THE THEORY OF THE RIEMANN ZETA-FUNCTION

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PREFACE TO THE SECOND EDITION

SINCE the first edition was written, a vast amount of further work has been done. This has been covered by the end-of-chapter notes. In most instances, restrictions on space have prohibited the inclusion of full proofs, but I have tried to give an indication of the methods used wherever possible. (Proofs of quite a few of the recent results described in the end of chapter notes may be found in the book by Ivic [**3**].) I have also corrected a number of minor errors, and made a few other small improvements to the text. A considerable number of recent references have been added.

In preparing this work I have had help from Professors J. B. Conrey, P. D. T. A. Elliott, A. Ghosh, S. M. Gonek, H. L. Montgomery, and S. J. Patterson. It is a pleasure to record my thanks to them.

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D. R. H.-B.

PREFACE TO FIRST EDITION

THIS book is a successor to my Cambridge Tract *The Zeta-Function of Riemann*, 1930, which is now out of print and out of date. It seems no longer practicable to give an account of the subject in such a small space as a Cambridge Tract, so that the present work, though on exactly the same lines as the previous one, is on a much larger scale. As before, I do not discuss general prime-number theory, though it has been convenient to include some theorems on primes.

Most of this book was compiled in the 1930's, when I was still researching on the subject. It has been brought partly up to date by including some of the work of A. Selberg and of Vinogradov, though a great deal of recent work is scantily represented.

The manuscript has been read by Dr. S. H. Min and by Prof. D. B. Sears, and my best thanks are due to them for correcting a large number of mistakes. I must also thank Prof. F. V. Atkinson and Dr. T. M. Fleet for their kind assistance in reading the proof-sheets.

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CONTENTS

I

THE FUNCTION $\zeta(s)$ AND THE DIRICHLET SERIES RELATED TO IT

1.1. Definition of $\zeta(s)$. The Riemann zeta-function $\zeta(s)$ has its origin in the identity expressed by the two formulae

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1.1.1}$$

where n runs through all integers, and

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \tag{1.1.2}$$

where p runs through all primes. Either of these may be taken as the definition of $\zeta(s)$; s is a complex variable, $s = \sigma + it$. The Dirichlet series (1.1.1) is convergent for $\sigma > 1$, and uniformly convergent in any finite region in which $\sigma \geqslant 1+\delta$, $\delta > 0$. It therefore defines an analytic function $\zeta(s)$, regular for $\sigma > 1$.

The infinite product is also absolutely convergent for $\sigma > 1$; for so is

$$\sum_p \left|\frac{1}{p^s}\right| = \sum_p \frac{1}{p^\sigma},$$

this being merely a selection of terms from the series $\sum n^{-\sigma}$. If we expand the factor involving p in powers of p^{-s}, we obtain

$$\prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \ldots\right).$$

On multiplying formally, we obtain the series (1.1.1), since each integer n can be expressed as a product of prime-powers p^m in just one way. The identity of (1.1.1) and (1.1.2) is thus an analytic equivalent of the theorem that the expression of an integer in prime factors is unique.

A rigorous proof is easily constructed by taking first a finite number of factors. Since we can multiply a finite number of absolutely convergent series, we have

$$\prod_{p \leqslant P} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \ldots\right) = 1 + \frac{1}{n_1^s} + \frac{1}{n_2^s} + \ldots,$$

where $n_1, n_2, \ldots,$ are those integers none of whose prime factors exceed P.

Since all integers up to P are of this form, it follows that, if $\zeta(s)$ is defined by (1.1.1),

$$\left|\zeta(s)-\prod_{p\leqslant P}\left(1-\frac{1}{p^s}\right)^{-1}\right| = \left|\zeta(s)-1-\frac{1}{n_1^s}-\frac{1}{n_2^s}-\ldots\right|$$
$$\leqslant \frac{1}{(P+1)^\sigma}+\frac{1}{(P+2)^\sigma}+\ldots.$$

This tends to 0 as $P\to\infty$, if $\sigma>1$; and (1.1.2) follows.

This fundamental identity is due to Euler, and (1.1.2) is known as Euler's product. But Euler considered it for particular values of s only, and it was Riemann who first considered $\zeta(s)$ as an analytic function of a complex variable.

Since a convergent infinite product of non-zero factors is not zero, we deduce that $\zeta(s)$ *has no zeros for* $\sigma>1$. This may be proved directly as follows. We have for $\sigma>1$

$$\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\ldots\left(1-\frac{1}{P^s}\right)\zeta(s) = 1+\frac{1}{m_1^s}+\frac{1}{m_2^s}+\ldots,$$

where $m_1, m_2,\ldots,$ are the integers all of whose prime factors exceed P. Hence

$$\left|\left(1-\frac{1}{2^s}\right)\ldots\left(1-\frac{1}{P^s}\right)\zeta(s)\right| \geqslant 1-\frac{1}{(P+1)^\sigma}-\frac{1}{(P+2)^\sigma}-\ldots>0$$

if P is large enough. Hence $|\zeta(s)|>0$.

The importance of $\zeta(s)$ in the theory of prime numbers lies in the fact that it combines two expressions, one of which contains the primes explicitly, while the other does not. The theory of primes is largely concerned with the function $\pi(x)$, the number of primes not exceeding x. We can transform (1.1.2) into a relation between $\zeta(s)$ and $\pi(x)$; for if $\sigma>1$,

$$\log\zeta(s) = -\sum_p \log\left(1-\frac{1}{p^s}\right) = -\sum_{n=2}^{\infty}\{\pi(n)-\pi(n-1)\}\log\left(1-\frac{1}{n^s}\right)$$
$$= -\sum_{n=2}^{\infty}\pi(n)\left\{\log\left(1-\frac{1}{n^s}\right)-\log\left(1-\frac{1}{(n+1)^s}\right)\right\}$$
$$= \sum_{n=2}^{\infty}\pi(n)\int_n^{n+1}\frac{s}{x(x^s-1)}\,dx = s\int_2^{\infty}\frac{\pi(x)}{x(x^s-1)}\,dx. \qquad (1.1.3)$$

The rearrangement of the series is justified since $\pi(n)\leqslant n$ and

$$\log(1-n^{-s}) = O(n^{-\sigma}).$$

Again
$$\frac{1}{\zeta(s)} = \prod_p\left(1-\frac{1}{p^s}\right),$$

and on carrying out the multiplication we obtain

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}\quad (\sigma>1), \qquad (1.1.4)$$

where $\mu(1)=1$, $\mu(n)=(-1)^k$ if n is the product of k different primes, and $\mu(n)=0$ if n contains any factor to a power higher than the first. The process is easily justified as in the case of $\zeta(s)$.

The function $\mu(n)$ is known as the Möbius function. It has the property

$$\sum_{d|q}\mu(d) = 1\ (q=1),\quad 0\ (q>1), \qquad (1.1.5)$$

where $d\,|\,q$ means that d is a divisor of q. This follows from the identity

$$1 = \sum_{m=1}^{\infty}\frac{1}{m^s}\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} = \sum_{q=1}^{\infty}\frac{1}{q^s}\sum_{d|q}\mu(d).$$

It also gives the 'Möbius inversion formula'

$$g(q) = \sum_{d|q}f(d), \qquad (1.1.6)$$

$$f(q) = \sum_{d|q}\mu\left(\frac{q}{d}\right)g(d), \qquad (1.1.7)$$

connecting two functions $f(n)$, $g(n)$ defined for integral n. If f is given and g defined by (1.1.6), the right-hand side of (1.1.7) is

$$\sum_{d|q}\mu\left(\frac{q}{d}\right)\sum_{r|d}f(r).$$

The coefficient of $f(q)$ is $\mu(1)=1$. If $r<q$, then $d=kr$, where $k\,|\,q/r$. Hence the coefficient of $f(r)$ is

$$\sum_{k|q/r}\mu\left(\frac{q}{kr}\right) = \sum_{k'|q/r}\mu(k') = 0$$

by (1.1.5). This proves (1.1.7). Conversely, if g is given, and f is defined by (1.1.7), then the right-hand side of (1.1.6) is

$$\sum_{d|q}\sum_{r|d}\mu\left(\frac{d}{r}\right)g(r),$$

and this is $g(q)$, by a similar argument. The formula may also be derived formally from the obviously equivalent relations

$$F(s)\zeta(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}, \qquad F(s) = \frac{1}{\zeta(s)} \sum_{n=1}^{\infty} \frac{g(n)}{n^s},$$

where

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

Again, on taking logarithms and differentiating (1.1.2), we obtain, for $\sigma > 1$,

$$\begin{aligned}\frac{\zeta'(s)}{\zeta(s)} &= -\sum_p \frac{\log p}{p^s}\left(1-\frac{1}{p^s}\right)^{-1} \\ &= -\sum_p \log p \sum_{m=1}^{\infty} \frac{1}{p^{ms}} \\ &= -\sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s}, \end{aligned} \tag{1.1.8}$$

where $\Lambda(n) = \log p$ if n is p or a power of p, and otherwise $\Lambda(n) = 0$. On integrating we obtain

$$\log \zeta(s) = \sum_{n=2}^{\infty} \frac{\Lambda_1(n)}{n^s} \quad (\sigma > 1), \tag{1.1.9}$$

where $\Lambda_1(n) = \Lambda(n)/\log n$, and the value of $\log \zeta(s)$ is that which tends to 0 as $\sigma \to \infty$, for any fixed t.

1.2. Various Dirichlet series connected with $\zeta(s)$. In the first place

$$\zeta^2(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s} \quad (\sigma > 1), \tag{1.2.1}$$

where $d(n)$ denotes the number of divisors of n (including 1 and n itself). For

$$\zeta^2(s) = \sum_{\mu=1}^{\infty} \frac{1}{\mu^s} \sum_{\nu=1}^{\infty} \frac{1}{\nu^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{\mu\nu=n} 1,$$

and the number of terms in the last sum is $d(n)$. And generally

$$\zeta^k(s) = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s} \quad (\sigma > 1), \tag{1.2.2}$$

where $k = 2, 3, 4,\ldots$, and $d_k(n)$ denotes the number of ways of expressing n as a product of k factors, expressions with the same factors in a different order being counted as different. For

$$\zeta^k(s) = \sum_{\nu_1=1}^{\infty} \frac{1}{\nu_1^s} \cdots \sum_{\nu_k=1}^{\infty} \frac{1}{\nu_k^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{\nu_1\ldots\nu_k=n} 1,$$

and the last sum is $d_k(n)$.

Since we have also

$$\zeta^2(s) = \prod_p \left(1-\frac{1}{p^s}\right)^{-2} = \prod_p \left(1+\frac{2}{p^s}+\frac{3}{p^{2s}}+\ldots\right), \tag{1.2.3}$$

on comparing the coefficients in (1.2.1) and (1.2.3) we verify the elementary formula

$$d(n) = (m_1+1)\ldots(m_r+1) \tag{1.2.4}$$

for the number of divisors of

$$n = p_1^{m_1} p_2^{m_2} \ldots p_r^{m_r}. \tag{1.2.5}$$

Similarly from (1.2.2)

$$d_k(n) = \frac{(k+m_1-1)!}{m_1!\,(k-1)!} \cdots \frac{(k+m_r-1)!}{m_r!\,(k-1)!}. \tag{1.2.6}$$

We next note the expansions

$$\frac{\zeta(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s} \quad (\sigma > 1), \tag{1.2.7}$$

where $\mu(n)$ is the coefficient in (1.1.4);

$$\frac{\zeta^2(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{2^{\nu(n)}}{n^s} \quad (\sigma > 1), \tag{1.2.8}$$

where $\nu(n)$ is the number of different prime factors of n;

$$\frac{\zeta^3(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{d(n^2)}{n^s} \quad (\sigma > 1), \tag{1.2.9}$$

and

$$\frac{\zeta^4(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{\{d(n)\}^2}{n^s} \quad (\sigma > 1). \tag{1.2.10}$$

To prove (1.2.7), we have

$$\frac{\zeta(s)}{\zeta(2s)} = \prod_p \frac{1-p^{-2s}}{1-p^{-s}} = \prod_p \left(1+\frac{1}{p^s}\right),$$

and this differs from the formula for $1/\zeta(s)$ only in the fact that the signs are all positive. The result is therefore clear. To prove (1.2.8), we have

$$\begin{aligned}\frac{\zeta^2(s)}{\zeta(2s)} &= \prod_p \frac{1-p^{-2s}}{(1-p^{-s})^2} = \prod_p \frac{1+p^{-s}}{1-p^{-s}} \\ &= \prod_p (1+2p^{-s}+2p^{-2s}+\ldots),\end{aligned}$$

and the result follows. To prove (1.2.9),

$$\frac{\zeta^3(s)}{\zeta(2s)} = \prod_p \frac{1-p^{-2s}}{(1-p^{-s})^3} = \prod_p \frac{1+p^{-s}}{(1-p^{-s})^2}$$
$$= \prod_p \{(1+p^{-s})(1+2p^{-s}+3p^{-2s}+...)\}$$
$$= \prod_p \{1+3p^{-s}+...+(2m+1)p^{-ms}+...\},$$

and the result follows, since, if n is (1.2.5),

$$d(n^2) = (2m_1+1)...(2m_r+1).$$

Similarly

$$\frac{\zeta^4(s)}{\zeta(2s)} = \prod_p \frac{1-p^{-2s}}{(1-p^{-s})^4} = \prod_p \frac{1+p^{-s}}{(1-p^{-s})^3}$$
$$= \prod_p (1+p^{-s})\{1+3p^{-s}+...+\tfrac{1}{2}(m+1)(m+2)p^{-ms}+...\}$$
$$= \prod_p \{1+4p^{-s}+...+(m+1)^2p^{-ms}+...\},$$

and (1.2.10) follows.

Other formulae are

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} \quad (\sigma > 1), \tag{1.2.11}$$

where $\lambda(n) = (-1)^r$ if n has r prime factors, a factor of degree k being counted k times;

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} \quad (\sigma > 2), \tag{1.2.12}$$

where $\phi(n)$ is the number of numbers less than n and prime to n; and

$$\frac{1-2^{1-s}}{1-2^{-s}}\zeta(s-1) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \quad (\sigma > 2), \tag{1.2.13}$$

where $a(n)$ is the greatest odd divisor of n. Of these, (1.2.11) follows at once from

$$\frac{\zeta(2s)}{\zeta(s)} = \prod_p \left(\frac{1-p^{-s}}{1-p^{-2s}}\right) = \prod_p \left(\frac{1}{1+p^{-s}}\right) = \prod_p (1-p^{-s}+p^{-2s}-...).$$

Also

$$\frac{\zeta(s-1)}{\zeta(s)} = \prod_p \left(\frac{1-p^{-s}}{1-p^{1-s}}\right) = \prod_p \left\{\left(1-\frac{1}{p^s}\right)\left(1+\frac{p}{p^s}+\frac{p^2}{p^{2s}}+...\right)\right\}$$
$$= \prod_p \left\{1+\left(1-\frac{1}{p}\right)\left(\frac{p}{p^s}+\frac{p^2}{p^{2s}}+...\right)\right\},$$

and (1.2.12) follows, since, if $n = p_1^{m_1}...p_r^{m_r}$,

$$\phi(n) = n\left(1-\frac{1}{p_1}\right)...\left(1-\frac{1}{p_r}\right).$$

Finally

$$\frac{1-2^{1-s}}{1-2^{-s}}\zeta(s-1) = \frac{1-2^{1-s}}{1-2^{-s}}\prod_p \frac{1}{1-p^{1-s}}$$
$$= \frac{1}{1-2^{-s}}\,\frac{1}{1-3^{1-s}}\,\frac{1}{1-5^{1-s}}...$$
$$= \left(1+\frac{1}{2^s}+\frac{1}{2^{2s}}+...\right)\left(1+\frac{3}{3^s}+\frac{3^2}{3^{2s}}+...\right)...,$$

and (1.2.13) follows.

Many of these formulae are, of course, simply particular cases of the general formula

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left\{1+\frac{f(p)}{p^s}+\frac{f(p^2)}{p^{2s}}+...\right\},$$

where $f(n)$ is a multiplicative function, i.e. is such that, if $n = p_1^{m_1}p_2^{m_2}...$, then

$$f(n) = f(p_1^{m_1})f(p_2^{m_2})....$$

Again, let $f_k(n)$ denote the number of representations of n as a product of k factors, each greater than unity when $n > 1$, the order of the factors being essential. Then clearly

$$\sum_{n=2}^{\infty} \frac{f_k(n)}{n^s} = \{\zeta(s)-1\}^k \quad (\sigma > 1). \tag{1.2.14}$$

Let $f(n)$ be the number of representations of n as a product of factors greater than unity, representations with factors in a different order being considered as distinct; and let $f(1) = 1$. Then

$$f(n) = \sum_{k=1}^{\infty} f_k(n).$$

Hence

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = 1+\sum_{k=1}^{\infty}\{\zeta(s)-1\}^k = 1+\frac{\zeta(s)-1}{1-\{\zeta(s)-1\}}$$
$$= \frac{1}{2-\zeta(s)}. \tag{1.2.15}$$

It is easily seen that $\zeta(s) = 2$ for $s = \alpha$, where α is a real number greater than 1; and $|\zeta(s)| < 2$ for $\sigma > \alpha$, so that (1.2.15) holds for $\sigma > \alpha$.

1.3. Sums involving $\sigma_a(n)$. Let $\sigma_a(n)$ denote the sum of the ath powers of the divisors of n. Then

$$\zeta(s)\zeta(s-a) = \sum_{\mu=1}^{\infty} \frac{1}{\mu^s} \sum_{\nu=1}^{\infty} \frac{\nu^a}{\nu^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{\mu\nu=n} \nu^a,$$

i.e.
$$\zeta(s)\zeta(s-a) = \sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^s} \quad (\sigma > 1,\ \sigma > \mathbf{R}(a)+1). \tag{1.3.1}$$

Since the left-hand side is, if $a \neq 0$,

$$\prod_p \left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\dots\right)\left(1+\frac{p^a}{p^s}+\frac{p^{2a}}{p^{2s}}+\dots\right)$$
$$= \prod_p \left(1+\frac{1+p^a}{p^s}+\frac{1+p^a+p^{2a}}{p^s}+\dots\right) = \prod_p \left(1+\frac{1-p^{2a}}{1-p^a}\frac{1}{p^s}+\dots\right)$$

we have
$$\sigma_a(n) = \frac{1-p_1^{(m_1+1)a}}{1-p_1^a} \cdots \frac{1-p_r^{(m_r+1)a}}{1-p_r^a}, \tag{1.3.2}$$

if n is (1.2.5), as is also obvious from elementary considerations.

The formula†

$$\frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)} = \sum_{n=1}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^s} \tag{1.3.3}$$

is valid for $\sigma > \max\{1, \mathbf{R}(a)+1, \mathbf{R}(b)+1, \mathbf{R}(a+b)+1\}$. The left-hand side is equal to

$$\prod_p \frac{1-p^{-2s+a+b}}{(1-p^{-s})(1-p^{-s+a})(1-p^{-s+b})(1-p^{-s+a+b})}.$$

Putting $p^{-s} = z$, the partial-fraction formula gives

$$\frac{1-p^{a+b}z^2}{(1-z)(1-p^az)(1-p^bz)(1-p^{a+b}z)}$$
$$= \frac{1}{(1-p^a)(1-p^b)}\left\{\frac{1}{1-z}-\frac{p^a}{1-p^az}-\frac{p^b}{1-p^bz}+\frac{p^{a+b}}{1-p^{a+b}z}\right\}$$
$$= \frac{1}{(1-p^a)(1-p^b)}\sum_{m=0}^{\infty}(1-p^{(m+1)a}-p^{(m+1)b}+p^{(m+1)(a+b)})z^m$$
$$= \frac{1}{(1-p^a)(1-p^b)}\sum_{m=0}^{\infty}(1-p^{(m+1)a})(1-p^{(m+1)b})z^m.$$

† Ramanujan (2), B. M. Wilson (1).

Hence

$$\frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)} = \prod_p \sum_{m=0}^{\infty} \frac{1-p^{(m+1)a}}{1-p^a}\,\frac{1-p^{(m+1)b}}{1-p^b}\,\frac{1}{p^{ms}},$$

and the result follows from (1.3.2). If $a = b = 0$, (1.3.3) reduces to (1.2.10).

Similar formulae involving $\sigma_a^{(q)}(n)$, the sum of the ath powers of those divisors of n which are qth powers of integers, have been given by Crum (1).

1.4. It is also easily seen that, if $f(n)$ is multiplicative, and

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

is a product of zeta-functions such as occurs in the above formulae, and k is a given positive integer, then

$$\sum_{n=1}^{\infty} \frac{f(kn)}{n^s}$$

can also be summed. An example will illustrate this point. The function $\sigma_a(n)$ is 'multiplicative', i.e. if m is prime to n

$$\sigma_a(mn) = \sigma_a(m)\sigma_a(n).$$

Hence
$$\sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^s} = \prod_p \sum_{m=0}^{\infty} \frac{\sigma_a(p^m)}{p^{ms}},$$

and, if $k = \prod p^l$,
$$\sum_{n=1}^{\infty} \frac{\sigma_a(kn)}{n^s} = \prod_p \sum_{m=0}^{\infty} \frac{\sigma_a(p^{l+m})}{p^{ms}}.$$

Hence
$$\sum_{n=1}^{\infty} \frac{\sigma_a(kn)}{n^s} = \zeta(s)\zeta(s-a) \prod_{p|k} \left\{\sum_{m=0}^{\infty} \frac{\sigma_a(p^{l+m})}{p^{ms}} \Big/ \sum_{m=0}^{\infty} \frac{\sigma_a(p^m)}{p^{ms}}\right\}.$$

Now if $a \neq 0$,
$$\sum_{m=0}^{\infty} \frac{\sigma_a(p^{l+m})}{p^{ms}} = \sum_{m=0}^{\infty} \frac{1-p^{(l+m+1)a}}{(1-p^a)p^{ms}} = \frac{1-p^{a-s}-p^{(l+1)a}+p^{(l+1)a-s}}{(1-p^a)(1-p^{-s})(1-p^{a-s})}.$$

Hence
$$\sum_{n=0}^{\infty} \frac{\sigma_a(kn)}{n^s} = \zeta(s)\zeta(s-a) \prod_{p|k} \frac{1-p^{a-s}-p^{(l+1)a}+p^{(l+1)a-s}}{1-p^a}. \tag{1.4.1}$$

Making $a \to 0$,
$$\sum_{n=0}^{\infty} \frac{d(kn)}{n^s} = \zeta^2(s) \prod_{p|k} (l+1-lp^{-s}). \tag{1.4.2}$$

1.5. Ramanujan's sums.† Let

$$c_k(n) = \sum_h e^{-2nh\pi i/k} = \sum_h \cos\frac{2nh\pi}{k}, \tag{1.5.1}$$

where h runs through all positive integers less than and prime to k. Many formulae involving these sums were proved by Ramanujan.

We shall first prove that

$$c_k(n) = \sum_{d|k,\, d|n} \mu\left(\frac{k}{d}\right) d. \tag{1.5.2}$$

The sum
$$\eta_k(n) = \sum_{m=0}^{k-1} e^{-2nm\pi i/k}$$

is equal to k if $k|n$ and 0 otherwise. Denoting by (r, d) the highest common factor of r and d, so that $(r, d) = 1$ means that r is prime to d,

$$\sum_{d|k} c_d(n) = \sum_{d|k} \sum_{(r,d)=1,\, r<d} e^{-2nr\pi i/d} = \eta_k(n).$$

Hence by the inversion formula of Möbius (1.1.7)

$$c_k(n) = \sum_{d|k} \mu\left(\frac{k}{d}\right)\eta_d(n),$$

and (1.5.2) follows. In particular

$$c_k(1) = \mu(k). \tag{1.5.3}$$

The result can also be written

$$c_k(n) = \sum_{dr=k,\, d|n} \mu(r) d.$$

Hence
$$\frac{c_k(n)}{k^s} = \sum_{dr=k,\, d|n} \frac{\mu(r)}{r^s} d^{1-s}.$$

Summing with respect to k, we remove the restriction on r, which now assumes all positive integral values. Hence‡

$$\sum_{k=1}^{\infty} \frac{c_k(n)}{k^s} = \sum_{r,\, d|n} \frac{\mu(r)}{r^s} d^{1-s} = \frac{\sigma_{1-s}(n)}{\zeta(s)}, \tag{1.5.4}$$

the series being absolutely convergent for $\sigma > 1$ since $|c_k(n)| \leqslant \sigma_1(n)$, by (1.5.2).

We have also

$$\sum_{n=1}^{\infty} \frac{c_k(n)}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{d|k\, d|n} \mu\left(\frac{k}{d}\right) d$$
$$= \sum_{d|k} \mu\left(\frac{k}{d}\right) d \sum_{m=1}^{\infty} \frac{1}{(md)^s} = \zeta(s) \sum_{d|k} \mu\left(\frac{k}{d}\right) d^{1-s}. \tag{1.5.5}$$

† Ramanujan (3), Hardy (5).
‡ Two more proofs are given by Hardy, *Ramanujan*, 137–41.

We can also sum series of the form†

$$\sum_{n=1}^{\infty} \frac{c_k(n) f(n)}{n^s},$$

where $f(n)$ is a multiplicative function. For example,

$$\sum_{n=1}^{\infty} \frac{c_k(n) d(n)}{n^s} = \sum_{n=1}^{\infty} \frac{d(n)}{n^s} \sum_{\delta|k,\, \delta|n} \delta\mu\left(\frac{k}{\delta}\right)$$
$$= \sum_{\delta|k} \delta\mu\left(\frac{k}{\delta}\right) \sum_{m=1}^{\infty} \frac{d(m\delta)}{(m\delta)^s}$$
$$= \zeta^2(s) \sum_{\delta|k} \delta^{1-s}\mu\left(\frac{k}{\delta}\right) \prod_{p|\delta} (l+1-lp^{-s})$$

if $\delta = \prod p^l$. If $k = \prod p^\lambda$ the sum is

$$k^{1-s} \prod_{p|k} (\lambda+1-\lambda p^{-s}) - \sum_{p|k} \left(\frac{k}{p}\right)^{1-s} \{\lambda-(\lambda-1)p^{-s}\} \prod_{\substack{p'|k \\ p'\neq p}} (\lambda+1-\lambda p'^{-s}) +$$
$$+ \sum_{pp'|k} \left(\frac{k}{pp'}\right)^{1-s} \{\lambda-(\lambda-1)p^{-s}\}\{\lambda-(\lambda-1)p'^{-s}\} \prod_{\substack{p''|k \\ p''\neq p,p'}} (\lambda+1-\lambda p''^{-s}) - \dots$$
$$= k^{1-s} \prod_{p|k} \left\{(\lambda+1-\lambda p^{-s}) - \frac{1}{p^{1-s}}\{\lambda-(\lambda-1)p^{-s}\}\right\}$$
$$= k^{1-s} \prod_{p|k} \left\{1-\frac{1}{p}+\lambda\left(1-\frac{1}{p^s}\right)\left(1-\frac{1}{p^{1-s}}\right)\right\}.$$

Hence

$$\sum_{n=1}^{\infty} \frac{c_k(n) d(n)}{n^s} = \zeta^2(s) k^{1-s} \prod_{p|k} \left\{1-\frac{1}{p}+\lambda\left(1-\frac{1}{p^s}\right)\left(1-\frac{1}{p^{1-s}}\right)\right\}. \tag{1.5.6}$$

We can also sum
$$\sum_{n=1}^{\infty} \frac{c_k(qn) f(n)}{n^s}.$$

For example, in the simplest case $f(n) = 1$, the series is

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{\delta|k,\, \delta|qn} \delta\mu\left(\frac{k}{\delta}\right).$$

For given δ, n runs through those multiples of δ/q which are integers. If δ/q in its lowest terms is δ_1/q_1, these are the numbers δ_1, $2\delta_1$,.... Hence the sum is

$$\sum_{\delta|k} \delta\mu\left(\frac{k}{\delta}\right) \sum_{r=1}^{\infty} \frac{1}{(r\delta_1)^s} = \zeta(s) \sum_{\delta|k} \delta\mu\left(\frac{k}{\delta}\right)\delta_1^{-s}.$$

† Crum (1).

Since $\delta_1 = \delta/(q,\delta)$, the result is

$$\sum_{n=1}^{\infty} \frac{c_k(qn)}{n^s} = \zeta(s) \sum_{\delta|k} \delta^{1-s}\mu\left(\frac{k}{\delta}\right)(q,\delta)^s. \tag{1.5.7}$$

1.6. There is another class of identities involving infinite series of zeta-functions. The simplest of these is†

$$\sum_p \frac{1}{p^s} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \zeta(ns). \tag{1.6.1}$$

We have
$$\log \zeta(s) = \sum_m \sum_p \frac{1}{mp^{ms}} = \sum_{m=1}^{\infty} \frac{P(ms)}{m},$$
where $P(s) = \sum p^{-s}$. Hence

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \zeta(ns) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \sum_{m=1}^{\infty} \frac{P(mns)}{m} = \sum_{r=1}^{\infty} \frac{P(rs)}{r} \sum_{n|r} \mu(n),$$

and the result follows from (1.1.5).

A closely related formula is

$$\sum_{n=1}^{\infty} \frac{\nu(n)}{n^s} = \zeta(s) \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \zeta(ns), \tag{1.6.2}$$

where $\nu(n)$ is defined under (1.2.8). This follows at once from (1.6.1) and the identity
$$\sum_{n=1}^{\infty} \frac{\nu(n)}{n^s} = \sum_{m=1}^{\infty} \frac{1}{m^s} \sum_p \frac{1}{p^s}.$$

Denoting by $b(n)$ the number of divisors of n which are primes or powers of primes, another identity of the same class is

$$\sum_{n=1}^{\infty} \frac{b(n)}{n^s} = \zeta(s) \sum_{n=1}^{\infty} \frac{\phi(n)}{n} \log \zeta(ns), \tag{1.6.3}$$

where $\phi(n)$ is defined under (1.2.12). For the left-hand side is equal to

$$\sum_{m=1}^{\infty} \frac{1}{m^s} \sum_p \left(\frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \dots\right),$$

and the series on the right is

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n} \sum_{m=1}^{\infty} \sum_p \frac{1}{mp^{mns}} = \sum_p \sum_\nu \frac{1}{\nu p^{\nu s}} \sum_{n|\nu} \phi(n).$$

Since
$$\sum_{n|\nu} \phi(n) = \nu,$$
the result follows.

† See Landau and Walfisz (1), Estermann (1), (2).

II

THE ANALYTIC CHARACTER OF $\zeta(s)$, AND THE FUNCTIONAL EQUATION

2.1. Analytic continuation and the functional equation, first method. Each of the formulae of Chapter I is proved on the supposition that the series or product concerned is absolutely convergent. In each case this restricts the region where the formula is proved to be valid to a half-plane. For $\zeta(s)$ itself, and in all the fundamental formulae of § 1.1, this is the half-plane $\sigma > 1$.

We have next to inquire whether the analytic function $\zeta(s)$ can be continued beyond this region. The result is

THEOREM 2.1. *The function $\zeta(s)$ is regular for all values of s except $s = 1$, where there is a simple pole with residue 1. It satisfies the functional equation*

$$\zeta(s) = 2^s\pi^{s-1} \sin \tfrac{1}{2}s\pi\, \Gamma(1-s)\zeta(1-s). \tag{2.1.1}$$

This can be proved in a considerable variety of different ways, some of which will be given in later sections. We shall first give a proof depending on the following summation formula.

Let $\phi(x)$ be any function with a continuous derivative in the interval $[a,b]$. Then, if $[x]$ denotes the greatest integer not exceeding x,

$$\sum_{a<n\leqslant b} \phi(n) = \int_a^b \phi(x)\,dx + \int_a^b (x-[x]-\tfrac{1}{2})\phi'(x)\,dx + \\ +(a-[a]-\tfrac{1}{2})\phi(a) - (b-[b]-\tfrac{1}{2})\phi(b). \tag{2.1.2}$$

Since the formula is plainly additive with respect to the interval $(a, b]$ it suffices to suppose that $n \leqslant a < b \leqslant n+1$. One then has

$$\int_a^b (x-n-\tfrac{1}{2})\phi'(x)\,dx = (b-n-\tfrac{1}{2})\phi(b) - (a-n-\tfrac{1}{2})\phi(a) - \int_a^b \phi(x)dx,$$

on integrating by parts. Thus the right hand side of (2.1.2) reduces to $([b]-n)\phi(b)$. This vanishes unless $b = n+1$, in which case it is $\phi(n+1)$, as required.

In particular, let $\phi(n) = n^{-s}$, where $s \neq 1$, and let a and b be positive integers. Then

$$\sum_{n=a+1}^{b} \frac{1}{n^s} = \frac{b^{1-s}-a^{1-s}}{1-s} - s\int_a^b \frac{x-[x]-\frac{1}{2}}{x^{s+1}}\,dx + \tfrac{1}{2}(b^{-s}-a^{-s}). \quad (2.1.3)$$

First take $\sigma > 1$, $a = 1$, and make $b \to \infty$. Adding 1 to each side, we obtain

$$\zeta(s) = s\int_1^\infty \frac{[x]-x+\frac{1}{2}}{x^{s+1}}\,dx + \frac{1}{s-1} + \frac{1}{2}. \quad (2.1.4)$$

Since $[x]-x+\frac{1}{2}$ is bounded, this integral is convergent for $\sigma > 0$, and uniformly convergent in any finite region to the right of $\sigma = 0$. It therefore defines an analytic function of s, regular for $\sigma > 0$. The right-hand side therefore provides the analytic continuation of $\zeta(s)$ up to $\sigma = 0$, and there is clearly a simple pole at $s = 1$ with residue 1.

For $0 < \sigma < 1$ we have

$$\int_0^1 \frac{[x]-x}{x^{s+1}}\,dx = -\int_0^1 x^{-s}\,dx = \frac{1}{s-1}, \qquad \frac{s}{2}\int_1^\infty \frac{dx}{x^{s+1}} = \frac{1}{2},$$

and (2.1.4) may be written

$$\zeta(s) = s\int_0^\infty \frac{[x]-x}{x^{s+1}}\,dx \quad (0 < \sigma < 1). \quad (2.1.5)$$

Actually (2.1.4) gives the analytic continuation of $\zeta(s)$ for $\sigma > -1$; for if

$$f(x) = [x]-x+\tfrac{1}{2}, \qquad f_1(x) = \int_1^x f(y)\,dy,$$

then $f_1(x)$ is also bounded, since, as is easily seen,

$$\int_k^{k+1} f(y)\,dy = 0$$

for any integer k. Hence

$$\int_{x_1}^{x_2} \frac{f(x)}{x^{s+1}}\,dx = \left[\frac{f_1(x)}{x^{s+1}}\right]_{x_1}^{x_2} + (s+1)\int_{x_1}^{x_2} \frac{f_1(x)}{x^{s+2}}\,dx,$$

which tends to 0 as $x_1 \to \infty$, $x_2 \to \infty$, if $\sigma > -1$. Hence the integral in (2.1.4) is convergent for $\sigma > -1$. Also it is easily verified that

$$s\int_0^1 \frac{[x]-x+\frac{1}{2}}{x^{s+1}}\,dx = \frac{1}{s-1} + \frac{1}{2} \quad (\sigma < 0).$$

Hence
$$\zeta(s) = s\int_0^\infty \frac{[x]-x+\frac{1}{2}}{x^{s+1}}\,dx \quad (-1 < \sigma < 0). \quad (2.1.6)$$

Now we have the Fourier series

$$[x]-x+\tfrac{1}{2} = \sum_{n=1}^{\infty} \frac{\sin 2n\pi x}{n\pi}, \quad (2.1.7)$$

where x is not an integer. Substituting in (2.1.6), and integrating term by term, we obtain

$$\begin{aligned}\zeta(s) &= \frac{s}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\int_0^\infty \frac{\sin 2n\pi x}{x^{s+1}}\,dx \\ &= \frac{s}{\pi}\sum_{n=1}^{\infty}\frac{(2n\pi)^s}{n}\int_0^\infty \frac{\sin y}{y^{s+1}}\,dy \\ &= \frac{s}{\pi}(2\pi)^s\{-\Gamma(-s)\}\sin\tfrac{1}{2}s\pi\,\zeta(1-s),\end{aligned}$$

i.e. (2.1.1). This is valid primarily for $-1 < \sigma < 0$. Here, however, the right-hand side is analytic for all values of s such that $\sigma < 0$. It therefore provides the analytic continuation of $\zeta(s)$ over the remainder of the plane, and there are no singularities other than the pole already encountered at $s = 1$.

We have still to justify the term-by-term integration. Since the series (2.1.7) is boundedly convergent, term-by-term integration over any finite range is permissible. It is therefore sufficient to prove that

$$\lim_{\lambda\to\infty} \sum_{n=1}^{\infty} \frac{1}{n}\int_\lambda^\infty \frac{\sin 2n\pi x}{x^{s+1}}\,dx = 0 \quad (-1 < \sigma < 0).$$

Now
$$\begin{aligned}\int_\lambda^\infty \frac{\sin 2n\pi x}{x^{s+1}}\,dx &= \left[-\frac{\cos 2n\pi x}{2n\pi x^{s+1}}\right]_\lambda^\infty - \frac{s+1}{2n\pi}\int_\lambda^\infty \frac{\cos 2n\pi x}{x^{s+2}}\,dx \\ &= O\left(\frac{1}{n\lambda^{\sigma+1}}\right) + O\left(\frac{1}{n}\int_\lambda^\infty \frac{dx}{x^{\sigma+2}}\right) = O\left(\frac{1}{n\lambda^{\sigma+1}}\right),\end{aligned}$$

and the desired result clearly follows.

The functional equation (2.1.1) may be written in a number of different ways. Changing s into $1-s$, it is

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos\tfrac{1}{2}s\pi\,\Gamma(s)\zeta(s). \tag{2.1.8}$$

It may also be written

$$\zeta(s) = \chi(s)\zeta(1-s), \tag{2.1.9}$$

where
$$\chi(s) = 2^s\pi^{s-1}\sin\tfrac{1}{2}s\pi\,\Gamma(1-s) = \pi^{s-\frac{1}{2}}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}s)}{\Gamma(\frac{1}{2}s)}, \tag{2.1.10}$$

and
$$\chi(s)\chi(1-s) = 1. \tag{2.1.11}$$

Writing
$$\xi(s) = \tfrac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s), \tag{2.1.12}$$

it is at once verified from (2.1.8) and (2.1.9) that

$$\xi(s) = \xi(1-s). \tag{2.1.13}$$

Writing
$$\Xi(z) = \xi(\tfrac{1}{2}+iz) \tag{2.1.14}$$

we obtain
$$\Xi(z) = \Xi(-z). \tag{2.1.15}$$

The functional equation is therefore equivalent to the statement that $\Xi(z)$ is an even function of z.

The approximation near $s = 1$ can be carried a stage farther; we have

$$\zeta(s) = \frac{1}{s-1}+\gamma+O(|s-1|), \tag{2.1.16}$$

where γ is Euler's constant. For by (2.1.4)

$$\begin{aligned}
\lim_{s\to 1}\left\{\zeta(s)-\frac{1}{s-1}\right\} &= \int_1^\infty \frac{[x]-x+\frac{1}{2}}{x^2}\,dx+\tfrac{1}{2} \\
&= \lim_{n\to\infty}\int_1^n \frac{[x]-x}{x^2}\,dx+1 \\
&= \lim_{n\to\infty}\left\{\sum_{m=1}^{n-1} m\int_m^{m+1}\frac{dx}{x^2}-\log n+1\right\} \\
&= \lim_{n\to\infty}\left\{\sum_{m=1}^{n-1}\frac{1}{m+1}+1-\log n\right\} = \gamma.
\end{aligned}$$

2.2. A considerable number of variants of the above proof of the functional equation have been given. A similar argument was applied by Hardy,† not to $\zeta(s)$ itself, but to the function

$$\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^s} = (1-2^{1-s})\zeta(s). \tag{2.2.1}$$

† Hardy (6).

This Dirichlet series is convergent for all real positive values of s, and so, by a general theorem on the convergence of Dirichlet series, for all values of s such that $\sigma > 0$. Here, of course, the pole of $\zeta(s)$ at $s = 1$ is cancelled by the zero of the other factor. These facts enable us to simplify the discussion in some respects.

Hardy's proof runs as follows. Let

$$f(x) = \sum_{n=0}^\infty \frac{\sin(2n+1)x}{2n+1}.$$

This series is boundedly convergent and

$$f(x) = (-1)^m\tfrac{1}{4}\pi \quad \text{for} \quad m\pi < x < (m+1)\pi \quad (m = 0, 1,\ldots).$$

Multiplying by x^{s-1} $(0 < s < 1)$, and integrating over $(0,\infty)$, we obtain

$$\begin{aligned}
\tfrac{1}{4}\pi\sum_{m=0}^\infty (-1)^m \int_{m\pi}^{(m+1)\pi} x^{s-1}\,dx &= \Gamma(s)\sin\tfrac{1}{2}s\pi\sum_{n=0}^\infty\frac{1}{(2n+1)^{s+1}} \\
&= \Gamma(s)\sin\tfrac{1}{2}s\pi(1-2^{-s-1})\zeta(s+1).
\end{aligned}$$

The term-by-term integration may be justified as in the previous proof. The series on the left is

$$\frac{\pi^s}{s}\left[1+\sum_{m=1}^\infty(-1)^m\{(m+1)^s-m^s\}\right].$$

This series is convergent for $s < 1$, and, as a little consideration of the above argument shows, uniformly convergent for $\mathbf{R}(s) \leqslant 1-\delta < 1$. Its sum is therefore an analytic function of s, regular for $\mathbf{R}(s) < 1$. But for $s < 0$ it is

$$2(1^s-2^s+3^s-\ldots) = 2(1-2^{s+1})\zeta(-s).$$

Its sum is therefore the same analytic function of s for $\mathbf{R}(s) < 1$. Hence, for $0 < s < 1$,

$$\frac{\pi^{s+1}}{2s}(1-2^{s+1})\zeta(-s) = \Gamma(s)\sin\tfrac{1}{2}s\pi(1-2^{-s-1})\zeta(s+1),$$

and the functional equation again follows.

2.3. Still another proof is based on Poisson's summation formula

$$\sum_{n=-\infty}^\infty f(n) = \sum_{-\infty}^\infty \int_{-\infty}^\infty f(u)\cos 2\pi nu\,du. \tag{2.3.1}$$

If we put $f(x) = |x|^{-s}$ and ignore all questions of convergence, we obtain the result formally at once. The proof may be established in various ways. If we integrate by parts to obtain integrals involving $\sin 2\pi nu$, we obtain a proof not fundamentally distinct from the first proof given here.† The formula can also be used to give a proof depending‡ on $(1-2^{1-s})\zeta(s)$.

Actually cases of Poisson's formula enter into several of the following proofs; (2.6.3) and (2.8.2) are both cases of Poisson's formula.

2.4. Second method. The whole theory can be developed in another way, which is one of Riemann's methods. Here the fundamental formula is

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x-1}\,dx \quad (\sigma > 1). \tag{2.4.1}$$

To prove this, we have for $\sigma > 0$

$$\int_0^\infty x^{s-1}e^{-nx}\,dx = \frac{1}{n^s}\int_0^\infty y^{s-1}e^{-y}\,dy = \frac{\Gamma(s)}{n^s}.$$

Hence

$$\Gamma(s)\zeta(s) = \sum_{n=1}^\infty \int_0^\infty x^{s-1}e^{-nx}\,dx = \int_0^\infty x^{s-1}\sum_{n=1}^\infty e^{-nx}\,dx = \int_0^\infty \frac{x^{s-1}}{e^x-1}\,dx$$

if the inversion of the order of summation and integration can be justified; and this is so by absolute convergence if $\sigma > 1$, since

$$\sum_{n=1}^\infty \int_0^\infty x^{\sigma-1}e^{-nx}\,dx = \Gamma(\sigma)\zeta(\sigma)$$

is convergent for $\sigma > 1$.

Now consider the integral

$$I(s) = \int_C \frac{z^{s-1}}{e^z-1}\,dz,$$

where the contour C starts at infinity on the positive real axis, encircles the origin once in the positive direction, excluding the points $\pm 2i\pi$, $\pm 4i\pi$,..., and returns to positive infinity. Here z^{s-1} is defined as

$$e^{(s-1)\log z}$$

when the logarithm is real at the beginning of the contour; thus $\mathbf{I}(\log z)$ varies from 0 to 2π round the contour.

We can take C to consist of the real axis from ∞ to ρ $(0 < \rho < 2\pi)$, the circle $|z| = \rho$, and the real axis from ρ to ∞. On the circle,

$$|z^{s-1}| = e^{(\sigma-1)\log|z| - t\arg z} \leqslant |z|^{\sigma-1}e^{2\pi|t|},$$

$$|e^z-1| > A|z|.$$

† Mordell (2).

‡ Ingham, *Prime Numbers*, 46.

Hence the integral round this circle tends to zero with ρ if $\sigma > 1$. On making $\rho \to 0$ we therefore obtain

$$\begin{aligned} I(s) &= -\int_0^\infty \frac{x^{s-1}}{e^x-1}\,dx + \int_0^\infty \frac{(xe^{2\pi i})^{s-1}}{e^x-1}\,dx \\ &= (e^{2\pi is}-1)\Gamma(s)\zeta(s) \\ &= \frac{2i\pi e^{i\pi s}}{\Gamma(1-s)}\zeta(s). \end{aligned}$$

Hence
$$\zeta(s) = \frac{e^{-i\pi s}\Gamma(1-s)}{2\pi i}\int_C \frac{z^{s-1}}{e^z-1}\,dz. \tag{2.4.2}$$

This formula has been proved for $\sigma > 1$. The integral $I(s)$, however, is uniformly convergent in any finite region of the s-plane, and so defines an integral function of s. Hence the formula provides the analytic continuation of $\zeta(s)$ over the whole s-plane. The only possible singularities are the poles of $\Gamma(1-s)$, viz. $s = 1, 2, 3,\ldots$. We know already that $\zeta(s)$ is regular at $s = 2, 3,\ldots$, and in fact it follows at once from Cauchy's theorem that $I(s)$ vanishes at these points. Hence the only possible singularity is a simple pole at $s = 1$. Here

$$I(1) = \int_C \frac{dz}{e^z-1} = 2\pi i,$$

and
$$\Gamma(1-s) = -\frac{1}{s-1}+\ldots.$$

Hence the residue at the pole is 1.

If s is any integer, the integrand in $I(s)$ is one-valued, and $I(s)$ can be evaluated by the theorem of residues. Since

$$\frac{z}{e^z-1} = 1-\tfrac{1}{2}z+B_1\frac{z^2}{2!}-B_2\frac{z^4}{4!}+\ldots,$$

where B_1, B_2,... are Bernoulli's numbers, we find the following values of $\zeta(s)$:

$$\zeta(0) = -\tfrac{1}{2}, \qquad \zeta(-2m) = 0, \qquad \zeta(1-2m) = \frac{(-1)^m B_m}{2m} \quad (m = 1, 2,\ldots). \tag{2.4.3}$$

To deduce the functional equation from (2.4.2), take the integral along the contour C_n consisting of the positive real axis from infinity to $(2n+1)\pi$, then round the square with corners $(2n+1)\pi(\pm 1 \pm i)$, and then back to infinity along the positive real axis. Between the contours

C and C_n the integrand has poles at the points $\pm 2i\pi, \ldots, \pm 2in\pi$. The residues at $2mi\pi$ and $-2mi\pi$ are together

$$(2m\pi e^{\frac{1}{2}i\pi})^{s-1}+(2m\pi e^{\frac{3}{2}i\pi})^{s-1} = (2m\pi)^{s-1}e^{i\pi(s-1)}2\cos\tfrac{1}{2}\pi(s-1)$$
$$= -2(2m\pi)^{s-1}e^{i\pi s}\sin\tfrac{1}{2}\pi s.$$

Hence by the theorem of residues

$$I(s) = \int_{C_n} \frac{z^{s-1}}{e^z-1}\,dz+4\pi ie^{i\pi s}\sin\tfrac{1}{2}\pi s\sum_{m=1}^{n}(2m\pi)^{s-1}.$$

Now let $\sigma < 0$ and make $n \to \infty$. The function $1/(e^z-1)$ is bounded on the contours C_n, and $z^{s-1} = O(|z|^{\sigma-1})$. Hence the integral round C_n tends to zero, and we obtain

$$I(s) = 4\pi ie^{i\pi s}\sin\tfrac{1}{2}\pi s\sum_{m=1}^{\infty}(2m\pi)^{s-1}$$
$$= 4\pi ie^{i\pi s}\sin\tfrac{1}{2}\pi s(2\pi)^{s-1}\zeta(1-s).$$

The functional equation now follows again.

Two minor consequences of the functional equation may be noted here. The formula

$$\zeta(2m) = 2^{2m-1}\pi^{2m}\frac{B_m}{(2m)!} \quad (m = 1, 2,\ldots) \tag{2.4.4}$$

follows from the functional equation (2.1.1), with $s = 1-2m$, and the value obtained above for $\zeta(1-2m)$. Also

$$\zeta'(0) = -\tfrac{1}{2}\log 2\pi. \tag{2.4.5}$$

For the functional equation gives

$$-\frac{\zeta'(1-s)}{\zeta(1-s)} = -\log 2\pi-\tfrac{1}{2}\pi\tan\tfrac{1}{2}s\pi+\frac{\Gamma'(s)}{\Gamma(s)}+\frac{\zeta'(s)}{\zeta(s)}.$$

In the neighbourhood of $s = 1$

$$\tfrac{1}{2}\pi\tan\tfrac{1}{2}s\pi = -\frac{1}{s-1}+O(|s-1|), \qquad \frac{\Gamma'(s)}{\Gamma(s)} = \frac{\Gamma'(1)}{\Gamma(1)}+\ldots = -\gamma+\ldots,$$

and
$$\frac{\zeta'(s)}{\zeta(s)} = \frac{-\{1/(s-1)^2\}+k+\ldots}{\{1/(s-1)\}+\gamma+k(s-1)+\ldots} = -\frac{1}{s-1}+\gamma+\ldots.$$

where k is a constant. Hence, making $s \to 1$, we obtain

$$-\frac{\zeta'(0)}{\zeta(0)} = -\log 2\pi,$$

and (2.4.5) follows.

2.5. Validity of (2.2.1) for all s. The original series (1.1.1) is naturally valid for $\sigma > 1$ only, on account of the pole at $s = 1$. The series (2.2.1) is convergent, and represents $(1-2^{1-s})\zeta(s)$, for $\sigma > 0$. This series ceases

to converge on $\sigma = 0$, but there is nothing in the nature of the function represented to account for this. In fact if we use summability instead of ordinary convergence the equation still holds to the left of $\sigma = 0$.

THEOREM 2.5. *The series* $\sum_{n=1}^{\infty}(-1)^{n-1}n^{-s}$ *is summable* (A) *to the sum* $(1-2^{1-s})\zeta(s)$ *for all values of* s.

Let $0 < x < 1$. Then

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}x^n = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{\Gamma(s)}\int_0^{\infty}e^{-nu}u^{s-1}\,du$$
$$= \frac{1}{\Gamma(s)}\int_0^{\infty}u^{s-1}\sum_{n=1}^{\infty}(-1)^{n-1}x^ne^{-nu}\,du = \frac{1}{\Gamma(s)}\int_0^{\infty}u^{s-1}\frac{xe^{-u}}{1+xe^{-u}}\,du.$$

This is justified by absolute convergence for $\sigma > 1$, and the result by analytic continuation for $\sigma > 0$.

We can now replace this by a loop-integral in the same way as (2.4.2) was obtained from (2.4.1). We obtain

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}x^n = \frac{e^{-i\pi s}\Gamma(1-s)}{2\pi i}\int_C w^{s-1}\frac{xe^{-w}}{1+xe^{-w}}\,dw,$$

when C encircles the origin as before, but excludes all zeros of $1+xe^{-w}$, i.e. the points $w = \log x+(2m+1)i\pi$.

It is clear that, as $x \to 1$, the right-hand side tends to a limit, uniformly in any finite region of the s-plane excluding positive integers; and, by the theory of analytic continuation, the limit must be $(1-2^{1-s})\zeta(s)$. This proves the theorem except if s is a positive integer, when the proof is elementary.

Similar results hold for other methods of summation.

2.6. Third method. This is also one of Riemann's original proofs. We observe that if $\sigma > 0$

$$\int_0^{\infty}x^{\frac{1}{2}s-1}e^{-n^2\pi x}\,dx = \frac{\Gamma(\frac{1}{2}s)}{n^s\pi^{\frac{1}{2}s}}.$$

Hence if $\sigma > 1$

$$\frac{\Gamma(\frac{1}{2}s)\zeta(s)}{\pi^{\frac{1}{2}s}} = \sum_{n=1}^{\infty}\int_0^{\infty}x^{\frac{1}{2}s-1}e^{-n^2\pi x}\,dx = \int_0^{\infty}x^{\frac{1}{2}s-1}\sum_{n=1}^{\infty}e^{-n^2\pi x}\,dx,$$

the inversion being justified by absolute convergence, as in § 2.4.

Writing

$$\psi(x)=\sum_{n=1}^{\infty}e^{-n^2\pi x} \tag{2.6.1}$$

we therefore have

$$\zeta(s)=\frac{\pi^{\frac{1}{2}s}}{\Gamma(\frac{1}{2}s)}\int_0^{\infty}x^{\frac{1}{2}s-1}\psi(x)\,dx \quad (\sigma>1). \tag{2.6.2}$$

Now it is known that, for $x>0$,

$$\sum_{n=-\infty}^{\infty}e^{-n^2\pi x}=\frac{1}{\sqrt{x}}\sum_{n=-\infty}^{\infty}e^{-n^2\pi/x},$$

$$2\psi(x)+1=\frac{1}{\sqrt{x}}\left\{2\psi\left(\frac{1}{x}\right)+1\right\}. \tag{2.6.3}$$

Hence (2.6.2) gives

$$\begin{aligned}\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s)&=\int_0^1 x^{\frac{1}{2}s-1}\psi(x)\,dx+\int_1^{\infty}x^{\frac{1}{2}s-1}\psi(x)\,dx\\&=\int_0^1 x^{\frac{1}{2}s-1}\left\{\frac{1}{\sqrt{x}}\psi\left(\frac{1}{x}\right)+\frac{1}{2\sqrt{x}}-\frac{1}{2}\right\}dx+\int_1^{\infty}x^{\frac{1}{2}s-1}\psi(x)\,dx\\&=\frac{1}{s-1}-\frac{1}{s}+\int_0^1 x^{\frac{1}{2}s-\frac{3}{2}}\psi\left(\frac{1}{x}\right)dx+\int_1^{\infty}x^{\frac{1}{2}s-1}\psi(x)\,dx\\&=\frac{1}{s(s-1)}+\int_1^{\infty}(x^{-\frac{1}{2}s-\frac{1}{2}}+x^{\frac{1}{2}s-1})\psi(x)\,dx.\end{aligned}$$

The last integral is convergent for all values of s, and so the formula holds, by analytic continuation, for all values of s. Now the right-hand side is unchanged if s is replaced by $1-s$. Hence

$$\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s)=\pi^{-\frac{1}{2}+\frac{1}{2}s}\Gamma(\tfrac{1}{2}-\tfrac{1}{2}s)\zeta(1-s), \tag{2.6.4}$$

which is a form of the functional equation.

2.7. Fourth method; proof by self-reciprocal functions. Still another proof of the functional equation is as follows. For $\sigma>1$, (2.4.1) may be written

$$\zeta(s)\Gamma(s)=\int_0^1\left(\frac{1}{e^x-1}-\frac{1}{x}\right)x^{s-1}\,dx+\frac{1}{s-1}+\int_1^{\infty}\frac{x^{s-1}\,dx}{e^x-1},$$

and this holds by analytic continuation for $\sigma>0$. Also for $0<\sigma<1$

$$\frac{1}{s-1}=-\int_1^{\infty}\frac{x^{s-1}}{x}\,dx.$$

Hence
$$\zeta(s)\Gamma(s)=\int_0^{\infty}\left(\frac{1}{e^x-1}-\frac{1}{x}\right)x^{s-1}\,dx \quad (0<\sigma<1). \tag{2.7.1}$$

Now it is known that the function

$$f(x)=\frac{1}{e^{x\sqrt{(2\pi)}}-1}-\frac{1}{x\sqrt{(2\pi)}} \tag{2.7.2}$$

is self-reciprocal for sine transforms, i.e. that

$$f(x)=\sqrt{\left(\frac{2}{\pi}\right)}\int_0^{\infty}f(y)\sin xy\,dy. \tag{2.7.3}$$

Hence, putting $x=\xi\sqrt{(2\pi)}$ in (2.7.1),

$$\begin{aligned}\zeta(s)\Gamma(s)&=(2\pi)^{\frac{1}{2}s}\int_0^{\infty}f(\xi)\xi^{s-1}\,d\xi\\&=(2\pi)^{\frac{1}{2}s}\sqrt{\left(\frac{2}{\pi}\right)}\int_0^{\infty}\xi^{s-1}\,d\xi\int_0^{\infty}f(y)\sin\xi y\,dy.\end{aligned}$$

If we can invert the order of integration, this is

$$\begin{aligned}&2^{\frac{1}{2}s+\frac{1}{2}}\pi^{\frac{1}{2}s-\frac{1}{2}}\int_0^{\infty}f(y)\,dy\int_0^{\infty}\xi^{s-1}\sin\xi y\,d\xi\\&=2^{\frac{1}{2}s+\frac{1}{2}}\pi^{\frac{1}{2}s-\frac{1}{2}}\int_0^{\infty}f(y)y^{-s}\,dy\int_0^{\infty}u^{s-1}\sin u\,du\\&=2^{\frac{1}{2}s+\frac{1}{2}}\pi^{\frac{1}{2}s-\frac{1}{2}}(2\pi)^{\frac{1}{2}s-\frac{1}{2}}\Gamma(1-s)\zeta(1-s)\,\frac{\pi}{2\cos\frac{1}{2}\pi s\Gamma(1-s)},\end{aligned}$$

and the functional equation again follows.

To justify the inversion, we observe that the integral

$$\int_0^{\infty}f(y)\sin\xi y\,dy$$

converges uniformly over $0<\delta\leqslant\xi\leqslant\Delta$. Hence the inversion of this part is valid, and it is sufficient to prove that

$$\lim_{\substack{\delta\to 0\\ \Delta\to\infty}}\int_0^{\infty}f(y)\,dy\left(\int_0^{\delta}+\int_{\Delta}^{\infty}\right)\xi^{s-1}\sin\xi y\,d\xi=0.$$

Now
$$\int_0^{\delta}\xi^{s-1}\sin\xi y\,d\xi=\int_0^{\delta}O(\xi^{\sigma-1}\,\xi y)\,d\xi=O(\delta^{\sigma+1}y)$$

and also
$$=y^{-s}\int_0^{\delta y}u^{s-1}\sin u\,du=O(y^{-\sigma}).$$

Since $f(y) = O(1)$ as $y \to 0$, and $= O(y^{-1})$ as $y \to \infty$, we obtain

$$\int_0^\infty f(y)\,dy \int_0^\delta \xi^{s-1}\sin \xi y\,d\xi$$

$$= \int_0^1 O(\delta^{\sigma+1}y)\,dy + \int_1^{1/\delta} O(\delta^{\sigma+1})\,dy + \int_{1/\delta}^\infty O(y^{-\sigma-1})\,dy = O(\delta^\sigma) \to 0.$$

A similar method shows that the integral involving Δ also tends to 0.

2.8. Fifth method. The process by which (2.7.1) was obtained from (2.4.1) can be extended indefinitely. For the next stage, (2.7.1) gives

$$\Gamma(s)\zeta(s) = \int_0^1 \left(\frac{1}{e^x-1} - \frac{1}{x} + \frac{1}{2}\right)x^{s-1}\,dx - \frac{1}{2s} + \int_1^\infty \left(\frac{1}{e^x-1} - \frac{1}{x}\right)x^{s-1}\,dx,$$

and this holds by analytic continuation for $\sigma > -1$. But

$$\int_1^\infty \tfrac{1}{2}x^{s-1}\,dx = -\frac{1}{2s} \quad (-1 < \sigma < 0).$$

Hence

$$\Gamma(s)\zeta(s) = \int_0^\infty \left(\frac{1}{e^x-1} - \frac{1}{x} + \frac{1}{2}\right)x^{s-1}\,dx \quad (-1 < \sigma < 0). \tag{2.8.1}$$

Now

$$\frac{1}{e^x-1} = \frac{1}{x} - \frac{1}{2} + 2x\sum_{n=1}^\infty \frac{1}{4n^2\pi^2+x^2}. \tag{2.8.2}$$

Hence

$$\Gamma(s)\zeta(s) = \int_0^\infty 2x\sum_{n=1}^\infty \frac{1}{4n^2\pi^2+x^2}x^{s-1}\,dx = 2\sum_{n=1}^\infty \int_0^\infty \frac{x^s}{4n^2\pi^2+x^2}\,dx$$

$$= 2\sum_{n=1}^\infty (2n\pi)^{s-1}\frac{\pi}{2\cos\frac{1}{2}s\pi} = \frac{2^{s-1}\pi^s}{\cos\frac{1}{2}s\pi}\zeta(1-s),$$

the functional equation. The inversion is justified by absolute convergence if $-1 < \sigma < 0$.

