\frac{9}{8}} \delta^{\sigma_1-\frac{1}{2}}\,d\sigma_1 < \int_0^\infty e^{-x\log 1/\delta}\,dx = \log^{-1}(1/\delta),$$
$$\int_{\frac{1}{2}}^{\frac{9}{8}} (\sigma_1-\tfrac{1}{2})\delta^{\sigma_1-\frac{1}{2}}\,d\sigma_1 < \int_0^\infty xe^{-x\log 1/\delta}\,dx = \log^{-2}(1/\delta).$$
As in § 14.5,
$$\sum_{|\gamma-t|\geq 1} e^{-A|\gamma-t|} = O(\log t).$$

Also, by (14.13.1), for $t-1 \leqslant t' \leqslant t+1$
$$N\left(t'+\frac{1}{\log\log t}\right)-N(t') = O\left(\frac{\log t}{\log\log t}\right).$$
Hence
$$\sum_{|\gamma-t|\leqslant 1/\log\log t} 1 = O\left(\frac{\log t}{\log\log t}\right),$$
and
$$\sum_{t+1/\log\log t<\gamma\leqslant t+1} \frac{1}{\gamma-t} = \sum_{m<\log\log t}\ \sum_{t+m/\log\log t\leqslant\gamma<t+(m+1)/\log\log t} \frac{1}{\gamma-t}$$
$$= \sum_{m<\log\log t} O\left(\frac{1}{m/\log\log t}\,\frac{\log t}{\log\log t}\right) = O(\log t\log\log\log t).$$
Hence
$$\sum_\rho = O\left(\frac{\log t}{\log 1/\delta\,\log\log t}\right)+O\left(\frac{\log t\log\log\log t}{\log^2 1/\delta}\right)+O\left(\frac{\log t}{\log^2 1/\delta}\right) = O\left(\frac{1}{\log\log t}\right)$$
for the given δ and t. This proves the theorem.

14.18. LEMMA 14.18. *If $a_n = O(1)$, $\delta < \frac{1}{2}$, then*
$$\frac{2}{T}\int_{\frac{1}{2}T}^{T}\left|\sum_{n=1}^\infty \frac{a_n}{n^s}e^{-\delta n}\right|^2 dt = \sum_{n=1}^\infty \frac{|a_n|^2}{n^{2\sigma}}e^{-2\delta n}+O\left(\frac{1}{T\delta}\log\frac{1}{\delta}\right)$$
uniformly for $\sigma \geqslant \frac{1}{2}$. Similarly, if $a_n = O(\log n)$, the formula holds with a remainder term
$$O\left(\frac{1}{T\delta}\log^3\frac{1}{\delta}\right).$$

The left-hand side is
$$\sum_{m=1}^\infty\sum_{n=1}^\infty \frac{2}{T}\int_{\frac{1}{2}T}^{T}\frac{a_m\bar a_n}{(mn)^\sigma}e^{-(m+n)\delta}\left(\frac{n}{m}\right)^{it}dt = \sum_{m=n}+\sum_{m\neq n}.$$
Clearly
$$\sum_{m=n} = \sum_{n=1}^\infty \frac{|a_n|^2}{n^{2\sigma}}e^{-2\delta n}.$$
Also
$$\sum_{m\neq n} = O\left(\frac{1}{T}\right)\sum_{m<n}\frac{e^{-n\delta}}{(mn)^{\frac{1}{2}}\log n/m}.$$

Now
$$\sum_{n=m+1}^{2m}\frac{e^{-n\delta}}{(mn)^{\frac{1}{2}}\log n/m} = O\left(\sum_{n=m+1}^{2m}\frac{e^{-m\delta}}{m\log\{1+(n-m)/m\}}\right)$$
$$= O\left(e^{-m\delta}\sum_{n=m+1}^{2m}\frac{1}{n-m}\right) = O(e^{-m\delta}\log m),$$
$$\sum_{n=2m+1}^{\infty}\frac{e^{-n\delta}}{(mn)^{\frac{1}{2}}\log n/m} = O\left(\frac{1}{m}\sum_{n=2m+1}^\infty e^{-n\delta}\right) = O\left(\frac{e^{-m\delta}}{m\delta}\right).$$
Hence
$$\sum_{m\neq n} = O\left(\frac{1}{T}\right)\sum_{m=1}^\infty\left(\log m+\frac{1}{m\delta}\right)e^{-m\delta} = O\left(\frac{1}{T\delta}\log\frac{1}{\delta}\right).$$
This proves the first part. In the second part we have a pair of logarithms running throughout the remainder terms, and this is easily seen to produce the extra $\log^2 1/\delta$ in the result.

14.19. THEOREM 14.19. *As $T\to\infty$,*
$$\frac{1}{T}\int_0^T\{S_1(t)\}^2\,dt \sim \frac{C^2}{\pi^2}+\frac{1}{2\pi^2}\sum_{n=2}^\infty\frac{\Lambda_1^2(n)}{n\log^2 n}.$$
Let
$$f(t) = Ce^{-\delta}-\sum_{n=2}^\infty\frac{\Lambda_1(n)\cos(t\log n)}{n^{\frac{1}{2}}\log n}e^{-\delta n}.$$
Then, as in the lemma,
$$\frac{2}{T}\int_{\frac{1}{2}T}^{T}\{f(t)\}^2\,dt = C^2e^{-2\delta}+\frac{1}{2}\sum_{n=2}^\infty\frac{\Lambda_1^2(n)}{n\log^2 n}e^{-2\delta n}+O\left(\frac{\log 1/\delta}{T\delta}\right),$$
and we can replace δ by 0 in the first two terms on the right with error
$$O(\delta)+O\left(\sum_{n=2}^\infty\frac{1-e^{-2\delta n}}{n\log^2 n}\right) = O(\delta)+O\left(\sum_{n\leqslant 1/\delta}\frac{\delta}{\log^2 n}\right)+O\left(\sum_{n>1/\delta}\frac{1}{n\log^2 n}\right)$$
$$= O\{1/\log(1/\delta)\}.$$
Hence, taking $\delta = T^{-\frac{1}{2}}$,
$$\frac{2}{T}\int_{\frac{1}{2}T}^{T}\{f(t)\}^2\,dt = C^2+\frac{1}{2}\sum_{n=2}^\infty\frac{\Lambda_1^2(n)}{n\log^2 n}+O\left(\frac{1}{\log T}\right).$$
Hence
$$\frac{2\pi^2}{T}\int_{\frac{1}{2}T}^{T}\{S_1(t)\}^2\,dt = \frac{2}{T}\int_{\frac{1}{2}T}^{T}\left\{f(t)+O\left(\frac{1}{\log\log T}\right)\right\}^2dt$$
$$= \frac{2}{T}\int_{\frac{1}{2}T}^{T}\{f(t)\}^2\,dt+O\left\{\frac{1}{T\log\log T}\int_{\frac{1}{2}T}^{T}|f(t)|\,dt\right\}+O\left\{\frac{1}{(\log\log T)^2}\right\},$$

and, since $\int_{\frac{1}{2}T}^{T} |f(t)|\,dt \leqslant \left[\int_{\frac{1}{2}T}^{T} dt \int_{\frac{1}{2}T}^{T} \{f(t)\}^2\,dt\right]^{\frac{1}{2}} = O(T)$,

it follows that

$$\frac{2}{T}\int_{\frac{1}{2}T}^{T} \{S_1(t)\}^2\,dt = \frac{C^2}{\pi^2}+\frac{1}{2\pi^2}\sum_{n=2}^{\infty}\frac{\Lambda_1^2(n)}{n\log^2 n}+O\left(\frac{1}{\log\log T}\right).$$

Replacing T by $\frac{1}{2}T$, $\frac{1}{4}T$,... and adding, we obtain the result.

14.20. The corresponding problem involving $S(t)$ is naturally more difficult, but it has recently been solved by A. Selberg (4). The solution depends on the following formula for $\zeta'(s)/\zeta(s)$

THEOREM 14.20. *Without any hypothesis*

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n<x^2}\frac{\Lambda_x(n)}{n^s}+\frac{x^{2(1-s)}-x^{1-s}}{(1-s)^2\log x}+$$
$$+\frac{1}{\log x}\sum_{q=1}^{\infty}\frac{x^{-2q-s}-x^{-2(2q+s)}}{(2q+s)^2}+\frac{1}{\log x}\sum_{\rho}\frac{x^{\rho-s}-x^{2(\rho-s)}}{(s-\rho)^2},$$

where

$$\Lambda_x(n) = \Lambda(n) \quad (1 \leqslant n \leqslant x), \qquad \frac{\Lambda(n)\log(x^2/n)}{\log x} \quad (x \leqslant n \leqslant x^2).$$

Let $\alpha = \max(2, 1+\sigma)$. Then

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{x^{z-s}-x^{2(z-s)}}{(z-s)^2}\frac{\zeta'(z)}{\zeta(z)}\,dz$$
$$= -\frac{1}{2\pi i}\sum_{n=1}^{\infty}\Lambda(n)\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{x^{z-s}-x^{2(z-s)}}{(z-s)^2 n^z}\,dz$$
$$= -\frac{1}{2\pi i}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}\int_{\alpha-\sigma-i\infty}^{\alpha-\sigma+i\infty}\frac{x^w-x^{2w}}{w^2 n^w}\,dw$$
$$= -\frac{1}{2\pi i}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}\left\{\int_{\alpha-\sigma-i\infty}^{\alpha-\sigma+i\infty}\left(\frac{x}{n}\right)^w\frac{dw}{w^2}-\int_{\alpha-\sigma-i\infty}^{\alpha-\sigma+i\infty}\left(\frac{x^2}{n}\right)^w\frac{dw}{w^2}\right\}$$
$$= -\sum_{n\leqslant x}\frac{\Lambda(n)}{n^s}\left(\log\frac{x}{n}-\log\frac{x^2}{n}\right)-\sum_{x<n\leqslant x^2}\frac{\Lambda(n)}{n^s}\left(-\log\frac{x^2}{n}\right)$$
$$= \log x\sum_{n\leqslant x^2}\frac{\Lambda_x(n)}{n^s}.$$

Now consider the residues obtained by moving the line of integration to the left. The residue at $z = s$ is $-\log x\,\zeta'(s)/\zeta(s)$; that at $z = 1$ is

$$-\frac{x^{1-s}-x^{2(1-s)}}{(1-s)^2};$$

those at $z = -2q$ and $z = \rho$ are

$$\frac{x^{-2q-s}-x^{2(-2q-s)}}{(-2q-s)^2}, \qquad \frac{x^{\rho-s}-x^{2(\rho-s)}}{(\rho-s)^2},$$

respectively. The result now easily follows.

