proof of the preceding theorem about factorization of functions in H_1. We need first of all to note the following analogue of a result already established for functions in H_1 or in H_∞:

Theorem. *If $f \in H_2$ and $f(t)$ is not a.e. zero on $\mathbb{R}$,*

$$\boxed{\int_{-\infty}^{\infty} \frac{\log^- |f(t)|}{1+t^2}\,dt \quad < \quad \infty.}$$

Also, for

$$f(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} f(t)\,dt,$$

one has

$$\log|f(z)| \quad \leqslant \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} \log|f(t)|\,dt$$

when $\Im z > 0$, the integral on the right converging absolutely.

Proof. Is very similar to that of the corresponding theorem in H_1.* Here, when considering the difference

$$\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} \log^+|f(t+ih)|\,dt \;-\; \int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} \log^+|f(t)|\,dt,$$

one first observes that it is bounded in absolute value by

$$\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} \big|\,|f(t+ih)| - |f(t)|\,\big|\,dt$$

and then applies Schwarz' inequality. The rest of the argument is the same as for H_1.

Corollary. *Unless $f(t) \in H_2$* vanishes *a.e.,* $|f(t)| > 0$ *a.e. on* $\mathbb{R}$.

Definition (Beurling). A function f in H_2 which is not a.e. zero on $\mathbb{R}$ is called *outer* if, for the function $f(z)$ of the above theorem we have

$$\log|f(z)| \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} \log|f(t)|\,dt$$

whenever $\Im z > 0$.

* One may also appeal directly to that theorem after noting that $f^2 \in H_1$.

Theorem. *Let $f \in H_2$, not a.e. zero on $\mathbb{R}$, be outer. Then the finite linear combinations of the $e^{i\lambda t}f(t)$ with $\lambda \geqslant 0$ are $\| \quad \|_2$ dense in H_2.*

Remark. This result is due to Beurling, who also established its *converse*. The latter will not be needed in our work; it is set at the end of this article as problem 42.

Proof of theorem. In order to show that the $e^{i\lambda t}f(t)$ with $\lambda \geqslant 0$ generate H_2, it suffices to verify that if φ is any element of L_2 such that

$$\int_{-\infty}^{\infty} e^{i\lambda t} f(t)\varphi(t)\,dt \quad = \quad 0$$

for all $\lambda \geqslant 0$, then

$$\int_{-\infty}^{\infty} g(t)\varphi(t)\,dt \quad = \quad 0$$

for each $g \in H_2$. This will follow if we can show that such a φ belongs to H_2, for then the products $g\varphi$ with $g \in H_2$ will be in H_1.

Since f and $\varphi \in L_2$, $f\varphi \in L_1$, and our assumed relation makes $f\varphi$ in H_1. The function

$$F(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} f(t)\varphi(t)\,dt$$

is thus *analytic* for $\Im z > 0$.

If $\varphi(t) \equiv 0$ a.e. there is nothing to prove, so we may assume that this is not the case. By the preceding corollary, $|f(t)| > 0$ a.e.; therefore $f(t)\varphi(t)$ is not a.e. zero on $\mathbb{R}$. Hence, by an earlier result,

$$\log|F(z)| \quad \leqslant \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} \log|f(t)\varphi(t)|\,dt$$

when $\Im z > 0$, with the right-hand integral *absolutely convergent.*

At the same time, for

$$f(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} f(t)\,dt$$

we have

$$\log|f(z)| \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} \log|f(t)\,dt$$

by hypothesis whenever $\Im z > 0$. The integral on the right is certainly $> -\infty$, being absolutely convergent, so $F(z)/f(z)$ is *analytic* in $\Im z > 0$.

For that ratio, the previous two relations give

$$\log\left|\frac{F(z)}{f(z)}\right| \quad \leqslant \quad \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2}\log|\varphi(t)|\,\mathrm{d}t, \qquad \Im z \; > \; 0.$$

Thence, by the inequality between arithmetic and geometric means,

$$\left|\frac{F(z)}{f(z)}\right|^2 \quad \leqslant \quad \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2}|\varphi(t)|^2\,\mathrm{d}t, \qquad \Im z \; > \; 0,$$

from which, by Fubini's theorem,

$$\int_{-\infty}^{\infty}\left|\frac{F(x+\mathrm{i}y)}{f(x+\mathrm{i}y)}\right|^2\mathrm{d}x \quad \leqslant \quad \|\varphi\|_2^2.$$

According to a previous theorem, there is hence a function $\psi\in H_2$ with

$$\frac{F(z)}{f(z)} \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2}\psi(t)\,\mathrm{d}t$$

for $\Im z \; > \; 0$, and

$$\frac{F(t+\mathrm{i}y)}{f(t+\mathrm{i}y)} \quad \longrightarrow \quad \psi(t) \quad \text{a.e.}$$

as $y \longrightarrow 0$.

We have, however, by the formula for $F(z)$,

$$F(t+\mathrm{i}y) \quad \longrightarrow \quad f(t)\varphi(t) \quad \text{a.e. as } y\longrightarrow 0,$$

and, for $f(z)$,

$$f(t+\mathrm{i}y) \quad \longrightarrow \quad f(t) \quad \text{a.e. as } y\longrightarrow 0.$$

Therefore, since $|f(t)| \; > \; 0$ a.e.,

$$\varphi(t) \quad = \quad \psi(t) \quad \text{a.e.},$$

i.e., $\varphi\in H_2$, as we needed to show.

The theorem is proved.

Remark. The function f in H_2 appearing near the end of the above factorization theorem for H_1 is outer. In general, given *any* function $M(t) \; \geqslant \; 0$ such that

$$\int_{-\infty}^{\infty}\frac{\log^- M(t)}{1+t^2}\mathrm{d}t \quad < \quad \infty$$

and

$$\int_{-\infty}^{\infty} (M(t))^2\, dt \quad < \quad \infty,$$

we can construct an outer function $f \in H_2$ for which

$$|f(t)| \;=\; M(t) \qquad \text{a.e. on } \mathbb{R}.$$

To do this, one first puts

$$F(z) \;=\; \exp\left\{\frac{1}{\pi i}\int_{-\infty}^{\infty}\left(\frac{1}{t-z} - \frac{t}{t^2+1}\right)\log M(t)\, dt\right\}$$

for $\Im z > 0$; the conditions on M ensure absolute convergence of the integral figuring on the right. We have

$$\log|F(z)| \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2}\log M(t)\, dt, \qquad \Im z > 0,$$

so that, in the first place,

$$\log|F(t+iy)| \longrightarrow \log M(t) \quad \text{a.e.}$$

for $y \longrightarrow 0$. In the second place, since geometric means do not exceed arithmetic means,

$$\int_{-\infty}^{\infty} |F(x+iy)|^2\, dx \quad \leqslant \quad \int_{-\infty}^{\infty} (M(t))^2\, dt$$

for $y > 0$, by an argument like one in the above proof. There is thus an $f \in H_2$ with

$$F(z) \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2} f(t)\, dt, \qquad \Im z > 0,$$

and

$$F(t+iy) \longrightarrow f(t) \quad \text{a.e.}$$

as $y \longrightarrow 0$.

Comparing the above two limit relations we see, first of all, that

$$|f(t)| \;=\; M(t) \qquad \text{a.e., } t \in \mathbb{R}.$$

Therefore

$$\log|F(z)| \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2}\log|f(t)|\, dt$$

for $\Im z > 0$. Here, our function $F(z)$ is in fact the $f(z)$ figuring in the proof of the last theorem. Hence f is *outer*.

This construction works in particular whenever $M(t) = |g(t)|$ with $g(t)$ in H_2 not a.e. zero on $\mathbb{R}$. Therefore, *any such g in H_2 coincides a.e. in modulus with an outer function in H_2.*

Problem 42

Prove the converse of the preceding result. Show, in other words, that if $f \in H_2$ is *not outer*, the $e^{i\lambda t}f(t)$ with $\lambda \geqslant 0$ *do not generate* H_2 (in norm $\| \quad \|_2$). (Hint: One may as well assume that $f(t)$ is not a.e. zero on $\mathbb{R}$. Take then the *outer* function $g \in H_2$ with $|g(t)| = |f(t)|$ a.e., furnished by the preceding remark. Show first that the ratio $\omega(t) = f(t)/g(t)$ – it is of modulus 1 a.e. – belongs to H_∞. For this purpose, one may look at $f(z)/g(z)$ in $\Im z > 0$.

Next observe that

$$\int_{-\infty}^{\infty} e^{i\lambda t} f(t)\overline{\omega(t)g(t)}\,dt = 0$$

for all $\lambda \geqslant 0$, so that it suffices to show that

$$\int_{-\infty}^{\infty} \varphi(t)\overline{\omega(t)g(t)}\,dt$$

cannot be zero for all $\varphi \in H_2$. Assume that *were* the case. Then

$$\int_{-\infty}^{\infty} e^{i\lambda t}\psi(t)\overline{\omega(t)g(t)}\,dt = 0,$$

i.e.,

$$\int_{-\infty}^{\infty} \overline{\omega(t)}\psi(t)e^{i\lambda t}g(t)\,dt = 0$$

for all $\lambda \geqslant 0$ and every $\psi \in H_2$.

Use now the preceding theorem (!) and another result to argue that

$$\int_{-\infty}^{\infty} \overline{\omega(t)}h(t)\,dt = 0$$

for all $h \in H_1$, making $\overline{\omega(t)}$ *also* in H_∞, together with $\omega(t)$. This means that

$$\omega(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\omega(t)\,dt$$

and $\overline{\omega(z)}$ are *both analytic* in $\Im z > 0$. Since f is *not* outer, however, $|\omega(z)| = |f(z)/g(z)| < 1$ for some such z. A contradiction is now easily obtained.)

Remark. The $\omega \in H_\infty$ figuring in the argument just indicated is called an *inner function.*

2. Statement of the problem, and simple reductions of it

Given a function $w \geqslant 0$ belonging to $L_1(\mathbb{R})$, we want to know whether there is an $\omega \geqslant 0$ defined on $\mathbb{R}$, *not a.e. zero*, such that

$$\int_{-\infty}^{\infty} |\tilde{U}(t)|^2 \omega(t)\,dt \quad \leqslant \quad \int_{-\infty}^{\infty} |U(t)|^2 w(t)\,dt$$

for the Hilbert transforms $\tilde{U}(t)$ (specified in some definite manner) of the functions $U(t)$ belonging to a certain class. Depending on that class, the answer is different for different specifications of $\tilde{U}(t)$.

Two particular specifications are in common use in analysis. The *first* is preferred when dealing with functions U for which only the convergence of

$$\int_{-\infty}^{\infty} \frac{|U(t)|}{1+t^2}\,dt$$

is assured; in that case one takes

$$\tilde{U}(x) \quad = \quad \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{1}{x-t} + \frac{t}{t^2+1} \right) U(t)\,dt.$$

The expression on the right – not really an integral – is a *Cauchy principal value*, defined for almost all real x. (At this point the reader should look again at §H.1, Chapter III and the second part of §C.1, Chapter VIII.)

A *second* definition of $\tilde{U}$ is adopted when, for $\delta > 0$, the integrals

$$\int_{|t-x| \geqslant \delta} \frac{U(t)}{x-t}\,dt$$

are *already absolutely convergent*. In that case, one drops the term $t/(t^2+1)$ figuring in the previous expression and simply takes

$$\tilde{U}(x) \quad = \quad \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{U(t)}{x-t}\,dt,$$

in other words, $1/\pi$ times the limit of the preceding integral for $\delta \longrightarrow 0$. This specification of $\tilde{U}$ was employed in §C.1 of Chapter VIII (see especially the scholium to that article). It is useful even in cases where the above integrals are *not absolutely convergent for $\delta > 0$ but merely exist as limits,* viz.,

$$\lim_{A \to \infty} \left(\int_{-A}^{x-\delta} + \int_{x+\delta}^{A} \right) \frac{U(t)}{x-t} \, dt.$$

This happens, for instance, with certain kinds of functions $U(t)$ bounded on $\mathbb{R}$ and *not* dying away to zero as $t \longrightarrow \pm\infty$. The Hilbert transforms thus obtained are the ones listed in various tables, such as those issued in the Bateman Project series.

If now our question is posed for the *first* kind of Hilbert transform, it turns out to have substance when the given class of functions U is so large as to *include all bounded ones.* In those circumstances, it is most readily treated by first making the substitution $t = \tan(\vartheta/2)$ and then working with functions $U(t)$ equal to *trigonometric polynomials in* ϑ and with certain auxiliary functions analytic in the unit disk. One finds in that way that the question has a positive answer (i.e., that a non-zero $\omega \geqslant 0$ exists) if and only if

$$\int_{-\infty}^{\infty} \frac{1}{(t^2+1)^2 w(t)} \, dt \quad < \quad \infty$$

(under the initial assumption that $w \in L_1(\mathbb{R})$); the reader will find this work set as problem 43 below, which may serve as a test of how well he or she has assimilated the procedures of the present §.

Except in problem 43, we do not consider the first kind of Hilbert transform any further. Instead, we turn to the *second* kind, taking $⨍_{-\infty}^{\infty}$ in its most general sense, as $\lim_{\delta \to \infty} \lim_{A \to \infty} (\int_{-A}^{x-\delta} + \int_{x+\delta}^{A})$. Then

$$\tilde{U}(x) \quad = \quad \frac{1}{\pi} ⨍_{-\infty}^{\infty} \frac{U(t)}{x-t} \, dt$$

is defined for $U(t)$ equal to $\sin \lambda t$ and $\cos \lambda t$, and hence for finite linear combinations of such functions (the so-called *trigonometric sums*). In the following articles, we restrict our attention to trigonometric sums $U(t)$, for which the definition of $\tilde{U}$ by means of the preceding formula presents no problem. As explained at the beginning of §D, one may think crudely of the collection of trigonometric sums as 'filling out' $L_\infty(\mathbb{R})$ 'for all practical purposes'.

By elementary contour integration, one readily finds that

$$\frac{1}{\pi} ⨍_{-\infty}^{\infty} \frac{e^{i\lambda t}}{x-t} \, dt \quad = \quad \begin{cases} -ie^{i\lambda x}, & \lambda > 0, \\ ie^{i\lambda x}, & \lambda < 0. \end{cases}$$

When $\lambda = 0$, the quantity on the left is *zero.* The reader should do this computation. One of the original applications made of contour integration by Cauchy, who *invented* it, was precisely the evaluation of such principal values! In terms of real valued functions, the formula just written goes as follows:

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos \lambda t}{x-t}\,dt \quad = \quad \sin \lambda t, \qquad \lambda > 0;$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin \lambda t}{x-t}\,dt \quad = \quad -\cos \lambda t, \qquad \lambda > 0.$$

From this we see already that our question (about the existence of non-zero $\omega \geqslant 0$) is *without substance* for the *present* specification of the Hilbert transform, when posed for *all trigonometric sums* U. *There can never be an* $\omega \geqslant 0$, *not a.e. zero, such that*

$$\int_{-\infty}^{\infty} |\tilde{U}(t)|^2 \omega(t)\,dt \quad \leqslant \quad \int_{-\infty}^{\infty} |U(t)|^2 w(t)\,dt$$

for all such U, *when* w *is integrable.* This follows immediately on taking

$$U(t) \;=\; \sin \lambda t, \qquad \tilde{U}(t) \;=\; -\cos \lambda t$$

in such a presumed relation and then making $\lambda \longrightarrow 0$; in that way one concludes by Fatou's lemma that

$$\int_{-\infty}^{\infty} \omega(t)\,dt \quad = \quad 0.$$

The same state of affairs prevails whenever our given class of functions U includes pure oscillations of arbitrary phase with frequencies tending to zero. For this reason, we should require the class of trigonometric sums $U(t)$ under consideration to *only contain terms involving frequencies bounded away from zero*, as we did in §D. The simplest non-trivial version of our problem thus has the following formulation:

Let $a > 0$. *Under what conditions on the given* $w \geqslant 0$ *belonging to* $L_1(\mathbb{R})$ *does there exist an* $\omega \geqslant 0$, *not a.e. zero, such that*

$$\int_{-\infty}^{\infty} |\tilde{U}(t)|^2 \omega(t)\,dt \quad \leqslant \quad \int_{-\infty}^{\infty} |U(t)|^2 w(t)\,dt$$

for all finite trigonometric sums

$$U(t) \quad = \sum_{|\lambda| \geqslant a} C_\lambda e^{i\lambda t} \quad ?$$

Here, we are dealing with the *second* kind of Hilbert transform, so, for the sum $U(t)$ just written,

$$\tilde{U}(t) \quad = \quad \sum_{|\lambda| \geqslant a} (-\mathrm{i}C_\lambda \operatorname{sgn} \lambda)\mathrm{e}^{\mathrm{i}\lambda t}.$$

Such functions $U(t)$ can, of course, also be expressed thus:

$$U(t) \quad = \quad \sum_{\lambda \geqslant a} (A_\lambda \cos \lambda t + B_\lambda \sin \lambda t).$$

Then

$$\tilde{U}(t) \quad = \quad \sum_{\lambda \geqslant a} (A_\lambda \sin \lambda t - B_\lambda \cos \lambda t).$$

This manner of writing our trigonometric sums will be preferred in the following discussion; it has the advantage of making the *real-valued* sums $U(t)$ be *precisely* the ones involving *only real coefficients* A_λ *and* B_λ.

We see in particular that if $U(t)$ is a complex-valued sum of the above kind, $\Re U(t)$ and $\Im U(t)$ are also sums of the same form. This means that our relation

$$\int_{-\infty}^{\infty} |\tilde{U}(t)|^2 \omega(t)\, \mathrm{d}t \quad \leqslant \quad \int_{-\infty}^{\infty} |U(t)|^2 w(t)\, \mathrm{d}t$$

holds for all complex-valued U *of the above form iff it holds for the real valued ones.*

Given any trigonometric sum $U(t)$ (real-valued or not) of the form in question, we have

$$U(t) \ + \ \mathrm{i}\tilde{U}(t) \quad = \quad \sum_{\lambda \geqslant a} C_\lambda \mathrm{e}^{\mathrm{i}\lambda t}$$

with certain coefficients C_λ. *Conversely*, if $F(t)$ is *any* finite sum like the one on the right,

$$\Re F(t) \ = \ U(t)$$

is a sum of the form under consideration, and then

$$\tilde{U}(t) \ = \ \Im F(t).$$

These statements are immediately verified by simple calculation.

Lemma. *Given* $w \geqslant 0$ *in* $L_1(\mathbb{R})$, *let* $a > 0$. *The relation*

$$\int_{-\infty}^{\infty} |\tilde{U}(t)|^2 \omega(t)\, \mathrm{d}t \quad \leqslant \quad \int_{-\infty}^{\infty} |U(t)|^2 w(t)\, \mathrm{d}t$$

holds for all trigonometric sums

$$U(t) \quad = \quad \sum_{\lambda \geqslant a} (A_\lambda \cos \lambda t + B_\lambda \sin \lambda t)$$

with the function $\omega(t) \geqslant 0$ *iff*

$$\left| \int_{-\infty}^{\infty} (w(t) + \omega(t))(F(t))^2 \, \mathrm{d}t \right| \quad \leqslant \quad \int_{-\infty}^{\infty} (w(t) - \omega(t)) |F(t)|^2 \, \mathrm{d}t$$

for all finite sums

$$F(t) \quad = \quad \sum_{\lambda \geqslant a} C_\lambda \mathrm{e}^{\mathrm{i}\lambda t}.$$

Proof. As remarked above, our relation holds for trigonometric sums U of the given form iff

$$\int_{-\infty}^{\infty} (\tilde{U}(t))^2 \omega(t) \, \mathrm{d}t \quad \leqslant \quad \int_{-\infty}^{\infty} (U(t))^2 w(t) \, \mathrm{d}t$$

for all such *real-valued* U. Multiply this relation by 2 and then add to both sides of the result the quantity

$$\int_{-\infty}^{\infty} \{ (\tilde{U}(t))^2 w(t) \; - \; (U(t))^2 w(t) \; - \; (\tilde{U}(t))^2 \omega(t) \; - \; (U(t))^2 \omega(t) \} \, \mathrm{d}t.$$

We obtain the relation

$$\int_{-\infty}^{\infty} (w(t) \; + \; \omega(t)) \{ (\tilde{U}(t))^2 \; - \; (U(t))^2 \} \, \mathrm{d}t$$

$$\leqslant \quad \int_{-\infty}^{\infty} (w(t) \; - \; \omega(t)) \{ (U(t))^2 \; + \; (\tilde{U}(t))^2 \} \, \mathrm{d}t$$

which must thus be *equivalent* to our original one (see the remark immediately following this proof).

In terms of $F(t) = U(t) + \mathrm{i}\tilde{U}(t)$, the last inequality becomes

$$-\Re \int_{-\infty}^{\infty} (w(t) + \omega(t)(F(t))^2 \, \mathrm{d}t \quad \leqslant \quad \int_{-\infty}^{\infty} (w(t) - \omega(t)) |F(t)|^2 \, \mathrm{d}t,$$

so, according to the statements preceding the lemma, our original relation holds with the trigonometric sums $U(t)$ iff the present one is valid for the finite sums

$$F(t) \quad = \quad \sum_{\lambda \geqslant a} C_\lambda \mathrm{e}^{\mathrm{i}\lambda t}.$$

If, however, $F(t)$ is of this form, so is $e^{i\gamma}F(t)$ for each real constant γ. The preceding condition is thus equivalent to the requirement that

$$-\Re e^{2i\gamma}\int_{-\infty}^{\infty}(w(t)+\omega(t))(F(t))^2\,dt \quad\leqslant\quad \int_{-\infty}^{\infty}(w(t)-\omega(t))|F(t)|^2\,dt$$

for each function F and all real γ, and that happens iff

$$\left|\int_{-\infty}^{\infty}(w(t)+\omega(t))(F(t))^2\,dt\right|$$

is $\leqslant$ the integral on the right for any such F. This last condition is hence equivalent to our original one, Q.E.D.

Remark. The argument just made tacitly assumes finiteness of $\int_{-\infty}^{\infty}(\tilde{U}(t))^2\omega(t)\,dt$ and $\int_{-\infty}^{\infty}(U(t))^2\omega(t)\,dt$, as well as that of $\int_{-\infty}^{\infty}(\tilde{U}(t))^2w(t)\,dt$ and $\int_{-\infty}^{\infty}(U(t))^2w(t)\,dt$. About the latter two quantities, there can be no question, w being assumed integrable. Then, however, the former two must also be finite, whether we suppose the *first relation* of the lemma to hold or the *second*. Indeed, if it is the *first* one that holds,

$$\int_{-\infty}^{\infty}\{(U(t))^2+(\tilde{U}(t))^2\}\omega(t)\,dt$$

$$\leqslant\quad \int_{-\infty}^{\infty}\{(U(t))^2+(\tilde{U}(t))^2\}w(t)\,dt \quad<\quad \infty,$$

since, for $F(t) = U(t) + i\tilde{U}(t)$, $\tilde{F}(t) = -iF(t)$. And, if the *second* holds, we surely have

$$\int_{-\infty}^{\infty}(w(t)-\omega(t))\big((U(t))^2+(\tilde{U}(t))^2\big)\,dt \quad\geqslant\quad 0.$$

Theorem. *Given $w \geqslant 0$ in $L_1(\mathbb{R})$ and $a > 0$, any $\omega \geqslant 0$ such that*

$$\int_{-\infty}^{\infty}|\tilde{U}(t)|^2\omega(t)\,dt \quad\leqslant\quad \int_{-\infty}^{\infty}|U(t)|^2w(t)\,dt$$

for all sums U of the form

$$U(t) \quad=\quad \sum_{\lambda\geqslant a}(A_\lambda\cos\lambda t+B_\lambda\sin\lambda t)$$

must satisfy

$$\omega(t) \;\leqslant\; w(t) \qquad \text{a.e. on } \mathbb{R}.$$

Proof. Such an ω must in the first place belong to $L_1(\mathbb{R})$. For, putting first

$U(t) = \sin at$, $\tilde{U}(t) = -\cos at$ in our relation, and then $U(t) = \sin 2at$, $\tilde{U}(t) = -\cos 2at$, we get

$$\int_{-\infty}^{\infty} (\cos^2 at + \cos^2 2at)\omega(t)\,dt < \infty.$$

Here,

$$\cos^2 at + \cos^2 2at = \tfrac{1}{2}(1 + \cos 2at + 2\cos^2 2at) \geqslant \tfrac{7}{16}$$

for $t \in \mathbb{R}$, so ω is integrable.

Knowing that w and ω are both integrable, we can prove the theorem by verifying that

$$\int_{-\infty}^{\infty} (w(t) - \omega(t))\varphi(t)\,dt \geqslant 0$$

for each continuous function $\varphi \geqslant 0$ of compact support.

Fix any such φ, and pick an $\varepsilon > 0$. Choose first an L so large that $\varphi(t)$ *vanishes identically* outside $(-L, L)$ and that

$$\|\varphi\|_\infty \cdot \int_{|t| \geqslant L} (w(t) + \omega(t))\,dt < \varepsilon;$$

since w and ω are in $L_1(\mathbb{R})$, such a choice is possible. Then expand $\sqrt{(\varphi(t))}$ in a Fourier series on $[-L, L]$:

$$\sqrt{(\varphi(t))} \sim \sum_{n=-\infty}^{\infty} a_n e^{\pi i n t/L}, \quad -L \leqslant t \leqslant L.$$

According to the rudiments of harmonic analysis, the Fejér means

$$s_N(t) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) a_n e^{\pi i n t/L}$$

of this Fourier series tend *uniformly* to $\sqrt{(\varphi(t))}$ on $[-L, L]$ as $N \longrightarrow \infty$. Also,

$$\|s_N\|_\infty \leqslant \|\sqrt{(\varphi)}\|_\infty.$$

We thus have

$$\int_{-\infty}^{\infty} (w(t) - \omega(t))\varphi(t)\,dt = \int_{-L}^{L} (w(t) - \omega(t))\varphi(t)\,dt$$

$$= \lim_{N \to \infty} \int_{-L}^{L} (w(t) - \omega(t))(s_N(t))^2\,dt.$$

And, for each N,

$$\int_{-L}^{L} (w(t)-\omega(t))(s_N(t))^2\,dt \;=\; \left(\int_{-\infty}^{\infty} - \int_{|t|\geq L}\right)(w(t)-\omega(t))(s_N(t))^2\,dt$$

$$\geq \int_{-\infty}^{\infty} (w(t)-\omega(t))(s_N(t))^2\,dt \;-\; \|s_N\|_\infty^2 \int_{|t|\geq L} (w(t)+\omega(t))\,dt$$

$$\geq \int_{-\infty}^{\infty} (w(t)-\omega(t))(s_N(t))^2\,dt \;-\; \varepsilon$$

by choice of L, since $\|s_N\|_\infty^2 \leq \|\varphi\|_\infty$.

Putting

$$F_N(t) \;=\; e^{iat}e^{\pi iNt/L}s_N(t),$$

we have $|F_N(t)|^2 = (s_N(t))^2$. $F_N(t)$, however, is of the form

$$\sum_{\lambda\geq a} C_\lambda e^{i\lambda t},$$

so

$$\int_{-\infty}^{\infty} (w(t)-\omega(t))(s_N(t))^2\,dt \;=\; \int_{-\infty}^{\infty} (w(t)-\omega(t))|F_N(t)|^2\,dt$$

is ≥ 0 by the lemma. Using this in the last member of the previous chain of inequalities, we see that

$$\int_{-L}^{L} (w(t)-\omega(t))(s_N(t))^2\,dt \;\geq\; -\varepsilon$$

for each N, so, by the above limit relation,

$$\int_{-\infty}^{\infty} (w(t)-\omega(t))\varphi(t)\,dt \;\geq\; -\varepsilon.$$

Squeezing ε, we see that the integral on the left is ≥ 0, which is what we needed to show to prove the theorem. Done.

