for $z \in \mathscr{D}$. This is a *formula* for solving the Dirichlet problem for $\mathscr{D}$, based on the conformal mapping function F. Knowledge of this formula will help us later on to get general qualitative information about the behaviour near $\partial\mathscr{D}$ of certain functions harmonic in $\mathscr{D}$ but *not* continuous up to $\partial\mathscr{D}$, even when $\mathscr{D}$ is *not* simply connected.

Let us return to the multiply connected domains $\mathscr{D}$ of the kind considered here. If φ is real and continuous on $\partial\mathscr{D}$ and U_φ, harmonic in $\mathscr{D}$ and continuous on $\bar{\mathscr{D}}$, agrees with φ on $\partial\mathscr{D}$, we have, by the principle of maximum,

$$-\|\varphi\|_\infty \leqslant U_\varphi(z) \leqslant \|\varphi\|_\infty$$

for each $z \in \mathscr{D}$; here we are writing

$$\|\varphi\|_\infty = \sup_{\zeta \in \partial\mathscr{D}} |\varphi(\zeta)|.$$

This shows in the first place that *there can only be one function U_φ corresponding to a given function φ*. We see, secondly, that *there must be a* (signed) *measure μ_z* on $\partial\mathscr{D}$ (depending, of course, on z) with

$$(*) \qquad U_\varphi(z) = \int_{\partial\mathscr{D}} \varphi(\zeta) \mathrm{d}\mu_z(\zeta).$$

The latter statement is simply a consequence of the Riesz representation theorem applied to the space $\mathscr{C}(\partial\mathscr{D})$. Since U_φ can be found for *every* $\varphi \in \mathscr{C}(\partial\mathscr{D})$ (i.e., the Dirichlet problem for $\mathscr{D}$ can be solved!) and since, corresponding to each given φ, there is only *one* U_φ, there can, for any $z \in \mathscr{D}$, be *only one* measure μ_z on $\partial\mathscr{D}$ such that $(*)$ is true with every $\varphi \in \mathscr{C}(\partial\mathscr{D})$. The measure μ_z is thus *a function of* $z \in \mathscr{D}$, and we proceed to make a gross examination of its dependence on z.

If $\varphi(\zeta) \geqslant 0$ we must have $U_\varphi(z) \geqslant 0$ throughout $\mathscr{D}$ by the principle of maximum. Referring to $(*)$, we see that *the measures μ_z must be positive.* Also, 1 is a harmonic function (!), so, if $\varphi(\zeta) \equiv 1$, $U_\varphi(z) \equiv 1$. Therefore

$$\int_{\partial\mathscr{D}} \mathrm{d}\mu_z(\zeta) = 1$$

for every $z \in \mathscr{D}$. Let $\zeta_0 \in \partial\mathscr{D}$ and consider any small fixed neighborhood $\mathscr{V}$ of ζ_0. Take any continuous function φ on $\partial\mathscr{D}$ such that $\varphi(\zeta_0) = 0$, $\varphi(\zeta) \equiv 1$ for $\zeta \notin \mathscr{V} \cap \partial\mathscr{D}$, and $0 \leqslant \varphi(\zeta) \leqslant 1$ on $\mathscr{V} \cap \partial\mathscr{D}$.

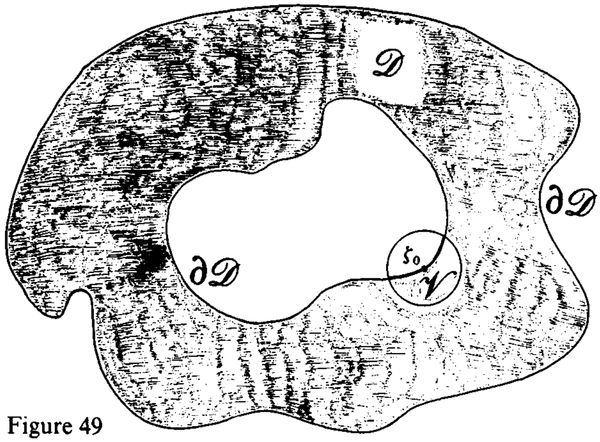

Figure 49

Since U_φ is a solution of the Dirichlet problem, we certainly have

$$U_\varphi(z) \longrightarrow \varphi(\zeta_0) = 0 \quad \text{for} \quad z \longrightarrow \zeta_0.$$

The positivity of the μ_z therefore makes

$$\int_{\zeta \notin \mathscr{V} \cap \partial\mathscr{D}} d\mu_z(\zeta) \longrightarrow 0$$

for $z \longrightarrow \zeta_0$. Because all the μ_z have total mass 1, we must also have

$$\int_{\mathscr{V} \cap \partial\mathscr{D}} d\mu_z(\zeta) \longrightarrow 1 \quad \text{for} \quad z \longrightarrow \zeta_0.$$

When z is near $\zeta_0 \in \partial\mathscr{D}$, μ_z has almost all of its total mass (1) *near* ζ_0 (on $\partial\mathscr{D}$). This is the so-called *approximate identity property* of the μ_z.

There is also a continuity property for the μ_z applying to variations of *z in the interior of $\mathscr{D}$.*

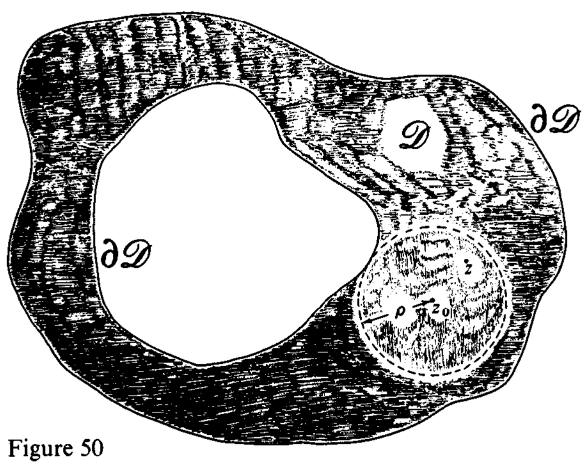

Figure 50

Take any $z_0 \in \mathscr{D}$, write $\rho = \mathrm{dist}(z_0, \partial\mathscr{D})$ and suppose that $|z - z_0| < \rho$. Then, if φ is *continuous and positive on* $\partial\mathscr{D}$,

$$\int_{\partial\mathscr{D}} \varphi(\zeta)\mathrm{d}\mu_z(\zeta)$$

lies between

$$\frac{\rho - |z - z_0|}{\rho + |z - z_0|} \int_{\partial\mathscr{D}} \varphi(\zeta)\mathrm{d}\mu_{z_0}(\zeta)$$

and

$$\frac{\rho + |z - z_0|}{\rho - |z - z_0|} \int_{\partial\mathscr{D}} \varphi(\zeta)\mathrm{d}\mu_{z_0}(\zeta).$$

This is nothing but *Harnack's inequality* applied to the circle $\{|z - z_0| < \rho\}$, $U_\varphi(z)$ being *harmonic and positive* in that circle. (The reader who does not recall Harnack's inequality may derive it very easily from the Poisson representation of positive harmonic functions for the unit disk given in Chapter III, §F.1.) These inequalities hold for *any* positive $\varphi \in \mathscr{C}(\partial\mathscr{D})$, so the signed measures

$$\mu_z \;-\; \frac{\rho - |z - z_0|}{\rho + |z - z_0|}\mu_{z_0},$$

$$\frac{\rho + |z - z_0|}{\rho - |z - z_0|}\mu_{z_0} \;-\; \mu_z$$

are in fact positive. This fact is usually expressed by the double inequality

$$\frac{\rho - |z - z_0|}{\rho + |z - z_0|}\mathrm{d}\mu_{z_0}(\zeta) \;\leqslant\; \mathrm{d}\mu_z(\zeta) \;\leqslant\; \frac{\rho + |z - z_0|}{\rho - |z - z_0|}\mathrm{d}\mu_{z_0}(\zeta).$$

What is important here is that we have a number $K(z, z_0)$, $0 < K(z, z_0) < \infty$, depending *only* on z and z_0 (and $\mathscr{D}$!), such that

$$\frac{1}{K(z, z_0)}\mathrm{d}\mu_{z_0}(\zeta) \;\leqslant\; \mathrm{d}\mu_z(\zeta) \;\leqslant\; K(z, z_0)\mathrm{d}\mu_{z_0}(\zeta).$$

Such an inequality in fact holds for *any* two points z, z_0 in $\mathscr{D}$; one needs only to join z to z_0 by a path lying in $\mathscr{D}$ and then take a chain of overlapping disks $\subseteq \mathscr{D}$ having their centres on that path, applying the previous special version of the inequality in each disk.

In order to indicate the dependence of the measures μ_z in (*) *on the domain* $\mathscr{D}$ *as well as on* $z \in \mathscr{D}$, we use a special notation for them which is now becoming standard. ***We write***

$$\mathrm{d}\omega_{\mathscr{D}}(\zeta, z) \quad \textit{for} \quad \mathrm{d}\mu_z(\zeta),$$

***so that* (*) *has this appearance*:**

$$\boxed{U_\varphi(z) = \int_{\partial\mathscr{D}} \varphi(\zeta)\mathrm{d}\omega_{\mathscr{D}}(\zeta, z).}$$

We call $\omega_{\mathscr{D}}(\ ,z)$ ***harmonic measure for*** $\mathscr{D}$ ***(or relative to*** $\mathscr{D}$***) as seen from*** z***.*** $\omega_{\mathscr{D}}(\ ,z)$ is a positive Radon measure on $\partial\mathscr{D}$, of total mass 1, which serves to recover functions harmonic in $\mathscr{D}$ and continuous on $\bar{\mathscr{D}}$ from their boundary values on $\partial\mathscr{D}$ by means of the boxed formula. That formula is just the analogue of *Poisson's* for our domains $\mathscr{D}$.

If E is a Borel set on $\partial\mathscr{D}$,

$$\omega_{\mathscr{D}}(E,z) = \int_E \mathrm{d}\omega_{\mathscr{D}}(\zeta, z)$$

is called *the harmonic measure of E relative to* $\mathscr{D}$ *(or in* $\mathscr{D}$*), seen from z.* We have, of course,

$$0 \leqslant \omega_{\mathscr{D}}(E,z) \leqslant 1.$$

Also, for fixed $E \subseteq \partial\mathscr{D}$, $\omega_{\mathscr{D}}(E,z)$ is a *harmonic function of z*. This almost obvious property may be verified as follows. Given $E \subseteq \partial\mathscr{D}$, take a sequence of functions $\varphi_n \in \mathscr{C}(\partial\mathscr{D})$ with $0 \leqslant \varphi_n \leqslant 1$ such that

$$\int_{\partial\mathscr{D}} |\chi_E(\zeta) - \varphi_n(\zeta)|\mathrm{d}\omega_{\mathscr{D}}(\zeta, z_0) \underset{n}{\longrightarrow} 0$$

for the characteristic function χ_E of E. Here, z_0 is *any fixed point of* $\mathscr{D}$ which may be chosen at pleasure. Since $\mathrm{d}\omega_{\mathscr{D}}(\zeta,z) \leqslant K(z,z_0)\mathrm{d}\omega_{\mathscr{D}}(\zeta,z_0)$ as we have seen above, the previous relation makes

$$U_{\varphi_n}(z) = \int_{\partial\mathscr{D}} \varphi_n(\zeta)\mathrm{d}\omega_{\mathscr{D}}(\zeta, z) \underset{n}{\longrightarrow} \omega_{\mathscr{D}}(E,z)$$

for every $z\in\mathscr{D}$; the convergence is even u.c.c. in $\mathscr{D}$ because $0 \leqslant U_{\varphi_n}(z) \leqslant 1$ there for each n. Therefore $\omega_{\mathscr{D}}(E,z)$ is harmonic in $z\in\mathscr{D}$ since the $U_{\varphi_n}(z)$ are.

Harmonic measure is also available for many *unbounded* domains $\mathscr{D}$. Suppose we have such a domain (perhaps of infinite connectivity) with a decent boundary $\partial\mathscr{D}$. The latter may consist of infinitely many pieces, but each individual piece should be nice, and they should not accumulate near any finite point in such a way as to cause trouble for the solution of the Dirichlet problem. In such case, $\partial\mathscr{D}$ is at least *locally compact* and, if $\varphi\in\mathscr{C}_0(\partial\mathscr{D})$ (the space of functions continuous on $\partial\mathscr{D}$ which *tend to zero* as one goes out towards ∞ thereon), there is one and *only* one function U_φ *harmonic and bounded* in $\mathscr{D}$, and continuous up to $\partial\mathscr{D}$, with $U_\varphi(\zeta) = \varphi(\zeta)$, $\zeta \in \partial\mathscr{D}$. (Here it is *absolutely necessary* to assume *boundedness* of U_φ in $\mathscr{D}$ in order to get *uniqueness*; look at the function y in $\Im z > 0$ which takes the value 0 on $\mathbb{R}$. Uniqueness of the *bounded* harmonic function with prescribed boundary values is a direct consequence of the *first* Phragmén–Lindelöf theorem in §C, Chapter III.) Riesz' representation theorem still holds in the present situation, and we will have (*) for $\varphi \in \mathscr{C}_0(\partial\mathscr{D})$. The examination of the μ_z carried out above goes through almost without change, and we write $d\mu_z(\zeta) = d\omega_{\mathscr{D}}(\zeta, z)$ as before, calling $\omega_{\mathscr{D}}(\ , z)$ the *harmonic measure for* $\mathscr{D}$, *as seen from* z. It serves to recover *bounded* functions harmonic in $\mathscr{D}$ and continuous up to $\partial\mathscr{D}$ from their boundary values, at least when the latter come from functions in $\mathscr{C}_0(\partial\mathscr{D})$.

Let us return for a moment to *bounded, finitely connected domains* $\mathscr{D}$. Suppose we are given a function $f(z)$, *analytic and bounded in* $\mathscr{D}$, and *continuous up to* $\partial\mathscr{D}$. An important problem in the theory of functions is to *obtain an upper bound for* $|f(z)|$ *when* $z \in \mathscr{D}$, *in terms of the boundary values* $f(\zeta)$, $\zeta \in \partial\mathscr{D}$. A very useful estimate is furnished by the

Theorem (on harmonic estimation). *For* $z \in \mathscr{D}$,

$$(\dagger) \qquad \log|f(z)| \;\leqslant\; \int_{\partial\mathscr{D}} \log|f(\zeta)|\, d\omega_{\mathscr{D}}(\zeta, z).$$

Proof. The result is really a generalization of Jensen's inequality. Take any $M > 0$. The function

$$V_M(z) \;=\; \max(\log|f(z)|,\ -M)$$

is continuous in $\bar{\mathscr{D}}$ and *subharmonic* in $\mathscr{D}$. Therefore the difference

$$V_M(z) \;-\; \int_{\partial\mathscr{D}} V_M(\zeta)\, d\omega_{\mathscr{D}}(\zeta, z)$$

is subharmonic in $\mathscr{D}$ and continuous up to $\partial\mathscr{D}$ where it takes the boundary value $V_M(\zeta) - V_M(\zeta) = 0$ everywhere. Hence that difference is $\leqslant 0$ throughout $\mathscr{D}$ by the principle of maximum, and

$$\log|f(z)| \;\leqslant\; V_M(z) \;\leqslant\; \int_{\partial\mathscr{D}} V_M(\zeta)\, d\omega_{\mathscr{D}}(\zeta, z)$$

for $z \in \mathscr{D}$. On making $M \to \infty$, the right side tends to

$$\int_{\partial\mathscr{D}} \log|f(\zeta)|\, d\omega_{\mathscr{D}}(\zeta, z)$$

by Lebesgue's monotone convergence theorem, since $\log|f(\zeta)|$, and hence the $V_M(\zeta)$, are *bounded above*, $|f(z)|$ being continuous and thus bounded on the compact set $\bar{\mathscr{D}}$. The proof is finished.

The result just established is true for *bounded* analytic functions in *unbounded* domains subject to the restrictions on such domains mentioned above. *Here the boundedness of $f(z)$ in $\mathscr{D}$ becomes crucial* (look at the functions e^{-inz} in $\Im z > 0$ with $n \to \infty$!). Verification of this proceeds very much as above, using the functions $V_M(\zeta)$. These are continuous and bounded (above *and* below) on $\partial\mathscr{D}$, so the functions

$$H_M(z) = \int_{\partial\mathscr{D}} V_M(\zeta) d\omega_{\mathscr{D}}(\zeta, z)$$

are harmonic and bounded in $\mathscr{D}$, and for each $\zeta_0 \in \mathscr{D}$ we can *check directly*, by using the *approximate identity* property of $\omega_{\mathscr{D}}(\ ,z)$ established in the above discussion, that

$$H_M(z) \longrightarrow V_M(\zeta_0) \quad \text{for} \quad z \longrightarrow \zeta_0.$$

(It is *not necessary* that $V_M(\zeta)$ belong to $\mathscr{C}_0(\partial\mathscr{D})$ in order to draw this conclusion; only that it be *continuous and bounded* on $\partial\mathscr{D}$.) The difference

$$V_M(z) - H_M(z)$$

is thus *subharmonic and bounded above* in $\mathscr{D}$, and tends to 0 as z tends to *any* point of $\partial\mathscr{D}$. We can therefore conclude by the first Phragmén–Lindelöf theorem of §C, Chapter III (or, rather, by its analogue for *subharmonic* functions), that $V_M(z) - H_M(z) \leqslant 0$ in $\mathscr{D}$. The rest of the argument is as above.

The inequality (†) has one very important consequence, called the ***theorem on two constants.*** *Let $f(z)$ be analytic and bounded in a domain $\mathscr{D}$ of the kind considered above, and continuous up to $\partial\mathscr{D}$. Suppose that $|f(\zeta)| \leqslant M$ on $\partial\mathscr{D}$, and that there is a Borel set $E \subseteq \partial\mathscr{D}$ with $|f(\zeta)| \leqslant$ some number m ($< M$) on E. Then, for $z \in \mathscr{D}$,*

$$\boxed{|f(z)| \leqslant m^{\omega_{\mathscr{D}}(E,z)} M^{1-\omega_{\mathscr{D}}(E,z)}.}$$

Deduction of this inequality from (†) is immediate.

Much of the importance of harmonic measure in analysis is due to this formula and to (†). For this reason, analysts have devoted (and continue to devote) considerable attention to the *estimation* of harmonic measure. We shall see some of this work later on in the present book. The systematic use of harmonic measure in analysis is mainly due to Nevanlinna, who also gave us the *name* for it. There are beautiful examples of its application in his book, *Eindeutige analytische Funktionen* (now translated into English), of which every analyst should own a copy.

Before ending our discussion of harmonic measure, let us describe a few more of its qualitative properties.

The first observation to be made is that the measures $\omega_{\mathscr{D}}(\ ,z)$ are *absolutely continuous with respect to arc length* on $\partial\mathscr{D}$ for the kind of domains considered here. This will follow if we can show that

$$\omega_{\mathscr{D}}(E_n, z_0) \underset{n}{\longrightarrow} 0 \quad \text{for} \quad z_0 \in \mathscr{D}$$

when the E_n *lie on any particular component* Γ of $\partial\mathscr{D}$ and

$$\int_\Gamma \chi_{E_n}(\zeta)|d\zeta| \underset{n}{\longrightarrow} 0.$$

(Here, χ_{E_n} denotes the *characteristic function* of E_n.) We do this by comparing $\omega_{\mathscr{D}}(\ ,z)$ with harmonic measure for a *simply connected domain*; the method is of independent interest and is frequently used.

Let $\mathscr{E}$ be the *simply connected domain on the Riemann sphere* (including perhaps ∞), *bounded by the component* Γ *of* $\partial\mathscr{D}$ *and including all the points of* $\mathscr{D}$.

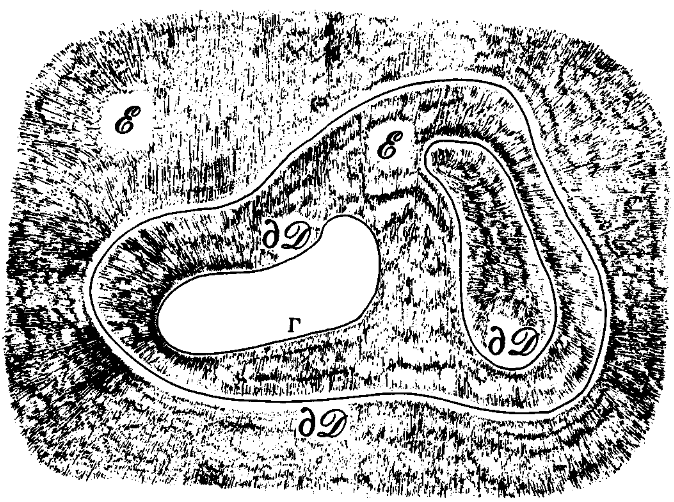

Figure 51

If $\varphi \in \mathscr{C}(\partial\mathscr{D})$ is *positive*, and *zero on all* the *components* of $\partial\mathscr{D}$ *save* Γ, we have

$$\int_\Gamma \varphi(\zeta)d\omega_{\mathscr{E}}(\zeta, z) \geqslant \int_{\partial\mathscr{D}} \varphi(\zeta)d\omega_{\mathscr{D}}(\zeta, z)$$

for $z \in \mathscr{D}$. Indeed, both integrals give us functions harmonic in $\mathscr{D}$ ($\subseteq \mathscr{E}$!), and continuous up to $\partial\mathscr{D}$. The right-hand function, $U_\varphi(z)$, equals $\varphi(\zeta)$ on Γ and *zero* on the *other* components of $\partial\mathscr{D}$. The left-hand one – call it $V(z)$ for the moment – *also* equals $\varphi(z)$ on Γ but is *surely* $\geqslant 0$ on the *other* components of $\partial\mathscr{D}$, because they lie in $\mathscr{E}$ and $\varphi \geqslant 0$. Therefore $V(z) \geqslant U_\varphi(z)$ throughout $\mathscr{D}$ by the principle of maximum, as claimed. This inequality holds for *every* function φ of the kind described above, whence, on Γ,

$$\boxed{\mathrm{d}\omega_{\mathscr{E}}(\zeta, z) \;\geqslant\; \mathrm{d}\omega_{\mathscr{D}}(\zeta, z) \quad \text{for} \quad z \in \mathscr{D}.}$$

This relation is an example of what Nevanlinna called ***the principle of extension of domain.***

Let us return to our sets

$$E_n \subseteq \Gamma \quad \text{with} \quad \int_\Gamma \chi_{E_n}(\zeta)|\mathrm{d}\zeta| \;\underset{n}{\longrightarrow}\; 0;$$

in order to verify that

$$\omega_{\mathscr{D}}(E_n, z) \underset{n}{\longrightarrow} 0, \quad z \in \mathscr{D},$$

it is enough, in virtue of the inequality just established, to check that $\omega_{\mathscr{E}}(E_n, z_0) \underset{n}{\longrightarrow} 0$ for each $z_0 \in \mathscr{E}$. Because $\mathscr{E}$ is *simply connected*, we may, however, use the formula derived near the beginning of the present article. Fixing $z_0 \in \mathscr{E}$, take a conformal mapping F of $\mathscr{E}$ onto $\{|w| < 1\}$ which sends z_0 to 0. From the formula just mentioned, it is clear that

$$\omega_{\mathscr{E}}(E_n, z_0) \;=\; \frac{1}{2\pi}\int_\Gamma \chi_{E_n}(\zeta)|\mathrm{d}F(\zeta)|.$$

The component Γ of $\partial\mathscr{D}$ is, however, *rectifiable*; a theorem of the brothers Riesz therefore guarantees that the mapping F from Γ onto the unit circumference is *absolutely continuous with respect to arc length.* For domains $\mathscr{D}$ whose boundary components are given explicitly and in fairly simple form (the sort we will be dealing with), that property can also be verified directly. We can hence write

$$\omega_{\mathscr{E}}(E_n, z_0) \;=\; \frac{1}{2\pi}\int_\Gamma \chi_{E_n}(\zeta)\left|\frac{\mathrm{d}F(\zeta)}{\mathrm{d}\zeta}\right||\mathrm{d}\zeta|$$

with

$$\left|\frac{\mathrm{d}F(\zeta)}{\mathrm{d}\zeta}\right| \quad \text{in} \quad L_1(\Gamma, |\mathrm{d}\zeta|),$$

and from this we see that $\omega_{\mathscr{E}}(E_n, z_0) \underset{n}{\longrightarrow} 0$ when

$$\int_{\Gamma} \chi_{E_n}(\zeta)|d\zeta| \underset{n}{\longrightarrow} 0.$$

The absolute continuity of $\omega_{\mathscr{D}}(\ , z)$ with respect to arc length on $\partial\mathscr{D}$ is thus verified.

The property just established makes it possible for us to write

$$\omega_{\mathscr{D}}(E, z) = \int_{\partial\mathscr{D}} \chi_E(\zeta) \frac{d\omega_{\mathscr{D}}(\zeta, z)}{|d\zeta|} \cdot |d\zeta|$$

for $E \subseteq \partial\mathscr{D}$ and $z \in \mathscr{D}$. It is important for us to be able to majorize the integral on the right by one of the form

$$K_z \int_{\partial\mathscr{D}} \chi_E(\zeta)|d\zeta|$$

(with K_z depending on z and, of course, on $\mathscr{D}$) when dealing with *certain kinds* of simply connected domains $\mathscr{D}$. In order to see for *which* kind, let us, for fixed $z_0 \in \mathscr{D}$, take a conformal mapping F of $\mathscr{D}$ onto $\{|w| < 1\}$ which sends z_0 to 0 and apply the formula used in the preceding argument, which here takes the form

$$\omega_{\mathscr{D}}(E, z_0) = \frac{1}{2\pi} \int_{\partial\mathscr{D}} \chi_E(\zeta) \left| \frac{dF(\zeta)}{d\zeta} \right| |d\zeta|.$$

If the boundary $\partial\mathscr{D}$ is an *analytic curve*, or merely has a *differentiably turning tangent*, the derivative $F'(z)$ of the conformal mapping function *will be* continuous up to $\partial\mathscr{D}$; in such circumstances $|dF(\zeta)/d\zeta|$ is bounded on $\partial\mathscr{D}$ (the bound depends evidently on z_0), and we *have* a majorization of the desired kind. This is even true when *$\partial\mathscr{D}$ has a finite number of corners* and is sufficiently smooth *away* from them, *provided that all those corners stick out.*

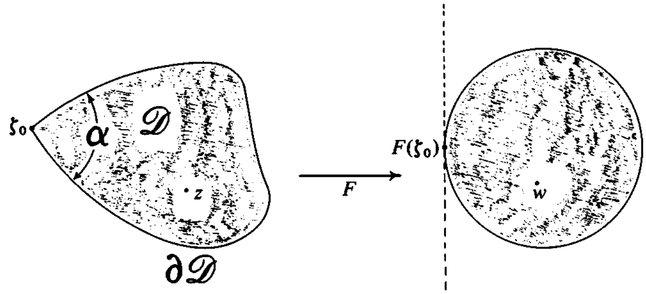

Figure 52

In this situation, where $\partial\mathscr{D}$ has a corner with internal angle α at ζ_0, $F(z) = F(\zeta_0) + (C + o(1))(z - \zeta_0)^{\pi/\alpha}$ for z in $\bar{\mathscr{D}}$ (*sic*!) *near* ζ_0; we see that $F'(\zeta_0) = 0$ if $\alpha < \pi$, and that $F'(\zeta)$ is *near* 0 if $\zeta \in \partial\mathscr{D}$ is *near* ζ_0 (sufficient smoothness of $\partial\mathscr{D}$ away from its corners is being assumed). In the present case, then, $|F'(\zeta)|$ *is* bounded on $\partial\mathscr{D}$, and an estimate

$$\omega_{\mathscr{D}}(E, z_0) \;\leqslant\; K_{z_0}\int_{\partial\mathscr{D}} \chi_E(\zeta)|d\zeta|$$

does hold good. It is *really necessary* that the corners *stick out*. If, for instance, $\alpha > \pi$, then $|F'(\zeta_0)| = \infty$, and $|F'(\zeta)|$ *tends* to ∞ for ζ on $\partial\mathscr{D}$ tending to ζ_0:

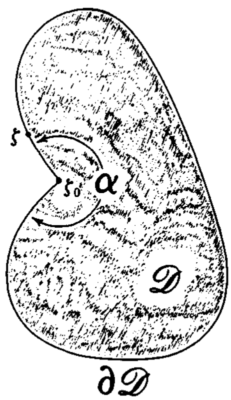

Figure 53

Here, we *do not* have

$$\omega_{\mathscr{D}}(E, z_0) \;\leqslant\; \text{const.}\int_{\partial\mathscr{D}} \chi_E(\zeta)|d\zeta|$$

for sets $E \subseteq \partial\mathscr{D}$ located near ζ_0.

Let us conclude with a general examination of the *boundary behaviour* of $\omega_{\mathscr{D}}(E, z)$ for $E \subseteq \partial\mathscr{D}$. Consider first of all the case where E is an *arc*, σ, *on one of the components* of $\partial\mathscr{D}$. Then the *simple approximate identity property* of $\omega_{\mathscr{D}}(\;\,, z)$ established above immediately shows that

$$\omega_{\mathscr{D}}(\sigma, z) \longrightarrow 1 \quad \text{if} \quad z \longrightarrow \zeta \in \sigma$$

and ζ is not an endpoint of σ, while

$$\omega_{\mathscr{D}}(\sigma, z) \longrightarrow 0 \quad \text{if} \quad z \longrightarrow \zeta \in \partial\mathscr{D} \sim \sigma$$

and ζ is not an endpoint of σ. If $z \in \mathscr{D}$ *tends* to an endpoint of σ, we cannot say much (in general) about $\omega_{\mathscr{D}}(\sigma, z)$, save that it remains between 0 and

1. These properties, however, *already suffice to determine the harmonic function* $\omega_{\mathscr{D}}(\sigma, z)$ (defined in $\mathscr{D}$) *completely*. This may be easily verified by using the principle of maximum together with an evident modification of the first Phragmén–Lindelöf theorem from §C, Chapter III; such verification is left to the reader. One sometimes *uses* this characterization in order to compute or estimate harmonic measure. Of course, once $\omega_{\mathscr{D}}(\sigma, z)$ is *known for arcs* $\sigma \subseteq \partial\mathscr{D}$, we can *get* $\omega_{\mathscr{D}}(E, z)$ for Borel sets E by the standard construction applying to all positive Radon measures.

What about the boundary behaviour of $\omega_{\mathscr{D}}(E, z)$ for a *more general* set E? We only consider *closed sets E lying on a single component* Γ of $\partial\mathscr{D}$; knowledge about this situation is all that is needed in practice.

Take, then, a closed subset E of the component Γ of $\partial\mathscr{D}$. In the first place, $\omega_{\mathscr{D}}(E, z) \leqslant \omega_{\mathscr{D}}(\Gamma, z)$. When z tends to any point of a component Γ' of $\partial\mathscr{D}$ *different from* Γ, $\omega_{\mathscr{D}}(\Gamma, z)$ tends to *zero* by the previous discussion (Γ is an arc without endpoints!) Hence $\omega_{\mathscr{D}}(E, z) \longrightarrow 0$ for $z \longrightarrow \zeta$ if $\zeta \in \partial\mathscr{D}$ belongs to a *component of the* latter *other than* Γ.

