as long as $\int_{-\infty}^{\infty}|f(u)|\mathrm{d}u < \infty$, which is the only situation we need consider (see proof of lemma in § H.1, Chapter III). Adding, we get

$$f(\mathrm{i}v) = \frac{1}{\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{uf(u)\mathrm{d}u}{u^2+v^2},$$

whence

$$\int_0^{\infty}|f(\mathrm{i}v)|\mathrm{d}v \leqslant \frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{|u||f(u)|}{u^2+v^2}\mathrm{d}v\mathrm{d}u = \frac{1}{2}\int_{-\infty}^{\infty}|f(u)|\mathrm{d}u.$$

Q.E.D.

Lemma. *Let $F(z)$ be analytic in a rectangle $\mathscr{D}$ and continuous up to $\bar{\mathscr{D}}$. If Λ is a straight line joining the midpoints of two opposite sides of $\mathscr{D}$, we have*

$$\int_{\Lambda}|F(z)||\mathrm{d}z| \leqslant \frac{1}{2}\int_{\partial\mathscr{D}}|F(\zeta)||\mathrm{d}\zeta|.$$

Proof.

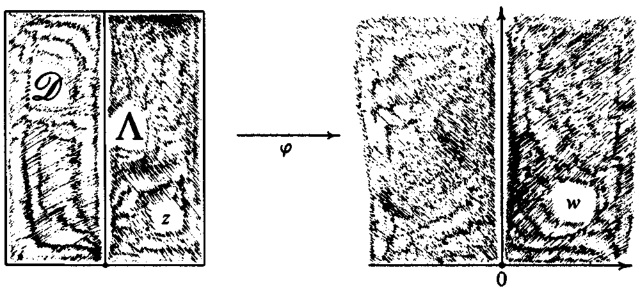

Figure 65

Let φ map $\mathscr{D}$ conformally onto $\Im w > 0$ in such a way that Λ goes onto the *positive imaginary axis*, and, for $z \in \mathscr{D}$ and $w = \varphi(z)$, put

$$f(w) = \frac{F(z)}{\varphi'(z)}.$$

When $w = \varphi(z) \longrightarrow \infty$, $\varphi'(z)$ must tend to ∞ (otherwise the upper half plane would be bounded!), so $f(w)$ must tend to *zero*, $F(z)$ being continuous on $\bar{\mathscr{D}}$. We may therefore apply the previous lemma to f. This yields

$$\int_{\Lambda}|F(z)||\mathrm{d}z| = \int_{\Lambda}\left|\frac{F(z)}{\varphi'(z)}\right||\varphi'(z)\mathrm{d}z| = \int_0^{\infty}|f(\mathrm{i}v)|\mathrm{d}v$$

$$\leqslant \frac{1}{2}\int_{-\infty}^{\infty}|f(u)|\mathrm{d}u = \frac{1}{2}\int_{\partial\mathscr{D}}\left|\frac{F(\zeta)}{\varphi'(\zeta)}\right||\varphi'(\zeta)\mathrm{d}\zeta| = \frac{1}{2}\int_{\partial\mathscr{D}}|F(\zeta)||\mathrm{d}\zeta|,$$

Q.E.D.

Lemma (Beurling). *Let $\mathscr{D}_0$ be the rectangle $\{-a < \Re z < a, 0 < \Im z < h\}$, and let $f \in \mathscr{S}_1(\mathscr{D}_0)$. Then, if $-a < x < a$,*

$$\int_0^h|f(x+\mathrm{i}y)|\mathrm{d}y \leqslant \left(1+\frac{h}{a-|x|}\right)\mathscr{s}_1(f).$$

Proof. Wlog, let $x \geqslant 0$. Taking any small $\delta > 0$ we let $\mathscr{D}_l$, for $0 < l < a - x$, be the rectangle shown in the figure:

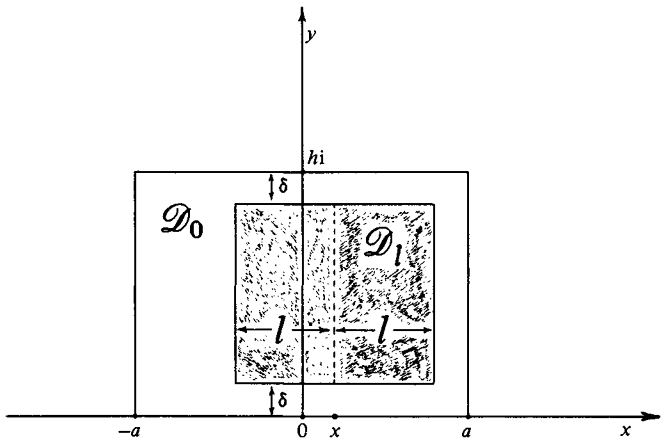

Figure 66

Applying the previous lemma to $\mathscr{D}_l$ we find that

$$\int_\delta^{h-\delta} |f(x+\mathrm{i}y)|\,\mathrm{d}y \;\leqslant\; \frac{1}{2}\int_{\partial\mathscr{D}_l} |f(\zeta)||\mathrm{d}\zeta|.$$

Multiply both sides by $\mathrm{d}l$ and integrate l from $\frac{1}{2}(a-x)$ to $a-x$! We get

$$\frac{a-x}{2}\int_\delta^{h-\delta} |f(x+\mathrm{i}y)|\,\mathrm{d}y \;\leqslant\; \frac{1}{2}\int_{(a-x)/2}^{a-x}\int_{\partial\mathscr{D}_l} |f(\zeta)||\mathrm{d}\zeta|\,\mathrm{d}l.$$

The *lower horizontal sides* of the $\mathscr{D}_l$ contribute at most

$$\frac{a-x}{4}\int_{-(a-x)}^{a-x} |f(x+\mathrm{i}\delta+\xi)|\,\mathrm{d}\xi \;\leqslant\; \frac{a-x}{4}\,\mathfrak{o}_1(f)$$

to the expression on the right, and the *top horizontal sides* of the $\mathscr{D}_l$ contribute a similar amount. The *right vertical sides* give

$$\frac{1}{2}\int_\delta^{h-\delta}\int_{(a-x)/2}^{a-x} |f(x+\mathrm{i}y+l)|\,\mathrm{d}l\,\mathrm{d}y$$

and the *left vertical sides* make a similar contribution. The sum of these last two amounts is

$$\leqslant\; \frac{1}{2}\int_\delta^{h-\delta}\int_{-(a-x)}^{(a-x)} |f(x+\mathrm{i}y+l)|\,\mathrm{d}l\,\mathrm{d}y \;\leqslant\; \tfrac{1}{2}(h-2\delta)\mathfrak{o}_1(f).$$

All told, we thus have

$$\frac{1}{2}\int_{(a-x)/2}^{a-x}\int_{\partial\mathscr{D}_l} |f(\zeta)||\mathrm{d}\zeta|\,\mathrm{d}l \;\leqslant\; \left(\frac{a-x}{2}+\frac{h-2\delta}{2}\right)\mathfrak{o}_1(f),$$

so by the previous relation we see that

$$\int_{\delta}^{h-\delta} |f(x+\mathrm{i}y)|\mathrm{d}y \;\leqslant\; \left(1+\frac{h-2\delta}{a-x}\right)\mathscr{I}_1(f).$$

Making $\delta \to 0$, we obtain the lemma for the case $x \geqslant 0$. Done.

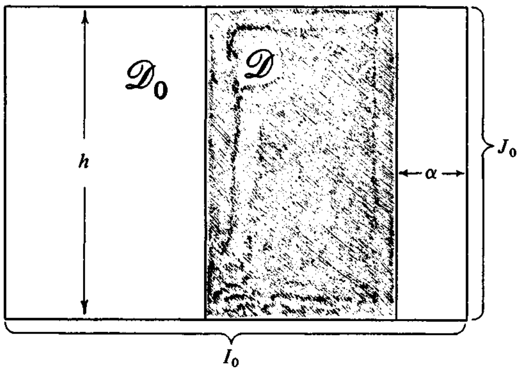

Figure 67

Let $f \in \mathscr{S}_1(\mathscr{D}_0)$, and let $\mathscr{D}$ be a rectangle lying in $\mathscr{D}_0$, in the manner shown – the vertical sides of $\mathscr{D}$ being at *positive distance*, say $\alpha > 0$, from those of $\mathscr{D}_0$. We proceed to investigate the boundary behaviour of f in $\mathscr{D}$.

In order to do this, it is convenient to *take* 0 *as the point of intersection of the diagonals of* $\mathscr{D}$. This setup makes it easy for us to imitate the discussion at the beginning of § F.1, Chapter III.

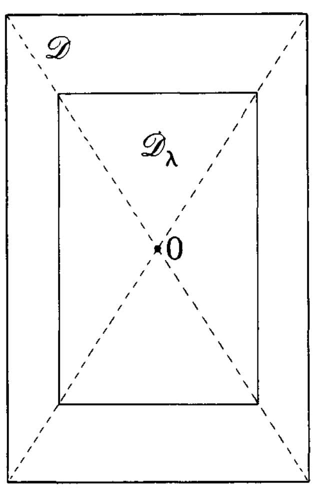

Figure 68

For $0<\lambda<1$ denote by $\mathscr{D}_\lambda$ the rectangle $\{\lambda z\colon z\in\mathscr{D}\}$ (see diagram). $\mathscr{D}_\lambda\subseteq\mathscr{D}$ which, in turn, has the above described disposition inside $\mathscr{D}_0$. Since $f\in\mathscr{S}_1(\mathscr{D}_0)$, we have, by the preceding lemma,

$$\int_{\partial\mathscr{D}_\lambda} |f(\zeta)||\mathrm{d}\zeta| \;\leqslant\; 2\sigma_1(f) \;+\; 2\left(1+\frac{h}{\alpha}\right)\sigma_1(f),$$

calling h the height of $\mathscr{D}_0$. In other words,

$$(*)\qquad \int_{\mathscr{D}} |f(\lambda\zeta)||\mathrm{d}\zeta| \;\leqslant\; \frac{K}{\lambda}$$

for $0<\lambda<1$, where K depends on $\mathscr{D}$ and on f.

