CHAPTER II

A DUALITY IN INTEGRAL GEOMETRY. GENERALIZED RADON TRANSFORMS AND ORBITAL INTEGRALS

§1 Homogeneous Spaces in Duality

The inversion formulas in Theorems 3.1, 3.5, 3.6 and 6.2, Ch. I suggest the general problem of determining a function on a manifold by means of its integrals over certain submanifolds. In order to provide a natural framework for such problems we consider the Radon transform $f \to \widehat{f}$ on $\mathbf{R}^n$ and its dual $\varphi \to \check{\varphi}$ from a group-theoretic point of view, motivated by the fact that the isometry group $\mathbf{M}(n)$ acts transitively both on $\mathbf{R}^n$ and on the hyperplane space $\mathbf{P}^n$. Thus

$$\mathbf{R}^n = \mathbf{M}(n)/\mathbf{O}(n)\,, \quad \mathbf{P}^n = \mathbf{M}(n)/\mathbb{Z}_2 \times \mathbf{M}(n-1)\,, \tag{1}$$

where $\mathbf{O}(n)$ is the orthogonal group fixing the origin $0 \in \mathbf{R}^n$ and $\mathbb{Z}_2 \times \mathbf{M}(n-1)$ is the subgroup of $\mathbf{M}(n)$ leaving a certain hyperplane ξ_0 through 0 stable. ($\mathbb{Z}_2$ consists of the identity and the reflection in this hyperplane.)

We observe now that a point $g_1\mathbf{O}(n)$ in the first coset space above lies on a plane $g_2(\mathbb{Z}_2 \times \mathbf{M}(n-1))$ in the second if and only if these cosets, considered as subsets of $\mathbf{M}(n)$, have a point in common. In fact

$$\begin{aligned} g_1 \cdot 0 \subset g_2 \cdot \xi_0 \quad &\Leftrightarrow \quad g_1 \cdot 0 = g_2 h \cdot 0 \text{ for some } h \in \mathbb{Z}_2 \times \mathbf{M}(n-1) \\ &\Leftrightarrow \quad g_1 k = g_2 h \text{ for some } k \in \mathbf{O}(n)\,. \end{aligned}$$

This leads to the following general setup.

Let G be a locally compact group, X and Ξ two left coset spaces of G,

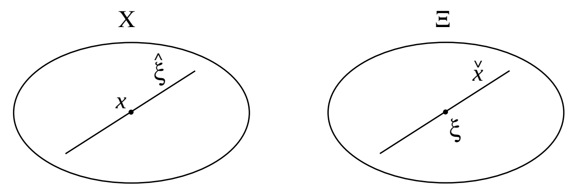
$$X = G/K\,, \quad \Xi = G/H\,, \tag{2}$$

where K and H are closed subgroups of G. Let $L = K \cap H$. We assume that the subset $KH \subset G$ is *closed*. This is automatic if one of the groups K or H is compact.

Two elements $x \in X$, $\xi \in \Xi$ are said to be *incident* if as cosets in G they intersect. We put (see Fig. II.1)

$$\begin{aligned} \check{x} &= \{\xi \in \Xi : x \text{ and } \xi \text{ incident}\} \\ \widehat{\xi} &= \{x \in X : x \text{ and } \xi \text{ incident}\}\,. \end{aligned}$$

Let $x_0 = \{K\}$ and $\xi_0 = \{H\}$ denote the origins in X and Ξ, respectively. If $\Pi : G \to G/H$ denotes the natural mapping then since $\check{x}_0 = K \cdot \xi_0$ we



$x \in \hat{\xi} \Leftrightarrow \xi \in \check{x}$

FIGURE II.1.

have

$$\Pi^{-1}(\Xi - \check{x}_0) = \{g \in G : gH \notin KH\} = G - KH\,.$$

In particular $\Pi(G - KH) = \Xi - \check{x}_0$ so since Π is an open mapping, $\check{x}_0$ is a closed subset of Ξ. This proves

Lemma 1.1. *Each $\check{x}$ and each $\widehat{\xi}$ is closed.*

Using the notation $A^g = gAg^{-1}$ $(g \in G, A \subset G)$ we have the following lemma.

Lemma 1.2. *Let $g, \gamma \in G$, $x \in gK$, $\xi = \gamma H$. Then*

$$\check{x} \textit{ is an orbit of } K^g, \ \widehat{\xi} \textit{ is an orbit of } H^\gamma\,,$$

and

$$\check{x} = K^g/L^g\,, \quad \widehat{\xi} = H^\gamma/L^\gamma\,.$$

Proof. By definition

$$\check{x} = \{\delta H \,:\, \delta H \cap gK \neq \emptyset\} = \{gkH \,:\, k \in K\} \tag{3}$$

which is the orbit of the point gH under gKg^{-1}. The subgroup fixing gH is $gKg^{-1} \cap gHg^{-1} = L^g$. Also (3) implies

$$\check{x} = g \cdot \check{x}_0 \qquad \widehat{\xi} = \gamma \cdot \widehat{\xi}_0\,,$$

where the dot $\cdot$ denotes the action of G on X and Ξ.

Lemma 1.3. *Consider the subgroups*

$$\begin{aligned} K_H &= \{k \in K \,:\, kH \cup k^{-1}H \subset HK\} \\ H_K &= \{h \in H \,:\, hK \cup h^{-1}K \subset KH\}\,. \end{aligned}$$

The following properties are equivalent:

(a) $K \cap H = K_H = H_K$.

(b) *The maps $x \to \check{x}\,(x \in X)$ and $\xi \to \widehat{\xi}\,(\xi \in \Xi)$ are injective.*

We think of property (a) as a kind of *transversality* of K and H.

Proof. Suppose $x_1 = g_1K$, $x_2 = g_2K$ and $\check{x}_1 = \check{x}_2$. Then by (3) $g_1 \cdot \check{x}_0 = g_1 \cdot \check{x}_0$ so $g \cdot \check{x}_0 = \check{x}_0$ if $g = g_1^{-1}g_2$. In particular $g \cdot \xi_0 \subset \check{x}_0$ so $g \cdot \xi_0 = k \cdot \xi_0$ for some $k \in K$. Hence $k^{-1}g = h \in H$ so $h \cdot \check{x}_0 = \check{x}_0$, that is $hK \cdot \xi_0 = K \cdot \xi_0$. As a relation in G, this means $hKH = KH$. In particular $hK \subset KH$. Since $h \cdot \check{x}_0 = \check{x}_0$ implies $h^{-1} \cdot \check{x}_0 = \check{x}_0$ we have also $h^{-1}K \subset KH$ so by (b) $h \in K$ which gives $x_1 = x_2$.

On the other hand, suppose the map $x \to \check{x}$ is injective and suppose $h \in H$ satisfies $h^{-1}K \cup hK \subset KH$. Then

$$hK \cdot \xi_0 \subset K \cdot \xi_0 \text{ and } h^{-1}K \cdot \xi_0 \subset K \cdot \xi_0 \,.$$

By Lemma 1.2, $h \cdot \check{x}_0 \subset \check{x}_0$ and $h^{-1} \cdot \check{x}_0 \subset \check{x}_0$. Thus $h \cdot \check{x}_0 = \check{x}_0$ whence by the assumption, $h \cdot x_0 = x_0$ so $h \in K$.

Thus we see that under the transversality assumption a) the elements ξ can be viewed as the subsets $\widehat{\xi}$ of X and the elements x as the subsets $\check{x}$ of Ξ. We say X and Ξ are *homogeneous spaces in duality*.

The maps are also conveniently described by means of the following *double fibration*

(4)
$$\begin{tikzcd} & G/L \arrow[dl, "p"'] \arrow[dr, "\pi"] & \\ G/K & & G/H \end{tikzcd}$$

where $p(gL) = gK$, $\pi(\gamma L) = \gamma H$. In fact, by (3) we have

$$\check{x} = \pi(p^{-1}(x)) \qquad \widehat{\xi} = p(\pi^{-1}(\xi))\,.$$

We now prove some group-theoretic properties of the incidence, supplementing Lemma 1.3.

Theorem 1.4. *(i) We have the identification*

$$G/L = \{(x,\xi) \in X \times \Xi \,:\, x \textit{ and } \xi \textit{ incident}\}$$

via the bijection $\tau \,:\, gL \to (gK, gH)$.

(ii) The property

$$KHK = G$$

is equivariant to the property:

Any two $x_1, x_2 \in X$ are incident to some $\xi \in \Xi$. A similar statement holds for $HKH = G$.

(iii) The property

$$HK \cap KH = K \cup H$$

is equivalent to the property:

For any two $x_1 \neq x_2$ in X there is at most one $\xi \in \Xi$ incident to both. By symmetry, this is equivalent to the property:

For any $\xi_1 \neq \xi_2$ in Ξ there is at most one $x \in X$ incident to both.

Proof. (i) The map is well-defined and injective. The surjectivity is clear because if $gK \cap \gamma H \neq \emptyset$ then $gk = \gamma h$ and $\tau(gkL) = (gK, \gamma H)$.

(ii) We can take $x_2 = x_0$. Writing $x_1 = gK$, $\xi = \gamma H$ we have

$$\begin{aligned} x_0, \xi \text{ incident } &\Leftrightarrow \gamma h &=& \; k \quad (\text{some } h \in H, k \in K) \\ x_1, \xi \text{ incident } &\Leftrightarrow \gamma h_1 &=& \; g_1 k_1 \quad (\text{some } h_1 \in H, k_1 \in K) \end{aligned}$$

Thus if x_0, x_1 are incident to ξ we have $g_1 = kh^{-1}h_1k_1^{-1}$. Conversely if $g_1 = k'h'k''$ we put $\gamma = k'h'$ and then x_0, x_1 are incident to $\xi = \gamma H$.

