so

$$(\widehat{f})^\vee = c^2 M^L M^L f\,. \tag{95}$$

Thus if X is not an even-dimensional projective space f is a constant multiple of $M^L P(L) M^L f$ which by (94) shows $f \to \widehat{f}$ surjective. For the remaining case $\mathbf{P}^n(\mathbf{R})$, n even, we use the expansion of $f \in \mathcal{E}(\mathbf{P}^n(\mathbf{R}))$ in spherical harmonics

$$f = \sum_{k,m} a_{km} S_{km} \quad (k \text{ even})\,.$$

Here $k \in \mathbb{Z}^+$, and $S_{km} (1 \leq m \leq d(k))$ is an orthonormal basis of the space of spherical harmonics of degree k. Here the coefficients a_{km} are rapidly decreasing in k. On the other hand, by (32) and (34),

$$\widehat{f} = \Omega_n M^{\frac{\pi}{2}} f = \Omega_n \sum_{k,m} a_{km} \varphi_k \left(\tfrac{\pi}{2}\right) S_{km} \quad (k \text{ even})\,. \tag{96}$$

The spherical function φ_k is given by

$$\varphi_k(s) = \frac{\Omega_{n-1}}{\Omega_n} \int_0^\pi (\cos\theta + i \sin\theta \cos\varphi)^k \sin^{n-2}\varphi \, d\varphi$$

so $\varphi_{2k}(\frac{\pi}{2}) \sim k^{-\frac{n-1}{2}}$. Thus every series $\sum_{k,m} b_{k,m} S_{2k,m}$ with $b_{2k,m}$ rapidly decreasing in k can be put in the form (96). This verifies the surjectivity of the map $f \to \widehat{f}$.

It remains to prove $(Lf)\widehat{\ } = \Lambda \widehat{f}$. For this we use (87), (vii), (41) and (94). By the definition of Λ we have

$$(\Lambda\varphi)(j(x)) = L(\varphi \circ j)(x)\,, \qquad x \in X, \varphi \in \mathcal{E}(X)\,.$$

Thus

$$(\Lambda\widehat{f})(j(x)) = (L(\widehat{f} \circ j))(x) = cL(M^L f)(x) = cM^L(Lf)(x) = (Lf)\widehat{\ }(j(x))\,.$$

This finishes our indication of the proof of Theorem 2.2.

Corollary 2.3. *Let X be a compact two-point homogeneous space and suppose f satisfies*

$$\int_\gamma f(x)\, ds(x) = 0$$

for each (closed) geodesic γ in X, ds being the element of arc-length. Then

(i) If X is a sphere, f is skew.

(ii) If X is not a sphere, $f \equiv 0$.

Taking a convolution with f we may assume f smooth. Part (i) is already contained in Theorem 1.7. For Part (ii) we use the classification; for $X = \mathbf{P}^{16}(\mathbf{Cay})$ the antipodal manifolds are totally geodesic spheres so using Part (i) we conclude that $\widehat{f} \equiv 0$ so by Theorem 2.2, $f \equiv 0$. For the remaining cases $\mathbf{P}^n(\mathbf{C})$ $(n = 4, 6, \ldots)$ and $\mathbf{P}^n(\mathbf{H}), (n = 8, 12, \ldots)$ (ii) follows similarly by induction as the initial antipodal manifolds, $\mathbf{P}^2(\mathbf{C})$ and $\mathbf{P}^4(\mathbf{H})$, are totally geodesic spheres.

Corollary 2.4. *Let B be a bounded open set in $\mathbf{R}^{n+1}$, symmetric and star-shaped with respect to 0, bounded by a hypersurface. Assume for a fixed k $(1 \leq k < n)$*

$$\textit{Area } (B \cap P) = \textit{ constant} \tag{97}$$

for all $(k+1)$-planes P through 0. Then B is an open ball.

In fact, we know from Theorem 1.7 that if f is a symmetric function on $X = \mathbf{S}^n$ with $\widehat{f}(\mathbf{S}^n \cap P)$ constant (for all P) then f is a constant. We apply this to the function

$$f(\theta) = \rho(\theta)^{k+1} \quad \theta \in \mathbf{S}^n$$

if $\rho(\theta)$ is the distance from the origin to each of the two points of intersection of the boundary of B with the line through 0 and θ; f is well defined since B is symmetric. If $\theta = (\theta_1, \ldots, \theta_k)$ runs through the k-sphere $\mathbf{S}^n \cap P$ then the point

$$x = \theta r \quad (0 \leq r < \rho(\theta))$$

runs through the set $B \cap P$ and

$$\text{Area } (B \cap P) = \int_{\mathbf{S}^n \cap P} d\omega(\theta) \int_0^{\rho(\theta)} r^k \, dr\,.$$

It follows that Area $(B \cap P)$ is a constant multiple of $\widehat{f}(\mathbf{S}^n \cap P)$ so (97) implies that f is constant. This proves the corollary.

§3 Noncompact Two-point Homogeneous Spaces

Theorem 2.2 has an analog for noncompact two-point homogeneous spaces which we shall now describe. By Tits' classification [1955], p. 183, of homogeneous manifolds L/H for which L acts transitively on the tangents to L/H it is known, in principle, what the noncompact two-point homogeneous spaces are. As in the compact case they turn out to be symmetric. A direct proof of this fact was given by Nagano [1959] and Helgason [1959]. The theory of symmetric spaces then implies that the noncompact two-point homogeneous spaces are the Euclidean spaces and the noncompact spaces $X = G/K$ where G is a connected semisimple Lie group with finite center and real rank one and K a maximal compact subgroup.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the direct decomposition of the Lie algebra of G into the Lie algebra $\mathfrak{k}$ of K and its orthogonal complement $\mathfrak{p}$ (with respect to the Killing form of $\mathfrak{g}$). Fix a 1-dimensional subspace $\mathfrak{a} \subset \mathfrak{p}$ and let

$$\mathfrak{p} = \mathfrak{a} + \mathfrak{p}_\alpha + \mathfrak{p}_{\alpha/2} \tag{98}$$

be the decomposition of $\mathfrak{p}$ into eigenspaces of T_H (in analogy with (86)). Let ξ_o denote the totally geodesic submanifold $\mathrm{Exp}(\mathfrak{p}_{\alpha/2})$; in the case $\mathfrak{p}_{\alpha/2} = 0$ we put $\xi_o = \mathrm{Exp}(\mathfrak{p}_\alpha)$. By the classification and duality for symmetric spaces we have the following complete list of the spaces G/K. In the list the superscript denotes the real dimension; for the lowest dimensions note that

$$\mathbf{H}^1(\mathbf{R}) = \mathbf{R}\,, \quad \mathbf{H}^2(\mathbf{C}) = \mathbf{H}^2(\mathbf{R})\,, \quad \mathbf{H}^4(\mathbf{H}) = \mathbf{H}^4(\mathbf{R})\,.$$

X		ξ_0
Real hyperbolic spaces	$\mathbf{H}^n(\mathbf{R}) (n = 2, 3, \ldots)$,	$\mathbf{H}^{n-1}(\mathbf{R})$
Complex hyperbolic spaces	$\mathbf{H}^n(\mathbf{C}) (n = 4, 6, \ldots)$,	$\mathbf{H}^{n-2}(\mathbf{C})$
Quaternian hyperbolic spaces	$\mathbf{H}^n(\mathbf{H}) (n = 8, 12, \ldots)$,	$\mathbf{H}^{n-4}(\mathbf{H})$
Cayley hyperbolic spaces	$\mathbf{H}^{16}(\mathbf{Cay})$,	$\mathbf{H}^8(\mathbf{R})$.

Let Ξ denote the set of submanifolds $g \cdot \xi_0$ of X as g runs through G; Ξ is given the canonical differentiable structure of a homogeneous space. Each $\xi \in \Xi$ has a measure m induced by the Riemannian structure of X and the Radon transform on X is defined by

$$\widehat{f}(\xi) = \int_\xi f(x)\, dm(x)\,, \quad f \in C_c(X)\,.$$

The dual transform $\varphi \to \check{\varphi}$ is defined by

$$\check{\varphi}(x) = \int_{\xi \ni x} \varphi(\xi)\, d\mu(\xi)\,, \quad \varphi \in C(\Xi)\,,$$

where μ is the invariant average on the set of ξ passing through x. Let L denote the Laplace-Beltrami operator on X, Riemannian structure being that given by the Killing form of $\mathfrak{g}$.

Theorem 3.1. *The Radon transform $f \to \widehat{f}$ is a one-to-one mapping of $\mathcal{D}(X)$ into $\mathcal{D}(\Xi)$ and, except for the case $X = \mathbf{H}^n(\mathbf{R})$, n even, is inverted by the formula $f = Q(L)((\widehat{f})^\vee)$. Here Q is given by*

$X = \mathbf{H}^n(\mathbf{R}), n$ *odd:*

$$Q(L) = \gamma \left(L + \tfrac{(n-2)1}{2n}\right) \left(L + \tfrac{(n-4)3}{2n}\right) \cdots \left(L + \tfrac{1(n-2)}{2n}\right).$$

$X = \mathbf{H}^n(\mathbf{C})$:

$$Q(L) = \gamma \left(L + \tfrac{(n-2)2}{2(n+2)}\right) \left(L + \tfrac{(n-4)4}{2(n+2)}\right) \cdots \left(L + \tfrac{2(n-2)}{2(n+2)}\right).$$

$X = \mathbf{H}^n(\mathbf{H})$:

$$Q(L) = \gamma \left(L + \tfrac{(n-2)4}{2(n+8)}\right)\left(L + \tfrac{(n-4)6}{2(n+8)}\right)\cdots\left(L + \tfrac{4(n-2)}{2(n+8)}\right).$$

$X = \mathbf{H}^{16}(\boldsymbol{Cay})$:

$$Q(L) = \gamma \left(L + \tfrac{14}{9}\right)^2 \left(L + \tfrac{15}{9}\right)^2.$$

The constants γ are obtained from the constants c in (90)–(93) by multiplication by the factor Ω_X which is the volume of the antipodal manifold in the compact space corresponding to X. This factor is explicitly determined for each X in [GGA], Chapter I, §4.

