we can obviously get *arbitrarily large* numbers $A > \beta_l$ which are well disposed with respect to β_l. We then have

$$\int_{\beta_l}^{A} \frac{\log|P(x)|}{x^2}\,dx \leqslant 5 \sum_{\beta_l < m < A} \frac{\log^+|P(m)|}{m^2}$$

for each such A by the first theorem of the preceding article, and need only make A tend to ∞.

Third stage. Replacement of the first few intervals I_k by a single one if $n(t)/t$ is not always $\leqslant p/(1-3p)$

Recall that the problem we are studying is as follows: we are presented with an unknown even polynomial $P(z)$ having only real roots and such that $P(0) = 1$, and *told* that

$$\sum_1^\infty \frac{\log^+|P(m)|}{m^2}$$

is small. We are *asked to obtain*, for $z \in \mathbb{C}$, a *bound on* $|P(z)|$ depending on that sum, but independent of P.

As a control on the size of $|P(z)|$ we will use the quantity

$$\sup_{t>0} \frac{n(t)}{t}.$$

A computation like the one at the end of §B, Chapter III, shows indeed that

$$\log|P(z)| \leqslant \pi|z| \sup_{t>0} \frac{n(t)}{t}.$$

We are therefore interested in obtaining an upper bound on $\sup_{t>0}(n(t)/t)$ from a suitable (small) one for

$$\sum_1^\infty \frac{\log^+|P(m)|}{m^2}.$$

Our procedure is to *work backwards*, assuming that $\sup_{t>0}(n(t)/t)$ is *not small* and thence deriving a *strictly positive lower bound for the sum.* We begin with the following simple

Lemma. *If $\sup_{t>0}(n(t)/t) > p/(1-3p)$, we have $|I_0|/\beta_0 \geqslant \frac{2}{3}$ for the interval $I_0 = [\alpha_0, \beta_0]$ arrived at in the previous stage of our construction.*

Proof. Let us examine carefully the *initial portion* of the last diagram given above:

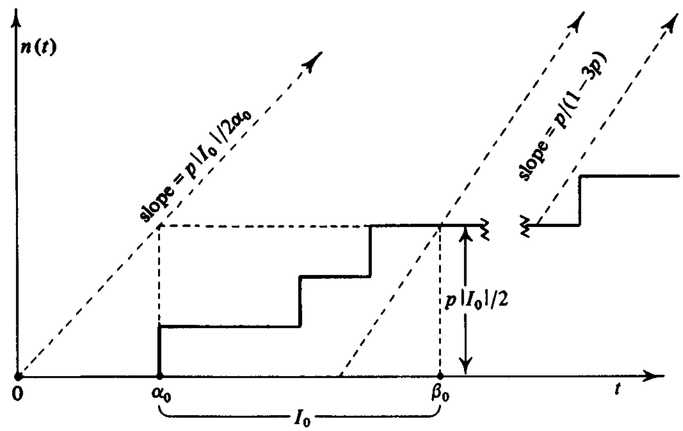

Figure 138

We see that, for $t > 0$,

$$\frac{n(t)}{t} \leqslant \max\left\{\frac{p}{1-3p},\ \frac{p|I_0|}{2\alpha_0}\right\},$$

whether or not the first term in curly brackets is less than the second.

Here,

$$\frac{p|I_0|}{2\alpha_0} = \frac{p}{2}\cdot\frac{|I_0|/\beta_0}{1-|I_0|/\beta_0},$$

and this is $< p < p/(1-3p)$ (making the above maximum *equal* to $p/(1-3p)$) if $|I_0|/\beta_0 < \frac{2}{3}$. Done.

Our construction of the intervals I_k involved the parameter p. We now bring in another quantity, η, which will continue to intervene during most of the articles of this §. For the time being, we require only that $0 < \eta < \frac{2}{3}$ and take the value of η as fixed during the work that follows. From time to time we will obtain various intermediate results whose validity will depend on η's having been chosen sufficiently small to begin with. A final decision about η's size will be made when we put together those results.

In accordance with the above indication of our procedure, we assume henceforth that

$$\sup_{t>0}\frac{n(t)}{t} > \frac{p}{1-3p}.$$

By the lemma we then certainly have

$$|I_0|/\beta_0 > \eta,$$

since we are taking $0 < \eta < \frac{2}{3}$. This being the case, we replace the first few intervals I_k by a single one, according to the following construction.

Let $\omega(x)$ be the continuous and piecewise linear function defined on $[0, \infty)$ which has slope 1 on each of the intervals I_k and slope *zero* elsewhere, and vanishes at the origin:

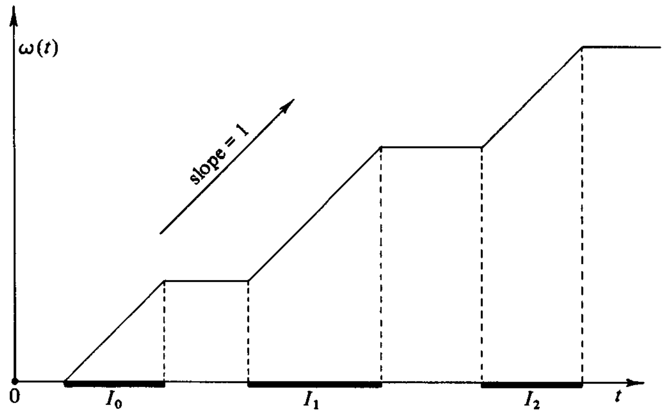

Figure 139

The ratio $\omega(t)/t$ is continuous and tends to zero as $t \to \infty$ since there are only a finite number of I_k. Clearly, $\omega(t)/t < 1$, so, if t belongs to the *interior* of an I_k,

$$\frac{d}{dt}\left(\frac{\omega(t)}{t}\right) = \frac{1}{t} - \frac{\omega(t)}{t^2} > 0;$$

i.e., *$\omega(t)/t$ is strictly increasing on each I_k.*

We have

$$\omega(\beta_0)/\beta_0 = |I_0|/\beta_0 > \eta,$$

so, in view of what has just been said, there must be a *largest* value of t $(> \beta_0)$ for which

$$\omega(t)/t = \eta$$

and that value *cannot lie in the interior, or be a left endpoint,* of any of the intervals I_k. Denote by d that value of t. Then, since $d > \beta_0$, there must be a *last interval I_k* – call it I_m – *lying entirely to the left* of d. If I_m is also the last of the intervals I_k we write

$$d_0 = d,$$
$$c_0 = (1-\eta)d,$$

and denote the interval $[c_0, d_0]$ by J_0. In this case all the positive zeros of $P(x)$ (discontinuities of $n(t)$) lie to the *left* of d_0.

It may be, however, that I_m is *not* the last of the I_k; then there is an interval

$$I_{m+1} = [\alpha_{m+1}, \beta_{m+1}],$$

and we must have $d < \alpha_{m+1}$ according to the above observation. Since $d > \beta_0 > 2/p > 40$ (remember that we are taking $0 < p < \frac{1}{20}$), we can apply the *second* theorem of article 1 to conclude that there is a d_0,

$$d - 3 < d_0 \leqslant d,$$

such that α_{m+1} is *well disposed with respect to* d_0. We then put

$$c_0 = d_0 - \eta d$$

and denote by J_0 the interval $[c_0, d_0]$. The intervals $I_{m+1}, I_{m+2}, \ldots$ are also relabeled as follows:

$$I_{m+1} = J_1,$$
$$I_{m+2} = J_2,$$

and we write $\alpha_{m+1} = c_1$, $\beta_{m+1} = d_1$, $\alpha_{m+2} = c_2$, $\beta_{m+2} = d_2$, and so forth, so as to have the uniform notation

$$J_k = [c_k, d_k], \qquad k = 0, 1, 2, \ldots$$

In the present case, $\beta_m \leqslant d < \alpha_{m+1}$ (*sic!*) so, referring to the previous (second) stage of our construction, we see that the part of the graph of $n(t)$ vs. t corresponding to values of $t \leqslant d$ lies entirely to the left of, or on, the line of slope p (*sic!*) through $(d, n(d))$. By an argument very much like the one near the end of the second stage, based on the fact that $n(t)$ increases by *at least* 1 at each of its jumps, this implies that d_0, *although it may lie to the left of* d, *still lies to the right of all the zeros of* $P(x)$ *in* $I_0, \ldots, I_m$, *and that*

$$n(d_0) - n(t) \leqslant \frac{p}{1-3p}(d_0 - t) \qquad \text{for } t \leqslant d_0.$$

(The diagram used here is obtained by *rotating through 180°* the one from the argument just referred to.)

We have, in the first place, $c_0 \geqslant (1-\eta)d - 3 > 0$, because $\eta < \frac{2}{3}$ and $d > \beta_0 > 40$.

In the second place,

$$|J_0|/d_0 \geqslant |J_0|/d = \eta,$$

by choice of d. Also,

$$\frac{|J_0|}{d_0} < \frac{|J_0|}{d-3} = \frac{\eta d}{d-3} < \frac{40\eta}{37},$$

since $d > 40$.

