Using the indicated value of k, we get

$$\iint_{\mathcal{O}_\sigma}(p(z))^2\,dx\,dy \;=\; \iint_{\Sigma}(p(z))^2\,dx\,dy \;=\; \int_{R_0}^{R}\int_{\Sigma(r)}\left(\frac{k}{r\theta(r)}\right)^2 r\,d\vartheta\,dr$$

$$= \int_{R_0}^{R}\left(\frac{k}{r\theta(r)}\right)^2 r\theta(r)\,dr \;=\; k^2\int_{R_0}^{R}\frac{dr}{r\theta(r)} \;=\; k,$$

whence

$$\Lambda(\mathcal{O}_\sigma, S) \;\leqslant\; k \;=\; 1\Big/\int_0^R \frac{dr}{r\theta(r)}$$

(since $\theta(r) = \infty$ for $0 < r < R_0$).

Substitution of the last relation into the inequality furnished by the preceding theorem now yields the desired result when $\Sigma(R)$ is made up of finitely many arcs.

The general case (with $\Sigma(R) \subseteq \mathcal{O}\cap\{|z-z_0| = R\}$ consisting of *countably many* open arcs) is easily reduced to the one just handled, which certainly obtains whenever $\partial\mathcal{O}$ is an *analytic* Jordan curve. One simply takes limits, working with an exhaustion of $\mathcal{O}$ by domains having analytic Jordan curve boundaries. To get the latter, map $\mathcal{O}$ conformally onto the unit disk and then take the preimages of smaller concentric disks. The details of this procedure are left to the reader.

Remark. Tsuji himself did not quite arrive at the inequality found here. In its place he got

$$\omega_{\mathcal{O}_R}(\Sigma(R),\, z_0) \;\leqslant\; \frac{3}{\sqrt{(1-\kappa)}}\,e^{-\pi\int_0^{\kappa R}\frac{dr}{r\theta(r)}},$$

with a parameter κ of arbitrary value $<$ 1. The estimate provided by our result is better, because in it the integration goes all the way out to R.

Instead of using extremal length to derive his formula, Tsuji worked with a differential inequality due to Carleman. That procedure is known as *Carleman's method*, and its application is not limited to simply connected domains. Therefore, since our definition of the function $\theta(r)$ makes sense when $\mathcal{O}$ is *multiply connected* (i.e., has 'islands' in it), it is *very likely true* that Tsuji's original estimate (with the parameter κ) *holds* for that more general situation. Regarding this possibility and most of the other material in the present § and in §E.1, the reader should consult the *Arkiv* paper by K. Haliste; I have not checked thoroughly to make sure that all the details of Tsuji's argument go through for multiply connected domains.

It would be very good if the result proved above (*without* the parameter

κ) could be shown to hold when $\mathcal{O}$ is multiply connected.* In at least *one special situation*, this is known to be so. That is when $\Sigma(R)$ is the *whole circle* $|z - z_0| = R$ and the rest of $\partial\mathcal{O}_R$ consists of *radial segments inside that circle*; the desired inequality then follows by a celebrated theorem of Beurling which may, for instance, be found in Nevanlinna's book. For Carleman's method, the reader should first consult the little book by Heins, going afterwards to those of Nevanlinna and of Tsuji. Haliste's paper, already referred to, is recommended to anyone who wishes to become more familiar with the whole circle of ideas just discussed.

For certain geometric configurations, *Tsuji's inequality can be deduced from the one of Ahlfors and Carleman* proved in §E.1. Without attempting to describe the most general circumstances whereunder this is possible (which would oblige us to enter into all kinds of fussy geometric considerations), let us restrict our attention to the simple situation where $\mathcal{O}$ consists of *a disk together with n* (unbranched) *arms extending outward therefrom:*

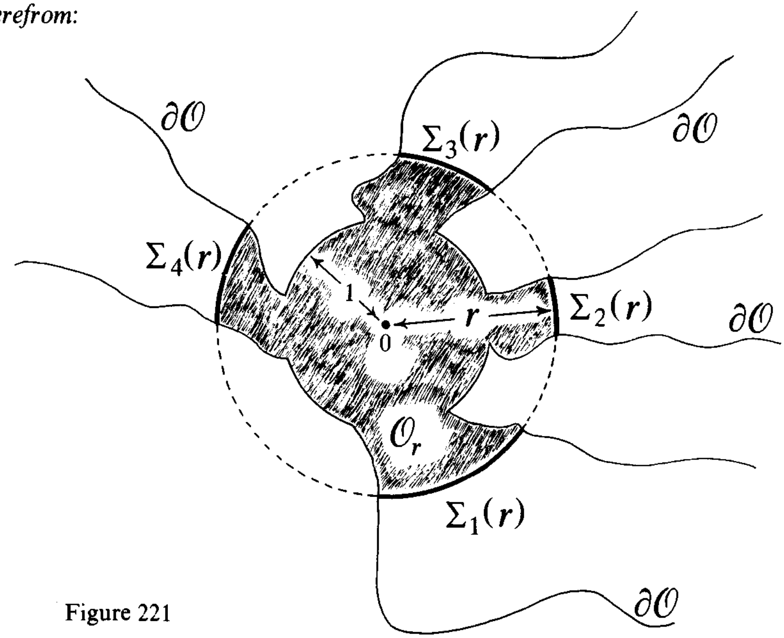

Figure 221

* Beurling's notes on extremal distance and harmonic measure have appeared in volume I of his *Collected Works*, published almost 5 years after the above lines were written. A version for *multiply connected* domains of the theorem given above on p. 149 is found in those notes on p. 372 (Theorem 3), and from it the analogue of Tsuji's inequality (without the κ) for branching channels with islands in them is readily derived (see pp 374–376 of the notes). Beurling's Theorem 3, and especially its elegant proof, were closely guarded secrets until his collected works came out.

Consider the case where $\mathcal{O}$'s central disk is just Δ. In order to keep things really simple, let us also suppose that any circle of radius $r > 1$ about the origin *intersects each of the arms along a single arc*, although this assumption is not really necessary (see the discussion of the arcs $S(R)$ and $\sigma(r)$ near the end of §E.1). *Say that* $\{|z| = r\}$ *intersects the* jth *arm of* $\mathcal{O}$ *along the arc* $\Sigma_j(r)$. In terms of the notation used with the previous theorem, we then have

$$\Sigma(r) = \Sigma_1(r) \cup \Sigma_2(r) \cup \cdots \cup \Sigma_n(r), \quad r > 1,$$

and if we put

$$\theta_j(r) = |\Sigma_j(r)|/r \quad \text{for } r > 1,$$

we have

$$\theta(r) = \theta_1(r) + \theta_2(r) + \cdots + \theta_n(r).$$

When $0 < r < 1$, *we put each of the* $\theta_j(r)$ *equal to* ∞.

Under the present circumstances, we clearly have

$$\omega_{\mathcal{O}_R}(\Sigma(R), 0) = \sum_{j=1}^{n} \omega_{\mathcal{O}_R}(\Sigma_j(R), 0),$$

and, for each j, *by the Ahlfors–Carleman inequality*,

$$\omega_{\mathcal{O}_R}(\Sigma_j(R), 0) \leqslant C\exp\left\{-\pi\int_0^R \frac{dr}{r\theta_j(r)}\right\}$$

with an absolute constant C. Therefore, wishing to show that

$$\omega_{\mathcal{O}_R}(\Sigma(R), 0) \leqslant K\exp\left\{-\pi\int_0^R \frac{dr}{r\theta(r)}\right\}$$

with a numerical constant K, it is *more than sufficient* to verify that

$$\boxed{\sum_{j=1}^{n} \exp\left\{-\pi\int_0^R \frac{dr}{r\theta_j(r)}\right\} \leqslant \exp\left\{-\pi\int_0^R \frac{dr}{r\theta(r)}\right\}}$$

for $\theta(r) = \sum_{j=1}^{n} \theta_j(r)$, when the $\theta_j(r)$ are $\geqslant 0$ and

$$\pi\int_0^R \frac{dr}{r\theta(r)} \geqslant 2.$$

(When this last condition is *violated*, the desired inequality for $\omega_{\mathcal{O}_R}(\Sigma(R), 0)$ is *true anyway* with $K = e^2$.)

Problem 37

Prove that under the given condition the boxed inequality holds. (Hint. One may, in the first place, assume all the quantities $\int_0^R \mathrm{d}r/r\theta_j(r)$ to be *finite*. Otherwise, if, say, the one with $j = n$ were *infinite*, we could *drop* the term corresponding to it in the sum

$$S(\theta_1, \theta_2, \ldots, \theta_n) \quad = \quad \sum_{j=1}^{n} \exp\left\{ -\pi \int_0^R \frac{\mathrm{d}r}{r\theta_j(r)} \right\}$$

and then set out to prove an inequality like the *boxed* one with $n - 1$ terms *on the left* and a *smaller* function $\theta(r) \geqslant 0$ on the *right*. The stated assumption now being *granted*, we *consider the function* $\theta(r)$ *to be fixed*. Let, wlog, the *largest among the quantities* $(\int_0^R \mathrm{d}r/r\theta_j(r))^2 \exp\{-\pi \int_0^R \mathrm{d}r/r\theta_j(r)\}$ *be the one corresponding to* $j = 1$. Taking any $k > 1$, make the variations $\delta\theta_1(r) = \eta\theta_k(r)$, $\delta\theta_k(r) = -\eta\theta_k(r)$ and $\delta\theta_j(r) = 0$ for $j \neq 1$, k in the functions $\theta_j(r)$; here η is an infinitesimal > 0. These variations are *allowable* because the functions $\theta_j(r) + \delta\theta_j(r)$ are *still all* $\geqslant 0$ and *still* add up to $\theta(r)$.

Show that, under the variations just described,

$$\delta S(\theta_1, \theta_2, \ldots, \theta_n) \quad = \quad \pi\eta \left\{ \int_0^R \frac{\theta_k(r)\,\mathrm{d}r}{r(\theta_1(r))^2} \exp\left(-\pi \int_0^R \frac{\mathrm{d}r}{r\theta_1(r)} \right) - \int_0^R \frac{\mathrm{d}r}{r\theta_k(r)} \exp\left(-\pi \int_0^R \frac{\mathrm{d}r}{r\theta_k(r)} \right) \right\},$$

and that the quantity in $\{\ \}$ *is certainly* > 0 *unless* $\theta_k(r) = \text{const}\,\theta_1(r)$. The *maximum value* of $S(\theta_1, \ldots, \theta_n)$ for $\theta_1 + \theta_2 + \cdots + \theta_n = \theta$ *is therefore attained when*, for each j, $\theta_j(r) = \lambda_j \theta(r)$ with a constant $\lambda_j \geqslant 0$. Here, $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$. Observe finally that when $M \geqslant 2$, $\mathrm{e}^{-M/\lambda}$ is an *increasing and convex function* of λ for $0 \leqslant \lambda \leqslant 1$.)

X

Why we want to have multiplier theorems

A. Meaning of term 'multiplier theorem' in this book

Suppose we have a function $W(x) \geqslant 1$ defined on $\mathbb{R}$. We will be interested in the *question of existence* of non-zero *entire functions* $\varphi(z)$ *of arbitrarily small exponential type* for which

$$W(x)\varphi(x)$$

is *bounded*, or *belongs to some* L_p *class, on the real axis.* The purpose of the present chapter is to discuss some of the *reasons* for this interest.

Existence of the entire functions φ is of course not guaranteed for *arbitrary* weights $W(x) \geqslant 1$. For us, *a multiplier theorem* is any result describing *conditions on W from which that existence must follow.* When such conditions are *realized,* we think of W as a weight that can be 'multiplied down' by the 'multipliers' φ. In those circumstances, we also say that $W(x)$ *admits multipliers.*

Warning. The term 'multiplier' is used *here* with *meaning entirely different* from that accepted in harmonic analysis and in the study of singular integrals.

The first restriction on a function $W(x) \geqslant 1$ which is to admit multipliers concerns its *size.* If $\varphi(z)$ is a non-zero entire function of exponential type with $W(x)|\varphi(x)| \leqslant C$ on $\mathbb{R}$, we have

$$\log|\varphi(x)| \leqslant \log C - \log W(x),$$

so, since

$$\int_{-\infty}^{\infty} \frac{\log^-|\varphi(x)|}{1+x^2}\,dx$$

must be *finite* by §G.2 of Chapter III, we *have to have*

$$\boxed{\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2}\,dx \quad < \quad \infty.}$$

The same condition on W must hold if we require, for example, that $W(x)\varphi(x) \in L_p(\mathbb{R})$ with $p > 0$.

Problem 38

Prove the assertion just made. (Hint: Insertion of the factor $1/(1+x^2)$ into the integrand makes the L_p integral smaller. Use the relation between arithmetic and geometric means.)

Things would be very simple if the boxed condition on W, *necessary* for that weight to admit multipliers, were *also sufficient*. That, unfortunately, is just not true, and *some additional restrictions on W's behaviour are needed*. A really adequate description of the minimal additional requirements to be imposed on a weight in order that it admit multipliers is not yet available; one has, on the one hand, some fairly straightforward *sufficient* conditions which are *more than* necessary, and, on the other, a criterion which is *both necessary and sufficient* for a very extensive class of weights, but at the same time *quite unwieldy*.

These matters will be taken up in the next chapter, the last one of this book. What we do in the present one is mainly to *show some applications* of multiplier theorems to various questions in analysis. In the following two articles we first review an elementary but quite useful such result already established in Chapter IV and then *state* a much deeper one, whose *proof* is deferred until Chapter XI.

1. The weight is even and increasing on the positive real axis

As we saw in §D of Chapter IV (see especially the corollary at the end of that §), a construction used in the study of quasi-analyticity also yields the following

Theorem. (Paley and Wiener) *If $W(x) \geqslant 1$ is even, and increasing for $x > 0$, W admits multipliers if and only if*

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2}\,dx \quad < \quad \infty.$$

Under the specified circumstances, then, convergence of the integral is both necessary and sufficient. This multiplier theorem is of considerable value in applications in spite of its elementary character. Levinson and Mandelbrojt have used it extensively, as did Paley and Wiener, and it will render considerable service in the construction of some important examples to be given in the next chapter. At that time, it will be helpful to refer to a *different derivation* of the result, independent of the special properties of the function $(\sin z)/z$. We proceed to give one now.

The basis for our argument here is the formula

$$\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right| \mathrm{d}\nu(t) \;=\; -x\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\left(\frac{\nu(t)}{t}\right), \qquad x>0,$$

valid for functions $\nu(t)$ positive and increasing on $[0,\infty)$ which are $\mathrm{O}(t)$ for both $t\to 0$ and $t\longrightarrow\infty$. This is just the first lemma of §B.4 in Chapter VIII. Starting with a weight $W(x)\geqslant 1$ whose logarithmic integral *converges*, we take a suitable constant multiple $m(t)$ of $\log W(t)$ (the constant will be specified later) and then, *fixing* an arbitrary $a>0$, put

$$\nu(t) \;=\; at \;-\; t\int_{\max(t,A)}^\infty \frac{m(\tau)}{\tau^2}\,\mathrm{d}\tau, \qquad t\geqslant 0,$$

where A is a large number depending on a in a manner to be described immediately.

We have

$$\nu'(t) \;=\; a \;-\; \int_A^\infty \frac{m(\tau)}{\tau^2}\,\mathrm{d}\tau, \qquad 0<t<A,$$

and

$$\nu'(t) \;=\; a \;-\; \int_t^\infty \frac{m(\tau)}{\tau^2}\,\mathrm{d}\tau \;+\; \frac{m(t)}{t}, \qquad t>A.$$

Therefore, if we choose A so as to make

$$\int_A^\infty \frac{m(\tau)}{\tau^2}\,\mathrm{d}\tau \;<\; a,$$

which is certainly *possible* thanks to our assumption on W, $\nu(t)$ will be an *increasing function* of t, with $0 < \nu(t) < at$ for $t>0$. It is also clear that

$$\frac{\nu(t)}{t}\longrightarrow a \qquad \text{for } t\longrightarrow\infty.$$

Using the function ν in the above formula gives us

$$\int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\nu(t) \;=\; -x\int_A^\infty \log\left|\frac{x+t}{x-t}\right| \frac{m(t)}{t^2}\,\mathrm{d}t.$$

Since, however, $m(t)$ is *increasing* according to our assumption on W, the *right side* of the last relation is

$$\leqslant \quad -m(x)\int_x^\infty \log\left|\frac{x+t}{x-t}\right| \frac{x\,\mathrm{d}t}{t^2}$$

for $x \geqslant A$. The substitution $\tau = t/x$ takes the integral figuring herein over to

$$\int_1^\infty \log\left|\frac{1+\tau}{1-\tau}\right| \frac{\mathrm{d}\tau}{\tau^2},$$

a certain (finite) *numerical quantity* – call it C – whose exact value we do not need to know. Thus,

$$\int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\nu(t) \;\leqslant\; -Cm(x) \qquad \text{for } x \geqslant A.$$

Write, for $\Im z \geqslant 0$,

$$U(z) \;=\; \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}\nu(t).$$

$U(z)$ is *harmonic* in the upper half plane and, for our present function $\nu(t)$, *continuous up to the real axis*, where it is certainly *bounded above* in view of the previous relation. Moreover,

$$U(z) \;\leqslant\; \int_0^\infty \log\left(1 + \frac{|z|^2}{t^2}\right) \mathrm{d}\nu(t),$$

and, after integrating by parts, the right side is easily seen to be $\sim \pi a|z|$ for $|z| \longrightarrow \infty$, keeping in mind that $\nu(t)/t \longrightarrow a$ as $t \longrightarrow \infty$. When $z = \mathrm{i}y$ with $y > 0$, the inequality just written becomes an *equality*, showing that $U(\mathrm{i}y)/y \longrightarrow \pi a$ for $y \longrightarrow \infty$.

These facts imply that

$$U(z) \;=\; \pi a \Im z \;+\; \frac{1}{\pi}\int_{-\infty}^\infty \frac{\Im z\, U(t)}{|z-t|^2}\,\mathrm{d}t \qquad \text{for } \Im z > 0$$

by an argument exactly like the one used to prove the theorem of §G.1, Chapter III. Plugging the previous relation into the integral on the right then gives

$$U(x+\mathrm{i}) \leqslant O(1) - \frac{1}{\pi}\int_{|t|\geq A} \frac{Cm(|t|)\,\mathrm{d}t}{(x-t)^2+1},$$

whence, since $m(|t|)$ increases with $|t|$,

$$U(x+\mathrm{i}) \leqslant O(1) - \frac{C}{2}m(|x|) \quad \text{for } |x| \geqslant A.$$

Taking a larger O(1) term of course ensures this estimate's validity for *all* real x.

The idea now is to observe that the integral

$$\int_0^\infty \log\left|1-\frac{z^2}{t^2}\right| \mathrm{d}\nu(t)$$

would represent the logarithm of the modulus of an entire function of exponential type, *if the increasing function* $\nu(t)$ *were integer-valued. Our* $\nu(t)$, of course, is *not* (it is absolutely continuous!), but one expects that

$$\int_0^\infty \log\left|1-\frac{z^2}{t^2}\right| \mathrm{d}[\nu(t)]$$

(with $[\nu(t)]$ designating the *greatest integer* $\leqslant \nu(t)$) should be *close* to

$$U(z) = \int_0^\infty \log\left|1-\frac{z^2}{t^2}\right| \mathrm{d}\nu(t).$$

This is indeed *true as soon as z gets away from the real axis*, and we have the following simple

Lemma. *If* $\nu(t)$ *is increasing and* $O(t)$ *on the positive real axis,*

$$\int_0^\infty \log\left|1-\frac{z^2}{t^2}\right| (\mathrm{d}[\nu(t)] - \mathrm{d}\nu(t))$$

$$\leqslant \log\left\{\frac{\max(|x|,|y|)}{2|y|} + \frac{|y|}{2\max(|x|,|y|)}\right\}$$

for $\Im z = y \neq 0$.

Proof. Assuming that $\Im z \neq 0$, integrate the left-hand member by parts. Because $\nu(t) = O(t)$, the integrated term vanishes, and we obtain

$$\int_0^\infty (\nu(t) - [\nu(t)])\frac{\partial}{\partial t}\log\left|1-\frac{z^2}{t^2}\right| \mathrm{d}t.$$

Fixing z, let us introduce the new variable $\zeta = z^2/t^2$. As t runs through $(0, \infty)$, ζ *moves in* along a certain *ray* $\mathscr{L}$ coming out from the origin. When $\Re z^2 \leqslant 0$, the distance $|1-\zeta|$ *decreases* as t *increases*, so, since $v(t)-[v(t)] \geqslant 0$, the expression just written is $\leqslant 0$.

If, however, $\Re z^2 > 0$, $|1-\zeta|$, for increasing t, *first decreases* to a *minimum value* $|\Im z^2|/|z|^2$ and *then increases, tending to* 1 as $t \longrightarrow \infty$:

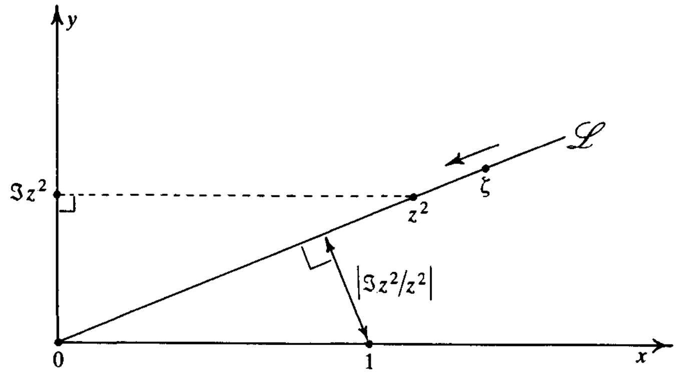

Figure 222

Hence, since $0 \leqslant v(t)-[v(t)] \leqslant 1$, the expression under consideration is

$$\leqslant \log(|z|^2/|\Im z^2|).$$

We see that $\int_0^\infty \log|1-(z^2/t^2)|(\mathrm{d}[v(t)]-\mathrm{d}v(t))$ is

$$\leqslant \begin{cases} 0, & |x| \leqslant |y|, \\ \log\left(\dfrac{|x|}{2|y|} + \dfrac{|y|}{2|x|}\right), & |x| > |y|. \end{cases}$$

The right-hand side can be represented by the single expression

$$\log\left(\frac{\max(|x|,|y|)}{2|y|} + \frac{|y|}{2\max(|x|,|y|)}\right).$$

The lemma is proved

Using the lemma with our function $U(z)$, we get

$$\int_0^\infty \log\left|1-\frac{(x+\mathrm{i})^2}{t^2}\right|\mathrm{d}[v(t)] \quad \leqslant \quad \log^+|x| \; + \; U(x+\mathrm{i}),$$

so, by the relation established above,

$$\int_0^\infty \log\left|1 - \frac{(x+\mathrm{i})^2}{t^2}\right| \mathrm{d}[\nu(t)] \quad \leqslant \quad \mathrm{O}(1) \;+\; \log^+|x| \;-\; \frac{C}{2}m(|x|), \quad x \in \mathbb{R}.$$

For each integer $k \geqslant 1$, denote by λ_k the positive value of t for which $\nu(t) = k$. Then, noting that $\nu(0) = 0$, we have

$$\int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}[\nu(t)] \quad = \quad \log\left|\prod_{k=1}^{\infty}\left(1 - \frac{z^2}{\lambda_k^2}\right)\right|,$$

so, putting

$$\varphi(z) \quad = \quad \prod_{k=2}^{\infty}\left(1 - \frac{z^2}{\lambda_k^2}\right) \quad (sic!),$$

the preceding inequality yields

$$\log|\varphi(x+\mathrm{i})| \quad \leqslant \quad \mathrm{O}(1) \;-\; \frac{C}{2}m(|x|), \qquad x \in \mathbb{R}.$$

Arguing as we did above for $U(z)$, we find without trouble that

$$\log|\varphi(z)| \;\leqslant\; \mathrm{O}(1) \;+\; \pi a|z|;$$

in other words, *$\varphi(z)$ is entire and of exponential type $\leqslant \pi a$.*

The relation involving $\varphi(x+\mathrm{i})$ and $m(|x|)$ can be rewritten

$$|\varphi(x+\mathrm{i})|\mathrm{e}^{Cm(|x|)/2} \quad \leqslant \quad \text{const.}, \qquad x \in \mathbb{R},$$

where C is a numerical constant independent of a and of $m(t)$. Going back to our even weight W, we now take

$$m(t) \quad = \quad \frac{2}{C}\log W(t), \qquad t \geqslant 0,$$

and the entire function φ of exponential type $\leqslant \pi a$ furnished by the construction just made satisfies

$$W(x)|\varphi(x+\mathrm{i})| \quad \leqslant \quad \text{const.}, \qquad x \in \mathbb{R}.$$

This φ is of course non-zero, so, since $a > 0$ is arbitrary, we have again arrived at the theorem stated at the beginning of the present article.

2. Statement of the Beurling–Malliavin multiplier theorem

The result just discussed implies in particular that if an entire function $F(z)$ of exponential type has, on the real axis, a *majorant*

$M(x) \geqslant 1$, *increasing* when $x \geqslant 0$ and *decreasing* for $x < 0$, such that

$$\int_{-\infty}^{\infty} \frac{\log M(x)}{1+x^2}\,dx \quad < \quad \infty,$$

then there are non-zero entire functions φ of arbitrarily small exponential type for which $F(x)\varphi(x)$ is bounded on $\mathbb{R}$. It suffices indeed to apply the theorem with the weight $W(x) = M(x)M(-x)$. In such circumstances, the function $F(z)$ is thus a *factor* of *other* non-zero entire functions, *having exponential type arbitrarily close to that of F* and *bounded* on $\mathbb{R}$.