Fix any $z_0\in\mathscr{D}$ and let $\lambda<1$. The function $f(\lambda z)$ is certainly analytic (hence *harmonic*!) in $\mathscr{D}$ and *continuous* up to $\partial\mathscr{D}$ (when z ranges over $\bar{\mathscr{D}}$, the argument of $f(\lambda z)$ actually ranges over $\bar{\mathscr{D}}_\lambda$). Therefore, *by the discussion of article* 1,

$$f(\lambda z_0) \;=\; \int_{\partial\mathscr{D}} f(\lambda\zeta)\,\mathrm{d}\omega_{\mathscr{D}}(\zeta, z_0),$$

denoting, as usual, harmonic measure for $\mathscr{D}$ by $\omega_{\mathscr{D}}(\quad, z)$. Since the corners of $\mathscr{D}$ makes angles (of 90°) *less than* 180° *from inside*, we know by article 1 that $\mathrm{d}\omega_{\mathscr{D}}(\zeta, z_0)/|\mathrm{d}\zeta|$ is *bounded* (and indeed *continuous*) on $\partial\mathscr{D}$, and the preceding formula can be rewritten thus:

$$f(\lambda z_0) \;=\; \int_{\partial\mathscr{D}} \frac{\mathrm{d}\omega_{\mathscr{D}}(\zeta, z_0)}{|\mathrm{d}\zeta|}\cdot f(\lambda\zeta)|\mathrm{d}\zeta|.$$

(In order to compute $\mathrm{d}\omega_{\mathscr{D}}(\zeta, z_0)/|\mathrm{d}\zeta|$ explicitly, we would have to resort to elliptic functions!)

We can now argue by $(*)$ that there is a certain complex valued measure μ on $\partial\mathscr{D}$ such that

$$f(\lambda\zeta)|\mathrm{d}\zeta| \longrightarrow \mathrm{d}\mu(\zeta) \quad \mathrm{w}^*$$

when $\lambda\to 1$ through a certain sequence of values, and thereby deduce from the previous relation that

$$(\dagger)\qquad f(z_0) = \int_{\partial\mathscr{D}} \frac{\mathrm{d}\omega_{\mathscr{D}}(\zeta, z_0)}{|\mathrm{d}\zeta|}\,\mathrm{d}\mu(\zeta).$$

(See proof of first theorem in § F.1, Chapter III.) This, of course, holds for any $z_0\in\mathscr{D}$.

Let φ be a conformal mapping of $\mathscr{D}$ onto $\{|w|<1\}$ and let the function F, analytic in the unit disk, be defined by the formula $F(\varphi(z)) = f(z)$, $z\in\mathscr{D}$. If ν is the complex measure on $\{|w|=1\}$ such that $\mathrm{d}\nu(\varphi(\zeta)) = \mathrm{d}\mu(\zeta)$ for ζ varying on $\partial\mathscr{D}$, (†) becomes

$$(\dagger\dagger) \qquad F(w) \;=\; \frac{1}{2\pi}\int_{|\omega|=1} \frac{1-|w|^2}{|w-\omega|^2}\,\mathrm{d}\nu(\omega),$$

$|w| < 1$. The integral on the right therefore represents an *analytic function of* w for $|w| < 1$. From *this* it follows by the celebrated *theorem of the brothers Riesz* that ν *must be absolutely continuous*, i.e.,

$$(\S) \qquad \mathrm{d}\nu(\omega) \;=\; \psi(\omega)|\mathrm{d}\omega|$$

with some L_1*-function* ψ *on the unit circumference.* By Chapter II, § B, and (††) we now have $F(w) \longrightarrow \psi(\omega)$ as $w \xrightarrow{\angle} \omega$ for almost every ω on the unit circumference. Write $g(\zeta) = \psi(\varphi(\zeta))$ for $\zeta \in \partial\mathscr{D}$. Then, going back to $\mathscr{D}$, we see by the discussion in article 1 that

$$f(z) \longrightarrow g(\zeta) \quad \text{as} \quad z \xrightarrow{\angle} \zeta$$

for almost every $\zeta \in \partial\mathscr{D}$.

Plugging (§) into (††) and then returning to (†), we find that

$$f(z_0) \;=\; \int_{\partial\mathscr{D}} \frac{\mathrm{d}\omega_{\mathscr{D}}(\zeta, z_0)}{|\mathrm{d}\zeta|}\, g(\zeta)|\mathrm{d}\zeta|.$$

We have already practically obtained the

Theorem. *Let* $f \in \mathscr{S}_1(\mathscr{D}_0)$. *Then*

$$\lim_{z \xrightarrow{\angle} \zeta} f(z) \text{ which we call } f(\zeta)$$

exists for almost every ζ *on the* horizontal sides *of* $\mathscr{D}_0$.

If $\mathscr{D}$ *is a rectangle in* $\mathscr{D}_0$, *disposed in the manner indicated above,*

$$\int_{\partial\mathscr{D}} |f(\zeta)||\mathrm{d}\zeta| \;<\; \infty,$$

and, for $z \in \mathscr{D}$,

$$f(z) \;=\; \int_{\partial\mathscr{D}} f(\zeta)\,\mathrm{d}\omega_{\mathscr{D}}(\zeta, z).$$

If B_1 *and* B_2 *denote the horizontal sides of* $\mathscr{D}_0$, *we have*

$$\int_{B_1} |f(z)|\mathrm{d}x \;\leqslant\; \mathscr{s}_1(f),$$

$$\int_{B_2} |f(z)|\mathrm{d}x \;\leqslant\; \mathscr{s}_1(f).$$

Proof.

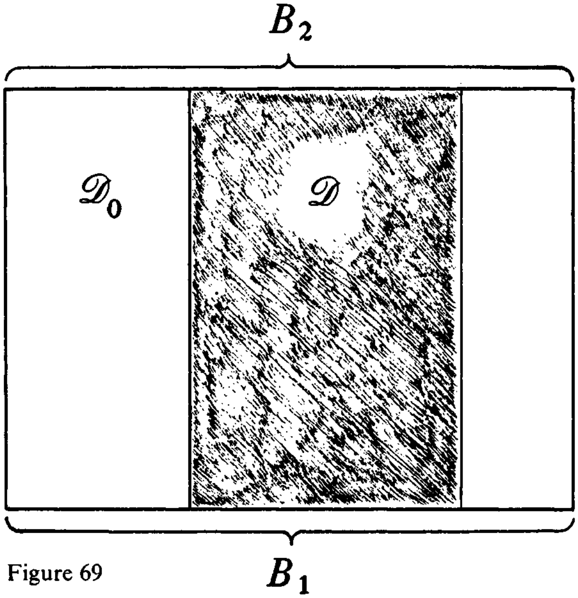

Figure 69

The *first* statement holds because $\lim_{z \angle\to \zeta} f(z)$ exists for almost all ζ on the boundary of *any* rectangle $\mathscr{D}$ lying in $\mathscr{D}_0$ in the manner shown; this we have just seen. Of course, if ζ lies on the *vertical sides* of such a rectangle $\mathscr{D}$, we *know anyway* that $\lim_{z\to\zeta} f(z)$ (without the angle mark!) exists and equals $f(\zeta)$, since those vertical sides lie in $\mathscr{D}_0$, where f is given as analytic. The *second* statement therefore follows from $(*)$ and the *first* one, by Fatou's lemma. (In using $(*)$, one must take 0 as the point of intersection of the diagonals of $\mathscr{D}$.)

In view of what has just been said, the *third* statement is merely another way of expressing the formula immediately preceding this theorem. There remains the *fourth* statement. Considering, for instance, the *upper horizontal side* B_2 of $\mathscr{D}_0$, we have $f(z - i/n) \underset{n}{\longrightarrow} f(z)$ for *almost all* $z \in B_2$ (*first* statement!). Therefore, by Fatou's lemma,

$$\int_{B_2} |f(z)|\,dz \;\leqslant\; \liminf_{n\to\infty} \int_{B_2} \left| f\left(z - \frac{i}{n}\right)\right| dx.$$

The integrals on the right are all $\leqslant \sigma_1(f)$ (by definition), at least as soon as $1/n <$ the height of $\mathscr{D}_0$. We are done.

Theorem. *Let I be any interval properly included within the base of $\mathscr{D}_0$, in the manner shown:*

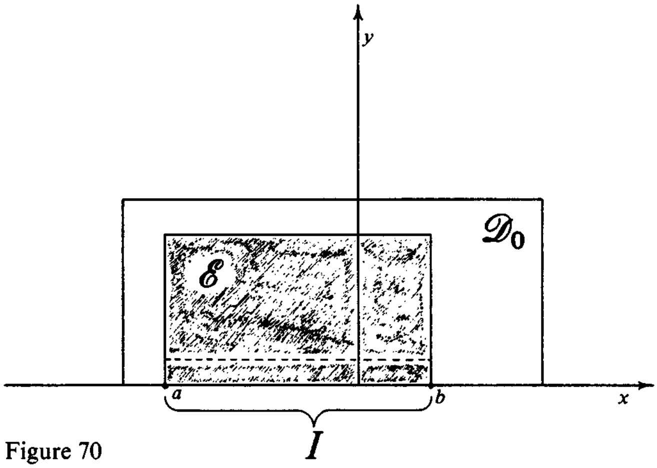

Figure 70

Then, if $f \in \mathscr{S}_1(\mathscr{D}_0)$,

$$\int_I |f(z+\mathrm{i}\delta) - f(z)|\,\mathrm{d}x \longrightarrow 0$$

as $\delta \to 0$.

Proof. To simplify the writing, we take the base of $\mathscr{D}_0$ to lie on the x-axis as shown in the figure.

In view of the preceding theorem, we may assume that, at the *endpoints* a and b of I, $\lim_{z \,\underset{\angle}{\to}\, a} f(z)$ and $\lim_{z \,\underset{\angle}{\to}\, b} f(z)$ *exist and are finite.* (Otherwise, just make I *a little bigger.*) Then, if we construct the rectangle $\mathscr{E} \subseteq \mathscr{D}_0$ with base on I, in the way shown in the figure, $f(z)$ *will be continuous on the top and two vertical sides of* $\mathscr{E}$, *right up to where the latter meet* I. And by *exactly the same argument* as the one used to establish the *third* statement of the preceding theorem, we can see that

$$f(z) = \int_{\partial\mathscr{E}} f(\zeta)\,\mathrm{d}\omega_{\mathscr{E}}(\zeta, z) \quad \text{for} \quad z \in \mathscr{E}.$$

Now let $\varepsilon > 0$ be given, and take a *continuous function* $g(\zeta)$ defined on $\partial\mathscr{E}$ which *coincides with* $f(\zeta)$ *on the top and vertical sides of* $\mathscr{E}$ and is specified on I in such a way that

$$\int_I |f(\xi) - g(\xi)|\,\mathrm{d}\xi < \varepsilon,$$

For $z \in \mathscr{E}$, put

$$g(z) = \int_{\partial\mathscr{E}} g(\zeta)\,\mathrm{d}\omega_{\mathscr{E}}(\zeta, z);$$

$g(z)$ is at least *harmonic* in $\mathscr{E}$ (N.B. *not* necessarily *analytic* there!), and, by the discussion in article 1, *continuous up to* $\partial\mathscr{E}$, *where it takes the boundary values* $g(\zeta)$.