§4 The X-ray Transform on a Symmetric Space

Let X be a complete Riemannian manifold of dimension > 1 in which any two points can be joined by a unique geodesic. The *X-ray transform* on X assigns to each continuous function f on X the integrals

$$\widehat{f}(\gamma) = \int_\gamma f(x)\, ds(x)\,, \tag{99}$$

γ being any complete geodesic in X and ds the element of arc-length. In analogy with the X-ray reconstruction problem on $\mathbf{R}^n$ (Ch.I, §7) one can consider the problem of inverting the X-ray transform $f \to \widehat{f}$. With d denoting the distance in X and $o \in X$ some fixed point we now define two subspaces of $C(X)$. Let

$$\begin{aligned} F(X) &= \{f \in C(X) : \sup_x d(o,x)^k |f(x)| < \infty \text{ for each } k \geq 0\} \\ \mathfrak{F}(X) &= \{f \in C(X) : \sup_x e^{kd(o,x)} |f(x)| < \infty \text{ for each } k \geq 0\}\,. \end{aligned}$$

Because of the triangle inequality these spaces do not depend on the choice of o. We can informally refer to $F(X)$ as the space of continuous *rapidly decreasing functions* and to $\mathfrak{F}(X)$ as the space of continuous *exponentially decreasing functions*. We shall now prove the analog of the support theorem (Theorem 2.6, Ch. I, Theorem 1.2, Ch. III) for the X-ray transform on a symmetric space of the noncompact type. This general analog turns out to be a direct corollary of the Euclidean case and the hyperbolic case, already done.

Corollary 4.1. *Let X be a symmetric space of the noncompact type, B any ball in M.*

(i) If a function $f \in \mathfrak{F}(X)$ satisfies

$$\widehat{f}(\xi) = 0 \quad \textit{whenever } \xi \cap B = \emptyset\,, \quad \xi \textit{ a geodesic,} \tag{100}$$

then

$$f(x) = 0 \quad \text{for } x \notin B\,. \tag{101}$$

In particular, the X-ray transform is one-to-one on $\mathfrak{F}(X)$.

(ii) If X *has rank greater than one statement (i) holds with* $\mathfrak{F}(X)$ *replaced by* $F(X)$.

Proof. Let o be the center of B, r its radius, and let γ be an arbitrary geodesic in X through o.

Assume first X has rank greater than one. By a standard conjugacy theorem for symmetric spaces γ lies in a 2-dimensional, flat, totally geodesic submanifold of X. Using Theorem 2.6, Ch. I on this Euclidean plane we deduce $f(x) = 0$ if $x \in \gamma, d(o,x) > r$. Since γ is arbitrary (101) follows.

Next suppose X has rank one. Identifying $\mathfrak{p}$ with the tangent space X_o let $\mathfrak{a}$ be the tangent line to γ. We can then consider the eigenspace decomposition (98). If $\mathfrak{b} \subset \mathfrak{p}_\alpha$ is a line through the origin then $S = \mathrm{Exp}(\mathfrak{a} + \mathfrak{b})$ is a totally geodesic submanifold of X (cf. (iv) in the beginning of §2). Being 2-dimensional and not flat, S is necessarily a hyperbolic space. From Theorem 1.2 we therefore conclude $f(x) = 0$ for $x \in \gamma$, $d(o,x) > r$. Again (101) follows since γ is arbitrary.

§5 Maximal Tori and Minimal Spheres in Compact Symmetric Spaces

Let $\mathfrak{u}$ be a compact semisimple Lie algebra, θ an involutive automorphism of $\mathfrak{u}$ with fixed point algebra $\mathfrak{k}$. Let U be the simply connected Lie group with Lie algebra $\mathfrak{u}$ and $\mathrm{Int}(\mathfrak{u})$ the adjoint group of $\mathfrak{u}$. Then θ extends to an involutive automorphism of U and $\mathrm{Int}(\mathfrak{u})$. We denote these extensions also by θ and let K and K_θ denote the respective fixed point groups under θ. The symmetric space $X_\theta = \mathrm{Int}(\mathfrak{u})/K_\theta$ is called the *adjoint space* of $(\mathfrak{u}, \theta)$ (Helgason [1978], p. 327), and is covered by $X = U/K$, this latter space being simply connected since K is automatically connected.

The flat totally geodesic submanifolds of X_θ of maximal dimension are permuted transitively by $\mathrm{Int}(\mathfrak{u})$ according to a classical theorem of Cartan. Let E_θ be one such manifold passing through the origin eK_θ in X_θ and let H_θ be the subgroup of $\mathrm{Int}(\mathfrak{u})$ preserving E_θ. We then have the pairs of homogeneous spaces

$$X_\theta = \mathrm{Int}(\mathfrak{u})/K_\theta\,, \quad \Xi_\theta = \mathrm{Int}(\mathfrak{u})/H_\theta\,. \tag{102}$$

The corresponding Radon transform $f \to \widehat{f}$ from $C(X_\theta)$ to $C(\Xi_\theta)$ amounts to

$$\widehat{f}(E) = \int_E f(x)\, dm(x)\,, \quad E \in \Xi_\theta\,, \tag{103}$$

E being any flat totally geodesic submanifold of X_θ of maximal dimension and dm the volume element. If X_θ has rank one, E is a geodesic and we are in the situation of Corollary 2.3. The transform (103) is often called the *flat Radon transform.*

Theorem 5.1. *Assume X_θ is irreducible. Then the flat Radon transform is injective.*

For a proof see Grinberg [1992].

The sectional curvatures of the space X lie in an interval $[0, \kappa]$. The space X contains totally geodesic spheres of curvature κ and all such spheres S of maximal dimension are conjugate under U (Helgason [1966b]). Fix one such sphere S_0 through the origin eK and let H be the subgroup of U preserving S_0. Then we have another double fibration

$$X = U/K\,, \quad \Xi = U/H$$

and the accompanying Radon transform

$$\widehat{f}(S) = \int_S f(x)\, d\sigma(x)\,.$$

$S \in \Xi$ being arbitrary and $d\sigma$ being the volume element on S.

It is proved by Grinberg [1994] that injectivity holds in many cases although the general question is not fully settled.

Bibliographical Notes

As mentioned earlier, it was shown by Funk [1916] that a function f on the two-sphere, symmetric with respect to the center, can be determined by the integrals of f over the great circles. When f is rotation-invariant (relative to a vertical axis) he gave an explicit inversion formula, essentially (78) in Proposition 1.16.

The Radon transform on hyperbolic and on elliptic spaces corresponding to k-dimensional totally geodesic submanifolds was defined in the author's paper [1959]. Here and in [1990] are proved the inversion formulas in Theorems 1.5, 1.7, 1.10 and 1.11. See also Semyanisty [1961] and Rubin [1998b]. The alternative version in (60) was obtained by Berenstein and Casadio Tarabusi [1991] which also deals with the case of $\mathbf{H}^k$ in $\mathbf{H}^n$ (where the regularization is more complex). Still another interesting variation of Theorem 1.10 (for $k = 1, n = 2$) is given by Lissianoi and Ponomarev [1997]. By calculating the dual transform $\check{\varphi}_p(z)$ they derive from (30) in Chapter II an inversion formula which has a formal analogy to (38) in Chapter II. The underlying reason may be that to each geodesic γ in $\mathbf{H}^2$ one can associate a pair of horocycles tangential to $|z| = 1$ at the endpoints of γ having the same distance from o as γ.

The support theorem (Theorem 1.2) was proved by the author ([1964], [1980b]) and its consequence, Cor. 4.1, pointed out in [1980d]. Interesting generalizations are contained in Boman [1991], Boman and Quinto [1987], [1993]. For the case of $\mathbf{S}^{n-1}$ see Quinto [1983] and in the stronger form of Theorem 1.17, Kurusa [1994]. The variation (82) of the Funk transform has also been considered by Abouelaz and Daher [1993] at least for K-invariant functions. The theory of the Radon transform for antipodal manifolds in compact two-point homogeneous spaces (Theorem 2.2) is from Helgason [1965a]. R. Michel has in [1972] and [1973] used Theorem 2.2 in establishing certain infinitesimal rigidity properties of the canonical metrics on the real and complex projective spaces. See also Guillemin [1976] and A. Besse [1978], Goldschmidt [1990], Estezet [1988].

CHAPTER IV

ORBITAL INTEGRALS AND THE WAVE OPERATOR FOR ISOTROPIC LORENTZ SPACES

In Chapter II, §3 we discussed the problem of determining a function on a homogeneous space by means of its integrals over generalized spheres. We shall now solve this problem for the *isotropic Lorentz spaces* (Theorem 4.1 below). As we shall presently explain these spaces are the Lorentzian analogs of the two-point homogeneous spaces considered in Chapter III.

§1 Isotropic Spaces

Let X be a manifold. A *pseudo-Riemannian structure* of signature (p, q) is a smooth assignment $y \to g_y$ where $y \in X$ and g_y is a symmetric nondegenerate bilinear form on $X_y \times X_y$ of signature (p, q). This means that for a suitable basis $Y_1, \ldots, Y_{p+q}$ of X_y we have

$$g_y(Y) = y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_{p+q}^2$$

if $Y = \sum_1^{p+q} y_i Y_i$. If $q = 0$ we speak of a *Riemannian* structure and if $p = 1$ we speak of a *Lorentzian* structure. Connected manifolds X with such structures g are called pseudo-Riemannian (respectively Riemannian, Lorentzian) manifolds.

A manifold X with a pseudo-Riemannian structure g has a differential operator of particular interest, the so-called Laplace-Beltrami operator. Let $(x_1, \ldots, x_{p+q})$ be a coordinate system on an open subset U of X. We define the functions g_{ij}, g^{ij}, and $\overline{g}$ on U by

$$g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right), \quad \sum_j g_{ij} g^{jk} = \delta_{ik}, \quad \overline{g} = |\det(g_{ij})|.$$

The *Laplace-Beltrami operator* L is defined on U by

$$Lf = \frac{1}{\sqrt{\overline{g}}}\left(\sum_k \frac{\partial}{\partial x_k}\left(\sum_i g^{ik}\sqrt{\overline{g}}\frac{\partial f}{\partial x_i}\right)\right)$$

for $f \in \mathcal{C}^\infty(U)$. It is well known that this expression is invariant under coordinate changes so L is a differential operator on X.

