(which will turn out to be convenient later on) *by merely taking K large enough* to begin with. In the present circumstances, $\tilde{\omega}(x)$ (like $\omega(x)$) is $\mathscr{C}_\infty$ on $\mathbb{R}$,* so $\tilde{\omega}'(x)$ is *bounded on any finite interval.* Hence, if K is chosen large enough in the first place, we will have

$$\tilde{\omega}'(x) \quad \leqslant \quad K \qquad \text{for } 0 \leqslant x \leqslant 1,$$

so that *any component* $(a_k,\ b_k)$ of the set $\mathcal{O}$ corresponding to this K which *lies entirely* in $(0,1)$ may be simply *thrown away* (and $p(x) = \pi v_1'(x)$ just taken *equal* to δ thereon) without in any way affecting the properties of $v_1(x)$ and $v_2(x)$ used up to now. There may, however, *still* be a component $(a_l,\ b_l)$ of $\mathcal{O}$ with $a_l < 1 < b_l$. In that event, b_l is certainly finite, and there is thus a $K' \geqslant K$ with

$$\tilde{\omega}'(x) \quad \leqslant \quad K' \qquad \text{for } 0 \leqslant x \leqslant b_l.$$

Then, if we *also throw away* $(a_l,\ b_l)$, what *remains* of $\mathcal{O}$ will be a certain open set $\mathcal{O}' \subseteq (1,\infty)$ composed of the intervals $(a_k,\ b_k)$ from $\mathcal{O}$ that *do not intersect* $[0,1]$. We will have

$$\tilde{\omega}'(x) \quad \leqslant \quad K' \qquad \text{for } x \in (0,\infty) \sim \mathcal{O}',$$

and for each of the $(a_k,\ b_k)$ making up $\mathcal{O}'$,

$$\int_{a_k}^{b_k} (\tilde{\omega}'(x) - K')^+ \, dx \quad \leqslant \quad \delta(b_k - a_k),$$

since the same relation holds with $K \leqslant K'$ standing in place of K'. *By increasing K to K'*, we thus ensure that *none of the intervals $(a_k,\ b_k)$ appearing in our construction intersect with* $[0,1]$. This merely amounts to *choosing a larger value initially for the number K corresponding to our given* δ, which *we henceforth assume as having been done.* The intervals $(a_k,\ b_k)$ involved in the formation of $v_1(x)$ and $v_2(x)$ are in such fashion guaranteed to *all lie in* $(1,\infty)$.

Having seen to this matter, we turn our attention to the behaviour of $\pi v_1(t)$ for $t \geqslant 0$. As we have already noted, $\pi v_1(t) = \delta t$ for $t \geqslant 0$ lying *outside* all the intervals $(a_k,\ b_k)$. When $a_k < t < b_k$, we have, since $v_1(t)$ increases,

$$\delta a_k \ \leqslant \ \pi v_1(t) \ \leqslant \ \delta b_k.$$

At the same time,

$$\delta a_k \ < \ \delta t \ < \ \delta b_k,$$

* cf. initial footnote to third lemma of article 1

so

$$|\pi\nu_1(t) - \delta t| \leqslant \delta(b_k - a_k) \qquad \text{for } a_k < t < b_k.$$

Thence,

$$\int_{a_k}^{b_k} \frac{|\pi\nu_1(t) - \delta t|}{t^2}\,dt \leqslant \delta\left(\frac{b_k - a_k}{a_k}\right)^2$$

which, with the preceding observation, implies that

$$\int_0^\infty \frac{|\pi\nu_1(t) - \delta t|}{t^2}\,dt < \infty$$

on account of the convergence of $\sum_k((b_k - a_k)/a_k)^2$; *it is here that we have made crucial use of that hypothesis.*

Let us, in the usual fashion, extend the increasing function $\nu_1(t)$ to all of $\mathbb{R}$ by making it *odd* there. Then the function

$$\Delta(t) = \nu_1(t) - \frac{\delta}{\pi}t$$

is *also* odd and, moreover, *zero* for $-1 < t < 1$ due to our having arranged that none of the intervals (a_k, b_k) intersect with $[0, 1]$. According to what we have just seen,

$$\int_{-\infty}^\infty \frac{|\Delta(t)|}{t^2}\,dt < \infty;$$

$\Delta(t)$ thus satisfies the hypothesis of the initial lemma in §B.2, *Chapter* X, with δ/π playing the rôle of the number D figuring there. That result gives us a function $q(t)$, *zero* for $-1 < t < 1$, having the other properties of the one there denoted by $\delta(t)$, corresponding to a value δ/π of the parameter η. (Here we write $q(t)$ instead of $\delta(t)$ because the letter δ is already in service.) Since our present function $\Delta(t)$ is *odd*, the one furnished by the lemma referred to may be taken to be *odd also*, and

$$\lambda(t) = \frac{\delta}{\pi}t + q(t)$$

is then *odd*, besides being *increasing* on $\mathbb{R}$. We have

$$\lambda(t) = \frac{\delta}{\pi}t \qquad \text{for } -1 < t < 1$$

and moreover,

$$\frac{\lambda(t)}{t} \longrightarrow \frac{\delta}{\pi} \quad \text{as } t \longrightarrow \pm\infty.$$

One may now apply the *first* theorem of §B.2, Chapter X, to the present functions $\Delta(t)$ and $q(t)$, and then do a calculation like the one used to prove the lemma of §B.1 there. On account of the oddness of $v_1(t)$ and $\lambda(t)$, that computation simplifies quite a bit*, and the final result is that

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} \left| \int_0^{\infty} \log\left|1 - \frac{x^2}{t^2}\right| d(v_1(t) + \lambda(t)) \right| dx \quad < \quad \infty.$$

Used with the previous boxed formula and the assumption (in the hypothesis) that $\omega(x) \geqslant 0$ enjoys property (i), this implies that

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} \left| \int_0^{\infty} \log\left|1 - \frac{x^2}{t^2}\right| d(v_2(t) + \lambda(t)) \right| dx \quad < \quad \infty.$$

Put now

$$V(z) \quad = \quad \int_0^{\infty} \log\left|1 - \frac{z^2}{t^2}\right| d(v_2(t) + \lambda(t));$$

the last relation can then be written

$$\int_{-\infty}^{\infty} \frac{|V(x)|}{1+x^2} dx \quad < \quad \infty.$$

We have

$$v_2(t) \ + \ \lambda(t) \quad \leqslant \quad \text{const.}\, t \qquad \text{for } t \geqslant 0,$$

so V also satisfies an inequality of the form

$$V(z) \quad \leqslant \quad \text{const.}\,|z|.$$

These two properties of V imply that

$$V(z) \quad = \quad B|\Im z| \quad + \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|\, V(t)}{|z-t|^2} dt$$

with a suitable constant $B \geqslant 0$, according to a version of the result from §G.1, Chapter III – the use of such a version here can be justified by an

* The usual partial integration is carried out with $\Delta(t) + q(t)$ playing the rôle of $v(t)$; then the relation $\int_{-\infty}^{\infty} \log|1 - (x^2/t^2)|\, dt = 0$ is used.

argument like one made while proving the second theorem of §B.1.* From this formula and the first of the two relations for V preceding it, we get

$$\int_{-\infty}^{\infty} \frac{|V(x+\mathrm{i})|}{1+x^2}\,\mathrm{d}x \quad < \quad \infty$$

in the usual way.

