A. The Hyperbolic Space

We take first the case of negative curvature, that is $\epsilon = -1$. The transform $f \to \widehat{f}$ is now given by

$$\widehat{f}(\xi) = \int_{\xi} f(x)\, dm(x) \tag{2}$$

ξ being any k-dimensional totally geodesic submanifold of X $(1 \leq k \leq n-1)$ with the induced Riemannian structure and dm the corresponding measure. From our description of the geodesics in X it is clear that any two points in X can be joined by a unique geodesic. Let d be a distance function on X, and for simplicity we write o for the origin x^o in X. Consider now geodesic polar-coordinates for X at o; this is a mapping

$$\mathrm{Exp}_o Y \to (r, \theta_1, \ldots, \theta_{n-1})\,,$$

where Y runs through the tangent space $X_o, r = |Y|$ (the norm given by the Riemannian structure) and $(\theta_1, \ldots, \theta_{n-1})$ are coordinates of the unit vector $Y/|Y|$. Then the Riemannian structure of X is given by

$$ds^2 = dr^2 + (\sinh\, r)^2\, d\sigma^2\,, \tag{3}$$

where $d\sigma^2$ is the Riemannian structure

$$\sum_{i,j=1}^{n-1} g_{ij}(\theta_1, \cdots, \theta_{n-1})\, d\theta_i\, d\theta_j$$

on the unit sphere in X_o. The surface area $A(r)$ and volume $V(r) = \int_o^r A(t)\, dt$ of a sphere in X of radius r are thus given by

$$A(r) = \Omega_n (\sinh\, r)^{n-1}\,, \quad V(r) = \Omega_n \int_o^r \sinh^{n-1} t\, dt \tag{4}$$

so $V(r)$ increases like $e^{(n-1)r}$. This explains the growth condition in the next result where $d(o, \xi)$ denotes the distance of o to the manifold ξ.

Theorem 1.2. *(The support theorem.) Suppose $f \in C(X)$ satisfies*

(i) For each integer $m > 0$, $f(x)e^{md(o,x)}$ is bounded.

(ii) There exists a number $R > 0$ such that

$$\widehat{f}(\xi) = 0 \quad \textit{for } d(o, \xi) > R\,.$$

Then

$$f(x) = 0 \quad \textit{for } d(o, x) > R\,.$$

Taking $R \to 0$ we obtain the following consequence.

Corollary 1.3. *The Radon transform $f \to \widehat{f}$ is one-to-one on the space of continuous functions on X satisfying condition (i) of "exponential decrease".*

Proof of Theorem 1.2. Using smoothing of the form

$$\int_G \varphi(g) f(g^{-1} \cdot x)\, dg$$

($\varphi \in \mathcal{D}(G)$, dg Haar measure on G) we can (as in Theorem 2.6, Ch. I) assume that $f \in \mathcal{E}(X)$.

We first consider the case when f in (2) is a radial function. Let P denote the point in ξ at the minimum distance $p = d(o, \xi)$ from o, let $Q \in \xi$ be arbitrary and let

$$q = d(o, Q)\,, \quad r = d(P, Q)\,.$$

Since ξ is totally geodesic $d(P, Q)$ is also the distance between P and Q in ξ. Consider now the totally geodesic plane π through the geodesics oP and oQ as given by Lemma 1.1 (Fig. III.1). Since a totally geodesic submanifold contains the geodesic joining any two of its points, π contains the geodesic PQ. The angle oPQ being 90° (see e.g. [DS], p. 77) we conclude by hyperbolic trigonometry, (see e.g. Coxeter [1957])

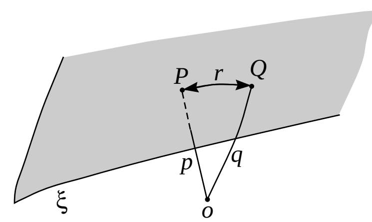

FIGURE III.1.

$$\cosh q = \cosh p \cosh r\,. \tag{5}$$

Since f is radial it follows from (5) that the restriction $f|\xi$ is constant on spheres in ξ with center P. Since these have area $\Omega_k (\sinh r)^{k-1}$ formula (2) takes the form

$$\widehat{f}(\xi) = \Omega_k \int_0^\infty f(Q)(\sinh r)^{k-1}\, dr\,. \tag{6}$$

Since f is a radial function it is invariant under the subgroup $K \subset G$ which fixes o. But K is not only transitive on each sphere $S_r(o)$ with center o, it is for each fixed k transitive on the set of k-dimensional totally geodesic submanifolds which are tangent to $S_r(o)$. Consequently, $\widehat{f}(\xi)$ depends only on the distance $d(o, \xi)$. Thus we can write

$$f(Q) = F(\cosh q)\,, \quad \widehat{f}(\xi) = \widehat{F}(\cosh p)$$

for certain 1-variable functions F and $\widehat{F}$, so by (5) we obtain

$$\widehat{F}(\cosh p) = \Omega_k \int_0^\infty F(\cosh p \cosh r)(\sinh r)^{k-1}\,dr\,. \tag{7}$$

Writing here $t = \cosh p$, $s = \cosh r$ this reduces to

$$\widehat{F}(t) = \Omega_k \int_1^\infty F(ts)(s^2-1)^{(k-2)/2}\,ds\,. \tag{8}$$

Here we substitute $u = (ts)^{-1}$ and then put $v = t^{-1}$. Then (8) becomes

$$v^{-1}\widehat{F}(v^{-1}) = \Omega_k \int_0^v \{F(u^{-1})u^{-k}\}(v^2-u^2)^{(k-2)/2}\,du\,.$$

This integral equation is of the form (19), Ch. I so we get the following analog of (20), Ch. I:

$$F(u^{-1})u^{-k} = cu\left(\frac{d}{d(u^2)}\right)^k \int_o^u (u^2-v^2)^{(k-2)/2}\widehat{F}(v^{-1})\,dv\,, \tag{9}$$

where c is a constant. Now by assumption (ii) $\widehat{F}(\cosh p) = 0$ if $p > R$. Thus

$$\widehat{F}(v^{-1}) = 0 \quad \text{if } 0 < v < (\cosh R)^{-1}\,.$$

From (9) we can then conclude

$$F(u^{-1}) = 0 \quad \text{if } u < (\cosh R)^{-1}$$

which means $f(x) = 0$ for $d(o,x) > R$. This proves the theorem for f radial.

Next we consider an arbitrary $f \in \mathcal{E}(X)$ satisfying (i), (ii) . Fix $x \in X$ and if dk is the normalized Haar measure on K consider the integral

$$F_x(y) = \int_K f(gk \cdot y)\,dk\,, \quad y \in X\,,$$

where $g \in G$ is an element such that $g \cdot o = x$. Clearly, $F_x(y)$ is the average of f on the sphere with center x, passing through $g \cdot y$. The function F_x satisfies the decay condition (i) and it is radial. Moreover,

$$\widehat{F}_x(\xi) = \int_K \widehat{f}(gk \cdot \xi)\,dk\,. \tag{10}$$

We now need the following estimate

$$d(o, gk \cdot \xi) \geq d(o,\xi) - d(o, g \cdot o)\,. \tag{11}$$

For this let x_o be a point on ξ closest to $k^{-1}g^{-1} \cdot o$. Then by the triangle inequality

$$\begin{aligned} d(o, gk \cdot \xi) = d(k^{-1}g^{-1} \cdot o, \xi) &\geq d(o, x_o) - d(o, k^{-1}g^{-1} \cdot o) \\ &\geq d(o,\xi) - d(o, g \cdot o)\,. \end{aligned}$$

Thus it follows by (ii) that

$$\widehat{F}_x(\xi) = 0 \ \text{ if } \ d(o,\xi) > d(o,x) + R\,.$$

Since F_x is radial this implies by the first part of the proof that

$$\int_K f(gk \cdot y)\, dk = 0 \tag{12}$$

if

$$d(o,y) > d(o,g \cdot o) + R\,. \tag{13}$$

But the set $\{gk \cdot y : k \in K\}$ is the sphere $S_{d(o,y)}(g \cdot o)$ with center $g \cdot o$ and radius $d(o,y)$; furthermore, the inequality in (13) implies the inclusion relation

$$B_R(o) \subset B_{d(o,y)}(g \cdot o) \tag{14}$$

for the balls. But considering the part in $B_R(o)$ of the geodesic through o and $g \cdot o$ we see that conversely relation (14) implies (13). Theorem 1.2 will therefore be proved if we establish the following lemma.

Lemma 1.4. *Let $f \in C(X)$ satisfy the conditions:*

(i) *For each integer $m > 0$, $f(x)e^{m\,d(o,x)}$ is bounded.*

(ii) *There exists a number $R > 0$ such that the surface integral*

$$\int_S f(s)\, d\omega(s) = 0\,,$$

whenever the spheres S encloses the ball $B_R(o)$.

Then

$$f(x) = 0 \ \text{ for } \ d(o,x) > R\,.$$

Proof. This lemma is the exact analog of Lemma 2.7, Ch. I, whose proof, however, used the vector space structure of $\mathbf{R}^n$. By using a special model of the hyperbolic space we shall nevertheless adapt the proof to the present situation. As before we may assume f is smooth, i.e., $f \in \mathcal{E}(X)$.

Consider the unit ball $\{x \in \mathbf{R}^n : \sum_1^n x_i^2 < 1\}$ with the Riemannian structure

$$ds^2 = \rho(x_1, \ldots, x_n)^2 (dx_1^2 + \cdots + dx_n^2) \tag{15}$$

where

$$\rho(x_1, \ldots, x_n) = 2(1 - x_1^2 - \ldots - x_n^2)^{-1}\,.$$

This Riemannian manifold is well known to have constant curvature -1 so we can use it for a model of X. This model is useful here because the spheres in X are the ordinary Euclidean spheres inside the ball. This fact is obvious for the spheres Σ with center 0. For the general statement it suffices to prove that if T is the geodesic symmetry with respect to a point (which we can take on the x_1-axis) then $T(\Sigma)$ is a Euclidean sphere. The unit disk D in the x_1x_2-plane is totally geodesic in X, hence invariant under T. Now the isometries of the non-Euclidean disk D are generated by the complex conjugation $x_1 + ix_2 \to x_1 - ix_2$ and fractional linear transformations so they map Euclidean circles into Euclidean circles. In particular $T(\Sigma \cap D) = T(\Sigma) \cap D$ is a Euclidean circle. But T commutes with the rotations around the x_1-axis. Thus $T(\Sigma)$ is invariant under such rotations and intersects D in a circle; hence it is a Euclidean sphere.