14.21. THEOREM 14.21. *For $t > 2$, $4 \leqslant x \leqslant t^2$,*

$$\sigma_1 = \frac{1}{2}+\frac{1}{\log x},$$

we have

$$S(t) = -\frac{1}{\pi}\sum_{n<x^3}\frac{\Lambda_x(n)}{n^{\sigma_1}}\frac{\sin(t\log n)}{\log n}+O\left(\frac{1}{\log x}\left|\sum_{n<x^3}\frac{\Lambda_x(n)}{n^{\sigma_1+it}}\right|\right)+O\left(\frac{\log t}{\log x}\right).$$

By the previous theorem,

$$\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)} = -\sum_{n<x^2}\frac{\Lambda_x(n)}{n^{\sigma+it}}+O\left\{\frac{x^{2(1-\sigma)}+x^{1-\sigma}}{t^2\log x}\right\}+\frac{2\omega x^{\frac{1}{2}-\sigma}}{\log x}\sum_{\gamma}\frac{1}{(\sigma_1-\frac{1}{2})^2+(t-\gamma)^2} \tag{14.21.1}$$

for $\sigma \geqslant \sigma_1$, where $|\omega| < 1$. Now

$$\frac{x^{2(1-\sigma)}+x^{1-\sigma}}{t^2\log x} \leqslant \frac{x^{1-2\sigma}+x^{-\sigma}}{\log x} < 2x^{\frac{1}{2}-\sigma}.$$

Hence

$$\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)} = -\sum_{n<x^2}\frac{\Lambda_x(n)}{n^{\sigma+it}}+2\omega x^{\frac{1}{2}-\sigma}\sum_{\gamma}\frac{\sigma_1-\frac{1}{2}}{(\sigma_1-\frac{1}{2})^2+(t-\gamma)^2}+O(x^{\frac{1}{2}-\sigma}). \tag{14.21.2}$$

Now by (2.12.7)

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right)+O(\log t).$$

Hence

$$\mathbf{R}\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)} = \sum_{\gamma}\left\{\frac{\sigma-\frac{1}{2}}{(\sigma-\frac{1}{2})^2+(t-\gamma)^2}+\frac{\frac{1}{2}}{\frac{1}{4}+\gamma^2}\right\}+O(\log t)$$
$$= \sum_{\gamma}\frac{\sigma-\frac{1}{2}}{(\sigma-\frac{1}{2})^2+(t-\gamma)^2}+O(\log t).$$

Taking real parts in (14.21.2), substituting this on the left, and taking $\sigma = \sigma_1$,

$$\sum_\gamma \frac{\sigma_1-\frac{1}{2}}{(\sigma_1-\frac{1}{2})^2+(t-\gamma)^2}+O(\log t)$$
$$= -\mathbf{R}\sum_{n<x^2}\frac{\Lambda_x(n)}{n^{\sigma_1+it}}+\frac{2\omega'}{e}\sum_\gamma \frac{\sigma_1-\frac{1}{2}}{(\sigma_1-\frac{1}{2})^2+(t-\gamma)^2} \quad (|\omega'|<1).$$

Hence

$$\left(1-\frac{2\omega'}{e}\right)\sum_\gamma \frac{\sigma_1-\frac{1}{2}}{(\sigma_1-\frac{1}{2})^2+(t-\gamma)^2} = -\mathbf{R}\sum_{n<x^2}\frac{\Lambda_x(n)}{n^{\sigma_1+it}}+O(\log t).$$

Here
$$1-\frac{2\omega'}{e}>1-\frac{2}{e}>\frac{1}{4}.$$

Hence

$$\sum_\gamma \frac{\sigma_1-\frac{1}{2}}{(\sigma_1-\frac{1}{2})^2+(t-\gamma)^2} = O\left|\sum_{n<x^2}\frac{\Lambda_x(n)}{n^{\sigma_1+it}}\right|+O(\log t). \quad (14.21.3)$$

Inserting this in (14.21.2), we get

$$\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)} = -\sum_{n<x^2}\frac{\Lambda_x(n)}{n^{\sigma+it}}+O\left\{x^{\frac{1}{2}-\sigma}\left|\sum_{n<x^2}\frac{\Lambda_x(n)}{n^{\sigma_1+it}}\right|\right\}+O(x^{\frac{1}{2}-\sigma}\log t). \quad (14.21.4)$$

Now

$$\arg\zeta(\tfrac{1}{2}+it) = -\int_{\frac{1}{2}}^{\infty}\mathbf{I}\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)}\,d\sigma$$
$$= -\int_{\sigma_1}^{\infty}\mathbf{I}\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)}\,d\sigma-(\sigma_1-\tfrac{1}{2})\mathbf{I}\frac{\zeta'(\sigma_1+it)}{\zeta(\sigma_1+it)}+$$
$$+\int_{\frac{1}{2}}^{\sigma_1}\mathbf{I}\left\{\frac{\zeta'(\sigma_1+it)}{\zeta(\sigma_1+it)}-\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)}\right\}d\sigma$$
$$= J_1+J_2+J_3.$$

By (14.21.4)

$$J_1 = \mathbf{I}\int_{\sigma_1}^{\infty}\sum_{n<x^2}\frac{\Lambda_x(n)}{n^{\sigma+it}}\,d\sigma+O\left\{\left|\sum_{n<x^2}\frac{\Lambda_x(n)}{n^{\sigma_1+it}}\right|\int_{\sigma_1}^{\infty}x^{\frac{1}{2}-\sigma}\,d\sigma\right\}+O\left\{\log t\int_{\sigma_1}^{\infty}x^{\frac{1}{2}-\sigma}\,d\sigma\right\}$$
$$= \mathbf{I}\sum_{n<x^2}\frac{\Lambda_x(n)}{n^{\sigma_1+it}\log n}+O\left\{\frac{1}{\log x}\left|\sum_{n<x^2}\frac{\Lambda_x(n)}{n^{\sigma_1+it}}\right|\right\}+O\left(\frac{\log t}{\log x}\right).$$

Also, by (14.21.4) with $\sigma = \sigma_1$,

$$|J_2| \leqslant (\sigma_1-\tfrac{1}{2})\left|\frac{\zeta'(\sigma_1+it)}{\zeta(\sigma_1+it)}\right|$$
$$= O\left\{(\sigma_1-\tfrac{1}{2})\left|\sum_{n<x^2}\frac{\Lambda_x(n)}{n^{\sigma_1+it}}\right|\right\}+O\{(\sigma_1-\tfrac{1}{2})\log t\}$$
$$= O\left\{\frac{1}{\log x}\left|\sum_{n<x^2}\frac{\Lambda_x(n)}{n^{\sigma_1+it}}\right|\right\}+O\left(\frac{\log t}{\log x}\right).$$

It remains to estimate J_3. For $\frac{1}{2}<\sigma\leqslant\sigma_1$,

$$\mathbf{I}\left\{\frac{\zeta'(\sigma_1+it)}{\zeta(\sigma_1+it)}-\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)}\right\} = \sum_\rho \mathbf{I}\left(\frac{1}{\sigma_1+it-\rho}-\frac{1}{\sigma+it-\rho}\right)+O(\log t)$$
$$= \sum_\gamma \frac{(t-\gamma)\{(\sigma-\frac{1}{2})^2-(\sigma_1-\frac{1}{2})^2\}}{\{(\sigma-\frac{1}{2})^2+(t-\gamma)^2\}\{(\sigma_1-\frac{1}{2})^2+(t-\gamma)^2\}}+O(\log t).$$

Hence

$$\left|\mathbf{I}\left\{\frac{\zeta'(\sigma_1+it)}{\zeta(\sigma_1+it)}-\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)}\right\}\right|$$
$$\leqslant \sum_\gamma \frac{|t-\gamma|(\sigma_1-\frac{1}{2})^2}{\{(\sigma-\frac{1}{2})^2+(t-\gamma)^2\}\{(\sigma_1-\frac{1}{2})^2+(t-\gamma)^2\}}+O(\log t).$$

Hence

$$|J_3| \leqslant \sum_\gamma \frac{(\sigma_1-\frac{1}{2})^2}{(\sigma_1-\frac{1}{2})^2+(t-\gamma)^2}\int_{\frac{1}{2}}^{\infty}\frac{|t-\gamma|\,d\sigma}{(\sigma-\frac{1}{2})^2+(t-\gamma)^2}+O\{(\sigma_1-\tfrac{1}{2})\log t\}$$
$$\leqslant \tfrac{1}{2}\pi(\sigma_1-\tfrac{1}{2})\sum_\gamma \frac{(\sigma_1-\frac{1}{2})}{(\sigma_1-\frac{1}{2})^2+(t-\gamma)^2}+O\{(\sigma_1-\tfrac{1}{2})\log t\}$$
$$= O\left\{\frac{1}{\log x}\left|\sum_{n<x^2}\frac{\Lambda_x(n)}{n^{\sigma_1+it}}\right|\right\}+O\left(\frac{\log t}{\log x}\right),$$

by (14.21.3). The theorem follows from these results.

Theorem 14.21 leads to an alternative proof of Theorem 14.13; for taking $x = \sqrt{(\log t)}$ we obtain

$$S(t) = O\left(\sum_{n\leqslant x^2}\frac{1}{n^{\frac{1}{2}}}\right)+O\left(\frac{1}{\log x}\sum_{n\leqslant x^2}\frac{\log n}{n^{\frac{1}{2}}}\right)+O\left(\frac{\log t}{\log x}\right)$$
$$= O(x)+O(x)+O\left(\frac{\log t}{\log x}\right)$$
$$= O\left(\frac{\log t}{\log\log t}\right).$$

14.22. THEOREM 14.22. *For*

$$T^a \leqslant x \leqslant T^{\frac{1}{2}} \quad (0 < a \leqslant \tfrac{1}{2})$$

$$\int\limits_{\frac{1}{2}T}^{T} \left\{S(t)+\frac{1}{\pi}\sum_{p<x^2}\frac{\sin(t\log p)}{\sqrt{p}}\right\}^2 dt = O(T).$$

We have

$$S(t)+\frac{1}{\pi}\sum_{p<x^2}\frac{\sin(t\log p)}{\sqrt{p}} = \frac{1}{\pi}\sum_{p<x^2}\frac{\Lambda(p)-\Lambda_x(p)p^{\frac{1}{2}-\sigma_1}}{p^{\frac{1}{2}}\log p}\sin(t\log p)+$$

$$+O\left\{\frac{1}{\log x}\left|\sum_{p<x^2}\frac{\Lambda_x(p)}{p^{\sigma_1+it}}\right|\right\}+O\left\{\left|\sum_{p^2<x^2}\frac{\Lambda_x(p^2)}{p^{2(\sigma_1+it)}\log p}\right|\right\}+$$

$$+O\left\{\frac{1}{\log x}\left|\sum_{p^2<x^2}\frac{\Lambda_x(p^2)}{p^{2(\sigma_1+it)}}\right|\right\}+O\left(\sum_{r>2}\sum_{p}\frac{1}{p^{\frac{1}{2}r}}\right)+O\left(\frac{\log t}{\log x}\right).$$

The last term is bounded if $\frac{1}{2}T \leqslant t \leqslant T$, $x \geqslant T^a$, where a is a fixed positive constant. The last term but one is

$$O\left(\sum_{p}\sum_{r=3}^{\infty}\frac{1}{p^{\frac{1}{2}r}}\right) = O\left(\sum_{p}\frac{p^{-\frac{3}{2}}}{1-p^{-\frac{1}{2}}}\right) = O(1).$$

Now consider the first term on the right. If $p \leqslant x$,

$$\Lambda(p)-\Lambda_x(p)p^{\frac{1}{2}-\sigma_1} = (1-p^{\frac{1}{2}-\sigma_1})\log p$$

$$= (1-p^{-1/\log x})\log p = (1-e^{-\log p/\log x})\log p = O\left(\frac{\log^2 p}{\log x}\right),$$

and, if $x < p \leqslant x^2$, it is

$$O\{\Lambda_x(p)\} = O\left\{\log p\frac{\log x^2/p}{\log x}\right\} = O\left(\frac{\log^2 p}{\log x}\right).$$