We desire to apply the *Theorem on the Multiplier* at this point, and for that an *entire function of exponential type* is needed. (It is not true here that $V(x) \geqslant 0$ on $\mathbb{R}$, so we are unable to directly adapt the *proof* of that theorem given in §C.2 to the function V.) Take, then, the entire function φ of exponential type given by the formula

$$\log|\varphi(z)| \quad = \quad \int_0^{\infty} \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}[\nu_2(t) + \lambda(t)].$$

By the lemma in §A.1, Chapter X,

$$\log|\varphi(x+\mathrm{i})| \quad \leqslant \quad V(x+\mathrm{i}) \;+\; \log^+|x| \qquad \text{for } x \in \mathbb{R},$$

whence, by the preceding inequality,

$$\int_{-\infty}^{\infty} \frac{\log^+|\varphi(x+\mathrm{i})|}{1+x^2}\,\mathrm{d}x \quad < \quad \infty.$$

The theorem on the multiplier thus gives us a non-zero entire function $\psi(z)$, of exponential type $\delta' \leqslant \delta$, bounded on $\mathbb{R}$ and with $|\varphi(x+\mathrm{i})\psi(x)|$ bounded on $\mathbb{R}$ as well. We may, of course, get such a ψ with $\delta' = \delta$ by simply multiplying the initial one by $\cos(\delta - \delta')z$.

We can also take $\psi(z)$ to be *even*, since φ is even, and, of course, can have $\psi(0) \neq 0$. The discussion following the first theorem of §B.1 shows furthermore that we can take $\psi(z)$ to have *real zeros only*, and thus be given in the form

$$\log|\psi(z)| \quad = \quad \int_0^{\infty} \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}\sigma(t),$$

with $\sigma(t)$ increasing, integer-valued, zero near the origin, and satisfying

$$\frac{\sigma(t)}{t} \;\longrightarrow\; \frac{\delta}{\pi} \qquad \text{for } t \longrightarrow \infty$$

* By its definition, $\nu_2(t)$ is absolutely continuous with $\nu_2'(t)$ bounded on finite intervals; $\lambda(t)$, on the other hand, has a graph similar to the one shown in fig. 226 (Chapter X, §B.2). These properties make $(V(z))^+$ continuous at the points of $\mathbb{R}$, and the arguments from §§E and G.1 of Chapter III may be used.

(by Levinson's theorem). By first dividing out four of the zeros of ψ if need be (it *has* infinitely many, being of exponential type $\delta > 0$ and bounded on $\mathbb{R}$!) we can finally ensure that in fact

$$|\varphi(x+\mathrm{i})\psi(x)| \quad \leqslant \quad \frac{\text{const.}}{(x^2+1)^2} \qquad \text{for } x \in \mathbb{R}$$

with (perhaps another) ψ of the kind described.

A relation between $V(x+\mathrm{i})$ and $\log|\varphi(x+\mathrm{i})|$ *opposite in sense to the above one* is now called for. To get it, observe that

$$V(z) \quad = \quad \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}(\min(\nu_2(t)+\lambda(t),\ 1))$$
$$+ \quad \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}(\nu_2(t)+\lambda(t)-1)^+.$$

Since $\lambda(t) = \delta t/\pi$ for $0 \leqslant t \leqslant 1$ and $\nu_2'(t)$ is certainly *bounded* there, the first integral on the right is

$$\leqslant \quad 2\log^+|z| \ + \ \text{const.}.$$

Therefore, when $x \in \mathbb{R}$,

$$V(x+\mathrm{i}) \leqslant 2\log^+|x| \ + \ \text{const.} \ + \int_0^\infty \log\left|1 - \frac{(x+\mathrm{i})^2}{t^2}\right| \mathrm{d}(\nu_2(t)+\lambda(t)-1)^+.$$

However, $(\nu_2(t)+\lambda(t)-1)^+ \leqslant [\nu_2(t)+\lambda(t)]$ for $t \geqslant 0$, so, by reasoning identical to that used in proving the lemma of §A.1, Chapter X, we find that the last right-hand integral is

$$\leqslant \quad \log|\varphi(x+\mathrm{i})| \ + \ \log^+|x|.$$

Thus,

$$V(x+\mathrm{i}) \quad \leqslant \quad \log|\varphi(x+\mathrm{i})| \ + \ 3\log^+|x| \ + \ \text{const.}, \qquad x \in \mathbb{R}.$$

Referring to the previous relation involving $\varphi(x+\mathrm{i})$ and $\psi(x)$, we thence obtain

$$V(x+\mathrm{i}) \ + \ \log|\psi(x)| \quad \leqslant \quad \text{const.}, \qquad x \in \mathbb{R}.$$

Clearly, $V(x) \leqslant V(x+\mathrm{i})$, so we have

$$V(x) \quad + \quad \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\sigma(t) \quad \leqslant \quad \text{const.} \qquad \text{for } x \in \mathbb{R}$$

by our formula for $\log|\psi(z)|$.

