$|f(x)-\sigma_n(x)| < \frac{1}{2}\epsilon$ throughout the interval. If we replace each sine and cosine in $\sigma_n(x)$ by a sufficiently large number of terms in its power series, we obtain a polynomial $p(x)$ such that $|\sigma_n(x)-p(x)| < \frac{1}{2}\epsilon$ throughout the interval. This proves the theorem.

13.34. Almost everywhere summability. As long as we restrict ourselves to ordinary convergence, we cannot show that the Fourier series of a function represents the function in general, without imposing some rather heavy restriction on the function. The theory of summability removes this defect.

The Fejér-Lebesgue theorem. *The Fourier series of $f(x)$ is summable $(C, 1)$ to the sum $f(x)$, for every value of x for which*

$$\int_0^t |f(x+u)-f(x)|\,du = o(t). \tag{1}$$

In particular, it is summable $(C, 1)$ to $f(x)$ almost everywhere.

We have shown in § 11.6 that the condition (1) is satisfied for almost all values of x, for any integrable function. The second part of the theorem therefore follows at once from the first.

Let x be a point where (1) is satisfied, and take $s = f(x)$ in the formulae of § 13.31. Then

$$\int_0^t |\phi(u)|\,du = \int_0^t |f(x+u)+f(x-u)-2f(x)|\,du$$

$$\leqslant \int_0^t |f(x+u)-f(x)|\,du + \int_0^t |f(x-u)-f(x)|\,du = o(t).$$

Let
$$\Phi(t) = \int_0^t |\phi(u)|\,du,$$

and, given ϵ, choose η so that $\Phi(t) < \epsilon t$ for $t \leqslant \eta$. We suppose that $n > 1/\eta$, and write

$$\int_0^\delta \frac{\sin^2\frac{1}{2}nu}{u^2}\phi(u)\,du = \int_0^{1/n} + \int_{1/n}^{\eta} + \int_{\eta}^{\delta} = J_1+J_2+J_3.$$

Then, since $\sin^2\theta \leqslant \theta^2$,

$$|J_1| \leqslant (\tfrac{1}{2}n)^2 \int_0^{1/n} |\phi(u)|\,du < \tfrac{1}{4}\epsilon n,$$

$$|J_2| \leqslant \int\limits_{1/n}^{\eta} \frac{|\phi(u)|}{u^2}\,du = \frac{\Phi(\eta)}{\eta^2} - n^2\Phi\left(\frac{1}{n}\right) + 2\int\limits_{1/n}^{\eta} \frac{\Phi(u)}{u^3}\,du$$

$$< \frac{\epsilon}{\eta} + 2\epsilon \int\limits_{1/n}^{\eta} \frac{du}{u^2} < \epsilon/\eta + 2\epsilon n < 3\epsilon n,$$

and obviously $$|J_3| < \frac{A}{\eta^2}.$$

Hence $$\left|\frac{1}{n}\int\limits_0^{\delta} \frac{\sin^2\frac{1}{2}nu}{u^2}\phi(u)\,du\right| < \tfrac{1}{4}\epsilon + 3\epsilon + \frac{A}{n\eta^2},$$

and the required result follows on choosing first ϵ, then η, and then n.

13.35. An immediate corollary is that *a trigonometrical series cannot be the Fourier series of two functions which differ in a set of positive measure.* For if it is the Fourier series of $f(x)$ and of $g(x)$, it is summable $(C, 1)$ both to $f(x)$ and to $g(x)$ almost everywhere. Hence $f(x) = g(x)$ almost everywhere.

13.4. A continuous function with a divergent Fourier series. While we have seen that the continuity of a function is a sufficient condition for its Fourier series to be summable $(C, 1)$, for convergence we have had to assume other conditions. That this is really in accordance with the facts is shown by the following example, due to Fejér,* of a Fourier series which is divergent at a point, although the function which gives rise to it is continuous.

13.41. We first require a lemma.

The sum

$$\phi(n,r,x) = \frac{\cos(r+1)x}{2n-1} + \frac{\cos(r+2)x}{2n-3} + \dots + \frac{\cos(r+n)x}{1} -$$
$$- \frac{\cos(r+n+1)x}{1} - \frac{\cos(r+n+2)x}{3} - \dots - \frac{\cos(r+2n)x}{2n-1}$$

is bounded for all values of n, r, and x.

* Fejér (2), (3), (4).

We have

$$\phi(n,r,x) = \sum_{\nu=1}^{n} \frac{\cos(r+n-\nu+1)x}{2\nu-1} - \sum_{\nu=1}^{n} \frac{\cos(r+n+\nu)}{2\nu-1}$$

$$= 2\sin(r+n+\tfrac{1}{2})x \sum_{\nu=1}^{n} \frac{\sin(\nu-\frac{1}{2})x}{2\nu-1}$$

$$= 2\sin(r+n+\tfrac{1}{2})x \left\{ \sum_{\lambda=1}^{2n} \frac{\sin\frac{1}{2}\lambda x}{\lambda} - \frac{1}{2} \sum_{\mu=1}^{n} \frac{\sin \mu x}{\mu} \right\},$$

and each of the sums in the bracket is bounded (§ 1.76).

13.42. Let G_n denote the group of $2n$ numbers

$$\frac{1}{2n-1}, \frac{1}{2n-3}, \ldots, \frac{1}{3}, 1, -1, -\frac{1}{3}, \ldots, -\frac{1}{2n-1}.$$

Let $\lambda_1, \lambda_2, \ldots$ denote an increasing sequence of integers. Take the numbers of the groups $G_{\lambda_1}, G_{\lambda_2}, \ldots$ in order, and multiply each of the numbers of the group G_{λ_ν} by ν^{-2}. We obtain the sequence

$$\frac{1}{1^2}\frac{1}{2\lambda_1-1}, \ldots, -\frac{1}{1^2}\frac{1}{2\lambda_1-1}, \frac{1}{2^2(2\lambda_2-1)}, \frac{1}{2^2(2\lambda_2-3)}, \ldots,$$

say $\alpha_1, \alpha_2, \ldots$.

Now consider the series

$$\sum_{n=1}^{\infty} \alpha_n \cos nx. \qquad (1)$$

Suppose first that the terms corresponding to each group G_{λ_ν} are bracketed together. The bracketed series is

$$\sum_{n=1}^{\infty} \frac{\phi(\lambda_n, 2\lambda_1+2\lambda_2+\ldots+2\lambda_{n-1}, x)}{n^2}, \qquad (2)$$

which is absolutely and uniformly convergent, by the lemma. The sum of the series (2), say $f(x)$, is therefore a continuous function.

We next observe that the series (1) is the Fourier series of $f(x)$. For since (2) is uniformly convergent, we may multiply it by $\cos mx$ or $\sin mx$ and integrate term by term. The integral of each term is zero, except that of the one containing the term $\alpha_m \cos mx$; and from this we obtain

$$\int_0^{2\pi} f(x)\cos mx\, dx = \pi\alpha_m.$$

The numbers α_m are therefore the Fourier cosine coefficients of $f(x)$.

We show finally that the numbers λ_ν can be chosen so that the series (1) is divergent at the point $x = 0$, i.e. that the series $\alpha_1+\alpha_2+...$ is divergent. Let s_n be its nth partial sum. Then

$$s_{2\lambda_1+2\lambda_2+...+2\lambda_{\nu-1}+\lambda_\nu} = \frac{1}{\nu^2}\left(\frac{1}{2\lambda_\nu-1}+\frac{1}{2\lambda_\nu-3}+...+\frac{1}{3}+1\right) \sim \frac{\log\lambda_\nu}{2\nu^2}.$$

If the numbers λ_ν tend to infinity sufficiently rapidly, e.g. if $\lambda_\nu = \nu^{\nu^2}$, it follows that $s_n \to \infty$ as $n \to \infty$ through a certain sequence of values. Hence the series is divergent.

13.43. Fejér's example, together with a simple argument depending on Dirichlet's integral, enables us to say how large the partial sums s_n of a Fourier series of a continuous function can be.

If $f(x)$ is continuous, then

$$s_n = o(\log n);$$

and no more is true, since, if $\psi(n)$ is a function which decreases steadily to zero, however slowly, there is a Fourier series of a continuous function for which

$$s_n > \psi(n)\log n$$

for arbitrarily large values of n.