It is very remarkable that *the monotoneity requirements on the majorant can be dispensed with here.* The *mere condition* that F be *entire and of exponential type* somehow implies *enough regularity* for the weight $1+|F(x)|$ so that *convergence* of the *logarithmic integral* associated with the latter *already ensures its admitting of multipliers.*

Theorem (Beurling and Malliavin, 1961 – called the ***theorem on the multiplier***). *Let $F(z)$ be entire and of exponential type. In order that the weight $|F(x)| + 1$ admit multipliers, it is necessary and sufficient that*

$$\int_{-\infty}^{\infty} \frac{\log^+|F(x)|}{1+x^2}\,dx \quad < \quad \infty.$$

This result is much deeper than the one of the preceding article, and there is so far no really simple way of arriving at it. A proof based on material from §C of Chapter VIII will be given in the next chapter. For the time being, the reader is only asked to take account of the theorem's *statement*. Some of its important consequences will be deduced in the following §§.

B. Completeness of sets of exponentials on finite intervals

We return to the study of completeness of collections of functions $e^{i\lambda_n t}$, begun in §D of the last chapter. There, we obtained a *lower bound* for the *completeness radius* associated with an arbitrary *real* sequence Λ of distinct frequencies λ_n; we wish now to show that that lower bound is *also* an *upper bound*, thus arriving at a full determination of the completeness radius. The reader should perhaps again look through the beginning of §D, Chapter IX, before continuing with the present discussion.

The lower bound just referred to is most conveniently expressed in terms of the Beurling–Malliavin *effective density* $\tilde{D}_\Lambda$ for the sequence Λ, defined in §D.2 of the previous chapter. According to a theorem in that

§, the completeness radius associated with Λ is $\geqslant \pi\tilde{D}_\Lambda$; the exponentials $e^{i\lambda_n t}$, $\lambda_n \in \Lambda$, are, in other words, *complete* (in any of the usual norms) on each interval of length $< 2\pi\tilde{D}_\Lambda$.

Showing this completeness radius to be *equal* to $\pi\tilde{D}_\Lambda$ was the first use made of the multiplier theorem stated in §A.2, which was indeed elaborated for that specific purpose. This application is given in the present §. Our task here is thus to prove that the completeness radius for Λ cannot be *larger* than $\pi\tilde{D}_\Lambda$; this amounts to establishing *incompleteness* of the $e^{i\lambda_n t}$, $\lambda_n \in \Lambda$, on any interval $[-L, L]$ with $L > \pi\tilde{D}_\Lambda$.

The known procedures for doing this are all based on the *duality argument* described at the beginning of §D, Chapter IX. Desiring, for instance, to prove that linear combinations of the $e^{i\lambda_n t}$ are *not dense* in $L_1(-L, L)$, one tries to obtain a *non-zero* g in the *dual* of that space – in this case, an element of $L_\infty(-L, L)$ – for which

$$\int_{-L}^{L} g(t)e^{i\lambda_n t}\,dt = 0, \qquad \lambda_n \in \Lambda.$$

Establishing incompleteness in this way thus involves proof of an existence theorem. That is why the determination of *upper bounds* on the completeness radius has always given much more difficulty than the search for *lower bounds*, which essentially depend on uniqueness theorems (based on various forms of Jensen's formula).

The idea is to arrive at the function g by constructing its *Fourier transform*

$$\int_{-L}^{L} e^{izt}g(t)\,dt.$$

Suppose, for instance, that we are able to construct a non-zero entire function $G(z)$ of exponential type $\leqslant L$, vanishing at the points of Λ, for which

$$\int_{-\infty}^{\infty} |G(x)|\,dx < \infty.$$

The function

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}G(x)\,dx$$

is then continuous, and an argument just like the one used in proving the Paley–Wiener theorem shows that $g(t) \equiv 0$ for $|t| > L$ (Chapter III, §D). We therefore certainly have $g \in L_1(\mathbb{R})$, and the Fourier inversion theorem for L_1 gives

$$G(x) = \int_{-L}^{L} e^{ixt} g(t)\,dt.$$

For each $\lambda_n \in \Lambda$, we then have

$$\int_{-L}^{L} e^{i\lambda_n t} g(t)\,dt = G(\lambda_n) = 0.$$

And $g(t) \not\equiv 0$ since $G \not\equiv 0$. Our aim can thus be accomplished by showing how to get such a function G when the sequence Λ and any $L > \pi \tilde{D}_\Lambda$ are given.

Beurling and Malliavin obtained a complete solution of this problem around 1961. Considerable effort had previously been expended on it by others who had succeeded in finding various constructions of entire functions G, subject always, however, to restrictive assumptions on the sequence Λ. This was done by Paley and Wiener and then by Levinson; later on, Redheffer obtained a number of results. I have worked on the question myself. Many of the methods devised for these investigations are still of interest even though they were not powerful enough to yield the final definitive conclusion; some of them indeed find service in the present book. The reader who wants to find out more about these matters should consult Redheffer's survey article (in *Advances in Math.*), which gives a very clear exposition of most of what has been done. There, the delicate question of completeness of the $e^{i\lambda_n t}$ on intervals of length *exactly equal* to $2\pi \tilde{D}_\Lambda$ is also discussed.

Before going on to article 1, let us indicate how the work will proceed. We are given a sequence $\Lambda \subseteq \mathbb{R}$ with $\tilde{D}_\Lambda < \infty$.* Picking any $\eta > 0$, we wish to construct a non-zero entire function $G(z)$ of exponential type $\leqslant \pi(\tilde{D}_\Lambda + 3\eta)$, say, such that

$$G(\lambda) = 0 \quad \text{for } \lambda \in \Lambda^\dagger$$

and

$$\int_{-\infty}^{\infty} |G(x)|\,dx < \infty.$$

Because the distribution of the $\lambda \in \Lambda$ may be very irregular, it is not

* It is best to allow Λ to have repeated points; that makes no difference for the constructions to follow.

† with, of course, appropriate multiplicity at the repeated points of Λ

advisable to start with the Hadamard product

$$\prod_{\substack{\lambda\in\Lambda\\ \lambda\neq 0}}\left(1-\frac{z}{\lambda}\right)e^{z/\lambda}.$$

Instead, we first turn to the *second* lemma of §D.2; Chapter IX, and to its corollary. Given $D > \tilde{D}_\Lambda$, these provide us with a real sequence $\Sigma \supset \Lambda$ for which

$$\int_{-\infty}^{\infty}\frac{|n_\Sigma(t)-Dt|}{1+t^2}\,dt \quad < \quad \infty.$$

Here, $n_\Sigma(t)$ denotes the number of points* of Σ in $[0,\ t]$ if $t \geqslant 0$, and *minus* the number of such points in $[t,\ 0)$ if $t < 0$. For our purposes, we take

$$D \;=\; \tilde{D}_\Lambda + \eta.$$

The points of Σ are already quite regularly distributed. Assuming, wlog, that $0 \notin \Sigma$, we form the function

$$F(z) \;=\; \prod_{\lambda\in\Sigma}\left(1-\frac{z}{\lambda}\right)e^{z/\lambda},$$

which turns out to be of exponential type. Its behaviour is worked out in article 1.

The next step (in article 2) is to prove what is called the *little multiplier theorem*. This result (which, strictly speaking, is *not* a multiplier theorem in the sense adopted for that term at the beginning of the present chapter) gives us a non-zero entire function $\varphi(z)$ of exponential type $\pi\eta$ such that

$$\int_{-\infty}^{\infty}\frac{\log^+|F(x)\varphi(x)|}{1+x^2}\,dx \quad < \quad \infty.$$

The theorem stated in §A.2 is finally applied to the product $F\varphi$ in order to obtain the function G. In this way, the completeness radius associated with Λ is seen to be $\leqslant \pi(\tilde{D}_\Lambda + 3\eta)$,† and the exact determination of the former quantity thus carried out for *real* sequences Λ (article 3).

It is somewhat remarkable that all the difficulties involved in the completeness problem for sets of exponentials $e^{i\lambda_n t}$ having *complex frequencies* λ_n already occur in the one about exponentials with *real frequencies*. The more general problem is rather easily reduced to the special one, and our solution of the latter made to yield one for the former.

* taking multiplicities of repeated points into account

† when the points of Λ are distinct

The completeness radius associated with arbitrary *complex* sequences Λ can thus be worked out. This result also is given in article 3.

1. The Hadamard product over Σ

Having fixed $\eta > 0$ and put $D = \tilde{D}_\Lambda + \eta$, we take a real sequence $\Sigma \supset \Lambda$ having, perhaps, repetitions, such that

$$\int_{-\infty}^{\infty} \frac{|n_\Sigma(t) - Dt|}{1+t^2} \mathrm{d}t \quad < \quad \infty,$$

being assured by §D.2 of Chapter IX that such Σ exist. During this article and the next one, we will assume that $n_\Sigma(t) \equiv 0$ for $|t| < 1$, i.e., that Σ (and hence surely our original Λ) has no points in $(-1, 1)$. Doing so simplifies some details in the work, but does not make the results obtained less applicable.

Let

$$F(z) \quad = \quad \prod_{\lambda \in \Sigma} \left(1 - \frac{z}{\lambda}\right) \mathrm{e}^{z/\lambda}.$$

Then we have the

Theorem. *If $n_\Sigma(t) = 0$ for $-1 < t < 1$ and*

$$\int_{-\infty}^{\infty} \frac{|n_\Sigma(t) - Dt|}{1+t^2} \mathrm{d}t \quad < \quad \infty,$$

$F(z)$ is of exponential type, and

$$\limsup_{r \to \infty} \frac{\log|F(r\mathrm{e}^{\mathrm{i}\vartheta})|}{r} \quad \leqslant \quad \pi D |\sin \vartheta| + c \cos \vartheta,$$

where c is a certain constant. When $\vartheta = \pm \pi/2$ the limit superior on the left is an actual limit, and equality holds.

Proof. The last lemma of §D.2, Chapter IX, tells us that

$$\frac{n_\Sigma(t)}{t} \longrightarrow D \quad \text{for } t \longrightarrow \pm\infty.$$

This, however, is not in itself enough to make $F(z)$ of exponential type; for that, boundedness of $|\int_{-R}^{R} (1/t) \mathrm{d}n_\Sigma(t)|$ as $R \longrightarrow \infty$ is also necessary – and sufficient – according to the Lindelöf theorems of §B, Chapter III. In the present circumstances, we have, however,

$$\int_{-R}^{R} \frac{\mathrm{d}n_\Sigma(t)}{t} = \frac{n_\Sigma(R) + n_\Sigma(-R)}{R} + \int_{1 \leqslant |t| \leqslant R} \frac{n_\Sigma(t)}{t^2}\,\mathrm{d}t$$

$$= \mathrm{o}(1) + \int_{1 \leqslant |t| \leqslant R} \frac{n_\Sigma(t) - Dt}{t^2}\,\mathrm{d}t = \mathrm{O}(1)$$

since

$$\int_{|t| \geqslant 1} \frac{|n_\Sigma(t) - Dt|}{t^2}\,\mathrm{d}t < \infty,$$

so F is of exponential type.

For y real, we have

$$\log|F(\mathrm{i}y)| = \frac{1}{2}\int_{-\infty}^{\infty} \log\left(1 + \frac{y^2}{t^2}\right)\mathrm{d}n_\Sigma(t).$$

After integrating by parts and then using the asymptotic behaviour of $n_\Sigma(t)$, we easily find that

$$\log|F(\mathrm{i}y)| \sim \pi D|y| \quad \text{for } y \longrightarrow \pm\infty.$$

To study the behaviour of $F(z)$ on the real axis, we take

$$c = \int_{|t| \geqslant 1} \frac{n_\Sigma(t) - Dt}{t^2}\,\mathrm{d}t$$

(the integral being absolutely convergent), and then look at

$$\mathrm{e}^{-cx}F(x)$$

for real x. Here, we are able to fall back on work already done for parts (a) – (d) of problem 29 (§B.1, Chapter IX). Denote

$$\Sigma \cap (0, \infty) \quad \text{by} \quad \Sigma_+$$

and

$$(-\Sigma) \cap (0, \infty) \ (sic!) \quad \text{by} \quad \Sigma_-.$$

Then, if $x > 0$, we can write

$$\log|F(x)| = \sum_{\lambda \in \Sigma_+} \log\left|1 - \frac{x^2}{\lambda^2}\right| + \left(\sum_{\lambda \in \Sigma_-} - \sum_{\lambda \in \Sigma_+}\right)\left(\log\left|1 + \frac{x}{\lambda}\right| - \frac{x}{\lambda}\right).$$

Since $\lim_{t\to\infty}(n_{\Sigma_+}(t)/t)$ *exists*, the *first* sum on the right is $\leqslant \mathrm{o}(x)$ for $x \longrightarrow \infty$ by problem 29(d). (For the solution of parts (a) – (e) of that problem, it is not necessary that the zeros of the function $C(z)$ considered there be *integers* – they need only be *real and positive.*)

What is left on the right side of the previous relation can be rewritten as

$$\int_0^\infty \left(\log\left(1+\frac{x}{t}\right) - \frac{x}{t}\right)(\mathrm{d}n_{\Sigma_-}(t) - \mathrm{d}n_{\Sigma_+}(t)).$$

This is integrated by parts, upon which all the integrated terms vanish ($n_{\Sigma_+}(t)$ and $n_{\Sigma_-}(t)$ are zero for $0 < t < 1$!), and we end with

$$\int_0^\infty \frac{x^2}{x+t}\,\frac{n_{\Sigma_+}(t) - n_{\Sigma_-}(t)}{t^2}\,\mathrm{d}t.$$

Since

$$\int_0^\infty \frac{n_{\Sigma_+}(t) - n_{\Sigma_-}(t)}{t^2}\,\mathrm{d}t \quad = \quad \int_{|t|\geqslant 1} \frac{n_\Sigma(t) - Dt}{t^2}\,\mathrm{d}t$$

with the right-hand integral absolutely convergent and equal to c, the left-hand integral is also absolutely convergent and equal to c, so the previous expression is $\sim\ cx$ for $x \longrightarrow \infty$.

We see that

$$\log|F(x)| \ \leqslant\ cx + \mathrm{o}(|x|) \qquad \text{for } x \longrightarrow \infty.$$

In like manner, the same is seen to hold for $x \longrightarrow -\infty$. The function $\mathrm{e}^{-cz}F(z)$ is thus in modulus $\leqslant\ \mathrm{e}^{\mathrm{o}(|x|)}$ on the *real* axis when x is large, and has the *same* growth as $F(z)$ on the *imaginary* axis; it is, moreover, of *exponential type.* Our desired result now follows by application of a Phragmén–Lindelöf theorem, as in part (e) of problem 29.

We shall have to look more closely at the behaviour of $|F(x)|$ on the real axis. Of course,

$$\log|F(x)| \quad = \quad \int_{-\infty}^\infty \left(\log\left|1 - \frac{x}{t}\right| + \frac{x}{t}\right)\mathrm{d}n_\Sigma(t), \qquad x \in \mathbb{R}.$$

Regarding integrals like the one on the right, one has the following generalization of the formula derived in problem 29(b):

Lemma. *Let $\nu(t)$, zero on a neighborhood of* 0 (*N.B.*!), *be increasing on* $(-\infty, \infty)$ *and* $\mathrm{O}(t)$ *there. Then*

$$\int_{-\infty}^\infty \left(\log\left|1 - \frac{x}{t}\right| + \frac{x}{t}\right)\mathrm{d}\nu(t) \quad = \quad ⨍_{-\infty}^\infty \frac{x^2}{x-t}\,\frac{\nu(t)}{t^2}\,\mathrm{d}t$$

at the $x \in \mathbb{R}$ where $v'(x)$ exists and is finite, and also at those where $v(t)$ has a jump discontinuity.

Remark. The expression on the right is a *Cauchy principal value*, viz.,

$$\lim_{\varepsilon \to 0} \int_{|t-x|>\varepsilon} \frac{x^2}{x-t} \frac{v(t)}{t^2} \, \mathrm{d}t.$$

See the end of §C.1 in Chapter VIII.

Proof. Taking an $\varepsilon > 0$, integrate

$$\int_{|t-x| \geqslant \varepsilon} \left(\log\left|1 - \frac{x}{t}\right| + \frac{x}{t} \right) \mathrm{d}v(t)$$

by parts. Under the given conditions, the integrated terms corresponding to $t = \pm\infty$ vanish, and, if $v'(x)$ exists and is finite, the *sum* of the ones corresponding to $t = x \pm \varepsilon$ tends to zero as $\varepsilon \longrightarrow 0$, leaving us with the right side of the identity in question. When v has a jump discontinuity at x, that identity is valid because each of its sides is then equal to $-\infty$.

Application of the lemma to our function F (formed from $n_\Sigma(t)$ which vanishes for $|t| < 1$) yields

$$\log|F(x)| \quad = \quad ⨍_{|t| \geqslant 1} \frac{x^2}{x-t} \frac{n_\Sigma(t)}{t^2} \, \mathrm{d}t.$$

In using this relation, we will want to take advantage of the condition

$$\int_{|t| \geqslant 1} \frac{|n_\Sigma(t) - Dt|}{1 + t^2} \, \mathrm{d}t \quad < \quad \infty,$$

and for that we will be helped by the formula

$$⨍_{|t| \geqslant 1} \frac{x^2}{x-t} \frac{\mathrm{d}t}{t} \quad = \quad -x \log\left|\frac{x+1}{x-1}\right|,$$

which is easily verified by direct calculation. From this and the previous, we get

$$\log|F(x)| \quad = \quad ⨍_{|t| \geqslant 1} \frac{x^2}{x-t} \frac{n_\Sigma(t) - Dt}{t^2} \, \mathrm{d}t \; - \; Dx \log\left|\frac{x+1}{x-1}\right|.$$

The second term on the right is $\geqslant 0$, and tends to $2D$ as $x \longrightarrow \pm\infty$; for it, we certainly have

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} Dx \log\left|\frac{x+1}{x-1}\right| \mathrm{d}x \quad < \quad \infty.$$

As far as we are concerned, then, the behaviour of $\log|F(x)|$ is governed by that of the Cauchy principal value on the right, involving the *integrable* function $(n_\Sigma(t) - Dt)/t^2$. It will be convenient in the next article to denote that principal value by $U(x)$, i.e.,

$$\boxed{U(x) \quad = \quad -\!\!\!\!\!\!\int_{|t|\geqslant 1} \frac{x^2}{x-t}\,\frac{n_\Sigma(t) - Dt}{t^2}\,dt.}$$

2. The little multiplier theorem

We proceed to construct a non-zero entire function $\varphi(z)$ of exponential type $\pi\eta$ which will make

$$\int_{-\infty}^{\infty} \frac{\log^+|F(x)\varphi(x)|}{1+x^2}\,dx \quad < \quad \infty,$$

F being the Hadamard product formed above. According to what was observed at the end of the last article, the relation just written will certainly hold if

$$\int_{-\infty}^{\infty} \frac{|U(x) + \log|\varphi(x)||}{1+x^2}\,dx \quad < \quad \infty$$

with the function $U(x)$ defined there. Let us write

$$\Delta(t) \quad = \quad \begin{cases} n_\Sigma(t) - Dt, & |t| \geqslant 1, \\ 0, & -1 < t < 1. \end{cases}$$

Then, as we have seen,

$$U(x) \;=\; \log|F(x)| \;+\; Dx\log\left|\frac{x+1}{x-1}\right|$$

is equal to

$$-\!\!\!\!\!\!\int_{-\infty}^{\infty} \frac{x^2}{x-t}\,\frac{\Delta(t)}{t^2}\,dt.$$

For the function φ we are seeking, $\log|\varphi(x)|$ will be related to a similar expression,

$$-\!\!\!\!\!\!\int_{-\infty}^{\infty} \frac{x^2}{x-t}\,\frac{\delta(t)}{t^2}\,dt,$$

involving a $\delta(t)$ obtained in a certain way from $\Delta(t)$. The property

$$\int_{-\infty}^{\infty} (|\Delta(t)|/t^2)\,\mathrm{d}t \quad < \quad \infty$$

and the fact that $\Delta(t) + Dt$ increases both play important rôles in that construction. Here is how it goes:

Lemma. *Given the function $\Delta(t)$, zero on $(-1,\ 1)$, fulfilling the conditions just mentioned, and a number $\eta > 0$, there are two sequences*

$$1 = x_1 < x_2 < x_3 < \cdots < x_k \xrightarrow[k]{} \infty,$$
$$-1 = x_{-1} > x_{-2} > x_{-3} > \cdots > x_{-k} \xrightarrow[k]{} -\infty,$$

and a function $\delta(t)$, zero on $(-1,\ 1)$, with the following properties:

(i) $$\sum_{k=1}^{\infty} \left(\frac{x_{k+1} - x_k}{x_k}\right)^2 + \sum_{k=-1}^{-\infty} \left(\frac{x_k - x_{k-1}}{x_k}\right)^2 \quad < \quad \infty;$$

(ii) *$\eta t + \delta(t)$ is increasing on $(-\infty, \infty)$;*

(iii) *$\delta(t)$ is $\mathrm{o}(t)$ for $t \longrightarrow \pm\infty$;*

(iv) $$\int_{-\infty}^{\infty} \frac{|\delta(t)|}{t^2}\,\mathrm{d}t \quad < \quad \infty;$$

(v) *for each $k \geqslant 1$ or < -1,*

$$\int_{x_k}^{x_{k+1}} \frac{\Delta(t) + \delta(t)}{t^2}\,\mathrm{d}t \quad = \quad 0;$$

(vi) *for $x_k \leqslant t \leqslant x_{k+1}$,*

$$|\Delta(t) + \delta(t)| \leqslant (D + \eta)(x_{k+2} - x_{k-1}),$$

where, to cover the cases $k = -2$ and $k = 1$, we put $x_0 = 0$.

Remark. Property (i) certainly implies that $x_k/x_{k-1} \longrightarrow 1$ for $k \longrightarrow \pm\infty$. Keeping this in mind, the reader familiar with the modern theory of H_1 and BMO will recognize in properties (v) and (vi) a stipulation that the functions

$$s_k(t) = \begin{cases} \dfrac{x_k^2(\Delta(t) + \delta(t))}{(D + \eta)(x_{k+2} - x_{k-1})(x_{k+1} - x_k)\,t^2}, & x_k \leqslant t \leqslant x_{k+1}, \\[2ex] 0, \quad t \notin [x_k,\ x_{k+1}] \end{cases}$$

be *atoms* for $\Re H_1$ (to within constant factors tending to 1 for $k \longrightarrow \pm\infty$). About this, more later on.

Proof of lemma. It suffices to show how to get the x_k with $k \geqslant 1$ and the function $\delta(t)$ when $t \geqslant 1$, the constructions on $(-\infty, -1]$ being exactly the same.

We start by putting $x_1 = 1$. Then, assuming that x_k has already been determined (and $\delta(t)$ specified on $[1, x_k)$ if $k > 1$), let us see how to find x_{k+1}, and how to define $\delta(t)$ for $x_k \leqslant t < x_{k+1}$.

As $x > x_k$ increases, the integral

$$\int_{x_k}^{x} \frac{t - x_k}{t^2}\,dt = \log\frac{x}{x_k} - \frac{x - x_k}{x}$$

tends to ∞, while

$$\int_{x_k}^{x} \frac{|\Delta(t)|}{t^2}\,dt$$

remains bounded, by hypothesis. Hence, unless the ratio

$$\int_{x_k}^{x} \frac{|\Delta(t)|}{t^2}\,dt \Big/ \int_{x_k}^{x} \frac{t - x_k}{t^2}\,dt$$

remains always $< \eta$ for $x > x_k$, there is a value of x for which it is *equal* to our given number η. If equality last obtains for a value $x > x_k + 1$ we call that value x_{k+1}; in any other case we put $x_{k+1} = x_k + 1$. We thus have $x_{k+1} \geqslant x_k + 1$ and also

$$\int_{x_k}^{x_{k+1}} \frac{|\Delta(t)|}{t^2}\,dt \leqslant \eta \int_{x_k}^{x_{k+1}} \frac{t - x_k}{t^2}\,dt,$$

with equality holding when $x_{k+1} > x_k + 1$.

We have

$$\int_{x_k}^{x_{k+1}} \frac{x_{k+1} - t}{t^2}\,dt - \int_{x_k}^{x_{k+1}} \frac{t - x_k}{t^2}\,dt > 0,$$

for the difference on the left can be rewritten as

$$-2\int_{-l}^{l} \frac{\tau}{(\tau + c)^2}\,d\tau = 2\int_{0}^{l}\left(\frac{1}{(c-\tau)^2} - \frac{1}{(c+\tau)^2}\right)\tau\,d\tau$$

with $c = (x_k + x_{k+1})/2$, $l = (x_{k+1} - x_k)/2$, and the new variable

$\tau = t - c$. Therefore, as x' *increases* from x_k to x_{k+1},

$$\eta \int_{x_k}^{x_{k+1}} \frac{x' - t}{t^2} \mathrm{d}t$$

increases from $-\eta\int_{x_k}^{x_{k+1}}((t - x_k)/t^2)\mathrm{d}t$ to $\eta\int_{x_k}^{x_{k+1}}((x_{k+1} - t)/t^2)\mathrm{d}t > \eta\int_{x_k}^{x_{k+1}}((t - x_k)/t^2)\mathrm{d}t$, and, by the previous relation, there must be an $x' \in [x_k, x_{k+1}]$ for which

$$\int_{x_k}^{x_{k+1}} \frac{\Delta(t)}{t^2} \mathrm{d}t + \eta \int_{x_k}^{x_{k+1}} \frac{x' - t}{t^2} \mathrm{d}t = 0.$$

We denote that value of x' by x'_k, and put

$$\delta(t) = \eta(x'_k - t) \quad \text{for } x_k \leqslant t < x_{k+1}.$$

In this way, the function $\delta(t)$ is defined piece by piece on the successive intervals $[x_k, x_{k+1})$ and thus on all of $[1, \infty)$, since our requirement that $x_{k+1} \geqslant x_k + 1$ ensures that $x_k \xrightarrow[k]{} \infty$.

