larly $\phi(x-0)=\phi(x)$ for all values of x. Hence the function is continuous.

If we put $x=\frac{1}{2}(x_1+x_2)$ in (1), we obtain

$$\phi(\tfrac{1}{2}x_1+\tfrac{1}{2}x_2)<\tfrac{1}{2}\{\phi(x_1)+\phi(x_2)\}. \tag{2}$$

This is sometimes taken as the definition of convexity* instead of (1). It is less restrictive than the definition adopted here, and does not involve continuity.

A sufficient condition for $\phi(x)$ to be convex is that $\phi''(x)>0$; for then $\phi'(x)$ is increasing, and

$$\frac{1}{x-x_1}\int_{x_1}^{x}\phi'(t)\,dt<\phi'(x)<\frac{1}{x_2-x}\int_{x}^{x_2}\phi'(t)\,dt \qquad (x_1<x<x_2),$$

which gives (1).

5.32. The three-circles theorem as a convexity theorem. Hadamard's three-circles theorem may be expressed by saying that $\log M(r)$ *is a convex function of* $\log r$. For we may write it in the form

$$\log M(r_2)\leqslant\frac{\log r_3-\log r_2}{\log r_3-\log r_1}\log M(r_1)+\frac{\log r_2-\log r_1}{\log r_3-\log r_1}\log M(r_3),$$

and the sign of equality occurs only if the function is a constant multiple of a power of z.

5.4. Mean values of $|f(z)|$. The mean values

$$I_1(r)=\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|\,d\theta, \qquad I_2(r)=\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^2\,d\theta,$$

have properties similar to those of $M(r)$.

5.41. *$I_2(r)$ increases steadily with r, and $\log I_2(r)$ is a convex function of $\log r$.*

Let
$$f(z)=\sum_{n=0}^{\infty}a_nz^n.$$

The fact that $I_2(r)$ is steadily increasing is then obvious from the formula
$$I_2(r)=\sum_{n=0}^{\infty}|a_n|^2r^{2n}$$
of § 2.5.

* e.g. in Pólya and Szegö, *Aufgaben*, 1, p. 52.

To prove convexity, let $u = \log r$, and let I_2', I_2'' denote derivatives with respect to u. Then

$$\frac{d^2}{du^2}(\log I_2) = \frac{I_2 I_2'' - I_2'^2}{I_2^2},$$

and by Schwarz's inequality

$$I_2'^2 = (\sum |a_n|^2 2n e^{2nu})^2 \leqslant (\sum |a_n|^2 e^{2nu})(\sum |a_n|^2 4n^2 e^{2nu}) = I_2 I_2''.$$

Hence the result.

5.42. *$I_1(r)$ increases steadily with r, and* $\log I_1(r)$ *is a convex function of* $\log r$.

It is possible to prove this in the same sort of way as the previous theorem,* but the proof is not so easy, since there is no simple expression for I_1 in terms of the coefficients a_n. So we adopt an entirely different method.†

Let $0 < r_1 < r_2 < r_3$, and let $k(\theta)$ and $F(z)$ be defined by

$$k(\theta) f(r_2 e^{i\theta}) = |f(r_2 e^{i\theta})| \qquad (0 \leqslant \theta \leqslant 2\pi),$$

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} f(ze^{i\theta}) k(\theta)\, d\theta.$$

Then $F(z)$ is regular for $|z| \leqslant r_3$, and attains its maximum in this circle on the boundary, say at $z = r_3 e^{i\lambda}$. Hence

$$I_1(r_2) = F(r_2) \leqslant |F(r_3 e^{i\lambda})| \leqslant I_1(r_3),$$

which proves the first part.

Now choose α so that

$$r_1^\alpha I_1(r_1) = r_3^\alpha I_1(r_3).$$

Then

$$r_2^\alpha I_1(r_2) = r_2^\alpha F(r_2) \leqslant \max_{r_1 \leqslant |z| \leqslant r_3} |z^\alpha F(z)| \leqslant r_1^\alpha I_1(r_1) = r_3^\alpha I_1(r_3),$$

and the result follows as in Hadamard's three-circles theorem.

5.5 Theorem of Borel and Carathéodory.‡ This result enables us to deduce an upper bound for the modulus of a function on a circle $|z| = r$, from bounds for its real or imaginary parts on a larger concentric circle $|z| = R$.

Let $f(z)$ be an analytic function regular for $|z| \leqslant R$, and let $M(r)$

* See Hardy (8), and Landau, *Ergebnisse der Funktionentheorie*, § 23.

† Pólya and Szegö, *Aufgaben*, Dritter Abschnitt, No. 308.

‡ See Borel, *Acta M.* 20, and Landau, *Ergebnisse*, § 24.

and $A(r)$ *denote, as usual, the maxima of* $|f(z)|$ *and* $\mathbf{R}\{f(z)\}$ *on* $|z| = r$. *Then for* $0 < r < R$

$$M(r) \leqslant \frac{2r}{R-r} A(R) + \frac{R+r}{R-r} |f(0)|.$$

The result is obvious if $f(z)$ is a constant. If $f(z)$ is not constant, suppose first that $f(0) = 0$. Then $A(R) > A(0) = 0$.

Let
$$\phi(z) = \frac{f(z)}{2A(R) - f(z)}.$$

Then $\phi(z)$ is regular for $|z| \leqslant R$, since the real part of the denominator does not vanish; $\phi(0) = 0$; and, if $f(z) = u + iv$,

$$|\phi(z)|^2 = \frac{u^2+v^2}{\{2A(R)-u\}^2+v^2} \leqslant 1$$

since $-2A(R)+u \leqslant u \leqslant 2A(R)-u$. Hence Schwarz's lemma gives
$$|\phi(z)| \leqslant \frac{r}{R}.$$

Hence
$$|f(z)| = \left| \frac{2A(R)\phi(z)}{1+\phi(z)} \right| \leqslant \frac{2A(R)r}{R-r},$$

and the result stated follows.

If $f(0)$ is not zero, we apply the result already obtained to $f(z) - f(0)$. Thus

$$|f(z)-f(0)| \leqslant \frac{2r}{R-r} \max_{|z|=R} \mathbf{R}\{f(z)-f(0)\} \leqslant \frac{2r}{R-r} \{A(R)+|f(0)|\},$$

and the result again follows. If $A(R) \geqslant 0$, we deduce

$$M(r) \leqslant \frac{R+r}{R-r} \{A(R)+|f(0)|\}.$$

By arguing with $-f(z)$, or with $\pm if(z)$ we obtain similar results in which $A(r)$ is replaced by $\min \mathbf{R}\{f(z)\}$, $\max \mathbf{I}\{f(z)\}$, or $\min \mathbf{I}\{f(z)\}$.

The inequality is thus proved. The form of the right-hand side may be varied to a certain extent. It must, however, contain, besides $A(R)$, a term involving $f(0)$, or we could falsify the result by replacing $f(z)$ by $f(z)+ik$, where k is a sufficiently large real number. Also it must contain a factor, such as $1/(R-r)$, which tends to infinity as $r \to R$. To show this, consider the function $f(z) = -i \log(1-z)$, and let $0 < r < R < 1$. Then $A(R) < \frac{1}{2}\pi$, however near R is to 1; and $f(0) = 0$. But $M(r) \to \infty$ as $r \to 1$.

5.51. The same principle can be extended to the derivatives of $f(z)$. *Under the conditions of the above theorem, with $A(R) \geqslant 0$,*

$$\max_{|z|=r} |f^{(n)}(z)| \leqslant \frac{2^{n+2} n! R}{(R-r)^{n+1}} \{A(R)+|f(0)|\}.$$

For
$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w-z)^{n+1}}\, dw, \tag{1}$$

where C is the circle with centre $w = z$ and radius $\delta = \frac{1}{2}(R-r)$. On this circle
$$|w| \leqslant r + \tfrac{1}{2}(R-r) = \tfrac{1}{2}(R+r),$$

so that Carathéodory's theorem gives

$$\max |f(w)| \leqslant \frac{R+\frac{1}{2}(R+r)}{R-\frac{1}{2}(R+r)} \{A(R)+|f(0)|\} < \frac{4R}{R-r} \{A(R)+|f(0)|\}.$$

Hence, by (1),

$$|f^{(n)}(z)| \leqslant \frac{n!}{\delta^n} \frac{4R}{(R-r)} \{A(R)+|f(0)|\} = \frac{2^{n+2} n! R}{(R-r)^{n+1}} \{A(R)+|f(0)|\}.$$

5.6. The theorems of Phragmén and Lindelöf.* The following important extension of the maximum-modulus theorem was given by Phragmén and Lindelöf:

Let C be a simple closed contour, and let $f(z)$ be regular inside and on C, except at one point P of C. Let $|f(z)| \leqslant M$ on C, except at P.

Suppose further that there is a function $\omega(z)$, regular and not zero in C, such that $|\omega(z)| \leqslant 1$ inside C, and such that, if ϵ is any given positive number, we can find a system of curves, arbitrarily near to P and connecting the two sides of C round P, on which

$$|\{\omega(z)\}^{\epsilon} f(z)| \leqslant M.$$

Then $|f(z)| \leqslant M$ at all points inside C.

