(b) Hence show that

$$\frac{\rho(M)}{M} \longrightarrow 0 \quad \text{as } M \longrightarrow \infty.$$

(Hint: $\rho(t) \geqslant -K(1+\log^+ t)$ for $t>0$, making

$$-\int_0^M \log\left|\frac{t+A}{t-A}\right| \frac{\rho(t)}{t}\mathrm{d}t \quad \leqslant \quad \text{const.} \log A$$

with a constant independent of M, for $A > \mathrm{e}$, say. Deduce that for fixed large A and $M \longrightarrow \infty$,

$$2\frac{\rho(M)}{M}A \quad \leqslant \quad \mathrm{O}(1) \; + \; \text{const.}\log A. \qquad)$$

(c) Then show that

$$\int_0^A \frac{U_\rho(x)}{x}\mathrm{d}x \quad = \quad \int_0^\infty \log\left|\frac{t+A}{t-A}\right| \frac{\rho(t)}{t}\mathrm{d}t.$$

(d) Show that for large $t>1$ we not only have $\rho(t) \geqslant -\text{const.}\log t$ but also $\rho(t) \leqslant \text{const.}\log t$. (Hint: Wlog, $\mathrm{d}\rho(t) \geqslant -\mathrm{d}t/t$ for $t>1$. Assuming that for some large A we have $\rho(A) \geqslant k\log A$ with a number $k>0$, it follows that

$$\rho(t) \quad \geqslant \quad k\log A \; - \; \log\frac{t}{A} \qquad \text{for } t > A.$$

At the same time, $\rho(t) \geqslant -\mathrm{O}(1) - \log^+ t$ for $0<t<A$. Use result of (c) with these relations to get a *lower bound* on $\int_0^A (U_\rho(x)/x)\mathrm{d}x$ involving k and $\log A$, thus arriving at an *upper bound* for k.)

Problem 67

Continuing with the material of the preceding problem, we now assume that

$$\lim_{A\to\infty} \int_0^A \frac{U_\rho(x)}{x}\mathrm{d}x$$

exists (and is finite). It is proposed to show by means of an elementary Tauberian argument that $\rho(t)$ then *also has a limit* (equal to $2/\pi^2$ times the preceding one) for $t \longrightarrow \infty$. Essentially this result was used by Beurling and Malliavin* in their original proof of the Theorem on the

* under the milder condition on ρ pointed out in the preceding footnote – they in fact assumed only that the measure ρ on $[0, \infty)$ satisfies $\mathrm{d}\rho(t) \geqslant -\text{const.}\,\mathrm{d}t/t$ there, but then the conclusion of problem 67 holds just as well because the existence

Multiplier.

(a) Show that for a and $b > 0$,

$$\int_0^\infty \log\left|\frac{x+a}{x-a}\right| \log\left|\frac{x+b}{x-b}\right| \mathrm{d}x \quad = \quad \pi^2 \min(a,\ b).$$

(Hint: We have

$$\frac{1}{\pi}\log\left|\frac{x+a}{x-a}\right| \quad = \quad \frac{1}{\pi}\int_{-a}^{a} \frac{\mathrm{d}t}{x-t}.$$

Apply the L_2 theory of Hilbert transforms sketched at the end of §C.1, Chapter VIII.)

(b) Hence derive the formula

$$\int_0^\infty \log\left|\frac{t+x}{t-x}\right| \left\{2\log\left|\frac{x+A}{x-A}\right| - \log\left|\frac{x+(1+\delta)A}{x-(1+\delta)A}\right| - \log\left|\frac{x+(1-\delta)A}{x-(1-\delta)A}\right|\right\} \mathrm{d}x$$

$$= \quad \pi^2(\delta A \;-\; |t-A|)^+,$$

valid for $t > 0$, $A > 0$ and $0 < \delta < 1$.

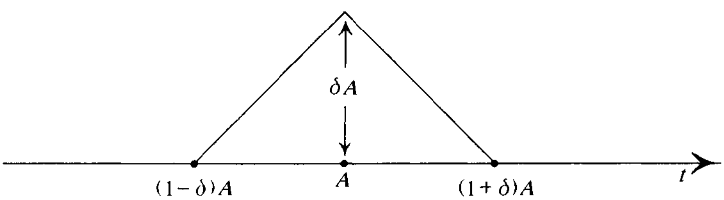

Figure 251

(c) Then, referring to part (c) of the previous problem, prove that

$$\frac{1}{A}\int_0^\infty \left(\int_0^x \frac{U_\rho(\xi)}{\xi}\mathrm{d}\xi\right)\left\{2\log\left|\frac{x+A}{x-A}\right| \;-\; \log\left|\frac{x+(1+\delta)A}{x-(1+\delta)A}\right|\right.$$

$$\left. -\; \log\left|\frac{x+(1-\delta)A}{x-(1-\delta)A}\right|\right\}\mathrm{d}x \quad = \quad \pi^2\delta\int_{(1-\delta)A}^{(1+\delta)A}\left(1 \;-\; \frac{|t-A|}{\delta A}\right)\frac{\rho(t)}{t}\mathrm{d}t.$$

(Hint: Use the formula obtained in (b) together with Fubini's theorem. In justifying application of the latter, the bound on $|\rho(t)|$ found at the end of the preceding problem comes in handy. One should also observe that the expression in $\{\quad\}$ involved in the left-hand integrand belongs to $L_1(0, \infty)$ and is in fact $O(1/x^3)$ for large x.)

of $\lim_{A\to\infty} \int_0^A (U_\rho(x)/x)\,\mathrm{d}x$ is not affected when ρ is replaced by its restriction to $[1, \infty)$

(d) Assume now that

$$\int_0^x \frac{U_\rho(\xi)}{\xi}\,\mathrm{d}\xi \quad \longrightarrow \quad l$$

for $x \longrightarrow \infty$. Show that then the *right side* of the relation establish in (c) *tends to a limit*, equal to $(2\delta^2 + O(\delta^3))l$, as $A \longrightarrow \infty$. (Hint: The *left side* of the relation referred to can be rewritten as

$$\int_0^\infty \left(\int_0^{uA} \frac{U_\rho(\xi)}{\xi}\,\mathrm{d}\xi \right) \varphi_\delta(u)\,\mathrm{d}u,$$

where $\varphi_\delta(u)$ is a certain L_1 function not involving A. To compute $\int_0^\infty \varphi_\delta(u)\,\mathrm{d}u$, look at $\int_0^M \varphi_\delta(u)\,\mathrm{d}u$. By making appropriate changes of variable, the last integral is thrown into the form

$$(1+\delta)\int_{M/(1+\delta)}^{M} \log\left|\frac{v+1}{v-1}\right| \mathrm{d}v \quad - \quad (1-\delta)\int_M^{M/(1-\delta)} \log\left|\frac{v+1}{v-1}\right| \mathrm{d}v,$$

and this is readily evaluated for large M by expanding the integrands in powers of $1/v$.)

(e) Hence show that under the assumption in (d),

$$\rho(t) \quad \longrightarrow \quad \frac{2}{\pi^2}l \qquad \text{for } t \longrightarrow \infty.$$

(Hint: Picking a small $\delta > 0$, assume that for some large A we have

$$\rho((1-\delta)A) \; > \; m,$$

a number $> 2l/\pi^2$. Recalling that $\mathrm{d}\rho(t) \geqslant -\mathrm{d}t/t$ for $t > 1$, we then get

$$\rho(t) \quad > \quad m \; - \; \log\left(\frac{1+\delta}{1-\delta}\right) \qquad \text{for } (1-\delta)A \;\leqslant\; t \;\leqslant\; (1+\delta)A,$$

and this will contradict the result in (d) if A is large, and m much bigger than $2l/\pi^2$.

In case we have

$$\rho((1+\delta)A) \; < \; m',$$

a number $< 2l/\pi^2$ for some large A, we get

$$\rho(t) \quad < \quad m' \; + \; \log\left(\frac{1+\delta}{1-\delta}\right) \qquad \text{for } (1-\delta)A \;\leqslant\; t \;\leqslant\; (1+\delta)A,$$

and then the same kind of argument can be made.)

Problem 68

Formulate and prove a theorem analogous to the one of this article for functions $\omega(x)$ of the form

$$\omega(x) \quad = \quad x \int_0^\infty \log \left| \frac{x+t}{x-t} \right| \, \mathrm{d}\rho(t),$$

where ρ is a measure subject to the assumption* stated above. (Hint: Work directly in terms of $\rho(t)$ and $\mathrm{d}\rho(t)$. *Boundedness* of $\rho(t)$ – needed to apply the second lemma of this article – is guaranteed by the last problem.)

4. Example. The finite energy condition not necessary

The construction starts out as in article 1; again we take

$$x_p \quad = \quad \exp(p^{1/3}) \qquad \text{for } p \; = \; 8, 9, 10, \ldots .$$

and put

$$\Delta_8 \quad = \quad x_8$$
$$\Delta_p \quad = \quad x_p - x_{p-1} \qquad \text{for } p \geqslant 9.$$

We also use a sequence of strictly positive numbers $\lambda_p < 1$ tending monotonically to 1, but at so slow a rate that

$$\sum_8^\infty \frac{(1-\lambda_p)^2}{p} \quad = \quad \infty .$$

It will turn out to be convenient to specify the λ_p explicitly near the end of this article.