For the first part, we have to prove that

$$\int\limits_0^\delta \frac{\sin(n+\frac{1}{2})u}{u}\phi(u)\,du = o(\log n)$$

if $\phi(u) \to 0$ as $u \to 0$. Suppose that $|\phi(u)| < \epsilon$ for $u \leqslant \eta$; and, if $n+\frac{1}{2} > 1/\eta$, put

$$\int\limits_0^\delta \frac{\sin(n+\frac{1}{2})u}{u}\phi(u)\,du = \int\limits_0^{1/(n+\frac{1}{2})} + \int\limits_{1/(n+\frac{1}{2})}^{\eta} + \int\limits_\eta^\delta = I_1+I_2+I_3.$$

Then $$|I_1| \leqslant (n+\tfrac{1}{2}) \int\limits_0^{1/(n+\frac{1}{2})} |\phi(u)|\,du < \epsilon,$$

$$|I_2| \leqslant \int\limits_{1/(n+\frac{1}{2})}^{\eta} \frac{\epsilon}{u}\,du < \epsilon\log(n+\tfrac{1}{2}),$$

and $$|I_3| \leqslant \frac{1}{\eta} \int_{\eta}^{\delta} |\phi(u)|\, du.$$

The result clearly follows from these inequalities.

The second part is obtained by taking λ_ν sufficiently large in Fejér's example. Suppose that $\lambda_\nu > 2\nu$, and let

$$n = 2\lambda_1 + 2\lambda_2 + \ldots + 2\lambda_{\nu-1} + \lambda_\nu.$$

Then $$\lambda_\nu < n < 2\nu\lambda_\nu < \lambda_\nu^2.$$

Now $s_n > \psi(n)\log n$ for sufficiently large values of ν, if

$$\frac{\log \lambda_\nu}{2\nu^2} > \psi(n)\log n;$$

and since $\psi(n)\log n < \psi(\lambda_\nu)\log \lambda_\nu^2$, this is true if

$$\psi(\lambda_\nu) < \frac{1}{4\nu^2};$$

and this will be so if the numbers λ_ν tend to infinity rapidly enough.

13.5. Integration of Fourier series. *Any Fourier series, whether convergent or not, may be integrated term by term between any limits; that is, the sum of the integrals of the separate terms is the integral of the function of which the series is the Fourier series.*

Let $f(x)$ have the Fourier coefficients a_n, b_n, and let

$$F(x) = \int_0^x \{f(t) - \tfrac{1}{2}a_0\}\, dt.$$

Then $F(x)$ is periodic, continuous, and of bounded variation. Hence it can be expanded in a Fourier series, say

$$F(x) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx),$$

convergent for all values of x. Here

$$A_n = \frac{1}{\pi} \int_0^{2\pi} F(x)\cos nx\, dx$$

$$= \frac{1}{\pi} \left[F(x)\frac{\sin nx}{n}\right]_0^{2\pi} - \frac{1}{n\pi} \int_0^{2\pi} \{f(x) - \tfrac{1}{2}a_0\}\sin nx\, dx$$

$$= -\frac{1}{n\pi}\int_0^{2\pi} f(x)\sin nx\,dx = -\frac{b_n}{n};$$

and $$B_n = \frac{1}{\pi}\int_0^{2\pi} F(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[-F(x)\frac{\cos nx}{n}\right]_0^{2\pi} + \frac{1}{n\pi}\int_0^{2\pi} \{f(x) - \tfrac{1}{2}a_0\}\cos nx\,dx$$

$$= \frac{1}{n\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{a_n}{n},$$

the integrated terms vanishing since $F(2\pi) = F(0) = 0$. Hence

$$F(x) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} \frac{a_n \sin nx - b_n \cos nx}{n}.$$

Putting $x = 0$, we obtain

$$\tfrac{1}{2}A_0 = \sum_{n=1}^{\infty} \frac{b_n}{n},$$

and, adding,

$$F(x) = \sum_{n=1}^{\infty} \frac{a_n \sin nx + b_n(1 - \cos nx)}{n}.$$

This proves the theorem.

13.51. An interesting particular case is that *the series*

$$\sum_{n=1}^{\infty} \frac{b_n}{n}$$

is convergent. This remark enables us to write down convergent trigonometrical series which are not Fourier series. A simple example is

$$\sum_{n=2}^{\infty} \frac{\sin nx}{\log n}.$$

This is convergent for all values of x, but it cannot be the Fourier series of its sum, since the series

$$\sum \frac{1}{n \log n}$$

is divergent. Actually the sum of this trigonometrical series is not integrable in the sense of Lebesgue, and it is easy to prove directly that the sum of the integrated series $\sum \cos nx/n \log n$ tends to infinity as $x \to 0$.

13.52. The following alternative proof of the above integration theorem is also interesting. We know that the series

$$\frac{\sin(x-t)}{1}+\frac{\sin 2(x-t)}{2}+\ldots = \phi(t)$$

is boundedly convergent. Hence we may multiply by $f(t)/\pi$ and integrate term by term over $(0, 2\pi)$. On the left we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n\pi} \int_0^{2\pi} \sin n(x-t)f(t)\,dt = \sum_{n=1}^{\infty} \frac{a_n \sin nx - b_n \cos nx}{n},$$

the integrated series. On the right we get

$$\frac{1}{\pi}\int_0^{2\pi} \phi(t)f(t)\,dt = \frac{1}{\pi}\int_{x-2\pi}^{x} \phi(t)f(t)\,dt = \frac{1}{\pi}\int_{x-2\pi}^{x} \tfrac{1}{2}(\pi-x+t)f(t)\,dt$$

$$= \frac{1}{2\pi}\left[(\pi-x+t)F(t)\right]_{x-2\pi}^{x} - \frac{1}{2\pi}\int_{x-2\pi}^{x} F(t)\,dt = F(x) - \frac{1}{2\pi}\int_0^{2\pi} F(t)\,dt,$$

since $F(x-2\pi) = F(x)$. The result now follows as before.

13.53. A similar method leads to the following more general integration theorem.

A Fourier series may be multiplied by any function of bounded variation and integrated term by term between any finite limits.

Let $$g(x) = \tfrac{1}{2}\alpha_0 + \sum_{n=1}^{\infty} (\alpha_n \cos nx + \beta_n \sin nx)$$

be a function of bounded variation. The series being boundedly convergent (§ 13.232), we may multiply by any integrable function $f(x)$ and integrate term by term over $(0, 2\pi)$. We obtain

$$\frac{1}{\pi}\int_0^{2\pi} f(x)g(x)\,dx = \tfrac{1}{2}a_0\alpha_0 + \sum_{n=1}^{\infty} (a_n\alpha_n + b_n\beta_n), \tag{1}$$

where a_n, b_n, are the Fourier coefficients of $f(x)$. This is the same result as we should have obtained by multiplying the Fourier series for $f(x)$ by $g(x)$ and integrating term by term over $(0, 2\pi)$.

A similar result may be obtained for other ranges of integration by replacing $g(x)$ by 0 outside the required range.

13.54. Parseval's theorem. If $f(x)$ is of bounded variation, we may put $g(x)=f(x)$ in 13.53 (1), and obtain

$$\frac{1}{\pi}\int_0^{2\pi}\{f(x)\}^2\,dx=\tfrac{1}{2}a_0^2+\sum_{n=1}^{\infty}(a_n^2+b_n^2).$$

This is known as Parseval's theorem. We shall show in § 13.63 that it is true under much more general conditions than those we have so far assumed.

13.6. Functions of the class L^2: Bessel's inequality. Let $f(x)$ be a function of the class $L^2(0,2\pi)$, with Fourier coefficients a_n, b_n. Then

$$\phi(x)=f(x)-\tfrac{1}{2}a_0-\sum_{m=1}^{n}(a_m\cos mx+b_m\sin mx)$$

also belongs to L^2; and

$$\frac{1}{\pi}\int_0^{2\pi}\{\phi(x)\}^2\,dx=\frac{1}{\pi}\int_0^{2\pi}\{f(x)\}^2\,dx+\tfrac{1}{2}a_0^2+\sum_{m=1}^{n}(a_m^2+b_m^2)-$$

$$-\frac{a_0}{\pi}\int_0^{2\pi}f(x)\,dx-\frac{2}{\pi}\sum_{m=1}^{n}\int_0^{2\pi}(a_m\cos mx+b_m\sin mx)f(x)\,dx$$

$$=\frac{1}{\pi}\int_0^{2\pi}\{f(x)\}^2\,dx-\tfrac{1}{2}a_0^2-\sum_{m=1}^{n}(a_m^2+b_m^2),$$

by the Euler-Fourier formulae. Since the left-hand side is not negative, it follows that

$$\tfrac{1}{2}a_0^2+\sum_{m=1}^{n}(a_m^2+b_m^2)\leqslant\frac{1}{\pi}\int_0^{2\pi}\{f(x)\}^2\,dx \tag{1}$$

for all values of n. This result is known as Bessel's inequality.