To prove this, consider the function

$$F(z) = \{\omega(z)\}^{\epsilon} f(z),$$

which is regular in C. If z_0 is any point inside C, we can, by the hypothesis about $\omega(z)$, find a curve surrounding z_0 on which

$$|F(z)| \leqslant M.$$

Hence $$|F(z_0)| \leqslant M,$$

and so $$|f(z_0)| \leqslant M |\omega(z_0)|^{-\epsilon}.$$

Making $\epsilon \to 0$, $$|f(z_0)| \leqslant M.$$

This proves the theorem.

* Phragmén and Lindelöf-(1).

It is not difficult to see that the exceptional point P may be replaced by any finite number, or even by an infinity, of points, provided that functions $\omega(z)$ corresponding to them with suitable properties can be found.

In the following sections we give a number of theorems of this type. Instead of actually using the above theorem, it is usually simpler to start again with a special auxiliary function adapted to the region considered. In practice the exceptional point P is always at infinity.

5.61. The above theorem gives many important results about the behaviour of a function in the neighbourhood of an essential singularity. By making a preliminary transformation, we can always suppose that the exceptional point is at infinity. The fundamental theorem then takes the following form:

Let $f(z)$ be an analytic function of $z = re^{i\theta}$, regular in the region D between two straight lines making an angle π/α at the origin, and on the lines themselves. Suppose that

$$|f(z)| \leqslant M \tag{1}$$

on the lines, and that, as $r \to \infty$,

$$f(z) = O(e^{r^\beta}), \tag{2}$$

where $\beta < \alpha$, uniformly in the angle. Then actually the inequality (1) *holds throughout the region D.*

We may suppose without loss of generality that the two lines are $\theta = \pm\frac{1}{2}\pi/\alpha$. Let

$$F(z) = e^{-\epsilon z^\gamma} f(z),$$

where $\beta < \gamma < \alpha$ and $\epsilon > 0$. Then

$$|F(z)| = e^{-\epsilon r^\gamma \cos\gamma\theta} |f(z)|. \tag{3}$$

On the lines $\theta = \pm\frac{1}{2}\pi/\alpha$, $\cos\gamma\theta > 0$, since $\gamma < \alpha$. Hence on these lines

$$|F(z)| \leqslant |f(z)| \leqslant M.$$

Also on the arc $|\theta| \leqslant \frac{1}{2}\pi/\alpha$ of the circle $|z| = R$,

$$|F(z)| \leqslant e^{-\epsilon R^\gamma \cos\frac{1}{2}\gamma\pi/\alpha} |f(z)| < Ae^{R^\beta - \epsilon R^\gamma \cos\frac{1}{2}\gamma\pi/\alpha},$$

and the right-hand side tends to 0 as $R \to \infty$. Hence, if R is sufficiently large, $|F(z)| \leqslant M$ on this arc also. Hence, by the maximum-modulus theorem, $|F(z)| \leqslant M$ throughout the interior of the region $|\theta| \leqslant \frac{1}{2}\pi/\alpha$, $r \leqslant R$; i.e., since R is arbitrarily large, throughout the region D. Hence, by (3),

$$|f(z)| \leqslant Me^{\epsilon r^{\gamma}}$$

in D; and making $\epsilon \to 0$ the result stated follows.

It is evidently unnecessary to suppose that the function $f(z)$ is regular in the region $|z| \leqslant r_0$, if there is an arc $|z| = r_1 > r_0$ on which (1) is satisfied. With this extension the theorem is significant for $\alpha < \frac{1}{2}$, the angle including part of the plane more than once, and the function not being necessarily one-valued. We can also replace the straight lines of the theorem by curves extending to infinity; the reader should have no difficulty in supplying the details of such extensions.

5.62. It is important to notice the relation between the 'angle' of the theorem, and the order of $f(z)$ at infinity. The wider the angle is, the smaller the order of $f(z)$ must be for the theorem to be true.

In the following theorem, the order is just not small enough for the previous proof to apply, and a more subtle argument is required.

The conclusion of the previous theorem still holds, if we are only given that

$$f(z) = O(e^{\delta r^{\alpha}})$$

for every positive δ, *uniformly in the angle.*

As before we take the angle to be $-\frac{1}{2}\pi/\alpha \leqslant \theta \leqslant \frac{1}{2}\pi/\alpha$. Let

$$F(z) = e^{-\epsilon z^{\alpha}} f(z).$$

Then $F(z)$ tends to zero on the real axis, and so has an upper bound M' on the real axis. Let

$$M'' = \max(M, M').$$

We may now apply the previous theorem to each of the two angles $(-\frac{1}{2}\pi/\alpha, 0)$ and $(0, \frac{1}{2}\pi/\alpha)$, and we thus find that

$$|F(z)| \leqslant M''$$

throughout the whole given angle.

But in fact $M' \leqslant M$; for $|F(z)|$ attains the value M' at a point of the real axis; hence, if $M' = M''$, $F(z)$ must reduce to a constant, and $M'' = M$. Otherwise $M' < M''$, so that $M'' = M$ in any case. It therefore follows that

$$|F(z)| \leqslant M.$$

Hence $$|f(z)| \leqslant M|e^{-\epsilon z^\alpha}|,$$
and the result follows on making $\epsilon \to 0$.

5.63. *If $f(z) \to a$ as $z \to \infty$ along two straight lines, and $f(z)$ is regular and bounded in the angle between them, then $f(z) \to a$ uniformly in the whole angle.*

We may suppose without loss of generality that the limit a is 0. We may also suppose that the angle between the two lines is less than π, since the general case can be reduced to this by a substitution of the form $z = w^k$. We may thus suppose that the lines are $\theta = \pm\theta'$, where $\theta' < \frac{1}{2}\pi$.

Let $$F(z) = \frac{z}{z+\lambda} f(z),$$
where $\lambda > 0$. Then
$$|F(z)| = \frac{r}{\sqrt{(r^2+2r\lambda\cos\theta+\lambda^2)}}|f(z)| < \frac{r}{\sqrt{(r^2+\lambda^2)}}|f(z)|.$$

Now $|f(z)| \leqslant M$, say, everywhere, and $|f(z)| < \epsilon$ for $r > r_1 = r_1(\epsilon)$ and $\theta = \pm\theta'$. Let $\lambda = r_1 M/\epsilon$. Then for $r \leqslant r_1$
$$|F(z)| < \frac{r}{\lambda} M < \epsilon$$
and $|F(z)| < |f(z)| < \epsilon$ for $r > r_1$ and $\theta = \pm\theta'$. Hence, by the main Phragmén-Lindelöf theorem, $|F(z)| \leqslant \epsilon$ in the whole region. Hence
$$|f(z)| \leqslant \left(1+\frac{\lambda}{r}\right)|F(z)| < 2\epsilon$$
if $r > \lambda$. This proves the theorem.

5.64. *If $f(z) \to a$ as $z \to \infty$ along a straight line, and $f(z) \to b$ as $z \to \infty$ along another straight line, and $f(z)$ is regular and bounded in the angle between, then $a = b$, and $f(z) \to a$ uniformly in the angle.*

Let $f(z) \to a$ along $\theta = \alpha$, and $f(z) \to b$ along $\theta = \beta$, where $\alpha < \beta$. The function $$\{f(z) - \tfrac{1}{2}(a+b)\}^2$$
is regular and bounded in the angle, and tends to $\frac{1}{4}(a-b)^2$ on each of the straight lines. Hence it tends to this limit uniformly in the angle; that is,
$$\{f(z)-\tfrac{1}{2}(a+b)\}^2 - \tfrac{1}{4}(a-b)^2 = \{f(z)-a\}\{f(z)-b\}$$

tends uniformly to zero. Thus to any ϵ corresponds an arc on which

$$|\{f(z)-a\}\{f(z)-b\}| \leqslant \epsilon.$$

At every point of this arc either $|f(z)-a| \leqslant \sqrt{\epsilon}$ or $|f(z)-b| \leqslant \sqrt{\epsilon}$ (or both), and we may suppose that the former inequality holds at $\theta = \alpha$, the latter at $\theta = \beta$; let θ_0 be the upper bound of values of θ for which the former holds; then θ_0 is a limit of points where the former holds, and is either a point where the latter holds, or a limit of such points; hence, since $f(z)$ is continuous, both inequalities hold at θ_0. Taking z to be this point, we have

$$|a-b| \leqslant |f(z)-a|+|f(z)-b| \leqslant 2\sqrt{\epsilon},$$

and, making $\epsilon \to 0$, it follows that $a = b$. Finally $f(z) \to a$ uniformly, by the previous theorem.

These theorems have obvious affinities with Montel's theorem (§ 5.23). But in Montel's theorem the line along which the function tends to a limit must be interior to the region of boundedness, so that these theorems become corollaries of Montel's only if we assume a slightly wider region of boundedness.

5.65. The Phragmén-Lindelöf theorem for other regions. The angle of the above theorem may be transformed into other regions, for example into a strip.