(iii) Suppose first $KH \cap HK = K \cup H$. Let $x_1 \neq x_2$ in X. Suppose $\xi_1 \neq \xi_2$ in Ξ are both incident to x_1 and x_2. Let $x_i = g_iK$, $\xi_j = \gamma_jH$. Since x_i is incident to ξ_j there exist $k_{ij} \in K$, $h_{ij} \in H$ such that

$$g_ik_{ij} = \gamma_jh_{ij} \qquad i = 1,2\,;\; j = 1,2\,. \tag{5}$$

Eliminating g_i and γ_j we obtain

$$k_{2\,2}^{-1}k_{2\,1}h_{2\,1}^{-1}h_{1\,1} = h_{2\,2}^{-1}h_{1\,2}k_{1\,2}^{-1}k_{1\,1}\,. \tag{6}$$

This being in $KH \cap HK$ it lies in $K \cup H$. If the left hand side is in K then $h_{2\,1}^{-1}h_{1\,1} \in K$ so

$$g_2K = \gamma_1h_{2\,1}K = \gamma_1h_{1\,1}K = g_1K\,,$$

contradicting $x_2 \neq x_1$. Similarly if expression (6) is in H then $k_{1\,2}^{-1}k_{1\,1} \in H$ so by (5) we get the contradiction

$$\gamma_2H = g_1k_{1\,2}H = g_1k_{1\,1}H = \gamma_1H\,.$$

Conversely, suppose $KH \cap HK \neq K \cup H$. Then there exist h_1, h_2, k_1, k_2 such that $h_1k_1 = k_2h_2$ and $h_1k_1 \notin K \cup H$. Put $x_1 = h_1K, \xi_2 = k_2H$. Then $x_1 \neq x_0, \xi_0 \neq \xi_2$, yet both ξ_0 and ξ_2 are incident to both x_0 and x_1.

Examples

(i) *Points outside hyperplanes.* We saw before that if in the coset space representation (1) $\mathbf{O}(n)$ is viewed as the isotropy group of 0 and $\mathbb{Z}_2\mathbf{M}(n-1)$ is viewed as the isotropy group of a hyperplane *through* 0 then the abstract incidence notion is equivalent to the naive one: $x \in \mathbf{R}^n$ is incident to $\xi \in \mathbf{P}^n$ if and only if $x \in \xi$.

On the other hand we can also view $\mathbb{Z}_2\mathbf{M}(n-1)$ as the isotropy group of a hyperplane ξ_δ at a distance $\delta > 0$ from 0. (This amounts to a different embedding of the group $\mathbb{Z}_2\mathbf{M}(n-1)$ into $\mathbf{M}(n)$.) Then we have the following generalization.

Proposition 1.5. *The point $x \in \mathbf{R}^n$ and the hyperplane $\xi \in \mathbf{P}^n$ are incident if and only if distance $(x, \xi) = \delta$.*

Proof. Let $x = gK$, $\xi = \gamma H$ where $K = \mathbf{O}(n), H = \mathbb{Z}_2\mathbf{M}(n-1)$. Then if $gK \cap \gamma H \neq \emptyset$, we have $gk = \gamma h$ for some $k \in K, h \in H$. Now the orbit $H \cdot 0$ consists of the two planes ξ'_δ and ξ''_δ parallel to ξ_δ at a distance δ from ξ_δ. The relation

$$g \cdot 0 = \gamma h \cdot 0 \in \gamma \cdot (\xi'_\delta \cup \xi''_\delta)$$

together with the fact that g and γ are isometries shows that x has distance δ from $\gamma \cdot \xi_\delta = \xi$.

On the other hand if distance $(x, \xi) = \delta$ we have $g \cdot 0 \in \gamma \cdot (\xi'_\delta \cup \xi''_\delta) = \gamma H \cdot 0$ which means $gK \cap \gamma H \neq \emptyset$.

(ii) *Unit spheres.* Let σ_0 be a sphere in $\mathbf{R}^n$ of radius one passing through the origin. Denoting by Σ the set of all *unit* spheres in $\mathbf{R}^n$ we have the dual homogeneous spaces

$$\mathbf{R}^n = \mathbf{M}(n)/\mathbf{O}(n)\,; \quad \Sigma = \mathbf{M}(n)/\mathbf{O}^*(n) \tag{7}$$

where $\mathbf{O}^*(n)$ is the set of rotations around the center of σ_0. Here a point $x = g\mathbf{O}(n)$ is incident to $\sigma_0 = \gamma\mathbf{O}^*(n)$ if and only if $x \in \sigma$.

§2 The Radon Transform for the Double Fibration

With K, H and L as in §1 we assume now that K/L and H/L have positive measures $d\mu_0 = dk_L$ and $dm_0 = dh_L$ invariant under K and H, respectively. This is for example guaranteed if L is compact.

Lemma 2.1. *Assume the transversality condition (a). Then there exists a measure on each $\check{x}$ coinciding with $d\mu_0$ on $K/L = \check{x}_0$ such that whenever $g \cdot \check{x}_1 = \check{x}_2$ the measures on $\check{x}_1$ and $\check{x}_2$ correspond under g. A similar statement holds for dm on $\widehat{\xi}$.*

Proof. If $\check{x} = g \cdot \check{x}_0$ we transfer the measure $d\mu_0 = dk_L$ over on $\check{x}$ by the map $\xi \to g \cdot \xi$. If $g \cdot \check{x}_0 = g_1 \cdot \check{x}_0$ then $(g \cdot x_0)^\vee = (g_1 \cdot x_0)^\vee$ so by Lemma 1.3, $g \cdot x_0 = g_1 \cdot x_0$ so $g = g_1 k$ with $k \in K$. Since $d\mu_0$ is K-invariant the lemma follows.

The measures defined on each $\check{x}$ and $\widehat{\xi}$ under condition (a) are denoted by $d\mu$ and dm, respectively.

Definition. The Radon transform $f \to \widehat{f}$ and its dual $\varphi \to \check{\varphi}$ are defined by

$$\widehat{f}(\xi) = \int_{\widehat{\xi}} f(x)\, dm(x)\,, \qquad \check{\varphi}(x) = \int_{\check{x}} \varphi(\xi)\, d\mu(\xi)\,. \tag{8}$$

whenever the integrals converge. Because of Lemma 1.1, this will always happen for $f \in C_c(X)$, $\varphi \in C_c(\Xi)$.

In the setup of Proposition 1.5, $\widehat{f}(\xi)$ is the integral of f over the two hyperplanes at distance δ from ξ and $\check{\varphi}(x)$ is the average of φ over the set of hyperplanes at distance δ from x. For $\delta = 0$ we recover the transforms of Ch. I, §1.

Formula (8) can also be written in the group-theoretic terms,

$$\widehat{f}(\gamma H) = \int_{H/L} f(\gamma h K)\, dh_L\,, \quad \check{\varphi}(gK) = \int_{K/L} \varphi(gkH)\, dk_L\,. \tag{9}$$

Note that (9) serves as a definition even if condition (a) in Lemma 1.3 is not satisfied. In this abstract setup the spaces X and Ξ have equal status. The theory in Ch. I, in particular Lemma 2.1, Theorems 2.4, 2.10, 3.1 thus raises the following problems:

Principal Problems:

A. Relate function spaces on X and on Ξ by means of the transforms $f \to \widehat{f}$, $\varphi \to \check{\varphi}$. In particular, determine their ranges and kernels.

B. Invert the transforms $f \to \widehat{f}$, $\varphi \to \check{\varphi}$ on suitable function spaces.

C. In the case when G is a Lie group so X and Ξ are manifolds let $\mathbf{D}(X)$ and $\mathbf{D}(\Xi)$ denote the algebras of G-invariant differential operators on X and Ξ, respectively. Is there a map $D \to \widehat{D}$ of $\mathbf{D}(X)$ into $\mathbf{D}(\Xi)$ and a map $E \to \check{E}$ of $\mathbf{D}(\Xi)$ into $\mathbf{D}(X)$ such that

$$(Df)\widehat{} = \widehat{D}\widehat{f}\,, \quad (E\varphi)^{\vee} = \check{E}\check{\varphi}\,?$$

Although weaker assumptions would be sufficient, we assume now that the groups G, K, H and L all have bi-invariant Haar measures dg, dk, dh and $d\ell$. These will then generate invariant measures dg_K, dg_H, dg_L, dk_L, dh_L on G/K, G/H, G/L, K/L, H/L, respectively. This means that

$$\int_G F(g)\, dg = \int_{G/K} \left(\int_K F(gk)\, dk \right) dg_K \tag{10}$$

and similarly dg and dh determine dg_H, etc. Then

$$\int_{G/L} Q(gL)\, dg_L = c \int_{G/K} dg_K \int_{K/L} Q(gkL)\, dk_L \tag{11}$$

for $Q \in C_c(G/L)$ where c is a constant. In fact, the integrals on both sides of (11) constitute invariant measures on G/L and thus must be proportional. However,

$$\int_G F(g)\, dg = \int_{G/L} \left(\int_L F(g\ell)\, d\ell \right) dg_L \tag{12}$$

and

$$\int_K F(k)\, dk = \int_{K/L} \left(\int_L F(k\ell)\, d\ell \right) dk_L\, . \tag{13}$$

Using (13) on (10) and combining with (11) we see that the constant c equals 1.

We shall now prove that $f \to \widehat{f}$ and $\varphi \to \check{\varphi}$ are adjoint operators. We write dx for dg_K and $d\xi$ for dg_H.

Proposition 2.2. *Let $f \in C_c(X)$, $\varphi \in C_c(\Xi)$. Then $\widehat{f}$ and $\check{\varphi}$ are continuous and*

$$\int_X f(x)\check{\varphi}(x)\, dx = \int_\Xi \widehat{f}(\xi)\varphi(\xi)\, d\xi\, .$$

Proof. The continuity statement is immediate from (9). We consider the function

$$P = (f \circ p)(\varphi \circ \pi)$$

on G/L. We integrate it over G/L in two ways using the double fibration (4). This amounts to using (11) and its analog with G/K replaced by G/H with $Q = P$. Since $P(gk\,L) = f(gK)\varphi(gkH)$ the right hand side of (11) becomes

$$\int_{G/K} f(gK)\check{\varphi}(gK)\, dg_K\, .$$

If we treat G/H similarly, the lemma follows.

The result shows how to define the Radon transform and its dual for measures and, in case G is a Lie group, for distributions.

Definition. Let s be a measure on X of compact support. Its Radon transform is the functional $\widehat{s}$ on $C_c(\Xi)$ defined by

$$\widehat{s}(\varphi) = s(\check{\varphi})\, . \tag{14}$$

Similarly $\check{\sigma}$ is defined by

$$\check{\sigma}(f) = \sigma(\widehat{f})\, , \qquad f \in C_c(X) \tag{15}$$

if σ is a compactly supported measure on Ξ.

Lemma 2.3. *(i) If s is a compactly supported measure on X, $\widehat{s}$ is a measure on Ξ.*

(ii) If s is a bounded measure on X and if $\check{x}_0$ has finite measure then $\widehat{s}$ as defined by (14) is a bounded measure.

Proof. (i) The measure s can be written as a difference $s = s^+ - s^-$ of two positive measures, each of compact support. Then $\widehat{s} = \widehat{s^+} - \widehat{s^-}$ is a difference of two positive *functionals* on $C_c(\Xi)$.

Since a positive functional is necessarily a measure, $\widehat{s}$ is a measure.