Finally,

$$\frac{n(d_0)}{p|J_0|} = \frac{1}{2}.$$

Indeed, both d_0 and d lie strictly between all the discontinuities of $n(t)$ in $I_0, I_1, \ldots, I_m$ and those in $I_{m+1}, I_{m+2}, \ldots$ (or to the *right* of the last I_k if our construction yields *only one* interval J_0), so

$$n(d_0) = \sum_{k=0}^{m} n(I_k) = \frac{1}{2}\sum_{k=0}^{m} p|I_k|$$

by property (ii) of the I_k. And

$$\sum_{k=0}^{m} |I_k| = \omega(d) = \eta d = |J_0|$$

by the choice of d and the definition of J_0. Thus $n(d_0) = \frac{1}{2}p|J_0|$, as claimed.

Denote by J the union of the J_k, and put for the moment

$$\tilde{\omega}(t) = |[0,t] \cap J|.$$

The function $\tilde{\omega}(t)$ is similar to $\omega(t)$, considered above, and differs from the latter only in that it increases (with constant slope 1) on each of the J_k instead of doing so on the I_k. The ratio $\tilde{\omega}(t)/t$ is therefore *increasing* on each J_k (see above), so in particular

$$\frac{\tilde{\omega}(t)}{t} \leqslant \frac{\tilde{\omega}(d_0)}{d_0} = \frac{|J_0|}{d_0} < \frac{40\eta}{37}$$

for $t \in J_0 = [c_0, d_0]$. This inequality remains (trivially) true for $0 \leqslant t < c_0$, since $\tilde{\omega}(t) = 0$ there. It also remains true for $d_0 \leqslant t \leqslant d$, for $\tilde{\omega}(t)$ is constant on that interval. And finally,

$$\tilde{\omega}(t) = \omega(t) \qquad \text{for } t > d,$$

so $\tilde{\omega}(t)/t = \omega(t)/t < \eta$ for such t by choice of d. Thus, we surely have

$$\frac{\tilde{\omega}(t)}{t} < 2\eta \qquad \text{for } t \geqslant 0.$$

The quantity on the left is, however, equal to $|J_0|/d_0 \geqslant \eta$ for $t = d_0$.

The purpose of the constructions in this article has been to arrive at the intervals J_k, and the remaining work of this § concerned with even polynomials having real zeros *deals exclusively with them.* The preceding discussions amount to a proof of the following

Theorem. *Let p, $0 < p < \frac{1}{20}$, and η, $0 < \eta < \frac{2}{3}$, be given, and suppose that*

$$\sup_t \frac{n(t)}{t} > \frac{p}{1-3p}.$$

Then there is a finite collection of intervals $J_k = [c_k, d_k]$, $k \geqslant 0$, lying in $(0, \infty)$, such that

(i) *all the discontinuities of $n(t)$ lie in $(0, d_0) \cup \bigcup_{k \geqslant 1} J_k$;*

(ii) $$\frac{n(d_0)}{p|J_0|} = \frac{n(J_k)}{p|J_k|} = \frac{1}{2} \quad \textit{for } k \geqslant 1$$

(if there are intervals J_k with $k \geqslant 1$);

(iii) *for $0 \leqslant t \leqslant d_0$,*

$$n(d_0) - n(t) \leqslant \frac{p}{1-3p}(d_0 - t),$$

whilst, for $c_k \leqslant t \leqslant d_k$ when $k \geqslant 1$,

$$n(t) - n(c_k) \leqslant \frac{p}{1-3p}(t - c_k)$$

and

$$n(d_k) - n(t) \leqslant \frac{p}{1-3p}(d_k - t);$$

(iv) *for $k \geqslant 1$, c_k is well disposed with respect to d_{k-1} (if there are J_k with $k \geqslant 1$);*

(v) *for $t \geqslant 0$,*

$$\frac{1}{t}\left|[0, t] \cap \bigcup_{k \geqslant 0} J_k\right| < 2\eta,$$

and the quantity on the left is $\geqslant \eta$ for $t = d_0$.

Remark 1. The J_k with $k \geqslant 1$ (if there are any) are just certain of the I_r from the *second stage*. So, for $k \geqslant 1$, the above property (ii) is just property (ii) for the I_r.

Remark 2. By property (iv) and the theorems of article 1, we have

$$\int_{d_{k-1}}^{c_k} \frac{\log|P(x)|}{x^2}\,dx \leqslant 5 \sum_{d_{k-1} < m < c_k} \frac{\log^+|P(m)|}{m^2}$$

for each of the intervals (d_{k-1}, c_k) with $k \geqslant 1$ (if there are any). And, if J_1

is the *last* of the J_k,

$$\int_{d_l}^{\infty} \frac{\log|P(x)|}{x^2}\,dx \;\leqslant\; 5 \sum_{d_l<m<\infty} \frac{\log^+|P(m)|}{m^2}.$$

See the end of the *second* stage of the preceding construction.

Here is a picture of the graph of $n(t)$ vs. t, showing the intervals J_k:

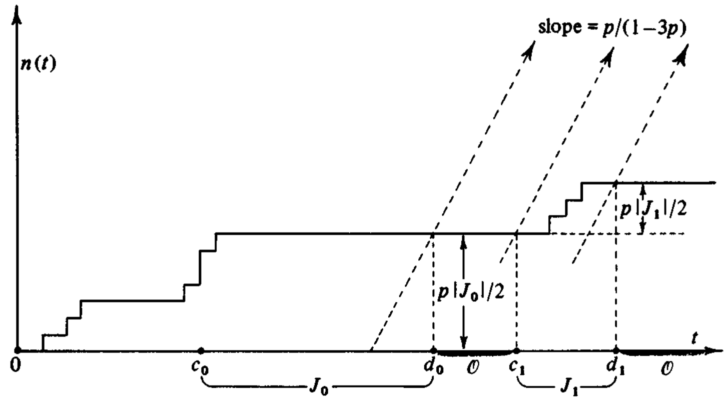

Figure 140

3. Replacement of the distribution $n(t)$ by a continuous one

Having chosen p, $0<p<1/20$, and η, $0<\eta<\frac{2}{3}$, we continue with our program, assuming that

$$\sup_{t>0} \frac{n(t)}{t} \;>\; \frac{p}{1-3p},$$

our aim being to obtain a *lower bound* for

$$\sum_{1}^{\infty} \frac{\log^+|P(m)|}{m^2}.$$

Our assumption makes it possible, by the work of the preceding article, to get the intervals

$$J_k \;=\; [c_k,\, d_k] \;\subset\; (0,\infty), \quad k=0,1,\ldots,$$

related to the (unknown) increasing function $n(t)$ in the manner described by the theorem at the end of that article.

Let J_l be the *last* of those J_k; *during this article we will denote the union*

$$(d_0,\, c_1)\cup(d_1,\, c_2)\cup\cdots\cup(d_{l-1},\, c_l)\cup(d_l,\, \infty)$$

by $\mathcal{O}$ – see the preceding diagram. (Note that this is *not* the same set $\mathcal{O}$ as the one used at the beginning of article 2!) Our idea is to estimate

$$\sum_{m\in\mathcal{O}} \frac{\log^+|P(m)|}{m^2}$$

from below, this quantity being certainly *smaller* than the one we are interested in. According to Remark 2 following the theorem about the J_k, we have

$$\sum_{m\in\mathcal{O}} \frac{\log^+|P(m)|}{m^2} \geqslant \frac{1}{5}\int_{\mathcal{O}} \frac{\log|P(x)|}{x^2}\,dx.$$

What we want, then, is a *lower bound* for *the integral* on the right. This is the form that our initial simplistic plan of 'replacing' sums by integrals finally assumes.

In terms of $n(t)$,

$$\log|P(x)| = \sum_k \log\left|1-\frac{x^2}{x_k^2}\right| = \int_0^\infty \log\left|1-\frac{x^2}{t^2}\right|dn(t),$$

so the object of our interest is the expression

$$\frac{1}{5}\int_{\mathcal{O}}\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right|dn(t)\frac{dx}{x^2}.$$

Here, $n(t)$ is constant on each component of $\mathcal{O}$, and increases only on that set's complement.