Lemma. *Given $w \geq 0$ in $L_1(\mathbb{R})$, let $a > 0$. A necessary and sufficient condition that there be an $\omega \geq 0$, not a.e. zero on $\mathbb{R}$, such that*

$$\int_{-\infty}^{\infty} |\tilde{U}(t)|^2\omega(t)\,dt \;\leq\; \int_{-\infty}^{\infty} |U(t)|^2 w(t)\,dt$$

for the finite sums

$$U(t) \;=\; \sum_{\lambda\geq a} (A_\lambda \cos\lambda t + B_\lambda \sin\lambda t),$$

is that there exist a function $\rho(t)$ not a.e. zero, $0 \leqslant \rho(t) \leqslant w(t)$, with

$$\left|\int_{-\infty}^{\infty} w(t)(F(t))^2\,dt\right| \leqslant \int_{-\infty}^{\infty} (w(t)-\rho(t))|F(t)|^2\,dt$$

for all functions F of the form

$$F(t) = \sum_{\lambda \geqslant a} C_\lambda e^{i\lambda t}.$$

When an ω fulfilling the above condition exists, ρ may be taken equal to it. When, on the other hand, a function ρ is known, the ω equal to $\frac{1}{2}\rho$ will work.

Proof. If a function ω with the stated properties exists, we know by the previous theorem that $0 \leqslant \omega(t) \leqslant w(t)$ a.e.. Therefore, if $U(t)$ is any sum of the above form,

$$\frac{1}{2}\int_{-\infty}^{\infty} |\tilde{U}(t)|^2\omega(t)\,dt \leqslant \frac{1}{2}\int_{-\infty}^{\infty} |U(t)|^2 w(t)\,dt$$
$$\leqslant \int_{-\infty}^{\infty} |U(t)|^2\left(w(t) - \tfrac{1}{2}\omega(t)\right)dt.$$

The first condition of the previous lemma is thus fulfilled with

$$\omega_1(t) = \tfrac{1}{2}\omega(t)$$

in place of $\omega(t)$ and

$$w_1(t) = w(t) - \tfrac{1}{2}\omega(t)$$

in place of $w(t)$. Hence, by that lemma,

$$\left|\int_{-\infty}^{\infty} (w_1(t)+\omega_1(t))(F(t))^2\,dt\right| \leqslant \int_{-\infty}^{\infty} (w_1(t)-\omega_1(t))|F(t)|^2\,dt$$

for the functions F of the form described. This relation goes over into the asserted one on taking $\rho(t) = \omega(t)$.

If, conversely, the relation involving functions F holds for some ρ, $0 \leqslant \rho(t) \leqslant w(t)$, we certainly have

$$\left|\int_{-\infty}^{\infty} \left(w(t) + \tfrac{1}{2}\rho(t)\right)(F(t))^2\,dt\right| \leqslant \int_{-\infty}^{\infty} (w(t)-\rho(t))|F(t)|^2\,dt$$
$$+ \frac{1}{2}\int_{-\infty}^{\infty} \rho(t)|F(t)|^2\,dt$$
$$= \int_{-\infty}^{\infty} \left(w(t) - \tfrac{1}{2}\rho(t)\right)|F(t)|^2\,dt$$

for such F, so, by the previous lemma,

$$\frac{1}{2}\int_{-\infty}^{\infty} |\tilde{U}(t)|^2 \rho(t)\,dt \quad \leqslant \quad \int_{-\infty}^{\infty} |U(t)|^2 w(t)\,dt$$

for the sums U. Our relation for the latter thus *holds* with $\omega(t) = \rho(t)/2$, and this is not a.e. zero if $\rho(t)$ is not. Done.

Theorem. *If, for given $w \geqslant 0$ in $L_1(\mathbb{R})$ and some $a > 0$ there is any $\omega \geqslant 0$, not a.e. zero, such that*

$$\int_{-\infty}^{\infty} |\tilde{U}(t)|^2 \omega(t)\,dt \quad \leqslant \quad \int_{-\infty}^{\infty} |U(t)|^2 w(t)\,dt$$

for the finite sums

$$U(t) \quad = \quad \sum_{\lambda \geqslant a} (A_\lambda \cos \lambda t + B_\lambda \sin \lambda t),$$

we have

$$\int_{-\infty}^{\infty} \frac{\log^- w(t)}{1+t^2}\,dt \quad < \quad \infty.$$

Remark. Of course,

$$\int_{-\infty}^{\infty} \frac{\log^+ w(t)}{1+t^2}\,dt \quad < \quad \infty$$

by the inequality between arithmetic and geometric means, w being in L_1.

Proof of theorem. If an ω having the stated properties exists, there is, by the preceding lemma, a function ρ, not a.e. zero, $0 \leqslant \rho(t) \leqslant w(t)$, such that

$$\left| \int_{-\infty}^{\infty} w(t)(F(t))^2\,dt \right| \quad \leqslant \quad \int_{-\infty}^{\infty} (w(t) - \rho(t))|F(t)|^2\,dt$$

for the functions

$$F(t) \quad = \quad \sum_{\lambda \geqslant a} C_\lambda e^{i\lambda t}.$$

Suppose now that

$$\int_{-\infty}^{\infty} \frac{\log^- w(t)}{1+t^2}\,dt \quad = \quad \infty;$$

then we will show that the function $\rho(t)$ figuring in the previous relation must be zero a.e., thus obtaining a contradiction. For this purpose, we use a variant of *Szegő's theorem* which, under our assumption on $\log^- w(t)$, gives us a sequence of functions $F_N(t)$, having the form just indicated, such that

$$\int_{-\infty}^{\infty} |1 - F_N(t)|^2 w(t)\,dt \xrightarrow[N]{} 0.$$

The reader should refer to Chapter II and to problem 2 at the end of it. *There*, Szegő's theorem was established for the weighted L_1 norm, and problem 2 yielded functions $F_N(t)$ of the above form for which

$$\int_{-\infty}^{\infty} |1 - F_N(t)|\,w(t)\,dt \xrightarrow[N]{} 0.$$

However, after making a simple modification in the argument of Chapter II, §A, which should be apparent to the reader, one obtains a proof of Szegő's theorem for weighted L_2 norms – indeed, for weighted L_p ones, where $1 \leqslant p < \infty$. There is then no difficulty in carrying out the steps of problem 2 for the weighted L_2 norm.

Once we have functions F_N satisfying the above relation, we see that

$$\int_{-\infty}^{\infty} w(t)(F_N(t))^2\,dt \xrightarrow[N]{} \int_{-\infty}^{\infty} w(t)\,dt.$$

Indeed, using Schwarz and the triangle inequality, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} w(t)\,|(F_N(t))^2 - 1|\,dt \;&=\; \int_{-\infty}^{\infty} w(t)\,|F_N(t) - 1|\,|F_N(t) + 1|\,dt \\
&\leqslant\; \sqrt{\left(\int_{-\infty}^{\infty} w(t)|F_N(t) + 1|^2\,dt \cdot \int_{-\infty}^{\infty} w(t)|F_N(t) - 1|^2\,dt\right)} \\
&\leqslant\; \left(\sqrt{\left(\int_{-\infty}^{\infty} w(t)|F_N(t) - 1|^2\,dt\right)} + \sqrt{\left(4\int_{-\infty}^{\infty} w(t)\,dt\right)}\right) \times \\
&\qquad \times \sqrt{\left(\int_{-\infty}^{\infty} w(t)|F_N(t) - 1|^2\,dt\right)},
\end{aligned}$$

and the last expression goes to zero as $N \longrightarrow \infty$.

We also see by this computation that

$$\int_{-\infty}^{\infty} w(t)\,|F_N(t)|^2\,dt \xrightarrow[N]{} \int_{-\infty}^{\infty} w(t)\,dt,$$

and again, since $0 \leqslant \rho(t) \leqslant w(t)$, that

$$\int_{-\infty}^{\infty} \rho(t)|F_N(t)|^2\,dt \xrightarrow[N]{} \int_{-\infty}^{\infty} \rho(t)\,dt.$$

Using these relations and making $N \longrightarrow \infty$ in the inequality

$$\left|\int_{-\infty}^{\infty} w(t)(F_N(t))^2\,dt\right| \leqslant \int_{-\infty}^{\infty} (w(t)-\rho(t))|F_N(t)|^2\,dt,$$

we get

$$\int_{-\infty}^{\infty} w(t)\,dt \leqslant \int_{-\infty}^{\infty} (w(t)-\rho(t))\,dt,$$

i.e.,

$$\rho(t) = 0 \quad \text{a.e.,}$$

since $\rho(t) \geqslant 0$.

We have reached our promised contradiction. This shows that the integral $\int_{-\infty}^{\infty}(\log^- w(t)/(1+t^2))\,dt$ must indeed be finite, as claimed. The theorem is proved.

3. Application of H_p space theory; use of duality

The last theorem of the preceding article shows that our problem can have a positive solution only when

$$\int_{-\infty}^{\infty} \frac{\log^- w(t)}{1+t^2}\,dt < \infty;$$

we may thus limit our further considerations to functions $w \geqslant 0$ in $L_1(\mathbb{R})$ fulfilling this condition. According to a remark at the end of article 1, there is, corresponding to any such w, an *outer function* φ in H_2 with

$$|\varphi(t)| = \sqrt{(w(t))} \qquad \text{a.e., } t \in \mathbb{R}.$$

Theorem. *Let $w \geqslant 0$, belonging to $L_1(\mathbb{R})$, satisfy the above condition on its logarithm, and let $a > 0$. In order that there exist an $\omega \geqslant 0$, not a.e. zero, such that*

$$\int_{-\infty}^{\infty} |\tilde{U}(t)|^2\omega(t)\,dt \leqslant \int_{-\infty}^{\infty} |U(t)|^2 w(t)\,dt$$

for the functions

$$U(t) \quad = \quad \sum_{\lambda \geqslant a} (A_\lambda \cos \lambda t + B_\lambda \sin \lambda t),$$

it is necessary and sufficient that there be a function $\sigma(t)$, not a.e. zero, with

$$0 \;\leqslant\; \sigma(t) \;\leqslant\; 1 \quad \text{a.e.}$$

and

$$\left| \int_{-\infty}^{\infty} e^{2iat} \frac{\overline{\varphi(t)}}{\varphi(t)} f(t)\,dt \right| \quad \leqslant \quad \int_{-\infty}^{\infty} (1 - \sigma(t)) |f(t)|\,dt$$

for all $f \in H_1$. Here, $\varphi(t)$ is any outer function in H_2 with

$$|\varphi(t)| \;=\; \sqrt{(w(t))} \qquad \text{a.e., } t \in \mathbb{R}.$$

If we have *a function ω for which the above relation holds, $\sigma(t)$ can be taken equal to $\omega(t)/w(t)$. If, on the other hand, a σ is furnished, $\omega(t)$ can be taken equal to $\sigma(t)w(t)/2$.*

Remark. Any two outer functions φ in H_2 with

$$|\varphi(t)| \;=\; \sqrt{(w(t))} \quad \text{a.e.}$$

differ by a constant factor of modulus 1.

Proof of theorem. Is based on an idea from the *Bologna Annali* paper of Helson and Szegő.

According to the second lemma of the preceding article, the existence of an ω having the properties in question is *equivalent* to that of a ρ not a.e. zero, $0 \leqslant \rho(t) \leqslant w(t)$, such that

$$\left| \int_{-\infty}^{\infty} w(t)(F(t))^2\,dt \right| \quad \leqslant \quad \int_{-\infty}^{\infty} (w(t) - \rho(t)) |F(t)|^2\,dt$$

for the functions

$$F(t) \quad = \quad \sum_{\lambda \geqslant a} C_\lambda e^{i\lambda t}.$$

This relation is in turn equivalent to the requirement that

$$\left| \int_{-\infty}^{\infty} w(t)P(t)Q(t)\,dt \right| \quad \leqslant \quad \frac{1}{2} \int_{-\infty}^{\infty} (w(t) - \rho(t))(|P(t)|^2 + |Q(t)|^2)\,dt$$

for all *pairs* P, Q of finite sums of the form

$$\sum_{\lambda \geqslant a} C_\lambda e^{i\lambda t}.$$

To see this, one notes in the first place that the present inequality goes over into the preceding one on taking $P = Q = F$. If, on the other hand, the preceding one always holds for our functions F, it is true both with

$$F = P + Q$$

and with

$$F = P - Q$$

whenever P and Q are two sums of the given form. Therefore,

$$\left|\int_{-\infty}^{\infty} w(t)P(t)Q(t)\,dt\right| = \frac{1}{4}\left|\int_{-\infty}^{\infty} w(t)\{(P(t)+Q(t))^2 - (P(t)-Q(t))^2\}\,dt\right|$$

$$\leqslant \frac{1}{4}\int_{-\infty}^{\infty} (w(t)-\rho(t))\{|P(t)+Q(t)|^2 + |P(t)-Q(t)|^2\}\,dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} (w(t)-\rho(t))(|P(t)|^2 + |Q(t)|^2)\,dt,$$

and we have the second of the two inequalities.

Take now an outer function $\varphi \in H_2$ such that $|\varphi(t)|^2 = w(t)$ a.e., and write

$$\sigma(t) = \rho(t)/w(t),$$

so that $0 \leqslant \sigma(t) \leqslant 1$ a.e.. Our condition on $\log w(t)$ makes

$$w(t) > 0 \quad \text{a.e.},$$

so the ratio $\sigma(t)$ will be > 0 on a set of positive measure iff $\rho(t)$ is. In terms of φ and σ, the relation involving P and Q can be rewritten

$$\left|\int_{-\infty}^{\infty} \frac{\overline{\varphi(t)}}{\varphi(t)}(\varphi(t)P(t))(\varphi(t)Q(t))\,dt\right|$$

$$\leqslant \frac{1}{2}\int_{-\infty}^{\infty} (1-\sigma(t))\{|\varphi(t)P(t)|^2 + |\varphi(t)Q(t)|^2\}\,dt.$$

Write now

$$P(t) = e^{iat}p(t), \qquad Q(t) = e^{iat}q(t),$$

so that $p(t)$ and $q(t)$ are sums of the form

$$\sum_{\lambda \geq 0} C_\lambda e^{i\lambda t}.$$

Then the last inequality becomes

$$\left| \int_{-\infty}^{\infty} e^{2iat} \frac{\overline{\varphi(t)}}{\varphi(t)} (\varphi(t)p(t))(\varphi(t)q(t))\, dt \right|$$
$$\leq \frac{1}{2} \int_{-\infty}^{\infty} (1 - \sigma(t)) \{ |\varphi(t)p(t)|^2 + |\varphi(t)q(t)|^2 \}\, dt.$$

Since $\varphi \in H_2$ is *outer*, the products $\varphi(t)p(t)$, $\varphi(t)q(t)$ are $\| \;\|_2$ dense in H_2 when p and q range through the collection of finite sums

$$\sum_{\lambda \geq 0} C_\lambda e^{i\lambda t},$$

according to the last theorem of article 1. The preceding relation is therefore *equivalent* to the condition that

$$\left| \int_{-\infty}^{\infty} e^{2iat} \frac{\overline{\varphi(t)}}{\varphi(t)} g(t)h(t)\, dt \right| \leq \frac{1}{2} \int_{-\infty}^{\infty} (1 - \sigma(t)) \{ |g(t)|^2 + |h(t)|^2 \}\, dt$$

for *all* g and h in H_2.

Suppose $f \in H_1$. According to the factorization theorem from article 1 and the remark thereto, there are functions $g, h \in H_2$ such that

$$f(t) = g(t)h(t) \quad \text{a.e.}$$

and

$$|g(t)| = |h(t)| = \sqrt{(|f(t)|)} \quad \text{a.e..}$$

Substituting these relations into the previous one, we get

$$\left| \int_{-\infty}^{\infty} e^{2iat} \frac{\overline{\varphi(t)}}{\varphi(t)} f(t)\, dt \right| \leq \int_{-\infty}^{\infty} (1 - \sigma(t)) |f(t)|\, dt.$$

This inequality is thus a *consequence* of the preceding one. But it also *implies* the latter. Let, indeed, g and h be in H_2. Then gh is in H_1 by a result of article 1, so, if the present relation holds,

$$\left| \int_{-\infty}^{\infty} e^{2iat} \frac{\overline{\varphi(t)}}{\varphi(t)} g(t)h(t)\, dt \right| \leq \int_{-\infty}^{\infty} (1 - \sigma(t)) |g(t)h(t)|\, dt$$
$$\leq \frac{1}{2} \int_{-\infty}^{\infty} (1 - \sigma(t)) \{ |g(t)|^2 + |h(t)|^2 \}\, dt.$$

Our final inequality, involving functions $f \in H_1$ and the quantity $\sigma(t)$, is hence *fully equivalent* with the initial one for our functions $F(t)$, involving the quantity $\rho(t)$. Since, as we have observed, $\rho(t)$ is > 0 on a set of positive measure iff $\sigma(t)$ is, the first and main conclusion of our theorem now follows directly from the second lemma of the preceding article. Again, since $\rho(t) = \sigma(t)w(t)$, the second conclusion also follows by that lemma. We are done.

In order to proceed further, we use the duality between $L_1(\mathbb{R})$ and $L_\infty(\mathbb{R})$. When one says that the latter space is the *dual* of the former, one means that each (bounded) linear functional Ψ on L_1 corresponds to a unique $\psi \in L_\infty$ such that

$$\Psi(F) \quad = \quad \int_{-\infty}^{\infty} F(t)\psi(t)\,\mathrm{d}t$$

for $F \in L_1$. Here, we need the linear functionals on the *closed subspace* H_1 of L_1. These can be described according to a well known recipe from functional analysis, in the following way.

Take the (w^*) closed subspace E of L_∞ consisting of the ψ therein for which

$$\int_{-\infty}^{\infty} f(t)\psi(t)\,\mathrm{d}t \quad = \quad 0$$

whenever $f \in H_1$; the *quotient space* L_∞/E can then be identified with the dual of H_1. This is how the identification goes: to each bounded linear functional Λ on H_1 corresponds precisely one subset of L_∞ of the form $\psi_0 + E$ (called a *coset* of E) such that

$$\Lambda(f) \quad = \quad \int_{-\infty}^{\infty} f(t)\psi(t)\,\mathrm{d}t$$

whenever $f \in H_1$ for any $\psi \in \psi_0 + E$, and *only* for those ψ.

From article 1, we know that E is H_∞. *The dual of* H_1 *can thus be identified with the quotient space* L_∞/H_∞. We want to use this fact to investigate the criterion furnished by the last result. For this purpose, we resort to a trick, consisting of the *introduction of new norms, equivalent to the usual ones, for* L_1 *and* L_∞. If the inequality in the conclusion of the last theorem *holds* with any function σ, $0 \leqslant \sigma(t) \leqslant 1$, *it certainly does so when* $\sigma(t)/2$ *stands in place of* $\sigma(t)$. According to that theorem, however, it is the *existence* of such functions σ *different from zero on a set of positive* *measure* which is of interest to us here. We may therefore limit our search for one for which the inequality is valid to those satisfying

$$0 \leqslant \sigma(t) \leqslant 1/2 \quad \text{a.e..}$$

This restriction on our functions σ we henceforth assume.

Given such a σ, we then put

$$\|f\|_1^\sigma = \int_{-\infty}^{\infty} (1-\sigma(t))|f(t)|\,dt$$

for $f \in L_1$; $\|f\|_1^\sigma$ is a norm equivalent to the usual one on L_1, because

$$\tfrac{1}{2}\|f\|_1 \leqslant \|f\|_1^\sigma \leqslant \|f\|_1.$$

On L_∞, we use the dual norm

$$\|\psi\|_\infty^\sigma = \operatorname*{ess\,sup}_{t \in \mathbb{R}} \frac{|\psi(t)|}{1-\sigma(t)};$$

here, the $1-\sigma(t)$ goes in the denominator although we multiply by it when defining $\| \ \|_1^\sigma$. We have

$$\|\psi\|_\infty \leqslant \|\psi\|_\infty^\sigma \leqslant 2\|\psi\|_\infty$$

for $\psi \in L_\infty$, so $\| \ \|_\infty^\sigma$ and $\| \ \|_\infty$ are equivalent on that space.

If $\psi \in L_\infty$, we have, for the functional

$$\Psi(f) = \int_{-\infty}^{\infty} f(t)\psi(t)\,dt$$

on L_1 corresponding to it,

$$|\Psi(f)| \leqslant \|\psi\|_\infty^\sigma \|f\|_1^\sigma.$$

Moreover, the *supremum* of $|\Psi(f)|$ for the $f \in L_1$ with $\|f\|_1^\sigma \leqslant 1$ is *precisely* $\|\psi\|_\infty^\sigma$. These facts are easily verified by writing

$$f(t)\psi(t) \quad \text{as} \quad (1-\sigma(t))f(t)\cdot\frac{\psi(t)}{1-\sigma(t)}$$

in the preceding integral.

For elements of the quotient space L_∞/H_∞ – these are just the cosets $\psi_0 + H_\infty$, $\psi_0 \in L_\infty$ – we write, following standard practice,

$$\|\psi_0 + H_\infty\|_\infty^\sigma = \inf\{\|\psi_0 + h\|_\infty^\sigma : \ h \in H_\infty\}.$$

We have already observed that to each such coset corresponds a linear functional Λ on H_1 given by the formula

$$\Lambda(f) \quad = \quad \int_{-\infty}^{\infty} f(t)\psi(t)\,dt, \quad f \in H_1,$$

where ψ is *any* element of $\psi_0 + H_\infty$. By choosing the $\psi \in \psi_0 + H_\infty$ to have $\|\psi\|_\infty^\sigma$ arbitrarily close to $\|\psi_0 + H_\infty\|_\infty^\sigma$, we see that

$$|\Lambda(f)| \quad \leqslant \quad \|\psi_0 + H_\infty\|_\infty^\sigma \, \|f\|_1^\sigma, \qquad f \in H_1.$$

It is important that this inequality is *sharp*. Even more than that is true:

> *The infimum appearing in the above formula for $\|\psi_0 + H_\infty\|_\infty^\sigma$ is actually attained for some $h \in H_\infty$, and is equal to the supremum of $|\Lambda(f)|$ for $f \in H_1$ and $\|f\|_1^\sigma \leqslant 1$.*

This statement is a straightforward consequence of results from elementary functional analysis. However, lest the reader suspect that something is being produced out of nothing here by mere juggling of notation, let us give the proof.

Denote the supremum of $|\Lambda(f)|$ for $f \in H_1$ and $\|f\|_1^\sigma \leqslant 1$ by M. *According to the Hahn–Banach theorem*, there is an extension Λ^* of the linear functional Λ *to all of* L_1, such that

$$|\Lambda^*(F)| \quad \leqslant \quad M\|F\|_1^\sigma$$

for $F \in L_1$. Corresponding to Λ^* there is, as observed earlier, a $\psi \in L_\infty$ with

$$\Lambda^*(F) \quad = \quad \int_{-\infty}^{\infty} F(t)\psi(t)\,dt, \qquad F \in L_1,$$

and, according to what was also noted above,

$$\|\psi\|_\infty^\sigma \quad = \quad \sup\{|\Lambda^*(F)|: \ F \in L_1 \text{ and } \|F\|_1^\sigma \leqslant 1\} \quad \leqslant \quad M.$$

Then, for $f \in H_1$,

$$\Lambda(f) \quad = \quad \Lambda^*(f) \quad = \quad \int_{-\infty}^{\infty} f(t)\psi(t)\,dt,$$

so $\psi \in \psi_0 + H_\infty$; there is, in other words, an $h \in H_\infty$ with $\psi = \psi_0 + h$, and

$$\|\psi_0 + h\|_\infty^\sigma \quad = \quad \|\psi\|_\infty^\sigma \quad \leqslant \quad M.$$

But, as we remarked previously,

$$|\Lambda(f)| \quad \leqslant \quad \|\psi_0 + H_\infty\|_\infty^\sigma \, \|f\|_1^\sigma, \qquad f \in H_1,$$

so $\|\psi_0 + h\|_\infty^\sigma \geqslant \|\psi_0 + H_\infty\|_\infty^\sigma \geqslant M$. Hence

$$\|\psi_0 + H_\infty\|_\infty^\sigma \quad = \quad \|\psi_0 + h\|_\infty^\sigma \quad = \quad M.$$

Once we are in possession of the above facts, it is easy to establish the following key result.

Theorem. *Let $w \geqslant 0$ in $L_1(\mathbb{R})$ and a number $a > 0$ be given. In order that there exist an $\omega \geqslant 0$, not a.e. zero, such that*

$$\int_{-\infty}^{\infty} |\tilde{U}(t)|^2 \omega(t)\,dt \quad \leqslant \quad \int_{-\infty}^{\infty} |U(t)|^2 w(t)\,dt$$

for the sums

$$U(t) \quad = \quad \sum_{\lambda \geqslant a} (A_\lambda \cos \lambda t + B_\lambda \sin \lambda t),$$

it is necessary and sufficient that, first of all,

$$\int_{-\infty}^{\infty} \frac{\log^- w(t)}{1+t^2}\,dt \quad < \quad \infty$$

and that then, if φ is any outer function in H_2 with

$$|\varphi(t)| \quad = \quad \sqrt{(w(t))} \quad \text{a.e.,}$$

we have

$$\left| e^{2iat}\frac{\overline{\varphi(t)}}{\varphi(t)} \; - \; h(t) \right| \quad \leqslant \quad 1 \qquad \text{a.e., } t \in \mathbb{R},$$

for some h, not a.e. zero, *belonging to H_∞.*

A function ω equal to a constant multiple of w will satisfy our conditions iff there is an $h \in H_\infty$ for which

$$\left| e^{2iat}\frac{\overline{\varphi(t)}}{\varphi(t)} \; - \; h(t) \right| \quad \leqslant \quad \text{const.} \quad < \quad 1 \quad \text{a.e..}$$

Proof. As we saw at the end of the last article, there can be no ω with the above properties unless

$$\int_{-\infty}^{\infty} \frac{\log^- w(t)}{1+t^2}\,dt \quad < \quad \infty.$$

Assuming, then, this condition, we take one of the outer functions φ specified in the statement, and see by the preceding theorem and discussion following it that the existence of an ω having the properties in question is *equivalent* to that of a σ, not a.e. zero,

$$0 \leqslant \sigma(t) \leqslant 1/2 \quad \text{a.e.},$$

such that

$$\left| \int_{-\infty}^{\infty} e^{2iat} \frac{\overline{\varphi(t)}}{\varphi(t)} f(t)\, dt \right| \leqslant \|f\|_1^\sigma, \qquad f \in H_1.$$

According to what we observed above, however, the last relation is equivalent to the existence of an $h \in H_\infty$ for which

$$\frac{|(e^{2iat}\overline{\varphi(t)}/\varphi(t)) - h(t)|}{1 - \sigma(t)} \leqslant 1 \qquad \text{a.e., } t \in \mathbb{R}.$$

Suppose in the first place that *there is no non-zero* $h \in H_\infty$ *for which*

$$|(e^{2iat}\overline{\varphi(t)}/\varphi(t)) - h(t)| \leqslant 1 \qquad \text{a.e.}.$$

Then, since $1/2 \leqslant 1 - \sigma(t) \leqslant 1$ a.e., no $h \in H_\infty$ other than the zero one could satisfy the previous relation. The latter must therefore reduce to

$$|e^{2iat}\overline{\varphi(t)}/\varphi(t)|/(1 - \sigma(t)) \leqslant 1 \qquad \text{a.e.},$$

i.e., $1 - \sigma(t) \geqslant 1$ a.e., so that

$$\sigma(t) \equiv 0 \quad \text{a.e., } t \in \mathbb{R}.$$

As has just been said, this means that *there can be no non-zero* ω fulfilling our conditions, and necessity is proved.