After these preliminaries we pass to the proof of Lemma 1.4. Let $S = S_r(y)$ be a sphere in X enclosing $B_r(o)$ and let $B_r(y)$ denote the corresponding ball. Expressing the exterior $X - B_r(y)$ as a union of spheres in X with center y we deduce from assumption (ii)

$$\int_{B_r(y)} f(x)\,dx = \int_X f(x)\,dx\,, \tag{16}$$

which is a constant for small variations in r and y. The Riemannian measure dx is given by

$$dx = \rho^n\,dx_o\,, \tag{17}$$

where $dx_o = dx_1 \dots dx_n$ is the Euclidean volume element. Let r_o and y_o, respectively, denote the Euclidean radius and Euclidean center of $S_r(y)$. Then $S_{r_o}(y_o) = S_r(y), B_{r_o}(y_o) = B_r(y)$ set-theoretically and by (16) and (17)

$$\int_{B_{r_0}(y_0)} f(x_0)\rho(x_0)^n\,dx_o = \text{const.}\,, \tag{18}$$

for small variations in r_o and y_o; thus by differentiation with respect to r_o,

$$\int_{S_{r_0}(y_0)} f(s_0)\rho(s_0)^n\,d\omega_o(s_o) = 0\,, \tag{19}$$

where $d\omega_o$ is the Euclidean surface element. Putting $f^*(x) = f(x)\rho(x)^n$ we have by (18)

$$\int_{B_{r_o}(y_o)} f^*(x_o)\,dx_o = \text{const.}\,,$$

so by differentiating with respect to y_o, we get

$$\int_{B_{r_o}(o)} (\partial_i f^*)(y_o + x_o)\,dx_o = 0\,.$$

Using the divergence theorem (26), Chapter I, §2, on the vector field $F(x_o) = f^*(y_o+x_o)\partial_i$ defined in a neighborhood of $B_{r_o}(0)$ the last equation implies

$$\int_{S_{r_o}(0)} f^*(y_o+s)s_i\, d\omega_o(s) = 0$$

which in combination with (19) gives

$$\int_{S_{r_o}(y_o)} f^*(s)s_i\, d\omega_o(s) = 0\,. \tag{20}$$

The Euclidean and the non-Euclidean Riemannian structures on $S_{r_o}(y_o)$ differ by the factor ρ^2. It follows that $d\omega = \rho(s)^{n-1}\, d\omega_o$ so (20) takes the form

$$\int_{S_r(y)} f(s)\rho(s)s_i\, d\omega(s) = 0\,. \tag{21}$$

We have thus proved that the function $x \to f(x)\rho(x)x_i$ satisfies the assumptions of the theorem. By iteration we obtain

$$\int_{S_r(y)} f(s)\rho(s)^k s_{i_1}\dots s_{i_k}\, d\omega(s) = 0\,. \tag{22}$$

In particular, this holds with $y = 0$ and $r > R$. Then $\rho(s) =$ constant and (22) gives $f \equiv 0$ outside $B_R(o)$ by the Weierstrass approximation theorem. Now Theorem 1.2 is proved.

Now let L denote the Laplace-Beltrami operator on X. (See Ch. IV, §1 for the definition.) Because of formula (3) for the Riemannian structure of X, L is given by

$$L = \frac{\partial^2}{\partial r^2} + (n-1)\coth r\frac{\partial}{\partial r} + (\sinh r)^{-2}L_S \tag{23}$$

where L_S is the Laplace-Beltrami operator on the unit sphere in X_0. We consider also for each $r \geq 0$ the mean value operator M^r defined by

$$(M^r f)(x) = \frac{1}{A(r)}\int_{S_r(x)} f(s)\, d\omega(s)\,.$$

As we saw before this can also be written

$$(M^r f)(g\cdot o) = \int_K f(gk\cdot y)\, dk \tag{24}$$

if $g \in G$ is arbitrary and $y \in X$ is such that $r = d(o,y)$. If f is an analytic function one can, by expanding it in a Taylor series, prove from (24) that M^r is a certain power series in L (cf. Helgason [1959], pp. 270-272). In particular we have the commutativity

$$M^r L = LM^r\,. \tag{25}$$

This in turn implies the "Darboux equation"

$$L_x(F(x,y)) = L_y(F(x,y)) \tag{26}$$

for the function $F(x,y) = (M^{d(o,y)}f)(x)$. In fact, using (24) and (25) we have if $g \cdot o = x$, $r = d(o,y)$

$$\begin{aligned} L_x(F(x,y)) &= (LM^r f)(x) = (M^r Lf)(x) \\ &= \int_K (Lf)(gk \cdot y)\, dk = \int_K (L_y(f(gk \cdot y)))\, dk \end{aligned}$$

the last equation following from the invariance of the Laplacian under the isometry gk. But this last expression is $L_y(F(x,y))$.

We remark that the analog of Lemma 2.13 in Ch. IV which also holds here would give another proof of (25) and (26).

For a fixed integer $k (1 \leq k \leq n-1)$ let Ξ denote the manifold of all k-dimensional totally geodesic submanifolds of X. If φ is a continuous function on Ξ we denote by $\check{\varphi}$ the point function

$$\check{\varphi}(x) = \int_{x \in \xi} \varphi(\xi)\, d\mu(\xi)\,,$$

where μ is the unique measure on the (compact) space of ξ passing through x, invariant under all rotations around x and having total measure one.

Theorem 1.5. *(The inversion formula.) For k even let Q_k denote the polynomial*

$$Q_k(z) = [z + (k-1)(n-k)][z + (k-3)(n-k+2)] \ldots [z + 1 \cdot (n-2)]$$

of degree $k/2$. The k-dimensional Radon transform on X is then inverted by the formula

$$cf = Q_k(L)\big((\widehat{f})^\vee\big)\,, \quad f \in \mathcal{D}(X)\,.$$

Here c is the constant

$$c = (-4\pi)^{k/2}\Gamma(n/2)/\Gamma((n-k)/2)\,. \tag{27}$$

The formula holds also if f satisfies the decay condition (i) in Corollary 4.1.

Proof. Fix $\xi \in \Xi$ passing through the origin $o \in X$. If $x \in X$ fix $g \in G$ such that $g \cdot o = x$. As k runs through K, $gk \cdot \xi$ runs through the set of totally geodesic submanifolds of X passing through x and

$$\check{\varphi}(g \cdot o) = \int_K \varphi(gk \cdot \xi)\, dk\,.$$

Hence

$$(\widehat{f})^\vee(g \cdot o) = \int_K \left(\int_\xi f(gk \cdot y)\, dm(y)\right)\, dk = \int_\xi (M^r f)(g \cdot o)\, dm(y)\,,$$

where $r = d(o, y)$. But since ξ is totally geodesic in X, it has also constant curvature -1 and two points in ξ have the same distance in ξ as in X. Thus we have

$$(\widehat{f})^\vee(x) = \Omega_k \int_0^\infty (M^r f)(x)(\sinh r)^{k-1}\, dr\,. \tag{28}$$

We apply L to both sides and use (23). Then

$$(L(\widehat{f})^\vee)(x) = \Omega_k \int_0^\infty (\sinh r)^{k-1} L_r(M^r f)(x)\, dr\,, \tag{29}$$

where L_r is the "radial part" $\frac{\partial^2}{\partial r^2} + (n-1)\coth r \frac{\partial}{\partial r}$ of L. Putting now $F(r) = (M^r f)(x)$ we have the following result.

Lemma 1.6. *Let m be an integer $0 < m < n = \dim X$. Then*

$$\int_0^\infty \sinh^m r L_r F\, dr =$$
$$(m+1-n)\left[m \int_0^\infty \sinh^m rF(r)\, dr + (m-1)\int_0^\infty \sinh^{m-2} rF(r)\, dr\right].$$

If $m = 1$ the term $(m-1)\int_0^\infty \sinh^{m-2} rF(r)\, dr$ should be replaced by $F(0)$.

This follows by repeated integration by parts.

From this lemma combined with the Darboux equation (26) in the form

$$L_x(M^r f(x)) = L_r(M^r f(x)) \tag{30}$$

we deduce

$$[L_x + m(n-m-1)]\int_0^\infty \sinh^m r(M^r f)(x)\, dr$$
$$= -(n-m-1)(m-1)\int_0^\infty \sinh^{m-2} r(M^r f)(p)\, dr\,.$$

Applying this repeatedly to (29) we obtain Theorem 1.5.

B. The Spheres and the Elliptic Spaces

Now let X be the unit sphere $\mathbf{S}^n(0) \subset \mathbf{R}^{n+1}$ and Ξ the set of k-dimensional totally geodesic submanifolds of X. Each $\xi \in \Xi$ is a k-sphere. We shall now invert the Radon transform

$$\widehat{f}(\xi) = \int_\xi f(x)\, dm(x)\,, \quad f \in \mathcal{E}(X)$$

where dm is the measure on ξ given by the Riemannian structure induced by that of X. In contrast to the hyperbolic space, each geodesic X through a point x also passes through the antipodal point A_x. As a result, $\widehat{f} = (f \circ A)\widehat{}$ and our inversion formula will reflect this fact. Although we state our result for the sphere, it is really a result for the *elliptic space*, that is the sphere with antipodal points identified. The functions on this space are naturally identified with symmetric functions on the sphere.

Again let

$$\check{\varphi}(x) = \int_{x \in \xi} \varphi(\xi)\, d\mu(\xi)$$

denote the average of a continuous function on Ξ over the set of ξ passing through x.

Theorem 1.7. *Let k be an integer, $1 \leq k < n = \dim X$.*

(i) The mapping $f \to \widehat{f}$ $(f \in \mathcal{E}(X))$ has kernel consisting of the skew function (the functions f satisfying $f + f \circ A = 0$).

(ii) Assume k even and let P_k denote the polynomial

$$P_k(z) = [z-(k-1)(n-k)][z-(k-3)(n-k+2)]\dots[z-1(n-2)]$$

of degree $k/2$. The k-dimensional Radon transform on X is then inverted by the formula

$$c(f + f \circ A) = P_k(L)((\widehat{f})^{\vee}), \quad f \in \mathcal{E}(X)$$

where c is the constant in (27).