Since the right-hand side of (1) is independent of n, it follows that *the series*

$$\tfrac{1}{2}a_0^2+\sum_{m=1}^{\infty}(a_m^2+b_m^2) \tag{2}$$

is convergent. Also

$$\tfrac{1}{2}a_0^2+\sum_{m=1}^{\infty}(a_m^2+b_m^2)\leqslant\frac{1}{\pi}\int_0^{2\pi}\{f(x)\}^2\,dx. \tag{3}$$

13.61. Parseval's theorem for continuous functions. We have seen that, for functions of bounded variation, the above inequality becomes an equality, viz. Parseval's theorem. The same result for continuous functions may be proved as follows. If $f(x)$ is continuous, $\sigma_n(x)$ tends uniformly to $f(x)$, and hence

$$\lim_{n\to\infty}\frac{1}{\pi}\int_0^{2\pi}\{f(x)-\sigma_n(x)\}f(x)\,dx=0.$$

Now $$\sigma_n(x)=\tfrac{1}{2}a_0+\sum_{m=1}^{n-1}(a_m\cos mx+b_m\sin mx)\left(1-\frac{m}{n}\right),$$

and, evaluating the integral as in the previous section, we obtain

$$\frac{1}{\pi}\int_0^{2\pi}\{f(x)\}^2\,dx-\tfrac{1}{2}a_0^2-\sum_{m=1}^{n-1}(a_m^2+b_m^2)\left(1-\frac{m}{n}\right)\to 0.$$

Parseval's formula therefore holds if the series is summed $(C, 1)$. Since by § 13.6 the series is convergent, it follows from § 13.3 that it holds in the ordinary sense.*

There is no difficulty in extending this proof to functions which have simple discontinuities. Actually Parseval's theorem holds for all functions of the class L^2. We shall prove this as a corollary of the theorem of the next section.

13.62. The Riesz-Fischer theorem. Let

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) \tag{1}$$

be any trigonometrical series with coefficients such that the series 13.6 (2) is convergent. Nothing that we have proved so far in this chapter enables us to decide whether such a series is a Fourier series. The problem is solved by means of the theory of mean convergence (§ 12.5). This theory was in fact originally constructed to deal with this very problem.

The following theorem was proved almost simultaneously by F. Riesz and Fischer.†

If the numbers a_n, b_n are such that the series 13.6 (2) *is convergent, then the series* (1) *is the Fourier series of a function* $f(x)$

* A number of different proofs under various conditions are given by Julia, *Exercices d'analyse*, 180–6.

† F. Riesz (1), Fischer (1).

of the class L^2. The partial sums of the series converge in mean to $f(x)$.

Denoting the nth partial sum of (1) by $s_n(x)$, we have

$$\int_0^{2\pi} \{s_n(x)-s_m(x)\}^2\,dx = \int_0^{2\pi} \Big\{\sum_{\nu=m+1}^{n} (a_\nu \cos\nu x + b_\nu \sin\nu x)\Big\}^2\,dx$$

$$= \pi \sum_{\nu=m+1}^{n} (a_\nu^2+b_\nu^2),$$

all the product terms disappearing on integration. The right-hand side tends to zero when m and n tend independently to infinity. Hence $s_n(x)$ converges in mean to a function, $f(x)$ say, of the class L^2.

Also, by § 12.53,

$$\lim_{n\to\infty} \int_0^{2\pi} s_n(x)\cos\nu x\,dx = \int_0^{2\pi} f(x)\cos\nu x\,dx.$$

But the integral on the left is equal to πa_ν, if $n \geqslant \nu$. Hence

$$a_\nu = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos\nu x\,dx,$$

i.e. a_ν is the νth Fourier cosine coefficient of $f(x)$. Similarly b_ν is the νth sine coefficient. Hence the given trigonometrical series is the Fourier series of the function $f(x)$.

It is important to observe that it is here that the Lebesgue integral first plays an indispensable part in the theory. Most of the previous analysis is true for Riemann integrals and elementary generalized absolutely convergent integrals. Here the result shows that the extension to Lebesgue integrals is really necessary.

13.63. Parseval's theorem for functions of the class L^2. Let $f(x)$ be any function belonging to $L^2(0, 2\pi)$, and let its Fourier series have the usual form. Then the series 13.6 (2) is convergent. Hence, by the Riesz-Fischer theorem, the partial sums $s_n(x)$ converge in mean to a function $g(x)$, of which the given series is the Fourier series. Hence, by § 13.35, $g(x) = f(x)$ almost everywhere. Also, by § 12.52,

$$\lim_{n\to\infty} \int_0^{2\pi} \{s_n(x)\}^2\,dx = \int_0^{2\pi} \{f(x)\}^2\,dx,$$

and on evaluating the integral on the left-hand side we obtain Parseval's formula.

The more general formula 13.53 (1) *also holds if* $f(x)$ *and* $g(x)$ *are any two functions of the class* L^2. For Parseval's formula holds for the functions $f(x)+g(x)$ and $f(x)-g(x)$, and the result stated follows on subtraction.

13.7. Properties of Fourier coefficients. Originally the Fourier coefficients were merely the material out of which the Fourier series was constructed. But the coefficients have some interesting properties of their own. In fact Bessel's inequality, and the theorems of Parseval and Riesz-Fischer call attention to the problem of the behaviour of the Fourier coefficients of given classes of functions and give some important information about it.

The first theorem of this kind (§ 13.22) is that *the Fourier coefficients of any integrable function tend to zero*. On the other hand, they do not tend to zero in any definite order; that is, any theorem such as '$a_n = O(1/\log n)$ for all integrable functions' is certainly false. For consider the series

$$\sum_{n=1}^{\infty} \frac{\cos k_n x}{n^2},$$

where k_n denotes a sequence of positive integers which tends to infinity rapidly as $n \to \infty$. The series is uniformly convergent, and so is the Fourier series of its sum; and

$$a_{k_n} = \frac{1}{n^2},$$

which falsifies any theorem of the kind suggested, if k_n tends to infinity rapidly enough.

13.71. Suppose next that $f(x)$ belongs to the class L^2. This does not enable us to prove any more about the order of the coefficients; in fact the function defined by the above series is clearly continuous, and so belongs to L^2. But we do obtain a definite result about the *average* order, viz. that

$$\sum (a_n^2+b_n^2)$$

is convergent (§ 13.6).

This result has been generalized so as to apply to other

Lebesgue classes; *if $f(x)$ belongs to L^p, where $1 < p \leqslant 2$, then the series*

$$\sum \left(|a_n|^{p/(p-1)} + |b_n|^{p/(p-1)}\right)$$

is convergent.

The proof of this theorem is, however, too long to be given here.*

There is also a corresponding extension of the Riesz-Fischer theorem: *if the series*

$$\sum \left(|a_n|^p + |b_n|^p\right),$$

where $1 < p \leqslant 2$, is convergent, then the numbers a_n, b_n are the Fourier coefficients of a function of the class $L^{p/(p-1)}$.

Both these theorems cease to be true if $p > 2$, so that they are not converses of each other unless $p = 2$.

13.72. If we make still more special assumptions about the function, we obtain new results about the coefficients. *Suppose that $f(x)$ satisfies a Lipschitz condition of order α, i.e. as $h \to 0$*

$$f(x+h) - f(x) = O(|h|^\alpha) \qquad (0 < \alpha \leqslant 1)$$

uniformly with respect to x. Then

$$a_n = O(n^{-\alpha}), \qquad b_n = O(n^{-\alpha}).$$

For

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx = -\frac{1}{\pi}\int_{-\pi/n}^{2\pi-\pi/n} f\left(\frac{\pi}{n}+t\right)\cos nt\, dt$$

$$= -\frac{1}{\pi}\int_0^{2\pi} f\left(\frac{\pi}{n}+t\right)\cos nt\, dt,$$

and hence also

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \left\{f(x) - f\left(\frac{\pi}{n}+x\right)\right\}\cos nx\, dx$$

$$= \int_0^{2\pi} O\left(\frac{1}{n^\alpha}\right) dx = O\left(\frac{1}{n^\alpha}\right);$$

and similarly for b_n.