Now by the previous boxed formula and our definition of the function V,

$$V(x) \quad = \quad \omega(x) \;-\; \omega(0) \;+\; \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}(\nu_1(t) + \lambda(t)).$$

Combination of this with the preceding thus yields

$$\omega(x) \quad + \quad \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}(\nu_1(t) + \lambda(t) + \sigma(t)) \quad \leqslant \quad \text{const.}, \qquad x \in \mathbb{R}$$

and hence, since $\log W(x) \leqslant \omega(x)$,

$$\log W(x) \quad + \quad \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}(\nu_1(t) + \lambda(t) + \sigma(t)) \quad \leqslant \quad \text{const.}, \quad x \in \mathbb{R}.$$

Here, $\lambda(t)/t$ and $\sigma(t)/t$ both tend to δ/π for $t \longrightarrow \infty$ as we have already noted. Again, since the ratios b_k/a_k corresponding to the intervals (a_k, b_k) used in the construction of $\nu_1(t)$ must tend to 1 for $k \longrightarrow \infty$, we also have*

$$\frac{\nu_1(t)}{t} \longrightarrow \frac{\delta}{\pi} \qquad \text{for } t \longrightarrow \infty$$

(look again at the above discussion of the behaviour of ν_1). For the increasing function

$$\rho(t) \quad = \quad \nu_1(t) \;+\; \lambda(t) \;+\; \sigma(t)$$

it is thus true that

$$\frac{\rho(t)}{t} \longrightarrow \frac{3\delta}{\pi} \qquad \text{as } t \longrightarrow \infty,$$

and that

$$\log W(x) \quad + \quad \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\rho(t) \quad \leqslant \quad \text{const.}, \qquad x \in \mathbb{R}.$$

The quantity $\delta > 0$ was, however, arbitrary. Therefore, since $W(x)$, by hypothesis, meets the local regularity requirement of §B.1, *it admits multipliers* according to the *second theorem* of that §, and *sufficiency is now established.*

Our result is completely proved.

* although a_k need not $\longrightarrow \infty$ for $k \longrightarrow \infty$, all *sufficiently large* a_k certainly *do* have *arbitrarily large* indices k.

Remark 1 (added in proof). In the *sufficiency* proof, fulfilment of our local regularity requirement is only used at the end; in the absence of that requirement one still gets functions $\rho(t)$, increasing and $O(t)$ on $[0, \infty)$, with $\limsup_{t\to\infty} (\rho(t)/t)$ arbitrarily small and

$$\log W(x) \quad + \quad \int_0^\infty \log|1 \; - \; (x^2/t^2)|\mathrm{d}\rho(t)$$

bounded above on $\mathbb{R}$. The *necessity* proof, on the other hand, actually goes through – see the footnotes to its first part – whenever $W(x) \geqslant 1$ is *continuous* and such $\rho(t)$ exist. *The existence of a majorant $\omega(x)$ having the properties specified by the theorem is therefore equivalent to the existence of such increasing functions ρ for continuous weights W.* Our theorem thus holds, in particular, for continuous weights meeting the milder regularity requirement from the scholium at the end of §B.1. Continuity, indeed, need not even be assumed for such weights; that is evident after a little thought about the abovementioned footnotes and the passage they refer to.

Remark 2. The proof for the *necessity* shows that if $W(x)$ *does* admit multipliers, a majorant $\omega(x)$ for $\log W(x)$ having the properties asserted by the theorem exists, with

$$\omega(x) \quad = \quad \omega(0) \quad - \quad \int_0^\infty \log\left|1 \; - \; \frac{x^2}{t^2}\right|\mathrm{d}\sigma(t),$$

where $\sigma(t)$ is *increasing* on $[0, \infty)$, *zero* for t close enough to 0, and $O(t)$ for $t \longrightarrow \infty$. Now look again at the example in §D.4 and the discussion in §D.5!

Remark 3. It was by thinking about the above result that I came upon the method explained in §§B.2, B.3 and used in §C, being led to it by way of the construction in problem 55 (near end of §B.2).

Remark 4. It seems possible to tie the theorem's property (ii) more closely to the *local* behaviour of $\omega(x)$. Referring to the remark following the statement of the theorem, we see that

$$\tilde{\omega}'(x) \quad = \quad \frac{1}{\pi}\int_0^{Y(x)} \frac{2\omega(x) - \omega(x+t) - \omega(x-t)}{t^2}\,\mathrm{d}t$$

$$+ \quad \frac{1}{\pi}\int_{Y(x)}^\infty \frac{2\omega(x) - \omega(x+t) - \omega(x-t)}{t^2}\,\mathrm{d}t,$$

where for $Y(x)$ we can take *any* positive quantity, depending on x in any way we want.

Because $\omega \geqslant 0$, the *second* of the two integrals on the right is

$$\leqslant \quad \frac{2}{\pi Y(x)}\omega(x);$$

it is, on the other hand,

$$\geqslant \quad -\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\omega(t)}{(x-t)^2+(Y(x))^2}\,dt.$$

For the present purpose this last expression's behaviour is adequately described by the 1967 lemma of Beurling and Malliavin given in §E.2 of Chapter IX. That result shows that for any given $\eta > 0$, the integral in question will lie between $-\eta$ and 0 for a function $Y(x) > 0$ with

$$\int_{-\infty}^{\infty}\int_0^{Y(x)}\frac{dy\,dx}{1+x^2+y^2} \quad < \quad \infty;$$

such a function is hence *not too large*.

Once a function $Y(x)$ is at hand, the set of $x > 0$ on which $\tilde{\omega}'(x)$ exceeds some large K seems to essentially be determined by the behaviour of $\omega(x)/Y(x)$ and of the integral

$$\frac{1}{\pi}\int_0^{Y(x)}\frac{2\omega(x)-\omega(x+t)-\omega(x-t)}{t^2}\,dt.$$

Both of these expressions involve *local behaviour* of ω.

I think an investigation along this line is worth trying, but have no time to undertake it now. *This book must go to press.*

Remark 5 (added in proof). We have been dealing with the notion of multiplier adopted in §B.1, using that term to desiquate a non-zero entire function of exponential type whose product with a given weight is *bounded* on $\mathbb{R}$. This specification of boundedness is largely responsible for our having had to introduce a local regularity requirement in §B.1.

Such requirements become to a certain extent irrelevant if we return to the broader interpretation of the term accepted in Chapter X and permit its use whenever the product in question belongs to some $L_p(\mathbb{R})$. This observation, already made by Beurling and Malliavin at the end of their 1962 article, is based on the following analogue of the second theorem in §B.2:

Lemma. *Let $\Omega(x) \geqslant 1$ be Lebesgue measurable. Suppose, given $A > 0$, that there is a function $\rho(t)$, increasing and $O(t)$ on $[0, \infty)$, with*

$$\limsup_{t\to\infty}(\rho(t)/t) \quad \leqslant \quad A/\pi$$

and

$$\log\Omega(x) \quad + \quad \int_0^\infty \log|1 \; - \; (x^2/t^2)|\,\mathrm{d}\rho(t) \quad \leqslant \quad \mathrm{O}(1) \qquad \text{a.e.}$$

on $\mathbb{R}$. *Then, if* $0 < p < \infty$, *there is a non-zero entire function* $\psi(z)$ *of exponential type* $\leqslant \; 4(p+2)A$ *such that*

$$\int_{-\infty}^\infty |\Omega(x)\psi(x)|^p \mathrm{d}x \quad < \quad \infty.$$

Proof. We consider the case $p = 1$; treatment for the other values of p is similar.