Proof. We first prove (ii) in a similar way as in the noncompact case. The Riemannian structure in (3) is now replaced by

$$ds^2 = dr^2 + \sin^2 r\, d\sigma^2 ;$$

the Laplace-Beltrami operator is now given by

$$L = \frac{\partial^2}{\partial r^2} + (n-1)\cot r \frac{\partial}{\partial r} + (\sin r)^{-2} L_S \tag{31}$$

instead of (23) and

$$(\widehat{f})^{\vee}(x) = \Omega_k \int_0^{\pi} (M^r f)(x) \sin^{k-1} r\, dr\,.$$

For a fixed x we put $F(r) = (M^r f)(x)$. The analog of Lemma 1.6 now reads as follows.

Lemma 1.8. *Let m be an integer, $0 < m < n = \dim X$. Then*

$$\int_0^\pi \sin^m r L_r F \, dr =$$
$$(n-m-1)\left[m\int_0^\pi \sin^m r F(r)\, dr - (m-1)\int_0^\pi \sin^{m-2} r F(r)\, dr\right].$$

If $m = 1$, the term $(m-1)\int_0^\pi \sin^{m-2} r F(r)\, dr$ should be replaced by $F(o) + F(\pi)$.

Since (30) is still valid the lemma implies

$$\begin{aligned}\big[L_x - m(n-m-1)\big] &\int_0^\pi \sin^m r (M^r f)(x)\, dr \\ = \quad &-(n-m-1)(m-1)\int_0^\pi \sin^{m-2} r (M^r f)(x)\, dr\end{aligned}$$

and the desired inversion formula follows by iteration since

$$F(0) + F(\pi) = f(x) + f(Ax).$$

In the case when k is even, Part (i) follows from (ii). Next suppose $k = n-1$, n even. For each ξ there are exactly two points x and Ax at maximum distance, namely $\frac{\pi}{2}$, from ξ and we write

$$\widehat{f}(x) = \widehat{f}(Ax) = \widehat{f}(\xi).$$

We have then

$$\widehat{f}(x) = \Omega_n (M^{\frac{\pi}{2}} f)(x). \tag{32}$$

Next we recall some well-known facts about spherical harmonics. We have

$$L^2(X) = \sum_0^\infty \mathcal{H}_s, \tag{33}$$

where the space $\mathcal{H}_s$ consist of the restrictions to X of the homogeneous harmonic polynomials on $\mathbf{R}^{n+1}$ of degree s.

(a) $Lh_s = -s(s+n-1)h_s \quad (h_s \in \mathcal{H}_s)$ for each $s \geq 0$. This is immediate from the decomposition

$$L_{n+1} = \frac{\partial^2}{\partial r^2} + \frac{n}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}L$$

of the Laplacian L_{n+1} of $\mathbf{R}^{n+1}$ (cf. (23)). Thus the spaces $\mathcal{H}_s$ are precisely the eigenspaces of L.

(b) Each $\mathcal{H}_s$ contains a function ($\not\equiv 0$) which is invariant under the group K of rotations around the vertical axis (the x_{n+1}-axis in $\mathbf{R}^{n+1}$). This function φ_s is nonzero at the North Pole o and is uniquely determined by the condition $\varphi_s(o) = 1$. This is easily seen since by (31) φ_s satisfies the ordinary differential equation

$$\frac{d^2\varphi_s}{dr^2} + (n-1)\cot r \frac{d\varphi_s}{dr} = -s(s+n-1)\varphi_s\,, \quad \varphi_s'(o) = 0\,.$$

It follows that $\mathcal{H}_s$ is irreducible under the orthogonal group $\mathbf{O}(n+1)$.

(c) Since the mean value operator $M^{\pi/2}$ commutes with the action of $\mathbf{O}(n+1)$ it acts as a scalar c_s on the irreducible space $\mathcal{H}_s$. Since we have

$$M^{\pi/2}\varphi_s = c_s\varphi_s\,, \quad \varphi_s(o) = 1\,,$$

we obtain

$$c_s = \varphi_s\left(\tfrac{\pi}{2}\right). \tag{34}$$

Lemma 1.9. *The scalar $\varphi_s(\pi/2)$ is zero if and only if s is odd.*

Proof. Let H_s be the K-invariant homogeneous harmonic polynomial whose restriction to X equals φ_s. Then H_s is a polynomial in $x_1^2 + \cdots + x_n^2$ and x_{n+1} so if the degree s is odd, x_{n+1} occurs in each term whence $\varphi_s(\pi/2) = H_s(1, 0, \ldots 0, 0) = 0$. If s is even, say $s = 2d$, we write

$$H_s = a_0(x_1^2 + \cdots + x_n^2)^d + a_1 x_{n+1}^2(x_1^2 + \cdots + x_n^2)^{d-1} + \cdots + a_d x_{n+1}^{2d}\,.$$

Using $L_{n+1} = L_n + \partial^2/\partial x_{n+1}^2$ and formula (31) in Ch. I the equation $L_{n+1}H_s \equiv 0$ gives the recursion formula

$$a_i(2d-2i)(2d-2i+n-2) + a_{i+1}(2i+2)(2i+1) = 0$$

($0 \leq i < d$). Hence $H_s(1, 0 \ldots 0)$, which equals a_0, is $\neq 0$; Q.e.d.

Now each $f \in \mathcal{E}(X)$ has a uniformly convergent expansion

$$f = \sum_0^\infty h_s \quad (h_s \in \mathcal{H}_s)$$

and by (32)

$$\widehat{f} = \Omega_n \sum_0^\infty c_s h_s\,.$$

If $\widehat{f} = 0$ then by Lemma 1.9, $h_s = 0$ for s even so f is skew. Conversely $\widehat{f} = 0$ if f is skew so Theorem 1.7 is proved for the case $k = n-1$, n even.

If k is odd, $0 < k < n-1$, the proof just carried out shows that $\widehat{f}(\xi)=0$ for all $\xi \in \Xi$ implies that f has integral 0 over every $(k+1)$-dimensional sphere with radius 1 and center o. Since $k+1$ is even and $< n$ we conclude by (ii) that $f + f \circ A = 0$ so the theorem is proved.

As a supplement to Theorems 1.5 and 1.7 we shall now prove an inversion formula for the Radon transform for general k (odd or even).

Let X be either the hyperbolic space $\mathbf{H}^n$ or the sphere $\mathbf{S}^n$ and Ξ the space of totally geodesic submanifolds of X of dimension k $(1 \leq k \leq n-1)$. We then generalize the transforms $f \to \widehat{f}, \varphi \to \check{\varphi}$ as follows. Let $p \geq 0$. We put

$$\widehat{f}_p(\xi) = \int_{d(x,\xi)=p} f(x)\, dm(x)\,, \quad \check{\varphi}_p(x) = \int_{d(x,\xi)=p} \varphi(\xi)\, d\mu(\xi)\,, \tag{35}$$

where dm is the Riemannian measure on the set in question and $d\mu$ is the average over the set of ξ at distance p from x. Let ξ_p be a fixed element of Ξ at distance p from 0 and H_p the subgroup of G leaving ξ_p stable. It is then easy to see that in the language of Ch. II, §1

$$x = gK\,, \quad \xi = \gamma H_p \text{ are incident } \Leftrightarrow d(x,\xi) = p\,. \tag{36}$$

This means that the transforms (35) are the Radon transform and its dual for the double fibration

$$\begin{tikzcd} & G/(K \cap H_p) \arrow[dl] \arrow[dr] & \\ G/K & & G/H_p \end{tikzcd}$$

For $X = \mathbf{S}^2$ the set $\{x : d(x,\xi) = p\}$ is two circles on $\mathbf{S}^2$ of length $2\pi \cos p$. For $X = \mathbf{H}^2$, the non-Euclidean disk, ξ a diameter, the set $\{x : d(x,\xi) = p\}$ is a pair of circular arcs with the same endpoints as ξ. Of course $\widehat{f}_0 = \widehat{f}\,, \quad \check{\varphi}_0 = \check{\varphi}$.

We shall now invert the transform $f \to \widehat{f}$ by invoking the more general transform $\varphi \to \check{\varphi}_p$. Consider $x \in X, \xi \in \Xi$ with $d(x,\xi) = p$. Select $g \in G$ such that $g \cdot o = x$. Then $d(o, g^{-1}\xi) = p$ so $\{kg^{-1} \cdot \xi : k \in K\}$ is the set of $\eta \in \Xi$ at distance p from o and $\{gkg^{-1} \cdot \xi : k \in K\}$ is the set of $\eta \in \Xi$ at distance p from x. Hence

$$\begin{aligned} (\widehat{f})^\vee_p(g \cdot o) &= \int_K \widehat{f}(gkg^{-1} \cdot \xi)\, dk = \int_K dk \int_\xi f(gkg^{-1} \cdot y)\, dm(y) \\ &= \int_\xi \left(\int_K f(gkg^{-1} \cdot y)\, dk \right) dm(y) \end{aligned}$$

so

$$(\widehat{f})^\vee_p(x) = \int_\xi (M^{d(x,y)} f)(x)\, dm(y)\,. \tag{37}$$

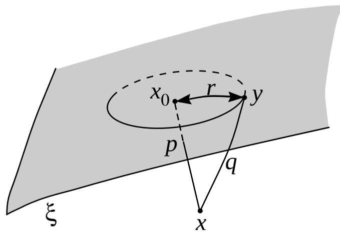

FIGURE III.2.

Let $x_0 \in \xi$ be a point at minimum distance (i.e., p) from x and let (Fig. III.2)

(38)
$$r = d(x_0, y)\,, \quad q = d(x, y)\,, \quad y \in \xi\,.$$

Since $\xi \subset X$ is totally geodesic, $d(x_o, y)$ is also the distance between x_o and y in ξ. In (37) the integrand $(M^{d(x,y)}f)(x)$ is constant in y on each sphere in ξ with center x_o.

Theorem 1.10. *The k-dimensional totally geodesic Radon transform $f \to \widehat{f}$ on the hyperbolic space $\mathbf{H}^n$ is inverted by*

$$f(x) = c\left[\left(\frac{d}{d(u^2)}\right)^k \int_0^u (\widehat{f})^\vee_{\operatorname{lm} v}(x)(u^2 - v^2)^{\frac{k}{2}-1}\, dv\right]_{u=1},$$

where $c^{-1} = (k-1)!\Omega_{k+1}/2^{k+1}$, $\operatorname{lm} v = \cosh^{-1}(v^{-1})$.

