for $z \in \mathscr{D}$. This is a *formula* for solving the Dirichlet problem for $\mathscr{D}$, based on the conformal mapping function F. Knowledge of this formula will help us later on to get general qualitative information about the behaviour near $\partial\mathscr{D}$ of certain functions harmonic in $\mathscr{D}$ but *not* continuous up to $\partial\mathscr{D}$, even when $\mathscr{D}$ is *not* simply connected.

Let us return to the multiply connected domains $\mathscr{D}$ of the kind considered here. If φ is real and continuous on $\partial\mathscr{D}$ and U_φ, harmonic in $\mathscr{D}$ and continuous on $\bar{\mathscr{D}}$, agrees with φ on $\partial\mathscr{D}$, we have, by the principle of maximum,

$$-\|\varphi\|_\infty \leqslant U_\varphi(z) \leqslant \|\varphi\|_\infty$$

for each $z \in \mathscr{D}$; here we are writing

$$\|\varphi\|_\infty = \sup_{\zeta \in \partial\mathscr{D}} |\varphi(\zeta)|.$$

This shows in the first place that *there can only be one function* U_φ corresponding to a *given* function φ. We see, secondly, that *there must be a* (signed) *measure* μ_z on $\partial\mathscr{D}$ (depending, of course, on z) with

$$(*) \qquad U_\varphi(z) = \int_{\partial\mathscr{D}} \varphi(\zeta)\mathrm{d}\mu_z(\zeta).$$

The latter statement is simply a consequence of the Riesz representation theorem applied to the space $\mathscr{C}(\partial\mathscr{D})$. Since U_φ can be found for *every* $\varphi \in \mathscr{C}(\partial\mathscr{D})$ (i.e., the Dirichlet problem for $\mathscr{D}$ can be solved!) and since, corresponding to each given φ, there is only *one* U_φ, there can, for any $z \in \mathscr{D}$, be *only one* measure μ_z on $\partial\mathscr{D}$ such that $(*)$ is true with every $\varphi \in \mathscr{C}(\partial\mathscr{D})$. The measure μ_z is thus *a function of* $z \in \mathscr{D}$, and we proceed to make a gross examination of its dependence on z.

If $\varphi(\zeta) \geqslant 0$ we must have $U_\varphi(z) \geqslant 0$ throughout $\mathscr{D}$ by the principle of maximum. Referring to $(*)$, we see that *the measures* μ_z *must be positive.* Also, 1 is a harmonic function (!), so, if $\varphi(\zeta) \equiv 1$, $U_\varphi(z) \equiv 1$. Therefore

$$\int_{\partial\mathscr{D}} \mathrm{d}\mu_z(\zeta) = 1$$

for every $z \in \mathscr{D}$. Let $\zeta_0 \in \partial\mathscr{D}$ and consider any small fixed neighborhood $\mathscr{V}$ of ζ_0. Take any continuous function φ on $\partial\mathscr{D}$ such that $\varphi(\zeta_0) = 0$, $\varphi(\zeta) \equiv 1$ for $\zeta \notin \mathscr{V} \cap \partial\mathscr{D}$, and $0 \leqslant \varphi(\zeta) \leqslant 1$ on $\mathscr{V} \cap \partial\mathscr{D}$.

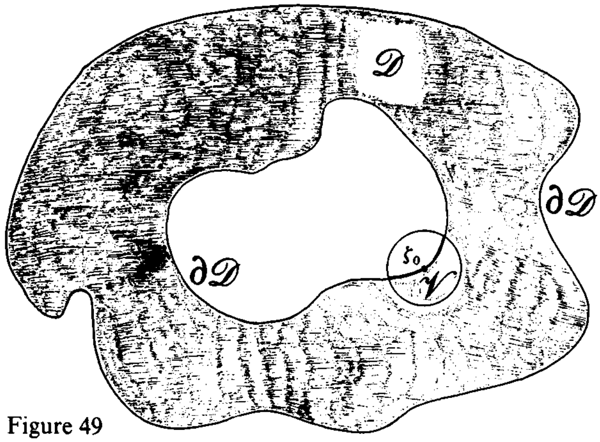

Figure 49

Since U_φ is a solution of the Dirichlet problem, we certainly have

$$U_\varphi(z) \longrightarrow \varphi(\zeta_0) = 0 \quad \text{for} \quad z \longrightarrow \zeta_0.$$

The positivity of the μ_z therefore makes

$$\int_{\zeta \notin \mathscr{V} \cap \partial\mathscr{D}} d\mu_z(\zeta) \longrightarrow 0$$

for $z \longrightarrow \zeta_0$. Because all the μ_z have total mass 1, we must also have

$$\int_{\mathscr{V} \cap \partial\mathscr{D}} d\mu_z(\zeta) \longrightarrow 1 \quad \text{for} \quad z \longrightarrow \zeta_0.$$

When z is near $\zeta_0 \in \partial\mathscr{D}$, μ_z has almost all of its total mass (1) *near* ζ_0 (on $\partial\mathscr{D}$). This is the so-called *approximate identity property* of the μ_z.

There is also a continuity property for the μ_z applying to variations of *z in the interior of $\mathscr{D}$*.

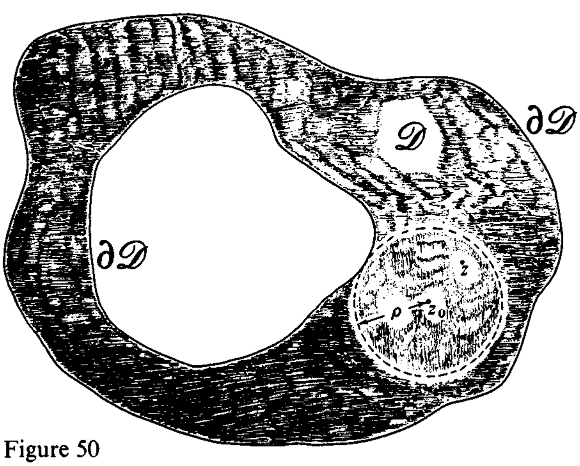

Figure 50

Take any $z_0 \in \mathscr{D}$, write $\rho = \operatorname{dist}(z_0, \partial\mathscr{D})$ and suppose that $|z - z_0| < \rho$. Then, if φ is *continuous and positive on $\partial\mathscr{D}$*,

$$\int_{\partial\mathscr{D}} \varphi(\zeta) \mathrm{d}\mu_z(\zeta)$$

lies between

$$\frac{\rho - |z - z_0|}{\rho + |z - z_0|} \int_{\partial\mathscr{D}} \varphi(\zeta) \mathrm{d}\mu_{z_0}(\zeta)$$

and

$$\frac{\rho + |z - z_0|}{\rho - |z - z_0|} \int_{\partial\mathscr{D}} \varphi(\zeta) \mathrm{d}\mu_{z_0}(\zeta).$$

This is nothing but *Harnack's inequality* applied to the circle $\{|z - z_0| < \rho\}$, $U_\varphi(z)$ being *harmonic and positive* in that circle. (The reader who does not recall Harnack's inequality may derive it very easily from the Poisson representation of positive harmonic functions for the unit disk given in Chapter III, §F.1.) These inequalities hold for *any* positive $\varphi \in \mathscr{C}(\partial\mathscr{D})$, so the signed measures

$$\mu_z \;-\; \frac{\rho - |z - z_0|}{\rho + |z - z_0|} \mu_{z_0},$$

$$\frac{\rho + |z - z_0|}{\rho - |z - z_0|} \mu_{z_0} \;-\; \mu_z$$

are in fact positive. This fact is usually expressed by the double inequality

$$\frac{\rho - |z - z_0|}{\rho + |z - z_0|} \mathrm{d}\mu_{z_0}(\zeta) \;\leqslant\; \mathrm{d}\mu_z(\zeta) \;\leqslant\; \frac{\rho + |z - z_0|}{\rho - |z - z_0|} \mathrm{d}\mu_{z_0}(\zeta).$$

What is important here is that we have a number $K(z, z_0)$, $0 < K(z, z_0) < \infty$, depending *only* on z and z_0 (and $\mathscr{D}$!), such that

$$\frac{1}{K(z, z_0)} \mathrm{d}\mu_{z_0}(\zeta) \;\leqslant\; \mathrm{d}\mu_z(\zeta) \;\leqslant\; K(z, z_0) \mathrm{d}\mu_{z_0}(\zeta).$$

Such an inequality in fact holds for *any* two points z, z_0 in $\mathscr{D}$; one needs only to join z to z_0 by a path lying in $\mathscr{D}$ and then take a chain of overlapping disks $\subseteq \mathscr{D}$ having their centres on that path, applying the previous special version of the inequality in each disk.

In order to indicate the dependence of the measures μ_z in $(*)$ *on the domain $\mathscr{D}$ as well as on $z \in \mathscr{D}$*, we use a special notation for them which is now becoming standard. ***We write***

$$\mathrm{d}\omega_{\mathscr{D}}(\zeta, z) \quad \textit{for} \quad \mathrm{d}\mu_z(\zeta),$$

***so that* (*) *has this appearance*:**

$$\boxed{U_\varphi(z) = \int_{\partial\mathscr{D}} \varphi(\zeta)\mathrm{d}\omega_{\mathscr{D}}(\zeta, z).}$$

We call $\omega_{\mathscr{D}}(\ ,z)$ harmonic measure for $\mathscr{D}$ (or relative to $\mathscr{D}$) as seen from z. $\omega_{\mathscr{D}}(\ ,z)$ is a positive Radon measure on $\partial\mathscr{D}$, of total mass 1, which serves to recover functions harmonic in $\mathscr{D}$ and continuous on $\bar{\mathscr{D}}$ from their boundary values on $\partial\mathscr{D}$ by means of the boxed formula. That formula is just the analogue of *Poisson's* for our domains $\mathscr{D}$.