2.9. Sixth method. The formula†

$$\zeta(s) = \frac{e^{i\pi s}}{2\pi i}\int_{c-i\infty}^{c+i\infty} \left\{\frac{\Gamma'(1+z)}{\Gamma(1+z)} - \log z\right\} z^{-s}\,dz \quad (-1 < c < 0) \tag{2.9.1}$$

is easily proved by the calculus of residues if $\sigma > 1$; and the integrand is $O(|z|^{-\sigma-1})$, so that the integral is convergent, and the formula holds by analytic continuation, if $\sigma > 0$.

† Kloosterman (1).

We may next transform this into an integral along the positive real axis after the manner of § 2.4. We obtain

$$\zeta(s) = -\frac{\sin\pi s}{\pi}\int_0^\infty \left\{\frac{\Gamma'(1+x)}{\Gamma(1+x)} - \log x\right\} x^{-s}\,dx \quad (0 < \sigma < 1). \tag{2.9.2}$$

To deduce the functional equation, we observe that†

$$\frac{\Gamma'(x)}{\Gamma(x)} = \log x - \frac{1}{2x} - 2\int_0^\infty \frac{t\,dt}{(t^2+x^2)(e^{2\pi t}-1)}.$$

Hence

$$\frac{\Gamma'(1+x)}{\Gamma(1+x)} - \log x = \frac{\Gamma'(x)}{\Gamma(x)} + \frac{1}{x} - \log x$$

$$= \frac{1}{2x} - 2\int_0^\infty \frac{t\,dt}{(t^2+x^2)(e^{2\pi t}-1)} = -2\int_0^\infty \frac{t}{t^2+x^2}\left(\frac{1}{e^{2\pi t}-1} - \frac{1}{2\pi t}\right)dt.$$

Hence (2.9.2) gives

$$\zeta(s) = \frac{2\sin\pi s}{\pi}\int_0^\infty x^{-s}\,dx \int_0^\infty \frac{t}{t^2+x^2}\left(\frac{1}{e^{2\pi t}-1} - \frac{1}{2\pi t}\right)dt$$

$$= \frac{2\sin\pi s}{\pi}\int_0^\infty \left(\frac{1}{e^{2\pi t}-1} - \frac{1}{2\pi t}\right)t\,dt \int_0^\infty \frac{x^{-s}}{t^2+x^2}\,dx$$

$$= \frac{\sin\pi s}{\cos\frac{1}{2}\pi s}\int_0^\infty \left(\frac{1}{e^{2\pi t}-1} - \frac{1}{2\pi t}\right)t^{-s}\,dt$$

$$= 2\sin\tfrac{1}{2}\pi s(2\pi)^{s-1}\int_0^\infty \left(\frac{1}{e^u-1} - \frac{1}{u}\right)u^{-s}\,du$$

$$= 2\sin\tfrac{1}{2}\pi s(2\pi)^{s-1}\Gamma(1-s)\zeta(1-s)$$

by (2.7.1). The inversion is justified by absolute convergence.

2.10. Seventh method. Still another method of dealing with $\zeta(s)$, due to Riemann, has been carried out in detail by Siegel.‡ It depends on the evaluation of the following infinite integral.

Let

$$\Phi(a) = \int_L \frac{e^{iw^2/(4\pi)+aw}}{e^w-1}\,dw, \tag{2.10.1}$$

where L is a straight line inclined at an angle $\frac{1}{4}\pi$ to the real axis, and

† Whittaker and Watson, § 12.32, example.

‡ Siegel (2).

intersecting the imaginary axis between O and $2\pi i$. The integral is plainly convergent for all values of a.

We have

$$\Phi(a+1)-\Phi(a) = \int\limits_L \frac{e^{\frac14 iw^2/\pi}}{e^w-1}(e^{(a+1)w}-e^{aw})\,dw$$

$$= \int\limits_L e^{\frac14 iw^2/\pi+aw}\,dw$$

$$= \int\limits_L e^{\frac14 i(w-2i\pi a)^2/\pi+i\pi a^2}\,dw$$

$$= e^{i\pi a^2}\int e^{\frac14 iW^2/\pi}\,dW,$$

where $W = w-2i\pi a$. Here we may move the contour to the parallel line through the origin, so that the last integral is

$$e^{\frac14 i\pi}\int\limits_{-\infty}^{\infty} e^{-\frac14\rho^2/\pi}\,d\rho = 2\pi e^{\frac14 i\pi}.$$

Hence $$\Phi(a+1)-\Phi(a) = 2\pi e^{i\pi(a^2+\frac14)}. \tag{2.10.2}$$

Next let L' be the line parallel to L and intersecting the imaginary axis at a distance 2π below its intersection with L. Then by the theorem of residues

$$\int\limits_{L'} \frac{e^{\frac14 iw^2/\pi+aw}}{e^w-1}\,dw - \int\limits_L \frac{e^{\frac14 iw^2/\pi+aw}}{e^w-1}\,dw = 2\pi i.$$

But

$$\int\limits_{L'} \frac{e^{\frac14 iw^2/\pi+aw}}{e^w-1}\,dw = \int\limits_L \frac{e^{\frac14 i(w-2\pi i)^2/\pi+a(w-2\pi i)}}{e^w-1}\,dw$$

$$= \int\limits_L \frac{e^{\frac14 iw^2/\pi+w-i\pi+a(w-2\pi i)}}{e^w-1}\,dw = -e^{-2\pi ia}\Phi(a+1).$$

Hence $$-e^{-2\pi ia}\Phi(a+1)-\Phi(a) = 2\pi i. \tag{2.10.3}$$

Eliminating $\Phi(a+1)$, we have

$$\Phi(a) = -\frac{2\pi i+2\pi e^{i\pi(a^2-2a+\frac14)}}{1+e^{-2\pi ia}}, \tag{2.10.4}$$

or $$\Phi(a) = 2\pi\frac{\cos\pi(\frac12 a^2-a-\frac18)}{\cos\pi a}e^{i\pi(\frac12 a^2-\frac38)}. \tag{2.10.5}$$

If $a = \frac12 iz/\pi+\frac12$, the result (2.10.4) takes the form

$$\int\limits_L \frac{e^{\frac14 iw^2/\pi+\frac12 izw/\pi+\frac12 w}}{e^w-1}\,dw = \frac{2\pi i}{e^z-1}-2\pi i\frac{e^{-\frac14 iz^2/\pi+\frac12 z}}{e^z-1}.$$

Multiplying by z^{s-1} ($\sigma > 1$), and integrating from 0 to $\infty e^{-\frac14 i\pi}$, we obtain

$$\int\limits_L \frac{e^{\frac14 iw^2/\pi+\frac12 w}}{e^w-1}\,dw\int\limits_0^{\infty e^{-\frac14 i\pi}} e^{\frac12 izw/\pi}z^{s-1}\,dz$$

$$= 2\pi i\Gamma(s)\zeta(s)-2\pi i\int\limits_0^{\infty e^{-\frac14 i\pi}}\frac{e^{-\frac14 iz^2/\pi+\frac12 z}}{e^z-1}z^{s-1}\,dz.$$

The inversion on the left-hand side is justified by absolute convergence; in fact $$w = -c+\rho e^{\frac14 i\pi}, \qquad z = re^{-\frac14 i\pi},$$

where $c > 0$, so that $\mathbf{R}(izw) = -cr/\sqrt{2}$.

Now

$$\int\limits_0^{\infty e^{-\frac14 i\pi}} e^{\frac12 izw/\pi}z^{s-1}\,dz = e^{\frac12 i\pi s}\int\limits_0^{\infty} e^{-\frac12 yw/\pi}y^{s-1}\,dy = e^{\frac12 i\pi s}\left(\frac{w}{2\pi}\right)^{-s}\Gamma(s),$$

and $$\int\limits_0^{\infty e^{-\frac14 i\pi}}\frac{e^{-\frac14 iz^2/\pi+\frac12 z}}{e^z-1}z^{s-1}\,dz = \frac{1}{1+e^{-is\pi}}\int\limits_{\bar L}\frac{e^{-\frac14 iz^2/\pi+\frac12 z}}{e^z-1}z^{s-1}\,dz,$$

where $\bar L$ is the reflection of L in the real axis. Hence

$$\zeta(s) = \frac{e^{\frac12 is\pi}(2\pi)^s}{2\pi i}\int\limits_L\frac{e^{\frac14 iw^2/\pi+\frac12 w}}{e^w-1}w^{-s}\,dw+\frac{1}{\Gamma(s)(1+e^{-is\pi})}\int\limits_{\bar L}\frac{e^{-\frac14 iz^2/\pi+\frac12 z}}{e^z-1}z^{s-1}\,dz,$$

or

$$\pi^{-\frac12 s}\Gamma(\tfrac12 s)\zeta(s) = e^{\frac12 i\pi(s-1)}2^{s-1}\pi^{\frac12 s-1}\Gamma(\tfrac12 s)\int\limits_L\frac{e^{\frac14 iw^2/\pi+\frac12 w}}{e^w-1}w^{-s}\,dw+$$

$$+e^{\frac12 i\pi s}2^{-s}\pi^{-\frac12 s-\frac12}\Gamma(\tfrac12-\tfrac12 s)\int\limits_{\bar L}\frac{e^{-\frac14 iz^2/\pi+\frac12 z}}{e^z-1}z^{s-1}\,dz. \tag{2.10.6}$$

This formula holds by the theory of analytic continuation for all values of s.

If $s = \frac12+it$, the two terms on the right are conjugates. Hence $f(s) = \pi^{-\frac12 s}\Gamma(\frac12 s)\zeta(s)$ is real on $\sigma = \frac12$. Hence

$$f(s) = f(\sigma+it) = \overline{f(1-\sigma+it)} = f(1-\sigma-it) = f(1-s),$$

the functional equation.

2.11. A general formula involving $\zeta(s)$. It was observed by Müntz† that several of the formulae for $\zeta(s)$ which we have obtained are particular cases of a formula containing an arbitrary function.

We have formally

$$\int_0^\infty x^{s-1}\sum_{n=1}^\infty F(nx)\,dx = \sum_{n=1}^\infty \int_0^\infty x^{s-1}F(nx)\,dx$$

$$= \sum_{n=1}^\infty \frac{1}{n^s}\int_0^\infty y^{s-1}F(y)\,dy$$

$$= \zeta(s)\int_0^\infty y^{s-1}F(y)\,dy,$$

where $F(x)$ is arbitrary; and the process is justifiable if $F(x)$ is bounded in any finite interval, and $O(x^{-\alpha})$, where $\alpha > 1$, as $x \to \infty$. For then

$$\sum_{n=1}^\infty \left|\frac{1}{n^s}\right| \int_0^\infty |y^{s-1}F(y)|\,dy$$

exists if $1 < \sigma < \alpha$, and the inversion is justified.

Suppose next that $F'(x)$ is continuous, bounded in any finite interval, and $O(x^{-\beta})$, where $\beta > 1$, as $x \to \infty$. Then as $x \to 0$

$$\sum_{n=1}^\infty F(nx) - \int_0^\infty F(ux)\,du = x\int_0^\infty F'(ux)(u-[u])\,du$$

$$= x\int_0^{1/x} O(1)\,du + x\int_{1/x}^\infty O\{(ux)^{-\beta}\}\,du = O(1),$$

i.e.
$$\sum_{n=1}^\infty F(nx) = \frac{1}{x}\int_0^\infty F(v)\,dv + O(1) = \frac{c}{x} + O(1),$$

say. Hence

$$\int_0^\infty x^{s-1}\sum_{n=1}^\infty F(nx)\,dx$$

$$= \int_0^1 x^{s-1}\left\{\sum_{n=1}^\infty F(nx) - \frac{c}{x}\right\}dx + \frac{c}{s-1} + \int_1^\infty x^{s-1}\sum_{n=1}^\infty F(nx)\,dx,$$

and the right-hand side is regular for $\sigma > 0$ (except at $s = 1$). Also for $\sigma < 1$

$$\frac{c}{s-1} = -c\int_1^\infty x^{s-2}\,dx.$$

† Müntz (1).

Hence we have Müntz's formula

$$\zeta(s)\int_0^\infty y^{s-1}F(y)\,dy = \int_0^\infty x^{s-1}\left\{\sum_{n=1}^\infty F(nx) - \frac{1}{x}\int_0^\infty F(v)\,dv\right\}dx, \quad (2.11.1)$$

valid for $0 < \sigma < 1$ if $F(x)$ satisfies the above conditions.

If $F(x) = e^{-x}$ we obtain (2.7.1); if $F(x) = e^{-\pi x^2}$ we obtain a formula equivalent to those of § 2.6; if $F(x) = 1/(1+x^2)$ we obtain a formula which is also obtained by combining (2.4.1) with the functional equation. If $F(x) = x^{-1}\sin \pi x$ we obtain a formula equivalent to (2.1.6), though this $F(x)$ does not satisfy our general conditions.

If $F(x) = 1/(1+x)^2$, we have

$$\sum_{n=1}^\infty F(nx) - \frac{1}{x}\int_0^\infty F(v)\,dv = \sum_{n=1}^\infty \frac{1}{(1+nx)^2} - \frac{1}{x}$$

$$= \frac{1}{x^2}\left[\frac{d^2}{d\xi^2}\log\Gamma(\xi+1)\right]_{\xi=1/x} - \frac{1}{x}.$$

Hence
$$\frac{(1-s)\pi}{\sin \pi s}\zeta(s) = \int_0^\infty \xi^{1-s}\left\{\frac{d^2}{d\xi^2}\log\Gamma(\xi+1) - \frac{1}{\xi}\right\}d\xi,$$

and on integrating by parts we obtain (2.9.2).

2.12. Zeros; factorization formulae.

THEOREM 2.12. *$\xi(s)$ and $\Xi(z)$ are integral functions of order* 1.

It follows from (2.1.12) and what we have proved about $\zeta(s)$ that $\xi(s)$ is regular for $\sigma > 0$, $(s-1)\zeta(s)$ being regular at $s = 1$. Since $\xi(s) = \xi(1-s)$, $\xi(s)$ is also regular for $\sigma < 1$. Hence $\xi(s)$ is an integral function.

Also

$$|\Gamma(\tfrac{1}{2}s)| = \left|\int_0^\infty e^{-u}u^{\frac{1}{2}s-1}\,du\right| \leqslant \int_0^\infty e^{-u}u^{\frac{1}{2}\sigma-1}\,du = \Gamma(\tfrac{1}{2}\sigma) = O(e^{A\sigma\log\sigma}) \quad (\sigma > 0), \quad (2.12.1)$$

and (2.1.4) gives for $\sigma \geqslant \frac{1}{2}$, $|s-1| > A$,

$$\zeta(s) = O\left(|s|\int_1^\infty \frac{du}{u^{\frac{3}{2}}}\right) + O(1) = O(|s|). \quad (2.12.2)$$

Hence (2.1.12) gives
$$\xi(s) = O(e^{A|s|\log|s|}) \quad (2.12.3)$$

for $\sigma \geqslant \frac{1}{2}$, $|s| > A$. By (2.1.13) this holds for $\sigma \leqslant \frac{1}{2}$ also. Hence $\xi(s)$ is of order 1 at most. The order is exactly 1 since as $s \to \infty$ by real values $\log \zeta(s) \sim 2^{-s}$, $\log \xi(s) \sim \frac{1}{2}s\log s$.

Hence also $\quad \Xi(z) = O(e^{A|z|\log|z|}) \quad (|z| > A),$

and $\Xi(z)$ is of order 1. But $\Xi(z)$ is an even function. Hence $\Xi(\sqrt{z})$ is also an integral function, and is of order $\frac{1}{2}$. It therefore has an infinity of zeros, whose exponent of convergence is $\frac{1}{2}$. Hence $\Xi(z)$ has an infinity of zeros, whose exponent of convergence is 1. The same is therefore true of $\xi(s)$. Let $\rho_1, \rho_2, \ldots$ be the zeros of $\xi(s)$.

We have already seen that $\zeta(s)$ has no zeros for $\sigma > 1$. It then follows from the functional equation (2.1.1) that $\zeta(s)$ has no zeros for $\sigma < 0$ except for simple zeros at $s = -2, -4, -6, \ldots$; for, in (2.1.1), $\zeta(1-s)$ has no zeros for $\sigma < 0$, $\sin\frac{1}{2}s\pi$ has simple zeros at $s = -2, -4, \ldots$ only, and $\Gamma(1-s)$ has no zeros.

The zeros of $\zeta(s)$ at $-2, -4, \ldots$, are known as the 'trivial zeros'. They do not correspond to zeros of $\xi(s)$, since in (2.1.12) they are cancelled by poles of $\Gamma(\frac{1}{2}s)$. It therefore follows from (2.1.12) that $\xi(s)$ has no zeros for $\sigma > 1$ or for $\sigma < 0$. Its zeros $\rho_1, \rho_2, \ldots$ therefore all lie in the strip $0 \leqslant \sigma \leqslant 1$; and they are also zeros of $\zeta(s)$, since $s(s-1)\Gamma(\frac{1}{2}s)$ has no zeros in the strip except that at $s = 1$, which is cancelled by the pole of $\zeta(s)$.

We have thus proved that $\zeta(s)$ has an infinity of zeros $\rho_1, \rho_2, \ldots$ in the strip $0 \leqslant \sigma \leqslant 1$. Since

$$(1-2^{1-s})\zeta(s) = 1-\frac{1}{2^s}+\frac{1}{3^s}-\ldots > 0 \quad (0 < s < 1) \qquad (2.12.4)$$

and $\zeta(0) \neq 0$, $\zeta(s)$ has no zeros on the real axis between 0 and 1. The zeros $\rho_1, \rho_2, \ldots$ are therefore all complex.

The remainder of the theory is largely concerned with questions about the position of these zeros. At this point we shall merely observe that they are in conjugate pairs, since $\zeta(s)$ is real on the real axis; and that, if ρ is a zero, so is $1-\rho$, by the functional equation, and hence so is $1-\bar{\rho}$. If $\rho = \beta+i\gamma$, then $1-\bar{\rho} = 1-\beta+i\gamma$. Hence the zeros either lie on $\sigma = \frac{1}{2}$, or occur in pairs symmetrical about this line.

Since $\xi(s)$ is an integral function of order 1, and $\xi(0) = -\zeta(0) = \frac{1}{2}$, Hadamard's factorization theorem gives, for all values of s,

$$\xi(s) = \tfrac{1}{2}e^{b_0 s}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho}, \qquad (2.12.5)$$

where b_0 is a constant. Hence

$$\zeta(s) = \frac{e^{bs}}{2(s-1)\Gamma(\frac{1}{2}s+1)}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho}, \qquad (2.12.6)$$

where $b = b_0+\frac{1}{2}\log\pi$. Hence also

$$\frac{\zeta'(s)}{\zeta(s)} = b-\frac{1}{s-1}-\frac{1}{2}\frac{\Gamma'(\frac{1}{2}s+1)}{\Gamma(\frac{1}{2}s+1)}+\sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right). \qquad (2.12.7)$$

Making $s \to 0$, this gives

$$\frac{\zeta'(0)}{\zeta(0)} = b+1-\frac{1}{2}\frac{\Gamma'(1)}{\Gamma(1)}.$$

Since $\zeta'(0)/\zeta(0) = \log 2\pi$ and $\Gamma'(1) = -\gamma$, it follows that

$$b = \log 2\pi-1-\tfrac{1}{2}\gamma. \qquad (2.12.8)$$

2.13. In this section† we shall show that the only function which satisfies the functional equation (2.1.1), and has the same general characteristics as $\zeta(s)$, is $\zeta(s)$ itself.

Let $G(s)$ be an integral function of finite order, $P(s)$ a polynomial, and $f(s) = G(s)/P(s)$, and let

$$f(s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s} \qquad (2.13.1)$$

be absolutely convergent for $\sigma > 1$. Let

$$f(s)\Gamma(\tfrac{1}{2}s)\pi^{-\frac{1}{2}s} = g(1-s)\Gamma(\tfrac{1}{2}-\tfrac{1}{2}s)\pi^{-\frac{1}{2}(1-s)}, \qquad (2.13.2)$$

where $$g(1-s) = \sum_{n=1}^{\infty}\frac{b_n}{n^{1-s}},$$

the series being absolutely convergent for $\sigma < -\alpha < 0$. Then $f(s) = C\zeta(s)$, where C is a constant.

We have, for $x > 0$,

$$\phi(x) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} f(s)\Gamma(\tfrac{1}{2}s)\pi^{-\frac{1}{2}s}x^{-\frac{1}{2}s}\,ds$$

$$= \sum_{n=1}^{\infty}\frac{a_n}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Gamma(\tfrac{1}{2}s)(\pi n^2x)^{-\frac{1}{2}s}\,ds$$

$$= 2\sum_{n=1}^{\infty}a_n e^{-\pi n^2 x}.$$

Also, by (2.13.2),

$$\phi(x) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} g(1-s)\Gamma(\tfrac{1}{2}-\tfrac{1}{2}s)\pi^{-\frac{1}{2}(1-s)}x^{-\frac{1}{2}s}\,ds.$$

We move the line of integration from $\sigma = 2$ to $\sigma = -1-\alpha$. We observe

† Hamburger (1)–(4), Siegel (1).

that $f(s)$ is bounded on $\sigma = 2$, and $g(1-s)$ is bounded on $\sigma = -1-\alpha$; since

$$\frac{\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1}{2}-\frac{1}{2}s)} = O(|t|^{\sigma-\frac{1}{2}}),$$

it follows that $g(1-s) = O(|t|^{\frac{3}{2}})$ on $\sigma = 2$. We can therefore, by the Phragmén-Lindelöf principle, apply Cauchy's theorem, and obtain

$$\phi(x) = \frac{1}{2\pi i}\int\limits_{-\alpha-1-i\infty}^{-\alpha-1+i\infty} g(1-s)\Gamma(\tfrac{1}{2}-\tfrac{1}{2}s)\pi^{-\frac{1}{2}(1-s)}x^{-\frac{1}{2}s}\,ds + \sum_{\nu=1}^{m} R_\nu,$$

where R_1, R_2,..., are the residues at the poles, say s_1,..., s_m. Thus

$$\sum_{\nu=1}^{m} R_\nu = \sum_{\nu=1}^{m} x^{-\frac{1}{2}s_\nu}Q_\nu(\log x) = Q(x),$$

where the $Q_\nu(\log x)$ are polynomials in $\log x$. Hence

$$\phi(x) = \frac{1}{\sqrt{x}}\sum_{n=1}^{\infty}\frac{b_n}{2\pi i}\int\limits_{-\alpha-1-i\infty}^{-\alpha-1+i\infty}\Gamma(\tfrac{1}{2}-\tfrac{1}{2}s)(\pi n^2/x)^{-\frac{1}{2}+\frac{1}{2}s}\,ds + Q(x)$$

$$= \frac{2}{\sqrt{x}}\sum_{n=1}^{\infty} b_n e^{-\pi n^2/x} + Q(x).$$

Hence
$$\sum_{n=1}^{\infty} a_n e^{-\pi n^2 x} = \frac{1}{\sqrt{x}}\sum_{n=1}^{\infty} b_n e^{-\pi n^2/x} + \tfrac{1}{2}Q(x).$$

Multiply by $e^{-\pi t^2 x}$ $(t > 0)$, and integrate over $(0, \infty)$. We obtain

$$\sum_{n=1}^{\infty}\frac{a_n}{\pi(t^2+n^2)} = \sum_{n=1}^{\infty}\frac{b_n}{t}e^{-2\pi nt} + \tfrac{1}{2}\int\limits_0^\infty Q(x)e^{-\pi t^2 x}\,dx,$$

and the last term is a sum of terms of the form

$$\int\limits_0^\infty x^a \log^b x\, e^{-\pi t^2 x}\,dx,$$

where the b's are integers and $\mathbf{R}(a) > -1$; i.e. it is a sum of terms of the form $t^\alpha \log^\beta t$.

Hence
$$\sum_{n=1}^{\infty} a_n\left(\frac{1}{t+in}+\frac{1}{t-in}\right) - \pi t H(t) = 2\pi\sum_{n=1}^{\infty} b_n e^{-2\pi nt},$$

where $H(t)$ is a sum of terms of the form $t^\alpha \log^\beta t$.

Now the series on the left is a meromorphic function, with poles at $\pm in$. But the function on the right is periodic, with period i. Hence (by analytic continuation) so is the function on the left. Hence the residues at ki and $(k+1)i$ are equal, i.e. $a_k = a_{k+1}$ $(k = 1, 2,...)$. Hence $a_k = a_1$ for all k, and the result follows.

2.14. Some series involving $\zeta(s)$. We have†

$$\zeta(s) - \frac{1}{s-1} = 1 - \tfrac{1}{2}s\{\zeta(s+1)-1\} - \frac{s(s+1)}{2.3}\{\zeta(s+2)-1\} - \ldots \quad (2.14.1)$$

for all values of s. For the right-hand side is

$$1 - \frac{1}{s-1}\sum_{n=2}^{\infty}\frac{1}{n^{s-1}}\left\{\frac{(s-1)s}{1.2}\frac{1}{n^2} + \frac{(s-1)s(s+1)}{1.2.3}\frac{1}{n^3} + \ldots\right\}$$

$$= 1 - \frac{1}{s-1}\sum_{n=2}^{\infty}\frac{1}{n^{s-1}}\left\{\left(1-\frac{1}{n}\right)^{1-s} - 1 - \frac{s-1}{n}\right\}$$

$$= 1 - \frac{1}{s-1}\sum_{n=2}^{\infty}\left\{\frac{1}{(n-1)^{s-1}} - \frac{1}{n^{s-1}} - \frac{s-1}{n^s}\right\}$$

$$= \zeta(s) - \frac{1}{s-1}.$$

The inversion of the order of summation is justified for $\sigma > 0$ by the convergence of

$$\sum_{n=2}^{\infty}\frac{1}{n^{\sigma-1}}\sum_{k=0}^{\infty}\frac{|s|\ldots(|s|+k)}{(k+1)!}\frac{1}{n^{k+2}} = \sum_{n=2}^{\infty}\frac{1}{n^\sigma}\left\{\left(1-\frac{1}{n}\right)^{-|s|} - 1\right\}.$$

The series obtained is, however, convergent for all values of s.

Another formula‡ which can be proved in a similar way is

$$(1-2^{1-s})\zeta(s) = s\frac{\zeta(s+1)}{2^{s+1}} + \frac{s(s+1)}{1.2}\frac{\zeta(s+2)}{2^{s+2}} + \ldots, \quad (2.14.2)$$

also valid for all values of s.

Either of these formulae may be used to obtain the analytic continuation of $\zeta(s)$ over the whole plane.

2.15. Some applications of Mellin's inversion formulae.§ Mellin's inversion formulae connecting the two functions $f(x)$ and $\mathfrak{F}(s)$ are

$$\mathfrak{F}(s) = \int\limits_0^\infty f(x)x^{s-1}\,dx, \qquad f(x) = \frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\mathfrak{F}(s)x^{-s}\,ds. \quad (2.15.1)$$

The simplest example is

$$f(x) = e^{-x}, \qquad \mathfrak{F}(s) = \Gamma(s) \quad (\sigma > 0). \quad (2.15.2)$$

From (2.4.1) we derive the pair

$$f(x) = \frac{1}{e^x-1}, \qquad \mathfrak{F}(s) = \Gamma(s)\zeta(s) \quad (\sigma > 1), \quad (2.15.3)$$

† Landau, *Handbuch*, 272. ‡ Ramaswami (1).

§ See E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, §§ 1.5, 1.29, 2.1, 2.7, 3.17.

and from (2.6.2) the pair

$$f(x) = \psi(x), \qquad \mathfrak{F}(s) = \pi^{-s}\Gamma(s)\zeta(2s) \quad (\sigma > \tfrac{1}{2}). \tag{2.15.4}$$

The inverse formulae are thus

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(s)\zeta(s)x^{-s}\,ds = \frac{1}{e^x-1} \quad (\sigma > 1) \tag{2.15.5}$$

and

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \pi^{-s}\Gamma(s)\zeta(2s)x^{-s}\,ds = \psi(x) \quad (\sigma > \tfrac{1}{2}). \tag{2.15.6}$$

Each of these can easily be proved directly by inserting the series for $\zeta(s)$ and integrating term-by-term, using (2.15.2).

As another example, (2.9.2), with s replaced by $1-s$, gives the Mellin pair

$$f(x) = \frac{\Gamma'(1+x)}{\Gamma(1+x)} - \log x, \qquad \mathfrak{F}(s) = -\frac{\pi\zeta(1-s)}{\sin\pi s} \quad (0 < \sigma < 1). \tag{2.15.7}$$

The inverse formula is thus

$$\frac{\Gamma'(1+x)}{\Gamma(1+x)} - \log x = -\frac{1}{2i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\zeta(1-s)}{\sin\pi s}x^{-s}\,ds. \tag{2.15.8}$$

Integrating with respect to x, and replacing s by $1-s$, we obtain

$$\log\Gamma(1+x) - x\log x + x = -\frac{1}{2i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\zeta(s)x^s}{s\sin\pi s}\,ds \quad (0 < \sigma < 1). \tag{2.15.9}$$

This formula is used by Whittaker and Watson to obtain the asymptotic expansion of $\log\Gamma(1+x)$.

Next, let $f(x)$ and $\mathfrak{F}(s)$ be related by (2.15.1), and let $g(x)$ and $\mathfrak{G}(s)$ be similarly related. Then we have, subject to appropriate conditions,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathfrak{F}(s)\mathfrak{G}(w-s)\,ds = \int_0^\infty f(x)g(x)x^{w-1}\,dx. \tag{2.15.10}$$

Take for example $\mathfrak{F}(s) = \mathfrak{G}(s) = \Gamma(s)\zeta(s)$, so that

$$f(x) = g(x) = 1/(e^x-1).$$

Then, if $\mathbf{R}(w) > 2$, the right-hand side is

$$\int_0^\infty \frac{x^{w-1}}{(e^x-1)^2}\,dx = \int_0^\infty (e^{-2x}+2e^{-3x}+3e^{-4x}+\ldots)x^{w-1}\,dx$$

$$= \left(\frac{1}{2^w}+\frac{2}{3^w}+\frac{3}{4^w}+\ldots\right)\Gamma(w) = \Gamma(w)\{\zeta(w-1)-\zeta(w)\}.$$

Thus if $1 < c < \mathbf{R}(w)-1$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\Gamma(w-s)\zeta(s)\zeta(w-s)\,ds = \Gamma(w)\{\zeta(w-1)-\zeta(w)\}. \tag{2.15.11}$$

Similarly, taking $\mathfrak{F}(s) = \mathfrak{G}(s) = \Gamma(s)\zeta(2s)$, so that

$$f(x) = g(x) = \psi(x/\pi) = \sum_{n=1}^\infty e^{-n^2x},$$

the right-hand side of (2.15.10) is, if $\mathbf{R}(w) > 1$,

$$\int_0^\infty \sum_{m=1}^\infty \sum_{n=1}^\infty e^{-(m^2+n^2)x}x^{w-1}\,dx = \Gamma(w)\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{1}{(m^2+n^2)^w}.$$

This may also be written

$$\Gamma(w)\left\{\frac{1}{4}\sum_{n=1}^\infty \frac{r(n)}{n^w} - \zeta(2w)\right\},$$

where $r(n)$ is the number of ways of expressing n as the sum of two squares; or as $\quad \Gamma(w)\{\zeta(w)\eta(w)-\zeta(2w)\}$,

where $\quad \eta(w) = 1^{-w}-3^{-w}+5^{-w}-\ldots.$

Hence† if $\frac{1}{2} < c < \mathbf{R}(w)-\frac{1}{2}$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\Gamma(w-s)\zeta(2s)\zeta(2w-2s)\,ds = \Gamma(w)\{\zeta(w)\eta(w)-\zeta(2w)\}. \tag{2.15.12}$$

2.16. Some integrals involving $\Xi(t)$. There are some cases‡ in which integrals of the form

$$\Phi(x) = \int_0^\infty f(t)\Xi(t)\cos xt\,dt$$

can be evaluated. Let $f(t) = |\phi(it)|^2 = \phi(it)\phi(-it)$, where ϕ is analytic. Writing $y = e^x$,

$$\Phi(x) = \tfrac{1}{2}\int_{-\infty}^\infty \phi(it)\phi(-it)\Xi(t)y^{it}\,dt$$

$$= \tfrac{1}{2}\int_{-\infty}^\infty \phi(it)\phi(-it)\xi(\tfrac{1}{2}+it)y^{it}\,dt$$

$$= \frac{1}{2i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \phi(s-\tfrac{1}{2})\phi(\tfrac{1}{2}-s)\xi(s)y^s\,ds$$

$$= \frac{1}{2i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \phi(s-\tfrac{1}{2})\phi(\tfrac{1}{2}-s)(s-1)\Gamma(1+\tfrac{1}{2}s)\pi^{-\frac{1}{2}s}\zeta(s)y^s\,ds.$$

† Hardy (4). A generalization is given by Taylor (1). ‡ Ramanujan (1).

Taking $\phi(s) = 1$, this is equal to

$$\frac{1}{i\sqrt{y}}\sum_{n=1}^{\infty}\int_{2-i\infty}^{2+i\infty}\{\Gamma(2+\tfrac{1}{2}s)-\tfrac{3}{2}\Gamma(1+\tfrac{1}{2}s)\}\left(\frac{y}{n\sqrt{\pi}}\right)^{s}ds$$

$$=\frac{1}{i\sqrt{y}}\sum_{n=1}^{\infty}\left\{2\int_{3-i\infty}^{3+i\infty}\Gamma(w)\left(\frac{y}{n\sqrt{\pi}}\right)^{2w-4}dw-3\int_{2-i\infty}^{2+i\infty}\Gamma(w)\left(\frac{y}{n\sqrt{\pi}}\right)^{2w-2}dw\right\}$$

$$=\frac{4\pi}{\sqrt{y}}\sum_{n=1}^{\infty}\left(\frac{y}{n\sqrt{\pi}}\right)^{-4}e^{-n^{2}\pi/y^{2}}-\frac{6\pi}{\sqrt{y}}\sum_{n=1}^{\infty}\left(\frac{y}{n\sqrt{\pi}}\right)^{-2}e^{-n^{2}\pi/y^{2}}.$$

Hence

$$\int_{0}^{\infty}\Xi(t)\cos xt\,dt = 2\pi^{2}\sum_{n=1}^{\infty}(2\pi n^{4}e^{-9x/2}-3n^{2}e^{-5x/2})\exp(-n^{2}\pi e^{-2x}). \tag{2.16.1}$$

Again, putting $\phi(s) = 1/(s+\frac{1}{2})$, we have

$$\Phi(x) = -\frac{1}{2i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{1}{s}\Gamma(1+\tfrac{1}{2}s)\pi^{-\frac{1}{2}s}\zeta(s)y^{s}\,ds$$

$$= -\frac{1}{4i\sqrt{y}}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Gamma(\tfrac{1}{2}s)\pi^{-\frac{1}{2}s}\zeta(s)y^{s}\,ds$$

$$= -\frac{\pi}{\sqrt{y}}\psi\left(\frac{1}{y^{2}}\right)+\tfrac{1}{2}\pi\sqrt{y}$$

in the notation of § 2.6. Hence

$$\int_{0}^{\infty}\frac{\Xi(t)}{t^{2}+\frac{1}{4}}\cos xt\,dt = \tfrac{1}{2}\pi\{e^{\frac{1}{2}x}-2e^{-\frac{1}{2}x}\psi(e^{-2x})\}. \tag{2.16.2}$$

The case $\phi(s) = \Gamma(\frac{1}{2}s-\frac{1}{4})$ was also investigated by Ramanujan, the result being expressed in terms of another integral.

2.17. The function $\zeta(s, a)$. A function which is in a sense a generalization of $\zeta(s)$ is the Hurwitz zeta-function, defined by

$$\zeta(s,a) = \sum_{n=0}^{\infty}\frac{1}{(n+a)^{s}} \quad (0 < a \leqslant 1,\ \sigma > 1).$$

This reduces to $\zeta(s)$ when $a = 1$, and to $(2^{s}-1)\zeta(s)$ when $a = \frac{1}{2}$. We shall obtain here its analytic continuation and functional equation, which are required later. This function, however, has no Euler product unless $a = \frac{1}{2}$ or $a = 1$, and so does not share the most characteristic properties of $\zeta(s)$.

As in § 2.4

$$\zeta(s,a) = \sum_{n=0}^{\infty}\frac{1}{\Gamma(s)}\int_{0}^{\infty}x^{s-1}e^{-(n+a)x}\,dx = \frac{1}{\Gamma(s)}\int_{0}^{\infty}\frac{x^{s-1}e^{-ax}}{1-e^{-x}}\,dx. \tag{2.17.1}$$

We can transform this into a loop integral as before. We obtain

$$\zeta(s,a) = \frac{e^{-i\pi s}\Gamma(1-s)}{2\pi i}\int_{C}\frac{z^{s-1}e^{-az}}{1-e^{-z}}\,dz. \tag{2.17.2}$$

This provides the analytic continuation of $\zeta(s,a)$ over the whole plane; it is regular everywhere except for a simple pole at $s = 1$ with residue 1.

Expanding the loop to infinity as before, the residues at $2m\pi i$ and $-2m\pi i$ are together

$$(2m\pi e^{\frac{1}{2}i\pi})^{s-1}e^{-2m\pi ia}+(2m\pi e^{\frac{3}{2}i\pi})^{s-1}e^{2m\pi ia}$$
$$= (2m\pi)^{s-1}e^{i\pi(s-1)}2\cos\{\tfrac{1}{2}\pi(s-1)+2m\pi a\}$$
$$= -2(2m\pi)^{s-1}e^{i\pi s}\sin(\tfrac{1}{2}\pi s+2m\pi a).$$

Hence, if $\sigma < 0$,

$$\zeta(s,a) = \frac{2\Gamma(1-s)}{(2\pi)^{1-s}}\left\{\sin\tfrac{1}{2}\pi s\sum_{m=1}^{\infty}\frac{\cos 2m\pi a}{m^{1-s}}+\cos\tfrac{1}{2}\pi s\sum_{m=1}^{\infty}\frac{\sin 2m\pi a}{m^{1-s}}\right\}. \tag{2.17.3}$$

If $a = 1$, this reduces to the functional equation for $\zeta(s)$.

NOTES FOR CHAPTER 2

2.18. Selberg [3] has given a very general method for obtaining the analytic continuation and functional equation of certain types of zeta-function which arise as the 'constant terms' of Eisenstein series. We sketch a form of the argument in the classical case. Let $\mathscr{H} = \{z = x+iy: y > 0\}$ be the upper half plane and define

$$E(z,s) = \sum_{\substack{c,d=-\infty \\ (c,d)=1}}^{\infty}\frac{y^{s}}{|cz+d|^{2s}} \quad (z\in\mathscr{H},\ \sigma > 1)$$

and

$$B(z,s) = \zeta(2s)E(z,s) = \sum_{\substack{c,d=-\infty \\ (c,d)\neq(0,0)}}^{\infty}\frac{y^{s}}{|cz+d|^{2s}} \quad (z\in\mathscr{H},\ \sigma > 1),$$

these series being absolutely and uniformly convergent in any compact subset of the region $\mathbf{R}(s) > 1$. Here $E(z, s)$ is an Eisenstein series, while $B(z, s)$ is, apart from the factor y^{s}, the Epstein zeta-function for the lattice generated by 1 and z. We shall find it convenient to work with $B(z, s)$ in preference to $E(z, s)$.

We begin with two basic observations. Firstly one trivially has

$$B(z+1, s) = B(-1/z, s) = B(z, s). \tag{2.18.1}$$

(Thus, in fact, $B(z, s)$ is invariant under the full modular group.) Secondly, if Δ is the Laplace–Beltrami operator

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right),$$

then

$$\Delta\left(\frac{y^s}{|cz+d|^{2s}}\right) = s(1-s)\frac{y^s}{|cz+d|^{2s}}, \tag{2.18.2}$$

whence

$$\Delta B(z, s) = s(1-s)B(z, s) \quad (\sigma > 1). \tag{2.18.3}$$

We proceed to obtain the Fourier expansion of $B(z, s)$ with respect to x. We have

$$B(z, s) = \sum_{-\infty}^{\infty} a_n(y, s)\, e^{2\pi inx},$$

where

$$a_n(y, s) = y^s \sum_{c,d} \int_0^1 \frac{e^{-2\pi inx}\,dx}{|cx+d+icy|^{2s}}$$

$$= 2\delta_n y^s \zeta(2s) + 2y^s \sum_{c=1}^{\infty}\sum_{d=-\infty}^{\infty}\int_0^1 \frac{e^{-2\pi inx}\,dx}{|cx+d+icy|^{2s}},$$

with $\delta_n = 1$ or 0 according as $n = 0$ or not. The d summation above is

$$\sum_{k=1}^{c}\sum_{j=-\infty}^{\infty}\int_0^1 \frac{e^{-2\pi inx}\,dx}{|c(x+j)+k+icy|^{2s}} = \sum_{k=1}^{c}\int_{-\infty}^{\infty}\frac{e^{-2\pi inx}\,dx}{|cx+k+icy|^{2s}}$$

$$= c^{-2s}y^{1-2s}\int_{-\infty}^{\infty}\frac{e^{-2\pi inyv}\,dv}{(v^2+1)^s}\sum_{k=1}^{c} e^{2\pi ink/c},$$

and the sum over k is c or 0 according as $c|n$ or not. Moreover

$$\int_{-\infty}^{\infty}\frac{dv}{(v^2+1)^s} = \frac{\pi^{\frac{1}{2}}\Gamma(s-\frac{1}{2})}{\Gamma(s)},$$

and

$$\int_{-\infty}^{\infty}\frac{e^{-2\pi inyv}}{(v^2+1)^s}\,dv = 2\pi^s(|n|y)^{s-\frac{1}{2}}\frac{K_{s-\frac{1}{2}}(2\pi|n|y)}{\Gamma(s)} \qquad (n \neq 0),$$

in the usual notation of Bessel functions†.

We now have

$$B(z, s) = \phi(s)y^s + \psi(s)y^{1-s} + B_0(z, s) \quad (\sigma > 1), \tag{2.18.4}$$

where

$$\phi(s) = 2\zeta(2s), \qquad \psi(s) = 2\pi^{\frac{1}{2}}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\zeta(2s-1)$$

and

$$B_0(z, s) = 8\pi^s y^{\frac{1}{2}}\sum_{n=1}^{\infty} n^{s-\frac{1}{2}}\sigma_{1-2s}(n)\cos(2\pi nx)\frac{K_{s-\frac{1}{2}}(2\pi ny)}{\Gamma(s)}. \tag{2.18.5}$$

We observe at this point that

$$K_u(t) \ll t^{-\frac{1}{2}}e^{-t} \quad (t \to \infty)$$

for fixed u, whence the series (2.18.5) is convergent for all s, and so defines an entire function. Moreover we have

$$B_0(z, s) \ll e^{-y} \quad (y \to \infty) \tag{2.18.6}$$

for fixed s. Similarly one finds

$$\frac{\partial B_0(z, s)}{\partial y} \ll e^{-y} \quad (y \to \infty). \tag{2.18.7}$$

We proceed to derive the 'Maass–Selberg' formula. Let $D = \{z \in \mathscr{H}: |z| \geqslant 1, |\mathbf{R}(z)| \leqslant \frac{1}{2}\}$ be the standard fundamental region for the modular group, and let $D_Y = \{z \in D: \mathbf{I}(z) \leqslant Y\}$, where $Y \geqslant 1$. Let $\mathbf{R}(s)$, $\mathbf{R}(w) > 1$ and write, for convenience, $F = B(z, s)$, $G = B(z, w)$. Then, according to (2.18.3), we have

$$\{s(1-s)-w(1-w)\}\iint_{D_Y} FG\frac{dx\,dy}{y^2} = \iint_{D_Y}(G\Delta F - F\Delta G)\frac{dx\,dy}{y^2}$$

$$= \iint_{D_Y}(F\nabla^2 G - G\nabla^2 F)\,dx\,dy$$

$$= \int_{\partial D_Y}(F\nabla G - G\nabla F)\cdot d\mathbf{n},$$

† see Watson, *Theory of Bessel functions* §6.16.

by Green's Theorem. The integrals along $x = \pm\frac{1}{2}$ cancel, since $F(z+1) = F(z)$, $G(z+1) = G(z)$ (see (2.18.1)). Similarly the integral for $|z| = 1$ vanishes, since $F(-1/z) = F(z)$, $G(-1/z) = G(z)$. Thus

$$\{s(1-s)-w(1-w)\}\iint_{D_Y} FG\frac{dxdy}{y^2} = \int_{-\frac{1}{2}}^{\frac{1}{2}}\left(F\frac{\partial G}{\partial y}(x, Y) - G\frac{\partial F}{\partial y}(x, Y)\right)dx. \tag{2.18.8}$$

The functions y^s and y^{1-s} also satisfy the eigenfunction equation (2.18.3) (by (2.18.2) with $c = 0$, $d = 1$) and thus, by (2.18.4) so too does $B_0(z, s)$. Consequently, if $Z \geqslant Y$, an argument analogous to that above yields

$$\{s(1-s)-w(1-w)\}\int_Y^Z\int_{-\frac{1}{2}}^{\frac{1}{2}} F_0G_0\frac{dx\,dy}{y^2}$$
$$= \int_{-\frac{1}{2}}^{\frac{1}{2}}\left(F_0\frac{\partial G_0}{\partial y}(x, Z) - G_0\frac{\partial F_0}{\partial y}(x, Z)\right)dx$$
$$- \int_{-\frac{1}{2}}^{\frac{1}{2}}\left(F_0\frac{\partial G_0}{\partial y}(x, Y) - G_0\frac{\partial F_0}{\partial y}(x, Y)\right)dx,$$

where $F_0 = B_0(z, s)$, $G_0 = B_0(z, w)$. Here we have used $F_0(z+1) = F_0(z)$ and $G_0(z+1) = G_0(z)$. (Note that we no longer have the corresponding relations involving $-1/z$.) We may now take $Z \to \infty$, using (2.18.6) and (2.18.7), so that the first integral on the right above vanishes. On adding the result to (2.18.8) we obtain the Maass–Selberg formula

$$[s(1-s)-w(1-w)]\iint_D \tilde{B}(z, s)\,\tilde{B}(z, w)\frac{dxdy}{y^2}$$
$$= \int_{-\frac{1}{2}}^{\frac{1}{2}}\left(F\frac{\partial G}{\partial y}(x, Y) - G\frac{\partial F}{\partial y}(x, Y)\right)dx$$
$$- \int_{-\frac{1}{2}}^{\frac{1}{2}}\left(F_0\frac{\partial G_0}{\partial y}(x, Y) - G_0\frac{\partial F_0}{\partial y}(x, Y)\right)dx$$
$$= (s-w)\{\psi(s)\,\psi(w)\,Y^{1-s-w} - \phi(s)\,\phi(w)\,Y^{s+w-1}\}$$
$$+ (1-s-w)\{\phi(s)\,\psi(w)\,Y^{s-w} - \psi(s)\,\phi(w)\,Y^{w-s}\}, \tag{2.18.9}$$

where

$$\tilde{B}(z, s) = \begin{cases} B(z, s) & (y \leqslant Y), \\ B_0(z, s) & (y > Y). \end{cases}$$

2.19. In the general case there are now various ways in which one can proceed in order to get the analytic continuation of ϕ and ψ. However one point is immediate: once the analytic continuation has been established one may take $w = 1-s$ in (2.18.9) to obtain the relation

$$\phi(s)\,\phi(1-s) = \psi(s)\,\psi(1-s), \tag{2.19.1}$$

which can be thought of as a weak form of the functional equation.

The analysis we shall give takes advantage of certain special properties not available in the general case. We shall take $Y = 1$ in (2.18.9) and expand the integral on the left to obtain

$$(s-w)\alpha(s+w)\psi(s)\psi(w) + \beta(s, w)\psi(s) + \gamma(s, w)\psi(w) + \delta(s, w) = 0, \tag{2.19.2}$$

where

$$\alpha(u) = (1-u)\iint_{D_1} y^{-u}dxdy - 1 = -2\int_0^{\frac{1}{2}}(1-x^2)^{\frac{1}{2}(1-u)}dx$$

and β, γ, δ involve the functions ϕ and B_0, but not ψ. If we know that $\zeta(s)$ has a continuation to the half plane $\mathbf{R}(s) > \sigma_0$ then $\phi(s)$ has a continuation to $\mathbf{R}(s) > \frac{1}{2}\sigma_0$, so that α, β, γ, δ are meromorphic there. If

$$(s-w)\alpha(s+w)\psi(w) + \beta(s, w) = 0 \tag{2.19.3}$$

identically for $\mathbf{R}(s)$, $\mathbf{R}(w) > 1$, then

$$\psi(w) = -\frac{\beta(s, w)}{(s-w)\alpha(s+w)}, \tag{2.19.4}$$

which gives the analytic continuation of $\psi(w)$ to $\mathbf{R}(w) > \frac{1}{2}\sigma_0$. Note that $(s-w)\alpha(s+w)$ does not vanish identically. If (2.19.3) does not hold for all s and w then (2.19.2) yields

$$\psi(s) = -\frac{\gamma(s, w)\psi(w) + \delta(s, w)}{(s-w)\alpha(s+w)\psi(w) + \beta(s, w)}, \tag{2.19.5}$$

which gives the analytic continuation of $\psi(s)$ to $\mathbf{R}(s) > \frac{1}{2}\sigma_0$, on choosing a suitable w in the region $\mathbf{R}(w) > 1$. In either case $\zeta(s)$ may be continued to $\mathbf{R}(s) > \sigma_0 - 1$. This process shows that $\zeta(s)$ has a meromorphic continuation to the whole complex plane.

Some information on possible poles comes from taking $w = \bar{s}$ in (2.18.9), so that $\tilde{B}(z, w) = \overline{B(z, s)}$. Then

$$(2\sigma-1)\iint_D |\tilde{B}(z,s)|^2 \frac{dxdy}{y^2} = \{|\phi(s)|^2 Y^{2\sigma-1} - |\psi(s)|^2 Y^{1-2\sigma}\}$$

$$+(2\sigma-1)\frac{\phi(s)\overline{\psi(s)}\,Y^{2it} - \psi(s)\overline{\phi(s)}\,Y^{-2it}}{2it}.$$

If $t \neq 0$ we may choose $Y \geqslant 1$ so that the second term on the right vanishes. It follows that

$$|\psi(s)|^2 Y^{1-2\sigma} \leqslant |\phi(s)|^2 Y^{2\sigma-1}$$

for $\sigma \geqslant \frac{1}{2}$. Thus ψ is regular for $\sigma \geqslant \frac{1}{2}$ and $t \neq 0$, providing that ϕ is. Hence $\zeta(s)$ has no poles for $\mathbf{R}(s) > 0$, except possibly on the real axis.

If we take $\frac{1}{2} < \mathbf{R}(s), \mathbf{R}(w) < 1$ in (2.19.5), so that $\phi(s)$ and $\phi(w)$ are regular, we see that $\psi(s)$ can only have a pole at a point s_0 for which the denominator vanishes identically in w. For such an s_0, (2.19.4) must hold. However $\alpha(u)$ is clearly non-zero for real u, whence $\psi(w)$ can have at most a single, simple pole for real $w > \frac{1}{2}$, and this is at $w = s_0$. Since it is clear that $\zeta(s)$ does in fact have a singularity at $s = 1$ we see that $s_0 = 1$.

Much of the inelegance of the above analysis arises from the fact that, in the general case where one uses the Eisenstein series rather than the Epstein zeta-function, one has a single function $\rho(s) = \psi(s)/\phi(s)$ rather than two separate ones. Here $\rho(s)$ will indeed have poles to the left of $\mathbf{R}(s) = \frac{1}{2}$. In our special case we can extract the functional equation for $\zeta(s)$ itself, rather than the weaker relation $\rho(s)\rho(1-s) = 1$ (see (2.19.1)), by using (2.18.4) and (2.18.5). We observe that

$$n^{s-1/2}\sigma_{1-2s}(n) = n^{1/2-s}\sigma_{2s-1}(n)$$

and that $K_u(z) = K_{-u}(z)$, whence $\pi^{-s}\Gamma(s)B_0(z, s)$ is invariant under the transformation $s \to 1-s$. It follows that

$$\pi^{-s}\Gamma(s)B(z,s) - \pi^{s-1}\Gamma(1-s)B(z,1-s)$$
$$= \{A(s) - A(\tfrac{1}{2}-s)\}y^s + \{A(s-\tfrac{1}{2}) - A(1-s)\}y^{1-s},$$

where we have written temporarily $A(s) = 2\pi^{-s}\Gamma(s)\zeta(2s)$. The left-hand side is invariant under the transformation $z \to -1/z$, by (2.18.1), and so, taking $z = iy$ for example, we see that $A(s) = A(\frac{1}{2}-s)$ and $A(s-\frac{1}{2}) = A(1-s)$. These produce the functional equation in the form (2.6.4) and

indeed yield

$$\pi^{-s}\Gamma(s)B(z,s) = \pi^{s-1}\Gamma(1-s)B(z,1-s).$$

2.20. An insight into the nature of the zeta-function and its functional equation may be obtained from the work of Tate [1]. He considers an algebraic number field k and a general zeta-function

$$\zeta(f, c) = \int f(a)c(a)\,d^*a,$$

where the integral on the right is over the ideles J of k. Here f is one of a certain class of functions and c is any quasi-character of J, (that is to say, a continuous homomorphism from J to $\mathbb{C}^\times$) which is trivial on $k^\times$. We may write $c(a)$ in the form $c_0(a)|a|^s$, where $c_0(a)$ is a character on J (i.e. $|c_0(a)| = 1$ for $a \in J$). Then $c_0(a)$ corresponds to χ, a 'Hecke character' for k, and $\zeta(f, c)$ differs from

$$\zeta(s, \chi) = \prod_P \{1 - \chi(P)(NP)^{-s}\}^{-1}$$

(where P runs over prime ideals of k), in only a finite number of factors. In particular, if $k = \mathbb{Q}$, then $\zeta(f, c)$ is essentially a Dirichlet L-series $L(s, \chi)$. Thus these are essentially the only functions which can be associated to the rational field in this manner.

Tate goes on to prove a Poisson summation formula in this idèlic setting, and deduces the elegant functional equation

$$\zeta(f, c) = \zeta(\hat{f}, \hat{c})$$

where $\hat{f}$ is the 'Fourier transform' of f, and $\hat{c}(a) = \overline{c_0(a)}|a|^{1-s}$. The functional equation for $\zeta(s, \chi)$ may be extracted from this. In the case $k = \mathbb{Q}$ we may take c_0 identically equal to 1, and make a particular choice $f = f_0$, such that $\hat{f}_0 = f_0$ and

$$\zeta(f_0, |\cdot|^s) = \pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s).$$

The functional equation (2.6.4) is then immediate. Moreover it is now apparent that the factor $\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s)$ should be viewed as the natural term to be included in the Euler product, to correspond to the real valuation of $\mathbb{Q}$.

2.21. It is remarkable that the values of $\zeta(s)$ for $s = 0, -1, -2, \ldots$, are all rational, and this suggests the possibility of a p-adic analogue of $\zeta(s)$, interpolating these numbers. In fact it can be shown that for any prime p and any integer n there is a unique meromorphic function $\zeta_{p,n}(s)$ defined for $s \in \mathbb{Z}_p$, (the p-adic integers) such that

$$\zeta_{p,n}(k) = (1-p^{-k})\zeta(k) \quad \text{for} \quad k \leqslant 0,\ k \equiv n \pmod{p-1}.$$

Indeed if $n \not\equiv 1 \pmod{p-1}$ then $\zeta_{p,n}(s)$ will be analytic on $\mathbb{Z}_p$, and if $n \equiv 1 \pmod{p-1}$ then $\zeta_{p,n}(s)$ will be analytic apart from a simple pole at $s = 1$, of residue $1-(1/p)$. These results are due to Leopoldt and Kubota [1]. While these p-adic zeta-functions seem to have little interest in the simple case above, their generalizations to Dirichlet L-functions yield important algebraic information about the corresponding cyclotomic fields.

III

THE THEOREM OF HADAMARD AND DE LA VALLÉE POUSSIN, AND ITS CONSEQUENCES

3.1. As we have already observed, it follows from the formula

$$\zeta(s) = \prod_p \left(1-\frac{1}{p^s}\right)^{-1} \quad (\sigma > 1) \tag{3.1.1}$$

that $\zeta(s)$ has no zeros for $\sigma > 1$. For the purpose of prime-number theory, and indeed to determine the general nature of $\zeta(s)$, it is necessary to extend as far as possible this zero-free region.

It was conjectured by Riemann that *all the complex zeros of* $\zeta(s)$ *lie on the 'critical line'* $\sigma = \frac{1}{2}$. This conjecture, now known as the Riemann hypothesis, has never been either proved or disproved.

The problem of the zero-free region appears to be a question of extending the sphere of influence of the Euler product (3.1.1) beyond its actual region of convergence; for examples are known of functions which are extremely like the zeta-function in their representation by Dirichlet series, functional equation, and so on, but which have no Euler product, and for which the analogue of the Riemann hypothesis is false. In fact the deepest theorems on the distribution of the zeros of $\zeta(s)$ are obtained in the way suggested. But the problem of extending the sphere of influence of (3.1.1) to the left of $\sigma = 1$ in any effective way appears to be of extreme difficulty.

By (1.1.4)
$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \quad (\sigma > 1),$$
where $|\mu(n)| \leqslant 1$. Hence for σ near to 1

$$\left|\frac{1}{\zeta(s)}\right| \leqslant \sum_{n=1}^{\infty} \frac{1}{n^\sigma} = \zeta(\sigma) < \frac{A}{\sigma-1},$$

i.e.
$$|\zeta(s)| > A(\sigma-1).$$

Hence if $\zeta(s)$ has a zero on $\sigma = 1$ it must be a simple zero. But to prove that there cannot be even simple zeros, a much more subtle argument is required.

It was proved independently by Hadamard and de la Vallée Poussin in 1896 *that* $\zeta(s)$ *has no zeros on the line* $\sigma = 1$. Their methods are similar in principle, and they form the main topic of this chapter.

The main object of both these mathematicians was to prove the prime-number theorem, that as $x \to \infty$

$$\pi(x) \sim \frac{x}{\log x}.$$

This had previously been conjectured on empirical grounds. It was shown by arguments depending on the theory of functions of a complex variable that the prime-number theorem is a consequence of the Hadamard–de la Vallée Poussin theorem. The proof of the prime-number theorem so obtained was therefore not elementary.

An elementary proof of the prime-number theorem, i.e. a proof not depending on the theory of $\zeta(s)$ and complex function theory, has recently been obtained by A. Selberg and Erdős. Since the prime-number theorem implies the Hadamard–de la Vallée Poussin theorem, this leads to a new proof of the latter. However, the Selberg–Erdős method does not lead to such good estimations as the Hadamard–de la Vallée Poussin method, so that the latter is still of great interest.

3.2. Hadamard's argument is, roughly, as follows. We have for $\sigma > 1$

$$\log\zeta(s) = \sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{ms}} = \sum_p \frac{1}{p^s} + f(s), \tag{3.2.1}$$

where $f(s)$ is regular for $\sigma > \frac{1}{2}$. Since $\zeta(s)$ has a simple pole at $s = 1$, it follows in particular that, as $\sigma \to 1$ $(\sigma > 1)$,

$$\sum_p \frac{1}{p^\sigma} \sim \log\frac{1}{\sigma-1}. \tag{3.2.2}$$

Suppose now that $s = 1+it_0$ is a zero of $\zeta(s)$. Then if $s = \sigma+it_0$, as $\sigma \to 1$ $(\sigma > 1)$

$$\sum_p \frac{\cos(t_0 \log p)}{p^\sigma} = \log|\zeta(s)| - \mathbf{R}f(s) \sim \log(\sigma-1). \tag{3.2.3}$$

Comparing (3.2.2) and (3.2.3), we see that $\cos(t_0 \log p)$ must, in some sense, be approximately -1 for most values of p. But then $\cos(2t_0 \log p)$ is approximately 1 for most values of p, and

$$\log|\zeta(\sigma+2it_0)| \sim \sum_p \frac{\cos(2t_0\log p)}{p^\sigma} \sim \sum_p \frac{1}{p^\sigma} \sim \log\frac{1}{\sigma-1},$$

so that $1+2it_0$ is a pole of $\zeta(s)$. Since this is false, it follows that $\zeta(1+it_0) \neq 0$.

To put the argument in a rigorous form, let

$$S = \sum_p \frac{1}{p^\sigma}, \qquad P = \sum_p \frac{\cos(t_0\log p)}{p^\sigma}, \qquad Q = \sum_p \frac{\cos(2t_0 \log p)}{p^\sigma}.$$

Let S', P', Q' be the parts of these sums for which

$$(2k+1)\pi - \alpha \leqslant t_0 \log p \leqslant (2k+1)\pi + \alpha$$

for any integer k, and α fixed, $0 < \alpha < \frac{1}{2}\pi$. Let S'', etc., be the remainders. Let $\lambda = S'/S$.