(ii) We have

$$\sup_x |\check{\varphi}(x)| \leq \sup_\xi |\varphi(\xi)|\, \mu_0(\check{x}_0)$$

so for a constant K,

$$|\widehat{s}(\varphi)| = |s(\check{\varphi})| \leq K \sup|\check{\varphi}| \leq K\mu_0(\check{x}_0) \sup|\varphi|\,,$$

and the boundedness of $\widehat{s}$ follows.

If G is a Lie group then (14), (15) with $f \in \mathcal{D}(X)\,,\quad \varphi \in \mathcal{D}(\Xi)$ serve to define the Radon transform $s \to \widehat{s}$ and the dual $\sigma \to \check{\sigma}$ for distributions s and σ of compact support. We consider the spaces $\mathcal{D}(X)$ and $\mathcal{E}(X)$ $(= \mathcal{C}^\infty(X))$ with their customary topologies (Chapter V,§1). The duals $\mathcal{D}'(X)$ and $\mathcal{E}'(X)$ then consist of the distributions on X and the distributions on X of compact support, respectively.

Proposition 2.4. *The mappings*

$$\begin{array}{rcl} f \in \mathcal{D}(X) & \to & \widehat{f} \in \mathcal{E}(\Xi) \\ \varphi \in \mathcal{D}(\Xi) & \to & \check{\varphi} \in \mathcal{E}(X) \end{array}$$

are continuous. In particular,

$$\begin{array}{rcl} s \in \mathcal{E}'(X) & \Rightarrow & \widehat{s} \in \mathcal{D}'(\Xi) \\ \sigma \in \mathcal{E}'(\Xi) & \Rightarrow & \check{\sigma} \in \mathcal{D}'(X)\,. \end{array}$$

Proof. We have

$$\widehat{f}(g \cdot \xi_0) = \int_{\widehat{\xi}_0} f(g \cdot x)\, dm_0(x)\,. \tag{16}$$

Let g run through a local cross section through e in G over a neighborhood of ξ_0 in Ξ. If $(t_1, \ldots, t_n)$ are coordinates of g and $(x_1, \ldots, x_m)$ the coordinates of $x \in \widehat{\xi}_0$ then (16) can be written in the form

$$\widehat{F}(t_1, \ldots, t_n) = \int F(t_1, \ldots, t_n\,; x_1, \ldots, x_m)\, dx_1 \ldots dx_m\,.$$

Now it is clear that $\widehat{f} \in \mathcal{E}(\Xi)$ and that $f \to \widehat{f}$ is continuous, proving the proposition.

The result has the following refinement.

Proposition 2.5. *Assume K compact. Then*

(i) $f \to \widehat{f}$ *is a continuous mapping of* $\mathcal{D}(X)$ *into* $\mathcal{D}(\Xi)$.

(ii) $\varphi \to \check{\varphi}$ *is a continuous mapping of* $\mathcal{E}(\Xi)$ *into* $\mathcal{E}(X)$.

A self-contained proof is given in the author's book [1994b], Ch. I, § 3. The result has the following consequence.

Corollary 2.6. *Assume K compact. Then* $\mathcal{E}'(X)\widehat{\ } \subset \mathcal{E}'(\Xi)$, $\mathcal{D}'(\Xi)^{\vee} \subset \mathcal{D}'(X)$.

In Chapter I we have given solutions to Problems A, B, C in some cases. Further examples will be given in § 4 of this chapter and Chapter III will include their solution for the antipodal manifolds for compact two-point homogeneous spaces.

The variety of the results for these examples make it doubtful that the individual results could be captured by a general theory. Our abstract setup in terms of homogeneous spaces in duality is therefore to be regarded as a framework for examples rather than as axioms for a general theory.

Nevertheless, certain general features emerge from the study of these examples. If $\dim X = \dim \Xi$ and $f \to \widehat{f}$ is injective the range consists of functions which are either arbitrary or at least subjected to rather weak conditions. As the difference $\dim \Xi - \dim X$ increases more conditions are imposed on the functions in the range. (See the example of the d-plane transform in $\mathbf{R}^n$.) In the case when G is a Lie group there is a group-theoretic explanation for this. Let $\mathbf{D}(G)$ denote the algebra of left-invariant differential operators on G. Since $\mathbf{D}(G)$ is generated by the left invariant vector fields on G, the action of G on X and on Ξ induces homomorphisms

$$\lambda : \mathbf{D}(G) \longrightarrow E(X)\,, \tag{17}$$

$$\Lambda : \mathbf{D}(G) \longrightarrow E(\Xi)\,, \tag{18}$$

where for a manifold M, $E(M)$ denotes the algebra of all differential operators on M. Since $f \to \widehat{f}$ and $\varphi \to \check{\varphi}$ commute with the action of G we have for $D \in \mathbf{D}(G)$,

$$(\lambda(D)f)\widehat{\ } = \Lambda(D)\widehat{f}\,, \quad (\Lambda(D)\varphi)^{\vee} = \lambda(D)\check{\varphi}\,. \tag{19}$$

Therefore $\Lambda(D)$ annihilates the range of $f \to \widehat{f}$ if $\lambda(D) = 0$. In some cases, including the case of the d-plane transform in $\mathbf{R}^n$, the range is characterized as the null space of these operators $\Lambda(D)$ (with $\lambda(D) = 0$).

§3 Orbital Integrals

As before let $X = G/K$ be a homogeneous space with origin $o = (K)$. Given $x_0 \in X$ let G_{x_0} denote the subgroup of G leaving x_0 fixed, i.e., the isotropy subgroup of G at x_0.

Definition. A *generalized sphere* is an orbit $G_{x_0} \cdot x$ in X of some point $x \in X$ under the isotropy subgroup at some point $x_0 \in X$.

Examples. (i) If $X = \mathbf{R}^n$, $G = \mathbf{M}(n)$ then the generalized spheres are just the spheres.

(ii) Let X be a locally compact subgroup L and G the product group $L \times L$ acting on L on the right and left, the element $(\ell_1, \ell_2) \in L \times L$ inducing action $\ell \to \ell_1 \ell \, \ell_2^{-1}$ on L. Let ΔL denote the diagonal in $L \times L$. If $\ell_0 \in L$ then the isotropy subgroup of ℓ_0 is given by

$$(L \times L)_{\ell_0} = (\ell_0, e) \Delta L (\ell_0^{-1}, e) \tag{20}$$

and the orbit of ℓ under it by

$$(L \times L)_{\ell_0} \cdot \ell = \ell_0 (\ell_0^{-1} \ell)^L .$$

that is the left translate by ℓ_0 of the conjugacy class of the element $\ell_0^{-1}\ell$. Thus the *generalized spheres in the group L are the left (or right) translates of its conjugacy classes*.

Coming back to the general case $X = G/K = G/G_0$ we assume that G_0, and therefore each G_{x_0}, is unimodular. But $G_{x_0} \cdot x = G_{x_0}/(G_{x_0})_x$ so $(G_{x_0})_x$ unimodular implies the orbit $G_{x_0} \cdot x$ has an invariant measure determined up to a constant factor. We can now consider the following general problem (following Problems A, B, C above).

D. *Determine a function f on X in terms of its integrals over generalized spheres.*

Remark 3.1. In this problem it is of course significant how the invariant measures on the various orbits are normalized.

(a) If G_0 is compact the problem above is rather trivial because each orbit $G_{x_0} \cdot x$ has finite invariant measure so $f(x_0)$ is given as the limit as $x \to x_0$ of the average of f over $G_{x_0} \cdot x$.

(b) Suppose that for each $x_0 \in X$ there is a G_{x_0}-invariant open set $C_{x_0} \subset X$ containing x_0 in its closure such that for each $x \in C_{x_0}$ the isotropy group $(G_{x_0})_x$ is compact. The invariant measure on the orbit $G_{x_0} \cdot x \, (x_0 \in X, x \in C_{x_0})$ can then be consistently normalized as follows: Fix a Haar measure dg_0 on G_0. If $x_0 = g \cdot o$ we have $G_{x_0} = gG_0g^{-1}$ and can carry dg_0 over to a measure dg_{x_0} on G_{x_0} by means of the conjugation $z \to gzg^{-1} \quad (z \in G_0)$.

Since dg_0 is bi-invariant, dg_{x_0} is independent of the choice of g satisfying $x_0 = g\cdot o$, and is bi-invariant. Since $(G_{x_0})_x$ is compact it has a unique Haar measure $dg_{x_0,x}$ with total measure 1 and now dg_{x_0} and $dg_{x_0,x}$ determine canonically an invariant measure μ on the orbit $G_{x_0}\cdot x = G_{x_0}/(G_{x_0})_x$. We can therefore state Problem D in a more specific form.

D′. *Express $f(x_0)$ in terms of integrals*

$$\int_{G_{x_0}\cdot x} f(p)\,d\mu(p) \qquad x \in C_{x_0}\,. \tag{21}$$

For the case when X is an *isotropic Lorentz manifold* the assumptions above are satisfied (with C_{x_0} consisting of the "timelike" rays from x_0) and we shall obtain in Ch. IV an explicit solution to Problem D' (Theorem 4.1, Ch. IV).

(c) If in Example (ii) above L is a semisimple Lie group Problem D is a basic step (Gelfand-Graev [1955], Harish-Chandra [1957]) in proving the Plancherel formula for the Fourier transform on L.

§4 Examples of Radon Transforms for Homogeneous Spaces in Duality

In this section we discuss some examples of the abstract formalism and problems set forth in the preceding sections §1–§2.

A. The Funk Transform.

This case goes back to Funk [1916] (preceding Radon's paper [1917]) where he proved that a symmetric function on $\mathbf{S}^2$ is determined by its great circle integrals. This is carried out in more detail and in greater generality in Chapter III, §1. Here we state the solution of Problem B for $X = \mathbf{S}^2$, Ξ the set of all great circles, both as homogeneous spaces of $\mathbf{O}(3)$. Given $p \geq 0$ let $\xi_p \in \Xi$ have distance p from the North Pole o, $H_p \subset \mathbf{O}(3)$ the subgroup leaving ξ_p invariant and $K \subset \mathbf{O}(3)$ the subgroup fixing o. Then in the double fibration

$$\begin{tikzcd} & \mathbf{O}(3)/(K\cap H_p) \arrow[dl] \arrow[dr] & \\ X = \mathbf{O}(3)/K & & \Xi = \mathbf{O}(3)/H_p \end{tikzcd}$$

$x \in X$ and $\xi \in \Xi$ are incident if and only if $d(x,\xi) = p$. The proof is the same as that of Proposition 1.5. In order to invert the Funk transform $f \to \widehat{f} \quad (= \widehat{f}_0)$ we invoke the transform $\varphi \to \check{\varphi}_p$. Note that $(\widehat{f})^{\vee}_p(x)$ is the

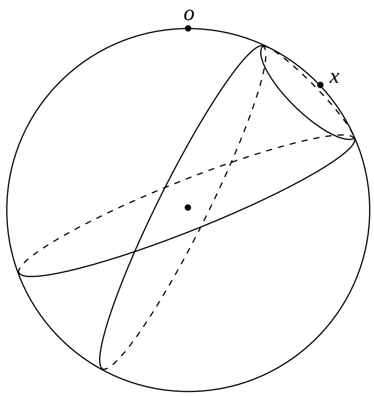


FIGURE II.2.