If E is a Borel set on $\partial\mathscr{D}$,

$$\omega_{\mathscr{D}}(E, z) = \int_E \mathrm{d}\omega_{\mathscr{D}}(\zeta, z)$$

is called *the harmonic measure of E relative to $\mathscr{D}$ (or in $\mathscr{D}$), seen from z.* We have, of course,

$$0 \leqslant \omega_{\mathscr{D}}(E, z) \leqslant 1.$$

Also, for fixed $E \subseteq \partial\mathscr{D}$, $\omega_{\mathscr{D}}(E, z)$ is a *harmonic function of z.* This almost obvious property may be verified as follows. Given $E \subseteq \partial\mathscr{D}$, take a sequence of functions $\varphi_n \in \mathscr{C}(\partial\mathscr{D})$ with $0 \leqslant \varphi_n \leqslant 1$ such that

$$\int_{\partial\mathscr{D}} |\chi_E(\zeta) - \varphi_n(\zeta)|\mathrm{d}\omega_{\mathscr{D}}(\zeta, z_0) \underset{n}{\longrightarrow} 0$$

for the characteristic function χ_E of E. Here, z_0 is *any fixed point of $\mathscr{D}$* which may be chosen at pleasure. Since $\mathrm{d}\omega_{\mathscr{D}}(\zeta,z) \leqslant K(z,z_0)\mathrm{d}\omega_{\mathscr{D}}(\zeta,z_0)$ as we have seen above, the previous relation makes

$$U_{\varphi_n}(z) = \int_{\partial\mathscr{D}} \varphi_n(\zeta)\mathrm{d}\omega_{\mathscr{D}}(\zeta, z) \underset{n}{\longrightarrow} \omega_{\mathscr{D}}(E, z)$$

for every $z \in \mathscr{D}$; the convergence is even u.c.c. in $\mathscr{D}$ because $0 \leqslant U_{\varphi_n}(z) \leqslant 1$ there for each n. Therefore $\omega_{\mathscr{D}}(E, z)$ is harmonic in $z \in \mathscr{D}$ since the $U_{\varphi_n}(z)$ are.

Harmonic measure is also available for many *unbounded* domains $\mathscr{D}$. Suppose we have such a domain (perhaps of infinite connectivity) with a decent boundary $\partial\mathscr{D}$. The latter may consist of infinitely many pieces, but each individual piece should be nice, and they should not accumulate near any finite point in such a way as to cause trouble for the solution of the Dirichlet problem. In such case, $\partial\mathscr{D}$ is at least *locally compact* and, if $\varphi \in \mathscr{C}_0(\partial\mathscr{D})$ (the space of functions continuous on $\partial\mathscr{D}$ which *tend to zero* as one goes out towards ∞ thereon), there is one and *only* one function U_φ *harmonic and bounded* in $\mathscr{D}$, and continuous up to $\partial\mathscr{D}$, with $U_\varphi(\zeta) = \varphi(\zeta)$, $\zeta \in \partial\mathscr{D}$. (Here it is *absolutely necessary* to assume *boundedness* of U_φ in $\mathscr{D}$ in order to get *uniqueness*; look at the function y in $\Im z > 0$ which takes the value 0 on $\mathbb{R}$. Uniqueness of the *bounded* harmonic function with prescribed boundary values is a direct consequence of the *first* Phragmén–Lindelöf theorem in §C, Chapter III.) Riesz' representation theorem still holds in the present situation, and we will have $(*)$ for $\varphi \in \mathscr{C}_0(\partial\mathscr{D})$. The examination of the μ_z carried out above goes through almost without change, and we write $d\mu_z(\zeta) = d\omega_{\mathscr{D}}(\zeta, z)$ as before, calling $\omega_{\mathscr{D}}(\ ,z)$ the *harmonic measure for* $\mathscr{D}$, *as seen from* z. It serves to recover *bounded* functions harmonic in $\mathscr{D}$ and continuous up to $\partial\mathscr{D}$ from their boundary values, at least when the latter come from functions in $\mathscr{C}_0(\partial\mathscr{D})$.

Let us return for a moment to *bounded, finitely connected domains* $\mathscr{D}$. Suppose we are given a function $f(z)$, *analytic and bounded in* $\mathscr{D}$, and *continuous up to* $\partial\mathscr{D}$. An important problem in the theory of functions is to *obtain an upper bound for* $|f(z)|$ *when* $z \in \mathscr{D}$, *in terms of the boundary values* $f(\zeta)$, $\zeta \in \partial\mathscr{D}$. A very useful estimate is furnished by the

Theorem (on harmonic estimation). *For* $z \in \mathscr{D}$,

$$(\dagger) \qquad \log|f(z)| \leqslant \int_{\partial\mathscr{D}} \log|f(\zeta)|\, d\omega_{\mathscr{D}}(\zeta, z).$$

Proof. The result is really a generalization of Jensen's inequality. Take any $M > 0$. The function

$$V_M(z) = \max(\log|f(z)|, -M)$$

is continuous in $\bar{\mathscr{D}}$ and *subharmonic* in $\mathscr{D}$. Therefore the difference

$$V_M(z) - \int_{\partial\mathscr{D}} V_M(\zeta) d\omega_{\mathscr{D}}(\zeta, z)$$

is subharmonic in $\mathscr{D}$ and continuous up to $\partial\mathscr{D}$ where it takes the boundary value $V_M(\zeta) - V_M(\zeta) = 0$ everywhere. Hence that difference is $\leqslant 0$ throughout $\mathscr{D}$ by the principle of maximum, and

$$\log|f(z)| \leqslant V_M(z) \leqslant \int_{\partial\mathscr{D}} V_M(\zeta) d\omega_{\mathscr{D}}(\zeta, z)$$

for $z \in \mathscr{D}$. On making $M \to \infty$, the right side tends to

$$\int_{\partial\mathscr{D}} \log|f(\zeta)|\, d\omega_{\mathscr{D}}(\zeta, z)$$

by Lebesgue's monotone convergence theorem, since $\log|f(\zeta)|$, and hence the $V_M(\zeta)$, are *bounded above*, $|f(z)|$ being continuous and thus bounded on the compact set $\bar{\mathscr{D}}$. The proof is finished.

The result just established is true for *bounded* analytic functions in *unbounded* domains subject to the restrictions on such domains mentioned above. *Here the boundedness of $f(z)$ in $\mathscr{D}$ becomes crucial* (look at the functions e^{-inz} in $\Im z > 0$ with $n \to \infty$!). Verification of this proceeds very much as above, using the functions $V_M(\zeta)$. These are continuous and bounded (above *and* below) on $\partial\mathscr{D}$, so the functions

$$H_M(z) = \int_{\partial\mathscr{D}} V_M(\zeta)\mathrm{d}\omega_{\mathscr{D}}(\zeta, z)$$

are harmonic and bounded in $\mathscr{D}$, and for each $\zeta_0 \in \mathscr{D}$ we can *check directly*, by using the *approximate identity* property of $\omega_{\mathscr{D}}(\ ,z)$ established in the above discussion, that

$$H_M(z) \longrightarrow V_M(\zeta_0) \quad \text{for} \quad z \longrightarrow \zeta_0.$$

(It is *not necessary* that $V_M(\zeta)$ belong to $\mathscr{C}_0(\partial\mathscr{D})$ in order to draw this conclusion; only that it be *continuous and bounded* on $\partial\mathscr{D}$.) The difference

$$V_M(z) - H_M(z)$$

is thus *subharmonic and bounded above* in $\mathscr{D}$, and tends to 0 as z tends to *any* point of $\partial\mathscr{D}$. We can therefore conclude by the first Phragmén–Lindelöf theorem of §C, Chapter III (or, rather, by its analogue for *subharmonic* functions), that $V_M(z) - H_M(z) \leqslant 0$ in $\mathscr{D}$. The rest of the argument is as above.

The inequality (†) has one very important consequence, called the ***theorem on two constants***. *Let $f(z)$ be analytic and bounded in a domain $\mathscr{D}$ of the kind considered above, and continuous up to $\partial\mathscr{D}$. Suppose that $|f(\zeta)| \leqslant M$ on $\partial\mathscr{D}$, and that there is a Borel set $E \subseteq \partial\mathscr{D}$ with $|f(\zeta)| \leqslant$ some number m ($< M$) on E. Then, for $z \in \mathscr{D}$,*

$$\boxed{|f(z)| \leqslant m^{\omega_{\mathscr{D}}(E,z)} M^{1-\omega_{\mathscr{D}}(E,z)}.}$$

Deduction of this inequality from (†) is immediate.

Much of the importance of harmonic measure in analysis is due to this formula and to (†). For this reason, analysts have devoted (and continue to devote) considerable attention to the *estimation* of harmonic measure. We shall see some of this work later on in the present book. The systematic use of harmonic measure in analysis is mainly due to Nevanlinna, who also gave us the *name* for it. There are beautiful examples of its application in his book, *Eindeutige analytische Funktionen* (now translated into English), of which every analyst should own a copy.

Before ending our discussion of harmonic measure, let us describe a few more of its qualitative properties.

The first observation to be made is that the measures $\omega_{\mathscr{D}}(\ ,z)$ are *absolutely continuous with respect to arc length* on $\partial\mathscr{D}$ for the kind of domains considered here. This will follow if we can show that

$$\omega_{\mathscr{D}}(E_n, z_0) \xrightarrow[n]{} 0 \quad \text{for} \quad z_0 \in \mathscr{D}$$

when the E_n *lie on any particular component* Γ of $\partial\mathscr{D}$ and

$$\int_\Gamma \chi_{E_n}(\zeta)|d\zeta| \xrightarrow[n]{} 0.$$

(Here, χ_{E_n} denotes the *characteristic function* of E_n.) We do this by comparing $\omega_{\mathscr{D}}(\ ,z)$ with harmonic measure for a *simply connected domain*; the method is of independent interest and is frequently used.

Let $\mathscr{E}$ be the *simply connected domain on the Riemann sphere* (including perhaps ∞), *bounded by the component* Γ *of* $\partial\mathscr{D}$ *and including all the points of* $\mathscr{D}$.