Based on these sequences, we form an increasing function $\nu(t)$, defined for $t \geqslant 0$ according to the rule

$$\nu(t) \quad = \quad \begin{cases} \lambda_8 t, & 0 \leqslant t < x_8, \\ x_{p-1} \; + \; \lambda_p(t - x_{p-1}) & \text{for } x_{p-1} \; \leqslant \; t \; < \; x_p \text{ with } p \geqslant 9. \end{cases}$$

This function has a jump of magnitude $(1-\lambda_p)\Delta_p$ at each of the points x_p; its behaviour is shown by the following figure:

* in its original form, including absolute continuity of ρ and boundedness of $\rho'(t)$ on finite intervals

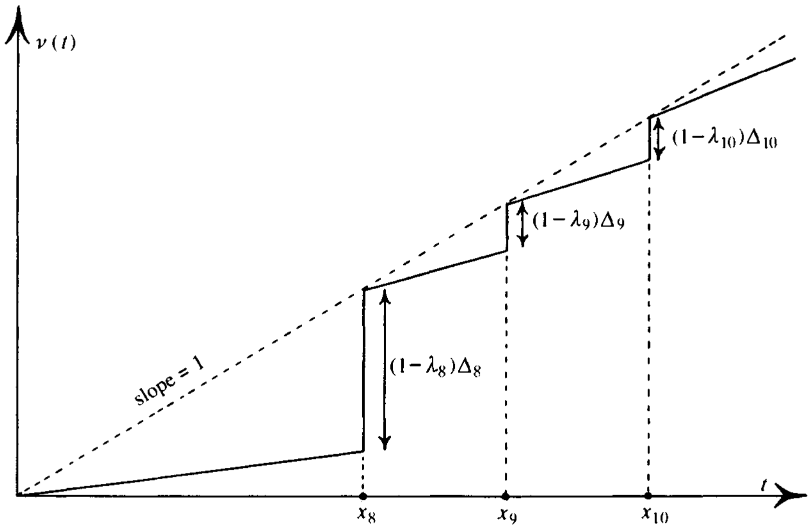

Figure 252

Obviously,

$$\nu(t) \leqslant t.$$

Also, since

$$\Delta_p \sim \tfrac{1}{3}p^{-2/3}x_p = \frac{x_p}{3(\log x_p)^2}$$

and

$$\frac{x_p}{x_{p-1}} \longrightarrow 1$$

as $p \longrightarrow \infty$, it is evident that

$$\nu(t) \geqslant t - \frac{t}{(\log t)^2} \qquad \text{for large } t.$$

Thence, putting

$$F_1(z) = \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| d\nu(t),$$

we see by computations just like those at the beginning of article 1 that

$$F_1(z) \leqslant F_1(\mathrm{i}|z|) = \pi|z| + \mathrm{o}(|z|)$$

for $|z|$ large and that moreover, for real values of x with sufficiently large absolute value,

$$F_1(x) \leqslant C\frac{|x|\log\log|x|}{(\log|x|)^2}$$

where C is a certain constant.

The right side of the last relation is an increasing function of $|x|$ when that quantity is large. Choosing, then, a large number l in a manner to be described presently, we take

$$T(x) = \begin{cases} 0, & 0 \leqslant x < l, \\ C\dfrac{x\log\log x}{(\log x)^2}, & x \geqslant l, \end{cases}$$

thus getting an increasing function T such that

$$\int_0^\infty \frac{T(x)}{x^2}\,dx < \infty.$$

By the above two inequalities for F_1, we then have

$$F_1(x) \leqslant T(|x|) + \text{const.} \qquad \text{for } x \in \mathbb{R}.$$

We now follow the procedure used in Chapter X, §A.1, to prove the elementary multiplier theorem of Paley and Wiener. Using a constant B to be determined shortly, write

$$\mu(t) = Bt\int_t^\infty \frac{T(\tau)}{\tau^2}\,d\tau$$

and then let

$$F_2(z) = \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| d\mu(t).$$

Since $T(t)$ is increasing, we have

$$\mu'(t) = B\int_t^\infty \frac{T(\tau)}{\tau^2}\,d\tau - B\frac{T(t)}{t} \geqslant 0$$

for $t > 0$, and at the same time,

$$\frac{\mu(t)}{t} \longrightarrow 0 \quad \text{as } t \longrightarrow \infty.$$

The first lemma of Chapter VIII, §B.4, may thus be applied to the right side of our formula for F_2, yielding

$$\begin{aligned} F_2(x) &= -x\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\left(\frac{\mu(t)}{t}\right) \\ &= Bx\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \frac{T(t)}{t^2}\mathrm{d}t \qquad \text{for } x \geqslant 0. \end{aligned}$$

Therefore, because $T(t)$ is increasing, we have

$$F_2(x) \geqslant BT(x)\int_1^\infty \log\left|\frac{1+\tau}{1-\tau}\right| \frac{\mathrm{d}\tau}{\tau^2}, \qquad x \geqslant 0.$$

The integral on the right is just a certain strictly positive numerical quantity. We can thus pick B large enough (independently of the value of the large number l used in the specification of T) so as to ensure that

$$F_2(x) \geqslant 2T(x) \qquad \text{for } x \geqslant 0.$$

Fix such a value of B – it will be clear later on why we want the coefficient 2 on the right. Then, taking

$$F(z) = F_1(z) - F_2(z),$$

we will have

$$F(x) \leqslant -T(|x|) + \text{const.} \leqslant \text{const.}$$

for real values of x.

The function F is given by the formula

$$F(z) = \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}(\nu(t) - \mu(t)),$$

in which $\nu(t) - \mu(t)$ is *increasing*, provided that the parameter l entering into the definition of T is chosen properly. Because $\nu(t)$ and $\mu(t)$ are each increasing, with the second function absolutely continuous, this may be verified by looking at $\nu'(t) - \mu'(t)$. For $x_{p-1} < t < x_p$ with $p > 8$, we have

$$\nu'(t) - \mu'(t) = \lambda_p + B\frac{T(t)}{t} - B\int_t^\infty \frac{T(\tau)}{\tau^2}\mathrm{d}\tau,$$

and an analogous relation holds in the interval $(0, x_8)$. *Choose*, therefore, *l large enough to make*

$$B\int_0^\infty \frac{T(\tau)}{\tau^2}\mathrm{d}\tau = BC\int_l^\infty \frac{\log\log\tau}{(\log\tau)^2\,\tau}\mathrm{d}\tau < \lambda_8.$$

Then, the sequence $\{\lambda_p\}$ being increasing, we will have $\nu'(t) - \mu'(t) > 0$ for $t > 0$ different from any of the points x_p, and $\nu(t) - \mu(t)$ will be increasing.

It is also clear that

$$\frac{\nu(t) - \mu(t)}{t} \longrightarrow 1 \quad \text{as } t \longrightarrow \infty.$$

Hence

$$F(z) \leqslant F(\mathrm{i}|z|) = \pi|z| + \mathrm{o}(|z|)$$

for large $|z|$. $F(x)$ is, on the other hand, *bounded above* for real x. From these two properties and the formula for $F(z)$ we can now deduce the representation of §G.1, Chapter III,

$$F(z) = \pi|\Im z| + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z| F(t)}{|z-t|^2}\,\mathrm{d}t,$$

by an argument like one used in the proof of the second theorem of §B.1.

Let K be any upper bound for $F(x)$ on $\mathbb{R}$, and then, proceeding much as in article 1, put

$$W(x) = \frac{\mathrm{e}^{\pi + K}}{\exp F(x+\mathrm{i})}, \qquad x \in \mathbb{R}.$$

From the preceding relation, we get

$$\log W(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(K - F(t))}{(x-t)^2 + 1}\,\mathrm{d}t,$$

and from this we see that

$$W(x) \geqslant 1,$$

besides which

$$\left|\frac{\mathrm{d}\log W(x)}{\mathrm{d}x}\right| \leqslant \log W(x),$$

making $\log\log W(x)$ uniformly Lip 1 on $\mathbb{R}$. *The present weight W thus meets the local regularity requirement from* §B.1, quoted at the beginning of this §. Since $F(t)$ is even, so is $W(x)$, and the relation $F(t) \leqslant -T(|t|) + \text{const.}$, together with $T(t)$'s tending to ∞ for $t \longrightarrow \infty$, implies that

$$W(x) \longrightarrow \infty \qquad \text{for } x \longrightarrow \pm\infty.$$

(*That's* why we chose B so as to have $F_2(x) \geqslant 2T(|x|)$ with a factor of 2.)

It will now be shown that $W(x)$ *admits multipliers*, but that *there can be no even function* $\Omega(x) \geqslant 1$ *with*

$$\int_0^\infty \frac{\log \Omega(x)}{x^2}\, dx \quad < \quad \infty$$

and $\log \Omega(x)/x$ *in* $\mathfrak{H}$ *such that*

$$W(x) \;\leqslant\; \Omega(x)$$

for large values of $|x|$.