Examination of the boundary of $\omega_{\mathscr{D}}(E, z)$ *for z near* Γ is more delicate.

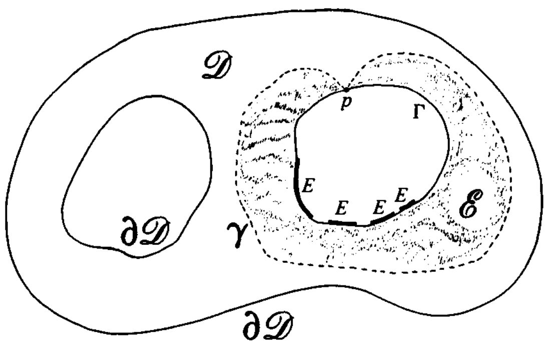

Figure 54

Take any point p on Γ lying outside the closed set E (if E were *all* of Γ, we could conclude by the case for *arcs* handled previously), and draw a curve γ lying in $\mathscr{D}$ like the one shown, with its two endpoints at p. Together, the curves γ and Γ bound a certain *simply connected* domain $\mathscr{E} \subseteq \mathscr{D}$.

We are going to derive the formula

$$\omega_{\mathscr{D}}(E, z) \;=\; \int_{\gamma} \omega_{\mathscr{D}}(E, \zeta)\, \mathrm{d}\omega_{\mathscr{E}}(\zeta, z) \;+\; \omega_{\mathscr{E}}(E, z),$$

valid for $z \in \mathscr{E}$. Take any finite union $\mathscr{U}$ of arcs on Γ containing the closed set E but avoiding a whole neighborhood of the point p, and let ψ be any function continuous on Γ with $0 \leqslant \psi(\zeta) \leqslant 1$, $\psi(\zeta) \equiv 0$ outside $\mathscr{U}$, and $\psi(\zeta) \equiv 1$ on E. Since ψ is zero on a neighborhood of p, the function

$$U_{\psi}(z) \;=\; \int_{\Gamma} \psi(\zeta)\,d\omega_{\mathscr{D}}(\zeta, z)$$

tends to *zero* as $z \to p$. Write $U_{\psi}(\zeta) = \psi(\zeta)$ for $\zeta \in \Gamma$; the function $U_{\psi}(\zeta)$ then becomes continuous on $\Gamma \cup \gamma = \partial\mathscr{E}$, so

$$V(z) \;=\; \int_{\partial\mathscr{E}} U_{\psi}(\zeta)\,d\omega_{\mathscr{E}}(\zeta, z)$$

is harmonic in $\mathscr{E}$ and continuous up to $\partial\mathscr{E}$, where it takes the boundary value $U_{\psi}(z)$. For this reason, the function

$$U_{\psi}(z) - V(z),$$

harmonic in $\mathscr{E}$, is *identically zero* therein, and we have

$$\int_{\gamma} U_{\psi}(\zeta)d\omega_{\mathscr{E}}(\zeta, z) + \int_{\Gamma} \psi(\zeta)\,d\omega_{\mathscr{E}}(\zeta, z) \;=\; V(z) \;=\; U_{\psi}(z)$$
$$= \int_{\Gamma} \psi(\zeta)\,d\omega_{\mathscr{D}}(\zeta, z)$$

for $z \in \mathscr{E}$. Making the covering $\mathscr{U}$ shrink down to E, we end with

$$\omega_{\mathscr{D}}(E, z) \;=\; \int_{\gamma} \omega_{\mathscr{D}}(E, \zeta)\,d\omega_{\mathscr{E}}(\zeta, z) \;+\; \omega_{\mathscr{E}}(E, z),$$

our desired relation.

The function $\omega_{\mathscr{D}}(E, \zeta)$ is continuous on γ and zero at p, because $\omega_{\mathscr{D}}(E, z) \leqslant$ each of the functions $U_{\psi}(z)$ considered above. The function harmonic in $\mathscr{E}$ with boundary values equal to $\omega_{\mathscr{D}}(E, \zeta)$ on γ and to *zero* on Γ is therefore *continuous on* $\gamma \cup \Gamma = \partial\mathscr{E}$, so

$$\int_{\gamma} \omega_{\mathscr{D}}(E, \zeta)\,d\omega_{\mathscr{E}}(\zeta, z)$$

tends to zero when $z \in \mathscr{E}$ *tends to any point of* Γ. Referring to the previous relation, we see that

$$(\ast) \qquad \omega_{\mathscr{D}}(E, z) - \omega_{\mathscr{E}}(E, z) \longrightarrow 0$$

whenever $z \in \mathscr{E}$ tends to any point of Γ. The *behaviour* of the *first term* on the left is thus *the same* as that of *the second*, for $z \longrightarrow \zeta_0 \in \Gamma$.

Because $\mathscr{E}$ is simply connected, we may use conformal mapping to study $\omega_{\mathscr{E}}(E, z)$'s boundary behaviour.

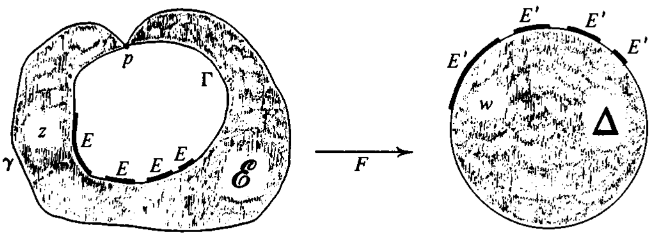

Figure 55

Let F map $\mathscr{E}$ conformally onto $\Delta = \{|w| < 1\}$; F takes $E \subseteq \Gamma$ onto a certain closed subset E' of the unit circumference, and we have

$$\omega_{\mathscr{E}}(E, z) = \omega_{\Delta}(E', F(z))$$

for $z \in \mathscr{E}$ (see the formula near the beginning of this article). *Assume that* Γ *is smooth, or at least that E lies on a smooth part of* Γ. Then it is a fact (easily verifiable directly in the cases which will interest us – the general result for curves with a tangent at every point being due to Lindelöf) that *F preserves angles right up to* Γ, as long as we stay away from p:

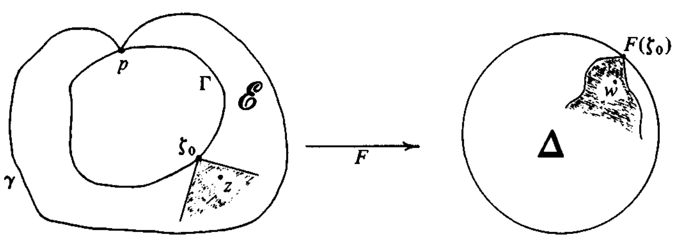

Figure 56

This means that if $z \in \mathscr{E}$ tends to any point ζ_0 of *E from within an acute angle with vertex at* ζ_0, *lying strictly in* $\mathscr{E}$ (we henceforth write this as '$z \xrightarrow{\angle} \zeta_0$'), *the image* $w = F(z)$ *will tend to* $F(\zeta_0) \in E'$ *from within such an angle lying in* Δ.

However,

$$\omega_{\Delta}(E', w) \;=\; \frac{1}{2\pi}\int_0^{2\pi} \frac{1 - |w|^2}{|w - e^{i\varphi}|^2}\chi_{E'}(e^{i\varphi})\,d\varphi.$$

A study of the boundary behaviour of the integral on the right was made in §B of Chapter II. According to the result proved there,

$$\omega_\Delta(E', w) \longrightarrow \chi_{E'}(\omega_0)$$

as $w \not\angle\!\!\longrightarrow \omega_0$, for almost every ω_0 on the unit circumference. In the present situation (E' *closed*) we even have

$$\omega_\Delta(E', w) \longrightarrow 0$$

whenever $w \to$ a point of the unit circumference *not* in E'. Under the conformal mapping F, sets of (arc length) measure zero on Γ *correspond precisely* to sets of measure zero on $\{|\omega| = 1\}$. (As before, one can verify this statement directly for the simple situations we will be dealing with. The general result is due to F. and M. Riesz.) In view of the angle preservation just described, we see, going back to $\mathscr{E}$, that, *for almost every* $\zeta_0 \in E$,

$$\omega_{\mathscr{E}}(E, z) \longrightarrow 1 \quad \text{as} \quad z \not\angle\!\!\longrightarrow \xi_0,$$

and that

$$\omega_{\mathscr{E}}(E, z) \longrightarrow 0 \quad \text{as} \quad z \longrightarrow \zeta_0$$

for $\zeta_0 \in \Gamma$ *not belonging to* E.

Now we bring in (⁑). According to what has just been shown, that relation tells us that

$$\omega_{\mathscr{D}}(E, z) \longrightarrow 1 \quad \text{as} \quad z \not\angle\!\!\longrightarrow \zeta_0$$

for *almost every* $\zeta_0 \in E$, whilst

$$\omega_{\mathscr{D}}(E, z) \longrightarrow 0 \quad \text{as} \quad z \longrightarrow \zeta_0$$

for $\zeta_0 \in \Gamma \sim E$, except possibly when $\zeta_0 = p$. By moving p slightly and taking a new curve γ (and new domain $\mathscr{E}$) we can, however, remove any doubt about that case. Referring to the already known boundary behaviour of $\omega_{\mathscr{D}}(E, z)$ at the *other* components of $\partial\mathscr{D}$, we have, finally,

$$\omega_{\mathscr{D}}(E, z) \longrightarrow \begin{cases} 0 & \text{as} \quad z \longrightarrow \zeta_0 \in \partial\mathscr{D} \sim E, \\ 1 & \text{as} \quad z \not\angle\!\!\longrightarrow \zeta_0 \text{ for } \textit{almost every } \zeta_0 \in E. \end{cases}$$

This completes our elementary discussion of harmonic measure.

2. Beurling's improvement of Levinson's theorem

We need two auxiliary results.

Lemma. *Let μ be a totally finite (complex) measure on $\mathbb{R}$, and put*

$$\hat{\mu}(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda t}\, d\mu(t)$$

(as usual). Suppose, for some real λ_0, *that*

$$\int_{\lambda_0}^{\infty} e^{-Y\lambda} e^{iX\lambda} \hat{\mu}(\lambda)\, d\lambda \;\equiv\; 0$$

and

$$\int_{-\infty}^{\lambda_0} e^{Y\lambda} e^{iX\lambda} \hat{\mu}(\lambda)\, d\lambda \;\equiv\; 0$$

for all $X \in \mathbb{R}$ *and all* $Y > 0$. *Then* $\mu \equiv 0$.

Proof. If we write $d\mu_{\lambda_0}(t) = e^{i\lambda_0 t} d\mu(t)$, we have $\hat{\mu}(\tau + \lambda_0) = \hat{\mu}_{\lambda_0}(\tau)$, and the identical vanishing of μ_{λ_0} clearly implies that of μ. In terms of $\hat{\mu}_{\lambda_0}$, the two relations from the hypothesis reduce to

$$\int_0^{\infty} e^{-Y\tau} e^{iX\tau} \hat{\mu}_{\lambda_0}(\tau)\, d\tau \;\equiv\; 0,$$

$$\int_{-\infty}^{0} e^{Y\tau} e^{iX\tau} \hat{\mu}_{\lambda_0}(\tau)\, d\tau \;\equiv\; 0,$$

valid for $X \in \mathbb{R}$ and $Y > 0$. Therefore, *if we prove the lemma for the case where* $\lambda_0 = 0$, we will *have* $\mu \equiv 0$. We thus proceed under the assumption that $\lambda_0 = 0$.

By direct calculation (!), for $X \in \mathbb{R}$ and $Y > 0$,

$$\frac{Y}{(X+t)^2 + Y^2} \;=\; \frac{1}{2}\int_{-\infty}^{\infty} e^{-Y|\lambda|} e^{iX\lambda} e^{i\lambda t}\, d\lambda.$$

The integral on the right is absolutely convergent, so, multiplying it by $d\mu(t)$, integrating with respect to t, and changing the order of integration, we find

$$\int_{-\infty}^{\infty} \frac{Y}{(X+t)^2 + Y^2}\, d\mu(t) \;=\; \frac{1}{2}\int_{-\infty}^{\infty} e^{-Y|\lambda|} e^{iX\lambda} \hat{\mu}(\lambda)\, d\lambda.$$

Under our assumption, the integral on the right vanishes identically for $X \in \mathbb{R}$ and $Y > 0$. Calling the one on the left $J_Y(X)$, we have, however,

$$J_Y(-X)\,dX \;\longrightarrow\; \pi\, d\mu(X) \quad w^*$$

for $Y \longrightarrow 0$. Therefore $d\mu(X) \equiv 0$, and we are done.

Lemma. *Let* $M(r) \geqslant 0$ *be increasing on* $[0, \infty)$, *and put*

$$M_*(r) = \min(r, M(r))$$

for $r \geqslant 0$. *Then, if*

$$\int_0^{\infty} \frac{M(r)}{1 + r^2}\, dr \;=\; \infty$$

we also have

$$\int_0^\infty \frac{M_*(r)}{1+r^2}\mathrm{d}r \;=\; \infty.$$

Proof. Is like that of the lemma in §A.3. The following diagram shows that $M(r) = M_*(r)$ outside of a certain open set $\mathcal{O}$, the union of disjoint intervals (a_n, b_n), on which $M_*(r) = r$.

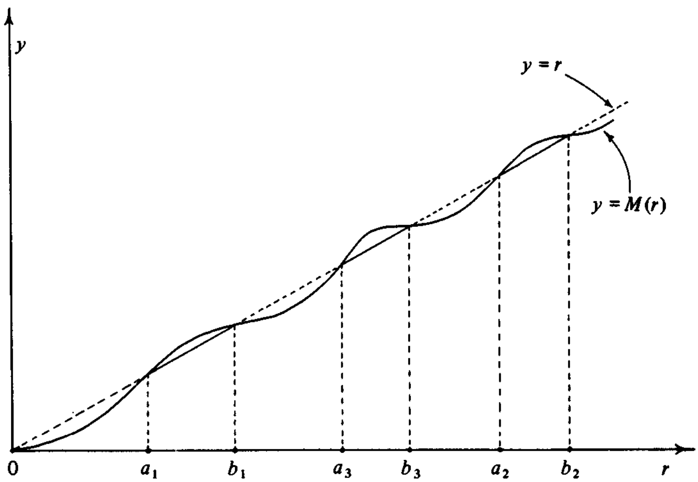

Figure 57

It is enough to show that

$$\int_1^\infty \frac{M_*(r)}{r^2}\mathrm{d}r \;=\; \infty.$$

If

$$\int_{\mathcal{O}\cap[1,\infty)} \frac{M_*(r)}{r^2}\mathrm{d}r \;=\; \infty,$$

we are already finished; let us therefore assume that this last integral is *finite*. We then sureiy have

$$\sum_{a_n\geqslant 1}\int_{a_n}^{b_n} \frac{M_*(r)}{r^2}\mathrm{d}r \;=\; \sum_{a_n\geqslant 1}\int_{a_n}^{b_n}\frac{\mathrm{d}r}{r} \;=\; \sum_{a_n\geqslant 1}\log\left(\frac{b_n}{a_n}\right) \;<\; \infty,$$

so $b_n/a_n \xrightarrow[n]{} 1$ which, fed back into the last relation, gives us

$$\sum_{a_n\geqslant 1}\frac{b_n - a_n}{a_n} \;<\; \infty.$$

Since, however, $M(r)$ is increasing, we see from the picture that

$$\int_{a_n}^{b_n} \frac{M(r)}{r^2}\,\mathrm{d}r \;\leqslant\; M(b_n)\int_{a_n}^{b_n} \frac{\mathrm{d}r}{r^2} \;=\; b_n\cdot\frac{b_n - a_n}{a_n b_n} \;=\; \frac{b_n - a_n}{a_n}.$$

Therefore

$$\sum_{a_n \geqslant 1} \int_{a_n}^{b_n} \frac{M(r)}{r^2}\,\mathrm{d}r \;<\; \infty$$

by the previous relation, so, since we are assuming

$$\int_0^\infty \frac{M(r)}{1+r^2}\,\mathrm{d}r \;=\; \infty$$

which implies

$$\int_1^\infty \frac{M(r)}{r^2}\,\mathrm{d}r \;=\; \infty$$

(M being increasing), we must have

$$\int_E \frac{M(r)}{r^2}\,\mathrm{d}r \;=\; \infty,$$

where

$$E \;=\; [1,\infty) \sim \bigcup_{a_n \geqslant 1} (a_n, b_n).$$

The set E is either *equal* to the complement of $\mathcal{O}$ in $[1, \infty)$ or else *differs* therefrom by an interval of the form $[1, b_k)$ where (a_k, b_k) is a component of $\mathcal{O}$ straddling the point 1 (in case there is one). Since $M_*(r) = M(r)$ *outside* $\mathcal{O}$, we thus have

$$\int_E \frac{M_*(r)}{r^2}\,\mathrm{d}r \;=\; \infty$$

(including in the possible situation where $b_k = \infty$), and therefore

$$\int_1^\infty \frac{M_*(r)}{r^2}\,\mathrm{d}r \;=\; \infty$$

as required.

Theorem (Beurling). *Let μ be a finite complex measure on $\mathbb{R}$ such that*

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\log\left(\frac{1}{\int_{-\infty}^{x}|\mathrm{d}\mu(t)|}\right)\mathrm{d}x \;=\; \infty.$$

If

$$\hat{\mu}(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda t}\,d\mu(t)$$

vanishes *on a set* $E \subseteq \mathbb{R}$ *of* positive measure, *then* $\mu \equiv 0$.

Proof. In the *complex* λ-plane, let $\mathscr{D}$ be the strip

$$\{0 < \Im\lambda < 1\};$$

we work with harmonic measure $\omega_{\mathscr{D}}(\ ,\lambda)$ *for* $\mathscr{D}$ (see article 1).

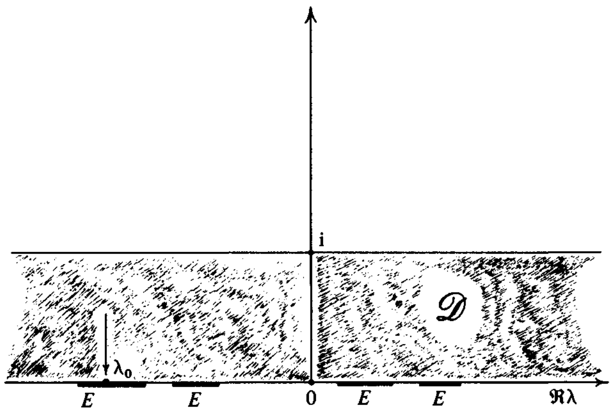

Figure 58

Because $|E| > 0$, E contains a compact set of *positive* Lebesgue measure; there is thus no loss of generality in assuming E *closed* and *bounded.* According to the discussion at the end of the previous article, we then have

$$\omega_{\mathscr{D}}(E,\lambda) \longrightarrow 1$$

as $\lambda \not\angle\!\!\longrightarrow \lambda_0$ *for almost every* λ_0 *in the set* E (*of positive measure*). There is thus certainly a $\lambda_0 \in E$ with

$$\omega_{\mathscr{D}}(E,\lambda_0 + i\tau) \longrightarrow 1$$

for $\tau \to 0+$; *we take such a* λ_0 *and fix it* throughout the rest of the proof.

We are going to show that

$$\int_{-\infty}^{\lambda_0} e^{Y\lambda} e^{iX\lambda} \hat{\mu}(\lambda)\,d\lambda \;\equiv\; 0$$

and

$$\int_{\lambda_0}^{\infty} e^{-Y\lambda} e^{iX\lambda} \hat{\mu}(\lambda)\,d\lambda \;\equiv\; 0$$

for $Y > 0$ and $X \in \mathbb{R}$; by the *first* of the above lemmas we will then have $\mu \equiv 0$ which is what we want to establish. The argument here is the same for any value of λ_0. *In order not to burden the exposition with a proliferation of symbols, we give it for the case where $\lambda_0 = 0$, which we henceforth assume. We have, then, $\hat{\mu}(\lambda) = 0$ on the closed set E, $0 \in E$, and $\omega_{\mathscr{D}}(E, \mathrm{i}\tau) \longrightarrow 1$ for $\tau \longrightarrow 0+$.*

Consider the *second* of the above two integrals. Under the present circumstances, it is equal to

$$\int_0^{\infty} \mathrm{e}^{\mathrm{i}Z\lambda}\hat{\mu}(\lambda)\,\mathrm{d}\lambda \;=\; F(Z),$$

say, where $Z = X + \mathrm{i}Y$. The function $F(Z)$ defined in this fashion is *analytic* for $\Im Z > 0$ and *bounded* in each half plane of the form $\Im Z \geqslant h > 0$. By §G.2 of Chapter III, we will therefore have $F(Z) \equiv 0$ for $\Im Z > 0$ provided that

$$(*) \qquad \int_0^{\infty} \frac{\log|F(X+\mathrm{i})|}{1+X^2}\,\mathrm{d}X \;=\; -\infty.$$

This relation we now proceed to establish.

Take a number $A > 0$ (later on, A will be made to depend on X), and write

$$\hat{\mu}(\lambda) \;=\; \hat{\mu}_A(\lambda) + \hat{\rho}_A(\lambda),$$

with

$$\hat{\mu}_A(\lambda) \;=\; \int_{-A}^{\infty} \mathrm{e}^{\mathrm{i}\lambda t}\,\mathrm{d}\mu(t)$$

and

$$\hat{\rho}_A(\lambda) \;=\; \int_{-\infty}^{-A} \mathrm{e}^{\mathrm{i}\lambda t}\,\mathrm{d}\mu(t).$$

The function $\hat{\mu}_A(\lambda)$ is actually defined for $\Im\lambda \geqslant 0$ and *analytic* when $\Im\lambda > 0$. $\hat{\rho}_A(\lambda)$ is not, in general, defined for $\Im\lambda > 0$; when A is large, it is, however, *very small* on the real axis in view of our assumption on

$$\int_{-\infty}^{x} |\mathrm{d}\mu(t)|$$

in the hypothesis. We think of $\hat{\rho}_A(\lambda)$ as a *correction* to $\hat{\mu}_A(\lambda)$ on $\mathbb{R}$.

Wlog,

$$\int_{-\infty}^{\infty} |\mathrm{d}\mu(t)| \;\leqslant\; 1.$$

Then, writing

$$\int_{-\infty}^{-A} |d\mu(t)| = e^{-M(A)},$$

we have

$$|\hat{\rho}_A(\lambda)| \leqslant e^{-M(A)}, \quad \lambda \in \mathbb{R},$$

with $M(A) \geqslant 0$ and increasing for $A \geqslant 0$. Going back to

$$F(X + i) = \int_0^\infty e^{-\lambda} e^{iX\lambda} \hat{\mu}(\lambda) d\lambda,$$

we see that the latter *differs* from

$$\int_0^\infty e^{i(X+i)\lambda} \hat{\mu}_A(\lambda) d\lambda$$

by a quantity in modulus

$$\leqslant \int_0^\infty e^{-\lambda} |\hat{\rho}_A(\lambda)| d\lambda \leqslant e^{-M(A)}.$$

As we have already remarked, this is very small when A is large. Showing that $|F(X + i)|$ gets small enough for (*) to hold thus turns out to reduce to the careful estimation of

$$\left| \int_0^\infty e^{i(X+i)\lambda} \hat{\mu}_A(\lambda) d\lambda \right|.$$

We use Cauchy's theorem for that.

Taking Γ as shown,

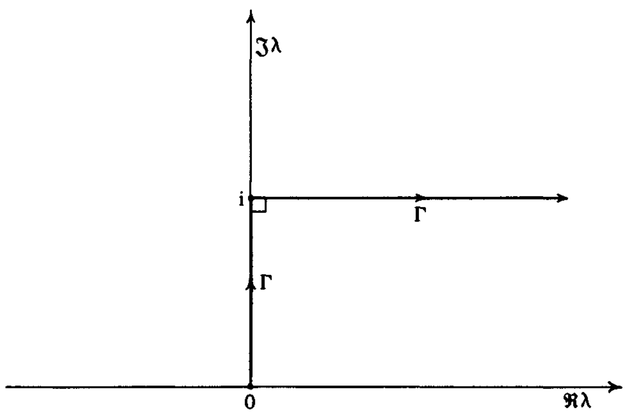

Figure 59

we have

$$\int_0^\infty e^{i\lambda(X+i)}\hat{\mu}_A(\lambda)\,d\lambda \;=\; \int_\Gamma e^{i\lambda(X+i)}\hat{\mu}_A(\lambda)\,d\lambda$$

because $\hat{\mu}_A(\lambda)$ is *analytic* for $\Im\lambda > 0$ and *bounded* in the strip

$$0 \leqslant \Im\lambda \leqslant 1,$$

with $|e^{i\lambda(X+i)}|$ going to zero like $e^{-\Re\lambda}$ there as $\Re\lambda \to \infty$. The integral along Γ breaks up as

$$i\int_0^1 e^{-i\tau}e^{-\tau X}\hat{\mu}_A(i\tau)\,d\tau \;+\; \int_0^\infty e^{i(\sigma X-1)}e^{-\sigma}e^{-X}\hat{\mu}_A(\sigma+i)\,d\sigma$$

$$= \;\mathrm{I} \;+\; \mathrm{II},$$

say.

Since

$$\int_{-\infty}^\infty |d\mu(t)| \;\leqslant\; 1,$$

we have

$$|e^{iA\lambda}\hat{\mu}_A(\lambda)| \;=\; \left|\int_{-A}^\infty e^{i\lambda(t+A)}\,d\mu(t)\right| \;\leqslant\; 1$$

for $\Im\lambda \geqslant 0$. In particular, for $\sigma\in\mathbb{R}$, $|\hat{\mu}_A(\sigma+i)| \leqslant e^A$, and

$$|\mathrm{II}| \;\leqslant\; \int_0^\infty e^{-(X-A)}e^{-\sigma}\,d\sigma \;=\; e^{A-X}.$$

To estimate I, we use the *theorem on two constants* given in the previous article. As we have just seen, $e^{iA\lambda}\hat{\mu}_A(\lambda)$ is in modulus $\leqslant 1$ on the closed strip $\bar{\mathscr{D}}$; it is also continuous there and analytic in $\mathscr{D}$. However, on $E \subseteq \mathbb{R}$, $\hat{\mu}(\lambda) = 0$ (by hypothesis!), so $\hat{\mu}_A(\lambda) = \hat{\mu}(\lambda) - \hat{\rho}_A(\lambda) = -\hat{\rho}_A(\lambda)$ there. Thus, for $\lambda\in E$,

$$|e^{iA\lambda}\hat{\mu}_A(\lambda)| \;=\; |\hat{\rho}_A(\lambda)| \;\leqslant\; e^{-M(A)}$$

According to the theorem on two constants we thus have

$$|e^{iA\lambda}\hat{\mu}_A(\lambda)| \;\leqslant\; e^{-M(A)\omega_{\mathscr{D}}(E,\lambda)}\cdot 1^{1-\omega_{\mathscr{D}}(E,\lambda)}$$

for $\lambda\in\mathscr{D}$, i.e.,

$$|\hat{\mu}_A(\lambda)| \;\leqslant\; e^{A\Im\lambda}\cdot e^{-M(A)\omega_{\mathscr{D}}(E,\lambda)}, \qquad \lambda\in\mathscr{D}.$$

Substituting this estimate into I, we find

$$|\mathrm{I}| \;\leqslant\; \int_0^1 e^{A\tau - X\tau - M(A)\omega_{\mathscr{D}}(E,i\tau)}\,d\tau.$$

Recall, however, that $\omega_{\mathscr{D}}(E, \mathrm{i}\tau) \longrightarrow 1$ for $\tau \to 0$. For this reason *$\tau + \omega_{\mathscr{D}}(E, \mathrm{i}\tau)$ has a strictly positive minimum – call it θ – for $0 \leqslant \tau \leqslant 1$; θ does not depend on A or X.*

Suppose $X > 0$. *Then take $A = X/2$.* With this value of A, the previous relation becomes

$$|\mathrm{I}| \leqslant \int_0^1 \mathrm{e}^{-(X/2)\tau - M(X/2)\omega_{\mathscr{D}}(E,\mathrm{i}\tau)} \mathrm{d}\tau \leqslant \mathrm{e}^{-\theta M_*(X/2)},$$

where $M_*(r) = \min(r, M(r))$.

At the same time,

$$|\mathrm{II}| \leqslant \mathrm{e}^{-X/2}$$

for $A = X/2$, according to the estimate made above. Therefore, for $X > 0$,

$$\int_0^\infty \mathrm{e}^{\mathrm{i}(X+\mathrm{i})\lambda} \hat{\mu}_{X/2}(\lambda)\,\mathrm{d}\lambda = \int_\Gamma \mathrm{e}^{\mathrm{i}(X+\mathrm{i})\lambda} \hat{\mu}_{X/2}(\lambda)\,\mathrm{d}\lambda$$

is in modulus

$$\leqslant |\mathrm{I}| + |\mathrm{II}| \leqslant \mathrm{e}^{-\theta M_*(X/2)} + \mathrm{e}^{-X/2}.$$

However, the *first* of the last two integrals differs from $F(X + \mathrm{i})$ by a quantity in modulus $\leqslant \mathrm{e}^{-M(X/2)}$ as we have seen. So, for $X > 0$,

$$|F(X+\mathrm{i})| \leqslant \mathrm{e}^{-\theta M_*(X/2)} + \mathrm{e}^{-X/2} + \mathrm{e}^{-M(X/2)}.$$

There is no loss of generality in assuming $\theta \leqslant 1$. Then we get

$$|F(X+\mathrm{i})| \leqslant 3\mathrm{e}^{-\theta M_*(X/2)}, \quad X > 0.$$

Returning to ($*$), which we are trying to prove, we see that

$$\int_0^\infty \frac{\log|F(X+\mathrm{i})|}{1+X^2}\,\mathrm{d}X \leqslant \int_0^\infty \frac{\log 3 - \theta M_*(X/2)}{1+X^2}\,\mathrm{d}X,$$

and the integral on the left will diverge to $-\infty$ if

$$(\ast\ast) \qquad \int_0^\infty \frac{M_*(X/2)}{1+X^2}\,\mathrm{d}X = \infty.$$

Here,

$$M(A) = \log\left(\frac{1}{\int_{-\infty}^{-A} |\mathrm{d}\mu(t)|}\right),$$

so $\int_0^\infty (M(A)/(1+A^2))\,\mathrm{d}A = \infty$ by the hypothesis. Therefore

$$\int_0^\infty \frac{M_*(A)}{1+A^2}\,\mathrm{d}A = \infty$$

for $M_*(A) = \min(A, M(A))$ by the second lemma, i.e.,

$$\int_0^\infty \frac{2M_*(X/2)}{4+X^2}\,\mathrm{d}X \;=\; \infty,$$

implying (⁑), since $M_*(A) \geqslant 0$.