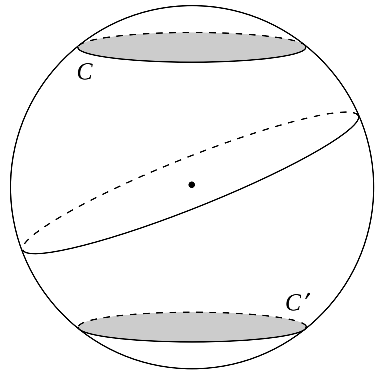


FIGURE II.3.

average of the integrals of f over the great circles ξ at distance p from x (see Figure II.2). As a special case of Theorem 1.11, Chapter III, we have the following inversion.

Theorem 4.1. *The Funk transform $f \to \widehat{f}$ is (for f even) inverted by*

$$f(x) = \frac{1}{2\pi}\left\{\frac{d}{du}\int_0^u (\widehat{f})^\vee_{\cos^{-1}(v)}(x)v(u^2-v^2)^{-\frac{1}{2}}\,dv\right\}_{u=1}. \tag{22}$$

Another inversion formula is

$$f = -\frac{1}{4\pi}LS((\widehat{f})^\vee) \tag{23}$$

(Theorem 1.15, Chapter III), where L is the Laplacian and S the integral operator given by (66)–(68), Chapter III. While (23) is short the operator S is only given in terms of a spherical harmonics expansion. Also Theorem 1.17, Ch. III shows that if f is even and if all its derivatives vanish on the equator then f vanishes outside the "arctic zones" C and C' if and only if $\widehat{f}(\xi) = 0$ for all great circles ξ disjoint from C and C' (Fig. II.3).

The Hyperbolic Plane $\mathbf{H}^2$.

This remarkable object enters into several fields in mathematics. In particular, it offers at least two interesting cases of Radon transforms. We take $\mathbf{H}^2$ as the disk $D : |z| < 1$ with the Riemannian structure

$$\langle u, v\rangle_z = \frac{(u,v)}{(1-|z|^2)^2}, \quad ds^2 = \frac{|dz|^2}{(1-|z|^2)^2} \tag{24}$$

if u and v are any tangent vectors at $z \in D$. Here (u, v) denotes the usual inner product on $\mathbf{R}^2$. The Laplace-Beltrami operator for (24) is given by

$$L = (1-x^2-y^2)^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

The group $G = \mathbf{SU}(1,1)$ of matrices

$$\left\{\begin{pmatrix} a & b \\ \overline{b} & \overline{a}\end{pmatrix} : |a|^2 - |b|^2 = 1\right\}$$

acts transitively on the unit disk by

$$\begin{pmatrix} a & b \\ \overline{b} & \overline{a}\end{pmatrix} \cdot z = \frac{az+b}{\overline{b}z + \overline{a}} \tag{25}$$

and leaves the metric (24) invariant. The length of a curve $\gamma(t)$ $(\alpha \leq t \leq \beta)$ is defined by

$$L(\gamma) = \int_\alpha^\beta (\langle \gamma'(t), \gamma'(t)\rangle_{\gamma(t)})^{1/2}\, dt\,. \tag{26}$$

If $\gamma(\alpha) = o$, $\gamma(\beta) = x \in \mathbf{R}$ and $\gamma_o(t) = tx \quad (0 \leq t \leq 1)$ then (26) shows easily that $L(\gamma) \geq L(\gamma_o)$ so γ_o is a geodesic and the distance d satisfies

$$d(o,x) = \int_0^1 \frac{|x|}{1-t^2x^2}\, dt = \tfrac{1}{2}\log\frac{1+|x|}{1-|x|}\,. \tag{27}$$

Since G acts conformally on D the *geodesics* in $\mathbf{H}^2$ are the circular arcs in $|z| < 1$ perpendicular to the boundary $|z| = 1$.

We consider now the following subgroups of G:

$$\begin{aligned}
K &= \{k_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix} : 0 \leq \theta < 2\pi\} \\
M &= \{k_0, k_\pi\}\,, \qquad M' = \{k_0, k_\pi\,, k_{-\frac{\pi}{2}}, k_{\frac{\pi}{2}}\} \\
A &= \{a_t = \begin{pmatrix} \operatorname{ch} t & \operatorname{sh} t \\ \operatorname{sh} t & \operatorname{ch} t\end{pmatrix} : t \in \mathbf{R}\}, \\
N &= \{n_x = \begin{pmatrix} 1+ix, & -ix \\ ix, & 1-ix\end{pmatrix} : x \in \mathbf{R}\} \\
\Gamma &= C\mathbf{SL}(2,\mathbb{Z})C^{-1}\,,
\end{aligned}$$

where C is the transformation $w \to (w-i)/(w+i)$ mapping the upper half-plane onto the unit disk.

The orbit $A \cdot o$ is the horizontal diameter and the orbits $N \cdot (a_t \cdot o)$ are the circles tangential to $|z| = 1$ at $z = 1$. Thus $NA \cdot o$ is the entire disk D so we see that $G = NAK$ and also $G = KAN$.

B. The X-ray Transform in $\mathbf{H}^2$.

The (unoriented) geodesics for the metric (24) were mentioned above. Clearly the group G permutes these geodesics transitively (Fig. II.4). Let

Ξ be the set of all these geodesics. Let o denote the origin in $\mathbf{H}^2$ and ξ_o the horizontal geodesic through o. Then

$$X = G/K\,, \quad \Xi = G/M'A\,. \tag{28}$$

We can also fix a geodesic ξ_p at distance p from o and write

$$X = G/K\,, \quad \Xi = G/H_p\,, \tag{29}$$

where H_p is the subgroup of G leaving ξ_p stable. Then for the homogeneous spaces (29), x and ξ are incident if and only if $d(x, \xi) = p$. The transform $f \to \widehat{f}$ is inverted by means of the dual transform $\varphi \to \check{\varphi}_p$ for (29). The inversion below is a special case of Theorem 1.10, Chapter III, and is the analog of (22). Note however the absence of v in the integrand. Observe also that the metric ds is renormalized by the factor 2 (so curvature is -1).

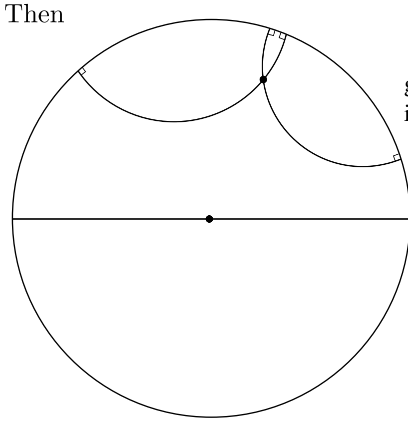


FIGURE II.4.

Theorem 4.2. *The X-ray transform in* $\mathbf{H}^2$ *with the metric*

$$ds^2 = \frac{4|dz|^2}{(1-|z|^2)^2}$$

is inverted by

$$f(z) = \frac{1}{\pi}\left\{\frac{d}{du}\int_0^u (\widehat{f})^{\vee}_{\operatorname{lm} v}(z)(u^2 - v^2)^{-\frac{1}{2}}\,dv\right\}_{u=1}, \tag{30}$$

where $\operatorname{lm} v = \cosh^{-1}(v^{-1})$.

Another inversion formula is

$$f = -\frac{1}{4\pi} LS((\widehat{f})^{\vee}), \tag{31}$$

where S is the operator of convolution on $\mathbf{H}^2$ with the function $x \to \coth(d(x,o)) - 1$, (Theorem 1.14, Chapter III).

C. The Horocycles in $\mathbf{H}^2$.

Consider a family of geodesics with the same limit point on the boundary B. The *horocycles* in $\mathbf{H}^2$ are by definition the orthogonal trajectories of such families of geodesics. Thus the horocycles are the circles tangential to $|z| = 1$ from the inside (Fig. II.5).

One such horocycle is $\xi_0 = N \cdot o$, the orbit of the origin o under the action of N. Since $a_t \cdot \xi$ is the horocycle with diameter $(\tanh t, 1)$ G acts transitively on the set Ξ of horocycles. Now we take $\mathbf{H}^2$ with the metric (24). Since $G = KAN$ it is easy to see that MN is the subgroup leaving ξ_o invariant. Thus we have here

(32)
$$X = G/K\,, \quad \Xi = G/MN\,.$$

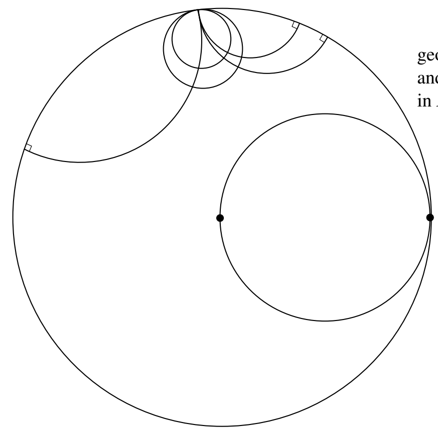


FIGURE II.5.

Furthermore each horocycle has the form $\xi = ka_t \cdot \xi_0$ where $kM \in K/M$ and $t \in \mathbf{R}$ are unique. Thus $\Xi \sim K/M \times A$, which is also evident from the figure.

We observe now that the maps

$$\psi : t \to a_t \cdot o\,, \quad \varphi : x \to n_x \cdot o$$

of $\mathbf{R}$ onto γ_0 and ξ_0, respectively, are isometries. The first statement follows from (27) because

$$d(o, a_t) = d(o, \tanh t) = t\,.$$

For the second we note that

$$\varphi(x) = x(x+i)^{-1}\,, \quad \varphi'(x) = i(x+i)^{-2}$$

so

$$\langle \varphi'(x), \varphi'(x) \rangle_{\varphi(x)} = (x^2+1)^{-4}(1 - |x(x+i)^{-1}|^2)^{-2} = 1\,.$$

Thus we give A and N the Haar measures $d(a_t) = dt$ and $d(n_x) = dx$.