To show that W *admits multipliers*, we start from the relation

$$\log W(x) \quad = \quad \pi + K - F_1(x+i) + F_2(x+i)$$

and deal separately with the terms $F_1(x+i)$ and $F_2(x+i)$ standing on the right. One handles each of those by first moving down to the real axis and working with $F_1(x)$ and $F_2(x)$; afterwards, one goes back up to the line $z = x+i$.

The function F_1 is easier to take care of on account of $\nu(t)$'s special form. Knowing that

$$-F_1(x) \quad = \quad -\int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\nu(t),$$

we proceed, for given arbitrary $\eta > 0$, to build an increasing $\sigma_1(t)$ with $\sigma_1(t)/t \leqslant \eta/2$ having *jumps* that will *cancel out* most of ν's, making, indeed, $\sigma_1(t) - \nu(t)$ a *constant multiple* of t for large values of that variable. The property that $\lambda_p \longrightarrow 1$ as $p \longrightarrow \infty$ enables us to do this.

Given the quantity $\eta > 0$, there is a number $p(\eta)$ such that

$$\lambda_p \;>\; 1 - \frac{\eta}{2} \qquad \text{for } p > p(\eta).$$

We put

$$\sigma_1(t) \quad = \quad \begin{cases} 0, \quad t < x_{p(\eta)}, \\ \dfrac{\eta}{2}x_{p-1} + \left\{\lambda_p - \left(1 - \dfrac{\eta}{2}\right)\right\}(t - x_{p-1}) \quad \text{for} \\ \qquad\qquad x_{p-1} \leqslant t < x_p \text{ with } p > p(\eta). \end{cases}$$

This increasing function $\sigma_1(t)$ is related to $\nu(t)$ in the way shown by the following diagram:

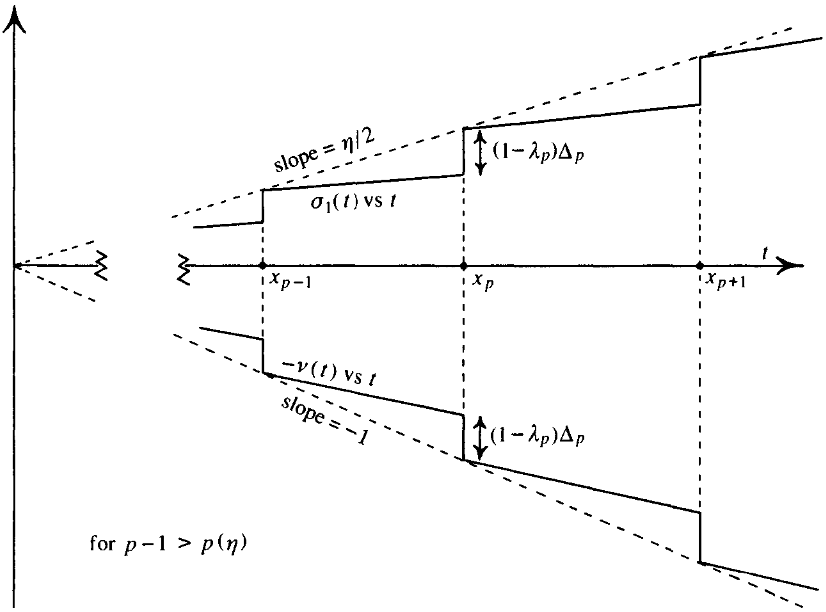

Figure 253

It is clear that

$$\sigma_1(t) \;-\; \nu(t) \;\;=\;\; -\left(1 \;-\; \frac{\eta}{2}\right)t \qquad \text{for } t \;\geqslant\; x_{p(\eta)}.$$

Take now

$$G_1(z) \;\;=\;\; \int_0^\infty \log\left|1 \;-\; \frac{z^2}{t^2}\right| \mathrm{d}\sigma_1(t).$$

We have

$$\frac{\sigma_1(t)}{t} \;\longrightarrow\; \frac{\eta}{2} \quad \text{as } t \longrightarrow \infty,$$

so for large values of $|z|$,

$$G_1(z) \;\;\leqslant\;\; G_1(\mathrm{i}|z|) \;\;=\;\; \frac{\pi\eta}{2}|z| \;+\; o(|z|).$$

The first lemma of §B.4, Chapter VIII, tells us that

$$G_1(x) \;-\; F_1(x) \;\;=\;\; \int_0^\infty \log\left|1 \;-\; \frac{x^2}{t^2}\right| \mathrm{d}(\sigma_1(t) - \nu(t))$$

$$=\;\; x\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\left(\frac{\nu(t) - \sigma_1(t)}{t}\right), \qquad x \in \mathbb{R}.$$

As we have just seen, $(\nu(t)-\sigma_1(t))/t$ is *constant* for $t \geqslant x_{p(\eta)}$; the last expression on the right thus reduces to

$$x\int_0^{x_{p(\eta)}} \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\left(\frac{\nu(t)-\sigma_1(t)}{t}\right).$$

This, however, is clearly bounded (above *and* below!) for $|x| \geqslant 2x_{p(\eta)}$, say. Therefore

$$G_1(x) \; - \; F_1(x) \quad \leqslant \quad \text{const.}, \qquad |x| \; \geqslant \; 2x_{p(\eta)}.$$

This relation does not hold *everywhere* on $\mathbb{R}$; $G_1(x) - F_1(x)$ is indeed *infinite* at each of the points $\pm x_p$ with $8 \leqslant p < p(\eta)$. But at those places (corresponding to the points where $\sigma_1(t)-\nu(t)$ jumps *downwards*) the infinities of $G_1(z) - F_1(z)$ are *logarithmic*, and hence *harmless* as far as we are concerned. Besides becoming $-\infty$ (logarithmically again) at $\pm x_{p(\eta)}$, the function $G_1(x) - F_1(x)$ is otherwise well behaved on $\mathbb{R}$, and belongs to $L_1(-2x_{p(\eta)}, 2x_{p(\eta)})$. We can now reason once again as in the proof of the second theorem, §B.1, and deduce from the properties of $G_1(x) - F_1(x)$ just noted, and from those of $G_1(z)$ and $F_1(z)$ in the complex plane, pointed out previously, that

$$G_1(z) \; - \; F_1(z) \quad = \quad -\pi\left(1 \; - \; \frac{\eta}{2}\right)|\Im z| \quad + \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|(G_1(t)-F_1(t))}{|z-t|^2}\,\mathrm{d}t.$$

Keeping in mind the behaviour of $G_1(t) - F_1(t)$ on the real axis, we see by this relation that

$$G_1(x+\mathrm{i}) \; - \; F_1(x+\mathrm{i}) \quad \leqslant \quad \text{const.}, \qquad x\in\mathbb{R}.$$

We turn to the function $F_2(x)$, equal, as we have seen, to

$$Bx\int_0^{\infty} \log\left|\frac{x+t}{x-t}\right| \frac{T(t)}{t^2}\,\mathrm{d}t$$

for $x\in\mathbb{R}$ (both $F_2(x)$ and this expression being even). Here we proceed just as in the passage from the function $Cx(\log\log x)/(\log x)^2$ to $F_2(z)$. A change of variable shows that

$$F_2(x) \quad = \quad B\int_0^{\infty} \log\left|\frac{1+\tau}{1-\tau}\right| \frac{T(x\tau)}{\tau^2}\,\mathrm{d}\tau \qquad \text{for } x \geqslant 0,$$

from which it is manifest that $F_2(x)$, like $T(x)$, is *increasing* on $[0,\infty)$. Again, by Fubini's theorem,

$$\int_0^{\infty} \frac{F_2(x)}{x^2}\,\mathrm{d}x \quad = \quad B\int_0^{\infty}\int_0^{\infty} \log\left|\frac{x+t}{x-t}\right| \frac{\mathrm{d}x}{x}\,\frac{T(t)}{t^2}\,\mathrm{d}t \quad =$$

$$= \frac{\pi^2 B}{2}\int_0^\infty \frac{T(t)}{t^2}\,dt \quad < \quad \infty.$$

Pick, then, a large number m (in a way to be described in a moment), and put

$$\Theta(t) \quad = \quad \begin{cases} 0, & 0 \leqslant t < m, \\ F_2(t), & t \geqslant m. \end{cases}$$

Bringing in once more the given quantity $\eta > 0$, we form the function

$$\sigma_2(t) \quad = \quad \frac{\eta}{2}t \quad - \quad Bt\int_t^\infty \frac{\Theta(\tau)}{\tau^2}\,d\tau$$

and observe that it is *increasing provided that m is chosen large enough* – verification of this statement is just like that of the corresponding one about $\nu(t) - \mu(t)$. *Fixing* once and for all such a value of m, we take

$$G_2(z) \quad = \quad \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| d\sigma_2(t).$$

The function $\sigma_2(t)$, besides being increasing, has the property that

$$\frac{\sigma_2(t)}{t} \longrightarrow \frac{\eta}{2} \qquad \text{as } t \longrightarrow \infty.$$

Thence,

$$G_2(z) \quad \leqslant \quad G_2(i|z|) \quad = \quad \frac{\pi\eta}{2}|z| \; + \; o(|z|)$$

for large values of $|z|$.