An *isometry* of a pseudo-Riemannian manifold X is a diffeomorphism preserving g. It is easy to prove that L is *invariant* under each isometry φ, that is $L(f \circ \varphi) = (Lf) \circ \varphi$ for each $f \in \mathcal{E}(X)$. Let $I(X)$ denote the group of all isometries of X. For $y \in X$ let $I(X)_y$ denote the subgroup of $I(X)$

fixing y (the isotropy subgroup at y) and let H_y denote the group of linear transformations of the tangent space X_y induced by the action of $I(X)_y$. For each $a \in \mathbf{R}$ let $\sum_a(y)$ denote the "sphere"

$$\Sigma_a(y) = \{Z \in X_y : g_y(Z, Z) = a\,, \quad Z \neq 0\}\,. \tag{1}$$

Definition. The pseudo-Riemannian manifold X is called *isotropic* if for each $a \in \mathbf{R}$ and each $y \in X$ the group H_y acts transitively on $\sum_a(y)$.

Proposition 1.1. *An isotropic pseudo-Riemannian manifold X is homogeneous; that is, $I(X)$ acts transitively on X.*

Proof. The pseudo-Riemannian structure on X gives an affine connection preserved by each isometry $g \in I(X)$. Any two points $y, z \in X$ can be joined by a curve consisting of finitely many geodesic segments $\gamma_i (1 \leq i \leq p)$. Let g_i be an isometry fixing the midpoint of γ_i and reversing the tangents to γ_i at this point. The product $g_p \cdots g_1$ maps y to z, whence the homogeneity of X.

A. The Riemannian Case

The following simple result shows that the isotropic spaces are natural generalizations of the spaces considered in the last chapter.

Proposition 1.2. *A Riemannian manifold X is isotropic if and only if it is two-point homogeneous.*

Proof. If X is two-point homogeneous and $y \in X$ the isotropy subgroup $I(X)_y$ at y is transitive on each sphere $S_r(y)$ in X with center y so X is clearly isotropic. On the other hand if X is isotropic it is homogeneous (Prop. 1.1) hence complete; thus by standard Riemannian geometry any two points in X can be joined by means of a geodesic. Now the isotropy of X implies that for each $y \in X, r > 0$, the group $I(X)_y$ is transitive on the sphere $S_r(y)$, whence the two-point homogeneity.

B. The General Pseudo-Riemannian Case

Let X be a manifold with pseudo-Riemannian structure g and curvature tensor R. Let $y \in X$ and $S \subset X_y$ a 2-dimensional subspace on which g_y is nondegenerate. The curvature of X along the section S spanned by Z and Y is defined by

$$K(S) = -\frac{g_p(R_p(Z,Y)Z,Y)}{g_p(Z,Z)g_p(Y,Y) - g_p(Z,Y)^2}$$

The denominator is in fact $\neq 0$ and the expression is independent of the choice of Z and Y.

We shall now construct isotropic pseudo-Riemannian manifolds of signature (p,q) and constant curvature. Consider the space $\mathbf{R}^{p+q+1}$ with the flat pseudo-Riemannian structure

$$B_e(Y) = y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_{p+q}^2 + e\, y_{p+q+1}^2\,, \quad (e = \pm 1)\,.$$

Let Q_e denote the quadric in $\mathbf{R}^{p+q+1}$ given by

$$B_e(Y) = e\,.$$

The orthogonal group $\mathbf{O}(B_e)$ $(= \mathbf{O}(p,q+1)$ or $\mathbf{O}(p+1,q))$ acts transitively on Q_e; the isotropy subgroup at $o = (0,\ldots,0,1)$ is identified with $\mathbf{O}(p,q)$.

Theorem 1.3. *(i) The restriction of B_e to the tangent spaces to Q_e gives a pseudo-Riemannian structure g_e on Q_e of signature (p,q).*

(ii) We have

(2) $$Q_{-1} \cong \mathbf{O}(p,q+1)/\mathbf{O}(p,q) \qquad \textit{(diffeomorphism)}$$

and the pseudo-Riemannian structure g_{-1} on Q_{-1} has constant curvature -1.

(iii) We have

(3) $$Q_{+1} = \mathbf{O}(p+1,q)/\mathbf{O}(p,q) \qquad \textit{(diffeomorphism)}$$

and the pseudo-Riemannian structure g_{+1} on Q_{+1} has constant curvature $+1$.

(iv) The flat space $\mathbf{R}^{p+q}$ with the quadratic form $g_o(Y) = \sum_1^p y_i^2 - \sum_{p+1}^{p+q} y_j^2$ and the spaces

$$\mathbf{O}(p,q+1)/\mathbf{O}(p,q)\,, \quad \mathbf{O}(p+1,q)/\mathbf{O}(p,q)$$

are all isotropic and (up to a constant factor on the pseudo-Riemannian structure) exhaust the class of pseudo-Riemannian manifolds of constant curvature and signature (p,q) except for local isometry.

Proof. If s_o denotes the linear transformation

$$(y_1,\ldots,y_{p+q},y_{p+q+1}) \to (-y_1,\ldots,-y_{p+q},y_{p+q+1})$$

then the mapping $\sigma : g \to s_o g s_o$ is an involutive automorphism of $\mathbf{O}(p,q+1)$ whose differential $d\sigma$ has fixed point set $\mathfrak{o}(p,q)$ (the Lie algebra of $\mathbf{O}(p,q)$). The (-1)-eigenspace of $d\sigma$, say $\mathfrak{m}$, is spanned by the vectors

(4) $$Y_i = E_{i,p+q+1} + E_{p+q+1,i} \qquad (1 \le i \le p)\,,$$

(5) $$Y_j = E_{j,p+q+1} - E_{p+q+1,j} \qquad (p+1 \le j \le p+q)\,.$$

Here E_{ij} denotes a square matrix with entry 1 where the i^{th} row and the j^{th} column meet, all other entries being 0.

The mapping $\psi : g\mathbf{O}(p,q) \to g \cdot o$ has a differential $d\psi$ which maps $\mathfrak{m}$ bijectively onto the tangent plane $y_{p+q+1} = 1$ to Q_{-1} at o and $d\psi(X) = X{\cdot}o$ $(X \in \mathfrak{m})$. Thus

$$d\psi(Y_k) = (\delta_{1k}, \dots, \delta_{p+q+1,k}), \quad (1 \leq k \leq p+q).$$

Thus

$$B_{-1}(d\psi(Y_k)) = 1 \quad \text{if} \quad 1 \leq k \leq p \text{ and } -1 \text{ if } p+1 \leq k \leq p+q,$$

proving (i). Next, since the space (2) is symmetric its curvature tensor satisfies

$$R_o(X,Y)(Z) = [[X,Y],Z],$$

where [,] is the Lie bracket. A simple computation then shows for $k \neq \ell$

$$K(\mathbf{R}Y_k + \mathbf{R}Y_\ell) = -1 \quad (1 \leq k, \ell \leq p+q)$$

and this implies (ii). Part (iii) is proved in the same way. For (iv) we first verify that the spaces listed are isotropic. Since the isotropy action of $\mathbf{O}(p,q+1)_o = \mathbf{O}(p,q)$ on $\mathfrak{m}$ is the ordinary action of $\mathbf{O}(p,q)$ on $\mathbf{R}^{p+q}$ it suffices to verify that $\mathbf{R}^{p+q}$ with the quadratic form g_o is isotropic. But we know $\mathbf{O}(p,q)$ is transitive on $g_e = +1$ and on $g_e = -1$ so it remains to show $\mathbf{O}(p,q)$ transitive on the cone $\{Y \neq 0 : g_e(Y) = 0\}$. By rotation in $\mathbf{R}^p$ and in $\mathbf{R}^q$ it suffices to verify the statement for $p = q = 1$. But for this case it is obvious. The uniqueness in (iv) follows from the general fact that a symmetric space is determined locally by its pseudo-Riemannian structure and curvature tensor at a point (see e.g. [DS], pp. 200–201). This finishes the proof.

The spaces (2) and (3) are the pseudo-Riemannian analogs of the spaces $\mathbf{O}(p,1)/\mathbf{O}(p)$, $\mathbf{O}(p+1)/\mathbf{O}(p)$ from Ch. III, §1. But the other two-point homogeneous spaces listed in Ch. III, §2–§3 have similar pseudo-Riemannian analogs (indefinite elliptic and hyperbolic spaces over **C**, **H** and **Cay**). As proved by Wolf [1967], p. 384, each non-flat isotropic pseudo-Riemannian manifold is locally isometric to one of these models.

We shall later need a lemma about the connectivity of the groups $\mathbf{O}(p,q)$. Let $I_{p,q}$ denote the diagonal matrix (d_{ij}) with

$$d_{ii} = 1 \quad (1 \leq i \leq p), \; d_{jj} = -1 \quad (p+1 \leq j \leq p+q)$$

so a matrix g with transpose ${}^t g$ belongs to $\mathbf{O}(p,q)$ if and only if

$$(6) \qquad {}^t g I_{p,q} g = I_{p,q}.$$

If $y \in \mathbf{R}^{p+q}$ let

$$y^T = (y_1, \dots, y_p, 0 \dots 0), y^S = (0, \dots, 0, y_{p+1}, \dots, y_{p+q})$$

and for $g \in \mathbf{O}(p,q)$ let g_T and g_S denote the matrices

$$\begin{array}{ll} (g_T)_{ij} = g_{ij} & (1 \le i,j \le p)\,, \\ (g_S)_{k\ell} = g_{k\ell} & (p+1 \le k,\ell \le p+q) \end{array} .$$

If $g_1, \dots, g_{p+q}$ denote the column vectors of the matrix g then (6) means for the scalar products

$$\begin{array}{ll} g_i^T \cdot g_i^T - g_i^S \cdot g_i^S = 1\,, & 1 \le i \le p\,, \\ g_j^T \cdot g_j^T - g_j^S \cdot g_j^S = -1\,, & p+1 \le j \le p+q\,, \\ g_j^T \cdot g_k^T = g_j^S \cdot g_k^S\,, & j \ne k\,. \end{array}$$

Lemma 1.4. *We have for each $g \in \mathbf{O}(p,q)$*

$$|\det(g_T)| \ge 1\,, \quad |\det(g_S)| \ge 1\,.$$

The components of $\mathbf{O}(p,q)$ are obtained by

(7)	$\det g_T \ge 1$,	$\det g_S \ge 1$;	*(identity component)*
(8)	$\det g_T \le -1$,	$\det g_S \ge 1$;	
(9)	$\det g_T \ge -1$,	$\det g_S \le -1$,	
(10)	$\det g_T \le -1$,	$\det g_S \le -1$.	