Take, then, the increasing function $\rho(t)$ furnished by the hypothesis and put

$$\nu(t) \;=\; 4\rho(t),$$

making

$$4\log\Omega(x) \quad + \quad \int_0^\infty \log|1 \; - \; (x^2/t^2)|\,\mathrm{d}\nu(t) \quad \leqslant \quad C \qquad \text{a.e., } x\in\mathbb{R}.$$

Since $\limsup_{t\to\infty}(\nu(t)/t) \;\leqslant\; 4A/\pi$, the entire function $\varphi(z)$ given by the formula

$$\log|\varphi(z)| \quad = \quad \int_0^\infty \log|1 \; - \; (z^2/t^2)|\,\mathrm{d}[\nu(t)]$$

is of exponential type $\leqslant\; 4A$; this may be checked by using partial integration to estimate $\log\varphi(|z|)$.

Putting

$$U(z) \quad = \quad \int_0^\infty \log|1 \; - \; (z^2/t^2)|\,\mathrm{d}\nu(t),$$

we have

$$(\Omega(x))^4 \exp U(x) \quad \leqslant \quad C \qquad \text{a.e., } x\in\mathbb{R}.$$

The idea behind our proof is that $|\varphi(x)|$ cannot be too much larger than $\exp U(x)$.

The usual integration by parts yields

$$\log|\varphi(x)| \; - \; U(x) \quad = \quad \int_0^\infty \log|1 \; - \; (x^2/t^2)|\,\mathrm{d}([\nu(t)] - \nu(t))$$

$$= \quad \int_0^\infty \frac{2x^2}{x^2 - t^2}\cdot\frac{[\nu(t)] - \nu(t)}{t}\,\mathrm{d}t$$

at every $x \in \mathbb{R}$ where $v'(x)$ exists and is finite, and hence almost everywhere (see the lemma in §B.1 of Chapter X). After extending v from $[0, \infty)$ to $\mathbb{R}$ by making it *odd* (which poses no problem, $v(t)$ being $O(t)$ for $t \geqslant 0$), we can rewrite the last integral as

$$\int_{-\infty}^{\infty} \frac{x}{x-t} \cdot \frac{[v(t)] - v(t)}{t} \mathrm{d}t \quad = \quad \int_{-\infty}^{\infty} \left(\frac{1}{x-t} + \frac{1}{t} \right) ([v(t)] - v(t)) \, \mathrm{d}t$$

$$= \quad b \quad + \quad \int_{-\infty}^{\infty} \left(\frac{1}{x-t} + \frac{t}{t^2+1} \right) ([v(t)] - v(t)) \, \mathrm{d}t,$$

where the quantity

$$b \quad = \quad \int_{-\infty}^{\infty} \frac{[v(t)] - v(t)}{t(t^2+1)} \mathrm{d}t$$

is finite. Hence, aside from the additive constant b, $\log|\varphi(x)| - U(x)$ is just the Hilbert transform of $\pi([v(x)] - v(x))$ which is, however, *bounded by* π *in absolute value*. Referring now to problem 45(*c*) (Chapter X, §F), we see that

$$\int_{-\infty}^{\infty} \frac{|\varphi(x)|^{1/4} \mathrm{e}^{-U(x)/4}}{1+x^2} \mathrm{d}x \quad < \quad \infty.$$

From this and the above relation involving $\Omega(x)$ and $U(x)$ we have, finally

$$\int_{-\infty}^{\infty} \frac{\Omega(x)|\varphi(x)|^{1/4}}{1+x^2} \mathrm{d}x \quad < \quad \infty.$$

Write

$$\psi(z) \quad = \quad \left(\frac{\sin Az}{z} \right)^8 \varphi(z);$$

$\psi(z)$ is entire, of exponential type $\leqslant 12A$, with $|\psi(x)| \leqslant \text{const.}|\varphi(x)|/(x^2+1)^4$ on the real axis. It thence follows by the preceding inequality that

$$\int_{-\infty}^{\infty} \Omega(x)|\psi(x)|^{1/4} \mathrm{d}x \quad < \quad \infty.$$

In order to conclude from this that

$$\int_{-\infty}^{\infty} \Omega(x)|\psi(x)| \mathrm{d}x \quad < \quad \infty$$

(thus proving the lemma in the case $p = 1$), it is enough to show that $\psi(x)$ is *bounded* on $\mathbb{R}$.

For that purpose, we note that $\int_{-\infty}^{\infty} |\psi(x)|^{1/4} \mathrm{d}x < \infty$ since $\Omega(x) \geqslant 1$, so surely

$$\int_{-\infty}^{\infty} \frac{\log^+ |\psi(x)|}{1 + x^2} \mathrm{d}x \quad < \quad \infty.$$

This gives us the right to use the theorem from §G.1 of Chapter III (the easier one in that chapter's §E would do just as well) to get

$$\log|\psi(x+\mathrm{i})| \quad \leqslant \quad 12A \quad + \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log|\psi(t)|}{(x-t)^2 + 1} \mathrm{d}t$$

for $x \in \mathbb{R}$. By the inequality between arithmetic and geometric means, the integral on the right is

$$\leqslant \quad 4\log\left(\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\psi(t)|^{1/4}}{(x-t)^2+1} \mathrm{d}t\right) \quad \leqslant \quad 4\log\left(\frac{1}{\pi}\int_{-\infty}^{\infty} |\psi(t)|^{1/4} \mathrm{d}t\right)$$

which, as we just observed, is finite. Therefore $\log|\psi(x+\mathrm{i})| \leqslant$ const., $x \in \mathbb{R}$. One can now conclude that $\psi(x)$ is bounded on $\mathbb{R}$, either by appealing to the third Phragmén–Lindelöf theorem from §C of Chapter III or by simply noting that $|\psi(x)| \leqslant |\psi(x+\mathrm{i})|$ on $\mathbb{R}$ for our function ψ (which has only real zeros). The proof is complete.