If ϵ is any positive number, it follows from (3.2.2) and (3.2.3) that

$$P < -(1-\epsilon)S$$

if $\sigma - 1$ is small enough. But

$$P' \geqslant -S' = -\lambda S$$

and $$P'' \geqslant -S''\cos\alpha = -(1-\lambda)S\cos\alpha.$$

Hence $$-\{\lambda+(1-\lambda)\cos\alpha\}S < -(1-\epsilon)S,$$

i.e. $$(1-\lambda)(1-\cos\alpha) < \epsilon.$$

Hence $\lambda \to 1$ as $\sigma \to 1$.

Also $$Q' \geqslant S'\cos 2\alpha, \qquad Q'' \geqslant -S'',$$

so that $$Q \geqslant S(\lambda\cos 2\alpha - 1 + \lambda).$$

Since $\lambda \to 1$, $S \to \infty$, it follows that $Q \to \infty$ as $\sigma \to 1$. Hence $1+2it_0$ is a pole, and the result follows as before.

The following form of the argument was suggested by Dr. F. V. Atkinson. We have

$$\left\{\sum_p \frac{\cos(t_0\log p)}{p^\sigma}\right\}^2 = \left\{\sum_p \frac{\cos(t_0 \log p)}{p^{\frac{1}{2}\sigma}}\frac{1}{p^{\frac{1}{2}\sigma}}\right\}^2$$
$$\leqslant \sum_p \frac{\cos^2(t_0\log p)}{p^\sigma} \sum_p \frac{1}{p^\sigma}$$
$$= \frac{1}{2}\sum_p \frac{1+\cos(2t_0\log p)}{p^\sigma} \sum_p \frac{1}{p^\sigma},$$

i.e. $$P^2 \leqslant \tfrac{1}{2}(S+Q)S.$$

Suppose now that, for some t_0, $P \sim \log(\sigma-1)$. Since $S \sim \log\{1/(\sigma-1)\}$, it follows that, for a given ϵ and $\sigma - 1$ small enough,

$$(1-\epsilon)^2\log^2\frac{1}{\sigma-1} \leqslant \frac{1}{2}\left\{(1+\epsilon)\log\frac{1}{\sigma-1} + Q\right\}(1+\epsilon)\log\frac{1}{\sigma-1},$$

i.e. $$Q \geqslant \left\{\frac{2(1-\epsilon)^2}{1+\epsilon} - 1 - \epsilon\right\}\log\frac{1}{\sigma-1}.$$

Hence $Q \to \infty$, and this involves a contradiction as before.

3.3. In de la Vallée Poussin's argument a relation between $\zeta(\sigma+it)$ and $\zeta(\sigma+2it)$ is also fundamental; but the result is now deduced from the fact that

$$3+4\cos\phi+\cos 2\phi = 2(1+\cos\phi)^2 \geqslant 0 \tag{3.3.1}$$

for all values of ϕ.

We have
$$\zeta(s) = \exp\sum_p\sum_{m=1}^{\infty}\frac{1}{mp^{ms}},$$
and hence
$$|\zeta(s)| = \exp\sum_p\sum_{m=1}^{\infty}\frac{\cos(mt\log p)}{mp^{m\sigma}}.$$
Hence
$$\zeta^3(\sigma)|\zeta(\sigma+it)|^4|\zeta(\sigma+2it)|$$
$$= \exp\left\{\sum_p\sum_{m=1}^{\infty}\frac{3+4\cos(mt\log p)+\cos(2mt\log p)}{mp^{m\sigma}}\right\}. \tag{3.3.2}$$
Since every term in the last sum is positive or zero, it follows that
$$\zeta^3(\sigma)|\zeta(\sigma+it)|^4|\zeta(\sigma+2it)| \geqslant 1 \quad (\sigma > 1). \tag{3.3.3}$$

Now, keeping t fixed, let $\sigma \to 1$. Then
$$\zeta^3(\sigma) = O\{(\sigma-1)^{-3}\},$$
and, if $1+it$ is a zero of $\zeta(s)$, $\zeta(\sigma+it) = O(\sigma-1)$. Also $\zeta(\sigma+2it) = O(1)$, since $\zeta(s)$ is regular at $1+2it$. Hence the left-hand side of (3.3.3) is $O(\sigma-1)$, giving a contradiction. This proves the theorem.

There are other inequalities of the same type as (3.3.1), which can be used for the same purpose; e.g. from
$$5+8\cos\phi+4\cos 2\phi+\cos 3\phi = (1+\cos\phi)(1+2\cos\phi)^2 \geqslant 0 \tag{3.3.4}$$
we deduce that
$$\zeta^5(\sigma)|\zeta(\sigma+it)|^8|\zeta(\sigma+2it)|^4|\zeta(\sigma+3it)| \geqslant 1. \tag{3.3.5}$$
This, however, has no particular advantage over (3.3.3).

3.4. Another alternative proof has been given by Ingham.† This depends on the identity
$$\frac{\zeta^2(s)\zeta(s+ai)\zeta(s-ai)}{\zeta(2s)} = \sum_{n=1}^{\infty}\frac{|\sigma_{ai}(n)|^2}{n^s} \quad (\sigma > 1), \tag{3.4.1}$$
where a is any real number other than zero, and
$$\sigma_{ai}(n) = \sum_{d|n} d^{ai}.$$

† Ingham (3).

This is the particular case of (1.3.3) obtained by putting ai for a and $-ai$ for b.

Let σ_0 be the abscissa of convergence of the series (3.4.1). Then $\sigma_0 \leqslant 1$, and (3.4.1) is valid by analytic continuation for $\sigma > \sigma_0$, the function $f(s)$ on the left-hand side being of necessity regular in this half-plane. Also, since all the coefficients in the Dirichlet series are positive, the real point of the line of convergence, viz. $s = \sigma_0$, is a singularity of the function.

Suppose now that $1+ai$ is a zero of $\zeta(s)$. Then $1-ai$ is also a zero, and these two zeros cancel the double pole of $\zeta^2(s)$ at $s = 1$. Hence $f(s)$ is regular on the real axis as far as $s = -1$, where $\zeta(2s) = 0$; and so $\sigma_0 = -1$. This is easily seen in various ways to be impossible; for example (3.4.1) would then give $f(\frac{1}{2}) \geqslant 1$, whereas in fact $f(\frac{1}{2}) = 0$.

3.5. In the following sections we extend as far as we can the ideas suggested by § 3.1.

Since $\zeta(s)$ has a finite number of zeros in the rectangle $0 \leqslant \sigma \leqslant 1$, $0 \leqslant t \leqslant T$ and none of them lie on $\sigma = 1$, it follows that there is a rectangle $1-\delta \leqslant \sigma \leqslant 1$, $0 \leqslant t \leqslant T$, which is free from zeros. Here $\delta = \delta(T)$ may, for all we can prove, tend to zero as $T \to \infty$; but we can obtain a positive lower bound for $\delta(T)$ for each value of T.

Again, since $1/\zeta(s)$ is regular for $\sigma = 1$, $1 \leqslant t \leqslant T$, it has an upper bound in the interval, which is a function of T. We also investigate the behaviour of this upper bound as $t \to \infty$. There is, of course, a similar problem for $\zeta(s)$, in which the distribution of the zeros is not immediately involved. It is convenient to consider all these problems together, and we begin with $\zeta(s)$.

Theorem 3.5. *We have*
$$\zeta(s) = O(\log t) \tag{3.5.1}$$
uniformly in the region
$$1-\frac{A}{\log t} \leqslant \sigma \leqslant 2 \quad (t > t_0),$$
where A is any positive constant. In particular
$$\zeta(1+it) = O(\log t). \tag{3.5.2}$$

In (2.1.3), take $\sigma > 1$, $a = N$, and make $b \to \infty$. We obtain
$$\zeta(s)-\sum_{n=1}^{N}\frac{1}{n^s} = s\int_N^{\infty}\frac{[x]-x+\frac{1}{2}}{x^{s+1}}\,dx+\frac{N^{1-s}}{s-1}-\tfrac{1}{2}N^{-s}, \tag{3.5.3}$$

the result holding by analytic continuation for $\sigma > 0$. Hence for $\sigma > 0$, $t > 1$,

$$\zeta(s)-\sum_{n=1}^{N}\frac{1}{n^s} = O\left(t\int_N^\infty \frac{dx}{x^{\sigma+1}}\right)+O\left(\frac{N^{1-\sigma}}{t}\right)+O(N^{-\sigma})$$

$$= O\left(\frac{t}{\sigma N^\sigma}\right)+O\left(\frac{N^{1-\sigma}}{t}\right)+O(N^{-\sigma}). \qquad (3.5.4)$$

In the region considered, if $n \leqslant t$,

$$|n^{-s}| = n^{-\sigma} = e^{-\sigma\log n} \leqslant \exp\left\{-\left(1-\frac{A}{\log t}\right)\log n\right\} \leqslant n^{-1}e^{A}.$$

Hence, taking $N = [t]$,

$$\zeta(s) = \sum_{n=1}^{N} O\left(\frac{1}{n}\right)+O\left(\frac{t}{N}\right)+O\left(\frac{1}{t}\right)+O\left(\frac{1}{N}\right)$$

$$= O(\log N)+O(1) = O(\log t).$$

This result will be improved later (Theorems 5.16, 6.14), but at the cost of far more difficult proofs.

It is also easy to see that

$$\zeta'(s) = O(\log^2 t) \qquad (3.5.5)$$

in the above region. For, differentiating (3.5.3),

$$\zeta'(s) = -\sum_{n=2}^{N}\frac{\log n}{n^s}+\int_N^\infty \frac{[x]-x+\frac{1}{2}}{x^{s+1}}(1-s\log x)\,dx-$$

$$-\frac{N^{1-s}\log N}{s-1}-\frac{N^{1-s}}{(s-1)^2}+\tfrac{1}{2}N^{-s}\log N,$$

and a similar argument holds, with an extra factor $\log t$ on the right-hand side. Similarly for higher derivatives of $\zeta(s)$.

We may note in passing that (3.5.3) shows the behaviour of the Dirichlet series (1.1.1) for $\sigma \leqslant 1$. If we take $\sigma = 1$, $t \neq 0$, we obtain

$$\zeta(1+it)-\sum_{1}^{N}\frac{1}{n^{1+it}} = (1+it)\int_N^\infty \frac{[x]-x+\frac{1}{2}}{x^{2+it}}\,dx+\frac{N^{-it}}{it}-\tfrac{1}{2}N^{-1-it},$$

which oscillates finitely as $N \to \infty$. For $\sigma < 1$ the series, of course, diverges (oscillates infinitely).

3.6. Inequalities for $1/\zeta(s)$, $\zeta'(s)/\zeta(s)$, and $\log\zeta(s)$. Inequalities of this type in the neighbourhood of $\sigma = 1$ can now be obtained by a slight elaboration of the argument of § 3.3. We have for $\sigma > 1$

$$\left|\frac{1}{\zeta(\sigma+it)}\right| \leqslant \{\zeta(\sigma)\}^{\frac{3}{4}}|\zeta(\sigma+2it)|^{\frac{1}{4}} = O\left\{\frac{\log^{\frac{1}{4}}t}{(\sigma-1)^{\frac{3}{4}}}\right\}. \qquad (3.6.1)$$

Also $$\zeta(1+it)-\zeta(\sigma+it) = -\int_1^\sigma \zeta'(u+it)\,du = O\{(\sigma-1)\log^2 t\} \qquad (3.6.2)$$

for $\sigma > 1-A/\log t$. Hence

$$|\zeta(1+it)| > A_1\frac{(\sigma-1)^{\frac{3}{4}}}{\log^{\frac{1}{4}}t}-A_2(\sigma-1)\log^2 t.$$

The two terms on the right are of the same order if $\sigma-1 = \log^{-9}t$. Hence, taking $\sigma-1 = A_3\log^{-9}t$, where A_3 is sufficiently small,

$$|\zeta(1+it)| > A\log^{-7}t. \qquad (3.6.3)$$

Next (3.6.2) and (3.6.3) together give, for $1-A\log t < \sigma < 1$,

$$|\zeta(\sigma+it)| > A\log^{-7}t-A(1-\sigma)\log^2 t, \qquad (3.6.4)$$

and the right-hand side is positive if $1-\sigma < A\log^{-9}t$. *Hence $\zeta(s)$ has no zeros in the region $\sigma > 1-A\log^{-9}t$*, and in fact, by (3.6.4),

$$\frac{1}{\zeta(s)} = O(\log^7 t) \qquad (3.6.5)$$

in this region.

Hence also, by (3.5.5),

$$\frac{\zeta'(s)}{\zeta(s)} = O(\log^9 t), \qquad (3.6.6)$$

and $$\log\zeta(s) = \int_2^\sigma \frac{\zeta'(u+it)}{\zeta(u+it)}\,du+\log\zeta(2+it) = O(\log^9 t), \qquad (3.6.7)$$

both for $\sigma > 1-A\log^{-9}t$.

We shall see later that all these results can be improved, but they are sufficient for some purposes.

3.7. The Prime-number Theorem. *Let $\pi(x)$ denote the number of primes not exceeding x. Then as $x \to \infty$*

$$\pi(x) \sim \frac{x}{\log x}. \qquad (3.7.1)$$

The investigation of $\pi(x)$ was, of course, the original purpose for which $\zeta(s)$ was studied. It is not our purpose to pursue this side of the theory farther than is necessary, but it is convenient to insert here a proof of the main theorem on $\pi(x)$.

We have proved in (1.1.3) that, if $\sigma > 1$,

$$\log\zeta(s) = s\int_2^\infty \frac{\pi(x)}{x(x^s-1)}\,dx.$$

We want an explicit formula for $\pi(x)$, i.e. we want to invert the above integral formula. We can reduce this to a case of Mellin's inversion formula as follows. Let

$$\omega(s) = \int_2^\infty \frac{\pi(x)}{x^{s+1}(x^s-1)}\,dx.$$

Then

$$\frac{\log\zeta(s)}{s} - \omega(s) = \int_2^\infty \frac{\pi(x)}{x^{s+1}}\,dx. \tag{3.7.2}$$

This is of the Mellin form, and $\omega(s)$ is a comparatively trivial function; in fact since $\pi(x) \leqslant x$ the integral for $\omega(s)$ converges uniformly for $\sigma \geqslant \frac{1}{2}+\delta$, by comparison with

$$\int_2^\infty \frac{dx}{x^{\frac{1}{2}+\delta}(x^{\frac{1}{2}+\delta}-1)}.$$

Hence $\omega(s)$ is regular and bounded for $\sigma \geqslant \frac{1}{2}+\delta$. Similarly so is $\omega'(s)$, since

$$\omega'(s) = \int_2^\infty \pi(x)\log x\,\frac{1-2x^s}{x^{s+1}(x^s-1)^2}\,dx.$$

We could now use Mellin's inversion formula, but the resulting formula is not easily manageable. We therefore modify (3.7.2) as follows. Differentiating with respect to s,

$$-\frac{\zeta'(s)}{s\zeta(s)} + \frac{\log\zeta(s)}{s^2} + \omega'(s) = \int_2^\infty \frac{\pi(x)\log x}{x^{s+1}}\,dx.$$

Denote the left-hand side by $\phi(s)$, and let

$$g(x) = \int_0^x \frac{\pi(u)\log u}{u}\,du, \qquad h(x) = \int_0^x \frac{g(u)}{u}\,du,$$

$\pi(x)$, $g(x)$, and $h(x)$ being zero for $x < 2$. Then, integrating by parts,

$$\phi(s) = \int_0^\infty g'(x)x^{-s}\,dx = s\int_0^\infty g(x)x^{-s-1}\,dx$$

$$= s\int_0^\infty h'(x)x^{-s}\,dx = s^2\int_0^\infty h(x)x^{-s-1}\,dx \quad (\sigma > 1),$$

or

$$\frac{\phi(1-s)}{(1-s)^2} = \int_0^\infty \frac{h(x)}{x}x^{s-1}\,dx.$$

Now $h(x)$ is continuous and of bounded variation in any finite interval; and, since $\pi(x) \leqslant x$, it follows that, for $x > 1$, $g(x) \leqslant x\log x$, and $h(x) \leqslant x\log x$. Hence $h(x)x^{k-2}$ is absolutely integrable over $(0,\infty)$ if $k < 0$. Hence

$$\frac{h(x)}{x} = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \frac{\phi(1-s)}{(1-s)^2}x^{-s}\,ds \quad (k < 0),$$

or

$$h(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\phi(s)}{s^2}x^s\,ds \quad (c > 1).$$

The integral on the right is absolutely convergent, since by (3.6.6) and (3.6.7) $\phi(s)$ is bounded for $\sigma \geqslant 1$, except in the neighbourhood of $s = 1$.

In the neighbourhood of $s = 1$

$$\phi(s) = \frac{1}{s-1} + \log\frac{1}{s-1} + \ldots,$$

and we may write

$$\phi(s) = \frac{1}{s-1} + \psi(s),$$

where $\psi(s)$ is bounded for $\sigma \geqslant 1$, $|s-1| \geqslant 1$, and $\psi(s)$ has a logarithmic infinity as $s \to 1$. Now

$$h(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^s}{(s-1)s^2}\,ds + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\psi(s)}{s^2}x^s\,ds.$$

The first term is equal to the sum of the residues on the left of the line $\mathbf{R}(s) = c$, and so is

$$x - \log x - 1.$$

In the other term we may put $c = 1$, i.e. apply Cauchy's theorem to the rectangle $(1 \pm iT,\ c \pm iT)$, with an indentation of radius ϵ round $s = 1$, and make $T \to \infty$, $\epsilon \to 0$. Hence

$$h(x) = x - \log x - 1 + \frac{x}{2\pi}\int_{-\infty}^{\infty} \frac{\psi(1+it)}{(1+it)^2}x^{it}\,dt.$$

The last integral tends to zero as $x \to \infty$, by the extension to Fourier integrals of the Riemann–Lebesgue theorem.† Hence

$$h(x) \sim x. \tag{3.7.3}$$

† See my *Introduction to the Theory of Fourier Integrals*, Theorem 1.

To get back to $\pi(x)$ we now use the following lemma:
Let $f(x)$ be be positive non-decreasing, and as $x \to \infty$ let

$$\int_1^x \frac{f(u)}{u}\,du \sim x.$$

Then $$f(x) \sim x.$$

If δ is a given positive number,

$$(1-\delta)x < \int_1^x \frac{f(t)}{t}\,dt < (1+\delta)x \quad (x > x_0(\delta)).$$

Hence for any positive ϵ

$$\int_x^{x(1+\epsilon)} \frac{f(u)}{u}\,du = \int_1^{x(1+\epsilon)} \frac{f(u)}{u}\,du - \int_1^x \frac{f(u)}{u}\,du$$
$$< (1+\delta)(1+\epsilon)x-(1-\delta)x$$
$$= (2\delta+\epsilon+\delta\epsilon)x.$$

But, since $f(x)$ is non-decreasing,

$$\int_x^{x(1+\epsilon)} \frac{f(u)}{u}\,du \geqslant f(x)\int_x^{x(1+\epsilon)} \frac{du}{u} > f(x)\int_x^{x(1+\epsilon)} \frac{du}{x(1+\epsilon)} = \frac{\epsilon}{1+\epsilon}f(x).$$

Hence $$f(x) < x(1+\epsilon)\left(1+\delta+\frac{2\delta}{\epsilon}\right).$$

Taking, for example, $\epsilon = \sqrt{\delta}$, it follows that

$$\overline{\lim}\,\frac{f(x)}{x} \leqslant 1.$$

Similarly, by considering $$\int_{x(1-\epsilon)}^x \frac{f(u)}{u}\,du,$$

we obtain $$\underline{\lim}\,\frac{f(x)}{x} \geqslant 1,$$

and the lemma follows.

Applying the lemma twice, we deduce from (3.7.3) that

$$g(x) \sim x,$$

and hence that $$\pi(x)\log x \sim x.$$

3.8. THEOREM 3.8. *There is a constant A such that $\zeta(s)$ is not zero for*

$$\sigma \geqslant 1-\frac{A}{\log t} \quad (t > t_0).$$

We have for $\sigma > 1$

$$-\mathbf{R}\left\{\frac{\zeta'(s)}{\zeta(s)}\right\} = \sum_{p,m} \frac{\log p}{p^{m\sigma}}\cos(mt\log p). \tag{3.8.1}$$

Hence, for $\sigma > 1$ and any real γ,

$$-3\frac{\zeta'(\sigma)}{\zeta(\sigma)} - 4\mathbf{R}\frac{\zeta'(\sigma+i\gamma)}{\zeta(\sigma+i\gamma)} - \mathbf{R}\frac{\zeta'(\sigma+2i\gamma)}{\zeta(\sigma+2i\gamma)}$$
$$= \sum_{p,m} \frac{\log p}{p^{m\sigma}}\{3+4\cos(m\gamma\log p)+\cos(2m\gamma\log p)\} \geqslant 0. \tag{3.8.2}$$

Now $$-\frac{\zeta'(\sigma)}{\zeta(\sigma)} < \frac{1}{\sigma-1}+O(1). \tag{3.8.3}$$

Also, by (2.12.7),

$$-\frac{\zeta'(s)}{\zeta(s)} = O(\log t) - \sum_\rho \left(\frac{1}{s-\rho}+\frac{1}{\rho}\right), \tag{3.8.4}$$

where $\rho = \beta+i\gamma$ runs through complex zeros of $\zeta(s)$. Hence

$$-\mathbf{R}\left\{\frac{\zeta'(s)}{\zeta(s)}\right\} = O(\log t) - \sum_\rho \left\{\frac{\sigma-\beta}{(\sigma-\beta)^2+(t-\gamma)^2}+\frac{\beta}{\beta^2+\gamma^2}\right\}.$$

Since every term in the last sum is positive, it follows that

$$-\mathbf{R}\left\{\frac{\zeta'(s)}{\zeta(s)}\right\} < O(\log t), \tag{3.8.5}$$

and also, if $\beta+i\gamma$ is a particular zero of $\zeta(s)$, that

$$-\mathbf{R}\left\{\frac{\zeta'(\sigma+i\gamma)}{\zeta(\sigma+i\gamma)}\right\} < O(\log\gamma) - \frac{1}{\sigma-\beta}. \tag{3.8.6}$$

From (3.8.2), (3.8.3), (3.8.5), (3.8.6) we obtain

$$\frac{3}{\sigma-1}-\frac{4}{\sigma-\beta}+O(\log\gamma) \geqslant 0,$$

or say $$\frac{3}{\sigma-1}-\frac{4}{\sigma-\beta} \geqslant -A_1\log\gamma.$$

Solving for β, we obtain

$$1-\beta \geqslant \frac{1-(\sigma-1)A_1\log\gamma}{3/(\sigma-1)+A_1\log\gamma}.$$

The right-hand side is positive if $\sigma-1 = \frac{1}{2}A_1/\log\gamma$, and then

$$1-\beta \geqslant \frac{A_2}{\log\gamma},$$

the required result.

3.9. There is an alternative method, due to Landau,† of obtaining results of this kind, in which the analytic character of $\zeta(s)$ for $\sigma \leqslant 0$ need not be known. It depends on the following lemmas.

LEMMA α. *If $f(s)$ is regular, and*

$$\left|\frac{f(s)}{f(s_0)}\right| < e^M \quad (M > 1)$$

in the circle $|s-s_0| \leqslant r$, then

$$\left|\frac{f'(s)}{f(s)} - \sum_\rho \frac{1}{s-\rho}\right| < \frac{AM}{r} \quad (|s-s_0| \leqslant \tfrac{1}{4}r),$$

where ρ runs through the zeros of $f(s)$ such that $|\rho-s_0| \leqslant \frac{1}{2}r$.

The function $g(s) = f(s)\prod_\rho (s-\rho)^{-1}$ is regular for $|s-s_0| \leqslant r$, and not zero for $|s-s_0| \leqslant \frac{1}{2}r$. On $|s-s_0| = r$, $|s-\rho| \geqslant \frac{1}{2}r \geqslant |s_0-\rho|$, so that

$$\left|\frac{g(s)}{g(s_0)}\right| = \left|\frac{f(s)}{f(s_0)}\prod\left(\frac{s_0-\rho}{s-\rho}\right)\right| \leqslant \left|\frac{f(s)}{f(s_0)}\right| < e^M.$$

This inequality therefore holds inside the circle also. Hence the function

$$h(s) = \log\left\{\frac{g(s)}{g(s_0)}\right\},$$

where the logarithm is zero at $s = s_0$, is regular for $|s-s_0| \leqslant \frac{1}{2}r$, and

$$h(s_0) = 0, \qquad \mathbf{R}\{h(s)\} < M.$$

Hence by the Borel–Carathéodory theorem‡

$$|h(s)| < AM \quad (|s-s_0| \leqslant \tfrac{3}{8}r), \tag{3.9.1}$$

and so, for $|s-s_0| \leqslant \frac{1}{4}r$,

$$|h'(s)| = \left|\frac{1}{2\pi i}\int_{|z-s|=\frac{1}{8}r} \frac{h(z)}{(z-s)^2}\,dz\right| < \frac{AM}{r}.$$

This gives the result stated.

LEMMA β. *If $f(s)$ satisfies the conditions of the previous lemma, and has no zeros in the right-hand half of the circle $|s-s_0| \leqslant r$, then*

$$-\mathbf{R}\left\{\frac{f'(s_0)}{f(s_0)}\right\} < \frac{AM}{r};$$

while if $f(s)$ has a zero ρ_0 between $s_0-\frac{1}{2}r$ and s_0, then

$$-\mathbf{R}\left\{\frac{f'(s_0)}{f(s_0)}\right\} < \frac{AM}{r} - \frac{1}{s_0-\rho_0}.$$

† Landau (14). ‡ Titchmarsh, *Theory of Functions*, § 5.5.

Lemma α gives

$$-\mathbf{R}\left\{\frac{f'(s_0)}{f(s_0)}\right\} < \frac{AM}{r} - \sum \mathbf{R}\frac{1}{s_0-\rho},$$

and since $\mathbf{R}\{1/(s_0-\rho)\} \geqslant 0$ for every ρ, both results follow at once.

LEMMA γ. *Let $f(s)$ satisfy the conditions of Lemma α, and let*

$$\left|\frac{f'(s_0)}{f(s_0)}\right| < \frac{M}{r}.$$

Suppose also that $f(s) \neq 0$ in the part $\sigma \geqslant \sigma_0 - 2r'$ of the circle $|s-s_0| \leqslant r$, where $0 < r' < \frac{1}{4}r$. Then

$$\left|\frac{f'(s)}{f(s)}\right| < A\frac{M}{r} \quad (|s-s_0| \leqslant r').$$

Lemma α now gives

$$-\mathbf{R}\left\{\frac{f'(s)}{f(s)}\right\} < A\frac{M}{r} - \sum \mathbf{R}\frac{1}{s-\rho} < A\frac{M}{r}$$

for all s in $|s-s_0| \leqslant \frac{1}{4}r$, $\sigma \geqslant \sigma_0 - 2r'$, each term of the sum being positive in this region. The result then follows on applying the Borel–Carathéodory theorem to the function $-f'(s)/f(s)$ and the circles $|s-s_0| = 2r'$, $|s-s_0| = r'$.

3.10. We can now prove the following general theorem, which we shall apply later with special forms of the functions $\theta(t)$ and $\phi(t)$.

THEOREM 3.10. *Let*

$$\zeta(s) = O(e^{\phi(t)})$$

as $t \to \infty$ in the region

$$1-\theta(t) \leqslant \sigma \leqslant 2 \quad (t \geqslant 0),$$

where $\phi(t)$ and $1/\theta(t)$ are positive non-decreasing functions of t for $t \geqslant 0$, such that $\theta(t) \leqslant 1$, $\phi(t) \to \infty$, and

$$\frac{\phi(t)}{\theta(t)} = o(e^{\phi(t)}). \tag{3.10.1}$$

Then there is a constant A_1 such that $\zeta(s)$ has no zeros in the region

$$\sigma \geqslant 1 - A_1\frac{\theta(2t+1)}{\phi(2t+1)}. \tag{3.10.2}$$

Let $\beta+i\gamma$ be a zero of $\zeta(s)$ in the upper half-plane. Let

$$1+e^{-\phi(2\gamma+1)} \leqslant \sigma_0 \leqslant 2,$$

$$s_0 = \sigma_0 + i\gamma, \qquad s_0' = \sigma_0 + 2i\gamma, \qquad r = \theta(2\gamma+1).$$

Then the circles $|s-s_0| \leqslant r$, $|s-s_0'| \leqslant r$ both lie in the region

$$\sigma \geqslant 1-\theta(t).$$

Now
$$\left|\frac{1}{\zeta(s_0)}\right| < \frac{A}{\sigma_0-1} < Ae^{\phi(2\gamma+1)},$$

and similarly for s_0'. Hence there is a constant A_2 such that

$$\left|\frac{\zeta(s)}{\zeta(s_0)}\right| < e^{A_2\phi(2\gamma+1)}, \qquad \left|\frac{\zeta(s)}{\zeta(s_0')}\right| < e^{A_2\phi(2\gamma+1)},$$

in the circles $|s-s_0| \leqslant r$, $|s-s_0'| \leqslant r$ respectively. We can therefore apply Lemma β with $M = A_2\phi(2\gamma+1)$. We obtain

$$-\mathbf{R}\left\{\frac{\zeta'(\sigma_0+2i\gamma)}{\zeta(\sigma_0+2i\gamma)}\right\} < \frac{A_3\phi(2\gamma+1)}{\theta(2\gamma+1)}, \tag{3.10.3}$$

and, if
$$\beta > \sigma_0-\tfrac{1}{2}r, \tag{3.10.4}$$

$$-\mathbf{R}\left\{\frac{\zeta'(\sigma_0+i\gamma)}{\zeta(\sigma_0+i\gamma)}\right\} < \frac{A_3\phi(2\gamma+1)}{\theta(2\gamma+1)} - \frac{1}{\sigma_0-\beta}. \tag{3.10.5}$$

Also as $\sigma_0 \to 1$
$$-\frac{\zeta'(\sigma_0)}{\zeta(\sigma_0)} \sim \frac{1}{\sigma_0-1}.$$

Hence
$$-\frac{\zeta'(\sigma_0)}{\zeta(\sigma_0)} < \frac{a}{\sigma_0-1}, \tag{3.10.6}$$

where a can be made as near 1 as we please by choice of σ_0.

Now (3.8.2), (3.10.3), (3.10.5), and (3.10.6) give

$$\frac{3a}{\sigma_0-1} + \frac{5A_3\phi(2\gamma+1)}{\theta(2\gamma+1)} - \frac{4}{\sigma_0-\beta} \geqslant 0,$$

$$\sigma_0-\beta \geqslant \left\{\frac{3a}{4(\sigma_0-1)} + \frac{5A_3}{4}\frac{\phi(2\gamma+1)}{\theta(2\gamma+1)}\right\}^{-1},$$

$$1-\beta \geqslant \left\{\frac{3a}{4(\sigma_0-1)} + \frac{5A_3}{4}\frac{\phi(2\gamma+1)}{\theta(2\gamma+1)}\right\}^{-1} - (\sigma_0-1)$$

$$= \left\{1-\frac{3a}{4}-\frac{5A_3}{4}\frac{\phi(2\gamma+1)(\sigma_0-1)}{\theta(2\gamma+1)}\right\}\Big/\left\{\frac{3a}{4(\sigma_0-1)} + \frac{5A_3}{4}\frac{\phi(2\gamma+1)}{\theta(2\gamma+1)}\right\}.$$

To make the numerator positive, take $a = \frac{5}{4}$, and

$$\sigma_0-1 = \frac{1}{40A_3}\frac{\theta(2\gamma+1)}{\phi(2\gamma+1)},$$

this being consistent with the previous conditions, by (3.10.1), if γ is large enough. It follows that

$$1-\beta \geqslant \frac{\theta(2\gamma+1)}{1240A_3\phi(2\gamma+1)}$$

as required. If (3.10.4) is not satisfied,

$$\beta \leqslant \sigma_0-\tfrac{1}{2}r = 1+\frac{1}{40A_3}\frac{\theta(2\gamma+1)}{\phi(2\gamma+1)} - \tfrac{1}{2}\theta(2\gamma+1),$$

which also leads to (3.10.2). This proves the theorem.

In particular, we can take $\theta(t) = \frac{1}{2}$, $\phi(t) = \log(t+2)$. This gives a new proof of Theorem 3.8.

3.11. THEOREM 3.11. *Under the hypotheses of Theorem* 3.10 *we have*

$$\frac{\zeta'(s)}{\zeta(s)} = O\left\{\frac{\phi(2t+3)}{\theta(2t+3)}\right\}, \qquad \frac{1}{\zeta(s)} = O\left\{\frac{\phi(2t+3)}{\theta(2t+3)}\right\} \tag{3.11.1), (3.11.2}$$

uniformly for
$$\sigma \geqslant 1-\frac{A_1}{4}\frac{\theta(2t+3)}{\phi(2t+3)}. \tag{3.11.3}$$

In particular

$$\frac{\zeta(1+it)}{\zeta'(1+it)} = O\left\{\frac{\phi(2t+3)}{\theta(2t+3)}\right\}, \qquad \frac{1}{\zeta(1+it)} = O\left\{\frac{\phi(2t+3)}{\theta(2t+3)}\right\}. \tag{3.11.4), (3.11.5}$$

We apply Lemma γ, with

$$s_0 = 1+\frac{A_1}{2}\frac{\theta(2t_0+3)}{\phi(2t_0+3)}+it_0, \qquad r = \theta(2t_0+3).$$

In the circle $|s-s_0| \leqslant r$

$$\frac{\zeta(s)}{\zeta(s_0)} = O\left\{\frac{e^{\phi(t)}}{\sigma_0-1}\right\} = O\left\{\frac{\phi(2t_0+3)}{\theta(2t_0+3)}e^{\phi(t_0+1)}\right\} = O\{e^{A\phi(2t_0+3)}\},$$

and
$$\frac{\zeta'(s_0)}{\zeta(s_0)} = O\left\{\frac{1}{\sigma_0-1}\right\} = O\left\{\frac{\phi(2t_0+3)}{\theta(2t_0+3)}\right\} = O\left\{\frac{\phi(2t_0+3)}{r}\right\}.$$

We can therefore take $M = A\phi(2t_0+3)$. Also, by the previous theorem, $\zeta(s)$ has no zeros for

$$t \leqslant t_0+1, \qquad \sigma \geqslant 1-A_1\frac{\theta\{2(t_0+1)+1\}}{\phi\{2(t_0+1)+1\}} = 1-A_1\frac{\theta(2t_0+3)}{\phi(2t_0+3)}.$$

Hence we can take
$$2r' = \frac{3A_1}{2}\frac{\theta(2t_0+3)}{\phi(2t_0+3)}.$$

Hence
$$\frac{\zeta'(s)}{\zeta(s)} = O\left\{\frac{\phi(2t_0+3)}{\theta(2t_0+3)}\right\}$$

for
$$|s-s_0| \leqslant \frac{3A_1}{4}\frac{\theta(2t_0+3)}{\phi(2t_0+3)},$$

and in particular for

$$t = t_0, \qquad \sigma \geqslant 1-\frac{A_1}{4}\frac{\theta(2t_0+3)}{\phi(2t_0+3)}.$$

This is (3.11.1), with t_0 instead of t.

Also, if $$1-\frac{A_1}{4}\frac{\theta(2t+3)}{\phi(2t+3)} \leqslant \sigma \leqslant 1+\frac{\theta(2t+3)}{\phi(2t+3)}, \tag{3.11.6}$$

$$\begin{aligned}\log\frac{1}{|\zeta(s)|} &= -\mathbf{R}\log\zeta(s)\\ &= -\mathbf{R}\log\zeta\left\{1+\frac{\theta(2t+3)}{\phi(2t+3)}+it\right\}+\int\limits_{\sigma}^{1+\frac{\theta(2t+3)}{\phi(2t+3)}}\mathbf{R}\frac{\zeta'(u+it)}{\zeta(u+it)}\,du\\ &\leqslant \log\zeta\left\{1+\frac{\theta(2t+3)}{\phi(2t+3)}\right\}+\int\limits_{\sigma}^{1+\frac{\theta(2t+3)}{\phi(2t+3)}} O\left\{\frac{\phi(2t+3)}{\theta(2t+3)}\right\}du\\ &< \log\frac{A\phi(2t+3)}{\theta(2t+3)}+O(1).\end{aligned}$$

Hence (3.11.2) follows if σ is in the range (3.11.6); and for larger σ it is trivial.

Since we may take $\theta(t)=\frac{1}{2}$, $\phi(t)=\log(t+2)$, it follows that

$$\frac{\zeta'(s)}{\zeta(s)}=O(\log t),\qquad \frac{1}{\zeta(s)}=O(\log t) \qquad (3.11.7),\ (3.11.8)$$

in a region $\sigma \geqslant 1-A/\log t$; and in particular

$$\frac{\zeta'(1+it)}{\zeta(1+it)}=O(\log t),\qquad \frac{1}{\zeta(1+it)}=O(\log t). \qquad (3.11.9),\ (3.11.10)$$

3.12. For the next theorem we require the following lemma.

LEMMA 3.12. *Let* $$f(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}\quad(\sigma>1),$$

where $a_n=O\{\psi(n)\}$, $\psi(n)$ *being non-decreasing, and*

$$\sum_{n=1}^{\infty}\frac{|a_n|}{n^\sigma}=O\left\{\frac{1}{(\sigma-1)^\alpha}\right\}$$

as $\sigma\to 1$. *Then if* $c>0$, $\sigma+c>1$, x *is not an integer, and* N *is the integer nearest to* x,

$$\sum_{n<x}\frac{a_n}{n^s}=\frac{1}{2\pi i}\int\limits_{c-iT}^{c+iT}f(s+w)\frac{x^w}{w}\,dw+O\left\{\frac{x^c}{T(\sigma+c-1)^\alpha}\right\}+ \\ +O\left\{\frac{\psi(2x)x^{1-\sigma}\log x}{T}\right\}+O\left\{\frac{\psi(N)x^{1-\sigma}}{T|x-N|}\right\}. \tag{3.12.1}$$

If x *is an integer, the corresponding result is*

$$\sum_{n=1}^{x-1}\frac{a_n}{n^s}+\frac{a_x}{2x^s}=\frac{1}{2\pi i}\int\limits_{c-iT}^{c+iT}f(s+w)\frac{x^w}{w}\,dw+O\left\{\frac{x^c}{T(\sigma+c-1)^\alpha}\right\}+ \\ +O\left\{\frac{\psi(2x)x^{1-\sigma}\log x}{T}\right\}+O\left\{\frac{\psi(x)x^{-\sigma}}{T}\right\}. \tag{3.12.2}$$

Suppose first that x is not an integer. If $n<x$, the calculus of residues gives

$$\frac{1}{2\pi i}\left(\int\limits_{-\infty-iT}^{c-iT}+\int\limits_{c-iT}^{c+iT}+\int\limits_{c+iT}^{-\infty+iT}\right)\left(\frac{x}{n}\right)^w\frac{dw}{w}=1.$$

Now

$$\begin{aligned}\int\limits_{-\infty+iT}^{c+iT}\left(\frac{x}{n}\right)^w\frac{dw}{w}&=\left[\frac{(x/n)^w}{w\log x/n}\right]_{-\infty+iT}^{c+iT}+\frac{1}{\log x/n}\int\limits_{-\infty+iT}^{c+iT}\left(\frac{x}{n}\right)^w\frac{dw}{w^2}\\ &=O\left\{\frac{(x/n)^c}{T\log x/n}\right\}+O\left\{\frac{(x/n)^c}{\log x/n}\int\limits_{-\infty}^{\infty}\frac{du}{u^2+T^2}\right\}\\ &=O\left\{\frac{(x/n)^c}{T\log x/n}\right\},\end{aligned}$$

and similarly for the integral over $(-\infty-iT, c-iT)$. Hence

$$\frac{1}{2\pi i}\int\limits_{c-iT}^{c+iT}\left(\frac{x}{n}\right)^w\frac{dw}{w}=1+O\left\{\frac{(x/n)^c}{T\log x/n}\right\}.$$

If $n>x$ we argue similarly with $-\infty$ replaced by $+\infty$, and there is no residue term. We therefore obtain a similar result without the term 1.

Multiplying by $a_n n^{-s}$ and summing,

$$\frac{1}{2\pi i}\int\limits_{c-iT}^{c+iT}f(s+w)\frac{x^w}{w}\,dw=\sum_{n<x}\frac{a_n}{n^s}+O\left\{\frac{x^c}{T}\sum_{n=1}^{\infty}\frac{|a_n|}{n^{\sigma+c}|\log x/n|}\right\}.$$

If $n<\frac{1}{2}x$ or $n>2x$, $|\log x/n|>A$, and these parts of the sum are

$$O\left(\sum_{n=1}^{\infty}\frac{|a_n|}{n^{\sigma+c}}\right)=O\left\{\frac{1}{(\sigma+c-1)^\alpha}\right\}.$$

If $N<n\leqslant 2x$, let $n=N+r$. Then

$$\log\frac{n}{x}\geqslant\log\frac{N+r}{N+\frac{1}{2}}>\frac{Ar}{N}>\frac{Ar}{x}.$$

Hence this part of the sum is

$$O\left\{\psi(2x)x^{1-\sigma-c}\sum_{1\leqslant r\leqslant x}\frac{1}{r}\right\} = O\{\psi(2x)x^{1-\sigma-c}\log x\}.$$

A similar argument applies to the terms with $\frac{1}{2}x \leqslant n < N$. Finally

$$\frac{|a_N|}{N^{\sigma+c}|\log x/N|} = O\left\{\frac{\psi(N)}{N^{\sigma+c}\log\{1+(x-N)/N\}}\right\} = O\left\{\frac{\psi(N)x^{1-\sigma-c}}{|x-N|}\right\}.$$

Hence (3.12.1) follows.

If x is an integer, all goes as before except for the term

$$\frac{a_x}{2\pi i x^s}\int_{c-iT}^{c+iT}\frac{dw}{w} = \frac{a_x}{2\pi i x^s}\log\frac{c+iT}{c-iT} = \frac{a_x}{2\pi i x^s}\left\{i\pi+O\left(\frac{1}{T}\right)\right\}.$$

Hence (3.12.2) follows.

3.13. THEOREM 3.13. *We have*

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}$$

at all points of the line $\sigma = 1$.

Take $a_n = \mu(n)$, $\alpha = 1$, $\sigma = 1$, in the lemma, and let x be half an odd integer. We obtain

$$\sum_{n<x}\frac{\mu(n)}{n^s} = \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{1}{\zeta(s+w)}\frac{x^w}{w}\,dw + O\left(\frac{x^c}{Tc}\right)+O\left(\frac{\log x}{T}\right).$$

The theorem of residues gives

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{1}{\zeta(s+w)}\frac{x^w}{w}\,dw = \frac{1}{\zeta(s)}+\frac{1}{2\pi i}\left(\int_{c-iT}^{-\delta-iT}+\int_{-\delta-iT}^{-\delta+iT}+\int_{-\delta+iT}^{c+iT}\right)$$

if δ is so small that $\zeta(s+w)$ has no zeros for

$$\mathbf{R}(w) \geqslant -\delta, \quad |\mathbf{I}(s+w)| \leqslant |t|+T.$$

By § 3.6 we can take $\delta = A\log^{-9}T$. Then

$$\int_{-\delta-iT}^{-\delta+iT}\frac{1}{\zeta(s+w)}\frac{x^w}{w}\,dw = O\left(x^{-\delta}\log^7 T\int_{-T}^{T}\frac{dv}{\sqrt{(\delta^2+v^2)}}\right)$$

$$= O\left\{x^{-\delta}\log^7 T\int_{-T/\delta}^{T/\delta}\frac{dv}{\sqrt{(1+v^2)}}\right\} = O(x^{-\delta}\log^8 T),$$

and

$$\int_{-\delta+iT}^{c+iT}\frac{1}{\zeta(s+w)}\frac{x^w}{w}\,dw = O\left(\frac{\log^7 T}{T}\int_{-\delta}^{c}x^u\,du\right) = O\left(\frac{x^c\log^7 T}{T}\right),$$

and similarly for the other integral. Hence

$$\sum_{n<x}\frac{\mu(n)}{n^s}-\frac{1}{\zeta(s)} = O\left(\frac{x^c}{Tc}\right)+O\left(\frac{\log x}{T}\right)+O\left(\frac{x^c\log^7 T}{T}\right)+O\left(\frac{\log^8 T}{x^\delta}\right).$$

Take $c = 1/\log x$, so that $x^c = e$; and take $T = \exp\{(\log x)^{1/10}\}$, so that $\log T = (\log x)^{1/10}$, $\delta = A(\log x)^{-9/10}$, $x^\delta = T^A$. Then the right-hand side tends to zero, and the result follows.

In particular $$\sum_{n=1}^{\infty}\frac{\mu(n)}{n} = 0.$$

3.14. *The series for* $\zeta'(s)/\zeta(s)$ *and* $\log\zeta(s)$ *on* $\sigma = 1$.

Taking† $a_n = \Lambda(n) = O(\log n)$, $\alpha = 1$, $\sigma = 1$, in the lemma, we obtain

$$\sum_{n<x}\frac{\Lambda(n)}{n^s} = -\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{\zeta'(s+w)}{\zeta(s+w)}\frac{x^w}{w}\,dw+O\left(\frac{x^c}{Tc}\right)+O\left(\frac{\log^2 x}{T}\right).$$

In this case there is a pole at $w = 1-s$, giving a residue term

$$\frac{\zeta'(s)}{\zeta(s)}-\frac{x^{1-s}}{1-s} \quad (s \neq 1), \qquad a-\log x \quad (s = 1),$$

where a is a constant. Hence if $s \neq 1$ we obtain

$$\sum_{n<x}\frac{\Lambda(n)}{n^s}+\frac{\zeta'(s)}{\zeta(s)}-\frac{x^{1-s}}{1-s} = O\left(\frac{x^c}{Tc}\right)+O\left(\frac{\log^2 x}{T}\right)+O\left(\frac{\log^{10}T}{x^\delta}\right)+O\left(\frac{x^c\log^9 T}{T}\right).$$

Taking $c = 1/\log x$, $T = \exp\{(\log x)^{1/10}\}$, we obtain as before

$$\sum_{n<x}\frac{\Lambda(n)}{n^s}+\frac{\zeta'(s)}{\zeta(s)}-\frac{x^{1-s}}{1-s} = o(1). \tag{3.14.1}$$

The term $x^{1-s}/(1-s)$ oscillates finitely, so that if $\mathbf{R}(s) = 1$, $s \neq 1$, the series $\sum\Lambda(n)n^{-s}$ is not convergent, but its partial sums are bounded.

If $s = 1$, we obtain

$$\sum_{n<x}\frac{\Lambda(n)}{n} = \log x+O(1), \tag{3.14.2}$$

or, since

$$\sum_{n<x}\frac{\Lambda(n)}{n} = \sum_{p<x}\frac{\log p}{p}+\sum_{m=2}^{\infty}\sum_{p^m<x}\frac{\log p}{p^m} = \sum_{p<x}\frac{\log p}{p}+O(1),$$

$$\sum_{p<x}\frac{\log p}{p} = \log x+O(1). \tag{3.14.3}$$

† See (1.1.8).

Since $\Lambda_1(n) = \Lambda(n)/\log n$, and $1/\log n$ tends steadily to zero, it follows that

$$\sum \frac{\Lambda_1(n)}{n^s}$$

is convergent on $\sigma = 1$, except for $t = 0$. Hence, by the continuity theorem for Dirichlet series, the equation

$$\log\zeta(s) = \sum_{n=2}^{\infty} \frac{\Lambda_1(n)}{n^s}$$

holds for $\sigma = 1$, $t \neq 0$.

To determine the behaviour of this series for $s = 1$ we have, as in the case of $1/\zeta(s)$,

$$\sum_{n<x} \frac{\Lambda_1(n)}{n} = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \log\zeta(w+1)\frac{x^w}{w}\,dw + O\left(\frac{\log x}{T}\right),$$

where $c = 1/\log x$, and T is chosen as before. Now

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \log\zeta(w+1)\frac{x^w}{w}\,dw = \frac{1}{2\pi i}\left(\int_{c-iT}^{-\delta-iT} + \int_{-\delta-iT}^{-\delta+iT} + \int_{-\delta+iT}^{c+iT}\right) + \frac{1}{2\pi i}\int_C,$$

where C is a loop starting and finishing at $s = -\delta$, and encircling the origin in the positive direction. Defining δ as before, the integral along $\sigma = -\delta$ is $O(x^{-\delta}\log^{10}T)$, and the integrals along the horizontal sides are $O(x^c T^{-1}\log^9 T)$, by (3.6.7). Since

$$\frac{1}{w}\left\{\log\zeta(w+1) - \log\frac{1}{w}\right\}$$

is regular at the origin, the last term is equal to

$$\frac{1}{2\pi i}\int_C \log\frac{1}{w}\frac{x^w}{w}\,dw.$$

Since
$$\frac{1}{2\pi i}\int_C \log\frac{1}{w}\frac{dw}{w} = -\frac{1}{4\pi i}\Delta_C \log^2 w$$

$$= -\frac{1}{4\pi i}\{\log^2(\delta e^{i\pi}) - \log^2(\delta e^{-i\pi})\} = -\log\delta,$$

this term is also equal to

$$\frac{1}{2\pi i}\int_C \log\frac{1}{w}\frac{x^w - 1}{w}\,dw - \log\delta.$$

Take C to be a circle with centre $w = 0$ and radius ρ ($\rho < \delta$), together with the segment $(-\delta, -\rho)$ of the real axis described twice. The integrals along the real segments together give

$$-\frac{1}{2\pi i}\int_\delta^\rho \log\left(\frac{1}{ue^{-i\pi}}\right)\frac{x^{-u}-1}{-u}\,du - \frac{1}{2\pi i}\int_\rho^\delta \log\left(\frac{1}{ue^{i\pi}}\right)\frac{x^{-u}-1}{-u}\,du$$

$$= -\int_\rho^\delta \frac{x^{-u}-1}{u}\,du = -\int_{\rho\log x}^{\delta\log x} \frac{e^{-v}-1}{v}\,dv$$

$$= \int_{\rho\log x}^{1} \frac{1-e^{-v}}{v}\,dv - \int_1^{\delta\log x} \frac{e^{-v}}{v}\,dv + \log(\delta\log x)$$

$$= \gamma + \log(\delta\log x) + o(1)$$

if $\rho\log x \to 0$ and $\delta\log x \to \infty$. Also

$$\int_{|w|=\rho} \log\frac{1}{w}\frac{x^w - 1}{w}\,dw = O\left(\rho\log\frac{1}{\rho}\log x\right).$$

Taking $\rho = 1/\log^2 x$, say, it follows that

$$\sum_{n<x} \frac{\Lambda_1(n)}{n} = \log\log x + \gamma + o(1). \tag{3.14.4}$$

The left-hand side can also be written in the form

$$\sum_{p<x} \frac{1}{p} + \sum_{m\geqslant 2} \sum_{p^m<x} \frac{1}{mp^m}.$$

As $x \to \infty$, the second term clearly tends to the limit

$$\sum_{m=2}^{\infty} \sum_p \frac{1}{mp^m}.$$

Hence
$$\sum_{p<x} \frac{1}{p} = \log\log x + \gamma - \sum_{m=2}^{\infty} \sum_p \frac{1}{mp^m} + o(1). \tag{3.14.5}$$

3.15. Euler's product on $\sigma = 1$. The above analysis shows that for $\sigma = 1$, $t \neq 0$,

$$\log\zeta(s) = \sum_p \frac{1}{p^s} + \sum_q \frac{\Lambda_1(q)}{q^s},$$

where p runs through primes and q through powers of primes. In fact the second series is absolutely convergent on $\sigma = 1$, since it is merely a rearrangement of

$$\sum_p \sum_{m=2}^{\infty} \frac{1}{mp^{ms}},$$

which is absolutely convergent by comparison with

$$\sum_p \sum_{m=2}^{\infty} \frac{1}{p^m} = \sum_p \frac{1}{p(p-1)}.$$

Hence also

$$\log \zeta(s) = \sum_p \frac{1}{p^s} + \sum_p \sum_{m=2}^{\infty} \frac{1}{mp^{ms}}$$

$$= \sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{ms}}$$

$$= \sum_p \log \frac{1}{1-p^{-s}} \quad (\sigma = 1,\ t \neq 0).$$

Taking exponentials,

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}}, \tag{3.15.1}$$

i.e. Euler's product holds on $\sigma = 1$, except at $t = 0$.

At $s = 1$ the product is, of course, not convergent, but we can obtain an asymptotic formula for its partial products, viz.

$$\prod_{p \leqslant x} \left(1 - \frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log x}. \tag{3.15.2}$$

To prove this, we have to prove that

$$f(x) = -\log \prod_{p \leqslant x} \left(1 - \frac{1}{p}\right) = \log\log x + \gamma + o(1).$$

Now we have proved that

$$g(x) = \sum_{n \leqslant x} \frac{\Lambda_1(n)}{n} = \log\log x + \gamma + o(1).$$

Also

$$f(x) - g(x) = \sum_{p \leqslant x} \sum_{m=1}^{\infty} \frac{1}{mp^m} - \sum\sum_{p^m \leqslant x} \frac{1}{mp^m}$$

$$= \frac{1}{2} \sum_{x^{\frac{1}{2}} \leqslant p < x} \frac{1}{p^2} + \frac{1}{3} \sum_{x^{\frac{1}{3}} < p \leqslant x} \frac{1}{p^3} + \ldots$$

$$< \sum_{\substack{p \\ p^m > x}} \sum_{m=2}^{\infty} \frac{1}{mp^m},$$

which tends to zero as $x \to \infty$, since the double series is absolutely convergent. This proves (3.15.2).

It will also be useful later to note that

$$\prod_{p \leqslant x} \left(1 + \frac{1}{p}\right) \sim \frac{6e^{\gamma} \log x}{\pi^2}. \tag{3.15.3}$$

For the left-hand side is

$$\prod_{p \leqslant x} \frac{1-1/p^2}{1-1/p} \sim e^{\gamma} \log x \prod_p \left(1 - \frac{1}{p^2}\right) = \frac{e^{\gamma} \log x}{\zeta(2)} = \frac{6e^{\gamma} \log x}{\pi^2}.$$

Note also that (3.14.3), (3.14.5) with error term $O(1)$, and (3.15.2) can be proved in an elementary way, i.e. without the theory of the Riemann zeta-function; see Hardy and Wright, *The Theory of Numbers* (5th edn), Theorems 425 and 427–429. Indeed the proof of Theorem 427 yields (3.14.5) with the error term $O\left(\frac{1}{\log x}\right)$.

NOTES FOR CHAPTER 3

3.16. The original elementary proofs of the prime number theorem may be found in Selberg [2] and Erdös [1], and a thorough survey of the ideas involved is given by Diamond [1]. The sharpest error term obtained by elementary methods to date is

$$\pi(x) = \mathrm{Li}(x) + O[x \exp\{-(\log x)^{\frac{1}{6}-\varepsilon}\}], \tag{3.16.1}$$

for any $\varepsilon > 0$, due to Lavrik and Sobirov [1]. Pintz [1] has obtained a very precise relationship between zero-free regions of $\zeta(s)$ and the error term in the prime-number theorem. Specifically, if we define

$$R(x) = \max\{|\pi(t) - \mathrm{Li}(t)| : 2 \leqslant t \leqslant x\},$$

then

$$\log \frac{x}{R(x)} \sim \min_{\rho} \{(1-\beta)\log x + \log|\gamma|\}, \quad (x \to \infty),$$

the minimum being over non-trivial zeros ρ of $\zeta(s)$. Thus (3.16.1) yields

$$(1-\beta)\log x + \log|\gamma| \gg (\log x)^{\frac{1}{6}-\varepsilon}$$

for any ρ and any x. Now, on taking $\log x = (1-\beta)^{-1}\log|\gamma|$ we deduce that

$$1 - \beta \gg (\log|\gamma|)^{-5-\varepsilon'},$$

for any $\varepsilon' > 0$. This should be compared with Theorem 3.8.

3.17. It may be observed in the proof of Theorem 3.10 that the bound $\zeta(s) = O(e^{\phi(t)})$ is only required in the immediate vicinity of s_0 and s_0'. It would be nice to eliminate consideration of s_0' and so to have a result of the strength of Theorem 3.10, giving a zero-free region around $1+it$ solely in terms of an estimate for $\zeta(s)$ in a neighbourhood of $1+it$.

Ingham's method in §3.4 is of special interest because it avoids any reference to the behaviour of $\zeta(s)$ near $1+2i\gamma$. It is possible to get quantitative zero-free regions in this way, by incorporating simple sieve estimates (Balasubramanian and Ramachandra [1]). Thus, for example, the analysis of §3.8 yields

$$\sum_{p,m} \frac{\log p}{p^{m\sigma}}\{1+\cos(m\gamma\log p)\} \leqslant \frac{1}{\sigma-1}-\frac{1}{\sigma-\beta}+O(\log\gamma).$$

However one can show that

$$\sum_{X<p\leqslant 2X}\{1+\cos(\gamma\log p)\} \gg \frac{X}{\log X}$$

for $X \geqslant \gamma^2$, by using a lower bound of Chebychev type for the number of primes $X<p\leqslant 2X$, coupled with an upper bound $O(h/\log h)$ for the number of primes in certain short intervals $X'<p\leqslant X'+h$. One then derives the estimate

$$\sum_{p\geqslant\gamma^2} \frac{\log p}{p^{\sigma}}\{1+\cos(\gamma\log p)\} \gg \frac{\gamma^{2(1-\sigma)}}{\sigma-1},$$

and an appropriate choice of $\sigma = 1+(A/\log\gamma)$ leads to the lower bound $1-\beta \gg (\log\gamma)^{-1}$.

3.18. Another approach to zero-free regions via sieve methods has been given by Motohashi [1]. This is distinctly complicated, but has the advantage of applying to the wider regions discussed in §§5.17, 6.15 and 6.19.

One may also obtain zero-free regions from a result of Montgomery [1; Theorem 11.2] on the proliferation of zeros. Let $n(t, w, h)$ denote the number of zeros $\rho = \beta+i\gamma$ of $\zeta(s)$ in the rectangle $1-w\leqslant\beta\leqslant 1$, $t-\frac{1}{2}h\leqslant\gamma\leqslant t+\frac{1}{2}h$. Suppose ρ is any zero with $\beta>\frac{1}{2}$, $\gamma>0$, and that δ satisfies $1-\beta\leqslant\delta\leqslant(\log\gamma)^{-\frac{1}{2}}$. Then there is some r with $\delta\leqslant r\leqslant 1$ for which

$$n(\gamma, r, r)+n(2\gamma, r, r) \gg \frac{r^3}{\delta^2(1-\beta)}. \tag{3.18.1}$$

Roughly speaking, this says that if $1-\beta$ is small, there must be many other zeros near either $1+i\gamma$ or $1+2i\gamma$. Montgomery gives a more precise version of this principle, as do Ramachandra [1] and Balasubramanian and Ramachandra [3]. To obtain a zero-free region

one couples hypotheses of the type used in Theorem 3.10 with Jensen's Theorem, to obtain an upper bound for $n(t, r, r)$. For example, the bound

$$\zeta(s) \ll (1+T^{1-\sigma})\log T, \qquad T = |t|+2,$$

which follows from Theorem 4.11, leads to

$$n(t, r, r) \ll r\log T+\log\log T+\log\frac{1}{r}. \tag{3.18.2}$$

On choosing $\delta = (\log\log\gamma)/(\log\gamma)$, a comparison of (3.18.1) and (3.18.2) produces Theorem 3.8 again.

One can also use the Epstein zeta-function of §2.18 and the Maass–Selberg formula (2.18.9) to prove the non-vanishing of $\zeta(s)$ for $\sigma = 1$. For, if $s = \frac{1}{2}+it$ and $\phi(s) = 2\zeta(2s) = 0$, then

$$|\psi(\tfrac{1}{2}+it)|^2 = \psi(s)\psi(1-s) = \phi(s)\phi(1-s) = |\phi(\tfrac{1}{2}+it)|^2 = 0,$$

by the functional equation (2.19.1). Thus (2.18.9) yields

$$\iint_D \tilde{B}(z, s)\tilde{B}(z, w)\frac{dx\,dy}{y^2} = 0$$

for any $w \neq s, 1-s$. This, of course, may be extended to $w = s$ or $w = 1-s$ by continuity. Taking $w = \frac{1}{2}-it = \bar{s}$ we obtain

$$\iint_D |\tilde{B}(z, s)|^2\frac{dx\,dy}{y^2} = 0$$

so that $\tilde{B}(z, s)$ must be identically zero. This however is impossible since the fourier coefficient for $n = 1$ is

$$8\pi^s y^{\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi y)/\Gamma(s),$$

according to (2.18.5), and this does not vanish identically. The above contradiction shows that $\zeta(2s) \neq 0$. One can get quantitative estimates by such methods, but only rather weak ones. It seems that the proof given here has its origins in unpublished work of Selberg.