Proof. Applying geodesic polar coordinates in ξ with center x_0 we obtain from (37)–(38),

(39)
$$(\widehat{f})^\vee_p(x) = \Omega_k \int_0^\infty (M^q f)(x) \sinh^{k-1} r\, dr\,.$$

Using the cosine relation on the right-angled triangle (xx_0y) we have by (38) and $d(x_0, x) = p$,

(40)
$$\cosh q = \cosh p \cosh r\,.$$

With x fixed we define F and $\widehat{F}$ by

(41)
$$F(\cosh q) = (M^q f)(x)\,, \quad \widehat{F}(\cosh p) = (\widehat{f})^\vee_p(x)\,.$$

Then by (39),

(42)
$$\widehat{F}(\cosh p) = \Omega_k \int_0^\infty F(\cosh p \cosh r) \sinh^{k-1} r\, dr\,.$$

Putting here $t = \cosh p$, $s = \cosh r$ this becomes

$$\widehat{F}(t) = \Omega_k \int_1^\infty F(ts)(s^2 - 1)^{\frac{k}{2}-1}\, ds\,,$$

which by substituting $u = (ts)^{-1}$, $v = t^{-1}$ becomes

$$v^{-1}\widehat{F}(v^{-1}) = \Omega_k \int_0^v F(u^{-1})u^{-k}(v^2 - u^2)^{\frac{k}{2}-1}\, du\,.$$

This is of the form (19), Ch. I, §2 and is inverted by

$$F(u^{-1})u^{-k} = cu\left(\frac{d}{d(u^2)}\right)^k \int_0^u (u^2 - v^2)^{\frac{k}{2}-1}\widehat{F}(v^{-1})\,dv\,, \tag{43}$$

where $c^{-1} = (k-1)!\,\Omega_{k+1}/2^{k+1}$. Defining $\operatorname{lm} v$ by $\cosh(\operatorname{lm} v) = v^{-1}$ and noting that $f(x) = F(\cosh 0)$ the theorem follows by putting $u = 1$ in (43).

For the sphere $X = \mathbf{S}^n$ we can proceed in a similar fashion. We assume f symmetric ($f(s) \equiv f(-s)$) because $\widehat{f} \equiv 0$ for f odd. Now formula (37) takes the form

$$(\widehat{f})^\vee_p(x) = 2\Omega_k \int_0^{\frac{\pi}{2}} (M^q f)(x)\sin^{k-1} r\,dr\,, \tag{44}$$

(the factor 2 and the limit $\pi/2$ coming from the symmetry assumption). This time we use spherical trigonometry on the triangle (xx_0y) to derive

$$\cos q = \cos p \cos r\,.$$

We fix x and put

$$F(\cos q) = (M^q f)(x)\,, \quad \widehat{F}(\cos p) = (\widehat{f})^\vee_p(x)\,. \tag{45}$$

and

$$v = \cos p\,, \quad u = v\cos r\,.$$

Then (44) becomes

$$v^{k-1}\widehat{F}(v) = 2\Omega_k \int_0^v F(u)(v^2 - u^2)^{\frac{k}{2}-1}\,du\,, \tag{46}$$

which is inverted by

$$F(u) = \frac{c}{2}u\left(\frac{d}{d(u^2)}\right)^k \int_0^u (u^2 - v^2)^{\frac{k}{2}-1} v^k \widehat{F}(v)\,dv\,,$$

c being as before. Since $F(1) = f(x)$ this proves the following analog of Theorem 1.10.

Theorem 1.11. *The k-dimensional totally geodesic Radon transform $f \to \widehat{f}$ on $\mathbf{S}^n$ is for f symmetric inverted by*

$$f(x) = \frac{c}{2}\left[\left(\frac{d}{d(u^2)}\right)^k \int_0^u (\widehat{f})^\vee_{\cos^{-1}(v)}(x) v^k (u^2 - v^2)^{\frac{k}{2}-1}\,dv\right]_{u=1}$$

where

$$c^{-1} = (k-1)!\Omega_{k+1}/2^{k+1}\,.$$

Geometric interpretation

In Theorems 1.10–1.11, $(\widehat{f})_p^\vee(x)$ is the average of the integrals of f over the k-dimensional totally geodesic submanifolds of X which have distance p from x.

We shall now look a bit closer at the geometrically interesting case $k = 1$. Here the transform $f \to \widehat{f}$ is called the *X-ray transform*.

We first recall a few facts about the *spherical transform* on the constant curvature space $X = G/K$, that is the hyperbolic space $\mathbf{H}^n = Q_-^+$ or the sphere $\mathbf{S}^n = Q_+$. A *spherical function* φ on G/K is by definition a K-invariant function which is an eigenfunction of the Laplacian L on X satisfying $\varphi(o) = 1$. Then the eigenspace of L containing φ consists of the functions f on X satisfying the functional equation

$$\int_K f(gk \cdot x)\, dk = f(g \cdot o)\varphi(x) \tag{47}$$

([GGA], p. 64). In particular, the spherical functions are characterized by

$$\int_K \varphi(gk \cdot x)\, dk = \varphi(g \cdot o)\varphi(x) \quad \varphi \not\equiv 0\,. \tag{48}$$

Consider now the case $\mathbf{H}^2$. Then the spherical functions are the solutions $\varphi_\lambda(r)$ of the differential equation

$$\frac{d^2\varphi_\lambda}{dr^2} + \coth r \frac{d\varphi_\lambda}{dr} = -(\lambda^2 + \tfrac{1}{4})\varphi_\lambda\,, \quad \varphi_\lambda(o) = 1\,. \tag{49}$$

Here $\lambda \in \mathbf{C}$ and $\varphi_{-\lambda} = \varphi_\lambda$. The function φ_λ has the integral representation

$$\varphi_\lambda(r) = \frac{1}{\pi}\int_0^\pi (\operatorname{ch} r - \operatorname{sh} r \cos\theta)^{-i\lambda + \frac{1}{2}}\, d\theta\,. \tag{50}$$

In fact, already the integrand is easily seen to be an eigenfunction of the operator L in (23) (for $n = 2$) with eigenvalue $-(\lambda^2 + 1/4)$.

If f is a radial function on X its *spherical transform* $\widetilde{f}$ is defined by

$$\widetilde{f}(\lambda) = \int_X f(x)\varphi_{-\lambda}(x)\, dx \tag{51}$$

for all $\lambda \in \mathbf{C}$ for which this integral exists. The continuous radial functions on X form a commutative algebra $C_c^\natural(X)$ under convolution

$$(f_1 \times f_2)(g \cdot o) = \int_G f_1(gh^{-1} \cdot o) f_2(h \cdot o)\, dh \tag{52}$$

and as a consequence of (48) we have

$$(f_1 \times f_2)^\sim(\lambda) = \widetilde{f_1}(\lambda)\widetilde{f_2}(\lambda)\,. \tag{53}$$

In fact,

$$\begin{aligned}(f_1 \times f_2)^{\sim}(\lambda) &= \int_G f_1(h\cdot o)\left(\int_G f_2(g\cdot o)\varphi_{-\lambda}(hg\cdot o)\,dg\right)dh\\ &= \int_G f_1(h\cdot o)\left(\int_G f_2(g\cdot o)\right)\left(\int_K \varphi_{-\lambda}(hkg\cdot o)\,dk\,dg\right)dh\\ &= \widetilde{f}_1(\lambda)\widetilde{f}_2(\lambda)\,.\end{aligned}$$

We know already from Corollary 1.3 that the Radon transform on $\mathbf{H}^n$ is injective and is inverted in Theorem 1.5 and Theorem 1.10. For the case $n = 2, k = 1$ we shall now obtain another inversion formula based on (53).

The spherical function $\varphi_\lambda(r)$ in (50) is the classical Legendre function $P_v(\cosh r)$ with $v = i\lambda - \frac{1}{2}$ for which we shall need the following result ([Prudnikov, Brychkov and Marichev], Vol. III, 2.17.8(2)).

Lemma 1.12.

$$2\pi\int_0^\infty e^{-pr}P_v(\cosh r)\,dr = \pi\frac{\Gamma(\frac{p-v}{2})\Gamma(\frac{p+v+1}{2})}{\Gamma(1+\frac{p+v}{2})\Gamma(\frac{1+p-v}{2})}\,, \tag{54}$$

for

$$Re(p-v) > 0\,, \quad Re(p+v) > -1\,. \tag{55}$$

We shall require this result for $p = 0, 1$ and λ real. In both cases, conditions (55) are satisfied.

Let τ and σ denote the functions

$$\tau(x) = \sinh\, d(o,x)^{-1}\,, \quad \sigma(x) = \coth(d(o,x)) - 1\,, \quad x \in X\,. \tag{56}$$

Lemma 1.13. *For $f \in \mathcal{D}(X)$ we have*

$$(\widehat{f})^\vee(x) = \pi^{-1}(f \times \tau)(x)\,. \tag{57}$$

Proof. In fact, the right hand side is

$$\int_X \sinh\, d(x,y)^{-1} f(y)\,dy = \int_0^\infty dr(\sinh\, r)^{-1}\int_{S_r(x)} f(y)\,dw(y)$$

so the lemma follows from (28).