Let us now refer to Remark 1, and once more to the *sufficiency* proof for the above theorem. The argument made there furnished, for each $A > 0$, a function $\rho(t)$ satisfying the hypothesis of the lemma with the weight $\Omega(x) = \exp\omega(x)$; comparison of $\omega(x)$ with $\log W(x)$ did not take place until the very end. We can thereby conclude that *the existence, for* $\log W(x)$, *of an* a.e. *majorant* $\omega(x)$ *having the other properties enumerated in the theorem implies, for each* $p < \infty$, *the existence of entire functions* $\psi(z) \not\equiv 0$ *of arbitrarily small exponential type with*

$$\int_{-\infty}^{\infty} |W(x)\psi(x)|^p \mathrm{d}x \quad < \quad \infty.$$

The function $\omega(x)$ with the stipulated properties does not even need to be an actual *majorant* of $\log W(x)$; as long as

$$\int_{-\infty}^{\infty} (\mathrm{e}^{-\omega(x)} W(x))^{r_0} \mathrm{d}x \quad < \quad \infty$$

for some $r_0 > 0$, *we will still, for each* $r < r_0$, *have entire functions* ψ *of the kind described with*

$$\int_{-\infty}^{\infty} |W(x)\psi(x)|^r \,\mathrm{d}x \quad < \quad \infty.$$

This also follows from the lemma; it suffices to take $\Omega(x) = \exp\omega(x)$ and $p = r_0/(r_0 - r)$, and then use Hölder's inequality.

The first of these results should be confronted with one going in the opposite direction that was already pointed out in Remark 1. That says that, *for a continuous weight* $W(x) \geqslant 1$, *the existence of entire functions* $\psi(z) \not\equiv 0$ *of arbitrarily small exponential type making* $W(x)\psi(x)$ *bounded on* $\mathbb{R}$ *implies existence of a majorant* $\omega(x)$ *for* $\log W(x)$ *with the properties specified by the theorem.* Thus, insofar as *continuous* weights are concerned, our theorem's majorant criterion is at the same time a *necessary* condition for the admittance of multipliers (in the narrow L_∞ sense) and a *sufficient* one, albeit in the broader L_p sense only. No additional regularity of the weight (beyond continuity) is involved here.

A very similar observation can be made about the last theorem in §B.3. *Any continuous weight* $W(x) \geqslant 1$ *will admit multipliers in the* L_p *sense* (*with* $p < \infty$) *provided that, for each* $A > 0$, *the smallest superharmonic majorant of*

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|\log W(t)}{|z-t|^2}\,\mathrm{d}t \quad - \quad A|\Im z|$$

is finite. This finiteness is, on the other hand, *necessary for the admittance of multipliers in the* L_∞ *sense by the weight* W. It is worthwhile in this connection to note, finally, the following fact: *for continuous weights* W, *finiteness of the smallest superharmonic majorants just mentioned is equivalent to the existence of an* $\omega(x)$ *enjoying all the properties described by the theorem.* That is an immediate consequence of the next-to-the-last theorem in §B.3 and Remark 1.

Scholium. One way of looking at the theorem on the multiplier is to view it as a *guarantee* of *admittance of multipliers* by smooth even weights $W(x) = \mathrm{e}^{\omega(x)} \geqslant 1$ with

$$\int_{-\infty}^{\infty} \frac{\omega(x)}{1+x^2}\,\mathrm{d}x \quad < \quad \infty$$

under the subsidiary condition that $\tilde{\omega}(x) - Kx$ be *decreasing* on $\mathbb{R}$ for some K, i.e., that

$$\tilde{\omega}'(x) \;\leqslant\; K.$$

As long as the *growth* of $\tilde{\omega}(x)$ is *thus limited, convergence of the logarithmic* *integral* of W is *in itself sufficient.** Referring, however, to the very elementary Paley–Wiener multiplier theorem from §A.1, Chapter X, we see that the convergence is *also sufficient* subject to a *similar requirement* on $\omega(x)$ *itself*, namely that $\omega(x)$ be *increasing* for $x \geqslant 0$.

Part of what this article's theorem does is to *generalize* the first result. As long as $W(x)$ meets the local regularity requirement, *more growth* of $\tilde{\omega}(x)$ is in fact *permissible*; the theorem tells us exactly *how much*. Could not then the Paley–Wiener result be generalized in the same way, so as to allow for a *certain amount of decrease* in $\omega(x)$ for $x \geqslant 0$?

What comes to mind is that perhaps an *analogous generalization* of the second result would carry over. In that way one is led to consider the following conjecture:

Let $W(x) = e^{\omega(x)}$ *with* $\omega(x) \geqslant 0$, $\mathscr{C}_\infty$ *and even. Suppose that*

$$\int_{-\infty}^{\infty} \frac{\omega(x)}{1+x^2}\,dx \quad < \quad \infty,$$

and that for a certain K,

$$\omega'(x) \quad \geqslant \quad -K$$

for all $x > 0$ *outside a set of disjoint intervals* $(a_k,\ b_k) \subseteq (0, \infty)$ *with*

$$\sum_k \left(\frac{b_k - a_k}{a_k}\right)^2 \quad < \quad \infty,$$

for each of which

$$\int_{a_k}^{b_k} (\omega'(x))^-\,dx \quad \leqslant \quad K(b_k - a_k).$$

Then $W(x)$ *admits multipliers.*

This conjecture is *true*. To prove it, one constructs a positive function $w(x)$, *uniformly* Lip 1 *on* $\mathbb{R}$, such that

$$w(x) \quad \geqslant \quad \omega(x)$$

* Without imposition of any local regularity requirement. Indeed, putting $Kt - \tilde{\omega}(t) = \pi\nu(t)$ and then $U(z) = \omega(0) + \int_0^\infty \log|1-(z/t)^2|\,d\nu(t)$, we have $\omega(x) = U(x) \leqslant U(x+i)$ (see p. 503 and the lemmas, p. 516 and 521). If $\varphi(z)$ is the entire function given by $\log|\varphi(z)| = \int_0^\infty \log|1-(z/t)^2|\,d[\nu(t)]$, $|\varphi(x+i)|$ *admits* multipliers in the present circumstances (see lemma, p. 521 and then pp. 546–7). But then $\exp U(x+i)$ does also (see p. 548), and so, finally, does $W(x) = \exp U(x)$.

there, and

$$\int_{-\infty}^{\infty} \frac{w(x)}{1+x^2}\,dx \quad < \quad \infty.$$

By the result in §C, Chapter X, it is *known* that $\exp w(x)$ *admits multipliers*. Hence $W(x) = \exp\omega(x)$ *must also*. The construction of $w(x)$ is outlined in the following two problems.