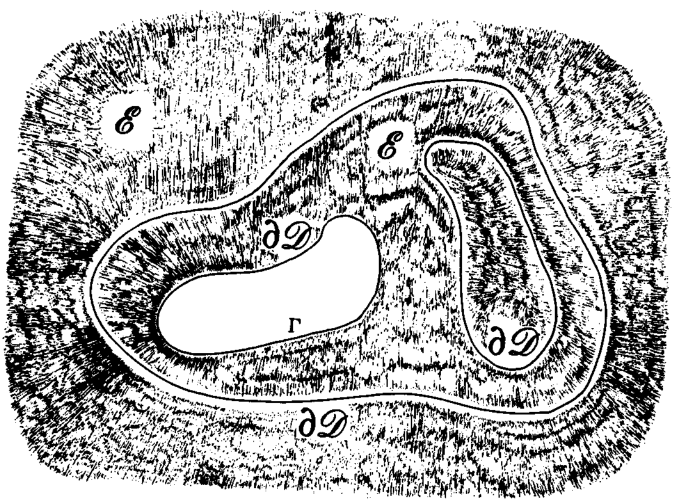

Figure 51

If $\varphi \in \mathscr{C}(\partial\mathscr{D})$ is *positive*, and *zero on all* the *components* of $\partial\mathscr{D}$ *save* Γ, we have

$$\int_\Gamma \varphi(\zeta)d\omega_{\mathscr{E}}(\zeta, z) \geqslant \int_{\partial\mathscr{D}} \varphi(\zeta)d\omega_{\mathscr{D}}(\zeta, z)$$

for $z \in \mathscr{D}$. Indeed, both integrals give us functions harmonic in $\mathscr{D}$ ($\subseteq \mathscr{E}$!), and continuous up to $\partial\mathscr{D}$. The right-hand function, $U_\varphi(z)$, equals $\varphi(\zeta)$ on Γ and *zero* on the *other* components of $\partial\mathscr{D}$. The left-hand one – call it $V(z)$ for the moment – *also* equals $\varphi(z)$ on Γ but is *surely* $\geqslant 0$ on the *other* components of $\partial\mathscr{D}$, because they lie in $\mathscr{E}$ and $\varphi \geqslant 0$. Therefore $V(z) \geqslant U_\varphi(z)$ throughout $\mathscr{D}$ by the principle of maximum, as claimed. This inequality holds for *every* function φ of the kind described above, whence, on Γ,

$$\boxed{d\omega_{\mathscr{E}}(\zeta, z) \;\geqslant\; d\omega_{\mathscr{D}}(\zeta, z) \quad \text{for} \quad z \in \mathscr{D}.}$$

This relation is an example of what Nevanlinna called ***the principle of extension of domain.***

Let us return to our sets

$$E_n \subseteq \Gamma \quad \text{with} \quad \int_\Gamma \chi_{E_n}(\zeta)\,|d\zeta| \;\underset{n}{\longrightarrow}\; 0;$$

in order to verify that

$$\omega_{\mathscr{D}}(E_n, z) \underset{n}{\longrightarrow} 0, \quad z \in \mathscr{D},$$

it is enough, in virtue of the inequality just established, to check that $\omega_{\mathscr{E}}(E_n, z_0) \underset{n}{\longrightarrow} 0$ for each $z_0 \in \mathscr{E}$. Because $\mathscr{E}$ is *simply connected*, we may, however, use the formula derived near the beginning of the present article. Fixing $z_0 \in \mathscr{E}$, take a conformal mapping F of $\mathscr{E}$ onto $\{|w| < 1\}$ which sends z_0 to 0. From the formula just mentioned, it is clear that

$$\omega_{\mathscr{E}}(E_n, z_0) \;=\; \frac{1}{2\pi}\int_\Gamma \chi_{E_n}(\zeta)\,|dF(\zeta)|.$$

The component Γ of $\partial\mathscr{D}$ is, however, *rectifiable*; a theorem of the brothers Riesz therefore guarantees that the mapping F from Γ onto the unit circumference is *absolutely continuous with respect to arc length.* For domains $\mathscr{D}$ whose boundary components are given explicitly and in fairly simple form (the sort we will be dealing with), that property can also be verified directly. We can hence write

$$\omega_{\mathscr{E}}(E_n, z_0) \;=\; \frac{1}{2\pi}\int_\Gamma \chi_{E_n}(\zeta)\left|\frac{dF(\zeta)}{d\zeta}\right||d\zeta|$$

with

$$\left|\frac{dF(\zeta)}{d\zeta}\right| \quad \text{in} \quad L_1(\Gamma,\ |d\zeta|),$$

and from this we see that $\omega_{\mathscr{E}}(E_n, z_0) \xrightarrow[n]{} 0$ when

$$\int_{\Gamma} \chi_{E_n}(\zeta)|d\zeta| \xrightarrow[n]{} 0.$$

The absolute continuity of $\omega_{\mathscr{D}}(\ , z)$ with respect to arc length on $\partial\mathscr{D}$ is thus verified.

The property just established makes it possible for us to write

$$\omega_{\mathscr{D}}(E, z) = \int_{\partial\mathscr{D}} \chi_E(\zeta) \frac{d\omega_{\mathscr{D}}(\zeta, z)}{|d\zeta|} \cdot |d\zeta|$$

for $E \subseteq \partial\mathscr{D}$ and $z \in \mathscr{D}$. It is important for us to be able to majorize the integral on the right by one of the form

$$K_z \int_{\partial\mathscr{D}} \chi_E(\zeta)|d\zeta|$$

(with K_z depending on z and, of course, on $\mathscr{D}$) when dealing with *certain kinds* of simply connected domains $\mathscr{D}$. In order to see for *which* kind, let us, for fixed $z_0 \in \mathscr{D}$, take a conformal mapping F of $\mathscr{D}$ onto $\{|w| < 1\}$ which sends z_0 to 0 and apply the formula used in the preceding argument, which here takes the form

$$\omega_{\mathscr{D}}(E, z_0) = \frac{1}{2\pi} \int_{\partial\mathscr{D}} \chi_E(\zeta) \left| \frac{dF(\zeta)}{d\zeta} \right| |d\zeta|.$$

If the boundary $\partial\mathscr{D}$ is an *analytic curve*, or merely has a *differentiably turning tangent*, the derivative $F'(z)$ of the conformal mapping function *will be* continuous up to $\partial\mathscr{D}$; in such circumstances $|dF(\zeta)/d\zeta|$ is bounded on $\partial\mathscr{D}$ (the bound depends evidently on z_0), and we *have* a majorization of the desired kind. This is even true when *$\partial\mathscr{D}$ has a finite number of corners* and is sufficiently smooth *away* from them, *provided that all those corners stick out.*

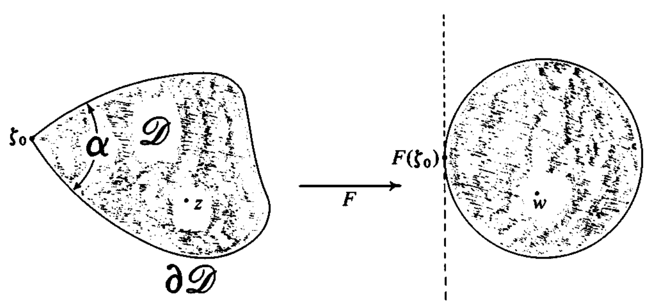

Figure 52

In this situation, where $\partial\mathscr{D}$ has a corner with internal angle α at ζ_0, $F(z) = F(\zeta_0) + (C + o(1))(z - \zeta_0)^{\pi/\alpha}$ for z in $\bar{\mathscr{D}}$ (*sic*!) *near* ζ_0; we see that $F'(\zeta_0) = 0$ if $\alpha < \pi$, and that $F'(\zeta)$ is *near* 0 if $\zeta \in \partial\mathscr{D}$ is *near* ζ_0 (sufficient smoothness of $\partial\mathscr{D}$ away from its corners is being assumed). In the present case, then, $|F'(\zeta)|$ *is* bounded on $\partial\mathscr{D}$, and an estimate

$$\omega_{\mathscr{D}}(E, z_0) \leqslant K_{z_0} \int_{\partial\mathscr{D}} \chi_E(\zeta)|d\zeta|$$

does hold good. It is *really necessary* that the corners *stick out*. If, for instance, $\alpha > \pi$, then $|F'(\zeta_0)| = \infty$, and $|F'(\zeta)|$ *tends* to ∞ for ζ on $\partial\mathscr{D}$ tending to ζ_0:

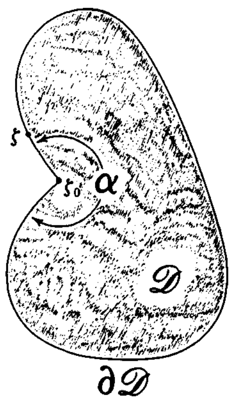

Figure 53

Here, we *do not* have

$$\omega_{\mathscr{D}}(E, z_0) \leqslant \text{const.} \int_{\partial\mathscr{D}} \chi_E(\zeta)|d\zeta|$$

for sets $E \subseteq \partial\mathscr{D}$ located near ζ_0.

Let us conclude with a general examination of the *boundary behaviour* of $\omega_{\mathscr{D}}(E, z)$ for $E \subseteq \partial\mathscr{D}$. Consider first of all the case where E is an *arc*, σ, *on one of the components* of $\partial\mathscr{D}$. Then the *simple approximate identity property* of $\omega_{\mathscr{D}}(\ \ , z)$ established above immediately shows that

$$\omega_{\mathscr{D}}(\sigma, z) \longrightarrow 1 \quad \text{if} \quad z \longrightarrow \zeta \in \sigma$$

and ζ is not an endpoint of σ, while

$$\omega_{\mathscr{D}}(\sigma, z) \longrightarrow 0 \quad \text{if} \quad z \longrightarrow \zeta \in \partial\mathscr{D} \sim \sigma$$

and ζ is not an endpoint of σ. If $z \in \mathscr{D}$ *tends* to an endpoint of σ, we cannot say much (in general) about $\omega_{\mathscr{D}}(\sigma, z)$, save that it remains between 0 and

1. These properties, however, *already suffice to determine the harmonic function* $\omega_{\mathscr{D}}(\sigma, z)$ (defined in $\mathscr{D}$) *completely*. This may be easily verified by using the principle of maximum together with an evident modification of the first Phragmén–Lindelöf theorem from §C, Chapter III; such verification is left to the reader. One sometimes *uses* this characterization in order to compute or estimate harmonic measure. Of course, once $\omega_{\mathscr{D}}(\sigma, z)$ is *known for arcs* $\sigma \subseteq \partial\mathscr{D}$, we can *get* $\omega_{\mathscr{D}}(E, z)$ for Borel sets E by the standard construction applying to all positive Radon measures.

What about the boundary behaviour of $\omega_{\mathscr{D}}(E, z)$ for a *more general* set E? We only consider *closed sets E lying on a single component* Γ of $\partial\mathscr{D}$; knowledge about this situation is all that is needed in practice.

Take, then, a closed subset E of the component Γ of $\partial\mathscr{D}$. In the first place, $\omega_{\mathscr{D}}(E, z) \leqslant \omega_{\mathscr{D}}(\Gamma, z)$. When z tends to any point of a component Γ' of $\partial\mathscr{D}$ *different from* Γ, $\omega_{\mathscr{D}}(\Gamma, z)$ tends to *zero* by the previous discussion (Γ is an arc without endpoints!) Hence $\omega_{\mathscr{D}}(E, z) \longrightarrow 0$ for $z \longrightarrow \zeta$ if $\zeta \in \partial\mathscr{D}$ belongs to a *component of the* latter *other than* Γ.

Examination of the boundary of $\omega_{\mathscr{D}}(E, z)$ *for z near* Γ is more delicate.