Geometrically, the Radon transform on X relative to the horocycles is defined by

(33)
$$\widehat{f}(\xi) = \int_\xi f(x)\, dm(x)\,,$$

where dm is the measure on ξ induced by (24). Because of our remarks about φ, (33) becomes

(34)
$$\widehat{f}(g \cdot \xi_0) = \int_N f(gn \cdot o)\, dn\,,$$

so the geometric definition (33) coincides with the group-theoretic one in (9). The dual transform is given by

$$\check{\varphi}(g \cdot o) = \int_K \varphi(gk \cdot \xi_o)\, dk\,, \qquad (dk = d\theta/2\pi)\,. \tag{35}$$

In order to invert the transform $f \to \widehat{f}$ we introduce the non-Euclidean analog of the operator Λ in Chapter I, §3. Let T be the distribution on $\mathbf{R}$ given by

$$T\varphi = \tfrac{1}{2} \int_{\mathbf{R}} (\operatorname{sh} t)^{-1} \varphi(t)\, dt\,, \quad \varphi \in \mathcal{D}(\mathbf{R})\,, \tag{36}$$

considered as the Cauchy principal value, and put $T' = dT/dt$. Let Λ be the operator on $\mathcal{D}(\Xi)$ given by

$$(\Lambda\varphi)(ka_t \cdot \xi_0) = \int_{\mathbf{R}} \varphi(ka_{t-s} \cdot \xi_0) e^{-s}\, dT'(s)\,. \tag{37}$$

Theorem 4.3. *The Radon transform $f \to \widehat{f}$ for horocycles in $\mathbf{H}^2$ is inverted by*

$$f = \frac{1}{\pi} (\Lambda \widehat{f})^{\vee}\,, \quad f \in \mathcal{D}(\mathbf{H}^2)\,. \tag{38}$$

We begin with a simple lemma.

Lemma 4.4. *Let τ be a distribution on $\mathbf{R}$. Then the operator $\widetilde{\tau}$ on $\mathcal{D}(\Xi)$ given by the convolution*

$$(\widetilde{\tau}\varphi)(ka_t \cdot \xi_0) = \int_{\mathbf{R}} \varphi(ka_{t-s} \cdot \xi_0)\, d\tau(s)$$

is invariant under the action of G.

Proof. To understand the action of $g \in G$ on $\Xi \sim (K/M) \times A$ we write $gk = k'a_{t'}n'$. Since each $a \in A$ normalizes N we have

$$gka_t \cdot \xi_0 = gka_tN \cdot o = k'a_{t'}n'a_tN \cdot o = k'a_{t+t'} \cdot \xi_0\,.$$

Thus the action of g on $\Xi \simeq (K/M) \times A$ induces this fixed translation $a_t \to a_{t+t'}$ on A. This translation commutes with the convolution by τ so the lemma follows.

Since the operators Λ, $\widehat{\ }$, $^{\vee}$ in (38) are all G-invariant it suffices to prove the formula at the origin o. We first consider the case when f is K-invariant, i.e., $f(k \cdot z) \equiv f(z)$. Then by (34)

$$\widehat{f}(a_t \cdot \xi_0) = \int_{\mathbf{R}} f(a_t n_x \cdot o)\, dx\,. \tag{39}$$

Because of (27) we have

$$|z| = \tanh\, d(o,z)\,, \quad \cosh^2\, d(o,z) = (1-|z|^2)^{-1}\,. \tag{40}$$

Since

$$a_t n_x \cdot o = (\operatorname{sh} t - ix\, e^t)/(\operatorname{ch} t - ix\, e^t)$$

(40) shows that the distance $s = d(o, a_t n_x \cdot o)$ satisfies

$$\operatorname{ch}^2 s = \operatorname{ch}^2 t + x^2 e^{2t}\,. \tag{41}$$

Thus defining F on $[1,\infty)$ by

$$F(\operatorname{ch}^2 s) = f(\tanh\, s)\,, \tag{42}$$

we have

$$F'(\operatorname{ch}^2 s) = f'(\tanh\, s)(2\operatorname{sh} s \operatorname{ch}^3 s)^{-1}$$

so, since $f'(0)=0$, $\lim_{u\to 1} F'(u)$ exists. The transform (39) now becomes (with $xe^t = y$)

$$e^t \widehat{f}(a_t \cdot \xi_0) = \int_{\mathbf{R}} F(\operatorname{ch}^2 t + y^2)\, dy\,. \tag{43}$$

We put

$$\varphi(u) = \int_{\mathbf{R}} F(u+y^2)\, dy$$

and invert this as follows:

$$\begin{aligned}\int_{\mathbf{R}} \varphi'(u+z^2)\, dz &= \int_{\mathbf{R}^2} F'(u+y^2+z^2)\, dy\, dz \\ &= 2\pi \int_0^\infty F'(u+r^2) r\, dr = \pi \int_0^\infty F'(u+\rho)\, d\rho\,,\end{aligned}$$

so

$$-\pi F(u) = \int_{\mathbf{R}} \varphi'(u+z^2)\, dz\,.$$

In particular,

$$\begin{aligned} f(o) &= -\frac{1}{\pi}\int_{\mathbf{R}} \varphi'(1+z^2)\, dz = -\frac{1}{\pi}\int_{\mathbf{R}} \varphi'(\operatorname{ch}^2\tau)\operatorname{ch}\tau\, d\tau\,, \\ &= -\frac{1}{\pi}\int_{\mathbf{R}}\int_{\mathbf{R}} F'(\operatorname{ch}^2 t + y^2)\, dy \operatorname{ch} t\, dt \end{aligned}$$

so

$$f(o) = -\frac{1}{2\pi}\int_{\mathbf{R}} \frac{d}{dt}(e^t \widehat{f}(a_t\cdot\xi_0))\frac{dt}{\operatorname{sh} t}\,.$$

Since $(e^t \widehat{f})(a_t \cdot \xi_0)$ is even (cf. (43)) its derivative vanishes at $t=0$ so the integral is well defined. With T as in (36), the last formula can be written

$$f(o) = \frac{1}{\pi} T'_t(e^t \widehat{f}(a_t \cdot \xi_0))\,, \tag{44}$$

the prime indicating derivative. If f is not necessarily K-invariant we use (44) on the average

$$f^{\natural}(z) = \int_K f(k \cdot z)\, dk = \frac{1}{2\pi} \int_0^{2\pi} f(k_\theta \cdot z)\, d\theta\,.$$

Since $f^{\natural}(o) = f(o)$, (44) implies

$$f(o) = \frac{1}{\pi} \int_{\mathbf{R}} [e^t (f^{\natural})\widehat{\ }(a_t \cdot \xi_0)]\, dT'(t)\,. \tag{45}$$

This can be written as the convolution at $t = 0$ of $(f^{\natural})\widehat{\ }(a_t \cdot \xi_0)$ with the image of the distribution $e^t T'_t$ under $t \to -t$. Since T' is even the right hand side of (45) is the convolution at $t = 0$ of $\widehat{f^{\natural}}$ with $e^{-t} T'_t$. Thus by (37)

$$f(o) = \frac{1}{\pi} (\Lambda \widehat{f^{\natural}})(\xi_0)\,.$$

Since Λ and $\widehat{\ }$ commute with the K action this implies

$$f(o) = \frac{1}{\pi} \int_K (\Lambda \widehat{f})(k \cdot \xi_0) = \frac{1}{\pi} (\Lambda \widehat{f})^{\vee}(o)$$

and this proves the theorem.

Theorem 4.3 is of course the exact analog to Theorem 3.6 in Chapter I, although we have not specified the decay conditions for f needed in generalizing Theorem 4.3.

D. The Poisson Integral as a Radon Transform.

Here we preserve the notation introduced for the hyperbolic plane $\mathbf{H}^2$. Now we consider the homogeneous spaces

$$X = G/MAN\,, \quad \Xi = G/K\,. \tag{46}$$

Then Ξ is the disk $D : |z| < 1$. On the other hand, X is identified with the boundary $B : |z| = 1$, because when G acts on B, MAN is the subgroup fixing the point $z = 1$. Since $G = KAN$, each coset $gMAN$ intersects eK. Thus each $x \in X$ is incident to each $\xi \in \Xi$. Our abstract Radon transform (9) now takes the form

$$\begin{aligned} \widehat{f}(gK) &= \int_{K/L} f(gkMAN)\, dk_L = \int_B f(g \cdot b)\, db\,, \\ &= \int_B f(b) \frac{d(g^{-1} \cdot b)}{db}\, db\,. \end{aligned} \tag{47}$$

Writing g^{-1} in the form

$$g^{-1} : \zeta \to \frac{\zeta - z}{-\overline{z}\zeta + 1}\,, \quad g^{-1} \cdot e^{i\theta} = e^{i\varphi}\,,$$

we have

$$e^{i\varphi} = \frac{e^{i\theta} - z}{-\overline{z}e^{i\theta} + 1}\,, \quad \frac{d\varphi}{d\theta} = \frac{1 - |z|^2}{|z - e^{i\theta}|}\,,$$

and this last expression is the classical Poisson kernel. Since $gK = z$, (47) becomes the classical Poisson integral

$$\widehat{f}(z) = \int_B f(b) \frac{1 - |z|^2}{|z - b|^2}\, db\,. \tag{48}$$

Theorem 4.5. *The Radon transform $f \to \widehat{f}$ for the homogeneous spaces (46) is the classical Poisson integral (48). The inversion is given by the classical Schwarz theorem*

$$f(b) = \lim_{z \to b} \widehat{f}(z)\,, \quad f \in C(B)\,, \tag{49}$$

solving the Dirichlet problem for the disk.

We repeat the geometric proof of (49) from our booklet [1981] since it seems little known and is considerably shorter than the customary solution in textbooks of the Dirichlet problem for the disk. In (49) it suffices to consider the case $b = 1$. Because of (47),

$$\begin{aligned} \widehat{f}(\tanh\, t) &= \widehat{f}(a_t \cdot 0) = \frac{1}{2\pi} \int_0^{2\pi} f(a_t \cdot e^{i\theta})\, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f\left(\frac{e^{i\theta} + \tanh\, t}{\tanh\, t\, e^{i\theta} + 1} \right) d\theta\,. \end{aligned}$$

Letting $t \to +\infty$, (49) follows by the dominated convergence theorem.