For $x \in I$ and small $\delta > 0$,

$$f(x+\mathrm{i}\delta) - f(x) = f(x+\mathrm{i}\delta) - g(x+\mathrm{i}\delta) + g(x+\mathrm{i}\delta) - g(x) + g(x) - f(x).$$

We are interested in showing that $\int_I |f(x+\mathrm{i}\delta) - f(x)|\mathrm{d}x$ is *small* if $\delta > 0$ is small enough. We already know that $\int_I |g(x) - f(x)|\mathrm{d}x < \varepsilon$, and, by *continuity* of g on $\bar{\mathscr{E}}$, $\int_I |g(x+\mathrm{i}\delta) - g(x)|\mathrm{d}x < \varepsilon$ if $\delta > 0$ is small. We will therefore be *done* if we verify that

$$\int_I |g(x+\mathrm{i}\delta) - f(x+\mathrm{i}\delta)|\mathrm{d}x \;<\; \varepsilon.$$

Since $f(\zeta) = g(\zeta)$ on $\partial\mathscr{E} \sim I$,

$$f(x+\mathrm{i}\delta) - g(x+\mathrm{i}\delta) = \int_I (f(\xi) - g(\xi))\mathrm{d}\omega_{\mathscr{E}}(\xi, x+\mathrm{i}\delta).$$

However, $\mathscr{E}$ lies in the upper half-plane and I on the real axis, so, by the *principle of extension of domain* used in article 1, for $x + \mathrm{i}\delta \in \mathscr{E}$,

$$\mathrm{d}\omega_{\mathscr{E}}(\xi, x+\mathrm{i}\delta) \;\leqslant\; \frac{1}{\pi}\frac{\delta\mathrm{d}\xi}{(x-\xi)^2 + \delta^2}$$

on I, the right-hand expression being the differential of *harmonic measure* for $\{\Im z > 0\}$ as seen from $x + \mathrm{i}\delta$. Thus, for $x \in I$,

$$|f(x+\mathrm{i}\delta) - g(x+\mathrm{i}\delta)| \;\leqslant\; \frac{1}{\pi}\int_I |f(\xi) - g(\xi)|\frac{\delta\mathrm{d}\xi}{(x-\xi)^2 + \delta^2}.$$

And

$$\begin{aligned} &\int_I |f(x+\mathrm{i}\delta) - g(x+\mathrm{i}\delta)|\mathrm{d}x \\ &\leqslant \frac{1}{\pi}\int_I \int_{-\infty}^{\infty} |f(\xi) - g(\xi)|\frac{\delta}{(x-\xi)^2 + \delta^2}\mathrm{d}x\,\mathrm{d}\xi \\ &= \int_I |f(\xi) - g(\xi)|\mathrm{d}\xi \;<\; \varepsilon. \end{aligned}$$

This does it.

Corollary. *Let* $f \in \mathscr{S}_1(\mathscr{D}_0)$ *and let* $G(z)$ *be any function analytic in a region including the closure of a rectangle* $\mathscr{E}$ *like the one used above lying in* $\mathscr{D}_0$*'s* *interior. Then*

$$\int_{\partial\mathscr{E}} G(\zeta)f(\zeta)\mathrm{d}\zeta \;=\; 0.$$

Proof. Use Cauchy's theorem for the rectangles with the dotted base together with the above result:

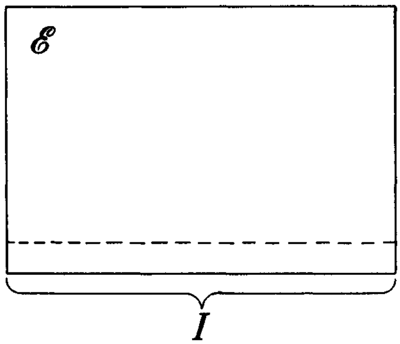

Figure 71

Note that the integrals along the *vertical sides* of $\mathscr{E}$ are absolutely convergent by the *third* lemma of this article.

We need one more result – a Jensen inequality for rectangles $\mathscr{E}$ like the one used above.

Theorem. *Let $f \in \mathscr{S}_1(\mathscr{D}_0)$, and let $\mathscr{E}$ be a rectangle like the one shown:*

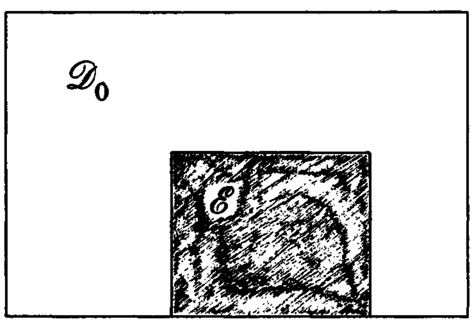

Figure 72

Then, for $z \in \mathscr{E}$,

$$\log|f(z)| \;\leqslant\; \int_{\partial\mathscr{E}} \log|f(\zeta)|\mathrm{d}\omega_{\mathscr{E}}(\zeta, z).$$

Proof. This would just be a restatement of the theorem on harmonic estimation from article 1, except that $f(z)$ is not necessarily continuous up to the base of $\mathscr{E}$. There are several ways of getting around the difficulty caused by this lack of continuity; in one such we first map $\mathscr{E}$ conformally onto the unit disk and then use properties of the space H_1. Functions in H_1 can be expressed as products of inner and outer factors, so Jensen's inequality holds for them.

In order to keep the exposition as nearly self-contained as possible, we give a different argument, based on *Szegő's theorem* (§A, Chapter II!), whose idea goes back to Helson and Lowdenslager.

Given $z_0 \in \mathscr{E}$, take a conformal mapping φ onto $\{|w|<1\}$ that sends z_0 to 0, and define a function $F(w)$ analytic in the unit disk by means of the formula

$$F(\varphi(z)) = f(z), \quad z \in \mathscr{E}.$$

The relation

$$f(z) \;=\; \int_{\partial\mathscr{E}} f(\zeta)\,\mathrm{d}\omega_{\mathscr{E}}(\zeta, z), \quad z \in \mathscr{E},$$

used in proving the above theorem, goes over into

$$F(w) \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-|w|^2}{|w-\mathrm{e}^{\mathrm{i}\tau}|^2}\,F(\mathrm{e}^{\mathrm{i}\tau})\,\mathrm{d}\tau,$$

with $F(\mathrm{e}^{\mathrm{i}\tau}) = f(\varphi^{-1}(\mathrm{e}^{\mathrm{i}\tau}))$ defined almost everywhere on the unit circumference and in L_1 (see discussion preceding the first theorem of this article).

From this last relation, we have

$$\int_0^{2\pi} |F(\rho\mathrm{e}^{\mathrm{i}\vartheta}) - F(\mathrm{e}^{\mathrm{i}\vartheta})|\,\mathrm{d}\vartheta \longrightarrow 0$$

as $\rho \to 1$. Also, for each $\rho < 1$, $\int_0^{2\pi} \mathrm{e}^{in\vartheta}F(\rho\mathrm{e}^{\mathrm{i}\vartheta})\,\mathrm{d}\vartheta = 0$ when $n = 1, 2, 3, \ldots$ by *Cauchy's theorem*. Hence

$$\int_0^{2\pi} \mathrm{e}^{in\vartheta}F(\mathrm{e}^{\mathrm{i}\vartheta})\,\mathrm{d}\vartheta \;=\; 0$$

for $n = 1, 2, 3, \ldots$, and, finally,

$$\frac{1}{2\pi}\int_0^{2\pi}\left(1 + \sum_{n>0} A_n\mathrm{e}^{in\vartheta}\right)F(\mathrm{e}^{\mathrm{i}\vartheta})\,\mathrm{d}\vartheta \;=\; F(0)$$

for *any* finite sum $\sum_{n>0} A_n\mathrm{e}^{in\vartheta}$.

Thus,

$$|F(0)| \;\leqslant\; \frac{1}{2\pi}\int_0^{2\pi}\left|1 + \sum_{n>0} A_n\mathrm{e}^{in\vartheta}\right| |F(\mathrm{e}^{\mathrm{i}\vartheta})|\,\mathrm{d}\vartheta$$

for all such finite sums. By Szegő's theorem, the *infimum* of the expressions on the right is

$$\exp\left(\frac{1}{2\pi}\int_0^{2\pi}\log|F(e^{i\vartheta})|\,d\vartheta\right).$$

Therefore,

$$\log|F(0)| \leqslant \frac{1}{2\pi}\int_0^{2\pi}\log|F(e^{i\vartheta})|\,d\vartheta,$$

or, in terms of f and $z_0 = \varphi^{-1}(0)$:

$$\log|f(z_0)| \leqslant \int_{\partial\mathscr{E}}\log|f(\zeta)|\,d\omega_{\mathscr{E}}(\zeta, z_0).$$

That's what we wanted to prove.

5. Beurling's quasianalyticity theorem for L_p approximation by functions in $\mathscr{S}_p(\mathscr{D}_0)$.

Being now in possession of the previous article's somewhat *ad hoc* material, we are able to look at approximation by functions in $\mathscr{S}_p(\mathscr{D}_0)$ $(p \geqslant 1)$ and to prove a result about such approximation analogous to the one of article 3.

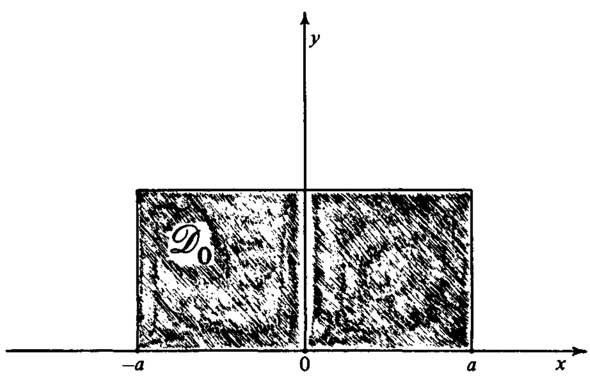

Figure 73

Throughout the following discussion, we work with a certain rectangular domain $\mathscr{D}_0$ whose base is an interval on the real axis which we take, wlog, as $[-a, a]$. If $p \geqslant 1$, $\mathscr{S}_p(\mathscr{D}_0) \subseteq \mathscr{S}_1(\mathscr{D}_0)$, so we know by the *first* theorem of the previous article that, for functions f in $\mathscr{S}_p(\mathscr{D}_0)$, the non-tangential boundary values $f(x)$ exist for almost every x on $[-a, a]$. As in the proof of that theorem we see by Fatou's lemma (there applied in the case $p = 1$) that

$$\int_{-a}^{a} |f(x)|^p \mathrm{d}x \;\leqslant\; (\mathscr{s}_p(f))^p, \qquad f \in \mathscr{S}_p(\mathscr{D}_0).$$

The 'restrictions' of functions $f \in \mathscr{S}_p(\mathscr{D}_0)$ to $[-a, a]$ thus belong to $L_p(-a, a)$, and we may use them to try to approximate arbitrary members of $L_p(-a, a)$ in the norm of that space.