For $x > 0$, by the first lemma of §B.4, Chapter VIII,

$$G_2(x) \quad = \quad -x\int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\sigma_2(t)}{t}\right)$$

$$= \quad -Bx\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \frac{\Theta(t)}{t^2}\,dt.$$

Thanks to our choice of B, the last quantity is $\leqslant -2\Theta(x)$ (cf. the previous examination of $F_2(x)$'s relation to $T(x)$). Thus,

$$G_2(x) \quad \leqslant \quad -2F_2(x) \qquad \text{for } |x| \geqslant m,$$

and we certainly have

$$G_2(x) \; + \; F_2(x) \quad \leqslant \quad 0$$

for such real x, $F_2(x)$ being clearly positive.

Since $\mu(t)/t \longrightarrow 0$ for $t \longrightarrow \infty$, with $\mu(t)$ increasing, we must have

$$F_2(z) \leqslant F_2(\mathrm{i}|z|) = \mathrm{o}(|z|)$$

for large $|z|$. Using this estimate and the corresponding one on $G_2(z)$ given above we deduce from the behaviour of $G_2(x) + F_2(x)$ on $\mathbb{R}$ just found that

$$G_2(z) + F_2(z) = \frac{\pi\eta}{2}|\Im z| + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|(G_2(t) + F_2(t))}{|z-t|^2}\,\mathrm{d}t;$$

this is done by the procedure followed twice already. It then follows from this relation and from the fact that $G_2(t) + F_2(t) \leqslant 0$ for $|t| \geqslant m$ that

$$G_2(x+\mathrm{i}) + F_2(x+\mathrm{i}) \leqslant \text{const.}, \qquad x \in \mathbb{R}.$$

Going back to $\log W(x)$ we find, recalling the above formula for it and using the two results now obtained, that

$$\log W(x) + G(x+\mathrm{i}) \leqslant \text{const.}, \qquad x \in \mathbb{R},$$

where

$$G(z) = G_1(z) + G_2(z) = \int_0^{\infty} \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}(\sigma_1(t) + \sigma_2(t)).$$

Consider now the entire function $\varphi(z)$ given by the formula

$$\log|\varphi(z)| = \int_0^{\infty} \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}[\sigma_1(t) + \sigma_2(t)].$$

Because

$$\frac{[\sigma_1(t) + \sigma_2(t)]}{t} = \frac{\sigma_1(t)}{t} + \frac{\sigma_2(t)}{t} + \mathrm{o}(1) \longrightarrow \eta$$

for $t \longrightarrow \infty$, we have

$$\log|\varphi(z)| \leqslant \log|\varphi(\mathrm{i}|z|)| = \pi\eta|z| + \mathrm{o}(|z|)$$

for z of large modulus, making $\varphi(z)$ *of exponential type* $\pi\eta$. The lemma from Chapter X, §A.1, now yields

$$\log|\varphi(x+\mathrm{i})| \leqslant G(x+\mathrm{i}) + \log^+|x|, \qquad x \in \mathbb{R},$$

which, with the previous relation, gives

$$W(x)|\varphi(x+\mathrm{i})| \leqslant \text{const.}\sqrt{(x^2+1)}, \qquad x \in \mathbb{R}.$$

However, $\sigma_1(t) + \sigma_2(t) \longrightarrow \infty$ for $t \longrightarrow \infty$, so $\varphi(z)$ certainly *has* zeros.

Dividing out any one of them then yields a new entire function, $\psi(z)$, *also of exponential type* $\pi\eta$, with

$$W(x)|\psi(x+\mathrm{i})| \leqslant \text{const.}, \qquad x \in \mathbb{R}.$$

The number $\eta > 0$ was, however, *arbitrary*. Our weight $W(x)$ therefore *admits multipliers*, as claimed.

To see that there is *no* even function $\Omega(x) \geqslant 1$ with $\log \Omega(x)/x$ in $\mathfrak{H}$ and

$$\int_0^\infty \frac{\log \Omega(x)}{x^2} \mathrm{d}x < \infty$$

such that

$$W(x) \leqslant \Omega(x)$$

for large $|x|$, we use the theorem from the last article. Because $W(0) \geqslant 1$, the relation just written would make

$$\frac{W(x)}{W(0)} \leqslant \Omega(x) \qquad \text{for } |x| \text{ large,}$$

so, since $W(x) \longrightarrow \infty$ for $x \longrightarrow \pm\infty$ as we have noted, the theorem referred to is applicable *provided that*

$$\log\left(\frac{W(x)}{W(0)}\right) = -\int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\rho(t),$$

where $\rho(t)$ is an *increasing*, *infinitely differentiable* odd function *defined on* $\mathbb{R}$, with $\rho(t)/t$ *bounded* for $t > 0$.

In our present circumstances,

$$\log\left(\frac{W(x)}{W(0)}\right) = F(\mathrm{i}) - F(x+\mathrm{i})$$

where, as already pointed out,

$$F(z) = \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}(\nu(t) - \mu(t))$$

with $\nu(t) - \mu(t)$ increasing and $O(t)$ on $[0, \infty)$. Taking note of the identity

$$\left|1 - \frac{z+\mathrm{i}}{t}\right| = \left|1 - \frac{z}{t-\mathrm{i}}\right|\left|1 - \frac{\mathrm{i}}{t}\right|, \qquad t \in \mathbb{R},$$

we see that

$$F(z+\mathrm{i}) \; - \; F(\mathrm{i}) \;\; = \;\; \int_0^{\infty} \log\left|\left(1 \; - \; \frac{z}{t-\mathrm{i}}\right)\left(1 \; + \; \frac{z}{t+\mathrm{i}}\right)\right| \mathrm{d}(\nu(t) - \mu(t)).$$

Now for any particular z, $\Im z \geqslant 0$, the function of w equal to $\log|1 + (z/w)|$ is *harmonic* for $\Im w > 0$. We can thence conclude, just as in proving the first lemma of §C.5, Chapter VIII, that the right-hand integral in the preceding relation is equal to

$$\int_0^{\infty} \log\left|1 \; - \; \frac{z^2}{t^2}\right| \mathrm{d}\rho(t) \qquad \text{for } \Im z \; \geqslant \; 0,$$

with an absolutely continuous increasing function $\rho(t)$ defined on $\mathbb{R}$, having there the derivative

$$\frac{\mathrm{d}\rho(t)}{\mathrm{d}t} \;\; = \;\; \frac{1}{\pi}\int_0^{\infty}\left(\frac{1}{(t-\tau)^2+1} \; + \; \frac{1}{(t+\tau)^2+1}\right)\mathrm{d}(\nu(\tau) - \mu(\tau)).$$

Infinite differentiability of $\rho(t)$ is manifest from the last formula. Taking

$$\rho(0) \;\; = \;\; 0$$

(which makes $\rho(t)$ *odd*), we can also verify boundedness of $\rho(t)/t$ in $(0, \infty)$ without much difficulty. One way is to simply refer to the *second* lemma of §C.5, Chapter VIII, using the formula

$$F(z+\mathrm{i}) \; - \; F(\mathrm{i}) \;\; = \;\; \int_0^{\infty} \log\left|1 \; - \; \frac{z^2}{t^2}\right| \mathrm{d}\rho(t), \qquad \Im z \; \geqslant \; 0,$$

just established together with the fact noted above that $F(z) \leqslant \pi|z| + \mathrm{o}(|z|)$ for large $|z|$. We see in this way that the *hypothesis of the theorem from the preceding article is fulfilled* for the function

$$\omega(x) \;\; = \;\; \log\left(\frac{W(x)}{W(0)}\right) \;\; = \;\; F(\mathrm{i}) \; - \; F(x+\mathrm{i}).$$

According to that theorem, *if an Ω having the properties described above did exist, we would have*

$$\int_0^{\infty} \frac{\omega(x)}{x^2}\,\mathrm{d}\rho(x) \quad < \quad \infty,$$

or, what comes to the same thing,

$$\int_1^{\infty} \frac{F(x+\mathrm{i})}{x^2}\,\mathrm{d}\rho(x) \quad > \quad -\infty,$$

$\rho(x)$ being increasing and $O(x)$. It thus *suffices to prove that*

$$\int_1^\infty \frac{F(x+\mathrm{i})}{x^2}\,\mathrm{d}\rho(x) \quad = \quad -\infty$$

in order to show that no such function Ω *can exist.*

For this purpose, we first obtain an *upper* bound on $F(x+\mathrm{i})$ for x near one of the points x_p, arguing somewhat as in article 1. Given $x > 0$ and $0 < r \leqslant x$, denote by $N(r,\ x+\mathrm{i})$ the quantity

$$\int_J \mathrm{d}(\nu(t) - \mu(t)),$$

where J is the *intersection* of the *disk* of radius r about $x+\mathrm{i}$ with the *real axis*:

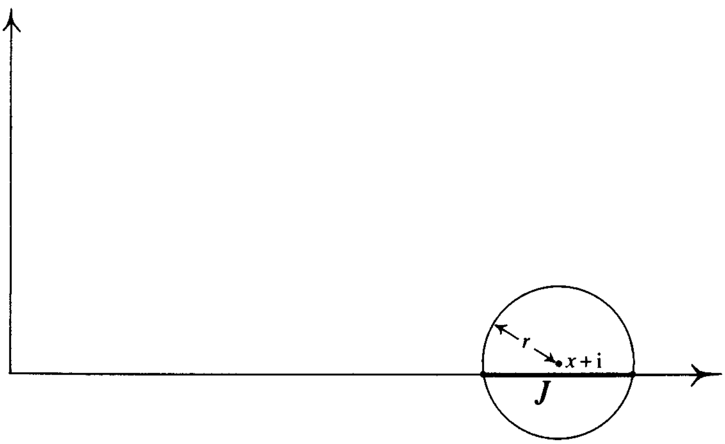

Figure 254

Keeping in mind the relation

$$F(z) \quad = \quad \int_0^\infty \log\left|1 \ - \ \frac{z^2}{t^2}\right| \mathrm{d}(\nu(t) - \mu(t)),$$

we see then, by an evident adaptation of Jensen's formula (cf. near the beginning of the proof of the *first* theorem, §B.3), that

$$F(x+\mathrm{i}) \quad \leqslant \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x+\mathrm{i}+R\mathrm{e}^{\mathrm{i}\vartheta})\,\mathrm{d}\vartheta \quad - \quad \int_0^R \frac{N(r,\ x+\mathrm{i})}{r}\,\mathrm{d}r$$

as long as $R \leqslant x$. (Some such restriction on R is *necessary* in order to ensure that the disk of radius R about $x+\mathrm{i}$ *not intersect* with the *negative real axis.*) We use this formula for

$$x_p - 1 \leqslant x \leqslant x_p + 1$$

with p large, remembering that the increasing function $\nu(t) - \mu(t)$ *jumps upwards by* $(1-\lambda_p)\Delta_p$ *units* at $t = x_p$. That makes

$$N(r,\ x+\mathrm{i}) \geqslant (1-\lambda_p)\Delta_p$$

for such x as soon as r exceeds $\sqrt{2}$. Since $\Delta_p/x_p \longrightarrow 0$ for $p \longrightarrow \infty$ we may, for large p, take $R = \sqrt{2}\,\Delta_p$ in the formula, which, in view of the relation just written, then yields

$$F(x+\mathrm{i}) \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x+\mathrm{i}+\sqrt{2}\,\Delta_p e^{\mathrm{i}\vartheta})\,\mathrm{d}\vartheta - (1-\lambda_p)\Delta_p \log \Delta_p$$

for $x_p - 1 \leqslant x \leqslant x_p + 1$.

From the relations $F(x) \leqslant$ const., $x \in \mathbb{R}$, and $F(z) \leqslant \pi|z| + o(|z|)$ it now follows by the third Phragmén–Lindelöf theorem of §C, Chapter III, that

$$F(x+\mathrm{i}+\sqrt{2}\,\Delta_p e^{\mathrm{i}\vartheta}) \leqslant \text{const.} + \sqrt{2}\,\pi\Delta_p|\sin\vartheta|.$$

Plugging this into the preceding inequality, we find that

$$F(x+\mathrm{i}) \leqslant \text{const.} + 2\sqrt{2}\,\Delta_p - (1-\lambda_p)\Delta_p \log \Delta_p$$

for $x_p - 1 \leqslant x \leqslant x_p + 1$, p being large.

We need also to see how much $\rho(t)$ increases on the interval $[x_p - 1,\ x_p + 1]$. Because $\nu(t) - \mu(t)$ has a jump of magnitude $(1-\lambda_p)\Delta_p$ at x_p, we see by the above formula for $\rho'(t)$ that

$$\frac{\mathrm{d}\rho(t)}{\mathrm{d}t} \geqslant \frac{(1-\lambda_p)\Delta_p}{\pi((t-x_p)^2+1)}.$$

Integrating, we get

$$\int_{x_p-1}^{x_p+1} \mathrm{d}\rho(x) \geqslant \frac{1}{2}(1-\lambda_p)\Delta_p.$$

We can simplify our work at this point by specifying the sequence $\{\lambda_p\}$ *in precise fashion. Take, namely,*

$$\lambda_p = 1 - \frac{1}{\sqrt{(\log p)}}, \qquad p \geqslant 8.$$

Then, for large p,

$$(1-\lambda_p)\Delta_p \log \Delta_p \quad \sim \quad \frac{\exp(p^{1/3})}{3p^{1/3}(\log p)^{1/2}}$$

is *much bigger* than

$$2\sqrt{2}\,\Delta_p \quad \sim \quad \frac{2\sqrt{2}\,\exp(p^{1/3})}{3p^{2/3}},$$

and thus *the third term in our last estimate for* $F(x+\mathrm{i})$ *will greatly outweigh both of the first two.* When $p \geqslant$ a certain p_0, the estimate therefore reduces to

$$F(x+\mathrm{i}) \quad \leqslant \quad -\frac{1}{2}(1-\lambda_p)\Delta_p \log \Delta_p, \qquad x_p - 1 \leqslant x \leqslant x_p + 1.$$

Use this relation together with the one just found involving ρ. That gives

$$\int_{x_p-1}^{x_p+1} F(x+\mathrm{i})\,\mathrm{d}\rho(x) \quad \leqslant \quad -\frac{1}{4}(1-\lambda_p)^2\,\Delta_p^2 \log \Delta_p, \qquad p \geqslant p_0.$$

Here, $F(x+\mathrm{i})$ is, as we know, bounded above for real x, and the increasing function $\rho(x)$ is $\mathrm{O}(x)$. The *divergence* of

$$\int_1^\infty \frac{F(x+\mathrm{i})}{x^2}\,\mathrm{d}\rho(x)$$

to $-\infty$ is hence *implied* by that of the *sum*

$$\sum_{p \geqslant p_0} \frac{1}{x_p^2} \int_{x_p-1}^{x_p+1} F(x+\mathrm{i})\,\mathrm{d}\rho(x).$$

By the preceding inequality,

$$\frac{1}{x_p^2}\int_{x_p-1}^{x_p+1} F(x+\mathrm{i})\,\mathrm{d}\rho(x) \quad \leqslant \quad -\frac{1}{4}(1-\lambda_p)^2\left(\frac{\Delta_p}{x_p}\right)^2 \log \Delta_p$$

for $p \geqslant p_0$, and the right side is

$$\sim \quad -\,\frac{1}{4}(1-\lambda_p)^2\cdot\frac{1}{9}p^{-4/3}\cdot p^{1/3} \quad = \quad -\,\frac{1}{36p\log p}$$

for $p \longrightarrow \infty$. So, since $\sum_p (1/p\log p)$ is divergent, we do indeed have

$$\int_1^\infty \frac{F(x+\mathrm{i})}{x^2}\,\mathrm{d}\rho(x) \quad = \quad -\infty.$$

Therefore, *no* even function $\Omega(x) \geqslant 1$ having the properties stated above and with

$$W(x) \leqslant \Omega(x)$$

for large $|x|$ *can exist. Nevertheless,* $W(x)$ *admits multipliers.*

5. Further discussion and a conjecture

At this point, the reader may have the impression that we have been merely raising up straw men in order to knock them down again, but that is not so. Considerable insight about the nature of the 'essential' condition we are seeking may be gained by studying the examples constructed above.

Suppose that we have an even weight $W(x) \geqslant 1$ meeting our local regularity requirement, with

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2}\,\mathrm{d}x \quad < \quad \infty.$$

Let us, for purposes of discussion, also assume $W(x)$ to be infinitely differentiable. In these circumstances, the odd function

$$u(x) \quad = \quad \frac{1}{x}\log\left(\frac{W(x)}{W(0)}\right)$$

has a $\mathscr{C}_\infty$ Hilbert transform*

$$\tilde{u}(x) \quad = \quad \frac{1}{\pi}\,⨍_{-\infty}^{\infty} \frac{u(t)}{x-t}\,\mathrm{d}t,$$

and it is frequently possible to justify the formula

$$u(x) \quad = \quad \frac{1}{\pi}\int_0^{\infty} \log\left|\frac{x+t}{x-t}\right| \tilde{u}'(t)\,\mathrm{d}t$$

by an argument like the one made near the end of the proof of the *last* theorem in §C.4. Provided that $|\tilde{u}(t)|$ does not get very big for $t \longrightarrow \infty$, further manipulation will yield

$$\log\left(\frac{W(x)}{W(0)}\right) \quad = \quad xu(x) \quad = \quad -\frac{1}{\pi}\int_0^{\infty} \log\left|1 \; - \; \frac{x^2}{t^2}\right| \mathrm{d}(t\tilde{u}(t)).$$

* regarding the infinite differentiability of $\tilde{u}(x)$, cf. initial footnote to the *third* lemma of §E.1 below

In this article, let us not worry further about the restrictions on W needed in order to justify these transformations; what we *have* is a representation of the form

$$\log W(x) = \log W(0) - \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\lambda(t)$$

for a fairly general collection of weights W, involving *signed* (and very smooth) measures λ on $[0, \infty)$. How is it for the admittance of multipliers by such weights?