Similarly we have

$$Sf = f \times \sigma\,, \tag{58}$$

where S is the operator

$$(Sf)(x) = \int_X (\coth(d(x,y)) - 1) f(y)\,dy\,. \tag{59}$$

Theorem 1.14. *The operator $f \to \widehat{f}$ is inverted by*

$$LS((\widehat{f})^\vee) = -4\pi f\,, \quad f \in \mathcal{D}(X)\,. \tag{60}$$

Proof. The operators $\widehat{}$, $^\vee$, S and L are all G-invariant so it suffices to verify (60) at o. Let $f^\natural(x) = \int_K f(k\cdot x)\,dk$. Then

$$(f \times \tau)^\natural = f^\natural \times \tau\,, \;\; (f\times\sigma)^\natural = f^\natural \times \sigma\,, \;\; (Lf)(o) = (Lf^\natural)(o)\,.$$

Thus by (57)–(58)

$$\begin{aligned} LS((\widehat{f})^\vee)(o) &= L(S((\widehat{f})^\vee))^\natural(o) = \pi^{-1}L(f\times\tau\times\sigma)^\natural(o) \\ &= LS(((f^\natural)\widehat{}\,)^\vee))(o)\,. \end{aligned}$$

Now, if (60) is proved for a radial function this equals $cf^\natural(o) = cf(o)$. Thus (60) would hold in general. Consequently, it suffices to prove

$$L(f\times\tau\times\sigma) = -4\pi^2 f\,, \quad f \text{ radial in } \mathcal{D}(X)\,. \tag{61}$$

Since f, $\tau\varphi_\lambda$ (λ real) and σ are all integrable on X, we have by the proof of (53)

$$(f\times\tau\times\sigma)^\sim(\lambda) = \widetilde{f}(\lambda)\widetilde{\tau}(\lambda)\widetilde{\sigma}(\lambda)\,. \tag{62}$$

Since $\coth r - 1 = e^{-r}/\sinh r$, and since $dx = \sinh r\,dr\,d\theta$, $\widetilde{\tau}(\lambda)$ and $\widetilde{\sigma}(\lambda)$ are given by the left hand side of (54) for $p=0$ and $p=1$, respectively. Thus

$$\begin{aligned} \widetilde{\tau}(\lambda) &= \pi\frac{\Gamma(\frac14 - \frac{i\lambda}{2})\Gamma(\frac{i\lambda}{2}+\frac14)}{\Gamma(\frac{i\lambda}{2}+\frac34)\Gamma(\frac34-\frac{i\lambda}{2})}\,, \\ \widetilde{\sigma}(\lambda) &= \pi\frac{\Gamma(\frac34 - \frac{i\lambda}{2})\Gamma(\frac{i\lambda}{2}+\frac34)}{\Gamma(\frac{i\lambda}{2}+\frac54)\Gamma(\frac54-\frac{i\lambda}{2})}\,. \end{aligned}$$

Using the identity $\Gamma(x+1) = x\Gamma(x)$ on the denominator of $\widetilde{\sigma}(\lambda)$ we see that

$$\widetilde{\tau}(\lambda)\widetilde{\sigma}(\lambda) = 4\pi^2(\lambda^2+\tfrac14)^{-1}\,. \tag{63}$$

Now

$$L(f\times\tau\times\sigma) = (Lf\times\tau\times\sigma)\,, \quad f\in\mathcal{D}^\natural(X)\,,$$

and by (49), $(Lf)^\sim(\lambda) = -(\lambda^2+\frac14)\widetilde{f}(\lambda)$. Using the decomposition $\tau = \varphi\tau + (1-\varphi)\tau$ where φ is the characteristic function of a ball $B(0)$ we see that $f\times\tau \in L^2(X)$ for $f\in\mathcal{D}^\sharp(X)$. Since $\sigma\in L'(X)$ we have $f\times\tau\times\sigma\in L^2(X)$. By the Plancherel theorem, the spherical transform is injective on $L^2(X)$ so we deduce from (62)–(63) that (60) holds with the constant $-4\pi^2$.

It is easy to write down an analog of (60) for $\mathbf{S}^2$. Let o denote the North Pole and put

$$\tau(x) = \sin d(o,x)^{-1} \quad x \in \mathbf{S}^2\,.$$

Then in analogy with (57) we have

$$(\widehat{f})^{\vee}(x) = \pi^{-1}(f \times \tau)(x)\,, \tag{64}$$

where $\times$ denotes the convolution on $\mathbf{S}^2$ induced by the convolution on G. The spherical functions on G/K are the functions

$$\varphi_n(x) = P_n(\cos\, d(o,x)) \quad n \geq 0\,,$$

where P_n is the Legendre polynomial

$$P_n(\cos\theta) = \frac{1}{2\pi}\int_0^{2\pi}(\cos\theta + i\sin\theta\cos u)^n\,du\,.$$

Since $P_n(\cos(\pi-\theta)) = (-1)^n P_n(\cos\theta)$, the expansion of τ into spherical functions

$$\tau(x) \sim \sum_{n=0}^{\infty}(4n+1)\widetilde{\tau}(2n)P_{2n}(\cos\, d(o,x))$$

only involves even indices. The factor $(4n+1)$ is the dimension of the space of spherical harmonics containing φ_{2n}. Here the Fourier coefficient $\widetilde{\tau}(2n)$ is given by

$$\widetilde{\tau}(2n) = \frac{1}{4\pi}\int_{\mathbf{S}^2}\tau(x)\varphi_{2n}(x)\,dx\,,$$

which, since $dx = \sin\theta\, d\theta\, d\varphi$, equals

$$\frac{1}{4\pi}2\pi\int_0^{\pi}P_{2n}(\cos\theta)\,d\theta = \frac{\pi}{2^{4n-1}}\binom{2n}{n}^2\,, \tag{65}$$

by *loc. cit.*, Vol. 2, 2.17.6 (11). We now define the functional σ on $\mathbf{S}^2$ by the formula

$$\sigma(x) = \sum_0^{\infty}(4n+1)a_{2n}P_{2n}(\cos\, d(o,x))\,, \tag{66}$$

where

$$a_{2n} = \frac{2^{4n}\pi}{\binom{2n}{n}^2 n(2n+1)}\,. \tag{67}$$

To see that (66) is well-defined note that

$$\begin{aligned}\binom{2n}{n} &= 2^n 1\cdot 3\cdots(2n-1)/n! \geq 2^n 1\cdot 2\cdot 4\cdots(2n-2)/n! \\ &\geq 2^{2n-1}/n\end{aligned}$$

so a_{2n} is bounded in n. Thus σ is a distribution on $\mathbf{S}^2$. Let S be the operator

$$Sf = f \times \sigma \,. \tag{68}$$

Theorem 1.15. *The operator $f \to \widehat{f}$ is inverted by*

$$LS((\widehat{f})^\vee) = -4\pi f \,. \tag{69}$$

Proof. Just as is the case with Theorem 1.14 it suffices to prove this for f K-invariant and there it is a matter of checking that the spherical transforms on both sides agree. For this we use (64) and the relation

$$L\varphi_{2n} = -2n(2n+1)\varphi_{2n} \,.$$

Since

$$(\tau \times \sigma)^\sim(2n) = \widetilde{\tau}(2n)a_{2n} \,.$$

the identity (69) follows.

A drawback of (69) is of course that (66) is not given in closed form. We shall now invert $f \to \widehat{f}$ in a different fashion on $\mathbf{S}^2$. Consider the spherical coordinates of a point $(x_1, x_2, x_3) \in \mathbf{S}^2$.

$$x_1 = \cos\varphi \sin\theta \,, \quad x_2 = \sin\varphi \sin\theta \,, \quad x_3 = \cos\theta \tag{70}$$

and let $k_\varphi = K$ denote the rotation by the angle φ around the x_3-axis. Then f has a Fourier expansion

$$f(x) = \sum_{n \in \mathbb{Z}} f_n(x) \,, \quad f_n(x) = \frac{1}{2\pi} \int_0^{2\pi} f(k_\varphi \cdot x) e^{-in\varphi} \, d\varphi \,.$$

Then

$$f_n(k_\varphi \cdot x) = e^{in\varphi} f_n(x) \,, \quad \widehat{f_n}(k_\varphi \cdot \gamma) = e^{in\varphi} \widehat{f_n}(\gamma) \tag{71}$$

for each great circle γ. In particular, f_n is determined by its restriction $g = f_n|_{x_1=0}$, i.e.,

$$g(\cos\theta) = f_n(0, \sin\theta, \cos\theta) \,.$$

Since f_n is even, (70) implies $g(\cos(\pi - \theta)) = (-1)^n g(\cos\theta)$, so

$$g(-u) = (-1)^n g(u) \,.$$

Let Γ be the set of great circles whose normal lies in the plane $x_1 = 0$. If $\gamma \in \Gamma$ let x_γ be the intersection of γ with the half-plane $x_1 = 0, x_2 > 0$ and let α be the angle from o to x_γ, (Fig. III.3). Since f_n is symmetric,

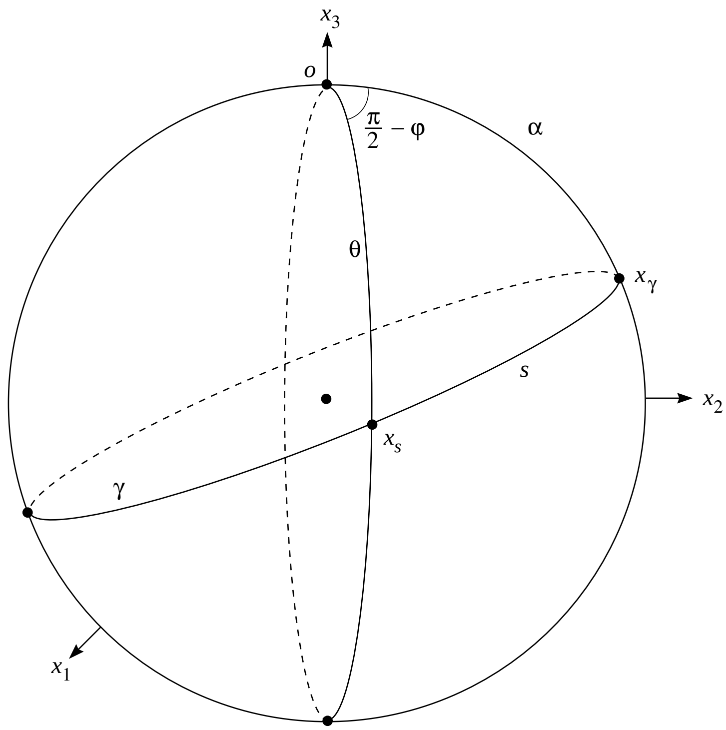

FIGURE III.3.

$$\widehat{f}_n(\gamma) = 2\int_0^{\pi} f_n(x_s)\, ds\,, \tag{72}$$

where x_s is the point on γ at distance s from x_γ (with $x_1(x_s) \geq 0$). Let φ and θ be the coordinates (70) of x_s. Considering the right angled triangle $x_s o x_\gamma$ we have

$$\cos\theta = \cos s\, \cos\alpha$$

and since the angle at o equals $\pi/2 - \varphi$, (71) implies

$$g(\cos\alpha) = f_n(x_\gamma) = e^{in(\pi/2-\varphi)} f_n(x_s)\,.$$

Writing

$$\widehat{g}(\cos\alpha) = \widehat{f}_n(\gamma) \tag{73}$$

equation (72) thus becomes

$$\widehat{g}(\cos\alpha) = 2(-i)^n \int_0^{\pi} e^{in\varphi} g(\cos\theta)\, ds\,.$$

Put $v = \cos\alpha\,, \quad u - v\cos s$, so

$$du = v(-\sin s)\, ds = -(v^2 - u^2)^{1/2}\, ds\,.$$

Then

$$(74) \qquad \widehat{g}(v) = 2(-i)^n \int_{-v}^{v} e^{in\varphi(u,v)} g(u)(v^2-u^2)^{-\frac{1}{2}}\, du\,,$$

where the dependence of φ on u and v is indicated (for $v \neq 0$).