In analogy with article 3, we define the L_p *approximation index* $M_p(A)$ *for any given* $\varphi \in L_p(-a, a)$ (and the rectangle $\mathscr{D}_0$) as follows:

$\mathrm{e}^{-M_p(A)}$ *is the infimum of* $\sqrt[p]{\int_{-a}^{a} |\varphi(x) - f(x)|^p \mathrm{d}x}$

for $f \in \mathscr{S}_p(\mathscr{D}_0)$ *with* $\mathscr{s}_p(f) \leqslant \mathrm{e}^A$.

$M_p(A)$ is obviously an *increasing* function of A, and we have the following

Theorem (Beurling). *Let* $\varphi \in L_p(-a, a)$, *and let its* L_p *approximation index* $M_p(A)$ (*for* $\mathscr{D}_0$) *satisfy*

$$\int_1^{\infty} \frac{M_p(A)}{A^2} \mathrm{d}A \;=\; \infty.$$

If $\varphi(x)$ *vanishes on a set of* positive measure *in* $[-a, a]$, *then* $\varphi(x) \equiv 0$ *a.e. on* $[-a, a]$.

Proof. We first carry out some *preliminary reductions.*

We have $\mathscr{S}_p(\mathscr{D}_0) \subseteq \mathscr{S}_1(\mathscr{D}_0)$, $L_p(-a, a) \subseteq L_1(-a, a)$, and, by Hölder's inequality, $\mathscr{s}_1(f) \leqslant a^{(p-1)/p}\mathscr{s}_p(f)$ and $\|\varphi - f\|_1 \leqslant a^{(p-1)/p}\|\varphi - f\|_p$ for $f \in \mathscr{S}_p(\mathscr{D}_0)$ and $\varphi \in L_p(-a, a)$. (We write $\| \;\; \|_p$ for the L_p norm on $[-a, a]$). From these facts it is clear that, if $\varphi \in L_p(-a, a)$ has L_p approximation index $M_p(A)$, *the* L_1 *approximation index* $M_1(A)$ *of* $a^{p/(p-1)}\varphi$ (*sic*!) *is* $\geqslant M_p(A)$. *It is therefore enough to prove the theorem for* $p = 1$, *for it will then follow for all values of* $p > 1$.

Suppose then that $\int_1^{\infty}(M_1(A)/A^2)\mathrm{d}A = \infty$ with $M_1(A)$ the L_1 approximation index for $\varphi \in L_1(-a, a)$, and that φ vanishes on a set of positive measure in $[-a, a]$. In order to prove that $\varphi \equiv 0$ a.e. on $[-a, a]$, it is *enough* to show that it *vanishes a.e. on some interval* $J \subseteq [-a, a]$ with positive length.

To see this, take any very small *fixed* $\eta > 0$ and write

$$\varphi_\eta(x) \;=\; \frac{1}{2\eta}\int_{-\eta}^{\eta} \varphi(x + t)\,\mathrm{d}t$$

for $-a+\eta \leqslant x \leqslant a-\eta$. $\varphi_\eta(x)$ is then *continuous* on the interval $[-a+\eta,\ a-\eta]$, and *vanishes identically on an interval of positive length therein* as long as $2\eta < |J|$. Corresponding to any $f \in \mathscr{S}_1(\mathscr{D}_0)$ we also form the function

$$f_\eta(z) \;=\; \frac{1}{2\eta}\int_{-\eta}^{\eta} f(z+t)\,\mathrm{d}t;$$

let us check that $f_\eta(z)$ is *analytic* in the rectangle $\mathscr{D}_\eta$ with base $[-a+2\eta,\ a-2\eta]$ having the same height as $\mathscr{D}_0$, and is *continuous* on $\mathscr{D}_\eta$.

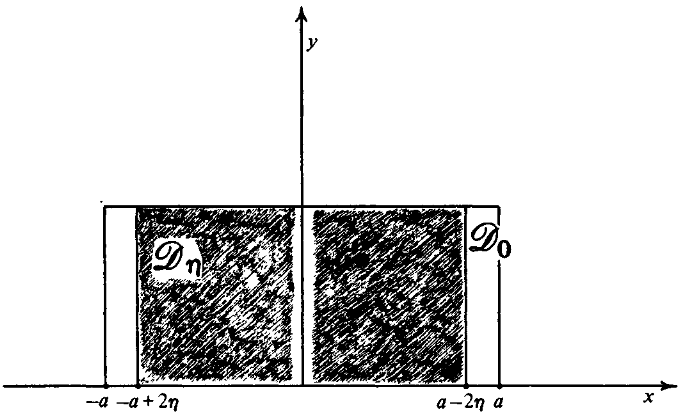

Figure 74

The *analyticity* of $f_\eta(z)$ in $\mathscr{D}_\eta$ is *clear*; so is *continuity* up to the *vertical sides* of $\mathscr{D}_\eta$. The boundary-value function $f(x)$ belongs to $L_1(-a,a)$, so $f_\eta(x)$ *is* continuous *on* $[-a+2\eta,\ a-2\eta]$. Let, then, $-a+2\eta \leqslant x_0 \leqslant a-2\eta$, and suppose that x, also on that closed interval, is *near* x_0 and that $y>0$ is *small*. We have $|f_\eta(x_0)-f_\eta(x+\mathrm{i}y)| \leqslant |f_\eta(x_0)-f_\eta(x)| + |f_\eta(x)-f_\eta(x+\mathrm{i}y)|$. The first term on the right is small if x is close enough to x_0. The second is

$$\leqslant\; \frac{1}{2\eta}\int_{x-\eta}^{x+\eta} |f(\xi)-f(\xi+\mathrm{i}y)|\,\mathrm{d}\xi \;\leqslant\; \frac{1}{2\eta}\int_{-a+\eta}^{a-\eta} |f(\xi)-f(\xi+\mathrm{i}y)|\,\mathrm{d}\xi$$

which, by the *second* theorem of the preceding article, tends to *zero* (independently of x!) as $y \to 0$. Thus $f_\eta(x+\mathrm{i}y) \longrightarrow f_\eta(x_0)$ as $x+\mathrm{i}y \longrightarrow x_0$ from within $\mathscr{D}_\eta$, and *continuity of f_η up to the lower horizontal side of $\mathscr{D}_\eta$ is established. Continuity* of f_η up to the *upper horizontal side* of $\mathscr{D}_\eta$ follows in like manner, so $f_\eta(z)$ is continuous on $\bar{\mathscr{D}}_\eta$.

The functions f_η are thus of the kind used in article 3 to *uniformly approximate continuous functions* given on $[-a+2\eta,\ a+2\eta]$. By definition of $M_1(A)$, we can find an f in $\mathscr{S}_1(\mathscr{D}_0)$ with $\mathfrak{o}_1(f) \leqslant e^A$ and $\int_{-a}^{a} |\varphi(x) - f(x)| dx \leqslant 2e^{-M_1(A)}$. With this f, $|f_\eta(z)| \leqslant (1/2\eta)e^A$ for $z \in \bar{\mathscr{D}}_\eta$ and

$$|\varphi_\eta(x) - f_\eta(x)| \;\leqslant\; \frac{1}{\eta} e^{-M_1(A)}$$

on $[-a+2\eta,\ a-2\eta]$. The *uniform approximation index* $M(A)$ *for* $\eta\varphi_\eta$ (and the domain $\mathscr{D}_\eta$) *is thus* $\geqslant M_1(A)$. Therefore, under the hypothesis of the present theorem,

$$\int_1^\infty \frac{M(A)}{A^2} dA \;=\; \infty,$$

so, since $\varphi_\eta(x)$ *vanishes identically* on an *interval of positive length* in $[-a+2\eta,\ a-2\eta]$ (when $\eta > 0$ is small enough) *we have*

$$\varphi_\eta(x) \equiv 0, \quad -a+2\eta \leqslant x \leqslant a-2\eta$$

by the theorem of article 3.

However, as $\eta \to 0$, $\varphi_\eta(x) \longrightarrow \varphi(x)$ a.e. on $(-a, a)$. From what has just been shown we conclude, then, that $\varphi(x) \equiv 0$ a.e. on $(-a, a)$ *if it vanishes* a.e. on an *interval* J of positive length lying therein, provided that

$$\int_1^\infty \frac{M_1(A)}{A^2} dA \;=\; \infty.$$

Our task has thus finally boiled down to the following one. *Given* $\varphi \in L_1(-a, a)$ *with* L_1 *approximation index* $M_1(A)$ (for $\mathscr{D}_0$) *such that*

$$\int_1^\infty \frac{M_1(A)}{A^2} dA \;=\; \infty,$$

prove that φ *vanishes a.e. on an interval of positive length in* $(-a, a)$ *if it vanishes on a set of positive measure therein.*

Let us proceed. It is easy to see that the *increasing* function $M_1(A)$ is continuous (in the extended sense) – that's because, if $\lambda < 1$ is close to 1, λf approximates φ *almost as well* as f does in $L_1(-a, a)$. Since $\int_1^\infty (M_1(A)/A^2) dA = \infty$ we may therefore, starting with a suitable $A_1 > 0$, get an increasing sequence of numbers A_n tending to ∞ such that

$$M_1(A_{n+1}) = 2M_1(A_n).*$$

Assume henceforth that $\varphi(x) = 0$ on the closed set $E_0 \subseteq [-a, a]$ with

* We are allowing for the possibility that $M_1(A) \equiv \infty$ for large values of A; this happens when $\varphi(x)$ actually coincides with a function in $\mathscr{S}_p(\mathscr{D}_0)$ on $(-a, a)$, and then it is necessary to take A_1 with $M_1(A_1) = \infty$. We will, in any event, need to have A_1 *large* – see the following page.

$|E_0| > 0$*. For each $A > 0$ there is an $f \in \mathscr{S}_1(\mathscr{D}_0)$ with $\mathfrak{s}_1(f) \leqslant e^A$ and

$$\int_{-a}^{a} |\varphi(x) - f(x)| dx \leqslant 2e^{-M_1(A)}$$

In particular,

$$\int_{E_0} |f(x)| dx \leqslant 2e^{-M_1(A)},$$

so, if

$$\Delta_A = \{x \in E_0 \colon |f(x)| > e^{-M_1(A)/2}\},$$

we have $|\Delta_A| \leqslant 2e^{-M_1(A)/2}$. Taking the sequence of numbers A_n just described, we thus get

$$\left|\bigcup_n \Delta_{A_n}\right| \leqslant 2\sum_n e^{-M_1(A_n)/2} = 2\sum_1^\infty e^{-2^{n-1}M_1(A_1)}.$$

We can choose A_1 large enough so that this sum is

$$< \frac{|E_0|}{2};$$

then the set

$$E = E_0 \sim \left(\bigcup_n \Delta_n\right)$$

has measure $> |E_0|/2$, and, by its *construction*, for each n there is an $f_n \in \mathscr{S}_1(\mathscr{D}_0)$ with $\mathfrak{s}_1(f_n) \leqslant e^{A_n}$,

$$\int_{-a}^{a} |\varphi(x) - f_n(x)| dx \leqslant 2e^{-M_1(A_n)},$$

and

$$|f_n(x)| \leqslant e^{-M_1(A_n)/2}$$

for $x \in E$.