Take, for example, the theorem of § 5.61, applied to the region $r \geqslant 1$, $|\theta| \leqslant \frac{1}{2}\pi/\alpha$, and put $s = i\log z$, $f(z) = \phi(s)$. If $s = \sigma+it$, the lines $\arg z = \pm\frac{1}{2}\pi/\alpha$ become parallel lines $\sigma = \pm\frac{1}{2}\pi/\alpha$, and $t = \log|z|$. Hence, *if $|\phi(s)| \leqslant M$ on the upper half of the two parallel lines and on the segment of the real axis joining them, while*

$$\phi(\sigma+it) = O(e^{e^{\rho t}}) \qquad (\rho < \alpha) \tag{1}$$

in the strip between them, then actually $|\phi(s)| \leqslant M$ throughout the strip.

Another theorem of this type, which we shall require in the theory of Dirichlet series, is as follows:

If $\phi(s)$ is regular and $O(e^{\epsilon|t|})$, for every positive ϵ, in the strip $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, and

$$\phi(\sigma_1+it) = O(|t|^{k_1}), \qquad \phi(\sigma_2+it) = O(|t|^{k_2}),$$

then

$$\phi(\sigma+it) = O(|t|^{k(\sigma)})$$

uniformly for $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, $k(\sigma)$ being the linear function of σ which takes the values k_1, k_2 for $\sigma = \sigma_1$, σ_2.

The result is true more generally if $\phi(s)$ satisfies a condition of the form (1). With the given condition it may be proved directly as follows.

Suppose first that $k_1 = 0$, $k_2 = 0$, so that $\phi(s)$ is bounded for $\sigma = \sigma_1$, $\sigma = \sigma_2$. Let M be the upper bound of $\phi(s)$ on these two lines and on the segment of the real axis between σ_1 and σ_2. Let

$$g(s) = e^{\epsilon si}\phi(s).$$

Then $$|g(s)| = e^{-\epsilon t}|\phi(s)| \leqslant |\phi(s)| \leqslant M$$

for $\sigma = \sigma_1$, $\sigma = \sigma_2$. Also $|g(s)| \to 0$ as $t \to \infty$ for $\sigma_1 \leqslant \sigma \leqslant \sigma_2$; and so, if T is large enough, $|g(s)| \leqslant M$ on $t = T$, $\sigma_1 \leqslant \sigma \leqslant \sigma_2$. Hence $|g(s)| \leqslant M$ at all points of the rectangle (σ_1, σ_2), $(0, T)$. Hence $|g(s)| \leqslant M$ at all points in the half-strip, i.e.

$$|\phi(s)| \leqslant e^{\epsilon t}M.$$

Making $\epsilon \to 0$, it follows that $|\phi(s)| \leqslant M$ for $t > 0$, and similarly for $t < 0$. This proves the theorem in the particular case considered.

In the general case, let

$$\psi(s) = (-is)^{k(s)} = e^{k(s)\log(-is)},$$

where the logarithm has its principal value. This function is regular for $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, $t \geqslant 1$; also, if $k(s) = as+b$,

$$\begin{aligned} \mathbf{R}\{k(s)\log(-is)\} &= \mathbf{R}[\{k(\sigma)+iat\}\log(t-i\sigma)] \\ &= k(\sigma)\log t + O(1). \end{aligned}$$

Hence $$|\psi(s)| = t^{k(\sigma)}e^{O(1)}.$$

The function $\Phi(s) = \phi(s)/\psi(s)$ therefore satisfies the same conditions as $\phi(s)$ did in the first part. Hence $\Phi(s)$ is bounded in the strip, and $$\phi(s) = O\{|\psi(s)|\} = O(t^{k(\sigma)}).$$

5.7. The Phragmén-Lindelöf function $h(\theta)$. In several of the preceding theorems we have been considering the way in which a function behaves as z tends to infinity in different directions. We shall now make a more systematic study of this question.

Consider first the function

$$f(z) = e^{(a+ib)z^\rho}.$$

Then $$|f(z)| = e^{r^\rho(a\cos\rho\theta - b\sin\rho\theta)}.$$

The behaviour of $\log|f(z)|$ depends in the first place on the factor r^ρ, which is independent of θ. The different behaviour in different directions is determined by the factor

$$h(\theta) = a\cos\rho\theta - b\sin\rho\theta = r^{-\rho}\log|f(z)|.$$

This is of course a very special case; but the general case is not so different from it as might be expected.

We shall suppose throughout the following sections that $f(z)$ is regular for $\alpha < \theta < \beta$, $|z| \geqslant r_0$, and that $f(z)$ is 'of order ρ' in this angle, i.e. that

$$\overline{\lim} \frac{\log|f(re^{i\theta})|}{r^{\rho+\epsilon}} = 0$$

uniformly in θ, for every positive value of ϵ, but not for any negative value. (For example, the above function is of order ρ.)

We define $h(\theta)$ in general as

$$h(\theta) = \overline{\lim_{r\to\infty}} \frac{\log|f(re^{i\theta})|}{V(r)},$$

where $V(r)$ depends on the function considered. We should naturally choose $V(r)$ so that $h(\theta)$ is finite and not identically zero. Here we shall consider the simplest case $V(r) = r^{\rho}$; but our argument would apply almost unchanged to any function such as

$$r^{\rho}(\log r)^{p}(\log\log r)^{q}\ldots.$$

5.701. It is convenient to introduce at this point an expression containing the word 'infinity', or the symbol ∞, which is not used in elementary analysis. We shall use $\lim \phi_n = \infty$ to mean the same thing as $\phi_n \to \infty$; and we shall say that $\phi(x)$ has an infinite value, or $\phi(x) = \infty$, if, and only if, $\phi(x)$ is defined as the limit of a sequence $\phi_n(x)$, and the sequence diverges to infinity for the particular value of x in question. We use $-\infty$ in the same way. For example, we might write

$$\int_0^1 \frac{dt}{t} = \infty$$

if the left-hand side is defined as $\lim_{\epsilon\to 0} \int_{\epsilon}^{1}$; and $h(\theta) = \infty$ means that $r^{-\rho}\log|f(re^{i\theta})|$ takes arbitrarily large values as $r \to \infty$.

The novelty consists in writing '$=\infty$', as if we had defined a number '∞'; but it should be remembered that we have not done so, and that 'infinity' remains an incomplete symbol.*

5.71. *Let $\alpha<\theta_1<\theta_2<\beta$, and $\theta_2-\theta_1<\pi/\rho$, and let*

$$h(\theta_1)\leqslant h_1, \qquad h(\theta_2)\leqslant h_2.$$

Let $H(\theta)$ be the function of the form $a\cos\rho\theta+b\sin\rho\theta$ which takes the values h_1, h_2 at θ_1, θ_2. Then

$$h(\theta)\leqslant H(\theta) \qquad (\theta_1\leqslant\theta\leqslant\theta_2).$$

It is easily seen that

$$H(\theta)=\frac{h_1\sin\rho(\theta_2-\theta)+h_2\sin\rho(\theta-\theta_1)}{\sin\rho(\theta_2-\theta_1)},$$

but we do not require this expression in the proof.

Let $$H_\delta(\theta)=a_\delta\cos\rho\theta+b_\delta\sin\rho\theta$$

be the H-function which is equal to $h_1+\delta$, $h_2+\delta$ $(\delta>0)$ for $\theta=\theta_1$, $\theta=\theta_2$ respectively. Let

$$F(z)=f(z)e^{-(a_\delta-ib_\delta)z^\rho}.$$

Then $$|F(z)|=|f(z)|e^{-H_\delta(\theta)r^\rho}, \tag{1}$$

and so, if r is large enough,

$$|F(re^{i\theta_1})|=O(e^{(h_1+\delta)r^\rho-H_\delta(\theta_1)r^\rho})=O(1).$$

A similar result holds for $F(re^{i\theta_2})$. Hence, by the theorem of § 5.61, $F(z)$ is bounded in the angle (θ_1,θ_2). Hence, by (1),

$$f(z)=O(e^{H_\delta(\theta)r^\rho}) \tag{2}$$

uniformly in the angle. Hence $h(\theta)\leqslant H_\delta(\theta)$ for $\theta_1\leqslant\theta\leqslant\theta_2$. Since $H_\delta(\theta)\to H(\theta)$ as $\delta\to 0$, the result follows.

5.711. As a particular case of the above theorem, one or both of $h(\theta_1)$, $h(\theta_2)$, may be $-\infty$. The conclusion is then that $h(\theta)=-\infty$ for $\theta_1<\theta<\theta_2$. The same proof still applies, one or both of the numbers h_1, h_2 now being arbitrarily large and negative.

5.712. *If $\alpha<\theta_1<\theta_2<\theta_3<\beta$, $\theta_2-\theta_1<\pi/\rho$, $\theta_3-\theta_2<\pi/\rho$; and $h(\theta_1)$, $h(\theta_2)$ are finite, and $H(\theta)$ is an H-function such that*

$$h(\theta_1)\leqslant H(\theta_1), \qquad h(\theta_2)=H(\theta_2),$$

then $$h(\theta_3)\geqslant H(\theta_3). \tag{1}$$

* *P.M.* § 55.