Hence the first term is the imaginary part of

$$\sum_{p<x^2}\frac{\alpha_p}{p^{\frac{1}{2}+it}},$$

where $$\alpha_p = \frac{\Lambda(p)-\Lambda_x(p)p^{\frac{1}{2}-\sigma_1}}{\log p} = O(\log p/\log x).$$

Now

$$\int\limits_{\frac{1}{2}T}^{T}\left|\sum_{p<x^2}\frac{\alpha_p}{p^{\frac{1}{2}+it}}\right|^2 dt = \sum_{p<x^2}\sum_{q<x^2}\frac{\alpha_p\alpha_q}{p^{\frac{1}{2}}q^{\frac{1}{2}}}\int\limits_{\frac{1}{2}T}^{T}\left(\frac{q}{p}\right)^{it}dt$$

$$= O\left(T\sum_{p<x^2}\frac{\alpha_p^2}{p}\right)+O\left(\sum_{p\neq q}\sum\frac{|\alpha_p\alpha_q|}{p^{\frac{1}{2}}q^{\frac{1}{2}}}\frac{1}{|\log p/q|}\right).$$

Since $$\alpha_p^2 = O\left(\frac{\log^2 p}{\log^2 x}\right) = O\left(\frac{\log p}{\log x}\right),$$

the first term is $$O\left(T\frac{1}{\log x}\sum_{p<x^2}\frac{\log p}{p}\right) = O(T),$$

by (3.14.3). The second term is

$$O\left(\sum_{p<x^2}\sum_{q\leqslant\frac{1}{2}p}\frac{|\alpha_p\alpha_q|}{p^{\frac{1}{2}}q^{\frac{1}{2}}}\right)+O\left(\sum_{p<x^2}\sum_{\frac{1}{2}p<q<p}\frac{|\alpha_p\alpha_q|}{p.(p-q)/p}\right)$$

$$= O\left(\sum_{p<x^2}\frac{\log p}{p^{\frac{1}{2}}\log x}\right)^2+O\left(\sum_{p<x^2}\frac{\log^2 p}{\log^2 x}.\log p\right)$$

$$= O\left(\sum_{p<x^2}\frac{1}{p^{\frac{1}{2}}}\right)^2+O\left(\sum_{p<x^2}\log p\right)$$

$$= O(x^2)+O(x^2) = O(T)$$

if $x \leqslant \sqrt{T}$.

A similar argument clearly applies to the second term. In the third term, the sum is of the form

$$\sum_{p<x}\frac{\alpha_p'}{p^{1+2it}},$$

where $\alpha_p' = O(1)$; and

$$\int\limits_{\frac{1}{2}T}^{T}\left|\sum_{p<x}\frac{\alpha_p'}{p^{1+2it}}\right|^2 dt = \sum_{p<x}\sum_{q<x}\frac{\alpha_p'\alpha_q'}{pq}\int\limits_{\frac{1}{2}T}^{T}\left(\frac{q}{p}\right)^{it}dt$$

$$= O\left(T\sum_{p<x}\frac{1}{p^2}\right)+O\left(\sum_{p\neq q}\sum\frac{1}{pq|\log p/q|}\right)$$

$$= O(T)+O\left(\sum_{p<x}\sum_{q\leqslant\frac{1}{2}p}\frac{1}{pq}\right)+O\left(\sum_{p<x}\sum_{\frac{1}{2}p<q<p}\frac{1}{p^2.(p-q)/p}\right)$$

$$= O(T)+O(\log^2 x)+O(\log^2 x) = O(T).$$

Similarly for the fourth term, and the result follows.

14.23. THEOREM 14.23. *If* $T^a \leqslant x \leqslant T^{\frac{1}{4}}$,

$$\int\limits_{\frac{1}{2}T}^{T}\left\{\sum_{p<x^2}\frac{\sin(t\log p)}{\sqrt{p}}\right\}^2 dt = \tfrac{1}{4}T\log\log T+O(T).$$

This is

$$\sum_{p<x^2}\sum_{q<x^2}\frac{1}{p^{\frac12}q^{\frac12}}\int\limits_{\frac12 T}^{T}\sin(t\log p)\sin(t\log q)\,dt$$
$$= \sum_{p<x^2}\frac{1}{p}\left\{\tfrac14 T+O\left(\frac{1}{\log p}\right)\right\}+O\left(\sum_{p\neq q}\sum\frac{1}{p^{\frac12}q^{\frac12}|\log p/q|}\right).$$

Now, by (3.14.5),

$$\sum_{p<x^2}\frac{1}{p} = \log\log x^2+O(1) = \log\log T+O(1)$$

and (since $p_n > An\log n$)

$$\sum\frac{1}{p\log p} = O(1).$$

Hence the first term is

$$\tfrac14 T\log\log T+O(T).$$

Also $$|\log p/q| > A/p > A/x^2.$$

Hence the remainder is

$$O\left\{x^2\left(\sum_{p<x^2}\frac{1}{p^{\frac12}}\right)^2\right\} = O(x^2.x^2) = O(x^4),$$

and the result follows if $x \leqslant T^{\frac14}$.

14.24. Theorem 14.24.

$$\int\limits_0^T \{S(t)\}^2\,dt \sim \frac{1}{2\pi^2}T\log\log T.$$

For

$$\int\limits_{\frac12 T}^{T}\{S(t)\}^2\,dt = \int\limits_{\frac12 T}^{T}\left\{S(t)+\frac{1}{\pi}\sum_{p<x^2}\frac{\sin(t\log p)}{\sqrt p}-\frac{1}{\pi}\sum_{p<x^2}\frac{\sin(t\log p)}{\sqrt p}\right\}^2 dt$$
$$= \int\limits_{\frac12 T}^{T}\left\{S(t)+\frac{1}{\pi}\sum_{p<x^2}\frac{\sin(t\log p)}{\sqrt p}\right\}^2 dt-$$
$$-\frac{2}{\pi}\int\limits_{\frac12 T}^{T}\left\{S(t)+\frac{1}{\pi}\sum_{p<x^2}\frac{\sin(t\log p)}{\sqrt p}\right\}\sum_{p<x^2}\frac{\sin(t\log p)}{\sqrt p}\,dt+$$
$$+\frac{1}{\pi^2}\int\limits_{\frac12 T}^{T}\left\{\sum_{p<x^2}\frac{\sin(t\log p)}{\sqrt p}\right\}^2 dt$$
$$= O(T)+O\{T(\log\log T)^{\frac12}\}+\frac{1}{4\pi^2}T\log\log T+O(T)$$

(using Schwarz's inequality on the middle term). The result then follows by addition.

It can be proved in a similar way that

$$\int\limits_0^T \{S(t)\}^{2k}\,dt \sim \frac{(2k)!}{k!(2\pi)^{2k}}T(\log\log T)^k \tag{14.24.1}$$

for every positive integer k.

14.25. The Dirichlet series for $1/\zeta(s)$. It was proved in § 3.13 that the formula

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s},$$

which is elementary for $\sigma > 1$, holds also for $\sigma = 1$. On the Riemann hypothesis we can go much farther than this.†

Theorem 14.25 (A). *The series*

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} \tag{14.25.1}$$

is convergent, and its sum is $1/\zeta(s)$, for every s with $\sigma > \frac12$.

In the lemma of § 3.12, take $a_n = \mu(n)$, $f(s) = 1/\zeta(s)$, $c = 2$, and x half an odd integer. We obtain

$$\sum_{n<x}\frac{\mu(n)}{n^s} = \frac{1}{2\pi i}\int\limits_{2-iT}^{2+iT}\frac{1}{\zeta(s+w)}\frac{x^w}{w}\,dw+O\left(\frac{x^2}{T}\right)$$
$$= \frac{1}{2\pi i}\left(\int\limits_{2-iT}^{\frac12-\sigma+\delta-iT}+\int\limits_{\frac12-\sigma+\delta-iT}^{\frac12-\sigma+\delta+iT}+\int\limits_{\frac12-\sigma+\delta+iT}^{2+iT}\right)\frac{1}{\zeta(s+w)}\frac{x^w}{w}\,dw+\frac{1}{\zeta(s)}+O\left(\frac{x^2}{T}\right),$$

where $0 < \delta < \sigma-\frac12$.

By (14.2.5), the first and third integrals are

$$O\left(T^{-1+\epsilon}\int\limits_{\frac12-\sigma+\delta}^{2}x^u\,du\right) = O(T^{-1+\epsilon}x^2),$$

and the second integral is

$$O\left\{x^{\frac12-\sigma+\delta}\int\limits_{-T}^{T}(1+|t|)^{-1+\epsilon}\,dt\right\} = O(x^{\frac12-\sigma+\delta}T^{\epsilon}).$$

† Littlewood (1).

Hence $$\sum_{n<x} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} + O(T^{-1+\epsilon}x^2) + O(T^{\epsilon}x^{\frac{1}{2}-\sigma+\delta}).$$

Taking, for example, $T = x^3$, the O-terms tend to zero as $x \to \infty$, and the result follows.

Conversely, if (14.25.1) is convergent for $\sigma > \frac{1}{2}$, it is uniformly convergent for $\sigma \geqslant \sigma_0 > \frac{1}{2}$, and so in this region represents an analytic function, which is $1/\zeta(s)$ for $\sigma > 1$ and so throughout the region. Hence the Riemann hypothesis is true. We have in fact

Theorem 14.25 (B). *The convergence of* (14.25.1) *for* $\sigma > \frac{1}{2}$ *is a necessary and sufficient condition for the truth of the Riemann hypothesis.*

We shall write $$M(x) = \sum_{n \leqslant x} \mu(n).$$

Then we also have

Theorem 14.25 (C). *A necessary and sufficient condition for the Riemann hypothesis is* $$M(x) = O(x^{\frac{1}{2}+\epsilon}). \tag{14.25.2}$$

The lemma of § 3.12 with $s = 0$, x half an odd integer, gives

$$\begin{aligned} M(x) &= \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \frac{1}{\zeta(w)} \frac{x^w}{w}\, dw + O\left(\frac{x^2}{T}\right) \\ &= \frac{1}{2\pi i} \left(\int_{2-iT}^{\frac{1}{2}+\delta-iT} + \int_{\frac{1}{2}+\delta-iT}^{\frac{1}{2}+\delta+iT} + \int_{\frac{1}{2}+\delta+iT}^{2+iT} \right) \frac{1}{\zeta(w)} \frac{x^w}{w}\, dw + O\left(\frac{x^2}{T}\right) \qquad (14.25.3) \\ &= O\left(\int_{-T}^{T} (1+|v|)^{-1+\epsilon} x^{\frac{1}{2}+\delta}\, dv \right) + O(T^{\epsilon-1}x^2) + O\left(\frac{x^2}{T}\right) \\ &= O(T^{\epsilon}x^{\frac{1}{2}+\delta}) + O(x^2 T^{\epsilon-1}), \end{aligned}$$

by (14.2.6). Taking $T = x^2$, (14.25.2) follows if x is half an odd integer, and so generally.

Conversely, if (14.25.2) holds, then by partial summation (14.25.1) converges for $\sigma > \frac{1}{2}$, and the Riemann hypothesis follows.