The range question A for $f \to \widehat{f}$ is also answered by classical results for the Poisson integral; for example, the classical characterization of the Poisson integrals of bounded functions now takes the form

$$L^\infty(B)\widehat{\ } = \{\varphi \in L^\infty(\Xi) \,:\, L\varphi = 0\}\,. \tag{50}$$

The range characterization (50) is of course quite analogous to the range characterization for the X-ray transform described in Theorem 6.9, Chapter I. Both are realizations of the general expectations at the end of §2 that when $\dim X < \dim \Xi$ the range of the transform $f \to \widehat{f}$ should be given as the kernel of some differential operators. The analogy between (50) and Theorem 6.9 is even closer if we recall Gonzalez' theorem [1990b] that if we view the X-ray transform as a Radon transform between two homogeneous spaces of $\mathbf{M}(3)$ (see next example) then the range (83) in Theorem 6.9, Ch. I, can be described as the null space of a differential operator which is invariant under $\mathbf{M}(3)$. Furthermore, the dual transform $\varphi \to \check{\varphi}$ maps $\mathcal{E}(\Xi)$ on $\mathcal{E}(X)$. (See Corollary 4.7 below.)

Furthermore, John's mean value theorem for the X-ray transform (Corollary 6.12, Chapter I) now becomes the exact analog of Gauss' mean value theorem for harmonic functions.

What is the dual transform $\varphi \to \check{\varphi}$ for the pair (46)? The invariant measure on $MAN/M = AN$ is the functional

$$\varphi \to \int_{AN} \varphi(an \cdot o)\, da\, dn\,. \tag{51}$$

The right hand side is just $\check{\varphi}(b_0)$ where $b_0 = eMAN$. If $g = a'n'$ the measure (51) is seen to be invariant under g. Thus it is a constant multiple of the surface element $dz = (1 - x^2 - y^2)^{-2}\, dx\, dy$ defined by (24). Since the maps $t \to a_t \cdot o$ and $x \to n_x \cdot o$ were seen to be isometries, this constant factor is 1. Thus the measure (51) is invariant under each $g \in G$. Writing $\varphi_g(z) = \varphi(g \cdot z)$ we know $(\varphi_g)^\vee = \check{\varphi}_g$ so

$$\check{\varphi}(g \cdot b_0) = \int_{AN} \varphi_g(an)\, da\, dn = \check{\varphi}(b_0)\,.$$

Thus the dual transform $\varphi \to \check{\varphi}$ assigns to each $\varphi \in \mathcal{D}(\Xi)$ its integral over the disk.

Table II.1 summarizes the various results mentioned above about the Poisson integral and the X-ray transform. The inversion formulas and the ranges show subtle analogies as well as strong differences. The last item in the table comes from Corollary 4.7 below for the case $n = 3$, $d = 1$.

E. The d-plane Transform.

We now review briefly the d-plane transform from a group theoretic standpoint. As in (1) we write

$$X = \mathbf{R}^n = \mathbf{M}(n)/\mathbf{O}(n)\,, \quad \Xi = \mathbf{G}(d,n) = \mathbf{M}(n)/(\mathbf{M}(d) \times \mathbf{O}(n-d))\,, \tag{52}$$

where $\mathbf{M}(d) \times \mathbf{O}(n-d)$ is the subgroup of $\mathbf{M}(n)$ preserving a certain d-plane ξ_0 through the origin. Since the homogeneous spaces

$$\mathbf{O}(n)/\mathbf{O}(n) \cap (\mathbf{M}(d) \times \mathbf{O}(n-d)) = \mathbf{O}(n)/(\mathbf{O}(d) \times \mathbf{O}(n-d))$$

and

$$(\mathbf{M}(d) \times \mathbf{O}(n-d))/\mathbf{O}(n) \cap (\mathbf{M}(d) \times \mathbf{O}(n-d)) = \mathbf{M}(d)/\mathbf{O}(d)$$

have unique invariant measures the group-theoretic transforms (9) reduce to the transforms (52), (53) in Chapter I. The range of the d-plane transform is described by Theorem 6.3 and the equivalent Theorem 6.5 in Chapter I. It was shown by Richter [1986a] that the differential operators in

	Poisson Integral	*X-ray Transform*
Coset spaces	$X = \mathbf{SU}(1,1)/MAN$ $\Xi = \mathbf{SU}(1,1)/K$	$X = \mathbf{M}(3)/\mathbf{O}(3)$ $\Xi = \mathbf{M}(3)/(\mathbf{M}(1) \times \mathbf{O}(2))$
$f \to \widehat{f}$	$\widehat{f}(z) = \int_B f(b) \frac{1-\|z\|^2}{\|z-b\|^2} db$	$\widehat{f}(\ell) = \int_\ell f(p)\, dm(p)$
$\varphi \to \check{\varphi}$	$\check{\varphi}(x) = \int_\Xi \varphi(\xi)\, d\xi$	$\check{\varphi}(x)$ = average of φ over set of ℓ through x
Inversion	$f(b) = \lim_{z \to b} \widehat{f}(z)$	$f = \frac{1}{\pi}(-L)^{1/2}((\widehat{f})^\vee)$
Range of $f \to \widehat{f}$	$L^\infty(X)\widehat{} =$ $\{\varphi \in L^\infty(\Xi) : L\varphi = 0\}$	$\mathcal{D}(X)\widehat{} =$ $\{\varphi \in \mathcal{D}(\Xi) : \Lambda(\|\xi - \eta\|^{-1}\varphi) = 0\}$
Range characterization	Gauss' mean value theorem	Mean value property for hyperboloids of revolution
Range of $\varphi \to \check{\varphi}$	$\mathcal{E}(\Xi)^\vee = \mathbf{C}$	$\mathcal{E}(\Xi)^\vee = \mathcal{E}(X)$

TABLE II.1. Analogies between the Poisson Integral and the X-ray Transform.

Theorem 6.5 could be replaced by $\mathbf{M}(n)$-induced second order differential operators and then Gonzalez [1990b] showed that the whole system could be replaced by a single fourth order $\mathbf{M}(n)$-invariant differential operator on Ξ.

Writing (52) for simplicity in the form

$$X = G/K\,, \quad \Xi = G/H \tag{53}$$

we shall now discuss the range question for the dual transform $\varphi \to \check{\varphi}$ by invoking the d-plane transform on $\mathcal{E}'(X)$.

Theorem 4.6. *Let $\mathcal{N}$ denote the kernel of the dual transform on $\mathcal{E}(\Xi)$. Then the range of $S \to \widehat{S}$ on $\mathcal{E}'(X)$ is given by*

$$\mathcal{E}'(X)\widehat{} = \{\Sigma \in \mathcal{E}'(\Xi) \,:\, \Sigma(\mathcal{N}) = 0\}\,.$$

The inclusion $\subset$ is clear from the definitions (14),(15) and Proposition 2.5. The converse is proved by the author in [1983a] and [1994b], Ch. I, §2 for $d = n-1$; the proof is also valid for general d.

For Fréchet spaces E and F one has the following classical result. A continuous mapping $\alpha : E \to F$ is surjective if the transpose ${}^t\alpha : F' \to E'$ is injective and has a closed image. Taking $E = \mathcal{E}(\Xi)$, $F = \mathcal{E}(X)$, α as the dual transform $\varphi \to \check{\varphi}$, the transpose ${}^t\alpha$ is the Radon transform on $\mathcal{E}'(X)$. By Theorem 4.6, ${}^t\alpha$ does have a closed image and by Theorem 5.5, Ch. I (extended to any d) ${}^t\alpha$ is injective. Thus we have the following result (Hertle [1984] for $d = n-1$) expressing the surjectivity of α.

Corollary 4.7. *Every $f \in \mathcal{E}(\mathbf{R}^n)$ is the dual transform $f = \check{\varphi}$ of a smooth d-plane function φ.*

F. Grassmann Manifolds.

We consider now the (affine) Grassmann manifolds $\mathbf{G}(p,n)$ and $\mathbf{G}(q,n)$ where $p + q = n - 1$. If $p = 0$ we have the original case of points and hyperplanes. Both are homogeneous spaces of the group $\mathbf{M}(n)$ and we represent them accordingly as coset spaces

$$X = \mathbf{M}(n)/H_p\,, \quad \Xi = \mathbf{M}(n)/H_q\,. \tag{54}$$

Here we take H_p as the isotropy group of a p-plane x_0 through the origin $0 \in \mathbf{R}^n$, H_q as the isotropy group of a q-plane ξ_0 through 0, *perpendicular to* x_0. Then

$$H_p \sim \mathbf{M}(p) \times \mathbf{O}(n-p)\,, \quad H_q = \mathbf{M}(q) \times \mathbf{O}(n-q)\,.$$

Also

$$H_q \cdot x_0 = \{x \in X \,:\, x \perp \xi_0, x \cap \xi_0 \neq \emptyset\}\,,$$

the set of p-planes intersecting ξ_0 orthogonally. It is then easy to see that

$$x \text{ is incident to } \xi \Leftrightarrow x \perp \xi\,, \quad x \cap \xi \neq \emptyset\,.$$

Consider as in Chapter I, §6 the mapping

$$\pi \,:\, \mathbf{G}(p,n) \to \mathbf{G}_{p,n}$$

given by parallel translating a p-plane to one such through the origin. If $\sigma \in \mathbf{G}_{p,n}$, the fiber $F = \pi^{-1}(\sigma)$ is naturally identified with the Euclidean space $\sigma^\perp$. Consider the linear operator $\square_p$ on $\mathcal{E}(\mathbf{G}(p,n))$ given by

$$(\square_p f)|F = L_F(f|F)\,. \tag{55}$$

Here L_F is the Laplacian on F and bar denotes restriction. Then one can prove that $\square_p$ is a differential operator on $\mathbf{G}(p,n)$ which is invariant under the action of $\mathbf{M}(n)$. Let $f \to \widehat{f}$, $\varphi \to \check{\varphi}$ be the Radon transform and its dual corresponding to the pair (54). Then $\widehat{f}(\xi)$ represents the integral of f over all p-planes x intersecting ξ under a right angle. For n odd this is inverted as follows (Gonzalez [1984, 1987]).

Theorem 4.8. *Let $p, q \in \mathbb{Z}^+$ such that $p+q+1 = n$ is odd. Then the transform $f \to \widehat{f}$ from $\mathbf{G}(p,n)$ to $\mathbf{G}(q,n)$ is inverted by the formula*

$$C_{p,q} f = ((\Box_q)^{(n-1)/2} \widehat{f})^{\vee}\,, \quad f \in \mathcal{D}(\mathbf{G}(p,n))$$

where $C_{p,q}$ is a constant.

If $p = 0$ this reduces to Theorem 3.6, Ch. I.