We can see already from the work of §C.2 that *negative measures λ are 'good'* insofar as this question is concerned. In the case of a weight with convergent logarithmic integral given by such a measure λ, one readily shows with help of the argument in §H.2, Chapter III, that the *increasing* function

$$-\lambda(t) = -\int_0^t d\lambda(\tau)$$

must be $O(t)$ on $[0, \infty)$. The proof of the Theorem on the Multiplier in §C.2 may then be taken over, essentially without change, to conclude that $W(x)$ admits multipliers.*

From this point of view, *positive measures λ are 'bad'*; the example in article 1 shows that weights with convergent logarithmic integrals given by *positive* λ's *need not* admit multipliers.

How bad is bad? The example in article 4 *does*, after all, furnish a weight admitting multipliers and given by a positive measure λ. The first thing to be observed is that absolutely continuous λ's with $\lambda'(t)$ *bounded above* on $[0, \infty)$ are *just as good* as the *negative* ones. For, since

$$\int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| dt = 0, \qquad x \in \mathbb{R},$$

we have, for any weight $W(x)$ given by such a λ with $\lambda'(t) \leqslant K$, say,

$$\log W(x) = \log W(0) - \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d(\lambda(t) - Kt),$$

showing that W is also given by the *negative measure* ρ with $d\rho(t) = d\lambda(t) - K\,dt$. Things can hence go wrong only for measures λ with $\lambda'(t)$ *very large* in certain places. It is therefore reasonable, when trying to find out *how* the *positive part* of a *signed measure* λ can bring about *failure of the weight given by it to admit multipliers*, to *slough off*

* See also the footnote on p. 556.

from λ its *portions having densities bounded above by ever larger constants,* and then *look each time at what is left.* That amounts to examining the behaviour of

$$\max(\lambda'(t),\ K) \;-\; K$$

on $[0, \infty)$ for larger and larger values of K.

The weights constructed in articles 1 and 4 (one *admitting* multipliers and the other *not*) are given by positive measures λ so similar in behaviour that something should be learned by treating those measures in the way described. It is better to first look at the measure giving the weight *of article 4.*

For that weight W we had

$$\log W(x) \;=\; \log W(0) \;-\; \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\rho(t)$$

with the absolutely continuous (indeed, $\mathscr{C}_\infty$) positive measure ρ furnished by the formula

$$\frac{\mathrm{d}\rho(t)}{\mathrm{d}t} \;=\; \frac{1}{\pi}\int_0^\infty \left(\frac{1}{(t-\tau)^2 + 1} + \frac{1}{(t+\tau)^2+1}\right)\mathrm{d}(\nu(\tau) - \mu(\tau)).$$

Here, $\nu(\tau)$ and $\mu(\tau)$, as well as the difference $\nu(\tau) - \mu(\tau)$ figuring in the integral, are increasing functions. The function $\mu(t)$, equal, in the notation of the last article, to

$$Bt\int_t^\infty \frac{T(\tau)}{\tau^2}\,\mathrm{d}\tau,$$

is *absolutely continuous,* with *bounded derivative,* and the behaviour of $\nu(t)$ is shown by the figure at the beginning of article 4. The latter consists of an *absolutely continuous part,* again with *bounded derivative,* together with a *singular* part having *jumps of magnitude* $(1-\lambda_p)\Delta_p$ *at the points* x_p, $p \geqslant 8$. The difference $\nu(t) - \mu(t)$ has therefore the *same description,* and, since $(1-\lambda_p)\Delta_p$ and $x_p - x_{p-1}$ both tend to ∞ with p in our example, the function

$$\rho(t) \;=\; \int_0^t \rho'(\tau)\,\mathrm{d}\tau,$$

really nothing but a regularized version of that difference, shows almost the same behaviour as the latter for large t, except for being somewhat smoother.

Thus, when K is big, a good representation of the graph of the residual function

$$\rho_K(t) = \int_0^t (\max(\rho'(\tau), K) - K)\,d\tau$$

will, for large values of t, be provided by one simply showing the *jumps* of $\nu(t)$ that go to make up the singular part of $\nu(t) - \mu(t)$.

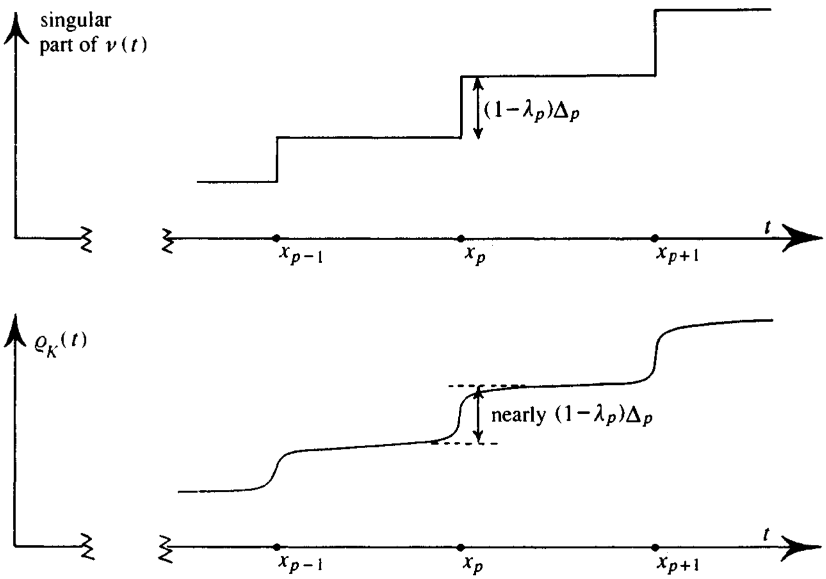

Figure 255

We turn to the weight W considered in *article 1*. In the notation of that article, it is given by the formula

$$W(x) = \frac{\text{const.}}{|F(x+i)\varphi(x+i)|},$$

where $F(z)$ and $\varphi(z)$ are certain even entire functions, of exponential type π and η respectively, having only real zeros. For the first of these, we had

$$\log|F(z)| = \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| dn(t)$$

with a function $n(t)$, increasing by a *jump* of magnitude $[\Delta_p]$ at each of the points x_p, $p \geqslant 8$, and constant on the intervals separating those points (as well as on $[0, x_8)$). The function $\varphi(z)$, obtained from §A.1 of

Chapter X, has the representation

$$\log|\varphi(z)| = \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}[s(t)],$$

with

$$s(t) = \left(\frac{\eta}{\pi}t - \mu_1(t) - 1\right)^+$$

an increasing function formed from a certain $\mu_1(t)$ very much like the $\mu(t)$ appearing in the example of article 4. Thus, although $[s(t)]$ is composed exclusively of *jumps*, it is *based* on the function $(\eta/\pi)t - \mu_1(t)$ which increases *quite uniformly*, having *derivative* between 0 and η/π in value at each $t \geqslant 0$.

Referring to the first lemma of §C.5, Chapter VIII, we see that for the weight W of article 1,

$$\log W(x) = \log W(0) - \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\sigma(t),$$

where $\sigma(t)$ is an absolutely continuous increasing function determined by the relation

$$\frac{\mathrm{d}\sigma(t)}{\mathrm{d}t} = \frac{1}{\pi}\int_0^\infty \left(\frac{1}{(t-\tau)^2+1} + \frac{1}{(t+\tau)^2+1}\right)\mathrm{d}(n(\tau) + [s(\tau)]).$$

By feeding just the increasing function $[s(\tau)]$ into the integral on the right (which has the effect of smoothing out the former's jumps), one obtains an *increasing function* having a *bounded derivative* (given by the integral in question), thanks to the moderate behaviour of $(\eta/\pi)t - \mu_1(t)$ just noted. Therefore, when K is *big*, the residual function

$$\sigma_K(t) = \int_0^t (\max(\sigma'(\tau, K) - K)\mathrm{d}\tau$$

acts, for large t, essentially like $n(t)$, which has the quite substantial jumps of height $[\Delta_p]$ at the points x_p. In this respect, the present situation is much like the one described above corresponding to the *weight from article 4*, involving the functions $\rho_K(t)$ and $v(t)$.