Thus $\mathbf{O}(p,q)$ has four components if $p \ge 1, q \ge 1$, two components if p or $q = 0$.

Proof. Consider the Gram determinant

$$\det \begin{pmatrix} g_1^T \cdot g_1^T & g_1^T \cdot g_2^T & \cdots & g_1^T \cdot g_p^T \\ g_2^T \cdot g_1^T & \cdot & & \\ \vdots & & & \\ g_p^T \cdot g_1^T & \cdots & & g_p^T \cdot g_p^T \end{pmatrix},$$

which equals $(\det g_T)^2$. Using the relations above it can also be written

$$\det \begin{pmatrix} 1 + g_1^S \cdot g_1^S & g_1^S \cdot g_2^S & \cdots & g_1^S \cdot g_p^S \\ g_2^S \cdot g_1^S & \cdot & \cdots & \\ \vdots & & & \\ g_p^S \cdot g_1^S & & & 1 + g_p^S \cdot g_p^S \end{pmatrix},$$

which equals 1 plus a sum of lower order Gram determinants each of which is still positive. Thus $(\det g_T)^2 \ge 1$ and similarly $(\det g_S)^2 \ge 1$. Assuming now $p \ge 1, q \ge 1$ consider the decomposition of $\mathbf{O}(p,q)$ into the four pieces (7), (8), (9), (10). Each of these is $\ne \emptyset$ because (8) is obtained from (7) by multiplication by $I_{1,p+q-1}$ etc. On the other hand, since the functions $g \to$

$\det(g_T)$, $g \to \det(g_S)$ are continuous on $\mathbf{O}(p,q)$ the four pieces above belong to different components of $\mathbf{O}(p,q)$. But by Chevalley [1946], p. 201, $\mathbf{O}(p,q)$ is homeomorphic to the product of $\mathbf{O}(p,q) \cap \mathbf{U}(p+q)$ with a Euclidean space. Since $\mathbf{O}(p,q) \cap \mathbf{U}(p+q) = \mathbf{O}(p,q) \cap \mathbf{O}(p+q)$ is homeomorphic to $\mathbf{O}(p) \times \mathbf{O}(q)$ it just remains to remark that $\mathbf{O}(n)$ has two components.

C. The Lorentzian Case

The isotropic Lorentzian manifolds are more restricted than one might at first think on the basis of the Riemannian case. In fact there is a theorem of Lichnerowicz and Walker [1945] (see Wolf [1967], Ch. 12) which implies that an isotropic Lorentzian manifold has constant curvature. Thus we can deduce the following result from Theorem 1.3.

Theorem 1.5. *Let X be an isotropic Lorentzian manifold (signature $(1,q)$, $q \geq 1$). Then X has constant curvature so (after a multiplication of the Lorentzian structure by a positive constant) X is locally isometric to one of the following:*

$$\begin{aligned}
&\mathbf{R}^{1+q}(\textit{flat, signature } (1,q))\,,\\
&Q_{-1} = \mathbf{O}(1,q+1)/\mathbf{O}(1,q):\ y_1^2 - y_2^2 - \cdots - y_{q+2}^2 = -1\,,\\
&Q_{+1} = \mathbf{O}(2,q)/\mathbf{O}(1,q):\ y_1^2 - y_2^2 - \cdots - y_{q+1}^2 + y_{q+2}^2 = 1\,,
\end{aligned}$$

the Lorentzian structure being induced by $y_1^2 - y_2^2 - \cdots \mp y_{q+2}^2$.

§2 Orbital Integrals

The orbital integrals for isotropic Lorentzian manifolds are analogs to the spherical averaging operator M^r considered in Ch. I, §1, and Ch. III, §1. We start with some geometric preparation.

For manifolds X with a Lorentzian structure g we adopt the following customary terminology: If $y \in X$ the cone

$$C_y = \{Y \in X_y : g_y(Y,Y) = 0\}$$

is called the *null cone* (or the *light cone*) in X_y with vertex y. A nonzero vector $Y \in X_y$ is said to be *timelike, isotropic* or *spacelike* if $g_y(Y,Y)$ is positive, 0, or negative, respectively. Similar designations apply to geodesics according to the type of their tangent vectors.

While the geodesics in $\mathbf{R}^{1+q}$ are just the straight lines, the geodesics in Q_{-1} and Q_{+1} can be found by the method of Ch. III, §1.

Proposition 2.1. *The geodesics in the Lorentzian quadrics Q_{-1} and Q_{+1} have the following properties:*

(i) The geodesics are the nonempty intersections of the quadrics with two-planes in $\mathbf{R}^{2+q}$ through the origin.

(ii) For Q_{-1} the spacelike geodesics are closed, for Q_{+1} the timelike geodesics are closed.

(iii) The isotropic geodesics are certain straight lines in $\mathbf{R}^{2+q}$.

Proof. Part (i) follows by the symmetry considerations in Ch. III, §1. For Part (ii) consider the intersection of Q_{-1} with the two-plane

$$y_1 = y_4 = \cdots = y_{q+2} = 0\,.$$

The intersection is the circle $y_2 = \cos t$, $y_3 = \sin t$ whose tangent vector $(0, -\sin t, \cos t, 0, \ldots, 0)$ is clearly spacelike. Since $\mathbf{O}(1, q+1)$ permutes the spacelike geodesics transitively the first statement in (ii) follows. For Q_{+1} we intersect similarly with the two-plane

$$y_2 = \cdots = y_{q+1} = 0\,.$$

For (iii) we note that the two-plane $\mathbf{R}(1, 0, \ldots, 0, 1) + \mathbf{R}(0, 1, \ldots, 0)$ intersects Q_{-1} in a pair of straight lines

$$y_1 = t, y_2 \pm 1, y_3 = \cdots = y_{q+1} = 0, y_{q+2} = t$$

which clearly are isotropic. The transitivity of $\mathbf{O}(1, q+1)$ on the set of isotropic geodesics then implies that each of these is a straight line. The argument for Q_{+1} is similar.

Lemma 2.2. *The quadrics Q_{-1} and Q_{+1} $(q \geq 1)$ are connected.*

Proof. The q-sphere being connected, the point $(y_1, \ldots, y_{q+2})$ on $Q_{\mp 1}$ can be moved continuously on $Q_{\mp 1}$ to the point

$$(y_1, (y_2^2 + \cdots + y_{q+1}^2)^{1/2}, 0, \ldots, 0, y_{q+2})$$

so the statement follows from the fact that the hyperboloids $y_1^2 - y_1^2 \mp y_3^2 = \mp 1$ are connected.

Lemma 2.3. *The identity components of $\mathbf{O}(1, q+1)$ and $\mathbf{O}(2, q)$ act transitively on Q_{-1} and Q_{+1}, respectively, and the isotropy subgroups are connected.*

Proof. The first statement comes from the general fact (see e.g [DS], pp. 121–124) that when a separable Lie group acts transitively on a connected manifold then so does its identity component. For the isotropy groups we use the description (7) of the identity component. This shows quickly that

$$\begin{aligned}
\mathbf{O}_o(1, q+1) \cap \mathbf{O}(1, q) &= \mathbf{O}_o(1, q)\,, \\
\mathbf{O}_o(2, q) \cap \mathbf{O}(1, q) &= \mathbf{O}_o(1, q)
\end{aligned}$$

the subscript o denoting identity component. Thus we have

$$\begin{aligned} Q_{-1} &= \mathbf{O}_o(1,q+1)/\mathbf{O}_o(1,q)\,, \\ Q_{+1} &= \mathbf{O}_o(2,q)/\mathbf{O}_o(1,q)\,, \end{aligned}$$

proving the lemma.

We now write the spaces in Theorem 1.5 in the form $X = G/H$ where $H = \mathbf{O}_o(1,q)$ and G is either $G^0 = \mathbf{R}^{1+q} \cdot \mathbf{O}_o(1,q)$ (semi-direct product) $G^- = \mathbf{O}_o(1,q+1)$ or $G^+ = \mathbf{O}_o(2,q)$. Let o denote the origin $\{H\}$ in X, that is

$$\begin{aligned} &o = (0,\dots,0) && \text{if } X = \mathbf{R}^{1+q} \\ &o = (0,\dots,0,1) && \text{if } X = Q_{-1} \text{ or } Q_{+1}\,. \end{aligned}$$

In the cases $X = Q_{-1}, X = Q_{+1}$ the tangent space X_o is the hyperplane $\{y_1,\dots,y_{q+1},1\} \subset \mathbf{R}^{2+q}$.

The timelike vectors at o fill up the "interior" C_o^o of the cone C_o. The set C_o^o consists of two components. The component which contains the timelike vector

$$v_o = (-1,0,\dots,0)$$

will be called the *solid retrograde cone* in X_o. It will be denoted by D_o. The component of the hyperboloid $g_o(Y,Y) = r^2$ which lies in D_o will be denoted $S_r(o)$. If y is any other point of X we define $C_y, D_y, S_r(y) \subset X_y$ by

$$C_y = g \cdot C_o\,,\ D_y = g \cdot D_o\,,\ S_r(y) = g \cdot S_r(o)$$

if $g \in G$ is chosen such that $g \cdot o = y$. This is a valid definition because the connectedness of H implies that $h \cdot D_o \subset D_o$. We also define

$$B_r(y) = \{Y \in D_y : 0 < g_y(Y,Y) < r^2\}\,.$$

If Exp denotes the exponential mapping of X_y into X, mapping rays through 0 onto geodesics through y we put

$$\begin{aligned} \mathbf{D}_y = \operatorname{Exp} D_y\,, \quad \mathbf{C}_y = \operatorname{Exp} C_y \\ \mathbf{S}_r(y) = \operatorname{Exp} S_r(y)\,, \quad \mathbf{B}_r(y) = \operatorname{Exp} B_r(y)\,. \end{aligned}$$

Again $\mathbf{C}_y$ and $\mathbf{D}_y$ are respectively called the *light cone* and *solid retrograde cone* in X with vertex y. For the spaces $X = Q_+$ we always assume $r < \pi$ in order that Exp will be one-to-one on $B_r(y)$ in view of Prop. 2.1(ii).