13.73. The next result of this kind is that *if $f(x)$ is of bounded variation, then*

$$a_n = O(1/n), \qquad b_n = O(1/n).$$

* W. H. Young (2), (3), (5), (6), Hausdorff (1), F. Riesz (4).

For $f(x)=f_1(x)-f_2(x)$, where $f_1(x)$ and $f_2(x)$ are positive and non-decreasing. Hence, by the second mean-value theorem,

$$\int_0^{2\pi} f_1(x)\cos nx\,dx = f_1(2\pi)\int_{\xi}^{2\pi}\cos nx\,dx \qquad (0<\xi<2\pi)$$

$$= -f_1(2\pi)\frac{\sin n\xi}{n} = O\left(\frac{1}{n}\right),$$

and a similar result holds for the other integral.

An alternative proof of Jordan's theorem (§ **13.232**) can be deduced from this result. If $f(x)$ is of bounded variation, its Fourier series is summable $(C, 1)$ to the sum $\frac{1}{2}\{f(x+0)+f(x-0)\}$, by § 13.32. Since $a_n = O(1/n)$, $b_n = O(1/n)$, the series actually converges to this sum (§ **13.3**, ex. (viii)).

If $f(x)$ is an integral, and has the period 2π, then

$$a_n = o(1/n), \qquad b_n = o(1/n).$$

For if
$$f(x)=f(0)+\int_0^x \phi(t)\,dt \qquad (x\geqslant 0),$$

then
$$\pi a_n = \left[f(x)\frac{\sin nx}{n}\right]_0^{2\pi} - \frac{1}{n}\int_0^{2\pi}\phi(x)\sin nx\,dx,$$

$$\pi b_n = \left[-f(x)\frac{\cos nx}{n}\right]_0^{2\pi} + \frac{1}{n}\int_0^{2\pi}\phi(x)\cos nx\,dx.$$

The integrated terms are zero, since $f(2\pi)=f(0)$ and the integrals on the right tend to zero, by the Riemann-Lebesgue theorem. This proves the theorem.

If $f'(x)$ satisfies special conditions, such as a Lipschitz condition, still further results of the same kind can, of course, be obtained.

13.8. Uniqueness of trigonometrical series. At the beginning of the chapter we associated with an integrable function a particular trigonometrical series, viz. the Fourier series of the function; and we have shown that the Fourier series does, in various ways, represent the function. The reader might, however, still contend that we had attached undue importance to Fourier series, and that there might be other types of trigonometrical series in which a given function could be expanded.

It is difficult to give a complete solution of this problem. If, however, we assume enough about the set of points where the series converge, we can show that, if a trigonometrical series converges to a given function, it is the only such series which does so; and therefore that, if the function can be expanded in a convergent Fourier series, it cannot be expanded in a convergent trigonometrical series of any other form.

The theory is due to Riemann, du Bois-Reymond, and Cantor. The theorem which we shall prove is as follows.

If two trigonometrical series converge to the same sum in the interval $(0, 2\pi)$, *with the possible exception of a finite number of points, then corresponding coefficients in the two series are equal, i.e. the series are identical.*

This is not all that is known, and more general theorems will be found, e.g., in Hobson's *Theory of Functions*, §§ 420–50. But some extensions which might naturally be suggested are not true; if we say 'are summable $(C, 1)$' instead of 'converge', the theorem becomes false, as is shown by § 13.3, ex. (iii).

The question whether a given trigonometrical series is a Fourier series is really a problem of integral equations. We are given numbers a_0, a_1, b_1,..., and it is required to determine whether there is an integrable function $f(x)$ such that the Euler-Fourier formulae § 13.1 (2), (3), are true. The question is not settled by mere convergence, since a trigonometrical series may be everywhere convergent without being a Fourier series (§ 13.51). But if it converges uniformly, or boundedly, or in mean with index p $(p \geqslant 1)$, then it is a Fourier series; and the theorems of §§ 13.62–13.71 enable us to state conditions for mean convergence, with $p \geqslant 2$.

Another theorem which would naturally suggest itself is that if a trigonometrical series converges almost everywhere to an integrable function, then it is the Fourier series of the function; but this is not necessarily true, and the state of affairs is rather complicated.

The proof of the theorem stated above depends on a number of lemmas.

13.81. Cantor's lemma. *If* $a_n \cos nx + b_n \sin nx$ *tends to* 0 *for all values of* x *in an interval, then* a_n *and* b_n *tend to* 0.

Suppose that $a_n \cos nx + b_n \sin nx \to 0$ in the interval (α, β).

If the lemma is false, there is a constant A and a sequence of values of n for which $a_n^2+b_n^2>A$. Hence, as $n\to\infty$ through this sequence, the function

$$f_n(x)=\frac{(a_n\cos nx+b_n\sin nx)^2}{a_n^2+b_n^2}$$

converges boundedly to 0 in (α,β). Hence, by the theorem of bounded convergence,

$$\int_\alpha^\beta f_n(x)\,dx\to 0.$$

But, evaluating the integral, we find that

$$\int_\alpha^\beta f_n(x)\,dx=\tfrac{1}{2}(\beta-\alpha)+O\left(\frac{1}{n}\right),$$

and this gives a contradiction. This proves the lemma.

13.82. Suppose now that the series

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) \tag{1}$$

converges to the sum $f(x)$ in $(0,2\pi)$, except possibly at a finite number of points. Let

$$F(x)=\tfrac{1}{4}a_0x^2-\sum_{n=1}^{\infty}\frac{a_n\cos nx+b_n\sin nx}{n^2}. \tag{2}$$

Since by Cantor's lemma a_n and b_n tend to zero, this series is uniformly convergent, and $F(x)$ is continuous, for all values of x. If we could differentiate twice term by term, we should have $F''(x)=f(x)$. We cannot necessarily do this, and instead have to proceed as follows.

Riemann's First Theorem. *If*

$$G(x,h)=\frac{F(x+2h)+F(x-2h)-2F(x)}{4h^2}, \tag{3}$$

then $G(x,h)\to f(x)$ as $h\to 0$, for all values of x for which the series (1) *converges to $f(x)$.*

We have

$$\cos n(x+2h)+\cos n(x-2h)-2\cos nx=-4\cos nx\sin^2 nh,$$
$$\sin n(x+2h)+\sin n(x-2h)-2\sin nx=-4\sin nx\sin^2 nh.$$

and hence

$$G(x,h) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\frac{\sin^2 nh}{n^2h^2}. \tag{4}$$

The nth term of (4) tends to the nth term of (1) as $h \to 0$. Hence it is sufficient to prove that the series (4) converges uniformly with respect to h. Let r_n denote the remainder of the series (1) after the term in $\sin nx$. Then $r_n \to 0$, say $|r_n| < \epsilon$ for $n \geqslant N$. Hence

$$\sum_{n=N}^{\infty}(a_n \cos nx + b_n \sin nx)\frac{\sin^2 nh}{n^2h^2} = \sum_{n=N}^{\infty}(r_n - r_{n+1})\left(\frac{\sin nh}{nh}\right)^2$$

$$= r_N\left(\frac{\sin Nh}{Nh}\right)^2 - \sum_{N+1}^{\infty} r_n\left[\left(\frac{\sin nh}{nh}\right)^2 - \left\{\frac{\sin(n+1)h}{(n+1)h}\right\}^2\right],$$

and the modulus of this does not exceed

$$\epsilon + \epsilon\sum_{N+1}^{\infty}\int_{nh}^{(n+1)h}\left|\frac{d}{dt}\left(\frac{\sin^2 t}{t^2}\right)\right| dt < \epsilon + \epsilon\int_0^{\infty}\left|\frac{d}{dt}\left(\frac{\sin^2 t}{t^2}\right)\right| dt,$$

the last integral being convergent. Hence (4) is uniformly convergent, and the result follows.

13.83. Riemann's Second Theorem. *If a_n and b_n tend to zero, then*

$$\lim_{h\to 0}\frac{F(x+2h)+F(x-2h)-2F(x)}{2h} = 0.$$

for all values of x.