Consider now the situation where there *is* a non-zero $h \in H_\infty$ making

$$|(e^{2iat}\overline{\varphi(t)}/\varphi(t)) - h(t)| \leqslant 1 \qquad \text{a.e.}.$$

Then; since

$$(e^{2iat}\overline{\varphi(t)}/\varphi(t)) - \tfrac{1}{2}h(t) = \tfrac{1}{2}\big(\{(e^{2iat}\overline{\varphi(t)}/\varphi(t)) - h(t)\} + e^{2iat}\overline{\varphi(t)}/\varphi(t)\big),$$

the expression on the *left* also has modulus $\leqslant 1$ a.e.. It is in fact of modulus < 1 *on a set of positive measure.* Indeed, the expression in curly brackets on the right has modulus $\leqslant 1$, and the remaining right-hand term (without the factor $1/2$) has modulus equal to 1. Therefore, since the unit circle is *strictly convex*, the whole right side cannot have modulus equal to 1 *unless* the expression in curly brackets and $e^{2iat}\overline{\varphi(t)}/\varphi(t)$ are *equal*, that is, *unless* $h(t) = 0$. We are, however, assuming that $h(t) \neq 0$

on a set of positive measure; the modulus in question must hence be < 1 on such a set.

Put

$$\sigma(t) \quad = \quad \min\left(1/2, \quad 1 \; - \; |(e^{2iat}\overline{\varphi(t)}/\varphi(t)) \; - \; (h(t)/2)|\right).$$

We then have $0 \leqslant \sigma(t) \leqslant 1/2$ a.e., and $\sigma(t) > 0$ on a set of positive measure by what we have just shown. Finally,

$$\frac{|(e^{2iat}\overline{\varphi(t)}/\varphi(t)) \; - \; (h(t)/2)|}{1 \; - \; \sigma(t)} \quad \leqslant \quad 1 \qquad \text{a.e.},$$

so there must, by the above equivalency statements, be a non-zero $\omega \geqslant 0$ for which our inequality on functions U is satisfied. Sufficiency is proved.

We have still to verify the last part of our theorem. It follows, however, from the last part of the preceding one that an ω *equal to a constant multiple of* w *will work* iff, in the relation involving H_1 functions and σ, we can take σ equal to some constant c, $0 < c < 1/2$. By the above discussion, this is *equivalent* to the existence of an $h \in H_\infty$ for which

$$\frac{|(e^{2iat}\overline{\varphi(t)}/\varphi(t)) \; - \; h(t)|}{1 \; - \; c} \quad \leqslant \quad 1 \qquad \text{a.e.},$$

and we have what was needed.

The theorem is completely proved.

Establishment of the remaining results in this § is based on the criterion furnished by the one just obtained. In that way, we get, first of all, the

Theorem. *Let $w \geqslant 0$ in $L_1(\mathbb{R})$ and a number $a > 0$ be given. If there is any $\omega \geqslant 0$ at all, different from zero on a set of positive measure, such that*

$$\int_{-\infty}^{\infty} |\tilde{U}(t)|^2 \omega(t)\, dt \quad \leqslant \quad \int_{-\infty}^{\infty} |U(t)|^2 w(t)\, dt$$

for the sums

$$U(t) \quad = \quad \sum_{\lambda \geqslant a} (A_\lambda \cos \lambda t + B_\lambda \sin \lambda t),$$

there is one with

$$\int_{-\infty}^{\infty} \frac{\log^- \omega(t)}{1 + t^2}\, dt \quad < \quad \infty.$$

Proof. If any ω enjoying the above properties exists, we know by the preceding result that

$$\int_{-\infty}^{\infty} \frac{\log^- w(t)}{1+t^2}\,dt \quad < \quad \infty,$$

and, taking an outer function $\varphi \in H_2$ with $\varphi(t) \;=\; \sqrt{(w(t))}$ a.e., that

$$\left| e^{2iat}\frac{\overline{\varphi(t)}}{\varphi(t)} \;-\; h(t) \right| \quad \leqslant \quad 1 \qquad \text{a.e.}$$

for some *non-zero* $h \in H_\infty$.

The proof of the sufficiency part of the last theorem shows, however, that *once we have such an h*, we can put

$$\sigma(t) \quad = \quad \min\left\{1/2, \quad 1 \;-\; |(e^{2iat}\overline{\varphi(t)}/\varphi(t)) \;-\; (h(t)/2)|\right\},$$

and then, with *this* function σ, the conditions of the *first* result in the present article will be satisfied, ensuring that

$$\omega(t) \quad = \quad \tfrac{1}{2}\sigma(t)w(t)$$

has the desired properties. It is claimed that

$$\int_{-\infty}^{\infty} \frac{\log^- \omega(t)}{1+t^2}\,dt \quad < \quad \infty$$

for *this* function ω. In view of the above condition on $\log^- w(t)$, it is enough to verify that

$$\int_{-\infty}^{\infty} \frac{\log \sigma(t)}{1+t^2}\,dt \quad > \quad -\infty$$

for our present function σ.

Write

$$\psi(t) \quad = \quad e^{-2iat}\varphi(t)/\overline{\varphi(t)};$$

then $|\psi(t)| \;=\; 1$ a.e., and the last inequality is implied by

$$\int_{-\infty}^{\infty} \frac{1}{1+t^2}\log(\,1 \;-\; |\,1 \;-\; (\psi(t)h(t)/2)|\,)\,dt \quad > \quad -\infty$$

which we proceed now to establish.

We have $|1 \;-\; \psi(t)h(t)| \;\leqslant\; 1$ a.e., so

$$1 \;-\; \tfrac{1}{2}\psi(t)h(t) \quad = \quad \tfrac{1}{2} \;+\; \tfrac{1}{2}(1 \;-\; \psi(t)h(t))$$

lies, for almost all t, in a circle of radius 1/2 about the point 1/2:

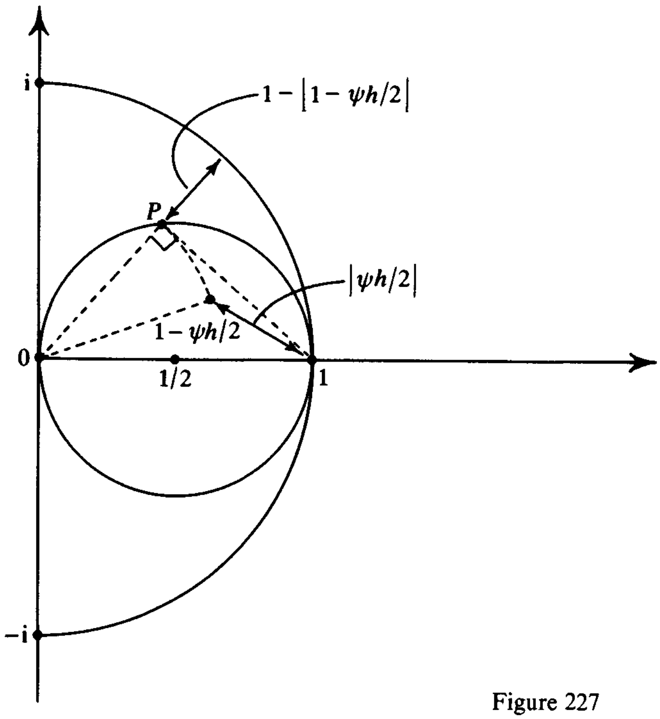

Figure 227

In this figure,

$$1 \; - \; |1 \; - \; (\psi(t)h(t)/2)| \;\; = \;\; 1 \; - \; \overline{OP},$$

where

$$\overline{OP}^2 \;\; \leqslant \;\; 1 \; - \; |\psi(t)h(t)/2|^2.$$

Therefore, for almost all t,

$$1 \; - \; |1 \; - \; (\psi(t)h(t)/2)|$$

$$\geqslant \; 1 \; - \; \sqrt{\left(1 \; - \; \frac{|\psi(t)h(t)|^2}{4}\right)} \;\; \geqslant \;\; \frac{|\psi(t)h(t)|^2}{8} \;\; = \;\; \frac{|h(t)|^2}{8}.$$

Thus,

$$\int_{-\infty}^{\infty} \frac{1}{1+t^2} \log(\,1 \; - \; |1 \; - \; (\psi(t)h(t)/2)|\,)\,\mathrm{d}t \;\; \geqslant \;\; \int_{-\infty}^{\infty} \frac{\log|(h(t))^2/8|}{1+t^2}\,\mathrm{d}t.$$

However, $h \in H_\infty$ is not a.e. zero. Therefore, *by a theorem from article* 1,

$$\int_{-\infty}^{\infty} \frac{\log|h(t)|}{1+t^2}\,\mathrm{d}t \;\; > \;\; -\infty.$$

The preceding integral is thus also $> -\infty$ and our desired relation is established. We are done.

4. Solution of our problem in terms of multipliers

We are now able to prove the

Theorem. *Let $w(t) \geqslant 0$ belonging to $L_1(\mathbb{R})$ and the number $a > 0$ be given. In order that there exist an $\omega \geqslant 0$, not a.e. zero, such that*

$$\int_{-\infty}^{\infty} |\tilde{U}(t)|^2 \omega(t)\,dt \quad \leqslant \quad \int_{-\infty}^{\infty} |U(t)|^2 w(t)\,dt$$

for the sums

$$U(t) \quad = \quad \sum_{\lambda \geqslant a} (A_\lambda \cos \lambda t + B_\lambda \sin \lambda t),$$

it is necessary and sufficient that there be a non-zero entire function $f(z)$ of exponential type $\leqslant a$ making

$$\int_{-\infty}^{\infty} \frac{|f(t)|^2}{(1+t^2)w(t)}\,dt \quad < \quad \infty.$$

Remark. We see that in the typical situation where $w(t)$ is bounded above and *very small for large values of* $|t|$, one has, under the conditions of the theorem, an entire function f of exponential type acting as a *multiplier* (in the sense adopted at the beginning of §A) for the *large* function $1/\sqrt{((1+t^2)w(t))}$.

Proof of theorem: necessity. Is based partly on a result from §D.

Suppose there *is* an ω having the properties in question. Then, by the last theorem of the preceding article, we have one for which

$$\int_{-\infty}^{\infty} \frac{\log^- \omega(t)}{1+t^2}\,dt \quad < \quad \infty.$$

Let $U(t)$ be any *real-valued* sum of the form indicated above, i.e., one for which the coefficients A_λ and B_λ are real. The function

$$F(t) \quad = \quad U(t) \ + \ i\tilde{U}(t)$$

is of the form

$$\sum_{\lambda \geqslant a} C_\lambda e^{i\lambda t}$$

and hence belongs to H_∞ according to a simple lemma in article 1. By the first theorem of article 2,

$$\omega(t) \leqslant w(t) \quad \text{a.e.},$$

so

$$\int_{-\infty}^{\infty} |F(t)|^2 \omega(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} \{(U(t))^2 + (\tilde{U}(t))^2\}\omega(t)\,\mathrm{d}t$$
$$\leqslant 2\int_{-\infty}^{\infty} (U(t))^2 w(t)\,\mathrm{d}t.$$

For $\Im z > 0$, put, as in article 1,

$$F(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} F(t)\,\mathrm{d}t;$$

then, since $F \in H_\infty$,

$$\log|F(z)| \leqslant \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} \log|F(t)|\,\mathrm{d}t$$

according to a result from that article. (Here, of course, $F(z)$ is just

$$\sum_{\lambda \geqslant a} C_\lambda \mathrm{e}^{\mathrm{i}\lambda z},$$

a function continuous up to $\mathbb{R}$, so one may, if one prefers, apply §G.2 of Chapter III directly.) The right side of the relation just written equals

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Im z \log(|F(t)|^2 \omega(t))}{|z-t|^2}\,\mathrm{d}t - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Im z \log \omega(t)}{|z-t|^2}\,\mathrm{d}t,$$

and this, by the inequality between arthmetic and geometric means, is

$$\leqslant \frac{1}{2}\log\left\{\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}|F(t)|^2\omega(t)\,\mathrm{d}t\right\} + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log^- \omega(t)\,\mathrm{d}t.$$

Here, the *first* term is

$$\leqslant \frac{1}{2}\log\left\{\frac{1}{\pi \Im z}\int_{-\infty}^{\infty} |F(t)|^2 \omega(t)\,\mathrm{d}t\right\}$$

which is in turn

$$\leqslant \frac{1}{2}\log\left\{\frac{2}{\pi \Im z}\int_{-\infty}^{\infty} (U(t))^2 w(t)\,\mathrm{d}t\right\}$$

by what we have already seen. The *second* term – call it $p(z)$ – is *finite* (and harmonic!) for $\Im z > 0$, thanks to the above condition on $\log^{-}\omega(t)$.

Our inequality for $\log|F(z)|$ thus boils down to the relation

$$|F(z)| \leq e^{p(z)}\sqrt{\left(\frac{2}{\pi\Im z}\int_{-\infty}^{\infty}(U(t))^2 w(t)\,dt\right)}, \qquad \Im z > 0,$$

whence, in the upper half-plane,

$$\left|\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2}U(t)\,dt\right| = |\Re F(z)| \leq e^{p(z)}\sqrt{\left(\frac{2}{\pi\Im z}\int_{-\infty}^{\infty}|U(t)|^2 w(t)\,dt\right)}$$

for the *real-valued* sums

$$U(t) = \sum_{\lambda\geq a}(A_\lambda\cos\lambda t + B_\lambda\sin\lambda t).$$

This last relation must indeed then hold for such *complex*-valued sums $U(t)$, because any of those can be written as $U_1(t) + iU_2(t)$ with *real-valued* ones U_1, U_2 of the same form.*

Invoke now the second theorem of §D! According to it, the estimate just obtained implies the existence of a non-zero entire function f of exponential type $\leq a$ such that

$$\int_{-\infty}^{\infty}\frac{|f(t)|^2}{(1+t^2)w(t)}\,dt < \infty.$$

Necessity is proved.

We turn to the ***sufficiency***. Suppose we *have* a non-zero entire function $f(z)$ of exponential type $\leq a$ with

$$\int_{-\infty}^{\infty}\frac{|f(t)|^2}{(1+t^2)w(t)}\,dt < \infty.$$

As in the sufficiency proof for the first theorem in §D, f may be taken to be *real* on $\mathbb{R}$. Also, by reasoning as in the sufficiency argument for the second theorem of that §, we see that $f(t)/(t+i)$ is *bounded* on $\mathbb{R}$.

The last statement certainly implies that

$$\int_{-\infty}^{\infty}\frac{\log^{+}|f(t)|}{1+t^2}\,dt < \infty,$$

* In that circumstance, $|\int_{-\infty}^{\infty}(\Im z\,U(t)/|z-t|^2)dt|^2 = \{\int_{-\infty}^{\infty}(\Im z\,U_1(t)/|z-t|^2)dt\}^2 + \{\int_{-\infty}^{\infty}(\Im z\,U_2(t)/|z-t|^2)dt\}^2$. Use of the inequality on each of the integrals on the right (for which it is already known to hold) yields the upper bound $(2\pi/\Im z)e^{2p(z)}\int_{-\infty}^{\infty}\{(U_1(t))^2 + (U_2(t))^2\}w(t)\,dt$.

so the first theorem of §G.2, Chapter III applies to our function f, and we have

$$\int_{-\infty}^{\infty} \frac{\log^- |f(t)|}{1+t^2} \, dt \quad < \quad \infty .$$

At the same time, by the inequality between arithmetic and geometric means,

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+t^2} \log\left(\frac{|f(t)|^2}{w(t)}\right) dt \quad \leqslant \quad \log\left(\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|f(t)|^2}{(1+t^2)w(t)} \, dt\right) \quad < \quad \infty$$

which, with the previous relation, yields

$$\int_{-\infty}^{\infty} \frac{\log^- w(t)}{1+t^2} \, dt \quad < \quad \infty .$$

This condition, however, gives us an outer function $\varphi \in H_2$ for which

$$|\varphi(t)| \quad = \quad \sqrt{(w(t))} \quad > \quad 0 \quad \text{a.e.}.$$

According, then, to the second theorem of the preceding article, a function $\omega \geqslant 0$ possessing the desired properties *will exist* provided that we can find a *non-zero* $h \in H_\infty$ such that

$$\left| e^{2iat} \frac{\overline{\varphi(t)}}{\varphi(t)} - h(t) \right| \quad \leqslant \quad 1 \quad \text{a.e.}.$$

We proceed to exhibit such an h.

For $\Im z > 0$, write, as in article 1,

$$\varphi(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} \varphi(t) \, dt;$$

by a theorem from that article, $\varphi(z)$ is *analytic* in the upper half-plane and

$$\varphi(t+iy) \quad \longrightarrow \quad \varphi(t) \quad \text{a.e.}$$

as $y \longrightarrow 0$. Saying that φ is *outer* means, as we recall, that

$$\log|\varphi(z)| \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} \log|\varphi(t)| \, dt, \qquad \Im z > 0;$$

$\varphi(z)$ has, in particular, *no zeros in the upper half plane*. The ratio

$$R(z) \quad = \quad e^{2iaz}\left(\frac{f(z)}{\varphi(z)}\right)^2$$

is thus *analytic* for $\Im z > 0$. Since $f(z)$ is *entire* and $|\varphi(t)| > 0$ a.e., $R(t + iy)$ approaches for almost every $t \in \mathbb{R}$ a *definite limit*,

$$R(t) = e^{2iat}\left(\frac{f(t)}{\varphi(t)}\right)^2,$$

as $y \longrightarrow 0$. Because our function f is real on $\mathbb{R}$,

$$R(t) \quad \text{and} \quad e^{2iat}\frac{\overline{\varphi(t)}}{\varphi(t)}$$

have *the same argument* there, and we see, referring to our requirement on h, that *if* $R(t)$ *were in* H_∞, we could take for h a suitable constant multiple of R. Usually, however, $R(t)$ is not bounded, so this will not be the case, and we have to do a supplementary construction.

We have

$$|R(t)| = \frac{(f(t))^2}{w(t)}, \qquad t \in \mathbb{R},$$

so by hypothesis,

$$\int_{-\infty}^{\infty} \frac{|R(t)|}{1+t^2}\,dt < \infty.$$

Following an idea from a paper of Adamian, Arov and Krein we now put

$$Q(z) = \frac{1}{\pi i}\int_{-\infty}^{\infty}\left(\frac{1}{t-z} - \frac{t}{t^2+1}\right)|R(t)|\,dt$$

for $\Im z > 0$; the previous relation guarantees absolute convergence of the integral on the right, and $Q(z)$ is *analytic* in the upper half plane, with $\Re Q(z) > 0$ there. The quotient $R(z)/Q(z)$ is thus analytic for $\Im z > 0$.

It is now claimed that

$$\left|\frac{R(z)}{Q(z)}\right| \leqslant 1, \qquad \Im z > 0.$$

The function f is of exponential type $\leqslant a$ and fulfills the above condition involving $\log^+|f(t)|$. Hence, by §G.2, Chapter III,

$$\log|f(z)| \leqslant a\Im z + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2}\log|f(t)|\,dt, \qquad \Im z > 0,$$

which, with the previous formula for $\log|\varphi(z)|$, yields

$$\log|R(z)| \leqslant \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2}\log|R(t)\,dt, \qquad \Im z > 0.$$

Returning to our function $Q(z)$, we get, by the inequality between arithmetic and geometric means,

$$|Q(z)| \geqslant \Re Q(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}|R(t)|\,dt$$

$$\geqslant \exp\left\{\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log|R(t)|\,dt\right\}.$$

The preceding relation says, however, that the right-hand member is $\geqslant |R(z)|$. We thus have $|Q(z)| \geqslant |R(z)|$ for $\Im z > 0$, and the above inequality is verified.

Thanks to that inequality we see, by a result from article 1, that there is an $h \in H_\infty$ with

$$\frac{R(z)}{Q(z)} = h(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}h(t)\,dt$$

for $\Im z > 0$, and that $\|h\|_\infty \leqslant 1$. This function h cannot be a.e. zero on $\mathbb{R}$ because $R(z)$ is not identically zero – the entire function $f(z)$ isn't! A result from article 1 therefore implies that

$$|h(t)| > 0 \qquad \text{a.e., } t \in \mathbb{R}.$$

As $y \longrightarrow 0$,

$$\frac{R(t+iy)}{Q(t+iy)} = h(t+iy) \longrightarrow h(t) \neq 0 \quad \text{a.e..}$$

At the same time,

$$R(t+iy) \longrightarrow R(t) \quad \text{a.e.,}$$

so $Q(t+iy)$ must approach a certain definite limit, $Q(t)$, for almost all $t \in \mathbb{R}$ as $y \longrightarrow 0$. (This also follows directly from §F.2 of Chapter III.) Since

$$\Re Q(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}|R(t)|\,dt,$$

we see, by the usual property of the Poisson kernel, that

$$\Re Q(t) = |R(t)| \quad \text{a.e..}$$

Finally, then,

$$h(t) = \frac{R(t)}{|R(t)| + i\Im Q(t)} \qquad \text{a.e., } t \in \mathbb{R}.$$

Recall that

$$R(t) \quad = \quad \mathrm{e}^{2iat}\left(\frac{f(t)}{\varphi(t)}\right)^2 \quad = \quad \mathrm{e}^{2iat}\frac{\overline{\varphi(t)}}{\varphi(t)}|R(t)|,$$

$f(t)$ being real. This and the preceding formula thus give

$$\left|\mathrm{e}^{2iat}\frac{\overline{\varphi(t)}}{\varphi(t)} - h(t)\right| \quad = \quad \left|1 - \frac{|R(t)|}{|R(t)| + \mathrm{i}\mathfrak{Z}Q(t)}\right| \qquad \text{a.e.},$$

and the right side is clearly $\leqslant 1$. Our function $h \in H_\infty$ therefore has the required properties, and our proof of sufficiency is finished.

We are done.

Remark. From this theorem and the second one of §D, we see that *only* by virtue of the existence of a non-zero $\omega \geqslant 0$ making

$$\int_{-\infty}^{\infty} |\tilde{U}(t)|^2 \omega(t)\,\mathrm{d}t \quad \leqslant \quad \int_{-\infty}^{\infty} |U(t)|^2 w(t)\,\mathrm{d}t$$

can the harmonic extension of a general sum

$$U(t) \quad = \quad \sum_{\lambda \geqslant a} (A_\lambda \cos \lambda t + B_\lambda \sin \lambda t)$$

to the upper half plane be controlled there by the integral on the right.

In the next problem, we consider bounded functions $u(\vartheta)$ defined on $[-\pi, \pi]$, using for them a Hilbert transform given by the formula

$$\tilde{u}(\vartheta) \quad = \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{u(\tau)}{\tan((\vartheta - \tau)/2)}\,\mathrm{d}\tau,$$

as is customary in the study of Fourier series. (The expression on the right is a Cauchy principal value.) If one puts $\tan(\vartheta/2) = x$, $\tan(\tau/2) = t$, and then writes $u(\tau) = U(t)$, the function $\tilde{u}(\vartheta)$ goes over into the *first kind* of Hilbert transform $\tilde{U}(x)$ for functions U defined on $\mathbb{R}$, described at the beginning of article 2.

If a function $w(\vartheta) \geqslant 0$ belonging to $L_1(-\pi, \pi)$ is given, one may ask whether there exists an $\omega(\vartheta) \geqslant 0$, not a.e. zero on $[-\pi, \pi]$, such that

$$\int_{-\pi}^{\pi} |\tilde{u}(\vartheta)|^2 \omega(\vartheta)\,\mathrm{d}\vartheta \quad \leqslant \quad \int_{-\pi}^{\pi} |u(\vartheta)|^2 w(\vartheta)\,\mathrm{d}\vartheta$$

for all bounded functions u. It is clear that any given ω has this property iff, with it, the relation just written holds for all u of the *special form*

$$u(\vartheta) \quad = \quad \sum_{-N}^{N} a_n e^{in\theta}.$$

(Here N is *finite*, but *arbitrary*.) Such a function u is called a *trigonometric polynomial*; for it we have

$$\tilde{u}(\vartheta) \quad = \quad -\mathrm{i}\sum_{-N}^{N} a_n \operatorname{sgn} n\, e^{in\theta}.$$

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Given $w \geqslant 0$ in $L_1(-\pi, \pi)$, one is to prove that there exists an $\omega \geqslant 0$, not a.e. zero on $[-\pi, \pi]$, such that

$$\int_{-\pi}^{\pi} |\tilde{u}(\vartheta)|^2 \omega(\vartheta)\, d\vartheta \quad \leqslant \quad \int_{-\pi}^{\pi} |u(\vartheta)|^2 w(\vartheta)\, d\vartheta$$

for all *trigonometric polynomials* u iff

$$\int_{-\pi}^{\pi} \frac{d\vartheta}{w(\vartheta)} \quad < \quad \infty.$$

(a) First prove Kolmogorov's theorem, which says that there is a sequence of *trigonometric polynomials* $u_k(\vartheta)$ *without constant term* (i.e., in which $a_0 = 0$) such that

$$\int_{-\pi}^{\pi} |1 \; - \; u_k(\vartheta)|^2 w(\vartheta)\, d\vartheta \;\xrightarrow[k]{}\; 0$$

iff

$$\int_{-\pi}^{\pi} \frac{d\vartheta}{w(\vartheta)} \quad = \quad \infty.$$

(Hint: Work with the inner product

$$\langle u, v\rangle_w \quad = \quad \int_{-\pi}^{\pi} u(\vartheta)\overline{v(\vartheta)} w(\vartheta)\, d\vartheta$$

and use orthogonality.)

(b) Show that the condition $\int_{-\pi}^{\pi}(d\vartheta/w(\vartheta)) < \infty$ is *necessary* for the existence of an ω enjoying the properties in question. (Hint: Let $u_0(\vartheta) = \sum_{n \neq 0} a_n e^{in\vartheta}$ be any trigonometric polynomial *without constant term*, and put

$$u_1(\vartheta) \; = \; 1 - u_0(\vartheta); \qquad u_2(\vartheta) \; = \; e^{-i\vartheta}(1 - u_0(\vartheta)).$$

Then observe that

$$e^{i\vartheta}\tilde{u}_2(\vartheta) - \tilde{u}_1(\vartheta) = i(1 - a_1 e^{i\vartheta}),$$

so, for an ω having the above properties, we would have

$$\int_{-\pi}^{\pi} |1 - a_1 e^{i\vartheta}|^2 \omega(\vartheta)\, d\vartheta \leqslant 4\int_{-\pi}^{\pi} |1 - u_0(\vartheta)|^2 w(\vartheta)\, d\vartheta.$$

We have

$$|1 - a_1 e^{i\vartheta}|^2 = (1 - |a_1|)^2 + 4|a_1|\sin^2\frac{\vartheta - \alpha}{2}$$

with $-\pi \leqslant \alpha \leqslant \pi$, so the integral on the left has a *strictly positive minimum* for $a_1 \in \mathbb{C}$ *unless* $\omega(\vartheta)$ *vanishes* a.e. on $[-\pi, \pi]$. Apply the result from (a).)