Figure IV.1 illustrates the situation for Q_{-1} in the case $q = 1$. Then Q_{-1} is the hyperboloid

$$y_1^2 - y_2^2 - y_3^2 = -1$$

and the y_1-axis is vertical. The origin o is

$$o = (0,0,1)$$

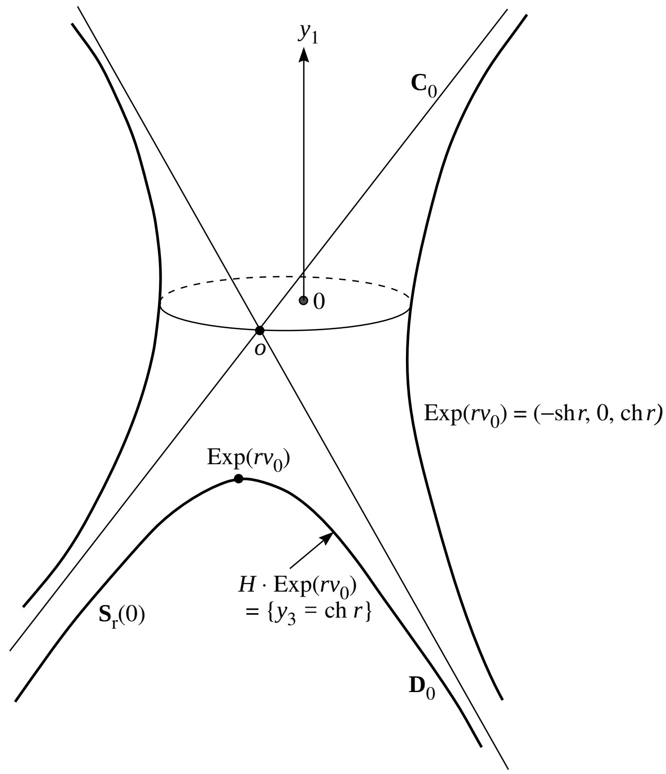

FIGURE IV.1.

and the vector $v_o = (-1, 0, 0)$ lies in the tangent space

$$(Q_{-1})_o = \{y : y_3 = 1\}$$

pointing downward. The mapping $\psi : gH \to g \cdot o$ has differential $d\psi : \mathfrak{m} \to (Q_{-1})_o$ and

$$d\psi(E_{1\,3} + E_{3\,1}) = -v_o$$

in the notation of (4). The geodesic tangent to v_o at o is

$$t \to \operatorname{Exp}(tv_o) = \exp(-t(E_{1\,3} + E_{3\,1})) \cdot o = (-\sinh t, 0, \cosh t)$$

and this is the section of Q_{-1} with the plane $y_2 = 0$. Note that since H preserves each plane $y_3 =$ const., the "sphere" $\mathbf{S}_r(o)$ is the plane section $y_3 = \cosh r, y_1 < 0$ with Q_{-1}.

Lemma 2.4. *The negative of the Lorentzian structure on $X = G/H$ induces on each $\mathbf{S}_r(y)$ a Riemannian structure of constant negative curvature $(q > 1)$.*

Proof. The manifold X being isotropic the group $H = \mathbf{O}_o(1,q)$ acts transitively on $\mathbf{S}_r(o)$. The subgroup leaving fixed the geodesic from o with tangent vector v_o is $\mathbf{O}_o(q)$. This implies the lemma.

Lemma 2.5. *The timelike geodesics from y intersect $\mathbf{S}_r(y)$ under a right angle.*

Proof. By the group invariance it suffices to prove this for $y = o$ and the geodesic with tangent vector v_o. For this case the statement is obvious.

Let $\tau(g)$ denote the translation $xH \to gxH$ on G/H and for $Y \in \mathfrak{m}$ let T_Y denote the linear transformation $Z \to [Y,[Y,Z]]$ of $\mathfrak{m}$ into itself. As usual, we identify $\mathfrak{m}$ with $(G/H)_o$.

Lemma 2.6. *The exponential mapping* $\mathrm{Exp} : \mathfrak{m} \to G/H$ *has differential*

$$d\mathrm{Exp}_Y = d\tau(\exp Y) \circ \sum_0^\infty \frac{T_Y^n}{(2n+1)!} \qquad (Y \in \mathfrak{m}) .$$

For the proof see [DS], p. 215.

Lemma 2.7. *The linear transformation*

$$A_Y = \sum_0^\infty \frac{T_Y^n}{(2n+1)!}$$

has determinant given by

$$\det A_Y = \left\{ \frac{\sinh(g(Y,Y))^{1/2}}{(g(Y,Y))^{1/2}} \right\}^q \quad \text{for } Q_{-1}$$

$$\det A_Y = \left\{ \frac{\sin(g(Y,Y))^{1/2}}{(g(Y,Y))^{1/2}} \right\}^q \quad \text{for } Q_{+1}$$

for Y timelike.

Proof. Consider the case of Q_{-1}. Since $\det(A_Y)$ is invariant under H it suffices to verify this for $Y = cY_1$ in (4), where $c \in \mathbf{R}$. We have $c^2 = g(Y,Y)$ and $T_{Y_1}(Y_j) = Y_j \quad (2 \le j \le q+1)$. Thus T_Y has the eigenvalue 0 and $g(Y,Y)$; the latter is a q-tuple eigenvalue. This implies the formula for the determinant. The case Q_{+1} is treated in the same way.

From this lemma and the description of the geodesics in Prop. 2.1 we can now conclude the following result.

Proposition 2.8. *(i) The mapping* $\mathrm{Exp} : \mathfrak{m} \to Q_{-1}$ *is a diffeomorphism of D_o onto $\mathbf{D}_o$.*

(ii) The mapping $\mathrm{Exp} : \mathfrak{m} \to Q_{+1}$ *gives a diffeomorphism of $B_\pi(o)$ onto $\mathbf{B}_\pi(o)$.*

Let dh denote a bi-invariant measure on the unimodular group H. Let $u \in \mathcal{D}(X)$, $y \in X$ and $r > 0$. Select $g \in G$ such that $g \cdot o = y$ and select $x \in \mathbf{S}_r(o)$. Consider the integral

$$\int_H u(gh \cdot x)\, dh\,.$$

Since the subgroup $K \subset H$ leaving x fixed is compact it is easy to see that the set

$$C_{g,x} = \{h \in H : gh \cdot x \in \text{ support } (u)\}$$

is compact; thus the integral above converges. By the bi-invariance of dh it is independent of the choice of g (satisfying $g \cdot o = y$) and of the choice of $x \in \mathbf{S}_r(o)$. In analogy with the Riemannian case (Ch. III, §1) we thus define the operator M^r (*the orbital integral*) by

$$(M^r u)(y) = \int_H u(gh \cdot x)\, dh\,. \tag{11}$$

If g and x run through suitable compact neighborhoods, the sets $C_{g,x}$ are enclosed in a fixed compact subset of H so $(M^r u)(y)$ depends smoothly on both r and y. It is also clear from (11) that the operator M^r is invariant under the action of G: if $m \in G$ and $\tau(m)$ denotes the transformation $nH \to mnH$ of G/H onto itself then

$$M^r(u \circ \tau(m)) = (M^r u) \circ \tau(m)\,.$$

If dk denotes the normalized Haar measure on K we have by standard invariant integration

$$\int_H u(h \cdot x)\, dh = \int_{H/K} d\dot{h} \int_K u(hk \cdot x)\, dk = \int_{H/K} u(h \cdot x)\, d\dot{h}\,,$$

where $d\dot{h}$ is an H-invariant measure on H/K. But if $d\mathbf{w}^r$ is the volume element on $\mathbf{S}_r(o)$ (cf. Lemma 2.4) we have by the uniqueness of H-invariant measures on the space $H/K \approx \mathbf{S}_r(o)$ that

$$\int_H u(h \cdot x)\, dh = \frac{1}{A(r)} \int_{\mathbf{S}_r(o)} u(z)\, d\mathbf{w}^r(z)\,, \tag{12}$$

where $A(r)$ is a positive scalar. But since g is an isometry we deduce from (12) that

$$(M^r u)(y) = \frac{1}{A(r)} \int_{\mathbf{S}_r(y)} u(z)\, d\mathbf{w}^r(z)\,.$$

Now we have to determine $A(r)$.

Lemma 2.9. *For a suitable fixed normalization of the Haar measure dh on H we have*

$$A(r) = r^q\,, \quad (\sinh r)^q\,, \quad (\sin r)^q$$

for the cases

$$\mathbf{R}^{1+q}\,, \quad \mathbf{O}(1,q+1)/\mathbf{O}(1,q)\,, \quad \mathbf{O}(2,q)/\mathbf{O}(1,q)\,,$$

respectively.

Proof. The relations above show that $dh = A(r)^{-1}\,d\mathbf{w}^r\,dk$. The mapping $\mathrm{Exp} : D_o \to \mathbf{D}_o$ preserves length on the geodesics through o and maps $S_r(o)$ onto $\mathbf{S}_r(o)$. Thus if $z \in S_r(o)$ and Z denotes the vector from 0 to z in X_o the ratio of the volume of elements of $\mathbf{S}_r(o)$ and $S_r(o)$ at z is given by $\det(d\mathrm{Exp}_Z)$. Because of Lemmas 2.6–2.7 this equals

$$1, \left(\frac{\sinh r}{r}\right)^q, \left(\frac{\sin r}{r}\right)^q$$

for the three respective cases. But the volume element $d\omega^r$ on $S_r(o)$ equals $r^q d\omega^1$. Thus we can write in the three respective cases

$$dh = \frac{r^q}{A(r)}\,d\omega^1\,dk\,, \quad \frac{\sinh^q r}{A(r)}\,d\omega^1\,dk\,, \quad \frac{\sin^q r}{A(r)}\,d\omega^1\,dk\,.$$

But we can once and for all normalize dh by $dh = d\omega^1\,dk$ and for this choice our formulas for $A(r)$ hold.