We have to prove that

$$a_0 h + 2\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\frac{\sin^2 nh}{n^2 h}$$

tends to zero. Given ϵ, we have

$$|a_n \cos nx + b_n \sin nx| < \epsilon \qquad (n > N).$$

Since $\sin^2 nh \leqslant n^2h^2$ for $n \leqslant 1/h$, the modulus of the sum does not exceed

$$A|h| + 2\sum_{n=1}^{N} A|h| + 2\sum_{N<n\leqslant 1/h}\epsilon h + 2\sum_{n>1/h}\frac{\epsilon}{n^2 h}$$

$$< AN|h| + 2\epsilon + \frac{2\epsilon}{h}\int_{1/h}^{\infty}\frac{du}{(u-1)^2} < AN|h| + A\epsilon,$$

and the result follows by choosing first ϵ and then h sufficiently small.

13.84. Schwarz's theorem. *If $F(x)$ is continuous in an interval (a, b), and*

$$\lim_{h\to 0}\frac{F(x+h)+F(x-h)-2F(x)}{h^2}=0$$

for all values of x in the interval, then $F(x)$ is a linear function.

The expression on the left is called the generalized second derivative of $F(x)$. If $F(x)$ has an ordinary second derivative, the generalized second derivative is equal to it, and the result follows at once.

To prove the theorem, consider the function

$$\phi(x)=F(x)-F(a)-\frac{x-a}{b-a}\{F(b)-F(a)\}.$$

We have $\phi(a)=0$ and $\phi(b)=0$. If $\phi(x)=0$ for all values of x, the result follows. Otherwise it takes values different from zero, say, for example, positive values. Suppose that $\phi(c)>0$. Let

$$\psi(x)=\phi(x)-\tfrac{1}{2}\epsilon(x-a)(b-x),$$

where ϵ is positive and so small that $\psi(c)>0$. Then $\psi(x)$ has a positive upper bound, say at $x=\xi$, which it attains, since it is continuous. Hence

$$\psi(\xi+h)+\psi(\xi-h)-2\psi(\xi)\leqslant 0.$$

But

$$\frac{\psi(\xi+h)+\psi(\xi-h)-2\psi(\xi)}{h^2}=\frac{F(\xi+h)+F(\xi-h)-2F(\xi)}{h^2}+\epsilon,$$

and the right-hand side tends to ϵ as $h\to 0$. This gives a contradiction. Similarly the supposition that $\phi(x)$ takes negative values leads to a contradiction. Hence $\phi(x)=0$ for all values of x, which is the desired result.

13.85. The proof of the main theorem now follows from Schwarz's theorem. It is sufficient to prove that, if a trigonometrical series converges to zero except at a finite number of points, then it must vanish identically. If the series 13.82 (1) has this property, the function $F(x)$ is continuous, and its generalized second derivative is zero except at a finite number of points. Hence $F(x)$ is linear in the interval between any two exceptional points, and the straight lines which form the graph join at the exceptional points. Now, taking x in Riemann's second theorem to be an exceptional point, it follows from the lemma that the slopes of the lines on the two sides of the exceptional point must be the same. Hence $F(x)$ is linear throughout the whole interval $(0, 2\pi)$, say

$$F(x) = ax+b.$$

Hence
$$\sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2} = \tfrac{1}{4}a_0 x^2 - ax - b.$$

Since the sum of the series is periodic, a_0 and a must be zero. Then, the series being uniformly convergent, we may multiply by $\cos mx$ or $\sin mx$ and integrate term by term; and we obtain

$$\frac{\pi a_m}{m^2} = -b \int_0^{2\pi} \cos mx \, dx = 0, \qquad \frac{\pi b_m}{m^2} = -b \int_0^{2\pi} \sin mx \, dx = 0,$$

for $m > 0$. This completes the proof.

13.9. Fourier series for any range. All our series so far have represented functions with the period 2π. A series of the form

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\frac{nx}{\lambda} + b_n \sin\frac{nx}{\lambda}\right)$$

represents a function with the period $2\pi\lambda$. Formulae for the coefficients may be calculated as before; we obtain

$$a_n = \frac{1}{\pi\lambda} \int_{-\pi\lambda}^{\pi\lambda} f(t)\cos\frac{nt}{\lambda}\, dt, \qquad b_n = \frac{1}{\pi\lambda} \int_{-\pi\lambda}^{\pi\lambda} f(t)\sin\frac{nt}{\lambda}\, dt.$$

Naturally the whole theory can be applied to series of this kind.

13.91. Fourier's integral formula. The above expansion may be written

$$f(x) = \frac{1}{2\pi\lambda} \int_{-\pi\lambda}^{\pi\lambda} f(t)\, dt + \sum_{n=1}^{\infty} \frac{1}{\pi\lambda} \int_{-\pi\lambda}^{\pi\lambda} f(t)\cos\frac{n(x-t)}{\lambda}\, dt.$$

Suppose now that $\lambda \to \infty$. Then the series on the right behaves very much like one of the sums by which a Riemann integral is defined. In fact, if we write $u_n = n/\lambda$, it is

$$\sum_{n=1}^{\infty} (u_{n+1} - u_n)\phi(u_n),$$

where
$$\phi(u) = \frac{1}{\pi}\int_{-\pi\lambda}^{\pi\lambda} f(t)\cos u(x-t)\,dt.$$

If, therefore, we make $\lambda \to \infty$, and ignore such difficulties as the fact that $\phi(u)$ depends on λ, and that the approximating sum is an infinite series, we obtain

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty} du \int_{-\infty}^{\infty} \cos u(x-t)f(t)\,dt.$$

This is Fourier's integral formula. It represents a function defined over $(-\infty, \infty)$ in the same way that a Fourier series represents a function with a finite period.

The difficulty of justifying a proof on these lines would be considerable. A direct consideration of the formula suggested is comparatively easy.

13.92. Suppose that $f(x)$ is integrable in the Lebesgue sense over $(-\infty, \infty)$. Then the integral

$$\int_{-\infty}^{\infty} \cos u(x-t)f(t)\,dt$$

converges uniformly with respect to u over any finite range. We may therefore integrate with respect to u over $(0, U)$, and invert the order of integration. Thus

$$\int_{0}^{U} du \int_{-\infty}^{\infty} \cos u(x-t)f(t)\,dt = \int_{-\infty}^{\infty} \frac{\sin U(x-t)}{x-t} f(t)\,dt.$$

Given ϵ, we can choose T so large that

$$\int_{-\infty}^{-T} |f(t)|\,dt < \epsilon, \qquad \int_{T}^{\infty} |f(t)|\,dt < \epsilon,$$

and we may suppose that $T > |x|+1$, x being supposed fixed. Then

$$\left|\int_{-\infty}^{-T} \frac{\sin U(x-t)}{x-t} f(t)\,dt\right| < \epsilon, \qquad \left|\int_{T}^{\infty} \frac{\sin U(x-t)}{x-t} f(t)\,dt\right| < \epsilon,$$

for all values of U. Having fixed T, the integrals

$$\int_{-T}^{x-\delta} \frac{\sin U(x-t)}{x-t} f(t)\,dt, \qquad \int_{x+\delta}^{T} \frac{\sin U(x-t)}{x-t} f(t)\,dt$$

tend to zero as $U \to \infty$, by the Riemann-Lebesgue lemma. Hence

$$\frac{1}{\pi}\int_0^U du \int_{-\infty}^{\infty} \cos u(x-t) f(t)\,dt = \frac{1}{\pi}\int_{x-\delta}^{x+\delta} \frac{\sin U(x-t)}{x-t} f(t)\,dt + o(1)$$

$$= \frac{1}{\pi}\int_0^{\delta} \frac{\sin Ut}{t}\{f(x+t)+f(x-t)\}\,dt + o(1).$$

The value of the limit, as $U \to \infty$, therefore depends only on the behaviour of $f(t)$ in the immediate neighbourhood of $t = x$; and the problem has been reduced to the discussion of an integral similar to Dirichlet's. Any of the convergence criteria of §§ 13.231–3 apply equally well to this problem. *In particular*

$$\lim_{U\to\infty} \frac{1}{\pi}\int_0^U du \int_{-\infty}^{\infty} \cos u(x-t) f(t)\,dt = \tfrac{1}{2}\{f(x+0)+f(x-0)\}$$

if $f(t)$ is integrable over $(-\infty, \infty)$, and of bounded variation in an interval including $t = x$.