If now we compare the graph of $\rho_K(t)$, corresponding to the weight *admitting* multipliers, with the one for $\sigma_K(t)$, corresponding to the *weight that does not*, only *one difference* is apparent, and that is in the *relative heights of the steps*. Wishing to arrive at a quantitative notion of this difference, one soon thinks of performing the F. Riesz construction on

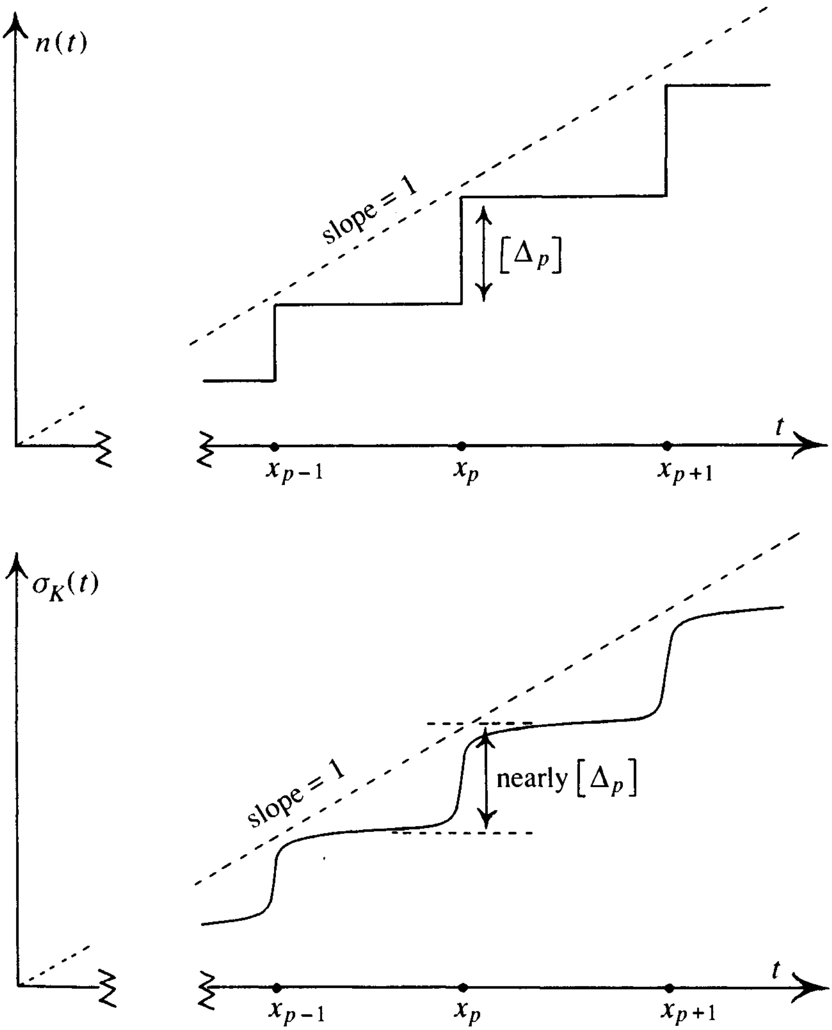

Figure 256

both graphs, letting light shine downwards on each of them from the right along a direction of small positive slope. On account of the great similarity just described between the graphs of $\sigma_K(t)$ and $n(t)$ for large t, and between those of $\rho_K(t)$ and the singular part of $\nu(t)$, it seems quite certain that we will (for large t) arrive at the same results by instead carrying out the Riesz construction for $n(t)$ and for the singular part of $\nu(t)$. This we do in order to save time, simply *assuming*, without bothering to verify the fact, that the results thus obtained really are the same as those that would be gotten (for large t), were the constructions to be made for $\sigma_K(t)$ and for $\rho_K(t)$. We are, after all, trying to *find* a theorem and not to *prove* one!

Taking, then, any small $\delta > 0$, we look at the set of *large* t with the property that

$$\frac{\text{sing. part of } \nu(t') \;-\; \text{sing. part of } \nu(t)}{t' - t} \;>\; \delta$$

for some $t' > t$ (depending, of course, on t). Here a crucial rôle is played by the fact that

$$\lambda_p \longrightarrow 1 \quad \text{as } p \longrightarrow \infty.$$

That makes $1 - \lambda_p < \delta$ for large enough p, and then the *jump* which $\nu(t)$ has at x_p, equal to $(1 - \lambda_p)\Delta_p$, will be $< \delta\Delta_p$, with $\Delta_p = x_p - x_{p-1}$, the distance from x_p to the *preceding* point of discontinuity for $\nu(t)$. Therefore the t fulfilling the last condition will, beyond a certain point, all lie in a collection of *disjoint intervals* $(x'_p,\ x_p)$ with

$$x_p - x'_p \;=\; \frac{1 - \lambda_p}{\delta}\Delta_p \;<\; \Delta_p$$

and

$$\frac{\text{sing. part of } \nu(x_p) \;-\; \text{sing. part of } \nu(x'_p)}{x_p - x'_p} \;=\; \delta.$$

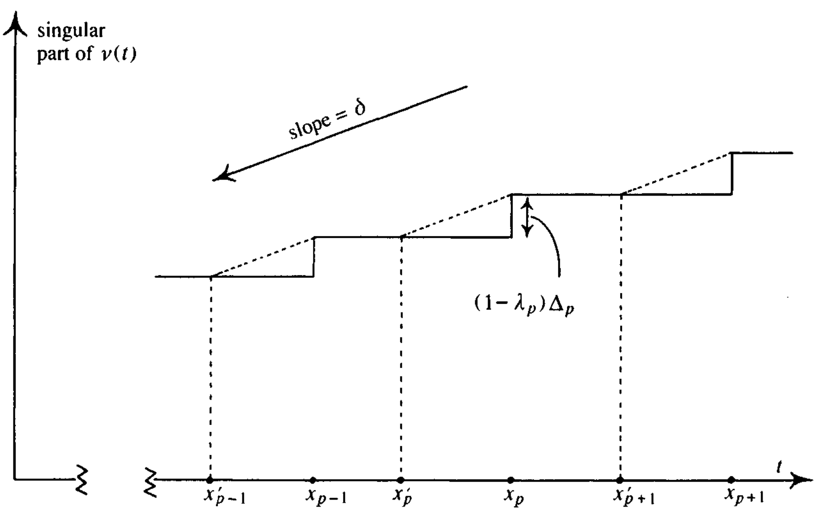

Figure 257

How *big* are the intervals $(x'_p,\ x_p)$? In the present circumstances,

$$\Delta_p \quad \sim \quad \frac{1}{3} p^{-2/3} x_p \qquad \text{for } p \longrightarrow \infty,$$

so then

$$\frac{x_p - x'_p}{x'_p} \quad < \quad \frac{\Delta_p}{x_p - \Delta_p} \quad \sim \quad \frac{1}{3} p^{-2/3},$$

and we have

$$\sum_p \left(\frac{x_p - x'_p}{x'_p} \right)^2 \quad < \quad \infty;$$

the intervals (x'_p, x_p) *satisfy the Beurling condition that has played such an important rôle in this book!*

What can we (with almost certain confidence) conclude from this about the residual functions $\rho_K(t)$? The function $\rho(t)$ is, after all, $\mathscr{C}_\infty$, so a large enough K will *swamp out the derivative* $\rho'(t)$ for *all save the very large values of* t. The residual $\rho_K(t)$ will, in other words, *stay equal to zero* until t gets so large that the singular part of $\nu(t)$ shows the behaviour just described; *thereafter*, however, $\rho_K(t)$ and the latter function have *almost the same behaviour*, as we have seen. This means that ***for given*** $\delta > 0$, ***we can, by making*** K ***sufficiently large, ensure that*** $\rho'_K(t) \leqslant \delta$ ***for all*** $t \geqslant 0$ ***save those belonging to a certain collection of disjoint intervals*** (a_n, b_n) ***(like the*** (x'_p, x_p) ***), with***

$$\frac{\rho_K(b_n) - \rho_K(a_n)}{b_n - a_n} \quad = \quad \delta$$

and

$$\sum_n \left(\frac{b_n - a_n}{a_n} \right)^2 \quad < \quad \infty.$$

Now what distinguishes the functions $\sigma_K(t)$ from the $\rho_K(t)$ is that *the analogous statement does not hold for the former when* $\delta < 1$. This is evident if we look at the graph of $n(t)$ which, for large enough t, is almost the same as that of any of the $\sigma_K(t)$. When $\delta < 1$, the Riesz construction, applied to $n(t)$, *will not even yield an infinite sequence of disjoint intervals like the* (x'_p, x_p); instead, one simply obtains a single big interval of infinite length. That's because at each x_p, $n(t)$, instead of jumping by a *small multiple* of Δ_p, *jumps by* $[\Delta_p]$, which is, for all intents and purposes, *the same as* $\Delta_p = x_p - x_{p-1}$ when p is large.