Choose θ_1' so that $\theta_3-\pi/\rho < \theta_1' < \theta_2$. Then $h(\theta_1') \leqslant H(\theta_1')$ by § 5.71. Hence, by § 5.711, $h(\theta_3)$ is not $-\infty$. If (1) is false, we can choose δ so that $h(\theta_3) \leqslant H(\theta_3)-\delta$. Let

$$H_\delta(\theta) = H(\theta)-\delta\frac{\sin\rho(\theta-\theta_1')}{\sin\rho(\theta_3-\theta_1')}.$$

Then

$$h(\theta_1') \leqslant H(\theta_1') = H_\delta(\theta_1'), \qquad h(\theta_3) \leqslant H(\theta_3-\delta) = H_\delta(\theta_3).$$

Hence
$$h(\theta_2) \leqslant H_\delta(\theta_2) < H(\theta_2),$$
contrary to hypothesis.

5.713. *If* $\theta_1 < \theta_2 < \theta_3$, $\theta_2-\theta_1 < \pi/\rho$, $\theta_3-\theta_2 < \pi/\rho$, *then*

$$h(\theta_1)\sin\rho(\theta_3-\theta_2)+h(\theta_2)\sin\rho(\theta_1-\theta_3)+h(\theta_3)\sin\rho(\theta_2-\theta_1) \geqslant 0.$$

For any $H(\theta)$

$$H(\theta_1)\sin\rho(\theta_3-\theta_2)+H(\theta_2)\sin\rho(\theta_1-\theta_3)+H(\theta_3)\sin\rho(\theta_2-\theta_1) = 0,$$

and choosing $H(\theta)$ so that $H(\theta_1) = h(\theta_1)$, $H(\theta_2) = h(\theta_2)$, and observing that, by the above theorem, $h(\theta_3) \geqslant H(\theta_3)$, we have the result stated.

The function $h(\theta)$ is continuous in any interval where it is finite.

Let $h(\theta)$ be finite in the interval $\theta_1 \leqslant \theta \leqslant \theta_3$, and let $\theta_1 < \theta_2 < \theta_3$. Let $H_{1,2}(\theta)$ be the h-function which takes the values $h(\theta_1)$, $h(\theta_2)$ at θ_1, θ_2; and define $H_{2,3}(\theta)$ similarly. Then by the above theorems

$$H_{2,3}(\theta) \leqslant h(\theta) \leqslant H_{1,2}(\theta) \qquad (\theta_1 \leqslant \theta \leqslant \theta_2)$$
$$H_{1,2}(\theta) \leqslant h(\theta) \leqslant H_{2,3}(\theta) \qquad (\theta_2 \leqslant \theta \leqslant \theta_3).$$

Hence, in whichever of these intervals θ lies,

$$\frac{H_{1,2}(\theta)-H_{1,2}(\theta_2)}{\theta-\theta_2} \leqslant \frac{h(\theta)-h(\theta_2)}{\theta-\theta_2} \leqslant \frac{H_{2,3}(\theta)-H_{2,3}(\theta_2)}{\theta-\theta_2}.$$

The extreme terms tend to limits as $\theta \to \theta_2$; hence the middle term is bounded, and so $h(\theta) \to h(\theta_2)$.

It also follows that $|f(re^{i\theta})| < \exp[r^\rho\{h(\theta)+\epsilon\}]$ uniformly for $r > r_0(\epsilon)$; (divide the θ-range into $n = n(\epsilon)$ parts).

5.72. Geometrical interpretation of the property of $h(\theta)$. In the case $\rho = 1$, the property of the function $h(\theta)$ has a simple geometrical interpretation.

For every value of θ in an interval where $h(\theta)$ is finite and positive, consider the radius vector of length $h(\theta)$ making an angle θ with an initial line, and the perpendicular to this radius vector at its end. (Consider, for example, the cases $f(z) = \cosh z$, $f(z) = \cos z + \cosh z$.)

Let h_1, h_2, h_3 be the values of $h(\theta)$ at θ_1, θ_2, θ_3, where $\theta_1 < \theta_2 < \theta_3$. Then the three perpendiculars are

$$x\cos\theta_1 + y\sin\theta_1 = h_1,$$
$$x\cos\theta_2 + y\sin\theta_2 = h_2,$$
$$x\cos\theta_3 + y\sin\theta_3 = h_3.$$

The first and third meet at a point (X, Y) given by

$$X = \frac{h_1\sin\theta_3 - h_3\sin\theta_1}{\sin(\theta_3-\theta_1)}, \qquad Y = \frac{h_3\cos\theta_1 - h_1\cos\theta_3}{\sin(\theta_3-\theta_1)}.$$

Now the condition that (X, Y) should lie on the opposite side from the origin of the second perpendicular (or on it) is

$$X\cos\theta_2 + Y\sin\theta_2 - h_2 \geqslant 0,$$

or

$$(h_1\sin\theta_3 - h_3\sin\theta_1)\cos\theta_2 + $$
$$+ (h_3\cos\theta_1 - h_1\cos\theta_3)\sin\theta_2 - h_2\sin(\theta_3-\theta_1) \geqslant 0,$$

or $$h_1\sin(\theta_3-\theta_2) + h_2\sin(\theta_1-\theta_3) + h_3\sin(\theta_2-\theta_1) \geqslant 0.$$

This is precisely the condition which the function $h(\theta)$ satisfies.

If the perpendiculars envelope a curve, then two tangents to it meet on the opposite side to the origin of any tangent at a point between them. It is easily seen geometrically that this means that *the curve is always concave to the origin.*

5.8. The following interesting applications of the Phragmén-Lindelöf principle are due to Carlson.*

Let $f(z)$ be regular and of the form $O(e^{k|z|})$ for $\mathbf{R}(z) \geqslant 0$; and let $f(z) = O(e^{-a|z|})$, where $a > 0$, on the imaginary axis. Then $f(z) = 0$ identically.

We apply the argument of § 5.71 to $f(z)$, with $\rho = 1$, $\theta_1 = 0$, $\theta_2 = \frac{1}{2}\pi$, $h_1 = k$, $h_2 = -a$; and here we can take $\delta = 0$ throughout the argument. Then § 5.71 (2) gives

$$f(z) = O\{e^{(k\cos\theta - a|\sin\theta|)r}\} \tag{1}$$

for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$; and a similar argument shows that (1) also holds for $-\frac{1}{2}\pi \leqslant \theta \leqslant 0$.

Let $$F(z) = e^{\omega z}f(z)$$

where ω is a (large) positive number. Then by (1) there is a

* In an Upsala thesis (1914). See M. Riesz (1), Hardy (14).

constant M, independent of ω, such that

$$|F(z)| \leqslant Me^{\{(k+\omega)\cos\theta - a|\sin\theta|\}r} \qquad (-\tfrac{1}{2}\pi \leqslant \theta \leqslant \tfrac{1}{2}\pi). \tag{2}$$

In particular we have

$$|F(z)| \leqslant M \tag{3}$$

for $\theta = \pm\frac{1}{2}\pi$ and $\theta = \pm\alpha$, where $\alpha = \arctan\{k+\omega)/a\}$.

We can now apply the theorem of § 5.61 to each of the three angles $(-\frac{1}{2}\pi, -\alpha)$, $(-\alpha, \alpha)$, and $(\alpha, \frac{1}{2}\pi)$. It follows that (3) actually holds for $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$.

Hence $|f(z)| \leqslant Me^{-\omega r\cos\theta}$, and making $\omega \to \infty$ it follows that $|f(z)| = 0$. This proves the theorem.

5.81. *If $f(z)$ is regular and of the form $O(e^{k|z|})$, where $k < \pi$, for $\mathbf{R}(z) \geqslant 0$, and $f(z) = 0$ for $z = 0, 1, 2, 3,\ldots$, then $f(z) = 0$ identically.*

Consider the function $F(z) = f(z)\operatorname{cosec}\pi z$. On the circles $|z| = n+\frac{1}{2}$, $\operatorname{cosec}\pi z$ is bounded. Hence $F(z) = O(e^{k|z|})$ on these circles, and also on the imaginary axis. Since $F(z)$ is regular it follows that, if $n-\frac{1}{2} < |z| < n+\frac{1}{2}$,

$$F(z) = O(e^{k(n+\frac{1}{2})}) = O(e^{k|z|}),$$

and so $F(z)$ is of this form throughout $\mathbf{R}(z) \geqslant 0$. Also

$$F(z) = O(e^{(k-\pi)|z|})$$

on the imaginary axis. The result therefore follows from the previous theorem.

MISCELLANEOUS EXAMPLES

1. A function $f(z)$ is regular inside and on a simple closed contour C, and $|f(z)| \leqslant M$ on C. Deduce from Cauchy's integral for $\{f(z)\}^n$ that, if z is inside C,

$$|f(z)|^n \leqslant KM^n,$$

where K is independent of n. Hence show that $|f(z)| \leqslant M$ inside C. [Landau.]