We conclude in this fashion that $(*)$ *holds*, whence $F(Z) \equiv 0$ for $\Im Z > 0$, i.e.,

$$\int_0^\infty \mathrm{e}^{-Y\lambda}\mathrm{e}^{\mathrm{i}X\lambda}\hat{\mu}(\lambda)\,\mathrm{d}\lambda \;\equiv\; 0$$

for $Y > 0$ and $X \in \mathbb{R}$.

One shows in like manner that $\int_{-\infty}^0 \mathrm{e}^{Y\lambda}\mathrm{e}^{\mathrm{i}X\lambda}\hat{\mu}(\lambda)\,\mathrm{d}\lambda \equiv 0$ for $Y > 0$ and $X \in \mathbb{R}$; here* one follows the above procedure to estimate

$$\left|\int_{-\infty}^0 \mathrm{e}^{\lambda}\mathrm{e}^{\mathrm{i}X\lambda}\hat{\mu}(\lambda)\,\mathrm{d}\lambda\right|$$

(*again* for $X > 0$!) using *this* contour:

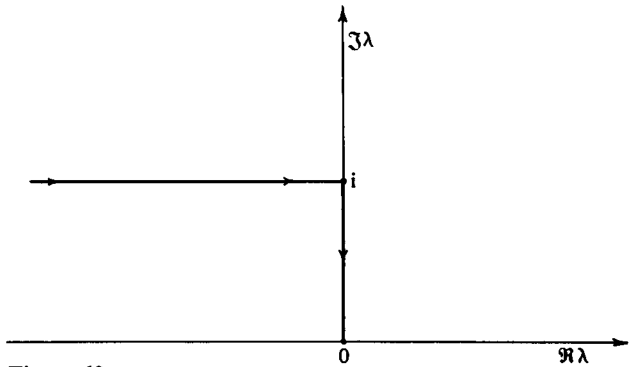

Figure 60

Aside from this change, the argument is like the one given.

The two integrals in question thus vanish identically for $Y > 0$ and $X \in \mathbb{R}$. This, as we remarked at the beginning of our proof, implies that $\mu \equiv 0$. We are done.

Remark 1. The use of the contour integral in the above argument goes back to Levinson, who assumed, however, that $\hat{\mu}(\lambda) = 0$ on an *interval* J instead of just on a set E with $|E| > 0$. In this way Levinson obtained his theorem, given in § A.5, which we now know how to prove much more easily using test functions. By bringing in harmonic measure, Beurling was able to replace the interval J by any measurable set E with $|E| > 0$, getting a qualitative improvement in Levinson's result.

* In which case the integral just written is an analytic function of $X - \mathrm{i}Y$

Remark 2. What about Beurling's gap theorem from §A.2, which says that if the measure μ *has no mass* on any of the intervals (a_n, b_n) with $0 < a_1 < b_1 < a_2 < b_2 < \cdots$ and $\sum_n((b_n - a_n)/a_n)^2 = \infty$, then $\hat{\mu}(\lambda)$ can't vanish identically *on an interval* J, $|J| > 0$, *unless* $\mu \equiv 0$? Can one improve *this* result so as to make it apply for *sets E of positive Lebesgue measure instead of just intervals J of positive length*? Contrary to what happens with Levinson's theorem, the answer *here* turns out to be *no*. This is shown by an example of P. Kargaev, to be given in §C.

3. Beurling's study of quasianalyticity

The argument used to establish the theorem of the preceding article can be applied in the investigation of a kind of quasianalyticity. Let γ be a nice Jordan arc, and look at functions $\varphi(\zeta)$ bounded and continuous on γ. A natural way of describing the *regularity* of such φ is to *measure how well* they can be *approximated* on γ by *certain analytic functions*. The regularity which we are able to specify in such fashion is not necessarily the same as differentiability; it is, however, relevant to the study of a quasianalyticity property considered by Beurling, namely, that of not being able to vanish on a subset of γ having positive (arc-length) measure without being identically zero.

A clue to the kind of regularity involved here comes from the observation that a function φ having a continuous analytic extension to a region *bordering on one side* of γ *possesses* the quasianalyticity property just described. We may thus think of such a φ as being *fully* regular. In order to make this notion of regularity quantitative, let us assume that the arc γ is part of the boundary $\partial\mathscr{D}$ of a simply connected region $\mathscr{D}$.

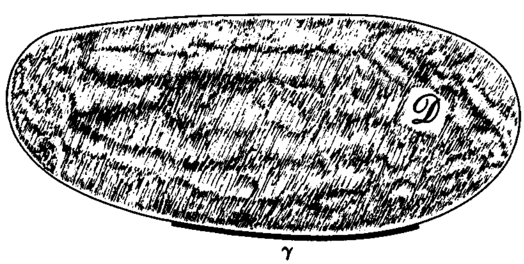

Figure 61

So as to avoid considerations foreign to the matter at hand, we take $\partial\mathscr{D}$ as 'nice' – piecewise analytic and rectifiable, for instance. Given φ bounded and continuous on γ, *define the approximation index $M(A)$ for φ by functions analytic in $\mathscr{D}$ as follows*:

$e^{-M(A)}$ *is the infimum of* $\sup_{\zeta\in\gamma}|\varphi(\zeta) - f(\zeta)|$ *for* f *analytic in* $\mathscr{D}$ *and continuous on* $\bar{\mathscr{D}}$ *such that* $|f(z)| \leqslant e^A$, $z\in\mathscr{D}$.

We *should* write $M_{\mathscr{D}}(A, \varphi)$ instead of $M(A)$ in order to show the dependence of the approximation index on φ and $\mathscr{D}$; we prefer, however, to use a simpler notation.

When A is made *larger*, we have *more* competing functions f with which to try to approximate φ on γ, so $e^{-M(A)}$ gets *smaller*. In other words, $M(A)$ *increases* with A and we take *the rapidity of this increase* as a *measure* of the *regularity* of φ. Note that if φ actually *has* a bounded continuous extension to $\bar{\mathscr{D}}$ which is *analytic* in $\mathscr{D}$, we have $M(A) = \infty$ beginning with a certain value of A. Such a function φ *cannot* vanish on a set of positive (arc-length) measure on γ without being identically zero, as we have already remarked (this comes, by the way, from two well-known results of F. and M. Riesz). We see that if $M(A)$ grows *rapidly enough*, φ will surely *have* the quasi-analyticity property in question.

The *approximation index* $M(A)$ is a *conformal invariant* in the following sense. Let F map $\mathscr{D}$ conformally onto $\tilde{\mathscr{D}}$, taking the arc γ of $\partial\mathscr{D}$ onto the arc $\tilde{\gamma} \subseteq \partial\tilde{\mathscr{D}}$, and let $\tilde{\varphi}$ be the function defined on $\tilde{\gamma}$ by the relation $\tilde{\varphi}(F(\zeta)) = \varphi(\zeta)$, $\zeta \in \gamma$. Then $\tilde{\varphi}$ has the *same* approximation index $M(A)$ for functions analytic in $\tilde{\mathscr{D}}$ as φ has for functions analytic in $\mathscr{D}$. *This is an evident consequence of the use of the sup-norm in defining* $M(A)$.

Our *quasianalyticity property is also* a *conformal invariant*. This follows from the famous theorem of F. and M. Riesz which says that as long as $\partial\mathscr{D}$ and $\partial\tilde{\mathscr{D}}$ are *both rectifiable*, a conformal mapping F of $\mathscr{D}$ onto $\tilde{\mathscr{D}}$ takes sets of arc-length measure zero on $\partial\mathscr{D}$ to such sets on $\partial\tilde{\mathscr{D}}$, and conversely. If $\partial\mathscr{D}$ and $\partial\tilde{\mathscr{D}}$ are really nice, that fact can also be verified directly.

Without further ado, we can now state the

Theorem (Beurling). *Suppose that, for a given bounded continuous* φ *on* $\gamma \subseteq \partial\mathscr{D}$, *the approximation index* $M(A)$ *for* φ *by functions analytic in* $\mathscr{D}$ *satisfies*

$$\int_1^\infty \frac{M(A)}{A^2}\,dA \;=\; \infty.$$

Then, if $E \subseteq \gamma$ *has positive (arc-length) measure, and* $\varphi(\zeta) \equiv 0$ *on* E, $\varphi \equiv 0$ *on* γ.

Proof. By the above statements on conformal invariance, it is enough to establish the result for the special case where $\mathscr{D}$ is the strip $0 < \Im\lambda < 1$ in the λ-plane and γ is the real axis. The fact that $\partial\mathscr{D}$ is *not rectifiable* here *makes no difference*. We need only show that a set of measure > 0 on the rectifiable boundary of our *original* nice domain corresponds under conformal mapping to a set of measure > 0 on the boundary of the *strip*. This may be checked by first mapping the original domain onto the unit disk Δ (whose boundary *is* rectifiable) and appealing to the theorem of F. and M. Riesz mentioned above. One then maps Δ conformally onto the strip; that mapping is, however, easily obtained *explicitly* and thus seen *by inspection* to take sets of measure > 0 on $\partial\Delta$ to sets of measure > 0 on the boundary of the strip.

We have, then, a function φ bounded and continuous on the *real axis*; wlog $|\varphi| \leqslant 1$ there.

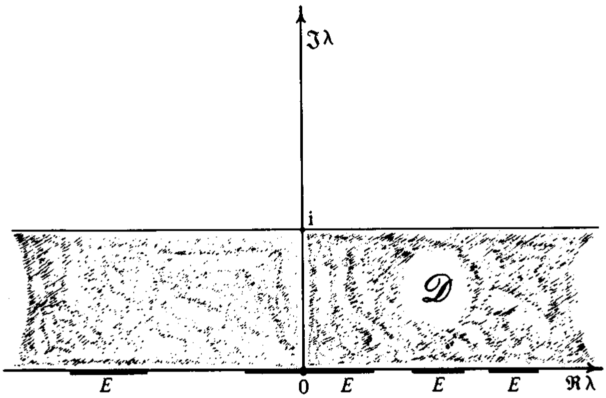

Figure 62

There is a set $E \subseteq \mathbb{R}$ (which we may wlog take to be *closed*) with $|E| > 0$* and $\varphi \equiv 0$ on E. According to the definition of $M(A)$ there is for each $A > 0$ a function $f_A(\lambda)$ analytic in $\mathscr{D}$ and continuous on $\bar{\mathscr{D}}$ with $|f_A(\lambda)| \leqslant e^A$ there and

(†) $\qquad |\varphi(\lambda) - f_A(\lambda)| \;\leqslant\; 2e^{-M(A)}, \quad \lambda \in \mathbb{R}.$

By the discussion at the end of article 1, we certainly have

$$\omega_{\mathscr{D}}(E, \lambda) \longrightarrow 1 \quad \text{as} \quad \lambda \xrightarrow{\angle} \lambda_0$$

for some $\lambda_0 \in E$. There is no loss of generality in taking $\lambda_0 = 0$ (we may arrive at this situation by sliding $\bar{\mathscr{D}}$ along the real axis!), and this we *henceforth assume*. We have, then,

$$\omega_{\mathscr{D}}(E, i\tau) \longrightarrow 1 \quad \text{as} \quad \tau \longrightarrow 0+,$$

just as in the proof of the theorem in article 2.

* We are denoting the Lebesgue measure of $E \subseteq \mathbb{R}$ by $|E|$.

In order to show that $\varphi(\lambda) \equiv 0$ on $\mathbb{R}$ it is enough to prove that

$$\int_{-\infty}^{\infty} e^{-Y|\lambda|} e^{i\lambda X} \varphi(\lambda)\,d\lambda = 0$$

for some $Y > 0$ and all real X, for then the function $e^{-Y|\lambda|}\varphi(\lambda)$ (which belongs to $L_1(\mathbb{R})$) must *vanish* a.e. on $\mathbb{R}$ by the *uniqueness theorem* for Fourier transforms. We do this by verifying separately that

$$\int_0^{\infty} e^{i\lambda(X+iY)} \varphi(\lambda)\,d\lambda = 0 \quad \text{for} \quad Y > 0 \quad \text{and} \quad X \in \mathbb{R},$$

and that

$$\int_{-\infty}^{0} e^{i\lambda(X+iY)} \varphi(\lambda)\,d\lambda = 0 \quad \text{for} \quad Y < 0 \quad \text{and} \quad X \in \mathbb{R}.$$

Considering the first relation, write, for $Y > 0$,

$$F(X+iY) = \int_0^{\infty} e^{i\lambda(X+iY)} \varphi(\lambda)\,d\lambda;$$

the function $F(Z)$ is analytic for $\Im Z > 0$ and *bounded* in each half plane $\Im Z \geqslant h > 0$. We want to conclude that $F(Z) \equiv 0$ for $\Im Z > 0$.

Beginning here, we can practically *copy* the proof of the theorem in the previous article. In that proof, we *replace*

$$\hat{\mu}(\lambda) \quad \text{by} \quad \varphi(\lambda),$$
$$\hat{\mu}_A(\lambda) \quad \text{by} \quad f_A(\lambda)$$

and $\hat{\rho}_A(\lambda)$ by $\varphi(\lambda) - f_A(\lambda)$. *Everything* will then be *the same*, almost *word* for *word*. True, instead of the inequality $|\hat{\rho}_A(\lambda)| \leqslant e^{-M(A)}$ used above, we *here* have (†), but the extra factor of 2 makes very little difference. We also have to find an inequality for $|f_A(\lambda)|$ in the strip $\bar{\mathscr{D}}$ which will play the rôle of the relation $|\hat{\mu}_A(\lambda)| \leqslant e^{A3\lambda}$ used previously. Our function f_A satisfies $|f_A(\lambda)| \leqslant e^A$ on $\bar{\mathscr{D}}$ and

$$|f_A(\lambda)| \leqslant |\varphi(\lambda)| + 2e^{-M(A)} \leqslant 1 + 2e^{-M(0)} \quad \text{for} \quad \lambda \in \mathbb{R}$$

by (†), $M(A)$ being increasing. Therefore

$$|e^{iA\lambda} f_A(\lambda)| \leqslant 1 + 2e^{-M(0)}, \qquad \lambda \in \partial\mathscr{D},$$

and we conclude that this inequality holds *throughout* $\bar{\mathscr{D}}$ by the extended principle of maximum (first theorem of § C, Chapter III). In other words,

$$|f_A(\lambda)| \leqslant (1 + 2e^{-M(0)})e^{A3\lambda}$$

for $\lambda \in \bar{\mathscr{D}}$, and this plays the same rôle as the abovementioned inequality on

$\hat{\mu}_A(\lambda)$, the constant factor $1 + 2e^{-M(0)}$ being without real importance.

Repeating in this way the argument from the previous article, we see that the hypothesis

$$\int_1^\infty \frac{M(A)}{A^2}\,dA \;=\; \infty$$

of our present theorem implies that $F(Z) \equiv 0$ for $\Im Z > 0$. The fact that

$$\int_{-\infty}^0 e^{i\lambda(X+iY)}\varphi(\lambda)\,d\lambda \;\equiv\; 0.$$

for $Y < 0$ and $X \in \mathbb{R}$ also follows by a simple modification of that argument, as indicated at the end of the proof we have been referring to. We are done.

Corollary. *Let μ be a finite measure on $\mathbb{R}$ such that*

$$\int_{-\infty}^0 \frac{1}{1+x^2}\log\left(\frac{1}{\int_{-\infty}^x |d\mu(t)|}\right)dx \;=\; \infty,$$

and suppose that $\psi(\lambda)$ is analytic *in a* rectangle

$$\{a < \Re\lambda < b, \quad 0 < \Im\lambda < h\},$$

and continuous on the closure of that rectangle.

If $E \subseteq (a,b)$, $|E| > 0$, and $\hat{\mu}(\lambda) = \int_{-\infty}^\infty e^{i\lambda t}d\mu(t)$ coincides with $\psi(\lambda)$ on E, then $\hat{\mu}(\lambda) \equiv \psi(\lambda)$ on the whole segment $[a,b]$.

Remark. For E an *interval* $\subseteq [a,b]$, this result was proved by Levinson.

Proof of corollary. Without loss of generality, assume that $h = 1$, that $|\psi(\lambda)| \leqslant \frac{1}{2}$ on the rectangle in question, and that $\int_{-\infty}^\infty |d\mu(t)| \leqslant \frac{1}{2}$.

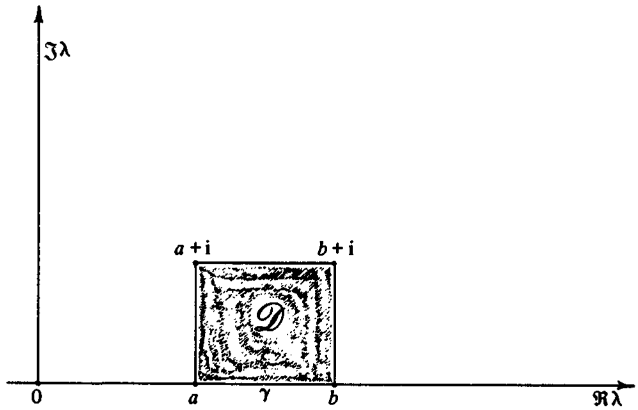

Figure 63

Take our *rectangle* as the domain $\mathscr{D}$ of the theorem, with (a,b) as the arc γ, and put

$$\varphi(\lambda) = \hat{\mu}(\lambda) - \psi(\lambda), \quad a < \lambda < b.$$

For $A > 0$, write

$$f_A(\lambda) = \int_{-A}^{\infty} e^{i\lambda t}\,d\hat{\mu}(t) - \psi(\lambda);$$

the function is analytic in $\mathscr{D}$ and continuous on $\bar{\mathscr{D}}$, and for $A > 0$,

$$|f_A(\lambda)| \leqslant \tfrac{1}{2}e^A + \tfrac{1}{2} \leqslant e^A, \qquad \lambda \in \bar{\mathscr{D}},$$

while for $a < \lambda < b$

$$|\varphi(\lambda) - f_A(\lambda)| \leqslant \int_{-\infty}^{-A} |d\mu(t)| = e^{-M(A)},$$

where $M(A) = \log(1/\int_{-\infty}^{-A}|d\mu(t)|)$.

According to our hypothesis, $\varphi(\lambda) \equiv 0$ on $E \subseteq (a,b) = \gamma$ with $|E| > 0$, and also

$$\int_1^{\infty} \frac{M(A)}{A^2}\,dA = \infty.$$

Therefore $\varphi(\lambda) \equiv 0$ on (a,b) by the theorem, and, by continuity, $\hat{\mu}(\lambda) \equiv \psi(\lambda)$ for $a \leqslant \lambda \leqslant b$. Q.E.D.

4. The spaces $\mathscr{S}_p(\mathscr{D}_0)$, especially $\mathscr{S}_1(\mathscr{D}_0)$

Suppose that $F(\vartheta) \sim \sum_{-\infty}^{\infty} a_n e^{in\vartheta}$ belongs to $L_2(-\pi,\pi)$ and that

$$\sum_{-\infty}^{-1} \frac{1}{n^2} \log\left(\frac{1}{\sum_{-\infty}^{n} |a_k|^2}\right) = \infty.$$

We would like, in analogy with the theorem of article 2, to be able to affirm that $F(\vartheta) \equiv 0$ a.e. if $F(\vartheta)$ vanishes on a set of positive measure. The trouble is that F is not necessarily bounded on $[-\pi,\pi]$, so we cannot work directly with the uniform norm used up to now in the present §. At least two ideas for getting around this difficulty come to mind; one of them is to establish L_p variants of the results in article 2 and 3. Such versions are no longer conformally invariant. Beurling gave one for *rectangular* domains; one could of course use his method to obtain similar results for other regions. In this and the next subsection we stick to rectangles.

Given a rectangle $\mathscr{D}_0$ with sides parallel to the axes, Beurling considers approximation in L_p norm by certain functions analytic in $\mathscr{D}_0$, belonging to a space $\mathscr{S}_p(\mathscr{D}_0)$ to be defined presently. We need some information about those functions which, strictly speaking, comes from the theory of H_p spaces. Although this is *not* a book about H_p spaces, we proceed to sketch that material. In various special situations (including the one mentioned at the beginning of this article), easier results would suffice, and the reader is encouraged to investigate the possibilities of such simplification.

If $\mathscr{D}_0$ is the rectangle $\{(x, y): x \in I_0, y \in J_0\}$,

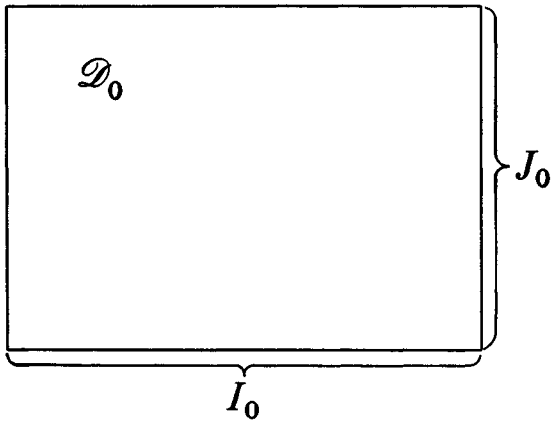

Figure 64

we denote by $\mathscr{S}_p(\mathscr{D}_0)$ the set of functions $f(z)$ analytic in $\mathscr{D}_0$ with

$$\sup_{y \in J_0} \int_{I_0} |f(x + \mathrm{i}y)|^p \, \mathrm{d}x$$

finite, and write

$$\sigma_p(f) \;=\; \sqrt[p]{\sup_{y \in J_0} \int_{I_0} |f(x + \mathrm{i}y)|^p \, \mathrm{d}x}$$

for such f. We are only interested in values of $p \geqslant 1$, and, for such p, $\sigma_p(\;\;)$ is a *norm*.

Note that the *compactness* of $\bar{I}_0$ makes $\mathscr{S}_p(\mathscr{D}_0) \subseteq \mathscr{S}_1(\mathscr{D}_0)$ for $p > 1$.

Lemma (Fejér and F. Riesz). *Let $f(w)$ be regular and bounded for $\mathfrak{J}w > 0$, continuous up to the real axis, and zero at ∞. Then*

$$\int_0^\infty |f(\mathrm{i}v)| \, \mathrm{d}v \;\leqslant\; \frac{1}{2} \int_{-\infty}^\infty |f(u)| \, \mathrm{d}u.$$

Proof. Under our assumptions on f, we have, for $v > 0$,

$$f(\mathrm{i}v) \;=\; \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^\infty \frac{f(u)\,\mathrm{d}u}{u - \mathrm{i}v},$$

$$0 \;=\; \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^\infty \frac{f(u)\,\mathrm{d}u}{u + \mathrm{i}v},$$

as long as $\int_{-\infty}^{\infty} |f(u)| du < \infty$, which is the only situation we need consider (see proof of lemma in § H.1, Chapter III). Adding, we get

$$f(iv) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{uf(u)du}{u^2 + v^2},$$

whence

$$\int_0^{\infty} |f(iv)| dv \leqslant \frac{1}{\pi} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{|u||f(u)|}{u^2+v^2} dv du = \frac{1}{2} \int_{-\infty}^{\infty} |f(u)| du.$$

Q.E.D.

Lemma. *Let $F(z)$ be analytic in a rectangle $\mathscr{D}$ and continuous up to $\bar{\mathscr{D}}$. If Λ is a straight line joining the midpoints of two opposite sides of $\mathscr{D}$, we have*

$$\int_{\Lambda} |F(z)||dz| \leqslant \frac{1}{2} \int_{\partial\mathscr{D}} |F(\zeta)||d\zeta|.$$

Proof.

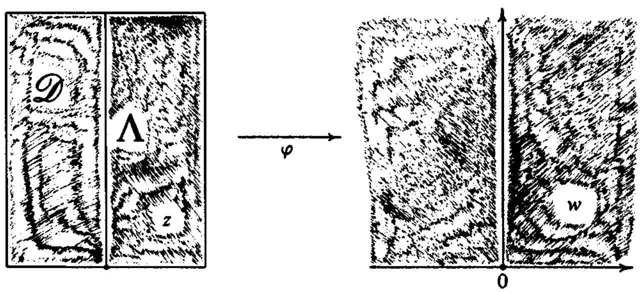

Figure 65

Let φ map $\mathscr{D}$ conformally onto $\Im w > 0$ in such a way that Λ goes onto the *positive imaginary axis*, and, for $z \in \mathscr{D}$ and $w = \varphi(z)$, put

$$f(w) = \frac{F(z)}{\varphi'(z)}.$$

When $w = \varphi(z) \longrightarrow \infty$, $\varphi'(z)$ must tend to ∞ (otherwise the upper half plane would be bounded!), so $f(w)$ must tend to *zero*, $F(z)$ being continuous on $\bar{\mathscr{D}}$. We may therefore apply the previous lemma to f. This yields

$$\int_{\Lambda} |F(z)||dz| = \int_{\Lambda} \left|\frac{F(z)}{\varphi'(z)}\right| |\varphi'(z) dz| = \int_0^{\infty} |f(iv)| dv$$

$$\leqslant \frac{1}{2} \int_{-\infty}^{\infty} |f(u)| du = \frac{1}{2} \int_{\partial\mathscr{D}} \left|\frac{F(\zeta)}{\varphi'(\zeta)}\right| |\varphi'(\zeta) d\zeta| = \frac{1}{2} \int_{\partial\mathscr{D}} |F(\zeta)||d\zeta|,$$

Q.E.D.

Lemma (Beurling). *Let $\mathscr{D}_0$ be the rectangle $\{-a < \Re z < a, 0 < \Im z < h\}$, and let $f \in \mathscr{S}_1(\mathscr{D}_0)$. Then, if $-a < x < a$,*

$$\int_0^h |f(x+iy)| dy \leqslant \left(1 + \frac{h}{a - |x|}\right) \mathscr{s}_1(f).$$

Proof. Wlog, let $x \geqslant 0$. Taking any small $\delta > 0$ we let $\mathscr{D}_l$, for $0 < l < a - x$, be the rectangle shown in the figure:

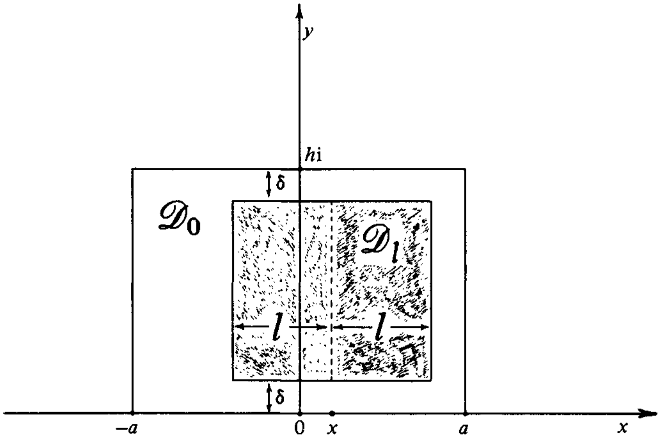

Figure 66

Applying the previous lemma to $\mathscr{D}_l$ we find that

$$\int_\delta^{h-\delta} |f(x+\mathrm{i}y)|\,\mathrm{d}y \;\leqslant\; \frac{1}{2}\int_{\partial\mathscr{D}_l} |f(\zeta)|\,|\mathrm{d}\zeta|.$$

Multiply both sides by $\mathrm{d}l$ and integrate l from $\frac{1}{2}(a-x)$ to $a-x$! We get

$$\frac{a-x}{2}\int_\delta^{h-\delta} |f(x+\mathrm{i}y)|\,\mathrm{d}y \;\leqslant\; \frac{1}{2}\int_{(a-x)/2}^{a-x}\int_{\partial\mathscr{D}_l} |f(\zeta)|\,|\mathrm{d}\zeta|\,\mathrm{d}l.$$

The *lower horizontal sides* of the $\mathscr{D}_l$ contribute at most

$$\frac{a-x}{4}\int_{-(a-x)}^{a-x} |f(x+\mathrm{i}\delta+\xi)|\,\mathrm{d}\xi \;\leqslant\; \frac{a-x}{4}\mathscr{s}_1(f)$$

to the expression on the right, and the *top horizontal sides* of the $\mathscr{D}_l$ contribute a similar amount. The *right vertical sides* give

$$\frac{1}{2}\int_\delta^{h-\delta}\int_{(a-x)/2}^{a-x} |f(x+\mathrm{i}y+l)|\,\mathrm{d}l\,\mathrm{d}y$$

and the *left vertical sides* make a similar contribution. The sum of these last two amounts is

$$\leqslant\; \frac{1}{2}\int_\delta^{h-\delta}\int_{-(a-x)}^{(a-x)} |f(x+\mathrm{i}y+l)|\,\mathrm{d}l\,\mathrm{d}y \;\leqslant\; \tfrac{1}{2}(h-2\delta)\mathscr{s}_1(f).$$

All told, we thus have

$$\frac{1}{2}\int_{(a-x)/2}^{a-x}\int_{\partial\mathscr{D}_l} |f(\zeta)|\,|\mathrm{d}\zeta|\,\mathrm{d}l \;\leqslant\; \left(\frac{a-x}{2}+\frac{h-2\delta}{2}\right)\mathscr{s}_1(f),$$

so by the previous relation we see that

$$\int_{\delta}^{h-\delta} |f(x+\mathrm{i}y)|\mathrm{d}y \;\leqslant\; \left(1+\frac{h-2\delta}{a-x}\right)\mathscr{I}_1(f).$$

Making $\delta \to 0$, we obtain the lemma for the case $x \geqslant 0$. Done.

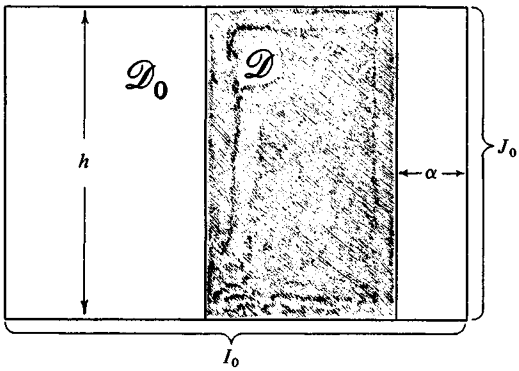

Figure 67

Let $f \in \mathscr{S}_1(\mathscr{D}_0)$, and let $\mathscr{D}$ be a rectangle lying in $\mathscr{D}_0$, in the manner shown – the vertical sides of $\mathscr{D}$ being at *positive distance*, say $\alpha > 0$, from those of $\mathscr{D}_0$. We proceed to investigate the boundary behaviour of f in $\mathscr{D}$.

In order to do this, it is convenient to *take* 0 *as the point of intersection of the diagonals of* $\mathscr{D}$. This setup makes it easy for us to imitate the discussion at the beginning of § F.1, Chapter III.