Take now a number b, $0 < b < a$, sufficiently close to a so that

$$|E \cap [-b, b]| > 0,$$

and construct the rectangle $\mathscr{D}$ with base on $[-b, b]$, lying within $\mathscr{D}_0$ in the manner shown:

* where $|E|$ denotes the Lebesgue measure of $E \subseteq \mathbb{R}$

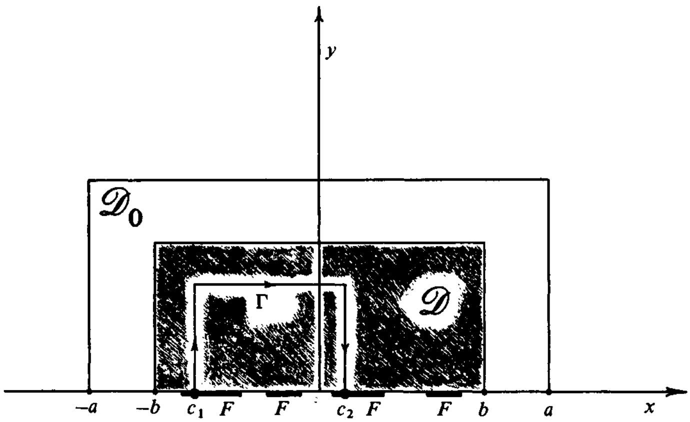

Figure 75

Take a *closed subset F of $E \cap (-b,\ b)$ having positive measure*; this set F will remain fixed during the following discussion.

As we saw at the end of article 1,

$$\omega_{\mathscr{D}}(F, x+\mathrm{i}y) \longrightarrow 1$$

as $y \to 0+$ for almost every $x \in F$. Let c_1 and c_2, $c_1 < c_2$, be *two such* x's *for which this is true*. We are going to show that $\varphi(x) = 0$ a.e. for $c_1 \leqslant x \leqslant c_2$; according to what has been said above, *this is all we need to do* to finish the proof of our theorem.

The desired vanishing of φ will follow if

$$\Phi(\lambda) \;=\; \int_{c_1}^{c_2} \mathrm{e}^{\mathrm{i}\lambda x}\varphi(x)\,\mathrm{d}x$$

is identically zero. Φ is, however, *an entire function of exponential type bounded on the real axis.* Hence, by §G.2 *of Chapter* III, $\Phi \equiv 0$ provided that

$$\int_1^\infty \frac{1}{\lambda^2}\log\left|\frac{1}{\Phi(\lambda)}\right|\mathrm{d}\lambda \;=\; \infty.$$

We proceed to verify this relation. The reasoning here resembles that of article 2, but is more complicated.

Take one of the functions f_n (later on, n will be made to depend on λ), and write

$$\Phi(\lambda) \;=\; \int_{c_1}^{c_2} \mathrm{e}^{\mathrm{i}\lambda x}(\varphi(x) - f_n(x))\,\mathrm{d}x + \int_{c_1}^{c_2} \mathrm{e}^{\mathrm{i}\lambda x} f_n(x)\,\mathrm{d}x \;=\; \mathrm{I} \;+\; \mathrm{II}, \text{ say.}$$

Here, for $\lambda > 0$,

$$|\mathrm{I}| \leqslant \int_{c_1}^{c_2} |\varphi(x) - f_n(x)| \mathrm{d}x \leqslant 2\mathrm{e}^{-M_1(A_n)},$$

and the real work is to estimate II.

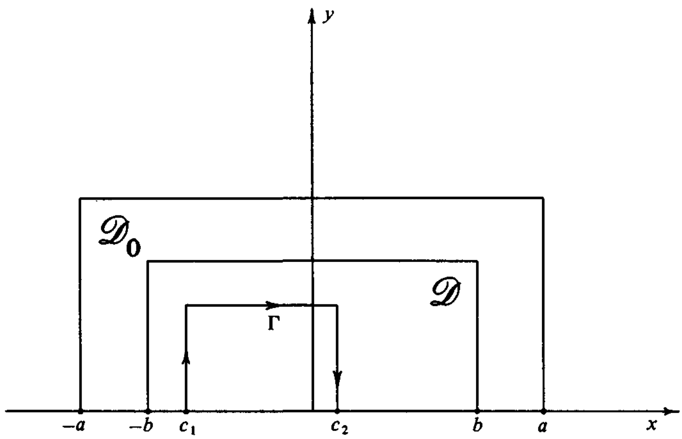

Figure 76

Let Γ be a fixed contour in $\mathscr{D}$ consisting of three sides of a rectangle with base on $[c_1, c_2]$. Because $f_n \in \mathscr{S}_1(\mathscr{D}_0)$, we have

$$\int_{c_1}^{c_2} \mathrm{e}^{\mathrm{i}\lambda x} f_n(x) \mathrm{d}x = \int_\Gamma \mathrm{e}^{\mathrm{i}\lambda z} f_n(z) \mathrm{d}z$$

by the *corollary* to the *second* theorem of the previous article. In order to estimate the integral on the right, we use the inequality

$$\log |f_n(z)| \leqslant \int_{\partial\mathscr{D}} \log |f_n(\zeta)| \mathrm{d}\omega_{\mathscr{D}}(\zeta, z), \qquad z \in \mathscr{D},$$

furnished by the *third* theorem in the preceding article. This we further break up so as to obtain the following for $z \in \mathscr{D}$:

$$(*) \qquad \log |f_n(z)| \leqslant \int_\Pi \log |f_n(\zeta)| \mathrm{d}\omega_{\mathscr{D}}(\zeta, z) + \int_F \log |f_n(\zeta)| \mathrm{d}\omega_{\mathscr{D}}(\zeta, z)$$
$$+ \int_{(-b,b)\sim F} \log |f_n(\zeta)| \mathrm{d}\omega_{\mathscr{D}}(\zeta, z).$$

Here, Π denotes $\partial\mathscr{D} \sim (-b, b)$, i.e., the *vertical* and *top horizontal sides* of $\mathscr{D}$:

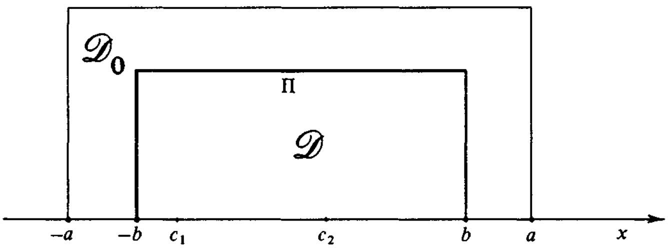

Figure 77

Consider the *first* integral on the right in $(*)$. It equals a certain function $u(z)$ harmonic in $\mathscr{D}$. Take any *harmonic conjugate* $v(z)$ of $u(z)$ for the region $\mathscr{D}$ and put

$$g_n(z) \;=\; e^{u(z)+iv(z)}, \qquad z\in\mathscr{D};$$

the function $g_n(z)$ is *analytic* in $\mathscr{D}$ and we have

$$\log|g_n(z)| \;=\; \int_\Pi \log|f_n(\zeta)|\,d\omega_{\mathscr{D}}(\zeta, z), \qquad z\in\mathscr{D}.$$

In the same way we get functions $h_n(z)$ and $k_n(z)$ analytic in $\mathscr{D}$ with

$$\log|h_n(z)| \;=\; \int_F \log|f_n(\zeta)|\,d\omega_{\mathscr{D}}(\zeta, z), \qquad z\in\mathscr{D},$$

and

$$\log|k_n(z)| \;=\; \int_{(-b,b)\sim F} \log|f_n(\zeta)|\,d\omega_{\mathscr{D}}(\zeta, z), \qquad z\in\mathscr{D}.$$

In terms of these functions, $(*)$ becomes

$$(\dagger) \qquad |f_n(z)| \;\leqslant\; |g_n(z)|\,|h_n(z)|\,|k_n(z)|, \qquad z\in\mathscr{D}.$$

Our idea now is to estimate $\sup_{z\in\Gamma}|g_n(z)|$, $\sup_{z\in\Gamma}|h_n(z)|$ and $\int_\Gamma|k_n(z)|\,|dz|$ in order to get a bound on $\int_\Gamma e^{i\lambda z}f_n(z)\,dz$ for $\lambda>0$. The *third* of these quantities will give us the most trouble.

We first look at $|g_n(z)|$, $z\in\Gamma$. For ζ on Π, the Poisson kernel $d\omega_{\mathscr{D}}(\zeta, z)/|d\zeta|$ goes to *zero* when $z\in\mathscr{D}$ tends to any point of $(-b, b)$, and does so *uniformly* for $\zeta\in\Pi$ and z tending to any point of $[c_1, c_2]$. From this we see, by reflecting the harmonic function $d\omega_{\mathscr{D}}(\zeta, z)/|d\zeta|$ of z across $(-b, b)$, that there is a certain constant C, depending *only* on the geometric configuration of Γ and $\mathscr{D}$, such that

$$\frac{d\omega_{\mathscr{D}}(\zeta, z)}{|d\zeta|} \;\leqslant\; C\,\Im z, \qquad z\in\Gamma, \quad \zeta\in\Pi.$$

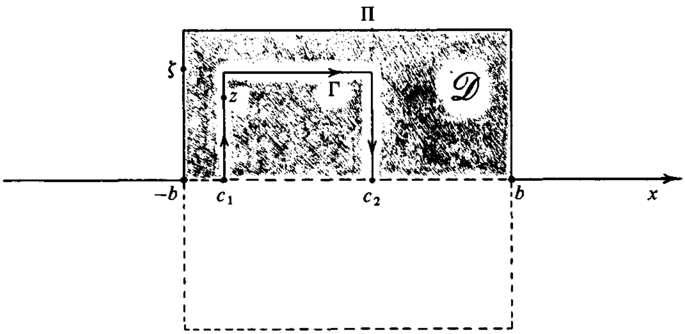

Figure 78

Substituting this into the above formula for $\log|g_n(z)|$, we get

$$\log|g_n(z)| \leqslant C\Im z \int_\Pi \log|f_n(\zeta)||d\zeta|$$

for $z \in \Gamma$, whence, by the inequality between arithmetic and geometric means,*

$$|g_n(z)| \leqslant \left(\frac{1}{|\Pi|}\right)^{|\Pi|C\Im z} \left(\int_\Pi |f_n(\zeta)||d\zeta|\right)^{|\Pi|C\Im z},$$

$z \in \Gamma$. Write now $|\Pi|C = B$. Then we have

$$|g_n(z)| \leqslant \text{const.}\left(\int_\Pi |f_n(\zeta)||d\zeta|\right)^{B\Im z}, \qquad z \in \Gamma,$$

where the constant is independent of n. Here, $f_n \in \mathscr{S}_1(\mathscr{D}_0)$ and $\mathscr{s}_1(f_n) \leqslant e^{A_n}$. Thence, by the *third* lemma of the preceding article, if h denotes the height of $\mathscr{D}_0$,

$$\int_\Pi |f_n(\zeta)||d\zeta| \leqslant \mathscr{s}_1(f_n) + \left\{1 + \frac{h}{a - |c_1|}\right\}\mathscr{s}_1(f_n)$$
$$+ \left\{1 + \frac{h}{a - |c_2|}\right\}\mathscr{s}_1(f_n) \leqslant Ke^{A_n}$$

with a constant K independent of n. Plugging this into the previous relation, we find that

$$|g_n(z)| \leqslant \text{const.}e^{BA_n\Im z}, \qquad z \in \Gamma,$$

the constant in front on the right being independent of n.