14.26. The finer theory of $M(x)$ is extremely obscure, and the results are not nearly so precise as the corresponding ones in the prime-number problem. The best O-result known is

Theorem 14.26 (A).†
$$M(x) = O\left\{x^{\frac{1}{2}} \exp\left(A \frac{\log x}{\log\log x}\right)\right\}. \tag{14.26.1}$$

To prove this, take
$$\delta = \frac{1}{2} + \frac{1}{\log\log T}, \qquad T = x^2,$$
in the formula (14.25.3). By (14.14.2),
$$\left|\frac{1}{\zeta(w)}\right| \leqslant \exp\left(A \frac{\log T}{\log\log T}\right)$$
on the horizontal sides of the contour. The contribution of these is therefore
$$O\left\{x^2 \frac{1}{T} \exp\left(A \frac{\log T}{\log\log T}\right)\right\} = O\left\{\exp\left(A \frac{\log x}{\log\log x}\right)\right\}.$$

On the vertical side, (14.14.2) gives
$$\left|\frac{1}{\zeta(\frac{1}{2}+\delta+iv)}\right| \leqslant \exp\left(A \frac{\log v}{\log\log v} \log \frac{2\log\log T}{\log\log v}\right)$$
for $v_0 \leqslant v \leqslant T$. Now it is easily seen that the right-hand side is a steadily increasing function of v in this interval. Hence
$$\left|\frac{1}{\zeta(w)}\right| \leqslant \exp\left(A \frac{\log T}{\log\log T}\right) \quad (v_0 \leqslant v \leqslant T).$$

Hence the integral along the vertical side is of the form
$$\begin{aligned} O(x^{\frac{1}{2}+\delta}) &+ O\left\{x^{\frac{1}{2}+\delta} \exp\left(A \frac{\log T}{\log\log T}\right) \int_{v_0}^{T} \frac{dv}{v}\right\} \\ &= O\left\{x^{\frac{1}{2}+\delta} \exp\left(A \frac{\log T}{\log\log T}\right) \log T\right\} = O\left\{x^{\frac{1}{2}} \exp\left(A \frac{\log x}{\log\log x}\right)\right\}. \end{aligned}$$

This proves the theorem.

Theorem 14.26 (B). $$M(x) = \Omega(x^{\frac{1}{2}}). \tag{14.26.2}$$

This is true without any hypothesis. For if the Riemann hypothesis is false, Theorem 14.25 (C) shows that
$$M(x) = \Omega(x^{a})$$

† Landau (13), Titchmarsh (3).

with some a greater than $\frac{1}{2}$. On the other hand, if the Riemann hypothesis is true, then for $\sigma > \frac{1}{2}$

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \sum_{n=1}^{\infty} M(n)\left\{\frac{1}{n^s} - \frac{1}{(n+1)^s}\right\} = s\int_1^{\infty} \frac{M(x)}{x^{s+1}}\,dx.$$

Suppose that (14.26.3)

$$|M(x)| \leqslant M_0 \;\; (1 \leqslant x < x_0), \quad \leqslant \delta x^{\frac{1}{2}} \;\; (x \geqslant x_0).$$

Then

$$\left|\frac{1}{\zeta(s)}\right| \leqslant |s|M_0 \int_1^{x_0} \frac{dx}{x^{\frac{3}{2}}} + |s|\delta \int_{x_0}^{\infty} \frac{dx}{x^{\sigma+\frac{1}{2}}}$$

$$< |s|M_0 \int_1^{\infty} \frac{dx}{x^{\frac{3}{2}}} + |s|\delta \int_1^{\infty} \frac{dx}{x^{\sigma+\frac{1}{2}}}$$

$$= 2|s|M_0 + \frac{|s|\delta}{\sigma - \frac{1}{2}}. \tag{14.26.4}$$

But if $\rho = \frac{1}{2}+i\gamma$ is a simple zero of $\zeta(s)$, and $s = \sigma+i\gamma$, $\sigma \to \frac{1}{2}$, then

$$\frac{1}{\zeta(s)} = \frac{1}{\zeta(s)-\zeta(\rho)} \sim \frac{1}{(\sigma-\frac{1}{2})\zeta'(\rho)}.$$

We therefore obtain a contradiction if

$$\delta < \frac{1}{|\rho\,\zeta'(\rho)|}.$$

This proves the theorem.

14.27. Formulae connecting the functions of prime-number theory with series of the form

$$\sum x^{\rho}, \quad \sum \frac{x^{\rho}}{\rho},$$

etc., are well known, and are discussed in the books of Landau and Ingham. Here we prove a similar formula for the function $M(x)$.

THEOREM 14.27. *There is a sequence T_ν, $\nu \leqslant T_\nu \leqslant \nu+1$, such that*

$$M(x) = \lim_{\nu\to\infty} \sum_{|\gamma|<T_\nu} \frac{x^{\rho}}{\rho\zeta'(\rho)} - 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2\pi/x)^{2n}}{(2n)!\,n\,\zeta(2n+1)} \tag{14.27.1}$$

if x is not an integer. If x is an integer, $M(x)$ is to be replaced by

$$M(x) - \tfrac{1}{2}\mu(x).$$

In writing the series we have supposed for simplicity that all the zeros of $\zeta(s)$ are simple; obvious modifications are required if this is not so.

For a fixed non-integral x, (3.12.1), with $a_n = \mu(n)$, $s = 0$, $c = 2$, and w replaced by s, gives

$$M(x) = \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \frac{x^s}{s} \frac{1}{\zeta(s)}\,ds + O\left(\frac{1}{T}\right).$$

If x is an integer, $\frac{1}{2}\mu(x)$ is to be subtracted from the left-hand side. By the calculus of residues, the first term on the right is equal to

$$\sum_{|\gamma|<T} \frac{x^{\rho}}{\rho\zeta'(\rho)} - 2 + \sum_{n=1}^{N} \frac{(-1)^{n-1}(2\pi/x)^{2n}}{(2n)!\,n\zeta(2n+1)} +$$

$$+\frac{1}{2\pi i}\left(\int_{2-iT}^{-2N-1-iT} + \int_{-2N-1-iT}^{-2N-1+iT} + \int_{-2N-1+iT}^{2+iT}\right) \frac{x^s}{s\zeta(s)}\,ds,$$

where T is not the ordinate of a zero. Now

$$\int_{-2N-1-iT}^{-2N-1+iT} \frac{x^s}{s\,\zeta(s)}\,ds = \int_{2N+2-iT}^{2N+2+iT} \frac{x^{1-s}}{(1-s)\zeta(1-s)}\,ds$$

$$= \int_{2N+2-iT}^{2N+2+iT} \frac{x^{1-s}}{1-s} \frac{2^{s-1}\pi^s}{\cos\frac{1}{2}s\pi\Gamma(s)} \frac{1}{\zeta(s)}\,ds.$$

Here

$$\frac{1}{\Gamma(s)} = O(|e^{s-(s-\frac{1}{2})\log s}|) = O(e^{\sigma-(\sigma-\frac{1}{2})\log|s|+\frac{1}{2}\pi|t|})$$

$$= O(e^{\sigma-(\sigma-\frac{1}{2})\log\sigma+\frac{1}{2}\pi|t|}).$$

Hence the integral is

$$O\left\{\int_{-T}^{T} \frac{1}{T}\left(\frac{2\pi}{x}\right)^{2N+2} e^{2N+2-(2N+\frac{3}{2})\log(2N+2)}\,dt\right\},$$

which tends to zero as $N \to \infty$, for a fixed T. Hence we obtain

$$\sum_{|\gamma|<T} \frac{x^{\rho}}{\rho\zeta'(\rho)} - 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2\pi/x)^{2n}}{(2n)!\,n\zeta(2n+1)} + \frac{1}{2\pi i}\left(\int_{2-iT}^{-\infty-iT} + \int_{-\infty+iT}^{2+iT}\right) \frac{x^s}{s\zeta(s)}\,ds.$$

Also

$$\int_{-\infty+iT}^{-1+iT} \frac{x^s}{s\zeta(s)}\,ds = \int_{2+iT}^{\infty+iT} \frac{x^{1-s}}{(1-s)\zeta(1-s)}\,ds = \int_{2+iT}^{\infty+iT} \frac{x^{1-s}}{1-s} \frac{2^{s-1}\pi^s}{\cos\frac{1}{2}s\pi\Gamma(s)} \frac{1}{\zeta(s)}\,ds$$

$$= O\left\{\int_2^{\infty} \frac{1}{T}\left(\frac{2\pi}{x}\right)^{\sigma} e^{\sigma-(\sigma-\frac{1}{2})\log\sigma}\,d\sigma\right\} = O\left(\frac{1}{T}\right).$$

Also by (14.16.2) we can choose $T = T_\nu$ ($\nu \leqslant T_\nu \leqslant \nu+1$) such that

$$\frac{1}{\zeta(s)} = O(t^\epsilon) \quad (\tfrac{1}{2} \leqslant \sigma \leqslant 2,\ t = T_\nu).$$

Hence for $-1 \leqslant \sigma \leqslant \frac{1}{2}$, $t = T_\nu$

$$\frac{1}{\zeta(s)} = O\left\{\frac{|t|^{\sigma-\frac{1}{2}}}{|\zeta(1-s)|}\right\} = O(t^\epsilon).$$

Hence
$$\int\limits_{-1+iT_\nu}^{2+iT_\nu} \frac{x^s}{s\,\zeta(s)}\,ds = O(T_\nu^{\epsilon-1}).$$

Similarly for the integral over $(2-iT, -\infty-iT)$, and the result stated follows.

It follows from the above theorem that

$$\sum \frac{1}{|\rho\zeta'(\rho)|}$$

is divergent; if it were convergent,

$$\sum \frac{x^\rho}{\rho\zeta'(\rho)}$$

would be uniformly convergent over any finite interval, and $M(x)$ would be continuous.

14.28. The Mertens hypothesis.† It was conjectured by Mertens, from numerical evidence, that

$$|M(n)| < \sqrt{n} \quad (n > 1). \tag{14.28.1}$$

This has not been proved or disproved. It implies the Riemann hypothesis, but is not apparently a consequence of it. A slightly less precise hypothesis would be

$$M(x) = O(x^{\frac{1}{2}}). \tag{14.28.2}$$

The problem has a certain similarity to that of the function $\psi(x)-x$ in prime-number theory, where

$$\psi(x) = \sum_{n \leqslant x} \Lambda(n).$$

On the Riemann hypothesis, $\psi(x)-x = O(x^{\frac{1}{2}+\epsilon})$, but it is not of the form $O(x^{\frac{1}{2}})$, and in fact

$$\psi(x)-x = \Omega(x^{\frac{1}{2}} \log\log\log x). \tag{14.28.3}$$

The influence of the factor $\log\log\log x$ is quite inappreciable as far as

† See references in Landau's *Handbuch*, and von Sterneck (1).

the calculations go, and it might be conjectured that (14.28.2) could be disproved similarly. We shall show, however, that there is an essential difference between the two problems, and that the proof of (14.28.3) cannot be extended to the other case, at any rate in any obvious way.