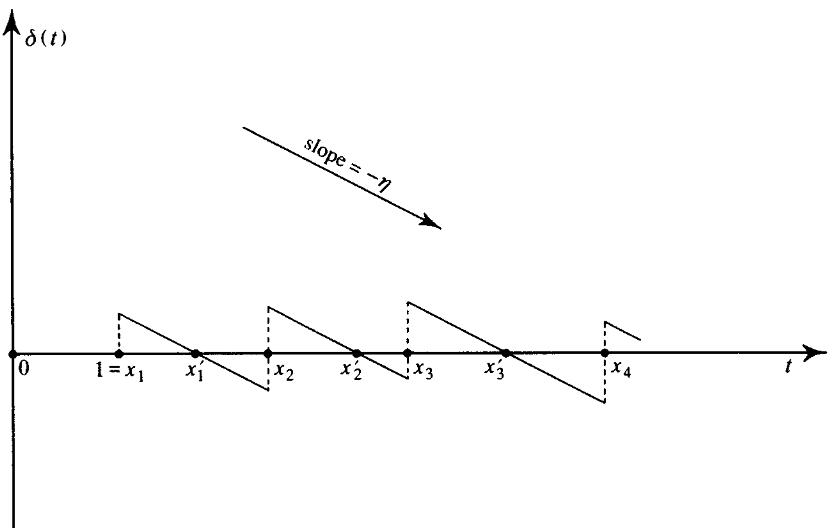

Figure 223

We must verify properties (i) – (vi) for this $\delta(t)$ and the sequence $\{x_k\}$.

Property (ii) is obvious, and (v) guaranteed by our choice of the x'_k. To check (i), observe that when $x_{k+1} > x_k + 1$,

$$\frac{\eta}{2} \frac{(x_{k+1} - x_k)^2}{x_{k+1}^2} < \eta \int_{x_k}^{x_{k+1}} \frac{t - x_k}{t^2} \mathrm{d}t = \int_{x_k}^{x_{k+1}} \frac{|\Delta(t)|}{t^2} \mathrm{d}t,$$

whence

$$\sum_{x_{k+1}>x_k+1}\left(\frac{x_{k+1}-x_k}{x_{k+1}}\right)^2 \quad < \quad \frac{2}{\eta}\cdot\int_1^\infty \frac{|\Delta(t)|}{t^2}\,dt \quad < \quad \infty.$$

The sum of the $(x_{k+1}-x_k)^2/x_{k+1}^2$ with $x_{k+1} = x_k+1$ is, on the other hand, obviously convergent, so we have

$$\sum_{k=1}^{\infty}\left(\frac{x_{k+1}-x_k}{x_{k+1}}\right)^2 \quad < \quad \infty.$$

This certainly implies that

$$\frac{x_{k+1}}{x_k} \longrightarrow 1, \qquad k \longrightarrow \infty,$$

and we must also have

$$\sum_{k=1}^{\infty}\left(\frac{x_{k+1}-x_k}{x_k}\right)^2 \quad < \quad \infty.$$

For (iii) and (iv), we use the fact that

$$|\delta(t)| \;=\; \eta|x_k'-t| \;\leqslant\; \eta(x_{k+1}-x_k) \quad \text{for } x_k \;\leqslant\; t \;<\; x_{k+1}.$$

Thence $|\delta(t)/t| \leqslant \eta(x_{k+1}-x_k)/x_k$ on $[x_k,\ x_{k+1})$, but, by what we have just seen, the right-hand quantity tends to zero for $k \longrightarrow \infty$. Again,

$$\int_{x_k}^{x_{k+1}} \frac{|\delta(t)|}{t^2}\,dt \quad < \quad \eta\frac{(x_{k+1}-x_k)^2}{x_k^2},$$

and the convergence of $\int_1^\infty(|\delta(t)|/t^2)\,dt$ follows from property (i), already verified.

We are left with property (vi). Given $k \geqslant 1$, we have

$$\int_{x_{k-1}}^{x_k} \frac{\Delta(t)+\delta(t)}{t^2}\,dt \quad = \quad \int_{x_{k+1}}^{x_{k+2}} \frac{\Delta(t)+\delta(t)}{t^2}\,dt \quad = \quad 0$$

by (v) and (for $k=1$) the fact that $\Delta(t) = \delta(t) = 0$ for $x_0 = 0 \leqslant t < 1 = x_1$. There are thus points t' and t'', in $[x_{k-1},\ x_k)$ and $[x_{k+1},\ x_{k+2})$ respectively, for which

$$\Delta(t') \;+\; \delta(t') \;\geqslant\; 0$$

$$\Delta(t'') \;+\; \delta(t'') \;\leqslant\; 0.$$

According to (ii), $\delta(t)+\eta t$ increases, and $\Delta(t)+Dt$ is increasing by hypothesis. Therefore, if $x_k \leqslant t \leqslant x_{k+1}$,

$$\Delta(t) + \delta(t) \geqslant \Delta(t') + \delta(t') - (D+\eta)(t-t') \geqslant -(D+\eta)(x_{k+1} - x_{k-1}),$$

and

$$\Delta(t) + \delta(t) \leqslant \Delta(t'') + \delta(t'') + (D+\eta)(t''-t) \leqslant (D+\eta)(x_{k+2} - x_k):$$

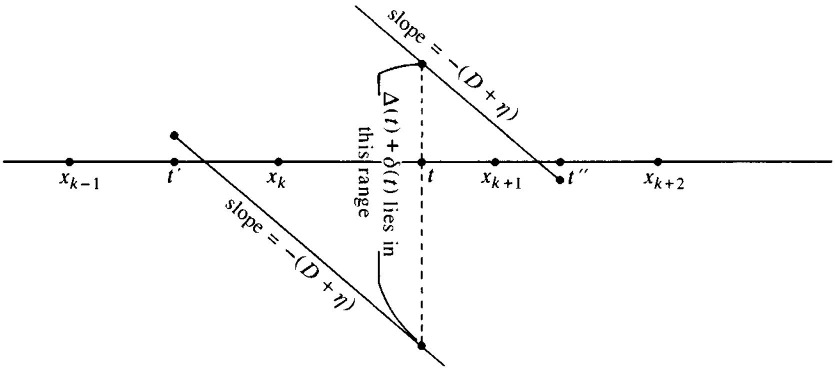

Figure 224

We see that for $x_k \leqslant t \leqslant x_{k+1}$,

$$|\Delta(t) + \delta(t)| \leqslant (D+\eta)\max\{(x_{k+2} - x_k),\ (x_{k+1} - x_{k-1})\},$$

more than what is asserted by (vi).

The lemma is proved.

Theorem. *If* $\Delta(t)$, *zero on* $(-1, 1)$ *and with* $\Delta(t) + Dt$ *increasing on* $\mathbb{R}$, *satisfies*

$$\int_{-\infty}^{\infty} \frac{|\Delta(t)|}{t^2} \mathrm{d}t < \infty$$

and if $\delta(t)$ *is the function furnished by the lemma, we have*

$$\int_{-\infty}^{\infty} \frac{|U(x) + v(x)|}{1 + x^2} \mathrm{d}x < \infty,$$

where

$$U(x) = \fint_{-\infty}^{\infty} \frac{x^2}{x-t} \frac{\Delta(t)}{t^2} \mathrm{d}t$$

and

$$v(x) \quad = \quad \int_{-\infty}^{\infty} \frac{x^2}{x-t}\,\frac{\delta(t)}{t^2}\,\mathrm{d}t.$$

Proof. Taking the sequences $x_1, x_2, x_3, \ldots$ and $x_{-1}, x_{-2}, x_{-3}, \ldots$ provided by the lemma, we can write

$$U(x) + v(x) \quad = \quad \sum_{k \neq -1,0} \int_{x_k}^{x_{k+1}} \frac{x^2}{x-t}\,\frac{\Delta(t)+\delta(t)}{t^2}\,\mathrm{d}t.$$

Putting, therefore,

$$V_k(x) \quad = \quad \int_{x_k}^{x_{k+1}} \frac{x^2}{x-t}\,\frac{\Delta(t)+\delta(t)}{t^2}\,\mathrm{d}t,$$

it is, in order to prove the theorem, more than sufficient to verify that

$$\sum_{k \neq -1,0} \int_{-\infty}^{\infty} \frac{|V_k(x)|}{x^2}\,\mathrm{d}t \quad < \quad \infty.$$

For this purpose, we use what has become a standard tool of singular integral theory.

Denote by I_k the interval $[x_k, \; x_{k+1})$ and by I_k^* the one having *the same midpoint as I_k, but twice its length*:

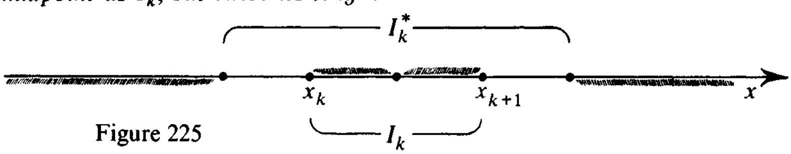

Figure 225

We then break up each separate expression

$$\int_{-\infty}^{\infty} \frac{|V_k(x)|}{x^2}\,\mathrm{d}x$$

as

$$\left(\int_{\mathbb{R} \sim I_k^*} + \int_{I_k^*} \right) \frac{|V_k(x)|}{x^2}\,\mathrm{d}x,$$

and estimate the last two integrals with the help of different techniques.

In considering the *first* one, we call on property (v) from the lemma. According to it, if $x \notin I_k^*$,

$$\begin{aligned} \frac{V_k(x)}{x^2} \quad &= \quad \int_{I_k} \frac{1}{x-t}\,\frac{\Delta(t)+\delta(t)}{t^2}\,\mathrm{d}t \\ &= \quad \int_{I_k} \left(\frac{1}{x-t} - \frac{1}{x-c} \right) \frac{\Delta(t)+\delta(t)}{t^2}\,\mathrm{d}t, \end{aligned}$$

where the constant c is arbitrary. We take c equal to the *abscissa* of I_k's *midpoint*, and thus find that

$$\int_{c+|I_k|}^{\infty} \frac{|V_k(x)|}{x^2}\,dx \quad \leqslant \quad \int_{c+|I_k|}^{\infty} \int_{I_k} \frac{|t-c|}{(x-t)(x-c)} \, \frac{|\Delta(t)+\delta(t)|}{t^2}\,dt\,dx$$

$$\leqslant \quad \int_{I_k} \frac{|\Delta(t)+\delta(t)|}{t^2}\,dt \int_{c+|I_k|}^{\infty} \frac{|I_k|/2}{(x-x_{k+1})^2}\,dx,$$

which boils down to just

$$\int_{I_k} \frac{|\Delta(t)+\delta(t)|}{t^2}\,dt \quad (!).$$

In exactly the same way, we get

$$\int_{-\infty}^{c-|I_k|} \frac{|V_k(x)|}{x^2}\,dx \quad \leqslant \quad \int_{I_k} \frac{|\Delta(t)+\delta(t)|}{t^2}\,dt,$$

so finally

$$\int_{\mathbb{R}\sim I_k^*} \frac{|V_k(x)|}{x^2}\,dx \quad \leqslant \quad 2\int_{I_k} \frac{|\Delta(t)+\delta(t)|}{t^2}\,dt.$$

To estimate the integral of $|V_k(x)|/x^2$ over I_k^*, we begin by using Schwarz:

$$\int_{I_k^*} \frac{|V_k(x)|}{x^2}\,dx \quad \leqslant \quad \sqrt{|I_k^*|}\left(\int_{I_k^*}\left(\frac{V_k(x)}{x^2}\right)^2 dx\right)^{1/2}.$$

At this point one must recognize that

$$\frac{V_k(x)}{x^2} \quad = \quad \pi\tilde{\psi}_k(x),$$

where

$$\tilde{\psi}_k(x) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{x-t}\,\psi_k(t)\,dt$$

is the *Hilbert transform* of the function

$$\psi_k(t) \quad = \quad \begin{cases} (\Delta(t)+\delta(t))/t^2, & t\in I_k, \\ 0, \quad t\notin I_k. \end{cases}$$

The L_2 theory of Hilbert transforms was discussed in the scholium at the end of §C.1, Chapter VIII, and according to that theory we have

$$\int_{-\infty}^{\infty} (\tilde{\psi}_k(x))^2\,dx \quad = \quad \int_{-\infty}^{\infty} (\psi_k(t))^2\,dt.$$

Thus,

$$\int_{-\infty}^{\infty}\left(\frac{V_k(x)}{x^2}\right)^2 \mathrm{d}x \quad = \quad \pi^2\int_{I_k}\left(\frac{\Delta(t)+\delta(t)}{t^2}\right)^2 \mathrm{d}t,$$

which, used in the previous relation, yields

$$\int_{I_k^*}\frac{|V_k(x)|}{x^2}\mathrm{d}x \quad \leqslant \quad \pi(2|I_k|)^{1/2}\left(\int_{I_k}\left(\frac{\Delta(t)+\delta(t)}{t^2}\right)^2 \mathrm{d}t\right)^{1/2}.$$

Combining this inequality with the one for the integral over $\mathbb{R}\sim I_k^*$, we get

$$\int_{-\infty}^{\infty}\frac{|V_k(x)|}{x^2}\mathrm{d}x \quad \leqslant \quad 2\int_{I_k}\frac{|\Delta(t)+\delta(t)|}{t^2}\mathrm{d}t$$

$$+ \quad \pi(2|I_k|)^{1/2}\left(\int_{I_k}\left(\frac{\Delta(t)+\delta(t)}{t^2}\right)^2 \mathrm{d}t\right)^{1/2}.$$

We can now obtain an estimate good enough for our purpose by using a very crude procedure on the right-hand integrals. We simply plug the inequality

$$|\Delta(t)+\delta(t)| \quad \leqslant \quad (D+\eta)(|I_{k-1}| \;+\; |I_k| \;+\; |I_{k+1}|), \qquad t\in I_k,$$

(property (vi) of the lemma) into each of them and find, for $k \geqslant 1$, the right side of the previous relation to be

$$\leqslant \quad (2+\sqrt{2}\,\pi)(D+\eta)\frac{(|I_{k-1}|+|I_k|+|I_{k+1}|)|I_k|}{x_k^2}$$

We therefore certainly have

$$\int_{-\infty}^{\infty}\frac{|V_k(x)|}{x^2}\mathrm{d}x \quad \leqslant \quad 7(D+\eta)\left(\frac{|I_{k-1}|+|I_k|+|I_{k+1}|}{x_k}\right)^2$$

for $k \geqslant 1$.

From this we see that

$$\sum_{k=1}^{\infty}\int_{-\infty}^{\infty}\frac{|V_k(x)|}{x^2}\mathrm{d}x \quad \leqslant \quad 7(D+\eta)\sum_{k=1}^{\infty}\left(\frac{x_{k+2}-x_{k-1}}{x_k}\right)^2.$$

We have

$$(x_{k+2}-x_{k-1})^2 \quad \leqslant \quad 3\{(x_k-x_{k-1})^2+(x_{k+1}-x_k)^2+(x_{k+2}-x_{k+1})^2\},$$

and, by property (i) of the lemma,

$$\sum_{k=1}^{\infty}\left(\frac{x_{k+1}-x_k}{x_k}\right)^2 \quad < \quad \infty,$$

so that $x_{k+1}/x_k \longrightarrow 1$ as $k \longrightarrow \infty$. The right-hand sum in the previous inequality is hence certainly convergent, and we conclude that

$$\sum_{k=1}^{\infty}\int_{-\infty}^{\infty}\frac{|V_k(x)|}{x^2}\,dx \quad < \quad \infty.$$

By the same reasoning, it is also shown that

$$\sum_{-\infty}^{-1}\int_{-\infty}^{\infty}\frac{|V_k(x)|}{x^2}\,dx \quad < \quad \infty.$$

Combination of this with the previous relation finally yields

$$\int_{-\infty}^{\infty}\frac{|U(x)+v(x)|}{1+x^2}\,dx \quad < \quad \infty$$

by the observation at the beginning of this proof. Q.E.D.

Remark. Thinking back to the remark immediately following the statement of the previous lemma, the reader acquainted with the modern theory of H_1 should recognize that in the argument just given, *exhibition of a specific atomic decomposition* for $(\Delta(t)+\delta(t))/t^2$ was used to show that that function belonged to $\Re H_1$. This reasoning was employed by Beurling and Malliavin some 13 years before Coifman brought atomic decomposition into H_p space theory* as a systematic tool in 1974. True, the work of Beurling and Malliavin was never widely circulated, and the version of it finally published by them in 1967 is very hard to understand.

It turns out that a function belongs to $\Re H_1$ *if and only if* it has an atomic decomposition like the one figuring in our proof (in general, with atoms having non-disjoint supporting intervals). Fefferman's celebrated theorem on the duality of $\Re H_1$ and BMO is easily seen to be equivalent to this result, whose *if* part was essentially verified in the course of the above reasoning. The *only if* part, which guarantees the *existence* of atomic decompositions for arbitrary functions in $\Re H_1$, is deeper. What Coifman did was to obtain a *direct proof* of that existence, and thus arrive at a new proof of the Fefferman duality theorem.

The reader who wishes to go into these matters should first look at

* For *martingale* H_1, he was preceded in this by Herz.

problem 11 on page 274 of Garnett's book* and then consult the articles referred to there.

We have now done most of the work needed to establish the

Little Multiplier Theorem (Beurling and Malliavin, 1961). *Let Σ be a sequence of (perhaps repeated) real numbers lying outside* $(-1,\ 1)$, *with*

$$\int_{-\infty}^{\infty} \frac{|n_\Sigma(t) - Dt|}{1+t^2}\,\mathrm{d}t \quad < \quad \infty$$

for some $D \geqslant 0$, and put

$$F(z) \quad = \quad \prod_{\lambda\in\Sigma}\left(1 - \frac{z}{\lambda}\right)\mathrm{e}^{z/\lambda}.$$

Given $\eta > 0$, there is a sequence S of real numbers lying outside $(-1,\ 1)$ *such that*

$$\frac{n_S(t)}{t} \longrightarrow \eta \qquad \textit{for } t \longrightarrow \pm\infty$$

and, for a suitable real number γ, the function

$$\varphi(z) \quad = \quad \mathrm{e}^{-\gamma z}\prod_{\lambda\in S}\left(1 - \frac{z}{\lambda}\right)\mathrm{e}^{z/\lambda}$$

satisfies

$$\int_{-\infty}^{\infty} \frac{\log^+|F(x)\varphi(x)|}{1+x^2}\,\mathrm{d}x \quad < \quad \infty.$$

$\varphi(z)$ is entire and of exponential type.

Remark. According to the theorem in article 1, $F(z)$ is of exponential type – so, then, is the product $F(z)\varphi(z)$. The limits of

$$\frac{\log|F(\mathrm{i}y)\varphi(\mathrm{i}y)|}{|y|}$$

for $y \longrightarrow \pm\infty$ both exist, and are equal to $\pi(D+\eta)$, *a quantity as close as we like to πD.* Because the product has a convergent logarithmic integral, one can show by the method of §B.2, Chapter VI, that in fact

$$|F(z)\varphi(z)| \quad \leqslant \quad C_\varepsilon \exp\bigl(\pi(D+\eta)|\Im z| + \varepsilon|z|\bigr)$$

* The one on bounded analytic functions – the recent publication by Garcia-Cuerva and Rubio de Francia is also (and especially!) called to the reader's attention.

for each $\varepsilon > 0$. At the same time, $F(z)\varphi(z)$ vanishes (with appropriate multiplicity) at each point of the given sequence Σ.

Proof of theorem. Fixing the number $\eta > 0$, we take the function $\delta(t)$ corresponding to it furnished by the lemma and $v(x)$, related to $\delta(t)$ as in the statement of the preceding theorem. The function $U(x)$ figuring in that result is, as we know, related to our F by the formula

$$U(x) \quad = \quad \log|F(x)| \; + \; Dx\log\left|\frac{x+1}{x-1}\right|.$$

Let us obtain a similar representation for $v(x)$.

Write

$$v(t) \quad = \quad \begin{cases} \eta t + \delta(t), & |t| \geqslant 1, \\ 0, & |t| < 1. \end{cases}$$

By property (ii) of the lemma, $v(t)$ is *increasing* on $(-\infty, \infty)$, and, by property (iii),

$$\frac{v(t)}{t} \longrightarrow \eta \qquad \text{as } t \longrightarrow \pm\infty.$$

$v(t)$ is in fact *piecewise constant*, with jump discontinuities at (and *only* at) the points x_k, $k = \pm 1, \pm 2, \ldots$ mentioned in the lemma's statement:

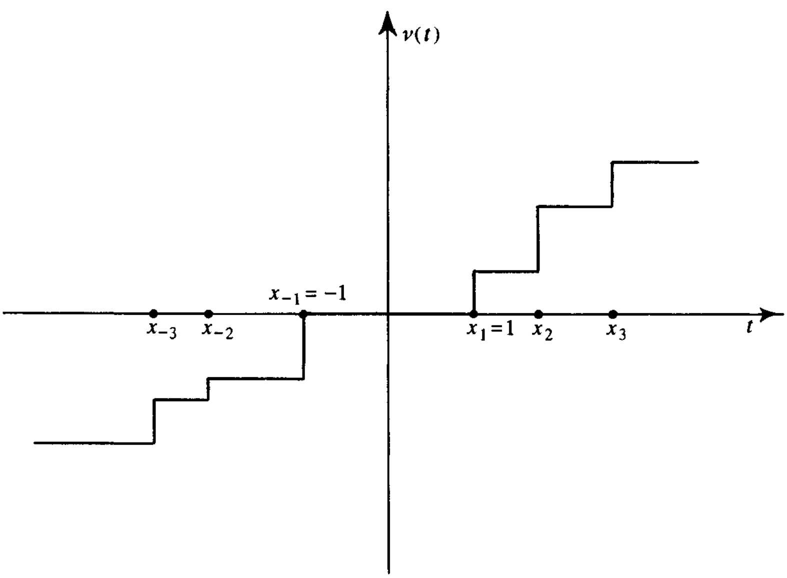

Figure 226

According to the lemma and discussion at the end of the last article, we have

$$v(x) \;=\; \int_{-\infty}^{\infty} \frac{x^2}{x-t}\,\frac{\delta(t)}{t^2}\,\mathrm{d}t \;=\; \int_{|t|\geqslant 1} \frac{x^2}{x-t}\,\frac{\nu(t)}{t^2}\,\mathrm{d}t \;+\; \eta x \log\left|\frac{x+1}{x-1}\right|$$

$$=\; \int_{-\infty}^{\infty} \left(\log\left|1-\frac{x}{t}\right| + \frac{x}{t}\right)\mathrm{d}\nu(t) \;+\; \eta x \log\left|\frac{x+1}{x-1}\right|.$$

In terms of

$$H(z) \;=\; \int_{-\infty}^{\infty} \left(\log\left|1-\frac{z}{t}\right| + \frac{\Re z}{t}\right)\mathrm{d}\nu(t),$$

we thus have our desired representation:

$$v(x) \;=\; H(x) \;+\; \eta x \log\left|\frac{x+1}{x-1}\right|.$$

Using this formula with the previous one for $U(x)$, we can reformulate the conclusion of the last theorem to get

$$\int_{-\infty}^{\infty} \frac{|\log|F(x)| + H(x)|}{1+x^2}\,\mathrm{d}x \;<\; \infty.$$

The use of

$$\int_{-\infty}^{\infty} \left(\log\left|1-\frac{z}{t}\right| + \frac{\Re z}{t}\right)\mathrm{d}[\nu(t)],$$

obviously the logarithm of the modulus of an entire function, in place of $H(z)$ comes now immediately to mind – one recalls the lemma of §A.1. (N.B. For $p \geqslant 0$, $[p]$ denotes, as usual, the greatest integer $\leqslant p$, *but when* $p < 0$, *we take* $[p]$ *as the least integer* $\geqslant p$, *so as to have* $[-p] = -[p]$.)

Here, one must be somewhat careful. The expression

$$\int_{-\infty}^{\infty} \left(\log\left|1-\frac{z}{t}\right| + \frac{\Re z}{t}\right)\mathrm{d}\nu(t)$$

is very sensitive to small changes in ν because of the term $\Re z/t$ in the integrand; replacement of $\nu(t)$ by $[\nu(t)]$ usually produces a new term linear in $\Re z = x$ which spoils the convergence of the integral involving $\log|F|$

and H. What we do have is a relation

$$\int_{-\infty}^{\infty}\left(\log\left|1-\frac{x+i}{t}\right|+\frac{x}{t}\right)(d[\nu(t)]-d\nu(t))$$
$$\leqslant\quad \gamma x \;+\; 2\log^+|x| \;+\; O(1),\qquad x\in\mathbb{R},$$

valid with a certain real constant γ.

To show this, a device from the proof of the theorem in article 1 is used.* Assuming, wlog, that $x \geqslant 0$, we observe that the *left-hand member* of the relation in question can be rewritten as

$$\int_0^{\infty}\log\left|1-\left(\frac{x+i}{t}\right)^2\right|(d[\nu(t)]-d\nu(t))$$
$$+\quad\int_0^{\infty}\left(\log\left|1+\frac{x+i}{t}\right|-\frac{x}{t}\right)(d[-\nu(-t)]+d\nu(-t)-d[\nu(t)]+d\nu(t)).$$

To estimate the *first* integral we fall back on the lemma from §A.1, according to which it is

$$\leqslant\quad \log^+|x| \;+\; O(1).$$

The *second* one we integrate by parts, remembering that $\nu(t) = 0$ for $|t| < 1$. When that is done, the integrated terms (involving the *differences* $(-\nu(-t))-[-\nu(-t)]$ and $\nu(t)-[\nu(t)]$) all vanish, leaving

$$x\int_1^{\infty}\frac{[\nu(t)]-\nu(t)+(-\nu(-t))-[-\nu(-t)]}{t^2}\,dt$$
$$+\quad\int_1^{\infty}\left(\frac{\partial}{\partial t}\log\left|1+\frac{x+i}{t}\right|\right)\{(-\nu(-t)-[-\nu(-t)]) \;-\; (\nu(t)-[\nu(t)])\}\,dt.$$

The first term here is just γx, where

$$\gamma\quad=\quad\int_1^{\infty}\frac{[\nu(t)]-\nu(t)+(-\nu(-t))-[-\nu(-t)]}{t^2}\,dt$$

(a quantity between -1 and 1). In the second term, the expression in $\{\ \}$ lies between -1 and 1, while

$$\frac{\partial}{\partial t}\log\left|1+\frac{x+i}{t}\right|\quad<\quad 0$$

* One may also work directly with the expression $\int_{|t|\geqslant 1}\{\log|1-(x+i)/t|+x/t\}d([\nu(t)]-\nu(t))$, adapting the proof of the lemma in §A.1 to the (improper) integral involving the first term in $\{\ \}$ and using partial integration on what remains.

for $1 \leqslant t < \infty$ since $x \geqslant 0$. The second term is therefore $\leqslant \log|1+x+\mathrm{i}|$ (cf. proof of the lemma in §A.1).