To estimate $|h_n(z)|$ on Γ we simply use the fact that

$$|f_n(\xi)| \leqslant e^{-M_1(A_n)/2} \quad \text{for} \quad \xi \in F \subseteq E$$

* in the following relation, $|\Pi|$ is used to designate the *linear measure* (length) of Π

and get

$$|h_n(z)| = \exp\left(\int_F \log|f_n(\zeta)|\,d\omega_{\mathscr{D}}(\zeta, z)\right) \leqslant e^{-\omega_{\mathscr{D}}(F,z)M_1(A_n)/2}, \qquad z\in\mathscr{D}.$$

Substituting the estimates for $|g_n(z)|$ and $|h_n(z)|$ which we have already found into (†), we obtain

$$(\ddagger) \qquad |e^{i\lambda z}f_n(z)| \leqslant \text{const.}e^{(BA_n-\lambda)\Im z}e^{-\omega_{\mathscr{D}}(F,z)M_1(A_n)/2}|k_n(z)|$$

for $z\in\Gamma$. Thus, in order to get a good upper bound for

$$|\mathrm{II}| = \left|\int_\Gamma e^{i\lambda z}f_n(z)\,dz\right|,$$

it suffices to find one for $\int_\Gamma |k_n(z)||dz|$ which is independent of n.

We have

$$\int_{-a}^{a} |\varphi(x) - f_n(x)|\,dx \leqslant 2e^{-M_1(A_n)}.$$

Wlog,

$$\int_{-a}^{a} |\varphi(x)|\,dx \leqslant \frac{1}{2},$$

therefore, for all sufficiently large n,

$$(\dagger\dagger) \qquad \int_{-a}^{a} |f_n(x)|\,dx \leqslant 1.$$

We henceforth limit our attention to the large values of n for which this relation is true.

The formula for $\log|k_n(z)|$ can be rewritten

$$\log|k_n(z)| = \int_{\partial\mathscr{D}} \log P(\zeta)\,d\omega_{\mathscr{D}}(\zeta, z),$$

where

$$P(\zeta) = \begin{cases} |f_n(\zeta)|, & \zeta\in(-b,b)\sim F, \\ 1 \text{ elsewhere on } \partial\mathscr{D}. \end{cases}$$

From this, by the inequality between arithmetic and geometric means, we get

$$|k_n(z)| \leqslant \int_{\partial\mathscr{D}} P(\zeta)d\omega_{\mathscr{D}}(\zeta, z) \leqslant 1 + \int_{-b}^{b} |f_n(\xi)|d\omega_{\mathscr{D}}(\xi, z), \qquad z\in\mathscr{D}.$$

However, for $-b<\xi<b$, we can apply the principle of extension of

domain to compare $d\omega_{\mathscr{D}}(\xi, z)$ with harmonic measure for $\{\Im z > 0\}$ as we did in proving the *second* theorem of the preceding article. This gives us

$$d\omega_{\mathscr{D}}(\xi, z) \;\leqslant\; \frac{1}{\pi}\frac{\Im z\, d\xi}{|z-\xi|^2}, \qquad -b < \xi < b,$$

so the previous inequality becomes

$$|k_n(z)| \;\leqslant\; 1 \;+\; \frac{1}{\pi}\int_{-b}^{b} \frac{\Im z}{|z-\xi|^2}\,|f_n(\xi)|\,d\xi, \qquad z \in \mathscr{D}.$$

Denoting by h' the *height* of $\mathscr{D}$, and using this last relation together with Fubini's theorem, we see that, for $0 < y < h'$,

$$\int_{-b}^{b} |k_n(x+iy)|\,dx \;\leqslant\; 2b + \int_{-b}^{b} |f_n(\xi)|\,d\xi \;\leqslant\; 2b+1$$

(in view of (††)).

In other words, $k_n(z) \in \mathscr{S}_1(\mathscr{D})$ (*sic!*), *and the $\mathscr{S}_1$-norm of k_n for $\mathscr{D}$ is* $\leqslant 2b+1$ *independently of n.*

Use now the *third* lemma of the previous article *for $\mathscr{D}$.*

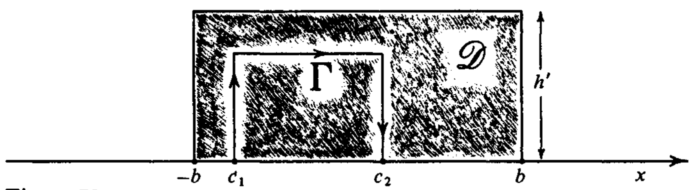

Figure 79

On account of what has just been said, we get

$$\int_{\Gamma} |k_n(z)|\,|dz| \;\leqslant\; (2b+1) + (2b+1)\left\{2 + \frac{h'}{b-|c_1|} + \frac{h'}{b-|c_2|}\right\},$$

i.e.

$$(\S) \qquad \int_{\Gamma} |k_n(z)|\,|dz| \;\leqslant\; \text{const.},$$

independently of n.

Let us return to (‡). It is at *this point* that we *choose n according to the value of* $\lambda > 0$. We are actually only interested in *large* values of λ. For any such one, we refer to the sequence $\{A_n\}$ described above, and take n as the integer for which $2BA_n \leqslant \lambda < 2BA_{n+1}$. For *this n*, (‡) becomes

$$|e^{i\lambda z} f_n(z)| \;\leqslant\; \text{const.}\,e^{-BA_n\Im z - (M_1(A_n)\omega_{\mathscr{D}}(F,z)/2)}|k_n(z)|, \qquad z \in \Gamma.$$

Recall that the two feet c_1 and c_2 of Γ were chosen so as to have

$$\lim_{y\to 0+} \omega_{\mathscr{D}}(F, c_1 + \mathrm{i}y) \;=\; \lim_{y\to 0+} \omega_{\mathscr{D}}(F, c_2 + \mathrm{i}y) \;=\; 1.$$

Therefore

$$B\Im z + \tfrac{1}{2}\omega_{\mathscr{D}}(F, z)$$

has a strictly positive minimum, say β, on Γ. β depends *only* on the geometric configuration of $\mathscr{D}$ and Γ. From the preceding relation, we have, then, when $2BA_n \leqslant \lambda < 2BA_{n+1}$, n being *large*,

$$|\mathrm{e}^{\mathrm{i}\lambda z} f_n(z)| \;\leqslant\; \mathrm{const.e}^{-\beta \min(A_n, M_1(A_n))}|k_n(z)|, \qquad z \in \Gamma.$$

Now use (§). We get

$$\left|\int_\Gamma \mathrm{e}^{\mathrm{i}\lambda z} f_n(z)\,\mathrm{d}z\right| \;\leqslant\; \mathrm{const.e}^{-\beta \min(A_n, M_1(A_n))}$$

for $2BA_n \leqslant \lambda < 2BA_{n+1}$; *this*, then, is our desired estimate for $|\mathrm{II}|$.

Now

$$|\Phi(\lambda)| \;=\; \left|\int_{c_1}^{c_2} \mathrm{e}^{\mathrm{i}\lambda x}\varphi(x)\,\mathrm{d}x\right| \;\leqslant\; |\mathrm{I}| + |\mathrm{II}|$$

where $|\mathrm{I}| \leqslant 2\mathrm{e}^{-M_1(A_n)}$, as we saw near the start of the present discussion. We may just as well take $\beta < 1$ (which is in fact *true* any way); then, by the estimate for $|\mathrm{II}|$ just found, we have, for *large n*,

$$|\Phi(\lambda)| \;\leqslant\; \mathrm{const.e}^{-\beta \min(A_n, M_1(A_n))}, \qquad 2BA_n \leqslant \lambda < 2BA_{n+1}.$$

Our aim here is to show that

$$\int_1^\infty \frac{1}{\lambda^2}\log\left|\frac{1}{\Phi(\lambda)}\right|\mathrm{d}\lambda \;=\; \infty,$$

or, what comes to the same thing, that

$$\int_{\lambda_0}^\infty \frac{1}{\lambda^2}\log\left|\frac{1}{\Phi(\lambda)}\right|\mathrm{d}\lambda \;=\; \infty$$

for some large λ_0. In view of the above inequality for $|\Phi(\lambda)|$, *this holds* if

$$\sum_n \int_{2BA_n}^{2BA_{n+1}} \frac{\min(A_n, M_1(A_n))}{\lambda^2}\mathrm{d}\lambda \;=\; \infty,$$

i.e., if

$$(\S\S) \qquad \sum_n \min(A_n, M_1(A_n))\left\{\frac{1}{A_n} - \frac{1}{A_{n+1}}\right\} \;=\; \infty.$$

We proceed to establish this relation. Our hypothesis says that

$$\int_1^\infty \frac{M_1(A)}{A^2}\,\mathrm{d}A = \infty.$$

The function $M_1(A)$ is increasing, so, by the *second* lemma of article 2, we also have

$$(\ddagger) \qquad \int_1^\infty \frac{\min(A, M_1(A))}{A^2}\,\mathrm{d}A = \infty.$$

Divide $\mathbb{N}$, the set of positive integers, into three disjoint subsets:

$$R = \{n \geqslant 1:\ A_{n+1} \leqslant 2A_n\},$$
$$S = \{n \geqslant 1:\ A_{n+1} > 2A_n \text{ and } A_n < M_1(A_n)\},$$
$$T = \{n \geqslant 1:\ A_{n+1} > 2A_n \text{ and } M_1(A_n) \leqslant A_n\}.$$

By ($\ddagger$), one of the three sums

$$\sum_{n\in R} \int_{A_n}^{A_{n+1}} \frac{\min(A, M_1(A))}{A^2}\,\mathrm{d}A,$$

$$\sum_{n\in S} \int_{A_n}^{A_{n+1}} \frac{\min(A, M_1(A))}{A^2}\,\mathrm{d}A,$$

$$\sum_{n\in T} \int_{A_n}^{A_{n+1}} \frac{\min(A, M_1(A))}{A^2}\,\mathrm{d}A$$

must be infinite.