(c) Suppose now that $w(\vartheta)$ and $1/w(\vartheta)$ both belong to $L_1(-\pi, \pi)$. For $|z| < 1$, put

$$\Omega(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\tau} + z}{e^{i\tau} - z}\,\frac{d\tau}{w(\tau)}.$$

$\Omega(z)$ is analytic for $\{|z| < 1\}$ and, by Chapter III, §F.2,

$$\lim_{r\to 1} \Omega(re^{i\vartheta}) = \Omega(e^{i\vartheta})$$

exists for almost all ϑ, and

$$\Re\Omega(e^{i\vartheta}) = 1/w(\vartheta) \quad \text{a.e.}.$$

This makes $|1/\Omega(e^{i\vartheta})| \leqslant w(\vartheta)$ a.e., so that $1/\Omega(e^{i\vartheta}) \in L_1(-\pi, \pi)$. *Show that*

$$\int_{-\pi}^{\pi} \frac{e^{in\vartheta}}{\Omega(e^{i\vartheta})}\, d\vartheta = 0 \quad \text{for } n = 1, 2, 3, \ldots.$$

(Hint: The reader familiar with the theory of H_p spaces for the unit disk may use Smirnov's theorem. Otherwise, one may start from scratch, arguing as follows. For $|z| < 1$, by the inequality between arithmetic and harmonic means,

$$\left|\frac{1}{\Omega(z)}\right| \leqslant \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - |z|^2}{|e^{i\tau} - z|^2}\,\frac{d\tau}{w(\tau)}\right)^{-1} \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - |z|^2}{|e^{i\tau} - z|^2}\, w(\tau)\, d\tau.$$

Use this relation to show, *firstly*, that there is a *complex measure* ν on $[-\pi, \pi]$ for which

$$\frac{1}{\Omega(z)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - |z|^2}{|e^{i\tau} - z|^2}\, d\nu(\tau), \qquad |z| < 1,$$

(see proof of *first* theorem in §F.1, Chapter III), and, *secondly*, that ν must be *absolutely continuous* (without appealing to the F. and M. Riesz theorem). These facts imply that

$$\frac{1}{\Omega(z)} \quad = \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-|z|^2}{|e^{i\tau}-z|^2}\,\frac{d\tau}{\Omega(e^{i\tau})}, \qquad |z|<1,$$

and from this the desired relation follows on taking an $r<1$, observing that

$$\int_{-\pi}^{\pi} (e^{in\vartheta}/\Omega(re^{i\vartheta}))\,d\vartheta \quad = \quad 0 \quad \text{for } n \; = \; 1, 2, 3, \ldots,$$

and then using Fubini's theorem.)

(d) Let $\Omega(e^{i\vartheta})$ be the function from (c). In analogy with what was done in article 3, put

$$\sigma(\vartheta) \quad = \quad 1 \quad - \quad \left|1 \; - \; \frac{1}{w(\vartheta)\Omega(e^{i\vartheta})}\right|.$$

Show that $0 \leqslant \sigma(\vartheta) \leqslant 1$ a.e. and that $\sigma(\vartheta)$ is *not* a.e. zero on $[-\pi,\ \pi]$.

(e) If $f(\vartheta)$ is a finite sum of the form $\sum_{n\geqslant 1} c_n e^{in\vartheta}$, show that

$$\Re\int_{-\pi}^{\pi} (f(\vartheta))^2 w(\vartheta)\,d\vartheta \quad \leqslant \quad \int_{-\pi}^{\pi} (1 \; - \; \sigma(\vartheta))w(\vartheta)|f(\vartheta)|^2\,d\vartheta.$$

(Hint: By (c), the integral figuring on the left equals

$$\int_{-\pi}^{\pi} w(\vartheta)\left(1 \; - \; \frac{1}{w(\vartheta)\Omega(e^{i\vartheta})}\right)(f(\vartheta))^2\,d\vartheta.$$

Refer to (d).)

(f) Hence show that

$$\int_{-\pi}^{\pi} \sigma(\vartheta)w(\vartheta)|\tilde{u}_0(\vartheta)|^2\,d\vartheta \quad \leqslant \quad 2\int_{-\pi}^{\pi} w(\vartheta)|u_0(\vartheta)|^2\,d\vartheta$$

for any trigonometric polynomial $u_0(\vartheta)$ *without constant term.*

(Hint: It is enough to do this for *real-valued* $u_0(\vartheta)$. Given such a one, use

$$f(\vartheta) \quad = \quad \tilde{u}_0(\vartheta) \; - \; iu_0(\vartheta)$$

in result from (e).)

(g) Show that

$$\int_{-\pi}^{\pi} \sigma(\vartheta)w(\vartheta)|\tilde{u}(\vartheta)|^2\,d\vartheta \quad \leqslant \quad C\int_{-\pi}^{\pi} w(\vartheta)|u(\vartheta)|^2\,d\vartheta$$

for *general* trigonometric polynomials $u(\vartheta)$, where C is a suitable constant. (Hint: If $u_0(\vartheta)$ denotes $u(\vartheta)$ minus its constant term, $\tilde{u}(\vartheta) = \tilde{u}_0(\vartheta)$. Use result from (a) to show that

$$\int_{-\pi}^{\pi} |u_0(\vartheta)|^2 w(\vartheta)\,d\vartheta \;\leqslant\; \text{const.} \int_{-\pi}^{\pi} |u(\vartheta)|^2 w(\vartheta)\,d\vartheta.\ )$$

(*h) Show that

$$\int_{-\pi}^{\pi} \log\big(\sigma(\vartheta)w(\vartheta)\big)\,d\vartheta \;>\; -\infty.$$

(Hint: Look at the proof of the last theorem in article 3; here $1/w(\vartheta)\Omega(e^{i\vartheta})$ already lies on the circle with diameter $[0, 1]$ for almost all $\vartheta \in [-\pi, \pi]$. Argument uses some H_p space theory for the unit disk.)

The result established in this problem was generalized to the case of weighted L_p norms ($1 < p < \infty$) by Carleson and Jones and, using a method different from theirs, by Rubio de Francia*. Related investigations have been made by Arocena, Cotlar, Sadoski, and their co-workers. A general result for operators in Hilbert space is due to Treil.

F. Relation of material in preceding § to the geometry of unit sphere in L_∞/H_∞

Combination of the second theorem in §E.3 with the one from §E.4 shows immediately that if we take any outer function $\varphi \in H_2$, the *existence* of a non-zero entire function f of exponential type $\leqslant a$ making

$$\int_{-\infty}^{\infty} \frac{|f(t)|^2}{(1+t^2)|\varphi(t)|^2}\,dt \;<\; \infty$$

is *equivalent* to that of a *non-zero* $h \in H_\infty$ such that

$$\left| e^{2iat}\frac{\overline{\varphi(t)}}{\varphi(t)} - h(t) \right| \;\leqslant\; 1 \quad \text{a.e..}$$

* Chapter VI of his recent book with Garcia-Cuerva has in it a rather general treatment of the corresponding question about singular integrals on $\mathbb{R}^n$.

The second of these two conditions has an interpretation in terms of the quotient space L_∞/H_∞ that deserves mention; we look at it briefly in the present §.

The space L_∞/H_∞ has already appeared in §E.3; as explained there, its elements are the *cosets*

$$\psi + H_\infty,$$

where ψ ranges over L_∞. Instead of the *ad hoc* norm $\| \ \|_\infty^\sigma$ employed for such cosets in §E.3, we will here use the standard one to which $\| \ \|_\infty^\sigma$ reduces when $\sigma = 0$, viz.

$$\|\psi + H_\infty\|_\infty = \inf\{\|\psi + h\|_\infty : \ h \in H_\infty\}.$$

Equipped with $\| \ \|_\infty$, L_∞/H_∞ becomes a Banach space, and we denote by Σ the *unit sphere* (unit ball) of that space; that is simply the collection of cosets $\psi + H_\infty$ for which

$$\|\psi + H_\infty\|_\infty \leqslant 1.$$

In §E.3, essential use was made of the fact that L_∞/H_∞ is the *dual* of H_1; now we observe that this makes Σ w^* *compact* by what boils down to Tychonoff's theorem.

A member P of Σ is called an *extreme point* of Σ if, whenever

$$P = \lambda Q + (1-\lambda)R$$

with Q and R in Σ and $0 < \lambda < 1$, we *must have* $Q = R = P$. Geometrically, this means that *there cannot be any straight segment lying in* Σ *and passing through* P (i.e., with P *strictly between* its endpoints).

If $0 < \|P\|_\infty < 1$ it is clear that P *cannot* be an extreme point of Σ; the *zero coset* cannot be one either, for, since $L_\infty \neq H_\infty$, Σ contains cosets P and $-P$ with $P \neq 0 + H_\infty$. Any extreme points that Σ can have must thus be included among the set of P with $\|P\|_\infty = 1$ which we may refer to as the *surface* of Σ. Knowledge about Σ's extreme points can be used to gain insight into the geometrical structure of that surface. Such an approach is familiar to functional analysts, and one may get an idea of some of its possibilities by consulting the *Proceedings* of the A.M.S. symposium on convexity. Phelps' beautiful little book is also recommended.

The convexity and w^* compactness of Σ ensure that it has lots of extreme points according to the celebrated Krein–Milman theorem. So many, in fact, that Σ is their w^* closed convex hull. From a theorem of Bishop and Phelps (about which more later) we can furthermore deduce a much stronger result in the present circumstances: the extreme points of Σ are actually $\| \ \|_\infty$ dense on its surface. We may thus think of that surface as being 'filled out, for all practical purposes' by Σ's extreme points.

For this very reason, it seems of interest to have a procedure for exhibiting points on Σ's surface which are *not* extreme points of Σ. An outer function $\varphi \in H_2$ satisfying *either* (and hence *both*) of the two conditions set down above will frequently *give* us such a point, thanks to the following simple

Lemma. *Let* $|u(t)| \equiv 1$ *a.e.. Then* $u + H_\infty$ *is an extreme point of* Σ, *the unit sphere of* L_∞/H_∞, *iff there is* no non-zero $h \in H_\infty$ *for which*

$$|u(t) + h(t)| \leqslant 1 \quad \text{a.e.}.$$

Proof. Since $\|u\|_\infty = 1$, $u + H_\infty$ is certainly in Σ.

Suppose in the first place that $u + H_\infty$ is *not* an extreme point of Σ, then there are two *different* cosets $v_1 + H_\infty$, $v_2 + H_\infty$, both of norm $\leqslant 1$, and a λ, $0 < \lambda < 1$, with

$$u + H_\infty = \lambda(v_1 + H_\infty) + (1-\lambda)(v_2 + H_\infty).$$

According to a result proved in §E.3 (recall that $\| \ \|_\infty^\sigma$ is just $\| \ \|_\infty$ when $\sigma = 0$!),

$$\inf\{\|v_1 + h\|_\infty : \ h \in H_\infty\} = \|v_1 + H_\infty\|_\infty$$

is actually realized for some $h \in H_\infty$. Therefore, since $v_1 + h + H_\infty = v_1 + H_\infty$, there is no loss of generality in assuming that $\|v_1\|_\infty \leqslant 1$. Similarly, we may suppose that $\|v_2\|_\infty \leqslant 1$.

The previous relation means that there is some $h_0 \in H_\infty$ for which

$$u + h_0 = \lambda v_1 + (1-\lambda)v_2.$$

Here, the right side has norm $\leqslant 1$. Therefore

$$\|u + h_0\|_\infty \leqslant 1.$$

In this relation, however, h_0 cannot be zero. Indeed, assuming it *were*, we would have

$$u(t) = \lambda v_1(t) + (1-\lambda)v_2(t) \quad \text{a.e.},$$

with $0 < \lambda < 1$, $|u(t)| = 1$, and $|v_1(t)| \leqslant 1$, $|v_2(t)| \leqslant 1$. *Strict convexity* of the unit circle would then make $v_1(t) = v_2(t)$ a.e., so the cosets $v_1 + H_\infty$ and $v_2 + H_\infty$ would be *equal*, contrary to our initial assumption.

We thus have a *non-zero* $h_0 \in H_\infty$ such that

$$|u(t) + h_0(t)| \leqslant 1 \quad \text{a.e.},$$

and our lemma is proved in one direction.

Going the other way, assume that there *is* a non-zero $h \in H_\infty$ such that $\|u + h\|_\infty \leqslant 1$. Put then

$$\sigma(t) = 1 - |u(t) + \tfrac{1}{2}h(t)|;$$

we have

$$0 \leqslant \sigma(t) \leqslant 1 \quad \text{a.e.},$$

and see, as in proving *sufficiency* for the second theorem of §E.3, that *$\sigma(t) > 0$ on a set of positive measure.*

We now have

$$|u(t) + \sigma(t) + \tfrac{1}{2}h(t)| \leqslant 1 \quad \text{a.e.}$$

and

$$|u(t) - \sigma(t) + \tfrac{1}{2}h(t)| \leqslant 1 \quad \text{a.e.},$$

so, since

$$u + H_\infty = \tfrac{1}{2}(u + \sigma + H_\infty) + \tfrac{1}{2}(u - \sigma + H_\infty),$$

it will follow that $u + H_\infty$ is *not* an extreme point of Σ as long as the two cosets on the right are *different*, i.e., as long as $\sigma \notin H_\infty$.

If, however, $\sigma \in H_\infty$,

$$\sigma(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z - t|^2}\sigma(t)\,\mathrm{d}t$$

is *analytic* in $\Im z > 0$ by §E.1; it is, at the same time, *real* there, since $\sigma(t)$ is real. This makes $\sigma(z)$ *constant* and hence $\sigma(t)$, equal a.e. to $\lim_{y\to 0}\sigma(t + \mathrm{i}y)$, *also constant.*

It thus follows from our present assumption that $u + H_\infty$ is *not* an extreme point of Σ, save perhaps in the case where $\sigma(t)$ is constant. *But then $u + H_\infty$ cannot be an extreme point either*, for the constant must be > 0, $\sigma(t)$ being > 0 on a set of positive measure. We have, in other words,

$$|u(t) + \tfrac{1}{2}h(t)| = 1 - c \quad \text{a.e.}$$

where $c > 0$, so $\|u + H_\infty\|_\infty < 1$. As already noted, such a coset $u + H_\infty$ is not an extreme point of Σ.

The lemma is proved.

This result and the equivalence noted at the beginning of the present § yield without further ado the

Theorem. *Let φ be an outer function in H_2. Given $a > 0$, the coset*

$$e^{2iat}\frac{\overline{\varphi(t)}}{\varphi(t)} \quad + \quad H_\infty$$

fails *to be an extreme point of Σ, the unit sphere in L_∞/H_∞, iff*

$$\int_{-\infty}^{\infty} \frac{|f(t)|^2}{(1+t^2)|\varphi(t)|^2}\,dt \quad < \quad \infty$$

for some non-zero *entire function f of exponential type $\leqslant a$.*

From the theorem we have the following recipe for obtaining points on Σ's surface that are *not* extreme points of Σ: take any outer $\varphi \in H_2$ such that, for some $a > 0$, an entire $f \not\equiv 0$ of exponential type $\leqslant a$ satisfying the relation in the statement exists. Then the point

$$P \quad = \quad e^{2iat}\frac{\overline{\varphi(t)}}{\varphi(t)} \quad + \quad H_\infty$$

will have the property in question as long as $\|P\|_\infty \; = \; 1$.

It will indeed *frequently happen* that $\|P\|_\infty \; = \; 1$. Should that *fail* to come about, in which case

$$\|(e^{2iat}\overline{\varphi(t)}/\varphi(t)) \; + \; H_\infty\|_\infty \; < \; 1,$$

the relation

$$\int_{-\infty}^{\infty} |\tilde{U}(t)|^2|\varphi(t)|^2\,dt \quad \leqslant \quad \text{const.}\int_{-\infty}^{\infty} |U(t)|^2|\varphi(t)|^2\,dt$$

will in fact hold for the sums

$$U(t) \quad = \quad \sum_{\lambda \geqslant a} (A_\lambda \cos \lambda t + B_\lambda \sin \lambda t)$$

according to the second theorem of §E.3. This can only occur for rather special φ in H_2, closely related to the entire functions of exponential type $\leqslant a$ of a particular kind, *integrable* on the real axis but at the same time *not too small* there. The possibility may be fully investigated by the method used in proving the Helson–Szegő theorem. In spite of the matter's relevance to the study of various questions, we cannot go further into it here; very similar material is taken up in the paper of Hruščev, Nikolskii and Pavlov.

Except in the circumstance just mentioned, an outer function $\varphi \in H_2$ will satisfy

$$\|(e^{2iat}\overline{\varphi(t)}/\varphi(t)) \; + \; H_\infty\|_\infty \; = \; 1.$$

Use of the above procedure with such φ can lead to interesting examples even when only the simple Paley–Wiener multiplier theorem from §A.1 is called on. The reader is encouraged to carry out one or two constructions in such fashion.

For any coset $u \; + \; H_\infty$ lying on Σ's surface but not an extreme point of Σ one has a beautiful parametric representation, due to Adamian, Arov and Krein, of the functions $h \in H_\infty$ with $\|u \; + \; h\|_\infty \; = \; 1$. This was first obtained by means of operator theory, but Garnett has since found an easier function-theoretic derivation, given in his book.*

The work in §E was originally done at the end of the 1960s, in hopes that the connection established by the last theorem would make possible a *proof* of the Beurling–Malliavin multiplier theorem (stated in §A.2) *based on* Banach space and Banach algebra techniques. That approach did not work, and I think now that it is probably not feasible. Whatever value the result may have seems rather to lie in the possibility of its helping us understand the structure of L_∞/H_∞ *when used with multiplier theorems* to construct various examples, according to the above scheme.

The quotient space L_∞/H_∞ is the foundation for the theory of Hankel and Toeplitz forms. Study of these is not really part of this book's subject matter, and the present § is included merely to show some ways of applying multiplier theorems therein. The reader interested in that study should first of all consult Sarason's Blacksburg notes. Wishing to go further, he or she should next take up the papers of that author and his co-workers, perusing, at the same time, a book on H_p spaces so as to get a good grounding in their theory. It is then essential that one become familiar with the remarkable papers of Adamian, Arov and Krein. Those make heavy use of Hilbert space operator theory.

There is, in general, much mutual interplay between operator theory and the investigation of L_∞/H_∞; so vast, indeed, is the region common to these two fields that it seems hopeless to try to furnish even sketchy references here. Let us at least mention the so-called *Nagy–Foiaş model*; the book about it by those two authors is well known. For more recent

* the one on bounded analytic functions

treatments, see Nikolskii's book and (especially) his survey article with Hruščev.

Before closing this § and the present chapter, let us see how the extreme points of Σ are related to its *support points*, to be defined in a moment. As we saw in §E.3, each coset $u + H_\infty$ in L_∞/H_∞ corresponds to a *linear functional* Λ on H_1 given by the formula

$$\Lambda(f) = \int_{-\infty}^{\infty} u(t)f(t)\,dt, \qquad f \in H_1,$$

and the *supremum* of $|\Lambda(f)|$ for the $f \in H_1$ with $\|f\|_1 \leqslant 1$ is equal to $\|u + H_\infty\|_\infty$. (The reader is again reminded that the norm $\|\ \ \|_\infty^\sigma$ used in §E.3 reduces to $\|\ \ \|_\infty$ when $\sigma(t) \equiv 0$.) We know that there *is* some $h \in H_\infty$ for which $\|u + h\|_\infty = \|u + H_\infty\|_\infty$; *that, however, does not mean that there need be an $f \in H_1$ of norm 1 with $\Lambda(f) = \|u + H_\infty\|_\infty$*. Since the space H_1 is not reflexive there is no reason why this should be the case; it is in fact *true* for *some* cosets $u + H_\infty$ and *false* for *others.*

Definition. A coset $u + H_\infty$ with $\|u + H_\infty\|_\infty = 1$ is said to be a *support point* for Σ if there is an $f \in H_1$ with

$$\int_{-\infty}^{\infty} u(t)f(t)\,dt = \|f\|_1 = 1.$$

There is then the

Theorem. *A support point of Σ is an extreme point of Σ.*

Proof. Let $u + H_\infty$ be a support point of Σ. There is, as we know, a $v \in u + H_\infty$ with $\|v\|_\infty = \|u + H_\infty\|_\infty = 1$; it is enough to show that such a v is of modulus 1 a.e. and *uniquely determined*, for then there can be *no* non-zero $h \in H_\infty$ with $\|v + h\|_\infty \leqslant 1$, and $u + H_\infty = v + H_\infty$ must hence be an extreme point of Σ by the previous lemma.

There is by definition an $f \in H_1$ with

$$\int_{-\infty}^{\infty} v(t)f(t)\,dt = \int_{-\infty}^{\infty} u(t)f(t)\,dt = \|f\|_1 = 1.$$

Here, $|v(t)| \leqslant 1$ a.e., so we must have

$$v(t)f(t) = |f(t)| \quad \text{a.e.}.$$

However, $|f(t)| > 0$ a.e. by §E.1, so we get

$$v(t) = \frac{|f(t)|}{f(t)} \quad \text{a.e.,}$$

which determines v makes it of modulus 1 a.e.. Done.

The *converse* of this theorem is *not true*; the example provided by problems 44 and 46 given below shows that. It is, however, true that the *surface* of Σ is *full of support points*; those are, indeed, $\| \ \|_\infty$ dense on that surface according to a remarkable theorem, due to Bishop and Phelps, whose proof may be found in the above mentioned A.M.S. volume on convexity. Since all these support points are extreme points by the result just obtained, it follows that *the extreme points of* Σ are $\| \ \|_\infty$ *dense on its surface.* This is a much higher concentration of extreme points than could be surmised from the Krein–Milman theorem. Still, there are lots of points on Σ's surface that are *not* extreme, and we have seen how to find many of them.

When a particular $u \in L_\infty$ with $\|u + H_\infty\|_\infty = 1$ is given, it is hard to tell by just looking at the qualitative behaviour of the function $u(t)$ whether $u + H_\infty$ is a support point of Σ or not. About all that is known *generally* is that $u + H_\infty$ must then be a support point if $u(t)$ is *continuous* on $\mathbb{R}$ *and tends to equal limits for* $t \longrightarrow \infty$ *and* $t \longrightarrow -\infty$. Proof of this fact, which depends on the F. and M. Riesz theorem, may be found in the more recent books about H_p spaces. If we merely require *uniform continuity* of $u(t)$ on $\mathbb{R}$, the conclusion may cease to hold. An example of this will be furnished by problems 44 and 46.

To work the following problems, a generalization of the Schwarz reflection principle due to Carleman will be needed. Suppose that we have a rectangle $\mathscr{D}_0$ in the upper half plane whose *base* is a segment of the real axis:

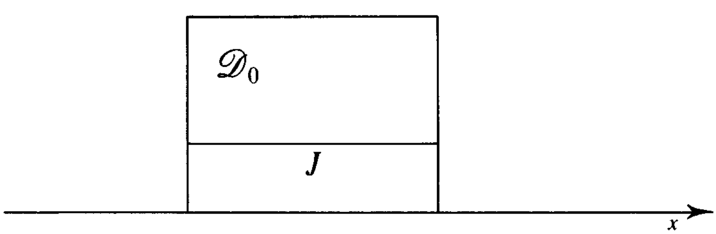


Figure 228

Carleman's result deals with functions $F(z)$ analytic in $\mathscr{D}_0$ for which

$$\int_J |F(z)||\mathrm{d}z| \quad \leqslant \quad \text{const.}$$

for all the *horizontal* line segments J running across the interior of $\mathscr{D}_0$. These functions are just those of the class $\mathscr{S}_1(\mathscr{D}_0)$ studied in §B.4, Chapter VII. The reader should refer again to that §. According to the first theorem proved there, when $F \in \mathscr{S}_1(\mathscr{D}_0)$, $\lim_{y\to x} F(x+\mathrm{i}y)$ exists for almost every x on the base of $\mathscr{D}_0$. Following standard practice, that limit is denoted by $F(x)$.

Lemma (Carleman). *If $F \in \mathscr{S}_1(\mathscr{D}_0)$ and $F(x)$ is* real *a.e. along the base of $\mathscr{D}_0$, $F(z)$ can be analytically continued across that base into $\mathscr{D}_0^*$, the reflection of $\mathscr{D}_0$ in the real axis, by putting $F(z) = \overline{F(\bar{z})}$ for $z \in \mathscr{D}_0^*$.*

Proof. Let I be any segment properly included in the base of $\mathscr{D}_0$ in the manner shown in the following figure, and take any rectangle $\mathscr{D}$, entirely contained in $\mathscr{D}_0$, having I as its base.

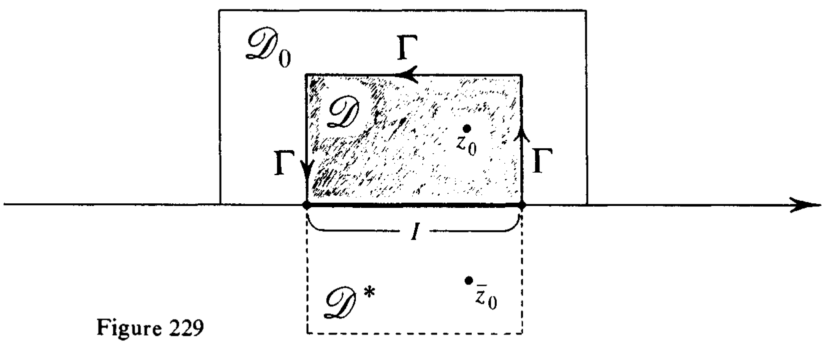


Figure 229

If $z_0 \in \mathscr{D}$, $1/(\zeta - \bar{z}_0)$ is analytic in ζ for ζ in $\mathscr{D}$, so, by the corollary to the *second* theorem of §B.4, Chapter VII,

$$\int_{\partial\mathscr{D}} \frac{F(\zeta)}{\zeta - \bar{z}_0}\mathrm{d}\zeta \quad = \quad 0;$$

here the integral is absolutely convergent according to the third lemma and first theorem* of that §.

The function $(F(z) - F(z_0))/(z - z_0)$ clearly belongs to $\mathscr{S}_1(\mathscr{D}_0)$ if F does.

* In Fig. 69, accompanying the proof of that theorem (p. 287 of vol I), B_1 and B_2 should have designated the horizontal sides of $\mathscr{D}_0$ and not of $\mathscr{D}$.

Hence, by the corollary just used, we also have

$$\int_{\partial\mathscr{D}} \frac{F(\zeta)-F(z_0)}{\zeta - z_0}\,d\zeta \quad = \quad 0,$$

from which the Cauchy formula

$$F(z_0) \quad = \quad \frac{1}{2\pi i}\int_{\partial\mathscr{D}} \frac{F(\zeta)}{\zeta - z_0}\,d\zeta$$

immediately follows on using the relation

$$\int_{\partial\mathscr{D}} \frac{d\zeta}{\zeta - z_0} \quad = \quad 2\pi i.$$

Combining our formula for $F(z_0)$ with the one preceding it, and then dropping the subscript on z, we get

$$F(z) \quad = \quad \frac{1}{2\pi i}\int_\Gamma \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - \bar{z}}\right) F(\zeta)\,d\zeta \;+\; \frac{1}{\pi}\int_I \frac{\Im z}{|\xi - z|^2} F(\xi)\,d\xi, \quad z \in \mathscr{D},$$

where Γ is the path consisting of the *top* of $\mathscr{D}$ together with its *two vertical sides* (see figure).