Let $\Box$ denote the *wave operator* on $X = G/H$, that is the Laplace-Beltrami operator for the Lorentzian structure g.

Lemma 2.10. *Let $y \in X$. On the solid retrograde cone $\mathbf{D}_y$, the wave operator $\Box$ can be written*

$$\Box = \frac{\partial^2}{\partial r^2} + \frac{1}{A(r)}\,\frac{dA}{dr}\,\frac{\partial}{\partial r} - L_{\mathbf{S}_r(y)}\,,$$

where $L_{\mathbf{S}_r(y)}$ is the Laplace-Beltrami operator on $\mathbf{S}_r(y)$.

Proof. We can take $y = o$. If $(\theta_1, \ldots, \theta_q)$ are coordinates on the "sphere" $S_1(o)$ in the flat space X_o then $(r\theta_1, \ldots, r\theta_q)$ are coordinates on $S_r(o)$. The Lorentzian structure on D_o is therefore given by

$$dr^2 - r^2\,d\theta^2\,,$$

where $d\theta^2$ is the Riemannian structure of $S_1(o)$. Since A_Y in Lemma 2.7 is a diagonal matrix with eigenvalues 1 and $r^{-1}A(r)^{1/q}$ (q-times) it follows from Lemma 2.6 that the image $\mathbf{S}_r(o) = \mathrm{Exp}(S_r(o))$ has Riemannian structure

$r^2\,d\theta^2$, $\sinh^2 r\,d\theta^2$ and $\sin^2 r\,d\theta^2$ in the cases $\mathbf{R}^{1+q}$, Q_{-1} and Q_{+1}, respectively. By the perpendicularity in Lemma 2.5 it follows that the Lorentzian structure on $\mathbf{D}_o$ is given by

$$dr^2 - r^2\,d\theta^2\,, \quad dr^2 - \sinh^2 r\,d\theta^2\,, \quad dr^2 - \sin^2 r\,d\theta^2$$

in the three respective cases. Now the lemma follows immediately.

The operator M^r is of course the Lorentzian analog to the spherical mean value operator for isotropic Riemannian manifolds. We shall now prove that in analogy to the Riemannian case (cf. (41), Ch. III) the operator M^r commutes with the wave operator $\Box$.

Theorem 2.11. *For each of the isotropic Lorentz spaces $X = G^-/H$, G^+/H or G^0/H the wave operator $\Box$ and the orbital integral M^r commute:*

$$\Box M^r u = M^r \Box u \qquad \textit{for } u \in \mathcal{D}(X)\,.$$

(For G^+/H we assume $r < \pi$.)

Given a function u on G/H we define the function $\widetilde{u}$ on G by $\widetilde{u}(g) = u(g\cdot o)$.

Lemma 2.12. *There exists a differential operator $\widetilde{\Box}$ on G invariant under all left and all right translations such that*

$$\widetilde{\Box}\widetilde{u} = (\Box u)^{\sim} \quad \textit{for } u \in \mathcal{D}(X)\,.$$

Proof. We consider first the case $X = G^-/H$. The bilinear form

$$K(Y,Z) = \tfrac{1}{2}\,\mathrm{Tr}(YZ)$$

on the Lie algebra $\mathfrak{o}(1, q+1)$ of G^- is nondegenerate; in fact K is nondegenerate on the complexification $\mathfrak{o}(q+2, \mathbf{C})$ consisting of all complex skew symmetric matrices of order $q+2$. A simple computation shows that in the notation of (4) and (5)

$$K(Y_1, Y_1) = 1\,, \quad K(Y_j, Y_j) = -1 \quad (2 \le j \le q+1)\,.$$

Since K is symmetric and nondegenerate there exists a unique left invariant pseudo-Riemannian structure $\widetilde{K}$ on G^- such that $\widetilde{K}_e = K$. Moreover, since K is invariant under the conjugation $Y \to gYg^{-1}$ of $\mathfrak{o}(1, q+1)$, $\widetilde{K}$ is also right invariant. Let $\widetilde{\Box}$ denote the corresponding Laplace-Beltrami operator on G^-. Then $\widetilde{\Box}$ is invariant under all left and right translations on G^-. Let $u = \mathcal{D}(X)$. Since $\widetilde{\Box}\widetilde{u}$ is invariant under all right translations from H there is a unique function $v \in \mathcal{E}(X)$ such that $\widetilde{\Box}\widetilde{u} = \widetilde{v}$. The mapping $u \to v$ is a differential operator which at the origin must coincide with $\Box$, that is $\widetilde{\Box}\widetilde{u}(e) = \Box u(o)$. Since, in addition, both $\Box$ and the operator $u \to v$ are invariant under the action of G^- on X it follows that they coincide. This proves $\widetilde{\Box}\widetilde{u} = (\Box u)^{\sim}$.

The case $X = G^+/H$ is handled in the same manner. For the flat case $X = G^0/H$ let

$$Y_j = (0, \ldots, 1, \ldots, 0)\,,$$

the j^{th} coordinate vector on $\mathbf{R}^{1+q}$. Then $\Box = Y_1^2 - Y_2^2 - \cdots - Y_{q+1}^2$. Since $\mathbf{R}^{1+q}$ is naturally embedded in the Lie algebra of G^0 we can extend Y_j to a left invariant vector field $\widetilde{Y}_j$ on G^0. The operator

$$\widetilde{\Box} = \widetilde{Y}_1^2 - \widetilde{Y}_2^2 - \cdots - \widetilde{Y}_{q+1}^2$$

is then a left and right invariant differential operator on G^0 and again we have $\widetilde{\Box}\widetilde{u} = (\Box u)^\sim$. This proves the lemma.

We can now prove Theorem 2.11. If $g \in G$ let $L(g)$ and $R(g)$, respectively, denote the left and right translations $\ell \to g\ell$, and $\ell \to \ell g$ on G. If $\ell \cdot o = x, x \in \mathbf{S}_r(o)$ $(r > 0)$ and $g \cdot o = y$ then

$$(M^r u)(y) = \int_H \widetilde{u}(gh\ell)\, dh$$

because of (11). As g and ℓ run through sufficiently small compact neighborhoods the integration takes place within a fixed compact subset of H as remarked earlier. Denoting by subscript the argument on which a differential operator is to act we shall prove the following result.

Lemma 2.13.

$$\widetilde{\Box}_\ell \left(\int_H \widetilde{u}(gh\ell)\, dh \right) = \int_H (\widetilde{\Box}\widetilde{u})(gh\ell)\, dh = \widetilde{\Box}_g \left(\int_H \widetilde{u}(gh\ell)\, dh \right).$$

Proof. The first equality sign follows from the left invariance of $\widetilde{\Box}$. In fact, the integral on the left is

$$\int_H (\widetilde{u} \circ L(gh))(\ell)\, dh$$

so

$$\widetilde{\Box}_\ell \left(\int_H \widetilde{u}(gh\ell)\, dh \right) = \int_H \left[\widetilde{\Box}(\widetilde{u} \circ L(gh)) \right] (\ell)\, dh$$
$$= \int_H \left[(\widetilde{\Box}\widetilde{u}) \circ L(gh) \right] (\ell)\, dh = \int_H (\widetilde{\Box}\widetilde{u})(gh\ell)\, dh\,.$$

The second equality in the lemma follows similarly from the right invariance of $\widetilde{\Box}$. But this second equality is just the commutativity statement in Theorem 2.11.

Lemma 2.13 also implies the following analog of the Darboux equation in Lemma 3.2, Ch. I.

Corollary 2.14. *Let $u \in \mathcal{D}(X)$ and put*

$$U(y,z) \quad = \quad (M^r u)(y) \quad \text{if } z \in \mathbf{S}_r(o)\,.$$

Then

$$\Box_y(U(y,z)) \quad = \quad \Box_z(U(y,z))\,.$$

Remark 2.15. In $\mathbf{R}^n$ the solutions to the Laplace equation $Lu = 0$ are characterized by the spherical mean-value theorem $M^r u = u$ (all r). This can be stated equivalently: $M^r u$ is a constant in r. In this latter form the mean value theorem holds for the solutions of the wave equation $\Box u = 0$ in an isotropic Lorentzian manifold: *If u satisfies $\Box u = 0$ and if u is suitably small at ∞ then $(M^r u)(o)$ is constant in r.* For a precise statement and proof see Helgason [1959], p. 289. For $\mathbf{R}^2$such a result had also been noted by Ásgeirsson.

§3 Generalized Riesz Potentials

In this section we generalize part of the theory of Riesz potentials (Ch. V, §5) to isotropic Lorentz spaces.