2. Use Poisson's integral to show that a function which is harmonic in a region cannot have a maximum at an interior point of the region.

3. If $f(z)$ is regular and $O(e^{r^{1-\epsilon}})$ for $\mathbf{R}(z) \geqslant 0$, $|f(z)| \leqslant M$ on the imaginary axis, and $f(1) = 0$, then for $x > 0$

$$|f(x+iy)| \leqslant \left\{\frac{(1-x)^2+y^2}{(1+x)^2+y^2}\right\}^{\frac{1}{2}} M.$$

[Consider $(1+z)/(1-z).f(z)$].

4. A function $f(z)$ is regular and satisfies the inequalities

$$e^{-r^{\rho+\epsilon}} < |f(z)| < e^{r^{\rho+\epsilon}}$$

in an angle $\theta_1 \leqslant \theta \leqslant \theta_2$, where $\theta_2 - \theta_1 < \pi/\rho$. As $r \to \infty$, $r^{-\rho}\log|f(z)|$ tends to the limits h_1 and h_2 for $\theta = \theta_1, \theta_2$. Let $H(\theta)$ be the function of the form $a \cos \rho\theta + b \sin \rho\theta$ which takes the values h_1, h_2 for $\theta = \theta_1, \theta_2$. Then $h(\theta) = H(\theta)$ throughout the interval $\theta_1 \leqslant \theta \leqslant \theta_2$.

5. Show that, if $f(z) = O(e^{r^{1+\epsilon}})$ in a given angle, the function

$$h(\theta) = \overline{\lim} \frac{\log|f(re^{i\theta})|}{r \log r}$$

has properties similar to those of the h-functions considered in the text.

Show that if $f(z) = 1/\Gamma(\frac{1}{2}+z)$, then $h(\theta) = -\cos\theta$ for all values of θ.

6. An analytic function $f(z)$ is regular and not zero in the half-strip defined by $a < x < b$, $y > 0$; $f(z) = O(y^A)$ as $y \to \infty$ uniformly in the strip, and $|\log f(z)|$ is bounded on the middle line $x = \frac{1}{2}(a+b)$. Prove that $\log f(z) = O(\log y)$ uniformly for $a+\delta < x < b-\delta$.

[Apply Carathéodory's theorem to $\log f(z)$ in a circle with centre at $\frac{1}{2}(a+b)+iy$.]

7. A function $f(z)$ is regular, and $|f(z)| \leqslant M$, for $\mathbf{R}(z) \geqslant 0$, and $f(z)$ has zeros at $z_1, z_2, \ldots$ in this half-plane. Prove that

$$|f(z)| \leqslant \left|\frac{z_1-z}{\bar{z}_1+z}\,\frac{z_2-z}{\bar{z}_2+z} \cdots \frac{z_n-z}{\bar{z}_n+z}\right| M$$

for $\mathbf{R}(z) > 0$; and deduce that, if $f(z)$ is not identically zero, the series

$$\sum \mathbf{R}\left(\frac{1}{z_n}\right)$$

is convergent. [See Pólya and Szegö, Absch. III, Nos. 295, 298.]

CHAPTER VI

CONFORMAL REPRESENTATION

6.1. Conformal representation. If w is an analytic function of z, then to values of z, which we represent as points in the z-plane, correspond values of w, which we represent as points in the w-plane. We also speak of the point in the w-plane representing its corresponding point in the z-plane; and of regions of the z-plane being represented, or mapped, on corresponding regions of the w-plane. The object of this chapter is to discuss in more detail the nature of this representation or mapping.

Let $w = f(z)$ be an analytic function of z, regular and one-valued in a region D of the z-plane. Let z_0 be an interior point of D; and let C_1 and C_2 be two continuous curves passing through z_0, and having definite tangents at this point, making angles α_1, α_2, say, with the real axis.

We have to discover what is the representation of this figure in the w-plane. Before we go any further, we shall make a restriction, the reason for which will appear in a moment. *We shall suppose that $f'(z_0)$ is not zero.*

Let z_1 and z_2 be points of the curves C_1 and C_2 near to z_0. We shall suppose that they are at the same distance r from z_0, so that we can write

$$z_1 - z_0 = re^{i\theta_1}, \qquad z_2 - z_0 = re^{i\theta_2}.$$

Then as $r \to 0$, $\theta_1 \to \alpha_1$, and $\theta_2 \to \alpha_2$.

The point z_0 corresponds to a point w_0 in the w-plane, and z_1 and z_2 correspond to points w_1 and w_2 which describe curves C_1' and C_2'. Let

$$w_1 - w_0 = \rho_1 e^{i\phi_1}, \qquad w_2 - w_0 = \rho_2 e^{i\phi_2}.$$

Then, by the definition of an analytic function,

$$\lim \frac{w_1 - w_0}{z_1 - z_0} = f'(z_0).$$

Since $f'(z_0)$ is not zero, we may write it in the form $Re^{i\delta}$. Then

$$\lim \frac{\rho_1 e^{i\phi_1}}{re^{i\theta_1}} = Re^{i\delta}.$$

Hence $\lim(\phi_1 - \theta_1) = \delta$, i.e.

$$\lim \phi_1 = \alpha_1 + \delta.$$

Hence the curve C_1' has a definite tangent at w_0, making an angle $\alpha_1+\delta$ with the real axis.

Similarly C_2' has a definite tangent at w_0, making an angle $\alpha_2+\delta$ with the real axis.

Hence the curves C_1', C_2' intersect at the same angle as the curves C_1, C_2. Also the angle between the curves has the same sense in the two figures.

Because of this property of the conservation of angles, an analytic representation is called 'conformal'. Any small figure in one plane corresponds to an approximately similar figure in the other plane, since all angles are approximately the same. To obtain one figure from the other we must rotate it through a certain angle—the angle $\delta = \arg\{f'(z_0)\}$ of the above notation —and subject it to a certain magnification, viz.

$$\lim \frac{\rho_1}{r} = R = |f'(z_0)|.$$

It is clear from the above analysis that the magnification is the same in all directions.

6.11. The case $f'(z)=0$. Suppose now that $f'(z)$ has a zero of order n at the point z_0. Then in the neighbourhood of this point

$$f(z) = f(z_0)+a(z-z_0)^{n+1}+\ldots$$

where $a \neq 0$. Hence

$$w_1-w_0 = a(z_1-z_0)^{n+1}+\ldots,$$

i.e.

$$\rho_1 e^{i\phi_1} = |a| r^{n+1} e^{i\{\delta+(n+1)\theta_1\}}+\ldots,$$

where $\delta = \arg a$. Hence

$$\begin{aligned}\lim \phi_1 &= \lim\{\delta+(n+1)\theta_1\}\\ &= \delta+(n+1)\alpha_1.\end{aligned}$$

Similarly

$$\lim \phi_2 = \delta+(n+1)\alpha_2.$$

Thus the curves C_1', C_2' still have definite tangents at w_0, but the angle between the tangents is

$$\lim(\phi_2-\phi_1) = (n+1)(\alpha_2-\alpha_1).$$

Also the linear magnification, $\lim \rho_1/r$, is zero. The conformal property therefore does not hold at such a point.

6.12. In the above conformal representations we have, not merely conservation of angles, but conservation of the sign of angles; if we get from C_1 to C_2 by a rotation through an angle

α in the positive sense, we also get from C_1' to C_2' by a rotation through α in the positive sense.

There are also conformal representations in which the magnitude of angles is conserved, but their sign is changed. Consider for example, the transformation

$$w = \bar{z},$$

where $\bar{z}$ is the complex number conjugate to z. This replaces every point by its reflection in the real axis. Hence angles are conserved, but their signs are changed. And this is true generally for every transformation of the form

$$w = f(\bar{z}),$$

where $f(z)$ is an analytic function of z; for this is the product of two transformations:

$$\text{(i)}\ Z = \bar{z}, \qquad \text{(ii)}\ w = f(Z).$$

In (i) angles are conserved, their signs changed. In (ii) angles and signs are conserved. Hence in the resulting transformation angles are conserved and their signs changed.

6.2. Linear* transformation. The function

$$w = \frac{az+b}{cz+d}$$

is called a linear function of z. We shall suppose that

$$ad-bc \neq 0,$$

for otherwise the numerator and denominator are proportional, and w is merely a constant.

To every value of z corresponds just one value of w. This is apparent except, in the case $c \neq 0$, for the value $z = -d/c$, which makes the denominator vanish. But as $z \to -d/c$, $|w| \to \infty$; and we may regard the point at infinity in the w-plane as corresponding to the point $z = -d/c$ in the z-plane.

If $c = 0$, then $$w = \frac{a}{d}z + \frac{b}{d}$$

and (since $a \neq 0$) the points at infinity in the two planes correspond.

Conversely $$z = \frac{dw-b}{-cw+a},$$

so that z is a linear function of w.

* Or bilinear.

