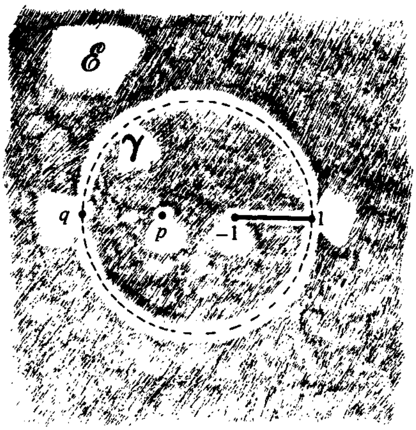

Figure 118

In this reduction, 0 goes to a point p on the real axis, $p < -1$, and the circle Γ goes to another, γ, having $[q, 1]$ as its diameter, where $q < p$. $g(z, 0)$ is of course *equal* to the Green's function for $\mathscr{E}$ with pole at p.

We have

$$|p| - 1 \;\leqslant\; \frac{|\mathscr{O}_0|}{|E_1|/2} \;\leqslant\; \frac{B}{\delta}$$

and

$$|q| - |p| \;\geqslant\; \frac{2|E_0|}{|E_1|} \;\geqslant\; \frac{2\delta}{\Delta}.$$

Therefore the Green's function for $\mathscr{E}$ with pole at p is bounded above on γ by some number α depending on B/δ and $2\delta/\Delta$. (The nature of this dependence could be worked out by mapping $\mathscr{E}$ conformally onto $\{1 < |w| \leqslant \infty\}$; we, however, do not need to know it.) This means that $g(z, 0) \leqslant \alpha$ on Γ and finally $G(z, 0) \leqslant \alpha$, $z \in \Gamma$.

This being verified, we take the centre m of Γ and, with R equal to that circle's radius, map the exterior of Γ conformally onto the domain $\mathscr{E}$ just considered by taking z to $w = \frac{1}{2}\{(z-m)/R + R/(z-m)\}$. That mapping takes Γ to the slit $E_1' = [-1, 1]$ *and each of the components* E_n of $\partial\mathscr{D}$, $n \neq 0$, 1, *onto segments* E_n' *on the real axis.* The function $\varphi(z) = \frac{1}{2}\{(z-m)/R + R/(z-m)\}$ thus takes

$$\mathscr{D}_\Gamma \;=\; \mathscr{D} \cap \{|z-m| > R\}$$

conformally onto a domain

$$\mathscr{D}' \;=\; \mathbb{C} \sim \bigcup_{-\infty}^{\infty}{}' E_n':$$

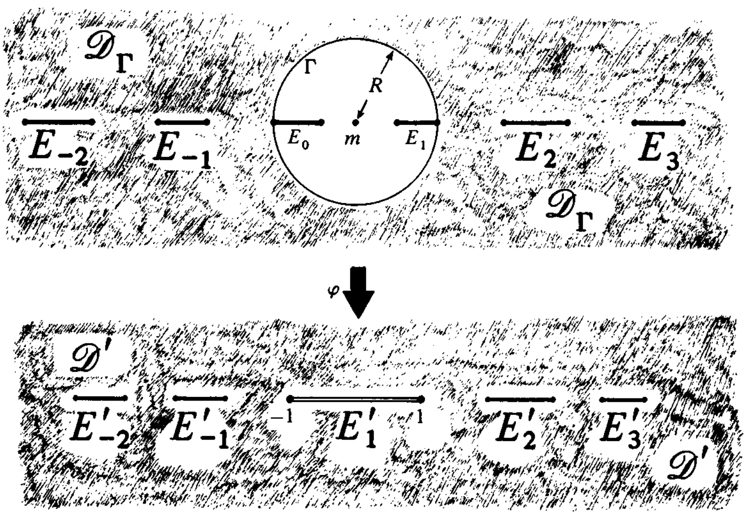

Figure 119

Define a harmonic function $\Omega(w)$ for $w \in \mathscr{D}'$ by putting

$$\Omega(\varphi(z)) \;=\; G(z, 0)$$

for $z \in \mathscr{D}_\Gamma$. $\Omega(w)$ is evidently bounded in $\mathscr{D}'$, and has boundary value *zero* on *each* of the components $E'_n (= \varphi(E_n))$ of $\partial\mathscr{D}'$, *save on* E'_1. $\Omega(w)$ *is*, however, continuous up to the latter one, and *on* it

$$\Omega(w) \;\leqslant\; \alpha,$$

since $E'_1 = \varphi(\Gamma)$ and $G(z,0) \leqslant \alpha$ on Γ. We therefore have $\Omega(w) \leqslant \alpha\omega_{\mathscr{D}'}(E'_1, w)$ in $\mathscr{D}'$, where $\omega_{\mathscr{D}'}(\ ,w)$ denotes *harmonic measure* for $\mathscr{D}'$.

The set

$$E' \;=\; \bigcup_{-\infty}^{\infty}{}' E'_n$$

has, however, the properties specified for our sets E at the beginning of this §. Indeed, for real x,

$$\varphi(x) \;=\; \frac{1}{2R}x - \frac{m}{2R} + O\left(\frac{1}{x}\right)$$

when $|x|$ is large, with R lying between the two numbers 2δ and $B/2 + 2\Delta$. Hence, each of the intervals $E'_n (n \neq 0)$ is of the form $[a'_n - \delta'_n,\ a'_n + \delta'_n]$, where, for certain numbers A', B', γ' and Δ' depending on the original A, B, δ and Δ,

$$0 < A' < a'_{n+1} - a'_n < B'$$

and

$$0 < \delta' < \delta'_n < \Delta'.$$

(Again, the exact *form* of the dependence does not concern us here.) *We can therefore apply Carleson's theorem from the previous article to the domain* $\mathscr{D}'$, and find that

$$\omega_{\mathscr{D}'}(E'_1, u) \leqslant \frac{C'}{1+u^2}, \qquad u \in \mathbb{R},$$

with a constant C' depending on A', B', δ' and Δ' and hence, finally, on A, B, δ and Δ. Thence, in view of the previous relation,

$$\Omega(u) \leqslant \frac{\alpha C'}{1+u^2} \quad \text{for} \quad u \in \mathbb{R},$$

i.e.

$$G(x, 0) \leqslant \frac{\alpha C'}{1+(\varphi(x))^2}$$

for real x lying *outside* the circle Γ. Using the fact that $0 \in \mathcal{O}_0$ (whence $|m| \leqslant B + 2\Delta$), the bounds on R given above, and the asymptotic formula for $\varphi(x)$, we see that the right side of the preceding inequality is in turn

$$\leqslant \frac{C}{1+x^2}$$

with a constant C depending only on A, B, δ and Δ. Thus

$$G(x, 0) \leqslant \frac{C}{1+x^2}$$

for real x outside of E_0, $\mathcal{O}_0$ and E_1. But this also holds on E_0 and E_1 since $G(x, 0) = 0$ on those sets! So it holds for real x outside of $\mathcal{O}_0$, which is what we had to prove. We are done.

Problem 16

Let $E \subseteq \mathbb{R}$ fulfill the conditions set forth at the beginning of this §, and assume that $0 \in \mathcal{O}_0$. Let $\omega_{\mathscr{D}}(\ , z)$ be *harmonic measure* for $\mathscr{D} = \mathbb{C} \sim E$. *Prove Benedicks' theorem*, which says that there is a constant C depending only on the four numbers A, B, δ and Δ associated with E, such that, for t in any component

$$E_n = [a_n - \delta_n,\ a_n + \delta_n]$$

of E, and $n \neq 0, 1$,

$$d\omega_{\mathscr{D}}(t,0) \leqslant \frac{C}{1+t^2}\cdot\frac{dt}{\sqrt{(\delta_n^2-(t-a_n)^2)}}.$$

(This is a most beautiful result, by the way!) (Hint: Let G be the Green's function for $\mathscr{D}$. According to a classical elementary formula, if, for instance, we consider points t_+ lying on the *upper edge* of E_n, we have

$$\frac{d\omega_{\mathscr{D}}(t_+,0)}{dt} = \frac{1}{2\pi}G_y(t_+,0) = \lim_{\Delta y\to 0+}\frac{G(t+i\Delta y,0)}{2\pi\Delta y},$$

since $G(t,0)=0$. (Green *introduced* the functions bearing his name for this very reason!) Take the ellipse Γ given by the equation

$$\frac{(x-a_n)^2}{2\delta_n^2}+\frac{y^2}{\delta_n^2} = 1$$

and compare $G(z,0)$ with

$$U(z) = \log\left|\frac{z-a_n}{\delta_n}+\sqrt{\left(\left(\frac{z-a_n}{\delta_n}\right)^2-1\right)}\right|$$

on Γ. Note that $G(x,0)$ and $U(x)$ *both vanish* on E_n, that $U(z)$ is *harmonic* in the region $\mathscr{E}$ between E_n and Γ, and that $G(z,0)$ is at least *subharmonic* there (not *necessarily* harmonic because some of the E_k with $k \neq n$ may intrude into $\mathscr{E}$).)

The work in Chapter VI frequently involved entire functions of exponential type bounded on the real axis. If $f(z)$ is such a function, of exponential type A say, and we know that

$$|f(x)| \leqslant 1, \quad x\in\mathbb{R},$$

we can deduce that

$$|f(z)| \leqslant e^{A|\Im z|}$$

for all z. This follows by the *third* Phragmén–Lindelöf theorem of Chapter III, §C, whose *proof* depends on the availability of the function $|\Im z|$, *harmonic and* $\geqslant 0$ *in each of the half planes* $\{\Im z>0\}$, $\{\Im z<0\}$, *and zero on the real axis.*

Suppose now that we are presented with such a function $f(z)$, known to be *bounded* (with, however, an *unknown bound*) on $\mathbb{R}$, such that, for some closed $E \subseteq \mathbb{R}$,

$$|f(x)| \leqslant 1, \quad x\in E.$$

If there is a function $Y(z)$, *harmonic in* $\mathscr{D} = \mathbb{C} \sim E$, *having boundary value zero on E and such that* $Y(z) \geqslant |\Im z|$, $z\in\mathscr{D}$, we can argue as in the proof of the

Phragmén–Lindelöf theorem just mentioned, and conclude that

$$|f(z)| \leqslant e^{AY(z)}$$

for $z \in \mathscr{D}$ if $f(z)$ is of exponential type A.* We are therefore interested in the *existence* of such functions $Y(z)$ for given closed sets $E \subseteq \mathbb{R}$.