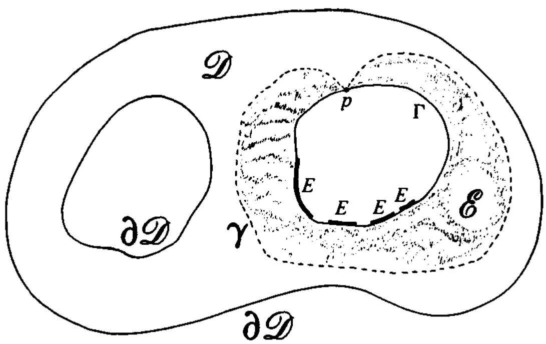

Figure 54

Take any point p on Γ lying outside the closed set E (if E were *all* of Γ, we could conclude by the case for *arcs* handled previously), and draw a curve γ lying in $\mathscr{D}$ like the one shown, with its two endpoints at p. Together, the curves γ and Γ bound a certain *simply connected* domain $\mathscr{E} \subseteq \mathscr{D}$.

We are going to derive the formula

$$\omega_{\mathscr{D}}(E, z) \;=\; \int_{\gamma} \omega_{\mathscr{D}}(E, \zeta)\, d\omega_{\mathscr{E}}(\zeta, z) \;+\; \omega_{\mathscr{E}}(E, z),$$

valid for $z \in \mathscr{E}$. Take any finite union $\mathscr{U}$ of arcs on Γ containing the closed set E but avoiding a whole neighborhood of the point p, and let ψ be any function continuous on Γ with $0 \leqslant \psi(\zeta) \leqslant 1$, $\psi(\zeta) \equiv 0$ outside $\mathscr{U}$, and $\psi(\zeta) \equiv 1$ on E. Since ψ is zero on a neighborhood of p, the function

$$U_\psi(z) = \int_\Gamma \psi(\zeta)\, d\omega_{\mathscr{D}}(\zeta, z)$$

tends to *zero* as $z \to p$. Write $U_\psi(\zeta) = \psi(\zeta)$ for $\zeta \in \Gamma$; the function $U_\psi(\zeta)$ then becomes continuous on $\Gamma \cup \gamma = \partial\mathscr{E}$, so

$$V(z) = \int_{\partial\mathscr{E}} U_\psi(\zeta)\, d\omega_{\mathscr{E}}(\zeta, z)$$

is harmonic in $\mathscr{E}$ and continuous up to $\partial\mathscr{E}$, where it takes the boundary value $U_\psi(z)$. For this reason, the function

$$U_\psi(z) - V(z),$$

harmonic in $\mathscr{E}$, is *identically zero* therein, and we have

$$\int_\gamma U_\psi(\zeta) d\omega_{\mathscr{E}}(\zeta, z) + \int_\Gamma \psi(\zeta)\, d\omega_{\mathscr{E}}(\zeta, z) = V(z) = U_\psi(z)$$
$$= \int_\Gamma \psi(\zeta)\, d\omega_{\mathscr{D}}(\zeta, z)$$

for $z \in \mathscr{E}$. Making the covering $\mathscr{U}$ shrink down to E, we end with

$$\omega_{\mathscr{D}}(E, z) = \int_\gamma \omega_{\mathscr{D}}(E, \zeta)\, d\omega_{\mathscr{E}}(\zeta, z) + \omega_{\mathscr{E}}(E, z),$$

our desired relation.

The function $\omega_{\mathscr{D}}(E, \zeta)$ is continuous on γ and zero at p, because $\omega_{\mathscr{D}}(E, z) \leqslant$ each of the functions $U_\psi(z)$ considered above. The function harmonic in $\mathscr{E}$ with boundary values equal to $\omega_{\mathscr{D}}(E, \zeta)$ on γ and to *zero* on Γ is therefore *continuous on* $\gamma \cup \Gamma = \partial\mathscr{E}$, so

$$\int_\gamma \omega_{\mathscr{D}}(E, \zeta)\, d\omega_{\mathscr{E}}(\zeta, z)$$

tends to zero when $z \in \mathscr{E}$ *tends to any point of* Γ. Referring to the previous relation, we see that

$$(\ddagger) \qquad \omega_{\mathscr{D}}(E, z) - \omega_{\mathscr{E}}(E, z) \longrightarrow 0$$

whenever $z \in \mathscr{E}$ tends to any point of Γ. The *behaviour* of the *first term* on the left is thus *the same* as that of *the second*, for $z \longrightarrow \zeta_0 \in \Gamma$.

Because $\mathscr{E}$ is simply connected, we may use conformal mapping to study $\omega_{\mathscr{E}}(E, z)$'s boundary behaviour.

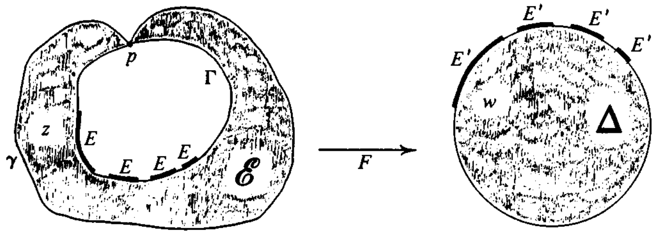

Figure 55

Let F map $\mathscr{E}$ conformally onto $\Delta = \{|w| < 1\}$; F takes $E \subseteq \Gamma$ onto a certain closed subset E' of the unit circumference, and we have

$$\omega_{\mathscr{E}}(E, z) = \omega_{\Delta}(E', F(z))$$

for $z \in \mathscr{E}$ (see the formula near the beginning of this article). *Assume that* Γ *is smooth, or at least that E lies on a smooth part of* Γ. Then it is a fact (easily verifiable directly in the cases which will interest us – the general result for curves with a tangent at every point being due to Lindelöf) that *F preserves angles right up to* Γ, as long as we stay away from p:

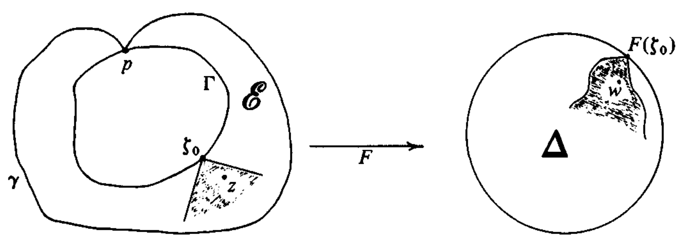

Figure 56

This means that if $z \in \mathscr{E}$ tends to any point ζ_0 of *E from within an acute angle with vertex at* ζ_0, *lying strictly in* $\mathscr{E}$ (we henceforth write this as '$z \xrightarrow{\angle} \zeta_0$'), *the image* $w = F(z)$ *will tend to* $F(\zeta_0) \in E'$ *from within such an angle lying in* Δ.

However,

$$\omega_{\Delta}(E', w) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |w|^2}{|w - e^{i\varphi}|^2} \chi_{E'}(e^{i\varphi}) d\varphi.$$

A study of the boundary behaviour of the integral on the right was made in §B of Chapter II. According to the result proved there,

$$\omega_\Delta(E', w) \longrightarrow \chi_{E'}(\omega_0)$$

as $w \not\angle\!\!\longrightarrow \omega_0$, for almost every ω_0 on the unit circumference. In the present situation (*E' closed*) we even have

$$\omega_\Delta(E', w) \longrightarrow 0$$

whenever $w \rightarrow$ a point of the unit circumference *not* in E'. Under the conformal mapping F, sets of (arc length) measure zero on Γ *correspond precisely* to sets of measure zero on $\{|\omega| = 1\}$. (As before, one can verify this statement directly for the simple situations we will be dealing with. The general result is due to F. and M. Riesz.) In view of the angle preservation just described, we see, going back to $\mathscr{E}$, that, *for almost every* $\zeta_0 \in E$,

$$\omega_{\mathscr{E}}(E, z) \longrightarrow 1 \quad \text{as} \quad z \not\angle\!\!\longrightarrow \xi_0,$$

and that

$$\omega_{\mathscr{E}}(E, z) \longrightarrow 0 \quad \text{as} \quad z \longrightarrow \zeta_0$$

for $\zeta_0 \in \Gamma$ *not belonging to E.*

Now we bring in $(\ddagger)$. According to what has just been shown, that relation tells us that

$$\omega_{\mathscr{D}}(E, z) \longrightarrow 1 \quad \text{as} \quad z \not\angle\!\!\longrightarrow \zeta_0$$

for *almost every* $\zeta_0 \in E$, whilst

$$\omega_{\mathscr{D}}(E, z) \longrightarrow 0 \quad \text{as} \quad z \longrightarrow \zeta_0$$

for $\zeta_0 \in \Gamma \sim E$, except possibly when $\zeta_0 = p$. By moving p slightly and taking a new curve γ (and new domain $\mathscr{E}$) we can, however, remove any doubt about that case. Referring to the already known boundary behaviour of $\omega_{\mathscr{D}}(E, z)$ at the *other* components of $\partial\mathscr{D}$, we have, finally,

$$\omega_{\mathscr{D}}(E, z) \longrightarrow \begin{cases} 0 & \text{as} \quad z \longrightarrow \zeta_0 \in \partial\mathscr{D} \sim E, \\ 1 & \text{as} \quad z \not\angle\!\!\longrightarrow \zeta_0 \text{ for } \textit{almost every } \zeta_0 \in E. \end{cases}$$

This completes our elementary discussion of harmonic measure.

2. Beurling's improvement of Levinson's theorem

We need two auxiliary results.

Lemma. *Let μ be a totally finite (complex) measure on $\mathbb{R}$, and put*

$$\hat{\mu}(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda t} \, d\mu(t)$$

(as usual). Suppose, for some real λ_0, that

$$\int_{\lambda_0}^{\infty} e^{-Y\lambda} e^{iX\lambda} \hat{\mu}(\lambda)\, d\lambda \;\equiv\; 0$$

and

$$\int_{-\infty}^{\lambda_0} e^{Y\lambda} e^{iX\lambda} \hat{\mu}(\lambda)\, d\lambda \;\equiv\; 0$$

for all $X \in \mathbb{R}$ and all $Y > 0$. Then $\mu \equiv 0$.

Proof. If we write $d\mu_{\lambda_0}(t) = e^{i\lambda_0 t}\, d\mu(t)$, we have $\hat{\mu}(\tau + \lambda_0) = \hat{\mu}_{\lambda_0}(\tau)$, and the identical vanishing of μ_{λ_0} clearly implies that of μ. In terms of $\hat{\mu}_{\lambda_0}$, the two relations from the hypothesis reduce to

$$\int_{0}^{\infty} e^{-Y\tau} e^{iX\tau} \hat{\mu}_{\lambda_0}(\tau)\, d\tau \;\equiv\; 0,$$

$$\int_{-\infty}^{0} e^{Y\tau} e^{iX\tau} \hat{\mu}_{\lambda_0}(\tau)\, d\tau \;\equiv\; 0,$$

valid for $X \in \mathbb{R}$ and $Y > 0$. Therefore, *if we prove the lemma for the case where $\lambda_0 = 0$*, we will *have $\mu \equiv 0$*. We thus proceed under the assumption that $\lambda_0 = 0$.