The proof of (14.28.3) depends on the fact that the real part of

$$\sum_{\gamma>0} \frac{e^{i\rho z}}{\rho}$$

is unbounded in the neighbourhood of $z = 0$. To deal with $M(x)$ in the same way, we should have to prove that the real part of

$$f(z) = \sum_{\gamma>0} \frac{e^{i\rho z}}{\rho\,\zeta'(\rho)} \quad (\mathbf{R}(z) > 0)$$

is unbounded in the neighbourhood of $z = 0$. This, however, is not the case. For consider the integral

$$\frac{1}{2\pi i} \int \frac{e^{isz}}{s\,\zeta(s)}\,ds$$

taken round the rectangle $(-1,\ 2,\ 2+iT_n,\ -1+iT_n)$, where the T_n are those of the previous section, and an indentation is made above $s = 0$. The integral along the upper side of the contour tends to 0 as $n \to \infty$, and we calculate that

$$f(z) = \frac{1}{2\pi i} \int\limits_{2}^{2+i\infty} \frac{e^{isz}}{s\,\zeta(s)}\,ds - \frac{1}{2\pi i} \int\limits_{-1}^{-1+i\infty} \frac{e^{isz}}{s\,\zeta(s)}\,ds + \frac{1}{2\pi i} \int\limits_{-1}^{2} \frac{e^{isz}}{s\,\zeta(s)}\,ds.$$

The last term tends to a finite limit as $z \to 0$. Also

$$|e^{isz}| = e^{y-xt} \leqslant e^y \quad (s = -1+it,\ z = x+iy,\ x > 0)$$

and $1/\zeta(-1+it) = O(t^{-\frac{3}{2}})$. The second term is therefore bounded for $\mathbf{R}(z) > 0$.

The first term is equal to

$$\frac{1}{2\pi i} \sum_{n=1}^{\infty} \mu(n) \int\limits_{2}^{2+i\infty} \frac{e^{isz}}{sn^s}\,ds.$$

Now, if $n > 1$,

$$\int\limits_{2}^{2+i\infty} \frac{e^{s(iz-\log n)}}{s}\,ds = \left[\frac{e^{s(iz-\log n)}}{s(iz-\log n)}\right]_2^{2+i\infty} + \frac{1}{iz-\log n} \int\limits_{2}^{2+i\infty} \frac{e^{s(iz-\log n)}}{s^2}\,ds$$

and
$$|e^{(2+it)(iz-\log n)}| = e^{-2y-2\log n-tx} \leqslant n^{-2}e^{-2y}.$$

Hence
$$\int\limits_{2}^{2+i\infty} \frac{e^{s(iz-\log n)}}{s}\,ds = O\left(\frac{1}{n^2\log n}\right)$$

uniformly in the neighbourhood of $z = 0$. Hence

$$\sum_{n=2}^{\infty} \mu(n) \int_{2}^{2+i\infty} \frac{e^{isz}}{sn^s}\, ds = O(1).$$

If $z = re^{i\theta}$, we have

$$\int_{2}^{2+i\infty} \frac{e^{isz}}{s}\, ds = \int_{2}^{e^{i(\frac{1}{2}\pi-\theta)}} + \int_{e^{i(\frac{1}{2}\pi-\theta)}}^{\infty e^{i(\frac{1}{2}\pi-\theta)}}$$

$$= O(1) + \int_{1}^{\infty} \frac{e^{-r\lambda}}{\lambda}\, d\lambda$$

$$= O(1) + \int_{r}^{\infty} \frac{e^{-x}}{x}\, dx$$

$$= O(1) + \int_{r}^{1} \frac{dx}{x} + \int_{r}^{1} \frac{e^{x}-1}{x}\, dx + \int_{1}^{\infty} \frac{e^{-x}}{x}\, dx$$

$$= \log\frac{1}{r} + O(1).$$

Hence

$$f(z) = \frac{1}{2\pi i}\log\frac{1}{r} + O(1),$$

and consequently $\mathbf{R}f(z)$ is bounded.

14.29. In this section we shall investigate the consequences of the hypothesis that

$$\int_{1}^{X} \left\{\frac{M(x)}{x}\right\}^2 dx = O(\log X). \tag{14.29.1}$$

This is less drastic than the Mertens hypothesis, since it clearly follows from (14.28.2). The corresponding formula with $M(x)$ replaced by $\psi(x)-x$ is a consequence of the Riemann hypothesis.†

THEOREM 14.29 (A). *If (14.29.1) is true, all the zeros of $\zeta(s)$ on the critical line are simple.*

By (14.26.3),

$$\left|\frac{1}{\zeta(s)}\right| \leqslant |s| \int_{1}^{\infty} \frac{|M(x)|}{x^{\sigma+1}}\, dx = |s| \int_{1}^{\infty} \frac{|M(x)|}{x^{\frac{1}{2}\sigma+\frac{3}{4}}} \frac{1}{x^{\frac{1}{2}\sigma+\frac{1}{4}}}\, dx$$

$$\leqslant |s| \left\{\int_{1}^{\infty} \frac{M^2(x)}{x^{\sigma+\frac{3}{2}}}\, dx \int_{1}^{\infty} \frac{dx}{x^{\sigma+\frac{1}{2}}}\right\}^{\frac{1}{2}} = \frac{|s|}{(\sigma-\frac{1}{2})^{\frac{1}{2}}} \left\{\int_{1}^{\infty} \frac{M^2(x)}{x^{\sigma+\frac{3}{2}}}\, dx\right\}^{\frac{1}{2}}.$$

† Cramér (5).

Let

$$f(X) = \int_{1}^{X} \left\{\frac{M(x)}{x}\right\}^2 dx.$$

Then

$$\int_{1}^{\infty} \frac{M^2(x)}{x^{\sigma+\frac{3}{2}}}\, dx = \int_{1}^{\infty} \frac{f'(x)}{x^{\sigma-\frac{1}{2}}}\, dx = (\sigma-\tfrac{1}{2}) \int_{1}^{\infty} \frac{f(x)}{x^{\sigma+\frac{1}{2}}}\, dx$$

$$= O\left\{(\sigma-\tfrac{1}{2}) \int_{1}^{\infty} \frac{\log x}{x^{\sigma+\frac{1}{2}}}\, dx\right\} = O\left(\int_{1}^{\infty} \frac{dx}{x^{\sigma+\frac{1}{2}}}\right) = O\left(\frac{1}{\sigma-\frac{1}{2}}\right).$$

Hence

$$\frac{1}{\zeta(s)} = O\left(\frac{|s|}{\sigma-\frac{1}{2}}\right).$$

Let ρ be a zero and $s = \rho+h$, where $h > 0$. Then $\sigma = \frac{1}{2}+h$, and hence

$$\frac{1}{\zeta(\rho+h)} = O\left(\frac{|\rho+h|}{h}\right). \tag{14.29.2}$$

This would be false for $h \to 0$ if ρ were a zero of order higher than the first, so that the result follows.

Multiplying each side of (14.29.2) by h, and making $h \to 0$, we obtain

$$\frac{1}{\zeta'(\rho)} = O(|\rho|). \tag{14.29.3}$$

We can, however, prove more than this.

THEOREM 14.29 (B). *If (14.29.1) is true,*

$$\sum \frac{1}{|\rho\zeta'(\rho)|^2} \tag{14.29.4}$$

is convergent.

This follows from an argument of the 'Bessel's inequality' type. We have

$$0 \leqslant \int_{1}^{X} \left\{\frac{M(x)}{x} - \sum_{|\gamma|<T} \frac{x^{\rho-1}}{\rho\zeta'(\rho)}\right\}^2 dx$$

$$= \int_{1}^{X} \left\{\frac{M(x)}{x}\right\}^2 dx + \sum_{|\gamma|<T} \sum_{|\gamma'|<T} \frac{1}{\rho\rho'\zeta'(\rho)\zeta'(\rho')} \int_{1}^{X} x^{\rho+\rho'-2}\, dx -$$

$$- 2 \sum_{|\gamma|<T} \frac{1}{\rho\zeta'(\rho)} \int_{1}^{X} M(x)x^{\rho-2}\, dx.$$

In the first sum, the terms with $\rho' = 1-\rho$ are

$$\sum_{|\gamma|<T} \frac{1}{\rho(1-\rho)\zeta'(\rho)\zeta'(1-\rho)} \int_1^X \frac{dx}{x} = \log X \sum_{|\gamma|<T} \frac{1}{|\rho\zeta'(\rho)|^2},$$

since $1-\rho$ is the conjugate of ρ. In the remaining terms, $\rho = \frac{1}{2}+i\gamma$, $\rho' = \frac{1}{2}+i\gamma'$, where $\gamma' \neq -\gamma$. Hence

$$\int_1^X x^{\rho+\rho'-2}\,dx = \frac{X^{\rho+\rho'-1}-1}{\rho+\rho'-1} = O\left(\frac{1}{|\gamma+\gamma'|}\right).$$

Hence the sum of these terms is less than $K_1 = K_1(T)$.

In the last sum we write

$$\int_1^X M(x)x^{\rho-2}\,dx = \int_1^X M(x)x^{\rho-2}\left(1-\frac{x}{X}\right)dx + \frac{1}{X}\int_1^X M(x)x^{\rho-1}\,dx.$$

The last term is

$$O\left\{\frac{1}{X}\int_1^X |M(x)|x^{-\frac{1}{2}}\,dx\right\} = O\left[\frac{1}{X}\left\{\int_1^X \frac{M^2(x)}{x^2}\,dx \int_1^X x\,dx\right\}^{\frac{1}{2}}\right]$$

$$= O\left\{\int_1^X \frac{M^2(x)}{x^2}\right\}^{\frac{1}{2}} = O(\log^{\frac{1}{2}}X),$$

by (14.29.1). Also

$$\int_1^X M(x)x^{\rho-2}\left(1-\frac{x}{X}\right)dx = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{X^{w+\rho-1}-1}{\zeta(w)w(w+\rho)(w+\rho-1)}\,dw. \quad (14.29.5)$$

To prove this, insert the Dirichlet series for $1/\zeta(w)$ on the right-hand side and integrate term by term. This is justified by absolute convergence. We obtain

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{1}{n^w}\,\frac{X^{w+\rho-1}-1}{w(w+\rho)(w+\rho-1)}\,dw.$$

Evaluating the integral in the usual way by the calculus of residues, we obtain

$$\sum_{n\leqslant X} \mu(n)\left\{\frac{X^{\rho-1}-n^{\rho-1}}{\rho-1} - \frac{X^{\rho}-n^{\rho}}{X\rho}\right\} = \sum_{n\leqslant X}\mu(n)\int_n^X \left(x^{\rho-2}-\frac{x^{\rho-1}}{X}\right)dx$$

$$= \int_1^X \sum_{n\leqslant X} \mu(n)x^{\rho-2}\left(1-\frac{x}{X}\right)dx = \int_1^X M(x)x^{\rho-2}\left(1-\frac{x}{X}\right)dx,$$

and (14.29.5) follows.

Let U be not the ordinate of a zero. Then the right-hand side of (14.29.5) is equal to

$$\frac{1}{2\pi i}\left(\int_{2-i\infty}^{2-iU} + \int_{2-iU}^{\frac{1}{4}-iU} + \int_{\frac{1}{4}-iU}^{\frac{1}{4}+iU} + \int_{\frac{1}{4}+iU}^{2+iU} + \int_{2+iU}^{2+i\infty}\right) +$$

$$+\text{sum of residues in } -U < \mathbf{I}(w) < U.$$

Let ρ'' run through zeros of $\zeta(s)$ with imaginary parts between $-U$ and U. Let $U > T$. Then there is a pole at $w = 1-\rho$, with residue

$$\frac{\log X}{(1-\rho)\zeta'(1-\rho)}.$$

At the other ρ'' the residues are

$$\frac{X^{\rho''+\rho-1}-1}{\zeta'(\rho'')\rho''(\rho''+\rho-1)(\rho''+\rho)} = O\left(\frac{1}{|(\rho''+\rho-1)(\rho''+\rho)|}\right) = O\left(\frac{1}{|\gamma''+\gamma|^2}\right),$$

by (14.29.3), and

$$\sum_{\substack{-U<\gamma''<U\\ \gamma''\neq-\gamma}} \frac{1}{|\gamma''+\gamma|^2} \leqslant \sum_{\gamma''\neq-\gamma} \frac{1}{|\gamma''+\gamma|^2} < K_2,$$

where K_2 depends on T, if $|\gamma| < T$, but not on U.