3.19. Lemma 3.12 is a version of Perron's formula. It is sometimes useful to have a form of this in which the error is bounded as $x \to N$.

LEMMA 3.19. *Under the hypotheses of Lemma* 3.12 *one has*

$$\sum_{n\leqslant x}\frac{a_n}{n^s}=\frac{1}{2\pi i}\int_{c-iT}^{c+iT} f(s+w)\frac{x^w}{w}\,dw+O\left\{\frac{x^c}{T(\sigma+c-1)^\alpha}\right\}$$
$$+O\left\{\frac{\psi(2x)x^{1-\sigma}\log x}{T}\right\}+O\left\{\psi(N)x^{-\sigma}\min\left(\frac{x}{T|x-N|},1\right)\right\}.$$

This follows at once from Lemma 3.12 unless $x-N=O(x/T)$. In the latter case one merely estimates the contribution from the term $n=N$ as

$$\int_{c-iT}^{c+iT}\frac{a_N}{N^s}\left(\frac{x}{N}\right)^w\frac{dw}{w}=\int_{c-iT}^{c+iT}\frac{a_N}{N^s}\left\{1+O\left(\frac{w}{T}\right)\right\}\frac{dw}{w}$$
$$=\frac{a_N}{N^s}\left\{\log\frac{c+iT}{c-iT}+O(1)\right\}$$
$$=O\{\psi(N)N^{-\sigma}\},$$

and the result follows.

IV

APPROXIMATE FORMULAE

4.1. IN this chapter we shall prove a number of approximate formulae for $\zeta(s)$ and for various sums related to it. We shall begin by proving some general results on integrals and series of a certain type.

4.2. LEMMA 4.2. *Let $F(x)$ be a real differentiable function such that $F'(x)$ is monotonic, and $F'(x)\geqslant m>0$, or $F'(x)\leqslant -m<0$, throughout the interval $[a,b]$. Then*

$$\left|\int_a^b e^{iF(x)}\,dx\right|\leqslant\frac{4}{m}. \tag{4.2.1}$$

Suppose, for example, that $F'(x)$ is positive increasing. Then by the second mean-value theorem

$$\int_a^b\cos\{F(x)\}\,dx=\int_a^b\frac{F'(x)\cos\{F(x)\}}{F'(x)}\,dx$$
$$=\frac{1}{F'(a)}\int_a^\xi F'(x)\cos\{F(x)\}\,dx=\frac{\sin\{F(\xi)\}-\sin\{F(a)\}}{F'(a)},$$

and the modulus of this does not exceed $2/m$. A similar argument applies to the imaginary part, and the result follows.

4.3. More generally, we have

LEMMA 4.3. *Let $F(x)$ and $G(x)$ be real functions, $G(x)/F'(x)$ monotonic, and $F'(x)/G(x)\geqslant m>0$, or $\leqslant -m<0$. Then*

$$\left|\int_a^b G(x)e^{iF(x)}\,dx\right|\leqslant\frac{4}{m}.$$

The proof is similar to that of the previous lemma.

The values of the constants in these lemmas are usually not of any importance.

4.4. LEMMA 4.4. *Let $F(x)$ be a real function, twice differentiable, and let $F''(x)\geqslant r>0$, or $F''(x)\leqslant -r<0$, throughout the interval $[a,b]$. Then*

$$\left|\int_a^b e^{iF(x)}\,dx\right|\leqslant\frac{8}{\sqrt{r}}. \tag{4.4.1}$$

Consider, for example, the first alternative. Then $F'(x)$ is steadily increasing, and so vanishes at most once in the interval (a, b), say at c. Let

$$I = \int_a^b e^{iF(x)}\,dx = \int_a^{c-\delta} + \int_{c-\delta}^{c+\delta} + \int_{c+\delta}^{b} = I_1 + I_2 + I_3,$$

where δ is a positive number to be chosen later, and it is assumed that $a+\delta \leqslant c \leqslant b-\delta$. In I_3

$$F'(x) = \int_c^x F''(t)\,dt \geqslant r(x-c) \geqslant r\delta.$$

Hence, by Lemma 4.2, $\quad |I_3| \leqslant \dfrac{4}{r\delta}.$

I_1 satisfies the same inequality, and $|I_2| \leqslant 2\delta$. Hence

$$|I| \leqslant \frac{8}{r\delta} + 2\delta.$$

Taking $\delta = 2r^{-\frac{1}{2}}$, we obtain the result. If $c < a+\delta$, or $c > b-\delta$, the argument is similar.

4.5. Lemma 4.5. *Let $F(x)$ satisfy the conditions of the previous lemma, and let $G(x)/F'(x)$ be monotonic, and $|G(x)| \leqslant M$. Then*

$$\left| \int_a^b G(x) e^{iF(x)}\,dx \right| \leqslant \frac{8M}{\sqrt{r}}.$$

The proof is similar to the previous one, but uses Lemma 4.3 instead of Lemma 4.2.

4.6. Lemma 4.6. *Let $F(x)$ be real, with derivatives up to the third order. Let*

$$0 < \lambda_2 \leqslant F''(x) < A\lambda_2, \tag{4.6.1}$$

or

$$0 < \lambda_2 \leqslant -F''(x) < A\lambda_2, \tag{4.6.2}$$

and

$$|F'''(x)| \leqslant A\lambda_3, \tag{4.6.3}$$

throughout the interval (a, b). Let $F'(c) = 0$, where

$$a \leqslant c \leqslant b. \tag{4.6.4}$$

Then in the case (4.6.1)

$$\int_a^b e^{iF(x)}\,dx = (2\pi)^{\frac{1}{2}} \frac{e^{\frac{1}{4}i\pi + iF(c)}}{|F''(c)|^{\frac{1}{2}}} + O(\lambda_2^{-\frac{4}{5}}\lambda_3^{\frac{1}{5}}) +$$

$$+ O\left\{\min\left(\frac{1}{|F'(a)|}, \lambda_2^{-\frac{1}{2}}\right)\right\} + O\left\{\min\left(\frac{1}{|F'(b)|}, \lambda_2^{-\frac{1}{2}}\right)\right\}. \tag{4.6.5}$$

In the case (4.6.2) *the factor $e^{\frac{1}{4}i\pi}$ is replaced by $e^{-\frac{1}{4}i\pi}$. If $F'(x)$ does not vanish on $[a, b]$ then* (4.6.5) *holds without the leading term.*

If $F'(x)$ does not vanish on $[a, b]$ the result follows from Lemmas 4.2 and 4.4. Otherwise either (4.6.1) or (4.6.2) shows that $F'(x)$ is monotonic, and so vanishes at only one point c. We put

$$\int_a^b e^{iF(x)}\,dx = \int_a^{c-\delta} + \int_{c-\delta}^{c+\delta} + \int_{c+\delta}^{b},$$

assuming that $a+\delta \leqslant c \leqslant b-\delta$. By (4.2.1)

$$\int_{c+\delta}^{b} = O\left\{\frac{1}{|F'(c+\delta)|}\right\} = O\left\{1 \Big/ \left|\int_c^{c+\delta} F''(x)\,dx\right|\right\} = O\left(\frac{1}{\delta\lambda_2}\right).$$

Similarly $\quad \displaystyle\int_a^{c-\delta} = O\left(\frac{1}{\delta\lambda_2}\right).$

Also

$$\int_{c-\delta}^{c+\delta} = \int_{c-\delta}^{c+\delta} \exp[i\{F(c) + (x-c)F'(c) + \tfrac{1}{2}(x-c)^2 F''(c) +$$

$$+ \tfrac{1}{6}(x-c)^3 F'''\{c+\theta(x-c)\}\}]\,dx$$

$$= e^{iF(c)} \int_{c-\delta}^{c+\delta} e^{\frac{1}{2}i(x-c)^2 F''(c)} [1 + O\{(x-c)^3\lambda_3\}]\,dx$$

$$= e^{iF(c)} \int_{c-\delta}^{c+\delta} e^{\frac{1}{2}i(x-c)^2 F''(c)}\,dx + O(\delta^4\lambda_3).$$

Supposing $F''(c) > 0$, and putting

$$\tfrac{1}{2}(x-c)^2 F''(c) = u,$$

the integral becomes

$$\frac{2^{\frac{1}{2}}}{\{F''(c)\}^{\frac{1}{2}}} \int_0^{\frac{1}{2}\delta^2 F''(c)} \frac{e^{iu}}{\sqrt{u}}\,du = \frac{2^{\frac{1}{2}}}{\{F''(c)\}^{\frac{1}{2}}} \left\{\int_0^\infty \frac{e^{iu}}{\sqrt{u}}\,du + O\left(\frac{1}{\delta\sqrt{\lambda_2}}\right)\right\}$$

$$= \frac{(2\pi)^{\frac{1}{2}} e^{\frac{1}{4}i\pi}}{\{F''(c)\}^{\frac{1}{2}}} + O\left(\frac{1}{\delta\lambda_2}\right).$$

Taking $\delta = (\lambda_2\lambda_3)^{-\frac{1}{5}}$, the result follows.

If $b-\delta < c \leqslant b$, there is also an error

$$e^{iF(c)} \int_b^{c+\delta} e^{\frac{1}{2}i(x-c)^2 F''(c)}\,dx = O\left\{\frac{1}{(b-c)\lambda_2}\right\} = O\left\{\frac{1}{|F'(b)|}\right\} \text{ and also } O(\lambda_2^{-\frac{1}{2}});$$

and similarly if $a \leqslant c \leqslant a+\delta$.

4.7. We now turn to the consideration of exponential sums, i.e. sums of the form

$$\sum e^{2\pi i f(n)},$$

where $f(n)$ is a real function. If the numbers $f(n)$ are the values taken by a function $f(x)$ of a simple kind, we can approximate to such a sum by an integral, or by a sum of integrals.

LEMMA 4.7.† *Let $f(x)$ be a real function with a continuous and steadily decreasing derivative $f'(x)$ in (a,b), and let $f'(b)=\alpha$, $f'(a)=\beta$. Then*

$$\sum_{a<n\leqslant b} e^{2\pi i f(n)} = \sum_{\alpha-\eta<\nu<\beta+\eta} \int_a^b e^{2\pi i\{f(x)-\nu x\}}\,dx + O\{\log(\beta-\alpha+2)\}, \tag{4.7.1}$$

where η is any positive constant less than 1.

We may suppose without loss of generality that $\eta-1<\alpha\leqslant\eta$, so that $\nu\geqslant 0$; for if k is the integer such that $\eta-1<\alpha-k\leqslant\eta$, and

$$h(x)=f(x)-kx,$$

then (4.7.1) is

$$\sum_{a<n\leqslant b} e^{2\pi i h(n)} = \sum_{\alpha'-\eta<\nu-k<\beta'+\eta} \int_a^b e^{2\pi i\{h(x)-(\nu-k)x\}}\,dx + O\{\log(\beta'-\alpha'+2)\},$$

where $\alpha'=\alpha-k$, $\beta'=\beta-k$, i.e. the same formula for $h(x)$.

In (2.1.2), let $\phi(x)=e^{2\pi i f(x)}$. Then

$$\sum_{a<n\leqslant b} e^{2\pi i f(n)} = \int_a^b e^{2\pi i f(x)}\,dx + \int_a^b (x-[x]-\tfrac{1}{2})2\pi i f'(x)e^{2\pi i f(x)}\,dx + O(1).$$

Also

$$x-[x]-\tfrac{1}{2} = -\frac{1}{\pi}\sum_{\nu=1}^{\infty}\frac{\sin 2\nu\pi x}{\nu}$$

if x is not an integer; and the series is boundedly convergent, so that we may multiply by an integrable function and integrate term-by-term. Hence the second term on the right is equal to

$$-2i\sum_{\nu=1}^{\infty}\int_a^b \frac{\sin 2\nu\pi x}{\nu} e^{2\pi i f(x)} f'(x)\,dx$$

$$= \sum_{\nu=1}^{\infty}\frac{1}{\nu}\int_a^b (e^{-2\pi i\nu x}-e^{2\pi i\nu x})e^{2\pi i f(x)}f'(x)\,dx.$$

The integral may be written

$$\frac{1}{2\pi i}\int_a^b \frac{f'(x)}{f'(x)-\nu}\,d(e^{2\pi i\{f(x)-\nu x\}}) - \frac{1}{2\pi i}\int_a^b \frac{f'(x)}{f'(x)+\nu}\,d(e^{2\pi i\{f(x)+\nu x\}}).$$

† van der Corput (1).

Since $\dfrac{f'(x)}{f'(x)+\nu}$ is steadily decreasing, the second term is

$$O\left(\frac{\beta}{\beta+\nu}\right),$$

by applying the second mean-value theorem to the real and imaginary parts. Hence this term contributes

$$O\left(\sum_{\nu=1}^{\infty}\frac{\beta}{\nu(\beta+\nu)}\right) = O\left(\sum_{\nu\leqslant\beta}\frac{1}{\nu}\right)+O\left(\sum_{\nu>\beta}\frac{\beta}{\nu^2}\right)$$

$$= O\{\log(\beta+2)\}+O(1).$$

Similarly the first term is $O\{\beta/(\nu-\beta)\}$ for $\nu\geqslant\beta+\eta$, and this contributes

$$O\left(\sum_{\nu\geqslant\beta+\eta}\frac{\beta}{\nu(\nu-\beta)}\right) = O\left(\sum_{\beta+\eta\leqslant\nu<2\beta}\frac{1}{\nu-\beta}\right)+O\left(\sum_{\nu\geqslant 2\beta}\frac{\beta}{\nu^2}\right)$$

$$= O\{\log(\beta+2)\}+O(1).$$

Finally

$$\sum_{\nu=1}^{\beta+\eta}\frac{1}{\nu}\int_a^b e^{2\pi i\{f(x)-\nu x\}}f'(x)\,dx = \sum_{\nu=1}^{\beta+\eta}\left[\frac{e^{2\pi i\{f(x)-\nu x\}}}{2\pi i\nu}\right]_a^b + \sum_{\nu=1}^{\beta+\eta}\int_a^b e^{2\pi i\{f(x)-\nu x\}}\,dx,$$

and the integrated terms are $O\{\log(\beta+2)\}$. The result therefore follows.

4.8. As a particular case, we have

LEMMA 4.8. *Let $f(x)$ be a real differentiable function in the interval $[a,b]$, let $f'(x)$ be monotonic, and let $|f'(x)|\leqslant\theta<1$. Then*

$$\sum_{a<n\leqslant b} e^{2\pi i f(n)} = \int_a^b e^{2\pi i f(x)}\,dx + O(1). \tag{4.8.1}$$

Taking $\eta<1-\theta$, the sum on the right of (4.7.1) either reduces to the single term $\nu=0$, or, if $f'(x)\geqslant\eta$ or $\leqslant-\eta$ throughout $[a,b]$, it is null, and

$$\int_a^b e^{2\pi i f(x)}\,dx = O(1)$$

by Lemma 4.2.

4.9. THEOREM 4.9.† *Let $f(x)$ be a real function with derivatives up to the third order. Let $f'(x)$ be steadily decreasing in $a\leqslant x\leqslant b$, and $f'(b)=\alpha$, $f'(a)=\beta$. Let x_ν be defined by*

$$f'(x_\nu)=\nu \quad (\alpha<\nu\leqslant\beta).$$

† van der Corput (2).

Let $\lambda_2 \leqslant |f''(x)| < A\lambda_2, \qquad |f'''(x)| < A\lambda_3.$

Then

$$\sum_{a<n\leqslant b} e^{2\pi i f(n)} = e^{-\frac{1}{4}\pi i} \sum_{\alpha<\nu\leqslant\beta} \frac{e^{2\pi i\{f(x_\nu)-\nu x_\nu\}}}{|f''(x_\nu)|^{\frac{1}{2}}} + O(\lambda_2^{-\frac{1}{2}}) +$$
$$+O[\log\{2+(b-a)\lambda_2\}]+O\{(b-a)\lambda_2^{\frac{1}{5}}\lambda_3^{\frac{1}{5}}\}.$$

We use Lemma 4.7, where now

$$\beta-\alpha = O\{(b-a)\lambda_2\}.$$

Also we can replace the limits of summation on the right-hand side by $(\alpha+1, \beta-1)$, with error $O(\lambda_2^{-\frac{1}{2}})$. Lemma 4.6. then gives

$$\sum_{\alpha+1<\nu<\beta-1} \int_a^b e^{2\pi i\{f(x)-\nu x\}}\,dx = e^{-\frac{1}{4}i\pi} \sum_{\alpha+1<\nu<\beta-1} \frac{e^{2\pi i\{f(x_\nu)-\nu x_\nu\}}}{|f''(x_\nu)|^{\frac{1}{2}}} +$$
$$+\sum_{\alpha+1<\nu<\beta-1} O(\lambda_2^{-\frac{4}{5}}\lambda_3^{\frac{1}{5}}) + \sum_{\alpha+1<\nu<\beta-1} \left\{O\left(\frac{1}{\nu-\alpha}\right)+O\left(\frac{1}{\beta-\nu}\right)\right\}.$$

The second term on the right is

$$O\{(\beta-\alpha)\lambda_2^{-\frac{4}{5}}\lambda_3^{\frac{1}{5}}\} = O\{(b-a)\lambda_2^{\frac{1}{5}}\lambda_3^{\frac{1}{5}}\},$$

and the last term is

$$O\{\log(2+\beta-\alpha)\} = O[\log\{2+(b-a)\lambda_2\}].$$

Finally we can replace the limits $(\alpha+1, \beta-1)$ by $(\alpha, \beta]$ with error $O(\lambda_2^{-\frac{1}{2}})$.

4.10. LEMMA 4.10. *Let $f(x)$ satisfy the same conditions as in Lemma 4.7, and let $g(x)$ be a real positive decreasing function, with a continuous derivative $g'(x)$, and let $|g'(x)|$ be steadily decreasing. Then*

$$\sum_{a<n\leqslant b} g(n)e^{2\pi i f(n)} = \sum_{\alpha-\eta<\nu<\beta+\eta} \int_a^b g(x)e^{2\pi i\{f(x)-\nu x\}}\,dx +$$
$$+O\{g(a)\log(\beta-\alpha+2)\}+O\{|g'(a)|\}.$$

We proceed as in § 4.7, but with

$$\phi(x) = g(x)e^{2\pi i f(x)}.$$

We encounter terms of the form

$$\int_a^b g(x)\frac{f'(x)}{f'(x)\pm\nu}\,d(e^{2\pi i(f(x)\pm\nu x)}),$$

and also

$$\int_a^b \frac{g'(x)}{f'(x)\pm\nu}\,d(e^{2\pi i(f(x)\pm\nu x)}).$$

The former lead to $O\{g(a)\log(\beta-\alpha+2)\}$ as before. The latter give, for example,

$$\sum_{\nu=1}^{\infty} \frac{|g'(a)|}{\nu^2} = O(|g'(a)|),$$

and the result follows.

4.11. We now come to the simplest theorem† on the approximation to $\zeta(s)$ in the critical strip by a partial sum of its Dirichlet series.

THEOREM 4.11. *We have*

$$\zeta(s) = \sum_{n\leqslant x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma}) \tag{4.11.1}$$

uniformly for $\sigma \geqslant \sigma_0 > 0$, $|t| < 2\pi x/C$, when C is a given constant greater than 1.

We have, by (3.5.3),

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} - \frac{N^{1-s}}{1-s} + s\int_N^\infty \frac{[u]-u+\frac{1}{2}}{u^{s+1}}\,du - \tfrac{1}{2}N^{-s}$$
$$= \sum_{n=1}^{N} \frac{1}{n^s} - \frac{N^{1-s}}{1-s} + O\left(\frac{|s|}{N^\sigma}\right) + O(N^{-\sigma}). \tag{4.11.2}$$

The sum $$\sum_{x<n\leqslant N} \frac{1}{n^s} = \sum_{x<n\leqslant N} \frac{n^{-it}}{n^\sigma}$$

is of the form considered in the above lemma, with $g(u) = u^{-\sigma}$, and

$$f(u) = -\frac{t\log u}{2\pi}, \qquad f'(u) = -\frac{t}{2\pi u}.$$

Thus $$|f'(u)| \leqslant \frac{t}{2\pi x} < \frac{1}{C}.$$

Hence $$\sum_{x<n\leqslant N} \frac{1}{n^s} = \int_x^N \frac{du}{u^s} + O(x^{-\sigma})$$
$$= \frac{N^{1-s}-x^{1-s}}{1-s} + O(x^{-\sigma}).$$

Hence $$\zeta(s) = \sum_{n\leqslant x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma}) + O\left(\frac{|s|+1}{N^\sigma}\right).$$

Making $N \to \infty$, the result follows.

† Hardy and Littlewood (3).

4.12. For many purposes the sum involved in Theorem 4.11 contains too many terms (at least $A|t|$) to be of use. We therefore consider the result of taking smaller values of x in the above formulae. The form of the result is given by Theorem 4.9, with an extra factor $g(n)$ in the sum. If we ignore error terms for the moment, this gives

$$\sum_{a<n\leqslant b} g(n)e^{2\pi i f(n)} \sim e^{-\frac{1}{4}\pi i} \sum_{\alpha<\nu\leqslant\beta} \frac{e^{2\pi i(f(x_\nu)-\nu x_\nu)}}{|f''(x_\nu)|^{\frac{1}{2}}} g(x_\nu).$$

Taking

$$g(u) = u^{-\sigma}, \qquad f(u) = \frac{t\log u}{2\pi},$$

$$f'(u) = \frac{t}{2\pi u}, \qquad f''(u) = -\frac{t}{2\pi u^2},$$

$$x_\nu = \frac{t}{2\pi\nu}, \qquad f''(x_\nu) = -\frac{2\pi\nu^2}{t},$$

and replacing a, b by x, N, and i by $-i$, we obtain

$$\sum_{x<n\leqslant N} \frac{1}{n^s} \sim e^{\frac{1}{4}\pi i} \sum_{t/2\pi N<\nu\leqslant t/2\pi x} \frac{e^{-2\pi i((t/2\pi)\log(t/2\pi\nu)-(t/2\pi))}}{(t/2\pi\nu)^\sigma(2\pi\nu^2/t)^{\frac{1}{2}}}$$

$$= \left(\frac{t}{2\pi}\right)^{\frac{1}{2}-\sigma} e^{\frac{1}{4}\pi i-it\log(t/2\pi e)} \sum_{t/2\pi N<\nu\leqslant t/2\pi x} \frac{1}{\nu^{1-s}}.$$

Now the functional equation is

$$\zeta(s) = \chi(s)\zeta(1-s),$$

where $\qquad \chi(s) = 2^{s-1}\pi^s \sec\tfrac{1}{2}s\pi/\Gamma(s).$

In any fixed strip $\alpha \leqslant \sigma \leqslant \beta$, as $t\to\infty$

$$\log\Gamma(\sigma+it) = (\sigma+it-\tfrac{1}{2})\log(it)-it+\tfrac{1}{2}\log 2\pi+O\left(\frac{1}{t}\right). \tag{4.12.1}$$

Hence

$$\Gamma(\sigma+it) = t^{\sigma+it-\frac{1}{2}}e^{-\frac{1}{2}\pi t-it+\frac{1}{2}i\pi(\sigma-\frac{1}{2})}(2\pi)^{\frac{1}{2}}\left\{1+O\left(\frac{1}{t}\right)\right\}, \tag{4.12.2}$$

$$\chi(s) = \left(\frac{2\pi}{t}\right)^{\sigma+it-\frac{1}{2}} e^{i(t+\frac{1}{4}\pi)}\left\{1+O\left(\frac{1}{t}\right)\right\}. \tag{4.12.3}$$

Hence the above relation is equivalent to

$$\sum_{x<n\leqslant N} \frac{1}{n^s} \sim \chi(s) \sum_{t/2\pi N<\nu\leqslant t/2\pi x} \frac{1}{\nu^{1-s}}.$$

The formulae therefore suggest that, with some suitable error terms,

$$\zeta(s) \sim \sum_{n\leqslant x}\frac{1}{n^s}+\chi(s)\sum_{\nu\leqslant y}\frac{1}{\nu^{1-s}},$$

where $2\pi xy = |t|$.

Actually the result is that

$$\zeta(s) = \sum_{n\leqslant x}\frac{1}{n^s}+\chi(s)\sum_{n\leqslant y}\frac{1}{n^{1-s}}+O(x^{-\sigma})+O(|t|^{\frac{1}{2}-\sigma}y^{\sigma-1}) \tag{4.12.4}$$

for $0<\sigma<1$. This is known as the *approximate functional equation*.†

4.13. Theorem 4.13. *If h is a positive constant,*

$$0<\sigma<1, \qquad 2\pi xy = t, \qquad x>h>0, \qquad y>h>0,$$

then

$$\zeta(s) = \sum_{n\leqslant x}\frac{1}{n^s}+\chi(s)\sum_{n\leqslant y}\frac{1}{n^{1-s}}+O(x^{-\sigma}\log|t|)+O(|t|^{\frac{1}{2}-\sigma}y^{\sigma-1}). \tag{4.13.1}$$

This is an imperfect form of the approximate functional equation in which a factor $\log|t|$ appears in one of the O-terms; but for most purposes it is quite sufficient. The proof depends on the same principle as Theorem 4.9, but Theorem 4.9 would not give a sufficiently good O-result, and we have to reconsider the integrals which occur in this problem. Let $t>0$. By Lemma 4.10

$$\sum_{x<n\leqslant N}\frac{1}{n^s} = \sum_{t/2\pi N-\eta<\nu\leqslant y+\eta}\int_x^N \frac{e^{2\pi i\nu u}}{u^s}\,du+O\left\{x^{-\sigma}\log\left(\frac{t}{x}-\frac{t}{N}+2\right)\right\},$$

and the last term is $O(x^{-\sigma}\log t)$. If $2\pi N\eta>t$, the first term is $\nu=0$, i.e.

$$\int_x^N \frac{du}{u^s} = \frac{N^{1-s}-x^{1-s}}{1-s}.$$

Hence by (4.11.2)

$$\zeta(s) = \sum_{n\leqslant x}\frac{1}{n^s}+\sum_{1\leqslant\nu\leqslant y+\eta}\int_x^N \frac{e^{2\pi i\nu u}}{u^s}\,du+O(x^{-\sigma}\log t)+O(tN^{-\sigma}),$$

since $\qquad x^{1-s}/(1-s) = O(x^{-\sigma}) = O(x^{-\sigma}\log t).$

Now

$$\int_0^\infty \frac{e^{2\pi i\nu u}}{u^s}\,du = \Gamma(1-s)\left(\frac{2\pi\nu}{i}\right)^{s-1},$$

and by Lemma 4.3

$$\int_N^\infty u^{-\sigma}e^{-2\pi i\{(t/2\pi)\log u-\nu u\}}\,du = O\left(\frac{N^{-\sigma}}{\nu-(t/2\pi N)}\right) = O\left(\frac{N^{-\sigma}}{\nu}\right),$$

$$\int_0^x u^{-s}e^{2\pi i\nu u}\,du = \left[\frac{u^{1-s}}{1-s}e^{2\pi i\nu u}\right]_0^x-\frac{2\pi i\nu}{1-s}\int_0^x u^{1-s}e^{2\pi i\nu u}\,du$$

$$= O\left(\frac{x^{1-\sigma}}{t}\right)+O\left(\frac{\nu}{t}\frac{x^{1-\sigma}}{\nu-(t/2\pi x)}\right).$$

† Hardy and Littlewood (3), (4), (6), Siegel (2).

Hence

$$\sum_{1\leqslant\nu\leqslant y-\eta}\int_x^N \frac{e^{2\pi i\nu u}}{u^s}\,du = \left(\frac{2\pi}{i}\right)^{s-1}\Gamma(1-s)\sum_{1\leqslant\nu\leqslant y-\eta}\frac{1}{\nu^{1-s}}+$$
$$+O(N^{-\sigma}\log y)+O\left(\frac{x^{1-\sigma}y}{t}\right)+O\left(\frac{x^{1-\sigma}}{t}\sum_{1\leqslant\nu\leqslant y-\eta}\frac{\nu}{\nu-y}\right)$$
$$=\left(\frac{2\pi}{i}\right)^{s-1}\Gamma(1-s)\sum_{1\leqslant\nu\leqslant y-\eta}\frac{1}{\nu^{1-s}}+O(N^{-\sigma}\log t)+O\left(\frac{x^{1-\sigma}y\log t}{t}\right).$$

There is still a possible term corresponding to $y-\eta<\nu\leqslant y+\eta$; for this, by Lemma 4.5,

$$\int_0^x u^{1-s}e^{2\pi i\nu u}\,du = O\left\{x^{1-\sigma}\left(\frac{t}{x^2}\right)^{-\frac{1}{2}}\right\},$$

giving a term

$$O\left\{\frac{y}{t}x^{1-\sigma}\left(\frac{t}{x^2}\right)^{-\frac{1}{2}}\right\} = O\left\{x^{-\sigma}\left(\frac{t}{x^2}\right)^{-\frac{1}{2}}\right\} = O(x^{1-\sigma}t^{-\frac{1}{2}}) = O(t^{\frac{1}{2}-\sigma}y^{\sigma-1}).$$

Finally we can replace $\nu\leqslant y-\eta$ by $\nu\leqslant y$ with error

$$O\left\{\left|\left(\frac{2\pi}{i}\right)^{s-1}\Gamma(1-s)\right|y^{\sigma-1}\right\} = O(t^{\frac{1}{2}-\sigma}y^{\sigma-1}).$$

Also for $t>0$
$$\chi(s) = 2^s\pi^{s-1}\sin\tfrac{1}{2}s\pi\,\Gamma(1-s)$$
$$= 2^s\pi^{s-1}\left\{-\frac{e^{-\frac{1}{2}is\pi}}{2i}+O(e^{-\frac{1}{2}\pi t})\right\}\Gamma(1-s)$$
$$= \left(\frac{2\pi}{i}\right)^{s-1}\Gamma(1-s)\{1+O(e^{-\pi t})\}.$$

Hence the result follows on taking N large enough.

It is possible to prove the full result by a refinement of the above methods. We shall not give the details here, since the result will be obtained by another method, depending on contour integration.

4.14. Complex-variable methods. An extremely powerful method of obtaining approximate formulae for $\zeta(s)$ is to express $\zeta(s)$ as a contour integral, and then move the contour into a position where it can be suitably dealt with. The following is a simple example.

Alternative proof of Theorem 4.11. We may suppose without loss of generality that x is half an odd integer, since the last term in the sum, which might be affected by the restriction, is $O(x^{-\sigma})$, and so is the possible variation in $x^{1-s}/(1-s)$.

Suppose first that $\sigma>1$. Then a simple application of the theorem of residues shows that

$$\zeta(s)-\sum_{n<x}n^{-s} = \sum_{n>x}n^{-s} = -\frac{1}{2i}\int_{x-i\infty}^{x+i\infty} z^{-s}\cot\pi z\,dz$$
$$= -\frac{1}{2i}\int_{x-i\infty}^{x}(\cot\pi z-i)z^{-s}\,dz-\frac{1}{2i}\int_{x}^{x+i\infty}(\cot\pi z+i)z^{-s}\,dz-\frac{x^{1-s}}{1-s}.$$

The final formula holds, by the theory of analytic continuation, for all values of s, since the last two integrals are uniformly convergent in any finite region. In the second integral we put $z=x+ir$, so that

$$|\cot\pi z+i| = \frac{2}{1+e^{2\pi r}} < 2e^{-2\pi r},$$

and
$$|z^{-s}| = |z|^{-\sigma}e^{t\arg z} < x^{-\sigma}e^{|t|\arctan(r/x)} < x^{-\sigma}e^{|t|r/x}.$$

Hence the modulus of this term does not exceed

$$x^{-\sigma}\int_0^\infty e^{-2\pi r+|t|r/x}\,dr = \frac{x^{-\sigma}}{2\pi-|t|/x}.$$

A similar result holds for the other integral, and the theorem follows.

It is possible to prove the approximate functional equation by an extension of this argument; we may write

$$-\cot\pi z-i = 2i\sum_{\nu=1}^{n}e^{2\nu\pi iz}+\frac{2ie^{2(n+1)\pi iz}}{1-e^{2\pi iz}}.$$

Proceeding as before, this leads to an O-term

$$O\left\{x^{-\sigma}\int_0^\infty e^{-2(n+1)\pi r-|t|r/x}\,dr\right\} = O\left(\frac{x^{-\sigma}}{2(n+1)\pi-|t|/x}\right),$$

and this is $O(x^{-\sigma})$ if $2(n+1)\pi-|t|/x>A$, i.e. for comparatively small values of x, if n is large. However, the rest of the argument suggested is not particularly simple, and we prefer another proof, which will be more useful for further developments.

4.15. THEOREM 4.15. *The approximate functional equation* (4.12.4) *holds for* $0\leqslant\sigma\leqslant 1$, $x>h>0$, $y>h>0$.

It is possible to extend the result to any strip $-k<\sigma<k$ by slight changes in the argument.

For $\sigma>1$
$$\zeta(s) = \sum_{n=1}^{m}\frac{1}{n^s}+\frac{1}{\Gamma(s)}\int_0^\infty\frac{x^{s-1}e^{-mx}}{e^x-1}\,dx.$$

Transforming the integral into a loop-integral as in § 2.4, we obtain

$$\zeta(s) = \sum_{n=1}^{m} \frac{1}{n^s} + \frac{e^{-i\pi s}\Gamma(1-s)}{2\pi i} \int_C \frac{w^{s-1}e^{-mw}}{e^w - 1}\, dw,$$

where C excludes the zeros of $e^w - 1$ other than $w = 0$. This holds for all values of s except positive integers.

Let $t > 0$ and $x \leqslant y$, so that $x \leqslant \sqrt{(t/2\pi)}$. Let $\sigma \leqslant 1$,

$$m = [x], \qquad y = t/(2\pi x), \qquad q = [y], \qquad \eta = 2\pi y.$$

We deform the contour C into the straight lines C_1, C_2, C_3, C_4 joining ∞, $c\eta + i\eta(1+c)$, $-c\eta + i\eta(1-c)$, $-c\eta - (2q+1)\pi i$, ∞, where c is an absolute constant, $0 < c \leqslant \frac{1}{2}$. If y is an integer, a small indentation is made above the pole at $w = i\eta$. We have then

$$\zeta(s) = \sum_{n=1}^{m} \frac{1}{n^s} + \chi(s) \sum_{n=1}^{q} \frac{1}{n^{1-s}} + \frac{e^{-i\pi s}\Gamma(1-s)}{2\pi i}\left(\int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}\right).$$

Let $w = u + iv = \rho e^{i\phi}$ $(0 < \phi < 2\pi)$. Then

$$|w^{s-1}| = \rho^{\sigma-1} e^{-t\phi}.$$

On C_4, $\phi \geqslant \frac{5}{4}\pi$, $\rho > A\eta$, and $|e^w - 1| > A$. Hence

$$\left|\int_{C_4}\right| = O\left(\eta^{\sigma-1} e^{-\frac{5}{4}\pi t} \int_{-c\eta}^{\infty} e^{-mu}\, du\right) = O(e^{mc\eta - \frac{5}{4}\pi t}) = O(e^{t(c - \frac{5}{4}\pi)}).$$

On C_3, $\phi \geqslant \frac{1}{2}\pi + \arctan\dfrac{c}{1-c} > \frac{1}{2}\pi + c + A$ where $A > 0$, since

$$\arctan\theta = \int_0^\theta \frac{d\mu}{1+\mu^2} > \int_0^\theta \frac{d\mu}{(1+\mu)^2} = \frac{\theta}{1+\theta}.$$

Hence

$$w^{s-1}e^{-mw} = O(\eta^{\sigma-1} e^{-t(\frac{1}{2}\pi + c + A) + mc\eta}) = O(\eta^{\sigma-1} e^{-t(\frac{1}{2}\pi + A)})$$

and $|e^w - 1| > A$. Hence

$$\int_{C_3} = O(\eta^{\sigma} e^{-t(\frac{1}{2}\pi + A)}).$$

On C_1, $|e^w - 1| > Ae^u$. Hence

$$\frac{w^{s-1}e^{-mw}}{e^w - 1} = O\left[\eta^{\sigma-1} \exp\left\{-t \arctan\frac{(1+c)\eta}{u} - (m+1)u\right\}\right].$$

Since $m + 1 \geqslant x = t/\eta$, and

$$\frac{d}{du}\left\{\arctan\frac{(1+c)\eta}{u} + \frac{u}{\eta}\right\} = -\frac{(1+c)\eta}{u^2 + (1+c)^2\eta^2} + \frac{1}{\eta} > 0,$$

we have

$$\arctan\frac{(1+c)\eta}{u} + \frac{u}{\eta} \geqslant \arctan\frac{1+c}{c} + c$$

$$= \tfrac{1}{2}\pi + c - \arctan\frac{c}{1+c} = \tfrac{1}{2}\pi + A,$$

since for $0 < \theta < 1$

$$\arctan\theta < \int_0^\theta \frac{d\mu}{(1-\mu)^2} = \frac{\theta}{1-\theta}.$$

Hence

$$\int_{C_1} = O\left(\eta^{\sigma-1} \int_0^{\pi\eta} e^{-(\frac{1}{2}\pi + A)t}\, du\right) + O\left(\eta^{\sigma-1} \int_{\pi\eta}^{\infty} e^{-xu}\, du\right)$$

$$= O(\eta^\sigma e^{-(\frac{1}{2}\pi + A)t}) + O(\eta^{\sigma-1} e^{-\pi\eta x}) = O(\eta^\sigma e^{-(\frac{1}{2}\pi + A)t}).$$

Finally consider C_2. Here $w = i\eta + \lambda e^{\frac{1}{4}i\pi}$, where λ is real, $|\lambda| \leqslant \sqrt{2}c\eta$. Hence

$$w^{s-1} = \exp[(s-1)\{\tfrac{1}{2}i\pi + \log(\eta + \lambda e^{-\frac{1}{4}i\pi})\}]$$

$$= \exp\left[(s-1)\left\{\tfrac{1}{2}i\pi + \log\eta + \frac{\lambda}{\eta}e^{-\frac{1}{4}i\pi} - \frac{1}{2}\frac{\lambda^2}{\eta^2}e^{-\frac{1}{2}i\pi} + O\left(\frac{\lambda^3}{\eta^3}\right)\right\}\right]$$

$$= O\left[\eta^{\sigma-1} \exp\left[\left\{-\tfrac{1}{2}\pi + \frac{\lambda}{\eta\sqrt{2}} - \frac{1}{2}\frac{\lambda^2}{\eta^2} + O\left(\frac{\lambda^3}{\eta^3}\right)\right\}t\right]\right].$$

Also

$$\frac{e^{-mw+xw}}{e^w - 1} = O\left(\frac{e^{(x-m-1)u}}{1 - e^{-u}}\right) \quad (u \geqslant 0), \quad = O\left(\frac{e^{(x-m)u}}{e^u - 1}\right) \quad (u < 0),$$

which is bounded for $u < -\frac{1}{2}\pi$ and $u > \frac{1}{2}\pi$; and

$$|e^{-xw}| = e^{-\lambda t/\eta\sqrt{2}}.$$

Hence the part with $|u| > \frac{1}{2}\pi$ is

$$O\left\{\eta^{\sigma-1} e^{-\frac{1}{2}\pi t} \int_{-c\eta\sqrt{2}}^{c\eta\sqrt{2}} \exp\left[\left\{-\frac{1}{2}\frac{\lambda^2}{\eta^2} + O\left(\frac{\lambda^3}{\eta^3}\right)\right\}t\right] d\lambda\right\}$$

$$= O\left\{\eta^{\sigma-1} e^{-\frac{1}{2}\pi t} \int_{-\infty}^{\infty} e^{-A\lambda^2\eta^{-2}t}\, d\lambda\right\} = O(\eta^\sigma t^{-\frac{1}{2}} e^{-\frac{1}{2}\pi t}).$$

The argument also applies to the part $|u| \leqslant \frac{1}{2}\pi$ if $|e^w - 1| > A$ on this part. If not, suppose, for example, that the contour goes too near to the pole at $w = 2q\pi i$. Take it round an arc of the circle $|w - 2q\pi i| = \frac{1}{2}\pi$. On this circle,

$$w = 2q\pi i + \tfrac{1}{2}\pi e^{i\theta}$$

and

$$\log(w^{s-1}e^{-mw}) = -\tfrac{1}{2}m\pi e^{i\theta}+(s-1)\{\tfrac{1}{2}i\pi+\log(2q\pi+\tfrac{1}{2}\pi e^{i\theta}/i)\}$$
$$= -\tfrac{1}{2}m\pi e^{i\theta}-\tfrac{1}{2}\pi t+(s-1)\log(2q\pi)+\frac{te^{i\theta}}{4q}+O(1).$$

Since
$$m\pi-\frac{t}{2q} = \frac{2mq\pi-t}{2q} = O(1),$$
this is
$$-\tfrac{1}{2}\pi t+(s-1)\log(2q\pi)+O(1).$$
Hence
$$|w^{s-1}e^{-mw}| = O(q^{\sigma-1}e^{-\frac{1}{2}\pi t}).$$
The contribution of this part is therefore
$$O(\eta^{\sigma-1}e^{-\frac{1}{2}\pi t}).$$

Since
$$e^{-i\pi s}\Gamma(1-s) = O(t^{\frac{1}{2}-\sigma}e^{\frac{1}{2}\pi t})$$
we have now proved that
$$\zeta(s) = \sum_{n=1}^{m}\frac{1}{n^s}+\chi(s)\sum_{n=1}^{q}\frac{1}{n^{1-s}}+O\{t^{\frac{1}{2}-\sigma}(e^{-At}+\eta^{\sigma}t^{-\frac{1}{2}}+\eta^{\sigma-1})\}.$$
The O-terms are
$$O(e^{-At})+O\left\{\left(\frac{t}{x}\right)^{\sigma}t^{-\sigma}\right\}+O\left\{t^{\frac{1}{2}-\sigma}\left(\frac{t}{x}\right)^{\sigma-1}\right\}$$
$$= O(e^{-At})+O(x^{-\sigma})+O(t^{-\frac{1}{2}}x^{1-\sigma}) = O(x^{-\sigma}).$$
This proves the theorem in the case considered.

To deduce the case $x \geqslant y$, change s into $1-s$ in the result already obtained. Then
$$\zeta(1-s) = \sum_{n\leqslant x}\frac{1}{n^{1-s}}+\chi(1-s)\sum_{n\leqslant y}\frac{1}{n^s}+O(x^{\sigma-1}).$$
Multiplying by $\chi(s)$, and using the functional equation and
$$\chi(s)\chi(1-s) = 1,$$
we obtain
$$\zeta(s) = \chi(s)\sum_{n\leqslant x}\frac{1}{n^{1-s}}+\sum_{n\leqslant y}\frac{1}{n^s}+O(t^{\frac{1}{2}-\sigma}x^{\sigma-1}).$$
Interchanging x and y, this gives the theorem with $x \geqslant y$.

4.16. Further approximations.† A closer examination of the above analysis, together with a knowledge of the formulae of § 2.10, shows that the O-terms in the approximate functional equation can be replaced by an asymptotic series, each term of which contains trigonometrical functions and powers of t only.

† Siegel (2).

We shall consider only the simplest case in which $x = y = \sqrt{(t/2\pi)}$, $\eta = \sqrt{(2\pi t)}$. In the neighbourhood of $w = i\eta$ we have
$$(s-1)\log\frac{w}{i\eta} = (s-1)\log\left(1+\frac{w-i\eta}{i\eta}\right)$$
$$= (\sigma+it-1)\left\{\frac{w-i\eta}{i\eta}-\frac{1}{2}\left(\frac{w-i\eta}{i\eta}\right)^2+\ldots\right\}$$
$$= \frac{\eta}{2\pi}(w-i\eta)+\frac{i}{4\pi}(w-i\eta)^2+\ldots.$$
Hence we write
$$e^{(s-1)\log(w/i\eta)} = e^{(\eta/2\pi)(w-i\eta)+(i/4\pi)(w-i\eta)^2}\phi\left(\frac{w-i\eta}{i\sqrt{(2\pi)}}\right),$$
where
$$\phi(z) = \exp\left\{(s-1)\log\left(1+\frac{z}{\sqrt{t}}\right)-iz\sqrt{t}+\tfrac{1}{2}iz^2\right\}$$
$$= \sum_{n=0}^{\infty}a_n z^n,$$
say. Now
$$\frac{d\phi}{dz} = \left(\frac{s-1}{z+\sqrt{t}}-i\sqrt{t}+iz\right)\phi(z) = \frac{\sigma-1+iz^2}{z+\sqrt{t}}\phi(z).$$
Hence
$$(z+\sqrt{t})\sum_{n=1}^{\infty}na_n z^{n-1} = (\sigma-1+iz^2)\sum_{n=0}^{\infty}a_n z^n,$$
and the coefficients a_n are determined in succession by the recurrence formula
$$(n+1)\sqrt{t}\,.\,a_{n+1} = (\sigma-n-1)a_n+ia_{n-2} \quad (n = 2, 3,\ldots),$$
this being true for $n = 0$, $n = 1$ also if we write $a_{-2} = a_{-1} = 0$. Thus
$$a_0 = 1, \qquad a_1 = \frac{\sigma-1}{\sqrt{t}}, \qquad a_2 = \frac{(\sigma-1)(\sigma-2)}{2t}, \qquad \ldots.$$
It follows that
$$a_n = O(t^{-\frac{1}{2}n+[\frac{1}{3}n]}) \tag{4.16.1}$$
(not uniformly in n); for if this is true up to n, then
$$a_{n+1} = O(t^{-\frac{1}{2}n+[\frac{1}{3}n]-\frac{1}{2}})+O(t^{-\frac{1}{2}(n-2)+[\frac{1}{3}(n-2)]-\frac{1}{2}}) = O(t^{-\frac{1}{2}(n+1)+[\frac{1}{3}(n+1)]}).$$
Hence (4.16.1) follows for all n by induction.

Now let
$$\phi(z) = \sum_{n=0}^{N-1}a_n z^n+r_N(z).$$
Then
$$r_N(z) = \frac{1}{2\pi i}\int_{\Gamma}\frac{\phi(w)z^N}{w^N(w-z)}\,dw,$$

where Γ is a contour including the points 0 and z. Now

$$\log\phi(w) = (s-1)\log\left(1+\frac{w}{\sqrt{t}}\right)+\tfrac{1}{2}iw^2-iw\sqrt{t}$$

$$= (\sigma-1)\log\left(1+\frac{w}{\sqrt{t}}\right)+iw^2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k+2}\left(\frac{w}{\sqrt{t}}\right)^k.$$

Hence for $|w| \leqslant \frac{3}{5}\sqrt{t}$ we have

$$\mathbf{R}\{\log\phi(w)\} \leqslant |\sigma-1|\log\frac{8}{5}+|w|^2\cdot\frac{5}{6}\frac{|w|}{\sqrt{t}}.$$

Let $|z| < \frac{4}{7}\sqrt{t}$, and let Γ be a circle with centre $w = 0$, radius ρ_N, where

$$\tfrac{21}{20}|z| \leqslant \rho_N \leqslant \tfrac{3}{5}\sqrt{t}.$$

Then $$r_N(z) = O(|z|^N\rho_N^{-N}e^{5\rho_N^3/6\sqrt{t}}).$$

The function $\rho^{-N}e^{5\rho^3/6\sqrt{t}}$ has the minimum $(5e/2N\sqrt{t})^{\frac{1}{3}N}$ for $\rho = (2N\sqrt{t}/5)^{\frac{1}{3}}$; ρ_N can have this value if

$$\frac{21}{20}|z| \leqslant \left(\frac{2N\sqrt{t}}{5}\right)^{\frac{1}{3}} \leqslant \frac{3}{5}\sqrt{t}.$$

Hence

$$r_N(z) = O\left\{|z|^N\left(\frac{5e}{2N\sqrt{t}}\right)^{\frac{1}{3}N}\right\} \qquad \left\{N \leqslant \frac{27}{50}t, \quad |z| \leqslant \frac{20}{21}\left(\frac{2N\sqrt{t}}{5}\right)^{\frac{1}{3}}\right\}.$$

For $|z| \leqslant \frac{4}{7}\sqrt{t}$ we can also take $\rho_N = \frac{21}{20}|z|$, giving

$$r_N(z) = O\left[\left(\frac{20}{21}\right)^N\left\{\exp\frac{5}{6\sqrt{t}}\left(\frac{21}{20}|z|\right)^3\right\}\right] = O\left\{\exp\left(\frac{14}{29}|z|^2\right)\right\} \quad (|z| \leqslant \tfrac{1}{2}\sqrt{t}).$$

Now consider the integral along C_2, and take $c = 2^{-\frac{3}{2}}$. Then

$$\int_{C_2}\frac{w^{s-1}e^{-mw}}{e^w-1}\,dw = \int_{C_2}(i\eta)^{s-1}\frac{e^{(i/4\pi)(w-i\eta)^2+(\eta/2\pi)(w-i\eta)-mw}}{e^w-1}\sum_{n=0}^{N-1}a_n\left(\frac{w-i\eta}{i\sqrt{(2\pi)}}\right)^n dw+$$

$$+\int_{C_2}(i\eta)^{s-1}\frac{e^{(i/4\pi)(w-i\eta)^2+(\eta/2\pi)(w-i\eta)-mw}}{e^w-1}r_N\left(\frac{w-i\eta}{i\sqrt{(2\pi)}}\right)dw.$$

If $|e^w-1| > A$ on C_2, the last integral is, as in the previous section,

$$O\left[\eta^{\sigma-1}e^{-\frac{1}{2}\pi t}\left\{\int_0^{A(N\sqrt{t})^{\frac{1}{3}}}e^{-\lambda^2/4\pi}\left(\frac{\lambda}{\sqrt{(2\pi)}}\right)^N\left(\frac{5e}{2N\sqrt{t}}\right)^{\frac{1}{3}N}d\lambda+\int_{A(N\sqrt{t})^{\frac{1}{3}}}^{\frac{1}{2}\eta}e^{-(\lambda^2/4\pi)+(7\lambda^2/29\pi)}\,d\lambda\right\}\right]$$

$$= O\left[\eta^{\sigma-1}e^{-\frac{1}{2}\pi t}\left\{\left(\frac{5e}{2N\sqrt{t}}\right)^{\frac{1}{3}N}2^{\frac{1}{2}N}\Gamma(\tfrac{1}{2}N+\tfrac{1}{2})+e^{-A(N\sqrt{t})^{\frac{1}{3}}}\right\}\right]$$

$$= O\left\{\eta^{\sigma-1}e^{-\frac{1}{2}\pi t}\left(\frac{AN}{t}\right)^{\frac{1}{6}N}\right\}$$

for $N < At$. The case where the contour goes near a pole gives a similar result, as in the previous section.

In the first N terms we now replace C_2 by the infinite straight line of which it is a part, C_2' say. The integral multiplying a_n changes by

$$O\left\{\eta^{\sigma-1}e^{-\frac{1}{2}\pi t}\int_{\frac{1}{2}\eta}^{\infty}e^{-(\lambda^2/4\pi)+(\eta\lambda/2\pi\sqrt{2})-(m+1)(\lambda/\sqrt{2})}\left(\frac{\lambda}{\sqrt{(2\pi)}}\right)^n d\lambda\right\}.$$

Since $m+1 \geqslant t/\eta = \eta/(2\pi)$, this is

$$O\left\{\eta^{\sigma-1}e^{-\frac{1}{2}\pi t}\int_{\frac{1}{2}\eta}^{\infty}e^{-\lambda^2/4\pi}\left(\frac{\lambda}{\sqrt{(2\pi)}}\right)^n d\lambda\right\}.$$

We can write the integrand as

$$e^{-\lambda^2/8\pi}\times e^{-\lambda^2/8\pi}\left(\frac{\lambda}{\sqrt{(2\pi)}}\right)^n,$$

and the second factor is steadily decreasing for $\lambda > 2\sqrt{(n\pi)}$, and so throughout the interval of integration if $n < N < At$ with A small enough. The whole term is then

$$O\left\{\eta^{\sigma-1}e^{-\frac{1}{2}\pi t-(\eta^2/32\pi^2)}\left(\frac{\eta}{2\sqrt{(2\pi)}}\right)^n\right\} = O\{\eta^{\sigma-1}e^{-\frac{1}{2}\pi t-(t/16\pi)}(\tfrac{1}{2}\sqrt{t})^n\}.$$

Also $$a_n = (r_n-r_{n+1})z^{-n} = O\left\{\left(\frac{5e}{2n\sqrt{t}}\right)^{\frac{1}{3}n}\right\}.$$

Hence the total error is

$$O\left\{\eta^{\sigma-1}e^{-\frac{1}{2}\pi t-(t/16\pi)}\sum_{n=0}^{N-1}(\tfrac{1}{2}\sqrt{t})^n\left(\frac{5e}{2n\sqrt{t}}\right)^{\frac{1}{3}n}\right\} = O\left\{\eta^{\sigma-1}e^{-\frac{1}{2}\pi t-(t/16\pi)}\sum_{n=0}^{N-1}\left(\frac{5et}{16n}\right)^{\frac{1}{3}n}\right\}.$$

Now $(t/n)^{\frac{1}{3}n}$ increases steadily up to $n = t/e$, and so if $n < At$, where $A < 1/e$, it is

$$O(e^{\frac{1}{3}tA\log 1/A}).$$

Hence if $N < At$, with A small enough, the whole term is

$$O(e^{-(\frac{1}{2}\pi+A)t}).$$

We have finally the sum

$$(i\eta)^{s-1}\sum_{n=0}^{N-1}\frac{a_n}{i^n(2\pi)^{\frac{1}{2}n}}\int_{C_2'}\frac{e^{(i/4\pi)(w-i\eta)^2+(\eta/2\pi)(w-i\eta)-mw}}{e^w-1}(w-i\eta)^n\,dw.$$

The integral may be expressed as

$$-\int_L\exp\left\{\frac{i}{4\pi}(w+2m\pi i-i\eta)^2+\frac{\eta}{2\pi}(w+2m\pi i-i\eta)-mw\right\}\times$$

$$\times\frac{(w+2m\pi i-i\eta)^n}{e^w-1}\,dw,$$

where L is a line in the direction $\arg w = \frac{1}{4}\pi$, passing between 0 and $2\pi i$.

This is $n!$ times the coefficient of ξ^n in

$$-\int_L \exp\left\{\frac{i}{4\pi}(w+2m\pi i-i\eta)^2+\frac{\eta}{2\pi}(w+2m\pi i-i\eta)-mw+\xi(w+2m\pi i-i\eta)\right\}\frac{dw}{e^w-1}$$

$$=-\exp\left\{i(2m\pi-\eta)\left(\frac{3\eta}{4\pi}-\tfrac{1}{2}m+\xi\right)\right\}\int_L \exp\left\{\frac{iw^2}{4\pi}+w\left(\frac{\eta}{\pi}-2m+\xi\right)\right\}\frac{dw}{e^w-1}$$

$$=-2\pi\Psi\left(\frac{\eta}{\pi}-2m+\xi\right)\exp\left\{\frac{i\pi}{2}\left(\frac{\eta}{\pi}-2m+\xi\right)^2-\frac{5i\pi}{8}+i(2m\pi-\eta)\left(\frac{3\eta}{4\pi}-\tfrac{1}{2}m+\xi\right)\right\},$$

where
$$\Psi(a)=\frac{\cos\pi(\frac{1}{2}a^2-a-\frac{1}{8})}{\cos\pi a},$$

$$=2\pi(-1)^{m-1}e^{-\frac{1}{2}it-(5i\pi/8)}\Psi\left(\frac{\eta}{\pi}-2m+\xi\right)e^{\frac{1}{2}i\pi\xi^2}$$

$$=2\pi(-1)^{m-1}e^{-\frac{1}{2}it-(5i\pi/8)}\sum_{\mu=0}^{\infty}\Psi^{(\mu)}\left(\frac{\eta}{\pi}-2m\right)\frac{\xi^\mu}{\mu!}\sum_{\nu=0}^{\infty}\frac{(\frac{1}{2}i\pi\xi^2)^\nu}{\nu!}.$$

Hence we obtain

$$e^{\frac{1}{2}i\pi(s-1)}(2\pi t)^{\frac{1}{2}s-\frac{1}{2}}2\pi(-1)^{m-1}e^{-\frac{1}{2}it-(5i\pi/8)}\sum_{n=0}^{N-1}\sum_{\nu\leqslant\frac{1}{2}n}\frac{n!\,i^{\nu-n}}{\nu!\,(n-2\nu)!\,2^n}\times\left(\frac{2}{\pi}\right)^{\frac{1}{2}n-\nu}a_n\Psi^{(n-2\nu)}\left(\frac{\eta}{\pi}-2m\right).$$

Denoting the last sum by S_N, we have the following result.

THEOREM 4.16. *If $0\leqslant\sigma\leqslant 1$, $m=[\sqrt{(t/2\pi)}]$, and $N<At$, where A is a sufficiently small constant,*

$$\zeta(s)=\sum_{n=1}^{m}\frac{1}{n^s}+\chi(s)\sum_{n=1}^{m}\frac{1}{n^{1-s}}+(-1)^{m-1}e^{-\frac{1}{2}i\pi(s-1)}(2\pi t)^{\frac{1}{2}s-\frac{1}{2}}e^{-\frac{1}{2}it-(i\pi/8)}\Gamma(1-s)\left\{S_N+O\left\{\left(\frac{AN}{t}\right)^{\frac{1}{6}N}\right\}+O(e^{-At})\right\}.$$

4.17. Special cases

In the approximate functional equation, let $\sigma=\frac{1}{2}$ and
$$x=y=\{t/(2\pi)\}^{\frac{1}{2}}.$$
Then (4.12.4) gives
$$\zeta(\tfrac{1}{2}+it)=\sum_{n\leqslant x}n^{-\frac{1}{2}-it}+\chi(\tfrac{1}{2}+it)\sum_{n\leqslant x}n^{-\frac{1}{2}+it}+O(t^{-\frac{1}{4}}). \tag{4.17.1}$$

This can also be put into another form which is sometimes useful. We have
$$\chi(\tfrac{1}{2}+it)\chi(\tfrac{1}{2}-it)=1,$$
so that
$$|\chi(\tfrac{1}{2}+it)|=1.$$
Let
$$\vartheta=\vartheta(t)=-\tfrac{1}{2}\arg\chi(\tfrac{1}{2}+it),$$
so that
$$\chi(\tfrac{1}{2}+it)=e^{-2i\vartheta}.$$
Let
$$Z(t)=e^{i\vartheta}\zeta(\tfrac{1}{2}+it)=\{\chi(\tfrac{1}{2}+it)\}^{-\frac{1}{2}}\zeta(\tfrac{1}{2}+it). \tag{4.17.2}$$
Since
$$\{\chi(\tfrac{1}{2}+it)\}^{-\frac{1}{2}}=\pi^{-\frac{1}{2}it}\left\{\frac{\Gamma(\frac{1}{4}+\frac{1}{2}it)}{\Gamma(\frac{1}{4}-\frac{1}{2}it)}\right\}^{\frac{1}{2}}=\frac{\pi^{-\frac{1}{2}it}\Gamma(\frac{1}{4}+\frac{1}{2}it)}{|\Gamma(\frac{1}{4}+\frac{1}{2}it)|},$$
we have also
$$Z(t)=-2\pi^{\frac{1}{4}}\frac{\Xi(t)}{(t^2+\frac{1}{4})|\Gamma(\frac{1}{4}+\frac{1}{2}it)|}. \tag{4.17.3}$$
The function $Z(t)$ is thus real for real t, and
$$|Z(t)|=|\zeta(\tfrac{1}{2}+it)|.$$
Multiplying (4.17.1) by $e^{i\vartheta}$, we obtain
$$Z(t)=e^{i\vartheta}\sum_{n\leqslant x}n^{-\frac{1}{2}-it}+e^{-i\vartheta}\sum_{n\leqslant x}n^{-\frac{1}{2}+it}+O(t^{-\frac{1}{4}})$$
$$=2\sum_{n\leqslant x}n^{-\frac{1}{2}}\cos(\vartheta-t\log n)+O(t^{-\frac{1}{4}}). \tag{4.17.4}$$
Again, in Theorem 4.16, let $N=3$. Then
$$S_3=a_0\Psi\left(\frac{\eta}{\pi}-2m\right)+\frac{1}{2i}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}a_1\Psi'\left(\frac{\eta}{\pi}-2m\right)-\frac{a_2}{2\pi}\Psi''\left(\frac{\eta}{\pi}-2m\right)+\frac{a_2}{2i}\Psi\left(\frac{\eta}{\pi}-2m\right)$$
$$=\Psi\left(\frac{\eta}{\pi}-2m\right)+O(t^{-\frac{1}{2}})$$
$$=\frac{\cos\{t-(2m+1)\sqrt{(2\pi t)}-\frac{1}{8}\pi\}}{\cos\sqrt{(2\pi t)}}+O(t^{-\frac{1}{2}}),$$
and the O-term gives, for $\zeta(s)$, a term $O(t^{-\frac{1}{2}\sigma-\frac{1}{2}})$. In the case $\sigma=\frac{1}{2}$ we obtain, on multiplying by $e^{i\vartheta}$ and proceeding as before,
$$Z(t)=2\sum_{n=1}^{m}\frac{\cos(\vartheta-t\log n)}{n^{\frac{1}{2}}}+(-1)^{m-1}\left(\frac{2\pi}{t}\right)^{\frac{1}{4}}\frac{\cos\{t-(2m+1)\sqrt{(2\pi t)}-\frac{1}{8}\pi\}}{\cos\sqrt{(2\pi t)}}+O(t^{-\frac{3}{4}}). \tag{4.17.5}$$

4.18. A different type of approximate formula has been obtained by Meulenbeld.† Instead of using finite partial sums of the original Dirichlet series, we can approximate to $\zeta(s)$ by sums of the form

$$\sum_{n\leqslant x}\frac{\phi(n/x)}{n^s},$$

where $\phi(u)$ decreases from 1 to 0 as u increases from 0 to 1. This reduces considerably the order of the error terms. The simplest result of this type is

$$\zeta(s) = 2\sum_{n\leqslant x}\frac{1-n/x}{n^s}+\chi(s)\sum_{n\leqslant y}\frac{1}{n^{1-s}}-$$

$$-\chi(s)\sum_{y<n<2y}\frac{1}{n^{1-s}}+\frac{2\chi(s-1)}{x}\sum_{y<n<2y}\frac{1}{n^{2-s}}+O\left(\frac{1}{t^{2\sigma}}+\frac{1}{x^{\sigma}t^{\frac{1}{2}}}+\frac{1}{x^{\sigma-1}t}\right),$$

valid for $2\pi xy = |t|$, $|t| \geqslant (x+1)^{\frac{1}{2}}$, $-2 < \sigma < 2$.