Of the two integrals on the right, the *first* certainly represents a complex-valued *harmonic* (*not* analytic!) function of z in any region disjoint from both Γ and its reflection in the real axis. That expression is, in particular, harmonic in the rectangle $\mathscr{D} \cup I \cup \mathscr{D}^*$, where $\mathscr{D}^*$ denotes the reflection of $\mathscr{D}$ in $\mathbb{R}$; its *imaginary part*, $V(z)$, is thus also harmonic in $\mathscr{D} \cup I \cup \mathscr{D}^*$.

The function $\Im F(z)$, harmonic in $\mathscr{D}$, is *equal* there to

$$V(z) \quad + \quad \frac{1}{\pi}\int_I \frac{\Im z}{|\xi - z|^2} \Im F(\xi)\,d\xi;$$

this, however, is just $V(z)$, since $\Im F(\xi) \;=\; 0$ a.e. on I by hypothesis. The function $\Im F(z)$ therefore has a *harmonic continuation* from $\mathscr{D}$ to the larger rectangle $\mathscr{D} \cup I \cup \mathscr{D}^*$. Its *harmonic conjugate*, $-\Re F(z)$, can thus also be continued harmonically into all of $\mathscr{D} \cup I \cup \mathscr{D}^*$, and then we obtain an *analytic continuation* of $F(z)$ into that larger rectangle by putting $F(z) \;=\; \Re F(z) \;+\; i\Im F(z)$.

This means, in particular, that $F(z)$ is *continuous* at the points of I (save perhaps at the endpoints). By hypothesis, however, $F(x)$ is *real* a.e. on I. Therefore it is *real everywhere* on I, besides being *continuous there*. Now we can apply the *classical* Schwarz reflection principle to conclude that $F(\bar{z}) \;=\; \overline{F(z)}$ for $z \in \mathscr{D}$.

Our choices of I, properly contained in $\mathscr{D}_0$'s base, and of $\mathscr{D}$, entirely included in $\mathscr{D}_0$, were arbitrary. The formula $F(\bar{z}) = \overline{F(z)}$ thus gives us an analytic continuation of F across the *whole* base of $\mathscr{D}_0$ into all of $\mathscr{D}_0^*$. Done.

Problem 44

Let Λ be any measurable sequence of distinct integers > 0, having (ordinary) density $D_\Lambda < \frac{1}{2}$ (refer to §E.3, Chapter VI).

Write

$$C(z) = \prod_{n\in\Lambda}\left(1 - \frac{z^2}{n^2}\right),$$

and then put

$$B(z) = \frac{C(z-\mathrm{i})}{C(z+\mathrm{i})};$$

$B(z)$ is just a *Blaschke product* for the upper half plane (see §G.3, Chapter III), having *zeros* at the points $\pm n + \mathrm{i}$, $n \in \Lambda$, and *poles* at $\pm n - \mathrm{i}$, $n\in\Lambda$. Consider the function

$$u(t) = \frac{\mathrm{e}^{\pi \mathrm{i} t}}{B(t)},$$

of modulus 1 on $\mathbb{R}$.

(a) Show that $u(t)$ is uniformly continuous on $\mathbb{R}$. (Hint: $B(t)$ is of the form $\exp(\mathrm{i}\varphi(t))$ where $\varphi(t)$ is real. Express $\varphi'(t)$ in terms of the $n\in\Lambda$.)

(b) Assume that $u + H_\infty$ *is a support point of* Σ; this means that there *is* an $f \in H_1$ with $\int_{-\infty}^{\infty} u(t)f(t)\,\mathrm{d}t = \|f\|_1 = 1$. For $\Im z > 0$, write, as usual,

$$f(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} f(t)\,\mathrm{d}t,$$

and then put

$$F(z) = \mathrm{e}^{\pi \mathrm{i} z}(C(z+\mathrm{i}))^2 f(z)$$

in the upper half plane. *Show that* $F(z)$ *can be continued analytically across* $\mathbb{R}$ by putting $F(\bar{z}) = \overline{F(z)}$. (Hint: Use Carleman's lemma.)

(c) Show that the entire function $F(z)$ obtained in (b) is of *exponential type*. (Hint: See proof of the first theorem in §F.4, Chapter VI.)

(d) Hence obtain a *contradiction with the assumption made in* (b) by showing that $F(z)$ must be identically zero. (Hint: Look at the behaviour of $F(z)$ on the imaginary axis, referring to problem 29(a) from §B.1 of Chapter IX.)

By making the right choice of the sequence Λ, various interesting examples can be obtained. We need another lemma, best given as

Problem 45

(a) Let $g(w)$ be analytic in $\{|w| < 1\}$, with $\Re g(w) \geqslant 0$ there. Show that for any $p < 1$, the integrals

$$\int_{-\pi}^{\pi} |g(re^{i\vartheta})|^p \, d\vartheta$$

are bounded for $r < 1$. (Hint: By the principle of conservation of domain, $g(w)$ can never be zero for $|w| < 1$, so we can define an analytic and single valued branch of $(g(w))^p$ there. Apply Cauchy's formula to the latter to get $(g(0))^p$, then take real parts and note that $\cos(p \arg g(w))$ is bounded away from 0.)

(b) If $g(w)$ is as in (a), show that $\lim_{r\to 1} g(re^{i\tau}) = g(e^{i\tau})$ exists a.e., and that for any $p < 1$,

$$(g(w))^p = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-|w|^2}{|w-e^{i\tau}|^2}(g(e^{i\tau}))^p \, d\tau, \qquad |w| < 1,$$

the integral on the right being absolutely convergent. (Hint: Fix a p', $p < p' < 1$, and apply the result from (a) to $(g(w))^{p'}$. Then argue as in the proof of the first theorem from §F.1, Chapter III, using the duality between the spaces $L_r(-\pi, \pi)$ and $L_s(-\pi, \pi)$, where $r = p'/p$ and $(1/r) + (1/s) = 1$. This will yield a function $G(\tau)$ in $L_r(-\pi, \pi)$ such that

$$(g(w))^p = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-|w|^2}{|w-e^{i\tau}|^2} G(\tau) \, d\tau, \qquad |w| < 1.$$

Appeal to standard results about the Poisson integral to describe the boundary behaviour of $(g(w))^p$ and relate $G(\tau)$ thereto.)

(c) Given $v(t)$ defined on $\mathbb{R}$ with $|v(t)| \leqslant \pi/2$ there, consider the function

$$\psi(z) = \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{t-z} - \frac{t}{t^2+1}\right)v(t)\,dt,$$

analytic in $\Im z > 0$. As we know,

$$\lim_{y\to 0} \psi(t+iy) = \psi(t)$$

exists a.e. on $\mathbb{R}$, with $\psi(t) = -\tilde{v}(t) + iv(t)$ a.e. there, $\tilde{v}(t)$ being the *first* kind of Hilbert transform described at the beginning of §E.2. *Show that, when* $p < 1$,

$$\frac{e^{p\psi(t)}}{(t+i)^2}$$

belongs to H_1. (Hint: By mapping the upper half plane conformally onto the unit disk and using the result from (b), show first of all that when $p < 1$,

$$e^{p\psi(z)} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} e^{p\psi(t)}\,dt$$

for $\Im z > 0$, the integral on the right being absolutely convergent. This representation gives us fairly good control on the *size* of $\exp(p\psi(z))$ in $\Im z > 0$ – cf. Chapter VI, §A.2 – A.3. Knowing this, show by integrating around suitable contours that if λ and $\delta > 0$,

$$\int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{(x+i)^{2+\delta}} \exp(p\psi(x+ih)\,dx = 0$$

for any $h > 0$ – cf. proof of theorem that the product of two H_2 functions is in H_1, §E.1. Now one may make $\delta \longrightarrow 0$ and use dominated convergence (*guaranteed* by our representation for $\exp(p\psi(z))$!) to get

$$\int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{(x+i)^2} \exp(p\psi(x+ih))\,dx = 0.$$

Plug the representation for $\exp(p\psi(z))$ into this result and use Fubini's theorem, noting that $e^{i\lambda t}/(t+i)^2$ is in H_∞. Finally, make $h \longrightarrow 0$.)

Let us now take a measurable sequence Λ of integers > 0 *having* (ordinary) *density zero*, whose Beurling–Malliavin *effective density* $\tilde{D}_\Lambda$ *is equal* to 1 (see §D.2 of Chapter IX). It is easy to construct such sequences. We need merely pick intervals $[a_k, b_k]$ with integral endpoints,

$$0 < a_1 < b_1 < a_2 < b_2 < a_3 < \cdots,$$

such that b_{k-1}/b_k and $(b_k - a_k)/b_k$ both tend to zero as $k \longrightarrow \infty$, while

$$\sum_1^\infty \left(\frac{b_k - a_k}{b_k}\right)^2 = \infty,$$

and then have Λ consist of the integers in the $[a_k, b_k]$.

Using such a sequence Λ, let us form the functions $C(z)$, $B(z)$ and $u(t)$ considered in problem 44. Then,

$u + H_\infty$ is an extreme point of Σ even though it is not a support point thereof.

This will follow from problem 44, the first lemma of the present §, and

Problem 46

To show that there can be *no* non-zero $h \in H_\infty$ with

$|u(t) - h(t)| \leqslant 1 \quad$ a.e..

(a) Assuming that there *is* such an h, show how to construct a function $v(t)$ defined a.e. on $\mathbb{R}$, with $|v(t)| \leqslant \pi/2$ and

$e^{-\pi i t} B(t) h(t)^{iv(t)} \geqslant 0 \quad$ a.e..

(b) Using the v found in (a), form $\psi(t) = -\tilde{v}(t) + iv(t)$ as in problem 45(c). Show that in the present circumstances,

$$\int_{-\infty}^{\infty} \frac{|h(t)\exp\psi(t)|}{1+t^2}\,dt < \infty.$$

(Hint: From the solution of problem 45(c),

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\exp(p\psi(t))}{1+t^2}\,dt = \exp(p\psi(i))$$

whenever $p < 1$. Take real parts and make $p \longrightarrow 1$; then what we have on the right tends to the *finite* value $\Re \exp\psi(i)$, so that

$$\int_{-\infty}^{\infty} \frac{e^{-\tilde{v}(t)}\cos v(t)}{1+t^2}\,dt \leqslant \pi \Re e^{\psi(i)} < \infty$$

by Fatou's lemma. If, however, we write

$g(t) = e^{-\pi i t} B(t) h(t)$,

we have the following diagram:

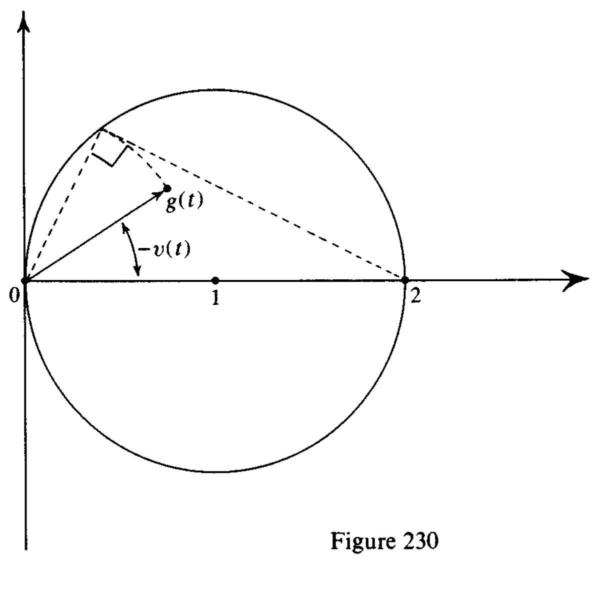


Figure 230

From this it is clear that $|g(t)| \leqslant 2\cos v(t)$.)

(c) Continuing with the notation used in (b), show that

$$\frac{B(t)h(t)\exp\psi(t)}{(t+\mathrm{i})^2}$$

belongs to H_1. (Hint: $Bh \in H_\infty$ and, for each $p < 1$, $\exp(p\psi(t)/(t+\mathrm{i})^2$ is in H_1 by problem 45(c). Thus, if $\lambda \geqslant 0$, we have

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\lambda t}\frac{B(t)h(t)\exp(p\psi(t))}{(t+\mathrm{i})^2}\,\mathrm{d}t \quad = \quad 0$$

by §E.1, whenever $p < 1$. In this relation, we may let $p \longrightarrow 1$ and use dominated convergence, referring to the result from (b).)

(d) Show that the function

$$F(z) \quad = \quad \mathrm{e}^{-\pi\mathrm{i}z}B(z)h(z)\mathrm{e}^{\psi(z)},$$

analytic in the upper half-plane, can be continued across the real axis, *yielding an entire function of exponential type* $\leqslant \pi$, by putting

$$F(\bar{z}) \quad = \quad \overline{F(z)}.$$

Here, as usual,

$$h(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2}h(t)\,\mathrm{d}t, \qquad \Im z \; > \; 0.$$

(Hint: See problem 44 again.)

(e) Hence show that the function $F(z)$ from (d) *is identically zero*, so that *in fact* $h(z) \equiv 0$, proving that the assumption made in (a) is *untenable*. (Hint: First apply the Riesz–Fejér theorem from §G.3 of Chapter III to get an entire function $f(z)$ of exponential type $\leqslant \pi/2$, *vanishing* at each of the points $\pm n + \mathrm{i}$, $n \in \Lambda$, such that

$$F(z) \quad = \quad f(z)\overline{f(\bar{z})}.$$

$f(z+\mathrm{i})$ then certainly vanishes at each point of Λ. Use the fact that $\tilde{D}_\Lambda = 1$, applying a suitable variant of the theorem quoted at the very beginning of §E, Chapter IX.)

Problem 47

Let $u(t)$ be as in problem 46. Show that $\bar{u} + H_\infty$ (*sic*!) is an extreme point, but not a support point, of Σ.

Remark. When Carleman's lemma is used in the above problems, one is actually dealing with functions $F \in \mathscr{S}_1(\mathscr{D}_0)$ whose boundary values are *positive* (and not merely real) along the base of $\mathscr{D}_0$. In this circumstance, analytic continuation across the base of $\mathscr{D}_0$ is possible under a *weaker* condition on F than that of membership in $\mathscr{S}_1(\mathscr{D}_0)$. It is enough that F belong to the space $H_{1/2}$ associated with each of the smaller rectangles $\mathscr{D}$ used in the proof of the lemma. That fact follows easily from an argument due to Neuwirth and Newman, and, independently, to Helson and Sarason. The reader is referred to the discussion accompanying problem 13 at the end of Chapter II in Garnett's book.*

* *Bounded Analytic Functions*

XI

Multiplier Theorems

It is time to prove the multiplier theorem stated in §A.2 of the preceding chapter and then applied there, in §§B and C. We desire also to establish another result of the same kind and finally to start working towards a *description* of the weights $W(x) \geqslant 1$ that *admit multipliers* (in the sense explained at the beginning of Chapter X). All this will require the use of some elementary material from potential theory.

There is a dearth of modern expositions of that theory accessible to readers having only a general background in analysis. Moreover, the books on it that do exist* are not so readily available. It therefore seems advisable to first explain the basic results we will use from the subject without, however, getting involved in any attempt at a systematic treatment of it. That is the purpose of the first § in this chapter. Other more special potential-theoretic results called for later on will be formulated and proved as they are needed.

A Some rudimentary potential theory

1. Superharmonic functions; their basic properties

A function $U(z)$ harmonic in a domain $\mathscr{D}$ enjoys the *mean value property* there: for $z \in \mathscr{D}$,

$$U(z) \quad = \quad \frac{1}{2\pi}\int_0^{2\pi} U(z+\rho e^{i\vartheta})\,d\vartheta, \qquad 0 \;<\; \rho \;<\; \operatorname{dist}(z,\, \partial\mathscr{D}).$$

* The books by Carleson, Tsuji, Kellogg, Helms and Landkof are in my possession, together with a copy of Frostman's thesis; most of the time I have been able to make do with just the *first three* of these.

To Gauss is due the important *converse* of this statement: among the functions $U(z)$ continuous in $\mathscr{D}$, the mean value property *characterizes* the ones harmonic there. The proof of this contains a key to the understanding of much of the work with superharmonic functions (defined presently) to concern us here; let us therefore recall how that proof goes.

An (apparently) more general result can in fact be established by the same reasoning. Suppose that a function $U(z)$, continuous in a domain $\mathscr{D}$, enjoys a *local mean value property there*; in other words, that to each $z \in \mathscr{D}$ corresponds an r_z, $0 < r_z \leqslant \operatorname{dist}(z, \partial\mathscr{D})$ (with, *a priori*, $r_z < \operatorname{dist}(z, \partial\mathscr{D})$) such that

$$U(z) \;=\; \frac{1}{2\pi}\int_0^{2\pi} U(z+\rho e^{i\vartheta})\,d\vartheta \qquad \text{for } 0 < \rho < r_z.$$

It is claimed that $U(z)$ is then *harmonic in* $\mathscr{D}$.

The main part of the argument consists in showing that the local mean value property implies the *strong maximum principle* for U on (connected) domains with compact closures lying in $\mathscr{D}$. Letting Ω be any such domain, we have to verify that for $z \in \Omega$,

$$U(z) \;<\; \sup_{\zeta\in\partial\Omega} U(\zeta)$$

unless $U(z) \equiv$ const on $\bar{\Omega}$. Here, $U(z)$ has on $\bar{\Omega}$ a *maximum* – call it M – and the statement in question amounts to the assertion that $U(z) \equiv M$ on $\bar{\Omega}$ if, for any $z_0 \in \Omega$, $U(z_0) = M$.

Suppose there is such a z_0. Then, for each sufficiently small $\rho > 0$,

$$\frac{1}{2\pi}\int_0^{2\pi} U(z_0+\rho e^{i\vartheta})\,d\vartheta \;=\; U(z_0) \;=\; M$$

with $U(z_0 + \rho e^{i\theta})$ continuous in ϑ and $\leqslant M$. This makes $U(z_0 + \rho e^{i\vartheta}) \equiv M$ for such ρ, so that $U(z) \equiv M$ in a *small disk* centered at z_0. The set

$$E \;=\; \{z_0 \in \Omega:\ U(z_0) = M\}$$

is thus *open*. That set is, however, *closed* in Ω's relative topology on account of the continuity of U. Hence $E = \Omega$ since Ω is connected, and $U(z) \equiv M$ in Ω – thus finally on $\bar{\Omega}$, thanks again to the continuity of U.

To complete the proof of Gauss' result, let us take any $z_0 \in \mathscr{D}$ and an $R < \operatorname{dist}(z_0, \partial\mathscr{D})$; it is enough to establish that

$$U(z_0+\rho e^{i\vartheta}) \;=\; \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-\rho^2}{R^2+\rho^2-2R\rho\cos(\vartheta-\tau)}\,U(z_0+Re^{i\tau})\,d\tau$$

for $0 \leqslant \rho < R$. Calling the expression on the right $V(z_0 + \rho e^{i\vartheta})$, we proceed to show first that

$$U(z) \leqslant V(z)$$

for $|z - z_0| < R$.

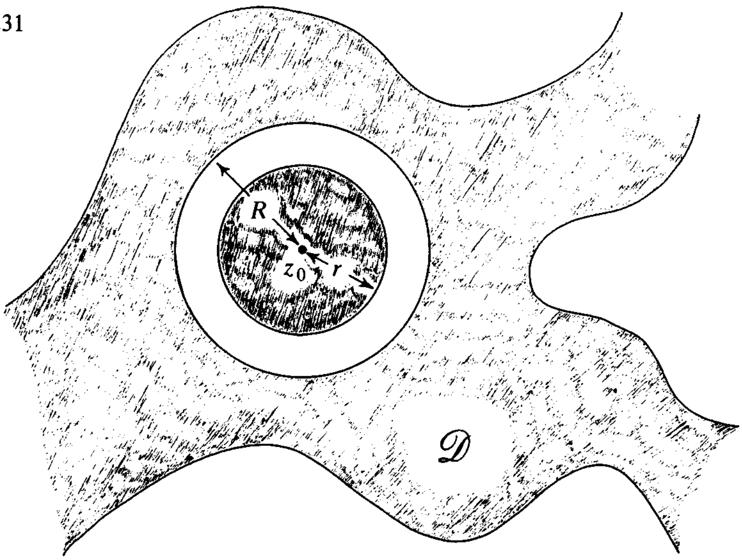
Fix any $\varepsilon > 0$. By continuity of U and the elementary properties of the Poisson kernel we know that

$$V(z_0 + re^{i\vartheta}) \longrightarrow U(z_0 + Re^{i\vartheta})$$

uniformly in ϑ for $r < R$ tending to R; the same is of course true if we replace V by U on the left. On the circles $|z - z_0| = r$ with radii $r < R$ sufficiently close to R we therefore have

$$U(z) - V(z) \leqslant \varepsilon.$$

Figure 231



Here, both $U(z)$ and the *harmonic* function $V(z)$ enjoy the local mean value property in the open disk $\{|z - z_0| < R\}$. Hence, by what has just been shown, we have the strong maximum principle for the *difference* $U(z) - V(z)$ on the smaller disks $\{|z - z_0| < r\}$. The preceding inequality thus implies that $U(z) - V(z) \leqslant \varepsilon$ on each of those disks, and finally that $U(z) - V(z) \leqslant \varepsilon$ for $|z - z_0| < R$. Squeezing ε, we see that

$$U(z) - V(z) \leqslant 0 \quad \text{for } |z - z_0| < R.$$

By working with the difference $V(z) - U(z)$ we can, however, prove the *reverse* inequality in the same fashion. This means that one must have $U(z) = V(z)$ for $|z - z_0| < R$, and our proof is finished. It is this *argument* that the reader will find helpful to keep in mind during the following development.

Next in importance to the harmonic functions as objects of interest in potential theory come those that are *subharmonic* or *superharmonic*. One can actually work exclusively with harmonic functions and the ones belonging to *either* of the last two categories; which of the latter is singled out makes very little difference. Logarithms of the moduli of analytic functions are subharmonic, but most writers on potential theory prefer (probably on account of the customary formulation of Riesz' theorem, to be given in article 2) to deal with *superharmonic functions*, and we follow their example here. The difference between the two kinds of functions is purely one of *sign*: a given $F(z)$ is *subharmonic* if and only if $-F(z)$ is *superharmonic*.

Definition. A function $U(z)$ defined in a domain $\mathscr{D}$ with $-\infty < U(z) \leqslant \infty$ there is said to be superharmonic in $\mathscr{D}$ provided that

(i) $\liminf_{z \to z_0} U(z) \geqslant U(z_0)$ for $z_0 \in \mathscr{D}$;

(ii) to each $z \in \mathscr{D}$ corresponds an r_z, $0 < r_z \leqslant \operatorname{dist}(z, \partial\mathscr{D})$, such that

$$\frac{1}{2\pi}\int_0^{2\pi} U(z + \rho e^{i\vartheta})\,d\vartheta \quad \leqslant \quad U(z) \qquad \text{for } 0 < \rho < r_z.$$

Superharmonic functions are thus permitted to assume the value $+\infty$ at certain points. Although authors on potential theory do not generally agree to call the function *identically equal* to $+\infty$ superharmonic, we will sometimes find it convenient to do so.

Assumption of the value $-\infty$, on the other hand, is not allowed. This restriction plays a serious rôle in the subject. By it, functions like

$$U(z) \quad = \quad \begin{cases} \Im z, & \Im z > 0, \\ -\infty, & \Im z \leqslant 0, \end{cases}$$

are excluded from consideration.

It may seem at first sight that an extensive theory could hardly be based on the definition just given. On thinking back, however, to the proof of

Gauss' result, one begins to suspect that the simple conditions figuring in the definition involve more structure than is immediately apparent. One notices, to begin with, that (i) and (ii) signify *opposite kinds* of local behaviour. The *first* guarantees that $U(z)$ *stays almost as large* as $U(z_0)$ on small neighborhoods of z_0, and the *second* gives us lots of points z in such neighborhoods at which $U(z) \leqslant U(z_0)$. Considerable use of the interplay between these two contrary effects will be made presently; for the moment, let us simply remark that together, they entail *equality* of $\liminf_{z \to z_0} U(z)$ and $U(z_0)$ at the $z_0 \in \mathscr{D}$.

It is probably best to start our work with superharmonic functions by seeing what can be deduced from the requirement that $U(z) > -\infty$ and condition (i), *taken by themselves*. The latter is nothing other than a prescription for *lower semicontinuity* in $\mathscr{D}$; as is well known, and easily verified by the reader, it implies that $U(z)$ has an *assumed minimum* on each compact subset of $\mathscr{D}$. Together with the requirement, that means that $U(z)$ *has a finite lower bound on every compact subset of* $\mathscr{D}$. This property will be used repeatedly. (I can never remember *which* of the two kinds of semicontinuity is *upper*, and which is *lower*, and suspect that some readers of this book may have the same trouble. That is why I systematically avoid using the *terms* here, and prefer instead to specify explicitly each time which behaviour is meant.)

A *monotonically increasing* sequence of functions continuous on a domain $\mathscr{D}$ tends to a limit $U(z) > -\infty$ satisfying (i) there. This is immediate; what is less apparent is a kind of *converse*:

Lemma. *If* $U(z) > -\infty$ *has property* (i) *in* $\mathscr{D}$ *there is, for any compact subset* K *of* $\mathscr{D}$, *a monotonically increasing sequence of functions* $\varphi_n(z)$ *continuous on* K *and tending to* $U(z)$ *there.*

Proof. For each $n \geqslant 1$ put, for $z \in K$,

$$\varphi_n(z) \quad = \quad \inf_{\zeta \in K}(U(\zeta) \; + \; n|z-\zeta|).$$

Since $U(\zeta)$ is bounded below on K by the above observation, the functions $\varphi_n(z)$ are all $> -\infty$. It is evident that $\varphi_n(z) \leqslant \varphi_{n+1}(z) \leqslant U(z)$ for $z \in K$ and each n.

To show continuity of φ_n at $z_0 \in K$, we remark that the function of ζ equal to $U(\zeta) + n|\zeta - z_0|$ enjoys, like $U(\zeta)$, property (i) and thus *assumes its minimum* on K. There is hence a $\zeta_0 \in K$ such that

$$\varphi_n(z_0) \quad = \quad n|\zeta_0 - z_0| \; + \; U(\zeta_0),$$

so, if $z \in K$,

$$\varphi_n(z) \;\leqslant\; n|z-\zeta_0| \;+\; U(\zeta_0) \;\leqslant\; n|z-z_0| \;+\; \varphi_n(z_0).$$

In the same way, we see that

$$\varphi_n(z_0) \;\leqslant\; n|z_0-z| \;+\; \varphi_n(z)$$

which, combined with the previous, yields

$$|\varphi_n(z)-\varphi_n(z_0)| \;\leqslant\; n|z-z_0| \quad \text{for } z_0 \text{ and } z \in K.$$

We proceed to verify that $\varphi_n(z_0) \underset{n}{\longrightarrow} U(z_0)$ at each $z_0 \in K$. Given such a z_0, take any number $V < U(z_0)$. By property (i) there is an $\eta > 0$ such that $U(\zeta) > V$ for $|\zeta - z_0| < \eta$. $U(\zeta)$ has, as just recalled, a finite lower bound, say $-M$, on K. Then, for $n > (V+M)/\eta$, we have $n|\zeta-z_0| + U(\zeta) > V$ for $\zeta \in K$ with $|\zeta - z_0| \geqslant \eta$. But when $|\zeta-z_0| < \eta$ we also have $n|\zeta - z_0| + U(\zeta) > V$. Therefore

$$\varphi_n(z_0) \geqslant V \quad \text{for } n > (M+V)/\eta$$

Since, on the other hand, $\varphi_n(z_0) \leqslant U(z_0)$, we see that the convergence in question holds, $V < U(z_0)$ being arbitrary.