In order to avoid situations involving irregular boundary points for the Dirichlet problem, whose investigation has nothing to do with the material of the present book, we *limit* the following discussion to closed sets E which can be expressed as *disjoint unions of segments on* $\mathbb{R}$ *not accumulating at any finite point*. We do *not*, however, assume in that discussion that the sets E have the form specified at the beginning of this §.

Definition. A Phragmén–Lindelöf function $Y(z)$ for $\mathscr{D} = \mathbb{C} \sim E$ is one *harmonic* in $\mathscr{D}$ and *continuous* up to E, such that

(i) $Y(x) = 0, \quad x \in E$

(ii) $Y(z) \geqslant |\Im z|, \quad z \in \mathscr{D}$,

(iii) $Y(iy) = |y| + o(|y|)$ for $y \to \pm\infty$.

It turns out that for given closed $E \subseteq \mathbb{R}$ of the form just described, the existence of $Y(z)$ is governed by the *behaviour of the Green's function* $G(z, w)$ for $\mathscr{D} = \mathbb{C} \sim E$. Before going into this matter, let us mention a simple example (not without its own usefulness) which the reader should keep in mind.

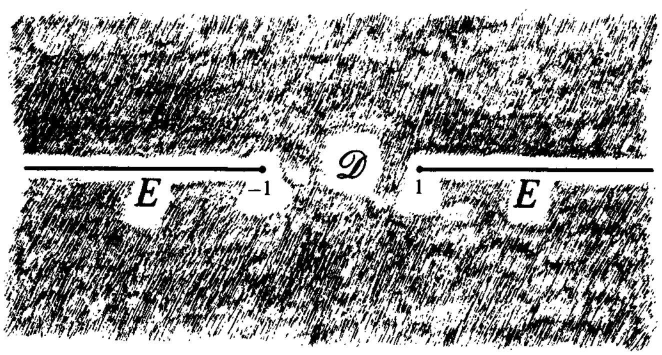

Figure 120

* In fact, *boundedness of* f on $\mathbb{R}$ is *not necessary* here. If $|f(x)| \leqslant 1$, $x \in E$, and $|f(z)| \leqslant C\exp(A|z|)$, the function $v(z) = \log|f(z)| - (A\sec\delta)\,Y(z)$ is subharmonic in $\mathscr{D}$ and bounded above on each of the lines $x = \pm y\tan\delta$ – here, $0 < \delta < \pi/2$. Since $v(z) \leqslant \text{const.}|z|$ in $\mathscr{D}$, the *second* Phragmén–Lindelöf theorem of Chapter III §C shows that v is bounded above in the vertical sectors $|x| < \pm y\tan\delta$. Because $v(x) \leqslant 0$ on E, the *proof* of that same theorem can be adapted without change to show that v is *also* bounded above in $\mathscr{D} \cap \{x > |y|\tan\delta\}$ and $\mathscr{D} \cap \{x < -|y|\tan\delta\}$, even though the latter domains are not full sectors. Therefore v is bounded above in $\mathscr{D}$, so by the *first* theorem of §C, Chapter III, $v(z) \leqslant 0$ in $\mathscr{D}$. Hence $|f(z)| \leqslant \exp(A\sec\delta \cdot Y(z))$, $z \in \mathscr{D}$, and, making $\delta \to 0$, we get $|f(z)| \leqslant \exp(AY(z))$.

Here, $E = (-\infty, -1] \cup [1, \infty)$. In $\mathscr{D} = \mathbb{C} \sim E$, we can put

$$Y(z) = \Im(\sqrt{(z^2 - 1)}),$$

using the branch of the square root which is *positive imaginary* for $z \in (-1, 1)$. It is easy to check that this $Y(z)$ is a Phragmén–Lindelöf function for $\mathscr{D}$.

The Green's function $G(z, w)$ for one of our domains $\mathscr{D}$ enjoys a *symmetry property*:

$$G(z, w) = G(w, z), \qquad z, w \in \mathscr{D}.$$

The reader who does not remember how this is proved may find a proof, general enough to cover our situation, at the end of this article. It is convenient to define $G(z, w)$ for all z and w in $\bar{\mathscr{D}}$ (which *here* is just $\mathbb{C}$!) by taking $G(z, w) = 0$ if *either* z or w belongs to $\partial\mathscr{D}$. Then we have

$$G(z, w) = G(w, z) \qquad \text{for} \quad z, w \in \bar{\mathscr{D}}.$$

(N.B. $G(z, w)$ as thus defined is not *quite* continuous from $\bar{\mathscr{D}} \times \bar{\mathscr{D}}$ to $[0, \infty]$ (*sic*!). We can take sequences $\{z_n\}$ and $\{w_n\}$ of points in $\mathscr{D}$, both tending to limits on $E = \partial\mathscr{D}$, but with $|z_n - w_n| \underset{n}{\longrightarrow} 0$ sufficiently rapidly to make $G(z_n, w_n) \underset{n}{\longrightarrow} \infty$.)

The connection between $G(z, w)$ and $Y(z)$ (when the latter exists) can be made to depend on the elementary formula

$$\lim_{R \to \infty} \int_{-R}^{R} \log\left|1 - \frac{z}{t}\right| dt = \pi |\Im z|,$$

which may be derived by contour integration. The reader is invited (nay, urged!) to do the computation. Here is the result.

Theorem. *A Phragmén–Lindelöf function $Y(z)$ exists for $\mathscr{D}$, a domain of the kind considered here, iff*

$$\int_{-\infty}^{\infty} G(z, t)\, dt < \infty$$

for some *$z \in \mathscr{D}$, G being the Green's function for that domain. If the integral just written converges for* any *such z, it converges for* all *$z \in \mathscr{D}$, and then*

$$Y(z) = |\Im z| + \frac{1}{\pi} \int_{-\infty}^{\infty} G(z, t)\, dt.$$

Remark. In his 1980 *Arkiv* paper, Benedicks has versions of this result for $\mathbb{R}^{n+1}$

Proof of theorem. The idea is very simple, and is expressed by the identity

$$|\Im z| + \frac{1}{\pi}\int_{-\infty}^{\infty} G(z,t)\,dt$$
$$= \frac{1}{\pi}\int_{-A}^{A}\left(\log\frac{1}{|t|} + \log|z-t| + G(z,t)\right)dt$$
$$+ \frac{1}{\pi}\int_{A}^{\infty}\log\left|1-\frac{z^2}{t^2}\right|dt + \frac{1}{\pi}\int_{|t|\geqslant A} G(z,t)\,dt,$$

an obvious consequence of the formula just mentioned. Here, $A > 0$ is arbitrary.

Suppose, indeed, that the left-hand integral is convergent. That integral then equals a positive harmonic function in each of the half planes $\{\Im z > 0\}$, $\{\Im z < 0\}$, and we can use Harnack to show that it is $o(|y|)$ for $z = iy$ and $y \to \pm\infty$. Denoting the left side of our identity by $Y(z)$, we thus see that $Y(z)$ is harmonic in the upper and lower half planes and has property (iii). It is clear that $Y(z)$ has the properties (i) and (ii). Only the harmonicity of $Y(z)$ at points of $\mathscr{D}\cap\mathbb{R}$ remains to be verified. This, however, can be checked in the neighborhood of any such point by taking $A > 0$ sufficiently large and looking at the *right side* of our identity. The *first* right-hand term will be harmonic in $\mathscr{D}\cap(-A,A)$, because, for each t therein, the logarithmic pole of $G(z,t)$ at t is cancelled by the term $\log|z-t|$. The *second* term on the right is *clearly* harmonic for $|z| < A$, and the *third* harmonic in $\mathscr{D}\cap(-A,A)$.

This explanation will probably satisfy the experienced analyst. The general mathematical reader may, however, well desire more justification, based if possible on general principles, so that he or she may avoid having to search through specialized books on potential theory. We proceed to furnish this justification. Its details make the following development somewhat long.

Let us begin with a preliminary remark. Suppose we have any open subset $\mathcal{O}$ of $\mathscr{D}$, and a compact $F \subseteq \bar{\mathscr{D}}$ *disjoint* from $\mathcal{O}$. By the symmetry of G, $G(z,w) = G(w,z)$ is, for each *fixed* $z\in\mathcal{O}$, *continuous* (as a function of w) on F, so, if μ is any finite positive measure on F, the integral

$$\int_F G(z,w)\,d\mu(w)$$

is obtainable as a limit of Riemann sums in the usual way. As a *function* of z, *any one* of those sums is *positive and harmonic in* $\mathcal{O}$. So the *integral*, being a *pointwise limit* of such functions (of z), *is itself* a *positive and harmonic function of* z in $\mathcal{O}$. We will make repeated use of this observation.

Suppose, now, that $\int_{-\infty}^{\infty} G(z,t)\mathrm{d}t < \infty$ for some *non-real* z, say wlog that

$$\int_{-\infty}^{\infty} G(\mathrm{i},t)\mathrm{d}t \;<\; \infty.$$

For each N, the function

$$H_N(z) \;=\; \int_{-N}^{N} G(z,t)\mathrm{d}t$$

is positive and harmonic in both $\{\Im z > 0\}$ and $\{\Im z < 0\}$ according to the remark just made. Therefore, since $G(z,t) \geqslant 0$, $H_{N+1}(z) \geqslant H_N(z)$, and

$$H(z) \;=\; \lim_{N\to\infty} H_N(z)$$

is either *harmonic* (and finite!) in $\{\Im z > 0\}$ or else *everywhere infinite* there. Because $H(\mathrm{i}) < \infty$, the first alternative holds, and $H(z)$ is then *also* finite (and harmonic) in $\Im z < 0$, since obviously

$$G(z,t) \;=\; G(\bar{z},t)$$

for real t, $E \;=\; \partial\mathscr{D}$ being on $\mathbb{R}$.