We are now able to render our problem more tractable by replacing $n(t)$ with another increasing function $\mu(t)$ of much more simple and regular behaviour, continuous and piecewise linear on $\mathbb{R}$ and constant on each of the intervals complementary to the J_k. The slope $\mu'(t)$ will take only *two* values, 0 and $p/(1-3p)$, and, on each J_k, $\mu(t)$ will increase by $p|J_k|/2$. What we have to do is find such a $\mu(t)$ which makes

$$\frac{1}{5}\int_{\mathcal{O}}\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right|d\mu(t)\frac{dx}{x^2}$$

smaller than the expression written above, yet *still* (we hope) *strictly positive.*

Part of our requirement on $\mu(t)$ is that $\mu(t) = n(t)$ for $t\in\mathcal{O}$, so we will have

$$\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right|d\mu(t) - \int_0^\infty \log\left|1-\frac{x^2}{t^2}\right|dn(t)$$

$$= \int_0^{d_0} \log\left|1-\frac{x^2}{t^2}\right|d(\mu(t)-n(t))$$

$$+ \sum_{k\geqslant 1}\int_{c_k}^{d_k} \log\left|1-\frac{x^2}{t^2}\right|d(\mu(t)-n(t)).$$

We are interested in values of x in $\mathcal{O}$, and for them, each of the above terms can be integrated by parts. Since $\mu(t) = n(t) = 0$ for t near 0 and $\mu(d_0) = n(d_0)$, $\mu(c_k) = n(c_k)$ and $\mu(d_k) = n(d_k)$ for $k \geqslant 1$, we obtain in this way the expression

$$\int_0^{d_0} \frac{2x^2}{x^2-t^2} \frac{\mu(t)-n(t)}{t} \mathrm{d}t + \sum_{k\geqslant 1} \int_{c_k}^{d_k} \frac{2x^2}{x^2-t^2} \frac{\mu(t)-n(t)}{t} \mathrm{d}t.$$

Therefore

$$\int_{\mathcal{O}} \int_0^{\infty} \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}(\mu(t) - n(t)) \frac{\mathrm{d}x}{x^2}$$

$$= \int_0^{d_0} \int_{\mathcal{O}} \frac{2\mathrm{d}x}{x^2-t^2} \cdot \frac{\mu(t)-n(t)}{t} \mathrm{d}t$$

$$+ \sum_{k\geqslant 1} \int_{c_k}^{d_k} \int_{\mathcal{O}} \frac{2\mathrm{d}x}{x^2-t^2} \cdot \frac{\mu(t)-n(t)}{t} \mathrm{d}t,$$

and we desire to find a function $\mu(t)$ fitting our requirements, for which each of the terms on the right comes out *negative.*

Put

$$F(t) = 2\int_{\mathcal{O}} \frac{\mathrm{d}x}{x^2-t^2}$$

for $t \notin \mathcal{O}$. We certainly have $F(t) > 0$ for $0 < t < d_0$, so the first right-hand term, which equals

$$\int_0^{d_0} F(t) \frac{\mu(t)-n(t)}{t} \mathrm{d}t$$

is $\leqslant 0$ if $\mu(t) \leqslant n(t)$ on $[0, d_0]$. Referring to the diagram at the end of the previous article, we see that this will happen if, for $0 \leqslant t \leqslant d_0$, $\mu(t)$ has the form shown here:

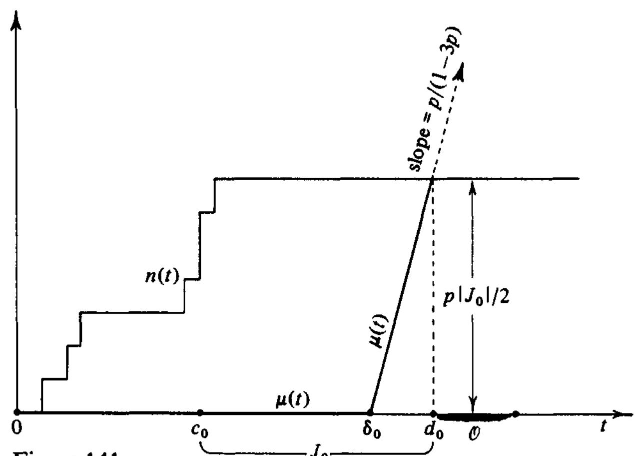

Figure 141

For $k \geqslant 1$, we need to define $\mu(t)$ on $[c_k, d_k]$ in a manner compatible with our requirements, so as to make

$$\int_{c_k}^{d_k} F(t)\frac{\mu(t)-n(t)}{t}\,dt \;\leqslant\; 0.$$

Here, $\mathscr{O}$ includes intervals of the form

$$(c_k - \delta,\ c_k)$$

and

$$(d_k,\ d_k + \delta)$$

where $\delta > 0$, so, when $t \in (c_k, d_k)$, $F(t) \to -\infty$ for $t \to c_k$ and $F(t) \to \infty$ for $t \to d_k$. Moreover, for such t,

$$F'(t) \;=\; 4t\int_{\mathscr{O}}\frac{dx}{(x^2-t^2)^2} \;>\; 0,$$

so there is *precisely one point* $t_k \in (c_k, d_k)$ *where* $F(t)$ *vanishes*, and $F(t) < 0$ for $c_k < t < t_k$, while $F(t) > 0$ for $t_k < t < d_k$. We see that in order to make

$$\int_{c_k}^{d_k} F(t)\frac{\mu(t)-n(t)}{t}\,dt \;\leqslant\; 0,$$

it is enough to define $\mu(t)$ so as to make

$$\mu(t) \;\geqslant\; n(t) \qquad \text{for } c_k \leqslant t < t_k$$

and

$$\mu(t) \;\leqslant\; n(t) \qquad \text{for } t_k \leqslant t \leqslant d_k.$$

The following diagram shows how to do this:

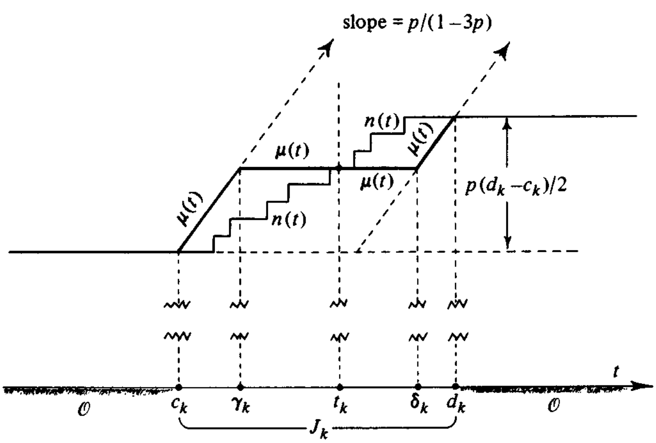

Figure 142

We carry out this construction on each of the J_k. When we are done we will have a function $\mu(t)$, defined for $t \geqslant 0$, with the following properties:

(i) $\mu(t)$ is piecewise linear and increasing, and constant on each interval component of

$$(0, \infty) \sim \bigcup_{k \geqslant 0} J_k;$$

(ii) on each of the intervals J_k, $\mu(t)$ increases by $p|J_k|/2$;

(iii) on J_0, $\mu(t)$ has *slope zero* for $c_0 < t < \delta_0$ and *slope* $p/(1-3p)$ for $\delta_0 < t < d_0$, where $(d_0 - \delta_0)/(d_0 - c_0) = (1-3p)/2$;

(iv) on each J_k, $k \geqslant 1$, $\mu(t)$ has *slope zero* for $\gamma_k < t < \delta_k$ and *slope* $p/(1-3p)$ in the intervals (c_k, γ_k) and (δ_k, d_k), where $c_k < \gamma_k < \delta_k < d_k$ and

$$\frac{\gamma_k - c_k + d_k - \delta_k}{d_k - c_k} = \frac{1-3p}{2};$$

(v) $$\int_{\mathcal{O}} \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\mu(t) \frac{dx}{x^2} \leqslant \int_{\mathcal{O}} \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| dn(t) \frac{dx}{x^2}.$$

Here is a drawing of the graph of $\mu(t)$ vs. t which the reader will do well to look at from time to time while reading the following articles:

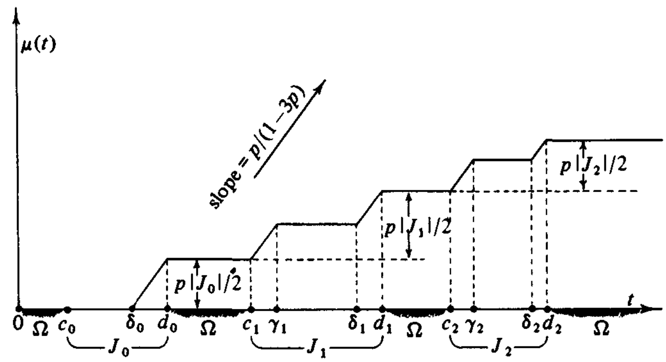

Figure 143

In what follows, we will in fact be working with integrals not over $\mathcal{O}$, but over the set $\Omega = (0, c_0) \cup \mathcal{O} = (0, \infty) \sim \bigcup_{k>0} J_k$ (see the diagram). Since our function $\mu(t)$ is *zero* for $t \leqslant c_0$, we certainly have

$$\int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\mu(t) \leqslant 0$$

for $0 < t < c_0$. Hence, by property (v),

$$\int_\Omega \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\mu(t) \frac{\mathrm{d}x}{x^2} \leqslant \int_\mathscr{O} \frac{\log|P(x)|}{x^2}\mathrm{d}x$$

for our polynomial P. And, as we have seen at the beginning of this article, the right-hand integral is in turn

$$\leqslant 5\sum_1^\infty \frac{\log^+|P(m)|}{m^2}.$$

What we have here is a

Theorem. *Let $0 < p < 1/20$ and $0 < \eta < \frac{2}{3}$, and suppose that*

$$\sup_{t>0} \frac{n(t)}{t} > \frac{p}{1-3p}.$$

Then there are intervals $J_k \subset (0, \infty)$, $k \geqslant 0$, fulfilling the conditions enumerated in the theorem of the preceding article, and a piecewise linear increasing function $\mu(t)$, related to those J_k in the manner just described, such that

$$\int_\Omega \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\mu(t) \frac{\mathrm{d}x}{x^2} \leqslant 5\sum_1^\infty \frac{\log^+|P(m)|}{m^2}$$

for the polynomial $P(x)$.