Example. Prove that in general there are two values of z ('invariant points') for which $w = z$, but that there is one only if

$$(a-d)^2+4bc = 0.$$

Show that, if there are distinct invariant points p and q, the transformation may be put in the form

$$\frac{w-p}{w-q} = k\frac{z-p}{z-q};$$

and that, if there is only one invariant point p, the transformation may be put in the form

$$\frac{1}{w-p} = \frac{1}{z-p}+k.$$

6.21. Circles. The equation

$$|z-z_0| = \rho$$

represents a circle with centre z_0 and radius ρ.

Two points p, q are said to be inverse with respect to the circle if they are collinear with the centre and on the same side of it, and if the product of their distances from the centre is equal to ρ^2. Thus, if

$$p = z_0+le^{i\lambda},$$

then

$$q = z_0+\frac{\rho^2}{l}e^{i\lambda}.$$

If

$$z = z_0+\rho e^{i\theta}$$

is any point of the circle, then

$$\left|\frac{z-p}{z-q}\right| = \left|\frac{\rho e^{i\theta}-le^{i\lambda}}{\rho e^{i\theta}-\rho^2 l^{-1}e^{i\lambda}}\right| = \frac{l}{\rho}\left|\frac{\rho e^{i\theta}-le^{i\lambda}}{le^{i\theta}-\rho e^{i\lambda}}\right| = \frac{l}{\rho}.$$

This is therefore a new form of the equation of the circle.

Conversely, any equation

$$\left|\frac{z-p}{z-q}\right| = k \qquad (k \neq 1)$$

represents a circle* with respect to which p and q are inverse points. For the equation gives

$$|z|^2-2\mathbf{R}(\bar{p}z)+|p|^2 = k^2\{|z|^2-2\mathbf{R}(\bar{q}z)+|q|^2\},$$

or

$$|z|^2-2\frac{\mathbf{R}\{(\bar{p}-k^2\bar{q})z\}}{1-k^2}+\frac{|p|^2-k^2|q|^2}{1-k^2} = 0,$$

or

$$\left|z-\frac{p-k^2q}{1-k^2}\right|^2 = \frac{|p-k^2q|^2}{(1-k^2)^2}-\frac{|p|^2-k^2|q|^2}{1-k^2}.$$

Since

$$|p-k^2q|^2-(1-k^2)(|p|^2-k^2|q|^2) = k^2|p-q|^2,$$

* The 'circle of Apollonius'.

as is easily verified, we obtain

$$\left|z-\frac{p-k^2q}{1-k^2}\right|=\frac{k|p-q|}{|1-k^2|}.$$

The equation therefore represents a circle, with centre

$$z_0=\frac{p-k^2q}{1-k^2},$$

and radius

$$\rho=\frac{k|p-q|}{|1-k^2|}.$$

Also

$$p-z_0=\frac{k^2(q-p)}{1-k^2}, \qquad q-z_0=\frac{q-p}{1-k^2},$$

so that $(p-z_0)/(q-z_0)$ is real and positive, and

$$|p-z_0||q-z_0|=\rho^2.$$

Hence p and q are inverse points.

In the particular case $k=1$, z is equidistant from the points p and q, and therefore lies on the perpendicular bisector of the line joining them.

6.22. Linear transformation of a circle. *In a linear transformation, a circle transforms into a circle, and inverse points transform into inverse points. In the particular case in which the circle becomes a straight line, inverse points become points symmetrical about the line.*

For let

$$\left|\frac{z-p}{z-q}\right|=k$$

be a circle (or straight line), with p and q as inverse points. Let

$$w=\frac{az+b}{cz+d}, \qquad z=\frac{dw-b}{-cw+a}.$$

Then the circle transforms into

$$\left|\frac{dw-b-p(-cw+a)}{dw-b-q(-cw+a)}\right|=k$$

or

$$\left|\frac{w-\dfrac{ap+b}{cp+d}}{w-\dfrac{aq+b}{cq+d}}\right|=k\left|\frac{cq+d}{cp+d}\right|.$$

The result is obvious from this equation.

Example. Prove that the linear transformation in which only one point p is invariant may be considered as the result of (i) an inversion in a circle (centre z_0, say), through the point p, (ii) an inversion in the circle with centre w_0 corresponding to z_0 in the transformation, and touching the previous circle at p.

6.23. *To find all linear transformations of the half-plane* $\mathbf{I}(z) \geqslant 0$ *into the unit circle* $|w| \leqslant 1$.

To points z, $\bar{z}$, symmetrical about the real z-axis correspond points w, $\dfrac{1}{\bar{w}}$, inverse with respect to the unit w-circle. In particular, the origin and the point at infinity in the w-plane correspond to conjugate values of z.

Let
$$w = \frac{az+b}{cz+d}$$
be the required transformation. Plainly $c \neq 0$, or the points at infinity would correspond. Now $w = 0$, $w = \infty$ correspond to $z = -b/a$, $-d/c$. Hence we may write
$$-\frac{b}{a} = \alpha, \qquad -\frac{d}{c} = \bar{\alpha},$$
and
$$w = \frac{a}{c}\frac{z-\alpha}{z-\bar{\alpha}}.$$
The point $z = 0$ must correspond to a point of the circle $|w| = 1$, so that
$$\left|\frac{a}{c}\cdot\frac{-\alpha}{-\bar{\alpha}}\right| = \left|\frac{a}{c}\right| = 1.$$
Hence we put
$$a = ce^{i\lambda},$$
where λ is real, and obtain
$$w = e^{i\lambda}\frac{z-\alpha}{z-\bar{\alpha}}. \tag{1}$$
Since $z = \alpha$ gives $w = 0$, α must be a point of the upper half-plane, i.e. $\mathbf{I}(\alpha) > 0$. With this condition the function (1) gives the required representation. For if z is real, obviously $|w| = 1$; and if $\mathbf{I}(z) > 0$, then z is nearer to α than to $\bar{\alpha}$, and $|w| < 1$.

There are three arbitrary constants in the transformation, λ, $\mathbf{R}(\alpha)$, and $\mathbf{I}(\alpha)$. We can therefore make three given points of the real axis correspond to three given points of the unit circle.

Example. The general linear transformation of the half-plane $\mathbf{R}(z) \geqslant 0$ on the circle $|w| \leqslant 1$ is
$$w = e^{i\lambda}\frac{z-\alpha}{z+\bar{\alpha}} \qquad (\mathbf{R}(\alpha) > 0).$$

6.24. *To find all linear transformations of the unit circle $|z| \leqslant 1$ into the unit circle $|w| \leqslant 1$.*

Let
$$w = \frac{az+b}{cz+d}.$$

Here $w = 0$, $w = \infty$, must correspond to inverse points $z = \alpha$, $z = 1/\bar{\alpha}$, where $|\alpha| < 1$. Hence
$$-\frac{b}{a} = \alpha, \qquad -\frac{d}{c} = \frac{1}{\bar{\alpha}},$$
$$w = \frac{a}{c}\,\frac{z-\alpha}{z-1/\bar{\alpha}} = \frac{a\bar{\alpha}}{c}\cdot\frac{z-\alpha}{\bar{\alpha}z-1}.$$

The point $z = 1$ corresponds to a point on $|w| = 1$. Hence
$$\left|\frac{a\bar{\alpha}}{c}\cdot\frac{1-\alpha}{\bar{\alpha}-1}\right| = \left|\frac{a\bar{\alpha}}{c}\right| = 1.$$

Hence
$$w = e^{i\lambda}\,\frac{z-\alpha}{\bar{\alpha}z-1},$$
where λ is real.

This is the required result; for if $z = e^{i\theta}$, $\alpha = be^{i\beta}$, then
$$|w| = \left|\frac{e^{i\theta}-be^{i\beta}}{be^{i(\theta-\beta)}-1}\right| = 1.$$

If $z = re^{i\theta}$, where $r < 1$, then
$$\begin{aligned} &|z-\alpha|^2-|\bar{\alpha}z-1|^2 \\ &\quad = r^2-2rb\cos(\theta-\beta)+b^2-\{b^2r^2-2br\cos(\theta-\beta)+1\} \\ &\quad = (r^2-1)(1-b^2) < 0. \end{aligned}$$

Hence $|w| < 1$.

If we are also given that $z = 0$ corresponds to $w = 0$, then $\alpha = 0$, and the transformation becomes
$$w = e^{i\lambda}z.$$

If also $\dfrac{dw}{dz} = 1$ at $z = 0$, then
$$w = z.$$

Example. *The general linear transformation of the circle $|z| \leqslant \rho$ into the circle $|w| \leqslant \rho'$ is*
$$w = \rho\rho' e^{i\lambda}\,\frac{z-\alpha}{\bar{\alpha}z-\rho^2} \qquad (|\alpha| < \rho).$$

6.25. *If $f(z)$ is regular for $|z| < 1$, $\mathbf{R}\{f(z)\} > 0$, and $f(0) = a > 0$, then $|f'(0)| \leqslant 2a$.*

A result of this type follows from Carathéodory's theorem and its corollary (§§ 5.5–5.51). The following argument is essentially the same, but can now be put in a form which throws some light on the general method.