Now $-u = v\cos(\pi - s)$ so by the geometry, $\varphi(-u,v) = -\varphi(u,v)$. Thus (74) splits into two Abel-type Volterra equations

$$(75) \quad \widehat{g}(v) \;=\; 4(-1)^{n/2} \int_0^v \cos(n\varphi(u,v))g(u)(v^2-u^2)^{-\frac{1}{2}}\, du\,, \quad n \text{ even}$$

$$(76) \quad \widehat{g}(v) \;=\; 4(-1)^{(n-1)/2} \int_0^v \sin(n\varphi(u,v))g(u)(v^2-u^2)^{-\frac{1}{2}}\, du\,, \quad n \text{ odd}\,.$$

For $n = 0$ we derive the following result from (43) and (75).

Proposition 1.16. *Let $f \in C^2(\mathbf{S}^2)$ be symmetric and K-invariant and $\widehat{f}$ its X-ray transform. Then the restriction $g(\cos d(o,x)) = f(x)$ and the function $\widehat{g}(\cos d(o,\gamma)) = \widehat{f}(\gamma)$ are related by*

$$(77) \qquad \widehat{g}(v) = 4\int_0^v g(u)(v^2-u^2)^{-\frac{1}{2}}\, du$$

and its inversion

$$(78) \qquad 2\pi g(u) = \frac{d}{du}\int_0^u \widehat{g}(v)(u^2-v^2)^{-\frac{1}{2}}\, v dv\,.$$

We shall now discuss the analog for $\mathbf{S}^n$ of the support theorem (Theorem 1.2) relative to the X-ray transform $f \to \widehat{f}$.

Theorem 1.17. *Let C be a closed spherical cap on $\mathbf{S}^n$, C' the cap on $\mathbf{S}^n$ symmetric to C with respect to the origin $0 \in \mathbf{R}^{n+1}$. Let $f \in C(\mathbf{S}^n)$ be symmetric and assume*

$$(79) \qquad \widehat{f}(\gamma) = 0$$

for every geodesic γ which does not enter the "arctic zones" C and C'. (See Fig. II.3.)

(i) If $n \geq 3$ then $f \equiv 0$ outside $C \cup C'$.

(ii) If $n = 2$ the same conclusion holds if all derivatives of f vanish on the equator.

Proof. (i) Given a point $x \in \mathbf{S}^n$ outside $C \cup C'$ we can find a 3-dimensional subspace ξ of $\mathbf{R}^{n+1}$ which contains x but does not intersect $C \cup C'$. Then $\xi \cap \mathbf{S}^n$ is a 2-sphere and f has integral 0 over each great circle on it. By Theorem 1.7, $f \equiv 0$ on $\xi \cap \mathbf{S}^n$ so $f(x) = 0$.

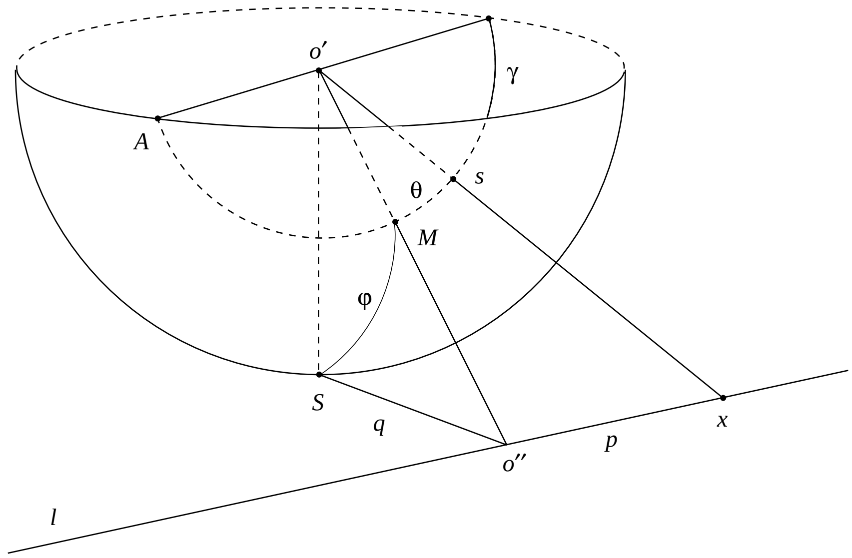

FIGURE III.4.

(ii) If f is K-invariant our statement follows quickly from Proposition 1.16. In fact, if C has spherical radius β, (79) implies $\widehat{g}(v) = 0$ for $0 < v < \cos\beta$ so by (78) $g(u) = 0$ for $0 < u < \cos\beta$ so $f \equiv 0$ outside $C \cup C'$.

Generalizing this method to f_n in (71) by use of (75)–(76) runs into difficulties because of the complexity of the kernel $e^{in\varphi(u,v)}$ in (74) near $v = 0$. However, if f is assumed $\equiv 0$ in a belt around the equator the theory of the Abel-type Volterra equations used on (75)–(76) does give the conclusion of (ii). The reduction to the K-invariant case which worked very well in the proof of Theorem 1.2 does not apply in the present compact case.

A better method, due to Kurusa, is to consider only the lower hemisphere $\mathbf{S}^2_-$ of the unit sphere and its tangent plane π at the South Pole S. The central projection μ from the origin is a bijection of $\mathbf{S}^2_-$ onto π which intertwines the two Radon transforms as follows: If γ is a (half) great circle on $\mathbf{S}^2_-$ and ℓ the line $\mu(\gamma)$ in π we have (Fig. III.4)

$$\cos d(S,\gamma)\widehat{f}(\gamma) = 2\int_\ell (f\circ\mu^{-1})(x)(1+|x|^2)^{-1}\,dm(x)\,. \tag{80}$$

The proof follows by elementary geometry: Let on Fig. III.4, $x = \mu(s)$, φ and θ the lengths of the arcs SM, Ms. The plane $o'So''$ is perpendicular to ℓ and intersects the semi-great circle γ in M. If $q = |So''|, p = |o''x|$ we have for $f \in C(\mathbf{S}^2)$ symmetric,

$$\widehat{f}(\gamma) = 2\int_\gamma f(s)d\theta = 2\int_\ell (f\circ\mu^{-1})(x)\frac{d\theta}{dp}\,dp\,.$$

Now

$$\tan\varphi = q\,, \quad \tan\theta = \frac{p}{(1+q^2)^{1/2}}\,, \quad |x|^2 = p^2 + q^2\,.$$

so

$$\frac{dp}{d\theta} = (1+q^2)^{1/2}(1+\tan^2\theta) = (1+|x|^2)/(1+q^2)^{1/2}\,.$$

Thus

$$\frac{dp}{d\theta} = (1+|x|^2)\cos\varphi$$

and since $\varphi = d(S,\gamma)$ this proves (80). Considering the triangle $o'xS$ we obtain

$$|x| = \tan d(S,s)\,. \tag{81}$$

Thus the vanishing of all derivatives of f on the equator implies rapid decrease of $f\circ\mu^{-1}$ at ∞.

Now if $\varphi > \beta$ we have by assumption, $\widehat{f}(\gamma) = 0$ so by (80) and Theorem 2.6 in Chapter I,

$$(f\circ\mu^{-1})(x) = 0 \quad \text{for } |x| > \tan\beta\,,$$

whence by (81),

$$f(s) = 0 \quad \text{for } d(S,s) > \beta\,.$$

Remark 1.18. Because of the example in Remark 2.9 in Chapter I the vanishing condition in (ii) cannot be dropped.

There is a generalization of (80) to d-dimensional totally geodesic submanifolds of $\mathbf{S}^n$ as well as of $\mathbf{H}^n$ (Kurusa [1992], [1994], Berenstein-Casadio Tarabusi [1993]). This makes it possible to transfer the range characterizations of the d-plane Radon transform in $\mathbf{R}^n$ (Chapter I, §6) to the d-dimensional totally geodesic Radon transform in $\mathbf{H}^n$. In addition to the above references see also Berenstein-Casadio Tarabusi-Kurusa [1997], Gindikin [1995] and Ishikawa [1997].

C. The Spherical Slice Transform

We shall now briefly consider a variation on the Funk transform and consider integrations over circles on $\mathbf{S}^2$ passing through the North Pole. This Radon transform is given by $f \to \widehat{f}$ where f is a function on $\mathbf{S}^2$,

$$\widehat{f}(\gamma) = \int_\gamma f(s)\,dm(s)\,, \tag{82}$$

γ being a circle on $\mathbf{S}^2$ passing through N and dm the arc-element on γ.

It is easy to study this transform by relating it to the X-ray transform on $\mathbf{R}^2$ by means of stereographic projection from N.