By direct calculation (!), for $X \in \mathbb{R}$ and $Y > 0$,

$$\frac{Y}{(X+t)^2 + Y^2} \;=\; \frac{1}{2}\int_{-\infty}^{\infty} e^{-Y|\lambda|} e^{iX\lambda} e^{i\lambda t}\, d\lambda.$$

The integral on the right is absolutely convergent, so, multiplying it by $d\mu(t)$, integrating with respect to t, and changing the order of integration, we find

$$\int_{-\infty}^{\infty} \frac{Y}{(X+t)^2 + Y^2}\, d\mu(t) \;=\; \frac{1}{2}\int_{-\infty}^{\infty} e^{-Y|\lambda|} e^{iX\lambda} \hat{\mu}(\lambda)\, d\lambda.$$

Under our assumption, the integral on the right vanishes identically for $X \in \mathbb{R}$ and $Y > 0$. Calling the one on the left $J_Y(X)$, we have, however,

$$J_Y(-X)\, dX \;\longrightarrow\; \pi\, d\mu(X) \quad w^*$$

for $Y \longrightarrow 0$. Therefore $d\mu(X) \equiv 0$, and we are done.

Lemma. *Let $M(r) \geqslant 0$ be increasing on $[0, \infty)$, and put*

$$M_*(r) = \min(r, M(r))$$

for $r \geqslant 0$. Then, if

$$\int_0^{\infty} \frac{M(r)}{1 + r^2}\, dr \;=\; \infty$$

we also have

$$\int_0^\infty \frac{M_*(r)}{1+r^2}\mathrm{d}r = \infty.$$

Proof. Is like that of the lemma in §A.3. The following diagram shows that $M(r) = M_*(r)$ outside of a certain open set $\mathcal{O}$, the union of disjoint intervals (a_n, b_n), on which $M_*(r) = r$.

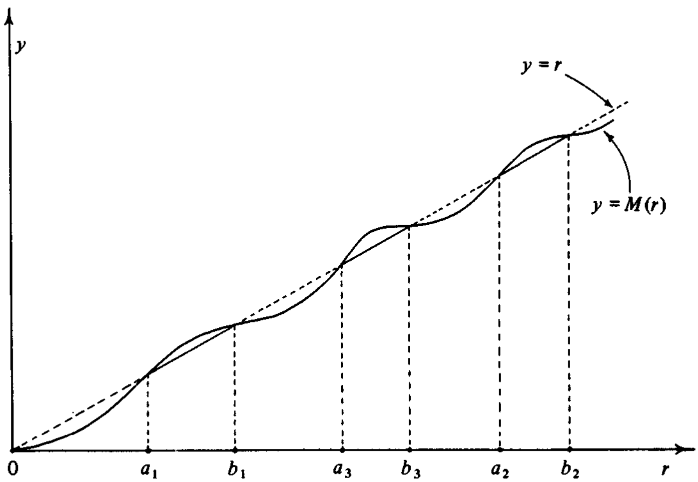

Figure 57

It is enough to show that

$$\int_1^\infty \frac{M_*(r)}{r^2}\mathrm{d}r = \infty.$$

If

$$\int_{\mathcal{O}\cap[1,\infty)} \frac{M_*(r)}{r^2}\mathrm{d}r = \infty,$$

we are already finished; let us therefore assume that this last integral is *finite*. We then sureiy have

$$\sum_{a_n\geqslant 1}\int_{a_n}^{b_n} \frac{M_*(r)}{r^2}\mathrm{d}r = \sum_{a_n\geqslant 1}\int_{a_n}^{b_n}\frac{\mathrm{d}r}{r} = \sum_{a_n\geqslant 1}\log\left(\frac{b_n}{a_n}\right) < \infty,$$

so $b_n/a_n \xrightarrow[n]{} 1$ which, fed back into the last relation, gives us

$$\sum_{a_n\geqslant 1}\frac{b_n - a_n}{a_n} < \infty.$$

Since, however, $M(r)$ is increasing, we see from the picture that

$$\int_{a_n}^{b_n} \frac{M(r)}{r^2}\,\mathrm{d}r \;\leqslant\; M(b_n)\int_{a_n}^{b_n} \frac{\mathrm{d}r}{r^2} \;=\; b_n\cdot\frac{b_n - a_n}{a_n b_n} \;=\; \frac{b_n - a_n}{a_n}.$$

Therefore

$$\sum_{a_n \geqslant 1} \int_{a_n}^{b_n} \frac{M(r)}{r^2}\,\mathrm{d}r \;<\; \infty$$

by the previous relation, so, since we are assuming

$$\int_0^\infty \frac{M(r)}{1+r^2}\,\mathrm{d}r \;=\; \infty$$

which implies

$$\int_1^\infty \frac{M(r)}{r^2}\,\mathrm{d}r \;=\; \infty$$

(M being increasing), we must have

$$\int_E \frac{M(r)}{r^2}\,\mathrm{d}r \;=\; \infty,$$

where

$$E \;=\; [1,\infty) \sim \bigcup_{a_n \geqslant 1} (a_n, b_n).$$

The set E is either *equal* to the complement of $\mathcal{O}$ in $[1,\infty)$ or else *differs* therefrom by an interval of the form $[1, b_k)$ where (a_k, b_k) is a component of $\mathcal{O}$ straddling the point 1 (in case there is one). Since $M_*(r) = M(r)$ *outside* $\mathcal{O}$, we thus have

$$\int_E \frac{M_*(r)}{r^2}\,\mathrm{d}r \;=\; \infty$$

(including in the possible situation where $b_k = \infty$), and therefore

$$\int_1^\infty \frac{M_*(r)}{r^2}\,\mathrm{d}r \;=\; \infty$$

as required.

Theorem (Beurling). *Let μ be a finite complex measure on $\mathbb{R}$ such that*

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\log\left(\frac{1}{\int_{-\infty}^{x}|\mathrm{d}\mu(t)|}\right)\mathrm{d}x \;=\; \infty.$$

If

$$\hat{\mu}(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda t} d\mu(t)$$

vanishes *on a set* $E \subseteq \mathbb{R}$ *of* positive measure, *then* $\mu \equiv 0$.

Proof. In the *complex* λ-plane, let $\mathscr{D}$ be the strip

$$\{0 < \Im\lambda < 1\};$$

we work with harmonic measure $\omega_{\mathscr{D}}(\ ,\lambda)$ *for* $\mathscr{D}$ (see article 1).

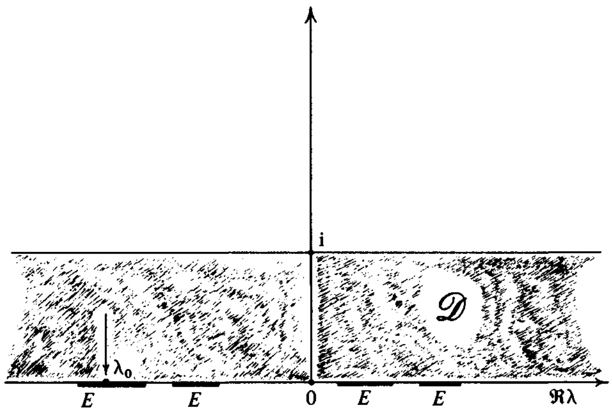

Figure 58

Because $|E| > 0$, E contains a compact set of *positive* Lebesgue measure; there is thus no loss of generality in assuming E *closed* and *bounded*. According to the discussion at the end of the previous article, we then have

$$\omega_{\mathscr{D}}(E,\lambda) \longrightarrow 1$$

as $\lambda \not\longrightarrow \lambda_0$ *for almost every* λ_0 *in the set* E (*of positive measure*). There is thus certainly a $\lambda_0 \in E$ with

$$\omega_{\mathscr{D}}(E,\lambda_0 + i\tau) \longrightarrow 1$$

for $\tau \to 0+$; *we take such a* λ_0 *and fix it* throughout the rest of the proof.

We are going to show that

$$\int_{-\infty}^{\lambda_0} e^{Y\lambda} e^{iX\lambda} \hat{\mu}(\lambda)\, d\lambda \equiv 0$$

and

$$\int_{\lambda_0}^{\infty} e^{-Y\lambda} e^{iX\lambda} \hat{\mu}(\lambda)\, d\lambda \equiv 0$$

for $Y > 0$ and $X \in \mathbb{R}$; by the *first* of the above lemmas we will then have $\mu \equiv 0$ which is what we want to establish. The argument here is the same for any value of λ_0. *In order not to burden the exposition with a proliferation of symbols, we give it for the case where $\lambda_0 = 0$, which we henceforth assume. We have, then, $\hat{\mu}(\lambda) = 0$ on the closed set E, $0 \in E$, and $\omega_{\mathscr{D}}(E, \mathrm{i}\tau) \longrightarrow 1$ for $\tau \longrightarrow 0+$.*

Consider the *second* of the above two integrals. Under the present circumstances, it is equal to

$$\int_0^\infty \mathrm{e}^{\mathrm{i}Z\lambda}\hat{\mu}(\lambda)\,\mathrm{d}\lambda \;=\; F(Z),$$

say, where $Z = X + \mathrm{i}Y$. The function $F(Z)$ defined in this fashion is *analytic* for $\Im Z > 0$ and *bounded* in each half plane of the form $\Im Z \geqslant h > 0$. By §G.2 of Chapter III, we will therefore have $F(Z) \equiv 0$ for $\Im Z > 0$ provided that

$$(*) \qquad \int_0^\infty \frac{\log|F(X+\mathrm{i})|}{1+X^2}\,\mathrm{d}X \;=\; -\infty.$$

This relation we now proceed to establish.

Take a number $A > 0$ (later on, A will be made to depend on X), and write

$$\hat{\mu}(\lambda) \;=\; \hat{\mu}_A(\lambda) + \hat{\rho}_A(\lambda),$$

with

$$\hat{\mu}_A(\lambda) \;=\; \int_{-A}^\infty \mathrm{e}^{\mathrm{i}\lambda t}\,\mathrm{d}\mu(t)$$

and

$$\hat{\rho}_A(\lambda) \;=\; \int_{-\infty}^{-A} \mathrm{e}^{\mathrm{i}\lambda t}\,\mathrm{d}\mu(t).$$

The function $\hat{\mu}_A(\lambda)$ is actually defined for $\Im\lambda \geqslant 0$ and *analytic* when $\Im\lambda > 0$. $\hat{\rho}_A(\lambda)$ is not, in general, defined for $\Im\lambda > 0$; when A is large, it is, however, *very small* on the real axis in view of our assumption on

$$\int_{-\infty}^x |\mathrm{d}\mu(t)|$$

in the hypothesis. We think of $\hat{\rho}_A(\lambda)$ as a *correction* to $\hat{\mu}_A(\lambda)$ on $\mathbb{R}$.