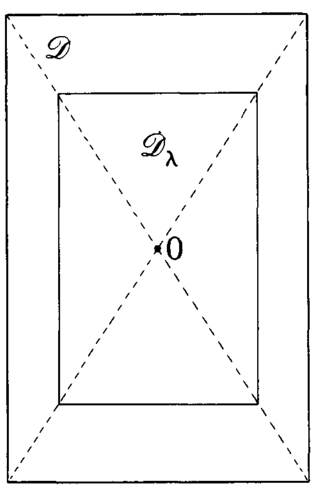

Figure 68

For $0<\lambda<1$ denote by $\mathscr{D}_\lambda$ the rectangle $\{\lambda z\colon z\in\mathscr{D}\}$ (see diagram). $\mathscr{D}_\lambda \subseteq \mathscr{D}$ which, in turn, has the above described disposition inside $\mathscr{D}_0$. Since $f\in\mathscr{S}_1(\mathscr{D}_0)$, we have, by the preceding lemma,

$$\int_{\partial\mathscr{D}_\lambda} |f(\zeta)||\mathrm{d}\zeta| \;\leqslant\; 2\sigma_1(f) \;+\; 2\left(1+\frac{h}{\alpha}\right)\sigma_1(f),$$

calling h the height of $\mathscr{D}_0$. In other words,

$$(*)\qquad \int_{\mathscr{D}} |f(\lambda\zeta)||\mathrm{d}\zeta| \;\leqslant\; \frac{K}{\lambda}$$

for $0<\lambda<1$, where K depends on $\mathscr{D}$ and on f.

Fix any $z_0\in\mathscr{D}$ and let $\lambda<1$. The function $f(\lambda z)$ is certainly analytic (hence *harmonic*!) in $\mathscr{D}$ and *continuous* up to $\partial\mathscr{D}$ (when z ranges over $\bar{\mathscr{D}}$, the argument of $f(\lambda z)$ actually ranges over $\bar{\mathscr{D}}_\lambda$). Therefore, *by the discussion of article* 1,

$$f(\lambda z_0) \;=\; \int_{\partial\mathscr{D}} f(\lambda\zeta)\,\mathrm{d}\omega_{\mathscr{D}}(\zeta, z_0),$$

denoting, as usual, harmonic measure for $\mathscr{D}$ by $\omega_{\mathscr{D}}(\ \ ,z)$. Since the corners of $\mathscr{D}$ makes angles (of 90°) *less than* 180° *from inside*, we know by article 1 that $\mathrm{d}\omega_{\mathscr{D}}(\zeta,z_0)/|\mathrm{d}\zeta|$ is *bounded* (and indeed *continuous*) on $\partial\mathscr{D}$, and the preceding formula can be rewritten thus:

$$f(\lambda z_0) \;=\; \int_{\partial\mathscr{D}} \frac{\mathrm{d}\omega_{\mathscr{D}}(\zeta,z_0)}{|\mathrm{d}\zeta|}\cdot f(\lambda\zeta)|\mathrm{d}\zeta|.$$

(In order to compute $\mathrm{d}\omega_{\mathscr{D}}(\zeta,z_0)/|\mathrm{d}\zeta|$ explicitly, we would have to resort to elliptic functions!)

We can now argue by $(*)$ that there is a certain complex valued measure μ on $\partial\mathscr{D}$ such that

$$f(\lambda\zeta)|\mathrm{d}\zeta| \longrightarrow \mathrm{d}\mu(\zeta) \quad \mathrm{w}^*$$

when $\lambda\to 1$ through a certain sequence of values, and thereby deduce from the previous relation that

$$(\dagger)\qquad f(z_0) = \int_{\partial\mathscr{D}} \frac{\mathrm{d}\omega_{\mathscr{D}}(\zeta,z_0)}{|\mathrm{d}\zeta|}\,\mathrm{d}\mu(\zeta).$$

(See proof of first theorem in § F.1, Chapter III.) This, of course, holds for any $z_0\in\mathscr{D}$.

Let φ be a conformal mapping of $\mathscr{D}$ onto $\{|w|<1\}$ and let the function F, analytic in the unit disk, be defined by the formula $F(\varphi(z)) = f(z)$, $z\in\mathscr{D}$. If ν is the complex measure on $\{|w|=1\}$ such that $\mathrm{d}\nu(\varphi(\zeta)) = \mathrm{d}\mu(\zeta)$ for ζ varying on $\partial\mathscr{D}$, (†) becomes

$$(\dagger\dagger) \qquad F(w) = \frac{1}{2\pi}\int_{|\omega|=1}\frac{1-|w|^2}{|w-\omega|^2}\mathrm{d}\nu(\omega),$$

$|w|<1$. The integral on the right therefore represents an *analytic function of* w for $|w|<1$. From *this* it follows by the celebrated *theorem of the brothers Riesz* that ν *must be absolutely continuous*, i.e.,

$$(\S) \qquad \mathrm{d}\nu(\omega) = \psi(\omega)|\mathrm{d}\omega|$$

with some L_1*-function* ψ *on the unit circumference.* By Chapter II, § B, and (††) we now have $F(w) \longrightarrow \psi(\omega)$ as $w \not\longrightarrow \omega$ for almost every ω on the unit circumference. Write $g(\zeta) = \psi(\varphi(\zeta))$ for $\zeta\in\partial\mathscr{D}$. Then, going back to $\mathscr{D}$, we see by the discussion in article 1 that

$$f(z) \longrightarrow g(\zeta) \quad \text{as} \quad z \not\longrightarrow \zeta$$

for almost every $\zeta\in\partial\mathscr{D}$.

Plugging (§) into (††) and then returning to (†), we find that

$$f(z_0) = \int_{\partial\mathscr{D}} \frac{\mathrm{d}\omega_{\mathscr{D}}(\zeta, z_0)}{|\mathrm{d}\zeta|} g(\zeta)|\mathrm{d}\zeta|.$$

We have already practically obtained the

Theorem. *Let* $f\in\mathscr{S}_1(\mathscr{D}_0)$*. Then*

$$\lim_{z \not\to \zeta} f(z) \text{ which we call } f(\zeta)$$

exists for almost every ζ *on the* horizontal sides *of* $\mathscr{D}_0$.

If $\mathscr{D}$ *is a rectangle in* $\mathscr{D}_0$*, disposed in the manner indicated above,*

$$\int_{\partial\mathscr{D}} |f(\zeta)||\mathrm{d}\zeta| < \infty,$$

and, for $z\in\mathscr{D}$*,*

$$f(z) = \int_{\partial\mathscr{D}} f(\zeta)\,\mathrm{d}\omega_{\mathscr{D}}(\zeta, z).$$

If B_1 *and* B_2 *denote the horizontal sides of* $\mathscr{D}_0$*, we have*

$$\int_{B_1} |f(z)|\mathrm{d}x \leqslant \mathfrak{s}_1(f),$$

$$\int_{B_2} |f(z)|\mathrm{d}x \leqslant \mathfrak{s}_1(f).$$

Proof.

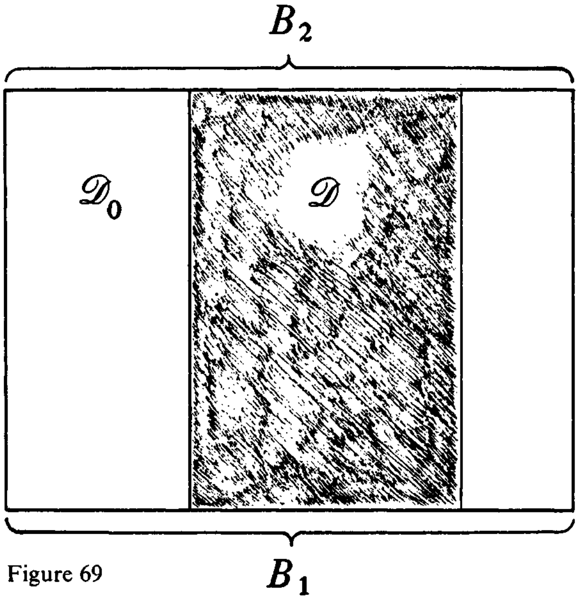

Figure 69

The *first* statement holds because $\lim_{z \,\angle\!\to \zeta} f(z)$ exists for almost all ζ on the boundary of *any* rectangle $\mathscr{D}$ lying in $\mathscr{D}_0$ in the manner shown; this we have just seen. Of course, if ζ lies on the *vertical sides* of such a rectangle $\mathscr{D}$, we *know anyway* that $\lim_{z\to\zeta} f(z)$ (without the angle mark!) exists and equals $f(\zeta)$, since those vertical sides lie in $\mathscr{D}_0$, where f is given as analytic. The *second* statement therefore follows from $(*)$ and the *first* one, by Fatou's lemma. (In using $(*)$, one must take 0 as the point of intersection of the diagonals of $\mathscr{D}$.)

In view of what has just been said, the *third* statement is merely another way of expressing the formula immediately preceding this theorem. There remains the *fourth* statement. Considering, for instance, the *upper horizontal side* B_2 of $\mathscr{D}_0$, we have $f(z - i/n) \underset{n}{\longrightarrow} f(z)$ for *almost all* $z \in B_2$ (*first* statement!). Therefore, by Fatou's lemma,

$$\int_{B_2} |f(z)|\,\mathrm{d}z \;\leqslant\; \liminf_{n\to\infty} \int_{B_2} \left| f\left(z - \frac{i}{n}\right) \right| \mathrm{d}x.$$

The integrals on the right are all $\leqslant \sigma_1(f)$ (by definition), at least as soon as $1/n <$ the height of $\mathscr{D}_0$. We are done.

Theorem. *Let I be any interval properly included within the base of $\mathscr{D}_0$, in the manner shown:*

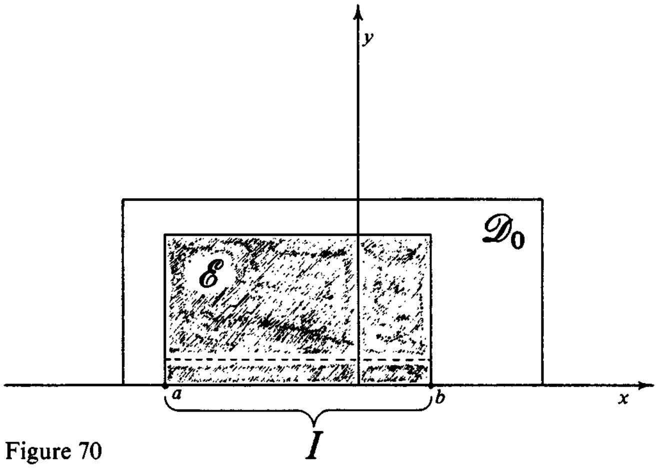

Figure 70

Then, if $f \in \mathscr{S}_1(\mathscr{D}_0)$,

$$\int_I |f(z+i\delta) - f(z)|\,dx \longrightarrow 0$$

as $\delta \to 0$.

Proof. To simplify the writing, we take the base of $\mathscr{D}_0$ to lie on the x-axis as shown in the figure.

In view of the preceding theorem, we may assume that, at the *endpoints* a and b of I, $\lim_{z \not\angle\!\to a} f(z)$ and $\lim_{z \not\angle\!\to b} f(z)$ *exist and are finite.* (Otherwise, just make *I a little bigger.*) Then, if we construct the rectangle $\mathscr{E} \subseteq \mathscr{D}_0$ with base on I, in the way shown in the figure, $f(z)$ *will be continuous on the top and two vertical sides of* $\mathscr{E}$, *right up to where the latter meet* I. And by *exactly the same argument* as the one used to establish the *third* statement of the preceding theorem, we can see that

$$f(z) = \int_{\partial\mathscr{E}} f(\zeta)\,d\omega_{\mathscr{E}}(\zeta, z) \quad \text{for} \quad z \in \mathscr{E}.$$

Now let $\varepsilon > 0$ be given, and take a *continuous function* $g(\zeta)$ defined on $\partial\mathscr{E}$ which *coincides with* $f(\zeta)$ *on the top and vertical sides of* $\mathscr{E}$ and is specified on I in such a way that

$$\int_I |f(\xi) - g(\xi)|\,d\xi < \varepsilon,$$

For $z \in \mathscr{E}$, put

$$g(z) = \int_{\partial\mathscr{E}} g(\zeta)\,d\omega_{\mathscr{E}}(\zeta, z);$$

$g(z)$ is at least *harmonic* in $\mathscr{E}$ (N.B. *not* necessarily *analytic* there!), and, by the discussion in article 1, *continuous up to* $\partial\mathscr{E}$, *where it takes the boundary values* $g(\zeta)$.

For $x \in I$ and small $\delta > 0$,

$$f(x+\mathrm{i}\delta)-f(x) = f(x+\mathrm{i}\delta)-g(x+\mathrm{i}\delta)+g(x+\mathrm{i}\delta) - g(x)+g(x)-f(x).$$

We are interested in showing that $\int_I |f(x+\mathrm{i}\delta)-f(x)|\mathrm{d}x$ is *small* if $\delta > 0$ is small enough. We already know that $\int_I |g(x)-f(x)|\mathrm{d}x < \varepsilon$, and, by *continuity* of g on $\bar{\mathscr{E}}$, $\int_I |g(x+\mathrm{i}\delta)-g(x)|\mathrm{d}x < \varepsilon$ if $\delta > 0$ is small. We will therefore be *done* if we verify that

$$\int_I |g(x+\mathrm{i}\delta)-f(x+\mathrm{i}\delta)|\mathrm{d}x < \varepsilon.$$

Since $f(\zeta) = g(\zeta)$ on $\partial\mathscr{E} \sim I$,

$$f(x+\mathrm{i}\delta)-g(x+\mathrm{i}\delta) = \int_I (f(\xi)-g(\xi))\mathrm{d}\omega_{\mathscr{E}}(\xi, x+\mathrm{i}\delta).$$

However, $\mathscr{E}$ lies in the upper half-plane and I on the real axis, so, by the *principle of extension of domain* used in article 1, for $x+\mathrm{i}\delta \in \mathscr{E}$,

$$\mathrm{d}\omega_{\mathscr{E}}(\xi, x+\mathrm{i}\delta) \leqslant \frac{1}{\pi}\frac{\delta\mathrm{d}\xi}{(x-\xi)^2+\delta^2}$$

on I, the right-hand expression being the differential of *harmonic measure* for $\{\Im z > 0\}$ as seen from $x+\mathrm{i}\delta$. Thus, for $x \in I$,

$$|f(x+\mathrm{i}\delta)-g(x+\mathrm{i}\delta)| \leqslant \frac{1}{\pi}\int_I |f(\xi)-g(\xi)|\frac{\delta\mathrm{d}\xi}{(x-\xi)^2+\delta^2}.$$

And

$$\int_I |f(x+\mathrm{i}\delta)-g(x+\mathrm{i}\delta)|\mathrm{d}x$$

$$\leqslant \frac{1}{\pi}\int_I\int_{-\infty}^{\infty} |f(\xi)-g(\xi)|\frac{\delta}{(x-\xi)^2+\delta^2}\mathrm{d}x\,\mathrm{d}\xi$$

$$= \int_I |f(\xi)-g(\xi)|\mathrm{d}\xi < \varepsilon.$$

This does it.

Corollary. *Let* $f \in \mathscr{S}_1(\mathscr{D}_0)$ *and let* $G(z)$ *be any function analytic in a region including the closure of a rectangle* $\mathscr{E}$ *like the one used above lying in* $\mathscr{D}_0$*'s* *interior. Then*

$$\int_{\partial\mathscr{E}} G(\zeta)f(\zeta)d\zeta \;=\; 0.$$

Proof. Use Cauchy's theorem for the rectangles with the dotted base together with the above result:

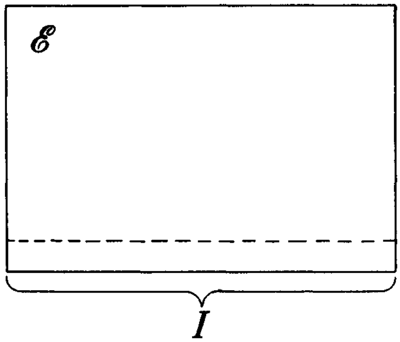

Figure 71

Note that the integrals along the *vertical sides* of $\mathscr{E}$ are absolutely convergent by the *third* lemma of this article.

We need one more result – a Jensen inequality for rectangles $\mathscr{E}$ like the one used above.

Theorem. *Let $f \in \mathscr{S}_1(\mathscr{D}_0)$, and let $\mathscr{E}$ be a rectangle like the one shown:*

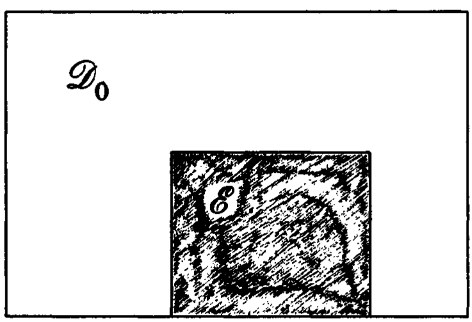

Figure 72

Then, for $z \in \mathscr{E}$,

$$\log|f(z)| \;\leqslant\; \int_{\partial\mathscr{E}} \log|f(\zeta)|d\omega_{\mathscr{E}}(\zeta, z).$$

Proof. This would just be a restatement of the theorem on harmonic estimation from article 1, except that $f(z)$ is not necessarily continuous up to the base of $\mathscr{E}$. There are several ways of getting around the difficulty caused by this lack of continuity; in one such we first map $\mathscr{E}$ conformally onto the unit disk and then use properties of the space H_1. Functions in H_1 can be expressed as products of inner and outer factors, so Jensen's inequality holds for them.

In order to keep the exposition as nearly self-contained as possible, we give a different argument, based on *Szegő's theorem* (§A, Chapter II!), whose idea goes back to Helson and Lowdenslager.

Given $z_0 \in \mathscr{E}$, take a conformal mapping φ onto $\{|w|<1\}$ that sends z_0 to 0, and define a function $F(w)$ analytic in the unit disk by means of the formula

$$F(\varphi(z)) = f(z), \quad z \in \mathscr{E}.$$

The relation

$$f(z) \;=\; \int_{\partial\mathscr{E}} f(\zeta)\, d\omega_{\mathscr{E}}(\zeta, z), \quad z \in \mathscr{E},$$

used in proving the above theorem, goes over into

$$F(w) \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-|w|^2}{|w-e^{i\tau}|^2}\, F(e^{i\tau})\, d\tau,$$

with $F(e^{i\tau}) = f(\varphi^{-1}(e^{i\tau}))$ defined almost everywhere on the unit circumference and in L_1 (see discussion preceding the first theorem of this article).

From this last relation, we have

$$\int_0^{2\pi} |F(\rho e^{i\vartheta}) - F(e^{i\vartheta})|\, d\vartheta \;\longrightarrow\; 0$$

as $\rho \to 1$. Also, for each $\rho < 1$, $\int_0^{2\pi} e^{in\vartheta} F(\rho e^{i\vartheta})\, d\vartheta = 0$ when $n = 1, 2, 3, \ldots$ by *Cauchy's theorem*. Hence

$$\int_0^{2\pi} e^{in\vartheta} F(e^{i\vartheta})\, d\vartheta \;=\; 0$$

for $n = 1, 2, 3, \ldots$, and, finally,

$$\frac{1}{2\pi}\int_0^{2\pi} \left(1 + \sum_{n>0} A_n e^{in\vartheta}\right) F(e^{i\vartheta})\, d\vartheta \;=\; F(0)$$

for *any* finite sum $\sum_{n>0} A_n e^{in\vartheta}$.

Thus,

$$|F(0)| \;\leqslant\; \frac{1}{2\pi}\int_0^{2\pi} \left|1 + \sum_{n>0} A_n e^{in\vartheta}\right| |F(e^{i\vartheta})|\, d\vartheta$$

for all such finite sums. By Szegő's theorem, the *infimum* of the expressions on the right is

$$\exp\left(\frac{1}{2\pi}\int_0^{2\pi}\log|F(e^{i\vartheta})|\,d\vartheta\right).$$

Therefore,

$$\log|F(0)| \leqslant \frac{1}{2\pi}\int_0^{2\pi}\log|F(e^{i\vartheta})|\,d\vartheta,$$

or, in terms of f and $z_0 = \varphi^{-1}(0)$:

$$\log|f(z_0)| \leqslant \int_{\partial\mathscr{E}}\log|f(\zeta)|\,d\omega_{\mathscr{E}}(\zeta, z_0).$$

That's what we wanted to prove.

5. Beurling's quasianalyticity theorem for L_p approximation by functions in $\mathscr{S}_p(\mathscr{D}_0)$.

Being now in possession of the previous article's somewhat *ad hoc* material, we are able to look at approximation by functions in $\mathscr{S}_p(\mathscr{D}_0)$ ($p \geqslant 1$) and to prove a result about such approximation analogous to the one of article 3.

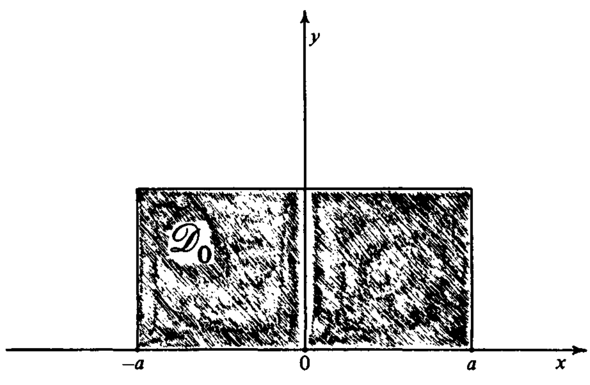

Figure 73

Throughout the following discussion, we work with a certain rectangular domain $\mathscr{D}_0$ whose base is an interval on the real axis which we take, wlog, as $[-a, a]$. If $p \geqslant 1$, $\mathscr{S}_p(\mathscr{D}_0) \subseteq \mathscr{S}_1(\mathscr{D}_0)$, so we know by the *first* theorem of the previous article that, for functions f in $\mathscr{S}_p(\mathscr{D}_0)$, the non-tangential boundary values $f(x)$ exist for almost every x on $[-a, a]$. As in the proof of that theorem we see by Fatou's lemma (there applied in the case $p = 1$) that

$$\int_{-a}^{a} |f(x)|^p \, dx \;\leqslant\; (\mathscr{s}_p(f))^p, \qquad f \in \mathscr{S}_p(\mathscr{D}_0).$$

The 'restrictions' of functions $f \in \mathscr{S}_p(\mathscr{D}_0)$ to $[-a, a]$ thus belong to $L_p(-a, a)$, and we may use them to try to approximate arbitrary members of $L_p(-a, a)$ in the norm of that space.

In analogy with article 3, we define the L_p *approximation index* $M_p(A)$ *for any given* $\varphi \in L_p(-a, a)$ (and the rectangle $\mathscr{D}_0$) as follows:

> $e^{-M_p(A)}$ *is the infimum of* $\sqrt[p]{\int_{-a}^{a} |\varphi(x) - f(x)|^p \, dx}$
>
> *for* $f \in \mathscr{S}_p(\mathscr{D}_0)$ *with* $\mathscr{s}_p(f) \leqslant e^A$.

$M_p(A)$ is obviously an *increasing* function of A, and we have the following

Theorem (Beurling). *Let* $\varphi \in L_p(-a, a)$, *and let its* L_p *approximation index* $M_p(A)$ (*for* $\mathscr{D}_0$) *satisfy*

$$\int_1^{\infty} \frac{M_p(A)}{A^2} \, dA \;=\; \infty.$$

If $\varphi(x)$ *vanishes on a set of* positive measure *in* $[-a, a]$, *then* $\varphi(x) \equiv 0$ *a.e. on* $[-a, a]$.

Proof. We first carry out some *preliminary reductions.*

We have $\mathscr{S}_p(\mathscr{D}_0) \subseteq \mathscr{S}_1(\mathscr{D}_0)$, $L_p(-a, a) \subseteq L_1(-a, a)$, and, by Hölder's inequality, $\mathscr{s}_1(f) \leqslant a^{(p-1)/p} \mathscr{s}_p(f)$ and $\|\varphi - f\|_1 \leqslant a^{(p-1)/p} \|\varphi - f\|_p$ for $f \in \mathscr{S}_p(\mathscr{D}_0)$ and $\varphi \in L_p(-a, a)$. (We write $\| \;\; \|_p$ for the L_p norm on $[-a, a]$). From these facts it is clear that, if $\varphi \in L_p(-a, a)$ has L_p approximation index $M_p(A)$, *the* L_1 *approximation index* $M_1(A)$ *of* $a^{p/(p-1)}\varphi$ (*sic*!) *is* $\geqslant M_p(A)$. *It is therefore enough to prove the theorem for* $p = 1$, *for it will then follow for all values of* $p > 1$.

Suppose then that $\int_1^{\infty} (M_1(A)/A^2) \, dA = \infty$ with $M_1(A)$ the L_1 approximation index for $\varphi \in L_1(-a, a)$, and that φ vanishes on a set of positive measure in $[-a, a]$. In order to prove that $\varphi \equiv 0$ a.e. on $[-a, a]$, it is *enough* to show that it *vanishes a.e. on some interval* $J \subseteq [-a, a]$ with positive length.

To see this, take any very small *fixed* $\eta > 0$ and write

$$\varphi_\eta(x) \;=\; \frac{1}{2\eta} \int_{-\eta}^{\eta} \varphi(x + t) \, dt$$

for $-a+\eta \leqslant x \leqslant a-\eta$. $\varphi_\eta(x)$ is then *continuous* on the interval $[-a+\eta,\ a-\eta]$, and *vanishes identically on an interval of positive length therein* as long as $2\eta < |J|$. Corresponding to any $f \in \mathscr{S}_1(\mathscr{D}_0)$ we also form the function

$$f_\eta(z) \;=\; \frac{1}{2\eta}\int_{-\eta}^{\eta} f(z+t)\,\mathrm{d}t;$$

let us check that $f_\eta(z)$ is *analytic* in the rectangle $\mathscr{D}_\eta$ with base $[-a+2\eta,\ a-2\eta]$ having the same height as $\mathscr{D}_0$, and is *continuous* on $\mathscr{D}_\eta$.

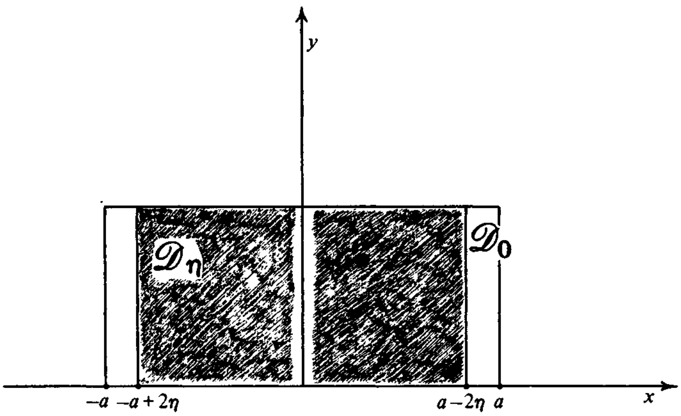

Figure 74

The *analyticity* of $f_\eta(z)$ in $\mathscr{D}_\eta$ is *clear*; so is *continuity* up to the *vertical sides* of $\mathscr{D}_\eta$. The boundary-value function $f(x)$ belongs to $L_1(-a,a)$, so $f_\eta(x)$ *is* continuous *on* $[-a+2\eta,\ a-2\eta]$. Let, then, $-a+2\eta \leqslant x_0 \leqslant a-2\eta$, and suppose that x, also on that closed interval, is *near* x_0 and that $y>0$ is *small*. We have $|f_\eta(x_0)-f_\eta(x+\mathrm{i}y)| \leqslant |f_\eta(x_0)-f_\eta(x)| + |f_\eta(x)-f_\eta(x+\mathrm{i}y)|$. The first term on the right is small if x is close enough to x_0. The second is

$$\leqslant\; \frac{1}{2\eta}\int_{x-\eta}^{x+\eta} |f(\xi)-f(\xi+\mathrm{i}y)|\,\mathrm{d}\xi \;\leqslant\; \frac{1}{2\eta}\int_{-a+\eta}^{a-\eta} |f(\xi)-f(\xi+\mathrm{i}y)|\,\mathrm{d}\xi$$

which, by the *second* theorem of the preceding article, tends to *zero* (independently of x!) as $y \to 0$. Thus $f_\eta(x+\mathrm{i}y) \longrightarrow f_\eta(x_0)$ as $x+\mathrm{i}y \longrightarrow x_0$ from within $\mathscr{D}_\eta$, and *continuity of f_η up to the lower horizontal side of $\mathscr{D}_\eta$ is established. Continuity* of f_η up to the *upper horizontal side* of $\mathscr{D}_\eta$ follows in like manner, so $f_\eta(z)$ is continuous on $\bar{\mathscr{D}}_\eta$.

The functions f_η are thus of the kind used in article 3 to *uniformly approximate continuous functions* given on $[-a+2\eta,\ a+2\eta]$. By definition of $M_1(A)$, we can find an f in $\mathscr{S}_1(\mathscr{D}_0)$ with $\mathfrak{s}_1(f) \leqslant e^A$ and $\int_{-a}^{a} |\varphi(x) - f(x)| dx \leqslant 2e^{-M_1(A)}$. With this f, $|f_\eta(z)| \leqslant (1/2\eta)e^A$ for $z \in \bar{\mathscr{D}}_\eta$ and

$$|\varphi_\eta(x) - f_\eta(x)| \;\leqslant\; \frac{1}{\eta} e^{-M_1(A)}$$

on $[-a + 2\eta,\ a - 2\eta]$. The *uniform approximation index* $M(A)$ *for* $\eta\varphi_\eta$ (and the domain $\mathscr{D}_\eta$) *is thus* $\geqslant M_1(A)$. Therefore, under the hypothesis of the present theorem,

$$\int_1^\infty \frac{M(A)}{A^2} dA \;=\; \infty,$$

so, since $\varphi_\eta(x)$ *vanishes identically* on an *interval of positive length* in $[-a + 2\eta,\ a - 2\eta]$ (when $\eta > 0$ is small enough) *we have*

$$\varphi_\eta(x) \equiv 0, \quad -a + 2\eta \leqslant x \leqslant a - 2\eta$$

by the theorem of article 3.

However, as $\eta \to 0$, $\varphi_\eta(x) \longrightarrow \varphi(x)$ a.e. on $(-a, a)$. From what has just been shown we conclude, then, that $\varphi(x) \equiv 0$ a.e. on $(-a, a)$ *if it vanishes* a.e. on an *interval* J of positive length lying therein, provided that

$$\int_1^\infty \frac{M_1(A)}{A^2} dA \;=\; \infty.$$

Our task has thus finally boiled down to the following one. *Given* $\varphi \in L_1(-a, a)$ *with* L_1 *approximation index* $M_1(A)$ (for $\mathscr{D}_0$) *such that*

$$\int_1^\infty \frac{M_1(A)}{A^2} dA \;=\; \infty,$$

prove that φ *vanishes a.e. on an interval of positive length in* $(-a, a)$ *if it vanishes on a set of positive measure therein.*

Let us proceed. It is easy to see that the *increasing* function $M_1(A)$ is continuous (in the extended sense) – that's because, if $\lambda < 1$ is close to 1, λf approximates φ *almost as well* as f does in $L_1(-a, a)$. Since $\int_1^\infty (M_1(A)/A^2) dA = \infty$ we may therefore, starting with a suitable $A_1 > 0$, get an increasing sequence of numbers A_n tending to ∞ such that

$$M_1(A_{n+1}) = 2M_1(A_n).*$$

Assume henceforth that $\varphi(x) = 0$ on the closed set $E_0 \subseteq [-a, a]$ with

* We are allowing for the possibility that $M_1(A) \equiv \infty$ for large values of A; this happens when $\varphi(x)$ actually coincides with a function in $\mathscr{S}_p(\mathscr{D}_0)$ on $(-a, a)$, and then it is necessary to take A_1 with $M_1(A_1) = \infty$. We will, in any event, need to have A_1 *large* – see the following page.