G. Half-lines in a Half-plane.

In this example X denotes the half-plane $\{(a,b) \in \mathbf{R}^2 \,:\, a > 0\}$ viewed as a subset of the plane $\{(a,b,1) \in \mathbf{R}^3\}$. The group G of matrices

$$(\alpha, \beta, \gamma) = \begin{pmatrix} \alpha & 0 & 0 \\ \beta & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix} \in \mathbf{GL}(3, \mathbf{R})\,, \quad \alpha > 0$$

acts transitively on X with the action

$$(\alpha, \beta, \gamma) \odot (a, b) = (\alpha a, \beta a + b + \gamma)\,.$$

This is the restriction of the action of $\mathbf{GL}(3, \mathbf{R})$ on $\mathbf{R}^3$. The isotropy group of the point $x_0 = (1, 0)$ is the group

$$K = \{(1, \beta - \beta) \,:\, \beta \in \mathbf{R}\}\,.$$

Let Ξ denote the set of half-lines in X which end on the boundary $\partial X = 0 \times \mathbf{R}$. These lines are given by

$$\xi_{v,w} = \{(t, v + tw) \,:\, t > 0\}$$

for arbitrary $v, w \in \mathbf{R}$. Thus Ξ can be identified with $\mathbf{R} \times \mathbf{R}$. The action of G on X induces a transitive action of G on Ξ which is given by

$$(\alpha, \beta, \gamma) \Diamond (v, w) = (v + \gamma, \frac{w + \beta}{\alpha})\,.$$

(Here we have for simplicity written (v, w) instead of $\xi_{v,w}$.) The isotropy group of the point $\xi_0 = (0, 0)$ (the x-axis) is

$$H = \{(\alpha, 0, 0) \,:\, \alpha > 0\} = \mathbf{R}^{\times}_{+}\,,$$

the multiplicative group of the positive real numbers. Thus we have the identifications

$$X = G/K\,, \quad \Xi = G/H\,. \tag{56}$$

The group $K \cap H$ is now trivial so the Radon transform and its dual for the double fibration in (56) are defined by

$$\widehat{f}(gH) \quad = \quad \int_H f(ghK)\, dh\,, \tag{57}$$

$$\check{\varphi}(gK) \quad = \quad \chi(g) \int_K \varphi(gkH)\, dk\,, \tag{58}$$

where χ is the homomorphism $(\alpha, \beta, \gamma) \to \alpha^{-1}$ of G onto $\mathbf{R}^\times_+$. The reason for the presence of χ is that we wish Proposition 2.2 to remain valid even if G is not unimodular. In (57) and (58) we have the Haar measures

$$dk_{(1,\beta-\beta)} = d\beta\,, \quad dh_{(\alpha,0,0)} = d\alpha/\alpha\,. \tag{59}$$

Also, if $g = (\alpha, \beta, \gamma)$, $h = (a, 0, 0)$, $k = (1, b, -b)$ then

$$\begin{aligned} gH = (\gamma, \beta/\alpha)\,, &\qquad ghK = (\alpha a, \beta a + \gamma) \\ gK = (\alpha, \beta + \gamma)\,, &\qquad gkH = (-b + \gamma, \tfrac{b+\beta}{\alpha}) \end{aligned}$$

so (57)–(58) become

$$\begin{aligned} \widehat{f}(\gamma, \beta/\alpha) \quad &= \quad \int_{\mathbf{R}^+} f(\alpha a, \beta a + \gamma) \frac{da}{a} \\ \check{\varphi}(\alpha, \beta + \gamma) \quad &= \quad \alpha^{-1} \int_{\mathbf{R}} \varphi(-b + \gamma, \tfrac{b+\beta}{\alpha})\, db\,. \end{aligned}$$

Changing variables these can be written

$$\widehat{f}(v, w) \quad = \quad \int_{\mathbf{R}^+} f(a, v + aw) \frac{da}{a}\,, \tag{60}$$

$$\check{\varphi}(a, b) \quad = \quad \int_{\mathbf{R}} \varphi(b - as, s)\, ds \qquad a > 0\,. \tag{61}$$

Note that in (60) the integration takes place over all points on the line $\xi_{v,w}$ and in (61) the integration takes place over the set of lines $\xi_{b-as,s}$ all of which pass through the point (a, b). This is an *a posteriori* verification of the fact that our incidence for the pair (56) amounts to $x \in \xi$.

From (60)–(61) we see that $f \to \widehat{f}, \varphi \to \check{\varphi}$ are adjoint relative to the measures $\frac{da}{a}\, db$ and $dv\, dw$:

$$\int_{\mathbf{R}} \int_{\mathbf{R}^\times_+} f(a, b) \check{\varphi}(a, b) \frac{da}{a}\, db = \int_{\mathbf{R}} \int_{\mathbf{R}} \widehat{f}(v, w) \varphi(v, w)\, dv\, dw\,. \tag{62}$$

The proof is a routine computation.

We recall (Chapter V) that $(-L)^{1/2}$ is defined on the space of rapidly decreasing functions on $\mathbf{R}$ by

$$((-L)^{1/2}\psi)^\sim\ (\tau) = |\tau| \widetilde{\psi}(\tau) \tag{63}$$

and we define Λ on $\mathcal{S}(\Xi)(=\mathcal{S}(\mathbf{R}^2))$ by having $(-L)^{1/2}$ only act on the second variable:

$$(\Lambda\varphi)(v,w) = ((-L)^{1/2}\varphi(v,\cdot))(w)\,. \tag{64}$$

Viewing $(-L)^{1/2}$ as the Riesz potential I^{-1} on $\mathbf{R}$ (Chapter V, §5) it is easy to see that if $\varphi_c(v,w)=\varphi(v,\frac{w}{c})$ then

$$\Lambda\varphi_c = |c|^{-1}(\Lambda\varphi)_c\,. \tag{65}$$

The Radon transform (57) is now inverted by the following theorem.

Theorem 4.9. *Let $f\in\mathcal{D}(X)$. Then*

$$f = \frac{1}{2\pi}(\Lambda\widehat{f})^\vee\,.$$

Proof. In order to use the Fourier transform $F\to\widetilde{F}$ on $\mathbf{R}^2$ and on $\mathbf{R}$ we need functions defined on all of $\mathbf{R}^2$. Thus we define

$$f^*(a,b) = \begin{cases} \frac{1}{a}f(\frac{1}{a}\,,\,\frac{-b}{a}) & a>0\,,\\ 0 & a\le 0\,.\end{cases}$$

Then

$$\begin{aligned} f(a,b) &= \frac{1}{a}f^*\left(\frac{1}{a}\,,\,-\frac{b}{a}\right)\\ &= a^{-1}(2\pi)^{-2}\iint \widetilde{f^*}(\xi,\eta)e^{i(\frac{\xi}{a}-\frac{b\eta}{a})}\,d\xi\,d\eta\\ &= (2\pi)^{-2}\iint \widetilde{f^*}(a\xi+b\eta,\eta)e^{i\xi}\,d\xi\,d\eta\\ &= a(2\pi)^{-2}\iint |\xi|\widetilde{f^*}((a+ab\eta)\xi,a\eta\xi)e^{i\xi}\,d\xi\,d\eta\,.\end{aligned}$$

Next we express the Fourier transform in terms of the Radon transform. We have

$$\begin{aligned} \widetilde{f^*}((a+ab\eta)\xi,a\eta\xi) &= \iint f^*(x,y)e^{-ix(a+ab\eta)\xi}e^{-iya\eta\xi}\,dx\,dy\\ &= \int_{\mathbf{R}}\int_{x\ge 0}\frac{1}{x}f\left(\frac{1}{x}\,,\,-\frac{y}{x}\right)e^{-ix(a+ab\eta)\xi}e^{-iya\eta\xi}\,dx\,dy\\ &= \int_{\mathbf{R}}\int_{x\ge 0} f\left(\frac{1}{x}\,,\,b+\frac{1}{\eta}+\frac{z}{x}\right)e^{iza\eta\xi}\,\frac{dx}{x}\,dz\,.\end{aligned}$$

This last expression is

$$\int_{\mathbf{R}}\widehat{f}(b+\eta^{-1},z)e^{iza\eta\xi}\,dz = (\widehat{f})^\sim(b+\eta^{-1},-a\eta\xi)\,,$$

where $\sim$ denotes the 1-dimensional Fourier transform (in the second variable). Thus

$$f(a,b) = a(2\pi)^{-2} \iint |\xi|(\widehat{f})^{\sim}(b+\eta^{-1}, -a\eta\xi)e^{i\xi}\, d\xi\, d\eta\,.$$

However $\widetilde{F}(c\xi) = |c|^{-1}(F_c)^{\sim}(\xi)$ so by (65)

$$\begin{aligned} f(a,b) &= a(2\pi)^{-2} \iint |\xi|((\widehat{f})_{a\eta})^{\sim}(b+\eta^{-1}, -\xi)e^{i\xi}\, d\xi |a\eta|^{-1}\, d\eta \\ &= (2\pi)^{-1} \int \Lambda((\widehat{f})_{a\eta})(b+\eta^{-1}, -1)|\eta|^{-1}\, d\eta \\ &= (2\pi)^{-1} \int |a\eta|^{-1}(\Lambda\widehat{f})_{a\eta}(b+\eta^{-1}, -1)|\eta|^{-1}\, d\eta \\ &= a^{-1}(2\pi)^{-1} \int (\Lambda\widehat{f})(b+\eta^{-1}, -(a\eta)^{-1})\eta^{-2}\, d\eta \end{aligned}$$

so

$$\begin{aligned} f(a,b) &= (2\pi)^{-1} \int_{\mathbf{R}} (\Lambda\widehat{f})(b-av, v)\, dv \\ &= (2\pi)^{-1}(\Lambda\widehat{f})^{\vee}(a,b)\,. \end{aligned}$$

proving the theorem.

Remark 4.10. It is of interest to compare this theorem with Theorem 3.6, Ch. I. If $f \in \mathcal{D}(X)$ is extended to all of $\mathbf{R}^2$ by defining it 0 in the left half plane then Theorem 3.6 does give a formula expressing f in terms of its integrals over half-lines in a strikingly similar fashion. Note however that while the operators $f \to \widehat{f}, \varphi \to \check{\varphi}$ are in the two cases defined by integration over the same sets (points on a half-line, half-lines through a point) the measures in the two cases are different. Thus it is remarkable that the inversion formulas look exactly the same.

H. Theta Series and Cusp Forms.

Let G denote the group $\mathbf{SL}(2,\mathbf{R})$ of 2×2 matrices of determinant one and Γ the *modular group* $\mathbf{SL}(2,\mathbb{Z})$. Let N denote the unipotent group $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ where $n \in \mathbf{R}$ and consider the homogeneous spaces

(66) $$X = G/N\,, \qquad \Xi = G/\Gamma\,.$$

Under the usual action of G on $\mathbf{R}^2$, N is the isotropy subgroup of $(1,0)$ so X can be identified with $\mathbf{R}^2 - (0)$, whereas Ξ is of course 3-dimensional.