Wlog,

$$\int_{-\infty}^\infty |\mathrm{d}\mu(t)| \;\leqslant\; 1.$$

Then, writing

$$\int_{-\infty}^{-A} |d\mu(t)| = e^{-M(A)},$$

we have

$$|\hat{\rho}_A(\lambda)| \leqslant e^{-M(A)}, \quad \lambda \in \mathbb{R},$$

with $M(A) \geqslant 0$ and increasing for $A \geqslant 0$. Going back to

$$F(X + i) = \int_0^{\infty} e^{-\lambda} e^{iX\lambda} \hat{\mu}(\lambda) d\lambda,$$

we see that the latter *differs* from

$$\int_0^{\infty} e^{i(X+i)\lambda} \hat{\mu}_A(\lambda) d\lambda$$

by a quantity in modulus

$$\leqslant \int_0^{\infty} e^{-\lambda} |\hat{\rho}_A(\lambda)| d\lambda \leqslant e^{-M(A)}.$$

As we have already remarked, this is very small when A is large. Showing that $|F(X + i)|$ gets small enough for $(*)$ to hold thus turns out to reduce to the careful estimation of

$$\left| \int_0^{\infty} e^{i(X+i)\lambda} \hat{\mu}_A(\lambda) d\lambda \right|.$$

We use Cauchy's theorem for that.

Taking Γ as shown,

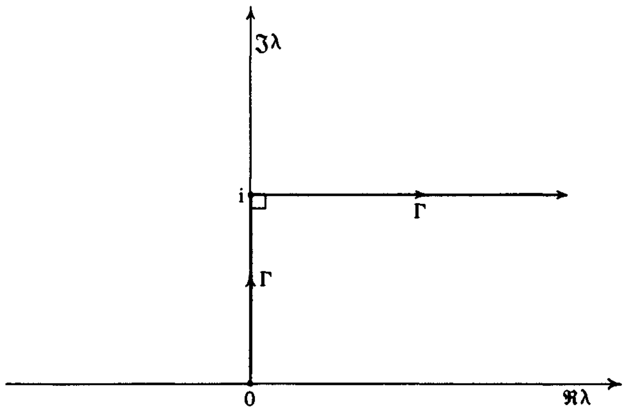

Figure 59

we have

$$\int_0^\infty e^{i\lambda(X+i)}\hat{\mu}_A(\lambda)\,d\lambda \;=\; \int_\Gamma e^{i\lambda(X+i)}\hat{\mu}_A(\lambda)\,d\lambda$$

because $\hat{\mu}_A(\lambda)$ is *analytic* for $\Im\lambda > 0$ and *bounded* in the strip

$$0 \leqslant \Im\lambda \leqslant 1,$$

with $|e^{i\lambda(X+i)}|$ going to zero like $e^{-\Re\lambda}$ there as $\Re\lambda \to \infty$. The integral along Γ breaks up as

$$i\int_0^1 e^{-i\tau}e^{-\tau X}\hat{\mu}_A(i\tau)\,d\tau \;+\; \int_0^\infty e^{i(\sigma X-1)}e^{-\sigma}e^{-X}\hat{\mu}_A(\sigma+i)\,d\sigma$$

$$= \; \mathrm{I} \;+\; \mathrm{II},$$

say.

Since

$$\int_{-\infty}^\infty |d\mu(t)| \;\leqslant\; 1,$$

we have

$$|e^{iA\lambda}\hat{\mu}_A(\lambda)| \;=\; \left|\int_{-A}^\infty e^{i\lambda(t+A)}\,d\mu(t)\right| \;\leqslant\; 1$$

for $\Im\lambda \geqslant 0$. In particular, for $\sigma\in\mathbb{R}$, $|\hat{\mu}_A(\sigma+i)| \leqslant e^A$, and

$$|\mathrm{II}| \;\leqslant\; \int_0^\infty e^{-(X-A)}e^{-\sigma}\,d\sigma \;=\; e^{A-X}.$$

To estimate I, we use the *theorem on two constants* given in the previous article. As we have just seen, $e^{iA\lambda}\hat{\mu}_A(\lambda)$ is in modulus $\leqslant 1$ on the closed strip $\bar{\mathscr{D}}$; it is also continuous there and analytic in $\mathscr{D}$. However, on $E \subseteq \mathbb{R}$, $\hat{\mu}(\lambda) = 0$ (by hypothesis!), so $\hat{\mu}_A(\lambda) = \hat{\mu}(\lambda) - \hat{\rho}_A(\lambda) = -\hat{\rho}_A(\lambda)$ there. Thus, for $\lambda \in E$,

$$|e^{iA\lambda}\hat{\mu}_A(\lambda)| \;=\; |\hat{\rho}_A(\lambda)| \;\leqslant\; e^{-M(A)}$$

According to the theorem on two constants we thus have

$$|e^{iA\lambda}\hat{\mu}_A(\lambda)| \;\leqslant\; e^{-M(A)\omega_{\mathscr{D}}(E,\lambda)} \cdot 1^{1-\omega_{\mathscr{D}}(E,\lambda)}$$

for $\lambda\in\mathscr{D}$, i.e.,

$$|\hat{\mu}_A(\lambda)| \;\leqslant\; e^{A\Im\lambda} e^{-M(A)\omega_{\mathscr{D}}(E,\lambda)}, \qquad \lambda\in\mathscr{D}.$$

Substituting this estimate into I, we find

$$|\mathrm{I}| \;\leqslant\; \int_0^1 e^{A\tau - X\tau - M(A)\omega_{\mathscr{D}}(E,i\tau)}\,d\tau.$$

Recall, however, that $\omega_{\mathscr{D}}(E, \mathrm{i}\tau) \longrightarrow 1$ for $\tau \to 0$. For this reason *$\tau + \omega_{\mathscr{D}}(E, \mathrm{i}\tau)$ has a strictly positive minimum – call it θ – for $0 \leqslant \tau \leqslant 1$; θ does not depend on A or X.*

Suppose $X > 0$. *Then take $A = X/2$.* With this value of A, the previous relation becomes

$$|\mathrm{I}| \;\leqslant\; \int_0^1 \mathrm{e}^{-(X/2)\tau - M(X/2)\omega_{\mathscr{D}}(E,\mathrm{i}\tau)}\,\mathrm{d}\tau \;\leqslant\; \mathrm{e}^{-\theta M_*(X/2)},$$

where $M_*(r) = \min(r, M(r))$.

At the same time,

$$|\mathrm{II}| \;\leqslant\; \mathrm{e}^{-X/2}$$

for $A = X/2$, according to the estimate made above. Therefore, for $X > 0$,

$$\int_0^\infty \mathrm{e}^{\mathrm{i}(X+\mathrm{i})\lambda}\hat{\mu}_{X/2}(\lambda)\,\mathrm{d}\lambda \;=\; \int_\Gamma \mathrm{e}^{\mathrm{i}(X+\mathrm{i})\lambda}\hat{\mu}_{X/2}(\lambda)\,\mathrm{d}\lambda$$

is in modulus

$$\leqslant\; |\mathrm{I}| + |\mathrm{II}| \;\leqslant\; \mathrm{e}^{-\theta M_*(X/2)} + \mathrm{e}^{-X/2}.$$

However, the *first* of the last two integrals differs from $F(X + \mathrm{i})$ by a quantity in modulus $\leqslant \mathrm{e}^{-M(X/2)}$ as we have seen. So, for $X > 0$,

$$|F(X+\mathrm{i})| \;\leqslant\; \mathrm{e}^{-\theta M_*(X/2)} + \mathrm{e}^{-X/2} + \mathrm{e}^{-M(X/2)}.$$

There is no loss of generality in assuming $\theta \leqslant 1$. Then we get

$$|F(X+\mathrm{i})| \;\leqslant\; 3\mathrm{e}^{-\theta M_*(X/2)}, \quad X > 0.$$

Returning to ($*$), which we are trying to prove, we see that

$$\int_0^\infty \frac{\log|F(X+\mathrm{i})|}{1+X^2}\,\mathrm{d}X \;\leqslant\; \int_0^\infty \frac{\log 3 - \theta M_*(X/2)}{1+X^2}\,\mathrm{d}X,$$

and the integral on the left will diverge to $-\infty$ if

$$(\ast\ast) \qquad \int_0^\infty \frac{M_*(X/2)}{1+X^2}\,\mathrm{d}X \;=\; \infty.$$

Here,

$$M(A) \;=\; \log\left(\frac{1}{\int_{-\infty}^{-A}|\mathrm{d}\mu(t)|}\right),$$

so $\int_0^\infty (M(A)/(1 + A^2))\,\mathrm{d}A = \infty$ by the hypothesis. Therefore

$$\int_0^\infty \frac{M_*(A)}{1+A^2}\,\mathrm{d}A \;=\; \infty$$

for $M_*(A) = \min(A, M(A))$ by the second lemma, i.e.,

$$\int_0^\infty \frac{2M_*(X/2)}{4+X^2}\,\mathrm{d}X \;=\; \infty,$$

implying ($\ddagger$), since $M_*(A) \geqslant 0$.

We conclude in this fashion that ($*$) *holds*, whence $F(Z) \equiv 0$ for $\Im Z > 0$, i.e.,

$$\int_0^\infty \mathrm{e}^{-Y\lambda}\mathrm{e}^{\mathrm{i}X\lambda}\hat{\mu}(\lambda)\,\mathrm{d}\lambda \;\equiv\; 0$$

for $Y > 0$ and $X \in \mathbb{R}$.

One shows in like manner that $\int_{-\infty}^0 \mathrm{e}^{Y\lambda}\mathrm{e}^{\mathrm{i}X\lambda}\hat{\mu}(\lambda)\,\mathrm{d}\lambda \equiv 0$ for $Y > 0$ and $X \in \mathbb{R}$; here* one follows the above procedure to estimate

$$\left|\int_{-\infty}^0 \mathrm{e}^{\lambda}\mathrm{e}^{\mathrm{i}X\lambda}\hat{\mu}(\lambda)\,\mathrm{d}\lambda\right|$$

(*again* for $X > 0$!) using *this* contour:

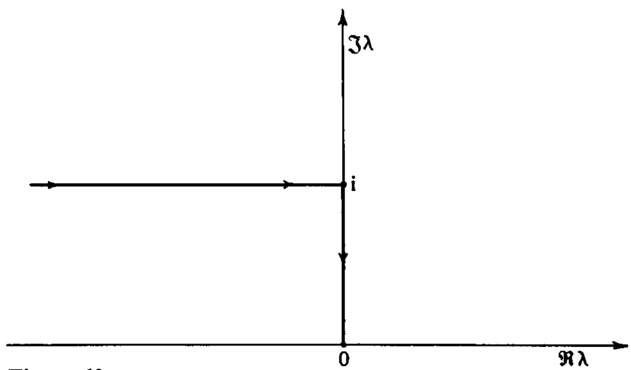

Figure 60

Aside from this change, the argument is like the one given.