Consider now some real $x_0 \notin E$. Take $A > \max(|x_0|, 1)$. The integrals $\int_{-A}^{A} G(x_0,t)\mathrm{d}t$ and $\int_{-A}^{A} G(\mathrm{i},t)\mathrm{d}t$ are *both finite*, so we can show that $\int_{-\infty}^{\infty} G(x_0,t)\mathrm{d}t$ and $\int_{-\infty}^{\infty} G(\mathrm{i},t)\mathrm{d}t$ are either both finite or else both infinite by comparing

$$\int_{|t|\geqslant A} G(x_0,t)\,\mathrm{d}t$$

and

$$\int_{|t|\geqslant A} G(\mathrm{i},t)\,\mathrm{d}t.$$

In $\mathscr{D}_A = \mathscr{D} \cap \{|z| < A\}$, the function $\int_{|t|\geqslant A} G(z,t)\mathrm{d}t$ is the limit of the increasing sequence of functions

$$\int_{A\leqslant|t|\leqslant N} G(z,t)\mathrm{d}t,$$

each of which is *positive and harmonic in* $\mathscr{D}_A$. So $\int_{|t|\geqslant A} G(z,t)\mathrm{d}t$ is either *harmonic* (and finite) in $\mathscr{D}_A$, or else everywhere infinite there. It is thus *finite* for $z = \mathrm{i}$ *if and only if* it is finite for $z = x_0$. We see that $\int_{-\infty}^{\infty} G(x_0,t)\mathrm{d}t < \infty$ iff $\int_{-\infty}^{\infty} G(\mathrm{i},t)\mathrm{d}t < \infty$, and, if this inequality holds,

$$H(z) \;=\; \int_{-\infty}^{\infty} G(x,t)\mathrm{d}t$$

is finite for every $z \in \mathscr{D}$.

If $H(z)$ *is* finite, let us show that

$$H(\mathrm{i}y) = \mathrm{o}(|y|) \quad \text{for} \quad y \to \pm\infty.$$

Pick any large N, and write

$$(*)\qquad H(\mathrm{i}y) \;=\; \int_{|t|\leqslant N} G(\mathrm{i}y,t)\mathrm{d}t \;+\; \int_{|t|>N} G(\mathrm{i}y,t)\mathrm{d}t.$$

Since $H(\mathrm{i}) < \infty$ we can, given any $\varepsilon > 0$, take N so large that $\int_{|t|\geqslant N} G(\mathrm{i},t)\,\mathrm{d}t < \varepsilon$. The function $\int_{|t|\geqslant N} G(z,t)\mathrm{d}t$ is, by the previous discussion, *positive and harmonic* in $\{\Im z > 0\}$. Therefore, by Harnack's theorem,*

$$\int_{|t|\geqslant N} G(\mathrm{i}y,t)\mathrm{d}t \;\leqslant\; y\int_{|t|\geqslant N} G(\mathrm{i},t)\mathrm{d}t \;<\; \varepsilon y$$

for $y > 1$. This takes care of the *second* term on the right in $(*)$.

The *first* term from the right side of $(*)$ remains; our claim is that it is *bounded*. This (and more) follows from a simple estimate which will be used several times in the proof.

Take any component E_0 of $E = \partial\mathscr{D}$, and put $\mathscr{D}_0 = (\mathbb{C} \sim E_0)\cup\{\infty\}$. If E_0 is of *infinite length*, replace it *by any segment of length* 2 *thereon* in this last expression. We have $\mathscr{D} \subseteq \mathscr{D}_0$, so, if $g(z,w)$ is the Green's function for $\mathscr{D}_0$,

$$G(z,w) \;\leqslant\; g(z,w), \qquad z, w\in\mathscr{D}$$

(cf. beginning of the proof of *first* theorem in this article). We compute $g(z,w)$ by first mapping $\mathscr{D}_0$ conformally onto the unit disk $\{|\zeta| < 1\}$, thinking of ζ as a new *coordinate variable* for $\mathscr{D}_0$:

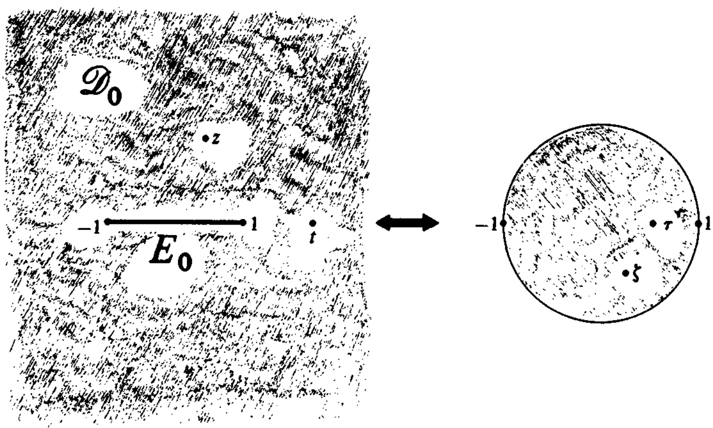

Figure 121

Say, for instance, that E_0 is $[-1, 1]$ so that we can use $z = \frac{1}{2}(\zeta + 1/\zeta)$. Then, if $t\in\mathscr{D}_0$ is

* Actually, by the Poisson representation for $\{\Im z > 0\}$ of functions positive and harmonic there. Using the ordinary form of Harnack's inequality gives us a factor of $2y$ instead of y on the right. That, of course, makes no difference in this discussion.

real, we can put $t = \frac{1}{2}(\tau + 1/\tau)$, where $-1 < \tau < 1$, and, in terms of ζ and τ,

$$g(z, t) = \log\left|\frac{1 - \tau\zeta}{\zeta - \tau}\right|,$$

the expression on the right being simply the Green's function for the unit disk.

If $N > 1$, we have

$$\int_{|t| \leqslant N} G(z, t)\mathrm{d}t \leqslant \int_{1 \leqslant |t| \leqslant N} g(z, t)\mathrm{d}t,$$

since $G(z, t)$ and $g(z, t)$ *vanish* for $t \in E_0 = [-1, 1]$. For $1 \leqslant |t| \leqslant N$, the parameter τ satisfies $C_N \leqslant |\tau| \leqslant 1$, $C_N > 0$ being a number depending on N which we need not calculate. Also, for such t,

$$\mathrm{d}t = -\frac{1}{2}\left(\frac{1 - \tau^2}{\tau^2}\right)\mathrm{d}\tau.$$

Therefore,

$$(\dagger) \qquad \int_{|t| \leqslant N} G(z, t)\mathrm{d}t \leqslant \frac{1}{2}\int_{C_N \leqslant |\tau| \leqslant 1} \log\left|\frac{1 - \tau\zeta}{\zeta - \tau}\right|\left(\frac{1 - \tau^2}{\tau^2}\right)\mathrm{d}\tau.$$

Since $C_N > 0$, the right side is clearly bounded for $|\zeta| < 1$; we see already that the *first right-hand term of* $(*)$ *is bounded*, verifying our claim.

As we have already shown, the *second* term on the right in $(*)$ will be $\leqslant \varepsilon y$ for $y \geqslant 1$ if N is large enough. Combining this result with the preceding, we have, from $(*)$,

$$H(\mathrm{i}y) \leqslant O(1) + \varepsilon y, \quad y > 1,$$

so, since $\varepsilon > 0$ is arbitrary, $H(\mathrm{i}y) = o(|y|)$, $y \to \infty$. Because $H(\bar{z}) = H(z)$, the same holds good for $y \to -\infty$.

Having established this fact, let us return for a moment to $(\dagger)$. For each τ, $C_N \leqslant |\tau| < 1$,

$$\log\left|\frac{1 - \tau\zeta}{\zeta - \tau}\right| \longrightarrow 0 \quad \text{as} \quad |\zeta| \longrightarrow 1.$$

Starting from this relation, one can, by a straightforward argument, check that

$$\int_{C_N \leqslant |\tau| \leqslant 1} \log\left|\frac{1 - \tau\zeta}{\zeta - \tau}\right|\left(\frac{1 - \tau^2}{\tau^2}\right)\mathrm{d}\tau \longrightarrow 0$$

as $|\zeta| \to 1$. (One may, for instance, break up the integral into two pieces.)

Problem 17 (a)
Carry out this verification

This means, by $(\dagger)$, that

$$\int_{|t| \leqslant N} G(z, t)\,\mathrm{d}t \longrightarrow 0$$

when $z \in \mathscr{D}$ *tends to any point of* E_0. We could, however, have taken E_0 to be *any of the components of* E with finite length, or any segment of length 2 on one of the unbounded ones (if there are any); that would not have essentially changed the above argument.* *Hence*

$$\int_{|t| \leq N} G(z,t)\,dt$$

tends to zero whenever z *tends to any point on* $E = \partial\mathscr{D}$ (besides being *bounded* in $\mathscr{D}$).

We can now prove that

$$H(z) \;=\; \int_{-\infty}^{\infty} G(z,t)\,dt \;\longrightarrow\; 0$$

whenever z tends to any point x_0 of E. *Given* such an x_0, take a circle γ about x_0 so small that *precisely one* of the components of E (the one containing x_0) cuts γ, passing into its inside:

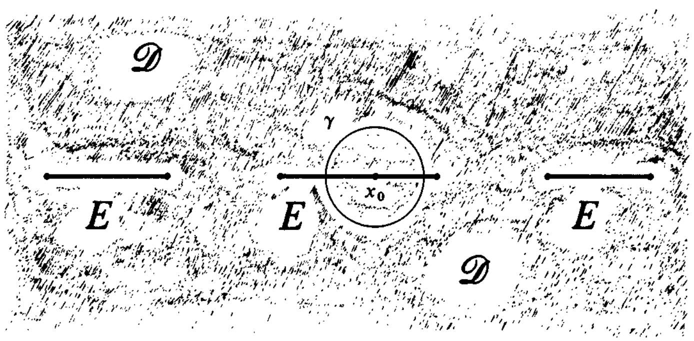

Figure 122

If x_0 is an endpoint of one of the components of E, our picture looks like this:

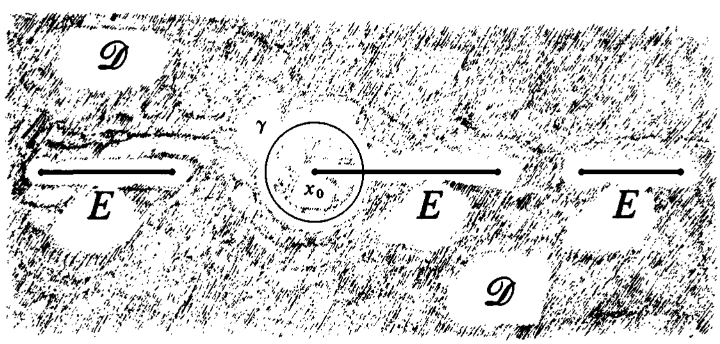

Figure 123

Call $\mathscr{D}_\gamma$ the part of $\mathscr{D}$ lying inside γ, and E_γ the part of E therein.