Here,

$$\Omega = (0, \infty) \sim \bigcup_{k \geqslant 0} J_k.$$

Our problem has thus boiled down to the *purely analytical one* of finding a positive lower bound for

$$\int_\Omega \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\mu(t) \frac{\mathrm{d}x}{x^2}$$

when $\mu(t)$ has the very special form shown in the above diagram. Note that here $|J_0|/d_0 \geqslant \eta$ according to the theorem of the preceding article.

4. Some formulas

The problem, formulated at the end of the last article, to which we have succeeded in reducing our original one seems at first glance to be rather easy – one feels that one can just sit down and *compute*

$$\int_\Omega \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\mu(t) \frac{\mathrm{d}x}{x^2}.$$

This, however, is far from being the case, and quite formidable difficulties still stand in our way. The trouble is that the intervals J_k to which μ is related may be exceedingly numerous, and we have no control over their positions relative to each other, nor on their relative lengths. To handle our task, we are going to need all the formulas we can muster.

Lemma. *Let $\nu(t)$ be increasing on $[0,\infty)$, with $\nu(0)=0$ and $\nu(t)=O(t)$ for $t\to 0$ and for $t\to\infty$. Then, for $x\in\mathbb{R}$,*

$$\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right| d\nu(t) \;=\; -x\int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu(t)}{t}\right).$$

Proof. Both sides are even functions of x and zero for $x=0$, so we may as well assume that $x>0$. If $\nu(t)$ has a (jump) discontinuity at x, both sides are clearly equal to $-\infty$, so we may suppose $\nu(t)$ continuous at x.

We have

$$\int_0^x \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu(t)}{t}\right) \;=\; \int_0^x \frac{1}{t}\log\left|\frac{x+t}{x-t}\right| d\nu(t)$$
$$-\int_0^x \frac{1}{t^2}\log\left|\frac{x+t}{x-t}\right| \nu(t)\,dt.$$

Using the identity

$$\int \frac{1}{t^2}\log\left|\frac{x+t}{x-t}\right| dt \;=\; -\frac{1}{t}\log\left|\frac{x+t}{x-t}\right| - \frac{1}{x}\log\left|1-\frac{x^2}{t^2}\right|,$$

we integrate the second term on the right by parts, obtaining for it the value

$$-\;\frac{2\nu(x)\log 2}{x} \;+\; \int_0^x \left(\frac{1}{t}\log\left|\frac{x+t}{x-t}\right| + \frac{1}{x}\log\left|1-\frac{x^2}{t^2}\right|\right) d\nu(t),$$

taking into account the given behaviour of $\nu(t)$ near 0. Hence

$$\int_0^x \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu(t)}{t}\right) \;=\; \frac{2\nu(x)\log 2}{x} \;-\; \frac{1}{x}\int_0^x \log\left|1-\frac{x^2}{t^2}\right| d\nu(t).$$

In the same way, we get

$$\int_x^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu(t)}{t}\right)$$
$$=\; -\left(\frac{2\nu(x)\log 2}{x}\right) \;-\; \frac{1}{x}\int_x^\infty \log\left|1-\frac{x^2}{t^2}\right| d\nu(t).$$

Adding these last two relations gives us the lemma.

Corollary. *Let* $\nu(t)$ *be* increasing and bounded *on* $[0,\infty)$, *and* zero *for all* t *sufficiently close to* 0. *Let* $\omega(x)$ *be increasing on* $[0,\infty)$, constant *for all sufficiently large* x, *and continuous at* 0. *Then*

$$\int_0^\infty\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right| d\nu(t)\frac{dx-d\omega(x)}{x^2}$$
$$= \int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu(t)}{t}\right)\frac{d\omega(x)}{x}.$$

Proof. By the lemma, the left-hand side equals

$$\int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu(t)}{t}\right)\frac{d\omega(x)-dx}{x}.$$

Our condition on ν makes

$$\int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu(t)}{t}\right)\frac{dx}{x}$$

absolutely convergent, so we can change the order of integration. For $t>0$,

$$\int_0^\infty \log\left|\frac{x+t}{x-t}\right|\frac{dx}{x}$$

assumes a constant value (equal to $\pi^2/2$ as shown by contour integration – see Problem 20), so, since in our present circumstances

$$\int_0^\infty d\left(\frac{\nu(t)}{t}\right) = 0,$$

the previous double integral vanishes, and the corollary follows.*

In our application of these results we will take

$$\nu(t) = \frac{1-3p}{p}\mu(t),$$

$\mu(t)$ being the function constructed in the previous article. This function $\nu(t)$ *increases with constant slope* 1 on each of the intervals $[\delta_k, d_k]$, $k\geqslant 0$, and $[c_k, \gamma_k]$, $k\geqslant 1$, and is *constant* on each of the intervals *complementary* to those. Therefore, if

$$\tilde{\Omega} = (0,\infty)\sim\bigcup_{k\geqslant 1}[c_k,\gamma_k]\sim\bigcup_{k\geqslant 0}[\delta_k,d_k]$$

* The two sides of the relation established may both be infinite, e.g., when $\nu(t)$ and $\omega(t)$ have some coinciding jumps. But the meaning of the two iterated integrals in question is always unambiguous; in the second one, for instance, the outer integral of the *negative part* of the inner one converges.

(note that this set $\tilde{\Omega}$ *includes* our Ω), we have

$$\int_{\tilde{\Omega}} \int_0^{\infty} \log\left|1-\frac{x^2}{t^2}\right| d\mu(t)\frac{dx}{x^2}$$
$$= \frac{p}{1-3p}\int_0^{\infty}\int_0^{\infty} \log\left|1-\frac{x^2}{t^2}\right| d\nu(t)\frac{dx - d\nu(x)}{x^2}.$$

The corollary shows that this expression (which we can think of as a first approximation to

$$\int_{\Omega} \int_0^{\infty} \log\left|1-\frac{x^2}{t^2}\right| d\mu(t)\frac{dx}{x^2} \;)$$

is equal to

$$\frac{p}{1-3p}\int_0^{\infty}\int_0^{\infty} \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu(t)}{t}\right)\frac{d\nu(x)}{x}.$$

This double integral can be given a symmetric form thanks to the

Lemma. *Let $\nu(t)$ be continuous, increasing, and piecewise continuously differentiable on $[0,\infty]$. Suppose, moreover, that $\nu(0) = 0$, that $\nu(t)$ is constant for t sufficiently large, and, finally*, that $(d/dt)(\nu(t)/t)$ remains bounded when $t \to 0+$. Then,*

$$\int_0^{\infty}\int_0^{\infty} \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu(t)}{t}\right)\frac{\nu(x)}{x^2}dx \;=\; -\frac{\pi^2}{4}(\nu'(0))^2.$$

Proof. Our assumptions on ν make reversal of the order of integrations in the left-hand expression legitimate, so it is equal to

$$\int_0^{\infty}\int_0^{\infty} \log\left|\frac{x+t}{x-t}\right| \frac{\nu(x)}{x^2}dx\, d\left(\frac{\nu(t)}{t}\right)$$
$$= \int_0^{\infty}\left(\int_0^{\infty} \log\left|\frac{\xi+1}{\xi-1}\right| \frac{\nu(t\xi)}{t\xi}\frac{d\xi}{\xi}\right) d\left(\frac{\nu(t)}{t}\right).$$

Since

$$\int_0^{\infty} \log\left|\frac{\xi+1}{\xi-1}\right| \frac{d\xi}{\xi} \;=\; \frac{\pi^2}{2}$$

(which may be verified by contour integration), we have

$$\int_0^{\infty} \log\left|\frac{\xi+1}{\xi-1}\right| \frac{\nu(t\xi)}{t\xi}\frac{d\xi}{\xi} \;\longrightarrow\; \frac{\pi^2}{2}\nu'(0)$$

for $t \to 0$, and integration by parts of the outer integral in the previous

* This last condition can be relaxed. See problem 28(b), p. 569.

expression yields the value

$$-\frac{\pi^2}{2}(v'(0))^2 - \int_0^\infty \frac{v(t)}{t}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^\infty \log\left|\frac{\xi+1}{\xi-1}\right|\frac{v(t\xi)}{t\xi}\frac{\mathrm{d}\xi}{\xi}\right)\mathrm{d}t.$$

Under the conditions of our hypothesis, the differentiation with respect to t can be carried out under the inner integral sign. The last expression thus becomes

$$-\frac{\pi^2}{2}(v'(0))^2 - \int_0^\infty \frac{v(t)}{t}\int_0^\infty \log\left|\frac{\xi+1}{\xi-1}\right|\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v(t\xi)}{t\xi}\right)\frac{\mathrm{d}\xi}{\xi}\,\mathrm{d}t$$

$$= -\frac{\pi^2}{2}(v'(0))^2 - \int_0^\infty \frac{v(t)}{t}\int_0^\infty \log\left|\frac{x+t}{x-t}\right|\frac{x}{t}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{v(x)}{x}\right)\frac{\mathrm{d}x}{x}\,\mathrm{d}t.$$

In other words

$$\int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right|\mathrm{d}\left(\frac{v(t)}{t}\right)\frac{v(x)}{x^2}\,\mathrm{d}x$$

$$= -\frac{\pi^2}{2}(v'(0))^2 - \int_0^\infty\int_0^\infty \log\left|\frac{t+x}{t-x}\right|\mathrm{d}\left(\frac{v(x)}{x}\right)\frac{v(t)}{t^2}\,\mathrm{d}t.$$

The second term on the right obviously equals the left-hand side, so the lemma follows.