Suppose the *first* of those sums is infinite. Recall that the A_n were chosen so as to have $M_1(A_{n+1}) = 2M_1(A_n)$. Therefore, if $n\in R$ and $A_n \leqslant A < A_{n+1}$,

$$\min(A, M_1(A)) \leqslant \min(A_{n+1}, M_1(A_{n+1})) \leqslant 2\min(A_n, M_1(A_n)),$$

i.e.,

$$\int_{A_n}^{A_{n+1}} \frac{\min(A, M_1(A))}{A^2}\,\mathrm{d}A \leqslant 2\min(A_n, M_1(A_n))\left\{\frac{1}{A_n} - \frac{1}{A_{n+1}}\right\}, \quad n\in R.$$

In the present case, then, we certainly have (§§).

If the *second* of the sums in question (the one over S) is infinite, the set S *cannot be finite*. However, for $n\in S$,

$$\min(A_n, M_1(A_n))\left\{\frac{1}{A_n} - \frac{1}{A_{n+1}}\right\} = \frac{A_{n+1} - A_n}{A_{n+1}} > \frac{1}{2},$$

so (§§) holds when S is infinite.

There remains the case where the *third* sum (over T) is infinite. Here, for $n \in T$ and $A_n \leqslant A < A_{n+1}$ we have

$$\min(A_n, M_1(A_n)) = M_1(A_n) = \tfrac{1}{2}M_1(A_{n+1}) \geqslant \tfrac{1}{2}M_1(A),$$

so, for such n,

$$\min(A_n, M_1(A_n))\left\{\frac{1}{A_n} - \frac{1}{A_{n+1}}\right\} \geqslant \frac{1}{2}\int_{A_n}^{A_{n+1}} \frac{M_1(A)}{A^2}\,\mathrm{d}A$$
$$\geqslant \frac{1}{2}\int_{A_n}^{A_{n+1}} \frac{\min(A, M_1(A))}{A^2}\,\mathrm{d}A.$$

Hence, if the sum of the right-hand integrals for $n \in T$ is infinite, so is that of the left-hand expressions, and (§§) holds.

The relation (§§) is thus proved. This, however, implies that

$$\int_1^\infty \frac{1}{\lambda^2} \log\left|\frac{1}{\Phi(\lambda)}\right| \mathrm{d}\lambda = \infty$$

as we have seen, which is what we needed to show. The theorem is completely proved, and we are done.

Corollary. *Let $f(\vartheta) \sim \sum_{-\infty}^{\infty} a_n \mathrm{e}^{\mathrm{i}n\vartheta}$ belong to $L_2(-\pi, \pi)$, and suppose that*

$$\sum_{-\infty}^{-1} \frac{1}{n^2} \log\left(\frac{1}{\sum_{-\infty}^{n} |a_k|^2}\right) = \infty.$$

If $f(\vartheta)$ vanishes on a set of positive measure, then $f \equiv 0$ a.e.

Let the reader deduce the corollary from the theorem. He or she is also encouraged to examine how some of the results from the previous article can be weakened (making their proofs simpler), leaving, however, enough to establish an L_2 version of the theorem which will yield the corollary.

C. Kargaev's example

In remark 2 following the proof of the Beurling gap theorem (§B.2), it was said that that result *cannot* be improved so as to apply to measure μ with $\hat{\mu}(\lambda)$ *vanishing on a set of positive measure, instead of on a whole interval.* This is shown by an example due to P. Kargaev which we give in the present §.

Kargaev's construction furnishes a measure μ with *gaps* (a_n, b_n) in its support, $0 < a_1 < b_1 < a_2 < b_2 < \cdots$, such that

$$\sum_1^\infty \left(\frac{b_n - a_n}{a_n}\right)^2 = \infty$$

while $\hat{\mu}(\lambda) = 0$ on a set E with $|E| > 0$. His method shows that *in fact the relative size*, $(b_n - a_n)/a_n$, *of the gaps in* μ's *support has no bearing on* $\hat{\mu}(\lambda)$'s *capability of vanishing on a set of positive measure without being identically zero*. It is possible to obtain such measures with $(b_n - a_n)/a_n \underset{n}{\longrightarrow} \infty$ *as rapidly as we please*. In view of Beurling's gap theorem, there is thus a *qualitative difference* between requiring that $\hat{\mu}(\lambda)$ *vanish on an interval* and merely having it *vanish on a set of positive measure*.

The measures obtained are *supported on the integers*, and their construction uses absolutely convergent Fourier series. The reasoning is elementary and somewhat reminiscent of the work of Smith, Pigno and McGehee on Littlewood's conjecture.

1. Two lemmas

Let us first introduce some *notation*. $\mathscr{A}$ denotes the collection of functions

$$f(\vartheta) = \sum_{-\infty}^{\infty} a_n e^{in\vartheta}$$

with the series on the right *absolutely convergent*. For such a function $f(\vartheta)$ we put

$$\|f\| = \sum_{-\infty}^{\infty} |a_n|$$

and frequently write $\hat{f}(n)$ instead of a_n (both of these notations are customary). $\mathscr{A}$, $\| \quad \|$ is a Banach space; in fact, a *Banach algebra* because, if f and $g \in \mathscr{A}$, then $f(\vartheta)g(\vartheta) \in \mathscr{A}$, and

$$\|fg\| \leqslant \|f\| \|g\|.$$

On account of this relation, $\Phi(f) \in \mathscr{A}$ for any *entire function* Φ if $f \in \mathscr{A}$.

We will be using some simple linear operators on $\mathscr{A}$.

Definition. If $f(\vartheta) = \sum_{-\infty}^{\infty} \hat{f}(n)e^{in\vartheta}$ belongs to $\mathscr{A}$,

$$(P_+f)(\vartheta) = \sum_{n=0}^{\infty} \hat{f}(n)e^{in\vartheta}$$

and $P_-f = f - P_+f$. We frequently write f_+ for P_+f and f_- for P_-f.

Observe that, for $f \in \mathscr{A}$, $\|P_+f\| \leqslant \|f\|$ and $\|P_-f\| \leqslant \|f\|$.

Definition. For N an integer $\geqslant 1$ and $f \in \mathscr{A}$,

$$(H_N f)(\vartheta) = f(N\vartheta).$$

(The H stands for 'homothety'.)

The following relations are obvious:

$$\begin{aligned} H_N(fg) &= (H_N f)(H_N g), \qquad f, g \in \mathscr{A}, \\ \|H_N f\| &= \|f\|, \\ P_+(H_N f) &= H_N(P_+ f), \end{aligned}$$

and $H_N\Phi(f) = \Phi(H_N f)$ for $f \in \mathscr{A}$ and Φ an entire function.

Lemma. *For each integer $N \geqslant 1$ and each $\delta > 0$ there is a linear operator $T_{N,\delta}$ on $\mathscr{A}$ together with a set $E_{N,\delta} \subseteq [0, 2\pi)$ such that:*

(i) *For each $f \in \mathscr{A}$, $g = T_{N,\delta} f$ has $\hat{g}(n) = 0$ for $-N \leqslant n < N$* (sic!);
(ii) *For each $f \in \mathscr{A}$, $(T_{N,\delta} f)(\vartheta) = f(\vartheta)$ for $\vartheta \in E_{N,\delta}$;*
(iii) $\|T_{N,\delta} f\| \leqslant C(\delta)\|f\|$ *with $C(\delta)$ depending* only *on δ and* not *on N;*
(iv) $|E_{N,\delta}| = 2\pi(1 - \delta)$.

Proof. The idea is as follows: starting with an $f \in \mathscr{A}$, we try to cook functions $g_+(\vartheta)$ and $g_-(\vartheta)$ in $\mathscr{A}$, the first having only positive frequencies and the second only negative ones, in such a way as to get

$$g_+(\vartheta)e^{iN\vartheta} + g_-(\vartheta)e^{-iN\vartheta}$$

'almost' equal to $f(\vartheta)$.

We take a certain $\psi \in \mathscr{A}$ (to be described in a moment) and write

$$(*) \qquad q = e^{i(\psi_+ - \psi_-)}.$$

According to the observations preceding the lemma, $q \in \mathscr{A}$. Our construction of $T_{N,\delta}$ and $E_{N,\delta}$ is based on the following *identity* valid for $f \in \mathscr{A}$:

$$\boxed{f = ((fq)_+ e^{-2i\psi_+})e^{i\psi} + ((fq)_- e^{2i\psi_-})e^{-i\psi}.}$$

To check this, just observe that the right-hand side is

$$\begin{aligned} &(fq)_+ e^{-i(\psi_+ - \psi_-)} + (fq)_- e^{i(\psi_- - \psi_+)} \\ &\quad = ((fq)_+ + (fq)_-)q^{-1} = fq \cdot q^{-1} = f. \end{aligned}$$

Here is the way we choose ψ. Take any 2π-periodic $\mathscr{C}_\infty$-function $\varphi_\delta(\vartheta)$ with a graph like this on the range $0 \leqslant \vartheta \leqslant 2\pi$:

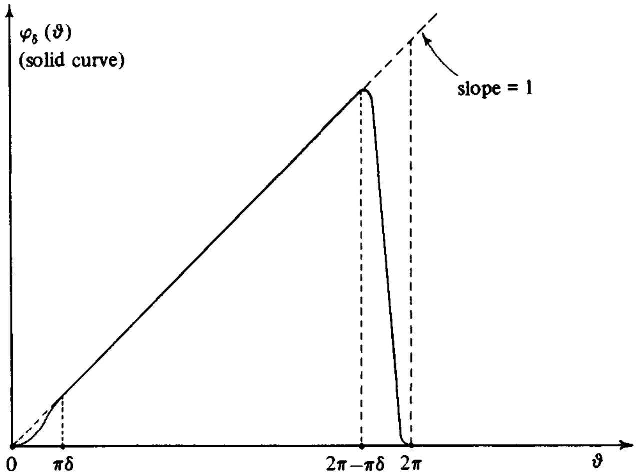

Figure 80

Then put $\psi = H_N\varphi_\delta$; ψ thus depends on N and δ. Note that $\varphi_\delta \in \mathscr{A}$ because φ_δ is infinitely differentiable ($|\hat{\varphi}_\delta(n)| \leqslant O(|n|^{-k})$ for *every* $k > 0$!). Therefore ψ belongs to $\mathscr{A}$.