The lemma is proved.

Remark. This result figures in some introductory treatments of the Lebesque integral.

Let us give some examples of superharmonic functions. The class of these includes, to begin with, all the *harmonic* functions. Gauss' result implies indeed that a function $U(z)$ defined on a domain $\mathscr{D}$ is *harmonic* there *if and only if* both $U(z)$ and $-U(z)$ are *superharmonic* in $\mathscr{D}$. The simplest kind of functions $U(z)$ superharmonic, but not harmonic, in $\mathscr{D}$ are those of the form

$$U(z) \;=\; \log\frac{1}{|z-z_0|} \quad \text{with } z_0 \in \mathscr{D}.$$

Positive linear combinations of these are also superharmonic, and so, finally, are the expressions

$$U(z) \;=\; \int_K \log\frac{1}{|z-\zeta|}\,d\mu(\zeta)$$

formed from positive measures μ supported on compact sets K. *The reader should not proceed further without verifying the last statement.* This involves the use of Fatou's lemma for property (i), and of the handy relation

$$\frac{1}{2\pi}\int_0^{2\pi} \log\frac{1}{|z+\rho e^{i\vartheta}-\zeta|}\,d\vartheta \quad = \quad \min\left(\log\frac{1}{|z-\zeta|},\ \log\frac{1}{\rho}\right)$$

(essentially the same as one appearing in the derivation of Jensen's formula, Chapter I!) for property (ii).

Integrals like the above one actually turn out to be practically capable of representing all superharmonic functions. In a sense made precise by Riesz' theorem, to be proved in article 2, the most general superharmonic function is equal to such an integral plus a harmonic function.

By such examples, one sees that superharmonic functions are far from being 'well behaved'. Consider, for instance

$$U(z) \quad = \quad \sum_n a_n \log\frac{1}{|z-z_n|},$$

formed with the z_n of modulus $< 1/2$ tending to 0 and numbers $a_n > 0$ chosen so as to make

$$\sum_n a_n \log\frac{1}{|z_n|} \quad < \quad \infty.$$

Here, $U(0) < \infty$ although U is infinite at each of the z_n. In more sophisticated versions of this construction, the z_n are *dense* in $\{|z| < 1/2\}$ and various sequences of $a_n > 0$ with $\sum_n a_n < \infty$ are used.

We now allow both properties from our definition to play their parts, (ii) as well as (i). In that way, we obtain the first general results pertaining specifically to superharmonic functions, among which the following *strong minimum principle* is probably the most important:

Lemma. *Let $U(z)$ be superharmonic in a domain $\mathscr{D}$. Then, if Ω is a (connected) domain with compact closure contained in $\mathscr{D}$,*

$$U(z) \quad > \quad \inf_{\zeta\in\partial\Omega} U(\zeta) \qquad \text{for } z\in\Omega$$

unless $U(z)$ is constant on $\bar{\Omega}$.

Proof. As we know, $U(z)$ attains its (finite) minimum, M, on $\bar{\Omega}$, and it is enough to show that if $U(z_0) = M$ at some $z_0\in\Omega$, we have $U(z) \equiv M$ on $\bar{\Omega}$. The reasoning here is like that followed in establishing the strong maximum principle for harmonic functions.

Assuming that there is such a z_0, we have, by property (ii),

$$M = U(z_0) \geqslant \frac{1}{2\pi}\int_0^{2\pi} U(z_0 + \rho e^{i\vartheta})\, d\vartheta$$

whenever $\rho > 0$ is sufficiently small. Here, $U(z_0 + \rho e^{i\vartheta}) \geqslant M$ and if, at any ϑ_0, we had $U(z_0 + \rho e^{i\vartheta_0}) > M$, $U(z_0 + \rho e^{i\vartheta})$ would be $> M$ for all ϑ belonging to some *open interval including* ϑ_0, by property (i). In that event, the above right-hand integral would also be $> M$, yielding a contradiction. We must therefore have $U(z_0 + \rho e^{i\vartheta}) \equiv M$ for small enough values of $\rho > 0$.

The rest of the proof is like that of the result for harmonic functions, with $E = \{z \in \Omega\colon\ U(z) = M\}$ closed in Ω's relative topology thanks to property (i). We are done.

Corollary. *Let $U(z)$ be superharmonic in a domain $\mathscr{D}$, and let $\mathscr{O}$ be an open set with compact closure lying in $\mathscr{D}$. Then, for $z \in \mathscr{O}$,*

$$U(z) \geqslant \inf_{\zeta \in \partial\mathscr{O}} U(\zeta).$$

Proof. Apply the lemma in each component of $\mathscr{O}$.

Corollary. *Let $U(z)$ be superharmonic in $\mathscr{D}$, a domain with compact closure. If* $\liminf_{z\to\zeta} U(z) \geqslant M$ *at each $\zeta \in \partial\mathscr{D}$, one has $U(z) \geqslant M$ in $\mathscr{D}$.*

Proof. Fix any $\varepsilon > 0$. Then, corresponding to each $\zeta \in \partial\mathscr{D}$ there is an r_ζ, $0 < r_\zeta < \varepsilon$, such that

$$U(z) \geqslant M - \varepsilon \quad \text{for } z \in \mathscr{D} \text{ and } |z - \zeta| \leqslant r_\zeta.$$

Here, $\partial\mathscr{D}$ is compact, so it can be covered by a *finite number* of the open disks

$$\{|z - \zeta| < r_\zeta\}.$$

Let $\mathscr{O}$ be the open set equal to the *complement*, in $\mathscr{D}$, of the *union of the closures* of *those particular disks.*

The closure $\bar{\mathscr{O}}$ is compact and contained in $\mathscr{D}$. If $z \in \partial\mathscr{O}$, we have $|z - \zeta| = r_\zeta$ for some $\zeta \in \partial\mathscr{D}$, so $U(z) \geqslant M - \varepsilon$. $U(z)$ is hence $\geqslant M - \varepsilon$ in $\mathscr{O}$ by the previous corollary. $\mathscr{O}$, however, certainly includes all points of $\mathscr{D}$ distant by more than ε from $\partial\mathscr{D}$. Our result thus follows on making $\varepsilon \longrightarrow 0$.

From these results we can deduce a useful *characterization* of superharmonic functions.

Theorem. *If $U(z)$ is $> -\infty$ and enjoys property* (i) *in a domain $\mathscr{D}$, it is superharmonic there provided that for each $z_0 \in \mathscr{D}$ and every disk Δ of sufficiently small radius with centre at z_0, one has*

$$U(z_0) \geqslant h(z_0)$$

for every function $h(z)$ harmonic in Δ and continuous up to $\partial\Delta$, satisfying

$$h(\zeta) \leqslant U(\zeta)$$

on $\partial\Delta$.

Conversely, if $U(z)$ is superharmonic in $\mathscr{D}$ and Ω is any domain having compact closure $\subseteq \mathscr{D}$, every function $h(z)$ harmonic in Ω and continuous up to $\partial\Omega$ is $\leqslant U(z)$ in Ω provided that $h(\zeta) \leqslant U(\zeta)$ on $\partial\Omega$.

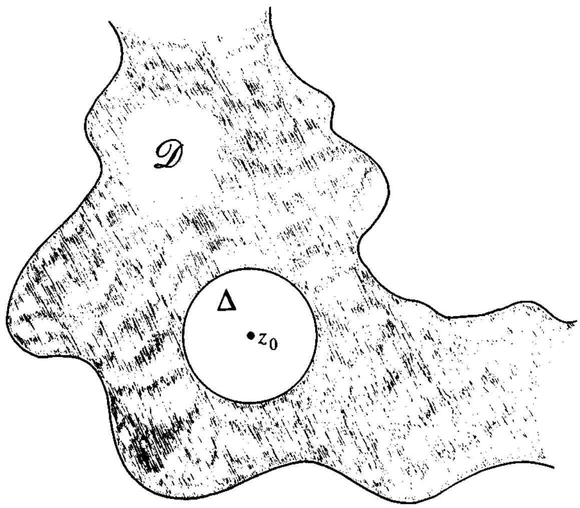


Figure 232

Proof. For the first part, we take any $z_0 \in \mathscr{D}$ and verify property (ii) for U there, assuming the hypothesis concerning disks Δ about z_0.

Let then $0 < r < \operatorname{dist}(z_0, \partial\mathscr{D})$. By the first lemma of this article, there is an *increasing sequence* of functions $u_n(\vartheta)$, *continuous* and of period 2π, such that

$$u_n(\vartheta) \underset{n}{\longrightarrow} U(z_0 + re^{i\vartheta}), \qquad 0 \leqslant \vartheta \leqslant 2\pi.$$

Put $h_n(z_0 + re^{i\vartheta}) = u_n(\vartheta)$, and, for $0 \leqslant \rho < r$, take

$$h_n(z_0 + \rho e^{i\vartheta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - \rho^2}{r^2 + \rho^2 - 2r\rho\cos(\vartheta - \tau)} u_n(\tau)\, d\tau.$$

Then each function $h_n(z)$ is harmonic in the disk Δ of radius r about z_0 and continuous up to $\partial\Delta$, where we of course have

$$h_n(\zeta) \leqslant U(\zeta).$$

If $r > 0$ is small enough, our assumption thus tells us that

$$h_n(z_0) \leqslant U(z_0)$$

for every n. Now Lebesgue's monotone convergence theorem ensures that

$$h_n(z_0) \xrightarrow[n]{} \frac{1}{2\pi}\int_0^{2\pi} U(z_0 + re^{i\tau})\,d\tau$$

as $n \longrightarrow \infty$. Hence

$$\frac{1}{2\pi}\int_0^{2\pi} U(z_0 + re^{i\tau})\,d\tau \leqslant U(z_0)$$

for all sufficiently small $r > 0$, and property (ii) holds.

The other part of the theorem is practically a restatement of the *second* of the above corollaries. Indeed, if $h(z)$, harmonic in Ω and continuous up to $\bar{\Omega} \subseteq \mathscr{D}$ satisfies $h(\zeta) \leqslant U(\zeta)$ on $\partial\Omega$, we certainly have

$$\liminf_{\substack{z\to\zeta \\ z\in\Omega}} (U(z) - h(z)) \geqslant 0$$

at each $\zeta \in \partial\Omega$ on account of property (i). At the same time, $U(z) - h(z)$ is superharmonic in Ω; it must therefore be $\geqslant 0$ there by the corollary in question.

This does it.

By combining the two arguments followed in the last proof, we immediately obtain the following inequality:

For $U(z)$ superharmonic in $\mathscr{D}$, $z_0 \in \mathscr{D}$, and $0 < r < \operatorname{dist}(z_0\ \partial\mathscr{D})$,

$$\boxed{U(z_0 + \rho e^{i\vartheta}) \geqslant \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - \rho^2}{r^2 + \rho^2 - 2r\rho\cos(\vartheta - \tau)} U(z_0 + re^{i\tau})\,d\tau, \quad 0 \leqslant \rho < r.}$$

This in turn gives us a result needed in article 2:

Lemma. *If $U(z)$ is superharmonic in a domain $\mathscr{D}$ and $z_0 \in \mathscr{D}$,*

$$\frac{1}{2\pi}\int_0^{2\pi} U(z_0 + re^{i\vartheta})\,d\vartheta$$

is a decreasing *function of r for* $0 < r < \operatorname{dist}(z_0,\ \partial\mathscr{D})$.

Proof. Integrate both sides of the boxed inequality with respect to ϑ and then use Fubini's theorem on the right.

Along these same lines, we have, finally, the

Theorem. *Let $U(z)$ be superharmonic in a domain $\mathscr{D}$, and suppose that $z_0 \in \mathscr{D}$ and that $0 < R < \operatorname{dist}(z_0, \partial\mathscr{D})$. Denoting by Δ the disk $\{|z - z_0| < R\}$, put $V(z) = U(z)$ for $z \in \mathscr{D} \sim \Delta$. In Δ, take*

$$V(z_0 + re^{i\vartheta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos(\vartheta - \tau)} U(z_0 + Re^{i\tau})\,d\tau$$

(for $0 \leqslant r < R$). Then $V(z) \leqslant U(z)$ and $V(z)$ is superharmonic in $\mathscr{D}$.

Proof. For $z \in \mathscr{D} \sim \Delta$, the relation $V(z) \leqslant U(z)$ is manifest, and for $z \in \Delta$ it is a consequence of the above boxed inequality.

To verify property (ii) for V, suppose first of all that $z \in \mathscr{D} \sim \Delta$. Then, for sufficiently small $\rho > 0$,

$$V(z) = U(z) \geqslant \frac{1}{2\pi}\int_0^{2\pi} U(z + \rho e^{i\vartheta})\,d\vartheta.$$

By the relation just considered, the right-hand integral is in turn

$$\geqslant \frac{1}{2\pi}\int_0^{2\pi} V(z + \rho e^{i\vartheta})\,d\vartheta;$$

V thus enjoys property (ii) at z.

We must also look at the points $z \in \Delta$. On $\partial\Delta$, the function $U(\zeta)$ is *bounded below*, according to an early observation in this article. The Poisson integral used above to define $V(z)$ in Δ is therefore *either infinite for every r, $0 \leqslant r < R$, or else convergent for each such r.* In the former case, $V(z) \equiv \infty$ for $z \in \Delta$, and V (trivially) possesses property (ii) at those z. In the latter case, $V(z)$ is actually *harmonic* in Δ and hence, for any given z therein, *equal to the previous mean value when $\rho < \operatorname{dist}(z, \partial\Delta)$.* Here also, V has property (ii) at z.

Verifications of the relation $V(z) > -\infty$ and of property (i) remain. The first of these is clear; it is certainly true in $\mathscr{D} \sim \Delta$ where V coincides with U, and also true in Δ where, as a Poisson integral,

$$V(z) \geqslant \inf_{|\zeta - z_0| = R} U(\zeta)$$

with the right side $> -\infty$, as we know.

We have, then, to check property (i). The only points at which this can present any difficulty must lie on $\partial\Delta$, for, *inside* Δ, V is either *harmonic* and thus *continuous* or else *everywhere infinite*, and *outside* $\bar{\Delta}$, V coincides (in $\mathscr{D}$) with U, a function *having* the semicontinuity in question. Let therefore $|z - z_0| = R$. Then we surely have

$$\liminf_{\substack{\zeta \to z \\ \zeta \notin \Delta}} V(\zeta) \quad = \quad \liminf_{\substack{\zeta \to z \\ \zeta \notin \Delta}} U(\zeta) \quad \geqslant \quad U(z) \quad = \quad V(z),$$

so we need only examine the behaviour of $V(\zeta)$ for ζ tending to z *from within* Δ. The relation just written holds in particular, however, for $\zeta = z_0 + Re^{i\tau}$ tending to z *on* $\partial\Delta$. Since $U(z_0 + Re^{i\tau})$ is also bounded below on $\partial\Delta$, we see by the elementary properties of the Poisson kernel that

$$\liminf_{\substack{\zeta \to z \\ \zeta \in \Delta}} V(\zeta) \quad \geqslant \quad U(z) \quad = \quad V(z).$$

We thus have

$$\liminf_{\zeta \to z} V(z) \quad \geqslant \quad V(z)$$

for the points z on $\partial\Delta$, as well as at the other $z \in \mathscr{D}$, and V has property (i).

The theorem is proved.

Our work will involve the consideration of certain *families* of superharmonic functions. Concerning these, one has two main results.

Theorem. *Let the $U_n(z)$ be superharmonic in a domain $\mathscr{D}$, with*

$$U_1(z) \leqslant U_2(z) \leqslant U_3(z) \leqslant \cdot \cdot \cdot \leqslant U_n(z) \leqslant \cdot \cdot \cdot$$

there. Then

$$U(z) \quad = \quad \lim_{n \to \infty} U_n(z)$$

is superharmonic (perhaps $\equiv \infty$) in $\mathscr{D}$.

Proof. Since $U_1(z) > -\infty$ in $\mathscr{D}$, the same is true for $U(z)$.

Verification of property (i) is almost automatic. Given $z_0 \in \mathscr{D}$, let M be any number $< U(z_0)$. Then, for some particular n, $U_n(z_0) > M$, so, since U_n enjoys property (i), $U_n(z) > M$ in a neighborhood of z_0. *A fortiori*, $U(z) > M$ in that same neighborhood, and $\liminf_{z \to z_0} U(z) \geqslant U(z_0)$ on account of the arbitrariness of M.

Property (ii) is a consequence of Lebesgue's monotone convergence theorem. Let $z_0 \in \mathscr{D}$ and fix any $\rho < \operatorname{dist}(z_0, \partial\mathscr{D})$. Then, by the above boxed inequality,

$$U_n(z_0) \quad \geqslant \quad \frac{1}{2\pi}\int_0^{2\pi} U_n(z_0 + \rho e^{i\vartheta})\, d\vartheta$$

for each n. Here $U_1(z_0+\rho e^{i\vartheta})$ is bounded below for $0 \leqslant \vartheta \leqslant 2\pi$, so the right-hand integral tends to

$$\frac{1}{2\pi}\int_0^{2\pi} U(z_0+\rho e^{i\vartheta})\,d\vartheta$$

as $n \longrightarrow \infty$ by the monotone convergence. At the same time, $U_n(z_0) \underset{n}{\longrightarrow} U(z_0)$, so property (ii) holds.

We are done.

A statement of opposite character is valid for *finite* collections of superharmonic functions. If, namely, $U_1(z)$, $U_2(z)$, ..., $U_N(z)$ *are superharmonic in a domain* $\mathscr{D}$, *so is* $\min_{1\leqslant k\leqslant N} U_k(z)$. This observation, especially useful when the functions $U_k(z)$ involved are *harmonic*, is easily verified directly.

WARNING. *The corresponding statement about* $\max_{1\leqslant k\leqslant N} U_k(z)$ *is* (in general) *false for superharmonic functions* U_k.

One has a version of the observation for *infinite* collections of superharmonic functions:

Theorem. *Let* $\mathscr{F}$ *be any family of functions superharmonic in a domain* $\mathscr{D}$. *For* $z\in\mathscr{D}$, *put*

$$W(z) \;=\; \inf\{U(z):\ U\in\mathscr{F}\},$$

and then let

$$V(z) \;=\; \liminf_{\zeta\to z} W(\zeta), \qquad z\in\mathscr{D}.$$

Then $V(z) \leqslant U(z)$ *in* $\mathscr{D}$ *for every* $U\in\mathscr{F}$, *and* (*especially*) *if* $V(z) > -\infty$ *in* $\mathscr{D}$, *it is superharmonic there.*

Remark. Something like the last condition is needed in order to avoid situations like the one where $\mathscr{D}=\mathbb{C}$ and $\mathscr{F}$ consists of the functions $n\Im z$, $n = 1, 2, 3, \ldots$. There, $V(z) = \Im z$ for $\Im z > 0$ but $V(z) = -\infty$ for $\Im z \leqslant 0$. Such functions V are not superharmonic.

Proof of theorem. First of all, $V(z) \leqslant W(z)$ in $\mathscr{D}$, i.e., $V(z) \leqslant U(z)$ there for each $U\in\mathscr{F}$. Indeed, since any such U is superharmonic in $\mathscr{D}$, $\liminf_{\zeta\to z} U(\zeta)$ is actually *equal* to $U(z)$ there, as observed earlier in this article (a result of playing properties (i) and (ii) against each other).

Therefore, whenever $U \in \mathscr{F}$,

$$V(z) \quad = \quad \liminf_{\zeta \to z} W(\zeta) \quad \leqslant \quad \liminf_{\zeta \to z} U(\zeta) \quad = \quad U(z), \qquad z \in \mathscr{D}.$$

Secondly, V has property (i) in $\mathscr{D}$. To see this, fix any $z_0 \in \mathscr{D}$ and pick* any $M < V(z_0)$; according to our definition of V, $W(z)$ is then $> M$ in some punctured open neighborhood of z_0 (i.e., an open neighborhood of z_0 with z_0 *deleted*). But this certainly makes $V(z) = \liminf_{\zeta \to z} W(\zeta) \geqslant M$ in that punctured neighborhood, so, since $M < V(z_0)$ was arbitrary, we have $\liminf_{z \to z_0} V(z) \geqslant V(z_0)$.

To complete verification of $V(z)$'s superharmonicity in $\mathscr{D}$ when that function is $> -\infty$ there, one may resort to the criterion provided by the *first* of the preceding theorems. According to the latter, it is enough to show that if $z_0 \in \mathscr{D}$ and Δ is any disk centred at z_0 with radius $< \operatorname{dist}(z_0, \partial\mathscr{D})$, we have $V(z_0) \geqslant h(z_0)$ for each function $h(z)$ *continuous* on $\bar{\Delta}$, *harmonic* in Δ, and satisfying $h(\zeta) \leqslant V(\zeta)$ on $\partial\Delta$. But for any such function h we certainly have $h(\zeta) \leqslant U(\zeta)$ on $\partial\Delta$ for every $U \in \mathscr{F}$, so, by the second part of the theorem referred to, $h(z) \leqslant U(z)$ in Δ for those U. Hence

$$h(z) \quad \leqslant \quad \inf_{U \in \mathscr{F}} U(z) \quad = \quad W(z)$$

in Δ, and finally, h being continuous at z_0 (the centre of Δ !),

$$h(z_0) \quad = \quad \lim_{z \to z_0} h(z) \quad \leqslant \quad \liminf_{z \to z_0} W(z) \quad = \quad V(z_0),$$

as required. We are done.

Remark. This theorem, together with the second of those preceding it, forms the basis for what is known as *Perron's method* of solution of the Dirichlet problem.

2. The Riesz representation of superharmonic functions

A superharmonic function can be approximated from below by others which are also infinitely differentiable. This is obvious for the function $U(z)$ identically infinite in a domain $\mathscr{D}$, that one being just the limit, as $n \longrightarrow \infty$, of the *constant* functions $U_n(z) = n$. We therefore turn to the construction of such approximations to functions $U(z)$ superharmonic and $\not\equiv \infty$ in $\mathscr{D}$.

Given such a U, one starts by forming the means

$$U_\rho(z) \quad = \quad \frac{1}{2\pi}\int_0^{2\pi} U(z + \rho e^{i\vartheta})\, d\vartheta;$$

* in the case where $V(z_0) > -\infty$; otherwise property (i) clearly does hold at z_0

when $\rho > 0$ is given, these are defined for the z in $\mathscr{D}$ with $\operatorname{dist}(z, \partial\mathscr{D}) > \rho$. According to property (ii) from our definition,

$$U_\rho(z) \leqslant U(z)$$

for all sufficiently small $\rho > 0$ (and in fact for *all* such $\rho < \operatorname{dist}(z, \partial\mathscr{D})$ by the boxed inequality near the end of the preceding article); on the other hand, $\liminf_{\rho\to 0} U_\rho(z) \geqslant U(z)$ by property (i). Thus, for each $z \in \mathscr{D}$,

$$U_\rho(z) \longrightarrow U(z) \quad \text{as } \rho \longrightarrow 0.$$

A lemma from the last article shows that this convergence is actually *monotone*; the $U_\rho(z)$ *increase* as ρ *diminishes* towards 0.

Concerning the U_ρ, we have the useful

Lemma. *If $U(z)$ is superharmonic in a (connected) domain $\mathscr{D}$ and not identically infinite there, the $U_\rho(z)$ are finite for $z \in \mathscr{D}$ and* $0 < \rho < \operatorname{dist}(z, \partial\mathscr{D})$.

Proof. Suppose that

$$U_r(z_0) = \frac{1}{2\pi}\int_0^{2\pi} U(z_0 + re^{i\tau})\,d\tau = \infty$$

for some $z_0 \in \mathscr{D}$ and an r with $0 < r < \operatorname{dist}(z_0, \partial\mathscr{D})$. It is claimed that then $U(z) \equiv \infty$ in $\mathscr{D}$.

By one of our first observations about superharmonic functions in the preceding article, $U(z_0 + re^{i\tau})$ is *bounded below* for $0 \leqslant \tau \leqslant 2\pi$. The above relation therefore makes the Poisson integrals occurring in the boxed inequality near the end of that article *infinite*, and we must have $U(z) \equiv \infty$ for $|z - z_0| < r$.

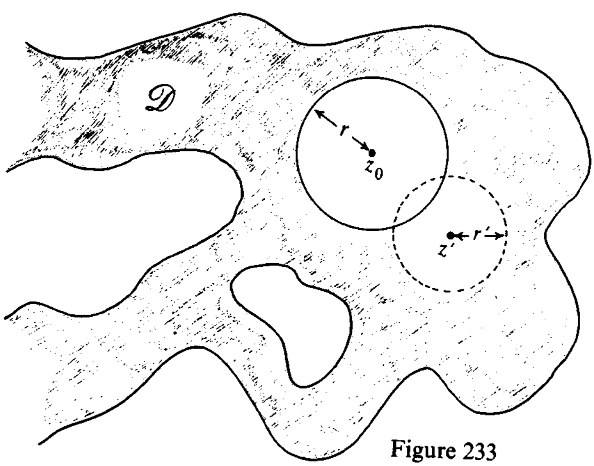


Figure 233

Let now z' be any point in $\mathscr{D}$ about which one can draw a circle – of radius r', say – *lying entirely* in $\mathscr{D}$ and also intersecting the open disk of radius r centred at z_0*. Since $U(z)$ is bounded below on that circle, we have $U_{r'}(z') = \infty$ so, by the argument just made, $U(z) \equiv \infty$ for $|z - z'| < r'$.

The process may evidently be continued indefinitely so as to gradually fill out the connected open region $\mathscr{D}$. In that way, one sees that $U(z) \equiv \infty$ therein, and the lemma is proved.

Corollary. *If $U(z)$ is superharmonic and not identically infinite in a (connected) domain $\mathscr{D}$, it is locally L_1 there (with respect to Lebesgue measure for $\mathbb{R}^2$).*

Proof. It is enough to verify that if $z_0 \in \mathscr{D}$ and $0 < r < \frac{1}{2}\text{dist}(z_0, \partial\mathscr{D})$, we have

$$\iint_{r \leqslant |z-z_0| \leqslant 2r} |U(z)|\,\mathrm{d}x\,\mathrm{d}y \quad < \quad \infty,$$

for, since each point of $\mathscr{D}$ lies in the *interior* of some annulus like the one over which the integral is taken, any compact subset of $\mathscr{D}$ can be covered by a finite number of such annuli.