There is also an approximate functional equation‡ for $\{\zeta(s)\}^2$. This is

$$\{\zeta(s)\}^2 = \sum_{n\leqslant x}\frac{d(n)}{n^s}+\chi^2(s)\sum_{n\leqslant y}\frac{d(n)}{n^{1-s}}+O(x^{\frac{1}{2}-\sigma}\log t), \tag{4.18.1}$$

where $0 \leqslant \sigma \leqslant 1$, $xy = (t/2\pi)^2$, $x \geqslant h > 0$, $y \geqslant h > 0$. The proofs of this are rather elaborate.

NOTES FOR CHAPTER 4

4.19. Lemmas 4.2 and 4.4 can be generalized by taking F to be k times differentiable, and satisfying $|F^{(k)}(x)| \geqslant \lambda > 0$ throughout $[a, b]$. By using induction, in the same way that Lemma 4.4 was deduced from Lemma 4.2, one finds that

$$\int_a^b e^{iF(x)}\,dx \ll_k \lambda^{-1/k}.$$

The error term $O(\lambda_2^{-\frac{4}{5}}\lambda_3^{-\frac{1}{5}})$ in Lemma 4.6 may be replaced by $O(\lambda_2^{-1}\lambda_3^{\frac{1}{3}})$, which is usually sharper in applications. To do this one chooses $\delta = \lambda_3^{-\frac{1}{3}}$ in the proof. It then suffices to show that

$$\int_{-\delta}^{\delta} e^{i\lambda x^2}(e^{if(x)}-1)\,dx \ll (\lambda\delta)^{-1}, \tag{4.19.1}$$

if f has a continuous first derivative and satisfies $f(x) \ll x^3\delta^{-3}$,

† Meulenbeld (1).
‡ Hardy and Littlewood (6), Titchmarsh (21).

$f'(x) \ll x^2\delta^{-3}$. Here we have written $\lambda = \frac{1}{2}F''(c)$ and

$$f(x) = F(x+c)-F(c)-\tfrac{1}{2}x^2F''(c).$$

If $\delta \leqslant (\lambda\delta)^{-1}$ then (4.19.1) is immediate. Otherwise we have

$$\int_{-\delta}^{\delta} = \int_{-\delta}^{-(\lambda\delta)^{-1}} + \int_{-(\lambda\delta)^{-1}}^{(\lambda\delta)^{-1}} + \int_{(\lambda\delta)^{-1}}^{\delta}.$$

The second integral on the right is trivially $O\{(\lambda\delta)^{-1}\}$, while the third, for example, is, on integrating by parts,

$$\int_{(\lambda\delta)^{-1}}^{\delta}(2i\lambda x e^{i\lambda x^2})\frac{e^{if(x)}-1}{2i\lambda x}\,dx =$$

$$\left[e^{i\lambda x^2}\frac{e^{if(x)}-1}{2i\lambda x}\right]_{(\lambda\delta)^{-1}}^{\delta} - \int_{(\lambda\delta)^{-1}}^{\delta} e^{i\lambda x^2}\frac{d}{dx}\left(\frac{e^{if(x)}-1}{2i\lambda x}\right)dx$$

$$\ll \max_{x=(\lambda\delta)^{-1},\delta}\left|\frac{f(x)}{\lambda x}\right| + \int_{(\lambda\delta)^{-1}}^{\delta}\left|\frac{xif'(x)e^{if(x)}-(e^{if(x)}-1)}{2i\lambda x^2}\right|dx$$

$$\ll (\lambda\delta)^{-1} + \int_{(\lambda\delta)^{-1}}^{\delta}\left|\frac{x^3\delta^{-3}}{2i\lambda x^2}\right|dx$$

$$\ll (\lambda\delta)^{-1}$$

as required. Similarly the error term $O\{(b-a)\lambda_2^{\frac{1}{5}}\lambda_3^{\frac{1}{5}}\}$ in Theorem 4.9 may be replaced by $O\{(b-a)\lambda_3^{\frac{1}{3}}\}$.

For further estimates along these lines see Vinogradov [2; pp. 86–91] and Heath-Brown [11; Lemmas 6 and 10]. These papers show that the error term $O((b-a)\lambda_2^{\frac{1}{5}}\lambda_3^{\frac{1}{5}})$ can be dropped entirely, under suitable conditions.

Lemmas 4.2 and 4.8 have the following corollary, which is sometimes useful.

LEMMA 4.19. *Let $f(x)$ be a real differentiable function on the interval $[a, b]$, let $f'(x)$ be monotonic, and let $0 < \lambda \leqslant |f'(x)| \leqslant \vartheta < 1$. Then*

$$\sum_{a<n\leqslant b} e^{2\pi i f(n)} \ll_{\vartheta} \lambda^{-1}.$$

4.20. Weighted approximate functional equations related to those mentioned in §4.18 have been given by Lavrik [1] and Heath-Brown [3; Lemma 1], [4; Lemma 1]. As a typical example one has

$$\zeta(s)^k = \sum_1^\infty d_k(n)n^{-s}w_s\left(\frac{n}{x}\right)+\chi(s)^k\sum_1^\infty d_k(n)n^{s-1}w_{1-s}\left(\frac{n}{y}\right)$$
$$+O(x^{1-\sigma}\log^k(2+x)e^{-t^2/4}) \qquad (4.20.1)$$

uniformly for $t \geqslant 1$, $|\sigma| \leqslant \frac{1}{2}t$, $xy = (t/2\pi)^k$, $x, y \gg 1$, for any fixed positive integer k. Here

$$w_s(u) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left((\tfrac{1}{2}t)^{-z/2}\frac{\Gamma\{\frac{1}{2}(s+z)\}}{\Gamma(\frac{1}{2}s)}\right)^k u^{-z}e^{z^2}\frac{dz}{z}$$
$$(c > \max(0, -\sigma)).$$

The advantage of (4.20.1) is the very small error term.

Although the weight $w_s(u)$ is a little awkward, it is easy to see, by moving the line of integration to $c = \pm 1$, for example, that

$$w_s(u) = \begin{cases} O(u^{-1}) & (u \geqslant 1), \\ 1+O(u)+O\left\{u^\sigma\left(\log\frac{2}{u}\right)^k e^{-\frac{1}{8}t^2}\right\} & (0 < u \leqslant 1), \end{cases}$$

uniformly for $0 \leqslant \sigma \leqslant 1$, $t \geqslant 1$. More accurate estimates are however possible.

To prove (4.20.1) one writes

$$\sum_1^\infty d_k(n)n^{-s}w_s\left(\frac{n}{x}\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left((\tfrac{1}{2}t)^{-\frac{1}{2}z}\frac{\Gamma\{\frac{1}{2}(s+z)\}}{\Gamma(\frac{1}{2}s)}\zeta(s+z)\right)^k x^z e^{z^2}\frac{dz}{z}$$
$$(c > \max(0, 1-\sigma)),$$

and moves the line of integration to $\mathbf{R}(z) = -d$, $d > \max(0, \sigma)$, giving

$$\frac{1}{2\pi i}\int_{-d-i\infty}^{-d+i\infty}\left((\tfrac{1}{2}t)^{-\frac{1}{2}z}\frac{\Gamma\{\frac{1}{2}(s+z)\}}{\Gamma(\frac{1}{2}s)}\zeta(s+z)\right)^k x^z e^{z^2}\frac{dz}{z}$$
$$+\zeta(s)^k+\mathrm{Res}(z=1-s).$$

The residue term is easily seen to be $O\{x^{1-\sigma}\log^k(2+x)e^{-t^2/4}\}$. In the integral we substitute $z = -w$, $x = (t/2\pi)^k y^{-1}$, and we apply the functional equation (2.6.4). This yields

$$\frac{1}{2\pi i}\int_{-d-i\infty}^{-d+i\infty}\left((\tfrac{1}{2}t)^{-z/2}\frac{\Gamma\{\frac{1}{2}(s+z)\}}{\Gamma(\frac{1}{2}s)}\zeta(s+z)\right)^k x^z e^{z^2}\frac{dz}{z}$$

$$= -\frac{\chi(s)^k}{2\pi i}\int_{d-i\infty}^{d+i\infty}\left((\tfrac{1}{2}t)^{-w/2}\frac{\Gamma\{(\frac{1}{2}(1-s+w)\}}{\Gamma\{\frac{1}{2}(1-s)\}}\zeta(1-s+w)\right)^k y^w e^{w^2}\frac{dw}{w}$$

$$= -\chi(s)^k\sum_1^\infty d_k(n)n^{s-1}w_{1-s}\left(\frac{n}{y}\right),$$

as required.

Another result of the same general nature is

$$|\zeta(\tfrac{1}{2}+it)|^{2k} = \sum_{m,n=1}^\infty d_k(m)d_k(n)m^{-\frac{1}{2}-it}n^{-\frac{1}{2}+it}W_t(mn)+O(e^{-t^2/2}) \qquad (4.20.2)$$

for $t \geqslant 1$, and any fixed positive integer k, where

$$W_t(u) = \frac{1}{\pi i}\int_{1-i\infty}^{1+i\infty}\left(\pi^{-z}\frac{\Gamma\{\frac{1}{2}(\frac{1}{2}+it+z)\}\,\Gamma\{\frac{1}{2}(\frac{1}{2}-it+z)\}}{\Gamma\{\frac{1}{2}(\frac{1}{2}+it)\}\,\Gamma\{\frac{1}{2}(\frac{1}{2}-it)\}}\right)^k u^{-z}e^{z^2}\frac{dz}{z}.$$

This type of formula has the advantage that the cross terms which would arise on multipling (4.20.1) by its complex conjugate are absent. By moving the line of integration to $\mathbf{R}(z) = \pm\frac{1}{2}$ one finds that

$$W_t(u) = 2+O\left\{u^{\frac{1}{2}}\log^k\left(\frac{2}{u}\right)\right\} \quad (0 < u \leqslant 1),$$

and $W_t(u) = O(u^{-\frac{1}{2}})$ for $u \geqslant 1$. Again better estimates are possible. The proof of (4.20.2) is similar to that of (4.20.1), and starts from the formula

$$\tfrac{1}{2}\sum_{m,n=1}^\infty d_k(m)d_k(n)m^{-\frac{1}{2}-it}n^{-\frac{1}{2}+it}W_t(mn)$$
$$= \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\left(\pi^{-z}\frac{\Gamma\{\frac{1}{2}(\frac{1}{2}+it+z)\}\,\Gamma\{\frac{1}{2}(\frac{1}{2}-it+z)\}}{\Gamma\{\frac{1}{2}(\frac{1}{2}+it)\}\,\Gamma\{\frac{1}{2}(\frac{1}{2}-it)\}}\right.$$
$$\left.\zeta(\tfrac{1}{2}+it+z)\zeta(\tfrac{1}{2}-it+z)\right)^k e^{z^2}\frac{dz}{z}.$$

4.21. We may write the approximate functional equation (4.18.1) in the form
$$\zeta(s)^2 = S(s,x)+\chi(s)^2 S(1-s,y)+R(s,x).$$
The estimate $R(s,x) \ll x^{\frac{1}{2}-\sigma}\log t$ has been shown by Jutila (see Ivic [3; §4.2]) to be best possible for
$$t^{\frac{1}{2}} \ll \left|x-\frac{t}{2\pi}\right| \ll t^{\frac{1}{2}}.$$
Outside this range however, one can do better. Thus Jutila (in work to appear) has proved that
$$R(s,x) \ll t^{\frac{1}{2}}x^{-\sigma}(\log t)\log\left(1+\frac{x}{t}\right)+t^{-1}x^{1-\sigma}(y^{\varepsilon}+\log t)$$
for $0 \leqslant \sigma \leqslant 1$ and $x \gg t \gg 1$. (The corresponding result for $x \ll t$ may be deduced from this, via the functional equation.) For the special case $x = y = t/2\pi$ one may also improve on (4.18.1). Motohashi [2], [3], and in work in the course of publication, has established some very precise results in this direction. In particular he has shown that
$$\chi(1-s)R\left(s,\frac{t}{2\pi}\right) = -\left(\frac{4\pi}{t}\right)^{\frac{1}{2}}\Delta\left(\frac{t}{2\pi}\right)+O(t^{-\frac{1}{4}}),$$
where $\Delta(x)$ is the remainder term in the Dirichlet divisor problem (see §12.1). Jutila, in the work to appear, cited above, gives another proof of this. In fact, for the special case $\sigma = \frac{1}{2}$, the result was obtained 40 years earlier by Taylor (1).

V

THE ORDER OF $\zeta(s)$ IN THE CRITICAL STRIP

5.1. The main object of this chapter is to discuss the order of $\zeta(s)$ as $t \to \infty$ in the 'critical strip' $0 \leqslant \sigma \leqslant 1$. We begin with a general discussion of the order problem. It is clear from the original Dirichlet series (1.1.1) that $\zeta(s)$ is bounded in any half-plane $\sigma \geqslant 1+\delta > 1$; and we have proved in (2.12.2) that
$$\zeta(s) = O(|t|) \quad (\sigma \geqslant \tfrac{1}{2}).$$
For $\sigma < \frac{1}{2}$, corresponding results follow from the functional equation
$$\zeta(s) = \chi(s)\zeta(1-s).$$
In any fixed strip $\alpha \leqslant \sigma \leqslant \beta$, as $t \to \infty$
$$|\chi(s)| \sim \left(\frac{t}{2\pi}\right)^{\frac{1}{2}-\sigma}$$
by (4.12.3). Hence
$$\zeta(s) = O(t^{\frac{1}{2}-\sigma}) \quad (\sigma \leqslant -\delta < 0), \tag{5.1.1}$$
and
$$\zeta(s) = O(t^{\frac{1}{2}+\delta}) \quad (\sigma \geqslant -\delta).$$
Thus in any half-plane $\sigma \geqslant \sigma_0$
$$\zeta(s) = O(|t|^k), \qquad k = k(\sigma_0),$$
i.e. $\zeta(s)$ is a function of finite order in the sense of the theory of Dirichlet series.†

For each σ we define a number $\mu(\sigma)$ as the lower bound of numbers ξ such that
$$\zeta(\sigma+it) = O(|t|^{\xi}).$$
It follows from the general theory of Dirichlet series‡ that, as a function of σ, $\mu(\sigma)$ is continuous, non-increasing, and convex downwards in the sense that no arc of the curve $y = \mu(\sigma)$ has any point above its chord; also $\mu(\sigma)$ is never negative.

Since $\zeta(s)$ is bounded for $\sigma \geqslant 1+\delta$ $(\delta > 0)$, it follows that
$$\mu(\sigma) = 0 \quad (\sigma > 1), \tag{5.1.2}$$
and then from the functional equation that
$$\mu(\sigma) = \tfrac{1}{2}-\sigma \quad (\sigma < 0). \tag{5.1.3}$$
These equations also hold by continuity for $\sigma = 1$ and $\sigma = 0$ respectively.

† See Titchmarsh, *Theory of Functions*, §§ 9.4, 9.41.
‡ Ibid., §§ 5.65, 9.41.

The chord joining the points $(0,\frac{1}{2})$ and $(1,0)$ on the curve $y=\mu(\sigma)$ is $y=\frac{1}{2}-\frac{1}{2}\sigma$. It therefore follows from the convexity property that

$$\mu(\sigma)\leqslant\tfrac{1}{2}-\tfrac{1}{2}\sigma \quad (0<\sigma<1). \tag{5.1.4}$$

In particular, $\mu(\frac{1}{2})\leqslant\frac{1}{4}$, i.e.

$$\zeta(\tfrac{1}{2}+it)=O(t^{\frac{1}{4}+\epsilon}) \tag{5.1.5}$$

for every positive ϵ.

The exact value of $\mu(\sigma)$ is not known for any value of σ between 0 and 1. It will be shown later that $\mu(\frac{1}{2})<\frac{1}{4}$, and the simplest possible hypothesis is that the graph of $\mu(\sigma)$ consists of two straight lines

$$\mu(\sigma)=\tfrac{1}{2}-\sigma \quad (\sigma\leqslant\tfrac{1}{2}),\quad 0 \quad (\sigma>\tfrac{1}{2}). \tag{5.1.6}$$

This is known as Lindelöf's hypothesis. It is equivalent to the statement that

$$\zeta(\tfrac{1}{2}+it)=O(t^{\epsilon}) \tag{5.1.7}$$

for every positive ϵ.

The approximate functional equation gives a slight refinement on the above results. For example, taking $\sigma=\frac{1}{2}$, $x=y=\sqrt{(t/2\pi)}$ in (4.12.4), we obtain

$$\begin{aligned}\zeta(\tfrac{1}{2}+it)&=\sum_{n\leqslant\sqrt{(t/2\pi)}}\frac{1}{n^{\frac{1}{2}+it}}+O(1)\sum_{n\leqslant\sqrt{(t/2\pi)}}\frac{1}{n^{\frac{1}{2}-it}}+O(t^{-\frac{1}{4}})\\&=O\Big(\sum_{n\leqslant\sqrt{(t/2\pi)}}\frac{1}{n^{\frac{1}{2}}}\Big)+O(t^{-\frac{1}{4}})\\&=O(t^{\frac{1}{4}}).\end{aligned} \tag{5.1.8}$$

5.2. To improve upon this we have to show that a certain amount of cancelling occurs between the terms of such a sum. We have

$$\sum_{n=a+1}^{b}n^{-s}=\sum_{n=a+1}^{b}n^{-\sigma}e^{-it\log n}$$

and we apply the familiar lemma of 'partial summation'. *Let*

$$b_1\geqslant b_2\geqslant\ldots\geqslant b_n\geqslant 0,$$

and

$$s_m=a_1+a_2+\ldots+a_m$$

where the a's are any real or complex numbers. Then if

$$|s_m|\leqslant M \quad (m=1,2,\ldots),$$

$$|a_1b_1+a_2b_2+\ldots+a_nb_n|\leqslant Mb_1. \tag{5.2.1}$$

For

$$\begin{aligned}a_1b_1+\ldots+a_nb_n&=b_1s_1+b_2(s_2-s_1)+\ldots+b_n(s_n-s_{n-1})\\&=s_1(b_1-b_2)+s_2(b_2-b_3)+\ldots+s_{n-1}(b_{n-1}-b_n)+s_nb_n.\end{aligned}$$

Hence

$$|a_1b_1+\ldots+a_nb_n|\leqslant M(b_1-b_2+\ldots+b_{n-1}-b_n+b_n)=Mb_1.$$

If $0\leqslant b_1\leqslant b_2\leqslant\ldots\leqslant b_n$ we obtain similarly

$$|a_1b_1+\ldots+a_nb_n|\leqslant 2Mb_n.$$

If $a_n=e^{-it\log n}$, $b_n=n^{-\sigma}$, where $\sigma\geqslant 0$, it follows that

$$\sum_{n=a+1}^{b}n^{-s}=O\Big(a^{-\sigma}\max_{a<c\leqslant b}\Big|\sum_{n=a+1}^{c}e^{-it\log n}\Big|\Big). \tag{5.2.2}$$

This raises the general question of the order of sums of the form

$$\Sigma=\sum_{n=a+1}^{b}e^{2\pi if(n)}, \tag{5.2.3}$$

when $f(n)$ is a real function of n. In the above case,

$$f(n)=\frac{-t\log n}{2\pi}.$$

The earliest method of dealing with such sums is that of Weyl,† largely developed by Hardy and Littlewood.‡ This is roughly as follows. We can reduce the problem of Σ to that of

$$S=\sum_{n=a+1}^{b}e^{2\pi ig(n)},$$

where $g(n)$ is a polynomial of sufficiently high degree, say of degree k. Now

$$\begin{aligned}|S|^2&=\sum_m\sum_n e^{2\pi i(g(m)-g(n))}=\sum_\nu\sum_n e^{2\pi i(g(n+\nu)-g(n))}\\&\leqslant\sum_\nu\Big|\sum_n e^{2\pi i(g(n+\nu)-g(n))}\Big|\end{aligned} \tag{5.2.4}$$

with suitable limits for the sums; and $g(n+\nu)-g(n)$ is of degree $k-1$. By repeating the process we ultimately obtain a sum of the form

$$S_k=\sum_{n=a+1}^{b}e^{2\pi i(\lambda n+\mu)}.$$

We can now actually carry out the summation. We obtain

$$|S_k|=\left|\frac{1-e^{2\pi i(b-a)\lambda}}{1-e^{2\pi i\lambda}}\right|\leqslant\frac{1}{|\sin\pi\lambda|}. \tag{5.2.5}$$

If $|\operatorname{cosec}\pi\lambda|$ is small compared with $b-a$, this is a favourable result, and can be used to give a non-trivial result for the original sum S.

An alternative method is due to van der Corput.§ In this method we approximate to the sum Σ by the corresponding integral

$$\int_a^b e^{2\pi if(x)}\,dx,$$

† Weyl (1), (2). ‡ Littlewood (2), Landau (15).
§ van der Corput (1)–(7), van der Corput and Koksma (1), Titchmarsh (8)–(12).

and then estimate the integral by the principle of stationary phase, or some such method. Actually the original sum is usually not suitable for this process, and intermediate steps of the form (5.2.4) have to be used.

Still another method has been introduced by Vinogradov. This is in some ways very complicated; but it avoids the k-fold repetition used in the Weyl–Hardy–Littlewood method, which for large k is very 'uneconomical'. An account of this method will be given in the next chapter.

5.3. The Weyl–Hardy–Littlewood method. The relation of the general sum to the sum involving polynomials is as follows:

LEMMA 5.3. *Let k be a positive integer,*

$$t \geqslant 1, \qquad \frac{b-a}{a} \leqslant \tfrac{1}{2}t^{-1/(k+1)},$$

and $$\left|\sum_{m=1}^{\mu} \exp\left\{-it\left(\frac{m}{a}-\frac{1}{2}\frac{m^2}{a^2}+\ldots+\frac{(-1)^{k-1}m^k}{ka^k}\right)\right\}\right| \leqslant M \quad (\mu \leqslant b-a).$$

Then $$\left|\sum_{n=a+1}^{b} e^{-it\log n}\right| < AM.$$

For

$$\left|\sum_{n=a+1}^{b} e^{-it\log n}\right| = \left|\sum_{m=1}^{b-a} e^{-it\log(a+m)}\right|$$

$$= \left|\sum_{m=1}^{b-a} \exp\left\{-it\left(\frac{m}{a}-\ldots+\frac{(-1)^{k-1}m^k}{ka^k}\right)-it\left(\frac{(-1)^k m^{k+1}}{(k+1)a^{k+1}}+\ldots\right)\right\}\right|$$

$$= \left|\sum_{m=1}^{b-a} \exp\left\{-it\left(\frac{m}{a}-\ldots+\frac{(-1)^{k-1}m^k}{ka^k}\right)\right\} \sum_{\nu=0}^{\infty} e_\nu(t)\left(\frac{m}{a}\right)^{\nu}\right|, \text{ say,}$$

$$= \left|\sum_{\nu=0}^{\infty} \frac{e_\nu(t)}{a^\nu} \sum_{m=1}^{b-a} m^\nu \exp\left\{-it\left(\frac{m}{a}-\ldots+\frac{(-1)^{k-1}m^k}{ka^k}\right)\right\}\right|$$

$$\leqslant 2M \sum_{\nu=0}^{\infty} |e_\nu(t)| \left(\frac{b-a}{a}\right)^{\nu}$$

$$\leqslant 2M \exp\left[t\left\{\frac{(b-a)^{k+1}}{(k+1)a^{k+1}}+\ldots\right\}\right]$$

$$\leqslant 2M \exp\left\{t\,\frac{(b-a)^{k+1}}{a^{k+1}}\bigg/\left(1-\frac{b-a}{a}\right)\right\} \leqslant 2Me^2.$$

5.4. The simplest case is that of $\zeta(\frac{1}{2}+it)$, and we begin by working this out. We require the case $k = 2$ of the above lemma, and also the following

LEMMA. *Let* $$S = \sum_{m=1}^{\mu} e^{2\pi i(\alpha m^2+\beta m)}.$$

Then $$|S|^2 \leqslant \mu+2\sum_{r=1}^{\mu-1} \min(\mu, |\operatorname{cosec} 2\pi\alpha r|).$$

For $$|S|^2 = \sum_{m=1}^{\mu}\sum_{m'=1}^{\mu} e^{2\pi i(\alpha m^2+\beta m-\alpha m'^2-\beta m')}.$$

Putting $m' = m-r$, this takes the form

$$\sum_m \sum_r e^{2\pi i(2\alpha mr-\alpha r^2+\beta r)} \leqslant \sum_{r=-\mu+1}^{\mu-1} \left|\sum_m e^{4\pi i\alpha mr}\right|,$$

where, corresponding to each value of r, m runs over at most μ consecutive integers. Hence, by (5.2.5),

$$|S|^2 \leqslant \sum_{r=-\mu+1}^{\mu-1} \min(\mu, |\operatorname{cosec} 2\pi\alpha r|)$$

$$= \mu+2\sum_{r=1}^{\mu-1} \min(\mu, |\operatorname{cosec} 2\pi\alpha r|).$$

5.5. THEOREM 5.5. $\zeta(\frac{1}{2}+it) = O(t^{\frac{1}{6}}\log^{\frac{3}{2}} t)$.

Let $2t^{\frac{1}{3}} \leqslant a < At$, $b \leqslant 2a$, and let

$$\mu = [\tfrac{1}{2}at^{-\frac{1}{3}}]. \tag{5.5.1}$$

Then

$$\Sigma = \sum_{n=a+1}^{b} e^{-it\log n} = \sum_{n=a+1}^{a+\mu} + \sum_{a+\mu+1}^{a+2\mu} + \ldots + \sum_{a+N\mu+1}^{b} = \Sigma_1+\Sigma_2+\ldots+\Sigma_{N+1},$$

where $$N = \left[\frac{b-a}{\mu}\right] = O\left(\frac{a}{\mu}\right) = O(t^{\frac{1}{3}}).$$

By § 5.3, $\Sigma_\nu = O(M)$, where M is the maximum of

$$S_\nu = \sum_{m=1}^{\mu'} \exp\left\{-it\left(\frac{m}{a+\nu\mu}-\frac{1}{2}\frac{m^2}{(a+\nu\mu)^2}\right)\right\}$$

for $\mu' \leqslant \mu$. By § 5.4 this is

$$O\left[\left\{\mu+\sum_{r=1}^{\mu-1} \min\left(\mu, \left|\operatorname{cosec}\frac{tr}{2(a+\nu\mu)^2}\right|\right)\right\}\right]^{\frac{1}{2}}.$$

Hence

$$\Sigma = O\{(N+1)\mu^{\frac12}\}+O\left[\left\{\sum_{\nu=1}^{N+1} 1 \sum_{\nu=1}^{N+1}\sum_{r=1}^{\mu-1} \min\left(\mu, \left|\operatorname{cosec}\frac{tr}{2(a+\nu\mu)^2}\right|\right)\right\}^{\frac12}\right]$$

$$= O\{(N+1)\mu^{\frac12}\}+O\left[(N+1)^{\frac12}\left\{\sum_{r=1}^{\mu-1}\sum_{\nu=1}^{N+1} \min\left(\mu, \left|\operatorname{cosec}\frac{tr}{2(a+\nu\mu)^2}\right|\right)\right\}^{\frac12}\right].$$

Now $$\frac{tr}{2(a+\nu\mu)^2}-\frac{tr}{2\{a+(\nu+1)\mu\}^2} = \frac{tr\mu\{2a+(2\nu+1)\mu\}}{2(a+\nu\mu)^2\{a+(\nu+1)\mu\}^2},$$

which, as ν varies, lies between constant multiples of $tr\mu/a^3$, or, by (5.5.1), of r/μ^2. Hence for the values of ν for which $\frac12 tr/(a+\nu\mu)^2$ lies in a certain interval $\{l\pi, (l\pm\frac12)\pi\}$, the least value but one of

$$\left|\sin\frac{tr}{2(a+\nu\mu)^2}\right|$$

is greater than Ar/μ^2, the least but two is greater than $2Ar/\mu^2$, the least but three is greater than $3Ar/\mu^2$, and so on to $O(N) = O(t^{\frac13})$ terms. Hence these values of ν contribute

$$\mu+O\left(\frac{\mu^2}{r}+\frac{\mu^2}{2r}+\ldots\right) = \mu+O\left(\frac{\mu^2}{r}\log t\right) = O\left(\frac{\mu^2}{r}\log t\right).$$

The number of such intervals $\{l\pi, (l\pm\frac12)\pi\}$ is

$$O\left\{(N+1)\frac{r}{\mu^2}+1\right\}.$$

Hence the ν-sum is

$$O\{(N+1)\log t\}+O\left(\frac{\mu^2}{r}\log t\right).$$

Hence

$$\Sigma = O\{(N+1)\mu^{\frac12}\}+O(N+1)^{\frac12}\left[\sum_{r=1}^{\mu-1}\left\{(N+1)\log t+\frac{\mu^2}{r}\log t\right\}\right]^{\frac12}$$

$$= O\{(N+1)\mu^{\frac12}\}+O\{(N+1)\mu^{\frac12}\log^{\frac12}t\}+O\{(N+1)^{\frac12}\mu\log t\}$$

$$= O(a^{\frac12}t^{\frac16}\log^{\frac12}t)+O(at^{-\frac16}\log t).$$

If $a = O(t^{\frac12})$, the second term can be omitted. Then by partial summation

$$\sum_{n=a+1}^{b}\frac{1}{n^{\frac12+it}} = O(t^{\frac16}\log^{\frac12}t)\quad (b\leqslant 2a).$$

By adding $O(\log t)$ sums of the above form, we get

$$\sum_{2t^{\frac13}\leqslant n\leqslant (t/2\pi)^{\frac12}}\frac{1}{n^{\frac12+it}} = O(t^{\frac16}\log^{\frac32}t).$$

Also $$\sum_{n<2t^{\frac13}}\frac{1}{n^{\frac12+it}} = O\left(\sum_{n<2t^{\frac13}}\frac{1}{n^{\frac12}}\right) = O(t^{\frac16}).$$

The result therefore follows from the approximate functional equation.

5.6. We now proceed to the general case. We require the following lemmas.

LEMMA 5.6. *Let* $$f(x) = \alpha x^k+\ldots$$ *be a polynomial of degree k with real coefficients. Let*

$$S = \sum e^{2\pi i f(m)}$$

where m ranges over at most μ consecutive integers. Let $K = 2^{k-1}$. Then for $k\geqslant 2$

$$|S|^K \leqslant 2^{2K}\mu^{K-1}+2^K\mu^{K-k}\sum_{r_1,\ldots,r_{k-1}}\min(\mu, |\operatorname{cosec}(\pi\alpha k!\,r_1\ldots r_{k-1})|)$$

where each r varies from 1 to $\mu-1$. For $k = 1$ the sum is replaced by the single term $\min(\mu, |\operatorname{cosec}\pi\alpha|)$.

We have $$|S|^2 = \sum_m\sum_{m'} e^{2\pi i(f(m)-f(m'))}$$

$$= \sum_m\sum_r e^{2\pi i(f(m)-f(m-r_1))}\quad (m' = m-r_1)$$

$$\leqslant \sum_{r_1=-\mu+1}^{\mu-1}|S_1|,$$

where $$S_1 = \sum_m e^{2\pi i(f(m)-f(m-r_1))} = \sum_m e^{2\pi i(\alpha k r_1 m^{k-1}+\ldots)}$$

and, for each r_1, m ranges over at most μ consecutive integers. Hence by Hölder's inequality

$$|S|^2 \leqslant \left(\sum_{r_1=-\mu+1}^{\mu-1}1\right)^{1-2/K}\left(\sum_{r_1=-\mu+1}^{\mu-1}|S_1|^{\frac12K}\right)^{2/K}$$

$$\leqslant (2\mu)^{1-2/K}\left(\mu^{\frac12K}+\sum_{r_1=-\mu+1}^{\mu-1}{}'|S_1|^{\frac12K}\right)^{2/K},$$

where the dash denotes that the term $r_1 = 0$ is omitted. Hence

$$|S|^K \leqslant (2\mu)^{\frac12K-1}\left(\mu^{\frac12K}+\sum_{r_1=-\mu+1}^{\mu-1}{}'|S_1|^{\frac12K}\right).$$

If the theorem is true for $k-1$, then

$$|S_1|^{\frac12K} \leqslant 2^K\mu^{\frac12K-1}+$$
$$+2^{\frac12K}\mu^{\frac12K-k+1}\sum_{r_2,\ldots,r_{k-1}}\min(\mu, |\operatorname{cosec}\{\pi(\alpha k r_1)(k-1)!\,r_2\ldots r_{k-1}\}|).$$

Hence

$$|S|^K \leqslant 2^{\frac{1}{2}K-1}\mu^{K-1}+$$
$$+2^{\frac{3}{2}K}\mu^{K-1}+2^K\mu^{K-k}\sum_{r_1,\ldots,r_{k-1}}\min(\mu,|\operatorname{cosec}(\pi\alpha k!\,r_1\ldots r_{k-1})|),$$

and the result for k follows. Since by § 5.4 the result is true for $k = 2$, it holds generally.

5.7. LEMMA 5.7. *For $a < b \leqslant 2a$, $k \geqslant 2$, $K = 2^{k-1}$, $a = O(t)$, $t > t_0$,*

$$\Sigma = \sum_{n=a+1}^{b} n^{-it} = O(a^{1-1/K}t^{1/((k+1)K)}\log^{1/K}t)+O(at^{-1/((k+1)K)}\log^{k/K}t).$$

If $a \leqslant 4t^{1/(k+1)}$, then

$$\Sigma = O(a) = O(a^{1-1/K}t^{1/((k+1)K)})$$

as required. Otherwise, let

$$\mu = [\tfrac{1}{2}at^{-1/(k+1)}],$$

and write

$$\Sigma = \sum_{n=a+1}^{a+\mu}+\sum_{a+\mu+1}^{a+2\mu}+\ldots+\sum_{a+N\mu+1}^{b} = \Sigma_1+\ldots+\Sigma_{N+1}.$$

Then $\Sigma_\nu = O(M)$, where M is the maximum, for $\mu' \leqslant \mu$, of

$$S_\nu = \sum_{m=1}^{\mu'}\exp\left\{-it\left(\frac{m}{a+\nu\mu}-\frac{1}{2}\frac{m^2}{(a+\nu\mu)^2}+\ldots+(-1)^{k-1}\frac{m^k}{k(a+\nu\mu)^k}\right)\right\}.$$

By Lemma 5.6

$$S_\nu = O(\mu^{1-1/K})+O\left[\mu^{1-k/K}\left\{\sum_{r_1,\ldots,r_{k-1}}\min\left(\mu,\left|\operatorname{cosec}\frac{t(k-1)!\,r_1\ldots r_{k-1}}{2(a+\nu\mu)^k}\right|\right)\right\}^{1/K}\right].$$

Hence

$$\Sigma = O\{(N+1)\mu^{1-1/K}\}+$$
$$+O\left[\mu^{1-k/K}\sum_{\nu=1}^{N+1}\left\{\sum_{r_1,\ldots,r_{k-1}}\min\left(\mu,\left|\operatorname{cosec}\frac{t(k-1)!\,r_1\ldots r_{k-1}}{2(a+\nu\mu)^k}\right|\right)\right\}^{1/K}\right]$$
$$= O\{(N+1)\mu^{1-1/K}\}+$$
$$+O\left[\mu^{1-k/K}(N+1)^{1-1/K}\left\{\sum_{\nu=1}^{N+1}\sum_{r_1,\ldots,r_{k-1}}\min\left(\mu,\left|\operatorname{cosec}\frac{t(k-1)!\,r_1\ldots r_{k-1}}{2(a+\nu\mu)^k}\right|\right)\right\}^{1/K}\right]$$

by Hölder's inequality.

Now as ν varies,

$$\frac{t(k-1)!\,r_1\ldots r_{k-1}}{2(a+\nu\mu)^k}-\frac{t(k-1)!\,r_1\ldots r_{k-1}}{2\{a+(\nu-1)\mu\}^k}$$

lies between constant multiples of $t(k-1)!\,r_1\ldots r_{k-1}\mu a^{-k-1}$, i.e. of $(k-1)!\,r_1\ldots r_{k-1}\mu^{-k}$. The number of intervals of the form $\{l\pi, (l\pm\frac{1}{2})\pi\}$ containing values of $\frac{1}{2}t(k-1)!\,r_1\ldots r_{k-1}(a+\nu\mu)^{-k}$ is therefore

$$O\{(N+1)(k-1)!\,r_1\ldots r_{k-1}\mu^{-k}+1\}.$$

The part of the ν-sum corresponding to each of these intervals is, as in the previous case,

$$\mu+O\left(\frac{\mu^k}{(k-1)!\,r_1\ldots r_{k-1}}\right)+O\left(\frac{\mu^k}{2.(k-1)!\,r_1\ldots r_{k-1}}\right)+\ldots$$
$$= \mu+O\left(\frac{\mu^k\log t}{(k-1)!\,r_1\ldots r_{k-1}}\right) = O\left(\frac{\mu^k\log t}{r_1\ldots r_{k-1}}\right).$$

Hence the ν-sum is

$$O\{(N+1)\log t\}+O\left(\frac{\mu^k\log t}{r_1\ldots r_{k-1}}\right).$$

Summing with respect to $r_1,\ldots, r_{k-1}$, we obtain

$$O\{(N+1)\mu^{k-1}\log t\}+O(\mu^k\log^k t).$$

Hence

$$\Sigma = O\{(N+1)\mu^{1-1/K}\}+O\{(N+1)\mu^{1-1/K}\log^{1/K}t\}+O\{(N+1)^{1-1/K}\mu\log^{k/K}t\}.$$

The first term on the right can be omitted, and since

$$N+1 = O\left(\frac{b-a}{\mu}+1\right) = O(t^{1/(k+1)})$$

the result stated follows.

5.8. THEOREM 5.8. *If l is a fixed integer greater than 2, and $L = 2^{l-1}$, then*

$$\zeta(s) = O(t^{1/((l+1)L)}\log^{1+1/L}t) \quad (\sigma = 1-1/L). \tag{5.8.1}$$

The second term in Lemma 5.7 can be omitted if

$$a \leqslant t^{2/(k+1)}\log^{1-k}t.$$

Taking $k = l$ and applying the result $O(\log t)$, times we obtain

$$\sum_{n\leqslant N} n^{-it} = O(N^{1-1/L}t^{1/\{(l+1)L\}}\log^{1/L}t), \tag{5.8.2}$$

for $N \leqslant t^{2/(l+1)}\log^{1-l}t$. Similarly, for $k < l$, we find

$$\sum_{t^{2/(k+2)}\log^{-k}t<n\leqslant N} n^{-it} = O(N^{1-1/K}t^{1/\{(k+1)K\}}\log^{1/K}t)$$

for $t^{2/(k+2)}\log^{-k}t < N \leqslant t^{2/(k+1)}\log^{1-k}t$. The error term here is at most $O(N^{1-1/L}t^{\alpha}\log^{\beta}t)$ with

$$\alpha = \left(\frac{1}{L}-\frac{1}{K}\right)\frac{2}{k+2}+\frac{1}{(k+1)K}, \quad \beta = -\left(\frac{1}{L}-\frac{1}{K}\right)k+\frac{1}{K}.$$

Thus $\beta \leqslant 1/L$. When $k = l-1$ we have

$$\alpha = \left(\frac{1}{L}-\frac{2}{L}\right)\frac{2}{l+1}+\frac{2}{lL} = \frac{2}{l(l+1)L} < \frac{1}{(l+1)L},$$

and for $2 \leqslant k \leqslant l-2$ we have

$$\alpha \leqslant \left(\frac{1}{4K}-\frac{1}{K}\right)\frac{2}{k+2}+\frac{1}{(k+1)K} = -\frac{k-1}{2(k+1)(k+2)K} \leqslant 0 < \frac{1}{(l+1)L}.$$

It therefore follows, on summing over k, that (5.8.2) holds for $N \leqslant t^{\frac{1}{2}}\log^{-1}t$. Hence, by partial summation, we have

$$\sum_{n \leqslant (t/2\pi)^{\frac{1}{2}}} n^{-s} = O(t^{1/\{(l+1)L\}}\log^{l+1/L}t),$$

$$\sum_{n \leqslant (t/2\pi)^{\frac{1}{2}}} n^{s-1} = O(t^{2\sigma-1+1/\{(l+1)L\}}\log^{1/L}t),$$

and the theorem follows from the approximate functional equation.

5.9. van der Corput's method. In this method we approximate to sums by integrals as in Chapter IV.

THEOREM 5.9. *If $f(x)$ is real and twice differentiable, and*

$$0 < \lambda_2 \leqslant f''(x) \leqslant h\lambda_2 \quad (or \quad \lambda_2 \leqslant -f''(x) \leqslant h\lambda_2)$$

throughout the interval $[a, b]$, and $b \geqslant a+1$, then

$$\sum_{a<n\leqslant b} e^{2\pi i f(n)} = O\{h(b-a)\lambda_2^{\frac{1}{2}}\}+O(\lambda_2^{-\frac{1}{2}}).$$

If $\lambda_2 \geqslant 1$ the result is trivial, since the sum is $O(b-a)$. Otherwise Lemmas 4.7 and 4.4 give

$$O\{(\beta-\alpha+1)\lambda_2^{-\frac{1}{2}}\}+O\{\log(\beta-\alpha+2)\},$$

where $\qquad \beta-\alpha = f'(a)-f'(b) = O\{(b-a)h\lambda_2\}.$

Since

$$\log(\beta-\alpha+2) = O(\beta-\alpha+2) = O\{(b-a)h\lambda_2\}+O(1) = O\{(b-a)h\lambda_2^{\frac{1}{2}}\}+O(1),$$

the result follows.

5.10. LEMMA 5.10. *Let $f(n)$ be a real function, $a < n \leqslant b$, and q a positive integer not exceeding $b-a$. Then*

$$\left|\sum_{a<n\leqslant b} e^{2\pi i f(n)}\right| < A\frac{b-a}{q^{\frac{1}{2}}}+A\left\{\frac{b-a}{q}\sum_{r=1}^{q-1}\left|\sum_{a<n\leqslant b-r} e^{2\pi i\{f(n+r)-f(n)\}}\right|\right\}^{\frac{1}{2}}.$$

For convenience in the proof, let $e^{2\pi i f(n)}$ denote 0 if $n \leqslant a$ or $n > b$. Then

$$\sum_n e^{2\pi i f(n)} = \frac{1}{q}\sum_n \sum_{m=1}^{q} e^{2\pi i f(m+n)},$$

the inner sum vanishing if $n \leqslant a-q$ or $n > b-1$. Hence

$$\left|\sum_n e^{2\pi i f(n)}\right| \leqslant \frac{1}{q}\sum_n\left|\sum_{m=1}^{q} e^{2\pi i f(m+n)}\right|$$

$$\leqslant \frac{1}{q}\left\{\sum_n 1 \sum_n\left|\sum_{m=1}^{q} e^{2\pi i f(m+n)}\right|^2\right\}^{\frac{1}{2}}.$$

Since there are at most $b-a+q \leqslant 2(b-a)$ values of n for which the inner sum does not vanish, this does not exceed

$$\frac{1}{q}\left\{2(b-a)\sum_n\left|\sum_{m=1}^{q} e^{2\pi i f(m+n)}\right|^2\right\}^{\frac{1}{2}}.$$

Now

$$\left|\sum_{m=1}^{q} e^{2\pi i f(m+n)}\right|^2 = \sum_{m=1}^{q}\sum_{\mu=1}^{q} e^{2\pi i\{f(m+n)-f(\mu+n)\}}$$

$$= q+\sum_{\mu<m}\sum e^{2\pi i\{f(m+n)-f(\mu+n)\}}+\sum_{m<\mu}\sum e^{2\pi i\{f(m+n)-f(\mu+n)\}}.$$

Hence

$$\sum_n\left|\sum_{m=1}^{q} e^{2\pi i f(m+n)}\right|^2 \leqslant 2(b-a)q+2\left|\sum_n\sum_{\mu<m}\sum e^{2\pi i\{f(m+n)-f(\mu+n)\}}\right|.$$

In the last sum, $f(m+n)-f(\mu+n) = f(\nu+r)-f(\nu)$, for given values of ν and r, $1 \leqslant r \leqslant q-1$, just $q-r$ times, namely $\mu = 1$, $m = r+1$, up to $\mu = q-r$, $m = q$, with a consequent value of n in each case. Hence the modulus of this sum is equal to

$$\left|\sum_{r=1}^{q-1}(q-r)\sum_\nu e^{2\pi i\{f(\nu+r)-f(\nu)\}}\right| \leqslant q\sum_{r=1}^{q-1}\left|\sum_\nu e^{2\pi i\{f(\nu+r)-f(\nu)\}}\right|. \tag{5.10.1}$$

Hence

$$\left|\sum_n e^{2\pi i f(n)}\right| \leqslant \frac{1}{q}\left\{4(b-a)^2q+4(b-a)q\sum_{r=1}^{q-1}\left|\sum_\nu e^{2\pi i\{f(\nu+r)-f(\nu)\}}\right|\right\}^{\frac{1}{2}},$$

and the result stated follows.

5.11. THEOREM 5.11. *Let $f(x)$ be real and have continuous derivatives up to the third order, and let $\lambda_3 \leqslant f'''(x) \leqslant h\lambda_3$, or $\lambda_3 \leqslant -f'''(x) \leqslant h\lambda_3$, and $b-a \geqslant 1$. Then*

$$\sum_{a<n\leqslant b} e^{2\pi i f(n)} = O\{h^{\frac{1}{2}}(b-a)\lambda_3^{\frac{1}{6}}\}+O\{(b-a)^{\frac{1}{2}}\lambda_3^{-\frac{1}{6}}\}.$$

Let
$$g(x) = f(x+r)-f(x).$$
Then
$$g''(x) = f''(x+r)-f''(x) = rf'''(\xi),$$
where $x < \xi < x+r$. Hence
$$r\lambda_3 \leqslant g''(x) \leqslant hr\lambda_3,$$
or the same for $-g''(x)$. Hence by Theorem 5.9
$$\sum_{a<n\leqslant b-r} e^{2\pi i g(n)} = O\{h(b-a)r^{\frac{1}{2}}\lambda_3^{\frac{1}{2}}\}+O(r^{-\frac{1}{2}}\lambda_3^{-\frac{1}{2}}).$$
Hence, by Lemma 5.10,
$$\sum_{a<n\leqslant b} e^{2\pi i f(n)} = O\left(\frac{b-a}{q^{\frac{1}{2}}}\right)+O\left[\frac{b-a}{q}\sum_{r=1}^{q-1}\{h(b-a)r^{\frac{1}{2}}\lambda_3^{\frac{1}{2}}+r^{-\frac{1}{2}}\lambda_3^{-\frac{1}{2}}\}\right]^{\frac{1}{2}}$$
$$= O\left(\frac{b-a}{q^{\frac{1}{2}}}\right)+O\{h(b-a)^2q^{\frac{1}{2}}\lambda_3^{\frac{1}{2}}+(b-a)q^{-\frac{1}{2}}\lambda_3^{-\frac{1}{2}}\}^{\frac{1}{2}}$$
$$= O\{(b-a)q^{-\frac{1}{2}}\}+O\{h^{\frac{1}{2}}(b-a)q^{\frac{1}{4}}\lambda_3^{\frac{1}{4}}\}+O\{(b-a)^{\frac{1}{2}}q^{-\frac{1}{4}}\lambda_3^{-\frac{1}{4}}\}.$$
The first two terms are of the same order in λ_3 if $q = [\lambda_3^{-\frac{1}{3}}]$ provided that $\lambda_3 \leqslant 1$. This gives
$$O\{h^{\frac{1}{2}}(b-a)\lambda_3^{\frac{1}{6}}\}+O\{(b-a)^{\frac{1}{2}}\lambda_3^{-\frac{1}{6}}\}$$
as stated. The theorem is plainly trivial if $\lambda_3 > 1$. The proof also requires that $q \leqslant b-a$. If this is not satisfied, then $b-a = O(\lambda_3^{-\frac{1}{3}})$,
$$b-a = O\{(b-a)^{\frac{1}{2}}\lambda_3^{-\frac{1}{6}}\},$$
and the result again follows.

5.12. THEOREM 5.12.
$$\zeta(\tfrac{1}{2}+it) = O(t^{\frac{1}{6}}\log t).$$

Taking $f(x) = -(2\pi)^{-1}t\log x$, we have
$$f'''(x) = -\frac{t}{\pi x^3}.$$
Hence if $b \leqslant 2a$ the above theorem gives
$$\sum_{a<n\leqslant b} n^{-it} = O\left\{a\left(\frac{t}{a^3}\right)^{\frac{1}{6}}\right\}+O\left\{a^{\frac{1}{2}}\left(\frac{t}{a^3}\right)^{-\frac{1}{6}}\right\}$$
$$= O(a^{\frac{1}{2}}t^{\frac{1}{6}})+O(at^{-\frac{1}{6}}),$$

and the second term can be omitted if $a \leqslant t^{\frac{2}{3}}$. Then by partial summation
$$\sum_{a<n\leqslant b}\frac{1}{n^{\frac{1}{2}+it}} = O(t^{\frac{1}{6}}). \tag{5.12.1}$$
Also, by Theorem 5.9,
$$\sum_{a<n\leqslant b} n^{-it} = O(t^{\frac{1}{2}})+O(at^{-\frac{1}{2}}),$$
and hence by partial summation
$$\sum_{a<n\leqslant b}\frac{1}{n^{\frac{1}{2}+it}} = O\left\{\left(\frac{t}{a}\right)^{\frac{1}{2}}\right\}+O\left\{\left(\frac{a}{t}\right)^{\frac{1}{2}}\right\}.$$
Hence (5.12.1) is also true if $t^{\frac{2}{3}} < a < t$. Hence, applying (5.12.1) $O(\log t)$ times, we obtain
$$\sum_{n<t}\frac{1}{n^{\frac{1}{2}+it}} = O(t^{\frac{1}{6}}\log t),$$
and the result follows.

5.13. THEOREM 5.13. *Let $f(x)$ be real and have continuous derivatives up to the k-th order, where $k \geqslant 4$. Let $\lambda_k \leqslant f^{(k)}(x) \leqslant h\lambda_k$ (or the same for $-f^{(k)}(x)$). Let $b-a \geqslant 1$, $K = 2^{k-1}$. Then*
$$\sum_{a<n\leqslant b} e^{2\pi i f(n)} = O\{h^{2/K}(b-a)\lambda_k^{1/(2K-2)}\}+O\{(b-a)^{1-2/K}\lambda_k^{-1/(2K-2)}\},$$
where the constants implied are independent of k.

If $\lambda_k \geqslant 1$ the theorem is trivial, as before. Otherwise, suppose the theorem true for all integers up to $k-1$. Let
$$g(x) = f(x+r)-f(x).$$
Then
$$g^{(k-1)}(x) = f^{(k-1)}(x+r)-f^{(k)}(x) = rf^{(k)}(\xi),$$
where $x < \xi < x+r$. Hence
$$r\lambda_k \leqslant g^{(k-1)}(x) \leqslant hr\lambda_k.$$
Hence the theorem with $k-1$ for k gives
$$\left|\sum_{a<n\leqslant b-r} e^{2\pi i g(n)}\right| < A_1 h^{4/K}(b-a)(r\lambda_k)^{1/(K-2)}+A_2(b-a)^{1-4/K}(r\lambda_k)^{-1/(K-2)}$$
(writing constants A_1, A_2 instead of the O's). Hence
$$\sum_{r=1}^{q-1}\left|\sum_{a<n\leqslant b-r} e^{2\pi i g(n)}\right| < A_1 h^{4/K}(b-a)q^{1+1/(K-2)}\lambda_k^{1/(K-2)}+$$
$$+2A_2(b-a)^{1-4/K}q^{1-1/(K-2)}\lambda_k^{-1/(K-2)}$$
since
$$\sum_{r=1}^{q-1} r^{-1/(K-2)} < \int_0^q r^{-1/(K-2)}\,dr = \frac{q^{1-1/(K-2)}}{1-1/(K-2)} \leqslant 2q^{1-1/(K-2)}$$

for $K \geqslant 4$. Hence, by Lemma 5.10,

$$\sum_{a<n\leqslant b} e^{2\pi i f(n)} \leqslant A_3(b-a)q^{-\frac{1}{2}}+A_4(b-a)^{\frac{1}{2}}q^{-\frac{1}{2}}\{A_1 h^{4/K}(b-a)q^{1+1/(K-2)}\lambda_k^{1/(K-2)}+$$
$$+2A_2(b-a)^{1-4/K}q^{1-1/(K-2)}\lambda_k^{-1/(K-2)}\}^{\frac{1}{2}}$$
$$\leqslant A_3(b-a)q^{-\frac{1}{2}}+A_4 A_1^{\frac{1}{2}} h^{2/K}(b-a)q^{1/(2K-4)}\lambda_k^{1/(2K-4)}+$$
$$+A_4(2A_2)^{\frac{1}{2}}(b-a)^{1-2/K}q^{-1/(2K-4)}\lambda_k^{-1/(2K-4)}.$$

To make the first two terms of the same order in λ_k, let

$$q = [\lambda_k^{-1/(K-1)}]+1.$$

Then

$$\lambda_k^{-1/(K-1)} \leqslant q \leqslant 2\lambda_k^{-1/(K-1)},$$
$$q^{1/(2K-4)}\lambda_k^{1/(2K-4)} \leqslant 2^{1/(2K-4)}\lambda_k^{1/(2K-4)(1-1/(K-1))} \leqslant 2\lambda_k^{1/(2K-2)},$$
$$q^{-1/(2K-4)}\lambda_k^{-1/(2K-4)} \leqslant \lambda_k^{-1/(2K-2)},$$

and we obtain

$$\left|\sum_{a<n\leqslant b} e^{2\pi i f(n)}\right| \leqslant (A_3+2A_4 A_1^{\frac{1}{2}})h^{2/K}(b-a)\lambda_k^{1/(2K-2)}+$$
$$+A_4(2A_2)^{\frac{1}{2}}(b-a)^{1-2/K}\lambda_k^{-1/(2K-2)}.$$

This gives the result for k; the constants are the same for k as for $k-1$ if

$$A_3+2A_4 A_1^{\frac{1}{2}} \leqslant A_1, \qquad A_4(2A_2)^{\frac{1}{2}} \leqslant A_2,$$

which are satisfied if A_1 and A_2 are large enough.

We have assumed in the proof that $q \leqslant b-a$, which is true if $2\lambda_k^{-1/(K-1)} \leqslant b-a$. Otherwise

$$\left|\sum_{a<n\leqslant b} e^{2\pi i f(n)}\right| \leqslant b-a \leqslant (b-a)^{\frac{1}{2}}(2\lambda_k^{-1/(K-1)})^{\frac{1}{2}} \leqslant 2^{\frac{1}{2}}(b-a)^{1-2/K}\lambda_k^{-1/(2K-2)},$$

and the result again holds.

5.14. THEOREM 5.14. *If $l \geqslant 3$, $L = 2^{l-1}$, $\sigma = 1-l/(2L-2)$,*

$$\zeta(s) = O(t^{1/(2L-2)}\log t). \tag{5.14.1}$$

We apply the above theorem with

$$f(x) = -\frac{t\log x}{2\pi}, \qquad f^{(k)}(x) = \frac{(-1)^k(k-1)!\,t}{2\pi x^k}.$$

If $a < n \leqslant b \leqslant 2a$, then

$$\frac{(k-1)!\,t}{2\pi(2a)^k} \leqslant |f^{(k)}(x)| \leqslant \frac{(k-1)!\,t}{2\pi a^k},$$

and we may apply the theorem with

$$\lambda_k = \frac{(k-1)!\,t}{2\pi(2a)^k}, \qquad h = 2^k.$$

Hence

$$\sum_{a<n\leqslant b} n^{-it} = O\left[2^{2k/K}a\left\{\frac{(k-1)!\,t}{2\pi(2a)^k}\right\}^{1/(2K-2)}\right]+O\left[a^{1-2/K}\left\{\frac{(k-1)!\,t}{2\pi(2a)^k}\right\}^{-1/(2K-2)}\right]$$
$$= O(a^{1-k/(2K-2)}t^{1/(2K-2)})+O(a^{1-2/K+k/(2K-2)}t^{-1/(2K-2)}). \tag{5.14.2}$$

The second term can be omitted if

$$a < At^{K/(kK-2K+2)}. \tag{5.14.3}$$

Hence by partial summation

$$\sum_{a<n\leqslant b} n^{-s} = O(a^{1-\sigma-k/(2K-2)}t^{1/(2K-2)}) \tag{5.14.4}$$

subject to (5.14.3). Taking $\sigma = 1-l/(2L-2)$,

$$\sum_{a<n\leqslant b} n^{-s} = O(a^{l/(2L-2)-k/(2K-2)}t^{1/(2K-2)}). \tag{5.14.5}$$

First take $k = l$. We obtain

$$\sum_{a<n\leqslant b} n^{-s} = O(t^{1/(2L-2)}) \quad (a < At^{L/(lL-2L+2)}).$$

Hence

$$\sum_{n\leqslant t^{L/(lL-2L+2)}} n^{-s} = \sum_{\frac{1}{2}t^{L/(lL-2L+2)}<n\leqslant t^{L/(lL-2L+2)}} +\ldots$$
$$= O(t^{1/(2L-2)})+O(t^{1/(2L-2)})+\ldots$$
$$= O(t^{1/(2L-2)}\log t). \tag{5.14.6}$$

Next

$$\sum_{t^{L/(lL-2L+2)}<n\leqslant t} \frac{1}{n^s} = \sum_{\frac{1}{2}t<n\leqslant t} + \sum_{\frac{1}{4}t<n\leqslant\frac{1}{2}t} +\ldots,$$

and to each term $\sum_{2^{-m}t<n\leqslant 2^{1-m}t}$ corresponds a $k < l$ such that

$$t^{K/\{(k+1)K-2K+1\}} < 2^{-m}t \leqslant t^{K/(kK-2K+2)}.$$

Then

$$\sum_{2^{-m}t<n\leqslant 2^{1-m}t} \frac{1}{n^s} = O\{t^{\{l/(2L-2)-k/(2K-2)\}K/\{(k+1)K-2K+1\}+1/(2K-2)}\}.$$

The index does not exceed that in (5.14.6) if

$$\left(\frac{l}{2L-2}-\frac{k}{2K-2}\right)\frac{K}{(k+1)K-2K+1}+\frac{1}{2K-2} \leqslant \frac{1}{2L-2},$$

which reduces to
$$L-K \geqslant (l-k)K,$$
i.e.
$$2^{l-k}-1 \geqslant l-k$$
which is true. Since there are again $O(\log t)$ terms,

$$\sum_{t^{L/(lL-2L+2)}<n\leqslant t} \frac{1}{n^s} = O(t^{1/(2L-2)}\log t).$$

The result therefore follows. Theorem 5.12 is the particular case $l = 3$, $L = 4$.