The two integrals in question thus vanish identically for $Y > 0$ and $X \in \mathbb{R}$. This, as we remarked at the beginning of our proof, implies that $\mu \equiv 0$. We are done.

Remark 1. The use of the contour integral in the above argument goes back to Levinson, who assumed, however, that $\hat{\mu}(\lambda) = 0$ on an *interval* J instead of just on a set E with $|E| > 0$. In this way Levinson obtained his theorem, given in § A.5, which we now know how to prove much more easily using test functions. By bringing in harmonic measure, Beurling was able to replace the interval J by any measurable set E with $|E| > 0$, getting a qualitative improvement in Levinson's result.

* In which case the integral just written is an analytic function of $X - \mathrm{i}Y$

Remark 2. What about Beurling's gap theorem from §A.2, which says that if the measure μ *has no mass* on any of the intervals (a_n, b_n) with $0 < a_1 < b_1 < a_2 < b_2 < \cdots$ and $\sum_n((b_n - a_n)/a_n)^2 = \infty$, then $\hat{\mu}(\lambda)$ can't vanish identically *on an interval* J, $|J| > 0$, *unless* $\mu \equiv 0$? Can one improve *this* result so as to make it apply for *sets E* of *positive Lebesgue measure instead of just intervals J of positive length*? Contrary to what happens with Levinson's theorem, the answer *here* turns out to be *no*. This is shown by an example of P. Kargaev, to be given in §C.

3. Beurling's study of quasianalyticity

The argument used to establish the theorem of the preceding article can be applied in the investigation of a kind of quasianalyticity. Let γ be a nice Jordan arc, and look at functions $\varphi(\zeta)$ bounded and continuous on γ. A natural way of describing the *regularity* of such φ is to *measure how well* they can be *approximated* on γ by *certain analytic functions*. The regularity which we are able to specify in such fashion is not necessarily the same as differentiability; it is, however, relevant to the study of a quasianalyticity property considered by Beurling, namely, that of not being able to vanish on a subset of γ having positive (arc-length) measure without being identically zero.

A clue to the kind of regularity involved here comes from the observation that a function φ having a continuous analytic extension to a region *bordering on one side* of γ *possesses* the quasianalyticity property just described. We may thus think of such a φ as being *fully* regular. In order to make this notion of regularity quantitative, let us assume that the arc γ is part of the boundary $\partial\mathscr{D}$ of a simply connected region $\mathscr{D}$.

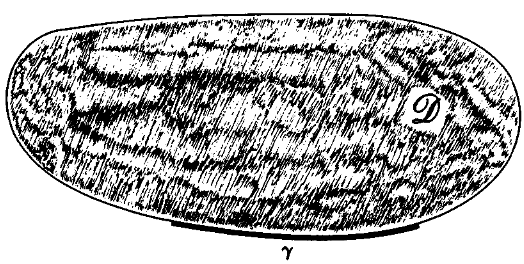

Figure 61

So as to avoid considerations foreign to the matter at hand, we take $\partial\mathscr{D}$ as 'nice' – piecewise analytic and rectifiable, for instance. Given φ bounded and continuous on γ, *define the approximation index* $M(A)$ *for* φ *by functions analytic in* $\mathscr{D}$ *as follows*:

> $e^{-M(A)}$ *is the infimum of* $\sup_{\zeta\in\gamma}|\varphi(\zeta)-f(\zeta)|$ *for* f *analytic in* $\mathscr{D}$ *and continuous on* $\bar{\mathscr{D}}$ *such that* $|f(z)|\leqslant e^A$, $z\in\mathscr{D}$.

We *should* write $M_{\mathscr{D}}(A,\varphi)$ instead of $M(A)$ in order to show the dependence of the approximation index on φ and $\mathscr{D}$; we prefer, however, to use a simpler notation.

When A is made *larger*, we have *more* competing functions f with which to try to approximate φ on γ, so $e^{-M(A)}$ gets *smaller*. In other words, $M(A)$ *increases* with A and we take *the rapidity of this increase* as a *measure* of the *regularity* of φ. Note that if φ actually *has* a bounded continuous extension to $\bar{\mathscr{D}}$ which is *analytic* in $\mathscr{D}$, we have $M(A)=\infty$ beginning with a certain value of A. Such a function φ *cannot* vanish on a set of positive (arc-length) measure on γ without being identically zero, as we have already remarked (this comes, by the way, from two well-known results of F. and M. Riesz). We see that if $M(A)$ grows *rapidly enough*, φ will surely *have* the quasi-analyticity property in question.

The *approximation index* $M(A)$ is a *conformal invariant* in the following sense. Let F map $\mathscr{D}$ conformally onto $\tilde{\mathscr{D}}$, taking the arc γ of $\partial\mathscr{D}$ onto the arc $\tilde{\gamma}\subseteq\partial\tilde{\mathscr{D}}$, and let $\tilde{\varphi}$ be the function defined on $\tilde{\gamma}$ by the relation $\tilde{\varphi}(F(\zeta))=\varphi(\zeta)$, $\zeta\in\gamma$. Then $\tilde{\varphi}$ has the *same* approximation index $M(A)$ for functions analytic in $\tilde{\mathscr{D}}$ as φ has for functions analytic in $\mathscr{D}$. *This is an evident consequence of the use of the sup-norm in defining* $M(A)$.

Our *quasianalyticity property is also* a *conformal invariant*. This follows from the famous theorem of F. and M. Riesz which says that as long as $\partial\mathscr{D}$ and $\partial\tilde{\mathscr{D}}$ are *both rectifiable*, a conformal mapping F of $\mathscr{D}$ onto $\tilde{\mathscr{D}}$ takes sets of arc-length measure zero on $\partial\mathscr{D}$ to such sets on $\partial\tilde{\mathscr{D}}$, and conversely. If $\partial\mathscr{D}$ and $\partial\tilde{\mathscr{D}}$ are really nice, that fact can also be verified directly.

Without further ado, we can now state the

Theorem (Beurling). *Suppose that, for a given bounded continuous* φ *on* $\gamma\subseteq\partial\mathscr{D}$, *the approximation index* $M(A)$ *for* φ *by functions analytic in* $\mathscr{D}$ *satisfies*

$$\int_1^\infty \frac{M(A)}{A^2}\,dA \;=\; \infty.$$

Then, if $E\subseteq\gamma$ *has positive (arc-length) measure, and* $\varphi(\zeta)\equiv 0$ *on* E, $\varphi\equiv 0$ *on* γ.

Proof. By the above statements on conformal invariance, it is enough to establish the result for the special case where $\mathscr{D}$ is the strip $0<\Im\lambda<1$ in the λ-plane and γ is the real axis. The fact that $\partial\mathscr{D}$ is *not rectifiable* here *makes no difference*. We need only show that a set of measure >0 on the rectifiable boundary of our *original* nice domain corresponds under conformal mapping to a set of measure > 0 on the boundary of the *strip*. This may be checked by first mapping the original domain onto the unit disk Δ (whose boundary *is* rectifiable) and appealing to the theorem of F. and M. Riesz mentioned above. One then maps Δ conformally onto the strip; that mapping is, however, easily obtained *explicitly* and thus seen *by inspection* to take sets of measure > 0 on $\partial\Delta$ to sets of measure > 0 on the boundary of the strip.

We have, then, a function φ bounded and continuous on the *real axis*; wlog $|\varphi| \leqslant 1$ there.

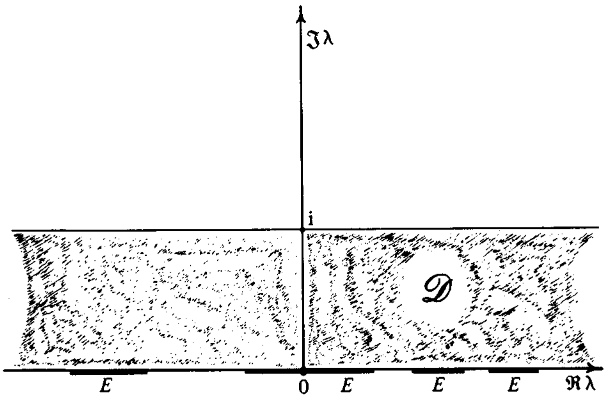

Figure 62

There is a set $E \subseteq \mathbb{R}$ (which we may wlog take to be *closed*) with $|E| > 0$* and $\varphi \equiv 0$ on E. According to the definition of $M(A)$ there is for each $A > 0$ a function $f_A(\lambda)$ analytic in $\mathscr{D}$ and continuous on $\bar{\mathscr{D}}$ with $|f_A(\lambda)| \leqslant e^A$ there and

(†) $\qquad |\varphi(\lambda) - f_A(\lambda)| \;\leqslant\; 2e^{-M(A)}, \quad \lambda \in \mathbb{R}.$

By the discussion at the end of article 1, we certainly have

$$\omega_{\mathscr{D}}(E, \lambda) \longrightarrow 1 \quad \text{as} \quad \lambda \xrightarrow{\angle} \lambda_0$$

for some $\lambda_0 \in E$. There is no loss of generality in taking $\lambda_0 = 0$ (we may arrive at this situation by sliding $\bar{\mathscr{D}}$ along the real axis!), and this we *henceforth assume*. We have, then,

$$\omega_{\mathscr{D}}(E, \mathrm{i}\tau) \longrightarrow 1 \quad \text{as} \quad \tau \longrightarrow 0+,$$

just as in the proof of the theorem in article 2.

* We are denoting the Lebesgue measure of $E \subseteq \mathbb{R}$ by $|E|$.