Putting these results together, we see that

$$\int_{-\infty}^{\infty}\left(\log\left|1-\frac{x+\mathrm{i}}{t}\right|+\frac{x}{t}\right)(\mathrm{d}[\nu(t)]-\mathrm{d}\nu(t))$$

$$\leqslant \quad \log^+|x| \ + \ O(1) \ + \ \gamma x \ + \ \log|1+x+\mathrm{i}|$$

for $x \geqslant 0$, proving the desired inequality when that is the case. A similar argument may be used when x is negative.

Having established our relation, we proceed to construct the entire function $\varphi(z)$. Take S as the sequence of points where $[\nu(t)]$ *jumps*, each of those being *repeated* a number of times equal to the *magnitude* of the *jump* corresponding to it; S simply consists of some of the points x_k, $k = \pm 1, \pm 2, \ldots$, with certain repetitions. We then put

$$\varphi(z) \quad = \quad \mathrm{e}^{-\gamma z}\prod_{\lambda\in S}\left(1-\frac{z}{\lambda}\right)\mathrm{e}^{z/\lambda},$$

taking care to repeat each of the factors on the right as many times as the λ corresponding to it is repeated in S. This function $\varphi(z)$ is entire, and clearly

$$\log|\varphi(z)| \quad = \quad -\gamma\Re z \ + \ \int_{-\infty}^{\infty}\left(\log\left|1-\frac{z}{t}\right|+\frac{\Re z}{t}\right)\mathrm{d}[\nu(t)].$$

The inequality proved above now yields

$$\log|\varphi(x+\mathrm{i})| \quad \leqslant \quad H(x+\mathrm{i}) \ + \ 2\log^+|x| \ + \ O(1), \qquad x\in\mathbb{R}.$$

Obviously, $n_S(t) \ = \ [\nu(t)]$, so

$$\frac{n_S(t)}{t} \longrightarrow \eta \quad \text{as } t \longrightarrow \pm\infty.$$

Also,

$$\int_{-\infty}^{\infty}\frac{|n_S(t)-\eta t|}{1+t^2}\mathrm{d}t \ \leqslant \ \int_{-\infty}^{\infty}\frac{|\nu(t)-\eta t|+1}{1+t^2}\mathrm{d}t \quad = \quad \int_{-\infty}^{\infty}\frac{|\delta(t)|+1}{1+t^2}\mathrm{d}t \ < \ \infty$$

by property (iv) of the lemma. The hypothesis of the theorem in article 1 therefore holds for the function $\mathrm{e}^{\gamma z}\varphi(z)$, so it – and hence $\varphi(z)$ – is of *exponential type*. That result (as well as the second Lindelöf theorem of §B, Chapter III, on which it depends) is also easily adapted so as to apply to functions like $H(z)$, and we thus find, reasoning as for $\mathrm{e}^{\gamma z}\varphi(z)$, that

$$H(z) \ \leqslant \ \text{const.}|z| \ + \ O(1).$$

$H(z)$ is, of course, *harmonic* in $\Im z > 0$. These properties of H are used to get a grip on $\log|F(x+\mathrm{i})| + H(x+\mathrm{i})$, the idea being to then make use of our relation between $\log|\varphi(x+\mathrm{i})|$ and $H(x+\mathrm{i})$.

Because $F(z)$ is of exponential type, we certainly have

$$\log|F(z)| \;+\; H(z) \quad\leqslant\quad \text{const.}|z| \;+\; O(1),$$

with the left side harmonic in $\Im z > 0$. The functions $|F(z)|$ and $e^{H(z)}$ are actually continuous right up to the real axis, as long as we take the value of the latter one to be *zero* at the points of S; moreover,

$$\int_{-\infty}^{\infty} \frac{(\log|F(x)| + H(x))_+}{1+x^2}\,dx \quad<\quad \infty$$

by the observations made at the beginning of this proof ($(a)_+$ denotes $\max(a,0)$ for real a). We can therefore use the theorem of §E, Chapter III (actually, a variant of it having, however, exactly the same proof) so as to conclude that (with an appropriate constant A)

$$\log|F(x+\mathrm{i})| \;+\; H(x+\mathrm{i}) \quad\leqslant\quad A \;+\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(\log|F(t)| + H(t))_+}{(x-t)^2+1}\,dt.$$

Now we bring in the above relation involving $|\varphi(x+\mathrm{i})|$ and $H(x+\mathrm{i})$, and get

$$\log|F(x+\mathrm{i})| \;+\; \log|\varphi(x+\mathrm{i})| \quad\leqslant\quad O(1) \;+\; 2\log^+|x|$$
$$+\quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\log|F(t)| + H(t)|}{(x-t)^2+1}\,dt,$$

from which we easily see that

$$\int_{-\infty}^{\infty} \frac{\log^+|F(x+\mathrm{i})\varphi(x+\mathrm{i})|}{x^2+1}\,dx \quad<\quad \infty,$$

following the procedure so often used in Chapter VI and elsewhere. Having arrived at this point, we may use the theorem from §E, Chapter III once more, this time in the half plane $\{\Im z \leqslant 1\}$ – the function $F(z)\varphi(z)$ *is* entire, and of exponential type. Doing that and then repeating the argument just referred to, we find that

$$\int_{-\infty}^{\infty} \frac{\log^+|F(x)\varphi(x)|}{1+x^2}\,dx \quad<\quad \infty,$$

which is what we wanted to prove.

We are done.

Remark. If the function $v(t) = \eta t + \delta(t)$ were known to be *integral valued*, $\log|\varphi(z)|$ could have been taken *equal* to $H(z)$ in the above proof, and the discussion about the effect of replacing $v(t)$ by $[v(t)]$ avoided. To realize this simplification, we would have had to modify the lemma's construction so as to make it yield a function $\delta(t)$ with $\delta(t) + \eta t$ integral valued. As a matter of fact, that can be done without too much difficulty, and one thus arrives at an alternative derivation of the preceding result. Such is the procedure followed by Redheffer in his survey article.

3. Determination of the completeness radius for real and complex sequences Λ

We are finally ready to apply the result stated in §A.2.

Theorem (Beurling and Malliavin, 1961). *Let Λ be a sequence of distinct real numbers having effective density $\tilde{D}_\Lambda < \infty$. Then the completeness radius associated with Λ is equal to $\pi\tilde{D}_\Lambda$.*

Proof. According to the discussion at the beginning of this §, it is enough to show that the $e^{i\lambda t}$, $\lambda \in \Lambda$, are not complete on any interval of length $> 2\pi\tilde{D}_\Lambda$, and for that purpose it suffices, as explained there, to establish, for arbitrary $\eta > 0$, the existence of a non-zero entire function $G(z)$ of exponential type $\leqslant \pi(\tilde{D}_\Lambda + 3\eta)$ with

$$\int_{-\infty}^{\infty} |G(x)|\,dx \quad < \quad \infty$$

and

$$G(\lambda) \;=\; 0 \quad \text{for } \lambda \in \Lambda.$$

Writing $D = \tilde{D}_\Lambda + \eta$, we take the real sequence $\Sigma \supseteq \Lambda$ such that

$$\int_{-\infty}^{\infty} \frac{|n_\Sigma(t) - Dt|}{1+t^2}\,dt \quad < \quad \infty,$$

used in article 1 at the start of our constructions. We then *throw away any points that Σ may have in $(-1, \; 1)$, so as to ensure that $n_\Sigma(t) = 0$ there.** This perhaps leaves us with a certain finite number of $\mu \in \Lambda$ *not belonging to* Σ (the points of Λ in $(-1, \; 1)$); those will be taken care of in a moment.

We next form

$$F(z) \;=\; \prod_{\substack{\lambda\in\Sigma \\ \lambda\notin(-1,1)}} \left(1 - \frac{z}{\lambda}\right) e^{z/\lambda}$$

as in article 1, and use the little multiplier theorem from article 2 to get

* That does not affect the preceding relation!

the non-zero entire function $\varphi(z)$ of exponential type described there, such that

$$\int_{-\infty}^{\infty} \frac{\log^+|F(x)\varphi(x)|}{1+x^2}\,dx \quad < \quad \infty.$$

As remarked just after the statement of the little multiplier theorem (and as one checks immediately), when $y \longrightarrow \pm\infty$,

$$\frac{\log|F(iy)\varphi(iy)|}{|y|} \quad \longrightarrow \quad \pi(D+\eta) \quad = \quad \pi(\tilde{D}_\Lambda + 2\eta).$$

Put now

$$F_0(z) \quad = \quad F(z)\prod_{\substack{\mu\in\Lambda\\ -1<\mu<1}}(z-\mu).$$

The relations just written obviously still hold with F_0 standing in place of F. The *theorem on the multiplier* enunciated in §A.2 therefore gives us a non-zero entire function $\psi(z)$ of exponential type $\leqslant \pi\eta$ with

$$|F_0(x)\varphi(x)\psi(x)| \leqslant \text{const.}, \qquad x\in\mathbb{R}.$$

In view of the previous relation, we clearly have

$$\limsup_{y\to\pm\infty} \frac{\log|F_0(iy)\varphi(iy)\psi(iy)|}{|y|} \quad \leqslant \quad \pi(\tilde{D}_\Lambda + 3\eta),$$

so the *boundedness* of $F_0\varphi\psi$ on the real axis implies that that product is *of exponential type* $\leqslant \pi(\tilde{D}_\Lambda + 3\eta)$ by the *third* Phragmén–Lindelöf theorem from §C of Chapter III.

The function $\varphi(z)$ furnished by the little multiplier theorem has a zero at each point of the real sequence S, with

$$\frac{n_S(t)}{t} \longrightarrow \eta \quad \text{for } t\longrightarrow \pm\infty.$$

We can thus certainly take *two different points* $s_1, s_2 \in S$ (!), and $\varphi(z)/(z-s_1)(z-s_2)$ will still be entire. Let, finally,

$$G(z) \quad = \quad \frac{F_0(z)\varphi(z)\psi(z)}{(z-s_1)(z-s_2)}.$$

This function is entire and of exponential type $\leqslant \pi(\tilde{D}_\Lambda + 3\eta)$, and

$$\int_{-\infty}^{\infty} |G(x)|\,dx \quad < \quad \infty.$$

Also, $G(z)$ vanishes at each point of Λ since $F_0(z)$ does. We are done.

Remark. If Λ has repeated points, the argument just made goes through without change. The entire function $G(z)$ thus obtained then vanishes with appropriate multiplicitly at each point of Λ.

In order to now obtain the completeness radius for sequences of *complex* numbers Λ, we proceed somewhat as in §H.3 of Chapter III.

Notation. For complex λ *with non-zero real part,* we write

$$\lambda' = 1/\Re(1/\lambda).$$

If Λ is any sequence of complex numbers, we let Λ' be the real sequence consisting of the λ' corresponding to the $\lambda \in \Lambda$ having non-zero real part. Should several members of Λ correspond to the *same* value for λ', we look on that value as *repeated an appropriate number of times* in Λ'.

We then have the

Theorem (Beurling and Malliavin, 1967). *Let* Λ *be any sequence of distinct complex numbers. If*

$$\sum_{\substack{\lambda\in\Lambda\\ \lambda\neq 0}} \frac{|\Im\lambda|}{|\lambda|^2} = \infty,$$

the exponentials $e^{i\lambda t}$, $\lambda \in \Lambda$, *are complete on any interval of finite length.*

Otherwise, they are complete on any interval of length $< 2\pi\tilde{D}_{\Lambda'}$, *and, if that quantity is finite, incomplete on any interval of length* $> 2\pi\tilde{D}_{\Lambda'}$.

Proof. If the $e^{i\lambda t}$ are incomplete on (say) the interval $[-L, L]$, there is (as at the beginning of §D, Chapter IX) a non-zero measure μ on $[-L, L]$ with

$$\int_{-L}^{L} e^{i\lambda t}\, d\mu(t) = 0, \qquad \lambda \in \Lambda.$$

The non-zero entire function

$$G(z) = \int_{-L}^{L} e^{izt}\, d\mu(t)$$

of exponential type $\leqslant L$ thus vanishes at each point of Λ, so, since G is bounded on the real axis, we have

$$\sum_{\substack{\lambda\in\Lambda\\ \lambda\neq 0}} \frac{|\Im\lambda|}{|\lambda|^2} < \infty$$

by §G.3 of Chapter III.

Let Σ denote the complete sequence of zeros of G (with repetitions according to multiplicities, as usual). The Hadamard representation for G is then

$$G(z) \quad = \quad Az^p e^{cz} \prod_{\substack{\mu \in \Sigma \\ \mu \neq 0}} \left(1 - \frac{z}{\mu}\right) e^{z/\mu}.$$

Denote by S the difference set

$$\Sigma \sim \{\lambda \in \Lambda:\ \Re\lambda \neq 0\};$$

then

$$G(z) \quad = \quad Az^p e^{cz} \prod_{\substack{\mu \in S \\ \mu \neq 0}} \left(1 - \frac{z}{\mu}\right) e^{z/\mu} \cdot \prod_{\substack{\lambda \in \Lambda \\ \Re\lambda \neq 0}} \left(1 - \frac{z}{\lambda}\right) e^{z/\lambda}.$$

Take now the function

$$G_0(z) \quad = \quad Az^p e^{cz} \prod_{\substack{\mu \in S \\ \mu \neq 0}} \left(1 - \frac{z}{\mu}\right) e^{z/\mu} \cdot \prod_{\lambda' \in \Lambda'} \left(1 - \frac{z}{\lambda'}\right) e^{z/\lambda}$$

(with exponentials $e^{z/\lambda}$ and *not* $e^{z/\lambda'}$ in the second product!). By work done in §H.3 of Chapter III, we see that $G_0(z)$ is of exponential type, and that

$$|G_0(x)| \ \leqslant\ |G(x)|, \qquad x \in \mathbb{R},$$

so that $G_0(x)$ is bounded on the real axis (like $G(x)$).

Write

$$B \quad = \quad \limsup_{y \to \infty} \frac{\log|G_0(iy)|}{y}$$

and

$$B' \quad = \quad \limsup_{y \to -\infty} \frac{\log|G_0(iy)|}{|y|}.$$

Observe also that

$$\log|G(z)| \quad \leqslant \quad L|\Im z| \ + \ O(1)$$

by the third Phragmén–Lindelöf theorem of §C, Chapter III, since G is of exponential type $\leqslant L$ and bounded on the real axis.

Apply now Levinson's theorem (Chapter III, §H.3) to the zero distribution for $G_0(z)$, and then use Jensen's formula on the one for $G(z)$ together with the estimate just written for the latter function. The two zero distributions are the same asymptotically,* so, by an argument just

* Refer to volume I, pp. 74–5.

like the one at the end of §H.3, Chapter III, it is found that

$$\frac{B+B'}{2} \leqslant L.$$

Once this is known, we have by §D of Chapter IX,

$$\pi\tilde{D}_{\Lambda'} \leqslant \frac{B+B'}{2} \leqslant L,$$

since $G_0(z)$ vanishes* at the points of $\Lambda' \subseteq \mathbb{R}$. *Incompleteness* of the $e^{i\lambda t}$, $\lambda \in \Lambda$, on $[-L, L]$ thus implies that $L \geqslant \pi\tilde{D}_{\Lambda'}$, and those exponentials must therefore be complete on any interval of length $< 2\pi\tilde{D}_{\Lambda'}$.

We must now show that if

$$\sum_{\substack{\lambda\in\Lambda \\ \lambda\neq 0}} \frac{|\Im\lambda|}{|\lambda|^2} < \infty$$

and $\tilde{D}_{\Lambda'} < \infty$, the $e^{i\lambda x}$, $\lambda \in \Lambda$, are *incomplete* on any interval of length $> 2\pi\tilde{D}_{\Lambda'}$. Fix any $\eta > 0$. The previous theorem and remark then give us an entire function $f(z) \not\equiv 0$ of exponential type $\leqslant \pi(\tilde{D}_{\Lambda'} + 3\eta)$, vanishing† at each point of Λ', and such that (wlog)

$$|f(x)| \leqslant 1, \qquad x \in \mathbb{R}.$$

Denote by Ξ the set of *non-zero, purely imaginary* $\mu \in \Lambda$, and then put

$$g(z) = f(z)\cdot\prod_{\lambda'\in\Lambda'}\left(\frac{1-z/\lambda}{1-z/\lambda'}\right)\cdot\prod_{\mu\in\Xi}\left(1-\frac{z}{\mu}\right).$$

Using the two Lindelöf theorems of §B, Chapter III we easily see that $g(z)$ is of exponential type, thanks to the convergence of the above sum of the $|\Im\lambda|/|\lambda|^2$. $g(z)$ vanishes at each $\lambda \in \Lambda$ (save that at the origin, in case $0 \in \Lambda$). By calculations like one made in §H.3, Chapter III, we also verify without difficulty that

$$\log\prod_{\lambda'\in\Lambda'}\left|\frac{1-iy/\lambda}{1-iy/\lambda'}\right| \leqslant o(|y|)$$

and

$$\log\prod_{\mu\in\Xi}\left|1-\frac{iy}{\mu}\right| \leqslant o(|y|)$$

* with the appropriate multiplicity at repeated points of Λ', which may well *have* some, although Λ does not

† with the appropriate multiplicity

as $y \longrightarrow \pm\infty$, the second on account of the convergence of $\sum_{\mu\in\Xi} 1/|\mu|$. Therefore, since $f(z)$ is of exponential type $\leqslant \pi(\tilde{D}_{\Lambda'} + 3\eta)$, we have

$$\limsup_{y\to\pm\infty} \frac{\log|g(iy)|}{|y|} \quad \leqslant \quad \pi(\tilde{D}_{\Lambda'} + 3\eta).$$

It is now claimed that

$$\int_{-\infty}^{\infty} \frac{\log^+|g(x)|}{1+x^2}\,dx \quad < \quad \infty.$$

Because $|f(x)| \leqslant 1$, this will follow from the convergence of

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} \log^+ \left| \prod_{\mu\in\Xi} \left(1 - \frac{x}{\mu}\right) \right| dx$$

and of

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} \log^+ \left| \prod_{\lambda'\in\Lambda'} \left(\frac{1 - x/\lambda}{1 - x/\lambda'}\right) \right| dx.$$

Using the pure imaginary character of the $\mu \in \Xi$ and the relation $1/\lambda' = \Re(1/\lambda)$, we see at once that the factors $1 - x/\mu$, $\mu \in \Xi$, $(1 - x/\lambda)/(1 - x/\lambda')$, $\lambda' \in \Lambda'$, are all in modulus $\geqslant 1$ for real x. *The* $\log^+$ *may therefore be replaced by* log *in these integrals.*

Once this is done, the resulting expressions are easily worked out explicitly using Poisson's formula for a half plane. In that way we find the *first* integral to be equal to

$$\pi \sum_{\mu\in\Xi} \log\left(1 + \frac{1}{|\mu|}\right)$$

and the *second* to be

$$\leqslant \quad \pi \sum_{\lambda'\in\Lambda'} \log\left(1 + \frac{|\Im\lambda|}{|\lambda|^2}\right).$$

Both of these sums, however, are *finite* since $\sum_{\substack{\lambda\in\Lambda \\ \lambda\neq 0}} |\Im\lambda|/|\lambda|^2 < \infty$. Convergence of the logarithmic integral involving g is thus established.

Thanks to that convergence, *the theorem from* §A.2 *applies to our entire function* g. There is, in other words, a non-zero entire function $\psi(z)$ of exponential type $\leqslant \pi\eta$ with $g(x)\psi(x)$ bounded on $\mathbb{R}$; reasoning as at the end of the preceding theorem's proof we can even choose ψ so as to have

$$\int_{-\infty}^{\infty} |xg(x)\psi(x)|\,dx \quad < \quad \infty.$$

Referring to the above estimate on $\log|g(iy)|/|y|$ and applying the usual Phragmén–Lindelöf theorem, we see that $zg(z)\psi(z)$ is of exponential type

$\leqslant \pi(\tilde{D}_{\Lambda'} + 4\eta)$. This function certainly vanishes at each $\lambda \in \Lambda$, so the $e^{i\lambda t}$, $\lambda \in \Lambda$, are *not* complete on $[-\pi(\tilde{D}_{\Lambda'} + 4\eta),\ \pi(\tilde{D}_{\Lambda'} + 4\eta)]$ according to the discussion at the beginning of this §.

The theorem is completely proved.

Problem 39

Let Λ be any sequence of distinct complex numbers. Show that the completeness radius associated with Λ is equal to π times the *infimum* of the numbers $c > 0$ with the following property: there exist *distinct integers* n_λ corresponding to the different non-zero λ in Λ such that

$$\sum_{\substack{\lambda\in\Lambda \\ \lambda\neq 0}} \left| \frac{1}{\lambda} - \frac{c}{n_\lambda} \right| \leqslant \infty.$$

This criterion is due to Redheffer. (Hint: Look again at the constructions in §D.2 of Chapter IX.)

C. The multiplier theorem for weights with uniformly continuous logarithms

The result of Beurling and Malliavin enunciated in §A.2 is broader in scope than may appear at first sight. One can, for instance, deduce from it another multiplier theorem for weights fulfilling a simple descriptive regularity condition. This is done in article 1 below; the work depends on some elementary material from Chapter VI and the first part of Chapter VII.

In article 2, the theorem of article 1 is used to extend a result obtained in problem 11 (Chapter VII, §A.2) to certain unbounded measures on $\mathbb{R}$.

1 The multiplier theorem

Theorem (Beurling and Malliavin, 1961). Let $W(x) \geqslant 1$, *and let* $\log W(x)$ *be uniformly continuous* on* $\mathbb{R}$. *Then* W *admits multipliers iff*

* Beurling and Malliavin require only that $\omega(s) = \operatorname{ess\,sup}_{x\in\mathbb{R}} |\log W(x+s) - \log W(x)|$ be *finite* for a set of $s \in \mathbb{R}$ having positive Lebesgue measure. To reduce the treatment under this less stringent assumption to that of the uniform Lip 1 case handled below, they observe that there must be some $M < \infty$ with $\omega(s) \leqslant M$ on a Lebesgue measurable set E with $|E| > 0$. But then $E - E$ includes a whole interval $(-h, h)$, $h > 0$, so $\omega(s) \leqslant 2M$ for $|s| < h$, ω being clearly *even* and *subadditive*. From this point, one proceeds as in the text, passing from W to W_h; the only changes are in the constants.

This argument is valid as long as $W(x) \geqslant 1$ is Lebesgue measurable, for then

$$\omega(s) = \lim_{p\to\infty} \left(\int_{-\infty}^{\infty} |\log W(x+s) - \log W(x)|^p e^{-2|x|}\, dx \right)^{1/p}$$

is also Lebesgue measurable (the *integrals* are by Tonelli's theorem, $\log W(x+s) - \log W(x)$ being Lebesgue measurable on $\mathbb{R}^2$).

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2}\,dx \quad < \quad \infty.$$

Remark. The weaker assumption that $\log\log W(x)$ is uniformly continuous on $\mathbb{R}$ *does not imply* that W admits multipliers when the above integral is convergent. For an example, see the following chapter.

Proof of theorem. As explained at the beginning of §A, convergence of the integral in question is certainly *necessary if* W is to admit multipliers; we therefore need only concern ourselves with the *sufficiency* of that convergence in the present circumstances.

We may, to begin with, replace the hypothesis of *uniform continuity* for $\log W(x)$ by the stronger one that

$$|\log W(x) - \log W(x')| \;\leqslant\; C|x-x'| \qquad \text{for } x, x' \in \mathbb{R},$$

i.e., that $\log W$ be *uniformly* Lip 1 on $\mathbb{R}$. Indeed, the former property gives us a fixed $h > 0$ such that

$$|\log W(x) - \log W(x')| \;\leqslant\; 1 \quad \text{whenever } |x-x'| \;\leqslant\; h.$$

Take any smooth positive function φ supported on $[-h, h]$ with

$$\int_{-h}^{h} \varphi(t)\,dt \;=\; 1,$$

and define a new weight $W_h(x)$ by putting

$$\log W_h(x) \;=\; \int_{-h}^{h} (\log W(x-t))\varphi(t)\,dt \;=\; \int_{-\infty}^{\infty} \varphi(x-s)\log W(s)\,ds.$$

Then, by our choice of h,

$$\log W(x) \;-\; 1 \;\leqslant\; \log W_h(x) \;\leqslant\; \log W(x) \;+\; 1.$$

Again,

$$\frac{d\log W_h(x)}{dx} \;=\; \int_{-\infty}^{\infty} \varphi'(x-s)\log W(s)\,ds \;=\; \int_{-h}^{h} \varphi'(t)\log W(x-t)\,dt.$$

Here, since $\varphi(-h) \;=\; \varphi(h) \;=\; 0$,

$$\int_{-h}^{h} \varphi'(t)\log W(x)\,dt \;=\; 0,$$

so

$$\frac{d\log W_h(x)}{dx} \;=\; \int_{-h}^{h} \varphi'(t)\big(\log W(x-t) - \log W(x)\big)\,dt.$$

By the choice of h, the integral on the right is in absolute value

$$\leqslant \quad \int_{-h}^{h} |\varphi'(t)|\,dt \quad = \quad C, \quad \text{say,}$$

so

$$\left|\frac{d \log W_h(x)}{dx}\right| \quad \leqslant \quad C, \qquad x \in \mathbb{R},$$

and $\log W_h(x)$ satisfies the Lipschitz condition written above.