Corollary. *Let $v(t)$ be increasing, continuous, and piecewise linear on* $[0,\infty)$, constant *for all sufficiently large t and* zero *for t near* 0. *Then*

$$\int_0^\infty\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right|\mathrm{d}v(t)\frac{\mathrm{d}x-\mathrm{d}v(x)}{x^2}$$

$$= \int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right|\mathrm{d}\left(\frac{v(t)}{t}\right)\mathrm{d}\left(\frac{v(x)}{x}\right).$$

Proof. By the previous corollary, the left-hand expression equals

$$\int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right|\mathrm{d}\left(\frac{v(t)}{t}\right)\frac{\mathrm{d}v(x)}{x}.$$

In the present circumstances, $v'(0)$ exists and equals *zero*. Therefore by the lemma

$$\int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right|\mathrm{d}\left(\frac{v(t)}{t}\right)\frac{v(x)}{x^2}\,\mathrm{d}x = 0,$$

and the previous expression is equal to

$$\int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right|\mathrm{d}\left(\frac{v(t)}{t}\right)\mathrm{d}\left(\frac{v(x)}{x}\right).$$

Problem 21

Prove the last lemma using contour integration. (Hint: For $\Im z > 0$, consider the analytic function

$$F(z) = \frac{1}{\pi}\int_0^\infty \log\left(\frac{z+t}{z-t}\right) \mathrm{d}\left(\frac{\nu(t)}{t}\right),$$

and examine the boundary values of $\Re F(z)$ and $\Im F(z)$ on the real axis. Then look at $\int_\Gamma ((F(z))^2/z)\,\mathrm{d}z$ for a suitable contour Γ.)

5. The energy integral

The expression, quadratic in $\mathrm{d}(\nu(t)/t)$, arrived at near the end of the previous article, namely,

$$\int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\left(\frac{\nu(t)}{t}\right)\mathrm{d}\left(\frac{\nu(x)}{x}\right),$$

has a simple physical interpretation. Let us assume that a flat metal plate of infinite extent, perpendicular to the z-plane, intersects the latter along the y-axis. This plate we suppose grounded. Let electric charge be continuously distributed on a very large thin sheet, made of non-conducting material, and intersecting the z-plane perpendicularly along the positive x-axis. Suppose the charge density on that sheet to be constant along lines perpendicular to the z-plane, and that the total charge contained in any rectangle of height 2 thereon, bounded by two such lines intersecting the x-axis at x and at $x + \Delta x$, is equal to the *net change* of $\nu(t)/t$ along $[x,\ x + \Delta x]$. This set-up will produce an *electric field* in the region lying to the right of the grounded metal plate; *near* the z-plane, the *potential function* for that field is equal, very nearly, to

$$u(z) = \int_0^\infty \log\left|\frac{z+t}{z-t}\right| \mathrm{d}\left(\frac{\nu(t)}{t}\right).$$

The quantity

$$\int_0^\infty u(x)\mathrm{d}\left(\frac{\nu(x)}{x}\right) = \int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\left(\frac{\nu(t)}{t}\right)\mathrm{d}\left(\frac{\nu(x)}{x}\right)$$

is then proportional to the *total energy* of the electric field generated by our distribution of electric charge (and inversely proportional to the height of the charged sheet). We therefore expect it to be *positive*, even though *charges of both sign* be present at different places on the non-conducting sheet, i.e., when $\mathrm{d}(\nu(t)/t)/\mathrm{d}t$ is *not of constant sign.*

Under quite general circumstances, the *positivity* of the quadratic form in question turns out to be *valid*, and plays a crucial rôle in the computations of the succeeding articles. In the present one, we derive two formulas, either of which makes that property evident.

The first formula is familiar from physics, and goes back to Gauss. It is convenient to write

$$\rho(t) \;=\; \frac{\nu(t)}{t}.$$

Lemma. *Let $\rho(t)$ be continuous on $[0,\infty)$, piecewise $\mathscr{C}_3$ there (say), and differentiable at 0. Suppose furthermore that $\rho(t)$ is uniformly* Lip 1 *on $[0,\infty)$ and $t\rho(t)$* constant *for sufficiently large t.*

If we write

$$u(z) \;=\; \int_0^\infty \log\left|\frac{z+t}{z-t}\right| \mathrm{d}\rho(t),$$

we have

$$\boxed{\int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t)\,\mathrm{d}\rho(x) \;=\; \frac{1}{\pi}\int_0^\infty\int_0^\infty \{(u_x(z))^2 + (u_y(z))^2\}\,\mathrm{d}x\,\mathrm{d}y.}$$

Remark 1. Note that we *do not require* that $\rho(t)$ *vanish* for t *near zero*, although $\rho(t) = \nu(t)/t$ has this property when $\nu(t)$ is the function introduced in the previous article.

Remark 2. The factor $1/\pi$ occurs on the right, and not $1/2\pi$ which one might expect from physics, because the right-hand integral is taken over the *first quadrant instead* of over the *whole right half plane* (where the 'electric field' is present). The right-hand expression is of course the *Dirichlet integral* of u over the first quadrant.

Remark 3. The function $u(z)$ is *harmonic* in each separate quadrant of the z-plane. Since

$$\log\left|\frac{z+\bar{w}}{z-w}\right|$$

is the Green's function for the right half plane, $u(z)$ is frequently referred to as the *Green potential of the charge distribution* $\mathrm{d}\rho(t)$ (for that half plane).

Proof of lemma. For $y > 0$, we have

$$u_y(z) \;=\; \int_0^\infty \left(\frac{y}{(x+t)^2+y^2} - \frac{y}{(x-t)^2+y^2}\right)\mathrm{d}\rho(t),$$

and, when $x > 0$ is not a point of discontinuity for $\rho'(t)$, the right side tends to $-\pi\rho'(x)$ as $y \to 0+$ by the usual (elementary) approximate identity property of the Poisson kernel. Thus,

$$u_y(x+\mathrm{i}0) \;=\; -\pi\rho'(x),$$

and

$$\int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t)\,\mathrm{d}\rho(x) \;=\; -\frac{1}{\pi}\int_0^\infty u(x)u_y(x+\mathrm{i}0)\,\mathrm{d}x.$$

At the same time, $u(\mathrm{i}y) = 0$ for $y > 0$, so the left-hand double integral from the previous relation is equal to

$$-\frac{1}{\pi}\int_0^\infty u(x)u_y(x+\mathrm{i}0)\,\mathrm{d}x \;-\; \frac{1}{\pi}\int_0^\infty u(\mathrm{i}y)u_x(\mathrm{i}y)\,\mathrm{d}y.$$

We have here a line integral around the boundary of the first quadrant. Applying Green's theorem to it in cook-book fashion, we get the value

$$\frac{1}{\pi}\int_0^\infty \int_0^\infty \left(\frac{\partial}{\partial y}(u(z)u_y(z)) + \frac{\partial}{\partial x}(u(z)u_x(z))\right)\mathrm{d}x\,\mathrm{d}y,$$

which reduces immediately to

$$\frac{1}{\pi}\int_0^\infty \int_0^\infty ((u_y(z))^2 + (u_x(z))^2)\,\mathrm{d}x\,\mathrm{d}y$$

(proving the lemma), since u is *harmonic* in the first quadrant, making $u\nabla^2 u = 0$ there.

We have, however, to justify our use of Green's theorem. The way to do that here is to adapt to our present situation the common 'non-rigorous' derivation of the theorem (using squares) found in books on engineering mathematics. Letting $\mathscr{D}_A$ denote the square with vertices at 0, A, $A+\mathrm{i}A$ and $\mathrm{i}A$, we verify in that way without difficulty (and without any being created by the discontinuities of $\rho'(x) = -u_y(x+\mathrm{i}0)/\pi$), that

$$\int_{\partial\mathscr{D}_A} (uu_x\mathrm{d}y - uu_y\mathrm{d}x) \;=\; \iint_{\mathscr{D}_A} (u_x^2 + u_y^2)\,\mathrm{d}x\,\mathrm{d}y.*$$

The line integral on the left equals

$$-\int_0^A u(x)u_y(x+\mathrm{i}0)\,\mathrm{d}x \;+\; \int_{\Gamma_A} (uu_x\mathrm{d}y - uu_y\mathrm{d}x),$$

where Γ_A denotes the *right side* and *top* of $\mathscr{D}_A$:

* The simplest procedure is to take $h > 0$ and write the corresponding relation involving $u(z+\mathrm{i}h)$ in place of $u(z)$, whose truth is certain here. Then one can make $h \to 0$. Cf the discussion on pp. 506–7.