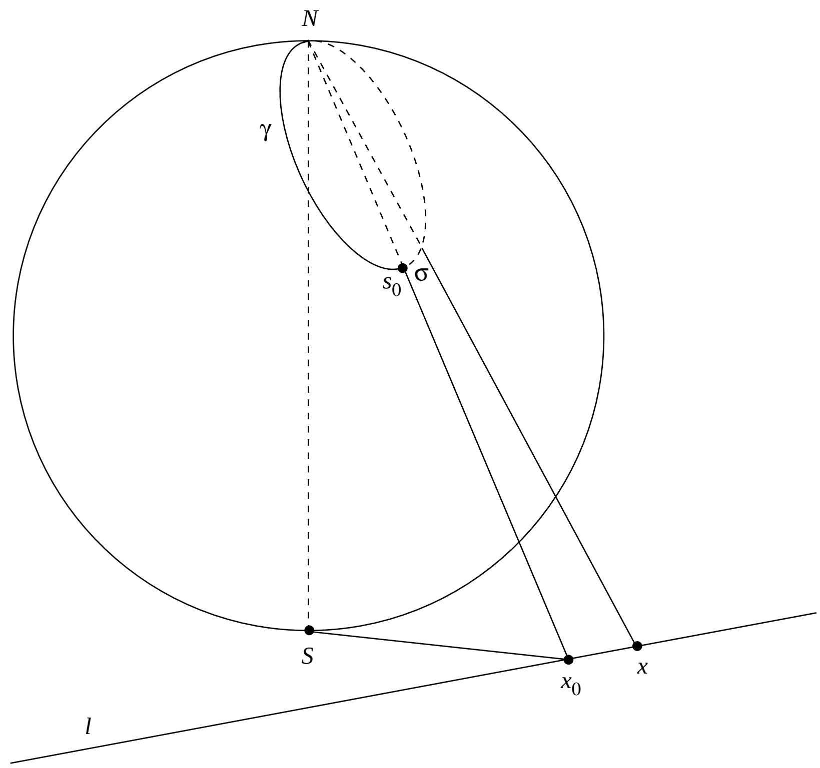

FIGURE III.5.

We consider a two-sphere $\mathbf{S}^2$ of *diameter* 1, lying on top of its tangent plane $\mathbf{R}^2$ at the South Pole. Let $\nu : \mathbf{S}^2 - N \to \mathbf{R}^2$ be the stereographic projection. The image $\nu(\gamma)$ is a line $\ell \subset \mathbf{R}^2$. (See Fig. III.5.) The plane through the diameter NS perpendicular to ℓ intersects γ in s_0 and ℓ in x_0. Then Ns_0 is a diameter in γ, and in the right angle triangle NSx_0, the line Ss_0 is perpendicular to Nx_0. Thus, d denoting the Euclidean distance in $\mathbf{R}^3$, and $q = d(S, x_0)$, we have

$$d(N, s_0) = (1 + q^2)^{-1/2}, \quad d(s_0, x_0) = q^2(1 + q^2)^{-1/2}. \tag{83}$$

Let σ denote the circular arc on γ for which $\nu(\sigma)$ is the segment (x_0, x) on ℓ. If θ is the angle between the lines Nx_0, Nx then

$$\sigma = (2\theta) \cdot \tfrac{1}{2}(1 + q^2)^{-1/2}, \quad d(x_0, x) = \tan\theta(1 + q^2)^{1/2}. \tag{84}$$

Thus, $dm(x)$ being the arc-element on ℓ,

$$\begin{aligned}\frac{dm(x)}{d\sigma} &= \frac{dm(x)}{d\theta}\cdot\frac{d\theta}{d\sigma} = (1+q^2)^{1/2}\cdot(1+\tan^2\theta)(1+q^2)^{1/2}\\ &= (1+q^2)\left(1+\frac{d(x_0,x)^2}{1+q^2}\right) = 1+|x|^2\,.\end{aligned}$$

Hence we have

$$\widehat{f}(\gamma) = \int_\ell (f\circ\nu^{-1})(x)(1+|x|^2)^{-1}\,dm(x)\,, \tag{85}$$

a formula quite similar to (80).

If f lies on $C^1(\mathbf{S}^2)$ and vanishes at N then $f\circ\nu^{-1} = 0(x^{-1})$ at ∞. Also of $f\in\mathcal{E}(\mathbf{S}^2)$ and all its derivatives vanish at N then $f\circ\nu^{-1}\in\mathcal{S}(\mathbf{R}^2)$. As in the case of Theorem 1.17 (ii) we can thus conclude the following corollaries of Theorem 3.1, Chapter I and Theorem 2.6, Chapter I.

Corollary 1.19. *The transform $f\to\widehat{f}$ is one-to-one on the space $C_0^1(\mathbf{S}^2)$ of C^1-functions vanishing at N.*

In fact, $(f\circ\nu^{-1})(x)/(1+|x|^2) = 0(|x|^{-3})$ so Theorem 3.1, Chapter I applies.

Corollary 1.20. *Let B be a spherical cap on $\mathbf{S}^2$ centered at N. Let $f\in\mathcal{C}^\infty(\mathbf{S}^2)$ have all its derivatives vanish at N. If*

$$\widehat{f}(\gamma) = 0 \textit{ for all } \gamma \textit{ through } N\,,\quad \gamma\subset B$$

then $f\equiv 0$ on B.

In fact $(f\circ\nu^{-1})(x) = 0(|x|^{-k})$ for each $k\geq 0$. The assumption on $\widehat{f}$ implies that $(f\circ\nu^{-1})(x)(1+|x|^2)^{-1}$ has line integral 0 for all lines outside $\nu(B)$ so by Theorem 2.6, Ch. I, $f\circ\nu^{-1}\equiv 0$ outside $\nu(B)$.

Remark 1.21. In Cor. 1.20 the condition of the vanishing of all derivatives at N cannot be dropped. This is clear from Remark 2.9 in Chapter I where the rapid decrease at ∞ was essential for the conclusion of Theorem 2.6.

If according to Remark 3.3, Ch. I $g\in\mathcal{E}(\mathbf{R}^2)$ is chosen such that $g(x) = 0(|x|^{-2})$ and all its line integrals are 0, the function f on $\mathbf{S}^2 - N$ defined by

$$(f\circ\nu^{-1})(x) = (1+|x|^2)g(x)$$

is bounded and by (85), $\widehat{f}(\gamma) = 0$ for all γ. This suggests, but does not prove, that the vanishing condition at N in Cor. 1.19 cannot be dropped.

§2 Compact Two-point Homogeneous Spaces. Applications

We shall now extend the inversion formula in Theorem 1.7 to compact two-point homogeneous spaces X of dimension $n > 1$. By virtue of Wang's classification [1952] these are also the compact symmetric spaces of rank one (see Matsumoto [1971] and Szabo [1991] for more direct proofs), so their geometry can be described very explicitly. Here we shall use some geometric and group theoretic properties of these spaces ((i)–(vii) below) and refer to Helgason ([1959], p. 278, [1965a], §5–6 or [DS], Ch. VII, §10) for their proofs.

Let U denote the group $I(X)$ of isometries X. Fix an origin $o \in X$ and let K denote the isotropy subgroup U_o. Let $\mathfrak{k}$ and $\mathfrak{u}$ be the Lie algebras of K and U, respectively. Then $\mathfrak{u}$ is semisimple. Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ and $\mathfrak{u}$ with respect to the Killing form B of $\mathfrak{u}$. Changing the distance function on X by a constant factor we may assume that the differential of the mapping $u \to u \cdot o$ of U onto X gives an isometry of $\mathfrak{p}$ (with the metric of $-B$) onto the tangent space X_o. This is the canonical metric X which we shall use.

Let L denote the diameter of X, that is the maximal distance between any two points. If $x \in X$ let A_x denote the set of points in X of distance L from x. By the two-point homogeneity the isotropy subgroup U_x acts transitively on A_x; thus $A_x \subset X$ is a submanifold, *the antipodal manifold* associated to x.

(i) *Each A_x is a totally geodesic submanifold of X; with the Riemannian structure induced by that of X it is another two-point homogeneous space.*

(ii) *Let Ξ denote the set of all antipodal manifolds in X; since U acts transitively on Ξ, the set Ξ has a natural manifold structure. Then the mapping $j : x \to A_x$ is a one-to-one diffeomorphism; also $x \in A_y$ if and only if $y \in A_x$.*

(iii) *Each geodesic in X has period $2L$. If $x \in X$ the mapping $\mathrm{Exp}_x : X_x \to X$ gives a diffeomorphism of the ball $B_L(0)$ onto the open set $X - A_x$.*

Fix a vector $H \in \mathfrak{p}$ of length L (i.e., $L^2 = -B(H,H)$). For $Z \in \mathfrak{p}$ let T_Z denote the linear transformation $Y \to [Z,[Z,Y]]$ of $\mathfrak{p}$, $[\,,\,]$ denoting the Lie bracket in $\mathfrak{u}$. For simplicity, we now write Exp instead of Exp_o. A point $Y \in \mathfrak{p}$ is said to be *conjugate* to o if the differential $d\mathrm{Exp}$ is singular at Y.

The line $\mathfrak{a} = \mathbf{R}H$ is a maximal abelian subspace of $\mathfrak{p}$. The eigenvalues of T_H are 0, $\alpha(H)^2$ and possibly $(\alpha(H)/2)^2$ where $\pm\alpha$ (and possibly $\pm\alpha/2$) are the roots of $\mathfrak{u}$ with respect to $\mathfrak{a}$. Let

$$\mathfrak{p} = \mathfrak{a} + \mathfrak{p}_\alpha + \mathfrak{p}_{\alpha/2} \tag{86}$$

be the corresponding decomposition of $\mathfrak{p}$ into eigenspaces; the dimensions $q = \dim(\mathfrak{p}_\alpha)$, $p = \dim(\mathfrak{p}_{\alpha/2})$ are called the *multiplicities* of α and $\alpha/2$, respectively.

(iv) *Suppose H is conjugate to o. Then* $\operatorname{Exp}(\mathfrak{a}+\mathfrak{p}_\alpha)$, *with the Riemannian structure induced by that of X, is a sphere, totally geodesic in X, having o and* $\operatorname{Exp}H$ *as antipodal points and having curvature $\pi^2 L^2$. Moreover*

$$A_{\operatorname{Exp}H} = \operatorname{Exp}(\mathfrak{p}_{\alpha/2})\,.$$

(v) *If H is not conjugate to o then $\mathfrak{p}_{\alpha/2} = 0$ and*

$$A_{\operatorname{Exp}H} = \operatorname{Exp}\,\mathfrak{p}_\alpha\,.$$

(vi) *The differential at Y of* Exp *is given by*

$$d\mathrm{Exp}_Y = d\tau(\exp Y) \circ \sum_0^\infty \frac{T_Y^k}{(2k+1)!}\,,$$

where for $u \in U$, $\tau(u)$ is the isometry $x \to u \cdot x$.

(vii) *In analogy with (23) the Laplace-Beltrami operator L on X has the expression*

$$L = \frac{\partial^2}{\partial r^2} + \frac{1}{A(r)} A'(r) \frac{\partial}{\partial r} + L_{S_r}\,,$$

where L_{S_r} is the Laplace-Beltrami operator on $S_r(o)$ and $A(r)$ its area.