5.15. Comparison between the Hardy–Littlewood result and the van der Corput result. The Hardy–Littlewood method shows that the function $\mu(\sigma)$ satisfies

$$\mu\left(1-\frac{1}{2^{k-1}}\right) \leqslant \frac{1}{(k+1)2^{k-1}}, \tag{5.15.1}$$

and the van der Corput method that

$$\mu\left(1-\frac{l}{2^l-2}\right) \leqslant \frac{1}{2^l-2}. \tag{5.15.2}$$

For a given k, determine l so that

$$1-\frac{l-1}{2^{l-1}-2} < 1-\frac{1}{2^{k-1}} \leqslant 1-\frac{l}{2^l-2}.$$

Then (5.15.2) and the convexity of $\mu(\sigma)$ give

$$\mu\left(1-\frac{1}{2^{k-1}}\right) \leqslant \frac{\dfrac{1}{2^{k-1}}-\dfrac{l}{2^l-2}}{\dfrac{l-1}{2^{l-1}-2}-\dfrac{l}{2^l-2}}\,\frac{1}{2^{l-1}-2}+\frac{\dfrac{l-1}{2^{l-1}-2}-\dfrac{1}{2^{k-1}}}{\dfrac{l-1}{2^{l-1}-2}-\dfrac{l}{2^l-2}}\,\frac{1}{2^l-2}$$

$$=\frac{2^{l-k}-1}{l2^{l-1}-2^l+2} \leqslant \frac{1}{(k+1)2^{k-1}}$$

if $\qquad (k+1)(2^{l-1}-2^{k-1}) \leqslant (l-2)2^{l-1}+2.$

Since $2^{k-1} > (2^{l-1}-2)/(l-1)$, this is true if

$$(k+1)\left(2^{l-1}-\frac{2^{l-1}-2}{l-1}\right) \leqslant (l-2)2^{l-1}+2,$$

i.e. if $\qquad k+1 \leqslant l-1.$

Now $\qquad 2^{k-1} \leqslant \dfrac{2^l-2}{l} < \dfrac{2^l}{l} \leqslant 2^{l-3}$

if $l \geqslant 8$. Hence the Hardy–Littlewood result follows from the van der Corput result if $l \geqslant 8$.

For $4 \leqslant l \leqslant 7$ the relevant values of $1-\sigma$ are

H.–L.	$\frac{1}{4}$,	$\frac{1}{8}$,	$\frac{1}{16}$.	
v. d. C.	$\frac{2}{7}$,	$\frac{1}{6}$,	$\frac{3}{31}$,	$\frac{1}{18}$.

The values of k and l in these cases are 3, 4, 5 and 5, 6, 7 respectively. Hence $k \leqslant l-2$ in all cases.

5.16. Theorem 5.16.

$$\zeta(1+it) = O\left(\frac{\log t}{\log\log t}\right).$$

We have to apply the above results with k variable; in fact it will be seen from the analysis of § 5.13 and § 5.14 that the constants implied in the O's are independent of k. In particular, taking $\sigma = 1$ in (5.14.4), we have

$$\sum_{a<n\leqslant b}\frac{1}{n^{1+it}} = O(a^{-k/(2K-2)}t^{1/(2K-2)}) \qquad (a<b\leqslant 2a)$$

uniformly with respect to k, subject to (5.14.3). If

$$t^{K/\{(k+1)K-2K+1\}} < a \leqslant t^{K/(kK-2K+2)}$$

it follows that

$$\sum_{a<n\leqslant b}\frac{1}{n^{1+it}} = O(t^{1/(2K-2)-kK/\{(2K-2)(kK-K+1)\}}) = O(t^{-1/\{2(k-1)K+2\}}).$$

Writing $\qquad \displaystyle\sum_{t^{R/\{(r-1)R+1\}}<n\leqslant t}\frac{1}{n^{1+it}} = \sum_{\frac{1}{2}t<n\leqslant t}+\sum_{\frac{1}{4}t<n\leqslant\frac{1}{2}t}+\dots,$

and applying the above result with $k = 2, 3,\dots,$ or r, we obtain, since there are $O(\log t)$ terms,

$$\sum_{t^{R/\{(r-1)R+1\}}<n\leqslant t}\frac{1}{n^{1+it}} = O(t^{-1/\{2(r-1)R+2\}}\log t). \tag{5.16.1}$$

Let $r = [\log\log t]$. Then

$$2R \leqslant 2^{\log\log t} = (\log t)^{\log 2},$$

and

$$t^{1/\{2(r-1)R+2\}} \geqslant \exp\left(\frac{\log t}{(\log t)^{\log 2}\log\log t+2}\right) > \exp\{(\log t)^{A}\} > A\log t.$$

Hence the above sum is bounded. Also

$$\sum_{n\leqslant t^{R/\{(r-1)R+1}}\frac{1}{n^{1+it}} = O(\log t^{R/\{(r-1)R+1\}}) = O\left\{\frac{R\log t}{(r-1)R+1}\right\}$$

$$= O\left(\frac{\log t}{r}\right) = O\left(\frac{\log t}{\log\log t}\right).$$

This proves the theorem.

The same result can also be deduced from the Weyl–Hardy–Littlewood analysis.

5.17. THEOREM 5.17. *For $t > A$*

$$\zeta(s) = O(\log^5 t), \qquad \sigma \geqslant 1-\frac{(\log\log t)^2}{\log t}, \tag{5.17.1}$$

$$\zeta(s) \neq 0, \qquad \sigma \geqslant 1-A_1\frac{\log\log t}{\log t} \tag{5.17.2}$$

(*with some A_1*), *and*
$$\frac{1}{\zeta(1+it)} = O\left(\frac{\log t}{\log\log t}\right), \tag{5.17.3}$$

$$\frac{\zeta'(1+it)}{\zeta(1+it)} = O\left(\frac{\log t}{\log\log t}\right). \tag{5.17.4}$$

We observe that (5.14.1) holds with a constant independent of l, and also, by the Phragmén–Lindelöf theorem, uniformly for

$$\sigma \geqslant 1-l/(2L-2).$$

Let t be given (sufficiently large), and let

$$l = \left[\frac{1}{\log 2}\log\left(\frac{\log t}{\log\log t}\right)\right].$$

Then
$$L \leqslant 2^{(1/\log 2)\log(\log t/\log\log t)-1} = \frac{1}{2}\frac{\log t}{\log\log t},$$

and similarly
$$L \geqslant \frac{1}{4}\frac{\log t}{\log\log t}.$$

Hence

$$\frac{l}{2L-2} \geqslant \frac{l}{2L} \geqslant \frac{\log\log t-\log\log\log t-\log 2}{\log 2}\frac{\log\log t}{\log t} \geqslant \frac{(\log\log t)^2}{\log t}$$

for $t > A$ (large enough). Hence if

$$\sigma \geqslant 1-\frac{(\log\log t)^2}{\log t}$$

then
$$\sigma \geqslant 1-\frac{l}{2L-2}.$$

Hence (5.14.1) is applicable, and gives

$$\zeta(s) = O(t^{1/(2L-2)}\log t) = O(t^{1/L}\log t)$$
$$= O(t^{(4\log\log t/\log t)}\log t) = O(\log^5 t).$$

This proves (5.17.1). The remaining results then follow from Theorems 3.10 and 3.11, taking (for $t > A$)

$$\theta(t) = \frac{(\log\log t)^2}{\log t}, \qquad \phi(t) = 5\log\log t.$$

5.18. In this section we reconsider the problem of the order of $\zeta(\frac{1}{2}+it)$. Small improvements on Theorem 5.12 have been obtained by various different methods. Results of the form

$$\zeta(\tfrac{1}{2}+it) = O(t^{\alpha}\log^{\beta}t)$$

with
$$\alpha = \frac{163}{988},\ \frac{27}{164},\ \frac{229}{1392},\ \frac{19}{116},\ \frac{15}{92}$$

were proved by Walfisz (1), Titchmarsh (9), Phillips (1), Titchmarsh (24), and Min (1) respectively.† We shall give here the argument which leads to the index $\frac{27}{164}$. The main idea of the proof is that we combine Theorem 5.13 with Theorem 4.9, which enables us to transform a given exponential sum into another, which may be easier to deal with.

THEOREM 5.18.
$$\zeta(\tfrac{1}{2}+it) = O(t^{27/164}).$$

Consider the sum

$$\Sigma_1 = \sum_{a<n\leqslant b} n^{-it} = \sum_{a<n\leqslant b} e^{-it\log n},$$

where $a < b \leqslant 2a$, $a < A\sqrt{t}$. By § 5.10

$$\Sigma_1 = O\left(\frac{a}{q^{\frac{1}{2}}}\right)+O\left\{\left(\frac{a}{q}\sum_{r=1}^{q-1}|\Sigma_2|\right)^{\frac{1}{2}}\right\}, \tag{5.18.1}$$

where $q \leqslant b-a$, and

$$\Sigma_2 = \sum_{a<n\leqslant b-r} e^{-it\{\log(n+r)-\log n\}}.$$

We now apply Theorem 4.9. to Σ_2. We have

$$f(x) = -\frac{t}{2\pi}\{\log(x+r)-\log x\}, \qquad f'(x) = \frac{tr}{2\pi x(x+r)},$$

$$f''(x) = -\frac{tr}{2\pi}\frac{2x+r}{x^2(x+r)^2}, \qquad f'''(x) = \frac{tr}{\pi}\frac{3x^2+3xr+r^2}{x^3(x+r)^3}.$$

We can therefore apply Theorem 4.9 with $\lambda_2 = tra^{-3}$, $\lambda_3 = tra^{-4}$. Thus

$$\Sigma_2 = e^{-\frac{1}{4}\pi i}\sum_{\alpha<\nu\leqslant\beta}\frac{e^{2\pi i\phi(\nu)}}{|f''(x_\nu)|^{\frac{1}{2}}}+O\left(\frac{a^{\frac{3}{2}}}{t^{\frac{1}{2}}r^{\frac{1}{2}}}\right)+O\left\{\log\left(2+\frac{tr}{a^2}\right)\right\}+O\left(\frac{t^{\frac{2}{5}}r^{\frac{2}{5}}}{a^{\frac{3}{5}}}\right), \tag{5.18.2}$$

where $\phi(\nu) = f(x_\nu)-\nu x_\nu$, $\alpha = f'(b-r)$, $\beta = f'(a)$. Actually the log-term can be omitted, since it is $O(t^{\frac{2}{5}}r^{\frac{2}{5}}a^{-\frac{4}{5}})$.

Consider next the sum

$$\sum_{\alpha<\nu\leqslant\gamma} e^{2\pi i\phi(\nu)} \quad (\alpha < \gamma \leqslant \beta).$$

† Note that the proof of the lemma in Titchmarsh (24) is incorrect. The lemma should be replaced by the corresponding theorem in Titchmarsh (16).

The numbers x_ν are given by

$$\frac{tr}{2\pi x_\nu(x_\nu+r)} = \nu, \quad \text{i.e.} \quad x_\nu = \frac{1}{2}\left(r^2+\frac{2tr}{\pi\nu}\right)^{\frac{1}{2}}-\tfrac{1}{2}r.$$

Hence

$$\phi'(\nu) = \{f'(x_\nu)-\nu\}\frac{dx_\nu}{d\nu}-x_\nu = -x_\nu = \tfrac{1}{2}r-\frac{1}{2}\left(r^2+\frac{2tr}{\pi\nu}\right)^{\frac{1}{2}},$$

$$\phi''(\nu) = \frac{tr}{2\pi\nu^2(r^2+2tr/\pi\nu)^{\frac{1}{2}}} = \frac{1}{2}\left(\frac{tr}{2\pi}\right)^{\frac{1}{2}}\left(\frac{1}{\nu^{\frac{3}{2}}}-\frac{\pi r}{4t}\frac{1}{\nu^{\frac{1}{2}}}+\ldots\right),$$

since $$r\nu \leqslant rf'(a) = \frac{tr^2}{2\pi a(a+r)} \leqslant \frac{tr^2}{2\pi a^2} \leqslant \frac{t}{2\pi}.$$

It follows that

$$\frac{K_1(tr)^{\frac{1}{2}}}{\nu^{k-\frac{1}{2}}} < |\phi^{(k)}(\nu)| < \frac{K_2(tr)^{\frac{1}{2}}}{\nu^{k-\frac{1}{2}}} \quad (t > t_k),$$

where K_1, K_2,..., and t_k depend on k only. We may therefore apply Theorem 5.13, with $h = O(1)$, and

$$\lambda_k = K_3(tr)^{\frac{1}{2}}(tr/a^2)^{\frac{1}{2}-k} = K_3(tr)^{1-k}a^{2k-1}.$$

Hence

$$\sum_{\alpha<\nu\leqslant\gamma} e^{2\pi i\phi(\nu)} = O\left\{\frac{tr}{a^2}\left(\frac{a^{2k-1}}{t^{k-1}r^{k-1}}\right)^{1/(2K-2)}\right\}+O\left\{\left(\frac{tr}{a^2}\right)^{1-2/K}\left(\frac{a^{2k-1}}{t^{k-1}r^{k-1}}\right)^{-1/(2K-2)}\right\}.$$

Also $|f''(x_\nu)|^{-\frac{1}{2}}$ is monotonic and of the form $O(t^{-\frac{1}{2}}r^{-\frac{1}{2}}a^{\frac{3}{2}})$. Hence by partial summation

$$\sum_{\alpha<\nu\leqslant\beta}\frac{e^{2\pi i\phi(\nu)}}{|f''(x_\nu)|^{\frac{1}{2}}} = O\{(tr)^{\frac{1}{2}-(k-1)/(2K-2)}a^{(2k-1)/(2K-2)-\frac{1}{2}}\}+$$
$$+O\{(tr)^{\frac{1}{2}-2/K+(k-1)/(2K-2)}a^{4/K-\frac{1}{2}-(2k-1)/(2K-2)}\}.$$

Hence

$$\frac{1}{q}\sum_{r=1}^{q-1}|\Sigma_2| = O\{(tq)^{\frac{1}{2}-(k-1)/(2K-2)}a^{(2k-1)/(2K-2)-\frac{1}{2}}\}+$$
$$+O\{(tq)^{\frac{1}{2}-2/K+(k-1)/(2K-2)}a^{4/K-\frac{1}{2}-(2k-1)/(2K-2)}\}+O\{(tq)^{-\frac{1}{2}}a^{\frac{3}{2}}\}+O\{(tq)^{\frac{2}{5}}a^{-\frac{3}{5}}\}.$$

Inserting this in (5.18.1), and using the inequality

$$(X+Y+\ldots)^{\frac{1}{2}} \leqslant X^{\frac{1}{2}}+Y^{\frac{1}{2}}+\ldots$$

we obtain

$$\Sigma_1 = O(aq^{-\frac{1}{2}})+O\{(tq)^{\frac{1}{4}-(k-1)/(4K-4)}a^{(2k-1)/(4K-4)+\frac{1}{4}}\}+$$
$$+O\{(tq)^{\frac{1}{4}-1/K+(k-1)/(4K-4)}a^{2/K+\frac{1}{4}-(2k-1)/(4K-4)}\}+O\{(tq)^{-\frac{1}{4}}a^{\frac{3}{4}}\}+O\{(tq)^{\frac{1}{5}}a^{\frac{3}{10}}\}.$$

The first two terms on the right are of the same order if

$$q = [a^{(3K-2k-2)/(3K-k-2)}t^{-(K-k)/(3K-k-2)}],$$

and they are then of the form

$$O(a^{(3K-2)/(2(3K-k-2))}t^{(K-k)/(2(3K-k-2))}) = O(t^{(5K-2k-2)/(4(3K-k-2))}) \quad (a < A\sqrt{t}).$$

For $k = 2, 3, 4, 5, 6,\ldots$, the index has the values

$$\frac{1}{2},\quad \frac{3}{7},\quad \frac{5}{12},\quad \frac{17}{41},\quad \frac{73}{176},\ldots$$

and of these $\frac{17}{41}$ is the smallest. We therefore take $k = 5$,

$$q = [a^{\frac{36}{41}}t^{-\frac{11}{41}}] \quad (a > t^{\frac{11}{36}}),$$

and obtain

$$\Sigma_1 = O(a^{\frac{23}{41}}t^{\frac{11}{82}})+O(a^{\frac{147}{328}}t^{\frac{61}{328}})+O(a^{\frac{169}{164}}t^{-\frac{15}{82}})+O(a^{\frac{39}{82}}t^{\frac{5}{41}}).$$

This also holds if $q \geqslant b-a$, since then

$$\Sigma_1 = O(b-a) = O(q) = O(a^{\frac{36}{41}}t^{-\frac{11}{41}}),$$

which is of smaller order than the third term in the above right-hand side.

It is easily seen that the last two terms are negligible compared with the first if $a = O(\sqrt{t})$. Hence by partial summation

$$\sum_{a<n\leqslant b}\frac{1}{n^{\frac{1}{2}+it}} = O(a^{\frac{5}{82}}t^{\frac{11}{82}})+O(a^{-\frac{17}{328}}t^{\frac{61}{328}}) \quad (a > t^{\frac{11}{36}}).$$

Applying this with $a = N$, $b = 2N-1$; $a = 2N$, $b = 4N-1$,... until $b = [A\sqrt{t}]$, we obtain

$$\sum_{N<n\leqslant A\sqrt{t}}\frac{1}{n^{\frac{1}{2}+it}} = O(t^{\frac{27}{164}})+O(N^{-\frac{17}{328}}t^{\frac{61}{328}})$$
$$= O(t^{\frac{27}{164}}) \quad (N > t^{\frac{7}{17}}).$$

We require a subsidiary argument for $n \leqslant t^{\frac{7}{17}}$, and in fact (5.14.2) with $k = 4$ gives

$$\sum_{a<n\leqslant b} n^{-it} = O(a^{\frac{5}{7}}t^{\frac{1}{14}}) \qquad (a < At^{\frac{4}{9}}),$$

$$\sum_{a<n\leqslant b}\frac{1}{n^{\frac{1}{2}+it}} = O(a^{\frac{3}{14}}t^{\frac{1}{14}}),$$

and by adding terms of this type as before

$$\sum_{n\leqslant t^{7/17}}\frac{1}{n^{\frac{1}{2}+it}} = O(t^{\frac{3}{14}\cdot\frac{7}{17}+\frac{1}{14}}) = O(t^{\frac{19}{119}}) = O(t^{\frac{27}{164}}).$$

The result therefore follows from the approximate functional equation.

NOTES FOR CHAPTER 5

5.19. Two more completely different arguments have been given, leading to the estimate

$$\mu(\tfrac{1}{2}) \leqslant \tfrac{1}{6}. \tag{5.19.1}$$

Firstly Bombieri, in unpublished work, has used a method related to that of §6.12, together with the bound

$$\int_0^1\int_0^1 \left|\sum_{1\leqslant x\leqslant P} \exp\{2\pi i(\alpha x+\beta x^2)\}\right|^6 d\alpha d\beta \ll P^3\log P,$$

to prove (5.19.1). Secondly, (5.19.1) follows from the mean-value bound (7.24.4) of Iwaniec [1]. (This deep result is described in §7.24.)

Heath-Brown [9] has shown that the weaker estimate $\mu(\frac{1}{2})\leqslant\frac{3}{16}$ follows from an argument analogous to Burgess's [1] treatment of character sums. Moreover the bound $\mu(\frac{1}{2})\leqslant\frac{7}{32}$, which is weaker still, but none the less non-trivial, follows from Heath-Brown's [4] fourth-power moment (7.21.1), based on Weil's estimate for the Kloosterman sum. Thus there are some extremely diverse arguments leading to non-trivial bounds for $\mu(\frac{1}{2})$.

5.20. The argument given in §5.18 is generalized by the 'method of exponent pairs' of van der Corput (1), (2) and Phillips (1). Let s, c be positive constants, and let $\mathscr{F}(s,c)$ be the set of quadruples (N, I, f, y) as follows:

(i) N and y are positive and satisfy $yN^{-s}\geqslant 1$,

(ii) I is a subinterval of $(N, 2N]$,

(iii) f is a real valued function on I, with derivatives of all orders, satisfying

$$\left|f^{(n+1)}(x)-\frac{d^n}{dx^n}(yx^{-s})\right|\leqslant c\left|\frac{d^n}{dx^n}(yx^{-s})\right|. \tag{5.20.1}$$

for $n\geqslant 0$.

We then say that (p, q) is an 'exponent pair' if $0\leqslant p\leqslant\frac{1}{2}\leqslant q\leqslant 1$ and if for each $s>0$ there exists a sufficiently small $c=c(p,q,s)>0$ such that

$$\sum_{n\in I}\exp\{2\pi i f(n)\}\ll_{p,q,s}(yN^{-s})^p N^q, \tag{5.20.2}$$

uniformly for $(N, I, f, y)\in\mathscr{F}(s,c)$.

We may observe that yN^{-s} is the order of magnitude of $f'(x)$. It is immediate that $(0, 1)$ is an exponent pair. Moreover Theorems 5.9, 5.11, and 5.13 correspond to the exponent pairs $(\frac{1}{2},\frac{1}{2})$, $(\frac{1}{6},\frac{2}{3})$, and

$$\left(\frac{1}{2^k-2},\ \frac{2^k-k-1}{2^k-2}\right).$$

By using Lemma 5.10 one may prove that

$$A(p,q)=\left(\frac{p}{2p+2},\ \frac{p+q+1}{2p+2}\right)$$

is an exponent pair whenever (p, q) is. Similarly from Theorem 4.9, as sharpened in §4.19, one may show that

$$B(p,q)=(q-\tfrac{1}{2}, p+\tfrac{1}{2})$$

is an exponent pair whenever (p, q) is, providing that $p+2q\geqslant\frac{3}{2}$. Thus one may build up a range of pairs by repeated applications of these A and B processes. In doing this one should note that $B^2(p,q)=(p,q)$. Examples of exponent pairs are:

$$B(0,1)=(\tfrac{1}{2},\tfrac{1}{2}),\qquad AB(0,1)=(\tfrac{1}{6},\tfrac{2}{3}),\qquad A^2B(0,1)=(\tfrac{1}{14},\tfrac{11}{14}),$$
$$A^3B(0,1)=(\tfrac{1}{30},\tfrac{26}{30}),\qquad BA^2B(0,1)=(\tfrac{2}{7},\tfrac{4}{7}),\qquad A^4B(0,1)=(\tfrac{1}{62},\tfrac{57}{62}),$$
$$BA^3B(0,1)=(\tfrac{11}{30},\tfrac{16}{30}),\qquad ABA^2B(1,0)=(\tfrac{2}{18},\tfrac{13}{18}),$$
$$BA^4B(0,1)=(\tfrac{13}{31},\tfrac{16}{31}),\qquad ABA^3B(0,1)=(\tfrac{11}{82},\tfrac{57}{82}),$$
$$A^2BA^2B(0,1)=(\tfrac{2}{40},\tfrac{33}{40}),\qquad BABA^2B(0,1)=(\tfrac{4}{18},\tfrac{11}{18}).$$

To estimate the sum Σ_1 of §5.18 we may take

$$f(x)=\frac{t}{2\pi}\log x,\qquad y=\frac{t}{2\pi},\qquad s=1,$$

so that (5.20.1) holds for any $c\geqslant 0$. The exponent pair $(\frac{11}{82},\frac{57}{82})$ then yields

$$\textstyle\sum_1\ll t^{\frac{11}{82}}a^{\frac{46}{82}}$$

whence

$$\sum_{a<n\leqslant b} n^{-\frac{1}{2}-it}\ll t^{\frac{11}{82}}a^{\frac{5}{82}}\ll t^{\frac{27}{164}}$$

for $a\ll t^{\frac{1}{2}}$. We therefore recover Theorem 5.18.

The estimate $\mu(\frac{1}{2})\leqslant\frac{229}{1392}$ of Phillips (1) comes from a better choice of exponent pair. In general we will have

$$\mu(\tfrac{1}{2})\leqslant\tfrac{1}{2}(p+q-\tfrac{1}{2}),$$

providing that $q\geqslant p+\frac{1}{2}$. Rankin [1] has shown that the infimum of $\frac{1}{2}(p+q-\frac{1}{2})$, over all pairs generated from (0, 1) by the A and B processes, is 0·16451067.... (Graham, in work in the course of publication, gives further details.) Note however that there are exponent pairs better for certain problems than any which can be got in this way, as we shall see in §§6.17–18. These unfortunately do not seem to help in the estimation of $\mu(\frac{1}{2})$.

5.21. The list of bounds for $\mu(\frac{1}{2})$ may be extended as follows.

$\frac{163}{988} = 0{\cdot}164979\ldots$	Walfisz (1),
$\frac{27}{164} = 0{\cdot}164634\ldots$	Titchmarsh (9),
$\frac{229}{1392} = 0{\cdot}164511\ldots$	Phillips (1)
$0{\cdot}164510\ldots$	Rankin [1]
$\frac{19}{116} = 0{\cdot}163793\ldots$	Titchmarsh (24)
$\frac{15}{92} = 0{\cdot}163043\ldots$	Min (1)
$\frac{6}{37} = 0{\cdot}162162\ldots$	Haneke [1]
$\frac{173}{1067} = 0{\cdot}162136\ldots$	Kolesnik [2]
$\frac{35}{216} = 0{\cdot}162037\ldots$	Kolesnik [4]
$\frac{139}{858} = 0{\cdot}162004\ldots$	Kolesnik [5].

The value $\frac{6}{37}$ was obtained by Chen [1], independently of Haneke, but a little later.

The estimates from Titchmarsh (24) onwards depend on bounds for multiple sums. In proving Lemma 5.10 the sum over r on the left of (5.10.1) is estimated trivially. However, there is scope for further savings by considering the sum over r and ν as a two-dimensional sum, and using two dimensional analogues of the A and B processes given by Lemma 5.10 and Theorem 4.9. Indeed since further variables are introduced each time an A process is used, higher-dimensional sums will occur. Srinivasan [1] has given a treatment of double sums, but it is not clear whether it is sufficiently flexible to give, for example, new exponent pairs for one-dimensional sums.

VI

VINOGRADOV'S METHOD

6.1. STILL another method of dealing with exponential sums is due to Vinogradov.† This has passed through a number of different forms of which the one given here is the most successful. In the theory of the zeta-function, the method gives new results in the neighbourhood of the line $\sigma = 1$.

Let
$$f(n) = \alpha_k n^k + \ldots + \alpha_1 n + \alpha_0$$
be a polynomial of degree $k \geqslant 2$ with real coefficients, and let a and q be integers,
$$S(q) = \sum_{a<n\leqslant a+q} e^{2\pi i f(n)},$$
$$J(q,l) = \int_0^1 \ldots \int_0^1 |S(q)|^{2l}\, d\alpha_1 \ldots d\alpha_k.$$

The question of the order of $J(q,l)$ as a function of q is important in the method.

Since $S(q) = O(q)$ we have trivially $J(q,l) = O(q^{2l})$. Less trivially, we could argue as follows. We have
$$\{S(q)\}^k = \sum_{n_1,\ldots,n_k} e^{2\pi i\alpha_k(n_1^k+\ldots+n_k^k)+\ldots}$$
$$|S(q)|^{2k} = \sum_{m_1,\ldots,n_k} e^{2\pi i\alpha_k(m_1^k+\ldots+m_k^k-n_1^k-\ldots-n_k^k)+\ldots}.$$

On integrating over the k-dimensional unit cube, we obtain a zero factor if any of the numbers
$$m_1^h+\ldots+m_k^h-n_1^h-\ldots-n_k^h \quad (h = 1,\ldots, k)$$
is different from zero. Hence $J(q,k)$ is equal to the number of solutions of the system of equations
$$m_1^h+\ldots+m_k^h = n_1^h+\ldots+n_k^h \quad (h = 1,\ldots, k),$$
where $a < m_\nu \leqslant a+q$, $a < n_\nu \leqslant a+q$.

But it follows from these equations that the numbers n_ν are equal (in some order) to the numbers m_ν. Hence only the m_ν can be chosen

† Vinogradov (1)–(4), Tchudakoff (1)–(5), Titchmarsh (20), Hua (1).

arbitrarily, and so the total number of solutions is $O(q^k)$. Hence

$$J(q,k) = O(q^k)$$

and
$$J(q,l) = O\{q^{2l-2k}J(q,k)\} = O(q^{2l-k}).$$

This, however, is not sufficient for the application (see Lemma 6.8).

For any integer l, $J(q,l)$ is equal to the number of solutions of the equations

$$m_1^h+...+m_l^h = n_1^h+...+n_l^h \quad (h = 1, 2,..., k),$$

where $a < m_\nu \leqslant a+q$, $a < n_\nu \leqslant a+q$. Actually $J(q,l)$ is independent of a; for putting $M_\nu = m_\nu-a$, $N_\nu = n_\nu-a$, we obtain

$$\sum_{\nu=1}^{l}(M_\nu+a)^h = \sum_{\nu=1}^{l}(N_\nu+a)^h \quad (h = 1,..., k),$$

which is equivalent to

$$\sum_{\nu=1}^{l}M_\nu^h = \sum_{\nu=1}^{l}N_\nu^h \quad (h = 1,..., k),$$

and $0 < M_\nu \leqslant q$, $0 < N_\nu \leqslant q$.

Clearly $J(q,l)$ is a non-decreasing function of q.

6.2. LEMMA 6.2. *Let $m_1,..., m_k$, $n_1,..., n_k$ be two sets of integers, let*

$$s_h = \sum_{\nu=1}^{k} m_\nu^h, \qquad s_h' = \sum_{\nu=1}^{k} n_\nu^h,$$

and let σ_h, σ_h' be the h-th elementary symmetric functions of the m_ν and n_ν respectively. If $|m_\nu| \leqslant q$, $|n_\nu| \leqslant q$, and

$$|s_h-s_h'| \leqslant q^{h-1} \quad (h = 1,..., k), \tag{6.2.1}$$

then
$$|\sigma_h-\sigma_h'| \leqslant \tfrac{3}{4}(2kq)^{h-1} \quad (h = 2,..., k). \tag{6.2.2}$$

Clearly
$$|s_h| \leqslant kq^h, \qquad |s_h'| \leqslant kq^h,$$

and
$$|\sigma_h'| \leqslant \tbinom{k}{h}q^h \leqslant k^h q^h.$$

Now
$$\sigma_2 = \tfrac{1}{2}(s_1^2-s_2).$$

Hence
$$\begin{aligned}|\sigma_2-\sigma_2'| &= \tfrac{1}{2}|(s_1^2-s_2)-(s_1'^2-s_2')| \\ &\leqslant \tfrac{1}{2}|(s_1-s_1')(s_1+s_1')|+\tfrac{1}{2}|s_2-s_2'| \\ &\leqslant kq+\tfrac{1}{2}q \leqslant \tfrac{3}{2}kq,\end{aligned}$$

the result stated for $h = 2$.

Now suppose that (6.2.2) holds with $h = 2,..., j-1$, where $3 \leqslant j \leqslant k$, so that
$$|\sigma_h-\sigma_h'| \leqslant (2kq)^{h-1} \quad (h = 1,..., j-1).$$

By a well-known theorem on symmetric functions

$$s_j-\sigma_1 s_{j-1}+\sigma_2 s_{j-2}-...+(-1)^j j\sigma_j = 0.$$

Hence

$$\begin{aligned}|\sigma_j-\sigma_j'| &\leqslant \frac{1}{j}|s_j-s_j'|+\frac{1}{j}\sum_{h=1}^{j-1}|\sigma_h s_{j-h}-\sigma_h' s_{j-h}'| \\ &= \frac{|s_j-s_j'|}{j}+\frac{1}{j}\sum_{h=1}^{j-1}|(\sigma_h-\sigma_h')s_{j-h}+\sigma_h'(s_{j-h}-s_{j-h}')| \\ &\leqslant \frac{q^{j-1}}{j}+\frac{1}{j}\sum_{h=1}^{j-1}\{(2kq)^{h-1}kq^{j-h}+(kq)^h q^{j-h-1}\} \\ &= \frac{q^{j-1}}{j}\Big\{1+\sum_{h=1}^{j-1}(2^{h-1}+1)k^h\Big\} \\ &\leqslant \frac{q^{j-1}}{j}\sum_{h=0}^{j-1}2^h k^h = \frac{q^{j-1}}{j}\frac{(2k)^j-1}{2k-1} \\ &\leqslant (2kq)^{j-1}\frac{2k}{j(2k-1)} \leqslant \tfrac{2}{3}(2kq)^{j-1} \leqslant \tfrac{3}{4}(2kq)^{j-1}\end{aligned}$$

since $2k/(2k-1) \leqslant 2$ and $j \geqslant 3$. This proves the lemma.

6.3. LEMMA 6.3. *Let $1 < G < q$, and let $g_1,..., g_k$ be integers satisfying*

$$1 < g_1 < g_2 < ... < g_k \leqslant G, \qquad g_\nu-g_{\nu-1} > 1. \tag{6.3.1}$$

For each value of ν $(1 \leqslant \nu \leqslant k)$ let m_ν be an integer lying in the interval

$$-a+(g_\nu-1)q/G < m_\nu \leqslant -a+g_\nu q/G,$$

where $0 \leqslant a \leqslant q$. Then the number of sets of such integers $m_1,..., m_k$ for which the values of s_h $(h = 1,..., k)$ lie in given intervals of lengths not exceeding q^{h-1}, is $\leqslant (4kG)^{\frac{1}{2}k(k-1)}$.

If x is any number such that $|x| \leqslant q$, the above lemma gives

$$\begin{aligned}|(x-m_1)...(x-m_k)-(x-n_1)...(x-n_k)| &\leqslant \sum_{h=1}^{k}|\sigma_h-\sigma_h'||x|^{k-h} \\ &\leqslant q^{k-1}\Big\{1+\tfrac{3}{4}\sum_{h=2}^{k}(2k)^{h-1}\Big\} \\ &= q^{k-1}\Big\{1+\frac{3}{4}\frac{(2k)^k-2k}{2k-1}\Big\} \leqslant (2kq)^{k-1}\end{aligned}$$

since $k \geqslant 2$. If $n_1,..., n_k$ satisfy the same conditions as $m_1,..., m_k$, then $|m_k-n_\nu| \geqslant q/G$ for $\nu = 1, 2,..., k-1$. Hence, putting $x = n_k$,

$$(q/G)^{k-1}|m_k-n_k| \leqslant (2kq)^{k-1},$$

i.e.
$$|m_k-n_k| \leqslant (2kG)^{k-1}.$$

Thus the number of numbers m_k satisfying the requirements of the theorem does not exceed

$$(2kG)^{k-1}+1 \leqslant (4kG)^{k-1}.$$

Next, for a given value of m_k, the numbers $m_1,\dots, m_{k-1}$ satisfy similar conditions with $k-1$ instead of k, and hence the number of values of m_{k-1} is at most $\{4(k-1)G\}^{k-2} < (4kG)^{k-2}$. Proceeding in this way, we find that the total number of sets does not exceed

$$(4kG)^{(k-1)+(k-2)+\dots} = (4kG)^{\frac{1}{2}k(k-1)}.$$

6.4. LEMMA 6.4. *Under the same conditions as in Lemma 6.3, the number of sets of integers $m_1,\dots, m_k$ for which the numbers s_h $(h = 1,\dots, k)$ lie in given intervals of lengths not exceeding $cq^{h(1-1/k)}$, where $c > 1$, does not exceed*

$$(2c)^k(4kG)^{\frac{1}{2}k(k-1)}q^{\frac{1}{2}(k-1)}.$$

We divide the hth interval into

$$1+\left[\frac{cq^{h(1-1/k)}}{q^{h-1}}\right] \leqslant 2cq^{1-h/k}$$

parts, and apply Lemma 6.3. Since

$$\prod_{h=1}^{k} (2cq^{1-h/k}) = (2c)^k q^{\frac{1}{2}(k-1)}$$

we have at most $(2c)^k q^{\frac{1}{2}(k-1)}$ sets of sub-intervals, each satisfying the conditions of Lemma 6.3. For each set there are at most $(4kG)^{\frac{1}{2}k(k-1)}$ solutions, so that the result follows.

6.5. LEMMA 6.5. *Let $k < l$, let $f(n)$ be as in §6.1, and let*

$$I = \int_0^1 \dots \int_0^1 |Z_{m,g_1} \dots Z_{m,g_k}|^2 |S(q^{1-1/k})|^{2(l-k)}\, d\alpha_1 \dots d\alpha_k,$$

where

$$Z_{m,g_\nu} = \sum_{(g_\nu-1)2^{-m}q<n\leqslant g_\nu 2^{-m}q} e^{2\pi i f(n)}$$

and the g_ν satisfy (6.3.1) *with $1 < G = 2^m < q$. Then*

$$I \leqslant 2^{3k+(m+2)\frac{1}{2}k(k-1)-mk}(l-k)^k k^{\frac{1}{2}k(k-1)} q^{\frac{3}{2}k-\frac{1}{2}} J(q^{1-1/k}, l-k).$$

We have

$$I = \sum_{N_1,\dots,N_k} \Psi(N_1,\dots, N_k) \int_0^1 \dots \int_0^1 e^{2\pi i(N_k\alpha_k+\dots+N_1\alpha_1)} |S(q^{1-1/k})|^{2(l-k)}\, d\alpha_1 \dots d\alpha_k$$

$$\leqslant \sum_{N_1,\dots,N_k} \Psi(N_1,\dots, N_k) \int_0^1 \dots \int_0^1 |S(q^{1-1/k})|^{2(l-k)}\, d\alpha_1 \dots d\alpha_k,$$

where $\Psi(N_1,\dots, N_k)$ is the number of solutions of the equations

$$m_1^h+\dots+m_k^h-n_1^h-\dots-n_k^h = N_h \quad (h = 1,\dots, k)$$

for m_ν and n_ν in the interval $(g_\nu-1)\,2^{-m}q < x \leqslant g_\nu 2^{-m}q$. Moreover N_h runs over those integers for which one can solve

$$N_h = n_1'^h+\dots+n_{l-k}'^h - m_1'^h - \dots - m_{l-k}'^h,$$

where m_ν' and n_ν' lie in an interval $(a, a+q^{1-1/k}]$. As in §6.1 we can shift each range through $-a$, i.e. replace a by 0. Then N_h ranges over at most $2(l-k)q^{h(1-1/k)}$ values. Hence by Lemma 6.4, for given values of $n_1,\dots, n_k$, the number of sets of $(m_1,\dots, m_k)$ does not exceed

$$\{4(l-k)\}^k(2^{m+2}k)^{\frac{1}{2}k(k-1)}q^{\frac{1}{2}(k-1)}.$$

Also $(n_1,\dots, n_k)$ takes not more than $(1+2^{-m}q)^k \leqslant (2^{1-m}q)^k$ values. Hence

$$\sum_{N_1,\dots,N_k} \Psi(N_1,\dots, N_k) \leqslant \{4(l-k)\}^k k^{\frac{1}{2}k(k-1)} 2^{(m+2)\frac{1}{2}k(k-1)-mk+k} q^{\frac{3}{2}k-\frac{1}{2}},$$

and the result follows.

6.6. LEMMA 6.6. *The result of Lemma 6.5 holds whether the g's satisfy,* (6.3.1) *or not, if m has the value*

$$M = \left[\frac{\log q}{k\log 2}\right]. \tag{6.6.1}$$

Since

$$|Z_{M,g_\nu}| \leqslant 2^{-M}q+1 \leqslant 2^{1-M}q,$$

$$|Z_{M,g_1} \dots Z_{M,g_k}|^2 \leqslant (2^{1-M}q)^{2k},$$

it is sufficient to prove that

$$(2^{1-M}q)^{2k} \leqslant 2^{3k+(M+2)\frac{1}{2}k(k-1)-Mk}(l-k)^k k^{\frac{1}{2}k(k-1)} q^{\frac{3}{2}k-\frac{1}{2}},$$

or that $$q^{\frac{1}{2}k+\frac{1}{2}} \leqslant 2^{(M+2)\frac{1}{2}k(k-1)+Mk} k^{\frac{1}{2}k(k-1)},$$

or that $$(\tfrac{1}{2}k+\tfrac{1}{2})\log q \leqslant \tfrac{1}{2}k(k+1)M\log 2+\tfrac{1}{2}k(k-1)\log 4k,$$

or that $$\log q \leqslant kM\log 2+\frac{k(k-1)}{k+1}\log 4k.$$

Since $$M \geqslant \frac{\log q}{k\log 2}-1,$$

this is true if $$k\log 2 \leqslant \frac{k(k-1)}{k+1}\log 4k,$$

or $$\log 2 \leqslant \frac{k-1}{k+1}\log 4k,$$

which is true for $k \geqslant 2$.

6.7. LEMMA 6.7. *The set of integers $(g_1,\dots, g_l)$, where $k < l$, and each g_ν ranges over $(1, G]$, is said to be well-spaced if there are at least k of them, say $g_{j_1},\dots, g_{j_k}$, satisfying*

$$g_{j_\nu}-g_{j_{\nu-1}} > 1 \quad (\nu = 2,\dots, k).$$

The number of sets which are not well-spaced is at most $l!\,3^l G^{k-1}$.

Let $g_1',\dots, g_l'$ denote $g_1,\dots, g_l$ arranged in increasing order, and let $f_\nu = g_\nu' - g_{\nu-1}'$. If the set is not well-spaced, there are at most $k-2$ of the numbers f_ν for which $f_\nu > 1$.

Consider those sets in which exactly h $(0 \leqslant h \leqslant k-2)$ of the numbers f_ν are greater than 1. The number of ways in which these h f_ν's can be chosen from the total $l-1$ is $\binom{l-1}{h}$. Also each of the h f_ν's can take at most G values, and each of the rest at most 2 values. Since g'_1 takes at most G values, the total number of sets of g'_ν arising in this way is at most

$$\binom{l-1}{h} G^{h+1} 2^{l-h-1}.$$

The total number of not well-spaced sets g'_ν is therefore

$$\leqslant \sum_{h=0}^{k-2} \binom{l-1}{h} G^{h+1} 2^{l-h-1} \leqslant G^{k-1} \sum_{h=0}^{k-2} \binom{l-1}{h} 2^{l-h-1}$$
$$\leqslant G^{k-1}(1+2)^{l-1} < 3^l G^{k-1}.$$

Since the number of sets g_ν corresponding to each set g'_ν is at most $l!$, the result follows.

6.8. Lemma 6.8. *If* $l \geqslant \frac14 k^2+\frac54 k$ *and* M *is defined by* (6.6.1), *then*

$$J(q,l) \leqslant \max(1,M) 48^{2l}(l!)^2 l^k k^{\frac12 k(k-1)} q^{2(l-k)/k+\frac32 k-\frac12} J(q^{1-1/k}, l-k).$$

Suppose first that M is not less than 2, i.e. that $q \geqslant 2^{2k}$. Let μ be a positive integer not greater than $M-1$. Then

$$\mu \leqslant \frac{\log q}{k \log 2}-1, \quad \text{i.e.} \quad 2^{\mu+1} \leqslant q^{1/k}.$$

Let
$$S(q) = \sum_{g=1}^{2^\mu} \sum_{(g-1)2^{-\mu}q<n\leqslant g2^{-\mu}q} e^{2\pi i f(n)} = \sum_{g=1}^{2^\mu} Z_{\mu,g},$$

say. Then
$$\{S(q)\}^l = \sum Z_{\mu,g_1} \dots Z_{\mu,g_l},$$

where each g_ν runs from 1 to 2^μ, and the sum contains $2^{\mu l}$ terms.

We denote those products $Z_{\mu,g_1} \dots Z_{\mu,g_l}$ with well-spaced g's by Z'_μ. The number of these, M_μ say, does not exceed $2^{\mu l}$. In the remaining terms we divide each factor into two parts, so that we obtain products of the type $Z_{\mu+1,g_1} \dots Z_{\mu+1,g_l}$, each g lying in $(1, 2^{\mu+1})$. The number of such terms, $M_{\mu+1}$ say, does not exceed $l!\,3^l 2^{\mu(k-1)} 2^l = l!\,6^l 2^{\mu(k-1)}$, by Lemma 6.7. The terms of this type with well-spaced g's we denote by $Z'_{\mu+1}$, and the rest we subdivide again. We proceed in this way until finally Z'_M denotes all the products of order M, whether containing well-spaced g's or not. We then have

$$\{S(q)\}^l = \sum_{m=\mu}^{M} \sum Z'_m,$$

$$|S(q)|^{2l} \leqslant M \sum_{m=\mu}^{M} \left|\sum Z'_m\right|^2 \leqslant M \sum_{m=\mu}^{M} M_m \sum |Z'_m|^2, \qquad (6.8.1)$$

where M_m is the number of terms in the sum $\sum Z'_m$. By Lemma 6.7,

$$M_m \leqslant l!\,3^l 2^{(m-1)(k-1)} 2^l = l!\,6^l 2^{(m-1)(k-1)} \quad (m > \mu).$$

Consider, for example, $\sum |Z'_\mu|^2$. The general Z'_μ can be written

$$Z_{\mu,g_1} \dots Z_{\mu,g_k} Z_{\mu,g_{k+1}} \dots Z_{\mu,g_l},$$

where $g_1,\dots, g_k$ satisfy (6.3.1) with $G = 2^\mu$. Now, since the geometric mean does not exceed the arithmetic mean,

$$|Z_{\mu,g_{k+1}} \dots Z_{\mu,g_l}|^2 \leqslant \frac{1}{l-k} \sum_{\nu=k+1}^{l} |Z_{\mu,g_\nu}|^{2(l-k)}.$$

We divide these Z_{μ,g_ν} into parts of length $q^{1-1/k}-1$ (or less). The number of such parts does not exceed

$$\left[\frac{2^{-\mu}q}{q^{1-1/k}-1}\right]+1 \leqslant \frac{2^{-\mu}q}{q^{1-1/k}-1}+2^{-\mu-1}q^{1/k} \leqslant \frac{2^{-\mu}q}{\frac34 q^{1-1/k}}+2^{-\mu-1}q^{1/k} \leqslant 2^{1-\mu}q^{1/k},$$

since $q^{1-1/k} \geqslant q^{1/k} \geqslant 2^M \geqslant 4$. Each part is of the form $S(q^{1-1/k})$, or with $q^{1-1/k}$ replaced by a smaller number. Hence by Hölder's inequality†

$$|Z_{\mu,g_\nu}|^{2(l-k)} \leqslant (2^{1-\mu}q^{1/k})^{2(l-k)-1} \sum |S(q^{1-1/k})|^{2(l-k)}.$$

Hence

$$\sum |Z'_\mu|^2 \leqslant \frac{(2^{1-\mu}q^{1/k})^{2(l-k)-1}}{l-k} \sum_{g_\nu} |Z_{\mu,g_1} \dots Z_{\mu,g_k}|^2 \sum_{\nu=k+1}^{l} \sum |S(q^{1-1/k})|^{2(l-k)}.$$

Hence by Lemma 6.5, and the non-decreasing property of $J(q,l)$ as a function of q,

$$\int_0^1 \dots \int_0^1 \sum |Z'_\mu|^2 \, d\alpha_1 \dots d\alpha_k \leqslant (2^{1-\mu}q^{1/k})^{2(l-k)-1} M_\mu \, 2^{1-\mu}q^{1/k} \times$$
$$\times 2^{3k+(\mu+2)\frac12 k(k-1)-\mu k}(l-k)^k k^{\frac12 k(k-1)} q^{\frac32 k-\frac12} J(q^{1-1/k}, l-k)$$
$$= 2^{\mu(\frac12 k^2+\frac12 k-2l)+2l+k^2+k} M_\mu (l-k)^k k^{\frac12 k(k-1)} q^{2(l-k)/k+\frac32 k-\frac32} J(q^{1-1/k}, l-k).$$

A similar argument applies to Z'_m, with μ replaced by m. Hence

$$J(q,l) \leqslant M \sum_{m=\mu}^{M} 2^{m(\frac12 k^2+\frac12 k-2l)} M_m^2 \times$$
$$\times 2^{2l+k^2+k}(l-k)^k k^{\frac12 k(k-1)} q^{2(l-k)/k+\frac32 k-\frac12} J(q^{1-1/k}, l-k).$$

Also

$$\sum_{m=\mu}^{M} 2^{m(\frac12 k^2+\frac12 k-2l)} M_m^2$$
$$\leqslant 2^{\mu(\frac12 k^2+\frac12 k-2l)+2\mu l} + \sum_{m=\mu+1}^{M} 2^{m(\frac12 k^2+\frac12 k-2l)} (l!)^2 6^{2l} 2^{2(m-1)(k-1)}$$
$$= 2^{\frac12 \mu(k^2+k)} + (l!)^2 6^{2l} \sum_{m=\mu+1}^{M} 2^{m(\frac12 k^2+\frac52 k-2l-2)-2(k-1)}$$
$$\leqslant 2^{2\mu l} + (l!)^2 6^{2l} \leqslant 2(l!)^2 6^{2l},$$

† Here $S(q^{1-1/k})$ denotes any sum of the form $S(p)$ with $p \leqslant q^{1-1/k}$.

since we can start with an integer μ such that $2^{\mu l} < l!$. (Indeed we may take $\mu = 1$.) Hence

$$J(q,l) \leqslant M 2^{2l+k^2+k+1}(l!)^2 6^{2l} l^k k^{\frac{1}{2}k(k-1)} q^{2(l-k)/k+\frac{3}{2}k-\frac{1}{2}} J(q^{1-1/k}, l-k),$$

and since $\qquad 2^{2l+k^2+k+1}6^{2l} \leqslant 2^{6l}6^{2l} = 48^{2l}$

the result follows.

If $M < 2$, i.e. $q < 2^{2k}$, divide $S(q)$ into four parts, each of the form $S(q')$, where $q' \leqslant \frac{1}{4}q \leqslant q^{1-1/k}$. By Hölder's inequality

$$|S(q)|^{2l} \leqslant 4^{2l-1} \sum |S(q')|^{2l} \leqslant 4^{2l-1} q^{2k(1-1/k)} \sum |S(q')|^{2(l-k)}.$$

Integrating over the unit hypercube,

$$J(q,l) \leqslant 4^{2l-1} q^{2k(1-1/k)} \sum J(q', l-k)$$
$$\leqslant 4^{2l} q^{2k(1-1/k)} J(q^{1-1/k}, l-k),$$

and the result again follows.

6.9. LEMMA 6.9. *If r is any non-negative integer, and $l \geqslant \frac{1}{4}k^2+\frac{1}{4}k+kr$, then*

$$J(q,l) \leqslant K^r \log^r q \,.\, q^{2l-\frac{1}{2}k(k+1)+\delta_r}$$

where $\qquad \delta_r = \frac{1}{2}k(k+1)\left(1-\frac{1}{k}\right)^r, \qquad K = 48^{2l}(l!)^2 l^k k^{\frac{1}{2}k(k-1)}.$

This is obvious if $r = 0$, since then $\delta_0 = \frac{1}{2}k(k+1)$ and $J(q,l) \leqslant q^{2l}$. Assuming then that it is true up to $r-1$, Lemma 6.8 (in which $M \leqslant \log q$) gives

$$J(q,l) \leqslant K \log q \,.\, q^{2(l-k)/k+\frac{3}{2}k-\frac{1}{2}} \,.\, K^{r-1} \log^{r-1}(q^{1-1/k}) \times$$
$$\times q^{(1-1/k)(2(l-k)-\frac{1}{2}k(k+1)+\delta_{r-1})},$$

and the index of q reduces to $2l-\frac{1}{2}k(k+1)+\delta_r$.

6.10. LEMMA 6.10. *If $l = [k^2\log(k^2+k)+\frac{1}{4}k^2+\frac{5}{4}k]+1$, $k \geqslant 7$,*

$$J(q,l) \leqslant e^{64lk\log^2 k} \log^{2l} q \,.\, q^{2l-\frac{1}{2}k(k+1)+\frac{1}{2}}.$$

We have $\delta_r \leqslant \frac{1}{2}$ if $\qquad k(k+1)\left(1-\frac{1}{k}\right)^r \leqslant 1,$

i.e. if $\qquad \log\{k(k+1)\} \leqslant r \log \dfrac{k}{k-1}.$

This is true if $\qquad \log\{k(k+1)\} \leqslant r/k,$

or if $\qquad r = [k\log(k^2+k)]+1.$

Since $\qquad r < k\log^3 k+1 < 4k\log k, \qquad l < k^3,$

and

$$\log K < 2l\log 48 + 2l\log l + k\log l + \tfrac{1}{2}k(k-1)\log k$$
$$< 5l\log l + l\log k < 16 l \log k,$$

the result follows.

6.11. LEMMA 6.11. *Let M and N be integers, $N > 1$, and let $\phi(n)$ be a real function of n, defined for $M \leqslant n \leqslant M+N-1$, such that*

$$\delta \leqslant \phi(n+1)-\phi(n) \leqslant c\delta \quad (M \leqslant n \leqslant M+N-2),$$

where $\delta > 0$, $c \geqslant 1$, $c\delta \leqslant \frac{1}{2}$. Let $W > 0$. Let $\bar{x}$ denote the difference between x and the nearest integer. Then the number of values of n for which $\overline{\phi(n)} \leqslant W\delta$ is less than

$$(Nc\delta+1)(2W+1).$$

Let α be a given real number, and let G be the number of values of n for which

$$\alpha+h < \phi(n) \leqslant \alpha+h+\delta$$

for some integer h. To each h corresponds at most one n, so that $G \leqslant h_2-h_1+1$, where h_1 and h_2 are the least and greatest values of h. But clearly

$$\phi(M) \leqslant \alpha+h_1+\delta, \qquad \alpha+h_2 < \phi(M+N-1),$$

whence $\qquad h_2-h_1-\delta < \phi(M+N-1)-\phi(M) \leqslant (N-1)c\delta,$

and $\qquad G \leqslant (N-1)c\delta+\delta+1 \leqslant Nc\delta+1.$

The result of the lemma now follows from the fact that an interval of length $2W\delta$ may be divided into $[2W+1]$ intervals of length less than δ ($< \frac{1}{2}$).

6.12. LEMMA 6.12. *Let k and Q be integers, $k \geqslant 7$, $Q \geqslant 2$, and let $f(x)$ be real and have continuous derivatives up to the $(k+1)$th order in $[P+1, P+Q]$; let $0 < \lambda < 1$ and*

$$\lambda \leqslant \frac{f^{(k+1)}(x)}{(k+1)!} \leqslant 2\lambda \quad (P+1 \leqslant x \leqslant P+Q) \tag{6.12.1}$$

or the same for $-f^{(k+1)}(x)$, and let

$$\lambda^{-\frac{1}{k}} \leqslant Q \leqslant \lambda^{-1}. \tag{6.12.2}$$

Then

$$\left|\sum_{n=P+1}^{P+Q} e^{2\pi i f(n)}\right| < A e^{33k\log^2 k} Q^{1-\rho}\log Q, \tag{6.12.3}$$

where $\qquad \rho = (24k^2\log k)^{-1}.$

Let $\qquad q = [\lambda^{-1/(k+1)}]+1,$

so that $\qquad 2 \leqslant q \leqslant [Q^{4/(k+1)}]+1 \leqslant Q,$

and write $\qquad S = \sum_{n=P+1}^{P+Q} e^{2\pi i f(n)},$

$$T(n) = \sum_{m=1}^{q} e^{2\pi i (f(m+n)-f(n))} \quad (P+1 \leqslant n \leqslant P+Q-q).$$

Then

$$\begin{aligned} q|S| &= \Big|\sum_{m=1}^{q}\sum_{n=P+1}^{P+Q} e^{2\pi i f(n)}\Big| \\ &\leqslant \Big|\sum_{m=1}^{q}\sum_{n=P+1+m}^{P+Q-q+m} e^{2\pi i f(n)}\Big| + \sum_{m=1}^{q} q \\ &= \Big|\sum_{m=1}^{q}\sum_{n=P+1}^{P+Q-q} e^{2\pi i f(m+n)}\Big| + q^2 \\ &= \Big|\sum_{n=P+1}^{P+Q-q}\sum_{m=1}^{q} e^{2\pi i f(m+n)}\Big| + q^2 \\ &\leqslant \sum_{n=P+1}^{P+Q-q} |T(n)| + q^2 \\ &\leqslant Q^{1-1/(2l)}\Big\{\sum_{n=P+1}^{P+Q-q} |T(n)|^{2l}\Big\}^{1/(2l)} + q^2, \end{aligned} \tag{6.12.4}$$

by Hölder's inequality, where l is any positive integer.

We now write $A_r = A_r(n) = f^{(r)}(n)/r!$ for $1 \leqslant r \leqslant k$, and define the k-dimensional region Ω_n by the inequalities

$$|\alpha_r - A_r| \leqslant \tfrac{1}{2} q^{-r} \quad (r = 1, \ldots, k). \tag{6.12.5}$$

If we set

$$\delta(m) = f(m+n) - f(n) - (\alpha_k m^k + \ldots + \alpha_1 m),$$

then, by partial summation, we will have

$$T(n) = S(q)e^{2\pi i \delta(q)} - 2\pi i \int_0^q S(p)\delta'(p)e^{2\pi i \delta(p)}\,dp.$$

However, by Taylor's theorem together with the bound (6.12.1) we obtain

$$\begin{aligned} \delta'(p) &= f'(p+n) - \sum_1^k r\alpha_r p^{r-1} \\ &= f'(n) + pf''(n) + \ldots + \frac{p^{k-1}}{(k-1)!} f^{(k)}(n) + \frac{p^k}{k!} f^{(k+1)}(n+\vartheta p) - \sum_1^k r\alpha_r p^{r-1} \\ &= \sum_1^k r(A_r - \alpha_r)p^{r-1} + 2(k+1)\lambda\vartheta' p^k, \end{aligned}$$

where $0 < \vartheta,\ \vartheta' \leqslant 1$. If (6.12.5) holds it follows that

$$|\delta'(p)| \leqslant \sum_1^k r\tfrac{1}{2}q^{-r}q^{r-1} + 3k\lambda q^k \leqslant \tfrac{1}{2}k^2 q^{-1} + 3k\lambda q^k \leqslant 2^{k+3}kq^{-1},$$

by our choice of q. We therefore have

$$|T(n)| \leqslant 2^{k+4}k\pi\Big(|S(q)| + \frac{1}{q}\int_0^q |S(p)|\,dp\Big) = 2^{k+4}k\pi S_0(q),$$

say. Integrating over the region Ω_n, and dividing by its volume, we obtain

$$|T(n)|^{2l} \leqslant (2^{k+4}k\pi)^{2l} q^{\frac{1}{2}k(k+1)} \int_{\Omega_n}\!\!\ldots\!\int |S_0(q)|^{2l}\,d\alpha_1\ldots d\alpha_k. \tag{6.12.6}$$

The integral of $|S_0(q)|^{2l}$ over Ω_n is equal to its integral taken over the region obtained by subtracting any integer from each coordinate. We say that such a region is congruent (mod 1) to Ω_n. Now let n, n' be two integers in the interval $[P+1,\ P+Q-q]$, and let Ω_n, Ω'_n be the corresponding regions defined by (6.12.5). A necessary condition that Ω_n should intersect with any region congruent (mod 1) to Ω'_n is that

$$\overline{A_k(n) - A_k(n')} \leqslant q^{-k} \leqslant \lambda q. \tag{6.12.7}$$

Let $\phi(n) = A_k(n) - A_k(n')$. Then

$$\phi(n+1) - \phi(n) = \frac{1}{k!}\{f^{(k)}(n+1) - f^{(k)}(n)\} = \frac{f^{(k+1)}(\xi)}{k!},$$

where $n < \xi < n+1$. The conditions of Lemma 6.11 are therefore satisfied, with $c = 2$ and $\delta = \lambda(k+1)$. Taking $W = q/(k+1)$, we see that the number of numbers n in $[P+1,\ P+Q-q]$ for which (6.12.7) is possible, does not exceed

$$\{2Q\lambda(k+1)+1\}\Big(\frac{2q}{k+1}+1\Big) \leqslant (2k+3)\Big(\frac{2q}{k+1}+1\Big) \leqslant 3kq.$$

Since this is independent of n', it follows that

$$\sum_{n=P+1}^{P+Q-q} \int_{\Omega_n}\!\!\ldots\!\int |S_0(q)|^{2l}\,d\alpha_1\ldots d\alpha_k \leqslant 3kq \int_0^1\!\!\ldots\!\int_0^1 |S_0(q)|^{2l}\,d\alpha_1\ldots d\alpha_k \leqslant 3kq2^{2l}J(q,l), \tag{6.12.8}$$

since

$$S_0(q)^{2l} \leqslant 2^{2l-1}\Big(|S(q)|^{2l} + \frac{1}{q}\int_0^q |S(p)|^{2l}\,dp\Big).$$

Defining l as in Lemma 6.10, we obtain from (6.12.4), (6.12.6), (6.12.8) and Lemma 10

$$\begin{aligned} |S| &\leqslant 2^{k+5}k\pi Q^{1-\frac{1}{2l}} q^{-1}\{q^{\frac{1}{2}k(k+1)} 3kq\,J(q,l)\}^{\frac{1}{2l}} + q \\ &\leqslant 2^{k+5}k\pi Q^{1-\frac{1}{2l}}\{3ke^{64lk\log^2 k} q^{\frac{3}{2}}\}^{\frac{1}{2l}} \log q + q. \end{aligned}$$

Now $q \leqslant 2\lambda^{-1/(k+1)} \leqslant 2Q^{4/(k+1)}$. Hence

$$|S| \leqslant Ae^{33k\log^2 k} Q^{1-\frac{1}{2l}+3/\{(k+1)l\}} \log Q + 2Q^{4/(k+1)}$$

and the result follows, since $\frac{1}{2l}-3/\{(k+1)l\} \geqslant \frac{1}{8l}$ and $l < 3k^2\log k$.