The *size* of these jumps of $n(t)$ was, by the way, the *key property ensuring* that the weight constructed in *article 1 did not admit multipliers of* *exponential type* $< \pi$. Cutting the jumps down to $(1-\lambda_p)\Delta_p$ for the construction in *article 4* was also what *made* the weight obtained there *admit multipliers*; it did so because $\lambda_p \longrightarrow 1$ as $p \longrightarrow \infty$. That, however, is just what guarantees the truth of the above statement about the $\rho_K(t)$! It thus seems likely that the distinction we have observed between behaviour of the $\rho_K(t)$ and that of the $\sigma_K(t)$ is the *source* of the corresponding two weights' difference in behaviour regarding the admittance of multipliers. *The 'essential' condition we have been seeking may well involve a requirement that the above statement hold for the* ρ_K *corresponding to a certain function* ρ, *associated with whatever weight one may have under consideration.*

Having been carried thus far by inductive reasoning, let us continue on grounds of pure speculation. We have been looking at $\mathscr{C}_\infty$ weights $W(x) \geqslant 1$ corresponding to monotone $\mathscr{C}_\infty$ functions $\lambda(t)$ according to the formula

$$\log W(x) \quad = \quad \log W(0) \quad - \quad \int_0^\infty \log\left|1 \; - \; \frac{x^2}{t^2}\right| \mathrm{d}\lambda(t).$$

Mostly, we have been considering *increasing* functions λ, and we have come around to the view that a weight W corresponding to one of these admits multipliers if (and, in *some* sense, *only if*) the above statement holds, with the functions

$$\lambda_K(t) \quad = \quad \int_0^t (\max(\lambda'(\tau),\ K) \; - \; K)\mathrm{d}\tau$$

standing in place of the $\rho_K(t)$. Insofar as *decreasing* functions $\lambda(t)$ were concerned, we simply observed near the beginning of this article that they were *good*, for a weight W given by any of *those admits* multipliers as long as

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2}\mathrm{d}x \quad < \quad \infty.$$

Let us now *drop* any requirement that the function $\lambda(t)$ be monotone, but *keep* the criterion that *the above statement hold for the* $\lambda_K(t)$.

The *increase* of $\lambda(t)$ is thereby *limited*, but *not* its *decrease*! Observe that for any $\mathscr{C}_\infty$ function $\omega(x)$ of the form

$$\omega(x) \quad = \quad \omega(0) \quad - \quad \int_0^\infty \log\left|1 \; - \; \frac{x^2}{t^2}\right| \mathrm{d}\lambda(t)$$

with

$$\int_{-\infty}^{\infty} \frac{|\omega(x)|}{1+x^2} dx \quad < \quad \infty,$$

the *Hilbert transform* $\tilde{\omega}(x)$ is defined (everywhere) and infinitely differentiable*, and *it differs from* $\pi\lambda(x)$ *by a constant multiple of* x. The statement involving the $\lambda_K(t)$ may thus be rephrased in terms of the Hilbert transform of $\log W(x)$, *eliminating* any direct reference to a *particular representation* for W.

Let us go one step further and guess at a criterion applicable to *any* weight $W(x)$ meeting the local regularity requirement of §B.1. Here, we *give up* trying to have the Hilbert transform of $\log W(x)$ fit the above statement. Instead, we let the latter apply to $\tilde{\omega}(x)$, where $\exp \omega(x)$ is *some even* $\mathscr{C}_\infty$ *majorant* of $W(x)$, as is in keeping with the guiding idea of this §. In that way, we arrive at the following

Conjecture. *A weight* $W(x) \geqslant 1$ *meeting the local regularity requirement admits multipliers iff it has an even* $\mathscr{C}_\infty$ *majorant* $\Omega(x)$ *with the following properties*:

(i) $$\int_{-\infty}^{\infty} \frac{\log \Omega(x)}{1+x^2} dx \quad < \quad \infty,$$

(ii) *To any* $\delta > 0$ *corresponds a* K *such that the* ($\mathscr{C}_\infty$) *Hilbert transform* $\tilde{\omega}(x)$ *of* $\omega(x) = \log \Omega(x)$ *has* derivative $\leqslant K + \delta$ *at all positive* x *save those contained in a set of* disjoint *intervals* (a_n, b_n), *with*

$$\sum_n \left(\frac{b_n - a_n}{a_n} \right)^2 \quad < \quad \infty$$

and

$$\int_{a_n}^{b_n} (\max(\tilde{\omega}'(x), K) - K) dx \quad \leqslant \quad \delta(b_n - a_n)$$

for each n.

E. A necessary and sufficient condition for weights meeting the local regularity requirement

The conjecture advanced at the end of the last § is true. One may look on its statement as an expression of the 'essential' condition for the admittance of multipliers that we had set out in that § to bring to light. A proof, which turns out to be not all that difficult, involves techniques

* cf. initial footnote to the *third* lemma of §E.1 below

like those employed in the determination of the completeness radius for a set of imaginary exponentials, carried out in Chapters IX and X. That proof is given below in article 2. Some auxiliary results are needed for it; we attend to those first.

1. Five lemmas

Lemma. *Let $v(t)$ be increasing on $[0, \infty)$ with $v(t)/t$ bounded for $t > 0$, and put*

$$F(x) \quad = \quad \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}v(t) \qquad \text{for } x \in \mathbb{R}.$$

Suppose that $\rho(\xi)$, positive and infinitely differentiable, has compact support in $(0, \infty)$. Then the function

$$F_\rho(x) \quad = \quad \int_0^\infty F(\xi x)\frac{\rho(\xi)}{\xi}\,\mathrm{d}\xi$$

is equal to

$$\int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}v_\rho(t),$$

where

$$v_\rho(t) \quad = \quad \int_0^\infty v(\xi t)\frac{\rho(\xi)}{\xi}\,\mathrm{d}\xi$$

is increasing and $\mathscr{C}_\infty$ in $(0, \infty)$, with

$$v'_\rho(t) \;\leqslant\; \text{const.} \quad \text{for } t > 0.$$

Proof. Is essentially an exercise about multiplicative convolution. Because the function $\log|1 - (x^2/t^2)|$ is neither bounded above nor below the justification of the transformations involved is a bit tricky.

Let us, as usual, write

$$F(z) \quad = \quad \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}v(t)$$

for complex z, noting that $F(x + \mathrm{i}y)$, for fixed $x \in \mathbb{R}$, is an *increasing* function of y when $y \geqslant 0$ (because $v(t)$ increases), and that

$$F(z) \;\leqslant\; \text{const.}\,|z|$$

(because $v(t)$ is also $\mathrm{O}(t)$ for $t \geqslant 0$). Supposing, then, that $\rho(\xi)$ has its support in $[a, b]$, $0 < a < b < \infty$, we have, for each fixed $x \in \mathbb{R}$,

$$F_\rho(x) \quad = \quad \int_a^b F(\xi x)\,\frac{\rho(\xi)}{\xi}\,d\xi \quad = \quad \lim_{y\to 0}\int_a^b F(\xi(x+iy))\,\frac{\rho(\xi)}{\xi}\,d\xi$$

by monotone convergence*.

For $z = x + iy$ with $y \neq 0$ it is easy, thanks to the properties of v, to show by partial integration that

$$F(z) \quad = \quad 2\Re\int_0^\infty \frac{z^2}{z^2-t^2}\,\frac{v(t)}{t}\,dt.$$

Hence

$$\int_a^b F(\xi z)\,\frac{\rho(\xi)}{\xi}\,d\xi \quad = \quad 2\Re\int_a^b\int_0^\infty \frac{\xi^2 z^2}{\xi^2 z^2 - t^2}\,\frac{v(t)}{t}\,\frac{\rho(\xi)}{\xi}\,dt\,d\xi.$$

On making the change of variable $t/\xi = \tau$, this becomes

$$2\Re\int_a^b\int_0^\infty \frac{z^2}{z^2-\tau^2}\,\frac{v(\xi\tau)}{\tau}\,\frac{\rho(\xi)}{\xi}\,d\tau\,d\xi.$$

Here, it is legitimate to change the order of integration, for the double integral is absolutely convergent. The last expression is thus equal to

$$2\Re\int_0^\infty \frac{z^2}{z^2-\tau^2}\,\frac{v_\rho(\tau)}{\tau}\,d\tau,$$

where

$$v_\rho(\tau) \quad = \quad \int_a^b \frac{v(\xi\tau)}{\xi}\,\rho(\xi)\,d\xi.$$

Since $\rho(\xi) \geqslant 0$ and $v(t)$ is increasing, $v_\rho(\tau)$ is also increasing. For $\tau > 0$, we can make the change of variable $\xi\tau = s$ in the preceding integral, getting

$$v_\rho(\tau) \quad = \quad \int_{a\tau}^{b\tau} \rho\left(\frac{s}{\tau}\right)\frac{v(s)}{s}\,ds.$$

Noting that $\rho(\xi)$ is $\mathscr{C}_\infty$ and vanishes (together with all its derivatives) for $\xi = a$ and $\xi = b$, we see that the expression on the right can be differentiated with respect to τ as many times as we want, making $v_\rho(\tau)$ $\mathscr{C}_\infty$ for $\tau > 0$. For the first derivative, we find

$$v_\rho'(\tau) \quad = \quad -\int_{a\tau}^{b\tau} \frac{s}{\tau^2}\,\rho'\left(\frac{s}{\tau}\right)\frac{v(s)}{s}\,ds, \qquad \tau > 0.$$

We have $v(s)/s \leqslant C$, say, for $s > 0$, so, denoting the maximum value of $|\rho'(\xi)|$ for $a \leqslant \xi \leqslant b$ by K, we get for the right side of the last relation a

* the integrand on the right being *bounded above* by the preceding inequality