Again

$$\int_{2+iU}^{2+i\infty} \frac{X^{w+\rho-1}-1}{\zeta(w)w(w+\rho)(w+\rho-1)}\,dw = O\left(X^{\frac{3}{2}}\int_U^{\infty} \frac{dv}{v(v+\gamma)^2}\right)$$

$$= O\left(\frac{X^{\frac{3}{2}}}{U(U+\gamma)}\right) = O\left(\frac{X^{\frac{3}{2}}}{U(U-T)}\right),$$

and similarly for the integral over $(2-i\infty, 2-iU)$. Also by (14.2.6) and the functional equation

$$\frac{1}{\zeta(\frac{1}{4}+it)} = O\left\{\frac{|t|^{-\frac{1}{4}}}{|\zeta(\frac{3}{4}-it)|}\right\} = O(|t|^{\epsilon-\frac{1}{4}}).$$

Hence, since $|w+\rho| \geqslant \frac{3}{4}$, $|w+\rho-1| \geqslant \frac{1}{4}$,

$$\int_{\frac{1}{4}-iU}^{\frac{1}{4}+iU} \frac{X^{w+\rho-1}-1}{\zeta(w)w(w+\rho)(w+\rho-1)}\,dw = O\left\{\int_{-U}^{U} \frac{|v|^{\epsilon-\frac{1}{4}}}{(\frac{1}{16}+v^2)^{\frac{1}{2}}}\,dv\right\} = O(1).$$

Finally, by Theorem 14.16, we can choose a sequence of values of U such that

$$\frac{1}{\zeta(w)} = O(|w|) \quad (t = U,\ \tfrac{1}{2} \leqslant \sigma \leqslant 2).$$

By the functional equation the same result then holds for $\frac{1}{4} \leqslant \sigma \leqslant \frac{1}{2}$ also. Hence

$$\int_{\frac{1}{4}+iU}^{2+iU} \frac{X^{w+\rho-1}-1}{\zeta(w)w(w+\rho)(w+\rho-1)}\,dw = O\left(\frac{X^{\frac{3}{2}}}{U^{1-\epsilon}(U+\gamma)^2}\right) = O\left\{\frac{X^{\frac{3}{2}}}{U^{1-\epsilon}(U-T)^2}\right\},$$

and similarly for the integral over $(2-iU, \frac{1}{4}-iU)$. Making $U \to \infty$, it follows that

$$\int_1^X M(x)x^{\rho-2}\left(1-\frac{x}{X}\right)dx = \frac{\log X}{(1-\rho)\zeta'(1-\rho)}+R,$$

where $|R| < K_3 = K_3(T)$ if $|\gamma| < T$.

Hence we obtain

$$0 \leqslant A\log X+\log X\sum_{|\gamma|<T}\frac{1}{|\rho\zeta'(\rho)|^2}-2\log X\sum_{|\gamma|<T}\frac{1}{|\rho\zeta'(\rho)|^2}+ +A\log^{\frac{1}{2}}X+K_4(T),$$

$$\sum_{|\gamma|<T}\frac{1}{|\rho\zeta'(\rho)|^2} \leqslant A+\frac{A}{\log^{\frac{1}{2}}X}+\frac{K_4(T)}{\log X}.$$

Making $X \to \infty$,
$$\sum_{|\gamma|<T}\frac{1}{|\rho\zeta'(\rho)|^2} \leqslant A.$$

Since the right-hand side is now independent of T, the result follows.

In particular
$$\frac{1}{\zeta'(\rho)} = o(|\rho|).$$

14.30. *If* (14.29.1) *is true,*†

$$\frac{1}{\zeta'(\frac{1}{2}+it)} = O\left\{\exp\left(\frac{A\log^2 t}{\log\log t}\right)\right\}.$$

Suppose that the interval $(t-t^{-3}, t+t^{-3})$ contains γ, the ordinate of a zero. By differentiating (2.1.4) twice,

$$\zeta''(\tfrac{1}{2}+it) = O(t).$$

Using this and (14.29.3), we obtain

$$|\zeta'(\tfrac{1}{2}+it)| = \left|\zeta'(\tfrac{1}{2}+i\gamma)+\int_{\frac{1}{2}+i\gamma}^{\frac{1}{2}+it}\zeta''(s)\,ds\right|$$

$$> \frac{A}{\gamma}-At|t-\gamma|$$

$$> \frac{A}{t}-\frac{A}{t^2} > \frac{A}{t}.$$

† Cramér and Landau (1).

Suppose on the contrary that $(t-t^{-3}, t+t^{-3})$ is free from ordinates of zeros. Theorem 14.15 gives

$$\log|\zeta(\tfrac{1}{2}+it)| = \sum_{|t-\gamma|\leqslant 1/\log\log t}\log|t-\gamma|+O\left(\frac{\log t\log\log\log t}{\log\log t}\right).$$

There are $O(\log t/\log\log t)$ terms in the sum, each being now $O(\log t)$. Hence

$$\log|\zeta(\tfrac{1}{2}+it)| = O\left(\frac{\log^2 t}{\log\log t}\right), \qquad \frac{1}{\zeta(\frac{1}{2}+it)} = O\left\{\exp\left(\frac{A\log^2 t}{\log\log t}\right)\right\}.$$

Now $\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s)\zeta(s)$ is real on $\sigma = \frac{1}{2}$. Hence

$$-\tfrac{1}{2}\log\pi+\frac{1}{2}\frac{\Gamma'(\frac{1}{2}s)}{\Gamma(\frac{1}{2}s)}+\frac{\zeta'(s)}{\zeta(s)}$$

is purely imaginary on $\sigma = \frac{1}{2}$. Hence, on $\sigma = \frac{1}{2}$,

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| \geqslant -\mathbf{R}\frac{\zeta'(s)}{\zeta(s)} = -\tfrac{1}{2}\log\pi+\tfrac{1}{2}\mathbf{R}\frac{\Gamma'(\frac{1}{2}s)}{\Gamma(\frac{1}{2}s)} = \tfrac{1}{2}\log t+O(1) \to \infty.$$

Hence (without any hypothesis)

$$|\zeta'(\tfrac{1}{2}+it)| \geqslant |\zeta(\tfrac{1}{2}+it)| \quad (t > t_0).$$

This proves the theorem.

14.31. *Let* $\frac{1}{2}+i\gamma$, $\frac{1}{2}+i\gamma'$ *be consecutive complex zeros of* $\zeta(s)$. *If* (14.29.1) *is true*

$$\gamma'-\gamma > \frac{A}{\gamma}\exp\left(-A\frac{\log\gamma}{\log\log\gamma}\right).$$

We have

$$0 = \int_\gamma^{\gamma'}\zeta'(\tfrac{1}{2}+it)\,dt = (\gamma'-\gamma)\zeta'(\tfrac{1}{2}+i\gamma)+\int_\gamma^{\gamma'}(\gamma'-t)\zeta''(\tfrac{1}{2}+it)\,dt.$$

Hence by (14.29.3)

$$\frac{\gamma'-\gamma}{\gamma} < A\left|\int_\gamma^{\gamma'}(\gamma'-t)\zeta''(\tfrac{1}{2}+it)\,dt\right|$$

$$< A\max_{\gamma\leqslant t\leqslant\gamma'}|\zeta''(\tfrac{1}{2}+it)|\int_\gamma^{\gamma'}(\gamma'-t)\,dt = A(\gamma'-\gamma)^2\max_{\gamma\leqslant t\leqslant\gamma'}|\zeta''(\tfrac{1}{2}+it)|.$$

Now

$$\zeta''(\tfrac{1}{2}+it) = \frac{1}{\pi i}\int_0^{2\pi}\frac{\zeta(\frac{1}{2}+it+re^{i\theta})}{(re^{i\theta})^3}ire^{i\theta}\,d\theta = O\left\{\frac{1}{r^2}\int_0^{2\pi}|\zeta(\tfrac{1}{2}+it+re^{i\theta})|\,d\theta\right\}$$

$$= O\left\{\frac{1}{r^2}\exp\left(A\frac{\log t}{\log\log t}\right)(1+t^{Ar})\right\}$$

by (14.14.1) and the functional equation. Taking $r = 1/\log t$,

$$\zeta''(\tfrac{1}{2}+it) = O\left\{\exp\left(A\frac{\log t}{\log\log t}\right)\right\}$$

and the result follows.

14.32. *Necessary and sufficient conditions for the Riemann hypothesis.* Two such conditions have been given in § 14.25. Other similar conditions occur in the prime-number problem.†

A different kind of condition was stated by M. Riesz.‡ Let

$$F(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{(k-1)!\,\zeta(2k)}. \tag{14.32.1}$$

Then a simple application of the calculus of residues gives

$$F(x) = \frac{i}{2}\int_{a-i\infty}^{a+i\infty} \frac{x^s}{\Gamma(s)\zeta(2s)\sin\pi s}\,ds = \frac{i}{2\pi}\int_{a-i\infty}^{a+i\infty} \frac{\Gamma(1-s)}{\zeta(2s)}x^s\,ds,$$

where $\frac{1}{2} < a < 1$. Taking a just greater than $\frac{1}{2}$, it clearly follows that

$$F(x) = O(x^{\frac{1}{2}+\epsilon}).$$

On the Riemann hypothesis we could move the line of integration to $a = \frac{1}{4}+\epsilon$ (using (14.2.5)) and obtain similarly

$$F(x) = O(x^{\frac{1}{4}+\epsilon}). \tag{14.32.2}$$

Conversely, by Mellin's inversion formula,

$$\frac{\Gamma(1-s)}{\zeta(2s)} = -\int_0^{\infty} F(x)x^{-1-s}\,ds.$$

If (14.32.2) holds, the integral converges uniformly for $\sigma \geqslant \sigma_0 > \frac{1}{4}$; the analytic function represented is therefore regular for $\sigma > \frac{1}{4}$, and the truth of the Riemann hypothesis follows. Hence (14.32.2) is a necessary and sufficient condition for the Riemann hypothesis.

A similar condition stated by Hardy and Littlewood§ is

$$\sum_{k=1}^{\infty} \frac{(-x)^k}{k!\,\zeta(2k+1)} = O(x^{-\frac{1}{4}}). \tag{14.32.3}$$

These conditions have a superficial attractiveness since they depend explicitly only on values taken by $\zeta(s)$ at points in $\sigma > 1$; but actually no use has ever been made of them.

† Landau, *Vorlesungen*, ii. 108–56. ‡ M. Riesz (1).
§ Hardy and Littlewood (2).

Conditions for the Riemann hypothesis also occur in the theory of Farey series. Let the fractions h/k with $0 < h \leqslant k$, $(h,k) = 1$, $k \leqslant N$, arranged in order of magnitude, be denoted by r_ν ($\nu = 1, 2,..., \Phi(N)$), where $\Phi(N) = \phi(1)+...+\phi(N)$). Let

$$\delta_\nu = r_\nu - \nu/\Phi(N)$$

be the distance between r_ν and the corresponding fraction obtained by dividing up the interval (0, 1) into $\Phi(N)$ equal parts. Then a necessary and sufficient condition for the Riemann hypothesis is‡

$$\sum_{\nu=1}^{\Phi(N)} \delta_\nu^2 = O\left(\frac{1}{N^{1-\epsilon}}\right). \tag{14.32.4}$$

An alternative necessary and sufficient condition is§

$$\sum_{\nu=1}^{\Phi(N)} |\delta_\nu| = O(N^{\frac{1}{2}+\epsilon}). \tag{14.32.5}$$

Details are given in Landau's *Vorlesungen*, ii. 167–77.