Suppose that we can find a linear transformation $g = \phi(f)$ such that $\mathbf{R}(f) = 0$ corresponds to $|g| = 1$, while $f = a$ corresponds to $g = 0$. Then we shall have $|g(z)| < 1$ for $\mathbf{R}\{f(z)\} > 0$, i.e. for $|z| < 1$, and $g(0) = 0$. In this form the data are easy to deal with. We have

$$|g'(0)| = \left|\frac{1}{2\pi i}\int\limits_{|z|=1-\epsilon} \frac{g(z)}{z^2}\,dz\right| \leqslant \frac{1}{1-\epsilon},$$

and hence, making $\epsilon \to 0$, $|g'(0)| \leqslant 1$.

We find, as in § 6.23, that the required linear transformation is

$$g(z) = \frac{f(z)-a}{f(z)+a},$$

or

$$f(z) = a\frac{1+g(z)}{1-g(z)}.$$

Hence

$$f'(z) = \frac{2ag'(z)}{\{1-g(z)\}^2},$$

and

$$|f'(0)| = 2a|g'(0)| \leqslant 2a.$$

6.3. Various transformations. We shall now consider some examples of functions which are not linear.

6.31. The function $w = z^2$. If $z = re^{i\theta}$ and $w = \rho e^{i\phi}$, then $\rho e^{i\phi} = r^2 e^{2i\theta}$, so that

$$\rho = r^2, \qquad \phi = 2\theta.$$

The distance from the origin is therefore squared, and the polar angle is doubled. An angular region $\alpha < \arg z < \beta$ is represented on an angular region $2\alpha < \arg w < 2\beta$; if $\beta - \alpha > \pi$, the angular region in the w-plane covers part of this plane twice. The ambiguity arising from this is removed if we replace the w-plane by the Riemann surface described in § 4.3.

If $z = x+iy$, $w = u+iv$, then

$$u+iv = (x+iy)^2 = x^2-y^2+2ixy,$$

or

$$u = x^2-y^2, \qquad v = 2xy.$$

Hence the straight lines $u = a$, $v = b$ correspond to the rectangular hyperbolas

$$x^2-y^2 = a, \qquad 2xy = b.$$

These cut at right angles except in the case $a = 0$, $b = 0$, when they intersect at the origin at an angle $\frac{1}{4}\pi$. Since dw/dz has a simple zero at the origin, this is in accordance with the general theorems on the transformation of angles.

Examples. (i) Prove that the straight lines x = const., y = const., correspond to systems of confocal parabolas.

(ii) Consider in the same way the function $w = z^n$ for $n = 3, 4, \ldots$.

6.32. The function $w = \dfrac{1}{2}\left(z+\dfrac{1}{z}\right)$. Here w becomes infinite at $z = 0$, while

$$\frac{dw}{dz} = \frac{1}{2}\left(1-\frac{1}{z^2}\right),$$

which vanishes at $z = \pm 1$. These points may therefore be expected to play a special part in the transformation.

Putting $z = re^{i\theta}$, $w = u+iv$, we have

$$u = \frac{1}{2}\left(r+\frac{1}{r}\right)\cos\theta, \qquad v = \frac{1}{2}\left(r-\frac{1}{r}\right)\sin\theta,$$

and, eliminating θ,

$$\frac{u^2}{\frac{1}{4}\left(r+\frac{1}{r}\right)^2} + \frac{v^2}{\frac{1}{4}\left(r-\frac{1}{r}\right)^2} = 1.$$

This is an ellipse in the w-plane, and it corresponds to each of the two circles $|z| = r$, $|z| = \dfrac{1}{r}$. As $r \to 1$, the major semi-axis of the ellipse tends to 1, while the minor semi-axis tends to zero. As $r \to 0$ or as $r \to \infty$, both semi-axes tend to infinity. From this it is clear that the inside and the outside of the unit circle in the z-plane both correspond to the whole w-plane, cut along the real axis from -1 to 1. The unit circle $|z| = 1$ itself corresponds to the straight line from -1 to 1 described twice.

On solving the equation for z, we see that the inverse function is a two-valued function of w. We can remove all ambiguity from the representation by replacing the w-plane by a Riemann surface of two sheets, each slit from -1 to 1, and joined cross-ways along the slit. If we pass round one or other of the points $w = \pm 1$, a different value of z is reached, but, if we pass round both, z returns to its original value.

Examples. To what curve in the w-plane does the line $x = 1$ correspond? Consider the result as an example of § 6.11.

6.33. The logarithmic function. *If $w = \log z$, the angular region $\alpha < \arg z < \beta$ corresponds to the infinite strip $\alpha < v < \beta$ in the w-plane.*

For if $z = re^{i\theta}$, then a value of w is

$$\log r + i\theta.$$

Hence $$u = \log r, \qquad v = \theta.$$

As r goes from 0 to ∞, u goes from $-\infty$ to ∞, and the result follows.

If we consider the general value of $\log z$,

$$w = \log r + i(\theta + 2k\pi),$$

where k is any integer, we obtain, not one strip, but an infinity of strips in the w-plane, corresponding to the infinity of values of the logarithm.

On the other hand, a strip $\alpha < v < \beta$ corresponds to an angle in the z-plane; but, if $\beta - \alpha > 2\pi$, part of the plane will be covered more than once. We can, however, avoid any ambiguity by replacing the single z-plane by a Riemann surface consisting of an infinity of sheets, each cut along the real axis from 0 to $-\infty$, and the upper half-plane of each joined to the lower half-plane of the next along the slit. Then a strip of the w-plane of breadth 2π corresponds to one complete sheet of the Riemann surface, and every point of the Riemann surface corresponds to just one point of the w-plane.

Examples. (i) Investigate the properties of the transformation $w = \tan z$ by considering it as the result of the two transformations

$$w = \frac{1}{i}\frac{\zeta - 1}{\zeta + 1}, \qquad \zeta = e^{2iz},$$

and hence obtain a Riemann surface for the inverse function

$$z = \arctan w.$$

[Hurwitz-Courant, *Funktionentheorie*, p. 293.]

(ii) Consider the properties of the transformation

$$w = z^{\alpha}$$

for general values of α.

[The function z^{α} is defined as $e^{\alpha \log z}$. Consider separately rational, irrational, and complex values of α.]

6.34. *If $w = \tan^2 \frac{1}{2}z$, the strip in the z-plane between the lines $x = 0$, $x = \frac{1}{2}\pi$ is represented on the interior of the unit circle in the w-plane, cut along the real axis from $w = -1$ to $w = 0$.*

We have $$w = \frac{1-\cos z}{1+\cos z}.$$

If $z = \frac{1}{2}\pi + iy$, then $\cos z = -i \sinh y$, and $|w| = 1$. It is easily seen that, as y goes from $-\infty$ to ∞, $\operatorname{am} w$ goes from $-\pi$ to π, so that w describes the whole unit circle once.

If $z = iy$, $\cos z = \cosh y$, and w is real. As y goes from $+\infty$ to 0, w goes from -1 to 0, and as y goes from 0 to $-\infty$, w retraces its path from 0 to -1.

The boundary of the strip therefore corresponds to the boundary of the cut circle, and there should be no difficulty in verifying that the interiors correspond.

Example. Prove that the line $x = \frac{1}{4}\pi$ corresponds to a loop of a closed curve, cutting the real axis at $w = -1$ and $w = 1/(3+2\sqrt{2})$.

6.4. Simple ('schlicht') functions.* We shall say that a function $f(z)$ is *simple* in a region D if it is analytic, one-valued, and does not take any value more than once in D.

The function $w = f(z)$ then represents the region D of the z-plane on a region D' of the w-plane, in such a way that there is a one-one correspondence between the points of the two regions.

If $f(z)$ is simple in D, $f'(z) \neq 0$ in D. For suppose, on the contrary, that $f'(z_0) = 0$. Then $f(z)-f(z_0)$ has a zero of order n $(n \geqslant 2)$ at z_0. Since $f(z)$ is not constant, we can find a circle $|z-z_0| = \delta$ on which $f(z)-f(z_0)$ does not vanish, and inside which $f'(z)$ has no zeros except z_0. Let m be the lower bound of $|f(z)-f(z_0)|$ on this circle. Then by Rouché's theorem, if $0 < |a| < m$, $f(z)-f(z_0)-a$ has n zeros in the circle (it cannot have a double zero, since $f'(z)$ has no other zeros in the circle). This is contrary to the hypothesis that $f(z)$ does not take any value more than once.

A simple function of a simple function is simple. If $f(z)$ is simple in D, and $F(w)$ in D', then $F\{f(z)\}$ is simple in D; for $F\{f(z_1)\} = F\{f(z_2)\}$ implies $f(z_1) = f(z_2)$, since F is simple; and this implies $z_1 = z_2$, since f is simple.

6.41. Inverse functions. In the above relationship, to every point of D' corresponds just one point of D. We can therefore consider z as a function of w, say $z = \phi(w)$. This is called the inverse function of $w = f(z)$.

* The German word is *schlicht*; Dienes, *The Taylor Series*, uses *biuniform*.