* As long as $E_0 \subseteq \{|t| < N\}$. If this is not so, we can *increase* N until the argument in the text applies. Since that only makes the integral in question *larger*, the one corresponding to the original value of N must (*a fortiori*!) have the asserted

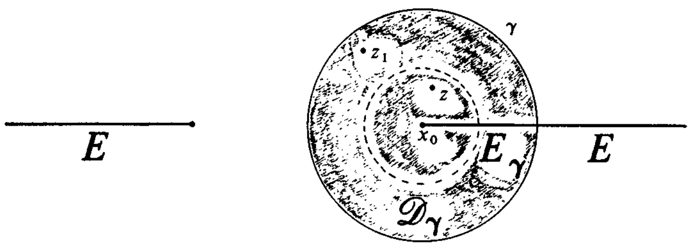

Figure 124

Fix any $z_1 \in \mathscr{D}_\gamma$. Then, there is a constant K, depending on z_1, such that, for any function $V(z)$, *positive* and *harmonic* in $\mathscr{D}_\gamma$ and continuous on its closure, *with* $V(x) = 0$ *on* E_γ, we have $V(z) \leqslant KV(z_1)$ for $|z - x_0| < \frac{1}{2}$ radius of γ.

Problem 17 (b)
Prove the statement just made.

This being granted, choose N so large that

$$\int_{|t| \geqslant N} G(z_1, t)\,\mathrm{d}t < \varepsilon/K,$$

ε being any number > 0. For each $M > N$, the function

$$V_M(z) = \int_{N \leqslant |t| \leqslant M} G(z, t)\,\mathrm{d}t$$

is positive and harmonic in $\mathscr{D}_\gamma$, and certainly continuous up to $\gamma \cap \mathscr{D}$. Also, $V_M(z) \leqslant \int_{|t| \leqslant M} G(z, t)$ which, *by the previous discussion*, tends to *zero* whenever z tends to any point of E. $V_M(z)$ is therefore continuous up to E_γ, *where it equals zero*. By the above statement, we thus have

$$V_M(z) \leqslant KV_M(z_1) \leqslant K \int_{|t| \geqslant N} G(z_1, t)\mathrm{d}t < \varepsilon$$

for $|z - x_0| < \frac{1}{2}$ radius of γ. This holds for all $M > N$, so making $M \to \infty$, we get $\int_{|t| \geqslant N} G(z, t)\,\mathrm{d}t \leqslant \varepsilon$ for $|z - x_0| < \frac{1}{2}$ radius of γ. Hence, since

$$H(z) = \int_{|t| \leqslant N} G(z, t)\,\mathrm{d}t + \int_{|t| \geqslant N} G(z, t)\,\mathrm{d}t,$$

and, as we already know, the first integral on the right tends to zero when $z \to x_0$, we must have $H(z) < 2\varepsilon$ for z close enough to x_0. This shows that $H(z) \to 0$ whenever z tends to any point of E.

We now see by the preceding arguments *that*

$$Y(z) \;=\; |\Im z| \;+\; \frac{1}{\pi}H(z)$$

enjoys the properties (i), (ii) *and* (iii) *required of Phragmén–Lindelöf functions, and is also harmonic in both the lower and upper half planes, and continuous everywhere.* Therefore, to complete the proof of the fact that $Y(z)$ *is* a Phragmén–Lindelöf function for $\mathscr{D}$, *we need only verify that it is harmonic at the points of* $\mathscr{D} \cap \mathbb{R}$.

For this purpose, we bring in the formula

$$|\Im z| \;=\; \lim_{A\to\infty} \frac{1}{\pi}\int_{-A}^{A} \log\left|1 \;-\; \frac{z}{t}\right| dt$$

mentioned earlier. From it, and the definition of $H(z)$, we get

$$(\ast)\qquad Y(z) \;=\; |\Im z| \;+\; \frac{1}{\pi}\int_{-\infty}^{\infty} G(z,t)\,dt$$

$$= \frac{1}{\pi}\int_{-A}^{A}\left(\log\frac{1}{|t|} + \log|z-t| + G(z,t)\right)dt$$

$$+\;\frac{1}{\pi}\int_{A}^{\infty}\log\left|1-\frac{z^2}{t^2}\right|dt \;+\; \frac{1}{\pi}\int_{|t|\geqslant A} G(z,t)\,dt.$$

The number $A > 0$ may be chosen at pleasure.

Let $x_0 \in \mathscr{D} \cap \mathbb{R}$; pick A larger than $|x_0|$. The function $\int_A^\infty \log|1 - z^2/t^2|\,dt$ is certainly harmonic near x_0; we have also seen previously that $\int_{|t|\geqslant A} G(z,t)\,dt$ is harmonic in $\mathscr{D} \cap \{|z| < A\}$, so, in particular, at x_0. Again, $\int_{-A}^{A} \log(1/|t|)\,dt$ is *finite*. Our task thus boils down to showing harmonicity of

$$\int_{-A}^{A} (\log|z-t| + G(z,t))\,dt$$

at x_0.

Take a $\delta > 0$ such that $(x_0 - 5\delta,\; x_0 + 5\delta) \subseteq \mathscr{D}$. According to observations already made,

$$\int_{\substack{|t-x_0|>\delta \\ |t|<A}} G(z,t)\,dt$$

is harmonic for $|z - x_0| < \delta$; so is (clearly)

$$\int_{\substack{|t-x_0|>\delta \\ |t|<A}} \log|z-t|\,dt.$$

We therefore need only check the harmonicity of

$$\int_{x_0-\delta}^{x_0+\delta} (\log|z-t| + G(z,t))\,dt.$$

Here, we must use the symmetry of $G(z,w)$. In order not to get bogged down in notation, *let us assume that* $x_0 = \alpha > 0$ *and that the segment* $[-2\alpha,\ 6\alpha]$ *lies entirely in* $\mathscr{D}$. The general situation can always be *reduced* to this one by *suitable translation.* It will be enough to show that

$$\int_0^{2\alpha} (\log|z-t| + G(z,t))\,\mathrm{d}t$$

is harmonic for $|z-\alpha| < \alpha$.

For each *fixed* z,

$$\log|z-w| + G(z,w) \;=\; \log|w-z| + G(w,z)$$

is a certain *harmonic function*, $h_z(w)$, of $w \in \mathscr{D}$; this is where the symmetry of G comes in. ($h_z(w)$ is harmonic in w *even at the point* z, for addition of the term $\log|w-z|$ *removes* the logarithmic singularity of $G(w,z)$ there.) Hence, if $\rho < \operatorname{dist}(w,E)$,

$$h_z(w) \;=\; \frac{1}{2\pi}\int_0^{2\pi} h_z(w+\rho e^{i\vartheta})\mathrm{d}\vartheta.$$

This relation makes a *trick* available. In it, put $w = t$ where $0 < t < 2\alpha$, and use $\rho = t + 2\alpha$.

We get

$$h_z(t) \;=\; \frac{1}{2\pi}\int_0^{2\pi} h_z(t+(t+2\alpha)e^{i\vartheta})\mathrm{d}\vartheta,$$

whence,

$$\int_0^{2\alpha}(\log|z-t| + G(z,t))\mathrm{d}t \;=\; \frac{1}{2\pi}\int_0^{2\alpha}\int_0^{2\pi} h_z(t+(t+2\alpha)e^{i\vartheta})\mathrm{d}\vartheta\,\mathrm{d}t.$$

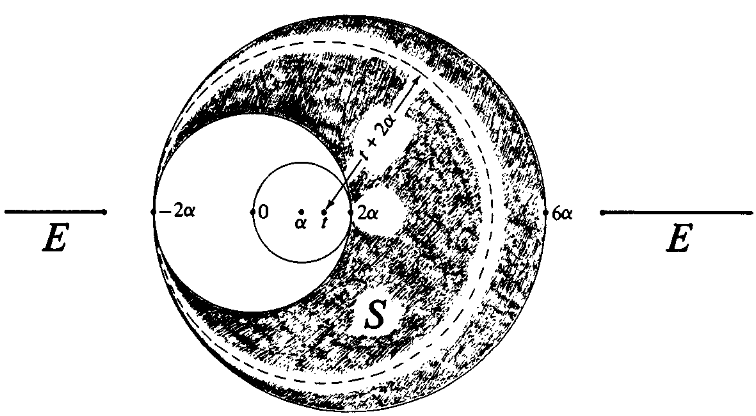

Figure 125

The double integral on the right can be expressed as one over the region

$$S = \{|\zeta - 2\alpha| < 4\alpha\} \cap \{|\zeta| > 2\alpha\}$$

shown in the above picture. Indeed, the mapping

$$(t, \vartheta) \longrightarrow (\xi, \eta)$$

given by $\xi + i\eta = \zeta = t + (t + 2\alpha)e^{i\vartheta}$ takes $\{0 < t < 2\alpha\} \times \{0 < \vartheta < 2\pi\}$ in one–one fashion onto S, and the Jacobian

$$\frac{\partial(\xi, \eta)}{\partial(t, \vartheta)}$$

works out to $(t + 2\alpha)(1 + \cos\vartheta) = \xi + 2\alpha$.
Hence,

$$\frac{1}{2\pi}\int_0^{2\alpha}\int_0^{2\pi} h_z(t + (t + 2\alpha)e^{i\vartheta})\,d\vartheta\,dt \;=\; \frac{1}{2\pi}\iint_S h_z(\zeta)\frac{d\xi\,d\eta}{\xi + 2\alpha},$$

so, by the previous relation,

$$\int_0^{2\alpha}(\log|z - t| + G(z, t))\,dt$$

$$= \frac{1}{2\pi}\iint_S \frac{\log|z - \zeta|}{\xi + 2\alpha}\,d\xi\,d\eta \;+\; \frac{1}{2\pi}\iint_S \frac{G(z, \zeta)}{\xi + 2\alpha}\,d\xi\,d\eta.$$

Here, we have

$$\iint_S \frac{d\xi\,d\eta}{\xi + 2\alpha} \;=\; \int_0^{2\alpha}\int_0^{2\pi} d\vartheta\,dt \;<\; \infty,$$

so both of the above double integrals must equal *harmonic functions of* z in the *disk* $\{|z - \alpha| < \alpha\}$, *disjoint from* $\bar{S}$. (This follows for the *second* of those integrals by the remark at the very beginning of this proof.) We see that the left-hand expression is *harmonic in* z *for* z *near* $x_0 = \alpha$. According, then, to ($*$) and the observations immediately following, *the same is true for* $Y(z)$.