Consider first the case

$$X = Q_{-1} = G^-/H = \mathbf{O}_o(1,n)/\mathbf{O}_o(1,n-1)$$

of dimension n and let $f \in \mathcal{D}(X)$ and $y \in X$. If $z = \mathrm{Exp}_y Y$ $(Y \in D_y)$ we put $r_{yz} = g(Y,Y)^{1/2}$ and consider the integral

$$(I_-^\lambda f)(y) = \frac{1}{H_n(\lambda)} \int_{\mathbf{D}_y} f(z) \sinh^{\lambda-n}(r_{yz})\, dz\,, \tag{13}$$

where dz is the volume element on X, and

$$H_n(\lambda) = \pi^{(n-2)/2} 2^{\lambda-1} \Gamma\left(\lambda/2\right) \Gamma\left((\lambda+2-n)/2\right)\,. \tag{14}$$

The integral converges for $\mathrm{Re}\,\lambda \geq n$. We transfer the integral in (13) over to D_y via the diffeomorphism $\mathrm{Exp}(= \mathrm{Exp}_y)$. Since

$$dz = dr d\mathbf{w}^r = dr \left(\frac{\sinh r}{r}\right)^{n-1} d\omega^r$$

and since $dr d\omega^r$ equals the volume element dZ on D_y we obtain

$$(I^\lambda f)(y) = \frac{1}{H_n(\lambda)} \int_{D_y} (f \circ \mathrm{Exp})(Z) \Big(\frac{\sinh r}{r}\Big)^{\lambda-1} r^{\lambda-n}\, dZ\,,$$

where $r = g(Z,Z)^{1/2}$. This has the form

$$\frac{1}{H_n(\lambda)} \int_{D_y} h(Z,\lambda) r^{\lambda-n}\, dZ\,, \tag{15}$$

where $h(Z, \lambda)$, as well as each of its partial derivatives with respect to the first argument, is holomorphic in λ and h has compact support in the first variable. The methods of Riesz [1949], Ch. III, can be applied to such integrals (15). In particular we find that the function $\lambda \to (I_-^\lambda f)(y)$ which by its definition is holomorphic for $\operatorname{Re}\lambda > n$ admits a holomorphic continuation to the entire λ-plane and that its value at $\lambda = 0$ is $h(0,0) = f(y)$. (In Riesz' treatment $h(Z,\lambda)$ is independent of λ, but his method still applies.) Denoting the holomorphic continuation of (13) by $(I_-^\lambda)f(y)$ we have thus obtained

$$I_-^0 f = f\,. \tag{16}$$

We would now like to differentiate (13) with respect to y. For this we write the integral in the form $\int_F f(z)K(y,z)\,dz$ over a bounded region F which properly contains the intersection of the support of f with the closure of $\mathbf{D}_y$. The kernel $K(y,z)$ is defined as $\sinh^{\lambda-n} r_{yz}$ if $z \in \mathbf{D}_y$, otherwise 0. For $\operatorname{Re}\lambda$ sufficiently large, $K(y,z)$ is twice continuously differentiable in y so we can deduce for such λ that $I_-^\lambda f$ is of class C^2 and that

$$(\Box I_-^\lambda f)(y) = \frac{1}{H_n(\lambda)} \int_{\mathbf{D}_y} f(z)\Box_y(\sinh^{\lambda-n} r_{yz})\,dz\,. \tag{17}$$

Moreover, given $m \in \mathbb{Z}^+$ we can find k such that $I_-^\lambda f \in C^m$ for $\operatorname{Re}\lambda > k$ (and all f). Using Lemma 2.10 and the relation

$$\frac{1}{A(r)}\frac{dA}{dr} = (n-1)\coth r$$

we find

$$\begin{aligned}\Box_y(\sinh^{\lambda-n} r_{yz}) &= \Box_z(\sinh^{\lambda-n} r_{yz}) \\ &= (\lambda-n)(\lambda-1)\sinh^{\lambda-n} r_{yz} + (\lambda-n)(\lambda-2)\sinh^{\lambda-n-2} r_{yz}\,.\end{aligned}$$

We also have

$$H_n(\lambda) = (\lambda-2)(\lambda-n)H_n(\lambda-2)$$

so substituting into (17) we get

$$\Box I_-^\lambda f = (\lambda-n)(\lambda-1)I_-^\lambda f + I_-^{\lambda-2} f\,.$$

Still assuming $\operatorname{Re}\lambda$ large we can use Green's formula to express the integral

$$\int_{\mathbf{D}_y} [f(z)\Box_z(\sinh^{\lambda-n} r_{yz}) - \sinh^{\lambda-n} r_{yz}(\Box f)(z)]\,dz \tag{18}$$

as a surface integral over a part of $\mathbf{C}_y$ (on which $\sinh^{\lambda-n} r_{yz}$ and its first order derivatives vanish) together with an integral over a surface inside $\mathbf{D}_y$

(on which f and its derivatives vanish). Hence the expression (18) vanishes so we have proved the relations

$$\square(I_-^\lambda f) = I_-^\lambda(\square f) \tag{19}$$

$$I_-^\lambda(\square f) = (\lambda-n)(\lambda-1)I_-^\lambda f + I_-^{\lambda-2}f \tag{20}$$

for $\operatorname{Re}\lambda > k$, k being some number (independent of f).

Since both sides of (20) are holomorphic in λ this relation holds for all $\lambda \in \mathbf{C}$. We shall now deduce that for each $\lambda \in \mathbf{C}$, we have $I_-^\lambda f \in \mathcal{E}(X)$ and (19) holds. For this we observe by iterating (20) that for each $p \in \mathbb{Z}^+$

$$I_-^\lambda f = I_-^{\lambda+2p}(Q_p(\square)f)\,, \tag{21}$$

Q_p being a certain p^{th}-degree polynomial. Choosing p arbitrarily large we deduce from the remark following (17) that $I_-^\lambda f \in \mathcal{E}(X)$; secondly (19) implies for $\operatorname{Re}\lambda + 2p > k$ that

$$\square I_-^{\lambda+2p}(Q_p(\square)f) = I_-^{\lambda+2p}(Q_p(\square)\square f)\,.$$

Using (21) again this means that (19) holds for all λ.

Putting $\lambda = 0$ in (20) we get

$$I_-^{-2} = \square f - nf\,. \tag{22}$$

Remark 3.1. In Riesz' paper [1949], p. 190, an analog I^α of the potentials in Ch. V, §5, is defined for any analytic Lorentzian manifold. These potentials I^α are however different from our I_-^λ and satisfy the equation $I^{-2}f = \square f$ in contrast to (22).

We consider next the case

$$X = Q_{+1} = G^+/H = \mathbf{O}_o(2, n-1)/\mathbf{O}_o(1, n-1)$$

and we define for $f \in \mathcal{D}(X)$

$$(I_+^\lambda f)(y) = \frac{1}{H_n(\lambda)} \int_{\mathbf{D}_y} f(z) \sin^{\lambda-n}(r_{yz})\,dz\,. \tag{23}$$

Again $H_n(\lambda)$ is given by (14) and dz is the volume element. In order to bypass the difficulties caused by the fact that the function $z \to \sin r_{yz}$ vanishes on $\mathbf{S}_\pi$ we assume that f has support disjoint from $\mathbf{S}_\pi(o)$. Then the support of f is disjoint from $\mathbf{S}_\pi(y)$ for all y in some neighborhood of o in X. We can then prove just as before that

$$(I_+^0 f)(y) = f(y) \tag{24}$$

$$(\square I_+^\lambda f)(y) = (I_+^\lambda \square f)(y) \tag{25}$$

$$(I_+^\lambda \square f)(y) = -(\lambda-n)(\lambda-1)(I_+^\lambda f)(y) + (I_+^{\lambda-2}f)(y) \tag{26}$$

for all $\lambda \in \mathbf{C}$. In particular

$$I_+^{-2} f = \Box f + n f\,. \tag{27}$$

Finally we consider the flat case

$$X = \mathbf{R}^n = G^0/H = \mathbf{R}^n \cdot \mathbf{O}_o(1, n-1)/\mathbf{O}_o(1, n-1)$$

and define

$$(I_o^\lambda f)(y) = \frac{1}{H_n(\lambda)} \int_{\mathbf{D}_y} f(z) r_{yz}^{\lambda - n}\, dz\,.$$

These are the potentials defined by Riesz in [1949], p. 31, who proved

$$I_o^0 f = f,\ \Box I_o^\lambda f = I_o^\lambda \Box f = I_o^{\lambda-2} f\,. \tag{28}$$

§4 Determination of a Function from its Integral over Lorentzian Spheres

In a Riemannian manifold a function is determined in terms of its spherical mean values by the simple relation $f = \lim_{r \to 0} M^r f$. We shall now solve the analogous problem for an even-dimensional isotropic Lorentzian manifold and express a function f in terms of its orbital integrals $M^r f$. Since the spheres $\mathbf{S}_r(y)$ do not shrink to a point as $r \to 0$ the formula (cf. Theorem 4.1) below is quite different.

For the solution of the problem we use the geometric description of the wave operator $\Box$ developed in §2, particularly its commutation with the orbital integral M^r, and combine this with the results about the generalized Riesz potentials established in §3.

We consider first the negatively curved space $X = G^-/H$. Let $n = \dim X$ and assume n even. Let $f \in \mathcal{D}(X)$, $y \in X$ and put $F(r) = (M^r f)(y)$. Since the volume element dz on $\mathbf{D}_y$ is given by $dz = dr\, d\mathbf{w}^r$ we obtain from (12) and Lemma 2.9 ,

$$(I_-^\lambda f)(y) = \frac{1}{H_n(\lambda)} \int_0^\infty \sinh^{\lambda-1} r F(r)\, dr\,. \tag{29}$$

Let $Y_1, \ldots, Y_n$ be a basis of X_y such that the Lorentzian structure is given by

$$g_y(Y) = y_1^2 - y_2^2 - \cdots - y_n^2\,, \quad Y = \sum_1^n y_i Y_i\,.$$

If $\theta_1, \ldots, \theta_{n-2}$ are geodesic polar coordinates on the unit sphere in $\mathbf{R}^{n-1}$ we put

$$\begin{aligned}
y_1 &= -r \cosh \zeta \qquad (0 \le \zeta < \infty, 0 < r < \infty)\\
y_2 &= r \sinh \zeta \cos \theta_1\\
&\ \vdots\\
y_n &= r \sinh \zeta \sin \theta_1 \ldots \sin \theta_{n-2}\,.
\end{aligned}$$

Then $(r,\zeta,\theta_1,\dots,\theta_{n-2})$ are coordinates on the retrograde cone D_y and the volume element on $S_r(y)$ is given by

$$d\omega^r = r^{n-1}\sinh^{n-2}\zeta\, d\zeta\, d\omega^{n-2}$$

where $d\omega^{n-2}$ is the volume element on the unit sphere in $\mathbf{R}^{n-1}$. It follows that

$$d\mathbf{w}^r = \sinh^{n-1} r\sinh^{n-2}\zeta\, d\zeta\, d\omega^{n-2}$$

and therefore

$$F(r) = \iint (f\circ \mathrm{Exp})(r,\zeta,\theta_1,\dots,\theta_{n-2})\sinh^{n-2}\zeta d\zeta d\omega^{n-2}\,, \tag{30}$$

where for simplicity

$$(r,\zeta,\theta_1,\dots,\theta_{n-2})$$

stands for

$$(-r\cosh\zeta, r\sinh\zeta\cos\theta_1,\dots,r\sinh\zeta\sin\theta_1\dots\sin\theta_{n-2})\,.$$