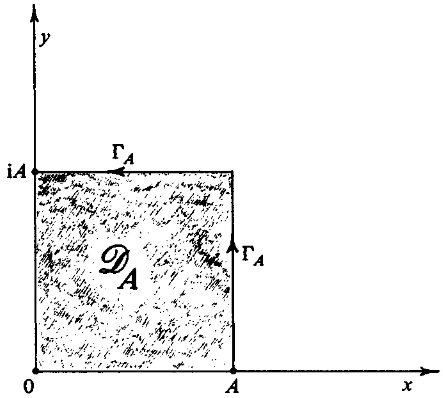

Figure 144

We will be done if we show that

$$\int_{\Gamma_A} (uu_x \mathrm{d}y - uu_y \mathrm{d}x) \longrightarrow 0 \qquad \text{for } A \longrightarrow \infty.$$

For this purpose, one may break up $u(z)$ as

$$\int_0^M \log\left|\frac{z+t}{z-t}\right| \mathrm{d}\rho(t) \;+\; \int_M^\infty \log\left|\frac{z+t}{z-t}\right| \mathrm{d}\rho(t),$$

M being chosen large enough so as to have $\rho(t) = C/t$ on $[M, \infty)$. Calling the *first* of these integrals $u_1(z)$, we easily find, for $|z| > M$ (by expanding the logarithm in powers of t/z), that

$$|u_1(z)| \;\leqslant\; \frac{\text{const.}}{|z|}$$

and that the first partial derivatives of $u_1(z)$ are $O(1/|z|^2)$.

Denote by $u_2(z)$ the *second* of the above integrals, which, by choice of M, is actually equal to

$$-C\int_M^\infty \log\left|\frac{z+t}{z-t}\right| \frac{\mathrm{d}t}{t^2}.$$

The substitution $t = |z|\tau$ enables us to see after very little calculation that this expression is in modulus

$$\leqslant \text{const.} \frac{\log|z|}{|z|}$$

for large $|z|$.

To investigate the partial derivatives of $u_2(z)$ in the open first quadrant, we take the function

$$F(z) \;=\; \int_M^\infty \log\left(\frac{z+t}{z-t}\right)\frac{\mathrm{d}t}{t^2},$$

analytic in that region, and note that by the Cauchy–Riemann equations,

$$\frac{\partial u_2(z)}{\partial x} - i\frac{\partial u_2(z)}{\partial y} = -CF'(z)$$

there. Here,

$$F'(z) = \int_M^\infty \frac{dt}{t^2(z+t)} - \int_M^\infty \frac{dt}{t^2(z-t)}.$$

The first term on the right is obviously $O(1/|z|)$ in modulus when $\Re z$ and $\Im z > 0$. The second works out to

$$\int_M^\infty \left(\frac{1}{zt^2} + \frac{1}{z^2 t} + \frac{1}{z^2(z-t)}\right)dt = \frac{1}{zM} + \frac{1}{z^2}\log\left(\frac{z-M}{M}\right),$$

using a suitable determination of the logarithm. This is evidently $O(1/|z|)$ for large $|z|$, so $|F'(z)| = O(1/|z|)$ for z with large modulus in the first quadrant. The same is thus true for the first partial derivatives of $u_2(z)$.

Combining the estimates just made on $u_1(z)$ and $u_2(z)$, we find for $u = u_1 + u_2$ that

$$|u(z)| \leqslant \text{const.}\frac{\log|z|}{|z|}$$

$$|u_x(z)| \leqslant \text{const.}\frac{1}{|z|}$$

$$|u_y(z)| \leqslant \text{const.}\frac{1}{|z|}$$

when $\Re z > 0$, $\Im z > 0$, $|z|$ being large. Therefore

$$\int_{\Gamma_A} (uu_x dy - uu_y dx) = O\left(\frac{\log A}{A}\right)$$

for large A, and the line integral tends to zero as $A \to \infty$. This is what was needed to finish the proof of the lemma. We are done.

Corollary. *If* $\rho(t)$ *is* real *and satisfies the hypothesis of the lemma,*

$$\int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\rho(t)\,d\rho(x) \geqslant 0.$$

Proof. Clear.

Notation. We write

$$\boxed{E(d\rho(t), d\sigma(t)) = \int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\rho(t)\,d\sigma(x)}$$

for *real* measures ρ and σ on $[0, \infty)$ without point mass at the origin making both of the integrals

$$\int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t)\,\mathrm{d}\rho(x), \qquad \int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\sigma(t)\,\mathrm{d}\sigma(x)$$

absolutely convergent. (Vanishing of $\rho(\{0\})$ and $\sigma(\{0\})$ is required because $\log|(x+t)/(x-t)|$ cannot be defined at $(0,0)$ so as to be continuous there.)

Note that, in the case of functions $\rho(t)$ and $\sigma(t)$ satisfying the hypothesis of the above lemma, the integrals just written *do* converge absolutely. In terms of $E(\mathrm{d}\rho(t), \mathrm{d}\sigma(t))$, we can state the very important

Corollary. *If $\rho(t)$ and $\sigma(t)$, defined and real valued on $[0, \infty)$, both satisfy the hypothesis of the lemma,*

$$|E(\mathrm{d}\rho(t), \mathrm{d}\sigma(t))| \leqslant \sqrt{(E(\mathrm{d}\rho(t), \mathrm{d}\rho(t)))}\cdot\sqrt{(E(\mathrm{d}\sigma(t), \mathrm{d}\sigma(t)))}.$$

Proof. Use the preceding corollary and proceed as in the usual derivation of Schwarz' inequality.

Remark. The result remains valid as long as ρ and σ, with $\rho(\{0\}) = \sigma(\{0\}) = 0$, are such that the abovementioned absolute convergence holds. We will see that at the end of the present article.

Scholium and warning. The results just given should not mislead the reader into believing that the energy integral corresponding to the *ordinary logarithmic potential* is necessarily positive. Example:

$$\int_0^{2\pi} \int_0^{2\pi} \log\frac{1}{|2\mathrm{e}^{\mathrm{i}\vartheta} - 2\mathrm{e}^{\mathrm{i}\varphi}|}\,\mathrm{d}\vartheta\,\mathrm{d}\varphi = \int_0^{2\pi} 2\pi\log\tfrac{1}{2}\,\mathrm{d}\varphi = -4\pi^2\log 2\ !$$

It is strongly recommended that the reader find out exactly where the argument used in the proof of the lemma *goes wrong*, when one attempts to adapt it to the potential

$$u(z) = \int_0^{2\pi} \log\frac{1}{|2\mathrm{e}^{\mathrm{i}\vartheta} - z|}\,\mathrm{d}\vartheta.$$

For 'nice' real measures μ of compact support, *it is true that*

$$\int_{\mathbb{C}} \int_{\mathbb{C}} \log\frac{1}{|z-w|}\,\mathrm{d}\mu(z)\,\mathrm{d}\mu(w) \geqslant 0$$

provided that $\int_{\mathbb{C}} \mathrm{d}\mu(z) = 0$. *The reader should verify this fact* by applying a suitable version of Green's theorem to the potential $\int_{\mathbb{C}} \log(1/|z-w|)\,\mathrm{d}\mu(w)$.

The formula for $E(\mathrm{d}\rho(t), \mathrm{d}\rho(t))$ furnished by the above lemma exhibits that quantity's positivity. The same service is rendered by an analogous relation involving the values of $\rho(t)$ on $[0, \infty)$. Such representations go back to Jesse Douglas; we are going to use one based on a beautiful identity of Beurling. In order to encourage the reader's participation, we set as a problem the derivation of Beurling's result.

Problem 22

(a) Let m be a real measure on $\mathbb{R}$. Suppose that $h > 0$ and that $\iint_{|\xi-\eta|\leq h} dm(\xi)\, dm(\eta)$ converges absolutely. Show that

$$\int_{-\infty}^{\infty} (m(x+h) - m(x))^2 \, dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (h - |\xi - \eta|)^+ \, dm(\xi) dm(\eta).$$

(Hint: Trick:

$$(m(x+h) - m(x))^2 \;=\; \int_x^{x+h}\int_x^{x+h} dm(\xi) dm(\eta).\ )$$

(b) Let $K(x)$ be even and positive, $\mathscr{C}_2$ and *convex* for $x > 0$, and such that $K(x) \to 0$ for $x \to \infty$. Show that, for $x \neq 0$,

$$K(x) \;=\; \int_0^{\infty} (h - |x|)^+ K''(h) dh.$$

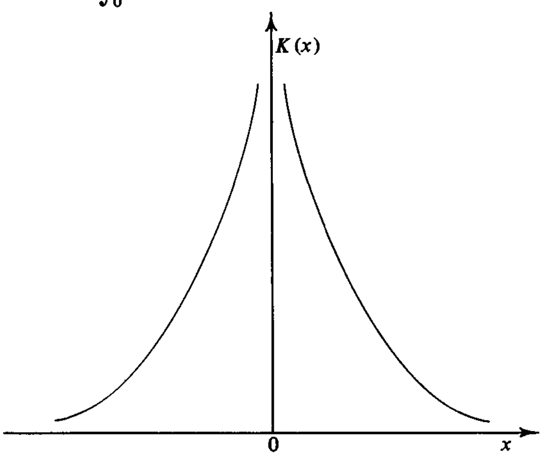

Figure 145

(Hint: First observe that $K'(x)$ must also $\to 0$ for $x \to \infty$.)