We have finished proving that $Y(z)$ *is a Phragmén–Lindelöf function for* $\mathscr{D}$ *if the integral* $\int_{-\infty}^{\infty} G(z, t)\,dt$ *is finite for any* z *therein.* The *second half* of our theorem thus remains to be established. That is easier.

In the second half, *we assume that* $\mathscr{D}$ *has a Phragmén–Lindelöf function* $Y(z)$, and set out to show that

$$\int_{-\infty}^{\infty} G(z, t)\,dt \;<\; \infty$$

for *each* $z \in \mathscr{D}$, G being that domain's Green's function.

Given any $A > 0$, consider the expression

$$\begin{aligned} Y_A(z) &= |\Im z| + \frac{1}{\pi}\int_{-A}^{A} G(z,t)\,\mathrm{d}t \\ &= \frac{1}{\pi}\int_{-A}^{A}\left(\log\frac{1}{|t|} + \log|z-t| + G(z,t)\right)\mathrm{d}t \\ &\qquad + \frac{1}{\pi}\int_{A}^{\infty}\log\left|1-\frac{z^2}{t^2}\right|\mathrm{d}t. \end{aligned}$$

From the preceding arguments, we know that the *first integral on the right* is *harmonic* for $z\in\mathscr{D}$ – proof of this fact *did not depend on the convergence of*

$$\int_{-\infty}^{\infty} G(z,t)\,\mathrm{d}t.$$

What we have already done also tells us that $Y_A(z)$ *tends to zero when z tends to any point of E* (again, *whether* $\int_{-\infty}^{\infty} G(z,t)\mathrm{d}t$ *converges or not*) and that, for any fixed A,

$$\int_{-A}^{A} G(z,t)\,\mathrm{d}t$$

is bounded in the complex plane. The expression

$$\int_{A}^{\infty}\log\left|1-\frac{z^2}{t^2}\right|\mathrm{d}t$$

is evidently *subharmonic* in the complex plane.

The function $Y_A(z)$ given by the above formula is thus *subharmonic*, and *zero on E*, and moreover,

$$Y_A(z) = |\Im z| + O(1), \quad z\in\mathscr{D}.$$

Our Phragmén–Lindelöf function $Y(z)$ (presumed to exist!) is, *however, harmonic and* $\geqslant|\Im z|$ *in* $\mathscr{D}$, *and zero on E.* The difference $Y_A(z) - Y(z)$ is therefore *subharmonic and bounded above in* $\mathscr{D}$, *and zero on E.* We can conclude by the *extended maximum principle* (subharmonic version of *first* theorem in § C, Chapter III) that $Y_A(z) - Y(z) \leqslant 0$ for $z\in\mathscr{D}$. In other words,

$$|\Im z| + \frac{1}{\pi}\int_{-A}^{A} G(z,t)\,\mathrm{d}t \leqslant Y(z).$$

Fixing $z\in\mathscr{D}$ and then making $A\to\infty$, we see that

$$\frac{1}{\pi}\int_{-\infty}^{\infty} G(z,t)\,\mathrm{d}t \leqslant Y(z) - |\Im z| < \infty.$$

This is what we wanted. The *second half of the theorem is proved.*

We are done.

We apply the result just proved to domains $\mathscr{D}$ of the special form described at the beginning of the present §, using the *first* theorem of this article. In that way we obtain the important

Theorem (Benedicks). *If E is a union of segments on $\mathbb{R}$ fulfilling the conditions given at the beginning of this* § (*involving the four constants A, B, δ and Δ*), there is *a Phragmén–Lindelöf function for the domain $\mathscr{D} = \mathbb{C} \sim E$.*

Proof. Assume wlog that $0 \in \mathscr{D}$, and call $\mathcal{O}_0$ the component of $\mathbb{R} \sim E$ containing 0. By the first theorem of the present article,

$$G(t,0) \;\leqslant\; \frac{C}{1+t^2}$$

for $t \in \mathcal{O}_0$, and clearly

$$G(t,0) \;\leqslant\; \log^+ \frac{1}{|t|} + O(1), \qquad t \in \mathcal{O}_0.$$

Therefore (symmetry again!)

$$\int_{-\infty}^{\infty} G(0,t)\,dt \;=\; \int_{-\infty}^{\infty} G(t,0)\,dt \;<\; \infty.$$

Now refer to the preceding theorem.

We are done.

This result will be applied to the study of weighted approximation on sets E in the next article. We cannot, however, end *this* one without keeping our promise about proving symmetry of the Green's function. So, here we go:

Theorem. *In $\mathscr{D} = \mathbb{C} \sim E$,*

$$G(z,w) \;=\; G(w,z).$$

Proof. Let us first treat the case where *E consists of a finite number of intervals, of finite or infinite length.* (If E contains *two* semi-infinite intervals at opposite ends of $\mathbb{R}$, we consider them as forming *one* interval passing through ∞.)

We first proceed as at the beginning of article 1, *and map $\mathscr{D}$* (or $\mathscr{D} \cup \{\infty\}$, if $\infty \notin E$) *conformally onto a bounded domain, bounded by a finite number of analytic Jordan curves.* This useful trick simplifies a lot of work; let us describe (in somewhat more detail than at the beginning of article 1) how it is done.

Suppose that $E_1, E_2, \ldots, E_N$ are the components of E. First map $(\mathbb{C} \cup \{\infty\}) \sim E_1$ conformally onto the disk $\{|z| < 1\}$; in this mapping, E_1 (which gets split down its middle, with its two edges spread apart) goes onto $\{|z| = 1\}$, and $E_2, \ldots, E_N$ are taken onto *analytic* Jordan arcs, $A_2, \ldots, A_N$ respectively, lying inside the unit disk. (Actually, in our situation, where the E_k lie on $\mathbb{R}$, we can choose the mapping of $(\mathbb{C} \cup \{\infty\}) \sim E_1$ onto $\{|z| < 1\}$ so that $\mathbb{R} \sim E_1$ is taken onto $(-1, 1)$. Then $A_2, \ldots, A_N$ will be *segments* on $(-1, 1)$.) In this fashion, $\mathscr{D}$ is mapped conformally onto

$$\{|z| < 1\} \sim A_2 \sim A_3 \sim \cdots \sim A_N.$$

Now map $(\mathbb{C} \cup \{\infty\}) \sim A_2$ conformally onto $\{|w| < 1\}$. In this transformation, $\{|z| = 1\}$ goes onto a certain analytic Jordan curve $\mathscr{C}_1$ lying inside the unit disk, A_2 (after having its two sides spread apart) goes onto $\{|w| = 1\}$, and, if $N > 2$, the arcs $A_3, \ldots, A_N$ go onto other analytic arcs $A'_3, \ldots, A'_N$, lying inside $\{|w| < 1\}$. ($A'_3, \ldots, A'_N$ are indeed *segments* on $(-1, 1)$ in our present situation, if this second conformal mapping is properly chosen.) So far, composition of our two mappings yields a conformal transformation of $\mathscr{D}$ onto the region lying in $\{|w| < 1\}$, bounded by the unit circumference, the analytic Jordan curve $\mathscr{C}_1$, and the analytic Jordan arcs $A'_3, \ldots, A'_N$ (in the case where $N > 2$).

It is evident how one may continue this process when $N > 2$. Do the same thing with A'_3 that was done with A_2, and so forth, until all the boundary components are used up. The final result is a conformal mapping of $\mathscr{D}$ onto a region bounded by the *unit circumference* and $N - 1$ *analytic Jordan curves situated within it.*

Under conformal mapping, Green's functions correspond to Green's functions. Therefore, in order to prove that $G(z, w) = G(w, z)$, we *may as well assume that G is the Green's function for a bounded domain* Ω like the one arrived at by the process just described, i.e., with $\partial\Omega$ *consisting of a finite number of analytic Jordan curves.* For such domains Ω we can establish symmetry using methods going back to Green himself. (Green's original proof – the result *is* due to him, by the way – is a little different from the one we are about to give. Adapted to two dimensions, it amounts to the observation that

$$\begin{aligned} G(z,w) &= \log\frac{1}{|z-w|} + \int_{\partial\Omega} \log|\zeta - w|\, d\omega_\Omega(\zeta, z) \\ &= \log\frac{1}{|z-w|} + \int_{\partial\Omega}\int_{\partial\Omega} \log|\zeta - \sigma|\, d\omega_\Omega(\sigma, w)\, d\omega_\Omega(\zeta, z), \end{aligned}$$

where $\omega_\Omega(\ \ , z)$ is the harmonic measure for Ω. This argument can easily be made rigorous for our domains Ω. The interested reader may want to consult Green's collected papers, reprinted by Chelsea in 1970.)