Still another condition‖ can be expressed in terms of the formulae of § 10.1. If $\Xi(t)$ and $\Phi(u)$ are related by (10.1.3), a necessary and sufficient condition that all the zeros of $\Xi(t)$ should be real is that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(\alpha)\Phi(\beta)e^{i(\alpha+\beta)x}e^{(\alpha-\beta)y}(\alpha-\beta)^2\,d\alpha d\beta \geqslant 0 \tag{14.32.6}$$

for all real values of x and y. But no method has been suggested of showing whether such criteria are satisfied or not.

A sufficient condition†† for the Riemann hypothesis is that the partial sums $\sum_{\nu=1}^{n} \nu^{-s}$ of the series for $\zeta(s)$ should have no zeros in $\sigma > 1$.

NOTES FOR CHAPTER 14

14.33. The argument of §14.5 may be extended to the strip $\frac{1}{2} \leqslant \sigma_0 \leqslant \sigma \leqslant \frac{3}{4}$, giving

$$\frac{\zeta'(s)}{\zeta(s)} = O\left(\frac{\delta^{\sigma-1}-1}{1-\sigma}\right) + O(\delta^{\sigma-\frac{1}{2}}\log t).$$

The choice $\delta = (\log t)^{-2}$ then yields

$$\frac{\zeta'(s)}{\zeta(s)} \ll \frac{(\log t)^{2-2\sigma}-1}{1-\sigma}$$

‡ Franel (1). § Landau (16). ‖ See Pólya (3), § 7. †† Turán (3).

uniformly for $\sigma_0 \leqslant \sigma \leqslant \frac{9}{8}$ and $t \geqslant 2$, and hence

$$\log \zeta(s) \ll \begin{cases} \log \dfrac{1}{\sigma - 1} & \text{if} \quad 1 + \dfrac{1}{\log\log t} \leqslant \sigma \leqslant \frac{9}{8}, \\ \dfrac{(\log t)^{2-2\sigma} - 1}{(1-\sigma)\log\log t} + \log\log\log t & \text{if} \quad \sigma_0 \leqslant \sigma \leqslant 1 + \dfrac{1}{\log\log t}. \end{cases}$$

These results, together with those of § 14.14 are the sharpest conditional order-estimates available at present.

14.34. The Ω-result given by Theorem 14.12(A) has been sharpened by Montgomery [3], to give

$$S(t) = \Omega_{\pm}\left(\frac{(\log t)^{\frac{1}{2}}}{(\log\log t)^{\frac{1}{2}}}\right)$$

on the Riemann hypothesis. A minor modification of his method also yields

$$S_1(t) = \Omega_{\pm}\left(\frac{(\log t)^{\frac{1}{2}}}{(\log\log t)^{\frac{3}{2}}}\right).$$

It may be conjectured that these are best possible.

Mueller [2] has shown, on the Riemann hypothesis, that if c is a suitable constant, then $S(t)$ changes sign in any interval $[T, T + c \log\log T]$.

Further results and conjectures on the vertical distribution of the zeros are given by Montgomery [2], who investigated the pair correlation function

$$F(\alpha, T) = \frac{1}{N(T)} \sum_{0 < \gamma, \gamma' \leqslant T} T^{i\alpha(\gamma - \gamma')} w(\gamma - \gamma'),$$

where $w(u) = 4/(4 + u^2)$. This is a real-valued, even, non-negative function of α, and satisfies

$$F(\alpha, T) = \alpha + T^{-2\alpha} \log T + O\left(\frac{1}{\log T}\right) + O(\alpha T^{\alpha - 1}) + O(T^{-\frac{1}{2}\alpha}) \tag{14.34.1}$$

for $\alpha \geqslant 0$, whence $F(\alpha, T) \to \alpha$ as $T \to \infty$, uniformly for $0 < \delta \leqslant \alpha \leqslant 1 - \delta$. Montgomery conjectured that in general

$$F(\alpha, T) \to \min(\alpha, 1) \tag{14.34.2}$$

uniformly for $0 \leqslant \delta \leqslant \alpha \leqslant A$. This is related to a number of conjectures on the distribution of prime numbers. (See Gallagher and Mueller [1], Heath-Brown [10], and joint work of Goldston and Montgomery in the course of publication.) From (14.34.2) one may deduce that

$$\#\left\{\gamma, \gamma' \in [0, T]: \frac{2\pi\alpha}{\log T} \leqslant \gamma - \gamma' \leqslant \frac{2\pi\beta}{\log T}\right\}$$

$$\sim N(T)\left\{\delta(\alpha, \beta) + \int_{\alpha}^{\beta} 1 - \left(\frac{\sin \pi u}{u}\right)^2 du\right\}$$

for fixed α, β, as $T \to \infty$. Here $\delta(\alpha, \beta) = 1$ or 0 according as $\alpha \leqslant 0 \leqslant \beta$ or not.

Using (14.34.1), Montgomery showed that

$$\sum_{0 < \gamma \leqslant T}' m(\rho)^2 \leqslant \{\tfrac{4}{3} + o(1)\} N(T),$$

where $m(\rho)$ is the multiplicity of ρ, and Σ' counts zeros without regard to multiplicity. One may deduce, in the notation of § 10.29, that

$$N^{(1)}(T) \geqslant \{\tfrac{2}{3} + o(1)\} N(T), \tag{14.34.3}$$

on the Riemann hypothesis. The conjecture (14.34.2) would indeed yield $N^{(1)}(T) \sim N(T)$, i.e. 'almost all' the zeros would be simple. Montgomery also used (14.34.1) to show that

$$\liminf_{n \to \infty} \frac{\gamma_{n+1} - \gamma_n}{(2\pi/\log \gamma_n)} \leqslant 0{\cdot}68; \tag{14.34.4}$$

here $2\pi/\log \gamma_n$ is the average spacing between zeros.

By using a different method, Conrey, Ghosh, and Gonek (in work in the course of publication) have improved (14.34.3). Their starting point is the observation that

$$\left|\sum_{0 < \gamma \leqslant T} M(\tfrac{1}{2} + i\gamma) \zeta'(\tfrac{1}{2} + i\gamma)\right|^2 \leqslant N^{(1)}(T) \sum_{0 < \gamma \leqslant T} |M(\tfrac{1}{2} + i\gamma) \zeta'(\tfrac{1}{2} + i\gamma)|^2, \tag{14.34.5}$$

by Cauchy's inequality. The function $M(s)$ is taken to be a mollifier

$$M(s) = \sum_{n \leqslant y} \mu(n) P\left(\frac{\log y/n}{\log y}\right) n^{-s}, \qquad y = T^{\frac{1}{2} - \epsilon},$$

where the polynomial $P(x)$ is chosen optimally as $\frac{3}{2}x - \frac{1}{2}x^2$. One may write the sums occurring in (14.34.5) as integrals

$$\frac{1}{2\pi i}\int_P \frac{\zeta'(s)}{\zeta(s)} M(s)\zeta'(s)\,ds$$

and

$$\frac{1}{2\pi i}\int_P \frac{\zeta'(s)}{\zeta(s)} M(s)M(1-s)\zeta'(s)\zeta'(1-s)\,ds,$$

taken around an appropriate rectangular path P. The estimation of these is long and complicated, but leads ultimately to the lower bound

$$N^{(1)}(T) \geqslant \{\tfrac{19}{27}+o(1)\}N(T).$$

The estimate (14.34.4) has also been improved, firstly by Montgomery and Odlyzko [1], and then by Conrey, Ghosh and Gonek [1]. The latter work produces the constant 0·5172. The corresponding lower bound

$$\limsup_{n\to\infty} \frac{\gamma_{n+1}-\gamma_n}{2\pi/\log\gamma_n} \geqslant \lambda > 1 \tag{14.34.6}$$

has been considered by Mueller [1], as well as in the two papers just cited. Here the best result known is that of Conrey, Ghosh, and Gonek [1], which has $\lambda = 2{\cdot}337$. Indeed, further work by Conrey, Ghosh, and Gonek, which is in the course of publication at the time of writing, yields $\lambda = 2{\cdot}68$ subject to the generalized Riemann hypothesis (i.e. a Riemann hypothesis for $\zeta(s)$ and all Dirichlet L-functions $L(s,\chi)$.) Moreover it seems likely that this condition may be relaxed to the ordinary Riemann hypothesis with further work.

If one asks for bounds of the form (14.34.4) and (14.34.6) which are satisfied by a positive proportion of zeros (as in § 9.25) then one may take constants 0·77 and 1·33 (Conrey, Ghosh, Goldston, Gonek, and Heath-Brown [1]).

14.35. It should be remarked in connection with § 14.24 that Selberg (4) proved Theorem 14.24 with error term $O(T)$, while the method here yields only $O\{T(\log\log T)^{\frac{1}{2}}\}$. Moreover he obtained the error term $O\{T(\log\log T)^{k-1}\}$ for (14.24.1).

14.36. The argument of the final paragraph of § 14.27 may be quantified, and then yields

$$\sum_{|\gamma|\leqslant T} |\zeta'(\tfrac{1}{2}+i\gamma)|^{-1} \gg T,$$

uniformly for $T \geqslant T_0$, assuming the Riemann hypothesis and that all the zeros are simple. However a slightly better result comes from combining

the asymptotic formula

$$\sum_{0<\gamma\leqslant T} |\zeta'(\tfrac{1}{2}+i\gamma)|^2 \sim \tfrac{1}{12}N(T)(\log T)^2$$

of Gonek [2] with the bound (14.34.3). Using Hölder's inequality one may then derive the estimate

$$\sum_{cT<\gamma<T}^{*} \frac{1}{|\zeta'(\frac{1}{2}+i\gamma)|} \gg T,$$

where Σ^* counts simple zero only, and $c > 0$ is a suitable numerical constant.

14.37. The Mertens hypothesis has been disproved by Odlyzko and te Riele [1], who showed that

$$\limsup_{x\to\infty} \frac{M(x)}{\sqrt{x}} > 1{\cdot}06$$

and

$$\liminf_{x\to\infty} \frac{M(x)}{\sqrt{x}} < -1{\cdot}009.$$

Their treatment is indirect, and produces no specific x for which $|M(x)| > x^{\frac{1}{2}}$. The method used is computational, and depends on solving numerically the inequalities occurring in Kronecker's theorem, so as to make the first few terms of (14.27.1) pull in the same direction. To this extent Odlyzko and te Riele follow the earlier work of Jurkat and Peyerimhoff [1], but they use a much more efficient algorithm for solving the Diophantine approximation problem.

14.38. Turán (3) conjectured that

$$\sum_{n\leqslant x} \frac{\lambda(n)}{n} \geqslant 0 \tag{14.38.1}$$

for all $x > 0$, where $\lambda(n)$ is the Liouville function, given by (1.2.11). He showed that his condition, given in § 14.32, implies the above conjecture, which in turn implies the Riemann hypothesis. However Haselgrove [2] proved that (14.38.1) is false in general, thereby showing that Turán's condition does not hold. Later Spira [1] found by calculation that

$$\sum_{n=1}^{19} n^{-s}$$

has a zero in the region $\sigma > 1$.