13.93. Fourier transforms. If $f(x)$ is an even function, Fourier's integral becomes

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \cos xu\,du \int_0^{\infty} \cos ut f(t)\,dt, \tag{1}$$

the term involving $\sin ut$ vanishing identically. This is Fourier's cosine formula. Similarly for an odd function we obtain Fourier's sine formula

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \sin xu\,du \int_0^{\infty} \sin ut f(t)\,dt. \tag{2}$$

If we write
$$g(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^{\infty} \cos xt f(t)\,dt, \tag{3}$$

then (1) gives
$$f(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^{\infty} \cos xt\, g(t)\,dt. \tag{4}$$

There is therefore a reciprocal relation between the functions $f(x)$ and $g(x)$; a pair of functions connected in one sense or another by these formulae are known as *Fourier cosine transforms* of one another. Thus, for example, if $f(x)$ belongs to $L(0,\infty)$, and is of bounded variation in any finite interval, then (3) is absolutely convergent, and (4) holds in the sense that the integral converges (not necessarily absolutely) to

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\}.$$

Similarly from (2) we obtain the reciprocal formulae

$$h(x)=\sqrt{\left(\frac{2}{\pi}\right)}\int\limits_0^\infty \sin xt f(t)\,dt, \qquad f(x)=\sqrt{\left(\frac{2}{\pi}\right)}\int\limits_0^\infty \sin xt\, h(t)\,dt, \quad (5)$$

and $f(x)$ and $h(x)$ are *Fourier sine transforms*.

13.94. Integration of Fourier integrals. It is convenient to notice at this point a theorem similar to that of § 13.5: *the formula obtained by integrating* 13.93 (1),

$$\int\limits_0^\xi f(x)\,dx=\frac{2}{\pi}\int\limits_0^\infty \frac{\sin\xi u}{u}\,du\int\limits_0^\infty \cos ut f(t)\,dt,$$

holds for any function $f(t)$ integrable over $(0,\infty)$.

For

$$\int\limits_0^U \frac{\sin\xi u}{u}\,du\int\limits_0^\infty \cos ut f(t)\,dt=\int\limits_0^\infty f(t)\,dt\int\limits_0^U \frac{\sin\xi u\cos ut}{u}\,du$$

by uniform convergence; and the inner integral on the right is bounded for all U and t; for it is equal to

$$\frac{1}{2}\int\limits_0^U \frac{\sin(\xi+t)u}{u}\,du+\frac{1}{2}\int\limits_0^U \frac{\sin(\xi-t)u}{u}\,du$$
$$=\frac{1}{2}\int\limits_0^{U(\xi+t)} \frac{\sin v}{v}\,dv\pm\frac{1}{2}\int\limits_0^{U|\xi-t|} \frac{\sin v}{v}\,dv,$$

the sign being that of $\xi-t$, and

$$\int\limits_0^V \frac{\sin v}{v}\,dv$$

is a bounded function of V. Hence, by Lebesgue's convergence theorem, we may make $U \to \infty$ under the integral sign. Since

$$\int_0^\infty \frac{\sin \xi u \cos ut}{u}\, du = \tfrac{1}{2}\pi \quad (t < \xi), \qquad = 0 \quad (t > \xi),$$

the result now follows.

A similar result may be obtained from Fourier's sine formula.

13.95. Fourier transforms of the class L^2. The analysis of § 13.93 gives conditions under which the reciprocal formulae connecting Fourier transforms hold; but they suffer from the defect that, while the formulae are symmetrical in $f(x)$ and $g(x)$, the conditions which these functions satisfy are quite different. An alternative set of conditions, which has perfect symmetry, can be obtained by considering functions of the class L^2, and using the theory of mean convergence.*

Let $f(x)$ belong to the class $L^2(0,\infty)$. Then the formulae for cosine transforms hold in the sense that, as $a \to \infty$, the integral

$$g_a(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^a \cos xt f(t)\, dt \tag{1}$$

converges in mean to a function $g(x)$ of the class $L^2(0,\infty)$; and

$$f_a(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^a \cos xt\, g(t)\, dt \tag{2}$$

converges in mean to $f(x)$.

We prove this by a method suggested by the formal process of § 13.92. Let

$$a_n = \int_{n/\lambda}^{(n+1)/\lambda} f(x)\, dx \qquad (n = 1, 2,\ldots).$$

Then, as $\lambda \to \infty$, the sum

$$\Phi_{m,n} = \sum_{\nu=m+1}^{n} a_\nu \cos \frac{\nu x}{\lambda}$$

tends to the integral

$$\int_a^b \cos ux f(u)\, du,$$

if $0 \leqslant a < b$, and $m = [\lambda a]$, $n = [\lambda b] - 1$; for the difference is

* Plancherel (1), (2), (3); Titchmarsh (1), (3); Hardy (12); Pollard (1).

$$\sum_{\nu=m+1}^{n} \int\limits_{\nu/\lambda}^{(\nu+1)/\lambda} \left(\cos ux - \cos\frac{\nu x}{\lambda}\right) f(u)\, du + \\ + \int\limits_{a}^{(m+1)/\lambda} \cos ux f(u)\, du + \int\limits_{(n+1)/\lambda}^{b} \cos ux f(u)\, du,$$

and
$$\left|\cos ux - \cos\frac{\nu x}{\lambda}\right| \leqslant \frac{|x|}{\lambda},$$

so that the sum is $O(1/\lambda)$, while the last two integrals plainly tend to zero. Further, the convergence is clearly uniform with respect to x for $0 \leqslant x \leqslant X$.

Now we can apply to $\Phi_{m,n}$ an argument similar to that used in proving the Riesz-Fischer theorem. We have

$$a_n^2 \leqslant \int\limits_{n/\lambda}^{(n+1)/\lambda} \{f(x)\}^2\, dx \int\limits_{n/\lambda}^{(n+1)/\lambda} dx = \frac{1}{\lambda} \int\limits_{n/\lambda}^{(n+1)/\lambda} \{f(x)\}^2\, dx;$$

and hence

$$\int\limits_0^{\pi\lambda} \Phi_{m,n}^2\, dx = \tfrac{1}{2}\pi\lambda \sum_{\nu=m+1}^{n} a_\nu^2 \leqslant \tfrac{1}{2}\pi \int\limits_{(m+1)/\lambda}^{(n+1)/\lambda} \{f(x)\}^2\, dx \leqslant \tfrac{1}{2}\pi \int\limits_a^b \{f(x)\}^2\, dx,$$

and *a fortiori*
$$\int\limits_0^X \Phi_{m,n}^2\, dx \leqslant \tfrac{1}{2}\pi \int\limits_a^b \{f(x)\}^2\, dx$$

if $\pi\lambda > X$. Keeping X fixed and making $\lambda \to \infty$, we obtain

$$\int\limits_0^X \{g_b(x) - g_a(x)\}^2\, dx \leqslant \int\limits_a^b \{f(x)\}^2\, dx,$$

and then, making $X \to \infty$,

$$\int\limits_0^\infty \{g_b(x) - g_a(x)\}^2\, dx \leqslant \int\limits_a^b \{f(x)\}^2\, dx. \tag{3}$$

Since the right-hand side tends to zero as $a \to \infty$, $b \to \infty$, so does the left-hand side; that is, $g_a(x)$ converges in mean to a function, $g(x)$ say, of the class $L^2(0,\infty)$.

The same argument now shows that the integral (2) converges in mean, to a function $\phi(x)$, say. We have to prove that $\phi(x) = f(x)$ almost everywhere, and for this it is sufficient to show that

$$\int\limits_0^\xi \phi(x)\, dx = \int\limits_0^\xi f(x)\, dx \tag{4}$$

for all values of ξ. Now

$$\int_0^\xi \phi(x)\,dx = \lim_{a\to\infty} \int_0^\xi f_a(x)\,dx = \lim_{a\to\infty} \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\xi dx \int_0^a \cos xt\, g(t)\,dt$$

$$= \lim_{a\to\infty} \sqrt{\left(\frac{2}{\pi}\right)} \int_0^a \frac{\sin \xi t}{t} g(t)\,dt = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \frac{\sin \xi t}{t} g(t)\,dt.$$

On the other hand, for $0 < \xi < a$,

$$\int_0^\xi f(x)\,dx = \frac{2}{\pi} \int_0^\infty \frac{\sin \xi u}{u}\,du \int_0^a \cos ut f(t)\,dt = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \frac{\sin \xi u}{u} g_a(u)\,du,$$

by § 13.94, $f(x)$ being integrable over $(0, a)$. Making $a \to \infty$, and observing that $\sin \xi u/u$ belongs to L^2, we obtain, by § 12.53,

$$\int_0^\xi f(x)\,dx = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \frac{\sin \xi u}{u} g(u)\,du.$$

This proves (4), and completes the proof of the theorem.