The inverse function is simple in D'. For it is one-valued; and it does not take any value more than once, since $f(z)$ is one-valued. Finally, it is analytic; for if $w_0 = f(z_0)$, then it is easily seen by considering $\int f'(z)/\{f(z)-w\}\,dz$ that, in any given neighbourhood of z_0, $f(z)$ takes every value w sufficiently near to w_0. Hence $z = \phi(w)$ is continuous, and

as $w \to w_0$, since $f'(z_0) \neq 0$.

6.42. Uniqueness of conformal transformation. *A simple function $w = f(z)$ which represents the unit circle on itself, so that the centre and a given direction through it remain unaltered, is the identical transformation $w = z$.*

We have $|f(z)| = 1$ for $|z| = 1$, and $f(0) = 0$. Hence, by Schwarz's lemma (§ 5.2),

$$|w| = |f(z)| \leqslant |z|.$$

But, applying Schwarz's lemma to the inverse function, we have $|z| \leqslant |w|$. Hence $|w| = |z|$, i.e.

$$|f(z)/z| = 1 \qquad (|z| \leqslant 1).$$

Since a function of constant modulus is itself constant, it follows that

$$f(z) = az,$$

where $|a| = 1$. The remaining conditions then show that $a = 1$.

Other functions, such as $w = z^2$, satisfy the conditions except that they are not simple.

A simple function which represents the unit circle on itself is a linear function.

If $w = f(z)$ represents the unit circle on itself, and $f(0) = w_0$, we can, by § 6.24, find a linear function $l(w)$ which represents the unit circle on itself, and is such that $l(w_0) = 0$. Then $l\{f(z)\}$ represents the unit circle on itself, and $l\{f(0)\} = 0$. Hence, by the above theorem, $l\{f(z)\} = az$. Since the inverse function of a linear function is linear, $f(z)$ is a linear function of z.

6.43. *Let $f(z)$ be regular at $z = 0$, and $f'(0) \neq 0$. Then $f(z)$ is simple in the immediate neighbourhood of $z = 0$; i.e. in the circle $|z| \leqslant \rho$, if ρ is small enough.*

We may suppose that $f(0) = 0$. Since $f'(0) \neq 0$, the origin is a zero of $f(z)$ of the first order. We can therefore find a circle

C, with centre $z = 0$, on which $f(z) \neq 0$, and inside which $f(z)$ has no zero other than $z = 0$. Let m be the lower bound of $|f(z)|$ on C.

Since $f(z)$ is continuous and vanishes at $z = 0$, we can find a circle $|z| = \rho$ inside which $|f(z)| < m$. Then $w = f(z)$ is simple in this circle. For let w' be any number such that $|w'| < m$. Then by Rouché's theorem (§ 3.42) the number of zeros of $f(z) - w'$ in C is the same as the number of zeros of $f(z)$, that is, one. Hence there is just one point z' in C corresponding to each such value of w'. The region consisting of these values of z' is therefore represented simply on the circle $|w| < m$; and this region includes the circle $|z| = \rho$.

An alternative proof may be obtained by considering the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

where $a_1 \neq 0$. If $f(z_1) = f(z_2)$,

$$\sum_{n=1}^{\infty} a_n(z_1^n - z_2^n) = 0,$$

i.e.

$$(z_1 - z_2)\left\{a_1 + \sum_{n=2}^{\infty} a_n(z_1^{n-1} + z_1^{n-2}z_2 + \ldots + z_2^{n-1})\right\} = 0.$$

If $|z_1| < \rho, |z_2| < \rho$, the modulus of the second factor is greater than

$$|a_1| - \sum_{n=2}^{\infty} n|a_n|\rho^{n-1},$$

which is positive if ρ is small enough. Hence $z_1 = z_2$, and the result follows.

6.44. *The limit of a uniformly convergent sequence of simple functions is either simple or constant. More precisely, if $f_n(z)$ is simple in D for each value of n, and $f_n(z) \to f(z)$ uniformly in D, then $f(z)$ is simple in D, or is a constant.*

The possibility of the limit being constant is shown by the example $f_n(z) = z/n$.

In any case, $f(z)$ is analytic and one-valued in D. If it is not simple, there are two points z_1 and z_2 at which $w = f(z)$ takes the same value w_0. Describe, with z_1 and z_2 as centres, two circles which lie in D, do not overlap, and such that $f(z) - w_0$ does not vanish on either circumference (this is possible unless $f(z)$ is a constant). Let m be the lower bound of $|f(z) - w_0|$ on the two circumferences. Then we can choose n so large that $|f(z)-f_n(z)|<m$ on the two circumferences. Hence, by Rouché's theorem, the function

$$f_n(z)-w_0=\{f(z)-w_0\}+\{f_n(z)-f(z)\}$$

has as many zeros in the circles as $f(z)-w_0$, that is, at least two. Hence $f_n(z)$ is not simple, contrary to hypothesis. This proves the theorem.

6.45. *Let C be a simple closed contour in the z-plane, enclosing a region D. Let $w=f(z)$ be an analytic function of z, regular in D and on C, and taking no value more than once on C. Then $f(z)$ is simple in D.*

The contour C corresponds to a contour C' in the w-plane. C' is closed, since $f(z)$ is one-valued; and it has no double points, since $f(z)$ does not take any value twice on C. Let D' be the region enclosed by C'.

We assume that $f(z)$ takes in D values other than those on C, say at z_0. Then, if Δ_C denotes variation round C,

$$\frac{1}{2\pi}\Delta_C \arg\{f(z)-f(z_0)\}$$

is equal to the number of zeros of $f(z)-f(z_0)$ in C; it is therefore a positive integer, since there is at least one such zero. But it is also equal to

$$\frac{1}{2\pi}\Delta_{C'} \arg(w-w_0),$$

where $w_0=f(z_0)$; and this is either 0, if w_0 is outside C', or ± 1, if w_0 is inside C', the sign depending on the direction in which C' is described. Hence it is equal to 1. Hence w_0 lies inside C', C' is described in the positive direction, and $f(z)$ takes the value w_0 just once in D. Thus D is represented simply on D'.

6.46. Extensions. The condition in the above analysis that the function $f(z)$ should be analytic on the contour can be relaxed to a certain extent. The state of affairs is not much altered if $f(z)$ is not analytic, but is continuous, at certain points of C. Suppose that there is a singularity at z_1 on the contour, and that C_1 is C with an indentation round z_1. Then the number of zeros of $f(z)-w_0$ inside C_1 is

$$\frac{1}{2\pi}\Delta_{C_1}\arg\{f(z)-w_0\}=\frac{1}{2\pi i}\int_{C_1}\frac{f'(z)}{f(z)-w_0}\,dz;$$

and, as the indentation is closed up, this tends to the corresponding integral round C, if $f(z)$ is continuous and $f'(z) = O(|z-z_1|^\alpha)$, where $\alpha > -1$. The argument of § 6.45 therefore still applies.

The function $f(z)$ may also have a pole on the contour; the region D' then extends to infinity. The theorem of § 6.45 still holds if the pole is of the first order, and the contour is a fairly ordinary one. Suppose that, by a preliminary change of variable, we take the pole at the origin, and that the direction can be taken so that $\mathbf{R}(z) \geqslant 0$ at all points of D. Let

$$w = f(z) = \frac{1}{z} + g(z),$$

where $g(z)$ is regular in D. Then, for z in D,

$$\mathbf{R}(w) \geqslant \min \mathbf{R}\{g(z)\} = a,$$

say. Let $b < a$. Then $|w-b| \geqslant a-b$ for z in D. Hence

$$\zeta = \frac{1}{w-b} = \frac{z}{1+zg(z)-bz}$$

is regular in D. The theorem applies directly to ζ, and so, since w is a simple function of ζ, to w.

The result is not necessarily true for poles of higher order.

Examples. (i) Let $$w = \frac{1}{i}\frac{z+1}{z-1}.$$

If $z = e^{i\theta}$, then $$w = \frac{1}{i}\frac{e^{i\theta}+1}{e^{i\theta}-1} = -\cot \tfrac{1}{2}\theta.$$

Hence, as z describes the unit circle from 0 to 2π, w describes the real axis from $-\infty$ to ∞. The only singularity on the z-boundary is a simple pole. Hence the unit circle in the z-plane is represented simply on the upper half w-plane. This of course is easily verified.

(ii) Let $$w = -\frac{1}{i}\left(\frac{z+1}{z-1}\right)^3.$$

Then, if $z = e^{i\theta}$, $$w = -\cot^3 \tfrac{1}{2}\theta.$$

Hence there is a one-one correspondence between the unit circle in the z-plane and the real w-axis. But since there is a triple pole on the boundary the areas do not necessarily correspond. In fact

$$w = i\left(\frac{x+iy+1}{x+iy-1}\right)^3 = i\frac{(x^2+y^2-1)^3-12(x^2+y^2-1)y^2}{\{(x-1)^2+y^2\}^3} + \ldots,$$

and $v = 0$ corresponds to

$$(x^2+y^2-1)(x^2+y^2-2\sqrt{3}y-1)(x^2+y^2+2\sqrt{3}y-1) = 0,$$

i.e. the equation of three circles. Hence $v > 0$ if z is outside each of these circles, or outside one and inside the other two.