Also
$$\int_0^\lambda |\phi_k(ixe^{-i\delta})|^2\,dx = \int_{1/\lambda}^\infty \left|\phi_k\left(\frac{1}{ixe^{-i\delta}}\right)\right|^2 \frac{dx}{x^2},$$
and by the above formula this should be approximately
$$\frac{(2\pi)^{k/(k-1)}}{k-1}\times$$
$$\times \int_{1/\lambda}^\infty \left|\sum_{n=1}^\infty \frac{d_k(n)}{n^{(\frac{1}{2}k-1)/(k-1)}}\exp\{-(k-1)i(2\pi)^{k/(k-1)}(nx)^{1/(k-1)}e^{-i\delta/(k-1)}\}\right|^2 \frac{dx}{x^{2-k/(k-1)}}.$$
Putting $x = \xi^{k-1}$, this is
$$(2\pi)^{k/(k-1)}\times$$
$$\times \int_{\lambda^{-1/(k-1)}}^\infty \left|\sum_{n=1}^\infty \frac{d_k(n)}{n^{(\frac{1}{2}k-1)/(k-1)}}\exp\{-(k-1)i(2\pi)^{k/(k-1)}n^{1/(k-1)}\xi e^{-i\delta/(k-1)}\}\right|^2 d\xi,$$
and we can integrate as before. We obtain
$$K\sum_{m=1}^\infty\sum_{n=1}^\infty \frac{d_k(m)d_k(n)}{(mn)^{(\frac{1}{2}k-1)/(k-1)}}\times$$
$$\times\frac{\exp[(k-1)(2\pi)^{k/(k-1)}\{(n^{1/(k-1)}-m^{1/(k-1)})i\cos\delta/(k-1)- -(m^{1/(k-1)}+n^{1/(k-1)})\sin\delta/(k-1)\}\lambda^{-1/(k-1)}]}{(n^{1/(k-1)}-m^{1/(k-1)})i\cos\delta/(k-1)-(m^{1/(k-1)}+n^{1/(k-1)})\sin\delta/(k-1)},$$
where K depends on k only.

The terms with $m = n$ are
$$O\left\{\frac{1}{\delta}\sum_{n=1}^\infty \frac{d_k^2(n)}{n}\exp(-K\delta n^{1/(k-1)}\lambda^{-1/(k-1)})\right\} = O\left\{\frac{1}{\delta}\frac{1}{(\lambda\delta)^\epsilon}\right\}.$$

The rest are
$$O\left\{\sum_{m>n}\sum \frac{1}{(mn)^{(\frac{1}{2}k-1)/(k-1)}}\frac{\exp(-K\delta m^{1/(k-1)}\lambda^{-1/(k-1)})}{m^{1/(k-1)}-n^{1/(k-1)}}\right\}.$$
Now
$$\sum_{n=1}^{m-1}\frac{1}{n^{(\frac{1}{2}k-1)/(k-1)}(m^{1/(k-1)}-n^{1/(k-1)})}$$
$$= O\left\{\sum_{n=1}^{\frac{1}{2}m}\frac{1}{n^{(\frac{1}{2}k-1)/(k-1)}m^{1/(k-1)}}+\sum_{\frac{1}{2}m}^{m-1}\frac{1}{m^{(\frac{1}{2}k-1)/(k-1)+1/(k-1)-1}(m-n)}\right\}$$
$$= O(m^{1-(\frac{1}{2}k-1)/(k-1)-1/(k-1)+\epsilon}) = O(m^{(\frac{1}{2}k-1)/(k-1)+\epsilon}).$$

Hence we obtain
$$O\left\{\sum_{m=2}^\infty m^\epsilon \exp(-K\delta m^{1/(k-1)}\lambda^{-1/(k-1)})\right\} = O\left\{\int_0^\infty x^\epsilon\exp(-K\delta x^{1/(k-1)}\lambda^{-1/(k-1)})\,dx\right\}$$
$$= O\left\{\left(\frac{\lambda}{\delta^{k-1}}\right)^{1+\epsilon}\right\}.$$
Altogether
$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^{2k}e^{-2\delta t}\,dt = O\left\{\frac{1}{(\lambda\delta)^{1+\epsilon}}\right\}+O\left\{\left(\frac{\lambda}{\delta^{k-1}}\right)^{1+\epsilon}\right\},$$
and taking $\lambda = \delta^{\frac{1}{2}k-1}$, we obtain
$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^{2k}e^{-2\delta t}\,dt = O(\delta^{-\frac{1}{2}k-\epsilon}) \quad (k \geqslant 2).$$
This index is what we should obtain from the approximate functional equation.

7.19. The attempt to obtain a non-trivial upper bound for
$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^{2k}e^{-\delta t}\,dt$$
for $k > 2$ fails. But we can obtain a lower bound† for it which may be somewhere near the truth; for in this problem we can ignore $\phi_k(ixe^{-i\delta})$ for small x, since by (7.13.5)
$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^{2k}e^{-2\delta t}\,dt > \int_1^\infty |\phi_k(ixe^{-i\delta})|^2\,dx+O(1), \tag{7.19.1}$$
and we can approximate to the right-hand side by the method already used.