There is, of course, a similar theorem for Fourier sine transforms.

13.96. We can also obtain a formula corresponding to Parseval's theorem. Putting $a = 0$ in 13.95 (3), we have

$$\int_0^\infty \{g_b(x)\}^2\,dx \leqslant \int_0^b \{f(x)\}^2\,dx \leqslant \int_0^\infty \{f(x)\}^2\,dx,$$

and making $b \to \infty$, by § 12.51

$$\int_0^\infty \{g(x)\}^2\,dx \leqslant \int_0^\infty \{f(x)\}^2\,dx.$$

But since the relation between $f(x)$ and $g(x)$ is reciprocal, the opposite inequality also holds. Hence in fact

$$\int_0^\infty \{g(x)\}^2\,dx = \int_0^\infty \{f(x)\}^2\,dx. \tag{1}$$

Finally, if $\phi(x)$ also belongs to L^2, and $\psi(x)$ is its transform, then $g(x)+\psi(x)$ is the transform of $f(x)+\phi(x)$. Hence

$$\int_0^\infty \{g(x)+\psi(x)\}^2\,dx = \int_0^\infty \{f(x)+\phi(x)\}^2\,dx,$$

and, subtracting (1) and the corresponding formula for ϕ and ψ, we obtain

$$\int_0^\infty g(x)\psi(x)\,dx = \int_0^\infty f(x)\phi(x)\,dx. \tag{2}$$

MISCELLANEOUS EXAMPLES

1. If $f(x)$ is first defined in $(0, \pi)$, then in $(-\pi, 0)$ by the equation $f(-x) = f(x)$, and elsewhere by periodicity, show that $f(x)$ has the Fourier cosine series

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx,$$

where
$$a_n = \frac{2}{\pi}\int_0^\pi f(t)\cos nt\,dt.$$

Similarly, if $f(-x) = -f(x)$, then $f(x)$ has the Fourier sine series

$$\sum_{n=1}^{\infty} b_n \sin nx,$$

where
$$b_n = \frac{2}{\pi}\int_0^\pi f(t)\sin nt\,dt.$$

2. Show that

$$e^{ax} = \frac{e^{2\pi a}-1}{\pi}\left\{\frac{1}{2a} + \sum_{n=1}^{\infty} \frac{a\cos nx - n\sin nx}{a^2+n^2}\right\} \qquad (0 < x < 2\pi),$$

$$e^{ax} = \frac{e^{a\pi}-1}{a\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} \{(-1)^n e^{a\pi}-1\}\frac{a\cos nx}{a^2+n^2} \qquad (0 < x < \pi),$$

$$e^{ax} = \frac{2}{\pi}\sum_{n=1}^{\infty} \{1-(-1)^n e^{a\pi}\}\frac{n\sin nx}{a^2+n^2} \qquad (0 < x < \pi).$$

Find the sums of the series when $x = 0$.

3. Sum the series

$$\sum_{n=1}^{\infty} \frac{a\cos nx}{a^2+n^2}, \qquad \sum_{n=1}^{\infty} \frac{n\sin nx}{a^2+n^2} \qquad (0 < x < 2\pi).$$

4. Expand in Fourier series valid over $(0, 2\pi)$, and also in Fourier cosine and sine series valid over $(0, \pi)$, the functions

$$1, \quad x, \quad x^2, \quad x^3, \quad \cos ax, \quad \sin ax, \quad \cosh ax, \quad \sinh ax,$$
$$e^{ax}\cos bx, \quad e^{ax}\sin bx, \quad [x/\pi], \quad [2x/\pi].$$

Consider the values of x for which the series converge to a value different from the value of the function expanded.

5. Prove that, if $-1 < r < 1$,

$$\frac{1-r^2}{1-2r\cos\theta+r^2} = 1+2\sum_{n=1}^{\infty} r^n \cos n\theta$$

for all values of θ.

6. If a_n, b_n denote the Fourier coefficients of $f(x)$, then for $-1 < r < 1$

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)r^n = \frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1-2r\cos(x-t)+r^2}f(t)\,dt.$$

7. Prove that

$$\lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1-2r\cos(x-t)+r^2}f(t)\,dt = \tfrac{1}{2}\{f(x+0)+f(x-0)\}$$

for all values of x for which the right-hand side exists.

[The discussion is similar to that of Fejér's integral.]

8. Show that, if $f(x)$ is bounded, then

$$s_n = O(\log n).$$

9. Show that, if $m \leqslant f(x) \leqslant M$, then

$$m \leqslant \sigma_n(x) \leqslant M$$

for all values of n and x.

10. Show that, if $m \leqslant f(x) \leqslant M$, and

$$|a_n| \leqslant \frac{A_1}{n}, \qquad |b_n| \leqslant \frac{A_2}{n},$$

then $$m-A_1-A_2 \leqslant s_n \leqslant M+A_1+A_2.$$

[Use the formula

$$s_n = \sigma_{n+1}-\frac{1}{n+1}\sum_{\nu=1}^{n}\nu(a_\nu\cos\nu x+b_\nu\sin\nu x).]$$

11. Show that

$$\frac{\pi-x}{2} = \frac{\sin x}{1}+\frac{\sin 2x}{2}+\frac{\sin 3x}{3}+\dots \qquad (0 < x < 2\pi),$$

and deduce that

$$\left|\frac{\sin x}{1}+\frac{\sin 2x}{2}+\dots+\frac{\sin nx}{n}\right| \leqslant \tfrac{1}{2}\pi+1$$

for all values of n and x.

[Compare § 1.76. The actual upper bound of the partial sums is $\int_0^{\pi}\frac{\sin x}{x}\,dx = 1.85\dots$; see Gronwall (1).]

12. Use Parseval's theorem to sum the series

$$\sum_{n=1}^{\infty}\frac{1}{n^2}, \qquad \sum_{n=1}^{\infty}\frac{1}{(a^2+n^2)^2}, \qquad \sum_{n=1}^{\infty}\frac{n^2}{(a^2+n^2)^2}.$$

13. A necessary and sufficient condition that

$$a_n = O(e^{-(k-\epsilon)n}), \qquad b_n = O(e^{-(k-\epsilon)n}),$$

where $k > 0$, for all positive values of ϵ, is that $f(x)$ should be almost everywhere equal to the value on the real axis of an analytic function $f(z)$, which is regular for $-k < y < k$, and has the period 2π.

14. Construct a Fourier series for which

$$s_n(0) > \frac{\log n}{\log\log n}$$

for arbitrarily large values of n.

15. Show that if, in the Fourier series of § 13.42, we substitute $\nu! x$ for x in the terms corresponding to the group of numbers G_{λ_ν}, we obtain a series which is also the Fourier series of a continuous function, and which diverges for all values of x such that x/π is a rational number.

16. Show that, if the series

$$\sum_{m=0}^{\infty} \alpha_m \cos(2^m x)$$

is a Fourier series, it is convergent for almost all values of x.

[In this case the formula used in example 10 becomes

$$\sigma_{2^k} - s_{2^k} = \frac{1}{2^k} \sum_{m=1}^{k-1} 2^m \alpha_m \cos(2^m x),$$

and since $\alpha_m \to 0$ the right-hand side tends to 0 for all values of x. Hence s_{2^k} tends to a limit wherever σ_{2^k} does, i.e. almost everywhere. See Kolmogoroff (1).]

17. If $f(x) = x^{-\alpha}$, where $0 < \alpha < 1$, for $0 < x \leqslant 2\pi$, show that, as $n \to \infty$,

$$a_n \sim \frac{n^{\alpha-1}}{2\Gamma(\alpha)\cos\frac{1}{2}\pi\alpha}, \qquad b_n \sim \frac{n^{\alpha-1}}{2\Gamma(\alpha)\sin\frac{1}{2}\pi\alpha}.$$

Show that $f(x)$ belongs to L^p if $p < 1/\alpha$, and that $\sum (|a_n|^q + |b_n|^q)$ is divergent if $q < 1/(1-\alpha)$.

[See Bromwich, *Infinite Series* (2nd ed.), § 174, Ex. 5, and Haslam-Jones (1). The result should be compared with the extended Riesz-Fischer theorem referred to in § 13.71. It shows that the exponent of convergence of the series of coefficients is the 'best possible'.]