$|E_0| > 0$*. For each $A > 0$ there is an $f \in \mathscr{S}_1(\mathscr{D}_0)$ with $\mathcal{s}_1(f) \leqslant e^A$ and

$$\int_{-a}^{a} |\varphi(x) - f(x)| dx \;\leqslant\; 2e^{-M_1(A)}$$

In particular,

$$\int_{E_0} |f(x)| dx \;\leqslant\; 2e^{-M_1(A)},$$

so, if

$$\Delta_A \;=\; \{x \in E_0 \colon \; |f(x)| > e^{-M_1(A)/2}\},$$

we have $|\Delta_A| \leqslant 2e^{-M_1(A)/2}$. Taking the sequence of numbers A_n just described, we thus get

$$\left|\bigcup_n \Delta_{A_n}\right| \;\leqslant\; 2\sum_n e^{-M_1(A_n)/2} \;=\; 2\sum_1^\infty e^{-2^{n-1}M_1(A_1)}.$$

We can choose A_1 large enough so that this sum is

$$< \frac{|E_0|}{2};$$

then the set

$$E \;=\; E_0 \sim \left(\bigcup_n \Delta_n\right)$$

has measure $> |E_0|/2$, and, by its *construction*, for each n there is an $f_n \in \mathscr{S}_1(\mathscr{D}_0)$ with $\mathcal{s}_1(f_n) \;\leqslant\; e^{A_n}$,

$$\int_{-a}^{a} |\varphi(x) - f_n(x)| dx \;\leqslant\; 2e^{-M_1(A_n)},$$

and

$$|f_n(x)| \;\leqslant\; e^{-M_1(A_n)/2}$$

for $x \in E$.

Take now a number b, $0 < b < a$, sufficiently close to a so that

$$|E \cap [-b, \; b]| \;>\; 0,$$

and construct the rectangle $\mathscr{D}$ with base on $[-b, \; b]$, lying within $\mathscr{D}_0$ in the manner shown:

* where $|E|$ denotes the Lebesgue measure of $E \subseteq \mathbb{R}$

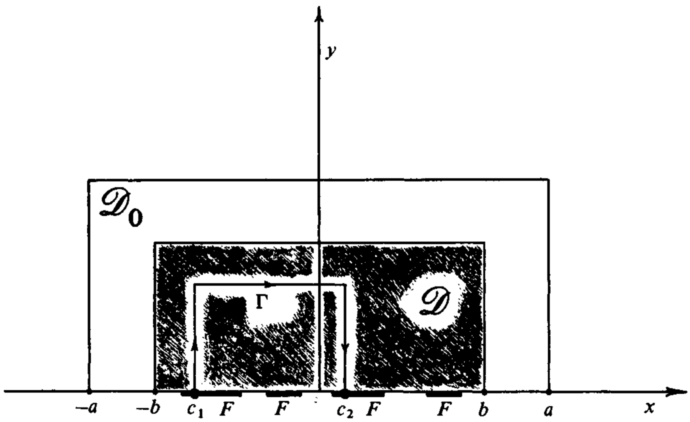

Figure 75

Take a *closed subset F of $E \cap (-b, b)$ having positive measure*; this set F will remain fixed during the following discussion.

As we saw at the end of article 1,

$$\omega_{\mathscr{D}}(F, x + iy) \longrightarrow 1$$

as $y \to 0+$ for almost every $x \in F$. Let c_1 and c_2, $c_1 < c_2$, be *two such x's for which this is true*. We are going to show that $\varphi(x) = 0$ a.e. for $c_1 \leqslant x \leqslant c_2$; according to what has been said above, *this is all we need to do* to finish the proof of our theorem.

The desired vanishing of φ will follow if

$$\Phi(\lambda) = \int_{c_1}^{c_2} e^{i\lambda x} \varphi(x)\, dx$$

is identically zero. Φ is, however, *an entire function of exponential type bounded on the real axis*. Hence, by §G.2 *of Chapter* III, $\Phi \equiv 0$ provided that

$$\int_1^\infty \frac{1}{\lambda^2} \log \left| \frac{1}{\Phi(\lambda)} \right| d\lambda = \infty.$$

We proceed to verify this relation. The reasoning here resembles that of article 2, but is more complicated.

Take one of the functions f_n (later on, n will be made to depend on λ), and write

$$\Phi(\lambda) = \int_{c_1}^{c_2} e^{i\lambda x}(\varphi(x) - f_n(x))\, dx + \int_{c_1}^{c_2} e^{i\lambda x} f_n(x)\, dx = \mathrm{I} + \mathrm{II}, \text{ say.}$$

Here, for $\lambda > 0$,

$$|\mathrm{I}| \leqslant \int_{c_1}^{c_2} |\varphi(x) - f_n(x)| \mathrm{d}x \leqslant 2\mathrm{e}^{-M_1(A_n)},$$

and the real work is to estimate II.

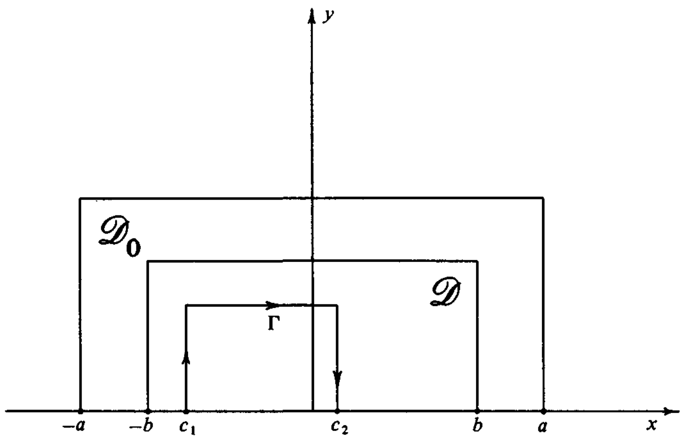

Figure 76

Let Γ be a fixed contour in $\mathscr{D}$ consisting of three sides of a rectangle with base on $[c_1, c_2]$. Because $f_n \in \mathscr{S}_1(\mathscr{D}_0)$, we have

$$\int_{c_1}^{c_2} \mathrm{e}^{\mathrm{i}\lambda x} f_n(x)\,\mathrm{d}x = \int_\Gamma \mathrm{e}^{\mathrm{i}\lambda z} f_n(z)\,\mathrm{d}z$$

by the *corollary* to the *second* theorem of the previous article. In order to estimate the integral on the right, we use the inequality

$$\log|f_n(z)| \leqslant \int_{\partial\mathscr{D}} \log|f_n(\zeta)|\,\mathrm{d}\omega_{\mathscr{D}}(\zeta, z), \qquad z \in \mathscr{D},$$

furnished by the *third* theorem in the preceding article. This we further break up so as to obtain the following for $z \in \mathscr{D}$:

$$(*) \qquad \log|f_n(z)| \leqslant \int_\Pi \log|f_n(\zeta)|\,\mathrm{d}\omega_{\mathscr{D}}(\zeta, z) + \int_F \log|f_n(\zeta)|\,\mathrm{d}\omega_{\mathscr{D}}(\zeta, z) + \int_{(-b,b)\sim F} \log|f_n(\zeta)|\,\mathrm{d}\omega_{\mathscr{D}}(\zeta, z).$$

Here, Π denotes $\partial\mathscr{D} \sim (-b, b)$, i.e., the *vertical* and *top horizontal sides* of $\mathscr{D}$:

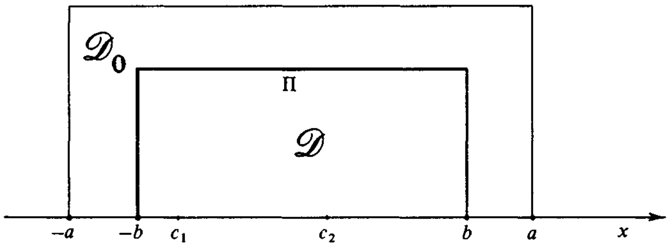

Figure 77

Consider the *first* integral on the right in $(*)$. It equals a certain function $u(z)$ harmonic in $\mathscr{D}$. Take any *harmonic conjugate* $v(z)$ of $u(z)$ for the region $\mathscr{D}$ and put

$$g_n(z) \;=\; \mathrm{e}^{u(z)+\mathrm{i}v(z)}, \qquad z\in\mathscr{D};$$

the function $g_n(z)$ is *analytic* in $\mathscr{D}$ and we have

$$\log|g_n(z)| \;=\; \int_\Pi \log|f_n(\zeta)|\,\mathrm{d}\omega_{\mathscr{D}}(\zeta, z), \qquad z\in\mathscr{D}.$$

In the same way we get functions $h_n(z)$ and $k_n(z)$ analytic in $\mathscr{D}$ with

$$\log|h_n(z)| \;=\; \int_F \log|f_n(\zeta)|\,\mathrm{d}\omega_{\mathscr{D}}(\zeta, z), \qquad z\in\mathscr{D},$$

and

$$\log|k_n(z)| \;=\; \int_{(-b,b)\sim F} \log|f_n(\zeta)|\,\mathrm{d}\omega_{\mathscr{D}}(\zeta, z), \qquad z\in\mathscr{D}.$$

In terms of these functions, $(*)$ becomes

$$(\dagger) \qquad |f_n(z)| \;\leqslant\; |g_n(z)|\,|h_n(z)|\,|k_n(z)|, \qquad z\in\mathscr{D}.$$

Our idea now is to estimate $\sup_{z\in\Gamma}|g_n(z)|$, $\sup_{z\in\Gamma}|h_n(z)|$ and $\int_\Gamma |k_n(z)|\,|\mathrm{d}z|$ in order to get a bound on $\int_\Gamma \mathrm{e}^{\mathrm{i}\lambda z} f_n(z)\,\mathrm{d}z$ for $\lambda > 0$. The *third* of these quantities will give us the most trouble.

We first look at $|g_n(z)|$, $z\in\Gamma$. For ζ on Π, the Poisson kernel $\mathrm{d}\omega_{\mathscr{D}}(\zeta, z)/|\mathrm{d}\zeta|$ goes to *zero* when $z\in\mathscr{D}$ tends to any point of $(-b, b)$, and does so *uniformly* for $\zeta\in\Pi$ and z tending to any point of $[c_1, c_2]$. From this we see, by reflecting the harmonic function $\mathrm{d}\omega_{\mathscr{D}}(\zeta, z)/|\mathrm{d}\zeta|$ of z across $(-b, b)$, that there is a certain constant C, depending *only* on the geometric configuration of Γ and $\mathscr{D}$, such that

$$\frac{\mathrm{d}\omega_{\mathscr{D}}(\zeta, z)}{|\mathrm{d}\zeta|} \;\leqslant\; C\,\Im z, \qquad z\in\Gamma, \quad \zeta\in\Pi.$$

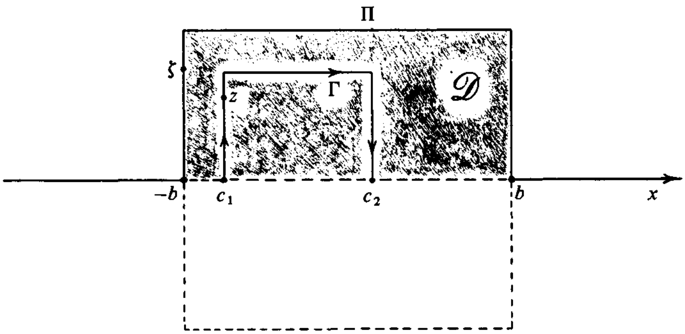

Figure 78

Substituting this into the above formula for $\log|g_n(z)|$, we get

$$\log|g_n(z)| \leqslant C\Im z \int_\Pi \log|f_n(\zeta)||d\zeta|$$

for $z\in\Gamma$, whence, by the inequality between arithmetic and geometric means,*

$$|g_n(z)| \leqslant \left(\frac{1}{|\Pi|}\right)^{|\Pi|C\Im z}\left(\int_\Pi |f_n(\zeta)||d\zeta|\right)^{|\Pi|C\Im z},$$

$z\in\Gamma$. Write now $|\Pi|C = B$. Then we have

$$|g_n(z)| \leqslant \text{const.}\left(\int_\Pi |f_n(\zeta)||d\zeta|\right)^{B\Im z}, \qquad z\in\Gamma,$$

where the constant is independent of n. Here, $f_n\in\mathscr{S}_1(\mathscr{D}_0)$ and $\sigma_1(f_n)\leqslant e^{A_n}$. Thence, by the *third* lemma of the preceding article, if h denotes the height of $\mathscr{D}_0$,

$$\int_\Pi |f_n(\zeta)||d\zeta| \leqslant \sigma_1(f_n) + \left\{1+\frac{h}{a-|c_1|}\right\}\sigma_1(f_n)$$
$$+\left\{1+\frac{h}{a-|c_2|}\right\}\sigma_1(f_n) \leqslant Ke^{A_n}$$

with a constant K independent of n. Plugging this into the previous relation, we find that

$$|g_n(z)| \leqslant \text{const.}e^{BA_n\Im z}, \qquad z\in\Gamma,$$

the constant in front on the right being independent of n.

To estimate $|h_n(z)|$ on Γ we simply use the fact that

$$|f_n(\xi)| \leqslant e^{-M_1(A_n)/2} \quad \text{for} \quad \xi\in F\subseteq E$$

* in the following relation, $|\Pi|$ is used to designate the *linear measure* (length) of Π

and get

$$|h_n(z)| = \exp\left(\int_F \log|f_n(\zeta)|\,d\omega_{\mathscr{D}}(\zeta, z)\right) \leqslant e^{-\omega_{\mathscr{D}}(F,z)M_1(A_n)/2}, \qquad z\in\mathscr{D}.$$

Substituting the estimates for $|g_n(z)|$ and $|h_n(z)|$ which we have already found into (†), we obtain

$$(\ddagger)\qquad |e^{i\lambda z}f_n(z)| \leqslant \text{const.}e^{(BA_n-\lambda)\Im z}e^{-\omega_{\mathscr{D}}(F,z)M_1(A_n)/2}|k_n(z)|$$

for $z\in\Gamma$. Thus, in order to get a good upper bound for

$$|\mathrm{II}| = \left|\int_\Gamma e^{i\lambda z}f_n(z)\,dz\right|,$$

it suffices to find one for $\int_\Gamma |k_n(z)||dz|$ which is independent of n.

We have

$$\int_{-a}^{a} |\varphi(x) - f_n(x)|\,dx \leqslant 2e^{-M_1(A_n)}.$$

Wlog,

$$\int_{-a}^{a} |\varphi(x)|\,dx \leqslant \frac{1}{2},$$

therefore, for all sufficiently large n,

$$(\dagger\dagger)\qquad \int_{-a}^{a} |f_n(x)|\,dx \leqslant 1.$$

We henceforth limit our attention to the large values of n for which this relation is true.

The formula for $\log|k_n(z)|$ can be rewritten

$$\log|k_n(z)| = \int_{\partial\mathscr{D}} \log P(\zeta)\,d\omega_{\mathscr{D}}(\zeta, z),$$

where

$$P(\zeta) = \begin{cases} |f_n(\zeta)|, & \zeta\in(-b, b)\sim F, \\ 1 \text{ elsewhere on } \partial\mathscr{D}. \end{cases}$$

From this, by the inequality between arithmetic and geometric means, we get

$$|k_n(z)| \leqslant \int_{\partial\mathscr{D}} P(\zeta)d\omega_{\mathscr{D}}(\zeta, z) \leqslant 1 + \int_{-b}^{b} |f_n(\xi)|\,d\omega_{\mathscr{D}}(\xi, z), \qquad z\in\mathscr{D}.$$

However, for $-b < \xi < b$, we can apply the principle of extension of domain to compare $d\omega_{\mathscr{D}}(\xi, z)$ with harmonic measure for $\{\Im z > 0\}$ as we did in proving the *second* theorem of the preceding article. This gives us

$$d\omega_{\mathscr{D}}(\xi, z) \;\leqslant\; \frac{1}{\pi}\frac{\Im z\, d\xi}{|z-\xi|^2}, \qquad -b < \xi < b,$$

so the previous inequality becomes

$$|k_n(z)| \;\leqslant\; 1 \;+\; \frac{1}{\pi}\int_{-b}^{b} \frac{\Im z}{|z-\xi|^2}\,|f_n(\xi)|\,d\xi, \qquad z \in \mathscr{D}.$$

Denoting by h' the *height* of $\mathscr{D}$, and using this last relation together with Fubini's theorem, we see that, for $0 < y < h'$,

$$\int_{-b}^{b} |k_n(x+iy)|\,dx \;\leqslant\; 2b + \int_{-b}^{b} |f_n(\xi)|\,d\xi \;\leqslant\; 2b+1$$

(in view of (††)).

In other words, $k_n(z) \in \mathscr{S}_1(\mathscr{D})$ (*sic!*), *and the $\mathscr{S}_1$-norm of k_n for $\mathscr{D}$ is $\leqslant 2b+1$ independently of n.*

Use now the *third* lemma of the previous article *for $\mathscr{D}$.*

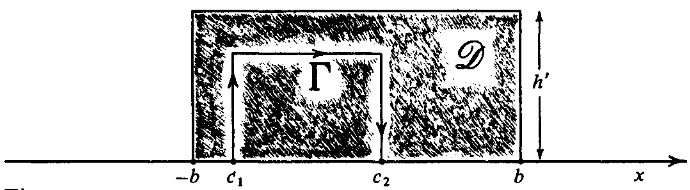

Figure 79

On account of what has just been said, we get

$$\int_{\Gamma} |k_n(z)|\,|dz| \;\leqslant\; (2b+1) + (2b+1)\left\{2 + \frac{h'}{b-|c_1|} + \frac{h'}{b-|c_2|}\right\},$$

i.e.

$$(\S) \qquad \int_{\Gamma} |k_n(z)|\,|dz| \;\leqslant\; \text{const.},$$

independently of n.

Let us return to $(\overset{*}{*})$. It is at *this point* that we *choose n according to the value of $\lambda > 0$*. We are actually only interested in *large* values of λ. For any such one, we refer to the sequence $\{A_n\}$ described above, and take n as the integer for which $2BA_n \leqslant \lambda < 2BA_{n+1}$. For *this* n, $(\overset{*}{*})$ becomes

$$|e^{i\lambda z} f_n(z)| \;\leqslant\; \text{const.}\,e^{-BA_n \Im z - (M_1(A_n)\omega_{\mathscr{D}}(F,z)/2)}\,|k_n(z)|, \qquad z \in \Gamma.$$

Recall that the two feet c_1 and c_2 of Γ were chosen so as to have

$$\lim_{y\to 0+} \omega_{\mathscr{D}}(F, c_1 + \mathrm{i}y) \;=\; \lim_{y\to 0+} \omega_{\mathscr{D}}(F, c_2 + \mathrm{i}y) \;=\; 1.$$

Therefore

$$B\mathfrak{J}z + \tfrac{1}{2}\omega_{\mathscr{D}}(F, z)$$

has a strictly positive minimum, say β, on Γ. β depends *only* on the geometric configuration of $\mathscr{D}$ and Γ. From the preceding relation, we have, then, when $2BA_n \leqslant \lambda < 2BA_{n+1}$, n being *large*,

$$|\mathrm{e}^{\mathrm{i}\lambda z} f_n(z)| \;\leqslant\; \mathrm{const.e}^{-\beta \min(A_n, M_1(A_n))} |k_n(z)|, \qquad z \in \Gamma.$$

Now use (§). We get

$$\left| \int_\Gamma \mathrm{e}^{\mathrm{i}\lambda z} f_n(z)\,\mathrm{d}z \right| \;\leqslant\; \mathrm{const.e}^{-\beta \min(A_n, M_1(A_n))}$$

for $2BA_n \leqslant \lambda < 2BA_{n+1}$; *this*, then, is our desired estimate for $|\mathrm{II}|$.

Now

$$|\Phi(\lambda)| \;=\; \left| \int_{c_1}^{c_2} \mathrm{e}^{\mathrm{i}\lambda x} \varphi(x)\,\mathrm{d}x \right| \;\leqslant\; |\mathrm{I}| + |\mathrm{II}|$$

where $|\mathrm{I}| \leqslant 2\mathrm{e}^{-M_1(A_n)}$, as we saw near the start of the present discussion. We may just as well take $\beta < 1$ (which is in fact *true* any way); then, by the estimate for $|\mathrm{II}|$ just found, we have, for *large* n,

$$|\Phi(\lambda)| \;\leqslant\; \mathrm{const.e}^{-\beta \min(A_n, M_1(A_n))}, \qquad 2BA_n \leqslant \lambda < 2BA_{n+1}.$$

Our aim here is to show that

$$\int_1^\infty \frac{1}{\lambda^2} \log \left| \frac{1}{\Phi(\lambda)} \right| \mathrm{d}\lambda \;=\; \infty,$$

or, what comes to the same thing, that

$$\int_{\lambda_0}^\infty \frac{1}{\lambda^2} \log \left| \frac{1}{\Phi(\lambda)} \right| \mathrm{d}\lambda \;=\; \infty$$

for some large λ_0. In view of the above inequality for $|\Phi(\lambda)|$, *this holds* if

$$\sum_n \int_{2BA_n}^{2BA_{n+1}} \frac{\min(A_n, M_1(A_n))}{\lambda^2} \mathrm{d}\lambda \;=\; \infty,$$

i.e., if

$$(\S\S) \qquad \sum_n \min(A_n, M_1(A_n)) \left\{ \frac{1}{A_n} - \frac{1}{A_{n+1}} \right\} \;=\; \infty.$$

We proceed to establish this relation. Our hypothesis says that

$$\int_1^\infty \frac{M_1(A)}{A^2}\,\mathrm{d}A \;=\; \infty.$$

The function $M_1(A)$ is increasing, so, by the *second* lemma of article 2, we also have

$$(\ddagger) \qquad \int_1^\infty \frac{\min(A, M_1(A))}{A^2}\,\mathrm{d}A \;=\; \infty.$$

Divide $\mathbb{N}$, the set of positive integers, into three disjoint subsets:

$$R = \{n \geqslant 1\colon\; A_{n+1} \leqslant 2A_n\},$$

$$S = \{n \geqslant 1\colon\; A_{n+1} > 2A_n \text{ and } A_n < M_1(A_n)\},$$

$$T = \{n \geqslant 1\colon\; A_{n+1} > 2A_n \text{ and } M_1(A_n) \leqslant A_n\}.$$

By ($\ddagger$), one of the three sums

$$\sum_{n\in R} \int_{A_n}^{A_{n+1}} \frac{\min(A, M_1(A))}{A^2}\,\mathrm{d}A,$$

$$\sum_{n\in S} \int_{A_n}^{A_{n+1}} \frac{\min(A, M_1(A))}{A^2}\,\mathrm{d}A,$$

$$\sum_{n\in T} \int_{A_n}^{A_{n+1}} \frac{\min(A, M_1(A))}{A^2}\,\mathrm{d}A$$

must be infinite.

Suppose the *first* of those sums is infinite. Recall that the A_n were chosen so as to have $M_1(A_{n+1}) = 2M_1(A_n)$. Therefore, if $n \in R$ and $A_n \leqslant A < A_{n+1}$,

$$\min(A, M_1(A)) \;\leqslant\; \min(A_{n+1}, M_1(A_{n+1})) \;\leqslant\; 2\min(A_n, M_1(A_n)),$$

i.e.,

$$\int_{A_n}^{A_{n+1}} \frac{\min(A, M_1(A))}{A^2}\,\mathrm{d}A \;\leqslant\; 2\min(A_n, M_1(A_n))\left\{\frac{1}{A_n} - \frac{1}{A_{n+1}}\right\}, \qquad n\in R.$$

In the present case, then, we certainly have (§§).

If the *second* of the sums in question (the one over S) is infinite, the set S *cannot be finite.* However, for $n \in S$,

$$\min(A_n, M_1(A_n))\left\{\frac{1}{A_n} - \frac{1}{A_{n+1}}\right\} \;=\; \frac{A_{n+1} - A_n}{A_{n+1}} \;>\; \frac{1}{2},$$

so (§§) holds when S is infinite.

There remains the case where the *third* sum (over T) is infinite. Here, for $n \in T$ and $A_n \leqslant A < A_{n+1}$ we have

$$\min(A_n, M_1(A_n)) \;=\; M_1(A_n) \;=\; \tfrac{1}{2}M_1(A_{n+1}) \;\geqslant\; \tfrac{1}{2}M_1(A),$$

so, for such n,

$$\min(A_n, M_1(A_n))\left\{\frac{1}{A_n} - \frac{1}{A_{n+1}}\right\} \;\geqslant\; \frac{1}{2}\int_{A_n}^{A_{n+1}} \frac{M_1(A)}{A^2}\,\mathrm{d}A$$

$$\geqslant\; \frac{1}{2}\int_{A_n}^{A_{n+1}} \frac{\min(A, M_1(A))}{A^2}\,\mathrm{d}A.$$

Hence, if the sum of the right-hand integrals for $n \in T$ is infinite, so is that of the left-hand expressions, and (§§) holds.

The relation (§§) is thus proved. This, however, implies that

$$\int_1^\infty \frac{1}{\lambda^2}\log\left|\frac{1}{\Phi(\lambda)}\right|\mathrm{d}\lambda \;=\; \infty$$

as we have seen, which is what we needed to show. The theorem is completely proved, and we are done.

Corollary. *Let $f(\vartheta) \sim \sum_{-\infty}^{\infty} a_n \mathrm{e}^{\mathrm{i}n\vartheta}$ belong to $L_2(-\pi, \pi)$, and suppose that*

$$\sum_{-\infty}^{-1} \frac{1}{n^2}\log\left(\frac{1}{\sum_{-\infty}^{n} |a_k|^2}\right) \;=\; \infty.$$

If $f(\vartheta)$ vanishes on a set of positive measure, then $f \equiv 0$ a.e.

Let the reader deduce the corollary from the theorem. He or she is also encouraged to examine how some of the results from the previous article can be weakened (making their proofs simpler), leaving, however, enough to establish an L_2 version of the theorem which will yield the corollary.

C. Kargaev's example

In remark 2 following the proof of the Beurling gap theorem (§B.2), it was said that that result *cannot* be improved so as to apply to measure μ with *$\hat{\mu}(\lambda)$ vanishing on a set of positive measure, instead of on a whole interval.* This is shown by an example due to P. Kargaev which we give in the present §.

Kargaev's construction furnishes a measure μ with *gaps* (a_n, b_n) in its support, $0 < a_1 < b_1 < a_2 < b_2 < \cdots$, such that

$$\sum_1^\infty \left(\frac{b_n - a_n}{a_n}\right)^2 \;=\; \infty$$

while $\hat{\mu}(\lambda) = 0$ on a set E with $|E| > 0$. His method shows that *in fact* the *relative size*, $(b_n - a_n)/a_n$, *of the gaps in* μ's *support has no bearing on* $\hat{\mu}(\lambda)$'s *capability of vanishing on a set of positive measure without being identically zero.* It is possible to obtain such measures with $(b_n - a_n)/a_n \xrightarrow[n]{} \infty$ *as rapidly as we please.* In view of Beurling's gap theorem, there is thus a *qualitative difference* between requiring that $\hat{\mu}(\lambda)$ *vanish on an interval* and merely having it *vanish on a set of positive measure.*

The measures obtained are *supported on the integers*, and their construction uses absolutely convergent Fourier series. The reasoning is elementary and somewhat reminiscent of the work of Smith, Pigno and McGehee on Littlewood's conjecture.

1. Two lemmas

Let us first introduce some *notation.* $\mathscr{A}$ denotes the collection of functions

$$f(\vartheta) = \sum_{-\infty}^{\infty} a_n e^{in\vartheta}$$

with the series on the right *absolutely convergent*. For such a function $f(\vartheta)$ we put

$$\|f\| = \sum_{-\infty}^{\infty} |a_n|$$

and frequently write $\hat{f}(n)$ instead of a_n (both of these notations are customary). $\mathscr{A}$, $\|\quad\|$ is a Banach space; in fact, a *Banach algebra* because, if f and $g \in \mathscr{A}$, then $f(\vartheta)g(\vartheta) \in \mathscr{A}$, and

$$\|fg\| \leqslant \|f\| \, \|g\|.$$

On account of this relation, $\Phi(f) \in \mathscr{A}$ for any *entire function* Φ if $f \in \mathscr{A}$.

We will be using some simple linear operators on $\mathscr{A}$.

Definition. If $f(\vartheta) = \sum_{-\infty}^{\infty} \hat{f}(n)e^{in\vartheta}$ belongs to $\mathscr{A}$,

$$(P_+ f)(\vartheta) = \sum_{n=0}^{\infty} \hat{f}(n)e^{in\vartheta}$$

and $P_- f = f - P_+ f$. We frequently write f_+ for $P_+ f$ and f_- for $P_- f$.

Observe that, for $f \in \mathscr{A}$, $\|P_+ f\| \leqslant \|f\|$ and $\|P_- f\| \leqslant \|f\|$.

Definition. For N an integer $\geqslant 1$ and $f \in \mathscr{A}$,

$$(H_N f)(\vartheta) = f(N\vartheta).$$

(The H stands for 'homothety'.)

The following relations are obvious:

$$\begin{aligned} H_N(fg) &= (H_N f)(H_N g), \qquad f, g \in \mathscr{A}, \\ \|H_N f\| &= \|f\|, \\ P_+(H_N f) &= H_N(P_+ f), \end{aligned}$$

and $H_N\Phi(f) = \Phi(H_N f)$ for $f \in \mathscr{A}$ and Φ an entire function.

Lemma. *For each integer $N \geqslant 1$ and each $\delta > 0$ there is a linear operator $T_{N,\delta}$ on $\mathscr{A}$ together with a set $E_{N,\delta} \subseteq [0, 2\pi)$ such that:*

(i) *For each $f \in \mathscr{A}$, $g = T_{N,\delta} f$ has $\hat{g}(n) = 0$ for $-N \leqslant n < N$* (sic!);
(ii) *For each $f \in \mathscr{A}$, $(T_{N,\delta} f)(\vartheta) = f(\vartheta)$ for $\vartheta \in E_{N,\delta}$;*
(iii) $\|T_{N,\delta} f\| \leqslant C(\delta)\|f\|$ *with $C(\delta)$ depending* only *on δ and* not *on N;*
(iv) $|E_{N,\delta}| = 2\pi(1 - \delta)$.