(viii) *The spherical mean-value operator M^r commutes with the Laplace-Beltrami operator.*

Lemma 2.1. *The surface area $A(r)$ $(0 < r < L)$ is given by*

$$A(r) = \Omega_n \lambda^{-p} (2\lambda)^{-q} \sin^p(\lambda r) \sin^q(2\lambda r)$$

where p and q are the multiplicities above and $\lambda = |\alpha(H)|/2L$.

Proof. Because of (iii) and (vi) the surface area of $S_r(o)$ is given by

$$A(r) = \int_{|Y|=r} \det\Big(\sum_0^\infty \frac{T_Y^k}{(2k+1)!}\Big)\, d\omega_r(Y)\,,$$

where $d\omega_r$ is the surface on the sphere $|Y| = r$ in $\mathfrak{p}$. Because of the two-point homogeneity the integrand depends on r only so it suffices to evaluate it for $Y = H_r = \frac{r}{L}H$. Since the nonzero eigenvalues of T_{H_r} are $\alpha(H_r)^2$ with multiplicity q and $(\alpha(H_r)/2)^2$ with multiplicity p, a trivial computation gives the lemma.

We consider now Problems A, B and C from Chapter II, §2 for the homogeneous spaces X and Ξ, which are acted on transitively by the same group U. Fix an element $\xi_o \in \Xi$ passing through the origin $o \in X$. If $\xi_o = A_0$, then an element $u \in U$ leaves ξ_o invariant if and only if it lies in the isotropy subgroup $K' = U_o$; we have the identifications

$$X = U/K\,, \quad \Xi = U/K'$$

and $x \in X$ and $\xi \in \Xi$ are incident if and only if $x \in \xi$.

On Ξ we now choose a Riemannian structure such that the diffeomorphism $j : x \to A_x$ from (ii) is an isometry. Let L and Λ denote the Laplacians on X and Ξ, respectively. With $\check{x}$ and $\widehat{\xi}$ defined as in Ch. II, §1, we have

$$\widehat{\xi} = \xi\,, \quad \check{x} = \{j(y) : y \in j(x)\}\,;$$

the first relation amounts to the incidence description above and the second is a consequence of the property $x \in A_y \Leftrightarrow y \in A_x$ listed under (ii).

The sets $\check{x}$ and $\widehat{\xi}$ will be given the measures $d\mu$ and dm, respectively, induced by the Riemannian structures of Ξ and X. The Radon transform and its dual are then given by

$$\widehat{f}(\xi) = \int_\xi f(x)\,dm(x)\,, \quad \check{\varphi}(x) = \int_{\check{x}} \varphi(\xi)\,d\mu(\xi)\,.$$

However

$$\check{\varphi}(x) = \int_{\check{x}} \varphi(\xi)\,d\mu(\xi) = \int_{y \in j(x)} \varphi(j(y))\,d\mu(j(y)) = \int_{j(x)} (\varphi \circ j)(y)\,dm(y)$$

so

$$\check{\varphi} = (\varphi \circ j)\widehat{\ } \circ j\,. \tag{87}$$

Because of this correspondence between the transforms $f \to \widehat{f}$, $\varphi \to \check{\varphi}$ it suffices to consider the first one. Let $\mathbf{D}(X)$ denote the algebra of differential operators on X, invariant under U. It can be shown that $\mathbf{D}(X)$ is generated by L. Similarly $\mathbf{D}(\Xi)$ is generated by Λ.

Theorem 2.2. *(i) The mapping $f \to \widehat{f}$ is a linear one-to-one mapping of $\mathcal{E}(X)$ onto $\mathcal{E}(\Xi)$ and*

$$(Lf)\widehat{\ } = \Lambda \widehat{f}\,.$$

(ii) Except for the case when X is an even-dimensional elliptic space

$$f = P(L)((\widehat{f})^\vee)\,, \quad f \in \mathcal{E}(X)\,,$$

where P is a polynomial, independent of f, explicitly given below, (90)–(93). In all cases

$$\textit{degree } P = \tfrac{1}{2} \textit{ dimension of the antipodal manifold}\,.$$

Proof. (Indication.) We first prove (ii). Let dk be the Haar measure on K such that $\int dk = 1$ and let Ω_X denote the total measure of an antipodal manifold in X. Then $\mu(\breve{o}) = m(A_o) = \Omega_X$ and if $u \in U$,

$$\breve{\varphi}(u \cdot o) = \Omega_X \int_K \varphi(uk \cdot \xi_o)\, dk\,.$$

Hence

$$(\widehat{f})^\vee(u \cdot o) = \Omega_X \int_K \left(\int_{\xi_o} f(uk \cdot y)\, dm(y) \right) dk = \Omega_X \int_{\xi_o} (M^r f)(u \cdot o)\, dm(y)\,,$$

where r is the distance $d(o, y)$ in the space X between o and y. If $d(o, y) < L$ there is a unique geodesic in X of length $d(o, y)$ joining o to y and since ξ_0 is totally geodesic, $d(o, y)$ is also the distance in ξ_0 between o and y. Thus using geodesic polar coordinates in ξ_0 in the last integral we obtain

$$(\widehat{f})^\vee(x) = \Omega_X \int_0^L (M^r f)(x) A_1(r)\, dr\,, \tag{88}$$

where $A_1(r)$ is the area of a sphere of radius r in ξ_0. By Lemma 2.1 we have

$$A_1(r) = C_1 \sin^{p_1}(\lambda_1 r) \sin^{q_1}(2\lambda_1 r)\,, \tag{89}$$

where C_1 and λ_1 are constants and p_1, q_1 are the multiplicities for the antipodal manifold. In order to prove (ii) on the basis of (88) we need the following complete list of the compact symmetric spaces of rank one and their corresponding antipodal manifolds:

X		A_0
Spheres	$\mathbf{S}^n (n = 1, 2, \ldots)$	point
Real projective spaces	$\mathbf{P}^n(\mathbf{R}) (n = 2, 3, \ldots)$	$\mathbf{P}^{n-1}(\mathbf{R})$
Complex projective spaces	$\mathbf{P}^n(\mathbf{C}) (n = 4, 6, \ldots)$	$\mathbf{P}^{n-2}(\mathbf{C})$
Quaternian projective spaces	$\mathbf{P}^n(\mathbf{H}) (n = 8, 12, \ldots)$	$\mathbf{P}^{n-4}(\mathbf{H})$
Cayley plane	$\mathbf{P}^{16}(\mathbf{Cay})$	$\mathbf{S}^8$

Here the superscripts denote the real dimension. For the lowest dimensions, note that

$$\mathbf{P}^1(\mathbf{R}) = \mathbf{S}^1\,, \quad \mathbf{P}^2(\mathbf{C}) = \mathbf{S}^2\,, \quad \mathbf{P}^4(\mathbf{H}) = \mathbf{S}^4\,.$$

For the case $\mathbf{S}^n$, (ii) is trivial and the case $X = \mathbf{P}^n(\mathbf{R})$ was already done in Theorem 1.7. The remaining cases are done by classification starting with (88). The mean value operator M^r still commutes with the Laplacian L

$$M^r L = L M^r$$

and this implies

$$L_x((M^r f)(x)) = L_r((M^r f)(x))\,,$$

where L_r is the radial part of L. Because of (vii) above and Lemma 2.1 it is given by

$$L_r = \frac{\partial^2}{\partial r^2} + \lambda\{p\cot(\lambda r) + 2q\cot(2\lambda r)\}\frac{\partial}{\partial r}\,.$$

For each of the two-point homogeneous spaces we prove (by extensive computations) the analog of Lemma 1.8. Then by the pattern of the proof of Theorem 1.5, part (ii) of Theorem 2.2 can be proved. The full details are carried out in Helgason ([1965a] or [GGA], Ch. I, §4).

The polynomial P is explicitly given in the list below. Note that for $\mathbf{P}^n(\mathbf{R})$ the metric is normalized by means of the Killing form so it differs from that of Theorem 1.7 by a nontrivial constant.

The polynomial P is now given as follows:

For $X = \mathbf{P}^n(\mathbf{R})$, n odd

$$\begin{aligned} P(L) &= c\left(L - \tfrac{(n-2)1}{2n}\right)\left(L - \tfrac{(n-4)3}{2n}\right)\cdots\left(L - \tfrac{1(n-2)}{2n}\right) \\ c &= \tfrac{1}{4}(-4\pi^2 n)^{\frac{1}{2}(n-1)}\,. \end{aligned} \tag{90}$$

For $X = \mathbf{P}^n(\mathbf{C})$, $n = 4, 6, 8, \ldots$

$$\begin{aligned} P(L) &= c\left(L - \tfrac{(n-2)2}{2(n+2)}\right)\left(L - \tfrac{(n-4)4}{2(n+2)}\right)\cdots\left(L - \tfrac{2(n-2)}{2(n+2)}\right) \\ c &= (-8\pi^2(n+2))^{1-\frac{n}{2}}\,. \end{aligned} \tag{91}$$

For $X = \mathbf{P}^n(\mathbf{H})$, $n = 8, 12, \ldots$

$$\begin{aligned} P(L) &= c\left(L - \tfrac{(n-2)4}{2(n+8)}\right)\left(L - \tfrac{(n-4)6}{2(n+8)}\right)\cdots\left(L - \tfrac{4(n-2)}{2(n+8)}\right) \\ c &= \tfrac{1}{2}[-4\pi^2(n+8)]^{2-n/2}\,. \end{aligned} \tag{92}$$

For $X = \mathbf{P}^{16}(\mathbf{Cay})$

$$P(L) = c\left(L - \tfrac{14}{9}\right)^2\left(L - \tfrac{15}{9}\right)^2\,, \quad c = 3^6\pi^{-8}2^{-13}\,. \tag{93}$$

That $f \to \widehat{f}$ is injective follows from (ii) except for the case $X = \mathbf{P}^n(\mathbf{R})$, n even. But in this exceptional case the injectivity follows from Theorem 1.7.

For the surjectivity we use once more the fact that the mean-value operator M^r commutes with the Laplacian (property (viii)). We have

$$\widehat{f}(j(x)) = c(M^L f)(x)\,, \tag{94}$$

where c is a constant. Thus by (87)

$$(\widehat{f})^\vee(x) = (\widehat{f}\circ j)\widehat{\ }(j(x)) = cM^L(\widehat{f}\circ j)(x)$$