If $\zeta \in \partial\Omega$ and the function F is $\mathscr{C}_1$ in a *neighborhood* of ζ, we denote by

$$\frac{\partial F(\zeta)}{\partial n_\zeta}$$

the *directional derivative of F in the direction of the unit outward normal* $\mathbf{n}_\zeta$ to $\partial\Omega$ at ζ:

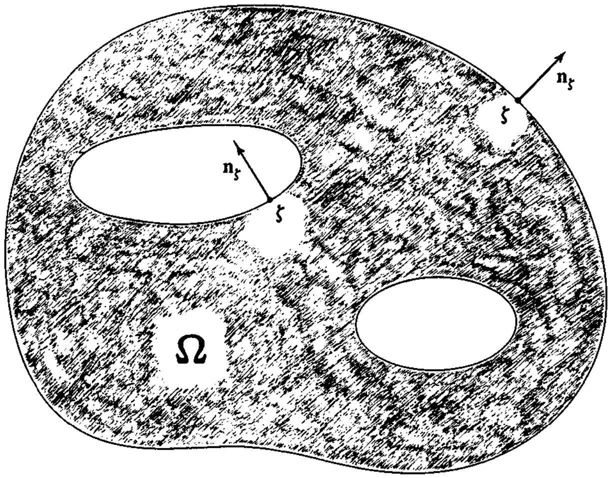

Figure 126

If $w \in \Omega$ is *fixed*, $G(z, w)$ is harmonic as a function of $z \in \Omega$ (for z away from w) and continuous up to $\partial\Omega$, where it equals *zero*. *Analyticity* of the components of $\partial\Omega$ means that, given any $\zeta_0 \in \partial\Omega$, we can find a *conformal mapping* of a small disk centered at ζ_0 which takes the *part of* $\partial\Omega$ *lying in that disk* to a *segment* σ *on the real axis*. If we compose $G(z, w)$ with this conformal mapping for $z \in \Omega$ near ζ_0, we see, by *Schwarz' reflection principle*, that the *composed* function is actually *harmonic in a neighborhood of* σ, and thence that $G(z, w)$ is *harmonic* (in z) *in a neighborhood of* ζ_0. $G(z, w)$ is, in particular, *a* $\mathscr{C}_\infty$ *function of* z *in the neighborhood of every point on* $\partial\Omega$.

This regularity, together with the smoothness of the components of $\partial\Omega$, makes it possible for us to apply *Green's theorem*. Given z and $w \in \Omega$ with $z \neq w$, take two small non-intersecting circles γ_z and γ_w lying in Ω, about z and w respectively. Call Ω' the domain obtained from Ω by *removing* therefrom the small disks bounded by γ_z and γ_w:

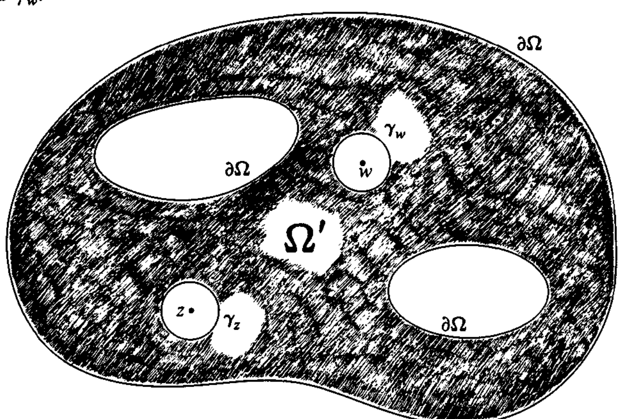

Figure 127

Denote by **grad** the *vector gradient* with respect to (ξ, η), where $\zeta = \xi + i\eta$, and by '$\cdot$' the *dot product* in $\mathbb{R}^2$. We have

$$\int_{\partial\Omega'} \left(G(\zeta, w) \frac{\partial G(\zeta, z)}{\partial n_\zeta} - G(\zeta, z) \frac{\partial G(\zeta, w)}{\partial n_\zeta} \right) |d\zeta|$$
$$= \int_{\partial\Omega'} (G(\zeta, w)\, \mathbf{grad}\, G(\zeta, z) - G(\zeta, z)\, \mathbf{grad}\, G(\zeta, w)) \cdot \mathbf{n}_\zeta |d\zeta|.$$

Since the vector-valued function

$$G(\zeta, w)\, \mathbf{grad}\, G(\zeta, z) - G(\zeta, z)\, \mathbf{grad}\, G(\zeta, w)$$

of ζ is $\mathscr{C}_\infty$ in and on $\bar{\Omega}'$, ($\mathscr{C}_\infty$ on $\partial\Omega$ by what was said above), we can apply *Green's theorem* to the second of these integrals, and find that it equals

$$\iint_{\Omega'} \operatorname{div} (G(\zeta, w)\, \mathbf{grad}\, G(\zeta, z) - G(\zeta, z)\, \mathbf{grad}\, G(\zeta, w))\, d\xi\, d\eta,$$

where div denotes *divergence* with respect to (ξ, η). However, by *Green's identity*,

$$\operatorname{div} (G(\zeta, w)\, \mathbf{grad}\, G(\zeta, z) - G(\zeta, z)\, \mathbf{grad}\, G(\zeta, w))$$
$$= G(\zeta, w) \nabla^2 G(\zeta, z) - G(\zeta, z) \nabla^2 G(\zeta, w),$$

where $\nabla^2 = \partial^2/\partial\xi^2 + \partial^2/\partial\eta^2$. Because $z \notin \Omega'$ and $w \notin \Omega'$, $G(\zeta, z)$ and $G(\zeta, w)$ are *harmonic* in ζ, $\zeta \in \Omega'$. Hence

$$\nabla^2 G(\zeta, z) = \nabla^2 G(\zeta, w) = 0, \qquad \zeta \in \Omega',$$

and the above double integral vanishes identically. Therefore the *first* of the above *line integrals* around $\partial\Omega'$ must be *zero.*

Now $\partial\Omega' = \partial\Omega \cup \gamma_z \cup \gamma_w$, and $G(\zeta, w) = G(\zeta, z) = 0$ for $\zeta \in \partial\Omega$. That line integral therefore reduces to

$$\left\{ \int_{\gamma_z} + \int_{\gamma_w} \right\} \left(G(\zeta, w) \frac{\partial G(\zeta, z)}{\partial n_\zeta} - G(\zeta, z) \frac{\partial G(\zeta, w)}{\partial n_\zeta} \right) |d\zeta|,$$

which must thus *vanish.* Near z, $G(\zeta, z)$ equals $\log(1/|\zeta - z|)$ plus a harmonic function of ζ; with this in mind we see that the integral around γ_z is *very nearly* $2\pi G(z, w)$ if the radius of γ_z is small. The integral around γ_w is seen in the same way to be very nearly equal to $-2\pi G(w, z)$ when that circle has small radius, so, making the radii of both γ_z and γ_w tend to *zero,* we find in the limit that

$$2\pi G(z, w) - 2\pi G(w, z) = 0,$$

i.e., $G(z, w) = G(w, z)$ for $z, w \in \Omega$. This same symmetry must then hold for the Green's functions belonging to *finitely connected domains* $\mathscr{D}$ of the kind we are considering. **How much must we admire George Green, self taught, who did such beautiful work isolated in provincial England at the beginning of the nineteenth century. One wonders what he might have done had he lived longer than he did.**

AN ESSAY

ON THE

APPLICATION

OF

MATHEMATICAL ANALYSIS TO THE THEORIES OF ELECTRICITY AND MAGNETISM.

BY

GEORGE GREEN.

Nottingham:

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1828.

Once the symmetry of Green's function for *finitely connected domains* $\mathscr{D}$ is *known*, we can establish that property *in the general case* by a limiting argument. By a slight modification of the following procedure, one can actually *prove existence* of the Green's function for infinitely connected domains $\mathscr{D} = \mathbb{C} \sim E$ of the kind being considered here, and the reader is invited to see how such a proof would go. Let us, however, content ourselves with what we set out to do.

Put $E_R = E \cap [-R, R]$ and take $\mathscr{D}_R = (\mathbb{C} \cup \{\infty\}) \sim E_R$. With our sets E, E_R consists of a finite number of intervals, so $\mathscr{D}_R$ is finitely connected, and, by what we have just shown, $G_R(z, w) = G_R(w, z)$ for the Green's function G_R belonging to $\mathscr{D}_R$. (Provided, of course, that R is large enough to make $|E_R| > 0$, so that $\mathscr{D}_R$ *has* a Green's function! *This we henceforth assume.*) We have $\mathscr{D}_R \supseteq \mathscr{D}$, whence, for z, $w \in \mathscr{D}$,

$$G(z, w) \leqslant G_R(z, w).$$

If we can show that

$$G_R(z, w) \longrightarrow G(z, w)$$

for z, $w \in \mathscr{D}$ as $R \to \infty$, we will obviously have $G(z, w) = G(w, z)$.

To verify this convergence, observe that

$$G_{R'}(z, w) \leqslant G_R(z, w)$$

for z and w in $\mathscr{D}_{R'}$ (hence *certainly* for z, $w \in \mathscr{D}$!) when $R' \geqslant R$, because then $\mathscr{D}_{R'} \subseteq \mathscr{D}_R$. The limit

$$\tilde{G}(z, w) = \lim_{R \to \infty} G_R(z, w)$$

thus *certainly exists* for $z, w \in \mathscr{D}$, and is $\geqslant 0$. If we can prove that $\tilde{G}(z, w) = G(z, w)$, we will be done.

Fix any $w \in \mathscr{D}$. Outside any small circle about w lying in $\mathscr{D}$, $\tilde{G}(z, w)$ is the limit of a decreasing sequence of positive harmonic functions of z, and is therefore itself harmonic in that variable. Let $x_0 \in E$. Take $R > |x_0|$; then, since $0 \leqslant \tilde{G}(z, w) \leqslant G_R(z, w)$ for $z \in \mathscr{D}$ and $G_R(z, w) \longrightarrow 0$ as $z \to x_0$, we have

$$\tilde{G}(z, w) \longrightarrow 0 \qquad \text{for } z \longrightarrow x_0.$$

If we fix any large R, we have, for $z \in \mathscr{D}$,

$$0 \leqslant \tilde{G}(z, w) \leqslant G_R(z, w) = \log \frac{1}{|z - w|} + O(1).$$

Therefore, since

$$G(z, w) = \log \frac{1}{|z - w|} + O(1),$$

we have

$$\tilde{G}(z, w) \leqslant G(z, w) + O(1), \qquad z \in \mathscr{D}.$$

However, this last inequality can be *turned around.* Indeed, for $z \in \mathscr{D}$ and *every* sufficiently large R,

$$G_R(z, w) \geqslant G(z, w),$$

from which we get

$$\tilde{G}(z, w) \geqslant G(z, w), \qquad z \in \mathscr{D}$$

on making $R \to \infty$.