Proof. The idea is as follows: starting with an $f \in \mathscr{A}$, we try to cook functions $g_+(\vartheta)$ and $g_-(\vartheta)$ in $\mathscr{A}$, the first having only positive frequencies and the second only negative ones, in such a way as to get

$$g_+(\vartheta)e^{iN\vartheta} + g_-(\vartheta)e^{-iN\vartheta}$$

'almost' equal to $f(\vartheta)$.

We take a certain $\psi \in \mathscr{A}$ (to be described in a moment) and write

$$(*) \qquad q = e^{i(\psi_+ - \psi_-)}.$$

According to the observations preceding the lemma, $q \in \mathscr{A}$. Our construction of $T_{N,\delta}$ and $E_{N,\delta}$ is based on the following *identity* valid for $f \in \mathscr{A}$:

$$\boxed{f = ((fq)_+ e^{-2i\psi_+})e^{i\psi} + ((fq)_- e^{2i\psi_-})e^{-i\psi}.}$$

To check this, just observe that the right-hand side is

$$\begin{aligned} &(fq)_+ e^{-i(\psi_+ - \psi_-)} + (fq)_- e^{i(\psi_- - \psi_+)} \\ &\quad = ((fq)_+ + (fq)_-)q^{-1} = fq \cdot q^{-1} = f. \end{aligned}$$

Here is the way we choose ψ. Take any 2π-periodic $\mathscr{C}_\infty$-function $\varphi_\delta(\vartheta)$ with a graph like this on the range $0 \leqslant \vartheta \leqslant 2\pi$:

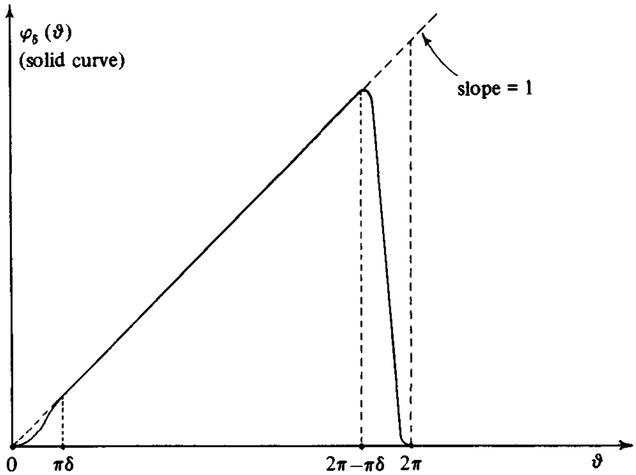

Figure 80

Then put $\psi = H_N\varphi_\delta$; ψ thus depends on N and δ. Note that $\varphi_\delta \in \mathscr{A}$ because φ_δ is infinitely differentiable ($|\hat{\varphi}_\delta(n)| \leqslant O(|n|^{-k})$ for *every* $k > 0$!). Therefore ψ belongs to $\mathscr{A}$.

With $q \in \mathscr{A}$ related by ($*$) to the ψ just specified, put, for $f \in \mathscr{A}$,

$$T_{N,\delta}f \;=\; ((fq)_+ \,e^{-2i\psi_+})e^{iN\vartheta} + ((fq)_- e^{2i\psi_-})e^{-iN\vartheta}.$$

$T_{N,\delta}$ obviously takes $\mathscr{A}$ into $\mathscr{A}$; let us show that there is a set $E_{N,\delta} \subseteq [0, 2\pi)$ independent of f such that (ii) holds.

The set

$$\Delta_{N,\delta} \;=\; \{\vartheta,\; 0 \leqslant \vartheta < 2\pi\colon\;\; 0 < N\vartheta < \pi\delta \bmod 2\pi \text{ or } 2\pi - \pi\delta < N\vartheta < 2\pi \bmod 2\pi\}$$

consists of $2N$ disjoint intervals, each of length $\pi\delta/N$, so $|\Delta_{N,\delta}| = 2\pi\delta$. Taking into account the 2π-periodicity of the function $\varphi_\delta(\vartheta)$ we see, by looking at its graph, that

$$e^{i\varphi_\delta(N\vartheta)} \;=\; e^{iN\vartheta}, \qquad \vartheta \in [0, 2\pi) \sim \Delta_{N,\delta};$$

i.e.,

$$e^{i\psi(\vartheta)} \;=\; e^{iN\vartheta}, \qquad \vartheta \in [0, 2\pi) \sim \Delta_{N,\delta}.$$

Put, therefore, $E_{N,\delta} = [0, 2\pi) \sim \Delta_{N,\delta}$; then, by comparing the formula for $T_{N,\delta}f$ with the boxed identity following ($*$), we see that $(T_{N,\delta}f)(\vartheta) = f(\vartheta)$ for $\vartheta \in E_{N,\delta}$, proving (ii).

We also have (iv), since

$$|E_{N,\delta}| \;=\; 2\pi - |\Delta_{N,\delta}| \;=\; 2\pi - 2\pi\delta.$$

We come to (i). The function $(fq)_+$ only has *non-negative frequencies* in its Fourier series. The same is true for $e^{-2i\psi_+}$. Indeed, the latter function equals

$$1 - 2i\psi_+ + \frac{(2i\psi_+)^2}{2!} - \frac{(2i\psi_+)^3}{3!} + \cdots$$

with the series *convergent in the norm* $\| \ \|$, and each *power* $(\psi_+)^n$ has a Fourier series involving only frequencies $\geqslant 0$. The Fourier series of the *product* $(fq)_+ e^{-2i\psi_+}$ thus only involves frequencies $\geqslant 0$, and finally, that for

$$((fq)_+ e^{-2i\psi_+})e^{iN\vartheta}$$

only has frequencies $\geqslant N$. One verifies in the same way that

$$((fq)_- e^{2i\psi_-})e^{-iN\vartheta}$$

has a Fourier series involving only the frequencies $< -N$, and (i) now follows from our definition of $T_{N,\delta}$.

There remains (iii). We have, for example,

$$\begin{aligned}\|(fq)_+ e^{-i\psi_+}\| &\leqslant \|(fq)_+\| \, \|e^{-i\psi_+}\| \\ &\leqslant \|fq\| \, \|e^{-2i\psi_+}\| \leqslant \|f\| \, \|q\| \, \|e^{-2i\psi_+}\|.\end{aligned}$$

Here,

$$e^{-2i\psi_+} = e^{-2iP + H_N\varphi_\delta} = e^{-2iH_N P + \varphi_\delta} = H_N e^{-2iP + \varphi_\delta},$$

according to the elementary relations preceding the lemma, so

$$\|e^{-2i\psi_+}\| = \|H_N e^{-2iP + \varphi_\delta}\| = \|e^{-2iP + \varphi_\delta}\|,$$

a *finite quantity, depending on* δ *but not on* N. In like manner,

$$\|q\| = \|e^{i(\psi_+ - \psi_-)}\| = \|H_N e^{i(P + \varphi_\delta - P_- \varphi_\delta)}\| = \|e^{i(P + \varphi_\delta - P_- \varphi_\delta)}\|,$$

a finite quantity depending on δ but independent of N. We thus have

$$\|(fq)_+ e^{-2i\psi_+} e^{iN\vartheta}\| = \|(fq)_+ e^{-2i\psi_+}\| \leqslant A_\delta \|f\|,$$

where A_δ depends only on δ.

The norm $\|(fq)_- e^{2i\psi_-} e^{-iN\vartheta}\|$ is handled in exactly the same way, and found to be $\leqslant B_\delta \|f\|$ with B_δ depending only on δ. Referring to the definition of $T_{N,\delta}$, we see that (iii) holds.

The lemma is thus proved.

We now take two positive integers L and N; N will usually be much larger than $2L$.

Definition.

$$\mathscr{M}(N,L) \;=\; \bigcup_{k=-2L-1}^{2L+1}{}' \;[Nk-L,\; Nk+L].$$

▶ Here, the prime next to the union sign means that the term corresponding to the value $k=0$ is *omitted.*

For $N>2L$, $\mathscr{M}(N,L)$ is the union of $4L+2$ separate intervals, each of length $2L$:

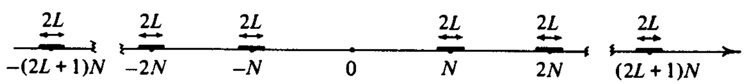

Figure 81

In proving the following lemma we use another linear operator on $\mathscr{A}$.

Definition. For $f\in\mathscr{A}$, put

$$(S_L f)(\vartheta) \;=\; \sum_{n=-L}^{L} \hat{f}(n)e^{in\vartheta}.$$

Observe that $\|S_L f\| \leqslant \|f\|$ and $\|f - S_L f\| \longrightarrow 0$ as $L\to\infty$ whenever $f\in\mathscr{A}$. We also have

$$P_+ S_L f \;=\; S_L P_+ f.$$

Lemma. *For each $\delta>0$ and pair N, L of positive integers there is a linear operator $T^{(L)}_{N,\delta}$ on $\mathscr{A}$ such that*

(1) *For any $f\in\mathscr{A}$, the Fourier coefficients $\hat{g}(n)$ of $g = T^{(L)}_{N,\delta}f$ are all* zero *when $n\notin\mathscr{M}(N,L)$;*
(2) *For $f\in\mathscr{A}$, $\|T^{(L)}_{N,\delta}f\| \leqslant C(\delta)\|f\|$ with $C(\delta)$* independent *of N and L;*
(3) *If $T_{N,\delta}$ is the operator furnished by the previous lemma, we have*

$$\|T_{N,\delta}f - T^{(L)}_{N,\delta}f\| \longrightarrow 0$$

uniformly *in N as $L\to\infty$, for each $f\in\mathscr{A}$ and $\delta>0$.*

Remarks. Actually, the spectrum of $T^{(L)}_{N,\delta}f$ is contained in a *smaller* set than $\mathscr{M}(N,L)$ when $f\in\mathscr{A}$. It is the *uniformity with respect to N* in property 3 which will turn out to be especially important in Kargaev's construction.

Proof of lemma. Fix $\delta>0$ and take the function φ_δ used in proving the preceding lemma – *here we just denote it by φ.* In terms of

$$q_0 \;=\; e^{i(\varphi_+ - \varphi_-)},$$

we observe that the definition of $T_{N,\delta}f$ given in the proof of the previous lemma can be rewritten thus:

$$T_{N,\delta}f \;=\; (fH_Nq_0)_+(H_N e^{-2i\varphi_+})\cdot e^{iN\vartheta} \;+\; (fH_Nq_0)_-(H_N e^{2i\varphi_-})\cdot e^{-iN\vartheta}.$$

Put

$$T^{(L)}_{N\delta}f \;=\; (S_Lf\cdot H_NS_Lq_0)_+(H_NS_L e^{-2i\varphi_+})\cdot e^{iN\vartheta} \\ \qquad +\; (S_Lf\cdot H_NS_Lq_0)_-(H_NS_L e^{2i\varphi_-})\cdot e^{-iN\vartheta}.$$

Since $\|g-S_Lg\|\longrightarrow 0$ as $L\to\infty$ for every $g\in\mathscr{A}$, $T^{(L)}_{N,\delta}f$ is clearly a kind of approximation to $T_{N,\delta}f$.

We proceed to verify property (1). The Fourier coefficients of S_Lf are all zero save for those with index in the set

$$\{-L,\ -L+1,\ \ldots,\ 0,\ 1,\ \ldots,\ L\}.$$

The non-zero Fourier coefficients of $H_NS_Lq_0$ have their indices in the set

$$\{-NL,\ -N(L-1),\ \ldots,\ -N,\ 0,\ N,\ \ldots,\ NL\}.$$

Therefore the Fourier coefficients of $(S_Lf\cdot H_NS_Lq_0)_+$ with index *outside* the set

$$\{0,\ 1,\ \ldots,\ L\}\ \cup\ \{N-L,\ N-L+1,\ \ldots,\ N,\ N+1,\ \ldots,\ N+L\} \\ \cup\ \{2N-L,\ 2N-L+1,\ \ldots,\ 2N+L\}\ \cup\ \cdots \\ \cup\ \{NL-L,\ NL-L+1,\ \ldots,\ NL+L\}$$

are *surely zero*.

Again, the Fourier coefficients of $H_NS_L e^{-2i\varphi_+}$ are all zero save for those with index in the set $\{0,\ N,\ 2N,\ \ldots,\ LN\}$. So, finally, the Fourier coefficients of

$$(S_Lf\cdot H_NS_Lq_0)_+(H_NS_L e^{-2i\varphi_+})e^{iN\vartheta}$$

(the *first* of the two terms making up $T^{(L)}_{N,\delta}f$) are *all zero, save for those with index in the union of intervals*

$$[N,\ N+L]\ \cup\ \bigcup_{k=2}^{2L+1}\,[Nk-L,\ Nk+L].$$

Treating the *second* term of $T^{(L)}_{N,\delta}f$ in the same way, we see that property 1 holds (and that indeed *more* is true regarding the spectrum of $T^{(L)}_{N,\delta}f$).

To check property (2), we have, for the *first term* of $T^{(L)}_{N,\delta}f$,

$$\|(S_Lf\cdot H_NS_Lq_0)_+(H_NS_L e^{-2i\varphi_+})\cdot e^{iN\vartheta}\| \\ \leqslant\ \|S_Lf\|\,\|H_NS_Lq_0\|\,\|H_NS_L e^{-2i\varphi_+}\| \\ \leqslant\ \|f\|\,\|S_Lq_0\|\,\|S_L e^{-2i\varphi_+}\|\ \leqslant\ \|f\|\,\|q_0\|\,\|e^{-2i\varphi_+}\|;$$

we have used the fact that $\|H_N g\| = \|g\|$ for $g \in \mathscr{A}$. In the extreme right-hand member of the chain of inequalities just written, the factors $\|q_0\|$ and $\|e^{-2i\varphi_+}\|$ are finite and *only involve* $\varphi = \varphi_\delta$; therefore they depend *only on* δ. The *second term* of $T^{(L)}_{N,\delta} f$ is handled in exactly the same fashion, and, putting together the estimates obtained for *both* terms, we arrive at property 2.

Verification of property (3) remains. This is somewhat long-winded. It is really nothing but an elaborate version of the argument presented in good elementary calculus courses to show that the limit of a product equals the product of the limits. In order not to lose sight of the main idea, let's just compare the *first terms* of $T_{N,\delta} f$ and $T^{(L)}_{N,\delta} f$. The *difference* of these first terms has norm equal to

$$\begin{aligned}
&\|(S_L f \cdot H_N S_L q_0)_+ (H_N S_L e^{-2i\varphi_+}) \cdot e^{iN\vartheta} \;-\; (f \cdot H_N q_0)_+ (H_N e^{-2i\varphi_+}) e^{iN\vartheta}\| \\
&\quad \leqslant \; \|(S_L f \cdot H_N S_L q_0)_+ - (f H_N q_0)_+\| \; \|H_N S_L e^{-2i\varphi_+}\| \\
&\qquad\qquad + \; \|(f H_N q_0)_+\| \; \|H_N e^{-2i\varphi_+} - H_N S_L e^{-2i\varphi_+}\| \\
&\quad \leqslant \; \|e^{-2i\varphi_+}\| \; \|(S_L f - f) H_N q_0 + (S_L f)(H_N S_L q_0 - H_N q_0)\| \\
&\qquad\qquad + \; \|f\| \; \|q_0\| \; \|e^{-2i\varphi_+} - S_L e^{-2i\varphi_+}\| \\
&\quad \leqslant \; \|e^{-2i\varphi_+}\| \{ \|S_L f - f\| \; \|q_0\| + \|f\| \; \|S_L q_0 - q_0\| \} \\
&\qquad\qquad + \; \|f\| \; \|q_0\| \; \|e^{-2i\varphi_+} - S_L e^{-2i\varphi_+}\|
\end{aligned}$$

This last expression *does not involve N at all*, and, for fixed $f \in \mathscr{A}$, tends to *zero* as $L \to \infty$. (It depends on δ through the functions φ and $q_0 = e^{i(\varphi_+ - \varphi_-)}$.)

The difference of the *second terms* of $T^{(L)}_{N,\delta} f$ and $T_{N,\delta} f$ is treated in the same way, and we see that property (3) holds. The lemma is proved, and we are done.

2. The example

Theorem (Kargaev). *Let* $\Lambda \subseteq \mathbb{Z}$. *Suppose that for each positive integer* L *there is some positive integer* N_L *with*

$$\Lambda \;\supseteq\; \mathscr{M}(N_L, L) \cap \mathbb{Z},$$

where the sets $\mathscr{M}(N, L)$ *are those defined in the previous article. Then, given* $\varepsilon > 0$ *and* $g \in \mathscr{A}$ *we can find a* $g_\varepsilon \in \mathscr{A}$ *such that*

(i) $\|g_\varepsilon\| \leqslant K_\varepsilon \|g\|$, *where* K_ε *depends* only *on* ε;
(ii) $\hat{g}_\varepsilon(n) = 0$ *for* $n \notin \Lambda$;
(iii) $g_\varepsilon(\vartheta) = g(\vartheta)$ *for* $\vartheta \in [0, 2\pi) \sim \Delta$, *where* $|\Delta| < 2\pi\varepsilon$.

Proof. Taking $\varepsilon > 0$, we put $\delta_n = \varepsilon/2^n$ and $\varepsilon_n = 1/2^n C(\delta_{n+1})$ with $C(\delta)$ from property (2) of the *second* lemma in the previous article. There is no harm in supposing that $C(\delta) > 1$; this we *do* in the following construction.

The function g_ε is obtained from a given $g \in \mathscr{A}$ by a process of successive approximations, using the operators $T_{N,\delta}$ and $T_{N,\delta}^{(L)}$ from the two lemmas of the preceding article.

According to the *second* of those lemmas, we can choose an L_1 such that

$$(*) \qquad \| T_{N,\delta_1}^{(L_1)} g - T_{N,\delta_1} g \| \;\leqslant\; \varepsilon_1 \| g \|$$

for all values of N simultaneously. If we take any positive integer N, the Fourier coefficients $\hat{h}(n)$ of $h = T_{N,\delta_1}^{(L_1)} g$ all vanish for $n \notin \mathscr{M}(N, L_1)$ by that second lemma. The hypothesis now furnishes a value of N such that

$$\mathscr{M}(N, L_1) \cap \mathbb{Z} \;\subseteq\; \Lambda.$$

Fix such a value of N, calling it N_1. Then, if we put $h_1 = T_{N_1,\delta_1}^{(L_1)} g$, we have $\hat{h}_1(n) = 0$ for $n \notin \Lambda$. Let us also write $r_1 = T_{N_1,\delta_1} g - h_1$. Then $(*)$ says that $\| r_1 \| \leqslant \varepsilon_1 \| g \|$, and, by the *first* lemma of the preceding article,

$$g(\vartheta) - h_1(\vartheta) - r_1(\vartheta) \;=\; g(\vartheta) - (T_{N_1,\delta_1} g)(\vartheta) \;=\; 0$$

for $\vartheta \in E_{N_1,\delta_1}$, a certain subset of $[0, 2\pi)$ with $|E_{N_1,\delta_1}| = 2\pi(1 - \delta_1)$.

We proceed, treating r_1 the way our given function g was just handled. First use the second lemma to get an L_2 such that

$$\| T_{N,\delta_2}^{(L_2)} r_1 - T_{N,\delta_2} r_1 \| \;\leqslant\; \varepsilon_2 \| g \|$$

for all positive N simultaneously, then choose (and fix) a value N_2 of N for which $\mathscr{M}(N_2, L_2) \cap \mathbb{Z} \subseteq \Lambda$, such choice being *possible according to the hypothesis.* Writing

$$h_2 \;=\; T_{N_2,\delta_2}^{(L_2)} r_1$$

and

$$r_2 \;=\; T_{N_2,\delta_2} r_1 - h_2,$$

we will have $\hat{h}_2(n) = 0$ for $n \notin \mathscr{M}(N_2, L_2)$ by the second lemma, hence, *a fortiori*, $\hat{h}_2(n) = 0$ for $n \notin \Lambda$. Our choice of L_2 makes

$$\| r_2 \| \leqslant \varepsilon_2 \| g \|,$$

and, by the first lemma, we have $r_1(\vartheta) = (T_{N_2,\delta_2} r_1)(\vartheta)$, i.e., $r_1(\vartheta) = h_2(\vartheta) + r_2(\vartheta)$ for $\vartheta \in E_{N_2,\delta_2}$, a subset of $[0, 2\pi)$ with $|E_{N_2,\delta_2}| = 2\pi(1 - \delta_2)$. *According to the preceding step,* we then have

$$g(\vartheta) \;=\; h_1(\vartheta) + h_2(\vartheta) + r_2(\vartheta) \qquad \text{for } \vartheta \in E_{N_1,\delta_1} \cap E_{N_2,\delta_2}.$$

And $\hat{h}_1(n) + \hat{h}_2(n) = 0$ for $n \notin \Lambda$.

Suppose that functions $h_1, h_2, \ldots, h_{k-1}$ and r_{k-1} (in $\mathscr{A}$) and positive integers $N_1, N_2, \ldots, N_{k-1}$ have been determined with $\|r_{k-1}\| \leqslant \varepsilon_{k-1}\|g\|$, $\hat{h}_j(n) = 0$ for $n \notin \Lambda$, $j = 1, 2, \ldots, k-1$, and $g = h_1 + h_2 + \cdots + h_{k-1} + r_{k-1}$ on the intersection $\bigcap_{j=1}^{k-1} E_{N_j,\delta_j}$. Then choose L_k in such a way that

$$\| T^{(L_k)}_{N,\delta_k} r_{k-1} \; - \; T_{N,\delta_k} r_{k-1} \| \;\leqslant\; \varepsilon_k \|g\|$$

simultaneously for all N (second lemma), and afterwards pick an N_k with $\mathscr{M}(N_k, L_k) \cap \mathbb{Z} \;\subseteq\; \Lambda$ (hypothesis). Putting

$$h_k \;=\; T^{(L_k)}_{N_k,\delta_k} r_{k-1}$$

and

$$r_k \;=\; T_{N_k,\delta_k} r_{k-1} - h_k,$$

we see that $\hat{h}_k(n) = 0$ for $n \notin \Lambda$, that $\|r_k\| \leqslant \varepsilon_k \|g\|$, and that $g = h_1 + h_2 + \cdots + h_{k-1} + h_k + r_k$ on $\bigcap_{j=1}^{k} E_{N_j,\delta_j}$ (first lemma).

Observe now that, by the second lemma, we *also* have

$$\begin{aligned} \|h_k\| \;=\; \|T^{(L_k)}_{N_k,\delta_k} r_{k-1}\| \;&\leqslant\; C(\delta_k)\|r_{k-1}\| \;\leqslant\; C(\delta_k)\varepsilon_{k-1}\|g\| \\ &\leqslant\; \|g\|/2^{k-1} \end{aligned}$$

for $k \geqslant 2$ on account of the way the numbers ε_k were rigged at the beginning of this proof. The series $h_1 + h_2 + h_3 + \cdots$ therefore *converges in the space* $\mathscr{A}$ (hence *uniformly* on $[0, 2\pi]$). Putting

$$g_\varepsilon(\vartheta) \;=\; \sum_{k=1}^{\infty} h_k(\vartheta),$$

we have

$$\|g_\varepsilon\| \;\leqslant\; (C(\delta_1) + \tfrac{1}{2} + \tfrac{1}{4} + \cdots)\|g\| \;=\; (1 + C(\varepsilon/2))\|g\|,$$

and $\hat{g}_\varepsilon(n) = 0$ for $n \notin \Lambda$ since, for such n, we have $\hat{h}_k(n) = 0$ for every k. Finally, since

$$|r_k(\vartheta)| \;\leqslant\; \|r_k\| \;\leqslant\; \varepsilon_k \|g\| \;\xrightarrow[k]{}\; 0,$$

we have

$$\sum_{j=1}^{k} h_j(\vartheta) \;+\; r_k(\vartheta) \;\xrightarrow[k]{}\; g_\varepsilon(\vartheta)$$

uniformly for $0 \leqslant \vartheta \leqslant 2\pi$, so $g_\varepsilon(\vartheta) = g(\vartheta)$ on the intersection

$$E \;=\; \bigcap_{j=1}^{\infty} E_{N_j,\delta_j}.$$

Here, since $|E_{N_j,\delta_j}| = 2\pi(1 - \delta_j)$ and the sets E_{N_j,δ_j} all lie in $[0, 2\pi)$, we have

$$\begin{aligned} |E| \;\geqslant\; 2\pi(1 - \delta_1 - \delta_2 - \delta_3 - \cdots) \;&=\; 2\pi\left(1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{4} - \cdots\right) \\ &=\; 2\pi(1 - \varepsilon). \end{aligned}$$

The theorem is proved.

Our example is now furnished by the following

Corollary. *There exists a non-zero measure* μ *having gaps* (a_n, b_n) *in its support, with*

$$0 < a_1 < b_1 < a_2 < b_2 < a_3 < \cdots$$

and

$$\sum_1^\infty \left(\frac{b_l - a_l}{a_l}\right)^2 = \infty$$

(*and the ratios* $(b_l - a_l)/a_l$ *even tending to* ∞ *as rapidly as we want!*), *while* $\hat{\mu}(\lambda) = 0$ *on a set of positive measure.*

Proof. For $l = 1, 2, 3, \ldots,$ take the sets

$$\mathscr{M}_l = \bigcup_{k=-2l-1}^{2l+1}{}' \; [N_l k - l, \; N_l k + l]$$

(term with $k = 0$ omitted), with the positive integers N_l so chosen that $N_l > 2l$ and that N_{l+1} is *much larger* than $(2l + 1)(N_l + 1)$. *There is no obstacle to our taking* N_{l+1} *as large as we wish in relation to* $(2l + 1)(N_l + 1)$ *for each* l.

Put

$$\Lambda = \bigcup_{l=1}^\infty (\mathscr{M}_l \cap \mathbb{Z});$$

it is clear that Λ satisfies the hypothesis of the theorem.

Choose any $g \in \mathscr{A}$ such that $g(\vartheta) > 0$ for $\pi/2 < \vartheta < 3\pi/2$ and $g(\vartheta) = 0$ for $0 \leqslant \vartheta \leqslant \pi/2$ and for $3\pi/2 \leqslant \vartheta \leqslant 2\pi$.

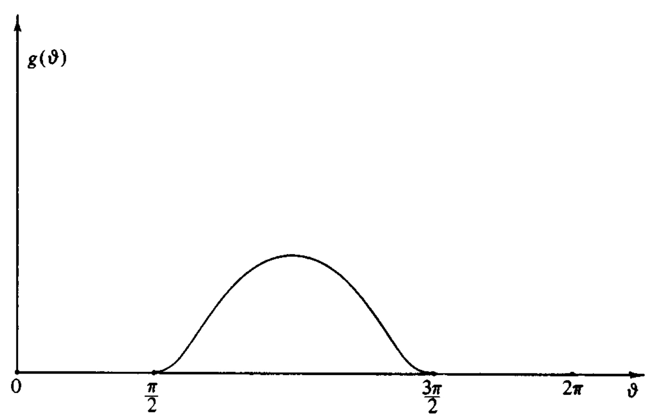

Figure 82

There are plenty of such functions g; we fix one of them.

Apply the theorem with $\varepsilon = \frac{1}{4}$, getting a function g_ε in $\mathscr{A}$ with $\hat{g}_\varepsilon(n) = 0$ for $n \notin \Lambda$ and $g_\varepsilon(\vartheta) = g(\vartheta)$ for all $\vartheta \in [0, 2\pi)$ outside a set of measure $\leqslant \pi/2$. Then *certainly* $g_\varepsilon(\vartheta)$ must be > 0 on a set in $[0, 2\pi)$ of measure $\geqslant \pi/2$ (*hence, in particular*, $g_\varepsilon \not\equiv 0$), while *at the same time* $g_\varepsilon(\vartheta) = 0$ *on a set of measure* $\geqslant \pi/2$ lying *in* $[0, 2\pi)$.

We have

$$g_\varepsilon(\vartheta) \;=\; \sum_{n\in\Lambda} \hat{g}_\varepsilon(n) e^{in\vartheta}$$

with

$$\sum_{n\in\Lambda} |\hat{g}_\varepsilon(n)| \;<\; \infty,$$

so, if we define a measure μ *supported on* $\Lambda \subseteq \mathbb{Z}$ by putting $\mu(E) = \sum_{n\in E} \hat{g}_\varepsilon(n)$, we have $\mu \not\equiv 0$, but $\hat{\mu}(\vartheta) = g_\varepsilon(\vartheta)$ *vanishes on a set of positive measure.*

The support ($\subseteq \Lambda$) of μ has the *gaps* $((2l+1)N_l + l,\ N_{l+1} - l - 1)$ *in it.* By choosing N_{l+1} *sufficiently large in relation to* $(2l+1)(N_l+1)$ for each l, *we can make the ratios*

$$\frac{(N_{l+1} - l - 1) - ((2l+1)N_l + l)}{(2l+1)N_l + l}$$

go to ∞ *as rapidly as we please for* $l \longrightarrow \infty$.

We are done.

D. Volberg's work

Let $f(\vartheta) \in L_1(-\pi, \pi)$; say

$$f(\vartheta) \;\sim\; \sum_{-\infty}^{\infty} a_n e^{in\vartheta}.$$

Suppose that the Fourier coefficients a_n with *negative* indices n are *small enough* to satisfy the relation

$$(*) \qquad \sum_{-\infty}^{-1} \frac{1}{n^2} \log\left(\frac{1}{\sum_{-\infty}^{n} |a_k|}\right) \;=\; \infty.$$

According to a corollary to Levinson's theorem (§ A.5), $f(\vartheta)$ then *cannot vanish on an interval of positive length* unless $f \equiv 0$. If we also assume (for instance) that $\sum_k |a_k| < \infty$, Beurling's improvement of Levinson's theorem (§B.2) shows that $f(\vartheta)$ *cannot even vanish on a set of positive measure* without being identically zero when $(*)$ holds.

It is therefore natural to ask *how small* $|f(\vartheta)|$ *can actually be* for a *non-*

zero f whose Fourier coefficients a_n satisfy ($*$), or something like it. Suppose for instance, that

$$|a_n| \leqslant e^{-M(|n|)}, \qquad n < 0,$$

with a regularly increasing $M(m)$ for which

$$\sum_1^\infty \frac{M(m)}{m^2} = \infty.$$

Volberg's surprising result is that if the behaviour of $M(m)$ is *regular enough*, then we *must have*

$$\int_{-\pi}^{\pi} \log|f(\vartheta)|\,d\vartheta > -\infty$$

unless $f \equiv 0$. *Very* loosely speaking, this amounts to saying that if $f \not\equiv 0$ and

$$\sum_{-\infty}^{-1} \frac{1}{n^2} \log\left|\frac{1}{\hat{f}(n)}\right| = \infty,$$

then

$$\int_{-\pi}^{\pi} \log|f(\vartheta)|\,d\vartheta > -\infty,$$

at least when the *decrease* of $|\hat{f}(n)|$ for $n \longrightarrow -\infty$ is *sufficiently regular. If one logarithmic integral* (the *sum*) *diverges, the other must converge*!

One could improve this result *only* by finding a way to relax the regularity conditions imposed on $M(m)$.