By the lower bound property already used so often, there is an $M < \infty$ such that $U(z) \geqslant -M$ when $|z - z_0| \leqslant 2r$. The preceding integral is therefore

$$\leqslant \quad \iint_{r \leqslant |z-z_0| \leqslant 2r} (\,U(z) + 2M\,)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \quad 6\pi r^2 M \quad + \quad \int_r^{2r}\int_0^{2\pi} U(z_0 + \rho e^{i\vartheta})\rho\,\mathrm{d}\vartheta\,\mathrm{d}\rho$$

$$= \quad 6\pi r^2 M \ + \ 2\pi \int_r^{2r} U_\rho(z_0)\rho\,\mathrm{d}\rho.$$

$U_\rho(z_0)$ is, as noted above, a *decreasing* function of ρ; the last expression is thus

$$\leqslant \quad 6\pi r^2 M \ + \ 3\pi r^2 U_r(z_0).$$

This, however, is finite by the lemma.

We are done.

With the means $U_\rho(z)$ at hand, we continue our construction of superharmonic $\mathscr{C}_\infty$ approximations to a given $U(z)$ superharmonic and $\not\equiv \infty$ in a domain $\mathscr{D}$. For this purpose, one chooses any function $\varphi(\rho)$ infinitely

* with its interior

differentiable on $(0, \infty)$, *identically zero outside* $(1, 2)$ *and* > 0 *on that interval*, normalized so as to make

$$\int_1^2 \varphi(\rho)\rho \, d\rho \quad = \quad 1 :$$

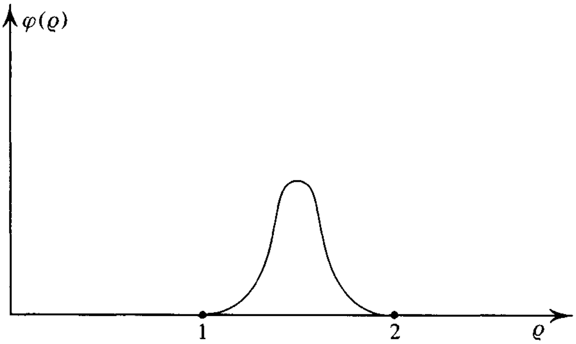


Figure 234

(As we shall see, it turns out to be convenient to work with a function $\varphi(\rho)$ *vanishing for the values of* ρ *near* 0 as well as for the large ones.) Using φ, one then forms *averages of the means* U_ρ :

$$(\Phi_r U)(z) \quad = \quad \frac{1}{r^2}\int_0^\infty U_\rho(z)\,\varphi(\rho/r)\rho \, d\rho;$$

for given $z \in \mathscr{D}$, these are defined when

$$0 \; < \; r \; < \; \tfrac{1}{2}\operatorname{dist}(z, \; \partial\mathscr{D}).$$

The $\Phi_r U$ are the approximations we set out to obtain; one has, namely, the

Theorem. *Given* $r > 0$, *denote by* $\mathscr{D}_r$ *the set of* $z \in \mathscr{D}$ *with* $\operatorname{dist}(z, \; \partial\mathscr{D}) \; > \; 2r$. *Let U be superharmonic in* $\mathscr{D}$, *then*:

$$(\Phi_r U)(z) \; \leqslant \; U(z) \quad \text{for } z \in \mathscr{D}_r;$$

$$(\Phi_r U)(z) \; \longrightarrow \; U(z) \quad \text{as } r \longrightarrow 0 \text{ for each } z \in \mathscr{D};$$

$$(\Phi_{2r} U)(z) \; \leqslant \; (\Phi_r U)(z) \quad \text{for } z \in \mathscr{D}_{2r}.$$

If also $U(z) \not\equiv \infty$ *in the (connected) domain* $\mathscr{D}$, *each* $(\Phi_r U)(z)$ *is infinitely differentiable in the corresponding* $\mathscr{D}_r$, *and superharmonic in each connected component thereof.*

Proof. The *first two* properties of the $\Phi_r U$ follow as direct consequences of the behaviour, noted above, of the $U_\rho(z)$ together with φ's normalization. The *third* is then assured by $\varphi(\rho)$'s being supported on the interval $(1, 2)$.

Passing to the *superharmonicity* of $\Phi_r U$, we first check property (ii) for that function in $\mathscr{D}_r$. This does not depend on the condition that $U(z) \not\equiv \infty$. Fix any $z \in \mathscr{D}_r$. For $0 < \sigma < \operatorname{dist}(z, \partial\mathscr{D}) - 2r$ we then have

$$\frac{1}{2\pi}\int_0^{2\pi} (\Phi_r U)(z + \sigma e^{i\psi})\, d\psi$$

$$= \frac{1}{4\pi^2 r^2}\int_0^{2\pi}\int_0^{\infty}\int_0^{2\pi} U(z + \rho e^{i\vartheta} + \sigma e^{i\psi})\,\varphi(\rho/r)\rho\, d\vartheta\, d\rho\, d\psi.$$

Since $\varphi(\rho/r)$ vanishes for $\rho \geqslant 2r$, $z + \rho e^{i\vartheta}$ lies in $\mathscr{D}$ and has distance $> \sigma$ from $\partial\mathscr{D}$ for all the values of ρ actually involved in the second expression. The argument $z + \rho e^{i\vartheta} + \sigma e^{i\psi}$ of U thus ranges over a *compact subset* of $\mathscr{D}$ in that triple integral, and on such a subset U is *bounded below*, as we know. This makes it permissible for us to *perform first the integration with respect to* ψ. Doing that, and using the boxed inequality from the preceding article, we obtain a value

$$\leqslant \frac{1}{2\pi r^2}\int_0^{\infty}\int_0^{2\pi} U(z + \rho e^{i\vartheta})\,\varphi(\rho/r)\rho\, d\vartheta\, d\rho \quad = \quad (\Phi_r U)(z),$$

showing that $\Phi_r U$ has property (ii) at z. Superharmonicity of $\Phi_r U$ in the components of $\mathscr{D}_r$ thus follows if it meets our definition's other two requirements there.

Satisfaction of the latter is, however, obviously guaranteed by the *infinite differentiability* of $\Phi_r U$ in $\mathscr{D}_r$, which we now proceed to verify for functions $U(z) \not\equiv \infty$ in $\mathscr{D}$.

The left-hand member of the last relation can be rewritten as

$$\frac{1}{2\pi r^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(z + \zeta)\,\varphi(|\zeta|/r)\, d\xi\, d\eta,$$

where, as usual, $\zeta = \xi + i\eta$. Putting $z + \zeta = \zeta' = \xi' + i\eta'$, this becomes

$$\frac{1}{2\pi r^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(\zeta')\,\varphi(|\zeta' - z|/r)\, d\xi'\, d\eta'.$$

Here, $\varphi(|\zeta' - z|/r)$ vanishes for $|\zeta' - z| \leqslant r$ and $|\zeta' - z| \geqslant 2r$. Looking, then, at values of z near some *fixed* $z_0 \in \mathscr{D}_r$ – to be definite, at those, say, with

$$|z - z_0| \quad < \quad \delta \quad = \quad \frac{1}{2}\min(r,\ \operatorname{dist}(z_0, \partial\mathscr{D}) \ - \ 2r),$$

we have

$$(\Phi_r U)(z) = \frac{1}{2\pi r^2} \iint_{r-\delta \leqslant |\zeta' - z_0| \leqslant 2r+\delta} U(\zeta')\,\varphi(|z-\zeta'|/r)\,d\xi'\,d\eta'.$$

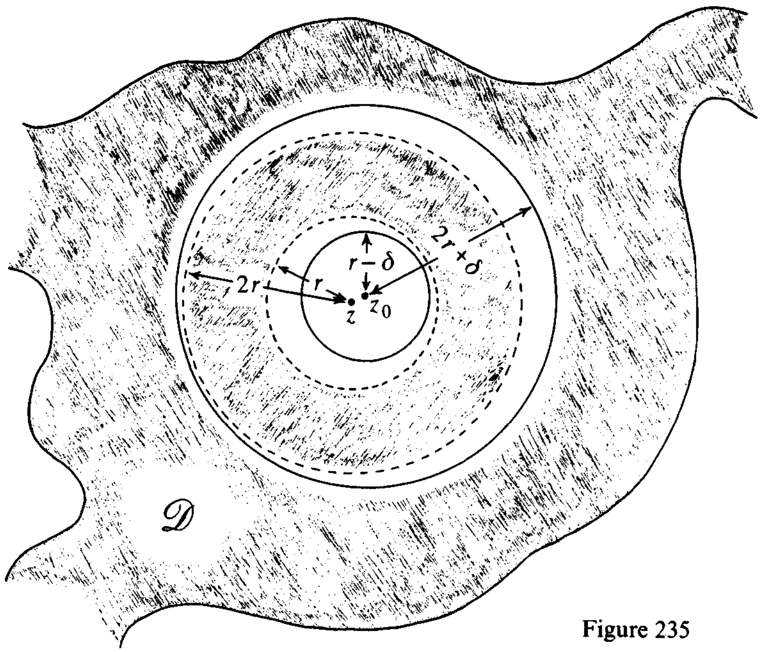


Figure 235

The annulus $K = \{\zeta' : r-\delta \leqslant |\zeta' - z_0| \leqslant 2r+\delta\}$ over which the integration is carried out here is a *compact* subset of $\mathscr{D}$ (independent of z !), so, for the function U under consideration we have

$$\iint_K |U(\zeta')|\,d\xi'\,d\eta' < \infty$$

by the above corollary. Infinite differentiability of $(\Phi_r U)(z)$ at z_0 can therefore be *read off by inspection* from the last formula, provided that $\varphi(|z-\zeta'|/r)$ enjoys the same property for *each* $\zeta' \in K$ (differentiation inside the integral signs). That, however, is indeed the case, as follows by the chain rule from infinite differentiability of φ and the fact that $|z_0 - \zeta'| \geqslant \delta > 0$ for each $\zeta' \in K$. (Here we have been helped by $\varphi(\rho)$'s vanishing for $0 < \rho < 1$.) $\Phi_r U$ is thus $\mathscr{C}_\infty$ in $\mathscr{D}_r$.

The theorem is proved.

The approximations $\Phi_r U$ to a given superharmonic function U are used in establishing the *Riesz representation* for the latter. That says essentially that a function $U(z)$ superharmonic and $\not\equiv \infty$ *in and on* a bounded domain $\mathscr{D}$ (i.e., in a domain *including* $\bar{\mathscr{D}}$) is given there by a *formula*

$$U(z) \quad = \quad \int_{\bar{\mathscr{D}}} \log \frac{1}{|z-\zeta|} \, \mathrm{d}\mu(\zeta) \quad + \quad H(z),$$

with μ a (finite) positive measure on $\bar{\mathscr{D}}$ and $H(z)$ *harmonic* in $\mathscr{D}$. (Conversely, expressions like the one on the right *are* always superharmonic in $\mathscr{D}$, according to the remarks following the first lemma of the preceding article.)

The representation is really of *local character*, for the restriction of the measure μ figuring in it to *any open disk* $\Delta \subseteq \mathscr{D}$ is *completely determined by the behaviour of* U *in* Δ (see problem 48 below), and at the same time, the function of z equal to

$$\int_{\mathscr{D} \sim \Delta} \log \frac{1}{|z-\zeta|} \; \mathrm{d}\mu(\zeta)$$

is certainly *harmonic* in Δ. The general form of the result can thus be obtained from *a special version of it for disks* by simply pasting some of those together so as to cover the given domain $\mathscr{D}$! In fact, *only the version for disks* will be required in the present chapter, so that is what we prove here. Passage from it to the more general form is left as an exercise to the reader (problem 49).

We proceed, then, to the derivation of the Riesz representation formula for disks. The idea is to first get it for $\mathscr{C}_\infty$ superharmonic functions by simple application of Green's theorem and then pass from those to the general ones with the help of the $\Phi_r U$. In this, an essential rôle is played by the classical

Lemma. *A function $V(z)$ infinitely differentiable in a domain $\mathscr{D}$ is superharmonic there if and only if*

$$\frac{\partial^2 V(z)}{\partial x^2} + \frac{\partial^2 V(z)}{\partial y^2} \quad \leqslant \quad 0 \quad \text{for } z \in \mathscr{D}.$$

Notation. The Laplacian $\partial^2/\partial x^2 + \partial^2/\partial y^2$ is denoted by ∇^2 (following earlier usage in this book).

Proof of lemma. Supposing that V is superharmonic in $\mathscr{D}$, we take any point z_0 therein. Then, by the third lemma of the preceding article,

$$\int_0^{2\pi} V(z_0 + \rho e^{i\vartheta}) \, \mathrm{d}\vartheta$$

is a *decreasing* function of ρ for $0 < \rho < \operatorname{dist}(z_0, \partial\mathscr{D})$. The $\mathscr{C}_\infty$ character of V makes it possible for us to differentiate this expression under the integral sign with respect to ρ, so we have

$$\int_0^{2\pi} \frac{\partial V(z_0 + \rho e^{i\vartheta})}{\partial \rho} d\vartheta \leqslant 0$$

for small positive values of that parameter.

By Green's theorem, however,

$$\iint_{|z-z_0|<\rho} (\nabla^2 V)(z) \, dx \, dy = \int_0^{2\pi} \frac{\partial V(z_0 + \rho e^{i\vartheta})}{\partial \rho} \rho \, d\vartheta;$$

the left-hand integral is thus *negative*. Finally,

$$(\nabla^2 V)(z_0) = \lim_{\rho \to 0} \frac{1}{\pi \rho^2} \iint_{|z-z_0|<\rho} (\nabla^2 V)(z) \, dx \, dy,$$

showing that $\nabla^2 V \leqslant 0$ at z_0.

Assuming, on the other hand, that $\nabla^2 V \leqslant 0$ in $\mathscr{D}$, we see by the second of the above relations that

$$\int_0^{2\pi} V(z_0 + \rho e^{i\vartheta}) \, d\vartheta$$

is a decreasing function of ρ for $0 < \rho < \operatorname{dist}(z_0, \partial\mathscr{D})$ when $z_0 \in \mathscr{D}$. At the same time,

$$V(z_0) = \lim_{\rho \to 0} \frac{1}{2\pi} \int_0^{2\pi} V(z_0 + \rho e^{i\vartheta}) \, d\vartheta$$

in the present circumstances, so $V(z_0)$ must be $\geqslant$ each of the means figuring on the right for the values of ρ just indicated. This establishes property (ii) for V at z_0 and hence the superharmonicity of V in $\mathscr{D}$.

The lemma is proved.

Here is the version of Riesz' result that we will be using. It is most convenient to obtain a representation differing slightly in appearance from the one written above, but equivalent to the latter. About this, more in the remark following the proof.

Theorem (F. Riesz). *Let $U(z)$ be superharmonic and $\not\equiv \infty$ in a domain $\mathscr{D}$, and suppose that $z_0 \in \mathscr{D}$ and $0 < r < \operatorname{dist}(z_0, \partial\mathscr{D})$. Then, for $|z - z_0| < r$, one has*

$$U(z) = \int_{|\zeta - z_0| \leqslant r} \log \left| \frac{r^2 - (\overline{\zeta - z_0})(z - z_0)}{r(z - \zeta)} \right| d\mu(\zeta) + h(z),$$

where μ is a finite positive measure on the closed disk $\{|z - z_0| \leqslant r\}$, and $h(z)$ a function harmonic for $|z - z_0| < r$.

Remark. In the integrand we simply have the *Green's function* associated with the disk $\{|z - z_0| < r\}$. The integral is therefore frequently referred to as a *pure Green potential* for that disk – 'pure' because the *measure* μ is *positive.*

Proof of theorem. To simplify the writing, we take $z_0 = 0$ and $r = 1$ – *that also frees the letter r for another use during this proof*! For some $R > 1$, the closed disk

$$\bar{\Delta} \;=\; \{|z| \leqslant R\}$$

lies in $\mathscr{D}$, and the averages $\Phi_r U$ introduced previously are hence *defined, infinitely differentiable* and *superharmonic in and on* $\bar{\Delta}$ when the parameter r (not to be confounded with the radius of the disk for which our representation is being derived!) is small enough. We *fix* such an r, and *denote* $\Phi_r U$ *by* V *for the time being* (again to help keep the notation clear).

Fix also any z, $|z| < R$, for the moment. The Green's function

$$\log\left|\frac{R^2 - \zeta\bar{z}}{R(\zeta - z)}\right|$$

is *harmonic* in ζ for $|\zeta| < R$ and $\zeta \neq z$; it is, besides, *zero* when $|\zeta| = R$. From this we see by applying Green's theorem in the region $\{\zeta\colon |\zeta - z| > \rho$ and $|\zeta| < R\}$ and afterwards causing ρ to tend to zero (cf. beginning of the proof of symmetry of the Green's function, end of

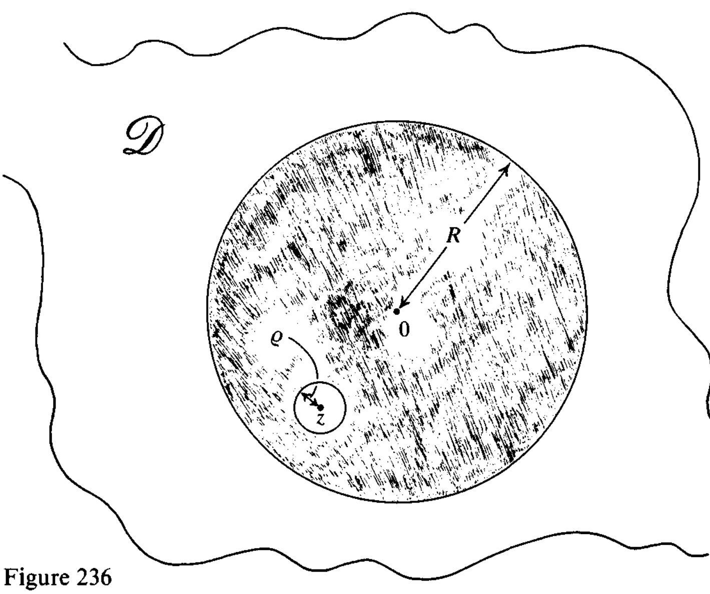


Figure 236

§A.2, Chapter VIII), that

$$V(z) \quad = \quad -\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial}{\partial\sigma}\log\left|\frac{R^2-\sigma e^{i\vartheta}\bar{z}}{R(\sigma e^{i\vartheta}-z)}\right|\right)_{\sigma=R} V(Re^{i\vartheta})R\,d\vartheta$$

$$-\quad \frac{1}{2\pi}\iint_{|\zeta|<R}\log\left|\frac{R^2-\zeta\bar{z}}{R(\zeta-z)}\right|(\nabla^2 V)(\zeta)\,d\xi\,d\eta.$$

Working out the partial derivative in the first integral on the right, we get

$$V(z) \quad = \quad \frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-|z|^2}{|z-Re^{i\vartheta}|^2}\,V(Re^{i\vartheta})\,d\vartheta$$

$$-\quad \frac{1}{2\pi}\iint_{|\zeta|<R}\log\left|\frac{R^2-\zeta\bar{z}}{R(\zeta-z)}\right|(\nabla^2 V)(\zeta)\,d\xi\,d\eta;$$

this, then, holds for each z of modulus $< R$.

Here, in the first integrand, we recognize the *Poisson kernel* for the disk $\{|z| < R\}$ (that's where the kernel *comes* from!); the *first* right-hand term is hence equal to a function *harmonic in that disk.* In the *second* term on the right, $(\nabla^2 V)(\zeta)$ is *negative according to the preceding lemma*, V being superharmonic in and on $\{|z| \leqslant R\}$. The last relation is therefore a *formula of the kind we are seeking to establish*, representing, however, the $\mathscr{C}_\infty$ superharmonic approximaton $V = \Phi_r U$ to our original superharmonic function U instead of U itself. We wish now to arrive at the desired formula for U by making $r \longrightarrow 0$.

With that in mind, we rearrange the preceding relation, writing $\Phi_r U$ in place of V:

$$-\frac{1}{2\pi}\iint_{|\zeta|<R}(\nabla^2\Phi_r U)(\zeta)\log\left|\frac{R^2-\zeta\bar{z}}{R(z-\zeta)}\right|d\xi\,d\eta$$

$$= \quad (\Phi_r U)(z) \quad - \quad \frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-|z|^2}{|z-Re^{i\vartheta}|^2}(\Phi_r U)(Re^{i\vartheta})\,d\vartheta, \qquad |z| \;<\; R.$$

From this, we proceed to deduce the *boundedness of*

$$-\iint_{|\zeta|\leqslant 1}(\nabla^2\Phi_r U)(\zeta)\,d\xi\,d\eta$$

for r *tending to zero.*

Here we use the *first* lemma of this article, according to which there are points z *arbitrarily close to* 0 *for which* $U(z) < \infty$, it having been given that $U \not\equiv \infty$ in $\mathscr{D}$. *Fixing* such a z, of modulus $< 1/2$, say, and denoting it by the letter c, we have, from the preceding theorem,

$$(\Phi_r U)(c) \;\leqslant\; U(c),$$

so, by the last formula,

$$\frac{1}{2\pi}\iint_{|\zeta|<R} (-\nabla^2\Phi_r U)(\zeta)\log\left|\frac{R^2-\zeta\bar{c}}{R(\zeta-c)}\right| d\xi\, d\eta$$
$$\leqslant \quad U(c) \quad - \quad \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-|c|^2}{|Re^{i\vartheta}-c|^2}(\Phi_r U)(Re^{i\vartheta})\,d\vartheta$$

for sufficiently small $r > 0$.

Now $(\Phi_r U)(Re^{i\vartheta})$ is by our construction an *average* of $U(\zeta)$ over the little annulus $r < |\zeta - Re^{i\vartheta}| < 2r$, and the union of these annuli for $0 \leqslant \vartheta \leqslant 2\pi$ is contained in the disk $\{|\zeta| \leqslant R+2r\}$. The latter, in turn, is contained in a *fixed* disk $\{|\zeta| \leqslant R'\}$ slightly larger than $\{|\zeta| \leqslant R\}$ for values of $r < (R'-R)/2$. R, however, was chosen so as to make the disk $\{|\zeta| \leqslant R\}$ lie in $\mathscr{D}$; we may thus take $R' > R$ close enough to R to ensure that $\{|\zeta| \leqslant R'\}$ is also in $\mathscr{D}$. Once this is done, we know there is a finite M with $U(\zeta) \geqslant -M$ for $|\zeta| \leqslant R'$; this, then, holds in particular on the little annuli first mentioned when $2r < R'-R$. For the averages $\Phi_r U$ corresponding to those values of r we therefore have

$$(\Phi_r U)(Re^{i\vartheta}) \quad \geqslant \quad -M, \qquad 0 \leqslant \vartheta \leqslant 2\pi.$$

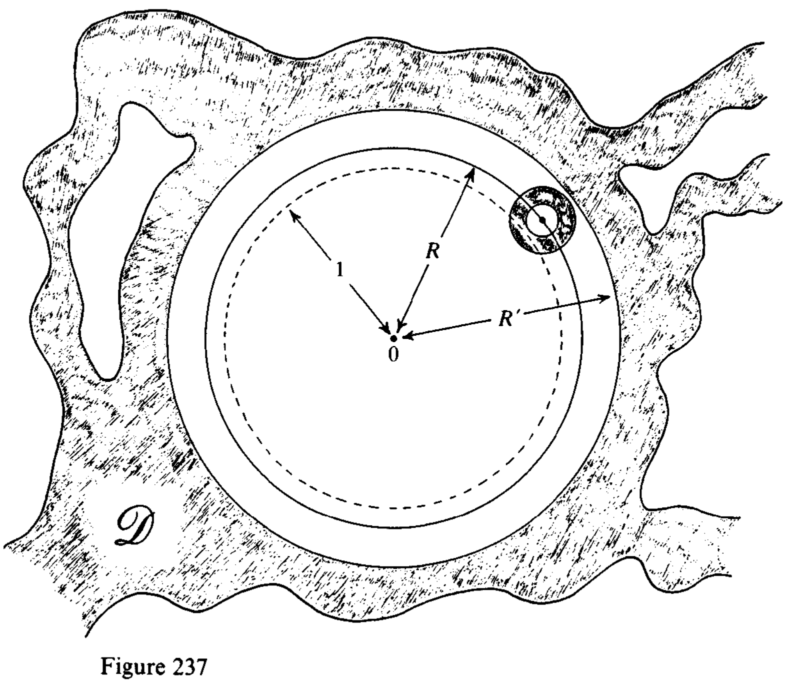


Figure 237

At the same time $R > 1$, so the expression

$$\log\left|\frac{R^2 - \zeta\bar{c}}{R(\zeta - c)}\right|,$$

positive for $|\zeta| < R$, is actually $\geqslant$ some $k > 0$ for $|\zeta| \leqslant 1$; meanwhile, $(-\nabla^2\Phi_r U)(\zeta) \geqslant 0$ for $|\zeta| \leqslant R$ as we know, when $r > 0$ is sufficiently small. Use these relations in the *left side* of the above inequality, and plug the previous one into the *right-hand* integral figuring in the latter. It is found that

$$\frac{k}{2\pi}\iint_{|\zeta|\leqslant 1}(-\nabla^2\Phi_r U)(\zeta)\,d\xi\,d\eta \quad\leqslant\quad U(c) \;+\; M,$$

a finite quantity, for $r > 0$ small enough. The integral on the left thus does remain bounded as $r \longrightarrow 0$.

By this boundedness we see, keeping positivity of the functions $-\nabla^2\Phi_r U$ in mind, that there is a certain *positive measure* μ *on* $\{|\zeta| \leqslant 1\}$ such that, *on the closed unit disk*,

$$-\frac{1}{2\pi}(\nabla^2\Phi_r U)(\zeta)\,d\xi\,d\eta \quad\longrightarrow\quad d\mu(\zeta) \qquad w^*$$

as $r \longrightarrow 0$ through a certain sequence of values r_n (cf. §F.1 of Chapter III, where the same kind of argument is used). There is no loss of generality in our taking $r_{n+1} < r_n/2$; this will permit us to take advantage of the relation $\Phi_{2r}U \leqslant \Phi_r U$.

Let us now rewrite *for the unit disk* the representation of the $\Phi_r U$ derived above for $\{|z| < R\}$. That takes the form

$$\begin{aligned}(\Phi_r U)(z) \quad &= \quad -\frac{1}{2\pi}\iint_{|\zeta|\leqslant 1}\log\left|\frac{1 - z\bar{\zeta}}{z - \zeta}\right|(\nabla^2\Phi_r U)(\zeta)\,d\xi\,d\eta \\ &+ \quad \frac{1}{2\pi}\int_0^{2\pi}\frac{1 - |z|^2}{|z - e^{i\vartheta}|^2}(\Phi_r U)(e^{i\vartheta})\,d\vartheta, \qquad |z| < 1\end{aligned}$$

(assuming, of course, as always that $r > 0$ is sufficiently small). Fixing any z of modulus < 1, we let r tend to 0 through the sequence $\{r_n\}$. According to the preceding theorem, $(\Phi_r U)(z)$ will then tend to $U(z)$, and, since $r_n > 2r_{n+1}$, $(\Phi_r U)(e^{i\vartheta})$ will, for each ϑ, *increase monotonically*, tending to $U(e^{i\vartheta})$. The *second* integral on the right will thus tend to

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1 - |z|^2}{|z - e^{i\vartheta}|^2}\,U(e^{i\vartheta})\,d\vartheta$$

by the monotone convergence theorem. We desire at this point to deduce simultaneous convergence of the *first* term on the right to

$$\int_{|\zeta|\leqslant 1} \log\left|\frac{1-z\bar{\zeta}}{z-\zeta}\right| d\mu(\zeta)$$

from the w^* convergence just described, since that would complete the proof.