With $q \in \mathscr{A}$ related by $(*)$ to the ψ just specified, put, for $f \in \mathscr{A}$,

$$T_{N,\delta}f \;=\; ((fq)_+ \mathrm{e}^{-2\mathrm{i}\psi_+})\mathrm{e}^{\mathrm{i}N\vartheta} + ((fq)_- \mathrm{e}^{2\mathrm{i}\psi_-})\mathrm{e}^{-\mathrm{i}N\vartheta}.$$

$T_{N,\delta}$ obviously takes $\mathscr{A}$ into $\mathscr{A}$; let us show that there is a set $E_{N,\delta} \subseteq [0, 2\pi)$ independent of f such that (ii) holds.

The set

$$\Delta_{N,\delta} \;=\; \{\vartheta,\ 0 \leqslant \vartheta < 2\pi:\quad 0 < N\vartheta < \pi\delta \bmod 2\pi \text{ or } 2\pi - \pi\delta < N\vartheta < 2\pi \bmod 2\pi\}$$

consists of $2N$ disjoint intervals, each of length $\pi\delta/N$, so $|\Delta_{N,\delta}| = 2\pi\delta$. Taking into account the 2π-periodicity of the function $\varphi_\delta(\vartheta)$ we see, by looking at its graph, that

$$\mathrm{e}^{\mathrm{i}\varphi_\delta(N\vartheta)} \;=\; \mathrm{e}^{\mathrm{i}N\vartheta}, \qquad \vartheta \in [0, 2\pi) \sim \Delta_{N,\delta};$$

i.e.,

$$\mathrm{e}^{\mathrm{i}\psi(\vartheta)} \;=\; \mathrm{e}^{\mathrm{i}N\vartheta}, \qquad \vartheta \in [0, 2\pi) \sim \Delta_{N,\delta}.$$

Put, therefore, $E_{N,\delta} = [0, 2\pi) \sim \Delta_{N,\delta}$; then, by comparing the formula for $T_{N,\delta}f$ with the boxed identity following $(*)$, we see that $(T_{N,\delta}f)(\vartheta) = f(\vartheta)$ for $\vartheta \in E_{N,\delta}$, proving (ii).

We also have (iv), since

$$|E_{N,\delta}| \;=\; 2\pi - |\Delta_{N,\delta}| \;=\; 2\pi - 2\pi\delta.$$

We come to (i). The function $(fq)_+$ only has *non-negative frequencies* in its Fourier series. The same is true for $e^{-2i\psi_+}$. Indeed, the latter function equals

$$1 - 2i\psi_+ + \frac{(2i\psi_+)^2}{2!} - \frac{(2i\psi_+)^3}{3!} + \cdots$$

with the series *convergent in the norm* $\| \ \|$, and each *power* $(\psi_+)^n$ has a Fourier series involving only frequencies $\geqslant 0$. The Fourier series of the *product* $(fq)_+e^{-2i\psi_+}$ thus only involves frequencies $\geqslant 0$, and finally, that for

$$((fq)_+e^{-2i\psi_+})e^{iN\vartheta}$$

only has frequencies $\geqslant N$. One verifies in the same way that

$$((fq)_-e^{2i\psi_-})e^{-iN\vartheta}$$

has a Fourier series involving only the frequencies $< -N$, and (i) now follows from our definition of $T_{N,\delta}$.

There remains (iii). We have, for example,

$$\begin{aligned}\|(fq)_+e^{-i\psi_+}\| &\leqslant \|(fq)_+\|\,\|e^{-i\psi_+}\| \\ &\leqslant \|fq\|\,\|e^{-2i\psi_+}\| \leqslant \|f\|\,\|q\|\,\|e^{-2i\psi_+}\|.\end{aligned}$$

Here,

$$e^{-2i\psi_+} = e^{-2iP_+H_N\varphi_\delta} = e^{-2iH_NP_+\varphi_\delta} = H_Ne^{-2iP_+\varphi_\delta},$$

according to the elementary relations preceding the lemma, so

$$\|e^{-2i\psi_+}\| = \|H_Ne^{-2iP_+\varphi_\delta}\| = \|e^{-2iP_+\varphi_\delta}\|,$$

a *finite quantity, depending on δ but not on* N. In like manner,

$$\|q\| = \|e^{i(\psi_+ - \psi_-)}\| = \|H_Ne^{i(P_+\varphi_\delta - P_-\varphi_\delta)}\| = \|e^{i(P_+\varphi_\delta - P_-\varphi_\delta)}\|,$$

a finite quantity depending on δ but independent of N. We thus have

$$\|(fq)_+e^{-2i\psi_+}e^{iN\vartheta}\| = \|(fq)_+e^{-2i\psi_+}\| \leqslant A_\delta\|f\|,$$

where A_δ depends only on δ.

The norm $\|(fq)_-e^{2i\psi_-}e^{-iN\vartheta}\|$ is handled in exactly the same way, and found to be $\leqslant B_\delta\|f\|$ with B_δ depending only on δ. Referring to the definition of $T_{N,\delta}$, we see that (iii) holds.

The lemma is thus proved.

We now take two positive integers L and N; N will usually be much larger than $2L$.

Definition.

$$\mathscr{M}(N,L) \;=\; \bigcup_{k=-2L-1}^{2L+1}{}' \; [Nk-L,\; Nk+L].$$

▶ Here, the prime next to the union sign means that the term corresponding to the value $k=0$ is *omitted.*

For $N>2L$, $\mathscr{M}(N,L)$ is the union of $4L+2$ separate intervals, each of length $2L$:

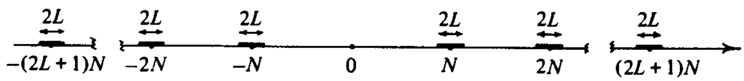

Figure 81

In proving the following lemma we use another linear operator on $\mathscr{A}$.

Definition. For $f\in\mathscr{A}$, put

$$(S_L f)(\vartheta) \;=\; \sum_{n=-L}^{L} \hat{f}(n)e^{in\vartheta}.$$

Observe that $\|S_L f\| \leqslant \|f\|$ and $\|f - S_L f\| \longrightarrow 0$ as $L\to\infty$ whenever $f\in\mathscr{A}$. We also have

$$P_+ S_L f \;=\; S_L P_+ f.$$

Lemma. *For each $\delta>0$ and pair N, L of positive integers there is a linear operator $T_{N,\delta}^{(L)}$ on $\mathscr{A}$ such that*

(1) *For any $f\in\mathscr{A}$, the Fourier coefficients $\hat{g}(n)$ of $g = T_{N,\delta}^{(L)}f$ are all* zero *when $n\notin\mathscr{M}(N,L)$;*
(2) *For $f\in\mathscr{A}$, $\|T_{N,\delta}^{(L)}f\| \leqslant C(\delta)\|f\|$ with $C(\delta)$* independent *of N and L;*
(3) *If $T_{N,\delta}$ is the operator furnished by the previous lemma, we have*

$$\|T_{N,\delta}f - T_{N,\delta}^{(L)}f\| \longrightarrow 0$$

uniformly *in N as $L\to\infty$, for each $f\in\mathscr{A}$ and $\delta>0$.*

Remarks. Actually, the spectrum of $T_{N,\delta}^{(L)}f$ is contained in a *smaller* set than $\mathscr{M}(N,L)$ when $f\in\mathscr{A}$. It is the *uniformity with respect to N* in property 3 which will turn out to be especially important in Kargaev's construction.

Proof of lemma. Fix $\delta>0$ and take the function φ_δ used in proving the preceding lemma – *here we just denote it by φ.* In terms of

$$q_0 \;=\; e^{i(\varphi_+ - \varphi_-)},$$

we observe that the definition of $T_{N,\delta}f$ given in the proof of the previous lemma can be rewritten thus:

$$T_{N,\delta}f = (fH_Nq_0)_+(H_N e^{-2i\varphi_+})\cdot e^{iN\vartheta} + (fH_Nq_0)_-(H_N e^{2i\varphi_-})\cdot e^{-iN\vartheta}.$$

Put

$$T_{N\delta}^{(L)}f = (S_Lf\cdot H_NS_Lq_0)_+(H_NS_L e^{-2i\varphi_+})\cdot e^{iN\vartheta} \\ + (S_Lf\cdot H_NS_Lq_0)_-(H_NS_L e^{2i\varphi_-})\cdot e^{-iN\vartheta}.$$

Since $\|g - S_Lg\| \longrightarrow 0$ as $L \to \infty$ for every $g \in \mathscr{A}$, $T_{N,\delta}^{(L)}f$ is clearly a kind of approximation to $T_{N,\delta}f$.

We proceed to verify property (1). The Fourier coefficients of S_Lf are all zero save for those with index in the set

$$\{-L,\ -L+1,\ \ldots,\ 0,\ 1,\ \ldots,\ L\}.$$

The non-zero Fourier coefficients of $H_NS_Lq_0$ have their indices in the set

$$\{-NL,\ -N(L-1),\ \ldots,\ -N,\ 0,\ N,\ \ldots,\ NL\}.$$

Therefore the Fourier coefficients of $(S_Lf\cdot H_NS_Lq_0)_+$ with index *outside* the set

$$\{0,\ 1,\ \ldots,\ L\} \cup \{N-L,\ N-L+1,\ \ldots,\ N,\ N+1,\ \ldots,\ N+L\} \\ \cup \{2N-L,\ 2N-L+1,\ \ldots,\ 2N+L\} \cup \cdots \\ \cup \{NL-L,\ NL-L+1,\ \ldots,\ NL+L\}$$

are *surely zero.*

Again, the Fourier coefficients of $H_NS_L e^{-2i\varphi_+}$ are all zero save for those with index in the set $\{0,\ N,\ 2N,\ \ldots,\ LN\}$. So, finally, the Fourier coefficients of

$$(S_Lf\cdot H_NS_Lq_0)_+(H_NS_L e^{-2i\varphi_+})e^{iN\vartheta}$$

(the *first* of the two terms making up $T_{N,\delta}^{(L)}f$) are *all zero, save for those with index in the union of intervals*

$$[N,\ N+L] \cup \bigcup_{k=2}^{2L+1} [Nk-L,\ Nk+L].$$

Treating the *second* term of $T_{N,\delta}^{(L)}f$ in the same way, we see that property 1 holds (and that indeed *more* is true regarding the spectrum of $T_{N,\delta}^{(L)}f$).

To check property (2), we have, for the *first term* of $T_{N,\delta}^{(L)}f$,

$$\|(S_Lf\cdot H_NS_Lq_0)_+(H_NS_L e^{-2i\varphi_+})\cdot e^{iN\vartheta}\| \\ \leqslant \|S_Lf\|\,\|H_NS_Lq_0\|\,\|H_NS_L e^{-2i\varphi_+}\| \\ \leqslant \|f\|\,\|S_Lq_0\|\,\|S_L e^{-2i\varphi_+}\| \leqslant \|f\|\,\|q_0\|\,\|e^{-2i\varphi_+}\|;$$