We also have

$$e^{-1} W(x) \;\leqslant\; W_h(x) \;\leqslant\; e\, W(x)$$

by the previous estimate, so

$$\int_{-\infty}^{\infty} \frac{\log W_h(x)}{1+x^2}\,dx \quad < \quad \infty$$

provided that the corresponding integral with W is finite. If, then, we can conclude that $W_h(x)$ admits multipliers, the left-hand half of the preceding double inequality shows that $W(x)$ also does so, and it is enough to establish the theorem for weights W with $\log W$ uniformly Lip 1 on $\mathbb{R}$.

Assuming henceforth this Lipschitz condition on $\log W(x)$ and the convergence of the corresponding logarithmic integral, we set out to show that W admits multipliers. Our idea is to produce an entire function $K(z)$ of exponential type such that

$$\int_{-\infty}^{\infty} \frac{\log^+ |K(x)|}{1+x^2}\,dx \quad < \quad \infty,$$

while

$$4K(x) \quad \geqslant \quad (W(x))^{\alpha} \qquad \text{for } x \in \mathbb{R}$$

with a certain constant $\alpha > 0$. Application of the theorem from §A.2 to $K(z)$ will then yield multipliers for W.

Following a procedure of Akhiezer used in Chapters VI and VII, we form the new weight

$$W_1(x) \;=\; \sup\{|f(x)|:\ f \text{ entire, of exponential type } \leqslant 1, \text{ bounded on } \mathbb{R}, \text{ and } |f(t)/W(t)| \leqslant 1 \text{ on } \mathbb{R}\}.$$

If $\log W(x)$ satisfies the Lipschitz condition written above (with Lipschitz constant C) we have, by the *first* theorem of §A.1, Chapter VII,

$$W_1(x) \quad \geqslant \quad \tfrac{1}{2}(W(x))^{1/\sqrt{(C^2+1)}} \qquad \text{for } x \in \mathbb{R}.$$

What we want, then, is an entire function $K(z)$ of exponential type with convergent logarithmic integral, such that

$$K(x) \geqslant (W_1(x))^2,$$

say, for $x \in \mathbb{R}$.

In order to obtain $K(z)$, we use Akhiezer's theory of weighted approximation by sums of exponentials, presented in Chapter VI. We work, however, with a weighted L_2 norm instead of the weighted uniform one used there. Taking

$$\Omega(x) = (1 + x^2)^{1/2} W(x),$$

let us consider approximation by finite linear combinations of the $e^{i\lambda x}$, $-1 \leqslant \lambda \leqslant 1$, in the norm $\| \quad \|_{\Omega,2}$ defined by

$$\|g\|_{\Omega,2} = \sqrt{\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\left|\frac{g(t)}{\Omega(t)}\right|^2 dt\right)}.$$

According to our assumed convergence of the logarithmic integral involving W, we have

$$\int_{-\infty}^{\infty} \frac{\log \Omega(x)}{1+x^2} dx < \infty.$$

Hence, by a version of T. Hall's theorem (the *first* one of §D, Chapter VI) appropriate to approximation in the norm $\| \quad \|_{\Omega,2}$ (see §§E.2 and G of Chapter VI), linear combinations of the $e^{i\lambda x}$, $-1 \leqslant \lambda \leqslant 1$, are *not* $\| \quad \|_{\Omega,2}$ dense in the space of functions for which that norm is finite. This, and the Akhiezer theorem (Chapter VI, §E.2) corresponding to the norm $\| \quad \|_{\Omega,2}$ (see again §G, Chapter VI) imply that

$$\int_{-\infty}^{\infty} \frac{\log \Omega_1(x)}{1+x^2} dx < \infty,$$

where

$$\Omega_1(x) = \sup\{|f(x)| : f \text{ entire, of exponential type } \leqslant 1, \\ \text{bounded on } \mathbb{R}, \text{ and } \|f\|_{\Omega,2} \leqslant 1\}.$$

Observe now that for any function f with $|f(t)/W(t)| \leqslant 1$ on $\mathbb{R}$, we certainly have $\|f\|_{\Omega,2} \leqslant 1$. Clearly, then,

$$\Omega_1(x) \geqslant W_1(x),$$

and, if we can show that $(\Omega_1(x))^2$ *coincides with an entire function of exponential type on the real axis*, we can simply take the latter as the function K we are seeking.

For that purpose we resort to a simple general argument. The space of Lebesgue measurable functions with finite $\| \quad \|_{\Omega,2}$ norm is certainly *separable*, so, since the entire functions of exponential type $\leqslant 1$ bounded on $\mathbb{R}$ *belong* to that space, we may choose a (countable!) *sequence* of those which is $\| \quad \|_{\Omega,2}$ *dense in the collection of all of them.* Using the inner product

$$\langle f, g \rangle_\Omega \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(x)\overline{g(x)}}{(\Omega(x))^2}\,dx,$$

one then applies *Schmidt's orthogonalization procedure* to that dense sequence, obtaining, after normalization, a sequence of entire functions $\varphi_n(z)$ of exponential type $\leqslant 1$, bounded on $\mathbb{R}$, with $\|\varphi_n\|_{\Omega,2} = 1$ and

$$\langle \varphi_n, \varphi_m \rangle_\Omega \; = \; 0 \quad \text{if } n \neq m.$$

Finite linear combinations of these φ_n are also $\| \quad \|_{\Omega,2}$ dense in the collection of all such entire functions.

Fixing any $x_0 \in \mathbb{R}$ and any N, we look at the finite linear combinations

$$S(x) \quad = \quad \sum_{k=1}^{N} a_k \varphi_k(x)$$

such that $\|S\|_{\Omega,2} \leqslant 1$, seeking the one which makes $|S(x_0)|$ *a maximum.* Since the φ_k are orthonormal with respect to $\langle \;,\; \rangle_\Omega$, the condition on $\|S\|_{\Omega,2}$ is equivalent to

$$\sum_{k=1}^{N} |a_k|^2 \quad \leqslant \quad 1,$$

so, by Schwarz' inequality,

$$|S(x_0)| \quad \leqslant \quad \sqrt{\left(\sum_{k=1}^{N} |\varphi_k(x_0)|^2\right)}.$$

For proper choice of the coefficients a_k, the two sides are the same; the *maximum value* of $|S(x_0)|$ for the sums S is *thus equal to the quantity on the right.*

Put

$$K_N(z) \quad = \quad \sum_{k=1}^{N} \varphi_k(z)\overline{\varphi_k(\bar{z})};$$

this function is entire, of exponential type $\leqslant 2$ (*sic* !), and bounded on the real axis, where it is also $\geqslant 0$. As we have just seen, for each given $x \in \mathbb{R}$, the maximum value of $|S(x)|$ for sums S of the kind just specified is equal to $\sqrt{(K_N(x))}$. It is now claimed that when $N \longrightarrow \infty$, the $K_N(z)$ converge u.c.c to a certain entire function $K(z)$ of exponential type $\leqslant 2$, and that

$$K(x) = (\Omega_1(x))^2, \qquad x \in \mathbb{R}.$$

Given any particular N, we have, for each of the sums S,

$$|S(x)| \leqslant \Omega_1(x)$$

by *definition* of the function on the right, so

$$0 \leqslant K_N(x) \leqslant (\Omega_1(x))^2.$$

As we already know,

$$\int_{-\infty}^{\infty} \frac{\log(\Omega_1(x))^2}{1+x^2}\,dx < \infty.$$

Therefore, since the $K_N(z)$ are of exponential type $\leqslant 2$, the last relation implies that they all satisfy a *uniform estimate* of the form

$$|K_N(z)| \leqslant C_\varepsilon \exp(2|\Im z| + \varepsilon|z|), \qquad z \in \mathbb{C}.$$

Here $\varepsilon > 0$ is arbitrary, and C_ε depends on it, but is *completely independent* of N. The statement just made is nothing other than an *adaptation*, to approximation in the norm $\|\ \ \|_{\Omega,2}$, of the *fourth* theorem in §E.2, Chapter VI, proved by the familiar Akhiezer argument of §B.2 in that chapter.

By the estimate just found, the K_N form a normal family in the complex plane, and any convergent sequence of them tends to an entire function for which the same estimate holds. However, $K_{N+1}(x) \geqslant K_N(x)$ on $\mathbb{R}$, so the entire sequence of the K_N is *already* convergent, and

$$K(z) = \lim_{N\to\infty} K_N(z)$$

is an entire function, obviously of exponential type $\leqslant 2$.

We still have to prove that

$$K(x) = (\Omega_1(x))^2$$

on $\mathbb{R}$. Of course, $0 \leqslant K(x) \leqslant (\Omega_1(x))^2$ since each $K_N(x)$ has that property, and it suffices to show the reverse inequality. Take any $x_0 \in \mathbb{R}$, and choose an entire function $f(z)$ of exponential type $\leqslant 1$, bounded on $\mathbb{R}$, with $\|f\|_{\Omega,2} \leqslant 1$ and at the same time $|f(x_0)|$ *close to* $\Omega_1(x_0)$. By our choice of the φ_n, the orthogonal series development

$$\sum_k \langle f, \varphi_k\rangle_\Omega \, \varphi_k(x)$$

converges in norm $\| \quad \|_{\Omega,2}$ to $f(x)$. For the partial sums

$$P_N(x) = \sum_{k=1}^{N} \langle f, \varphi_k \rangle_\Omega \varphi_k(x)$$

we have, however,

$$\|P_N\|_{\Omega,2} \leqslant \|f\|_{\Omega,2} \leqslant 1,$$

so by definition,

$$|P_N(x)| \leqslant \Omega_1(x), \quad x \in \mathbb{R}.$$

Hence, since the P_N are of exponential type $\leqslant 1$, another application of our version of the fourth theorem from §E.2, Chapter VI, gives us the uniform estimate

$$|P_N(z)| \leqslant \tilde{C}_\varepsilon \exp(|\Im z| + \varepsilon|z|), \qquad z \in \mathbb{C},$$

on them. (Again, $\varepsilon > 0$ is arbitrary and $\tilde{C}_\varepsilon$ depends on it, but is independent of N.) The function $f(z)$ of course satisfies the same kind of estimate, and u.c.c convergence of the $P_N(z)$ to $f(z)$ now follows from the relation

$$\|f - P_N\|_{\Omega,2} \xrightarrow[N]{} 0$$

by a simple normal family argument. We see in particular that

$$P_N(x_0) \xrightarrow[N]{} f(x_0).$$

Since $\|f\|_{\Omega,2} \leqslant 1$, however,

$$|P_N(x_0)| \leqslant \sqrt{\left(\sum_{k=1}^{N} |\langle f, \varphi_k \rangle_\Omega|^2\right)} \sqrt{(K_N(x_0))} \leqslant \sqrt{(K_N(x_0))},$$

so

$$\sqrt{(K_N(x_0))} \geqslant |f(x_0)| - \varepsilon$$

with arbitrary $\varepsilon > 0$ for large enough N. Thence,

$$\sqrt{(K(x_0))} \geqslant |f(x_0)|.$$

But we chose f with $|f(x_0)|$ *close* to $\Omega_1(x_0)$ – indeed, *as close as we like*. Finally, then,

$$\sqrt{(K(x_0))} \geqslant \Omega_1(x_0),$$

whence

$$K(x) = (\Omega_1(x))^2, \qquad x \in \mathbb{R},$$

the reverse inequality having already been noted.

We are at this point essentially done. The entire function $K(z)$ of exponential type $\leqslant 2$ satisfies the relation just written. We have

$$\Omega_1(x) \;\geqslant\; W_1(x) \;\geqslant\; \tfrac{1}{2}(W(x))^{1/\sqrt{(C^2+1)}}$$

on $\mathbb{R}$, where $W(x) \geqslant 1$, so $4K(x) \geqslant 1$, $x \in \mathbb{R}$, and it follows from

$$\int_{-\infty}^{\infty} \frac{\log \Omega_1(x)}{1+x^2}\,dx \;<\; \infty$$

that

$$\int_{-\infty}^{\infty} \frac{\log^+ K(x)}{1+x^2}\,dx \;<\; \infty.$$

The theorem of Beurling and Malliavin from §A.2 now gives us, for any $\eta > 0$, an entire function $\psi(z) \not\equiv 0$ of exponential type $\leqslant \eta$ with

$$4K(x)|\psi(x)| \;\leqslant\; 1, \qquad x \in \mathbb{R},$$

i.e.,

$$|\psi(x)|(W(x))^{2/\sqrt{(C^2+1)}} \;\leqslant\; 1, \qquad x \in \mathbb{R}.$$

Taking any fixed integer m with

$$\frac{1}{m} \;<\; \frac{2}{\sqrt{(C^2+1)}},$$

we get

$$W(x)|(\psi(x))^m| \;\leqslant\; 1 \quad \text{on } \mathbb{R}.$$

Here $(\psi(z))^m$ is entire, of exponential type $\leqslant m\eta$, and not identically zero. Hence $W(x)$ admits multipliers, $\eta > 0$ being arbitrary. The theorem is proved.

Remark. Beurling and Malliavin did not derive this result from their theorem stated in §A.2. Instead, they gave an independent proof similar to the one furnished by them for the latter result. See the end of §C.5 in Chapter XI.

2. A theorem of Beurling

In order to indicate the location and extent of the intervals on $\mathbb{R}$ where a complex measure μ has little or no mass, Beurling, in his Stanford lectures, used a function $\sigma(x)$ related to μ by the formula

$$e^{-\sigma(x)} \;=\; \int_{-\infty}^{\infty} e^{-|x-t|}\,|d\mu(t)|.$$

(Truth to tell, Beurling wrote $\sigma(x)$ where we write $-\sigma(x)$. Some of the formulas used in working with this function look a little simpler if the minus sign is taken in the exponent as we do here.)

For *finite* measures μ, $\sigma(x)$ is bounded below – $\sigma(x)$ is *positive* if $\int_{\mathbb{R}}|d\mu(t)| \leqslant 1$. *Large* values of $\sigma(x)$ then correspond to the abscissae near which μ has *very little mass.* In problem 11 (§A.2, Chapter VII) the reader was asked to show that if the function $\sigma(x)$ associated with a finite complex measure μ is *so large that*

$$\int_{-\infty}^{\infty}\frac{\sigma(x)}{1+x^2}dx \quad = \quad \infty,$$

then the Fourier–Stieltjes transform

$$\hat{\mu}(\lambda) \quad = \quad \int_{-\infty}^{\infty} e^{i\lambda t}\,d\mu(t)$$

cannot vanish over any interval of positive length without μ's vanishing identically. Beurling originally established the gap theorem in §A.2, Chapter VII, with the help of this result, which is also due to him.

The multiplier theorem from the preceding article may be used to show that the result quoted is, in a certain sense, *best possible.* This application, set as problem 40, may be found at the end of the present article. Right now, we have in mind another application of that multiplier theorem, namely, *Beurling's extension of his result to certain unbounded complex measures μ.* This is also from his Stanford lectures.

When μ is *unbounded*, we can still define $\sigma(x)$ by means of the formula

$$e^{-\sigma(x)} \quad = \quad \int_{-\infty}^{\infty} e^{-|x-t|}\,|d\mu(t)|$$

as long as we admit the possibility that $\sigma(x) = -\infty$. In the case, however, that $\sigma(x) > -\infty$ for *any value* of x, *it is* $> -\infty$ *for all.* The reason for this is that $\sigma(x)$, if it is $> -\infty$ anywhere on $\mathbb{R}$, *is uniformly* Lip 1 there. To see that, we need only note that

$$|x'-t| \quad \geqslant \quad |x-t| \; - \; |x-x'|, \qquad t\in\mathbb{R},$$

whence

$$\int_{-\infty}^{\infty} e^{-|x'-t|}|d\mu(t)| \quad \leqslant \quad e^{|x'-x|}\int_{-\infty}^{\infty} e^{-|x-t|}\,|d\mu(t)|,$$

and

$$\sigma(x') \quad \geqslant \quad \sigma(x) \; - \; |x'-x|.$$

Interchanging x and x', we find that

$$\boxed{|\sigma(x') - \sigma(x)| \leqslant |x' - x|,}$$

a relation used several times in the following discussion.

Let us consider an unbounded μ for which $\sigma(x) > -\infty$. In this more general situation, $\sigma(x)$ is usually *not* bounded below (as it was for finite μ), and we need to look separately at

$$\sigma^+(x) = \max(\sigma(x), 0)$$

and

$$\sigma^-(x) = -\min(\sigma(x), 0) \qquad (sic!).$$

Since $\sigma(x)$ is uniformly Lip 1, *so are* $\sigma^+(x)$ *and* $\sigma^-(x)$.

The functions σ^+ and σ^- serve different purposes. *Large values of* $\sigma^+(x)$ correspond (as in the case of $\sigma(x)$ when dealing with finite measures) to the abscissae near which μ *has very little mass*. $\sigma^-(x)$, on the other hand, is *large near the places where* μ *has a great deal of mass*. With unbounded μ, one expects to come upon more and more such places (where $\sigma^-(x)$ assumes ever larger values) as x goes out to $+\infty$ or $-\infty$ along the real axis.

Beurling considered unbounded measures μ having *growth limited in such a way as to make*

$$\boxed{\int_{-\infty}^{\infty} \frac{\sigma^-(x)}{1+x^2}\,dx < \infty.}$$

Lemma. *Under the boxed condition on* σ^-, $\sigma^-(x)$ *is* $o(|x|)$ *for* $x \longrightarrow \pm\infty$.

Proof. Let $0 < c < 1$, and suppose that for any large x_0, we have

$$\sigma^-(x_0) \geqslant 2cx_0.$$

Then, by the Lip 1 property of σ^-,

$$\sigma^-(x) \geqslant cx_0 \quad \text{for } (1-c)x_0 \leqslant x \leqslant (1+c)x_0,$$

so

$$\int_{(1-c)x_0}^{(1+c)x_0} \frac{\sigma^-(x)}{x^2}\,dx \geqslant \frac{2c^2}{(1+c)^2}.$$

If the boxed relation holds, this cannot happen for arbitrarily large values of x_0, and $\sigma^-(x)$ must be $o(|x|)$ for $x \longrightarrow \infty$. Similarly for $x \longrightarrow -\infty$.

Lemma

$$\int_{x-1}^{x+1} e^{-\sigma^-(t)}|d\mu(t)| \quad \leqslant \quad e^2.$$

Proof. By definition,

$$e^{-\sigma(x)} \quad \geqslant \quad \int_{x-1}^{x+1} e^{-|x-t|}|d\mu(t)| \quad \geqslant \quad e^{-1}\int_{x-1}^{x+1}|d\mu(t)|.$$

This holds *a fortiori* if $\sigma(x)$ is replaced by $-\sigma^-(x) \leqslant \sigma(x)$. The Lip 1 property of σ^- now makes

$$\sigma^-(t) \quad \geqslant \quad \sigma^-(x) - 1, \qquad x-1 \leqslant t \leqslant x+1,$$

so we have

$$\int_{x-1}^{x+1} e^{-\sigma^-(t)}|d\mu(t)| \quad \leqslant \quad e\int_{x-1}^{x+1} e^{-\sigma^-(x)}|d\mu(t)| \quad \leqslant \quad e^2.$$

Done.

From these two lemmas we see that if the function σ^- corresponding to an unbounded complex measure μ fulfills the above boxed condition,

$$\int_{-\infty}^{\infty} e^{-\delta|t|}|d\mu(t)|$$

is convergent for every $\delta > 0$. In that circumstance, *the Fourier–Stieltjes transforms*

$$\hat{\mu}_\delta(\lambda) \quad = \quad \int_{-\infty}^{\infty} e^{i\lambda t}e^{-\delta|t|}\,d\mu(t)$$

are available. The $\hat{\mu}_\delta$ are nothing but the Abel means frequently used in harmonic analysis to try to give meaning to the expression

$$\int_{-\infty}^{\infty} e^{i\lambda t}\,d\mu(t)$$

when the integral is not absolutely convergent. It is often possible to interpret the latter as a limit (in some sense) of the $\hat{\mu}_\delta(\lambda)$ as $\delta \longrightarrow 0$ for certain (or sometimes even all) values of $\lambda \in \mathbb{R}$. We have examples of such treatment in the first lemma and theorem of §H.1, Chapter VI.

Definition. If $\int_{-\infty}^{\infty} e^{-\delta|t|}|d\mu(t)| < \infty$ for each $\delta > 0$, we say that "$\hat{\mu}(\lambda)$" ($\hat{\mu}(\lambda)$ itself is in general not defined!) vanishes on a closed interval $[a, b]$ of $\mathbb{R}$ provided that

$$\hat{\mu}_\delta(\lambda) \longrightarrow 0 \textit{ uniformly } \text{for } a \leqslant \lambda \leqslant b$$

as $\delta \longrightarrow 0$.

In terms of this notion, Beurling's extension of the result established in problem 11 takes the following form:

Theorem (Beurling) *Let μ be a complex-valued Radon measure on $\mathbb{R}$ for which*

$$\int_{-\infty}^{\infty} \frac{\sigma^-(x)}{1+x^2} dx \quad < \quad \infty,$$

but at the same time,

$$\int_{-\infty}^{\infty} \frac{\sigma^+(x)}{1+x^2} dx \quad = \quad \infty,$$

σ^- and σ^+ being the functions related to μ in the manner described above. If also "$\hat{\mu}(\lambda)$" vanishes on an interval of positive length, μ is identically zero.

Remark. According to the previous discussion, the condition on $\sigma^-(x)$ means that μ, although (perhaps) unbounded, *does not accumulate too much mass* anywhere. The one involving $\sigma^+(x)$ means that there are also *large parts* of $\mathbb{R}$ where μ has *very little mass.*

Proof of theorem. Let, wlog, "$\hat{\mu}(\lambda)$" vanish on $[-A, A]$, where $A > 0$. The function

$$\sigma^-(x) \;+\; \log(1+x^2)$$

is uniformly Lip 1 on $\mathbb{R}$, so, *by the theorem of the preceding article*, the integral condition on σ^- makes it possible for us to get a non-zero entire function $f(z)$ of exponential type $a < A$ such that

$$|f(x)| \quad \leqslant \quad \frac{e^{-\sigma^-(x)}}{1+x^2}.$$

From this, the second of the above lemmas yields

$$\int_{x-1}^{x+1} |f(t)|\,|d\mu(t)| \quad \leqslant \quad \frac{1}{1+(|x|-1)^2}\int_{x-1}^{x+1} e^{-\sigma^-(t)}|d\mu(t)|$$

$$\leqslant \quad \frac{e^2}{1+(|x|-1)^2} \qquad \text{for } |x| \;\geqslant\; 1,$$

so by summation over integer values of x, we get

$$\int_{-\infty}^{\infty} |f(t)||d\mu(t)| \quad < \quad \infty,$$

making

$$d\nu(t) \;=\; f(t)d\mu(t)$$

a totally finite measure on $\mathbb{R}$.

To ν we now apply the result from problem 11. Write

$$e^{-\tau(x)} \quad = \quad \int_{-\infty}^{\infty} e^{-|x-t|}|d\nu(t)|;$$

$\tau(x)$ is just the analogue of the function $\sigma(x)$ corresponding to the finite measure ν. Without loss of generality, $\int_{-\infty}^{\infty}|d\nu(t)| \leqslant 1$, so $\tau(x) \geqslant 0$. Also, $|f(x)| \leqslant 1$, so $|d\nu(t)| \leqslant |d\mu(t)|$ and

$$e^{-\tau(x)} \quad \leqslant \quad e^{-\sigma(x)},$$

i.e., $\tau(x) \geqslant \sigma(x)$. Combining this with the previous inequality, we get

$$\tau(x) \;\geqslant\; \sigma^{+}(x),$$

whence

$$\int_{-\infty}^{\infty} \frac{\tau(x)}{1+x^2}dx \quad = \quad \infty$$

by hypothesis.

It is now claimed that

$$\int_{-\infty}^{\infty} e^{i\lambda_0 t}d\nu(t) \;=\; 0 \quad \text{for } |\lambda_0| \;\leqslant\; A-a.$$

By the Paley–Wiener theorem, we have

$$f(t) \quad = \quad \int_{-a}^{a} e^{it\lambda}\varphi(\lambda)d\lambda,$$

where φ is (under the present circumstances) a continuous function on $[-a,\ a]$. Therefore,

$$\begin{aligned}\int_{-\infty}^{\infty} e^{i\lambda_0 t}d\nu(t) \quad &= \quad \lim_{\delta\to 0}\int_{-\infty}^{\infty} e^{-\delta|t|}e^{i\lambda_0 t}f(t)d\mu(t) \\ &= \quad \lim_{\delta\to 0}\int_{-\infty}^{\infty}\int_{-a}^{a} e^{i\lambda_0 t}e^{it\lambda}\varphi(\lambda)e^{-\delta|t|}d\lambda\, d\mu(t) \\ &= \quad \lim_{\delta\to 0}\int_{-a}^{a}\int_{-\infty}^{\infty} e^{-\delta|t|}e^{i(\lambda_0+\lambda)t}d\mu(t)\varphi(\lambda)\,d\lambda;\end{aligned}$$

here, for each $\delta > 0$, absolute convergence holds throughout. The last limit is just

$$\lim_{\delta\to 0}\int_{-a}^{a} \varphi(\lambda)\hat{\mu}_\delta(\lambda+\lambda_0)\mathrm{d}x$$

which is, however, *zero* when $|\lambda_0| \leqslant A-a$ since then $\hat{\mu}_\delta(\lambda+\lambda_0)\longrightarrow 0$ uniformly for $|\lambda| \leqslant a$ as $\delta\longrightarrow 0$.