Now select A such that $f\circ\mathrm{Exp}$ vanishes outside the sphere $y_1^2+\cdots+y_n^2 = A^2$ in X_y. Then, in the integral (30), the range of ζ is contained in the interval $(0,\zeta_o)$ where

$$r^2\cosh^2\zeta_o + r^2\sinh^2\zeta_o = A^2\,.$$

Then

$$r^{n-2}F(r) = \int_{\mathbf{S}^{n-2}}\int_0^{\zeta_o}(f\circ\mathrm{Exp})(r,\zeta,(\theta))(r\sinh\zeta)^{n-2}\,d\zeta\, d\omega^{n-2}\,.$$

Since

$$|r\sinh\zeta| \le re^{\zeta} \le 2A \text{ for } \zeta\le\zeta_o$$

this implies

$$|r^{n-2}(M^r f)(y)| \le CA^{n-2}\sup|f|\,, \tag{31}$$

where C is a constant independent of r. Also substituting $t = r\sinh\zeta$ in the integral above, the ζ-integral becomes

$$\int_0^k \varphi(t)t^{n-2}(r^2+t^2)^{-1/2}\,dt\,,$$

where $k = [(A^2-r^2)/2]^{1/2}$ and φ is bounded. Thus if $n>2$ the limit

$$a = \lim_{r\to 0}\sinh^{n-2} rF(r) \quad n>2 \tag{32}$$

exist and is $\not\equiv 0$. Similarly, we find for $n=2$ that the limit

$$b = \lim_{r\to 0}(\sinh r)F'(r) \qquad (n=2) \tag{33}$$

exists.

Consider now the case $n > 2$. We can rewrite (29) in the form

$$(I_-^\lambda f)(y) = \frac{1}{H_n(\lambda)} \int_0^A \sinh^{n-2} r F(r) \sinh^{\lambda-n+1} r \, dr \,,$$

where $F(A) = 0$. We now evaluate both sides for $\lambda = n-2$. Since $H_n(\lambda)$ has a simple pole for $\lambda = n-2$ the integral has at most a simple pole there and the residue is

$$\lim_{\lambda \to n-2} (\lambda - n+2) \int_0^A \sinh^{n-2} r F(r) \sinh^{\lambda-n+1} r \, dr \,.$$

Here we can take λ real and greater than $n-2$. This is convenient since by (32) the integral is then absolutely convergent and we do not have to think of it as an implicitly given holomorphic extension. We split the integral in two parts

$$(\lambda-n+2) \int_0^A (\sinh^{n-2} r F(r) - a) \sinh^{\lambda-n+1} r \, dr$$
$$+ a(\lambda - n + 2) \int_0^A \sinh^{\lambda-n+1} r \, dr \,.$$

For the last term we use the relation

$$\lim_{\mu \to 0+} \mu \int_0^A \sinh^{\mu-1} r \, dr = \lim_{\mu \to 0+} \mu \int_0^{\sinh A} t^{\mu-1} (1+t^2)^{-1/2} \, dt = 1$$

by (38) in Chapter V. For the first term we can for each $\epsilon > 0$ find a $\delta > 0$ such that

$$|\sinh^{n-2} r F(r) - a| < \epsilon \quad \text{for } 0 < r < \delta \,.$$

If $N = \max |\sinh^{n-2} r F(r)|$ we have for $n-2 < \lambda < n-1$ the estimate

$$\left|(\lambda-n+2) \int_\delta^A (\sinh^{n-2} r F(r) - a) \sinh^{\lambda-n+1} r \, dr\right|$$
$$\leq (\lambda-n+2)(N + |a|)(A - \delta)(\sinh \delta)^{\lambda-n+1} \,;$$

$$\left|(\lambda+n-2) \int_0^\delta (\sinh^{n-2} r F(r) - a) \sinh^{\lambda-n+1} r \, dr\right|$$
$$\leq \epsilon(\lambda - n + 2) \int_0^\delta r^{\lambda-n+1} \, dr = \epsilon \delta^{\lambda-n+2} \,.$$

Taking $\lambda - (n-2)$ small enough the right hand side of each of these inequalities is $< 2\epsilon$. We have therefore proved

$$\lim_{\lambda \to n-2} (\lambda - n + 2) \int_0^\infty \sinh^{\lambda-1} r F(r) \, dr = \lim_{r \to 0} \sinh^{n-2} r \, F(r) \,.$$

Taking into account the formula for $H_n(\lambda)$ we have proved for the integral (29):

$$I_-^{n-2} f = (4\pi)^{(2-n)/2} \frac{1}{\Gamma((n-2)/2)} \lim_{r\to o} \sinh^{n-2} r\; M^r f\,. \tag{34}$$

On the other hand, using formula (20) recursively we obtain for $u \in \mathcal{D}(X)$

$$I_-^{n-2}(Q(\Box)u) = u$$

where

$$Q(\Box) = (\Box + (n-3)2)(\Box + (n-5)4)\cdots(\Box + 1(n-2))\,.$$

We combine this with (34) and use the commutativity $\Box M^r = M^r \Box$. This gives

$$u = (4\pi)^{(2-n)/2} \frac{1}{\Gamma((n-2)/2)} \lim_{r\to 0} \sinh^{n-2} r\, Q(\Box) M^r u\,. \tag{35}$$

Here we can for simplicity replace $\sinh r$ by r.

For the case $n = 2$ we have by (29)

$$(I_-^2 f)(y) = \frac{1}{H_2(2)} \int_0^\infty \sinh r F(r)\, dr\,. \tag{36}$$

This integral which in effect only goes from 0 to A is absolutely convergent because our estimate (31) shows (for $n = 2$) that $rF(r)$ is bounded near $r = 0$. But using (20), Lemma 2.10, Theorem 2.11 and Cor. 2.14, we obtain for $u \in \mathcal{D}(X)$

$$\begin{aligned} u &= I_-^2 \Box u = \tfrac{1}{2} \int_0^\infty \sinh r M^r \Box u\, dr \\ &= \tfrac{1}{2} \int_0^\infty \sinh r \Box M^r u\, dr = \tfrac{1}{2} \int_0^\infty \sinh r \left(\frac{d^2}{dr^2} + \coth r \frac{d}{dr} \right) M^r u\, dr \\ &= \tfrac{1}{2} \int_0^\infty \frac{d}{dr} \left(\sinh r \frac{d}{dr} M^r u \right) dr = -\tfrac{1}{2} \lim_{r\to 0} \sinh r \frac{d(M^r u)}{dr}\,. \end{aligned}$$

This is the substitute for (35) in the case $n = 2$.

The spaces G^+/H and G^o/H can be treated in the same manner. We have thus proved the following principal result of this chapter.

Theorem 4.1. *Let X be one of the isotropic Lorentzian manifolds G^-/H, G^o/H, G^+/H. Let κ denote the curvature of X ($\kappa = -1, 0, +1$) and assume $n = \dim X$ to be even, $n = 2m$. Put*

$$Q(\Box) = (\Box - \kappa(n-3)2)(\Box - \kappa(n-5)4)\cdots(\Box - \kappa 1(n-2))\,.$$

Then if $u \in \mathcal{D}(X)$

$$\begin{aligned} u &= c \lim_{r\to 0} r^{n-2} Q(\square)(M^r u)\,, \qquad (n \neq 2) \\ u &= \tfrac{1}{2} \lim_{r\to 0} r \frac{d}{dr}(M^r u) \qquad (n = 2)\,. \end{aligned}$$

Here $c^{-1} = (4\pi)^{m-1}(m-2)!$ *and* $\square$ *is the Laplace-Beltrami operator on* X.

§5 Orbital Integrals and Huygens' Principle

We shall now write out the limit in (35) and thereby derive a statement concerning Huygens' principle for $\square$. As $r \to 0$, $\mathbf{S}_r(o)$ has as limit the boundary $C_R = \partial\mathbf{D}_o - \{o\}$ which is still an H-orbit. The limit

$$\lim_{r\to 0} r^{n-2}(M_r v)(o) \qquad v \in C_c(X - o) \tag{37}$$

is by (31)–(32) a positive H-invariant functional with support in the H-orbit C_R, which is closed in $X - o$. Thus the limit (37) only depends on the restriction $v|C_R$. Hence it is "the" H-invariant measure on C_R and we denote it by μ. Thus

$$\lim_{r\to 0} r^{n-2}\,(M_r v)(o) = \int_{C_R} v(z)\, d\mu(z)\,. \tag{38}$$

To extend this to $u \in \mathcal{D}(X)$, let $A > 0$ be arbitrary and let φ be a "smoothed out" characteristic function of $\operatorname{Exp} B_A$. Then if

$$u_1 = u\varphi\,,\ u_2 = u(1-\varphi)$$

we have

$$\begin{aligned} &\left| r^{n-2}(M^r u)(o) - \int_{C_R} u(z)\, d\mu(z) \right| \\ &\quad \leq \left| r^{n-2}(M^r u_1)(o) - \int_{C_R} u_1(z)\, d\mu(z) \right| + \left| r^{n-2}(M^r u_2)(o) - \int_{C_R} u_2(z)\, d\mu(z) \right|\,. \end{aligned}$$

By (31) the first term on the right is $O(A)$ uniformly in r and by (38) the second tends to 0 as $r \to 0$. Since A is arbitrary (38) holds for $u \in \mathcal{D}(X)$.

Corollary 5.1. *Let* $n = 2m\,(m > 1)$ *and* δ *the delta distribution at* o. *Then*

$$\delta = c\, Q(\square)\mu\,, \tag{39}$$

where $c^{-1} = (4\pi)^{m-1}(m-2)!$.