(c) If $K(x)$ is as in (b) and m is a real measure on $\mathbb{R}$ with $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(|\xi - \eta|) dm(\xi)\, dm(\eta)$ absolutely convergent, that integral is equal to

$$\int_{-\infty}^{\infty}\int_0^{\infty} [m(x+h) - m(x)]^2 K''(h) \, dh \, dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [m(y) - m(x)]^2 K''(|x-y|) \, dy \, dx.$$

(Hint: The assumed absolute convergence guarantees that m fulfills, for each $h > 0$, the condition required in part (a). The order of integration in

$$\int_{-\infty}^{\infty}\int_{0}^{\infty} K''(h)(m(x+h)-m(x))^2\,\mathrm{d}h\,\mathrm{d}x$$

may be reversed, yielding, by part (a), an iterated triple integral. Here, that triple integral is absolutely convergent and we may conclude by the help of part (b).)

Lemma. *Let the real measure ρ on $[0,\infty)$, without point mass at the origin, be such that*

$$\int_0^{\infty}\int_0^{\infty}\log\left|\frac{x+t}{x-t}\right|\mathrm{d}\rho(t)\,\mathrm{d}\rho(x)$$

is absolutely convergent. Then

$$\boxed{\int_0^{\infty}\int_0^{\infty}\log\left|\frac{x+t}{x-t}\right|\mathrm{d}\rho(t)\,\mathrm{d}\rho(x) = \int_0^{\infty}\int_0^{\infty}\left(\frac{\rho(x)-\rho(y)}{x-y}\right)^2\frac{x^2+y^2}{(x+y)^2}\,\mathrm{d}x\,\mathrm{d}y.}$$

Proof. The left-hand double integral is of the form

$$\int_0^{\infty}\int_0^{\infty} k\left(\frac{x}{t}\right)\mathrm{d}\rho(x)\,\mathrm{d}\rho(t),$$

where

$$k(\tau) = \log\left|\frac{1+\tau}{1-\tau}\right| = k\left(\frac{1}{\tau}\right),$$

so we can reduce that integral to one figuring in Problem 22(c) by making the substitutions $x = \mathrm{e}^{\xi}$, $t = \mathrm{e}^{\eta}$, $\rho(x) = m(\xi)$, $\rho(t) = m(\eta)$, and

$$k\left(\frac{x}{t}\right) = K(\xi-\eta) = \log\left|\coth\left(\frac{\xi-\eta}{2}\right)\right|.$$

$K(h)$, besides being obviously even and positive, tends to zero for $h \to \infty$. Also

$$K'(h) = \frac{1}{2}\tanh\frac{h}{2} - \frac{1}{2}\coth\frac{h}{2},$$

and

$$K''(h) = \frac{1}{4}\operatorname{sech}^2\frac{h}{2} + \frac{1}{4}\operatorname{cosech}^2\frac{h}{2} > 0,$$

so $K(h)$ is convex for $h > 0$. The application of Beurling's formula from problem 22(c) is therefore legitimate, and yields

$$\int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t)\,\mathrm{d}\rho(x)$$

$$= \int_{-\infty}^\infty\int_{-\infty}^\infty K(|\xi-\eta|)\mathrm{d}m(\eta)\mathrm{d}m(\xi)$$

$$= \frac{1}{2}\int_{-\infty}^\infty\int_{-\infty}^\infty K''(|\xi-\eta|)[m(\xi)-m(\eta)]^2\,\mathrm{d}\xi\,\mathrm{d}\eta$$

(note that the first of these integrals, *and hence the second*, is absolutely convergent by hypothesis).

Here,

$$K''(|\xi-\eta|) = \frac{1}{4}\,\frac{\sinh^2\left(\dfrac{\xi-\eta}{2}\right) + \cosh^2\left(\dfrac{\xi-\eta}{2}\right)}{\sinh^2\left(\dfrac{\xi-\eta}{2}\right)\cosh^2\left(\dfrac{\xi-\eta}{2}\right)}$$

$$= \frac{\cosh(\xi-\eta)}{\sinh^2(\xi-\eta)} = 2\mathrm{e}^\xi\mathrm{e}^\eta\,\frac{\mathrm{e}^{2\xi}+\mathrm{e}^{2\eta}}{(\mathrm{e}^{2\xi}-\mathrm{e}^{2\eta})^2},$$

so the third of the above expressions reduces to

$$\int_{-\infty}^\infty\int_{-\infty}^\infty \frac{\mathrm{e}^{2\xi}+\mathrm{e}^{2\eta}}{(\mathrm{e}^\xi+\mathrm{e}^\eta)^2}\left(\frac{m(\xi)-m(\eta)}{\mathrm{e}^\xi-\mathrm{e}^\eta}\right)^2 \mathrm{e}^\xi\mathrm{e}^\eta\,\mathrm{d}\xi\,\mathrm{d}\eta$$

$$= \int_0^\infty\int_0^\infty \frac{x^2+t^2}{(x+t)^2}\left(\frac{\rho(x)-\rho(t)}{x-t}\right)^2 \mathrm{d}x\,\mathrm{d}t.$$

We are done.

Remark. This certainly implies that the *first* of the above corollaries is true for *any real measure ρ with $\rho(\{0\}) = 0$ rendering absolutely convergent* the double integral used to define $E(\mathrm{d}\rho(t), \mathrm{d}\rho(t))$. The *second* corollary is then *also true* for such real measures ρ and σ.

The formula provided by this second lemma is one of the main ingredients in our treatment of the question discussed in the present §. It is the basis for the important calculation carried out in the next article.

6. A lower estimate for $\displaystyle\int_{\tilde{\Omega}}\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right| d\mu(t)\,\frac{dx}{x^2}$

We return to where we left off near the end of article 4, focusing our attention on the quantity

$$\int_{\tilde{\Omega}}\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right| d\mu(t)\,\frac{dx}{x^2},$$

where $\mu(t)$ is the function constructed in article 3 and

$$\tilde{\Omega} \;=\; (0,\ \infty) \;\sim\; \{x\colon\ \mu'(x)>0\}.$$

Before going any further, the reader should refer to the graph of $\mu(t)$ found near the end of article 3. As explained in article 4, we prefer to work not with $\mu(t)$, but with

$$\nu(t) \;=\; \frac{1-3p}{p}\,\mu(t);$$

the graph of $\nu(t)$ looks just like that of $\mu(t)$, save that its slanting portions all have slope 1, and not $p/(1-3p)$. Those slanting portions lie over certain intervals $[c_k,\ \gamma_k]$, $k\geqslant 1$, $[\delta_k,\ d_k]$, $k\geqslant 0$, contained in the $J_k=[c_k,\ d_k]$, and

$$\tilde{\Omega} \;=\; (0,\infty)\sim\bigcup_{k\geqslant 0}\,[\delta_k, d_k]\sim\bigcup_{k\geqslant 1}\,[c_k,\gamma_k].$$

This set $\tilde{\Omega}$ is obtained from the one Ω shown on the graph of $\mu(t)$ by adjoining to the latter the intervals $(c_0,\ \delta_0)\subseteq J_0$ and $(\gamma_k,\ \delta_k)\subseteq J_k,\ k\geqslant 1$.

By the corollary at the end of article 4,

$$\begin{aligned}
&\int_{\tilde{\Omega}}\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right| d\mu(t)\,\frac{dx}{x^2}\\
&\quad= \frac{p}{1-3p}\int_0^\infty\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right| d\nu(t)\,\frac{dx-d\nu(x)}{x^2}\\
&\quad= \frac{p}{1-3p}\int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu(t)}{t}\right)d\left(\frac{\nu(x)}{x}\right),
\end{aligned}$$

and this is just

$$\frac{p}{1-3p}\,E\left(d\left(\frac{\nu(t)}{t}\right),d\left(\frac{\nu(x)}{x}\right)\right),$$

$E(\ \ ,\ \ )$ being the bilinear form defined and studied in the previous article.