In order to show that $\varphi(\lambda) \equiv 0$ on $\mathbb{R}$ it is enough to prove that

$$\int_{-\infty}^{\infty} e^{-Y|\lambda|} e^{i\lambda X} \varphi(\lambda)\, d\lambda \;=\; 0$$

for some $Y > 0$ and all real X, for then the function $e^{-Y|\lambda|}\varphi(\lambda)$ (which belongs to $L_1(\mathbb{R})$) must *vanish* a.e. on $\mathbb{R}$ by the *uniqueness theorem* for Fourier transforms. We do this by verifying separately that

$$\int_{0}^{\infty} e^{i\lambda(X+iY)} \varphi(\lambda)\, d\lambda \;=\; 0 \quad \text{for} \quad Y > 0 \quad \text{and} \quad X \in \mathbb{R},$$

and that

$$\int_{-\infty}^{0} e^{i\lambda(X+iY)} \varphi(\lambda)\, d\lambda \;=\; 0 \quad \text{for} \quad Y < 0 \quad \text{and} \quad X \in \mathbb{R}.$$

Considering the first relation, write, for $Y > 0$,

$$F(X+iY) \;=\; \int_{0}^{\infty} e^{i\lambda(X+iY)} \varphi(\lambda)\, d\lambda;$$

the function $F(Z)$ is analytic for $\Im Z > 0$ and *bounded* in each half plane $\Im Z \geqslant h > 0$. We want to conclude that $F(Z) \equiv 0$ for $\Im Z > 0$.

Beginning here, we can practically *copy* the proof of the theorem in the previous article. In that proof, we *replace*

$$\hat{\mu}(\lambda) \quad \text{by} \quad \varphi(\lambda),$$
$$\hat{\mu}_A(\lambda) \quad \text{by} \quad f_A(\lambda)$$

and $\hat{\rho}_A(\lambda)$ by $\varphi(\lambda) - f_A(\lambda)$. *Everything* will then be *the same*, almost *word* for *word*. True, instead of the inequality $|\hat{\rho}_A(\lambda)| \leqslant e^{-M(A)}$ used above, we *here* have (†), but the extra factor of 2 makes very little difference. We also have to find an inequality for $|f_A(\lambda)|$ in the strip $\bar{\mathscr{D}}$ which will play the rôle of the relation $|\hat{\mu}_A(\lambda)| \leqslant e^{A\Im\lambda}$ used previously. Our function f_A satisfies $|f_A(\lambda)| \leqslant e^A$ on $\bar{\mathscr{D}}$ and

$$|f_A(\lambda)| \;\leqslant\; |\varphi(\lambda)| + 2e^{-M(A)} \;\leqslant\; 1 + 2e^{-M(0)} \quad \text{for} \quad \lambda \in \mathbb{R}$$

by (†), $M(A)$ being increasing. Therefore

$$|e^{iA\lambda} f_A(\lambda)| \;\leqslant\; 1 + 2e^{-M(0)}, \qquad \lambda \in \partial\mathscr{D},$$

and we conclude that this inequality holds *throughout* $\bar{\mathscr{D}}$ by the extended principle of maximum (first theorem of § C, Chapter III). In other words,

$$|f_A(\lambda)| \;\leqslant\; (1 + 2e^{-M(0)}) e^{A\Im\lambda}$$

for $\lambda \in \bar{\mathscr{D}}$, and this plays the same rôle as the abovementioned inequality on

$\hat{\mu}_A(\lambda)$, the constant factor $1 + 2e^{-M(0)}$ being without real importance.

Repeating in this way the argument from the previous article, we see that the hypothesis

$$\int_1^\infty \frac{M(A)}{A^2}\,dA = \infty$$

of our present theorem implies that $F(Z) \equiv 0$ for $\Im Z > 0$. The fact that

$$\int_{-\infty}^0 e^{i\lambda(X+iY)}\varphi(\lambda)\,d\lambda \equiv 0.$$

for $Y < 0$ and $X \in \mathbb{R}$ also follows by a simple modification of that argument, as indicated at the end of the proof we have been referring to. We are done.

Corollary. *Let μ be a finite measure on $\mathbb{R}$ such that*

$$\int_{-\infty}^0 \frac{1}{1+x^2}\log\left(\frac{1}{\int_{-\infty}^x |d\mu(t)|}\right)dx = \infty,$$

and suppose that $\psi(\lambda)$ is analytic *in a* rectangle

$$\{a < \Re\lambda < b, \quad 0 < \Im\lambda < h\},$$

and continuous on the closure of that rectangle.

If $E \subseteq (a,b)$, $|E| > 0$, and $\hat{\mu}(\lambda) = \int_{-\infty}^\infty e^{i\lambda t}d\mu(t)$ coincides with $\psi(\lambda)$ on E, then $\hat{\mu}(\lambda) \equiv \psi(\lambda)$ on the whole segment $[a,b]$.

Remark. For E an *interval* $\subseteq [a,b]$, this result was proved by Levinson.

Proof of corollary. Without loss of generality, assume that $h = 1$, that $|\psi(\lambda)| \leqslant \frac{1}{2}$ on the rectangle in question, and that $\int_{-\infty}^\infty |d\mu(t)| \leqslant \frac{1}{2}$.

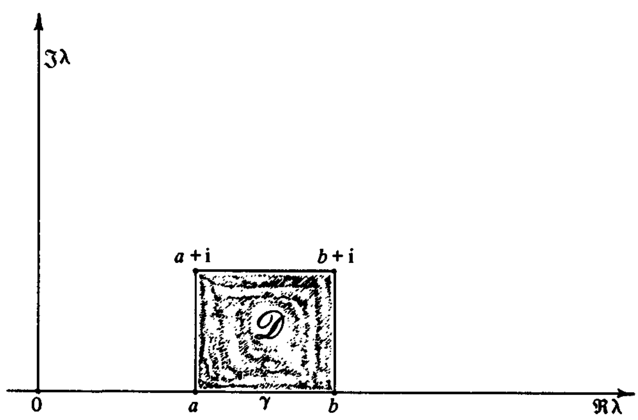

Figure 63

Take our *rectangle* as the domain $\mathscr{D}$ of the theorem, with (a,b) as the arc γ, and put

$$\varphi(\lambda) = \hat{\mu}(\lambda) - \psi(\lambda), \quad a < \lambda < b.$$

For $A > 0$, write

$$f_A(\lambda) \;=\; \int_{-A}^{\infty} e^{i\lambda t}\, d\hat{\mu}(t) \;-\; \psi(\lambda);$$

the function is analytic in $\mathscr{D}$ and continuous on $\bar{\mathscr{D}}$, and for $A > 0$,

$$|f_A(\lambda)| \;\leqslant\; \tfrac{1}{2}e^A + \tfrac{1}{2} \;\leqslant\; e^A, \qquad \lambda \in \bar{\mathscr{D}},$$

while for $a < \lambda < b$

$$|\varphi(\lambda) - f_A(\lambda)| \;\leqslant\; \int_{-\infty}^{-A} |d\mu(t)| \;=\; e^{-M(A)},$$

where $M(A) = \log(1/\int_{-\infty}^{-A} |d\mu(t)|)$.

According to our hypothesis, $\varphi(\lambda) \equiv 0$ on $E \subseteq (a,b) = \gamma$ with $|E| > 0$, and also

$$\int_1^{\infty} \frac{M(A)}{A^2}\, dA \;=\; \infty.$$

Therefore $\varphi(\lambda) \equiv 0$ on (a,b) by the theorem, and, by continuity, $\hat{\mu}(\lambda) \equiv \psi(\lambda)$ for $a \leqslant \lambda \leqslant b$. Q.E.D.

4. The spaces $\mathscr{S}_p(\mathscr{D}_0)$, especially $\mathscr{S}_1(\mathscr{D}_0)$

Suppose that $F(\vartheta) \sim \sum_{-\infty}^{\infty} a_n e^{in\vartheta}$ belongs to $L_2(-\pi,\pi)$ and that

$$\sum_{-\infty}^{-1} \frac{1}{n^2} \log\left(\frac{1}{\sum_{-\infty}^{n} |a_k|^2}\right) \;=\; \infty.$$

We would like, in analogy with the theorem of article 2, to be able to affirm that $F(\vartheta) \equiv 0$ a.e. if $F(\vartheta)$ vanishes on a set of positive measure. The trouble is that F is not necessarily bounded on $[-\pi,\pi]$, so we cannot work directly with the uniform norm used up to now in the present §. At least two ideas for getting around this difficulty come to mind; one of them is to establish L_p variants of the results in article 2 and 3. Such versions are no longer conformally invariant. Beurling gave one for *rectangular* domains; one could of course use his method to obtain similar results for other regions. In this and the next subsection we stick to rectangles.

Given a rectangle $\mathscr{D}_0$ with sides parallel to the axes, Beurling considers approximation in L_p norm by certain functions analytic in $\mathscr{D}_0$, belonging to a space $\mathscr{S}_p(\mathscr{D}_0)$ to be defined presently. We need some information about those functions which, strictly speaking, comes from the theory of H_p spaces. Although this is *not* a book about H_p spaces, we proceed to sketch that material. In various special situations (including the one mentioned at the beginning of this article), easier results would suffice, and the reader is encouraged to investigate the possibilities of such simplification.

If $\mathscr{D}_0$ is the rectangle $\{(x, y): x \in I_0, y \in J_0\}$,

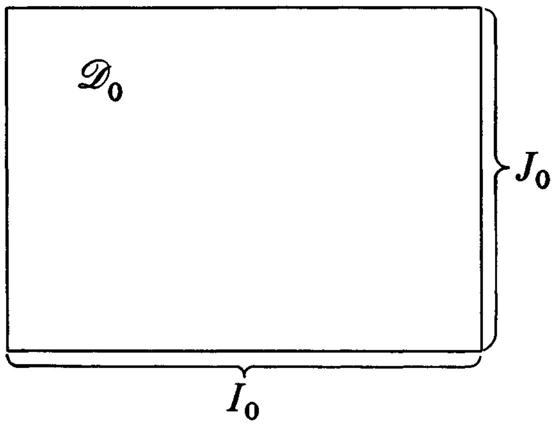

Figure 64

we denote by $\mathscr{S}_p(\mathscr{D}_0)$ the set of functions $f(z)$ analytic in $\mathscr{D}_0$ with

$$\sup_{y \in J_0} \int_{I_0} |f(x+iy)|^p \, dx$$

finite, and write

$$\sigma_p(f) = \sqrt[p]{\sup_{y \in J_0} \int_{I_0} |f(x+iy)|^p \, dx}$$

for such f. We are only interested in values of $p \geqslant 1$, and, for such p, $\sigma_p(\ \ )$ is a *norm*.

Note that the *compactness* of $\bar{I}_0$ makes $\mathscr{S}_p(\mathscr{D}_0) \subseteq \mathscr{S}_1(\mathscr{D}_0)$ for $p > 1$.

Lemma (Fejér and F. Riesz). *Let $f(w)$ be regular and bounded for $\Im w > 0$, continuous up to the real axis, and zero at ∞. Then*

$$\int_0^{\infty} |f(iv)| \, dv \leqslant \frac{1}{2} \int_{-\infty}^{\infty} |f(u)| \, du.$$

Proof. Under our assumptions on f, we have, for $v > 0$,

$$f(iv) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(u) \, du}{u - iv},$$

$$0 = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(u) \, du}{u + iv},$$