That, however, involves a slight difficulty, for, as a function of ζ,

$$\log\left|\frac{1-\bar{\zeta}z}{\zeta-z}\right|$$

is discontinuous at $\zeta = z$. To deal with this, we first break up the preceding formula for $\Phi_r U$ in the following way:

$$\begin{aligned}(\Phi_r U)(z') \quad = \quad & -\frac{1}{2\pi}\iint_{|\zeta|\leqslant 1} \log\frac{1}{|z'-\zeta|}\,(\nabla^2\Phi_r U)(\zeta)\,d\xi\,d\eta \\ & - \frac{1}{2\pi}\iint_{|\zeta|\leqslant 1} \log|1-z'\bar{\zeta}|\,(\nabla^2\Phi_r U)(\zeta)\,d\xi\,d\eta \\ & + \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z'|^2}{|z'-e^{i\vartheta}|^2}\,(\Phi_r U)(e^{i\vartheta})\,d\vartheta, \qquad |z'| < 1.\end{aligned}$$

Keeping z, of modulus < 1, fixed, we take z' in this relation equal to $z + \rho e^{i\psi}$, with $0 < \rho < 1-|z|$, and then integrate with respect to ψ on both sides, from 0 to 2π.

When $|\zeta| \leqslant 1$, $\log|1-z'\bar{\zeta}|$ is *harmonic* in z' for $|z'| < 1$; we thus have

$$\frac{1}{2\pi}\int_0^{2\pi} \log|1 - (z+\rho e^{i\psi})\bar{\zeta}|\,d\psi \quad = \quad \log|1-z\bar{\zeta}|$$

for each such ζ and $0 < \rho < 1-|z|$. In like manner,

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1 - |z+\rho e^{i\psi}|^2}{|z + \rho e^{i\psi} - e^{i\vartheta}|^2}\,d\psi \quad = \quad \frac{1-|z|^2}{|z-e^{i\vartheta}|^2}$$

for the indicated values of ρ. There is, finally, the elementary formula

$$\frac{1}{2\pi}\int_0^{2\pi} \log\frac{1}{|z+\rho e^{i\psi}-\zeta|}\,d\psi \quad = \quad \min\left(\log\frac{1}{|z-\zeta|},\ \log\frac{1}{\rho}\right)$$

already mentioned in the last article.

With the help of these relations we find by integration of the parameter

ψ that

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi}(\Phi_r U)(z+\rho e^{i\psi})\,d\psi \\
&= \; -\frac{1}{2\pi}\iint_{|\zeta|\leq 1}\min\left(\log\frac{1}{|z-\zeta|},\ \log\frac{1}{\rho}\right)(\nabla^2\Phi_r U)(\zeta)\,d\xi\,d\eta \\
&\quad -\ \frac{1}{2\pi}\iint_{|\zeta|\leq 1}\log|1-z\bar{\zeta}|\,(\nabla^2\Phi_r U)(\zeta)\,d\xi\,d\eta \\
&\quad +\ \frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|z-e^{i\vartheta}|^2}(\Phi_r U)(e^{i\vartheta})\,d\vartheta
\end{aligned}$$

for $|z| < 1$, $0 < \rho < 1-|z|$, and r sufficiently small.

Fix now such values of z and ρ, and make $r \longrightarrow 0$ through the sequence of values r_n. The *third* integral on the right in the present relation then tends to

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|z-e^{i\vartheta}|^2}U(e^{i\vartheta})\,d\vartheta$$

as observed above, and the *left side* tends to

$$\frac{1}{2\pi}\int_0^{2\pi}U(z+\rho e^{i\psi})\,d\psi \quad = \quad U_\rho(z)$$

for the same reason (monotone convergence). Here, the functions of ζ involved in the *first two integrals* on the right *are* continuous on the closed unit disk. This allows us to conclude from the w^* convergence described above that those integrals tend respectively to

$$\iint_{|\zeta|\leq 1}\min\left(\log\frac{1}{|z-\zeta|},\ \log\frac{1}{\rho}\right)d\mu(\zeta)$$

and to

$$\iint_{|\zeta|\leq 1}\log|1-z\bar{\zeta}|\,d\mu(\zeta).$$

Putting these observations together, we see that

$$\begin{aligned}
U_\rho(z) \quad = \quad &\iint_{|\zeta|\leq 1}\min\left(\log\left|\frac{1-z\bar{\zeta}}{z-\zeta}\right|,\ \log\frac{|1-z\bar{\zeta}|}{\rho}\right)d\mu(\zeta) \\
&+\ \frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|z-e^{i\vartheta}|^2}U(e^{i\vartheta})\,d\vartheta
\end{aligned}$$

for $|z| < 1$ and $0 < \rho < 1-|z|$.

We finally let $\rho \longrightarrow 0$, continuing to hold z fixed. Then, as noted at the very beginning of this article, $U_\rho(z) \longrightarrow U(z)$. At the same time, the first right-hand integral in the formula just written tends to

$$\iint_{|\zeta| \leqslant 1} \log \left| \frac{1 - z\bar{\zeta}}{z - \zeta} \right| d\mu(\zeta)$$

by monotone convergence! We therefore have

$$U(z) \quad = \quad \iint_{|\zeta| \leqslant 1} \log \left| \frac{1 - z\bar{\zeta}}{z - \zeta} \right| d\mu(\zeta) \quad + \quad \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|z - e^{i\vartheta}|^2} U(e^{i\vartheta}) \, d\vartheta$$

for $|z| < 1$. The function

$$h(z) \quad = \quad \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|z - e^{i\vartheta}|^2} U(e^{i\vartheta}) \, d\vartheta$$

is *harmonic* in the open unit disk. Our theorem is thus proved.

Remark. The representation just obtained is frequently written differently. Taking, to simplify the notation, $z_0 = 0$, what we have so far reads

$$U(z) \quad = \quad \int_{|\zeta| \leqslant r} \log \left| \frac{r^2 - z\bar{\zeta}}{r(z - \zeta)} \right| d\mu(\zeta) \quad + \quad h(z), \qquad |z| \leqslant r,$$

with $h(z)$ a certain function *harmonic* in $\{|z| < r\}$. Under the circumstances of the theorem (U superharmonic in a *slightly larger* disk), we even have an explicit formula for h,

$$h(z) \quad = \quad \frac{1}{2\pi} \int_0^{2\pi} \frac{r^2 - |z|^2}{|z - re^{i\vartheta}|^2} U(re^{i\vartheta}) \, d\vartheta, \qquad |z| < r,$$

found at the end of the above proof.

The integral

$$\int_{|\zeta| \leqslant r} \log |r^2 - z\bar{\zeta}| \; d\mu(\zeta),$$

however, is itself a harmonic function of z for $|z| < r$. The preceding relation can be thus rewritten as

$$\boxed{U(z) \quad = \quad \int_{|\zeta| \leqslant r} \log \frac{1}{|z - \zeta|} \, d\mu(\zeta) \quad + \quad H(z), \qquad |z| < r,}$$

with

$$H(z) \quad = \quad h(z) \quad + \quad \int_{|\zeta| \leqslant r} \left(\log |r^2 - z\bar{\zeta}| \quad + \quad \log \frac{1}{r} \right) \mathrm{d}\mu(\zeta)$$

also harmonic in the disk $\{|z| < r\}$. Here we recognize on the right the familiar *logarithmic potential* (corresponding to the (here finite) positive measure μ) which has already played a rôle in §F of Chapter IX. The simplicity of this version in comparison with the original one is somewhat offset by a drawback: $H(z)$, unlike $h(z)$, is no longer determined by the boundary values $U(re^{i\vartheta})$ alone. It is often easier, nevertheless, to work with the former rather than the latter.

As they stand, the two forms of the representation are *equivalent*, with the above relation between the harmonic functions h and H serving to pass from one to the other. As long as the (finite) measure μ is *positive*, and the function $H(z)$ *harmonic* in $\{|z| < r\}$, the boxed formula does give us a function $U(z)$ superharmonic there according to observations in the preceding article; this is also true of the other formula under the same circumstances regarding μ and h. Concerning, however, such a function U, with $H(z)$, say, known *only* to be harmonic for $|z| < r$, we can say nothing about the boundary values $U(re^{i\vartheta})$ (not even as regards their existence), and thus lose the above representation for the function $h(z)$ corresponding to H as a Poisson integral in $\{|z| < r\}$. In order to have the latter, some additional information about U is necessary, its superharmonicity in a *larger* disk, for instance (this in turn implied by harmonicity of H in such a disk).

Regarding the measure μ appearing in either version of Riesz' result one has the important

Theorem. *In the representation*

$$U(z) \quad = \quad \int_{|\zeta - z_0| \leqslant r} \log \frac{1}{|z - \zeta|} \, \mathrm{d}\mu(\zeta) \quad + \quad H(z), \qquad |z - z_0| \; < \; r,$$

of a function $U(z)$ superharmonic in and on $|z - z_0| \leqslant r$ (with μ positive on that disk and $H(z)$ harmonic in its interior), the measure μ has no mass in any open subset of the disk where $U(z)$ is harmonic.

Proof. Let $|z - z_0| < r$ and suppose that $U(z')$ is harmonic in and on the *closed* disk $|z' - z| \leqslant \rho$, where $0 < \rho < r - |z - z_0|$. By the mean value property we then have

$$\frac{1}{2\pi}\int_0^{2\pi} U(z+\rho e^{i\psi})\,d\psi \quad = \quad U(z).$$

$H(z)$, however, has also the mean value property. Hence, using once again the formula

$$\frac{1}{2\pi}\int_0^{2\pi} \log\frac{1}{|z+\rho e^{i\psi}-\zeta|}\,d\psi \quad = \quad \min\left(\log\frac{1}{|z-\zeta|},\ \log\frac{1}{\rho}\right)$$

together with the given representation for U, we see that the left-hand integral in the preceding relation equals

$$\int_{|\zeta-z_0|\leqslant r} \min\left(\log\frac{1}{|z-\zeta|},\ \log\frac{1}{\rho}\right)d\mu(\zeta) \quad + \quad H(z).$$

Subtracting $U(z)$ from this, we get

$$\int_{|\zeta-z_0|\leqslant r} \log^+\frac{\rho}{|\zeta-z|}\,d\mu(\zeta) \quad = \quad 0.$$

Therefore $\mu(\{|\zeta-z| \ < \ \rho\}) \ = \ 0,$ μ being positive. This does it.

Problem 48

(a) In the Riesz representation

$$U(z) \quad = \quad \int_{|\zeta-z_0|\leqslant r} \log\frac{1}{|z-\zeta|}\,d\mu(\zeta) \quad + \quad H(z), \qquad |z-z_0| \ < \ r,$$

of a function $U(z)$ superharmonic in and on $|\zeta-z_0| \leqslant r$ (with the measure μ positive and $H(z)$ harmonic for $|\zeta-z_0| < r$), the *restriction of μ to the open disk $|\zeta-z_0| < r$ is unique.* (Hint: If $F(z)$ is any continuous function supported on a *compact subset* of the open disk in question, we have

$$F(\zeta) \quad = \quad \lim_{\rho\to 0}\frac{2}{\pi\rho^2}\int\!\!\int_{|z-z_0|<r} F(z)\log^+\frac{\rho}{|\zeta-z|}\,dx\,dy$$

uniformly for $|\zeta-z_0| \ < \ r.$)

(b) In the representation for U written in (a), the function $H(z)$ harmonic for $|z-z_0| < r$ is *not unique* – it can be *altered* by letting μ have *more mass on the circle* $|\zeta-z_0| = r$. *Show uniqueness* for the function $h(z)$, harmonic in $\{|z-z_0| < r\}$, figuring in the *original form* of the Riesz representation of U in that disk:

$$U(z) \quad = \quad \int_{|\zeta - z_0| \leq r} \log\left|\frac{r^2 - (z-z_0)\overline{(\zeta - z_0)}}{r(z - \zeta)}\right| \mathrm{d}\mu(\zeta) \quad + \quad h(z).$$

(c) Let $U(z)$, superharmonic in a domain $\mathscr{D}$, have the Riesz representation

$$U(z) \quad = \quad \int_{|\zeta - z_0| \leq r_0} \log\frac{1}{|z-\zeta|}\,\mathrm{d}\mu_0(\zeta) \quad + \quad H_0(z),$$

$$U(z) \quad = \quad \int_{|\zeta - z_1| \leq r_1} \log\frac{1}{|z-\zeta|}\,\mathrm{d}\mu_1(\zeta) \quad + \quad H_1(z),$$

with $H_0(z)$ and $H_1(z)$ harmonic, in the respective disks $\{|z-z_0| < r_0\}$, $\{|z-z_1| < r_1\}$, whose *closures* lie in $\mathscr{D}$. Show that the positive measures μ_0 and μ_1 *agree* on the *intersection* of those *open disks* as long as it is non-empty. (Hint: The method followed in part (a) may be used.)

The last part of this problem gives us a procedure for extending the Riesz representation from disks to more general domains – the pasting argument referred to earlier.

Problem 49

Let $U(z)$ be superharmonic in a domain $\mathscr{D}$, and let Ω be any smaller domain with *compact closure* lying in $\mathscr{D}$. Corresponding to each open disk Δ whose closure lies in $\mathscr{D}$ we have, by the Riesz representation, a positive measure μ_Δ on $\bar{\Delta}$ and a function H_Δ harmonic in Δ such that

$$U(z) \quad = \quad \int_{\bar{\Delta}} \log\frac{1}{|z-\zeta|}\,\mathrm{d}\mu_\Delta(\zeta) \quad + \quad H_\Delta(z)$$

for $z \in \Delta$.

(a) Show that there is a (finite) positive measure μ on $\bar{\Omega}$ agreeing in each intersection $\bar{\Omega} \cap \Delta$ with the corresponding measure μ_Δ. (Hint: Use a finite covering of $\bar{\Omega}$ by some of the disks Δ and then refer to the result from problem 48(c).)

(b) Hence show that

$$U(z) \quad = \quad \int_{\bar{\Omega}} \log\frac{1}{|z-\zeta|}\,\mathrm{d}\mu(\zeta) \quad + \quad H(z)$$

for $z \in \Omega$, where $H(z)$ is harmonic in that domain and μ is the measure obtained in (a). (Hint: It suffices to show that for each $z_0 \in \Omega$,

$$U(z) \quad - \quad \int_{\Delta_0} \log\frac{1}{|z-\zeta|}\,\mathrm{d}\mu(\zeta)$$

is harmonic in Δ_0, a disk contained in Ω with centre at z_0.)

3. A maximum principle for pure logarithmic potentials. Continuity of such a potential when its restriction to generating measure's support has that property

Consider a function $U(z)$ superharmonic in and on $\{|z| \leqslant 1\}$ and thus having a Riesz representation

$$U(z) \quad = \quad \int_{|\zeta| \leqslant 1} \log \frac{1}{|z-\zeta|} \, d\mu(\zeta) \quad + \quad H(z)$$

(with μ positive and $H(z)$ harmonic) in the *open* unit disk. If U is actually *harmonic* in an open subset $\mathcal{O}$ of the latter, μ is in fact supported on the compact set

$$K \quad = \quad \{|\zeta| \leqslant 1\} \sim \mathcal{O}$$

according to the last theorem of the preceding article.

One is frequently interested in the *continuity* of $U(z)$ for $|z| < 1$. Because $H(z)$ is even harmonic for such z, the property in question is governed by the continuity of

$$\int_K \log \frac{1}{|z-\zeta|} \, d\mu(\zeta)$$

there. An important result of Evans and Vasilesco given in the present article guarantees the continuity of such a logarithmic potential (everywhere!), provided that *its restriction to the support K of μ* enjoys that property. This enables one to *exclude from consideration the open set $\mathcal{O}$ in which U is known to be harmonic* when checking for that function's continuity in the open unit disk.

The result referred to is based on a *version of the maximum principle*, of considerable interest in its own right.

Maria's theorem. *Let the (finite) positive measure μ be supported on a compact set K, and suppose that*

$$V(z) \quad = \quad \int_K \log \frac{1}{|z-\zeta|} \, d\mu(\zeta).$$

Then, if $V(z) \leqslant M$ at each $z \in K$, one has $V(z) \leqslant M$ in $\mathbb{C}$.

Remark. $V(z)$ *is*, of course *harmonic* in $\Omega = \mathbb{C} \sim K$ (and tends to $-\infty$ as $z \longrightarrow \infty$, unless $\mu \equiv 0$), but the theorem *does not follow without further work* from the ordinary maximum principle for harmonic functions. For $\zeta \in \partial\Omega \subseteq K$, *all* that the *elementary properties* of superharmonic functions tell us *directly* is that

$$\liminf_{z\to\zeta} V(z) \quad = \quad V(\zeta) \quad \leqslant \quad M.$$

If we only had limsup *on the left instead of* liminf, there would be no problem, but that's not what stands there! Such pitfalls abound in this subject.

Proof of theorem. We need only consider the situation where $M < \infty$, since otherwise the result is trivial. In that event, the quantities

$$V_\rho(z) \quad = \quad \int_K \min\left(\log\frac{1}{|z-\zeta|},\ \log\frac{1}{\rho}\right) \mathrm{d}\mu(\zeta)$$

increase, for *each* $z \in K$, to the *finite* limit $V(z)$ as $\rho \longrightarrow 0$. Given $\varepsilon > 0$, there is thus by *Egorov's theorem* a compact $E \subseteq K$ such that

$$V_\rho(z) \longrightarrow V(z) \quad \textit{uniformly} \text{ for } z \in E$$

as $\rho \longrightarrow 0$, and

$$\mu(K \sim E) \quad < \quad \varepsilon.$$

Since $|z-\zeta| \leqslant \operatorname{diam} K$ for z and ζ in K, the *second* condition makes

$$\int_E \log\frac{1}{|z-\zeta|}\,\mathrm{d}\mu(\zeta) \quad \leqslant \quad \int_K \log\frac{1}{|z-\zeta|}\,\mathrm{d}\mu(\zeta) \quad + \quad (\log\operatorname{diam} K)\,\mu(K \sim E)$$

$$\leqslant \quad V(z) \ + \ \varepsilon \log\operatorname{diam} K \quad \leqslant \quad M \ + \ \varepsilon\log\operatorname{diam} K$$

for $z \in K$, hence certainly for $z \in E$. By choosing $\varepsilon > 0$ small enough, we can ensure that the last quantity on the right, $M + \varepsilon\log\operatorname{diam} K$, *denoted henceforth by* M', is as close as we like to M.

At the same time, when $z \notin K$,

$$\int_{K\sim E} \log\frac{1}{|z-\zeta|}\,\mathrm{d}\mu(\zeta)$$

lies between

$$\varepsilon \log \frac{1}{\operatorname{dist}(z,K) + \operatorname{diam} K} \qquad \text{and} \qquad \varepsilon\log\frac{1}{\operatorname{dist}(z,K)}.$$

For any such fixed z, then,

$$\int_E \log\frac{1}{|z-\zeta|}\,\mathrm{d}\mu(\zeta)$$

will be arbitrarily close to $V(z)$ when $\varepsilon > 0$ is sufficiently small (depending on z). We see, $\varepsilon > 0$ being arbitrary, that we will have $V(z) \leqslant M$ at each

$z \notin K$ (thus proving the theorem) *if we can deduce that*

$$\int_E \log \frac{1}{|z-\zeta|}\, d\mu(\zeta) \quad \leqslant \quad M'$$

outside E knowing that this holds everywhere on E.

The last implication looks just like the one affirmed by the theorem, so it may seem as though nothing has been gained. We nevertheless have more of a toehold here on account of the *first* condition on our set E, according to which

$$\int_K \log^+ \frac{\rho}{|z-\zeta|}\, d\mu(\zeta) \quad = \quad \int_K \log \frac{1}{|z-\zeta|}\, d\mu(\zeta)$$
$$- \quad \int_K \min\left(\log \frac{1}{|z-\zeta|},\ \log \frac{1}{\rho}\right) d\mu(\zeta)$$

tends to zero *uniformly for* $z \in E$ as $\rho \longrightarrow 0$. Thence, *a fortiori* (!),

$$\int_E \log^+ \frac{\rho}{|z-\zeta|}\, d\mu(\zeta) \quad \longrightarrow \quad 0 \quad \textit{uniformly for} \quad z \in E$$

as $\rho \longrightarrow 0$. This uniformity plays an essential rôle in the following argument.

It will be convenient to write

$$U(z) \quad = \quad \int_E \log \frac{1}{|z-\zeta|}\, d\mu(\zeta).$$

The proof of our theorem has boiled down to showing that if

$$U(z) \ \leqslant\ M' \quad \text{for} \quad z \in E,$$

then $U(z)$ is *also* $\leqslant\ M'$ at each $z \notin E$.

This is where we use the maximum principle for harmonic functions. In $\mathbb{C} \sim E$, U is harmonic; also,

$$U(z) \quad \longrightarrow \quad -\infty \quad \text{as} \quad z \longrightarrow \infty$$

unless $\mu(E) = 0$, in which case the desired conclusion is obviously true. The principle of maximum will therefore make $U(z) \leqslant M'$ in $\mathbb{C} \sim E$ provided that $\limsup_{z \to z_0} U(z) \leqslant M'$ for each $z_0 \in E$ (cf. second corollary to the second lemma in article 1).

Take any $\delta > 0$; we wish to show that at each $z_0 \in E$,

$$U(z) \ <\ M' \ +\ 7\delta$$

for the points z in a neighborhood of z_0. Thanks to the uniformity arrived at in the preceding construction, we can *fix* a $\rho > 0$ such that

$$\int_E \log^+ \frac{\rho}{|z-\zeta|}\,d\mu(\zeta) \quad < \quad \delta$$

whenever $z \in E$. With such a ρ, which we can also take to be < 1, we have

$$U(z) \quad = \quad \int_E \min\left(\log\frac{1}{|z-\zeta|},\ \log\frac{1}{\rho}\right) d\mu(\zeta) \quad + \quad \int_E \log^+ \frac{\rho}{|z-\zeta|}\,d\mu(\zeta).$$

The first integral on the right is $\leqslant U(z)$ and hence $\leqslant M'$ for $z \in E$; it is, moreover, *continuous* in z. *That integral is therefore* $< M' + \delta$ *whenever* z *is sufficiently close to any* $z_0 \in E$; our task thus reduces to verifying that

$$\int_E \log^+ \frac{\rho}{|z-\zeta|}\,d\mu(\zeta) \quad < \quad 6\delta$$

for z close enough to such a z_0. The last relation holds in fact at *all* points z, as we now proceed to show with the help of an ingenious device used in Carleson's little book. The latter has the advantage of being applicable when the logarithmic potential kernel $\log(1/|z-\zeta|)$ is replaced by fairly general ones of the form $k(|z-\zeta|)$, and it can be used in $\mathbb{R}^n$ for $n > 2$ as well as in $\mathbb{R}^2$.

Fix any z. If $z \in E$, the integral in question is even $< \delta$ by choice of ρ, so we may suppose that $z \notin E$. Then, using z as vertex, we partition the complex plane into six sectors, each of $60°$ opening, and denote by $E_1, E_2, \ldots, E_6$ the respective intersections of E with those sectors (so as to have $E_1 \cup E_2 \cup \cdots \cup E_6 = E$).

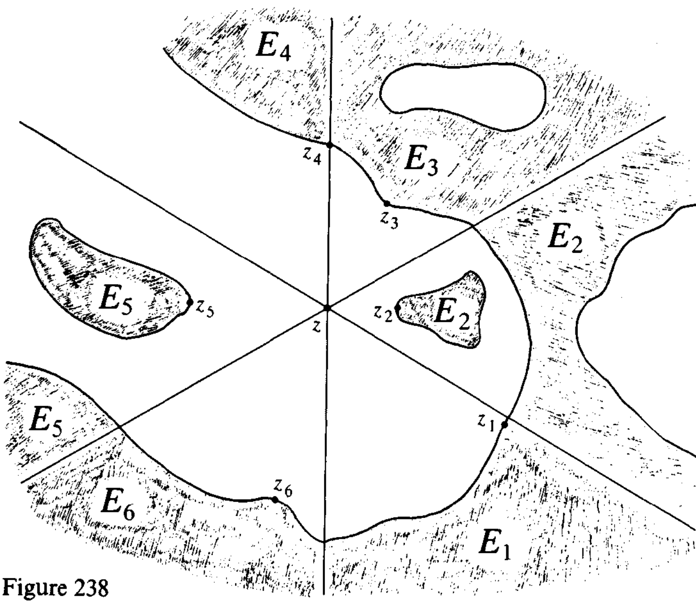


Figure 238

In each non-empty closure $\bar{E}_k$, $k = 1, 2, \ldots, 6$, pick a point z_k for which

$$|z_k - z| = \operatorname{dist}(z, E_k).$$

We have

$$\int_E \log^+ \frac{\rho}{|z-\zeta|}\, d\mu(\zeta) \leqslant \sum_{k=1}^{6} \int_{E_k} \log^+ \frac{\rho}{|z-\zeta|}\, d\mu(\zeta)$$

(with $\leqslant$ here and not $=$, because the E_k may intersect along the edges of the sectors*). However, for each k,

$$\int_{E_k} \log^+ \frac{\rho}{|z-\zeta|}\, d\mu(\zeta) \leqslant \int_{E_k} \log^+ \frac{\rho}{|z_k-\zeta|}\, d\mu(\zeta),$$

since

$$|z-\zeta| \geqslant |z_k - \zeta| \quad \text{when } \zeta \in E_k,$$

as one sees from the following diagram, drawn for $k = 6$:

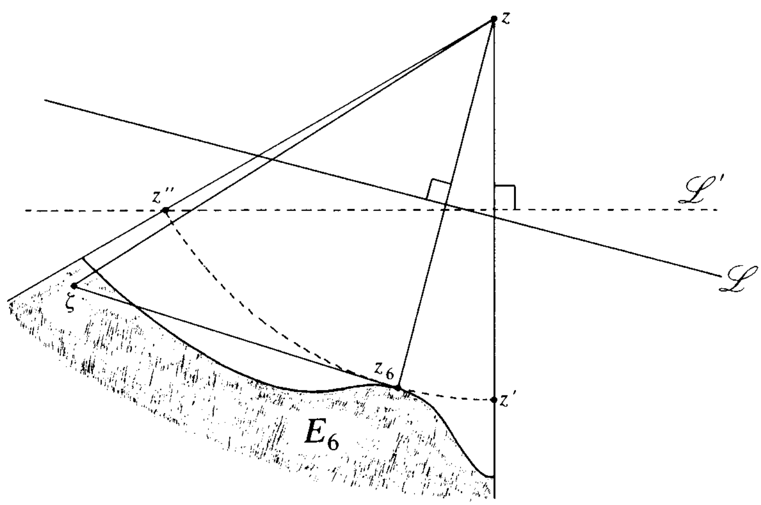


Figure 239

Here, $\mathscr{L}$ is the perpendicular bisector of the segment $[z, z_6]$ and $\mathscr{L}'$ that of $[z, z']$. Any point ζ in E_6 lies *on the same side* of $\mathscr{L}$ as z_6 and *on the opposite side thereof* from z, so the last inequality must hold. (By imagining z_6 to coincide with z' – in which case $\mathscr{L}$, coinciding with $\mathscr{L}'$, would pass

* if, for instance, we work with *closed* 60° sectors (which we may just as well do), in which case the sets E_k are *already* closed.