In number theory one is interested in decomposing the space $L^2(G/\Gamma)$ into G-invariant irreducible subspaces. We now give a rough description of this by means of the transforms $f \to \widehat{f}$ and $\varphi \to \check{\varphi}$.

As customary we put $\Gamma_\infty = \Gamma \cap N$; our transforms (9) then take the form

$$\widehat{f}(g\Gamma) = \sum_{\Gamma/\Gamma_\infty} f(g\gamma N)\,, \quad \check{\varphi}(gN) = \int_{N/\Gamma_\infty} \varphi(gn\Gamma)\,dn_{\Gamma_\infty}\,.$$

Since N/Γ_∞ is the circle group, $\check{\varphi}(gN)$ is just the constant term in the Fourier expansion of the function $n\Gamma_\infty \to \varphi(gn\Gamma)$. The null space $L^2_d(G/\Gamma)$ in $L^2(G/\Gamma)$ of the operator $\varphi \to \check{\varphi}$ is called the space of *cusp forms* and the series for $\widehat{f}$ is called *theta series*. According to Prop. 2.2 they constitute the orthogonal complement of the image $C_c(X)\widehat{\ }$.

We have now the G-invariant decomposition

$$L^2(G/\Gamma) = L^2_c(G/\Gamma) \oplus L^2_d(G/\Gamma)\,, \tag{67}$$

where ($-$ denoting closure)

$$L^2_c(G/\Gamma) = (C_c(X)\widehat{\ })^- \tag{68}$$

and as mentioned above,

$$L^2_d(G/\Gamma) = (C_c(X)\widehat{\ })^\perp\,. \tag{69}$$

It is known (cf. Selberg [1962], Godement [1966]) that the representation of G on $L^2_c(G/\Gamma)$ is the *continuous* direct sum of the irreducible representations of G from the principal series whereas the representation of G on $L^2_d(G/\Gamma)$ is the *discrete* direct sum of irreducible representations each occurring with finite multiplicity.

In conclusion we note that the determination of a function in $\mathbf{R}^n$ in terms of its integrals over unit spheres (John [1955]) can be regarded as a solution to the first half of Problem B in §2 for the double fibration (4).

Bibliographical Notes

The Radon transform and its dual for a double fibration

$$\begin{tikzcd} & Z = G/(K\cap H) \arrow[dl] \arrow[dr] & \\ X = G/K & & \Xi = G/H \end{tikzcd} \tag{70}$$

was introduced in the author's paper [1966a]. The results of §1–§2 are from there and from [1994b]. The definition uses the concept of *incidence* for $X = G/K$ and $\Xi = G/H$ which goes back to Chern [1942]. Even when the elements of Ξ can be viewed as subsets of X and vice versa (Lemma 1.3) it can be essential for the inversion of $f \to \widehat{f}$ not to restrict the incidence to the naive one $x \in \xi$. (See for example the classical case $X = \mathbf{S}^2, \Xi =$ set of great circles where in Theorem 4.1 a more general incidence is essential.) The double fibration in (1) was generalized in Gelfand, Graev and Shapiro [1969], by relaxing the homogeneity assumption.

For the case of geodesics in constant curvature spaces (Examples A, B in §4) see notes to Ch. III.

The proof of Theorem 4.3 (a special case of the author's inversion formula in [1964], [1965b]) makes use of a method by Godement [1957] in another context. Another version of the inversion (38) for $\mathbf{H}^2$ (and $\mathbf{H}^n$) is given in Gelfand-Graev-Vilenkin [1966]. A further inversion of the horocycle transform in $\mathbf{H}^2$ (and $\mathbf{H}^n$), somewhat analogous to (30) for the X-ray transform, is given by Berenstein and Tarabusi [1994].

The analogy suggested above between the X-ray transform and the horocycle transform in $\mathbf{H}^2$ goes even further in $\mathbf{H}^3$. There the 2-dimensional transform for totally geodesic submanifolds has *the same* inversion formula as the horocycle transform (Helgason [1994b], p. 209).

For a treatment of the horocycle transform on a Riemannian symmetric space see the author's monograph [1994b], Chapter II, where Problems A, B, C in §2 are discussed in detail along with some applications to differential equations and group representations. See also Quinto [1993a] and Gonzalez and Quinto [1994] for new proofs of the support theorem.

Example G is from Hilgert's paper [1994], where a related Fourier transform theory is also established. It has a formal analogy to the Fourier analysis on $\mathbf{H}^2$ developed by the author in [1965b] and [1972].

CHAPTER III

THE RADON TRANSFORM ON TWO-POINT HOMOGENEOUS SPACES

Let X be a complete Riemannian manifold, x a point in X and X_x the tangent space to X at x. Let Exp_x denote the mapping of X_x into X given by $\mathrm{Exp}_x(u) = \gamma_u(1)$ where $t \to \gamma_u(t)$ is the geodesic in X through x with tangent vector u at $x = \gamma_u(0)$.

A connected submanifold S of a Riemannian manifold X is said to be *totally geodesic* if each geodesic in X which is tangential to S at a point lies entirely in S.

The totally geodesic submanifolds of $\mathbf{R}^n$ are the planes in $\mathbf{R}^n$. Therefore, in generalizing the Radon transform to Riemannian manifolds, it is natural to consider integration over totally geodesic submanifolds. In order to have enough totally geodesic submanifolds at our disposal we consider in this section Riemannian manifolds X which are *two-point homogeneous* in the sense that for any two-point pairs $p, q \in X$ $p', q' \in X$, satisfying $d(p, q) = d(p', q')$, (where d = distance), there exists an isometry g of X such that $g \cdot p = p',\ g \cdot q = q'$. We start with the subclass of Riemannian manifolds with the richest supply of totally geodesic submanifolds, namely the spaces of constant curvature.

While §1, which constitutes most of this chapter, is elementary, §2–§5 will involve a bit of Lie group theory.

§1 Spaces of Constant Curvature. Inversion and Support Theorems

Let X be a simply connected complete Riemannian manifold of dimension $n \geq 2$ and constant sectional curvature.

Lemma 1.1. *Let $x \in X$, V a subspace of the tangent space X_x. Then* $\mathrm{Exp}_x(V)$ *is a totally geodesic submanifold of X.*

Proof. For this we choose a specific embedding of X into $\mathbf{R}^{n+1}$, and assume for simplicity the curvature is $\epsilon(= \pm 1)$. Consider the quadratic form

$$B_\epsilon(x) = x_1^2 + \cdots + x_n^2 + \epsilon x_{n+1}^2$$

and the quadric Q_ϵ given by $B_\epsilon(x) = \epsilon$. The orthogonal group $\mathbf{O}(B_\epsilon)$ acts transitively on Q_ϵ. The form B_ϵ is positive definite on the tangent space $\mathbf{R}^n \times (0)$ to Q_ϵ at $x^0 = (0, \ldots, 0, 1)$; by the transitivity B_ϵ induces a positive definite quadratic form at each point of Q_ϵ, turning Q_ϵ into a

Riemannian manifold, on which $\mathbf{O}(B_\epsilon)$ acts as a transitive group of isometries. The isotropy subgroup at the point x^0 is isomorphic to $\mathbf{O}(n)$ and its acts transitively on the set of 2-dimensional subspaces of the tangent space $(Q_\epsilon)_{x^0}$. It follows that all sectional curvatures at x^0 are the same, namely ϵ, so by homogeneity, Q_ϵ has constant curvature ϵ. In order to work with connected manifolds, we replace Q_{-1} by its intersection Q_{-1}^+ with the half-space $x_{n+1} > 0$. Then Q_{+1} and Q_{-1}^+ are simply connected complete Riemannian manifolds of constant curvature. Since such manifolds are uniquely determined by the dimension and the curvature it follows that we can identify X with Q_{+1} or Q_{-1}^+.

The geodesic in X through x^0 with tangent vector $(1, 0, \ldots, 0)$ will be left point-wise fixed by the isometry

$$(x_1, x_2, \ldots, x_n, x_{n+1}) \to (x_1, -x_2, \ldots, -x_n, x_{n+1}).$$

This geodesic is therefore the intersection of X with the two-plane $x_2 = \cdots = x_n = 0$ in $\mathbf{R}^{n+1}$. By the transitivity of $\mathbf{O}(n)$ all geodesics in X through x^0 are intersections of X with two-planes through 0. By the transitivity of $\mathbf{O}(Q_\epsilon)$ it then follows that the geodesics in X are precisely the nonempty intersections of X with two-planes through the origin.

Now if $V \subset X_{x^0}$ is a subspace, $\mathrm{Exp}_{x^0}(V)$ is by the above the intersection of X with the subspace of $\mathbf{R}^{n+1}$ spanned by V and x^0. Thus $\mathrm{Exp}_{x^0}(V)$ is a quadric in $V + \mathbf{R}x^0$ and its Riemannian structure induced by X is the same as induced by the restriction $B_\epsilon | (V + \mathbf{R}x^0)$. Thus, by the above, the geodesics in $\mathrm{Exp}_{x^0}(V)$ are obtained by intersecting it with two-planes in $V + \mathbf{R}x^0$ through 0. Consequently, the geodesics in $\mathrm{Exp}_{x^0}(V)$ are geodesics in X so $\mathrm{Exp}_{x^0}(V)$ is a totally geodesic submanifold of X. By the homogeneity of X this holds with x^0 replaced by an arbitrary point $x \in X$. The lemma is proved.

In accordance with the viewpoint of Ch. II we consider X as a homogeneous space of the identity component G of the group $\mathbf{O}(Q_\epsilon)$. Let K denote the isotropy subgroup of G at the point $x^0 = (0, \ldots, 0, 1)$. Then K can be identified with the special orthogonal group $\mathbf{SO}(n)$. Let k be a fixed integer, $1 \leq k \leq n-1$; let $\xi_0 \subset X$ be a fixed totally geodesic submanifold of dimension k passing through x^0 and let H be the subgroup of G leaving ξ_0 invariant. We have then

$$X = G/K\,, \quad \Xi = G/H\,, \tag{1}$$

Ξ denoting the set of totally geodesic k-dimensional submanifolds of X. Since $x^0 \in \xi_0$ it is clear that the abstract incidence notion boils down to the naive one, in other words: The cosets $x = gK$ $\xi = \gamma H$ have a point in common if and only if $x \in \xi$. In fact

$$x \in \xi \Leftrightarrow x^0 \in g^{-1}\gamma \cdot \xi_0 \Leftrightarrow g^{-1}\gamma \cdot \xi_0 = k \cdot \xi_0 \quad \text{for some } k \in K\,.$$