Indeed, if $p(\vartheta) \geqslant 0$ is *any* function in $L_1(-\pi, \pi)$ with

$$\int_{-\pi}^{\pi} \log p(\vartheta)\,d\vartheta > -\infty,$$

we can *get* a function

$$f(\vartheta) \sim \sum_0^\infty a_n e^{in\vartheta}$$

such that $|f(\vartheta)| = p(\vartheta)$ a.e. by putting

$$f(\vartheta) = \lim_{\substack{z \to e^{i\vartheta} \\ |z|<1}} \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \log p(t)\,dt\right\}$$

(see Chapter II, § A). Here, the Fourier coefficients of *negative* index *are all zero*, i.e., for $n < 0$,

$$|a_n| \leqslant e^{-M(|n|)}$$

with $M(|n|) \equiv \infty$. This means that from the condition

$$|a_n| \leqslant e^{-M(|n|)}, \qquad n<0,$$

with

$$\sum_1^\infty \frac{M(m)}{m^2} = \infty$$

one can *never hope to deduce a more stringent restriction* on the *smallness* of $|f(\vartheta)|$ than

$$\int_{-\pi}^{\pi} \log|f(\vartheta)|d\vartheta > -\infty.$$

Also, if $M(m)$ is *increasing*, from a *less* stringent condition than

$$\sum_1^\infty \frac{M(m)}{m^2} = \infty$$

one can *never hope to deduce any limitation on the smallness of* $|f(\vartheta)|$ for functions $f \not\equiv 0$ with $|a_n| \leqslant e^{-M(|n|)}$, $n<0$. That is the content of

Problem 12

Let $M(m) > 0$ be increasing for $m>0$, and such that

$$\sum_1^\infty \frac{M(m)}{m^2} < \infty.$$

Given h, $0<h<\pi$, show that there is a function $f(\vartheta)$, continuous and of period 2π, with $f(\vartheta)=0$ for $h \leqslant |\vartheta| \leqslant \pi$ but $f \not\equiv 0$, such that

$$|a_n| \leqslant e^{-M(|n|)}, \quad n \neq 0\,(sic!),$$

for the Fourier coefficients a_n of $f(\vartheta)$. (Hint: Use the theorems of Chapter IV, § D and Chapter III, § D. Take a suitable convolution.)

It is important to note that Volberg's theorem *relates specifically to the unit circle; its analogue for the real line is false*. Take, namely, $F(x) = e^{-x^2}$, so that

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} \log F(x)\,dx = -\infty.$$

Here,

$$\hat{F}(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda x} e^{-x^2} dx = \sqrt{\left(\frac{\pi}{2}\right)} e^{-\lambda^2/4},$$

so

$$\int_{-\infty}^{0} \frac{1}{1+\lambda^2} \log\left(\frac{1}{\hat{F}(\lambda)}\right) d\lambda = \infty.$$

and even

$$\int_{-\infty}^{0} \frac{1}{1+\lambda^2} \log\left(\frac{1}{\int_{-\infty}^{\lambda} \hat{F}(t)\,dt}\right) d\lambda = \infty.$$

This example shows that a *function and its Fourier transform can both get very small on* $\mathbb{R}$ (in terms of the logarithmic integral).

1. The planar Cauchy transform

Notation. If $G(z)$ is differentiable as a function of x and y we write

$$\frac{\partial G(z)}{\partial z} = G_z(z) = \frac{\partial G(z)}{\partial x} - i\frac{\partial G(z)}{\partial y}$$

and

$$\frac{\partial G(z)}{\partial \bar{z}} = G_{\bar{z}}(z) = \frac{\partial G(z)}{\partial x} + i\frac{\partial G(z)}{\partial y}.$$

Nota bene. Nowadays, most people take $\partial G/\partial z$ and $\partial G/\partial \bar{z}$ as *one-half* of the respective right-hand quantities.

Remark. If $G = U + iV$ with *real* functions U and V, the equation $G_{\bar{z}} = 0$ reduces to

$$\begin{cases} U_x = V_y, \\ U_y = -V_x, \end{cases}$$

i.e., the *Cauchy–Riemann equations* for U and V. The condition that $G_{\bar{z}} \equiv 0$ in a domain $\mathscr{D}$ is thus *equivalent to analyticity of* $G(z)$ *in* $\mathscr{D}$.

Theorem. *Let $F(z)$ be bounded and $\mathscr{C}_1$ in a bounded domain $\mathscr{D}$, and put*

$$G(z) = \frac{1}{2\pi}\iint_{\mathscr{D}} \frac{F(\zeta)\,d\xi\,d\eta}{z-\zeta},$$

where, as usual, $\zeta = \xi + i\eta$. Then $G(z)$ is $\mathscr{C}_1$ in $\mathscr{D}$ and

$$\frac{\partial G(z)}{\partial \bar{z}} = F(z), \qquad z \in \mathscr{D}.$$

Remark. The integral in question *converges absolutely* for each z, as is seen by going over to the polar coordinates (ρ, ψ) with

$$\zeta - z = \rho e^{i\psi}.$$

$G(z)$ is called the *planar Cauchy transform of* $F(z)$.

Proof of theorem. We first establish the differentiability of $G(z)$ in $\mathscr{D}$.

Let $z_0 \in \mathscr{D}$ with $\operatorname{dist}(z_0, \partial\mathscr{D}) = 3\rho$, say. Take any infinitely differentiable function $\varphi(\zeta)$ of ζ with $0 \leqslant \varphi(\zeta) \leqslant 1$ and

$$\varphi(\zeta) \;=\; \begin{cases} 1, & |\zeta - z_0| \leqslant \rho, \\ 0, & |\zeta - z_0| \geqslant 2\rho. \end{cases}$$

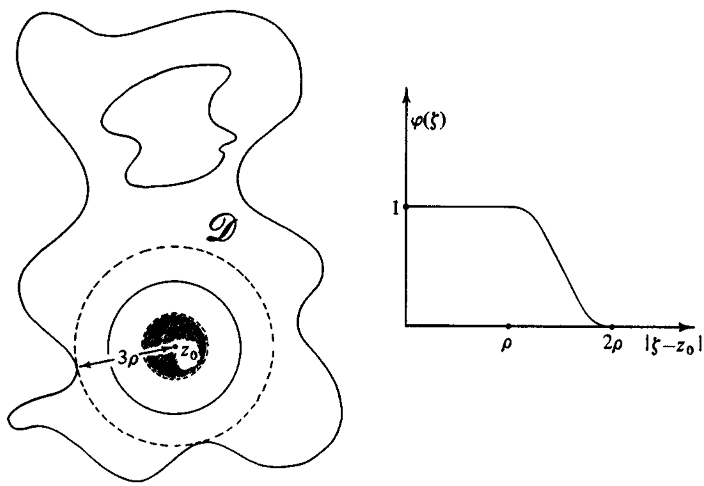

Figure 83

We can write

$$G(z) \;=\; \frac{1}{2\pi}\iint_{|\zeta - z_0| \leqslant 2\rho} \frac{\varphi(\zeta)F(\zeta)}{z-\zeta}\,d\xi\,d\eta$$
$$+\frac{1}{2\pi}\iint_{\substack{|\zeta - z_0| > \rho \\ \zeta\in\mathscr{D}}} \frac{(1-\varphi(\zeta))F(\zeta)}{z-\zeta}\,d\xi\,d\eta.$$

The *second* integral on the right is obviously a $\mathscr{C}_\infty$ function of z for $|z - z_0| < \rho$; it remains to consider the *first* one. After a change of variable, the *latter can be rewritten as*

$$\frac{1}{2\pi}\iint_{\mathbb{C}} \frac{F_1(z-w)}{w}\,du\,dv$$

(where $w = u + \mathrm{i}v$, as usual) with $F_1(\zeta) = \varphi(\zeta)F(\zeta)$. Here, $F_1(\zeta)$ is *of compact support*, and has as much differentiability as $F(\zeta)$. Hence, since

$$\iint_{|w|<R} \frac{\mathrm{d}u\,\mathrm{d}v}{|w|} \;<\; \infty$$

for any finite R, we can differentiate $(1/2\pi)\iint_{\mathbb{C}}(F_1(z-w)/w)\,\mathrm{d}u\,\mathrm{d}v$ with respect to x and y *under the integral sign*, and thus see that that expression is $\mathscr{C}_1$ in those variables.

We have shown that $G(z)$ is $\mathscr{C}_1$ in the neighborhood of any $z_0 \in \mathscr{D}$; there remains the evaluation of $G_{\bar{z}}(z_0)$ in terms of F. This turns out to be surprisingly difficult if we try to do it directly, and we resort to the following dodge.

Let $r > 0$ be small, and $z_0 \in \mathscr{D}$. By the differentiability of $G(z)$ at z_0,

$$\begin{aligned} G(z_0 + re^{\mathrm{i}\vartheta}) &= G(z_0) + G_x(z_0)r\cos\vartheta + G_y(z_0)r\sin\vartheta + o(r) \\ &= G(z_0) + \tfrac{1}{2}G_z(z_0)re^{\mathrm{i}\vartheta} + \tfrac{1}{2}G_{\bar{z}}(z_0)re^{-\mathrm{i}\vartheta} + o(r). \end{aligned}$$

Multiplying the last expression by $e^{\mathrm{i}\vartheta}\,\mathrm{d}\vartheta$ and integrating ϑ from 0 to 2π, we find the value $\pi r G_{\bar{z}}(z_0) + o(r)$; therefore

$$G_{\bar{z}}(z_0) \;=\; \lim_{r\to 0} \frac{1}{\pi r}\int_0^{2\pi} G(z_0 + re^{\mathrm{i}\vartheta})e^{\mathrm{i}\vartheta}\,\mathrm{d}\vartheta.$$

Plugging in the expression for G in terms of F and changing the order of integration, this becomes

$$G_{\bar{z}}(z_0) \;=\; \lim_{r\to 0} \frac{1}{2\pi^2 r}\iint_{\mathscr{D}}\int_0^{2\pi} \frac{F(\zeta)e^{\mathrm{i}\vartheta}}{z_0 + re^{\mathrm{i}\vartheta} - \zeta}\,\mathrm{d}\vartheta\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

However,

$$\int_0^{2\pi} \frac{\mathrm{i}re^{\mathrm{i}\vartheta}\,\mathrm{d}\vartheta}{re^{\mathrm{i}\vartheta} - (\zeta - z_0)} \;=\; \begin{cases} 2\pi\mathrm{i}, & |\zeta - z_0| < r, \\ 0, & |\zeta - z_0| > r. \end{cases}$$

Therefore, by the previous relation, we have

$$G_{\bar{z}}(z_0) \;=\; \lim_{r\to 0} \frac{1}{\pi r^2}\iint_{|\zeta - z_0| < r} F(\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta \;=\; F(z_0),$$

F having been assumed to be $\mathscr{C}_1$ in $\mathscr{D}$. We are done.

Corollary. *Let $\mathscr{D}$ be a bounded domain. Suppose that $F(z)$ is $\mathscr{C}_2$ in $\mathscr{D}$, that $|F(z)| > 0$ there, and that there is a constant C such that*

$$\left|\frac{\partial F(z)}{\partial \bar{z}}\right| \;\leqslant\; C|F(z)|, \qquad z \in \mathscr{D}.$$

Then

$$\Phi(z) \;=\; F(z)\exp\left\{\frac{1}{2\pi}\iint_{\mathscr{D}}\frac{F_{\bar{\zeta}}(\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta}{F(\zeta)(\zeta - z)}\right\}$$

is analytic in $\mathscr{D}$, and $|\Phi(z)|$ lies between two constant multiples of $|F(z)|$ therein.

Proof. $F_{\bar{z}}(z)/F(z)$ is $\mathscr{C}_1$ in $\mathscr{D}$ and bounded there by hypothesis, so we can apply the theorem, which tells us first of all that $\Phi(z)$ is differentiable in $\mathscr{D}$, and secondly that

$$\frac{\partial\Phi(z)}{\partial\bar{z}} \;=\; \left(F_{\bar{z}}(z) - F(z)\frac{F_{\bar{z}}(z)}{F(z)}\right)\exp\left(\frac{1}{2\pi}\iint_{\mathscr{D}}\frac{F_{\bar{\zeta}}(\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta}{F(\zeta)(\zeta - z)}\right) \;=\; 0$$

there. The *Cauchy–Riemann equations for $\mathfrak{R}\Phi(z)$ and $\mathfrak{I}\Phi(z)$ are thus satisfied* (see remark at the beginning of this article), so $\Phi(z)$ is *analytic* in $\mathscr{D}$.

If R is the *diameter* of $\mathscr{D}$, we easily check that

$$\mathrm{e}^{-CR}|F(z)| \;\leqslant\; |\Phi(z)| \;\leqslant\; \mathrm{e}^{CR}|F(z)|$$

for $z\in\mathscr{D}$. This does it.

The corollary has been extensively used by Lipman Bers and by Vekua in the study of partial differential equations. Volberg also uses it so as to bring analytic functions into his treatment.

Problem 13

Show that the condition that $|F(z)| > 0$ in $\mathscr{D}$ can be *dropped* from the hypothesis of the corollary, provided that we *maintain* the assumption that $|F_{\bar{z}}(z)| \leqslant C|F(z)|$, $z\in\mathscr{D}$, and define the ratio $F_{\bar{z}}(z)/F(z)$ in a satisfactory way on the set where $F(z) = 0$. Hence show that a function F satisfying the inequality $|F_{\bar{z}}(z)| \leqslant C|F(z)|$ *can have only isolated zeros in $\mathscr{D}$*, unless $F \equiv 0$ there. (Hint. On $E = \{z\in\mathscr{D}\colon F(z) = 0\}$, assign any constant value to the ratio $F_{\bar{z}}(z)/F(z)$. The function $\Phi(z)$ defined in the statement of the corollary is surely *analytic* in $\mathscr{D} \sim E$; it is also analytic in E° (if that set is non-empty) because it vanishes identically there. To check *existence* of

$$\Phi'(z_0) \;=\; \lim_{z\to z_0}\frac{\Phi(z) - \Phi(z_0)}{z - z_0}$$

at a point $z_0 \in \partial E \cap \mathscr{D}$, note that both $F(z_0)$ and $F_{\bar{z}}(z_0)$ must vanish, so, *near* z_0,

$$F(z) \;=\; \tfrac{1}{2}F_z(z_0)(z - z_0) + \mathrm{o}(|z - z_0|).$$

If $F_z(z_0) = 0$, $\Phi'(z_0)$ *exists* and equals *zero*. If $F_z(z_0) \neq 0$, $|F(z)| > 0$ in some punctured neighborhood $0 < |z - z_0| < \eta$ of z_0, so such a punctured neighborhood is included in $\mathscr{D} \sim E$.)

2. The function $M(\nu)$ and its Legendre transform $h(\xi)$

As explained at the beginning of this chapter, Volberg's work deals with functions

$$f(\vartheta) \sim \sum_{-\infty}^{\infty} a_n e^{in\vartheta}$$

for which the a_n with negative index are very small; more precisely,

$$|a_{-n}| \leqslant e^{-M(n)}, \quad n > 0,$$

where $M(n)$ is *increasing* and such that

$$\sum_1^{\infty} \frac{M(n)}{n^2} = \infty.$$

► It will be convenient to assume throughout this § that $M(\nu)$ *is defined for all real values of* $\nu \geqslant 0$ *and not just the integral ones, and is increasing on* $[0, \infty)$. We do not, to begin with, exclude the possibility that $M(0) = -\infty$. Whether this happens or not will turn out to make no difference as far as our final result is concerned.

Volberg's treatment makes essential use of a *weight* $w(r) > 0$ defined for $0 < r < 1$ by means of the formula

$$\log\left(\frac{1}{w(r)}\right) = \sup_{\nu > 0}\left(M(\nu) - \nu \log \frac{1}{r}\right).$$

It is therefore necessary to make a study of the relation between $M(\nu)$ and the function

$$\boxed{h(\xi) = \sup_{\nu > 0}(M(\nu) - \nu\xi),}$$

defined for $\xi > 0$, and to find out how various properties of $M(\nu)$ are connected to others of $h(\xi)$. We take up these matters in the present article.

The formula for the function $h(\xi)$ (sometimes called the *Legendre transform* of $M(\nu)$) is reminiscent of material discussed extensively in Chapter IV, beginning with § A.2 therein. It is perhaps a good idea to start by showing how the situation now under consideration is related to that of Chapter IV, and especially how it *differs* from the latter.

Our present function $M(\nu)$ can be interpreted as $\log T(\nu)$, where $T(r)$ is the Ostrowski function used in Chapter IV. ($M(n)$ is *not*, as the similarity in letters might lead one to believe, a version of the $\{M_n\}$ – or of $\log M_n$ –

from Chapter IV!) Suppose indeed that

$$f(\vartheta) \sim \sum_{-\infty}^{\infty} a_n e^{in\vartheta}$$

is infinitely differentiable and in the class $\mathscr{C}(\{M_n\})$ considered in Chapter IV – in order to simplify matters, let us say that

$$|f^{(n)}(\vartheta)| \leqslant M_n, \quad n \geqslant 0.$$

We have

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\vartheta} f(\vartheta)\,d\vartheta,$$

and the right side, after k integrations by parts, becomes

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} (in)^{-k} f^{(k)}(\vartheta)\,d\vartheta$$

when $n \neq 0$. Using the above inequality on the derivatives of $f(\vartheta)$ in this integral, we see that

$$|a_n| \leqslant \inf_{k \geqslant 0} \frac{M_k}{|n|^k} = \frac{1}{T(|n|)}$$

where, as in Chapter IV,

$$T(r) = \sup_{k \geqslant 0} \frac{r^k}{M_k} \quad \text{for} \quad r > 0.$$

On putting $T(v) = e^{M(v)}$, we get

$$|a_n| \leqslant e^{-M(|n|)}.$$

This connection makes it possible to apply the final result of the present § to certain classes $\mathscr{C}(\{M_n\})$ of periodic functions, of period 2π. But that application does not show its real scope. The inequality for the a_n obtained by assuming that $f \in \mathscr{C}(\{M_n\})$ is a *two-sided* one; it shows that the a_n go to zero rapidly as $n \longrightarrow \pm\infty$. The *hypothesis* for the theorem on the logarithmic integral is, however, *one-sided*; it is only necessary to assume that

$$|a_{-n}| \leqslant e^{-M(n)}$$

for $n > 0$ in order to reach the desired conclusion.

There is another essential difference between our present situation and that of Chapter IV. *Here* we look at the function

$$h(\xi) = \sup_{v > 0} (M(v) - v\xi),$$

i.e., in terms of $T(v)$,

$$h(\xi) = \sup_{v>0}(\log T(v) - v\xi).$$

There we used the convex logarithmic regularisation $\{\underline{M}_n\}$ given by

$$\log \underline{M}_n = \sup_{v>0}((\log v)n - \log T(v)).$$

There is, first of all, a *change in sign.* Besides this, the former expression involves terms $v\xi$, *linear in the parameter* v, where the latter has terms *linear in* $\log v$. On account of these differences it usually turns out that the function $h(\xi)$ considered here *tends to* ∞ *for* $\xi \to 0$, whereas $\log \underline{M}_n$ *usually tended to* ∞ *for* $n \to \infty$.

Let us begin our examination of $h(\xi)$ by verifying the statement just made about its behaviour for $\xi \to 0$.

Lemma. *If* $M(v) \to \infty$ *for* $v \to \infty$, $h(\xi) \to \infty$ *for* $\xi \to 0$.

Proof. Take any v_0. Then, if $0 < \xi < \frac{1}{2}M(v_0)/v_0$,

$$h(\xi) \geqslant M(v_0) - v_0\xi > \tfrac{1}{2}M(v_0).$$ Q.E.D.

The function

$$h(\xi) = \sup_{v>0}(M(v) - v\xi),$$

as the supremum of *decreasing* functions of ξ, is *decreasing*. As the *supremum* of *linear functions* of ξ, *it is convex*. The *upper supporting line of slope* ξ *to the graph of* $M(v)$ vs v *has ordinate intercept equal to* $h(\xi)$:

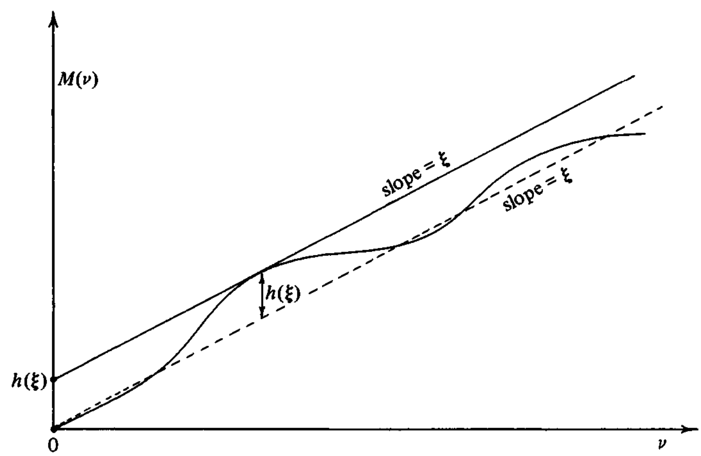

Figure 84

From this picture, we see immediately that

$$M^*(\nu) \;=\; \inf_{\xi>0}\,(h(\xi)+\xi\nu)$$

is the smallest concave increasing function which is $\geqslant M(\nu)$. Therefore, *if* $M(\nu)$ is *also concave*, $M^*(\nu) = M(\nu)$. We will come back to this relation later on.

Here is a graph dual to the one just drawn:

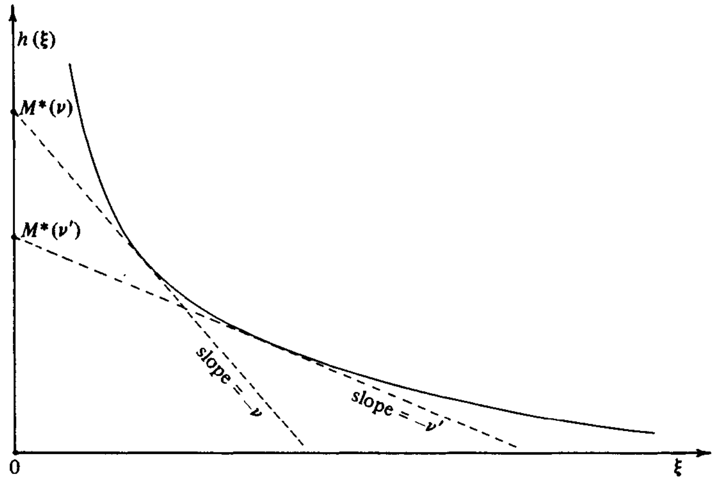

Figure 85

We see that $M^*(\nu)$ is the *ordinate intercept of the* (lower) *supporting line to the convex graph of* $h(\xi)$ having slope $-\nu$.

Volberg's construction depends in an essential way on a theorem of Dynkin, to be proved in the next article, which requires *concavity* of the function $M(\nu)$. Insofar as inequalities of the form

$$|a_{-n}| \;\leqslant\; e^{-M(n)}$$

are concerned, this concavity is pretty much *equivalent* to the cruder property that $M(\nu)/\nu$ be *decreasing*. It is, first of all, fairly evident that *the concavity of* $M(\nu)$ *makes* $M(\nu)/\nu$ *decreasing* (and even *strictly* decreasing, save in the trivial case where $M(\nu)/\nu \equiv \text{const.}$) *for all sufficiently large* ν. We have, in the other direction, the following

Theorem. *Let* $M(\nu)$ *be* >0 *and increasing for* $\nu>0$, *and denote by* $M^*(\nu)$ *the* smallest concave majorant *of* $M(\nu)$. *If* $M(\nu)/\nu$ *is* decreasing.

$$M^*(\nu) \;<\; 2M(\nu).$$

Problem 14(a)

Prove this result. (Hint: The graph of $M^*(v)$ vs v coincides with that of $M(v)$, save on certain open intervals (a_n, b_n) on each of which $M^*(v)$ is *linear*, with $M^*(a_n) = M(a_n)$ and $M^*(b_n) = M(b_n)$:

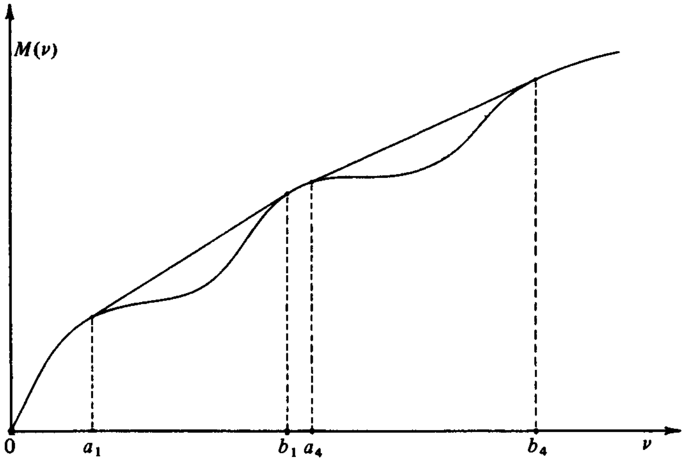

Figure 86

The (a_n, b_n) *may*, of course, be disposed like the contiguous intervals to the Cantor set, for instance. Consider any one of them, say, wlog, (a_1, b_1):

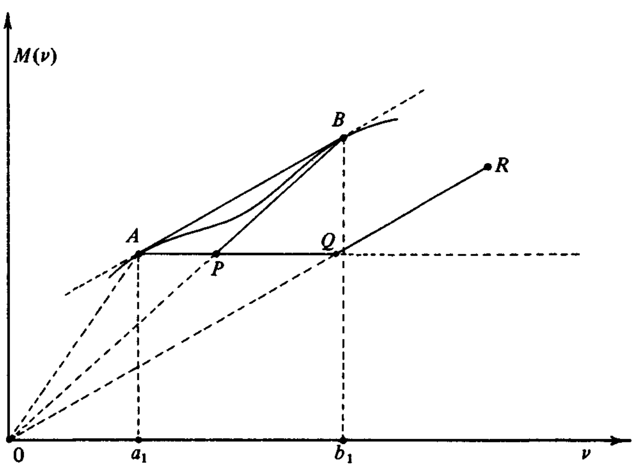

Figure 87

For $a_1 \leqslant v \leqslant b_1$, $(v, M(v))$ must lie *above* the broken line path APB, and $(v, M^*(v))$ lies on the segment $\overline{AB}$. Work with the broken line path $\overline{AQR}$, where OR is a line through the origin parallel to $\overline{AB}$.)

Because of this fact, the Fourier coefficients a_n of a given function which satisfy an inequality of the form

$$|a_{-n}| \leqslant e^{-M(n)}, \qquad n \geqslant 1,$$

with an *increasing* $M(v) > 0$ *such that* $M(v)/v$ *decreases* also satisfy

$$|a_{-n}| \leqslant e^{-M^*(n)/2}, \qquad n \geqslant 1$$

with the *concave majorant* $M^*(v)$ of $M(v)$. Clearly, $\sum_1^\infty M^*(n)/n^2 = \infty$ if $\sum_1^\infty M(n)/n^2 = \infty$. This circumstance makes it possible to simplify much of the computational work by *supposing to begin with that* $M(v)$ *is concave as well as increasing.*

A further (really, mainly formal) simplification results if we consider only functions $M(v)$ for which $M(v)/v \longrightarrow 0$ as $v \rightarrow \infty$ (see the next lemma). *As far as Volberg's work is concerned, this entails no restriction.* Since we will be assuming (at least) that $M(v)/v$ is decreasing, $\lim_{v\to\infty}(M(v)/v)$ *certainly exists. In case that limit is strictly positive,* the inequalities

$$|a_{-n}| \leqslant e^{-M(n)}, \qquad n \geqslant 1,$$

imply that

$$F(z) = \sum_{-\infty}^{\infty} a_n z^n$$

is *analytic in some annulus* $\{\rho < |z| < 1\}$, $\rho < 1$. This makes it possible for us to apply the *theorem on harmonic estimation* (§B.1), at least when $F(z)$ is continuous up to $\{|z| = 1\}$ (which will be the case in our version of Volberg's result). We find in this way that

$$\int_{-\pi}^{\pi} \log|F(e^{i\vartheta})| d\vartheta > -\infty$$

unless $F(z) \equiv 0$, using a simple estimate for harmonic measure in an annulus. (If the reader has any trouble working out that estimate, he or she may find it near the *very end* of the proof of Volberg's theorem in article 6 below.) The *conclusion* of Volberg's theorem is thus *verified* in the special case that $\lim_{v\to\infty}(M(v)/v) > 0$.

For this reason, we will mostly only consider functions $M(v)$ *for which* $\lim_{v\to\infty}(M(v)/v) = 0$ *in the present §.*

Once we decide to work with *concave* functions $M(v)$, it costs but little to *further restrict our attention* to *strictly concave infinitely differentiable* $M(v)$'s. Given any concave increasing $M(v)$, we may, first of all, add to it a *bounded strictly concave* increasing function (with second derivative < 0

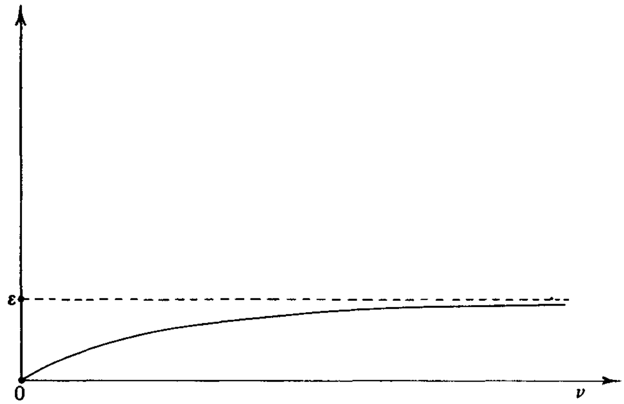

Figure 88

on $(0, \infty)$) whose graph has a *horizontal asymptote* of height ε, and thus obtain a new *strictly concave* increasing function $M_1(v)$, with $M_1''(v) < 0$, differing by at most ε from $M(v)$. We may then take an infinitely differentiable positive function φ supported on $[0, 1]$ and having $\int_0^1 \varphi(t)\,dt = 1$,

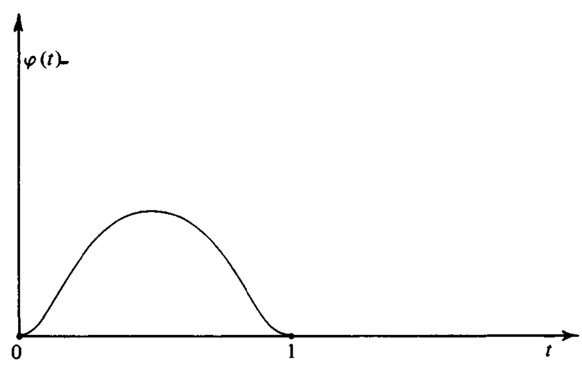

Figure 89

and form the function

$$M_2(v) = \frac{1}{h}\int_0^h M_1(v+\tau)\varphi(\tau/h)\,d\tau,$$

using a small value of $h > 0$. $M_2(v)$ will also be strictly concave with

$M_2''(v) < 0$ on $(0, \infty)$, and increasing, and *infinitely differentiable besides* for $0 < v < \infty$. It will differ by less than ε from $M_1(v)$ for $v \geqslant a$ when a is any given number > 0, if $h > 0$ is small enough (depending on a). That's because $0 \leqslant M_1'(v) \leqslant M_1'(a) < \infty$ for $v \geqslant a$.