The claim just established and the integral condition on $\tau(x)$ *now make* $\nu \equiv 0$ *by problem* 11. That is,

$$f(x)\mathrm{d}\mu(x) \equiv 0.$$

If the function f vanishes at all on $\mathbb{R}$, it does so only at certain points x_n isolated from each other, for f is *entire* and *not identically zero.* What we have just proved is that μ, *if not identically zero*, has all its mass distributed on the points x_n. Then there must be *one* of those points, say x_0, for which

$$\mu(\{x_0\}) \neq 0.$$

That, however, *cannot happen.* If, for instance, x_0 is a k-fold zero of $f(z)$, we may *repeat the above argument using the entire function*

$$f_0(z) = \frac{f(z)}{(z-x_0)^k}$$

instead of f; doing so, we then find that

$$f_0(t)\mathrm{d}\mu(t) \equiv 0.$$

Since $f_0(x_0) \neq 0$, we thus have

$$\mu(\{x_0\}) = 0,$$

a contradiction.

The measure μ must hence vanish identically. We are done.

Problem 40

Show that the result from problem 11 is best possible in the following sense:

If μ is a finite complex measure and

$$e^{-\sigma(x)} = \int_{-\infty}^{\infty} e^{-|x-t|}|\mathrm{d}\mu(t)|,$$

then, in the case that

$$\int_{-\infty}^{\infty} \frac{\sigma(x)}{1+x^2}\,dx \quad < \quad \infty,$$

there *is* a finite non-zero complex measure ν on $\mathbb{R}$ such that

$$\int_{-\infty}^{\infty} e^{-|x-t|}|d\nu(t)| \quad \leqslant \quad e^{-\sigma(x)}$$

but $\hat{\nu}(\lambda) \;\equiv\; 0$ outside some finite interval.

(Hint. One takes $d\nu(t) = f(t)dt$ where $f(t)$ is an entire function of exponential type chosen to satisfy $|f(x)| \;\leqslant\; e^{-\sigma(x)}/\pi(1+x^2)$.)

Yet another application of the result from article 1 is found near the end of Louis de Branges' book.

D. Poisson integrals of certain functions having given weighted quadratic norms

A condition involving the existence of multipliers is encountered when one desires to estimate certain harmonic functions whose boundary data are controlled by weighted norms. As a very simple example, let us consider the problem of estimating

$$U(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\,U(t)\,dt, \qquad \Im z \;>\; 0,$$

when it is known that

$$\int_{-\infty}^{\infty} |U(t)|^2 w(t)\,dt \quad \leqslant \quad 1$$

with some given function $w(t) \;\geqslant\; 0$ belonging to $L_1(\mathbb{R})$. We may, if we like, require that

$$|U(t)| \;\leqslant\; \text{some } M \quad \text{for } t \in \mathbb{R},$$

where M is *unknown and beyond our control.* Is it possible, in these circumstances, to say anything about the magnitude of $|U(z)|$?

If the bounded function $U(t)$ is permitted to be *arbitrary*, a simple condition on w is both necessary and sufficient for the existence of an estimate on $U(z)$. Then, we may wlog take $z \;=\; i$, and rewrite $\pi U(i)$ as

$$\int_{-\infty}^{\infty} \frac{1}{w(t)(t^2+1)}\,U(t)w(t)\,dt.$$

The very rudiments of analysis now tell us that this integral is bounded for

$$\int_{-\infty}^{\infty} |U(t)|^2 w(t)\mathrm{d}t \quad \leqslant \quad 1$$

if and only if

$$\int_{-\infty}^{\infty} \frac{1}{w(t)(t^2+1)^2}\mathrm{d}t \quad < \quad \infty.$$

It is for such w, then, and only for them, that the estimate in question (with arbitrary z having $\Im z > 0$) is available.

The situation alters when we restrict the *spectrum* of the functions $U(t)$ under consideration. In order not to get bogged down here in questions of harmonic analysis not really germane to the matter at hand, let us simply say that we look at *arbitrary finite sums*

$$S(t) \quad = \quad \sum_{\lambda\in\Sigma} A_\lambda e^{i\lambda t}$$

with some prescribed closed $\Sigma \subseteq \mathbb{R}$ — the *spectrum* for those sums. When $\Sigma = \mathbb{R}$, such sums are of course w^* dense in $L_\infty(\mathbb{R})$, because an L_1 function whose Fourier transform is everywhere zero must vanish identically. In that case we may think crudely of the collection of sums S as filling out the set of bounded functions U 'for all practical purposes', and our problem boils down to the simple one with the solution just described. It is thus natural to ask what happens when $\Sigma \neq \mathbb{R}$, and the simplest situation in which this occurs is the one where $\mathbb{R} \sim \Sigma$ *consists of one finite interval.* Then, we may take the complementary interval to be symmetric about 0, and it is possible to describe completely the functions w for which estimates of the above kind on the sums $S(t)$ exist. The description is in terms of multipliers.

Theorem. *Let $w \geqslant 0$ belong to $L_1(\mathbb{R})$ and let $a > 0$. A necessary and sufficient condition for the existence of a β, $\Im\beta > 0$, and corresponding constant C_β, such that*

$$\left|\int_{-\infty}^{\infty} \frac{\Im\beta}{|t-\beta|^2} S(t)\mathrm{d}t\right| \quad \leqslant \quad C_\beta\sqrt{\left(\int_{-\infty}^{\infty} |S(t)|^2 w(t)\mathrm{d}t\right)}$$

for all finite sums

$$S(t) \quad = \quad \sum_{|\lambda|\geqslant a} A_\lambda e^{i\lambda t},$$

is that there exist a non-zero entire function $\varphi(t)$ of exponential type $\leqslant a$

which makes

$$\int_{-\infty}^{\infty} \frac{|\varphi(t)|^2}{w(t)(t^2+1)^2}\,\mathrm{d}t \quad < \quad \infty.$$

Proof*: *Necessity. It is convenient to work with the Hilbert space norm

$$\|f\| \quad = \quad \sqrt{\left(\int_{-\infty}^{\infty} |f(t)|^2 w(t)\,\mathrm{d}t\right)}$$

and corresponding inner product

$$\langle f, g\rangle \quad = \quad \int_{-\infty}^{\infty} f(t)\overline{g(t)}w(t)\,\mathrm{d}t.$$

Assuming, then, that for some β, $\Im\beta > 0$, we have

$$\left|\int_{-\infty}^{\infty} \frac{\Im\beta}{|\beta-t|^2} S(t)\,\mathrm{d}t\right| \quad \leqslant \quad C_\beta \|S\|$$

for all sums S of the given form, there must, by the Hahn–Banach theorem*, be a measurable $k(t)$ for which

$$\|k\| \quad < \quad \infty$$

and

$$\int_{-\infty}^{\infty} \frac{\Im\beta}{|\beta-t|^2} S(t)\,\mathrm{d}t \quad = \quad \langle S, k\rangle$$

for such S. Taking just $S(t) = \mathrm{e}^{\mathrm{i}\lambda t}$ with $|\lambda| \geqslant a$, we find that

$$\int_{-\infty}^{\infty} \left(w(t)\overline{k(t)} \; - \; \frac{\Im\beta}{|\beta-t|^2}\right)\mathrm{e}^{\mathrm{i}\lambda t}\,\mathrm{d}t \quad = \quad 0$$

for such λ.

This shows, to begin with, that $w(t)\overline{k(t)}$ *is certainly not a.e. zero.* Since $\|k\| < \infty$ and $w \in L_1(\mathbb{R})$, we see by Schwarz' inequality that $w(t)\overline{k(t)} \in L_1(\mathbb{R})$. According to the last relation, then, the Fourier transform of the integrable function

$$w(t)\overline{k(t)} \; - \; \frac{\Im\beta}{|\beta-t|^2}$$

vanishes outside $[-a, a]$, so the latter must coincide a.e. on the real axis with $\psi(t)$, where ψ *is an entire function of exponential type* $\leqslant a$.

* or rather that theorem's special and elementary version for *Hilbert space*

We have

$$w(t)\overline{k(t)} \quad = \quad \psi(t) \; + \; \frac{\Im\beta}{(t-\beta)(t-\bar{\beta})} \qquad \text{a.e., } t \in \mathbb{R}.$$

Here,

$$\varphi(t) \quad = \quad \psi(t)(t-\beta)(t-\bar{\beta}) \; + \; \Im\beta$$

is *also* entire and of exponential type $\leqslant a$, and

$$\varphi(t) \quad = \quad w(t)\overline{k(t)}\,|t-\beta|^2 \qquad \text{a.e., } t \in \mathbb{R},$$

so $\varphi \not\equiv 0$. Also,

$$\int_{-\infty}^{\infty} \frac{|\varphi(t)|^2}{w(t)|t-\beta|^4}\,dt \quad = \quad \|k\|^2 \quad < \quad \infty,$$

whence

$$\int_{-\infty}^{\infty} \frac{|\varphi(t)|^2}{w(t)(t^2+1)^2}\,dt \quad < \quad \infty.$$

Sufficiency. We continue to use the norm symbol $\|\quad\|$ introduced above with the same meaning as before. Suppose there *is* a non-zero entire function φ of exponential type $\leqslant a$ such that

$$\int_{-\infty}^{\infty} \frac{|\varphi(t)|^2}{w(t)(t^2+1)^2}\,dt \quad < \quad \infty.$$

One may, to begin with, *exclude* the case where $\varphi(t)$ is *constant*, for then we would have

$$\int_{-\infty}^{\infty} \frac{dt}{w(t)(t^2+1)^2} \quad < \quad \infty,$$

making

$$\left|\int_{-\infty}^{\infty} \frac{\Im\beta}{|t-\beta|^2}\,U(t)\,dt\right| \quad \leqslant \quad C_\beta \|U\|$$

for *any* β, $\Im\beta > 0$, and *all* bounded U, by the discussion at the beginning of this §.

One may also take $\varphi(t)$ to be *real-valued* on $\mathbb{R}$. Indeed,

$$\varphi(z) \quad = \quad \frac{\varphi(z)+\overline{\varphi(\bar{z})}}{2} \; + \; \frac{\varphi(z)-\overline{\varphi(\bar{z})}}{2},$$

and *one* of the two functions on the right must be $\not\equiv 0$. *Both* are entire and of exponential type $\leqslant a$, and, on the *real axis*, the *first* coincides with $\Re\varphi(t)$ and the *second* with $\mathrm{i}\Im\varphi(t)$. The *first* one, or else the *second* one divided by i, will thus do the job.

Let $\Im\beta > 0$. Then

$$\frac{\varphi(t)}{(t-\beta)(t-\bar{\beta})} \quad = \quad \frac{\varphi(t) \; - \; \dfrac{\varphi(\beta)}{\beta-\bar{\beta}}(t-\bar{\beta}) \; - \; \dfrac{\varphi(\bar{\beta})}{\bar{\beta}-\beta}(t-\beta)}{(t-\beta)(t-\bar{\beta})}$$

$$+ \quad \frac{\varphi(\beta)}{(\beta-\bar{\beta})(t-\beta)} \quad + \quad \frac{\varphi(\bar{\beta})}{(\bar{\beta}-\beta)(t-\bar{\beta})}.$$

The first term on the right, $\psi(t)$, is a certain *entire function of exponential type* $\leqslant a$, and, after multiplying by $\beta-\bar{\beta}$ and collecting terms, we get

$$\frac{(\Im\beta)\varphi(t)}{|t-\beta|^2} \quad = \quad (\Im\beta)\psi(t) \quad + \quad \frac{1}{2\mathrm{i}}\left(\frac{\varphi(\beta)}{t-\beta} \; - \; \frac{\varphi(\bar{\beta})}{t-\bar{\beta}}\right),$$

that is,

$$\frac{(\Im\beta)\varphi(t)}{|t-\beta|^2} \quad = \quad (\Im\beta)\psi(t) \quad + \quad (\Im\varphi(\beta))\frac{t-\Re\beta}{|t-\beta|^2} \quad + \quad (\Re\varphi(\beta))\frac{\Im\beta}{|t-\beta|^2},$$

since $\varphi(\bar{\beta}) \; = \; \overline{\varphi(\beta)}$, $\;\varphi$ being real on $\mathbb{R}$.

Suppose we can choose β *in such a way that* $\Im\varphi(\beta) \; = \; 0$ *but* $\Re\varphi(\beta) \; \neq \; 0$. *Then*

$$\left|\int_{-\infty}^{\infty}\frac{\Im\beta}{|t-\beta|^2}S(t)\,\mathrm{d}t\right| \quad \leqslant \quad C_\beta\|S\|$$

for the sums

$$S(t) \quad = \quad \sum_{|\lambda|\geqslant a} A_\lambda \mathrm{e}^{\mathrm{i}\lambda t}.$$

Indeed,

$$\int_{-\infty}^{\infty}\frac{|\varphi(t)|}{|t-\beta|^2}\mathrm{d}t \quad \leqslant \quad \sqrt{\left(\int_{-\infty}^{\infty}\frac{|\varphi(t)|^2}{w(t)|t-\beta|^4}\mathrm{d}t\int_{-\infty}^{\infty}w(t)\,\mathrm{d}t\right)},$$

a finite quantity, so the *left side* of the last identity is in $L_1(\mathbb{R})$. On the *right side*, the term

$$(\Im\varphi(\beta))\frac{t-\Re\beta}{|t-\beta|^2}$$

is *absent*, so, since $(\Re\varphi(\beta))\cdot\Im\beta/|t-\beta|^2$ is in $L_1(\mathbb{R})$, *so is* $\psi(t)$. ψ being *entire and of exponential type* $\leqslant a$, we have, however,

$$\int_{-\infty}^{\infty}\psi(t)S(t)\mathrm{d}t \quad = \quad 0$$

for each of the sums S. Therefore, keeping in mind that $\Im\varphi(\beta) = 0$, we have for the latter

$$(\Re\varphi(\beta))\int_{-\infty}^{\infty}\frac{\Im\beta}{|t-\beta|^2}S(t)\mathrm{d}t \quad = \quad (\Im\beta)\int_{-\infty}^{\infty}\frac{\varphi(t)}{|t-\beta|^2}S(t)\,\mathrm{d}t.$$

The right side is in modulus

$$\leqslant \quad \Im\beta\sqrt{\left(\int_{-\infty}^{\infty}\frac{|\varphi(t)|^2}{w(t)|t-\beta|^4}\mathrm{d}t\right)}\cdot\|S\|,$$

so we are *done* as long as $\Re\varphi(\beta) \neq 0$.

We need therefore only show that there are β, $\Im\beta > 0$, with $\Im\varphi(\beta) = 0$ but $\Re\varphi(\beta) \neq 0$. It is claimed in the first place that there are β, $\Im\beta > 0$, with $\Im\varphi(\beta) = 0$. *Otherwise*, the harmonic function $\Im\varphi(z)$ *would be of one sign*, say $\Im\varphi(z) \geqslant 0$, for $\Im z > 0$. In such case, the *second* theorem of §F.1, Chapter III, gives us a number $\alpha \geqslant 0$ and a positive measure μ on $\mathbb{R}$ with

$$\Im\varphi(z) \quad = \quad \alpha\Im z \;+\; \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2}\mathrm{d}\mu(t), \qquad \Im z > 0.$$

Our function $\Im\varphi(z)$ is continuous right up to $\mathbb{R}$, φ being entire, so we readily see in the usual way that

$$\mathrm{d}\mu(t) \quad = \quad (\Im\varphi(t))\mathrm{d}t.$$

In our circumstances, however, $\Im\varphi(t) \equiv 0$, since we took $\varphi(t)$ to be *real on* $\mathbb{R}$. Hence $\Im\varphi(z) = \alpha\Im z$ for $\Im z > 0$ and finally

$$\varphi(z) \quad = \quad \alpha z + C.$$

Here, *we cannot have* $\alpha > 0$. For, if that were so, we would get

$$\int_{-\infty}^{\infty}\frac{|\alpha t+C|^2}{w(t)(t^2+1)^2}\mathrm{d}t \quad = \quad \int_{-\infty}^{\infty}\frac{|\varphi(t)|^2}{w(t)(t^2+1)^2}\mathrm{d}t \quad < \quad \infty,$$

whence, for A larger than $|C/\alpha|$,

$$\int_{A}^{\infty}\frac{\mathrm{d}t}{w(t)(t^2+1)} \quad < \quad \infty.$$

From this, however, it would follow that

$$\int_A^\infty \frac{dt}{\sqrt{(t^2+1)}} \quad \leqslant \quad \sqrt{\left(\int_A^\infty \frac{dt}{w(t)(t^2+1)} \int_A^\infty w(t)\,dt\right)} \quad < \quad \infty,$$

which is nonsense.

Thus, $\alpha = 0$ and $\varphi(z)$ reduces to a constant C. This possibility was, however, excluded at the very beginning of the present argument – *our* φ is *not constant*. The function $\Im\varphi(z)$, then, *cannot be of one sign* in $\Im z > 0$, and we *have* points β in that half plane for which $\Im\varphi(\beta) = 0$.

Take any one of those – call it β_0. Since $\Im\varphi(z)$ is harmonic (everywhere!), we have

$$\int_{-\pi}^{\pi} \Im\varphi(\beta_0 + \rho e^{i\vartheta})d\vartheta \quad = \quad 2\pi\Im\varphi(\beta_0) \quad = \quad 0$$

for $\rho > 0$, and there must be a point on *each circle about* β_0 where $\Im\varphi$ also vanishes. There is thus a sequence of points $\beta_n \neq \beta_0$ in the upper half plane with $\beta_n \xrightarrow[n]{} \beta_0$ and $\Im\varphi(\beta_n) = 0$ for each n. If now $\Re\varphi(\beta_n)$ also vanished for each n, we would have $\varphi(z) \equiv 0$. But $\varphi \not\equiv 0$. Hence,

$$\Im\varphi(\beta_n) \;=\; 0 \quad \text{but } \Re\varphi(\beta_n) \;\neq\; 0$$

for some n, and, taking that β_n as our β, we have what was needed. The *sufficiency* of our condition on w is thus established.

We are done.

When dealing with the harmonic functions

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} S(t)\,dt,$$

one usually needs estimates on them in the *whole* upper half plane, and not just for *certain values* of z therein. The availability of these for sums $S(t)$ like those figuring in the last theorem with

$$\int_{-\infty}^{\infty} |S(t)|^2 w(t)\,dt \quad \leqslant \quad 1$$

is governed by a different condition on w.

Theorem. *Let* $w(t) \geqslant 0$ *belong to* $L_1(\mathbb{R})$, *and let* $a > 0$. *In order that the finite sums*

$$S(t) \quad = \quad \sum_{|\lambda| \geqslant a} A_\lambda e^{i\lambda t}$$

satisfy a relation

$$\left|\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} S(t)\mathrm{d}t\right| \quad \leqslant \quad K_z \sqrt{\left(\int_{-\infty}^{\infty} |S(t)|^2 w(t)\mathrm{d}t\right)}$$

for every z, $\Im z > 0$, *it is necessary and sufficient that there exist a non-zero entire function* ψ *of exponential type* $\leqslant a$ *such that*

$$\int_{-\infty}^{\infty} \frac{|\psi(t)|^2}{w(t)(t^2+1)}\mathrm{d}t \quad < \quad \infty.$$

When this condition is met, the numbers K_z *can be taken to be bounded above on compact subsets of* $\{\Im z > 0\}$.

Proof: Sufficiency. If there *is* such a function ψ, we have, for any complex β,

$$\frac{\psi(t)}{t-\beta} \quad = \quad f_\beta(t) \; + \; \frac{\psi(\beta)}{t-\beta},$$

with

$$f_\beta(t) \quad = \quad \frac{\psi(t)-\psi(\beta)}{t-\beta}$$

entire and of exponential type $\leqslant a$. As long as $\beta \notin \mathbb{R}$, we may take $\psi(\beta)$ to be $\neq 0$. Indeed, we may assume that both $|\psi(\beta)|$ and $|\psi(\bar{\beta})|$ are bounded away from zero when β ranges over any compact subset E of $\{\Im z > 0\}$. To see this, observe that ψ has at most a finite number of zeros on $E \cup E^*$, where E^* is the reflection of E in $\mathbb{R}$. Calling those $z_1, z_2, \ldots, z_n$ (repetitions according to multiplicities, as usual), we may work with

$$\psi_E(t) \quad = \quad \frac{\psi(t)}{(t-z_1)(t-z_2)\cdots(t-z_n)}$$

instead of ψ. This function is entire, of exponential type $\leqslant a$, and bounded away from zero in modulus on $E \cup E^*$. And

$$\int_{-\infty}^{\infty} \frac{|\psi_E(t)|^2}{w(t)(t^2+1)}\mathrm{d}t \quad < \quad \infty$$

because none of the z_k, $1 \leqslant k \leqslant n$, are on the real axis.*

All this being granted, we fix a β with $\Im\beta \neq 0$ and look at the entire function f_β figuring in the above relation. It is claimed that $f_\beta(t)$ is in

* $\psi(z)$ may *need* to vanish at some points on the real axis in order to offset certain zeros that the given weight w might have there! See the scholium at the end of this §.

$L_2(\mathbb{R})$. We have

$$\int_{-\infty}^{\infty} \frac{|\psi(t)|}{|t-\beta|}\mathrm{d}t \quad \leqslant \quad \sqrt{\left(\int_{-\infty}^{\infty} \frac{|\psi(t)|^2}{w(t)|t-\beta|^2}\mathrm{d}t \int_{-\infty}^{\infty} w(t)\mathrm{d}t\right)} \quad < \quad \infty,$$

i.e., $|\psi(t)|/|t-\beta|$ is in $L_1(\mathbb{R})$. This ratio is also *bounded* on the real axis.

Indeed, if ψ had no zeros at all its Hadamard factorization would reduce to $\psi(t) = Ce^{\gamma t}$ with constants C and γ. Then, however, $|\psi(t)|/|t-\beta|$ could not be integrable over $\mathbb{R}$. Consequently, ψ *has* a zero, say at z_0, and then $\psi(t)/(t-z_0)$ is *entire, of exponential type* $\leqslant a$, and in $L_1(\mathbb{R})$ since $\psi(t)/(t-\beta)$ is. A simple version of the Paley–Wiener theorem (Chapter III, §D) now shows that

$$\frac{\psi(t)}{t-z_0} \quad = \quad \int_{-a}^{a} e^{it\lambda}p(\lambda)\,\mathrm{d}\lambda,$$

with (here) $p(\lambda)$ some *continuous* function on $[-a, a]$. By this formula, we see at once that $\psi(t)/(t-z_0)$ is bounded on $\mathbb{R}$ – so, then, is $\psi(t)/(t-\beta)$.

The ratio $|\psi(t)|/|t-\beta|$ is thus both in $L_1(\mathbb{R})$ and bounded on $\mathbb{R}$. Therefore it is in $L_2(\mathbb{R})$. So, however, is $\psi(\beta)/(t-\beta)$. The difference

$$f_\beta(t) \quad = \quad \frac{\psi(t)}{t-\beta} - \frac{\psi(\beta)}{t-\beta}$$

must hence also be square integrable.

Because f_β is entire and of exponential type $\leqslant a$, we now have

$$\underset{A\to\infty}{\mathrm{l.i.m.}} \int_{-A}^{A} e^{i\lambda t} f_\beta(t)\,\mathrm{d}t \quad = \quad 0$$

for almost all $\lambda \notin [-a, a]$ by the L_2 form of the Paley–Wiener theorem (Chapter III, §D). At the same time, when $A \longrightarrow \infty$, the integrals

$$\int_{-A}^{A} e^{i\lambda t} f_\beta(t)\,\mathrm{d}t$$

tend, for $\lambda \neq 0$, to a certain function of λ *continuous* on $\mathbb{R} \sim \{0\}$. This, indeed, is certainly true if, in those integrals, we replace $f_\beta(t)$ by $\psi(t)/(t-\beta) \in L_1(\mathbb{R})$. *Direct verification* shows that the same holds good when $f_\beta(t)$ is replaced by $\psi(\beta)/(t-\beta)$. The statement therefore holds for the difference $f_\beta(t)$ of these functions.

The (continuous) pointwise limit of the expressions

$$\int_{-A}^{A} e^{i\lambda t} f_\beta(t)\,dt, \qquad \lambda \neq 0,$$

for $A \longrightarrow \infty$ must, however, coincide a.e. with their limit in mean, known to be zero a.e. for $|\lambda| \geqslant a$, as we have just seen. Hence

$$\lim_{A\to\infty} \int_{-A}^{A} e^{i\lambda t} f_\beta(t)\,dt = 0, \qquad |\lambda| \geqslant a.$$

That is,

$$\lim_{A\to\infty} \int_{-A}^{A} \frac{\psi(\beta)}{t-\beta} e^{i\lambda t}\,dt = \int_{-\infty}^{\infty} \frac{\psi(t)e^{i\lambda t}}{(t-\beta)}\,dt$$

for $|\lambda| \geqslant a$, and finally,

$$\lim_{A\to\infty} \int_{-A}^{A} \frac{\psi(\beta)S(t)}{t-\beta}\,dt = \int_{-\infty}^{\infty} \frac{\psi(t)S(t)}{t-\beta}\,dt$$

for each of our sums $S(t)$.