If k is any positive integer, and $\sigma > 1$,
$$\zeta^k(s) = \prod_p\left(1-\frac{1}{p^s}\right)^{-k} = \prod_p\sum_{m=0}^\infty\frac{(k+m-1)!}{(k-1)!\,m!}\frac{1}{p^{ms}} = \sum_{n=1}^\infty\frac{d_k(n)}{n^s}.$$
If we replace the coefficient of each term p^{-ms} by its square, the coefficient of each n^{-s} is replaced by its square. Hence if
$$F_k(s) = \sum_{n=1}^\infty\frac{d_k^2(n)}{n^s},$$
then
$$F_k(s) = \prod_p\sum_{m=0}^\infty\left\{\frac{(k+m-1)!}{(k-1)!\,m!}\right\}^2\frac{1}{p^{ms}} = \prod_p f_k(p^{-s}),$$

† Titchmarsh (4).

say. Thus
$$f_k\left(\frac{1}{p^s}\right) = 1+\frac{k^2}{p^s}+\ldots,$$
and
$$\left(1-\frac{1}{p^s}\right)^{k^2} f_k\left(\frac{1}{p^s}\right) = \left(1-\frac{k^2}{p^s}+\ldots\right)\left(1+\frac{k^2}{p^s}+\ldots\right) = 1+O\left(\frac{1}{p^{2\sigma}}\right).$$
Hence the product
$$\prod_p \left(1-\frac{1}{p^s}\right)^{k^2} f_k\left(\frac{1}{p^s}\right)$$
is absolutely convergent for $\sigma > \frac{1}{2}$, and so represents an analytic function, $g(s)$ say, regular for $\sigma > \frac{1}{2}$, and bounded in any half-plane $\sigma \geqslant \frac{1}{2}+\delta$; and
$$F_k(s) = \zeta^{k^2}(s)g(s).$$
Now
$$\sum_{n=1}^{\infty} d_k^2(n)e^{-2n\sin\delta} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Gamma(s)F_k(s)(2\sin\delta)^{-s}\,ds.$$
Moving the line of integration just to the left of $\sigma = 1$, and evaluating the residue at $s = 1$, we obtain in the usual way
$$\sum_{n=1}^{\infty} d_k^2(n)e^{-2n\sin\delta} \sim \frac{C_k'}{\delta}\log^{k^2-1}\frac{1}{\delta}.$$
Similarly
$$\sum_{n=1}^{\infty} \frac{d_k^2(n)}{n}e^{-2n\sin\delta} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Gamma(s)F_k(s+1)(2\sin\delta)^{-s}\,ds \sim C_k\log^{k^2}\frac{1}{\delta},$$
since here there is a pole of order k^2+1 at $s = 0$.

We can now prove

THEOREM 7.19. *For any fixed integer k, and $0 < \delta \leqslant \delta_0 = \delta_0(k)$,*
$$\int_0^{\infty} |\zeta(\tfrac{1}{2}+it)|^{2k}e^{-\delta t}\,dt \geqslant \frac{C_k}{\delta}\log^{k^2}\frac{1}{\delta}.$$

The integral on the right of (7.19.1) is equal to (7.18.1) with $\lambda = 1$; and
$$\Sigma_1 \sim \frac{C_k}{2\delta}\log^{k^2}\frac{1}{\delta},$$
while
$$\Sigma_2+\Sigma_3 = O\left(\frac{1}{\delta}\log^{k^2-1}\frac{1}{\delta}\right).$$
The result therefore follows.

NOTES FOR CHAPTER 7

7.20. When applied (with care) to a general Dirichlet polynomial, the proof of the first lemma of §7.2 leads to
$$\int_0^T \left|\sum_1^N a_n n^{-it}\right|^2 dt = \sum_1^N |a_n|^2\{T+O(n\log 2n)\}.$$
However Montgomery and Vaughan [1] have given a superior result, namely
$$\int_0^T \left|\sum_1^N a_n n^{-it}\right|^2 dt = \sum_1^N |a_n|^2\{T+O(n)\}. \tag{7.20.1}$$
Ramachandra [2] has given an alternative proof of this result. Both proofs are more complicated than the argument leading to (7.2.1). However (7.20.1) has the advantage of dealing with the mean value of $\zeta(s)$ uniformly for $\sigma \geqslant \frac{1}{2}$. Suppose for example that $\sigma = \frac{1}{2}$. One takes $x = 2T$ in Theorem 4.11, whence
$$\zeta(\tfrac{1}{2}+it) = \sum_{n\leqslant 2T} n^{-\frac{1}{2}-it} + O(T^{-\frac{1}{2}}) = Z+O(T^{-\frac{1}{2}}),$$
say, for $T \leqslant t \leqslant 2T$. Then
$$\int_T^{2T} |Z|^2\,dt = \sum_{n\leqslant 2T} n^{-1}\{T+O(n)\} = T\log T+O(T).$$
Moreover $Z \ll T^{\frac{1}{2}}$, whence
$$\int_T^{2T} |Z|\,T^{-\frac{1}{2}}\,dt \ll T.$$
Then, since
$$\int_T^{2T} O(T^{-\frac{1}{2}})^2\,dt = O(1),$$
we conclude that
$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^2\,dt = T\log T+O(T),$$

and Theorem 7.3 follows (with error term $O(T)$) on summing over $\frac{1}{2}T, \frac{1}{4}T, \ldots$. In particular we see that Theorem 4.11 is sufficient for this purpose, contrary to Titchmarsh's remark at the beginning of §7.3.