Our function $M_2(v)$, infinitely differentiable, increasing, and strictly concave, thus differs by less than 2ε from $M(v)$ when v is large. This, however, means that $h_2(\xi) = \sup_{v>0}(M_2(v) - v\xi)$ differs by less than 2ε from

$$h(\xi) \;=\; \sup_{v>0}(M(v) - v\xi)$$

for small values of $\xi > 0$, the suprema in question being attained for *large values* of v if ξ is *small*:

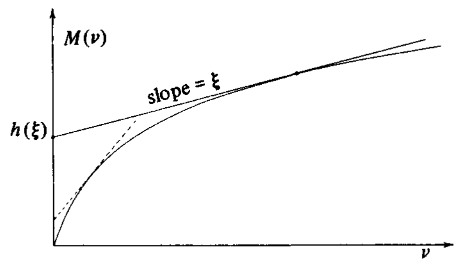

Figure 90

Hence, in studying the *order of magnitude of* $h(\xi)$ *for* ξ *near zero* (which is what we will be mainly concerned with in this §), *we may as well assume to begin with that* $M(v)$ is *strictly concave* and *infinitely differentiable.*

When this restriction holds, one can obtain some useful relations in connection with the duality between $M(v)$ and $h(\xi)$.*

Lemma. *If* $M(v)$ *is strictly concave and increasing with* $M(v)/v \longrightarrow 0$ *for* $v \to \infty$, *there is for each* $\xi > 0$ *a* unique $v = v(\xi)$ *such that*

$$h(\xi) \;=\; M(v) - v\xi.$$

$h(\xi)$ *has a* derivative *for* $\xi > 0$, *and* $h'(\xi) = -\,v(\xi)$.

Proof. Since $M(v)/v \longrightarrow 0$ as $v \to \infty$, the supporting line of slope ξ to the graph of $M(v)$ vs v *does* touch that graph *somewhere* (see preceding diagram), say at $(v_1, M(v_1))$. Thus,

$$h(\xi) \;=\; M(v_1) - v_1\xi.$$

* In the following 3 lemmas, it is tacitly assumed that $\xi > 0$ ranges over some *small* interval with left endpoint at the origin, for they will be used only for such values of ξ. This eliminates our having to worry about the behaviour of $M(v)$ for small v.

Suppose that $v_2 \neq v_1$ and also

$$h(\xi) = M(v_2) - v_2\xi;$$

wlog say that $v_2 > v_1$. Then

$$M(v_2) = M(v_1) + \xi(v_2 - v_1).$$

Therefore, for $v_1 < v < v_2$, by *strict* concavity of $M(v)$,

$$M(v) > M(v_1) + \xi(v - v_1),$$

i.e.,

$$M(v) - v\xi > M(v_1) - v_1\xi = h(\xi).$$

This, however, contradicts the definition of $h(\xi)$, so there can *be* no $v_2 \neq v_1$ with

$$h(\xi) = M(v_2) - v_2\xi.$$

Since $M(v)$ is already concave, it is *equal* to its smallest concave majorant, $M^*(v)$, i.e.,

$$M(v) = \inf_{\xi > 0} (h(\xi) + \xi v).$$

The function $h(\xi)$ is convex, so if it does *not* have a derivative at a point $\xi_0 > 0$, it has a *corner* there, with two *different* supporting lines, of slopes $-v_1$ and $-v_2$, touching the graph of $h(\xi)$ vs ξ at $(\xi_0, h(\xi_0))$:

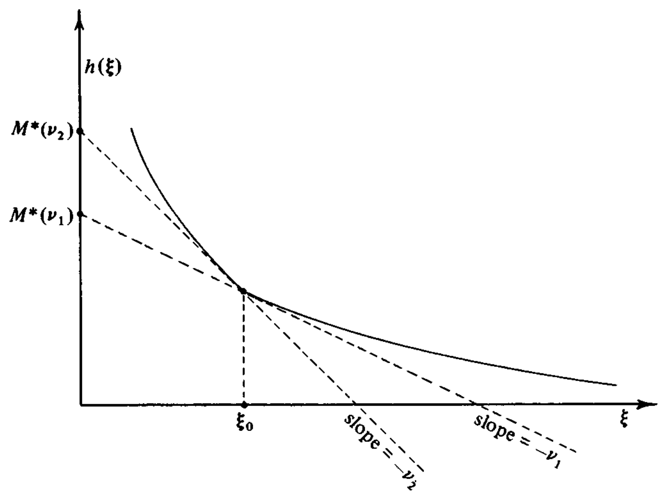

Figure 91

Those two supporting lines have ordinate intercepts equal to $M^*(v_1)$ and $M^*(v_2)$, i.e., to $M(v_1)$ and $M(v_2)$. But then $h(\xi_0) = M(v_1) - v_1\xi_0 = M(v_2) - v_2\xi_0$, which we have already seen to be impossible. $h'(\xi_0)$ must therefore *exist*, and it is now clear that derivative must have the value $-v(\xi_0)$, the slope of the *unique* supporting line to the graph of $h(\xi)$ vs ξ at the point $(\xi_0, h(\xi_0))$.

Lemma. *If $M(v)$ is differentiable and strictly concave and $M(v)/v \longrightarrow 0$ for $v \to \infty$,*

$$\frac{\mathrm{d}M(v)}{\mathrm{d}v} = \xi \quad \textit{for} \quad v = v(\xi).$$

Proof. $v(\xi)$ is the abscissa at which the supporting line of slope ξ to the graph of $M(v)$ vs v touches that graph.

Recall that, for the strictly concave functions $M(v)$ we are dealing with here, *we actually have* $M''(v) < 0$ on $(0, \infty)$ – refer to the above construction of $M_1(v)$ and $M_2(v)$ from $M(v)$.

Lemma. *If $M(v)$ is twice continuously differentiable and $M''(v) < 0$ on $(0, \infty)$, and if $M(v)/v \longrightarrow 0$ for $v \to \infty$, $h''(\xi)$ exists for $\xi > 0$.*

Proof. $M(v)$ is certainly strictly concave, so, by the preceding two lemmas, $h'(\xi)$ exists and we have the implicit relation

$$M'(-h'(\xi)) = \xi.$$

Since $M''(v)$ exists, is continuous, and is < 0, we can apply the *implicit function theorem* to conclude that $h''(\xi)$ exists and equals $-1/M''(-h'(\xi))$.

Volberg's construction, besides depending (through Dynkin's theorem) on the concavity of $M(v)$, makes essential use of one *additional special property*, namely, that

$$\boxed{\xi^{-K} \leqslant \text{const.}h(\xi)}$$

for some $K > 1$ as $\xi \to 0$. Let us express this in terms of $M(v)$.

Lemma. *For concave $M(v)$, the preceding boxed relation holds with some $K > 1$ for $\xi \to 0$ iff*

$$M(v) \geqslant \text{const.}v^{K/(K+1)} \quad \textit{for large } v.$$

Proof. Since $M(v)$ is concave, it is equal to $\inf_{\xi>0}(h(\xi) + v\xi)$. If the boxed relation holds and v is large, this expression is $\geqslant \inf_{\xi>0}(\text{const.}\xi^{-K} + v\xi)$

whose value is readily seen to be of the form const.$\nu^{K/(K+1)}$.

To go the other way, compute $\sup_{\nu>0}(\text{const.}\nu^{K/(K+1)} - \nu\xi)$.

Remark. One *might think* that the *concavity of* $M(\nu)$ and the fact that

$$\sum_1^\infty M(n)/n^2 \;=\; \infty$$

together imply that $M(\nu) \geqslant \nu^\rho$ with *some* positive ρ (say $\rho = \frac{1}{2}$) for large ν. That, however, is *not so.* A counter example may easily be constructed by building the graph of $M(\nu)$ vs ν out of exceedingly long straight segments chosen one after the other so as to alternately cut the graph of ν^ρ vs. ν *from below* and *from above.*

Here is one more rather trivial fact which we will have occasion to use.

Lemma. *For* increasing $M(\nu)$,

$$h(\xi) \;\geqslant\; M(0) \quad \textit{for} \quad \xi > 0$$

and hence $\lim_{\xi\to\infty} h(\xi)$ *is* finite *if* $M(0) > -\infty$.

Proof. $h(\xi)$ is *decreasing*, so $\lim_{\xi\to\infty} h(\xi)$ *exists*, but is perhaps equal to $-\infty$. The rest is clear.

The principal result on the connection between $M(\nu)$ and $h(\xi)$ was published independently by Beurling and by Dynkin in 1972. It says that, if $a > 0$ is sufficiently small (so that $\log h(\xi) > 0$ for $0 < \xi \leqslant a$), the convergence of $\int_0^a \log h(\xi)\,\mathrm{d}\xi$ is equivalent to that of $\int_1^\infty (M(\nu)/\nu^2)\,\mathrm{d}\nu$ (compare with the material in §C of Chapter IV). More precisely:

Theorem. *If* $M(\nu)$ *is* increasing *and* concave, *and*

$$h(\xi) \;=\; \sup_{\nu>0}(M(\nu) - \nu\xi),$$

there is an $a > 0$ *such that*

$$\int_0^a \log h(\xi)\,\mathrm{d}\xi \;<\; \infty$$

iff

$$\int_1^\infty \frac{M(\nu)}{\nu^2}\,\mathrm{d}\nu \;<\; \infty.$$

Proof. In the first place, if $\lim_{\nu\to\infty} M(\nu)/\nu = c > 0$, the function $h(\xi) = \sup_{\nu>0}(M(\nu) - \nu\xi)$ is *infinite* for $0 < \xi < c$. In this case, the integrals involved in the theorem *both diverge.* For the remainder of the proof we may thus suppose that $M(\nu)/\nu \longrightarrow 0$ as $\nu \to \infty$.

Again, by the *first* lemma of this article, $h(\xi) \longrightarrow \infty$ for $\xi \to 0$ unless $M(\nu)$ is bounded for $v \to \infty$, and in that case both of the integrals in question are obviously finite. There is thus no loss of generality in supposing that $h(\xi) \longrightarrow \infty$ for $\xi \to 0$, and we may take an $a > 0$ with $h(a) \geqslant 2$, say.

These things being granted, let us, as in the previous discussion, approximate $M(v)$ to within ε on $[A, \infty)$, $A > 0$, by an infinitely differentiable *strictly* concave function $M_\varepsilon(v)$, with $M_\varepsilon''(v) < 0$. If $\varepsilon > 0$ and $A > 0$ are small enough, the corresponding function

$$h_\varepsilon(\xi) \;=\; \sup_{v>0}(M_\varepsilon(v) - v\xi)$$

approximates $h(\xi)$ *to within* 1 *unit* (say) on $(0, a]$. But then

$$\int_0^a \log h(\xi)\,d\xi \quad \text{and} \quad \int_0^a \log h_\varepsilon(\xi)\,d\xi$$

converge simultaneously, and the *same* is true for the integrals

$$\int_1^\infty \frac{M(v)}{v^2}\,dv \quad \text{and} \quad \int_1^\infty \frac{M_\varepsilon(v)}{v^2}\,dv.$$

It is therefore enough to establish the theorem for $M_\varepsilon(v)$ and $h_\varepsilon(\xi)$; in other words, we may, wlog, *assume to begin with that $M(v)$ is infinitely differentiable and strictly concave, with $M''(v) < 0$, and that $M(v)/v \longrightarrow 0$ for $v \to \infty$.*

In these circumstances, we can use the relations furnished by the preceding lemmas. It is convenient to work with $\log|h'(\xi)|$ instead of $\log h(\xi)$, so for this purpose let us first show that

$$\int_0^a \log h(\xi)\,d\xi \quad \text{and} \quad \int_0^a \log|h'(\xi)|\,d\xi$$

converge simultaneously. First of all,

$$h(\xi) \;\leqslant\; h(a) + (a-\xi)|h'(\xi)| \;\leqslant\; h(a) + a|h'(\xi)| \quad \text{for} \quad 0 < \xi < a$$

by the *convexity* of $h(\xi)$, as the following diagram shows:

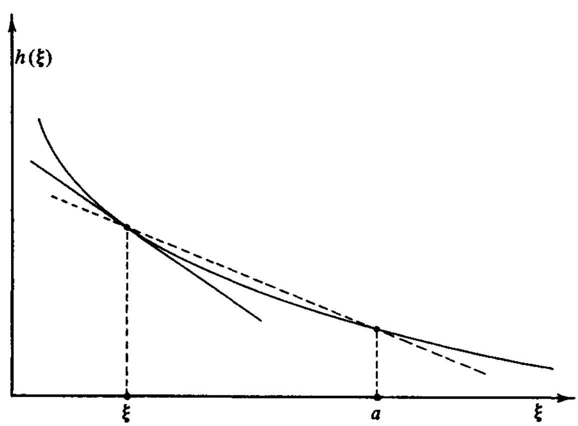

Figure 92

Therefore convergence of the *second* integral implies that of the *first*. Again, for $0 < \xi < a$, $h(\xi) \geqslant 2$, so

$$h\left(\frac{\xi}{2}\right) \;\geqslant\; 2 + \frac{\xi}{2}|h'(\xi)| \;\geqslant\; \frac{\xi}{2}|h'(\xi)|$$

for such ξ:

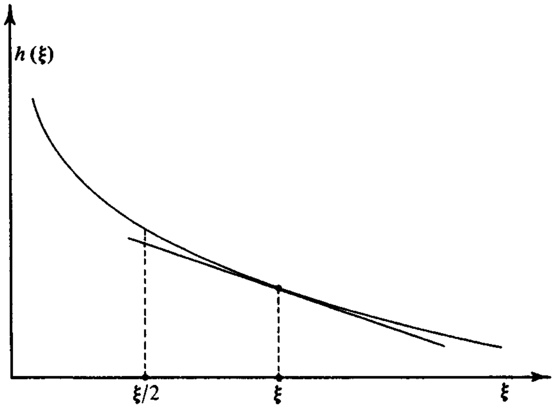

Figure 93

So, since $\int_0^a |\log \xi| \mathrm{d}\xi < \infty$, convergence of the *first* integral implies that of the *second*.

We have

$$h(\xi) \;=\; M(\nu(\xi)) - \xi\nu(\xi)$$

with $\nu(\xi) = -h'(\xi)$, and $M'(\nu(\xi)) = \xi$. Therefore

$$\xi \mathrm{d} \log |h'(\xi)| \;=\; M'(\nu(\xi))\frac{\mathrm{d}\nu(\xi)}{\nu(\xi)}.$$

Taking a number b, $0 < b < a$, and integrating by parts, we find that

$$\int_b^a \xi \mathrm{d} \log |h'(\xi)| \;=\; a \log |h'(a)| - b \log |h'(b)| - \int_b^a \log |h'(\xi)| \mathrm{d}\xi$$

$$= \frac{M(\nu(a))}{\nu(a)} - \frac{M(\nu(b))}{\nu(b)} + \int_{\nu(b)}^{\nu(a)} \frac{M(\nu)}{\nu^2} \mathrm{d}\nu.$$

Here, $\nu(\xi)$ is decreasing, so $\nu(b) \geqslant \nu(a)$. Turning things around, we thus have

$$\int_b^a \log|h'(\xi)|\mathrm{d}\xi + b\log|h'(b)| - a\log|h'(a)|$$

$$= \frac{M(\nu(b))}{\nu(b)} - \frac{M(\nu(a))}{\nu(a)} + \int_{\nu(a)}^{\nu(b)} \frac{M(\nu)}{\nu^2} \mathrm{d}\nu.$$

$M(\nu)/\nu$ is decreasing (concavity of $M(\nu)$!) and, as $b \to 0$, $\nu(b) \to \infty$. We see, then, that

$$\int_0^a \log|h'(\xi)|\,d\xi \quad < \quad \infty$$

if

$$\int_{\nu(a)}^{\infty} \frac{M(\nu)}{\nu^2}\,d\nu \quad < \quad \infty.$$

Also, $|h'(\xi)|$ *decreases*, so $b\log|h'(b)| \leqslant \int_0^b \log|h'(\xi)|\,d\xi$. Therefore

$$\int_{\nu(a)}^{\nu(b)} \frac{M(\nu)}{\nu^2}\,d\nu$$

is bounded above for $b \to 0$ if $\int_0^a \log|h'(\xi)|\,d\xi < \infty$, i.e., $\int_{\nu(a)}^{\infty}(M(\nu)/\nu^2)\,d\nu < \infty$. We are done.

Problem 14(b)

Let $H(\xi)$ be decreasing for $\xi > 0$ with $H(\xi) \to \infty$ for $\xi \to 0$, and denote by $h(\xi)$ the *largest convex minorant* of $H(\xi)$. Show that, if, for some small $a > 0$, $\int_0^a \log h(\xi)\,d\xi < \infty$, then $\int_0^a \log H(\xi)\,d\xi < \infty$. Hint: Use the following picture:

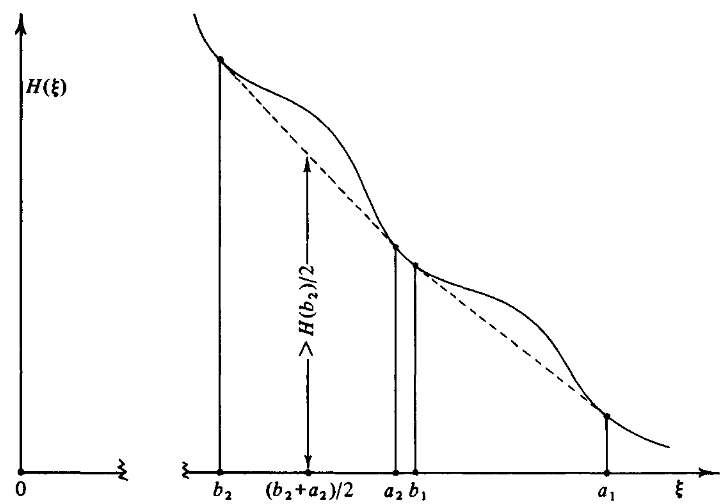

Figure 94

Problem 14(c)

If $M(\nu)$ is increasing, it is in general *false* that $\int_1^{\infty}(M(\nu)/\nu^2)\,d\nu < \infty$ makes $\int_1^{\infty}(M^*(\nu)/\nu^2)\,d\nu < \infty$ for the *smallest concave majorant* $M^*(\nu)$ of $M(\nu)$. (Hint: In one counter example, $M^*(\nu)$ has a broken line graph with vertices on the one of $\nu/\log\nu$ (ν large).)

Theorem. *Let $H(\xi)$ be decreasing for $\xi > 0$ and tend to ∞ as $\xi \to 0$. For $v > 0$, put*

$$M(v) \;=\; \inf_{\xi > 0} (H(\xi) + \xi v).$$

Then

$$\int_0^a \log H(\xi)\,\mathrm{d}\xi \;<\; \infty$$

for some (and hence for all) arbitrarily small values of $a > 0$ iff

$$\int_1^\infty \frac{M(v)}{v^2}\,\mathrm{d}v \;<\; \infty.$$

Proof. As the infimum of linear functions of v, $M(v)$ *is* concave; it is obviously *increasing.* The function

$$h(\xi) \;=\; \sup_{v>0} (M(v) - v\xi)$$

is the largest *convex minorant* of $H(\xi)$ because its height at any abscissa ξ is the supremum of the heights of all the (lower) supporting lines with slopes $-v < 0$ to the graph of H:

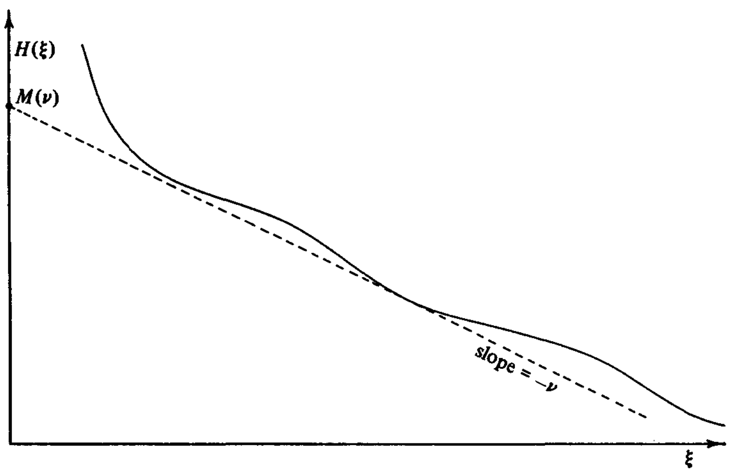

Figure 95

Therefore $\int_0^a \log H(\xi)\mathrm{d}\xi = \infty$ makes $\int_0^a \log h(\xi)\mathrm{d}\xi = \infty$ *by problem* 14(b), so in that case $\int_1^\infty (M(v)/v^2)\mathrm{d}v = \infty$ *by the preceding theorem.* If, on the other hand, $\int_1^\infty (M(v)/v^2)\mathrm{d}v$ *does diverge,* $\int_0^a \log h(\xi)\mathrm{d}\xi = \infty$ for each small enough $a > 0$ by that same theorem, so certainly $\int_0^a \log H(\xi)\mathrm{d}\xi = \infty$ for such a. This does it.

3. Dynkin's extension of $F(e^{i\vartheta})$ to $\{|z| \leqslant 1\}$ with control on $|F_{\bar{z}}(z)|$.

As stated near the beginning of the previous article, a very important role in Volberg's construction is played by a weight $w(r) > 0$ defined for $0 < r < 1$ by the formula

$$w(r) = \exp\left(-h\left(\log\frac{1}{r}\right)\right),$$

where, for $\xi > 0$,

$$h(\xi) = \sup_{v>0}(M(v) - v\xi).$$

Here $M(v)$ is an increasing (usually concave) function such that $\int_1^\infty (M(v)/v^2)\,dv = \infty$; this makes $h(\xi)$ increase to ∞ rather rapidly as ξ decreases towards 0, so that $w(r)$ *decreases very rapidly towards zero* as $r \to 1$.

A typical example of the kind of functions $M(v)$ figuring in Volberg's theorem is obtained by putting

$$M(v) = \frac{v}{\log v}$$

for $v \geqslant e^2$, say, and defining $M(v)$ in any convenient fashion for $0 \leqslant v < e^2$ so as to keep it increasing and concave on that range. Here we find without difficulty that

$$h(\xi) \sim \frac{\xi^2}{e}e^{1/\xi} \quad \text{for} \quad \xi \longrightarrow 0,$$

and $w(r)$ decreases towards zero like

$$\exp\left(-\frac{(1-r)^2}{e}e^{1/(1-r)}\right)$$

as $r \to 1$; this is *really* fast. It is good to keep this example in mind during the following development.

Lemma. *Let $M(v)$ be increasing and strictly concave for $v > 0$ with $M(v)/v \longrightarrow 0$ for $v \to \infty$, put*

$$h(\xi) = \sup_{v>0}(M(v) - v\xi),$$

and write $w(r) = \exp(-h(\log(1/r)))$ for $0 < r < 1$. Then

$$\int_0^1 r^{n+2}w(r)\,dr > \frac{\text{const.}}{n}e^{-M(n)}$$

for $n \geqslant 1$.

Proof. In terms of $\xi = \log(1/r)$, $r^n w(r) = \exp(-h(\xi) - \xi n)$. Since $M(v)$ is strictly concave, we have, by the previous article,

$$\inf_{\xi > 0} (h(\xi) + \xi n) \;=\; M(n),$$

the infimum being attained at the value $\xi = \xi_n = M'(n)$. Put $r_n = e^{-\xi_n}$. Then, $r_n^n w(r_n) = e^{-M(n)}$. Because $w(r)$ decreases, we now see that

$$\int_0^1 r^{n+2} w(r)\,dr \;\geqslant\; w(r_n)\int_0^{r_n} r^{n+2}\,dr \;=\; \frac{r_n^3}{n+3}(r_n^n w(r_n))$$
$$=\; \frac{r_n^3}{n+3} e^{-M(n)}.$$

Here,

$$r_n^3 \;=\; e^{-3M'(n)},$$

and this is $\geqslant e^{-3M'(1)}$ since $M'(v)$ decreases, when $n \geqslant 1$. From the previous relation, we thus find that

$$\int_0^1 r^{n+2} w(r)\,dr \;\geqslant\; \frac{e^{-3M'(1)}}{4n} e^{-M(n)}$$

for $n \geqslant 1$, Q.E.D.

Theorem (Dynkin (the younger), 1972). *Let $M(v)$ be increasing on $(0, \infty)$, as well as strictly concave and infinitely differentiable on $(0, \infty)$, with $M''(v) < 0$ there and $M(v)/v \longrightarrow 0$ for $v \to \infty$. Let $M(0) > -\infty$.*

For $0 < r < 1$, put

$$w(r) \;=\; \exp\left(-h\left(\log\frac{1}{r}\right)\right),$$

where $h(\xi)$ is related to $M(v)$ in the usual fashion.

Suppose that

$$F(e^{i\vartheta}) \;\sim\; \sum_{-\infty}^{\infty} a_n e^{in\vartheta}$$

is continuous on the unit circumference, and that

$$\sum_1^{\infty} |n^2 a_{-n}| e^{M(n)} \;<\; \infty.$$

Then F has a continuous extension $F(z)$ onto $\{|z| \leqslant 1\}$ with $F(z)$ continuously differentiable for $|z| < 1$ and $|\partial F(z)/\partial \bar{z}| \leqslant$ const.$w(|z|)$, $|z| < 1$.

Remark. The sense of Dynkin's theorem is that *rapid growth* of $M(n)$

to ∞ for $n \to \infty$ (which corresponds to *rapid growth* of $h(\xi)$ to ∞ for ξ tending to 0) makes it possible to extend F continuously to $\{|z| \leqslant 1\}$ in such a way as to have $|\partial F(z)/\partial \bar{z}|$ *dropping off to zero very quickly for* $|z| \to 1$.

Proof of theorem. We start by taking a continuously differentiable function $\Omega(e^{it})$, to be determined presently, and putting*

$$(*) \qquad G(z) \;=\; \sum_0^{\infty} a_n z^n \;+\; \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 \frac{\Omega(e^{it}) r^2 w(r) r\,dr\,dt}{re^{it} - z}$$

for $|z| \leqslant 1$. The reason for using the factor r^2 with $w(r)$ will soon be apparent.

The idea now is to specify $\Omega(e^{it})$ in such fashion as to make $G(e^{i\vartheta})$ have the same Fourier series as $F(e^{i\vartheta})$. If we can do that, the function $G(z)$ will be a continuous extension of $F(e^{i\vartheta})$ to $\{|z| \leqslant 1\}$.

To see this, observe that our hypothesis certainly makes the trigonometric series

$$\sum_{-\infty}^{-1} a_n e^{in\vartheta}$$

absolutely convergent, so, since $F(e^{i\vartheta})$ is continuous,

$$\sum_0^{\infty} a_n e^{in\vartheta}$$

must also be the Fourier series of some continuous function, and hence the power series on the right in $(*)$ a continuous function of z for $|z| \leqslant 1$. According to a lemma from the previous article, the property $M(0) > -\infty$ makes $h(\xi)$ *bounded below* for $\xi > 0$ and hence $w(r)$ *bounded above* in $(0, 1)$. The right-hand integral in $(*)$ is thus of the form

$$\frac{1}{2\pi}\iint_{|\zeta|<1} \frac{b(\zeta)\,d\xi\,d\eta}{\zeta - z}$$

with a *bounded function* $b(\zeta)$. (Here, we are writing $\zeta = \xi + i\eta$ which *conflicts* with our frequent use of ξ to denote $\log(1/r)$. *No confusion should thereby result.*) It is well known that such an integral gives a *continuous* function of z; that's because it's a *convolution* on $\mathbb{R}^2$, with

$$\iint_{|\zeta|<R} \frac{d\xi\,d\eta}{|\zeta|}$$

finite for each finite R. We see in this way that the function $G(z)$ given by $(*)$ *will* be continuous for $|z| \leqslant 1$. If also the Fourier series of $G(e^{i\vartheta})$ and $F(e^{i\vartheta})$ coincide, those two functions must obviously be equal.

We wish to apply the theorem of article 1 to the right-hand integral in $(*)$. In order to stay honest, we should therefore check *continuous*

* The power series on the right in $(*)$ may not actually be *convergent* for $|z| = 1$, but *does* represent a *continuous function* for $|z| \leqslant 1$, as will be clear in a moment.

differentiability (for $|\zeta| < 1$) of the function $b(\zeta)$ figuring in that double integral, viz.,

$$b(re^{it}) = r^2 w(r)\Omega(e^{it}),$$

because continuous differentiability (at least) is required in the hypothesis of the theorem. It is for this purpose that the factor r^2 has been included; that factor ensures differentiability of $b(\zeta)$ at 0. Thanks to it, the desired property of $b(\zeta)$ follows from the continuous differentiability of $\Omega(e^{it})$ together with the continuity of $rw'(r)$ on $[0, 1)$ which we now verify.

We have $rw'(r) = h'(\log(1/r))w(r)$ for $0 < r < 1$. By a lemma in the previous article, $h''(\xi)$ exists for each $\xi > 0$ since $M''(v) < 0$. This certainly makes $h'(\log(1/r))$ continuous for $0 < r < 1$, so $rw'(r)$ is continuous for such r. When $r \to 0$, $w(r)$ *increases* and tends to a *finite* limit (since $M'(0) > -\infty$), and $h'(\log(1/r))$ *increases* (convexity of $h'(\xi)$), remaining, however, always $\leqslant 0$. Hence $rw'(r)$ *tends to a finite limit* as $r \to 0$, and (with obvious definition of $rw'(r)$ *at* the origin) is thus continuous at 0.

Having justified the application of the theorem from article 1 by this rather fussy argument, we see through its use that

$$\frac{\partial G(z)}{\partial \bar{z}} = -|z|^2 w(|z|)\Omega(e^{i\vartheta})$$

for

$$z = |z|e^{i\vartheta}, \quad |z| < 1,$$

after taking account of the fact that

$$\frac{\partial}{\partial \bar{z}}\left(\sum_0^\infty a_n z^n\right) = 0, \qquad |z| < 1.$$

This relation certainly makes

$$\left|\frac{\partial G(z)}{\partial \bar{z}}\right| \leqslant \text{const.} w(|z|)$$

for $|z| < 1$, so, if $G(z)$ *does* coincide with $F(z)$ for $|z| = 1$, *we will* have the theorem on putting $F(z) = G(z)$ for $|z| < 1$.

Everything thus depends on our being able to determine a continuously differentiable $\Omega(e^{it})$ which will make

$$(*) \qquad \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 \frac{\Omega(e^{it})r^2 w(r) r\,dr\,dt}{re^{it} - e^{i\vartheta}}$$

have the Fourier series

$$\sum_{-\infty}^{-1} a_n e^{in\vartheta}.$$