Let us continue to write $\| \;\|$ for the norm appearing in the proof of the preceding theorem. In terms of this notation, we have for the modulus of the *right-hand* member of the last relation the upper bound

$$\sqrt{\left(\int_{-\infty}^{\infty} \frac{|\psi(t)|^2}{w(t)|t-\beta|^2}\,dt\right)} \cdot \|S\|.$$

The condition that

$$\int_{-\infty}^{\infty} \frac{|\psi(t)|^2}{w(t)(t^2+1)}\,dt < \infty$$

clearly makes the square root $\leqslant$ a quantity C_β, *bounded above* when β ranges over *compact subsets of* $\mathbb{C} \sim \mathbb{R}$. We thus have

$$\left|\lim_{A\to\infty} \int_{-A}^{A} \frac{S(t)}{t-\beta}\,dt\right| \leqslant \frac{C_\beta}{|\psi(\beta)|}\|S\|$$

for the sums S, when $\Im\beta > 0$. In like manner,

$$\left|\lim_{A\to\infty} \int_{-A}^{A} \frac{S(t)}{t-\bar{\beta}}\,dt\right| \leqslant \frac{C_{\bar{\beta}}}{|\psi(\bar{\beta})|}\|S\|,$$

so finally, since

$$\frac{1}{t-\beta} - \frac{1}{t-\bar{\beta}} = \frac{2i\Im\beta}{|t-\beta|^2},$$

$$\left|\int_{-\infty}^{\infty} \frac{\Im\beta}{|t-\beta|^2} S(t)\,dt\right| \leqslant K_\beta \|S\|.$$

If $E \subseteq \{\Im z > 0\}$ is compact,

$$K_\beta = \frac{C_\beta}{2|\psi(\beta)|} + \frac{C_{\bar{\beta}}}{2|\psi(\bar{\beta})|}$$

may be taken to be bounded above on E, since, as explained at first, we can choose ψ with $|\psi(\beta)|$ and $|\psi(\bar{\beta})|$ bounded away from 0 on E. Sufficiency is proved.

Necessity. Suppose that for *every* β, $\Im\beta > 0$, the last inequality (at the end of the preceding discussion) *holds*, with some finite K_β. Then, by the previous theorem, we certainly have a non-zero entire function φ of exponential type $\leqslant a$, with

$$\int_{-\infty}^{\infty} \frac{|\varphi(t)|^2}{w(t)(t^2+1)^2}\,dt < \infty.$$

As we saw at the beginning of the *sufficiency* part of that theorem's proof, we may take $\varphi(t)$ to be *real* on $\mathbb{R}$.

Let $\Im\beta \neq 0$. We have an identity

$$\frac{(\Im\beta)\varphi(t)}{|t-\beta|^2} = (\Im\beta)g_\beta(t) + (\Im\varphi(\beta))\frac{t-\Re\beta}{|t-\beta|^2} + (\Re\varphi(\beta))\frac{\Im\beta}{|t-\beta|^2}$$

like the one used in establishing the preceding theorem, where g_β is an entire function of exponential type $\leqslant a$.

The relation involving $\varphi(t)$ and w implies that $\varphi(t)/(t-\beta)^2 \in L_1(\mathbb{R})$ by the usual application of Schwarz' inequality. Then, if φ is *not* a pure exponential (when it *is*, it must be *bounded* on $\mathbb{R}$), it must have *at least two zeros*, for, if it had only *one*, $\varphi(t)/(t-\beta)^2$ would, by φ's resulting Hadamard factorization, be prevented from being in $L_1(\mathbb{R})$. This being the case, an argument like the one made during the preceding sufficiency proof shows that $\varphi(t)/(t-\beta)^2$ is *bounded* on $\mathbb{R}$. The conclusion is that $\varphi(t)/(t-\beta)^2$ is also in $L_2(\mathbb{R})$, and, referring to the above formula, we see that $g_\beta(t)$ is square integrable.

The function $g_\beta(t)$ is, however, entire and of exponential type $\leqslant a$, so we may essentially *repeat* the reasoning followed above, based on the

Paley–Wiener theorem, to conclude from the previous formula that

$$\Im\varphi(\beta)\cdot\lim_{A\to\infty}\int_{-A}^{A}\frac{t-\Re\beta}{|t-\beta|^2}S(t)\,\mathrm{d}t \;+\; \Re\varphi(\beta)\cdot\int_{-\infty}^{\infty}\frac{\Im\beta}{|t-\beta|^2}S(t)\,\mathrm{d}t$$

$$=\quad \Im\beta\cdot\int_{-\infty}^{\infty}\frac{\varphi(t)}{|t-\beta|^2}S(t)\,\mathrm{d}t$$

for the sums $S(t)$.

Here, we have

$$\int_{-\infty}^{\infty}\frac{|\varphi(t)|^2}{w(t)|t-\beta|^4}\,\mathrm{d}t \quad<\quad \infty,$$

so the *right side* of the preceding relation is in modulus

$$\leqslant\quad \text{const.}\,\|S\|,$$

where $\|\ \|$ has the same meaning as before. At the same time, our *assumption* is that

$$\left|\int_{-\infty}^{\infty}\frac{\Im\beta}{|t-\beta|^2}S(t)\,\mathrm{d}t\right| \quad\leqslant\quad \text{const.}\,\|S\|$$

for our sums S (with the constant depending, of course, on β). This may now be combined with the result just found to yield

$$\left|\Im\varphi(\beta)\cdot\lim_{A\to\infty}\int_{-A}^{A}\frac{t-\Re\beta+\mathrm{i}\Im\beta}{|t-\beta|^2}\,S(t)\,\mathrm{d}t\right| \quad\leqslant\quad \text{const.}\,\|S\|.$$

For each β, then, with $\Im\beta\ >\ 0$ there is a finite L_β such that

$$\left|\Im\varphi(\beta)\cdot\lim_{A\to\infty}\int_{-A}^{A}\frac{S(t)}{t-\beta}\,\mathrm{d}t\right| \quad\leqslant\quad L_\beta\|S\|$$

for the sums S.

Suppose that $\Im\varphi(\beta)\ \neq\ 0$ *for some* β *with* $\Im\beta\ >\ 0$. *Then we are done.* We can, indeed, argue as at the very *start* of the previous theorem's proof to obtain, thanks to the last relation, a $k(t)$ with $\|k\|\ <\ \infty$ (and hence $w(t)k(t)\ \in\ L_1(\mathbb{R})$ by Schwarz) such that

$$\lim_{A\to\infty}\int_{-A}^{A}\left(w(t)k(t)\ -\ \frac{1}{t-\beta}\right)\mathrm{e}^{\mathrm{i}\lambda t}\,\mathrm{d}t \quad=\quad 0$$

for $|\lambda|\ \geqslant\ a$.

Here, the integrable function $w(t)k(t)$ must also be in $L_2(\mathbb{R})$. Indeed, the (*bounded*!) Fourier transform

$$\int_{-\infty}^{\infty} e^{i\lambda t} w(t)k(t)\,dt$$

coincides with the L_2 Fourier transform of $1/(t-\beta)$ for large $|\lambda|$, and is thus itself in L_2. Then, however, $w(t)k(t) \in L_2(\mathbb{R})$ by Plancherel's theorem.

We may now apply the L_2 Paley–Wiener theorem (Chapter III, §D) to the function

$$w(t)k(t) \;-\; \frac{1}{t-\beta}$$

and conclude from the preceding relation that it coincides a.e. on $\mathbb{R}$ with an entire function $f(t)$ of exponential type $\leqslant a$. The function

$$\psi(t) \;=\; (t-\beta)f(t) \;+\; 1$$

is also entire and of exponential type a, and

$$\psi(t) \;=\; (t-\beta)w(t)k(t) \qquad \text{a.e., } t\in\mathbb{R}.$$

The above integral relation clearly implies that $w(t)k(t)$ cannot vanish a.e., so $\psi \not\equiv 0$. Finally,

$$\int_{-\infty}^{\infty} \frac{|\psi(t)|^2}{w(t)|t-\beta|^2}\,dt \;=\; \|k\|^2 \;<\; \infty,$$

so

$$\int_{-\infty}^{\infty} \frac{|\psi(t)|^2}{w(t)(t^2+1)}\,dt \;<\; \infty.$$

The *necessity is thus established* provided that for some β, $\Im\beta > 0$, the *original entire function* φ has *non-zero imaginary part at* β. If, however, there is *no* such β, we are *also* finished! *Then*, $\Im\varphi(\beta) \equiv 0$ for $\Im\beta > 0$, so $\varphi(z)$ must be *constant*, wlog, $\varphi(z) \equiv 1$. This means that

$$\int_{-\infty}^{\infty} \frac{1}{w(t)(t^2+1)^2}\,dt \;<\; \infty.$$

In that case,

$$\int_{-\infty}^{\infty} \frac{|\psi(t)|^2}{w(t)(t^2+1)}\,dt \;<\; \infty$$

with, e.g., $\psi(t) = \sin at/at$, and *this* function ψ is entire, of exponential type a, and $\not\equiv 0$.

The theorem is completely proved.

Scholium. The discrepancy between the conditions on w involved in the above two theorems is annoying. How can there be a $w \geqslant 0$ such that the sums

$$S(t) = \sum_{|\lambda| \geqslant a} A_\lambda e^{i\lambda t}$$

satisfying

$$\int_{-\infty}^{\infty} |S(t)|^2 w(t)\,dt \leqslant 1$$

yield harmonic functions

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} S(t)\,dt$$

with values bounded at *some* points z in the upper half plane, but not at *each* of those points? If there is a non-constant entire function $\varphi \not\equiv 0$ of exponential type $\leqslant a$ for which

$$\int_{-\infty}^{\infty} \frac{|\varphi(t)|^2}{w(t)(t^2+1)^2}\,dt < \infty,$$

can we not *divide out* one of the *zeros* of φ to get another such function ψ making

$$\int_{-\infty}^{\infty} \frac{|\psi(t)|^2}{w(t)(t^2+1)}\,dt < \infty \ ?$$

The present situation illustrates the care that must be taken in the investigation of such matters, straightforward though they may appear. The conditions involved in the two results are *not* equivalent, and there *really do* exist functions $w \geqslant 0$ *satisfying one, but not the other.* None of the zeros of φ can be divided out if they are all needed to cancel those of $w(t)$!

Here is a simple example. Let

$$w(t) = \frac{\sin^2 \pi t}{t^2+1}.$$

The condition

$$\int_{-\infty}^{\infty} \frac{|\varphi(t)|^2}{w(t)(t^2+1)^2}\,dt < \infty$$

is satisfied here with

$$\varphi(t) \quad = \quad \sin \pi t,$$

an entire function of exponential type π. *The kind of estimate furnished by the first theorem* is therefore *available* for the sums

$$S(t) \;=\; \sum_{|\lambda| \geqslant \pi} A_\lambda e^{i\lambda t}.$$

Here, however, the estimates provided by the second theorem are not all valid!

To see this, consider the functions

$$T_\eta(t) \quad = \quad \frac{i}{\sin \pi(t + i\eta)},$$

where η is a small parameter > 0. We have

$$\Re T_\eta(t) \quad = \quad \frac{\Im \sin \pi(t+i\eta)}{|\sin \pi(t+i\eta)|^2} \quad = \quad \frac{\sinh \pi\eta \cos \pi t}{\sin^2 \pi t + \sinh^2 \pi\eta}.$$

Clearly $\Re T_\eta(t+2) \;=\; \Re T_\eta(t)$ and $\Re T_\eta$ is $\mathscr{C}_\infty$ on the real axis, so

$$\Re T_\eta(t) \quad = \quad \sum_{-\infty}^{\infty} a_n e^{\pi i n t}, \qquad t \in \mathbb{R},$$

the series being absolutely convergent. Since $\Re T_\eta(t - \frac{1}{2})$ and $\Re T_\eta(t + \frac{1}{2})$ are *odd* functions of t, we have

$$a_0 \quad = \quad \frac{1}{2}\int_{-1}^{1} (\Re T_\eta)(t)\,dt \quad = \quad 0,$$

and $\Re T_\eta(t)$ is a (uniform!) limit of sums

$$\sum_{1 \leqslant |n| \leqslant N} a_n e^{\pi i n t},$$

each of the form

$$\sum_{|\lambda| \geqslant \pi} A_\lambda e^{i\lambda t}.$$

If the estimates furnished by the second theorem held for the present w *and for* $a \;=\; \pi$, we would now have

$$\left| \int_{-\infty}^{\infty} \frac{(\Re T_\eta)(t)}{1+t^2}\,dt \right| \quad \leqslant \quad C\sqrt{\left(\int_{-\infty}^{\infty} (\Re T_\eta(t))^2 w(t)\,dt\right)}$$

with a constant C *independent* of $\eta > 0$. That, however, is not the case.

Because

$$(\Re T_\eta(t))^2 w(t) \quad \leqslant \quad |T_\eta(t)|^2 w(t) \quad = \quad |T_\eta(t)|^2 \frac{\sin^2 \pi t}{t^2+1} \quad \leqslant \quad \frac{1}{t^2+1}$$

and

$$\Re T_\eta(t) \longrightarrow 0 \qquad \text{as } \eta \longrightarrow 0$$

for $t \neq 0, \pm 1, \pm 2, \ldots$, we have

$$\int_{-\infty}^{\infty} (\Re T_\eta(t))^2 w(t)\,\mathrm{d}t \longrightarrow 0$$

when $\eta \longrightarrow 0$, by dominated convergence.

At the same time, since each of the functions

$$T_\eta(z) \quad = \quad \frac{\mathrm{i}}{\sin \pi(z + \mathrm{i}\eta)}$$

is *analytic* and *bounded* in $\Im z > 0$,

$$\int_{-\infty}^{\infty} \frac{(\Re T_\eta)(t)}{1+t^2}\mathrm{d}t \quad = \quad \pi \Re T_\eta(\mathrm{i}) \quad = \quad \frac{\pi}{\sinh \pi(1+\eta)} \quad \longrightarrow \quad \frac{\pi}{\sinh \pi} \quad > \quad 0$$

as $\eta \longrightarrow 0$. This does it.

It is not hard to see that here, for the sums

$$S(t) \quad = \sum_{|\lambda| \geqslant \pi} A_\lambda \mathrm{e}^{\mathrm{i}\lambda t},$$

the condition

$$\int_{-\infty}^{\infty} |S(t)|^2 w(t)\,\mathrm{d}t \quad \leqslant \quad 1$$

gives us control on the integrals

$$\int_{-\infty}^{\infty} \frac{\Im \beta}{|t-\beta|^2} S(t)\,\mathrm{d}t$$

when $\Re \beta = \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \ldots,$ $\Im \sin \pi\beta$ vanishing precisely for such values of β.

One may pose a problem similar to the one discussed in this §, but with the sums

$$S(t) \quad = \sum_{|\lambda| \geqslant a} A_\lambda \mathrm{e}^{\mathrm{i}\lambda t}$$

replaced by others of the form

$$\sum_{|\lambda| \leqslant a} A_\lambda e^{i\lambda t}$$

(i.e., by entire functions of exponential type $\leqslant a$ bounded on $\mathbb{R}$!). That seems *harder*. Some of the material in the first part of de Branges' book is relevant to it.

E. Hilbert transforms of certain functions having given weighted quadratic norms.

We continue along the lines of the preceding §'s discussion. Taking, as we did there, some fixed $w \geqslant 0$ belonging to $L_1(\mathbb{R})$, let us suppose that we are given a certain class of functions $U(t)$, bounded on the real axis, whose harmonic extensions

$$U(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} U(t)\,dt$$

to the upper half plane are controlled by the weighted norm

$$\sqrt{\left(\int_{-\infty}^{\infty} |U(t)|^2 w(t)\,dt\right)}.$$

A suitably defined *harmonic conjugate* $\tilde{U}(z)$ of each of our functions $U(z)$ will then also be controlled by that norm. As we have seen in Chapter III, §F.2 and in the scholium to §H.1 of that chapter, the $\tilde{U}(z)$ have well defined non-tangential boundary values a.e. on $\mathbb{R}$ and thereby give rise to Lebesgue measurable functions $\tilde{U}(t)$ of the real variable t. Each of the latter is a *Hilbert transform* of the corresponding original bounded function $U(t)$; we say *a* Hilbert transform because that object, like the harmonic conjugate, is really only defined to within an additive constant. The reader can arrive at a fairly clear idea of these transforms by referring first to the §§ mentioned above and then to the middle of §C.1, Chapter VIII, and the scholium at the end of it.

Whatever specification is adopted for the Hilbert transforms $\tilde{U}(t)$ of our functions U, one may ask whether their *size* is governed by the weighted norm in question when that is the case for the harmonic extensions $U(z)$. To be more definite, let us ask whether there is some integrable function $\omega(t) \geqslant 0$, not a.e. zero on $\mathbb{R}$, such that

$$\int_{-\infty}^{\infty} |\tilde{U}(t)|^2 \omega(t)\,dt \quad \leqslant \quad \int_{-\infty}^{\infty} |U(t)|^2 w(t)\,dt$$

for the particular class of functions U under consideration. In the present

§, we study this question for the exponential sums

$$U(t) = \sum_{|\lambda| \geqslant a} A_\lambda e^{i\lambda t}$$

worked with in §D. Although the problem, as formulated, no longer refers directly to the harmonic extensions $U(z)$, it will turn out to have a positive solution (for given w) precisely when the latter are controlled by $\int_{-\infty}^{\infty} |U(t)|^2 w(t)\,dt$ in $\{\Im z > 0\}$ (and only then). For this reason, multipliers will again be involved in our discussion.

The work will require some material from the theory of H_p spaces. In order to save the reader the trouble of digging up that material elsewhere, we give it (and no more) in the next article, starting from scratch. This is *not* a book about H_p spaces, and anyone wishing to really learn about them should refer to such a book. Several are now available, including (and *why* not!) my own.*

1. H_p spaces for people who don't want to really learn about them

We will need to know some things about H_1, H_∞ and H_2, and proceed to take up those spaces in that order. Most of the real work involved here has actually been done already in various parts of the present book.

For our purposes, it is most convenient to use the

Definition. $H_1(\mathbb{R})$, or, as we usually write, H_1, is the set of f in $L_1(\mathbb{R})$ for which the Fourier transform

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda t} f(t)\,dt$$

vanishes for all $\lambda \geqslant 0$.

* As much as I want that book to sell, I should warn the reader that there are a fair number of misprints and also some actual mistakes in it. The *statement* of the lemma on p. 104 is inaccurate; boundedness only holds for r away from 0 when $F(0) = 0$. *Statement* of the lemma on p. 339 is wrong; ν may also contain a point mass at 0. That, however, makes no difference for the subsequent application of the lemma. The *argument* at the bottom of p. 116 is nonsense. *Instead*, one should say that if $\mathrm{Б}|B_\alpha$ and $d\sigma' \leqslant d\sigma_\alpha$ for each α, then every f_α is in ΩH_2, where Ω is given by the formula displayed there. Hence $\omega H_2 = E$ is $\subseteq \Omega H_2$, so $\mathrm{Б}|B$ and $d\sigma' \leqslant d\sigma$ by reasoning like that at the top of p. 116. There are confusing *misprints* in the proof of the first theorem on p 13; near the end of that proof, F should be replaced by G.

Lemma. *If* $f \in H_1$, $e^{i\lambda t} f(t) \in H_1$ *for each* $\lambda \geqslant 0$.

Proof. Clear.

Lemma. *If* $f \in H_1$ *and* $\Im z > 0$, $f(t)/(t-\bar{z}) \in H_1$.

Proof. For $\Im z > 0$ (i.e., $\Re(-i\bar{z}) < 0$), we have

$$\frac{i}{t-\bar{z}} = \int_0^\infty e^{-i\bar{z}\lambda} e^{i\lambda t}\, d\lambda, \qquad t \in \mathbb{R}.$$

Therefore, if $f \in H_1$,

$$i\int_{-\infty}^\infty \frac{f(t)}{t-\bar{z}}\, dt = \int_{-\infty}^\infty \int_0^\infty e^{-i\bar{z}\lambda} e^{i\lambda t} f(t)\, d\lambda\, dt.$$

The double integral on the right is absolutely convergent, and hence can be rewritten as

$$\int_0^\infty \int_{-\infty}^\infty e^{-i\bar{z}\lambda} e^{i\lambda t} f(t)\, dt\, d\lambda = \int_0^\infty e^{-i\bar{z}\lambda} \hat{f}(\lambda)\, d\lambda = 0.$$

If $\alpha \geqslant 0$ and $f \in H_1$, $e^{i\alpha t} f(t)$ is also in H_1 by the preceding lemma, so, using it in place of $f(t)$ in the computation just made, we get

$$\int_{-\infty}^\infty e^{i\alpha t} \frac{f(t)}{t-\bar{z}}\, dt = 0.$$

$f(t)/(t-\bar{z})$ is thus in H_1 by definition.

Theorem. *If, for* $f \in H_1$, *we write*

$$f(z) = \frac{1}{\pi}\int_{-\infty}^\infty \frac{\Im z}{|z-t|^2} f(t)\, dt$$

for $\Im z > 0$, *the function* $f(z)$ *is* analytic *in the upper half plane.*

Proof. We have

$$f(z) = \frac{1}{2\pi i}\int_{-\infty}^\infty \left(\frac{1}{t-z} - \frac{1}{t-\bar{z}}\right) f(t)\, dt.$$

By the last lemma, the right side equals

$$\frac{1}{2\pi i}\int_{-\infty}^\infty \frac{f(t)}{t-z}\, dt$$

for $\Im z > 0$, and this expression is clearly analytic in the upper half plane. We are done.

Theorem. *The function $f(z)$ defined in the statement of the preceding result has the following properties:*

(i) *$f(z)$ is continuous and bounded in each half plane $\{\Im z \geqslant h\}$, $h > 0$, and tends to 0 as $z \longrightarrow \infty$ in any one of those;*

(ii) $$\int_{-\infty}^{\infty} |f(x+iy)|\,dx \leqslant \|f\|_1 \quad \text{for } y > 0;$$

(iii) $$\int_{-\infty}^{\infty} |f(t+iy) - f(t)|\,dt \longrightarrow 0 \quad \text{as } y \longrightarrow 0;$$

(iv) $f(t+iy) \longrightarrow f(t)$ a.e. as $y \longrightarrow 0$.

Remark. Properties (iii) and (iv) justify our denoting

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} f(t)\,dt$$

by $f(z)$.

Proof of theorem. Property (i) is verified by inspection; (ii) and (iii) hold because the Poisson kernel is a (positive) approximate identity. Property (iv) comes out of the discussion beginning in Chapter II, §B and then continuing in §F.2 of Chapter III and in the scholium to §H.1 of that chapter. These ideas have already appeared frequently in the present book.

Theorem. *If $f(t) \in H_1$ is not zero a.e. on $\mathbb{R}$, we have*

$$\boxed{\int_{-\infty}^{\infty} \frac{\log^- |f(t)|}{1+t^2}\,dt \quad < \quad \infty,}$$

and, for each z, $\Im z > 0$,

$$\log|f(z)| \quad \leqslant \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} \log|f(t)|\,dt,$$

the integral on the right being absolutely convergent. Here, $f(z)$ has the same meaning as in the preceding two results.

Proof. For each $h > 0$ we can apply the results from Chapter III, §G.2 to $f(z+ih)$ in the half plane $\Im z > 0$, thanks to property (i), guaranteed by the last theorem. In this way we get

$$\log|f(z+ih)| \quad \leqslant \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} \log|f(t+ih)|\,dt$$

for $\Im z > 0$, with the integral on the right absolutely convergent.

Fix for the moment any z, $\Im z > 0$, for which $f(z) \neq 0$. The *left* side of the relation just written then tends to a limit $> -\infty$ as $h \longrightarrow 0$. At the same time, the *right* side is equal to

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log^+|f(t+ih)|\,dt \quad - \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log^-|f(t+ih)|\,dt,$$

where

$$\int_{-\infty}^{\infty} \left|\,\log^+|f(t+ih)| - \log^+|f(t)|\,\right| dt \quad \leqslant \quad \int_{-\infty}^{\infty} \left|\,|f(t+ih)| - |f(t)|\,\right| dt,$$

which tends to zero as h does, according to property (iii) in the preceding result. Therefore

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log^+|f(t+ih)|\,dt \quad \longrightarrow \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log^+|f(t)|\,dt,$$

a finite quantity (by the inequality between arithmetic and geometric means), as $h \longrightarrow 0$.

From property (iv) in the preceding theorem and Fatou's lemma, we have, however,

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log^-|f(t)|\,dt \quad \leqslant \quad \liminf_{h\to 0} \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log^-|f(t+ih)|\,dt.$$

Using this and the preceding relation we see, by making $h \longrightarrow 0$ in our initial one, that

$$-\infty \quad < \quad \log|f(z)| \quad \leqslant \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log^+|f(t)|\,dt$$
$$- \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log^-|f(t)|\,dt.$$

Since the *first* integral on the right is finite, the *second* must also be so. That, however, is equivalent to the relation

$$\int_{-\infty}^{\infty} \frac{\log^-|f(t)|}{1+t^2}\,dt \quad < \quad \infty.$$

Putting the two right-hand integrals together, we see that

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log|f(t)|\,dt$$

is absolutely convergent for our particular z, and hence for any z with

$\Im z > 0$. That quantity is $\geqslant \log|f(z)|$ as we have just seen, provided that $|f(z)| > 0$. It is of course $> \log|f(z)|$ in case $f(z) = 0$. We are done.

Corollary. *If $f(t) \in H_1$ is not a.e. zero, $|f(t)|$ is necessarily > 0 a.e.*.

Proof. The theorem's boxed inequality makes $\log^- |f(t)| > -\infty$ a.e..

Definition. $H_\infty(\mathbb{R})$, or, as we frequently write, H_∞, is the collection of g in $L_\infty(\mathbb{R})$ satisfying

$$\int_{-\infty}^{\infty} g(t)f(t)\,dt = 0$$

for all $f \in H_1$.

H_∞ is thus the subspace of L_∞, dual of L_1, consisting of functions *orthogonal* to the closed subspace H_1 of L_1. As such, it is closed, and even w^* closed, in L_∞.

By definition of H_1 we have the

Lemma. *Each of the functions $e^{i\lambda t}$, $\lambda \geqslant 0$, belongs to H_∞.*

Corollary. *A function $f \in L_1(\mathbb{R})$ belongs to H_1 iff*

$$\int_{-\infty}^{\infty} g(t)f(t)\,dt = 0$$

for all $g \in H_\infty$.