18. A function $f(x)$ is equal to $\nu^\alpha \cos(\nu^2 x)$ in the intervals

$$\frac{\pi}{(\nu+1)^\beta} < x \leqslant \frac{\pi}{\nu^\beta}, \qquad \nu = 1, 2, \ldots,$$

where $0 < \alpha < \beta < 1$, and is defined in $(-\pi, 0)$ by the relation

$$f(-x) = -f(x).$$

Show that $f(x)$ is integrable in the Lebesgue sense, and that its Fourier sine coefficients satisfy

$$b_n = O(n^{\frac{1}{2}\alpha - \frac{1}{2}} \log n).$$

By taking α small enough and β/α near enough to 1, show that the convergence of $\sum |b_n|^q$, where $q > 2$, is not sufficient to ensure that $f(x)$ shall belong to L^p, where $p = p(q) > 1$.

[The point of the example is that if $q = 2$ the convergence of $\sum |b_n|^q$ *does* imply that $f(x)$ belongs to L^2, and there is an abrupt change in the state of affairs when q becomes greater than 2.

We have

$$b_n = \frac{1}{\pi}\sum_{\nu=1}^{\infty} \nu^{\alpha} \int\limits_{\pi/(\nu+1)^{\beta}}^{\pi/\nu^{\beta}} \{\sin(n+\nu^2)x + \sin(n-\nu^2)x\}\, dx.$$

The terms for which $\sqrt{n}-2 \leqslant \nu \leqslant \sqrt{n}+2$ are

$$O(\nu^{\alpha-\beta-1}) = O(n^{\frac{1}{2}(\alpha-\beta-1)});$$

the terms for which $\nu \leqslant \sqrt{n}-2$ are

$$O\left(\sum_{\nu\leqslant\sqrt{n}-2} \frac{\nu^{\alpha}}{n-\nu^2}\right) = O\left(n^{\frac{1}{2}\alpha}\int\limits_0^{\sqrt{n}-1} \frac{du}{n-u^2}\right)$$
$$= O\left(n^{\frac{1}{2}\alpha-\frac{1}{2}}\int\limits_0^{1-n^{-\frac{1}{2}}} \frac{dv}{1-v^2}\right) = O(n^{\frac{1}{2}\alpha-\frac{1}{2}}\log n),$$

and a similar result holds for the remaining terms. See also Titchmarsh (2).]

19. Show that the function

$$f(x) = -x + \lim_{m\to\infty}\int\limits_0^x (1+\cos t)(1+\cos 4t)\ldots(1+\cos 4^{m-1}t)\, dt$$

is continuous and of bounded variation, and has the period 2π; but that, if b_n is its nth Fourier sine coefficient, nb_n does not tend to zero so that $f(x)$ is not an integral.

[This example is due to F. Riesz (3). Let $\tau_m(x)$ denote the integrand. It is a cosine polynomial of order

$$1+4+\ldots+4^{m-1} = \tfrac{1}{3}(4^m-1).$$

On multiplying by $1+\cos 4^m x$, the first new term involves

$$\cos\{4^m - \tfrac{1}{3}(4^m-1)\}x = \cos\tfrac{1}{3}(2.4^m+1)x,$$

which is of higher order than any of the terms in $\tau_m(x)$. Hence $\tau_{m+1}(x)$ is obtained by adding new terms to $\tau_m(x)$ without altering the existing ones. Also it is easily seen that all the coefficients lie between 0 and 1.

Let α_m be the number of non-vanishing terms in τ_m. The recurrence relation $\alpha_{m+1} = 3\alpha_m - 1$ is easily verified. Hence $\alpha_{m+1}-\alpha_m = 3(\alpha_m-\alpha_{m-1})$, $\alpha_{m+1}-\alpha_m = 3^m$. Hence, if $0 < x \leqslant 2\pi$,

$$\left|\int\limits_0^x \{\tau_{m+1}(t)-\tau_m(t)\}\, dt\right| \leqslant 2\pi\frac{3^m}{\frac{1}{3}(2.4^m+1)}.$$

Hence $\int\limits_0^x \tau_m(t)\, dt$ tends uniformly to a limit, i.e. $f(x)$ is continuous. Also

$\tau_m(x)$ is non-decreasing, and so is its limit. Hence $f(x)$ is of bounded variation. Finally $b_{4^m} = 1/4^m$.]

20. Show that, if $a_n \cos nx + b_n \sin nx \to 0$ in a set of positive measure, then $a_n \to 0$ and $b_n \to 0$.

21. Show that the reciprocal formulae

$$F(x) = \int_{-\infty}^{\infty} e^{ixt} f(t)\,dt, \qquad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixt} F(t)\,dt,$$

hold under the same conditions as Fourier's integral.

22. Show that Mellin's inversion formulae

$$\phi(s) = \int_0^{\infty} x^{s-1}\psi(x)\,dx, \qquad \psi(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \phi(s)x^{-s}\,ds,$$

may, with suitable conditions, be deduced from the formulae of the previous example.

23. Show that the functions

$$x^{-\frac{1}{2}}, \qquad e^{-\frac{1}{2}x^2}, \qquad \operatorname{sech} x\sqrt{(\tfrac{1}{2}\pi)}$$

are their own Fourier cosine transforms, and that

$$x^{-\frac{1}{2}}, \qquad xe^{-\frac{1}{2}x^2}, \qquad \frac{1}{e^{x\sqrt{(2\pi)}}-1} - \frac{1}{x\sqrt{(2\pi)}}$$

are their own sine transforms.

24. Express $e^{-a|x|}$, where $a > 0$, as a Fourier integral. Verify the formula 13.96 (2) in the case where $f(x) = e^{-ax}$, $\phi(x) = e^{-bx}$.

25. Evaluate the integral

$$\int_0^{\infty} \frac{\sin ax \sin bx}{x^2}\,dx$$

by means of the formula 13.96 (2).

26. Let $f(x)$ belong to $L(0, \infty)$, and be continuous and steadily decreasing to zero as $x \to \infty$ (or be the difference between two functions of this type). Let $\alpha > 0$, $\alpha\beta = 2\pi$, and let $g(x)$ be the Fourier cosine transform of $f(x)$. Then

$$\sqrt{\alpha}\left\{\tfrac{1}{2}f(0) + \sum_{n=1}^{\infty} f(n\alpha)\right\} = \sqrt{\beta}\left\{\tfrac{1}{2}g(0) + \sum_{n=1}^{\infty} g(n\beta)\right\}.$$

[This is known as Poisson's formula. It is easily verified that

$$\sqrt{\beta}\left\{\tfrac{1}{2}g(0) + \sum_{m=1}^{n} g(m\beta)\right\}$$

$$= \frac{\sqrt{\alpha}}{2\pi} \int_0^{\pi} f\left(\frac{t}{\beta}\right) \frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt + \frac{\sqrt{\alpha}}{2\pi} \sum_{m=1}^{\infty} \int_{(2m-1)\pi}^{(2m+1)\pi} f\left(\frac{t}{\beta}\right) \frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt.$$

This differs from the left-hand side of Poisson's formula by

$$\frac{\sqrt{\alpha}}{2\pi}\int_0^\pi \left\{f\left(\frac{t}{\beta}\right)-f(0)\right\}\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt+$$
$$+\frac{\sqrt{\alpha}}{2\pi}\sum_{m=1}^{\infty}\int_{(2m-1)\pi}^{(2m+1)\pi}\left\{f\left(\frac{t}{\beta}\right)-f\left(\frac{2m\pi}{\beta}\right)\right\}\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt.$$

The given conditions ensure that this series converges uniformly with respect to n; in fact, it is easily seen from the second mean-value theorem that the general term is $O[f\{(2m-1)\pi/\beta\}]$ independently of n; and each term tends to zero as $n \to \infty$ (as in the proof of Jordan's test), and the result follows.

For other conditions for the formula see Linfoot (1), Mordell (2).]

27. Verify Poisson's formula for the function $f(x) = 1/(1+x^2)$. [The result is equivalent to that of § 3.22, ex. (iii).]

28. Deduce from Poisson's formula that if $x > 0$

$$\sum_{n=-\infty}^{\infty} e^{-n^2x^2} = \frac{\sqrt{\pi}}{x}\sum_{n=-\infty}^{\infty} e^{-n^2\pi^2/x^2}.$$

29. Sum the series $\sum_{n=1}^{\infty} n^{-\nu}J_\nu(n\beta)$, where $\beta > 0$, $\nu > \frac{1}{2}$, by means of Poisson's formula and the first result of Ch. 1, ex. 5.