6.13. LEMMA 6.13. *If $f(x)$ satisfies the conditions of Lemma 6.12 in an interval $[P+1, P+N]$, where $N \leqslant Q$, and*

$$\lambda^{-\frac{1}{3}} \leqslant Q \leqslant \lambda^{-1}, \tag{6.13.1}$$

then

$$\left|\sum_{n=P+1}^{P+N} e^{2\pi i f(n)}\right| < Ae^{33k\log^2 k}Q^{1-\rho}\log Q. \tag{6.13.2}$$

If $\lambda^{-\frac{1}{4}} \leqslant N$, the conditions of the previous theorem are satisfied when Q is replaced by N, and (6.13.2) follows at once from (6.12.3). On the other hand, if $\lambda^{-\frac{1}{4}} > N$, then

$$\left|\sum_{n=P+1}^{P+N} e^{2\pi i f(n)}\right| \leqslant N < \lambda^{-\frac{1}{4}} \leqslant Q^{\frac{3}{4}} \leqslant Q^{1-\rho},$$

and (6.13.1) again follows.

6.14. THEOREM 6.14.

$$\zeta(1+it) = O\{(\log t \log\log t)^{\frac{3}{4}}\}.$$

Let
$$f(x) = -\frac{t\log x}{2\pi}, \qquad f^{(k+1)}(x) = \frac{(-1)^{k+1}k!\,t}{2\pi x^{k+1}}.$$

Let $a < x \leqslant b \leqslant 2a$. Since $(-1)^{k+1}f^{(k+1)}(x)$ is steadily decreasing, we can divide the interval $[a, b]$ into not more than $k+1$ intervals, in each of which inequalities of the form (6.12.1) hold, where λ depends on the particular interval, and satisfies

$$\frac{t}{2\pi(k+1)(2a)^{k+1}} \leqslant \lambda \leqslant \frac{t}{4\pi(k+1)a^{k+1}}. \tag{6.14.1}$$

Let $Q = a \leqslant t$, $\log a > 2\log^{\frac{1}{2}}t$, and

$$k = \left[\frac{\log t}{\log a}\right]+1.$$

Then
$$Q < a^{k+1}t^{-1} \leqslant Q^2.$$

Clearly $\lambda \leqslant Q^{-1}$, while $\lambda \geqslant Q^{-3}$ if $Q \geqslant 2^{k+2}\pi(k+1)$, or if

$$\log a \geqslant \left(\frac{\log t}{\log a}+3\right)\log 2+\log\left(\frac{\log t}{\log a}+2\right)+\log\pi,$$

and this is true if t is large enough. It follows from Lemma 6.13 that

$$\sum_{a<n\leqslant b} e^{-it\log n} = O(ke^{33k\log^2 k}a^{1-\rho}\log a),$$

where ρ is defined as in § 6.12. Hence

$$\sum_{a<n\leqslant b}\frac{1}{n^{1+it}} = O(ke^{33k\log^2 k}a^{-\rho}\log a)$$
$$= O\left\{\log t\exp\left(33k\log^2 k-\frac{\log a}{24k^2\log k}\right)\right\}.$$

Suppose that
$$k\log k < A\log^{\frac{1}{3}}a,$$
with a sufficiently small A, or
$$\log a > A(\log t\log\log t)^{\frac{3}{4}}$$
with a sufficiently large A. Then

$$\sum_{a<n\leqslant b}\frac{1}{n^{1+it}} = O\left\{\log t\exp\left(\frac{-A\log^3 a}{\log^2 t\log\log t}\right)\right\}$$
$$= O[\log t\exp\{-A\log^{\frac{1}{4}}t(\log\log t)^{\frac{3}{4}}\}],$$

and the sum of $O(\log t)$ such terms is bounded.

Since $k \geqslant 7$, we also require that $a \leqslant t^{\frac{1}{7}}$. Using (5.16.1) with $r = 8$, and writing $\beta = t^{128/(7\times 128+1)}$, we obtain

$$\zeta(1+it) = \sum_{n\leqslant\beta}\frac{1}{n^{1+it}}+O(1) = O(\log\alpha)+\sum_{\alpha<n\leqslant\beta}\frac{1}{n^{1+it}}+O(1).$$

The last sum is bounded if

$$\log\alpha = A(\log t\log\log t)^{\frac{3}{4}}$$

with a suitable A, and the theorem follows.

6.15. If $0 < \sigma < 1$, we obtain similarly

$$\sum_{\alpha<n\leqslant\beta}\frac{1}{n^{\sigma+it}} = O[\alpha^{1-\sigma}\exp\{-A\log^{\frac{1}{4}}t(\log\log t)^{\frac{3}{4}}\}\log t],$$

and this is bounded if

$$1-\sigma < \frac{A(\log\log t)^{\frac{1}{2}}}{\log^{\frac{1}{2}}t} = 1-\sigma_0,$$

with a sufficiently small A. Hence in this region

$$\zeta(s) = O\left(\sum_{n\leqslant\alpha}\frac{1}{n^{\sigma_0}}\right)+O(1)$$
$$= O\left(\frac{\alpha^{1-\sigma_0}}{1-\sigma_0}\right)+O(1)$$
$$= O\left[\exp\{A\log^{\frac{1}{2}}t(\log\log t)^{\frac{3}{4}}\}\frac{\log^{\frac{1}{2}}t}{(\log\log t)^{\frac{1}{2}}}\right].$$

We can now apply Theorem 3.10, with

$$\theta(t) = \frac{A(\log\log t)^{\frac{1}{2}}}{\log^{\frac{1}{2}}t}, \qquad \phi(t) = A\log^{\frac{1}{2}}t(\log\log t)^{\frac{3}{4}}.$$

Hence there is a region

$$\sigma \geqslant 1-\frac{A}{\log^{\frac{3}{4}}t(\log\log t)^{\frac{3}{4}}} \tag{6.15.1}$$

which is free from zeros of $\zeta(s)$; and by Theorem 3.11 we have also

$$\frac{1}{\zeta(1+it)} = O\{\log^{\frac{2}{3}}t(\log\log t)^{\frac{1}{3}}\}, \qquad \frac{\zeta'(1+it)}{\zeta(1+it)} = O\{\log^{\frac{2}{3}}t(\log\log t)^{\frac{1}{3}}\}. \tag{6.15.2), (6.15.3}$$

NOTES FOR CHAPTER 6

6.16. Further improvements have been made in the estimation of $J(q, l)$. The most important of these is due to Karatsuba [**2**] who used a p-adic analogue of the argument given here, thereby producing a considerable simplication of the proof. Moreover, as was shown by Steckin [**1**], one is then able to sharpen Lemma 6.9 to yield the bound

$$J(q, l) \leqslant C^{k^3 \log k} q^{2l - \frac{1}{2}k(k+1) + \delta_r},$$

for $l \geqslant kr$, where $k \geqslant 2$, r is a positive integer, C is an absolute constant, and $\delta_r = \frac{1}{2}k^2(1-1/k)^r$. Here one has a smaller value for δ_r than formerly, but more significantly, the condition $l \geqslant \frac{1}{4}k^2 + \frac{1}{4}k + kr$ has been relaxed.

6.17. One can use Lemma 6.13 to obtain exponent pairs. To avoid confusion of notation, we take f to be defined on $(a, b]$, with $a < b \leqslant 2a$ and $\lambda^{-\frac{1}{3}} \leqslant a \leqslant \lambda^{-1}$. Then

$$\left| \sum_{a<n\leqslant b} e^{2\pi i f(n)} \right| \leqslant A e^{33k \log^2 k} a^{1-\rho} \log a.$$

Now suppose that (N, I, f, y) is in the set $\mathscr{F}(s, \frac{1}{4})$ of §5.20, whence

$$\tfrac{3}{4}\alpha_k x^{-s-k} \leqslant \frac{|f^{(k+1)}(x)|}{(k+1)!} \leqslant \tfrac{5}{4}\alpha_k x^{-s-k}$$

with

$$\alpha_k = y\frac{s(s+1)\dots(s+k-1)}{(k+1)!}.$$

We may therefore break up I into $O(s+k)$ subintervals $(a, b]$ with $b \leqslant (\frac{6}{5})^{1/(s+k)} a$, on each of which one has

$$\lambda \leqslant \frac{|f^{(k+1)}(x)|}{(k+1)!} \leqslant 2\lambda,$$

with $\lambda = \frac{5}{8}\alpha_k a^{-s-k}$. We now choose k so that $\lambda^{-\frac{1}{3}} \leqslant N \leqslant 2N \leqslant \lambda^{-1}$ for all a in the range $N \leqslant a \leqslant 2N$. To do this we take $k \geqslant 7$ such that

$$\frac{N^{k-1}}{\alpha_{k-1}} < \tfrac{5}{4}N^{1-s} \leqslant \frac{N^k}{\alpha_k}. \tag{6.17.1}$$

Note that N^k/α_k tends to infinity with k, if $N \geqslant 2$, so this is always possible, providing that

$$\frac{N^6}{\alpha_6} < \tfrac{5}{4}N^{1-s}. \tag{6.17.2}$$

The estimate (6.17.1) ensures that $2N \leqslant \lambda^{-1}$, and hence, incidentally, that $\lambda < 1$. Moreover we also have

$$N^k < \tfrac{5}{4}\alpha_{k-1}N^{2-s} \leqslant \tfrac{5}{8}\alpha_k 2^{-s-k}N^{3-s}$$

if $N \geqslant 2^{s+k+2}$, and so $\lambda^{-\frac{1}{3}} \leqslant N$. It follows that

$$\sum_{n\in I} e^{2\pi i f(n)} \ll_s k e^{33k^2 \log k} N^{1-\rho} \log N \tag{6.17.3}$$

for $N \geqslant 2^{s+k+2}$, subject to (6.17.2).

We shall now show that

$$(p, q) = \left(\frac{1}{25(m-2)m^2 \log m}, 1 - \frac{1}{25m^2 \log m}\right) \tag{6.17.4}$$

is an exponent pair whenever $m \geqslant 3$. If $yN^{2-s-m} \geqslant 1$ then $(yN^{-s})^p N^q \geqslant N$, and the required bound (5.20.2) is trivial. If (6.17.2) fails, then $yN^{-s} \ll_s N^5$ and, using the exponent pair $(\frac{1}{126}, \frac{120}{126}) = A^5B(0, 1)$ (in the notation of §5.20) we have

$$\sum_{n\in I} e^{2\pi i f(n)} \ll_s (yN^{-s})^{\frac{1}{126}} N^{\frac{120}{126}} \ll_s N^{\frac{125}{126}} \ll N^q \ll (yN^{-s})^p N^q$$

as required. We may therefore assume that $yN^{2-s-m} < 1$, and that (6.17.2) holds. Let us suppose that $N \geqslant \max(2^{s+m+2}, 2(\frac{1}{2}s+1)^m)$. Then (6.17.1) yields

$$N^{k-1} < \frac{5}{4}\cdot\frac{s}{2}\cdot\frac{s+1}{3}\cdot\frac{s+2}{4}\cdot\ldots\cdot\frac{s+k-2}{k}yN^{1-s}$$

$$\leqslant \tfrac{5}{4}\left(\max\left(\frac{s}{2}, 1\right)\right)^{k-1} yN^{1-s} < 2(\tfrac{1}{2}s+1)^{k-1}N^{m-1},$$

whence

$$\left(\frac{N}{\frac{1}{2}s+1}\right)^{k-m} < 2(\tfrac{1}{2}s+1)^{m-1}.$$

Since $N \geqslant 2(\frac{1}{2}s+1)^m$ we deduce that $k \leqslant m$. Moreover we then have $N \geqslant 2^{s+m+2} \geqslant 2^{s+k+2}$, so that (6.17.3) applies. Since k is bounded in terms of p, q and s, it follows that

$$\sum_{n \in I} e^{2\pi i f(n)} \ll_{p,q,s} N^{1-p} \log N \ll_{p,q,s} N^q$$

if $N \gg_{p,q,s} 1$, and the required estimate (5.20.2) follows.

6.18. We now show that the exponent pair (6.17.4) is better than any pair (α, β) obtainable by the A and B processes from (0, 1), if $m \geqslant 10^6$. By this we mean that there is no pair (α, β) with both $p \geqslant \alpha$ and $q \geqslant \beta$. To do this we shall show that

$$\beta + 5\alpha^{\frac{3}{4}} \geqslant 1. \tag{6.18.1}$$

Then, since $5^4 25 m^2 \log m < (m-2)^3$ for $m \geqslant 10^6$, we have $q + 5p^{\frac{3}{4}} < 1$, and the result will follow. Certainly (6.18.1) holds for (0, 1). Thus it suffices to prove (6.18.1) by induction on the number of A and B processes needed to obtain (α, β). Since $B^2(\alpha, \beta) = (\alpha, \beta)$ and $A(0, 1) = (0, 1)$, we may suppose that either $(\alpha, \beta) = A(\gamma, \delta)$ or $(\alpha, \beta) = BA(\gamma, \delta)$, where (γ, δ) satisfies (6.18.1). In the former case we have

$$\beta + 5\alpha^{\frac{3}{4}} = \frac{\gamma+\delta+1}{2\gamma+2} + 5\left(\frac{\gamma}{2\gamma+2}\right)^{\frac{3}{4}} \geqslant \frac{\gamma+2-5\gamma^{\frac{3}{4}}}{2\gamma+2} + 5\left(\frac{\gamma}{2\gamma+2}\right)^{\frac{3}{4}} \geqslant 1$$

for $0 \leqslant \gamma \leqslant \frac{1}{2}$, and in the latter case

$$\beta + 5\alpha^{\frac{3}{4}} = \frac{2\gamma+1}{2\gamma+2} + 5\left(\frac{\delta}{2\gamma+2}\right)^{\frac{3}{4}} \geqslant \frac{2\gamma+1}{2\gamma+2} + 5\left(\frac{\frac{1}{2}}{2\gamma+2}\right)^{\frac{3}{4}} \geqslant 1$$

for $0 \leqslant \gamma \leqslant \frac{1}{2}$. This completes the proof of our assertion.

The exponent pairs (6.17.4) are not likely to be useful in practice. The purpose of the above analysis is to show that Lemma 6.13 is sufficiently general to apply to any function for which the exponent pairs method can be used, and that there do exist exponent pairs not obtainable by the A and B processes.

6.19. Different ways of using $J(q, l)$ to estimate exponential sums have been given by Korobov [1] and Vinogradov [1] (see Walfisz [1; Chapter 2] for an alternative exposition). These methods require more information about f than a bound (6.12.1) for a single derivative, and so we shall give the result for partial sums of the zeta-function only. The two methods give qualitatively similar estimates, but Vinogradov's is slightly simpler, and is quantitatively better. Vinogradov's result, as given by Walfisz [1], is

$$\sum_{a < n \leqslant b} n^{-it} \ll a^{1-\rho} \tag{6.19.1}$$

for $a < b \leqslant 2a$, $t \geqslant 1$, where

$$t^{1/k} \leqslant a \leqslant t^{1/(k-1)},$$

$k \geqslant 19$, and

$$\rho = \frac{1}{60000k^2}.$$

The implied constant is absolute. Richert [3] has used this to show that

$$\zeta(\sigma+it) \ll (1 + t^{100(1-\sigma)^{\frac{3}{2}}})(\log t)^{\frac{2}{3}}, \tag{6.19.2}$$

uniformly for $0 \leqslant \sigma \leqslant 2$, $t \geqslant 2$. The choices

$$\theta(t) = \left(\frac{\log\log t}{100 \log t}\right)^{\frac{2}{3}}, \ \phi(t) = \log\log t$$

in Theorems 3.10 and 3.11 therefore give a region

$$\sigma \geqslant 1 - A(\log t)^{-\frac{2}{3}}(\log\log t)^{-\frac{1}{3}}$$

free of zeros, and in which

$$\frac{\zeta'(s)}{\zeta(s)} \ll (\log t)^{\frac{2}{3}}(\log\log t)^{\frac{1}{3}},$$

$$\frac{1}{\zeta(s)} \ll (\log t)^{\frac{2}{3}}(\log\log t)^{\frac{1}{3}}.$$

The superiority of (6.19.1) over Lemma 6.13 lies mainly in the elimination of the term $\exp(33k^2 \log k)$, rather than in the improvement in the exponent ρ.

Various authors have reduced the constant 100 in (6.19.2), and the best result to date appears to be one in which 100 is replaced by 18.8 (Heath-Brown, unpublished).

6.20. We shall sketch the proof of Vinogradov's bound. The starting point is an estimate of the form (6.12.4), but with

$$\sum_{u, v=1}^{q} e^{2\pi i \{f(uv+n) - f(n)\}} \tag{6.20.1}$$

in place of $T(n)$. One replaces $f(uv+n) - f(n)$ by a polynomial

$$F(uv) = A_1 uv + \ldots + A_k u^k v^k$$

as in §6.12, and then uses Hölder's inequality to obtain

$$\left|\sum e^{2\pi iF(uv)}\right|^{l} \leqslant q^{l-1}\sum_{v}\left|\sum_{u} e^{2\pi iF(uv)}\right|^{l}$$

$$= q^{l-1}\sum_{v}\eta(v)\left(\sum_{u} e^{2\pi iF(uv)}\right)^{l}$$

$$= q^{l-1}\sum_{\sigma_1,\ldots,\sigma_k} n(\sigma_1,\ldots,\sigma_k)\sum_{v}\eta(v)e^{2\pi iG(\sigma_1,\ldots,\sigma_k;v)},$$

where $|\eta(v)| = 1$, $n(\sigma_1,\ldots,\sigma_k)$ denotes the number of solutions of

$$u_1{}^h + \ldots + u_l{}^h = \sigma_h \quad (1 \leqslant h \leqslant k),$$

and

$$G(\sigma_1,\ldots,\sigma_k;v) = A_1\sigma_1 v + \ldots + A_k\sigma_k v^k.$$

Now, by Hölder's inequality again, one has

$$\left|\sum e^{2\pi iF(uv)}\right|^{2l^2} \leqslant q^{2l(l-1)}\left(\sum n(\sigma_1,\ldots,\sigma_k)\right)^{2l-2}\times\left(\sum n(\sigma_1,\ldots,\sigma_k)^2\right)$$

$$\times\left(\sum_{\sigma_1,\ldots,\sigma_k}\left|\sum_{v}\eta(v)e^{2\pi iG(\sigma_1,\ldots,\sigma_k;v)}\right|^{2l}\right).$$

Here

$$\sum_{\sigma_1,\ldots,\sigma_k} n(\sigma_1,\ldots,\sigma_k) = q^l,$$

and

$$\sum_{\sigma_1,\ldots,\sigma_k} n(\sigma_1,\ldots,\sigma_k)^2 = J(q,l).$$

Moreover

$$\sum_{\sigma_1,\ldots,\sigma_k}\left|\sum_{v}\eta(v)e^{2\pi iG}\right|^{2l} = \sum_{\tau_1,\ldots,\tau_k} n^*(\tau_1,\ldots,\tau_k)\sum_{\sigma_1,\ldots,\sigma_k} e^{2\pi iH(\sigma_1,\ldots,\sigma_k;\tau_1,\ldots,\tau_k)},$$

where

$$H(\sigma_1,\ldots,\sigma_k;\tau_1,\ldots,\tau_k) = A_1\sigma_1\tau_1 + \ldots + A_k\sigma_k\tau_k,$$

and $n^*(\tau_1,\ldots,\tau_k)$ is the sum of $\eta(v_1)\ldots\eta(v_{2l})$ subject to

$$v_1{}^h + \ldots + v_l{}^h - v_{l+1}{}^h - \ldots - v_{2l}{}^h = \tau_h \quad (1 \leqslant h \leqslant k).$$

Since $|n^*(\tau_1,\ldots,\tau_k)| \leqslant J(q,l)$, it follows that

$$\left|\sum e^{2\pi iF(uv)}\right|^{2l^2} \leqslant q^{4l^2-4l}J(q,l)^2\prod_{h=1}^{k}\left(\sum_{\tau_h}\left|\sum_{\sigma_h}\exp(2\pi iA_h\sigma_h\tau_h)\right|\right)$$

$$\leqslant q^{4l^2-4l}J(q,l)^2\prod_{h=1}^{k}\left(\sum_{\tau_h}\min(lq^h, |\csc\pi A_h\tau_n|)\right).$$

At this point one estimates the sum over τ_h, getting a non-trivial bound whenever $q^{-2h} \ll |A_h| \ll 1$. This leads to an appropriate result for the original sum (6.20.1), on taking $l = [ck^2]$ with a suitable constant c. If we use Lemma 6.9, for example, to estimate $J(q,l)$, then

$$(K^{2r})^{(2l^2)^{-1}} \ll 1.$$

One therefore sees that the implied constant in (6.19.1) is indeed independent of k.

VII

MEAN-VALUE THEOREMS

7.1. The problem of the order of $\zeta(s)$ in the critical strip is, as we have seen, unsolved. The problem of the average order, or mean-value, is much easier, and, in its simplest form, has been solved completely. The form which it takes is that of determining the behaviour of

$$\frac{1}{T}\int_1^T |\zeta(\sigma+it)|^2\,dt$$

as $T \to \infty$, for any given value of σ. We also consider mean values of other powers of $\zeta(s)$.

Results of this kind have applications in the problem of the zeros, and also in problems in the theory of numbers. They could also be used to prove O-results if we could push them far enough; and they are closely connected with the Ω-results which are the subject of the next chapter.

We begin by recalling a general mean-value theorem for Dirichlet series.

THEOREM 7.1. *Let*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \qquad g(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$$

be absolutely convergent for $\sigma > \sigma_1$, $\sigma > \sigma_2$ respectively. Then for $\alpha > \sigma_1$, $\beta > \sigma_2$,

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(\alpha+it)g(\beta-it)\,dt = \sum_{n=1}^{\infty} \frac{a_n b_n}{n^{\alpha+\beta}}. \tag{7.1.1}$$

For

$$f(\alpha+it)g(\beta-it) = \sum_{m=1}^{\infty}\frac{a_m}{m^{\alpha+it}}\sum_{n=1}^{\infty}\frac{b_n}{n^{\beta-it}} = \sum_{n=1}^{\infty}\frac{a_n b_n}{n^{\alpha+\beta}} + \sum_{m\neq n}\sum \frac{a_m b_n}{m^\alpha n^\beta}\left(\frac{n}{m}\right)^{it},$$

the series being absolutely convergent, and uniformly convergent in any finite t-range. Hence we may integrate term-by-term, and obtain

$$\frac{1}{2T}\int_{-T}^{T} f(\alpha+it)g(\beta-it)\,dt = \sum_{n=1}^{\infty}\frac{a_n b_n}{n^{\alpha+\beta}} + \sum_{m\neq n}\sum \frac{a_m b_n}{m^\alpha n^\beta}\,\frac{2\sin(T\log n/m)}{2T\log n/m}.$$

The factor involving T is bounded for all T, m, and n, so that the double series converges uniformly with respect to T; and each term tends to zero as $T \to \infty$. Hence the sum also tends to zero, and the result follows.

In particular, taking $b_n = \bar{a}_n$ and $\alpha = \beta = \sigma$, we obtain

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(\sigma+it)|^2\,dt = \sum_{n=1}^{\infty}\frac{|a_n|^2}{n^{2\sigma}} \quad (\sigma > \sigma_1). \tag{7.1.2}$$

These theorems have immediate applications to $\zeta(s)$ in the half-plane $\sigma > 1$. We deduce at once

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\zeta(\sigma+it)|^2\,dt = \zeta(2\sigma) \quad (\sigma > 1), \tag{7.1.3}$$

and generally

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\zeta^{(\mu)}(\alpha+it)\zeta^{(\nu)}(\beta-it)\,dt = \zeta^{(\mu+\nu)}(\alpha+\beta) \quad (\alpha > 1,\ \beta > 1). \tag{7.1.4}$$

Taking $a_n = d_k(n)$, we obtain

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\zeta(\sigma+it)|^{2k}\,dt = \sum_{n=1}^{\infty}\frac{d_k^2(n)}{n^{2\sigma}} \quad (\sigma > 1). \tag{7.1.5}$$

By (1.2.10), the case $k = 2$ is

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\zeta(\sigma+it)|^4\,dt = \frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} \quad (\sigma > 1). \tag{7.1.6}$$

The following sections are mainly concerned with the attempt to extend these formulae to values of σ less than or equal to 1. The attempt is successful for $k \leqslant 2$, only partially successful for $k > 2$.

7.2. We require the following lemmas.

LEMMA. *We have*

$$\sum_{0<m<n<T}\sum \frac{1}{m^\sigma n^\sigma \log n/m} = O(T^{2-2\sigma}\log T) \tag{7.2.1}$$

for $\frac{1}{2} \leqslant \sigma < 1$, and uniformly for $\frac{1}{2} \leqslant \sigma \leqslant \sigma_0 < 1$.

Let Σ_1 denote the sum of the terms for which $m < \frac{1}{2}n$, Σ_2 the remainder. In Σ_1, $\log n/m > A$, so that

$$\Sigma_1 < A\sum_{m<n<T}\sum m^{-\sigma}n^{-\sigma} < A\Big(\sum_{n<T} n^{-\sigma}\Big)^2 < AT^{2-2\sigma}.$$

In Σ_2 we write $m = n-r$, where $1 \leqslant r \leqslant \frac{1}{2}n$, and then

$$\log n/m = -\log(1-r/n) > r/n.$$

Hence

$$\Sigma_2 < A\sum_{n<T}\sum_{r\leqslant\frac{1}{2}n}\frac{(n-r)^{-\sigma}n^{-\sigma}}{r/n} < A\sum_{n<T} n^{1-2\sigma}\sum_{r\leqslant\frac{1}{2}n}\frac{1}{r} < AT^{2-2\sigma}\log T.$$

LEMMA. $$\sum_{0<m<n<\infty}\sum \frac{e^{-(m+n)\delta}}{m^\sigma n^\sigma \log n/m} = O\left(\delta^{2\sigma-2}\log\frac{1}{\delta}\right). \tag{7.2.2}$$

Dividing up as before, we obtain

$$\Sigma_1 = O\left[\left(\sum_1^\infty n^{-\sigma}e^{-\delta n}\right)^2\right] = O(\delta^{2\sigma-2}),$$

and $$\Sigma_2 = O\left(\sum_{n=1}^\infty n^{1-2\sigma}e^{-\delta n}\sum_{r=1}^{\frac{1}{2}n}\frac{1}{r}\right) = O\left(\delta^{2\sigma-2}\log\frac{1}{\delta}\right).$$

THEOREM 7.2. $$\lim_{T\to\infty}\frac{1}{T}\int_1^T |\zeta(\sigma+it)|^2\,dt = \zeta(2\sigma) \quad (\sigma > \tfrac{1}{2}).$$

We have already accounted for the case $\sigma > 1$, so that we now suppose that $\frac{1}{2} < \sigma \leqslant 1$. Since $t \geqslant 1$, Theorem 4.11, with $x = t$, gives

$$\zeta(s) = \sum_{n<t} n^{-s} + O(t^{-\sigma}) = Z + O(t^{-\sigma}),$$

say. Now

$$\int_1^T |Z|^2\,dt = \int_1^T \left[\sum_{m<t} m^{-\sigma-it}\sum_{n<t} n^{-\sigma+it}\right] dt$$

$$= \sum_{m<T}\sum_{n<T} m^{-\sigma}n^{-\sigma}\int_{T_1}^T \left(\frac{n}{m}\right)^{it} dt \quad (T_1 = \max(m,n))$$

$$= \sum_{n<T} n^{-2\sigma}(T-n) + \sum_{m\neq n}\sum m^{-\sigma}n^{-\sigma}\frac{(n/m)^{iT}-(n/m)^{iT_1}}{i\log n/m}$$

$$= T\sum_{n<T} n^{-2\sigma} - \sum_{n<T} n^{1-2\sigma} + O\left(\sum_{0<m<n<T}\sum \frac{1}{m^\sigma n^\sigma \log n/m}\right)$$

$$= T\{\zeta(2\sigma)+O(T^{1-2\sigma})\} + O(T^{2-2\sigma}) + O(T^{2-2\sigma}\log T),$$

provided that $\sigma < 1$. If $\sigma = 1$, we can replace the σ of the last two terms by $\frac{3}{4}$, say. In either case

$$\int_1^T |Z|^2\,dt \sim T\,\zeta(2\sigma).$$

Hence

$$\int_1^T |\zeta(s)|^2\,dt = \int_1^T |Z|^2\,dt + O\left(\int_1^T |Z|t^{-\sigma}\,dt\right) + O\left(\int_1^T t^{-2\sigma}\,dt\right)$$

$$= \int_1^T |Z|^2\,dt + O\left(\int_1^T |Z|^2\,dt \int_1^T t^{-2\sigma}\,dt\right)^{\frac{1}{2}} + O(\log T)$$

$$= \int_1^T |Z|^2\,dt + O\{(T\log T)^{\frac{1}{2}}\} + O(\log T),$$

and the result follows.

It will be useful later to have a result of this type which holds uniformly in the strip. It is†

THEOREM 7.2 (A).

$$\int_1^T |\zeta(\sigma+it)|^2\,dt < AT\min\left(\log T, \frac{1}{\sigma-\frac{1}{2}}\right)$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 2$.

Suppose first that $\frac{1}{2} \leqslant \sigma \leqslant \frac{3}{4}$. Then we have, as before,

$$\int_1^T |Z|^2\,dt < T\sum_{n<T} n^{-2\sigma} + O(T^{2-2\sigma}\log T)$$

uniformly in σ. Now

$$\sum_{n<T} n^{-2\sigma} \leqslant \sum_{n<T} n^{-1} < A\log T$$

and also $$\leqslant 1 + \int_1^\infty u^{-2\sigma}\,du < \frac{A}{\sigma-\frac{1}{2}}.$$

Similarly $$T^{2-2\sigma}\log T \leqslant T\log T,$$

and also, putting $x = (2\sigma-1)\log T$,

$$T^{2-2\sigma}\log T = \tfrac{1}{2}Txe^{-x}/(\sigma-\tfrac{1}{2}) \leqslant \tfrac{1}{2}T/(\sigma-\tfrac{1}{2}).$$

This gives the result for $\sigma \leqslant \frac{3}{4}$, the term $O(t^{-\sigma})$ being dealt with as before.

If $\frac{3}{4} \leqslant \sigma \leqslant 2$, we obtain

$$\int_1^T |Z|^2\,dt < T\sum_{n<T} n^{-\frac{3}{2}} + O(T^{\frac{1}{2}}\log T),$$

and the result follows at once.

7.3. The particular case $\sigma = \frac{1}{2}$ of the above theorem is

$$\int_1^T |\zeta(\tfrac{1}{2}+it)|^2\,dt = O(T\log T).$$

We can improve this O-result to an asymptotic equality.‡ But Theorem 4.11 is not sufficient for this purpose, and we have to use the approximate functional equation.

THEOREM 7.3. *As* $T\to\infty$

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^2\,dt \sim T\log T.$$

† Littlewood (4). ‡ Hardy and Littlewood (2), (4).

In the approximate functional equation (4.12.4), take $\sigma = \frac{1}{2}$, $t > 2$, and $x = t/(2\pi\sqrt{\log t})$, $y = \sqrt{\log t}$. Then, since $\chi(\frac{1}{2}+it) = O(1)$,

$$\zeta(\tfrac{1}{2}+it) = \sum_{n<x} n^{-\frac{1}{2}-it} + O\Big(\sum_{n<y} n^{-\frac{1}{2}}\Big) + O(t^{-\frac{1}{4}}\log^{\frac{1}{4}}t) + O(\log^{-\frac{1}{4}}t)$$
$$= \sum_{n<x} n^{-\frac{1}{2}-it} + O(\log^{\frac{1}{4}}t)$$
$$= Z + O(\log^{\frac{1}{4}}t),$$

say. Since $\int\limits_2^T (\log^{\frac{1}{4}}t)^2\,dt = O(T\log^{\frac{1}{2}}T) = o(T\log T)$,

it is, as in the proof of Theorem 7.2, sufficient to prove that

$$\int\limits_0^T |Z|^2\,dt \sim T\log T.$$

Now $$\int\limits_0^T |Z|^2\,dt = \int\limits_0^T \sum_{m<x} m^{-\frac{1}{2}-it} \sum_{n<x} n^{-\frac{1}{2}+it}\,dt.$$

In inverting the order of integration and summation, it must be remembered that x is a function of t. The term in (m,n) occurs if

$$x > \max(m,n) = T_1/(2\pi\sqrt{\log T_1})$$

say, where $T_1 = T_1(m,n)$. Hence, writing $X = T/(2\pi\sqrt{\log T})$,

$$\int\limits_0^T |Z|^2\,dt = \sum_{m,n<X}\sum \int\limits_{T_1}^T m^{-\frac{1}{2}-it} n^{-\frac{1}{2}+it}\,dt$$
$$= \sum_{n<X} \frac{T-T_1(n,n)}{n} + \sum_{m\neq n}\sum \frac{1}{\sqrt{(mn)}} \int\limits_{T_1}^T \left(\frac{n}{m}\right)^{it} dt$$
$$= T\sum_{n<X}\frac{1}{n} + O\Big(\sum_{n<X}\frac{T_1(n,n)}{n}\Big) + O\Big(\sum_{m<n<X}\sum \frac{1}{\sqrt{(mn)}\log n/m}\Big).$$

The first term is

$$T\log X + O(T) = T\log T + o(T\log T).$$

The second term is

$$O\Big(\sum_{n<X}\sqrt{\log n}\Big) = O(X\sqrt{\log X}) = O(T),$$

and, by the first lemma of § 7.2, the last term is

$$O(X\log X) = O(T\sqrt{\log T}).$$

This proves the theorem.

7.4. We shall next obtain a more precise form of the above mean-value formula.†

THEOREM 7.4.

$$\int\limits_0^T |\zeta(\tfrac{1}{2}+it)|^2\,dt = T\log T + (2\gamma-1-\log 2\pi)T + O(T^{\frac{1}{2}+\epsilon}). \quad (7.4.1)$$

We first prove the following lemma.

LEMMA. *If* $n < T/2\pi$,

$$\frac{1}{2\pi i}\int\limits_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \chi(1-s)n^{-s}\,ds = 2 + O\left(\frac{1}{n^{\frac{1}{2}}\log(T/2\pi n)}\right) + O\left(\frac{\log T}{n^{\frac{1}{2}}}\right). \quad (7.4.2)$$

If $n > T/2\pi$, $c > \frac{1}{2}$,

$$\frac{1}{2\pi i}\int\limits_{c-iT}^{c+iT} \chi(1-s)n^{-s}\,ds = O\left(\frac{T^{c-\frac{1}{2}}}{n^c\log(2\pi n/T)}\right) + O\left(\frac{T^{c-\frac{1}{2}}}{n^c}\right). \quad (7.4.3)$$

We have

$$\chi(1-s) = 2^{1-s}\pi^{-s}\cos\tfrac{1}{2}s\pi\,\Gamma(s) = \frac{2^{1-s}\pi^{1-s}}{2\sin\frac{1}{2}s\pi\,\Gamma(1-s)}.$$

This has poles at $s = -2\nu$ ($\nu = 0, 1,\ldots$) with residues

$$\frac{(-1)^\nu 2^{1+2\nu}\pi^{2\nu}}{(2\nu)!}.$$

Also, by Stirling's formula, for $-\pi+\delta < \arg(-s) < \pi-\delta$

$$\chi(1-s) = \left(\frac{2\pi}{-s}\right)^{\frac{1}{2}-s} \frac{e^{-s}}{2\sin\frac{1}{2}s\pi}\left\{1+O\left(\frac{1}{|s|}\right)\right\}.$$

The calculus of residues therefore gives

$$\frac{1}{2\pi i}\left(\int\limits_{-\infty-iT_1}^{\frac{1}{2}-iT_1} + \int\limits_{\frac{1}{2}-iT_1}^{\frac{1}{2}+iT_1} + \int\limits_{\frac{1}{2}+iT_1}^{-\infty+iT_1}\right)\chi(1-s)n^{-s}\,ds$$
$$= \sum_{\nu=0}^{\infty} \frac{(-1)^\nu 2^{1+2\nu}\pi^{2\nu}n^{2\nu}}{(2\nu)!}$$
$$= 2\cos 2\pi n = 2.$$

Also, since $$e^{is\arg(-s)} = O(e^{\frac{1}{2}\pi t}),$$

† Ingham (1) obtained the error term $O(T^{\frac{1}{2}}\log T)$; the method given here is due to Atkinson (1).

$$\int_{\frac12+iT_1}^{-\infty+iT_1} \chi(1-s)n^{-s}\,ds = O\left\{\int_{-\infty}^{\frac12}\left(\frac{2\pi}{|\sigma+iT_1|}\right)^{\frac12-\sigma} e^{-\sigma}n^{-\sigma}\,d\sigma\right\}$$

$$= O\left\{n^{-\frac12}\int_{-\infty}^{\frac12}\left(\frac{T_1}{2\pi e n}\right)^{\sigma-\frac12}d\sigma\right\} = O\left(\frac{1}{n^{\frac12}\log(T_1/2\pi e n)}\right),$$

and similarly for the integral over $(-\infty-iT_1, \frac12-iT_1)$.

Again, for a fixed σ,

$$\chi(1-s) = \left(\frac{2\pi}{t}\right)^{\frac12-\sigma-it} e^{it-\frac14 i\pi}\left\{1+O\left(\frac1t\right)\right\}\quad (t\geqslant 1).$$

Hence

$$\int_{\frac12+iT}^{\frac12+iT_1}\chi(1-s)n^{-s}\,ds = n^{-\frac12}e^{-\frac14 i\pi}\int_T^{T_1} e^{iF(t)}\,dt + O(n^{-\frac12}\log T_1),$$

where

$$F(t) = t\log t - t(\log 2\pi+1+\log n),$$
$$F'(t) = \log t - \log 2\pi n.$$

Hence by Lemma 4.2, the last integral is of the form

$$O\left(\frac{1}{\log(T/2\pi n)}\right)$$

uniformly with respect to T_1. Taking, for example, $T_1 = 2eT > 4\pi e n$, we obtain (7.4.2). Again

$$\int_{c+i}^{c+iT}\chi(1-s)n^{-s}\,ds = n^{-c}e^{-\frac14 i\pi}\int_1^T\left(\frac{t}{2\pi}\right)^{c-\frac12}e^{iF(t)}\,dt + O\left(n^{-c}\int_1^T t^{c-\frac32}\,dt\right),$$

and (7.4.3) follows from Lemma 4.3.

In proving (7.4.1) we may suppose that $T/2\pi$ is half an odd integer; for a change of $O(1)$ in T alters the left-hand side by $O(T^{\frac12})$, since $\zeta(\frac12+it) = O(t^{\frac14})$, and the leading terms on the right-hand side by $O(\log T)$. Now the left-hand side is

$$\frac12\int_{-T}^{T}|\zeta(\tfrac12+it)|^2\,dt = \frac12\int_{-T}^{T}\zeta(\tfrac12+it)\zeta(\tfrac12-it)\,dt$$

$$=\frac1{2i}\int_{\frac12-iT}^{\frac12+iT}\zeta(s)\zeta(1-s)\,ds = \frac1{2i}\int_{\frac12-iT}^{\frac12+iT}\chi(1-s)\zeta^2(s)\,ds$$

$$=\frac1{2i}\int_{\frac12-iT}^{\frac12+iT}\chi(1-s)\sum_{n\leqslant T/2\pi}\frac{d(n)}{n^s}\,ds + \frac1{2i}\int_{\frac12-iT}^{\frac12+iT}\chi(1-s)\left(\zeta^2(s)-\sum_{n\leqslant T/2\pi}\frac{d(n)}{n^s}\right)ds$$

$$= I_1+I_2, \text{ say.}$$

By (7.4.2),

$$I_1 = 2\pi\sum_{n\leqslant T/2\pi}d(n)+O\left(\sum_{n\leqslant T/2\pi}\frac{d(n)}{n^{\frac12}\log(T/2\pi n)}\right)+O\left(\log T\sum_{n\leqslant T/2\pi}\frac{d(n)}{n^{\frac12}}\right).$$

The first term is†

$$2\pi\left\{\frac{T}{2\pi}\log\frac{T}{2\pi}+(2\gamma-1)\frac{T}{2\pi}+O(T^{\frac12})\right\}$$
$$= T\log T+(2\gamma-1-\log 2\pi)T+O(T^{\frac12}).$$

Since‡ $d(n) = O(n^{\epsilon})$, the second term is

$$O\left(\sum_{n\leqslant T/4\pi}\frac{1}{n^{\frac12-\epsilon}}\right)+O\left\{T^{\frac12+\epsilon}\sum_{T/4\pi<n\leqslant T/2\pi}\frac{1}{(T/2\pi)-n}\right\} = O(T^{\frac12+\epsilon}).$$

The last term is also clearly of this form. Hence

$$I_1 = T\log T+(2\gamma-1-\log 2\pi)T+O(T^{\frac12+\epsilon}).$$

Next, if $c > 1$,

$$I_2 = \frac1{2i}\left(\int_{\frac12-iT}^{c-iT}+\int_{c+iT}^{\frac12+iT}\right)\chi(1-s)\left(\zeta^2(s)-\sum_{n\leqslant T/2\pi}\frac{d(n)}{n^s}\right)ds +$$
$$+\frac1{2i}\sum_{n>T/2\pi}d(n)\int_{c-iT}^{c+iT}\chi(1-s)n^{-s}\,ds - A,$$

A being the residue of $\pi\chi(1-s)\zeta^2(s)$ at $s = 1$.

Since $\chi(1-s) = O(t^{\sigma-\frac12})$, and $\zeta^2(\sigma+iT)$ and $\sum_{n\leqslant T/2\pi} d(n)n^{-s}$ are both of the form

$$O(T^{1-\sigma+\epsilon})\quad(\sigma\leqslant 1),\qquad O(T^{\epsilon})\quad(\sigma>1),$$

the first term is $\quad O(T^{\frac12+\epsilon})+O(T^{c-\frac12+\epsilon}).$

By (7.4.3), the second term is

$$O\left\{T^{c-\frac12}\sum_{n>T/2\pi}\frac{d(n)}{n^c}\left(\frac{1}{\log(2\pi n/T)}+1\right)\right\}$$

$$=O\left(T^{\frac12+\epsilon}\sum_{T/2\pi<n\leqslant T/\pi}\frac{1}{n-(T/2\pi)}\right)+O\left(T^{c-\frac12}\sum_{n>T/\pi}\frac{1}{n^{c-\epsilon}}\right)$$

$$=O(T^{\frac12+\epsilon}).$$

Since c may be as near to 1 as we please, this proves the theorem.

A more precise form of the above argument shows that the error-term in (7.4.1) is $O(T^{\frac12}\log^2 T)$. But a more complicated argument,§

† See § 12.1, or Hardy and Wright, *An Introduction to the Theory of Numbers*, Theorem 320.

‡ Ibid. Theorem 315.

§ Titchmarsh (12).

depending on van der Corput's method, shows that it is $O(T^{\frac{5}{12}}\log^2 T)$; and presumably further slight improvements could be made by the methods of the later sections of Chapter V.

7.5. We now pass to the more difficult, but still manageable, case of $|\zeta(s)|^4$. We first prove†

THEOREM 7.5.

$$\lim_{T\to\infty}\frac{1}{T}\int_1^T|\zeta(\sigma+it)|^4\,dt=\frac{\zeta^4(2\sigma)}{\zeta(4\sigma)}\quad(\sigma>\tfrac{1}{2}).$$

Take $x=y=\sqrt{(t/2\pi)}$ and $\sigma>\frac{1}{2}$ in the approximate functional equation. We obtain

$$\zeta(s)=\sum_{n<\sqrt{(t/2\pi)}}\frac{1}{n^s}+\chi(s)\sum_{n<\sqrt{(t/2\pi)}}\frac{1}{n^{1-s}}+O(t^{-\frac{1}{4}})=Z_1+Z_2+O(t^{-\frac{1}{4}}),\tag{7.5.1}$$

say. Now

$$|Z_1|^4=\sum\frac{1}{m^{\sigma+it}}\sum\frac{1}{n^{\sigma+it}}\sum\frac{1}{\mu^{\sigma-it}}\sum\frac{1}{\nu^{\sigma-it}}$$
$$=\sum\frac{1}{(mn\mu\nu)^\sigma}\left(\frac{\mu\nu}{mn}\right)^{it},$$

where each variable runs over $\{1,\sqrt{(t/2\pi)}\}$. Hence

$$\int_1^T|Z_1|^4\,dt=\int_1^T\sum\frac{1}{(mn\mu\nu)^\sigma}\left(\frac{\mu\nu}{mn}\right)^{it}dt$$
$$=\sum_{m,n,\mu,\nu<\sqrt{(T/2\pi)}}\frac{1}{(mn\mu\nu)^\sigma}\int_{T_1}^T\left(\frac{\mu\nu}{mn}\right)^{it}dt,$$

where $T_1=2\pi\max(m^2,n^2,\mu^2,\nu^2)$

$$=\sum_{mn=\mu\nu}\frac{T-T_1}{(mn)^{2\sigma}}+\sum_{mn\neq\mu\nu}O\left(\frac{1}{(mn\mu\nu)^\sigma}\,\frac{1}{|\log(\mu\nu/mn)|}\right).$$

The number of solutions of the equations $mn=\mu\nu=r$ is $\{d(r)\}^2$ if $r<\sqrt{(T/2\pi)}$, and in any case does not exceed $\{d(r)\}^2$. Hence

$$T\sum_{mn=\mu\nu}\frac{1}{(mn)^{2\sigma}}=T\sum_{r<\sqrt{(T/2\pi)}}\frac{\{d(r)\}^2}{r^{2\sigma}}+O\left(T\sum_{\sqrt{(T/2\pi)}\leqslant r<T/2\pi}\frac{\{d(r)\}^2}{r^{2\sigma}}\right)$$
$$\sim T\sum_{r=1}^\infty\frac{\{d(r)\}^2}{r^{2\sigma}}=T\frac{\zeta^4(2\sigma)}{\zeta(4\sigma)}.\tag{7.5.2}$$

† Hardy and Littlewood (4).

Next
$$\sum_{mn=\mu\nu}\frac{T_1}{(mn)^{2\sigma}}<\sum_{mn=\mu\nu}\frac{2\pi(m^2+n^2+\mu^2+\nu^2)}{(mn\mu\nu)^\sigma},$$

and the right-hand side, by considerations of symmetry, is

$$8\pi\sum_{mn=\mu\nu}\frac{m^2}{(mn\mu\nu)^\sigma}\leqslant 8\pi\sum\frac{m^2\,d(mn)}{(mn)^{2\sigma}}=O(T^\epsilon\sum m^{2-2\sigma}\sum n^{-2\sigma})$$
$$=O\{T^\epsilon(T^{\frac{1}{2}(3-2\sigma)}+1)\log T\}=O(T^{\frac{3}{2}-\sigma+\epsilon})+O(T^\epsilon).$$

The remaining sum is

$$O\left(\sum_{0<q<r<T/2\pi}\frac{d(q)d(r)}{(qr)^\sigma\log(r/q)}\right)=O\left(T^\epsilon\sum\frac{1}{(qr)^\sigma\log(r/q)}\right)=O(T^{2-2\sigma+\epsilon}),$$

by the lemma of § 7.2. Hence

$$\int_1^T|Z_1|^4\,dt\sim T\frac{\zeta^4(2\sigma)}{\zeta(4\sigma)}.$$

Now let
$$j(T)=\int_1^T\Big|\sum_{n<\sqrt{(t/2\pi)}}n^{s-1}\Big|^4\,dt.$$

The calculations go as before, but with σ replaced by $1-\sigma$. The term corresponding to (7.5.2) is

$$T\sum_{r<AT}\frac{O(r^\epsilon)}{r^{2-2\sigma}}=O(T^{2\sigma+\epsilon}),$$

and the other two terms are $O(T^{\frac{1}{2}+\sigma+\epsilon})$ and $O(T^{2\sigma+\epsilon})$ respectively. Hence

$$j(T)=O(T^{2\sigma+\epsilon}),$$

and, since $\chi(s)=O(t^{\frac{1}{2}-\sigma})$,

$$\int_1^T|Z_2|^4\,dt<A\int_1^T t^{2-4\sigma}j'(t)\,dt$$
$$=A\big[t^{2-4\sigma}j(t)\big]_1^T+A(4\sigma-2)\int_1^T t^{1-4\sigma}j(t)\,dt$$
$$=O(T^{2-2\sigma+\epsilon})+O\left(\int_1^T t^{1-2\sigma+\epsilon}\,dt\right)=O(T^{2-2\sigma+\epsilon}).$$

The theorem now follows as in previous cases.

7.6. The problem of the mean value of $|\zeta(\frac{1}{2}+it)|^4$ is a little more difficult. If we follow out the above argument, with $\sigma=\frac{1}{2}$, as accurately as possible, we obtain

$$\int_1^T|\zeta(\tfrac{1}{2}+it)|^4\,dt=O(T\log^4 T),\tag{7.6.1}$$

but fail to obtain an asymptotic equality. It was proved by Ingham† by means of the functional equation for $\{\zeta(s)\}^2$ that

$$\int_1^T |\zeta(\tfrac{1}{2}+it)|^4\,dt = \frac{T\log^4 T}{2\pi^2}+O(T\log^3 T). \tag{7.6.2}$$

The relation
$$\int_1^T |\zeta(\tfrac{1}{2}+it)|^4\,dt \sim \frac{T\log^4 T}{2\pi^2} \tag{7.6.3}$$

is a consequence of a result obtained later in this chapter (Theorem 7.16).

7.7. We now pass to still higher powers of $\zeta(s)$. In the general case our knowledge is very incomplete, and we can state a mean-value formula in a certain restricted range of values of σ only.

THEOREM 7.7. *For every positive integer $k > 2$*

$$\lim_{T\to\infty}\frac{1}{T}\int_1^T |\zeta(\sigma+it)|^{2k}\,dt = \sum_{n=1}^{\infty}\frac{d_k^2(n)}{n^{2\sigma}} \qquad \left(\sigma > 1-\frac{1}{k}\right). \tag{7.7.1}$$

This can be proved by a straightforward extension of the argument of § 7.5. Starting again from (7.5.1), we have

$$|Z_1|^{2k} = \sum \frac{1}{(m_1\ldots m_k n_1\ldots n_k)^\sigma}\left(\frac{n_1\ldots n_k}{m_1\ldots m_k}\right)^{it},$$

where each variable runs over $\{1, \sqrt{(t/2\pi)}\}$. The leading term goes in the same way as before, $d(r)$ being replaced by $d_k(r)$. The main O-term is of the form

$$O\left(T^\epsilon \sum_{0<q<r<AT^{\frac{1}{2}k}}\sum \frac{1}{(qr)^\sigma \log r/q}\right) = O(T^{k(1-\sigma)+\epsilon}).$$

The corresponding term in

$$j(T) = \int_1^T \Big|\sum_{n<\sqrt{(t/2\pi)}} n^{s-1}\Big|^{2k}\,dt$$

is
$$O(T^{k\sigma+\epsilon}),$$

and since $|\chi|^{2k} = O(t^{k-2k\sigma})$, we obtain $O(T^{k(1-\sigma)+\epsilon})$ again. These terms are $o(T)$ if $\sigma > 1-1/k$, and the theorem follows as before.

7.8. It is convenient to introduce at this point the following notation. For each positive integer k and each σ, let $\mu_k(\sigma)$ be the lower bound of positive numbers ξ such that

$$\frac{1}{T}\int_1^T |\zeta(\sigma+it)|^{2k}\,dt = O(T^\xi).$$

† Ingham (1).

Each $\mu_k(\sigma)$ has the same general properties as the function $\mu(\sigma)$ defined in § 5.1. By (7.1.5), $\mu_k(\sigma) = 0$ for $\sigma > 1$. Further, as a function of σ, $\mu_k(\sigma)$ is continuous, non-increasing, and convex downwards. We shall deduce this from a general theorem on mean-values of analytic functions.†

Let $f(s)$ be an analytic function of s, real for real s, regular for $\sigma \geqslant \alpha$ except possibly for a pole at $s = s_0$, and $O(e^{\epsilon|t|})$ as $|t|\to\infty$ for every positive ϵ and $\sigma \geqslant \alpha$. Let $\alpha < \beta$, and suppose that for all $T > 0$

$$\int_0^T |f(\alpha+it)|^2\,dt \leqslant C(T^a+1), \tag{7.8.1}$$

$$\int_0^T |f(\beta+it)|^2\,dt \leqslant C'(T^b+1), \tag{7.8.2}$$

where $a \geqslant 0$, $b \geqslant 0$, and C, C' depend on $f(s)$. Then for $\alpha < \sigma < \beta$, $T \geqslant 2$,

$$\int_{\frac{1}{2}T}^T |f(\sigma+it)|^2\,dt \leqslant K(CT^a)^{(\beta-\sigma)/(\beta-\alpha)}(C'T^b)^{(\sigma-\alpha)/(\beta-\alpha)}, \tag{7.8.3}$$

where K depends on a, b, α, β only, and is bounded if these are bounded.

We may suppose in the proof that $\alpha \geqslant \frac{1}{2}$, since otherwise we could apply the argument to $f(s+\frac{1}{2}-\alpha)$. Suppose first that $f(s)$ is regular for $\sigma \geqslant \alpha$. Let

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \Gamma(s)f(s)z^{-s}\,ds = \phi(z) \quad (\sigma \geqslant \alpha,\ |\arg z| < \tfrac{1}{2}\pi).$$

Putting $z = ixe^{-i\delta}$ $(0 < \delta < \frac{1}{2}\pi)$, we find that

$$\Gamma(\sigma+it)f(\sigma+it)e^{-i(\sigma+it)(\frac{1}{2}\pi-\delta)}, \qquad \phi(ixe^{-i\delta})$$

are Mellin transforms. Let

$$I(\sigma) = \int_{-\infty}^{\infty} |\Gamma(\sigma+it)f(\sigma+it)|^2 e^{(\pi-2\delta)t}\,dt.$$

Then, using Parseval's formula and Hölder's inequality, we obtain

$$\begin{aligned} I(\sigma) &= 2\pi\int_0^\infty |\phi(ixe^{-i\delta})|^2 x^{2\sigma-1}\,dx \\ &\leqslant 2\pi\left(\int_0^\infty |\phi|^2 x^{2\alpha-1}\,dx\right)^{(\beta-\sigma)/(\beta-\alpha)}\left(\int_0^\infty |\phi|^2 x^{2\beta-1}\,dx\right)^{(\sigma-\alpha)/(\beta-\alpha)} \\ &= \{I(\alpha)\}^{(\beta-\sigma)/(\beta-\alpha)}\{I(\beta)\}^{(\sigma-\alpha)/(\beta-\alpha)}. \end{aligned}$$

† Hardy, Ingham, and Pólya (1), Titchmarsh (23).

Writing $$F(T) = \int_0^T |f(\alpha+it)|^2\,dt \leqslant C(T^a+1)$$

we have by Stirling's theorem (with various values of K)

$$I(\alpha) < K\int_0^\infty (t^{2\alpha-1}+1)|f(\alpha+it)|^2 e^{-2\delta t}\,dt$$

$$= K\int_0^\infty F(t)\{2\delta(t^{2\alpha-1}+1)-(2\alpha-1)t^{2\alpha-2}\}e^{-2\delta t}\,dt$$

$$< KC\int_0^\infty (t^a+1)2\delta(t^{2\alpha-1}+1)e^{-2\delta t}\,dt$$

$$< KC\int_0^\infty (t^{a+2\alpha-1}+1)\delta e^{-2\delta t}\,dt$$

$$= KC\int_0^\infty \left\{\left(\frac{u}{\delta}\right)^{a+2\alpha-1}+1\right\}e^{-2u}\,du$$

$$< KC(\delta^{-a-2\alpha+1}+1) < KC\delta^{-a-2\alpha+1}.$$

Similarly for $I(\beta)$. Hence

$$I(\sigma) < K(C\delta^{-a-2\alpha+1})^{(\beta-\sigma)/(\beta-\alpha)}(C'\delta^{-b-2\beta+1})^{(\sigma-\alpha)/(\beta-\alpha)}$$
$$= K\delta^{-2\sigma+1}(C\delta^{-a})^{(\beta-\sigma)/(\beta-\alpha)}(C'\delta^{-b})^{(\sigma-\alpha)/(\beta-\alpha)}.$$

Also
$$I(\sigma) > K\int_{1/2\delta}^{1/\delta} |f(\sigma+it)|^2 t^{2\sigma-1}\,dt > K\delta^{-2\sigma+1}\int_{1/2\delta}^{1/\delta}|f(\sigma+it)|^2\,dt.$$

Putting $\delta = 1/T$, the result follows.

If $f(s)$ has a pole of order k at s_0, we argue similarly with $(s-s_0)^k f(s)$; this merely introduces a factor T^{2k} on each side of the result, so that (7.8.3) again follows.

Replacing T in (7.8.3) by $\frac{1}{2}T, \frac{1}{4}T,\ldots$, and adding, we obtain the result:

If
$$\int_0^T |f(\alpha+it)|^2\,dt = O(T^a), \qquad \int_0^T |f(\beta+it)|^2\,dt = O(T^b),$$

then
$$\int_0^T |f(\sigma+it)|^2\,dt = O\{T^{(a(\beta-\sigma)+b(\sigma-\alpha))/(\beta-\alpha)}\}.$$

Taking $f(s) = \zeta^k(s)$, the convexity of $\mu_k(\sigma)$ follows.

7.9. An alternative method of dealing with these problems is due to Carlson.† His main result is

THEOREM 7.9. *Let σ_k be the lower bound of numbers σ such that*

$$\frac{1}{T}\int_1^T |\zeta(\sigma+it)|^{2k}\,dt = O(1). \tag{7.9.1}$$

Then $$\sigma_k \leqslant \max\left(1-\frac{1-\alpha}{1+\mu_k(\alpha)}, \tfrac{1}{2}, \alpha\right)$$
for $0 < \alpha < 1$.

We first prove the following lemma.

LEMMA. *Let $f(s) = \sum_{n=1}^\infty a_n n^{-s}$ be absolutely convergent for $\sigma > 1$. Then*

$$\sum_{n=1}^\infty \frac{a_n}{n^s}e^{-\delta n} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(w-s)f(w)\delta^{s-w}\,dw$$

for $\delta > 0$, $c > 1$, $c > \sigma$.

For the right-hand side is

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(w-s)\sum_{n=1}^\infty \frac{a_n}{n^w}\delta^{s-w}\,dw = \sum_{n=1}^\infty \frac{a_n}{n^s}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(w-s)(\delta n)^{s-w}\,dw$$
$$= \sum_{n=1}^\infty \frac{a_n}{n^s}\frac{1}{2\pi i}\int_{c-\sigma-i\infty}^{c-\sigma+i\infty}\Gamma(w')(\delta n)^{-w'}\,dw'$$
$$= \sum_{n=1}^\infty \frac{a_n}{n^s}e^{-\delta n}.$$

The inversion is justified by the convergence of

$$\int_{-\infty}^{\infty} |\Gamma\{c-\sigma+i(v-t)\}|\sum_{n=1}^\infty \frac{|a_n|}{n^c}\delta^{\sigma-c}\,dv.$$

Taking $a_n = d_k(n)$, $f(s) = \zeta^k(s)$, $c = 2$, we obtain

$$\sum_{n=1}^\infty \frac{d_k(n)}{n^s}e^{-\delta n} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Gamma(w-s)\zeta^k(w)\delta^{s-w}\,dw \quad (\sigma < 2).$$

Moving the contour to $\mathbf{R}(w) = \alpha$, where $\sigma-1 < \alpha < \sigma$, we pass the pole of $\Gamma(w-s)$ at $w = s$, with residue $\zeta^k(s)$, and the pole of $\zeta^k(w)$ at $w = 1$, where the residue is a finite sum of terms of the form

$$K_{m,n}\Gamma^{(m)}(1-s)\log^n\delta\,.\,\delta^{s-1}.$$

† Carlson (2), (3).