We see finally that $0 \leqslant \tilde{G}(z, w) - G(z, w) \leqslant O(1)$ for $z \in \mathscr{D}$ (at least when $z \neq w$); the difference in question is, moreover, *harmonic in* z (for $z \in \mathscr{D}$, $z \neq w$) and *tends,* according to what we have shown above, to *zero when* z *tends to any point of* $E = \partial\mathscr{D}$. Hence

$$\tilde{G}(z, w) - G(z, w) = 0, \qquad z \in \mathscr{D},$$

i.e.,

$$G_R(z, w) \longrightarrow G(z, w)$$

for $z \in \mathscr{D}$ when $R \to \infty$, which is what we needed to establish the symmetry of $G(z, w)$.
We are done.

3. Weighted approximation on the sets E

Let E be a closed set on $\mathbb{R}$, having infinite extent in both directions and consisting of (at most) countably many closed intervals not accumulating at any finite point. Suppose that we are given a function $W(x) \geqslant 1$, defined and continuous on E, such that $W(x) \to \infty$ for $x \to \pm\infty$ in E. Then, in analogy with Chapter VI, we make the

Definition. $\mathscr{C}_W(E)$ is the set of functions φ defined and continuous on E, such that

$$\frac{\varphi(x)}{W(x)} \longrightarrow 0 \quad \text{for} \quad x \longrightarrow \pm\infty \textit{ in } E.$$

And we put

$$\|\varphi\|_{W,E} = \sup_{x \in E} \left|\frac{\varphi(x)}{W(x)}\right|$$

for $\varphi \in \mathscr{C}_W(E)$.

For $A > 0$, we denote by $\mathscr{C}_W(A, E)$ the $\| \;\|_{W,E}$-closure in $\mathscr{C}_W(E)$ of the collection of finite sums of the form

$$\sum_{-A \leqslant \lambda \leqslant A} C_\lambda e^{i\lambda x}.$$

Also, if, for every $n > 0$,

$$\frac{x^n}{W(x)} \longrightarrow 0 \quad \text{as} \quad x \longrightarrow \pm\infty \text{ in } E,$$

we denote by $\mathscr{C}_W(0, E)$ the $\| \;\|_{W,E}$-closure in $\mathscr{C}_W(E)$ of the set of *polynomials*.

We are interested in obtaining criteria for equality of the $\mathscr{C}_W(A, E)$, $A > 0$, (and of $\mathscr{C}_W(0, E)$) with $\mathscr{C}_W(E)$. One *can*, of course, reduce our present situation to the one considered in Chapter VI by putting $W(x) \equiv \infty$ on $\mathbb{R} \sim E$ and working with the space $\mathscr{C}_W(\mathbb{R})$. The equality in question is then governed by Akhiezer's theorems found in §§B and E of Chapter VI, according to the remark in §B.1 of that chapter (see also the corollary at the end of §E.2 therein). In this way, one arrives at results in which the set E does not figure explicitly. Our aim, however, already mentioned at the beginning of the present chapter, is to show how the *form*

$$\int_{-\infty}^{\infty} \frac{\log W_*(x)}{1+x^2}\,\mathrm{d}x,$$

occurring in Akhiezer's first theorem, can, in the present situation, be replaced by

$$\int_E \frac{\log W_*(x)}{1+x^2}\,\mathrm{d}x$$

when dealing with certain kinds of sets E. That is the subject of the following discussion. Our results will depend strongly on those of the preceding two articles.

Lemma. *Let $A > 0$, and suppose that there is a finite M such that*

$$(*) \qquad \int_E \frac{\log|S(x)|}{1+x^2}\,\mathrm{d}x \;\leqslant\; M$$

for all finite sums $S(x)$ of the form

$$\sum_{-A\leqslant\lambda\leqslant A} a_\lambda \mathrm{e}^{\mathrm{i}\lambda x}$$

with $\|S\|_{W,E} \leqslant 1$. Then there is a finite M' such that

$$(\S) \qquad \int_E \frac{\log^+|S(x)|}{1+x^2}\,\mathrm{d}x \;\leqslant\; M'$$

for such S with $\|S\|_{W,E} \leqslant 1$.

Proof. Given a sum $S(x)$ of the specified form with $\|S\|_{W,E} \leqslant 1$, we wish to show that (§) holds for some M' independent of S. Let us assume, to begin with, that the *exponents λ figuring in the sum $S(x)$ are in arithmetic progression*, more precisely, that

$$S(x) \;=\; \sum_{n=-N}^{N} C_n \mathrm{e}^{\mathrm{i}nhx}$$

where $h = A/N$, N being some large integer. There is then another sum

$$T(x) \;=\; \sum_{n=-N}^{N} a_n \mathrm{e}^{\mathrm{i}nhx}$$

(which is thus *also* of the form $\sum_{-A\leqslant\lambda\leqslant A} C_\lambda \mathrm{e}^{\mathrm{i}\lambda x}$) such that

$$1 + S(x)\overline{S(x)} \;=\; T(x)\overline{T(x)} \qquad \text{for } x\in\mathbb{R}.$$

This we can see by an elementary argument, going back to Fejér and

Riesz. For $x \in \mathbb{R}$, we have

$$1 + S(x)\overline{S(x)} \quad = \quad \sum_{n=-2N}^{2N} \gamma_n e^{inhx}$$

with certain coefficients γ_n. Write, for the moment,

$$e^{ihx} = \zeta;$$

then

$$1 + S(x)\overline{S(x)} \quad = \quad R(\zeta),$$

where

$$R(\zeta) \quad = \quad \sum_{n=-2N}^{2N} \gamma_n \zeta^n$$

is a certain *rational* function of ζ. We have $R(\zeta) \geqslant 1$ for $|\zeta| = 1$, so, by the Schwarz reflection principle,

$$R(\overline{1/\zeta}) \quad = \quad \overline{R(\zeta)}.$$

Therefore, if α, $0 < |\alpha| < 1$, is a zero of $R(\zeta)$, so is $1/\bar{\alpha}$, and the latter has the same multiplicity as α. Also, if $-m$ denotes the *least* integer n for which $\gamma_n \neq 0$, we must have $\gamma_n = 0$ for $n > m$ (*sic*!), as follows on comparing the orders of magnitude of $R(\zeta)$ for $\zeta \to 0$ and for $\zeta \to \infty$.

The *polynomial* $\zeta^m R(\zeta)$ is thus of degree $2m$, and of the form

$$\text{const.} \prod_{k=1}^{m} (\zeta - \alpha_k)\left(\zeta - \frac{1}{\bar{\alpha}_k}\right).$$

Thence,

$$R(\zeta) \quad = \quad C \prod_{k=1}^{m} (\zeta - \alpha_k)\left(\frac{1}{\zeta} - \bar{\alpha}_k\right),$$

and $C > 0$ since $R(\zeta) \geqslant 1$ for $|\zeta| = 1$. Going back to the real variable x, we see that

$$1 + S(x)\overline{S(x)} \quad = \quad C \prod_{k=1}^{m} (e^{ihx} - \alpha_k)(e^{-ihx} - \bar{\alpha}_k) \quad = \quad T(x)\overline{T(x)},$$

where

$$T(x) \quad = \quad C^{\frac{1}{2}} e^{-iNhx} \prod_{k=1}^{m} (e^{ihx} - \alpha_k)$$

is of the form

$$\sum_{n=-N}^{N} a_n e^{inhx},$$

since $m \leqslant 2N$.

Once this is known, it is easy to deduce (§) for sums $S(x)$ of the special form just considered with $\|S\|_{W,E} \leqslant 1$. Take any such S; we have another sum $T(x)$ of the same kind with $1 + |S(x)|^2 = |T(x)|^2$ on $\mathbb{R}$. Since $W(x) \geqslant 1$ on E, the condition $\|S\|_{W,E} \leqslant 1$ implies that $\|T\|_{W,E} \leqslant \sqrt{2}$, i.e.,

$$\|T/\sqrt{2}\|_{W,E} \leqslant 1.$$

For this reason, $T(x)/\sqrt{2}$ satisfies $(*)$, by hypothesis. Hence

$$\int_E \frac{\log|T(x)|}{1+x^2}\,dx \leqslant M + \int_E \frac{\log\sqrt{2}}{1+x^2}\,dx.$$

But $\log|T(x)| = \log\sqrt{(1+|S(x)|^2)} \geqslant \log^+|S(x)|$. Therefore

$$\int_E \frac{\log^+|S(x)|}{1+x^2}\,dx \leqslant M + \pi\log\sqrt{2},$$

and we have obtained (§).

We must still consider the case where the exponents λ in the *finite* sum

$$\sum_{-A\leqslant\lambda\leqslant A} a_\lambda e^{i\lambda x} = S(x)$$

are *not* in arithmetic progression, the condition $\|S\|_{W,E} \leqslant 1$ being, however, satisfied. Here, we may associate to each λ figuring in the expression just written a *rational multiple* λ' *of* A, with $|\lambda' - \lambda|$ exceedingly small. Since $W(x) \to \infty$ for $x \to \pm\infty$ in E, the sum

$$S'(x) = \sum_{-A\leqslant\lambda\leqslant A} a_\lambda e^{i\lambda' x}$$

will then be *as close as we like* in $\|\ \ \|_{W,E}$-norm to $S(x)$ (depending on the closeness of the individual λ' to their corresponding λ). In this way, we can get a sequence of sums $S_n(x)$ of the form in question, *each one having its exponents in arithmetic progression*, such that $\|S_n\|_{W,E} \leqslant 1$ and $S_n(x) \xrightarrow[n]{} S(x)$ u.c.c. on $\mathbb{R}$. By what we have already shown, $\int_E(\log^+|S_n(x)|/(1+x^2))\,dx \leqslant M + \pi\log\sqrt{2}$ for each n. Therefore

$$\int_E \frac{\log^+|S(x)|}{1+x^2}\,dx \leqslant M + \pi\log\sqrt{2}$$

by Fatou's lemma. We are done.