Lemma. *If $f \in H_1$ and $g \in H_\infty$, $g(t)f(t) \in H_1$.*

Proof. First of all, $gf \in L_1$. Also, when $\lambda \geqslant 0$, $e^{i\lambda t}f(t) \in H_1$ by a previous lemma, so by definition of H_∞,

$$\int_{-\infty}^{\infty} g(t)e^{i\lambda t}f(t)\,dt = 0,$$

i.e.,

$$\int_{-\infty}^{\infty} e^{i\lambda t}g(t)f(t)\,dt = 0$$

for *each* $\lambda \geqslant 0$. Therefore $gf \in H_1$.

Lemma. *If g and h belong to H_∞, $g(t)h(t)$ does also.*

Proof. If f is any member of H_1, gf is also in H_1 by the previous lemma. Therefore

$$\int_{-\infty}^{\infty} h(t)\cdot g(t)f(t)\,dt = 0.$$

This, holding for all $f \in H_1$, makes $hg \in H_\infty$ by definition.

Theorem. *Let $g \in H_\infty$. Then the function*

$$g(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\, g(t)\,dt$$

is analytic for $\Im z > 0$.

Proof. Fix z, $\Im z > 0$, and, for the moment, a large $A > 0$. The function

$$f(t) = \frac{1}{t-\bar{z}}\,\frac{iA}{t+iA}$$

belongs to H_1. This is easily verified directly by showing that

$$\int_{-\infty}^{\infty} e^{i\lambda t} f(t)\,dt = 0$$

for $\lambda \geqslant 0$ using contour integration. One takes large semi-circular contours in the upper half plane with base on the real axis; the details are left to the reader.

By definition of H_∞, we thus have

$$\int_{-\infty}^{\infty} g(t)\frac{1}{t-\bar{z}}\,\frac{iA}{iA+t}\,dt = 0.$$

Subtracting the left side from

$$\int_{-\infty}^{\infty} g(t)\cdot\frac{1}{t-z}\,\frac{iA}{iA+t}\,dt$$

and then dividing by 2i, we see that

$$\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\,\frac{iA}{iA+t}\,g(t)\,dt = \frac{1}{2i}\int_{-\infty}^{\infty}\frac{1}{t-z}\,\frac{iA}{iA+t}\,g(t)\,dt.$$

For each $A > 0$, then,

$$g_A(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\,\frac{iA}{iA+t}\,g(t)\,dt$$

is analytic for $\Im z > 0$ (by inspection).

As $A \longrightarrow \infty$, the functions $g_A(z)$ tend u.c.c. in $\{\Im z > 0\}$ to

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\, g(t)\, \mathrm{d}t \quad = \quad g(z).$$

The latter is therefore also analytic there.

Remark. For the function $g(z)$ figuring in the above theorem we have, for each z, $\Im z > 0$,

$$|g(z)| \;\leqslant\; \|g\|_\infty,$$

where the L_∞ norm on the right is taken for $g(t)$ on $\mathbb{R}$. This is evident by inspection. The same reasoning which shows that

$$f(t+\mathrm{i}y) \;\longrightarrow\; f(t) \quad \text{a.e. as } y \longrightarrow 0$$

for functions f in H_1 also applies here, yielding the result that

$$g(t+\mathrm{i}y) \;\longrightarrow\; g(t) \quad \text{a.e. as } y \longrightarrow 0$$

when $g \in H_\infty$. Unless $g(t)$ is uniformly continuous, however, we do *not* have

$$\|g(t+\mathrm{i}y) \;-\; g(t)\|_\infty \;\longrightarrow\; 0$$

for $y \longrightarrow 0$. Instead, we are only able to affirm that $g(t+\mathrm{i}y)$ tends w^* to $g(t)$ (in $L_\infty(\mathbb{R})$) as $y \longrightarrow 0$.

The theorem just proved has an important *converse*:

Theorem. *Let* $G(z)$ *be analytic and bounded for* $\Im z > 0$. *Then there is a* $g \in H_\infty$ *such that*

$$G(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\, g(t)\, \mathrm{d}t$$

for $\Im z > 0$, *and*

$$\|g\|_\infty \quad = \quad \sup_{\Im z > 0} |G(z)|.$$

Proof. It is claimed first of all that each of the functions $G(t+\mathrm{i}h)$, $h > 0$, belongs to H_∞ (as a function of t). Take any $f \in H_1$, and put

$$f(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\, f(t)\, \mathrm{d}t, \qquad \Im z \;>\; 0.$$

Our definition of H_∞ requires us to verify that

$$\int_{-\infty}^{\infty} G(t+\mathrm{i}h) f(t)\, \mathrm{d}t \quad = \quad 0.$$

Since

$$\|f(t+ib) - f(t)\|_1 \longrightarrow 0$$

as $b \longrightarrow 0$, it is enough to show that

$$\int_{-\infty}^{\infty} G(t+ih)f(t+ib)\,dt = 0$$

for each $b > 0$.

Fix any such b. According to a previous result, $f(z+ib)$ is then analytic and *bounded* for $\Im z > 0$, and continuous up to the real axis. The same is true for $G(z+ih)$. These properties make it easy for us to see by contour integration that

$$\int_{-\infty}^{\infty} \left(\frac{iA}{iA+t}\right)^2 G(t+ih)f(t+ib)\,dt = 0$$

for $A > 0$; one just integrates

$$\left(\frac{iA}{iA+z}\right)^2 G(z+ih)f(z+ib)$$

around large semi-circles in $\Im z \geqslant 0$ having their diameters on the real axis. Since $f(t+ib) \in L_1(\mathbb{R})$, we may now make $A \longrightarrow \infty$ in the relation just found to get

$$\int_{-\infty}^{\infty} G(t+ih)f(t+ib)\,dt = 0$$

and thus ensure that $G(t+ih) \in H_\infty(\mathbb{R})$.

For each $h > 0$ the first lemma of §H.1, Chapter III, makes

$$G(z+ih) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} G(t+ih)\,dt$$

when $\Im z > 0$. Here,

$$|G(t+ih)| \leqslant \sup_{\Im z > 0} |G(z)| < \infty.$$

Hence, since L_∞ is the *dual* of L_1, a procedure just like the one used in establishing the first theorem of §F.1, Chapter III, gives us a sequence of numbers $h_n > 0$ tending to zero and a g in L_∞ with

$$G(t+ih_n) \longrightarrow g(t) \quad w^*$$

as $n \longrightarrow \infty$. From this we see, referring to the preceding formula, that

$$G(z) = \lim_{n\to\infty} G(z+ih_n) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} g(t)\,dt$$

for $\Im z > 0$. By the w^* convergence we also have

$$\|g\|_\infty \leqslant \liminf_{n\to\infty} \|G(t+ih_n)\|_\infty \leqslant \sup_{\Im z>0} |G(z)|.$$

However, the representation just found for $G(z)$ implies the reverse inquality, so

$$\|g\|_\infty = \sup_{\Im z>0} |G(z)|.$$

As we have seen, each of the functions $G(t+ih_n)$ is in H_∞. Their w^* limit $g(t)$ must then also be in H_∞.

The theorem is proved.

Remark. An analogous theorem is true about H_1. Namely, if $F(z)$, analytic for $\Im z > 0$, is such that the integrals

$$\int_{-\infty}^{\infty} |F(x+iy)|\,dx$$

are *bounded* for $y > 0$, there *is* an $f \in H_1$ for which

$$F(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} f(t)\,dt, \qquad \Im z > 0.$$

This result will not be needed in the present §; it is deeper than the one just found because $L_1(\mathbb{R})$ is *not the dual* of any Banach space. The F. and M. Riesz theorem is required for its proof; see §B.4 of Chapter VII.

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Let $g \in H_\infty$, and write

$$g(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} g(t)\,dt$$

for $\Im z > 0$.

(a) If $\Im c > 0$, both functions

$$\frac{g(t)-g(c)}{t-\bar{c}} \quad \text{and} \quad \frac{g(t)-g(c)}{t-c}$$

belong to H_∞. (Hint: In considering the first function, begin by noting that $1/(t-\bar{c}) \in H_\infty$ according to the second lemma about H_1. To investigate the second function, look at $(g(z)-g(c))/(z-c)$ in the upper half plane.)

(b) Hence show that if $f \in H_1$ and

$$f(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} f(t)\,dt$$

for $\Im z > 0$, one has

$$f(c)g(c) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im c}{|c-t|^2} f(t)g(t)\,dt$$

for each c with $\Im c > 0$.

(c) If, for the $f(z)$ of part (b) one has $f(c) = 0$ for some c, $\Im c > 0$, show that $f(t)/(t-c)$ belongs to H_1. (Hint: Follow the argument of (b) using the function $g(t) = e^{i\lambda t}$, where $\lambda \geqslant 0$ is arbitrary.)

Theorem. *If $g(t) \in H_\infty$ is* not *a.e. zero on $\mathbb{R}$, we have*

$$\boxed{\int_{-\infty}^{\infty} \frac{\log^{-}|g(t)|}{1+t^2}\,dt < \infty,}$$

and, for $\Im z > 0$,

$$\log|g(z)| \leqslant \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} \log|g(t)|\,dt,$$

the integral on the right being absolutely convergent. Here, $g(z)$ has its usual meaning:

$$g(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} g(t)\,dt.$$

Proof. By the first of the preceding two theorems, $g(z)$ is analytic (and of course bounded) for $\Im z > 0$. Therefore, by the results of §G.2 in Chapter III, for each $h > 0$,

$$\log|g(z+ih)| \leqslant \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} \log|g(t+ih)|\,dt$$

when $\Im z > 0$.

We may, wlog, take $\|g\|_\infty$ to be $\leqslant 1$, so that $|g(z)| \leqslant 1$ and $\log|g(z)| \leqslant 0$ for $\Im z > 0$. As $h \longrightarrow 0$, $g(t+ih) \longrightarrow g(t)$ a.e. according to a previous remark, so, by Fatou's lemma,

$$\limsup_{h\to 0} \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} \log|g(t+ih)|\,dt \leqslant \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} \log|g(t)|\,dt.$$

The right-hand quantity must thus be $\geqslant \log|g(z)|$ by the previous relation, proving the second inequality of our theorem.

In case $g(t)$ is not a.e. zero, there must be some z, $\Im z > 0$, with $g(z) \neq 0$, again because $g(t+ih) \longrightarrow g(t)$ a.e. for $h \longrightarrow 0$. Using this z in the inequality just proved, we see that

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} \log|g(t)|\,dt \quad > \quad -\infty,$$

whence

$$\int_{-\infty}^{\infty} \frac{\log^-|g(t)|}{1+t^2}\,dt \quad < \quad \infty,$$

and the former integral is actually absolutely convergent for all z with $\Im z > 0$, whether $g(z) \neq 0$ or not.

We are done.

Come we now to the space H_2.

Definition. A function $f \in L_2(\mathbb{R})$ belongs to $H_2(\mathbb{R})$, usually designated as H_2, iff

$$\int_{-\infty}^{\infty} \frac{f(t)}{t-\bar{z}}\,dt \quad = \quad 0$$

for all z with $\Im z > 0$.

H_2 is clearly a closed subspace of $L_2(\mathbb{R})$.

Theorem. *If $f \in H_2$, the function*

$$f(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} f(t)\,dt$$

is analytic for $\Im z > 0$.

Proof. Is like that of the corresponding result for H_1.

Theorem. *If $f \in H_2$, the function $f(z)$ in the preceding theorem has the following properties:*

(i) $|f(z)| \leqslant \|f\|_2/\sqrt{(\pi \Im z)}, \qquad \Im z > 0;$

(ii) $\displaystyle\int_{-\infty}^{\infty} |f(x+iy)|^2 dx \leqslant \|f\|_2^2$ for $y > 0$;

(iii) $\displaystyle\int_{-\infty}^{\infty} |f(t+iy)-f(t)|^2\,dt \longrightarrow 0$ as $y \longrightarrow 0$;

(iv) $f(t+iy) \longrightarrow f(t)$ a.e. as $y \longrightarrow 0$.

Proof. Property (i) follows by applying Schwarz' inequality to the formula for $f(z)$. The remaining properties are verified by arguments like those used in proving the corresponding theorem about H_1, given above.

As is the case for H_∞ (and for H_1), these results have a converse:

Theorem. *Let $F(z)$ be analytic for $\Im z > 0$, and suppose that*

$$\int_{-\infty}^{\infty} |F(x+iy)|^2\,dx$$

is bounded for $y > 0$. Then there is an $f \in H_2$ with

$$F(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} f(t)\,dt, \qquad \Im z > 0,$$

and

$$\|f\|_2^2 = \sup_{y>0} \int_{-\infty}^{\infty} |F(x+iy)|^2\,dx.$$

Proof. For each $h > 0$, put

$$F_h(z) = \frac{1}{2h}\int_{-h}^{h} F(z+s)\,ds, \qquad \Im z > 0.$$

By Schwarz' inequality,

$$|F_h(z)| \leqslant (2h)^{-1/2}\sqrt{\left(\int_{-\infty}^{\infty} |F(z+s)|^2\,ds\right)} \leqslant \frac{C}{\sqrt{(2h)}}$$

where C is independent of z or h; each function $F_h(z)$ is therefore *bounded* in $\Im z > 0$, besides being analytic there.

A previous theorem therefore gives us functions $f_h \in H_\infty$ such that

$$F_h(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} f_h(t)\,dt, \qquad \Im z > 0,$$

and, as already remarked,

$$F_h(t+iy) \longrightarrow f_h(t) \quad \text{a.e. for } y \longrightarrow 0.$$

We have, for each h and $y > 0$,

$$\int_{-\infty}^{\infty} |F_h(x+\mathrm{i}y)|^2\,\mathrm{d}x \;\leqslant\; \frac{1}{2h}\int_{-\infty}^{\infty}\int_{-h}^{h} |F(x+s+\mathrm{i}y)|^2\,\mathrm{d}s\,\mathrm{d}x$$

$$= \int_{-\infty}^{\infty} |F(x+\mathrm{i}y)|^2\,\mathrm{d}x$$

by Schwarz' inequality and Fubini. Since the right side is bounded by a quantity $M < \infty$ independent of y (and h), the limit relation just written guarantees that

$$\|f_h\|_2^2 \;\leqslant\; M$$

for $h > 0$, according to Fatou's lemma.

Once it is known that the norms $\|f_h\|_2$ are bounded we can, as in the proof of the corresponding theorem about H_∞, get a sequence of $h_n > 0$ tending to zero for which the f_{h_n} converge *weakly*, *this time* in L_2, to some $f \in L_2(\mathbb{R})$. Then, for each z, $\Im z > 0$,

$$F_{h_n}(z) \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} f_{h_n}(t)\,\mathrm{d}t \;\longrightarrow\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} f(t)\,\mathrm{d}t$$

as $n \longrightarrow \infty$. At the same time,

$$F_{h_n}(z) \;\xrightarrow[n]{}\; F(z),$$

so we *have* our desired representation of $F(z)$ if we can show that $f \in H_2$.

For this purpose, it is enough to verify that when $\Im z > 0$,

$$\int_{-\infty}^{\infty} \frac{f_h(t)}{t-\bar{z}}\,\mathrm{d}t \;=\; 0,$$

since the f_{h_n} tend to f weakly in L_2. However, the f_h belong to H_∞, and, when $\Im z > 0$ and $A > 0$, the function

$$\frac{1}{t-\bar{z}}\,\frac{\mathrm{i}A}{\mathrm{i}A+t}$$

belongs to H_1, as we have noted during the proof of a previous result. Hence

$$\int_{-\infty}^{\infty} \frac{\mathrm{i}A}{\mathrm{i}A+t}\,\frac{f_h(t)}{t-\bar{z}}\,\mathrm{d}t \;=\; 0.$$

Here, $f_h(t)/(t-\bar{z})$ belongs to L_1, so we may make $A \longrightarrow \infty$ in this relation, which yields the desired one.

We still need to show that $\|f\|_2^2 = \sup_{y>0}\int_{-\infty}^{\infty}|F(x+iy)|^2\,dx$. Here, we now know that the function $F(z)$ is nothing but the $f(z)$ figuring in the preceding theorem. The statement in question thus follows from properties (ii) and (iii) of that result.

We are done.

Remark. Using the theorems just proved, one readily verifies that H_2 consists precisely of the functions $u(t)+i\tilde{u}(t)$, with u an arbitrary real-valued member of $L_2(\mathbb{R})$ and $\tilde{u}$ its L_2 Hilbert transform – the one studied in the scholium to §C.1 of Chapter VIII. The reader should carry out this verification.

Our use of the space H_2 in the following articles of this § is based on a relation between H_2 and H_1, established by the following two results.

Theorem. *If f and g belong to H_2, $f\cdot g$ is in H_1.*

Proof. Certainly $fg \in L_1$, so the quantity

$$\int_{-\infty}^{\infty} e^{i\lambda t} f(t)\,g(t)\,dt$$

varies continuously with λ. It is therefore enough to show that it vanishes for $\lambda > 0$ (*sic*) in order to prove that $fg \in H_1$.

Let, as usual,

$$f(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} f(t)\,dt$$

for $\Im z > 0$, and

$$g(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} g(t)\,dt$$

there.

Using the facts that $\|f(t+ih)-f(t)\|_2 \longrightarrow 0$ and $\|g(t+ih)-g(t)\|_2 \longrightarrow 0$ for $h \longrightarrow 0$ (property (iii) in the first of the preceding two theorems) and applying Schwarz' inequality to the identity

$$\begin{aligned} &f(t+ih)g(t+ih) \quad - \quad f(t)g(t) \\ &\quad = \quad [f(t+ih)-f(t)]\,g(t) \;+\; f(t+ih)\,[g(t+ih)-g(t)], \end{aligned}$$

one readily sees that

$$\|f(t+ih)g(t+ih) \;-\; f(t)g(t)\|_1 \;\longrightarrow\; 0$$

as $h \longrightarrow 0$. It is therefore sufficient to check that

$$\int_{-\infty}^{\infty} e^{i\lambda t} f(t+ih)\, g(t+ih)\, dt \quad = \quad 0$$

for each $h > 0$ when $\lambda > 0$.

Fix any such h. By property (i) from the result just referred to,

$$|f(z+ih)| \quad \leqslant \quad \frac{\text{const.}}{\sqrt{h}} \qquad \text{for } \Im z \;\geqslant\; 0.$$

Also, since $f(t) \in L_2(\mathbb{R})$, the function

$$f(z+ih) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z + h}{|z+ih-t|^2} f(t)\, dt$$

tends uniformly to zero for z tending to ∞ in any *fixed strip* $0 \leqslant \Im z \leqslant L$. The function $g(z+ih)$ has the same behaviour.

These properties make it possible for us to now virtually *copy* the contour integral argument made in proving the Paley–Wiener theorem, Chapter III, §D, replacing the function $f_h(z)$ figuring there* by $f(z+ih)g(z+ih)$. In that way we find that

$$\int_{-\infty}^{\infty} e^{i\lambda t} f(t+ih)\, g(t+ih)\, dt \quad = \quad 0$$

for $\lambda > 0$, the relation we needed.

The theorem is proved.

The last result has an important converse:

Theorem. *Given $\varphi \in H_1$, there are functions f and g in H_2 with $\varphi = fg$ and* $\|f\|_2 \;=\; \|g\|_2 \;=\; \sqrt{(\|\varphi\|_1)}$.

Proof. There is no loss of generality in assuming that $\varphi(t)$ is not a.e. zero on $\mathbb{R}$, for otherwise our theorem is trivial. Putting, then,

$$\varphi(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\varphi(t)\, dt$$

for $\Im z \;>\; 0$, we know by previous results that $\varphi(z)$ is analytic in the upper

* Here, the condition $\lambda > 0$ plays the rôle that the relation $\lambda > A$ did in the discussion referred to.

half plane and that

$$\log|\varphi(z)| \quad \leqslant \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log|\varphi(t)|\,\mathrm{d}t$$

there, the integral on the right being absolutely convergent.

Thanks to the absolute convergence, we can define a function $F(z)$ analytic for $\Im z > 0$ by writing

$$F(z) \quad = \quad \exp\left\{\frac{1}{2\pi \mathrm{i}}\int_{-\infty}^{\infty}\left(\frac{1}{t-z} - \frac{t}{t^2+1}\right)\log|\varphi(t)|\,\mathrm{d}t\right\};$$

the idea here is that $F(z) \neq 0$ for $\Im z > 0$, with

$$\log|F(z)| \quad = \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log|\varphi(t)|\,\mathrm{d}t,$$

one half the right side of the preceding inequality. The ratio

$$G(z) \quad = \quad \frac{\varphi(z)}{F(z)}$$

is then *analytic* for $\Im z > 0$, and we have

$$\log|G(z)| \quad \leqslant \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log|\varphi(t)|\,\mathrm{d}t \quad = \quad \log|F(z)|,$$

i.e.,

$$|G(z)| \leqslant |F(z)|, \qquad \Im z > 0.$$

By the inequality between arithmetic and geometric means,

$$|F(z)|^2 \quad \leqslant \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}|\varphi(t)|\,\mathrm{d}t,$$

so, for each $y > 0$,

$$\int_{-\infty}^{\infty} |G(x+\mathrm{i}y)|^2\,\mathrm{d}x \quad \leqslant \quad \int_{-\infty}^{\infty} |F(x+\mathrm{i}y)|^2\,\mathrm{d}x$$

$$\leqslant \quad \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{y|\varphi(t)|}{(x-t)^2+y^2}\,\mathrm{d}t\,\mathrm{d}x \quad = \quad \|\varphi\|_1.$$

According to a previous theorem, there are thus functions f and g in H_2 with

$$F(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} f(t)\,\mathrm{d}t, \qquad \Im z > 0,$$

$$G(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} g(t)\,\mathrm{d}t, \qquad \Im z \ > \ 0,$$

and

$$\|g\|_2^2 \quad \leqslant \quad \|f\|_2^2 \quad \leqslant \quad \|\varphi\|_1.$$

For $\Im z \ > \ 0$, we have

$$\varphi(z) \quad = \quad F(z)G(z),$$

However, when $y \longrightarrow 0$,

$$\varphi(t + \mathrm{i}y) \quad \longrightarrow \quad \varphi(t) \quad \text{a.e.}$$

while at the same time

$$F(t + \mathrm{i}y) \quad \longrightarrow \quad f(t) \quad \text{a.e.}$$

and

$$G(t + \mathrm{i}y) \quad \longrightarrow \quad g(t) \quad \text{a.e..}$$

Therefore,

$$\varphi(t) \quad = \quad f(t)g(t) \qquad \text{a.e.,} \ \ t \in \mathbb{R},$$

our desired factorization.

Schwarz' inequality now yields

$$\|\varphi\|_1 \quad \leqslant \quad \|f\|_2 \|g\|_2.$$

We already know, however, that

$$\|g\|_2 \quad \leqslant \quad \|f\|_2 \quad \leqslant \quad \sqrt{(\|\varphi\|_1)}.$$

Hence $\|g\|_2 \quad = \quad \|f\|_2 \quad = \quad \sqrt{(\|\varphi\|_1)}$.

We are done.

Remark. For the function $F(z)$ used in the above proof, we have

$$\log|F(z)| \quad = \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} \log|\varphi(t)|\,\mathrm{d}t,$$

so

$$\log|F(t + \mathrm{i}y)| \quad \longrightarrow \left| \tfrac{1}{2}\log|\varphi(t)| \quad \text{a.e.} \right.$$

as $y \longrightarrow 0$ by the property of the Poisson kernel already used frequently in this article. This means, however, that

$$|F(t + \mathrm{i}y)| \quad \longrightarrow \quad \sqrt{(|\varphi(t)|)} \quad \text{a.e.}$$

for $y \longrightarrow 0$. At the same time,

$$F(t + \mathrm{i}y) \longrightarrow f(t) \quad \text{a.e.},$$

so we have

$$|f(t)| = \sqrt{(|\varphi(t)|)} \qquad \text{a.e., } t \in \mathbb{R},$$

for the H_2 function f furnished by the last theorem.

Since $\varphi \in H_1$, we must have $|\varphi(t)| > 0$ a.e. by a previous corollary (unless $\varphi(t) \equiv 0$ a.e., a trivial special case which we are excluding). The H_2 function g with $fg = \varphi$ must then also satisfy

$$|g(t)| = \sqrt{(|\varphi(t)|)} \qquad \text{a.e., } t \in \mathbb{R}.$$

In spite of the fact that the H_2 functions f and g involved here have a.e. the same moduli on $\mathbb{R}$, they are in general essentially different. It is usually true that their extensions F and G to the upper half plane satisfy

$$|G(z)| < |F(z)|$$

there.

Later on in this §, our work will involve the products

$$\mathrm{e}^{\mathrm{i}\lambda t} f(t)$$

with $\lambda \geqslant 0$, where f is a given function in H_2. Our first observation about these is the

Lemma. *If $f \in H_2$ and $\lambda \geqslant 0$, $\mathrm{e}^{\mathrm{i}\lambda t} f(t) \in H_2$.*

Proof. If $\Im z > 0$, the function $1/(t - \bar{z})$ belongs to H_2. This is most easily checked by referring to the definition of H_2 and doing a contour integral; such verification is left to the reader. According to a previous theorem, then, $f(t)/(t - \bar{z})$ belongs to H_1. Hence

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\lambda t} f(t)}{t - \bar{z}} \, \mathrm{d}t = 0$$

for each $\lambda \geqslant 0$. Here, z with $\Im z > 0$ is arbitrary, so the functions $\mathrm{e}^{\mathrm{i}\lambda t} f(t)$ with $\lambda \geqslant 0$ belong to H_2 by definition. Done.

When $f \in H_2$, finite linear combinations of the products $\mathrm{e}^{\mathrm{i}\lambda t} f(t)$ with $\lambda \geqslant 0$ form, by the lemma just proved, a certain vector subspace of H_2. We want to know when the L_2 closure of that subspace is *all* of H_2. This question was answered by Beurling. His argument uses material from the