This residue is therefore of the form $O(\delta^{\sigma-1+\epsilon}e^{-A|t|})$, and, if $\delta > |t|^{-A}$, it is of the form $O(e^{-A|t|})$. Hence

$$\zeta^k(s) = \sum_{n=1}^{\infty}\frac{d_k(n)}{n^s}e^{-\delta n}-\frac{1}{2\pi i}\int\limits_{\alpha-i\infty}^{\alpha+i\infty}\Gamma(w-s)\zeta^k(w)\delta^{s-w}\,dw+O(e^{-A|t|}).$$

Let us call the first two terms on the right Z_1 and Z_2. Then, as in previous proofs, if $\sigma > \frac{1}{2}$,

$$\int\limits_{\frac{1}{2}T}^{T}|Z_1|^2\,dt = O\left(T\sum_{n=1}^{\infty}\frac{d_k^2(n)}{n^{2\sigma}}e^{-2\delta n}\right)+O\left(\sum\sum_{m\neq n}\frac{d_k(m)d_k(n)e^{-(m+n)\delta}}{m^{\sigma}n^{\sigma}|\log m/n|}\right)$$

$$= O(T)+O\left(\sum\sum_{m\neq n}\frac{e^{-(m+n)\delta}}{m^{\sigma-\epsilon}n^{\sigma-\epsilon}|\log m/n|}\right)$$

$$= O(T)+O(\delta^{2\sigma-2-\epsilon})$$

by (7.2.2). Also, putting $w = \alpha+iv$,

$$|Z_2| \leqslant \frac{\delta^{\sigma-\alpha}}{2\pi}\int\limits_{-\infty}^{\infty}|\Gamma(w-s)\zeta^k(w)|\,dv$$

$$\leqslant \frac{\delta^{\sigma-\alpha}}{2\pi}\left(\int\limits_{-\infty}^{\infty}|\Gamma(w-s)|\,dv\int\limits_{-\infty}^{\infty}|\Gamma(w-s)\zeta^{2k}(w)|\,dv\right)^{\frac{1}{2}}.$$

The first integral is $O(1)$, while for $|t| \leqslant T$

$$\left(\int\limits_{-\infty}^{-2T}+\int\limits_{2T}^{\infty}\right)|\Gamma(w-s)\zeta^{2k}(w)|\,dv = \left(\int\limits_{-\infty}^{-2T}+\int\limits_{2T}^{\infty}\right)e^{-A|v-t|}|v|^{Ak}\,dv = O(e^{-AT}).$$

Hence

$$\int\limits_{\frac{1}{2}T}^{T}|Z_2|^2\,dt = O\left\{\delta^{2\sigma-2\alpha}\int\limits_{-2T}^{2T}|\zeta(w)|^{2k}\,dv\int\limits_{\frac{1}{2}T}^{T}|\Gamma(w-s)|\,dt\right\}+O(\delta^{2\sigma-2\alpha})$$

$$= O\left\{\delta^{2\sigma-2\alpha}\int\limits_{-2T}^{2T}|\zeta(\alpha+iv)|^{2k}\,dv\right\}+O(\delta^{2\sigma-2\alpha})$$

$$= O(\delta^{2\sigma-2\alpha}T^{1+\mu_k(\alpha)+\epsilon}).$$

Hence

$$\int\limits_{\frac{1}{2}T}^{T}|\zeta(s)|^{2k}\,dt = O(T)+O(\delta^{2\sigma-2-\epsilon})+O(\delta^{2\sigma-2\alpha}T^{1+\mu_k(\alpha)+\epsilon}).$$

Let $\delta = T^{-\frac{1}{2}(1+\mu_k(\alpha))/(1-\alpha)}$, so that the last two terms are of the same order, apart from ϵ's. These terms are then $O(T)$ if

$$\sigma > 1-\frac{1-\alpha}{1+\mu_k(\alpha)}.$$

For such values of σ, replacing T by $\frac{1}{2}T$, $\frac{1}{4}T$,..., and adding, it follows that (7.9.1) holds. Hence σ_k is less than any such σ, and the theorem follows.

A similar argument shows that, if we define σ'_k to be the lower bound of numbers σ such that

$$\frac{1}{T}\int\limits_{1}^{T}|\zeta(\sigma+it)|^{2k}\,dt = O(T^{\epsilon}), \tag{7.9.2}$$

then actually $\sigma'_k = \sigma_k$. For clearly $\sigma'_k \leqslant \sigma_k$; and the above argument shows that, if $\alpha > \sigma'_k$, and $\sigma < \alpha$, then

$$\int\limits_{\frac{1}{2}T}^{T}|\zeta(\sigma+it)|^{2k}\,dt = O(T)+O(\delta^{2\sigma-2-\epsilon})+O(\delta^{2\sigma-2\alpha}T^{1+\epsilon}).$$

Taking $\delta = T^{-\lambda}$, where $0 < \lambda < 1/(2-2\sigma)$, the right-hand side is $O(T)$. Hence $\sigma_k \leqslant \alpha$, and so $\sigma_k \leqslant \sigma'_k$.

It is also easily seen that

$$\frac{1}{T}\int\limits_{1}^{T}|\zeta(\sigma+it)|^{2k}\,dt \sim \sum_{n=1}^{\infty}\frac{d_k^2(n)}{n^{2\sigma}} \quad (\sigma > \sigma_k).$$

For the term $O(T)$ of the above argument is actually

$$\tfrac{1}{2}T\sum_{n=1}^{\infty}\frac{d_k^2(n)}{n^{2\sigma}}e^{-2\delta n} = \tfrac{1}{2}T\sum_{n=1}^{\infty}\frac{d_k^2(n)}{n^{2\sigma}}+o(T),$$

and the result follows by obvious modifications of the argument. This is a case of a general theorem on Dirichlet series.†

THEOREM 7.9 (A). *If $\mu(\sigma)$ is the μ-function defined in § 5.1,*

$$1-\sigma_k \geqslant \frac{1-\sigma_{k-1}}{1+2\mu(\sigma_{k-1})}$$

for $k = 1, 2,\ldots$.

Since $\zeta(\sigma+it) = O(t^{\mu(\sigma)+\epsilon})$,

$$\int\limits_{1}^{T}|\zeta(\sigma+it)|^{2k}\,dt = O\left\{T^{2\mu(\sigma)+\epsilon}\int\limits_{1}^{T}|\zeta(\sigma+it)|^{2k-2}\,dt\right\},$$

and hence $\qquad \mu_k(\sigma) \leqslant 2\mu(\sigma)+\mu_{k-1}(\sigma).$

Since $\mu_{k-1}(\sigma_{k-1}) = 0$, this gives $\mu_k(\sigma_{k-1}) \leqslant 2\mu(\sigma_{k-1})$, and the result follows on taking $\alpha = \sigma_{k-1}$ in the previous theorem.

† See E. C. Titchmarsh, *Theory of Functions*, § 9.51.

These formulae may be used to give alternative proofs of Theorems 7.2, 7.5, and 7.7. It follows from the functional equation that

$$\mu_k(1-\sigma) = \mu_k(\sigma)+2k(\sigma-\tfrac{1}{2}).$$

Since $\mu_k(\sigma_k) = 0$, $\mu_k(1-\sigma_k) \geqslant 0$, it follows that $\sigma_k \geqslant \frac{1}{2}$. Hence, putting $\alpha = 1-\sigma_k$ in Theorem 7.9, we obtain either $\sigma_k = \frac{1}{2}$ or

$$\sigma_k \leqslant 1-\frac{\sigma_k}{1+2k(\sigma_k-\frac{1}{2})},$$

i.e.

$$2\sigma_k-1 \leqslant 2k(\sigma_k-\tfrac{1}{2})(1-\sigma_k).$$

Hence $\sigma_k = \frac{1}{2}$, or

$$1 \leqslant k(1-\sigma_k), \qquad \sigma_k \leqslant 1-\frac{1}{k}. \tag{7.9.3}$$

For $k = 2$ we obtain $\sigma_2 = \frac{1}{2}$, but for $k > 2$ we must take the weaker alternative (7.9.2).

7.10. The following refinement† on the above results uses the theorems of Chapter V on $\mu(\sigma)$.

THEOREM 7.10. *Let k be an integer greater than 1, and let ν be determined by*

$$(\nu-1)2^{\nu-2}+1 < k \leqslant \nu 2^{\nu-1}+1. \tag{7.10.1}$$

Then

$$\sigma_k \leqslant 1-\frac{\nu+1}{2k+2^\nu-2}. \tag{7.10.2}$$

The theorem is true for $k = 2$ ($\nu = 1$). We then suppose it true for all l with $1 < l < k$, and deduce it for k.

Take $l = (\nu-1)2^{\nu-2}+1$, where ν is determined by (7.10.1). Then $\mu_l(\alpha) = 0$, provided that

$$\alpha > 1-\frac{\nu}{2l+2^{\nu-1}-2} = 1-\frac{1}{2^{\nu-1}}.$$

Taking $\alpha = 1-2^{-\nu+1}+\epsilon$, we have, since

$$\frac{1}{T}\int_1^T |\zeta(\alpha+it)|^{2k}\,dt \leqslant \max_{1\leqslant t\leqslant T} |\zeta(\alpha+it)|^{2k-2l}\frac{1}{T}\int_1^T |\zeta(\alpha+it)|^{2l}\,dt,$$

$$\begin{aligned}\mu_k(\alpha) &\leqslant 2(k-l)\mu(\alpha)+\mu_l(\alpha)\\ &= 2(k-l)\mu(\alpha)\\ &\leqslant \frac{2\{k-(\nu-1)2^{\nu-2}-1\}}{(\nu+1)2^{\nu-1}}\end{aligned}$$

† Davenport (1), Haselgrove (1).

by Theorem 5.8. Hence, by Theorem 7.9,

$$\sigma_k \leqslant 1-2^{-\nu+1}\left(\frac{2k+2^\nu-2}{(\nu+1)2^{\nu-1}}\right)^{-1} = 1-\frac{\nu+1}{2k+2^\nu-2}.$$

The theorem therefore follows by induction.

For example, if $k = 3$, then $\nu = 2$, and we obtain

$$\sigma_3 \leqslant \tfrac{5}{8}$$

instead of the result $\sigma_3 \leqslant \frac{2}{3}$ given by Theorem 7.7.

7.11. For integral k, $d_k(n)$ denotes the number of decompositions of n into k factors. If k is not an integer, we can define $d_k(n)$ as the coefficient of n^{-s} in the Dirichlet series for $\zeta^k(s)$, which converges for $\sigma > 1$.

We can now extend Theorem 7.7 to certain non-integral values of k.

THEOREM† 7.11. *For $0 < k \leqslant 2$*

$$\lim_{T\to\infty}\frac{1}{T}\int_1^T |\zeta(\sigma+it)|^{2k}\,dt = \sum_{n=1}^{\infty}\frac{d_k^2(n)}{n^{2\sigma}} \quad (\sigma > \tfrac{1}{2}). \tag{7.11.1}$$

This is the formula already proved for $k = 1$, $k = 2$; we now take $0 < k < 2$. Let

$$\zeta_N(s) = \prod_{p<N}\frac{1}{1-p^{-s}}, \qquad \eta_N(s) = \zeta(s)/\zeta_N(s).$$

The proof depends on showing (i) that the formula corresponding to (7.11.1) with ζ_N instead of ζ is true; and (ii) that $\zeta_N(s)$, though it does not converge to $\zeta(s)$ for $\sigma \leqslant 1$, still approximates to it in a certain average sense in this strip.

We have, if $\lambda > 0$,

$$\{\zeta_N(s)\}^\lambda = \prod_{p<N}(1-p^{-s})^{-\lambda} = \sum_{n=1}^{\infty}\frac{d'_\lambda(n)}{n^s},$$

say, where the series on the right converges absolutely for $\sigma > 0$, and $d'_\lambda(n) = d_\lambda(n)$ if $n < N$, and $0 \leqslant d'_\lambda(n) \leqslant d_\lambda(n)$ for all n. Hence

$$\lim_{T\to\infty}\frac{1}{T}\int_1^T |\zeta_N(\sigma+it)|^{2\lambda}\,dt = \sum_{n=1}^{\infty}\frac{\{d'_\lambda(n)\}^2}{n^{2\sigma}} \quad (\sigma > 0), \tag{7.11.2}$$

and

$$\lim_{N\to\infty}\lim_{T\to\infty}\frac{1}{T}\int_1^T |\zeta_N(\sigma+it)|^{2\lambda}\,dt = \sum_{n=1}^{\infty}\frac{\{d_\lambda(n)\}^2}{n^{2\sigma}} \quad (\sigma > \tfrac{1}{2}). \tag{7.11.3}$$

† Ingham (4); proof by Davenport (1).

We shall next prove that

$$\lim_{N\to\infty}\lim_{T\to\infty}\frac{1}{T}\int_1^T|\zeta(\sigma+it)-\zeta_N(\sigma+it)|^{2k}\,dt = 0 \quad (\sigma > \tfrac{1}{2}). \tag{7.11.4}$$

By Hölder's inequality

$$\frac{1}{T}\int_1^T|\zeta-\zeta_N|^{2k}\,dt \leqslant \left\{\frac{1}{T}\int_1^T|\eta_N-1|^4\,dt\right\}^{\frac{1}{2}k}\left\{\frac{1}{T}\int_1^T|\zeta_N|^{4k/(2-k)}\,dt\right\}^{\frac{1}{2}(2-k)}. \tag{7.11.5}$$

Now $\{\eta_N(s)-1\}^2$ is regular everywhere except for a pole at $s = 1$, and is of finite order in t. Also, for $\sigma > \frac{1}{2}$,

$$\int_1^T|\eta_N(\sigma+it)-1|^4\,dt \leqslant \int_1^T\{1+2^N|\zeta(\sigma+it)|\}^4\,dt = O(T).$$

Hence, by a theorem of Carlson,†

$$\lim_{T\to\infty}\frac{1}{T}\int_1^T|\eta_N(\sigma+it)-1|^4\,dt = \sum_{n=1}^{\infty}\frac{\rho_N^2(n)}{n^{2\sigma}}$$

for $\sigma > \frac{1}{2}$, where ρ_N is the coefficient of n^{-s} in the Dirichlet series of $\{\eta_N(s)-1\}^2$. Now $\rho_N(n) = 0$ for $n < N$, and $0 \leqslant \rho_N(n) \leqslant d(n)$ for all n. Since $\sum d^2(n)n^{-2\sigma}$ converges, it follows that

$$\lim_{N\to\infty}\lim_{T\to\infty}\frac{1}{T}\int_1^T|\eta_N(\sigma+it)-1|^4\,dt = 0; \tag{7.11.6}$$

(7.11.4) now follows from (7.11.5), (7.11.6), and (7.11.3).

We can now deduce (7.11.1) from (7.11.3) and (7.11.4). We have‡

$$\left\{\int_1^T|\zeta|^{2k}\,dt\right\}^R = \left\{\int_1^T|\zeta_N+\zeta-\zeta_N|^{2k}\,dt\right\}^R$$
$$\leqslant \left\{\int_1^T|\zeta_N|^{2k}\,dt\right\}^R+\left\{\int_1^T|\zeta-\zeta_N|^{2k}\,dt\right\}^R,$$

where $R = 1$ if $0 < 2k \leqslant 1$, $R = 1/2k$ if $2k > 1$. Similarly

$$\left\{\int_1^T|\zeta_N|^{2k}\,dt\right\}^R \leqslant \left\{\int_1^T|\zeta|^{2k}\,dt\right\}^R+\left\{\int_1^T|\zeta-\zeta_N|^{2k}\,dt\right\}^R,$$

and (7.11.1) clearly follows.

† See Titchmarsh, *Theory of Functions*, § 9.51.
‡ Hardy, Littlewood, and Pólya, *Inequalities*, Theorem 28.

7.12. An alternative set of mean-value theorems.† Instead of considering integrals of the form

$$I(T) = \int_0^T|\zeta(\sigma+it)|^{2k}\,dt$$

where T is large, we shall now consider integrals of the form

$$J(\delta) = \int_0^\infty|\zeta(\sigma+it)|^{2k}e^{-\delta t}\,dt$$

where δ is small.

The behaviour of these two integrals is very similar. If $J(\delta) = O(1/\delta)$, then

$$I(T) < e\int_0^T|\zeta(\sigma+it)|^{2k}e^{-t/T}\,dt < eJ(1/T) = O(T).$$

Conversely, if $I(T) = O(T)$, then

$$J(\delta) = \int_0^\infty I'(t)e^{-\delta t}\,dt = [I(t)e^{-\delta t}]_0^\infty+\delta\int_0^\infty I(t)e^{-\delta t}\,dt$$
$$= O\left(\delta\int_0^\infty te^{-\delta t}\,dt\right) = O(1/\delta).$$

Similar results plainly hold with other powers of T, and with other functions, such as powers of T multiplied by powers of $\log T$.

We have also more precise results; for example, *if* $I(T) \sim CT$, *then* $J(\delta) \sim C/\delta$, *and conversely.*

If $I(T) \sim CT$, let $|I(T)-CT| \leqslant \epsilon T$ for $T \geqslant T_0$. Then

$$J(\delta) = \delta\int_0^{T_0}I(t)e^{-\delta t}\,dt+\delta\int_{T_0}^\infty\{I(t)-Ct\}e^{-\delta t}\,dt+C\delta\int_{T_0}^\infty te^{-\delta t}\,dt.$$

The last term is $Ce^{-\delta T_0}(T_0+1/\delta)$, and the modulus of the previous term does not exceed $\epsilon(T_0+1/\delta)$. That $J(\delta) \sim C/\delta$ plainly follows on choosing first T_0 and then δ.

The converse deduction is the analogue for integrals of the well-known Tauberian theorem of Hardy and Littlewood,‡ viz. that *if* $a_n \geqslant 0$, *and*

$$\sum_{n=0}^\infty a_n x^n \sim \frac{1}{1-x} \quad (x \to 1)$$

then

$$\sum_{n=0}^N a_n \sim N.$$

† Titchmarsh (1), (19).
‡ See Titchmarsh, *Theory of Functions*, §§ 7.51–7.53.

The theorem for integrals is as follows:

If $f(t) \geqslant 0$ for all t, and

$$\int_0^\infty f(t)e^{-\delta t}\,dt \sim \frac{1}{\delta} \tag{7.12.1}$$

as $\delta \to 0$, then

$$\int_0^T f(t)\,dt \sim T \tag{7.12.2}$$

as $T \to \infty$.

We first show that, if $P(x)$ is any polynomial,

$$\int_0^\infty f(t)e^{-\delta t}P(e^{-\delta t})\,dt \sim \frac{1}{\delta}\int_0^1 P(x)\,dx.$$

It is sufficient to prove this for $P(x) = x^k$. In this case the left-hand side is

$$\int_0^\infty f(t)e^{-(k+1)\delta t}\,dt \sim \frac{1}{(k+1)\delta} = \frac{1}{\delta}\int_0^1 x^k\,dx.$$

Next, we deduce that

$$\int_0^\infty f(t)e^{-\delta t}g(e^{-\delta t})\,dt \sim \frac{1}{\delta}\int_0^1 g(x)\,dx \tag{7.12.3}$$

if $g(x)$ is continuous, or has a discontinuity of the first kind. For, given ϵ, we can† construct polynomials $p(x)$, $P(x)$, such that

$$p(x) \leqslant g(x) \leqslant P(x)$$

and

$$\int_0^1 \{g(x)-p(x)\}\,dx \leqslant \epsilon, \qquad \int_0^1 \{P(x)-g(x)\}\,dx \leqslant \epsilon.$$

Then

$$\varlimsup_{\delta\to 0} \delta \int_0^\infty f(t)e^{-\delta t}g(e^{-\delta t})\,dt \leqslant \varlimsup_{\delta\to 0} \delta \int_0^\infty f(t)e^{-\delta t}P(e^{-\delta t})\,dt$$

$$= \int_0^1 P(x)\,dx < \int_0^1 g(x)\,dx+\epsilon,$$

and making $\epsilon \to 0$ we obtain

$$\varlimsup_{\delta\to 0} \delta \int_0^\infty f(t)e^{-\delta t}g(e^{-\delta t})\,dt \leqslant \int_0^1 g(x)\,dx.$$

Similarly, arguing with $p(x)$, we obtain

$$\varliminf_{\delta\to 0} \delta \int_0^\infty f(t)e^{-\delta t}g(e^{-\delta t})\,dt \geqslant \int_0^1 g(x)\,dx,$$

and (7.12.3) follows.

† See Titchmarsh, *Theory of Functions*, § 7.53.

Now let

$$g(x) = 0 \quad (0 \leqslant x < e^{-1}), \qquad = 1/x \quad (e^{-1} \leqslant x \leqslant 1).$$

Then

$$\int_0^\infty f(t)e^{-\delta t}g(e^{-\delta t})\,dt = \int_0^{1/\delta} f(t)\,dt$$

and

$$\int_0^1 g(x)\,dx = \int_{1/e}^1 \frac{dx}{x} = 1.$$

Hence

$$\int_0^{1/\delta} f(t)\,dt \sim \frac{1}{\delta},$$

which is equivalent to (7.12.2).

If $f(t) \geqslant 0$ for all t, and, for a given positive m,

$$\int_0^\infty f(t)e^{-\delta t}\,dt \sim \frac{1}{\delta}\log^m\frac{1}{\delta}, \tag{7.12.4}$$

then

$$\int_0^T f(t)\,dt \sim T\log^m T. \tag{7.12.5}$$

The proof is substantially the same. We have

$$\int_0^\infty f(t)e^{-(k+1)\delta t}\,dt \sim \frac{1}{(k+1)\delta}\log^m\left\{\frac{1}{(k+1)\delta}\right\} \sim \frac{1}{(k+1)\delta}\log^m\frac{1}{\delta},$$

and the argument runs as before, with $\frac{1}{\delta}$ replaced by $\frac{1}{\delta}\log^m\frac{1}{\delta}$.

We shall also use the following theorem:

If

$$\int_1^\infty f(t)e^{-\delta t}\,dt \sim C\delta^{-\alpha} \quad (\alpha > 0), \tag{7.12.6}$$

then

$$\int_1^\infty t^{-\beta}f(t)e^{-\delta t}\,dt \sim C\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\delta^{\beta-\alpha} \quad (0 < \beta < \alpha). \tag{7.12.7}$$

Multiplying (7.12.6) by $(\delta-\eta)^{\beta-1}$ and integrating over (η, ∞), we obtain

$$\int_1^\infty f(t)\,dt \int_\eta^\infty e^{-\delta t}(\delta-\eta)^{\beta-1}\,d\delta = C\int_\eta^\infty \{\delta^{-\alpha}+o(\delta^{-\alpha})\}(\delta-\eta)^{\beta-1}\,d\delta.$$

Now
$$\int_\eta^\infty e^{-\delta t}(\delta-\eta)^{\beta-1}\,d\delta = e^{-\eta t}\int_0^\infty e^{-xt}x^{\beta-1}\,dx = e^{-\eta t}t^{-\beta}\Gamma(\beta),$$
$$\int_\eta^\infty \delta^{-\alpha}(\delta-\eta)^{\beta-1}\,d\delta = \int_0^\infty \frac{x^{\beta-1}}{(\eta+x)^\alpha}\,dx = \eta^{\beta-\alpha}\frac{\Gamma(\beta)\Gamma(\alpha-\beta)}{\Gamma(\alpha)},$$
and the remaining term is plainly $o(\eta^{\beta-\alpha})$ as $\eta \to 0$. Hence the result.

7.13. We can approximate to integrals of the form $J(\delta)$ by means of Parseval's formula. If $\mathbf{R}(z) > 0$, we have
$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Gamma(s)\zeta^k(s)z^{-s}\,ds = \sum_{n=1}^\infty \frac{d_k(n)}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Gamma(s)(nz)^{-s}\,ds = \sum_{n=1}^\infty d_k(n)e^{-nz},$$
the inversion being justified by absolute convergence. Now move the contour to $\sigma = \alpha$ $(0<\alpha<1)$. Let $R_k(z)$ be the residue at $s = 1$, so that $R_k(z)$ is of the form
$$\frac{1}{z}(a_0^{(k)}+a_1^{(k)}\log z+...+a_{k-1}^{(k)}\log^{k-1}z).$$

Let
$$\phi_k(z) = \sum_{n=1}^\infty d_k(n)e^{-nz}-R_k(z).$$

Then
$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\Gamma(s)\zeta^k(s)z^{-s}\,ds = \phi_k(z). \tag{7.13.1}$$

Putting $z = ixe^{-i\delta}$, where $0<\delta<\frac{1}{2}\pi$, we see that
$$\phi_k(ixe^{-i\delta}),\qquad \Gamma(s)\zeta^k(s)e^{-i(\frac{1}{2}\pi-\delta)s} \tag{7.13.2}$$
are Mellin transforms. Hence the Parseval formula gives
$$\frac{1}{2\pi}\int_{-\infty}^\infty |\Gamma(\sigma+it)\zeta^k(\sigma+it)|^2e^{(\pi-2\delta)t}\,dt = \int_0^\infty |\phi_k(ixe^{-i\delta})|^2x^{2\sigma-1}\,dx. \tag{7.13.3}$$

Now as $|t| \to \infty$
$$|\Gamma(\sigma+it)| = e^{-\frac{1}{2}\pi|t|}|t|^{\sigma-\frac{1}{2}}\sqrt{(2\pi)}\{1+O(t^{-1})\}.$$
Hence the part of the t-integral over $(-\infty, 0)$ is bounded as $\delta \to 0$, and we obtain, for $\frac{1}{2}<\sigma<1$,
$$\int_0^\infty t^{2\sigma-1}\{1+O(t^{-1})\}|\zeta(\sigma+it)|^{2k}e^{-2\delta t}\,dt = \int_0^\infty |\phi_k(ixe^{-i\delta})|^2x^{2\sigma-1}\,dx+O(1). \tag{7.13.4}$$

In the case $\sigma = \frac{1}{2}$, we have
$$|\Gamma(\tfrac{1}{2}+it)|^2 = \pi\operatorname{sech}\pi t = 2\pi e^{-\pi|t|}+O(e^{-3\pi|t|}).$$
The integral over $(-\infty, 0)$, and the contribution of the O-term to the whole integral, are now bounded, and in fact are analytic functions of δ, regular for sufficiently small $|\delta|$. Hence we have
$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^{2k}e^{-2\delta t}\,dt = \int_0^\infty |\phi_k(ixe^{-i\delta})|^2\,dx+O(1). \tag{7.13.5}$$

7.14. We now apply the above formulae to prove

THEOREM 7.14. *As* $\delta \to 0$
$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^2e^{-\delta t}\,dt \sim \frac{1}{\delta}\log\frac{1}{\delta}. \tag{7.14.1}$$

In this case $R_1(z) = 1/z$, and
$$\phi_1(z) = \sum_{n=1}^\infty e^{-nz}-\frac{1}{z} = \frac{1}{e^z-1}-\frac{1}{z}.$$
Hence (7.13.5) gives
$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^2e^{-2\delta t}\,dt = \int_0^\infty \left|\frac{1}{\exp(ixe^{-i\delta})-1}-\frac{1}{ixe^{-i\delta}}\right|^2dx+O(1). \tag{7.14.2}$$
The x-integrand is bounded uniformly in δ over $(0,\pi)$, so that this part of the integral is $O(1)$. The remainder is
$$\int_\pi^\infty \left\{\frac{1}{\exp(ixe^{-i\delta})-1}-\frac{1}{ixe^{-i\delta}}\right\}\left\{\frac{1}{\exp(-ixe^{i\delta})-1}+\frac{1}{ixe^{i\delta}}\right\}dx$$
$$= \int_\pi^\infty \frac{dx}{\{\exp(ixe^{-i\delta})-1\}\{\exp(-ixe^{i\delta})-1\}}+ie^{i\delta}\int_\pi^\infty \frac{1}{\exp(-ixe^{i\delta})-1}\frac{dx}{x}-$$
$$-ie^{-i\delta}\int_\pi^\infty \frac{1}{\exp(ixe^{-i\delta})-1}\frac{dx}{x}+\int_\pi^\infty \frac{dx}{x^2}. \tag{7.14.3}$$

The last term is a constant. In the second term, turn the line of integration round to $(\pi, \pi+i\infty)$. The integrand is then regular on the contour for sufficiently small $|\delta|$, and is $O\{x^{-1}\exp(-x\cos\delta)\}$ as $x \to \infty$. This integral is therefore bounded; and similarly so is the third term.

The first term is

$$\int_{\pi}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\exp(-imxe^{-i\delta}+inxe^{i\delta})\,dx$$

$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\exp\{-(m+n)\pi\sin\delta-i(m-n)\pi\cos\delta\}}{(m+n)\sin\delta+i(m-n)\cos\delta}$$

$$=\sum_{n=1}^{\infty}\frac{e^{-2n\pi\sin\delta}}{2n\sin\delta}+2\sum_{m=2}^{\infty}\sum_{n=1}^{m-1}\frac{(m+n)\sin\delta\cos\{(m-n)\pi\cos\delta\}}{(m+n)^2\sin^2\delta+(m-n)^2\cos^2\delta}e^{-(m+n)\pi\sin\delta}-$$

$$-2\sum_{m=2}^{\infty}\sum_{n=1}^{m-1}\frac{(m-n)\cos\delta\sin\{(m-n)\pi\cos\delta\}}{(m+n)^2\sin^2\delta+(m-n)^2\cos^2\delta}e^{-(m+n)\pi\sin\delta}$$

$$=\Sigma_1+\Sigma_2+\Sigma_3,$$

the series of imaginary parts vanishing identically. Now

$$\Sigma_1=\frac{1}{2\sin\delta}\log\frac{1}{1-e^{-2\pi\sin\delta}}\sim\frac{1}{2\delta}\log\frac{1}{\delta},$$

$$|\Sigma_2|<2\sum_{m=2}^{\infty}\sum_{n=1}^{m-1}\frac{2m\sin\delta}{(m-n)^2\cos^2\delta}e^{-m\pi\sin\delta}=O\Big(\delta\sum_{m=2}^{\infty}me^{-m\pi\sin\delta}\Big)=O\Big(\frac{1}{\delta}\Big),$$

and, since $|\sin\{(m-n)\pi\cos\delta\}|=|\sin\{2(m-n)\pi\sin^2\frac{1}{2}\delta\}|=O\{(m-n)\delta^2\}$,

$$\Sigma_3=O\Big(\delta^2\sum_{m=2}^{\infty}\sum_{n=1}^{m-1}e^{-m\pi\sin\delta}\Big)=O\Big(\delta^2\sum_{m=2}^{\infty}me^{-m\pi\sin\delta}\Big)=O(1).$$

This proves the theorem.

The case $\frac{1}{2}<\sigma<1$ can be dealt with in a similar way. The leading term is

$$\int_{\pi}^{\infty}\sum_{n=1}^{\infty}e^{-2nx\sin\delta}x^{2\sigma-1}\,dx=\int_{\pi}^{\infty}\frac{x^{2\sigma-1}}{e^{2x\sin\delta}-1}\,dx$$

$$=\frac{1}{(2\sin\delta)^{2\sigma}}\int_{2\pi\sin\delta}^{\infty}\frac{y^{2\sigma-1}}{e^y-1}\,dy\sim\frac{1}{(2\delta)^{2\sigma}}\int_{0}^{\infty}\frac{y^{2\sigma-1}}{e^y-1}\,dy$$

$$=\frac{1}{(2\delta)^{2\sigma}}\Gamma(2\sigma)\zeta(2\sigma).$$

Also (turning the line of integration through $-\frac{1}{2}\pi$)

$$\int_{\pi}^{\infty}e^{-\{(m+n)\sin\delta+i(m-n)\cos\delta\}x}x^{2\sigma-1}\,dx$$

$$=O\Big\{e^{-(m+n)\pi\sin\delta}\int_{0}^{\infty}e^{-(m-n)y\cos\delta}(\pi^{2\sigma-1}+y^{2\sigma-1})\,dy\Big\}$$

$$=O\Big(\frac{e^{-(m+n)\pi\sin\delta}}{m-n}\Big),\qquad(7.14.4)$$

and the terms with $m\neq n$ give

$$O\Big(\sum_{m=2}^{\infty}e^{-m\pi\sin\delta}\sum_{n=1}^{m-1}\frac{1}{m-n}\Big)=O\Big(\frac{1}{\delta}\log\frac{1}{\delta}\Big).$$

Hence

$$\int_{0}^{\infty}t^{2\sigma-1}|\zeta(\sigma+it)|^2e^{-2\delta t}\,dt\sim\frac{\Gamma(2\sigma)\zeta(2\sigma)}{2^{2\sigma}\delta^{2\sigma}}.\qquad(7.14.5)$$

Hence by (7.12.6), (7.12.7)

$$\int_{0}^{\infty}|\zeta(\sigma+it)|^2e^{-\delta t}\,dt\sim\frac{\zeta(2\sigma)}{\delta}.\qquad(7.14.6)$$

7.15. We shall now show that we can approximate to the integral (7.14.1) by an asymptotic series in positive powers of δ.

We first require†

THEOREM 7.15. *As $z\to 0$ in any angle $|\arg z|\leqslant\lambda$, where $\lambda<\frac{1}{2}\pi$,*

$$\sum_{n=1}^{\infty}d(n)e^{-nz}=\frac{\gamma-\log z}{z}+\frac{1}{4}+\sum_{n=0}^{N-1}b_nz^{2n+1}+O(|z|^{2N}),\qquad(7.15.1)$$

where the b_n are constants.

Near $s=1$

$$\Gamma(s)\zeta^2(s)z^{-s}=\{1-\gamma(s-1)+...\}\Big(\frac{1}{s-1}+\gamma+...\Big)^2\frac{1}{z}\{1-(s-1)\log z+...\}$$

$$=\frac{1}{z(s-1)^2}+\frac{\gamma-\log z}{z}\frac{1}{s-1}+....$$

Hence by (7.13.1), with $k=2$,

$$\sum_{n=1}^{\infty}d(n)e^{-nz}=\frac{\gamma-\log z}{z}+\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\Gamma(s)\zeta^2(s)z^{-s}\,ds\quad(0<\alpha<1).$$

Here we can move the line of integration to $\sigma=-2N$, since $\Gamma(s)=O(|t|^Ke^{-\frac{1}{2}\pi|t|})$, $\zeta^2(s)=O(|t|^K)$ and $z^{-s}=O(r^{-\sigma}e^{\lambda|t|})$. The residue at $s=0$ is $\zeta^2(0)=\frac{1}{4}$. The poles of $\Gamma(s)$ at $s=-2n$ are cancelled by zeros of $\zeta^2(s)$. The poles of $\Gamma(s)$ at $s=-2n-1$ give residues

$$\frac{-1}{(2n+1)!}\zeta^2(-2n-1)z^{2n+1}=-\frac{B_{n+1}^2}{(2n+1)!(2n+2)^2}z^{2n+1}.$$

The remaining integral is $O(|z|^{2N})$, and the result follows.

The constant implied in the O, of course, depends on N, and the series taken to infinity is divergent, since the function $\sum d(n)\,e^{-nz}$ cannot be continued analytically across the imaginary axis.

† Wigert (1).

We can now prove†

THEOREM 7.15 (A). *As $\delta \to 0$, for every positive N,*

$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^2 e^{-2\delta t}\,dt = \frac{\gamma-\log 4\pi\delta}{2\sin\delta}+\sum_{n=0}^{N} c_n\,\delta^n+O(\delta^{N+1})$$

the constant of the O depending on N, and the c_n being constants.

We observe that the term $O(1)$ in (7.14.2) is

$$\tfrac{1}{2}\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^2 e^{-(\pi+2\delta)t}\operatorname{sech}\pi t\,dt-\tfrac{1}{2}\int_{-\infty}^0 |\zeta(\tfrac{1}{2}+it)|^2 e^{(\pi-2\delta)t}\operatorname{sech}\pi t\,dt,$$

and is thus an analytic function of δ, regular for $|\delta| < \pi$. Also

$$\int_0^\pi \left\{\frac{1}{\exp(ixe^{-i\delta})-1}-\frac{1}{ixe^{-i\delta}}\right\}\left\{\frac{1}{\exp(-ixe^{i\delta})-1}+\frac{1}{ixe^{i\delta}}\right\}dx$$

is analytic for sufficiently small $|\delta|$. We dissect the remainder of the integral on the right of (7.14.2) as in (7.14.3). As before

$$\int_\pi^\infty \frac{1}{\exp(-ixe^{i\delta})-1}\frac{dx}{x} = \int_\pi^{\pi+i\infty} \frac{1}{\exp(-ize^{i\delta})-1}\frac{dz}{z},$$

and the integrand is regular on the new line of integration for sufficiently small $|\delta|$, and, if $\delta = \xi+i\eta$, $z = \pi+iy$, it is $O\{y^{-1}\exp(-y\cos\xi e^{-\eta})\}$ as $y \to \infty$. The integral is therefore regular for sufficiently small $|\delta|$. Similarly for the third term on the right of (7.14.3); and the fourth term is a constant.

By the calculus of residues, the first term is equal to

$$2i\pi\sum_{n=1}^\infty \frac{1}{ie^{-i\delta}}\frac{1}{\exp(-2in\pi e^{2i\delta})-1}+$$
$$+\int_0^\infty \frac{dy}{[\exp\{(i\pi-y)e^{-i\delta}\}-1][\exp\{(-i\pi+y)e^{i\delta}\}-1]}.$$

As before, the y-integral is an analytic function of δ, regular for $|\delta|$ small enough. Expressing the series as a power series in $\exp(2i\pi e^{2i\delta})$, we therefore obtain

$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^2 e^{-2\delta t}\,dt = 2\pi e^{i\delta}\sum_{n=1}^\infty d(n)\exp(2in\pi e^{2i\delta})+\sum_{n=0}^\infty a_n\,\delta^n \quad (7.15.2)$$

for $|\delta|$ small enough and $\mathbf{R}(\delta) > 0$.

† Kober (4), Atkinson (1).

Let $z = 2i\pi(1-e^{2i\delta})$ in (7.15.1). Multiplying by $2\pi e^{i\delta}$, we obtain

$$2\pi e^{i\delta}\sum_{n=1}^\infty d(n)\exp(2in\pi e^{2i\delta}) = \frac{\gamma-\log(4\pi e^{i\delta}\sin\delta)}{2\sin\delta}+\frac{1}{4}+$$
$$+\sum_{n=0}^{N-1} b_n\{2i\pi(1-e^{2i\delta})\}^{2n+1}+O(\delta^{2N}),$$

and the result now easily follows.

7.16. The next case is that of $|\zeta(\tfrac{1}{2}+it)|^4$.

In (7.14.2) the contribution of the x-integral for small x was negligible. We now take (7.13.5) with $k = 2$, and

$$\phi_2(z) = \sum_{n=1}^\infty d(n)e^{-nz}-\frac{\gamma-\log z}{z}. \quad (7.16.1)$$

In this case the contribution of small x is not negligible, but is substantially the same as that of the other part. We have

$$\phi_2\left(\frac{1}{z}\right) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \Gamma(s)\zeta^2(s)z^s\,ds \quad (0 < \alpha < 1)$$
$$= \frac{1}{2\pi i}\int_{1-\alpha-i\infty}^{1-\alpha+i\infty} \Gamma(1-s)\zeta^2(1-s)z^{1-s}\,ds$$
$$= \frac{z}{2\pi i}\int_{1-\alpha-i\infty}^{1-\alpha+i\infty} \frac{\Gamma(1-s)}{\chi^2(s)}\zeta^2(s)z^{-s}\,ds.$$

Now

$$\Gamma(1-s)/\chi^2(s) = 2^{2-2s}\pi^{-2s}\cos^2\tfrac{1}{2}s\pi\,\Gamma^2(s)\Gamma(1-s)$$
$$= 2^{1-2s}\pi^{1-2s}\cot\tfrac{1}{2}s\pi\,\Gamma(s)$$
$$= 2^{1-2s}\pi^{1-2s}\left\{-i+O\left(\frac{e^{-\frac{1}{2}\pi t}}{|\sin\frac{1}{2}s\pi|}\right)\right\}\Gamma(s) \quad (t \to \pm\infty).$$

If $z = ixe^{-i\delta}$ ($x > x_0$, $0 < \delta < \tfrac{1}{2}\pi$), the O term is

$$O\left\{x\int_{1-\alpha-i\infty}^{1-\alpha+i\infty} \frac{e^{-\frac{1}{2}\pi t}}{|\sin\frac{1}{2}s\pi|}|\Gamma(s)|e^{(\frac{1}{2}\pi-\delta)t}(1+|t|)x^{\alpha-1}\,dt\right\} = O(x^\alpha),$$

uniformly for small δ. Hence

$$\phi_2\left(\frac{1}{z}\right) = \frac{-iz}{2\pi i}\int_{1-\alpha-i\infty}^{1-\alpha+i\infty} 2^{1-2s}\pi^{1-2s}\Gamma(s)\zeta^2(s)z^{-s}\,ds+O(x^\alpha)$$
$$= -2\pi iz\phi_2(4\pi^2 z)+O(x^\alpha), \quad (7.16.2)$$

where α may be as near zero as we please.

We also use the results

$$\sum_{n=1}^{\infty} d^2(n)e^{-n\eta} = O\left(\frac{1}{\eta}\log^3\frac{1}{\eta}\right), \tag{7.16.3}$$

$$\sum_{n=1}^{\infty} n^2 d^2(n)e^{-n\eta} = O\left(\frac{1}{\eta^3}\log^3\frac{1}{\eta}\right) \tag{7.16.4}$$

as $\eta \to 0$. By (1.2.10)

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Gamma(s)\frac{\zeta^4(s)}{\zeta(2s)}\,\eta^{-s}\,ds = \frac{1}{2\pi i}\sum_{n=1}^{\infty} d^2(n)\int_{2-i\infty}^{2+i\infty}\Gamma(s)(n\eta)^{-s}\,ds$$
$$= \sum_{n=1}^{\infty} d^2(n)e^{-n\eta}. \tag{7.16.5}$$

Hence

$$\sum_{n=1}^{\infty} d^2(n)e^{-n\eta} = R+\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\frac{\zeta^4(s)}{\zeta(2s)}\,\eta^{-s}\,ds \quad (\tfrac{1}{2}<c<1)$$
$$= R+O(\eta^{-c}),$$

where R is the residue at $s = 1$; and

$$R = \frac{1}{\eta}\left(a\log^3\frac{1}{\eta}+b\log^2\frac{1}{\eta}+c\log\frac{1}{\eta}+d\right),$$

where a, b, c, d are constants, and in fact

$$a = \frac{1}{3!\,\zeta(2)} = \frac{1}{\pi^2}.$$

This proves (7.16.3); and (7.16.4) can be proved similarly by first differentiating (7.16.5) twice with respect to η.

We can now prove†

THEOREM 7.16. *As* $\delta \to 0$

$$\int_0^{\infty}|\zeta(\tfrac{1}{2}+it)|^4 e^{-\delta t}\,dt \sim \frac{1}{2\pi^2}\frac{1}{\delta}\log^4\frac{1}{\delta}.$$

Using (7.13.5), we have

$$\int_0^{\infty}|\zeta(\tfrac{1}{2}+it)|^4 e^{-2\delta t}\,dt = \int_0^{\infty}|\phi_2(ixe^{-i\delta})|^2\,dx+O(1),$$

and it is sufficient to prove that

$$\int_{2\pi}^{\infty}|\phi_2(ixe^{-i\delta})|^2\,dx \sim \frac{1}{8\pi^2}\frac{1}{\delta}\log^4\frac{1}{\delta}.$$

† Titchmarsh (1).

For then, by (7.16.2),

$$\int_0^{2\pi}|\phi_2(ixe^{-i\delta})|^2\,dx = \int_{1/2\pi}^{\infty}\left|\phi_2\left(\frac{ie^{-i\delta}}{x}\right)\right|^2\frac{dx}{x^2} = \int_{1/2\pi}^{\infty}\left|\phi_2\left(\frac{1}{ixe^{-i\delta}}\right)\right|^2\frac{dx}{x^2}$$

$$= \int_{1/2\pi}^{\infty}|2\pi xe^{-i\delta}\phi_2(4\pi^2 ixe^{-i\delta})+O(x^{\alpha})|^2\frac{dx}{x^2} \quad (0<\alpha<\tfrac{1}{2})$$

$$= \int_{2\pi}^{\infty}|\phi_2(ixe^{-i\delta})+O(x^{\alpha-1})|^2\,dx$$

$$= \int_{2\pi}^{\infty}|\phi_2(ixe^{-i\delta})|^2\,dx+O\left(\int_{2\pi}^{\infty}|\phi_2(ixe^{-i\delta})|^2\,dx\int_{2\pi}^{\infty}x^{2\alpha-2}\,dx\right)^{\frac{1}{2}}+$$
$$+O\left(\int_{2\pi}^{\infty}x^{2\alpha-2}\,dx\right)$$

$$= \frac{1}{8\pi^2}\frac{1}{\delta}\log^4\frac{1}{\delta}+O\left(\frac{1}{\sqrt{\delta}}\log^2\frac{1}{\delta}\right)+O(1),$$

and the result clearly follows.

It is then sufficient to prove that

$$\int_{2\pi}^{\infty}\left|\sum_{n=1}^{\infty}d(n)\exp(-inxe^{-i\delta})\right|^2dx \sim \frac{1}{8\pi^2}\frac{1}{\delta}\log^4\frac{1}{\delta},$$

for the remainder of (7.16.1) will then contribute $O(\delta^{-\frac{1}{2}}\log^2 1/\delta)$.

As in the previous proof, the left-hand side is equal to

$$\sum_{n=1}^{\infty}d^2(n)\frac{e^{-4n\pi\sin\delta}}{2n\sin\delta}+$$
$$+2\sum_{m=2}^{\infty}\sum_{n=1}^{m-1}d(m)d(n)\frac{(m+n)\sin\delta\cos\{2(m-n)\pi\cos\delta\}}{(m+n)^2\sin^2\delta+(m-n)^2\cos^2\delta}e^{-2(m+n)\pi\sin\delta}-$$
$$-2\sum_{m=2}^{\infty}\sum_{n=1}^{m-1}d(m)d(n)\frac{(m-n)\cos\delta\sin\{2(m-n)\pi\cos\delta\}}{(m+n)^2\sin^2\delta+(m-n)^2\cos^2\delta}e^{-2(m+n)\pi\sin\delta}$$
$$= \Sigma_1+\Sigma_2+\Sigma_3.$$

Now

$$\Sigma_1 = \frac{1}{2\sin\delta}(1-e^{-4\pi\sin\delta})\sum_{n=1}^{\infty}e^{-4\pi n\sin\delta}\sum_{\nu=1}^{n}\frac{d^2(\nu)}{\nu}$$

$$\sim 2\pi\sum_{n=1}^{\infty}e^{-4\pi n\sin\delta}\frac{\log^4 n}{4\pi^2} \sim \frac{1}{2\pi}\int_0^{\infty}e^{-4\pi x\sin\delta}\log^4 x\,dx$$

$$= \frac{1}{8\pi^2\sin\delta}\int_0^{\infty}e^{-y}\log^4\left(\frac{y}{4\pi\sin\delta}\right)dy \sim \frac{1}{8\pi^2\delta}\log^4\frac{1}{\delta},$$

$$|\Sigma_2| \leqslant 2\sum_{m=2}^{\infty}\sum_{n=1}^{m-1} d(m)d(n)\frac{2m\sin\delta}{(m-n)^2\cos^2\delta}e^{-2\pi m\sin\delta}$$
$$= \frac{4\sin\delta}{\cos^2\delta}\sum_{m=2}^{\infty} m\,d(m)\,e^{-2\pi m\sin\delta}\sum_{r=1}^{m-1}\frac{d(m-r)}{r^2}$$
$$= \frac{4\sin\delta}{\cos^2\delta}\sum_{r=1}^{\infty}\frac{1}{r^2}\sum_{m=r+1}^{\infty} m\,d(m)d(m-r)e^{-2\pi m\sin\delta}.$$

The square of the inner sum does not exceed

$$\sum_{m=r+1}^{\infty} m^2d^2(m)e^{-2\pi m\sin\delta}\sum_{m=r+1}^{\infty} d^2(m-r)e^{-2\pi m\sin\delta}$$
$$\leqslant \sum_{m=1}^{\infty} m^2d^2(m)e^{-2\pi m\sin\delta}\sum_{m=1}^{\infty} d^2(m)e^{-2\pi m\sin\delta}$$
$$= O\left(\frac{1}{\delta^3}\log^3\frac{1}{\delta}\right)O\left(\frac{1}{\delta}\log^3\frac{1}{\delta}\right) = O\left(\frac{1}{\delta^4}\log^6\frac{1}{\delta}\right)$$

by (7.16.3) and (7.16.4). Hence

$$\Sigma_2 = O\left(\frac{1}{\delta}\log^3\frac{1}{\delta}\right).$$

Finally (as in the previous proof)

$$\Sigma_3 = O\left(\delta^2\sum_{m=2}^{\infty} m^{1+\epsilon}e^{-2m\pi\sin\delta}\right) = O(\delta^{-\epsilon}).$$

This proves the theorem.

It has been proved by Atkinson (2) that

$$\int_0^{\infty} |\zeta(\tfrac{1}{2}+it)|^4e^{-\delta t}\,dt$$
$$= \frac{1}{\delta}\left(A\log^4\frac{1}{\delta}+B\log^3\frac{1}{\delta}+C\log^2\frac{1}{\delta}+D\log\frac{1}{\delta}+E\right)+O\left\{\left(\frac{1}{\delta}\right)^{\frac{13}{14}+\epsilon}\right\},$$

where $\quad A = \dfrac{1}{2\pi^2}, \qquad B = -\dfrac{1}{\pi^2}\left(2\log 2\pi-6\gamma+\dfrac{24\zeta'(2)}{\pi^2}\right).$

A method is also indicated by which the index $\frac{13}{14}$ could be reduced to $\frac{8}{9}$.

7.17. The method of residues used in § 7.15 for $|\zeta(\frac{1}{2}+it)|^2$ suggests still another method of dealing with $|\zeta(\frac{1}{2}+it)|^4$. This is primarily a question of approximating to

$$\int_{2\pi}^{\infty}\left|\sum_{n=1}^{\infty} d(n)\exp(-inxe^{-i\delta})\right|^2 dx = \int_{2\pi}^{\infty}\left|\sum_{n=1}^{\infty}\frac{1}{\exp(inxe^{-i\delta})-1}\right|^2 dx$$
$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_{2\pi}^{\infty}\frac{dx}{\{\exp(imxe^{-i\delta})-1\}\{\exp(-inxe^{i\delta})-1\}}.$$

In the terms with $n \geqslant m$, put $x = \xi/m$. We get

$$\sum_{m=1}^{\infty}\frac{1}{m}\sum_{n=m}^{\infty}\int_{2\pi m}^{\infty}\frac{d\xi}{\{\exp(i\xi e^{-i\delta})-1\}\{\exp(-inm^{-1}\xi e^{i\delta})-1\}}.$$

Approximating to the integral by a sum obtained from the residues of the first factor, as in § 7.15, we obtain as an approximation to this

$$2\pi e^{i\delta}\sum_{m=1}^{\infty}\frac{1}{m}\sum_{n=m}^{\infty}\sum_{r=m}^{\infty}\frac{1}{\exp\{-2i(nr/m)\pi e^{2i\delta}\}-1}$$
$$= 2\pi e^{i\delta}\sum_{m=1}^{\infty}\frac{1}{m}\sum_{n=m}^{\infty}\sum_{r=m}^{\infty}\sum_{q=1}^{\infty}\exp\left(2i\frac{nqr}{m}\pi e^{2i\delta}\right)$$
$$= 2\pi e^{i\delta}\sum_{m=1}^{\infty}\frac{1}{m}\sum_{r=m}^{\infty}\sum_{q=1}^{\infty}\frac{\exp(2iqr\pi e^{2i\delta})}{1-\exp\{2i(qr/m)\pi e^{2i\delta}\}}$$
$$= O\left(\sum_{m=1}^{\infty}\frac{1}{m}\sum_{r=m}^{\infty}\sum_{q=1}^{\infty}\frac{e^{-2qr\pi\sin 2\delta}}{|1-\exp\{2i(qr/m)\pi e^{2i\delta}\}|}\right)$$
$$= O\left(\sum_{m=1}^{\infty}\frac{1}{m}\sum_{\nu=m}^{\infty}\frac{d(\nu)e^{-2\nu\pi\sin 2\delta}}{|1-\exp(2i\nu m^{-1}\pi e^{2i\delta})|}\right).$$

The terms with $m\,|\,\nu$ are

$$O\left(\sum_{m=1}^{\infty}\frac{1}{m}\sum_{\nu=m}^{\infty}\frac{d(\nu)e^{-2\nu\pi\sin 2\delta}}{\nu m^{-1}\delta}\right) = O\left(\frac{1}{\delta}\sum\sum_{m|\nu}\frac{d(\nu)}{\nu}e^{-2\nu\pi\sin 2\delta}\right)$$
$$= O\left(\frac{1}{\delta}\sum_{\nu=1}^{\infty}\frac{d^2(\nu)}{\nu}e^{-2\nu\pi\sin 2\delta}\right) = O\left(\frac{1}{\delta}\log^4\frac{1}{\delta}\right).$$

The remaining terms are

$$O\left(\sum_{m=1}^{\infty}\frac{1}{m}\sum_{k=1}^{\infty}\sum_{l=1}^{m-1}\frac{d(km+l)e^{-2(km+l)\pi\sin 2\delta}}{l/m}\right)$$
$$= O\left(\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}e^{-2km\pi\sin 2\delta}\sum_{l=1}^{m-1}\frac{d(km+l)}{l}\right)$$
$$= O\left(\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}e^{-2km\pi\sin 2\delta}\sum_{l=1}^{km}\frac{d(km+l)}{l}\right)$$
$$= O\left(\sum_{\nu=1}^{\infty}d(\nu)e^{-2\nu\pi\sin 2\delta}\sum_{l=1}^{\nu}\frac{d(\nu+l)}{l}\right)$$
$$= O\left(\sum_{l=1}^{\infty}\frac{1}{l}\sum_{\nu=l}^{\infty}d(\nu)d(\nu+l)e^{-2\nu\pi\sin 2\delta}\right)$$
$$= O\left(\sum_{l=1}^{\infty}\frac{e^{-l\pi\sin 2\delta}}{l}\sum_{\nu=l}^{\infty}d(\nu)d(\nu+l)e^{-(2\nu-l)\pi\sin 2\delta}\right).$$

Using Schwarz's inequality and (7.16.3) we obtain

$$O\left(\sum_{l=1}^{\infty}\frac{e^{-l\pi\sin 2\delta}}{l}\frac{1}{\delta}\log^3\frac{1}{\delta}\right)=O\left(\frac{1}{\delta}\log^4\frac{1}{\delta}\right).$$

Actually it follows from a theorem of Ingham (1) that this term is

$$O\left(\frac{1}{\delta}\log^3\frac{1}{\delta}\right).$$

7.18. There are formulae similar to those of § 7.16 for larger values of k, though in the higher cases they fail to give the desired mean-value formula.†

We have

$$\phi_k\left(\frac{1}{z}\right)=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\Gamma(s)\zeta^k(s)z^s\,ds$$

$$=\frac{1}{2\pi i}\int_{1-\alpha-i\infty}^{1-\alpha+i\infty}\Gamma(1-s)\zeta^k(1-s)z^{1-s}\,ds$$

$$=\frac{z}{2\pi i}\int_{1-\alpha-i\infty}^{1-\alpha+i\infty}\frac{\Gamma(1-s)}{\chi^k(s)}\zeta^k(s)z^{-s}\,ds.$$

Now

$$\Gamma(1-s)\chi^{-k}(s)=2^{k-ks}\pi^{-ks}\cos^k\tfrac{1}{2}s\pi\,\Gamma^k(s)\Gamma(1-s)$$

$$=2^{k-ks}\pi^{1-ks}\cos^k\tfrac{1}{2}s\pi\operatorname{cosec}\pi s\,\Gamma^{k-1}(s).$$

For large s $\quad \Gamma^{k-1}(s)\sim s^{(k-1)(s-\frac{1}{2})}e^{-(k-1)s}(2\pi)^{\frac{1}{2}(k-1)}.$

Now $\quad \Gamma\{(k-1)s-\tfrac{1}{2}k+1\}\sim\{(k-1)s\}^{(k-1)s-\frac{1}{2}k+\frac{1}{2}}e^{-(k-1)s}(2\pi)^{\frac{1}{2}}.$

Hence we may expect to be able to replace $\Gamma^{k-1}(s)$ by

$$(k-1)^{-(k-1)s+\frac{1}{2}k-\frac{1}{2}}(2\pi)^{\frac{1}{2}(k-2)}\Gamma\{(k-1)s-\tfrac{1}{2}k+1\}.$$

Also, in the upper half-plane,

$$\cos^k\tfrac{1}{2}s\pi\operatorname{cosec}s\pi\sim(\tfrac{1}{2}e^{-\frac{1}{2}is\pi})^k\frac{-2i}{e^{-is\pi}}=-2^{1-k}ie^{-is\pi(\frac{1}{2}k-1)}.$$

We should thus replace $\Gamma(1-s)\chi^{-k}(s)$ by

$$-i\,.\,2^{1-ks}\pi^{1-ks}e^{-is\pi(\frac{1}{2}k-1)}(k-1)^{-(k-1)s+\frac{1}{2}k-\frac{1}{2}}(2\pi)^{\frac{1}{2}(k-2)}\Gamma\{(k-1)s-\tfrac{1}{2}k+1\}.$$

Hence an approximation to $\phi_k(1/z)$ should be

$$\psi_k\left(\frac{1}{z}\right)=-i(2\pi)^{\frac{1}{2}k}\sum_{n=1}^{\infty}d_k(n)\frac{z}{2\pi i}\times$$

$$\times\int_{1-\alpha-i\infty}^{1-\alpha+i\infty}\Gamma\{(k-1)s-\tfrac{1}{2}k+1\}(k-1)^{-(k-1)s+\frac{1}{2}k-\frac{1}{2}}e^{-is\pi(\frac{1}{2}k-1)}(2^k\pi^knz)^{-s}\,ds.$$

† See also Bellman (3).

Putting $s=(w+\frac{1}{2}k-1)/(k-1)$, the integral is

$$-i(2\pi)^{\frac{1}{2}k}\frac{z}{2\pi i}\int\Gamma(w)(k-1)^{-w-\frac{1}{2}}e^{-i\pi(\frac{1}{2}k-1)(w+\frac{1}{2}k-1)/(k-1)}(2^k\pi^knz)^{-(w+\frac{1}{2}k-1)/(k-1)}\,dw$$

$$=-i(2\pi)^{\frac{1}{2}k}z(k-1)^{-\frac{1}{2}}e^{-i\pi(\frac{1}{2}k-1)^2/(k-1)}(2^k\pi^knz)^{-(\frac{1}{2}k-1)/(k-1)}\times$$

$$\times\exp\{-(k-1)e^{i\pi(\frac{1}{2}k-1)/(k-1)}2^{k/(k-1)}\pi^{k/(k-1)}(nz)^{1/(k-1)}\}.$$

Putting $z=ixe^{-i\delta}$, we obtain

$$(2\pi)^{k/(2k-2)}(k-1)^{-\frac{1}{2}}x^{k/\{2(k-1)\}}n^{-(\frac{1}{2}k-1)/(k-1)}\times$$

$$\times C_k\exp\{-(k-1)e^{\frac{1}{2}i\pi}2^{k/(k-1)}\pi^{k/(k-1)}(nx)^{1/(k-1)}e^{-i\delta/(k-1)}\},$$

where $|C_k|=1$.

We have, by (7.13.5),

$$\int_0^{\infty}|\zeta(\tfrac{1}{2}+it)|^{2k}e^{-2\delta t}\,dt=\int_0^{\infty}|\phi_k(ixe^{-i\delta})|^2\,dx+O(1)$$

$$=\int_0^{\lambda}|\phi_k(ixe^{-i\delta})|^2\,dx+\int_{\lambda}^{\infty}|\phi_k(ixe^{-i\delta})|^2\,dx+O(1).$$

As in the above cases, the integral over (λ,∞) is

$$\sum_{n=1}^{\infty}d_k^2(n)\frac{e^{-2\lambda n\sin\delta}}{2n\sin\delta}+$$

$$+2\sum_{m=2}^{\infty}\sum_{n=1}^{m-1}d_k(m)d_k(n)\frac{(m+n)\sin\delta\cos\{\lambda(m-n)\cos\delta\}}{(m+n)^2\sin^2\delta+(m-n)^2\cos^2\delta}e^{-\lambda(m+n)\sin\delta}-$$

$$-2\sum_{m=2}^{\infty}\sum_{n=1}^{m-1}d_k(m)d_k(n)\frac{(m-n)\cos\delta\sin\{\lambda(m-n)\cos\delta\}}{(m+n)^2\sin^2\delta+(m-n)^2\cos^2\delta}e^{-\lambda(m+n)\sin\delta}$$

$$=\Sigma_1+\Sigma_2+\Sigma_3.\qquad(7.18.1)$$

Using the relation $d_k(n)=O(n^\epsilon)$, we obtain

$$\Sigma_1=O\left(\frac{1}{\delta}\frac{1}{(\lambda\delta)^\epsilon}\right),$$

and, since $(m+n)^2\sin^2\delta+(m-n)^2\cos^2\delta>A\delta(m+n)(m-n)$,

$$\Sigma_2=O\left(\sum_{m=2}^{\infty}m^\epsilon e^{-\lambda m\sin\delta}\sum_{n=1}^{m-1}\frac{1}{m-n}\right)=O\left(\sum_{m=2}^{\infty}m^\epsilon e^{-\lambda m\sin\delta}\right)=O\left(\frac{1}{(\lambda\delta)^{1+\epsilon}}\right),$$

$$\Sigma_3=O\left(\sum_{m=2}^{\infty}m^\epsilon e^{-\lambda m\sin\delta}\sum_{n=1}^{m-1}\frac{1}{m-n}\right)=O\left(\frac{1}{(\lambda\delta)^{1+\epsilon}}\right).$$

Hence, for $\lambda<A$,

$$\int_{\lambda}^{\infty}|\phi_k(ixe^{-i\delta})|^2\,dx=O\left(\frac{1}{(\lambda\delta)^{1+\epsilon}}\right).$$