This identification is a key step in our work. It, and the results of article

5, enable us to see that

$$\int_{\tilde{\Omega}}\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right| d\mu(t)\frac{dx}{x^2}$$

is at least *positive* (until now, we were not even sure of this). The second lemma of article 5 actually makes it possible for us to estimate that integral from below in terms of a sum,

$$\sum_{k\geqslant 1}\left(\frac{\gamma_k - c_k}{\gamma_k}\right)^2 \;+\; \sum_{k\geqslant 0}\left(\frac{d_k-\delta_k}{d_k}\right)^2,$$

like one which occurred previously in Chapter VII, §A.2. In our estimate, that sum is affected with a certain *coefficient.*

On account of the theorem of article 3, we are *really* interested in

$$\int_{\Omega}\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right| d\mu(t)\frac{dx}{x^2}$$

rather than the quantity considered here. It will turn out later on that the passage from integration over $\tilde{\Omega}$ to that over Ω involves a serious *loss*, in whose evaluation the sum just written again figures. For this reason we have to take care to get *a large enough numerical value* for the coefficient mentioned above. That circumstance requires us to be somewhat fussy in the computation made to derive the following result. *From now on, in* ▶ *order to make the notation more uniform, we will write*

$$\gamma_0 \;=\; c_0.$$

Theorem. *If* $\nu(t) = ((1-3p)/p)\mu(t)$ *with the function* $\mu(t)$ *from article* 3, *and the parameter* $\eta > 0$ *used in the construction of the* J_k *(see the theorem, end of article* 2) *is* sufficiently small, *we have*

$$E\left(d\left(\frac{\nu(t)}{t}\right), d\left(\frac{\nu(t)}{t}\right)\right)$$
$$\geqslant\; (\tfrac{3}{2}-\log 2 - K\eta)\sum_{k\geqslant 0}\left\{\left(\frac{\gamma_k-c_k}{\gamma_k}\right)^2 + \left(\frac{d_k-\delta_k}{d_k}\right)^2\right\}.$$

Here, K is a purely numerical constant, independent of p or the configuration of the J_k.

Remark. Later on, we will need the numerical value

$$\boxed{\tfrac{3}{2}-\log 2 \;=\; 0.80685\ldots.}$$

Proof of theorem. By the second lemma of article 5 and brute force. The lemma gives

$$E\left(\mathrm{d}\left(\frac{v(t)}{t}\right), \mathrm{d}\left(\frac{v(t)}{t}\right)\right)$$

$$= \int_0^\infty\int_0^\infty \left(\frac{\dfrac{v(x)}{x}-\dfrac{v(y)}{y}}{x-y}\right)^2 \frac{x^2+y^2}{(x+y)^2}\,\mathrm{d}x\,\mathrm{d}y$$

$$\geqslant \frac{1}{2}\sum_{k\geqslant 0}\int_{J_k}\int_{J_k}\left(\frac{\dfrac{v(x)}{x}-\dfrac{v(y)}{y}}{x-y}\right)^2 \mathrm{d}x\,\mathrm{d}y.$$

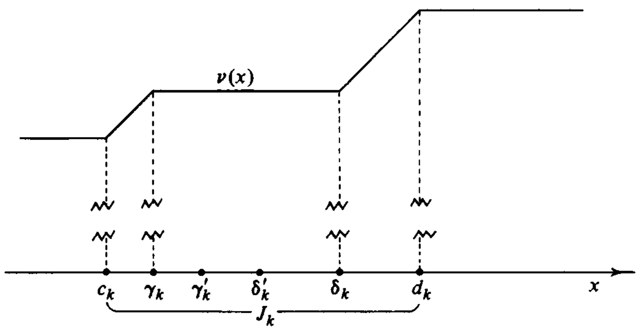

Figure 146

On each interval $J_k = [c_k, d_k]$ we take

$$\begin{aligned}\gamma_k' &= c_k + 2(\gamma_k - c_k)\\ \delta_k' &= d_k - 2(d_k - \delta_k)\end{aligned}$$

(see figure). Since

$$\frac{\gamma_k - c_k + d_k - \delta_k}{d_k - c_k} = \frac{1-3p}{2} < \frac{1}{2}$$

(properties (iii), (iv) of the description near the end of article 3) we have $\gamma_k' < \delta_k'$. Therefore, for each k,

$$\int_{J_k}\int_{J_k}\left(\frac{\dfrac{v(x)}{x}-\dfrac{v(y)}{x}}{x-y}\right)^2 \mathrm{d}x\,\mathrm{d}y$$

$$\geqslant \left\{\int_{c_k}^{\gamma_k'}\int_{c_k}^{\gamma_k'} + \int_{\delta_k'}^{d_k}\int_{\delta_k'}^{d_k}\right\}\left(\frac{\dfrac{v(x)}{x}-\dfrac{v(y)}{y}}{x-y}\right)^2 \mathrm{d}x\,\mathrm{d}y.$$

We estimate the *second* of the integrals on the right – the other one is handled similarly.

We begin by writing

$$\int_{\delta_k'}^{d_k}\int_{\delta_k'}^{d_k}\left(\frac{\frac{v(x)}{x}-\frac{v(x)}{y}}{x-y}\right)^2 \mathrm{d}x\,\mathrm{d}y$$

$$\geqslant \left\{\int_{\delta_k}^{d_k}\int_{\delta_k}^{d_k}+\int_{\delta_k'}^{\delta_k}\int_{\delta_k}^{d_k}+\int_{\delta_k}^{d_k}\int_{\delta_k'}^{\delta_k}\right\}\left(\frac{\frac{v(x)}{x}-\frac{v(y)}{y}}{x-y}\right)^2 \mathrm{d}x\,\mathrm{d}y.$$

Of the three double integrals on the right, the *first* is easiest to evaluate. Things being bad enough as they are, *let us lighten the notation by dropping, for the moment, the subscript k, putting*

δ' for δ_k',

δ for δ_k

and

d for d_k.

Since $v'(x)=1$ for $\delta_k=\delta<x<d=d_k$,

$$\frac{v(x)}{x} = 1+\frac{v(\delta)-\delta}{x}, \qquad \delta\leqslant x\leqslant d.$$

Using this, we easily find that

$$\int_{\delta}^{d}\int_{\delta}^{d}\left(\frac{\frac{v(x)}{x}-\frac{v(y)}{y}}{x-y}\right)^2 \mathrm{d}x\,\mathrm{d}y = \left(1-\frac{v(\delta)}{\delta}\right)^2\left(\frac{d-\delta}{d}\right)^2.$$

In terms of

$$\tilde{J} = \bigcup_{k\geqslant 0}((c_k,\gamma_k)\cup(\delta_k,d_k))$$

and

$$J = \bigcup_{k\geqslant 0} J_k,$$

we have clearly

$$v(t) = |[0,\,t]\cap\tilde{J}| \leqslant |[0,\,t]\cap J|, \qquad t>0.$$

The right-hand quantity is, however, $\leqslant 2\eta t$ by construction of the J_k (property (v) in the theorem at the end of article 2). Therefore

$v(\delta)/\delta \;=\; v(\delta_k)/\delta_k \;\leqslant\; 2\eta$, and the integral just evaluated is

$$\geqslant\; (1-2\eta)^2\left(\frac{d-\delta}{d}\right)^2.$$

We pass now to the *second* of the three double integrals in question, continuing to omit the subscript k. To simplify the work, we make the changes of variable

$$x = \delta + s, \qquad y = \delta - t,$$

and denote $d-\delta = \delta-\delta'$ by Δ. Then

$$\int_{\delta'}^{\delta}\int_{\delta}^{d}\left(\frac{\dfrac{v(x)}{x}-\dfrac{v(y)}{y}}{x-y}\right)^2 \mathrm{d}x\,\mathrm{d}y \;=\; \int_0^{\Delta}\int_0^{\Delta}\left(\frac{\dfrac{v(\delta)+s}{\delta+s}-\dfrac{v(\delta)}{\delta-t}}{s+t}\right)^2 \mathrm{d}s\,\mathrm{d}t,$$

since $v(y) = v(\delta)$ for $\delta' \leqslant y \leqslant \delta$ (see the above figure). The expression on the right simplifies to

$$\int_0^{\Delta}\int_0^{\Delta}\left(\frac{s}{(\delta+s)(t+s)}-\frac{v(\delta)}{(\delta-t)(\delta+s)}\right)^2 \mathrm{d}s\,\mathrm{d}t$$

which in turn is

$$\geqslant\; \frac{1}{d^2}\int_0^{\Delta}\int_0^{\Delta}\left(\frac{s}{t+s}\right)^2 \mathrm{d}t\,\mathrm{d}s \;-\; 2\frac{v(\delta)}{\delta}\cdot\frac{1}{\delta'\delta}\int_0^{\Delta}\int_0^{\Delta}\frac{s}{t+s}\mathrm{d}s\,\mathrm{d}t$$

$$\geqslant\; \frac{\Delta^2}{d^2}(1-\log 2) - \frac{4\eta\Delta^2}{\delta'\delta}$$

(we have again used the fact that $v(\delta)/\delta \leqslant 2\eta$). We have $v(d) \geqslant v(d) - v(\delta) = d-\delta = \Delta$, so, since $v(d)/d \leqslant 2\eta$,

$$\delta \;=\; d-\Delta \;\geqslant\; (1-2\eta)d$$

and

$$\delta' \;=\; d-2\Delta \;\geqslant\; (1-4\eta)d.$$

By the computation just made we thus have

$$\int_{\delta'}^{\delta}\int_{\delta}^{d}\left(\frac{\dfrac{v(x)}{x}-\dfrac{v(y)}{y}}{x-y}\right)^2 \mathrm{d}x\,\mathrm{d}y$$

$$\geqslant\; \left(1-\log 2-\frac{4\eta}{(1-2\eta)(1-4\eta)}\right)\left(\frac{d-\delta}{d}\right)^2.$$

For the *third* of our three double integrals we have exactly the same