We may, first of all, ensure that all the intervals (a_k, b_k) lie in $(1, \infty)$ by taking K large enough to begin with (see discussion in first half of the proof of sufficiency for the above theorem). This detail being settled, we take a function $\varphi(x) \geqslant 0$ defined on $[0, \infty)$ as follows:

$$\varphi(x) \quad = \quad K \; - \; (\omega'(x))^- \qquad \text{for} \quad x \in [0,\infty) \;\sim\; \bigcup_k (a_k, b_k);$$

$$\varphi(x) \quad = \quad K \; - \; \frac{1}{b_k - a_k}\int_{a_k}^{b_k} (\omega'(t))^-\,dt \qquad \text{for } a_k \;<\; x \;<\; b_k.$$

We then put

$$P(x) \quad = \quad \int_0^x \{(\omega'(t))^+ \; + \; \varphi(t)\}\,dt$$

and

$$N(x) \quad = \quad \int_0^x \{(\omega'(t))^- \; + \; \varphi(t)\}\,dt$$

getting, in this way, two continuous functions $P(x)$ and $N(x)$, both *increasing* on $[0, \infty]$, with

$$\omega(x) \quad = \quad P(x) \; - \; N(x), \qquad x \geqslant 0.$$

Note that

$$N(x) \; = \; Kx \qquad \text{for } x \in [0,\infty) \;\sim\; \bigcup_k (a_k, b_k);$$

in particular, $N(x) = Kx$ for $0 \leqslant x \leqslant 1$.

Fix now any number $M > K$ and consider the open set

$$\Omega \quad = \quad \Big\{x > 0\colon \; \frac{N(x) - N(\xi)}{x - \xi} \; > \; M(x - \xi)$$

$$\text{for some positive } \xi < x \; (sic!)\Big\}.$$

Ω can be obtained by shining light *up from underneath* the graph of $N(x)$ vs x *from the left*, in a direction of slope M:

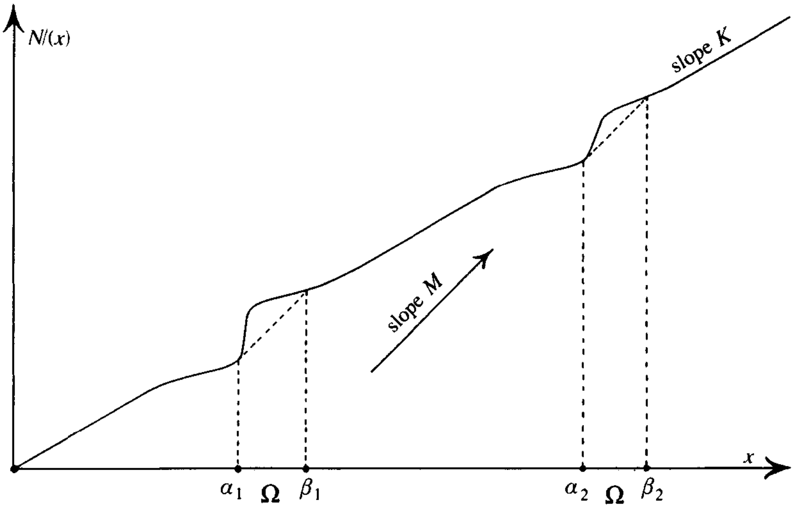

Figure 262

Ω is a *disjoint union* of certain open intervals $(\alpha_l, \beta_l) \subseteq (0, \infty)$ (not to be confounded with the given intervals (a_k, b_k)), and for $x \in (0, \infty) \sim \Omega$, $N'(x) \leqslant M$.

Problem 69

(a) Show that

$$\int_0^\infty \frac{|N(x) - Kx|}{x^2}\,dx \quad < \quad \infty.$$

(Hint: cf. the examination of $\pi v_1(t)$ in the proof of sufficiency for the above theorem.)

(b) Show that the intervals (α_l, β_l) actually lie in $(1, \infty)$.

For the rest of this problem, we make the following construction. Considering any one of the intervals (α_l, β_l), denote by $\mathscr{L}_l$ the line of slope M through the points $(\alpha_l, N(\alpha_l))$ and $(\beta_l, N(\beta_l))$. Then denote by γ_l the *abscissa* of the point where $\mathscr{L}_l$ and the *line of slope K through the origin intersect* (cf. proof of *third* lemma in §D.2, Chapter IX). Note that γ_l may well coincide with α_l or β_l, or even lie *outside* $[\alpha_l, \beta_l]$.

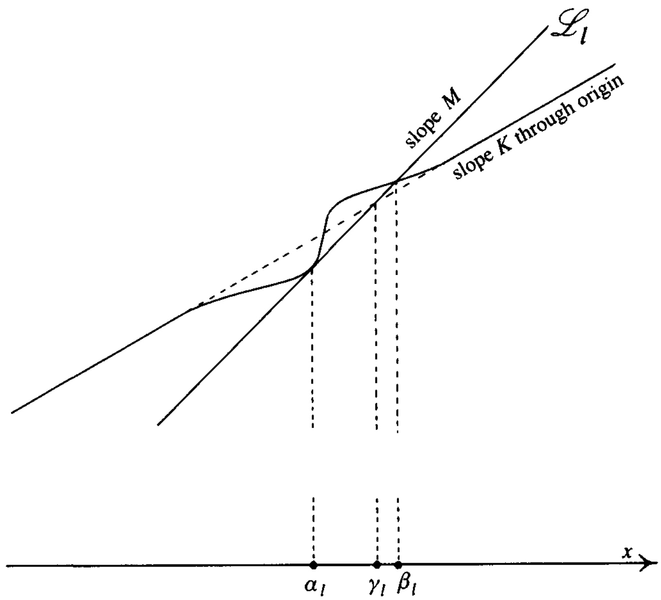

Figure 263

Let R be the set of *indices* l for which γ_l *lies to the right of the midpoint of* $(\alpha_l,\ \beta_l)$, and S the set of those indices for which γ_l lies *to the left* of that midpoint.

(c) Show that $\sum_{l\in S}((\beta_l - \alpha_l)/\alpha_l)^2 < \infty$. (Hint: cf. proof of third lemma, Chapter IX, §D.2. Note that the *difference* between our present construction and the one used there is that *left and right have exchanged rôles*, as have *above and below*!)