XV

CALCULATIONS RELATING TO THE ZEROS

15.1. It is possible to verify by means of calculation that all the complex zeros of $\zeta(s)$ up to a certain point lie exactly (not merely approximately) on the critical line. As a simple example we shall find roughly the position of the first complex zero in the upper half-plane, and show that it lies on the critical line.

We consider the function $Z(t) = e^{i\vartheta}\zeta(\frac{1}{2}+it)$ defined in § 4.17. This is real for real values of t, so that, if $Z(t_1)$ and $Z(t_2)$ have opposite signs, $Z(t)$ vanishes between t_1 and t_2, and so $\zeta(s)$ has a zero on the critical line between $\frac{1}{2}+it_1$ and $\frac{1}{2}+it_2$.

It follows from (2.2.1) that $\zeta(\frac{1}{2}) < 0$, then from (2.1.12) that $\xi(\frac{1}{2}) > 0$, i.e. that $\Xi(0) > 0$; and then from (4.17.3) that $Z(0) < 0$.

We shall next consider the value $t = 6\pi$. Now the argument of § 4.14 shows that, if x is half an odd integer,

$$\left|\zeta(s)-\sum_{n<x}\frac{1}{n^s}\right| \leqslant \frac{x^{1-\sigma}}{|1-s|}+\frac{2x^{-\sigma}}{2\pi-|t|/x}. \tag{15.1.1}$$

Hence, taking $t > 0$,

$$\left|Z(t)-\sum_{n<x}\frac{\cos(t\log n-\vartheta)}{n^{\frac{1}{2}}}\right| \leqslant \frac{x^{\frac{1}{2}}}{t}+\frac{2x^{\frac{1}{2}}}{2\pi x-t}. \tag{15.1.2}$$

For $x = \frac{9}{2}$, $t = 6\pi$, the right-hand side is about 0·6.

We next require an approximation to ϑ. We have

$$e^{-2i\vartheta} = \chi(\tfrac{1}{2}+it) = \pi^{it}\frac{\Gamma(\frac{1}{4}-\frac{1}{2}it)}{\Gamma(\frac{1}{4}+\frac{1}{2}it)},$$

so that

$$\vartheta = -\tfrac{1}{2}t\log\pi+\mathbf{I}\log\Gamma(\tfrac{1}{4}+\tfrac{1}{2}it)$$

$$= \tfrac{1}{2}t\log\frac{t}{2\pi}-\tfrac{1}{2}t-\tfrac{1}{8}\pi+O\!\left(\frac{1}{t}\right).$$

It may be verified that the term $O(1/t)$ is negligible in the calculations.

Writing $\vartheta = 2\pi K$, and using the values

$$\log 2 = 0{\cdot}6931, \qquad \log 3 = 1{\cdot}0986,$$

it is found that

$$K = 0{\cdot}1166, \qquad 3\log 3-K = 3{\cdot}179$$

$$3\log 2-K = 1{\cdot}963, \qquad 3\log 4-K = 4{\cdot}042,$$

approximately. Hence the cosines in (15.1.2) are all positive, and $\cos 2\pi K = 0{\cdot}74\ldots$. Hence $Z(6\pi) > 0$.

There is therefore one zero at least on the critical line between $t = 0$ and $t = 6\pi$.

Again, the formulae of § 9.3 give

$$N(T) = 1+2K+\frac{1}{\pi}\Delta\arg\zeta(s),$$

where Δ denotes variation along $(2, 2+iT, \frac{1}{2}+iT)$. Now $\mathbf{R}\,\zeta(s) > 0$ on $\sigma = 2$, and an argument similar to that already used, but depending on (15.1.1), shows that $\mathbf{R}\,\zeta(s) > 0$ on $(2+iT, \frac{1}{2}+iT)$, if $T = 6\pi$. Hence $|\Delta\arg\zeta(s)| < \frac{1}{2}\pi$, and

$$N(6\pi) < \tfrac{3}{2}+2K < 2.$$

Hence there is at most one complex zero with imaginary part less than 6π, and so in fact just one, namely the one on the critical line.

15.2. It is plain that the above process can be continued as long as the appropriate changes of sign of the function $Z(t)$ occur. Defining $K = K(t)$, as before, let t_ν be such that

$$K(t_\nu) = \tfrac{1}{2}\nu-1 \quad (\nu = 1, 2,\ldots). \tag{15.2.1}$$

Then (15.1.2) gives

$$Z(t_\nu) \sim (-1)^\nu \sum_{n<x}\frac{\cos(t_\nu\log n)}{n^{\frac{1}{2}}}.$$

If the sum is dominated by its first term, it is positive, and so $Z(t_\nu)$ has the sign of $(-1)^\nu$. If this is true for ν and $\nu+1$, $Z(t)$ has a zero in the interval $(t_\nu, t_{\nu+1})$.

The value $t = 6\pi$ in the above argument is a rough approximation to t_2.

The ordinates of the first six zeros are

$$14{\cdot}13,\quad 21{\cdot}02,\quad 25{\cdot}01,\quad 30{\cdot}42,\quad 32{\cdot}93,\quad 37{\cdot}58$$

to two decimal places.† Some of these have been calculated with great accuracy.

15.3. The calculations which the above process requires are very laborious if t is at all large. A much better method is to use the formula (4.17.5) arising from the approximate functional equation. Let us write $t = 2\pi u$,

$$\alpha_n = \alpha_n(u) = n^{-\frac{1}{2}}\cos 2\pi(K-u\log n),$$

and

$$h(\xi) = \frac{\cos 2\pi(\xi^2-\xi-\frac{1}{16})}{\cos 2\pi\xi}.$$

† See the references Gram (6), Lindelöf (3), in Landau's *Handbuch*.

Then (4.17.5) gives

$$Z(2\pi u) = 2\sum_{n=1}^{m} \alpha_n(u) + (-1)^{m-1}u^{-\frac{1}{4}}h(\sqrt{u}-m) + R(u),$$

where $m = [\sqrt{u}]$, and $R(u) = O(u^{-\frac{3}{4}})$. The $\alpha_n(u)$ can be found, for given values of u, from a table of the function $\cos 2\pi x$. In the interval $0 \leqslant \xi \leqslant \frac{1}{2}$, $h(\xi)$ decreases steadily from 0·92388 to 0·38268, and $h(1-\xi) = h(\xi)$.

For the purpose of calculation we require a numerical upper bound for $R(u)$. A rather complicated formula of this kind is obtained in Titchmarsh (17), Theorem 2. For values of u which are not too small it can be much simplified, and in fact it is easy to deduce that

$$|R(u)| < \frac{3}{2u^{\frac{3}{4}}} \quad (u > 125).$$

This inequality is sufficient for most purposes.

Occasionally, when $Z(2\pi u)$ is too small, a second term of the Riemann–Siegel asymptotic formula has to be used.

The values of u for which the calculations are performed are the solutions of (15.2.1), since they make α_1 alternately 1 and -1. In the calculations described in Titchmarsh (17), I began with

$$u = 1{\cdot}6, \qquad K = -0{\cdot}04865$$

and went as far as

$$u = 62{\cdot}785, \qquad K = 98{\cdot}5010.$$

The values of u were obtained in succession, and are rather rough approximations to the u_ν, so that the K's are not quite integers or integers and a half.

It was shown in this way that the first 198 zeros of $\zeta(s)$ above the real axis all lie on the line $\sigma = \frac{1}{2}$.

The calculations were carried a great deal farther by Dr. Comrie.† Proceeding on the same lines, it was shown that the first 1,041 zeros of $\zeta(s)$ above the real axis all lie on the critical line, in the interval $0 < t < 1{,}468$.

One interesting point which emerges from these calculations is that $Z(t_\nu)$ does not always have the same sign as $(-1)^\nu$. A considerable number of exceptional cases were found; but in each of these cases there is a neighbouring point t'_ν such that $Z(t'_\nu)$ has the sign of $(-1)^\nu$, and the succession of changes of sign of $Z(t)$ is therefore not interrupted.

15.4. As far as they go, these calculations are all in favour of the truth of the Riemann hypothesis. Nevertheless, it may be that they do

† See Titchmarsh (18).

not go far enough to reveal the real state of affairs. At the end of the table constructed by Dr. Comrie there are only fifteen terms in the series for $Z(t)$, and this is a very small number when we are dealing with oscillating series of this kind. Indeed there is one feature of the table which may suggest a change in its character farther on. In the main, the result is dominated by the first term α_1, and later terms more or less cancel out. Occasionally (e.g. at $K = 435$) all, or nearly all, the numbers α_n have the same sign, and $Z(t)$ has a large maximum or minimum. As we pass from this to neighbouring values of t, the first few α_n undergo violent changes, while the later ones vary comparatively slowly. The term α_n appears when $u = n^2$, and here

$$\cos 2\pi(K - u\log n) = \cos \pi\{u \log(u/n^2) - u - \tfrac{1}{8} + \ldots\}$$
$$= \cos \pi(n^2 + \tfrac{1}{8} + \ldots) = (-1)^n \cos \tfrac{1}{8}\pi + \ldots,$$

and

$$\frac{d}{du}(K - u\log n) = \tfrac{1}{2}\log u - \log n - \frac{1}{192\pi^2u^2} + \ldots \sim -\frac{1}{192\pi^2u^2}.$$

At its first appearance in the table α_n will therefore be approximately $(-1)^n n^{-\frac{1}{2}} \cos \frac{1}{8}\pi$, and it will vary slowly for some time after its appearance.

It is conceivable that if t, and so the number of terms, were large enough, there might be places where the smaller slowly varying terms would combine to overpower the few quickly varying ones, and so prevent the graph of $Z(t)$ from crossing the zero line between successive maxima. There are too few terms in the table already constructed to test this possibility.

There are, of course, relations between the numbers α_n which destroy any too simple argument of this kind. If the Riemann hypothesis is true, there must be some relation, at present hidden, which prevents the suggested possibility from ever occurring at all.

No doubt the whole matter will soon be put to the test of modern methods of calculation. Naturally the Riemann hypothesis cannot be proved by calculation, but, if it is false, it could be disproved by the discovery of exceptions in this way.

NOTES FOR CHAPTER 15

15.5. A number of workers have checked the Riemann hypothesis over increasingly large ranges. At the time of writing the most extensive calculation is that of van de Lune and te Riele (as reported in Odlyzko and te Riele [1]), who have found that the first $1{\cdot}5 \times 10^9$ non-trivial zeros are simple and lie on the critical line.

ORIGINAL PAPERS

[*This list includes that given in my Cambridge tract; it does not include papers referred to in Landau's* Handbuch der Lehre von der Verteilung der Primzahlen, *1909.*]

ABBREVIATIONS

A.M.	*Acta Mathematica.*
C.R.	*Comptes rendus de l'Académie des sciences* (Paris).
J.L.M.S.	*Journal of the London Mathematical Society.*
J.M.	*Journal für die reine und angewandte Mathematik.*
M.A.	*Mathematische Annalen.*
M.Z.	*Mathematische Zeitschrift.*
P.C.P.S.	*Proceedings of the Cambridge Philosophical Society.*
P.L.M.S.	*Proceedings of the London Mathematical Society.*
Q.J.O.	*Quarterly Journal of Mathematics* (Oxford Series).

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