For the sets E described at the beginning of the present § we can establish analogues, involving integrals over E, of the Akhiezer and Pollard theorems given in Chapter VI, §§E.2 and E.4. These are included in the following theorem which, in one direction, assumes as little as possible and concludes that $\mathscr{C}_W(A, E) \subset \mathscr{C}_W(E)$ properly. In the other, it assumes the proper inclusion and asserts as much as possible.

For $z \in \mathbb{C}$, *denote by* $W_{A,E}(z)$ *the supremum of* $|S(z)|$ *for the finite sums*

$$S(z) \;=\; \sum_{-A \leqslant \lambda \leqslant A} a_\lambda e^{i\lambda z}$$

with $\|S\|_{W,E} \leqslant 1$. (If we agree that $W(x) \equiv \infty$ on $\mathbb{R} \sim E$, $W_{A,E}(z)$ and $\mathscr{C}_W(A, E)$ reduce respectively to the function $W_A(z)$ and the space $\mathscr{C}_W(A)$ already considered in Chapter VI, §§E.2ff.)

Theorem. *Let E be one of the sets described at the beginning of this §, the conditions involving the four numbers* A, B, δ *and* Δ *being fulfilled. If, for some* $C > 0$, *the supremum of*

$$\int_E \frac{\log|S(x)|}{1+x^2}\,dx$$

for all finite sums

$$S(x) \;=\; \sum_{-C \leqslant \lambda \leqslant C} a_\lambda e^{i\lambda x}$$

with $\|S\|_{W,E} \leqslant 1$ *is* finite, *then* $\mathscr{C}_W(C, E) \subset \mathscr{C}_W(E)$ properly.

If, conversely, that proper inclusion holds, then

$$\int_E \frac{\log W_{C,E}(x)}{1+x^2}\,dx \;<\; \infty.$$

Proof. All the work here is in the establishment of the *first* part of the statement.

Define $W(x)$ on all of $\mathbb{R}$ by putting it equal to ∞ on $\mathbb{R} \sim E$. This makes it possible for us to apply results about $\mathscr{C}_W(C)$ from Chapter VI, §E, in the present situation. According to the Pollard theorem of Chapter VI, §E.4, and the remark thereto, we will have $\mathscr{C}_W(C) \neq \mathscr{C}_W(\mathbb{R})$ as soon as $W_C(i) < \infty$; in other words, $\mathscr{C}_W(C, E) \neq \mathscr{C}_W(E)$ provided that

$$W_{C,E}(i) \;<\; \infty.$$

We proceed to show this inequality, using the results of articles 1 and 2.

According to the lemma, our hypothesis for the first part of the theorem implies that

$$(*) \qquad \int_E \frac{\log^+|S(x)|}{1+x^2}\,dx \;\leqslant\; M' \;<\; \infty$$

for all sums $S(x)$ of the stipulated form with $\|S\|_{W,E} \leqslant 1$.

We have

$$E = \bigcup_{n=-\infty}^{\infty} [a_n - \delta_n, \ a_n + \delta_n],$$

where

$$0 < A < a_{n+1} - a_n < B,$$
$$0 < \delta < \delta_n < \Delta.$$

Given any finite sum

$$S(z) = \sum_{-C \leqslant \lambda \leqslant C} a_\lambda e^{i\lambda z}$$

with $\|S\|_{W,E} \leqslant 1$, let us put

$$v_S(z) = \frac{1}{\delta}\int_{-\delta/2}^{\delta/2} \log^+ |S(z+t)|\,dt$$

(using the δ associated to E by the above inequalities). The function $v_S(z)$ is then *continuous* and *subharmonic* in the complex plane. Obviously,

$$|S(z)| \leqslant \text{const.} e^{C|\Im z|}$$

(the constant may be *enormous*, but we don't care!), so

(†) $\qquad v_S(z) \leqslant O(1) + C|\Im z|, \quad z \in \mathbb{C}.$

Put now

$$E' = \bigcup_{n=-\infty}^{\infty} \left[a_n - \frac{\delta_n}{2}, \ a_n + \frac{\delta_n}{2}\right].$$

On the component

$$E'_n = \left[a_n - \frac{\delta_n}{2}, \ a_n + \frac{\delta_n}{2}\right]$$

of E' we have

$$v_S(x) \leqslant \frac{1}{\delta}\int_{E_n} \log^+ |S(t)|\,dt,$$

where (as usual)

$$E_n = [a_n - \delta_n, \ a_n + \delta_n].$$

Denoting the right side of the previous relation by v_n, we have

(§§) $\qquad v_S(x) \leqslant v_n, \quad x \in E'_n.$

The set E' (like E) is one of the kind specified at the beginning of the present §; the numbers A, B, $\delta/2$ and $\Delta/2$ are associated to it. *The* *results of the previous two articles are therefore valid for E' and the domain $\mathscr{D}' = \mathbb{C} \sim E'$.* We can, in particular, apply Carleson's theorem from article 1. Assume, wlog, that

$$0 \in \mathcal{O}'_0 = \left(a_0 + \frac{\delta_0}{2},\ a_1 - \frac{\delta_1}{2}\right).$$

Then, if we denote by $\omega'_n(z)$ the *harmonic measure of* E'_n in $\mathscr{D}'$ (as seen from z), that theorem tells us that

$$\omega'_n(0) \leqslant \frac{K}{1+n^2}$$

with a constant K depending on A, B, $\delta/2$ and $\Delta/2$. By Harnack's theorem, there is thus a function $K(z)$, *continuous in* $\mathscr{D}'$, such that

$$\omega'_n(z) \leqslant \frac{K(z)}{1+n^2}, \qquad z \in \mathscr{D}'$$

(see discussion near the beginning of §B.1, Chapter VII).

Using properties (i) and (ii) from the beginning of this § we see that the quantities v_n introduced above satisfy

$$\frac{v_n}{1+n^2} \leqslant \alpha \int_{E_n} \frac{\log^+ |S(t)|}{1+t^2} \mathrm{d}t,$$

α being a certain constant depending only on the set E. Combined with the previous estimate, this yields

$$v_n \omega'_n(z) \leqslant \alpha K(z) \int_{E_n} \frac{\log^+ |S(t)|}{1+t^2} \mathrm{d}t, \qquad z \in \mathscr{D}',$$

so, for the sum

$$P(z) = \sum_{n=-\infty}^{\infty} v_n \omega'_n(z),$$

we have

$$(\dagger\dagger) \qquad P(z) \leqslant \alpha M' K(z), \qquad z \in \mathscr{D}',$$

by virtue of ($\ast\!\ast$).

Because $|S(x)|$ is bounded on $\mathbb{R}$, the v_n (which are $\geqslant 0$, by the way) are bounded. The series used to define $P(z)$ is therefore u.c.c. convergent, so that function is *continuous up to* $E' = \partial\mathscr{D}'$ as well as being *positive* and *harmonic* in $\mathscr{D}'$. On the component E'_n of E', $P(x)$ takes the constant value v_n. Hence the function

$$v_S(z) - P(z)$$

is *subharmonic in* $\mathscr{D}'$ and continuous up to E', where it is $\leqslant 0$ by (§§). It is, moreover, $\leqslant O(1) + C|\Im z|$ by (†).

Now according to *Benedicks' theorem* (article 2), a *Phragmén–Lindelöf function* $Y(z)$ is available for $\mathscr{D}'$. The function

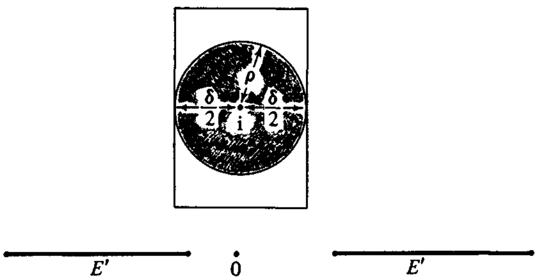
$$v_S(z) - P(z) - CY(z)$$

is *subharmonic and bounded above* in $\mathscr{D}'$ and continuous up to $E' = \partial\mathscr{D}'$ where it is $\leqslant 0$. It is thence $\leqslant 0$ *throughout* $\mathscr{D}'$ by the extended maximum principle (Chapter III, §C). Referring to (††), we see that

$$(\ddagger) \qquad v_S(z) \leqslant \alpha M'K(z) + CY(z), \qquad z \in \mathscr{D}'.$$

Let $\rho = \min(\frac{1}{2}, \delta/2)$. Since $\log^+|S(z)|$ is *subharmonic*, we have

$$\log^+|S(\mathrm{i})| \leqslant \frac{1}{\pi\rho^2}\iint_{|z-\mathrm{i}|<\rho} \log^+|S(z)|\,\mathrm{d}x\,\mathrm{d}y.$$



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The integral on the right is

$$\leqslant \frac{1}{\pi\rho^2}\int_{1-\rho}^{1+\rho}\int_{-\delta/2}^{\delta/2} \log^+|S(x+\mathrm{i}y)|\,\mathrm{d}x\,\mathrm{d}y = \frac{\delta}{\pi\rho^2}\int_{1-\rho}^{1+\rho} v_S(\mathrm{i}y)\,\mathrm{d}y.$$

Plugging (‡) into the last expression, we obtain

$$\log^+|S(\mathrm{i})| \leqslant \frac{\delta}{\pi\rho^2}\int_{1-\rho}^{1+\rho}(\alpha M'K(\mathrm{i}y) + CY(\mathrm{i}y))\,\mathrm{d}y.$$

This, then, is valid for *any* finite sum $S(z)$ of the form

$$\sum_{-C\leqslant\lambda\leqslant C} a_\lambda e^{\mathrm{i}\lambda z}$$

with $\|S\|_{W,E} \leqslant 1$.

The *right side* of the inequality just found is a *finite quantity*, dependent on M' and C, and on the set E (through α, $K(\mathrm{i}y)$ and $Y(\mathrm{i}y)$); it is, however,