(d) Show that if $\eta > 0$, there cannot be infinitely many indices l in R for which $\beta_l - \alpha_l > \eta\alpha_l$. (Hint: It is enough to consider η with

$$0 < \frac{M-K}{2K}\eta < 1.$$

If $(\alpha_l,\ \beta_l)$ is any interval corresponding to an $l \in R$ with $\beta_l - \alpha_l > \eta\alpha_l$, write

$$\alpha_l' = \left(1 - \frac{M-K}{2K}\eta\right)\alpha_l$$

and then estimate

$$\int_{\alpha_l'}^{\alpha_l} \frac{Kx - N(x)}{x^2}\,dx$$

from below. Note that if this situation arises for infinitely many l in R, there must still be infinitely many of those indices for which the intervals (α_l', α_l) are *disjoint.*)

(e*) Show that $\sum_{l\in R}((\beta_l - \alpha_l)/\alpha_l)^2 < \infty$. (Hint: by (d) we may wlog suppose that $((M-K)/2K)(\beta_l - \alpha_l) < \frac{1}{2}\alpha_l$ for all $l \in R$. For those l we then put

$$\alpha_l^* = \alpha_l - \frac{M-K}{2K}(\beta_l - \alpha_l)$$

and estimate each of the integrals

$$\int_{\alpha_l^*}^{\alpha_l} \frac{Kx - N(x)}{x^2}\,dx$$

from below. Starting, then, with an arbitrarily large *finite* subset R' of R, we go *first* to the *rightmost* of the (α_l, β_l) with $l \in R'$, and then make a covering argument like the one in the proof of the third lemma, §D.2, Chapter IX (used when considering the sums *over* S' figuring there), moving, however, *back towards the left instead of towards the right,* and working with the intervals (α_l^*, α_l). This gives a bound on

$$\sum_{l\in R'} \left(\frac{\beta_l - \alpha_l}{\alpha_l}\right)^2$$

independent of the size of R'.)

To finish this problem, we define a function $N_0(x)$ by putting

$$N_0(x) = N(x) \quad \text{for} \quad x \in [0, \infty) \sim \bigcup_l (\alpha_l, \beta_l)$$

and

$$N_0(x) = N(\alpha_l) + M(x - \alpha_l) \quad \text{for} \quad \alpha_l < x < \beta_l.$$

This makes

$$N_0(x) \leqslant N(x) \quad \text{for} \quad x \geqslant 0$$

and

$$N_0'(x) \leqslant M.$$

(f) Show that

$$\int_0^\infty \frac{N(x) - N_0(x)}{x^2}\,dx < \infty.$$

Carrying through the steps of the last problem has given us the increasing functions $P(x)$, $N(x)$ and $N_0(x)$, having the properties indicated above.

Let now

$$w_0(x) \quad = \quad P(x) \;-\; N_0(x) \qquad \text{for } x \geqslant 0.$$

Then

$$w_0(x) \quad \geqslant \quad P(x) \;-\; N(x) \quad = \quad \omega(x), \qquad x \geqslant 0$$

while

$$w_0'(x) \quad \geqslant \quad -N_0'(x) \quad \geqslant \quad -M.$$

At the same time, since

$$\int_0^\infty \frac{\omega(x)}{1+x^2}\,\mathrm{d}x \quad < \quad \infty,$$

we have

$$\int_0^\infty \frac{w_0(x)}{1+x^2}\,\mathrm{d}x \quad < \quad \infty$$

by part (f) of the problem, since

$$w_0(x) \;-\; \omega(x) \quad = \quad N(x) \;-\; N_0(x).$$

Problem 70

Denote by $w(x)$ the *smallest majorant* of $w_0(x)$ on $[0, \infty)$ having the property that

$$|w(x) - w(x')| \;\leqslant\; M|x - x'| \qquad \text{for } x \text{ and } x' \;\geqslant\; 0.$$

The object of this problem is to prove that

$$\int_0^\infty \frac{w(x)}{1+x^2}\,\mathrm{d}x \quad < \quad \infty.$$

(a) Given $\eta > 0$, show that one cannot have $w_0(x) \;>\; \eta x$ for arbitrarily large x. (Hint: Given any such $x > 0$, estimate

$$\int_x^{(1+(\eta/2M))x} \frac{w_0(t)}{t^2}\,\mathrm{d}t$$

from below. Cf. problem 69(d).)

(b) Hence show that $w(x) \;<\; \infty$ for $x \geqslant 0$ and that in $(0, \infty)$, $w(x) > w_0(x)$ on a certain set of disjoint *bounded* open intervals lying therein.

Continuing with this problem we take *just the intervals from* (b) *that lie in* $(1, \infty)$, and denote them by (A_n, B_n), with $n \;=\; 1, 2, 3, \ldots$. In order to verify the desired property of $w(x)$, it is enough to show that

$$\int_{A_0}^{\infty} \frac{w(x)}{x^2}\,\mathrm{d}x \quad < \quad \infty,$$

where $A_0 \;=\; \inf_{n\geqslant 1} A_n,\;$ a quantity $\geqslant 1$. In

$$(A_0,\ \infty) \;\sim\; \bigcup_{n=1}^{\infty} (A_n,\ B_n)$$

we have $w(x) \;=\; w_0(x)$, where, as we know

$$\int_{1}^{\infty} \frac{w_0(x)}{x^2}\,\mathrm{d}x \quad < \quad \infty.$$

It is therefore only necessary for us to prove that

$$\sum_{n\geqslant 1} \int_{A_n}^{B_n} \frac{w(x)}{x^2}\,\mathrm{d}x \quad < \quad \infty.$$

Note that for each $n \geqslant 1$, we have

$$w(A_n) \;=\; w_0(A_n),$$
$$w(B_n) \;=\; w_0(B_n)$$

and

$$w(x) \;=\; w_0(A_n) \;+\; M(x-A_n) \qquad \text{for } A_n \;\leqslant\; x \;\leqslant\; B_n.$$

(c) Show that $B_n/A_n \;\longrightarrow\; 1$ as $n \longrightarrow \infty$. (Hint: If $\eta > 0$ and there are infinitely many n with $B_n/A_n \;\geqslant\; 1+\eta$, the corresponding A_n must tend to ∞ since the $(A_n,\ B_n)$ are disjoint. Observe that for such n, since $w_0(x) \;\geqslant\; \omega(x) \;\geqslant\; 0$,

$$w_0(B_n) \;\geqslant\; M\eta B_n/(1+\eta).$$

Refer to part (a).)

(d) For each $n \geqslant 1$, write

$$B_n^* \;=\; B_n \;+\; (B_n - A_n).$$

Show then that

$$\int_{A_n}^{B_n} \frac{w(x)}{x^2}\,\mathrm{d}x \quad \leqslant \quad \left(\frac{B_n^*}{A_n}\right)^2 \int_{B_n}^{B_n^*} \frac{w_0(x)}{x^2}\,\mathrm{d}x.$$

Hint:

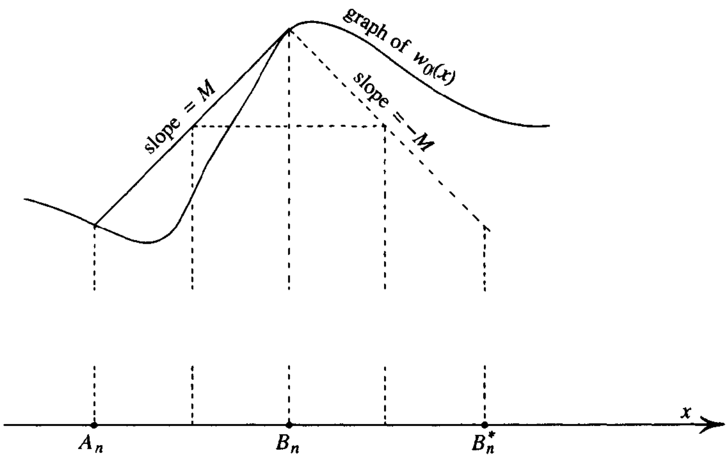

Figure 264

(e) Let us agree to call an interval (A_n, B_n) *special* if

$$w(A_n) \quad \geqslant \quad M(B_n - A_n).$$

Show then that if $(A_n,\ B_n)$ is special,

$$\int_{A_n}^{B_n} \frac{w(x)}{x^2}\,\mathrm{d}x \quad \leqslant \quad 3\left(\frac{B_n}{A_n}\right)^2 \int_{A_n}^{B_n} \frac{w_0(x)}{x^2}\,\mathrm{d}x.$$

Hint:

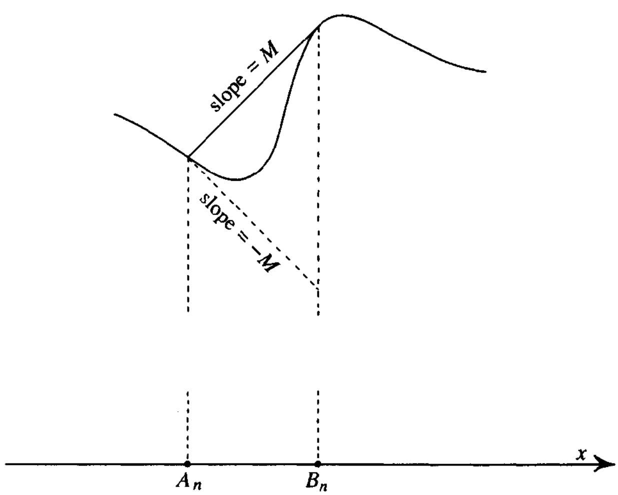

Figure 265

(f) Given any *finite* set T of integers $\geqslant 1$, obtain an upper bound *independent of* T on

$$\sum_{n\in T}\int_{A_n}^{B_n}\frac{w(x)}{x^2}\,dx,$$

hence showing that

$$\sum_{n\geqslant 1}\int_{A_n}^{B_n}\frac{w(x)}{x^2}\,dx \quad < \quad \infty.$$

(Procedure: Reindex the $(A_n,\ B_n)$ with $n\in T$ so as to have *n increase* from 1 up to some finite value as those intervals go *towards the right*. By (c), the ratios B_n^*/A_n must be bounded above by a quantity independent of T. Use then the result from (d) to estimate

$$\int_{A_1}^{B_1}\frac{w(x)}{x^2}\,dx.$$

Show next that *any interval* $(A_n,\ B_n)$ *entirely contained in* $(B_1,\ B_1^*)$ *must be special.* For such intervals, the result from (e) may be used to estimate

$$\int_{A_n}^{B_n}\frac{w(x)}{x^2}\,dx.$$

If there is an interval $(A_m,\ B_m)$ *intersecting* with $(B_1,\ B_1^*)$ but *not* lying *therein* ($m\in T$), $(B_1,\ B_1^*)$ and $(B_m,\ B_m^*)$ are certainly *disjoint*, and we may again use the result of (d) to estimate

$$\int_{A_m}^{B_m}\frac{w(x)}{x^2}\,dx.$$

Then look to see if there are any $(A_n,\ B_n)$ entirely contained in $(B_m,\ B_m^*)$ and keep on going in this fashion, *moving steadily towards the right*, until all the $(A_n,\ B_n)$ with $n\in T$ are accounted for.)

The function $w(x)$ furnished by the constructions of these two problems is finally extended from $[0,\infty)$ to all of $\mathbb{R}$ by making it even. Then we will have

$$|w(x)-w(x')| \;\leqslant\; M|x-x'| \qquad \text{for } x \text{ and } x' \text{ in } \mathbb{R},$$

$$w(x) \;\geqslant\; \omega(x) \qquad \text{on } \mathbb{R},$$

and

$$\int_{-\infty}^{\infty}\frac{w(x)}{1+x^2}\,dx \quad < \quad \infty,$$

this last by problem 70. Our w thus has the properties we needed, and $W(x) = \exp\omega(x)$ admits multipliers, as explained at the beginning of this scholium.

One might hope to turn *around* the result just obtained and somehow show, in parallel to the necessity part of this article's theorem, that, for admittance of multipliers by a weight $W(x) \geqslant 1$ meeting the local regularity requirement, existence of a $\mathscr{C}_\infty$ even ω with

$$e^{\omega(x)} \quad \geqslant \quad W(x) \qquad \text{on } \mathbb{R}$$

enjoying the other properties enumerated in the conjecture is *necessary*.

Problem 71

Show that such a proposition would be *false*. (Hint: Were such an ω to exist, the preceding constructions would give us an even uniformly Lip 1 $w(x) \geqslant \omega(x)$ for which

$$\int_{-\infty}^{\infty} \frac{w(x)}{1+x^2}\,dx \quad < \quad \infty.$$

Modify $w(x)$ in smooth fashion near 0 so as to obtain a new uniformly Lip 1 even function $w_1(x) \geqslant 0$, equal to zero at the origin and $O(x^2)$ *near* there, agreeing with $w(x)$ for $|x| \geqslant 1$, say. Then

$$\int_0^{\infty} \frac{w_1(x)}{x^2}\,dx \quad < \quad \infty.$$

Refer to problem 62 (end of §C.4) and then to the example of §D.4.)

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prime minister of Sweden, on
the Friday, February 28th
preceding.

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