We now write

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^2\,dt = T\log\left(\frac{T}{2\pi}\right)+(2\gamma-1)\,T+E(T).$$

Much further work has been done concerning the error term $E(T)$. It has been shown by Balasubramanian [1] that $E(T) \ll T^{\frac{1}{3}+\varepsilon}$. A different proof was given by Heath-Brown [4]. The estimate may be improved slightly by using exponential sums, and Ivic [3; Corollary 15.4] has sketched the argument leading to the exponent $\frac{35}{108}+\varepsilon$, using a lemma due to Kolesnik [4]. It is no coincidence that this is twice the exponent occurring in Kolesnik's estimate for $\mu(\frac{1}{2})$, since one has the following result.

LEMMA 7.20. *Let k be a fixed positive integer and let $t \geqslant 2$. Then*

$$\zeta(\tfrac{1}{2}+it)^k \ll (\log t)\left(1+\int_{-\log^2 t}^{\log^2 t} |\zeta(\tfrac{1}{2}+it+iu)|^k e^{-|u|}\,du\right). \tag{7.20.2}$$

This is a trivial generalization of Lemma 3 of Heath-Brown [2], which is the case $k=2$. It follows that

$$\zeta(\tfrac{1}{2}+it)^2 \ll (\log t)^4+(\log t)\max E\{t\pm(\log t)^2\}. \tag{7.20.3}$$

Thus, if μ is the infimum of those α for which $E(T) \ll T^{\alpha}$, then $\mu(\frac{1}{2}) \leqslant \frac{1}{2}\mu$. On the other hand, an examination of the initial stages of the process for estimating $\zeta(\frac{1}{2}+it)$ by van der Corput's method shows that one is, in effect, bounding the mean square of $\zeta(\frac{1}{2}+it)$ over a short range $(t-\Delta, t+\Delta)$. Thus it appears that one can hope for nothing better for $\mu(\frac{1}{2})$, by this method, than is given by (7.20.3).

The connection between estimates for $\zeta(\frac{1}{2}+it)$ and those for $E(T)$ should not be pushed too far however, for Good [1] has shown that $E(T) = \Omega(T^{\frac{1}{4}})$. Indeed Heath-Brown [1] later gave the asymptotic formula

$$\int_0^T E(t)^2\,dt = \tfrac{2}{3}(2\pi)^{-\frac{1}{2}}\frac{\zeta(\frac{3}{2})^4}{\zeta(3)}T^{\frac{3}{2}}+O(T^{\frac{5}{4}}\log^2 T) \tag{7.20.4}$$

from which the above Ω-result is immediate. It is perhaps of interest to

note that the error term of (7.20.4) must be $\Omega\{T^{\frac{3}{4}}(\log T)^{-1}\}$, since any estimate $O\{F(T)\}$ readily yields $E(T) \ll \{F(T)\log T\}^{\frac{1}{3}}$, by an argument analogous to that used in the proof of Lemma α in 14.13. It would be nice to reduce the error term in (7.20.4) to $O(T^{1+\varepsilon})$ so as to include Balasubramanian's bound $E(T) \ll T^{\frac{1}{3}+\varepsilon}$.

Higher mean-values of $E(T)$ have been investigated by Ivic [1] who showed, for example, that

$$\int_0^T E(t)^8\,dt \ll T^{3+\varepsilon}. \tag{7.20.5}$$

This readily implies the estimate $E(T) \ll T^{\frac{3}{8}+\varepsilon}$.

The mean-value theorems of Heath-Brown and Ivic depend on a remarkable formula for $E(T)$ due to Atkinson [1]. Let $0 < A < A'$ be constants and suppose $AT \leqslant N \leqslant A'T$. Put

$$N' = N'(T) = \frac{T}{2\pi}+\frac{N}{2}-\left(\frac{NT}{2\pi}+\frac{N^2}{4}\right)^{\frac{1}{2}}.$$

Then $E(T) = \Sigma_1+\Sigma_2+O(\log^2 T)$, where

$$\Sigma_1 = 2^{-\frac{1}{2}}\sum_{n\leqslant N}(-1)^n d(n)\left(\frac{nT}{2\pi}+\frac{n^2}{4}\right)^{-\frac{1}{4}}\left\{\sinh^{-1}\left(\frac{\pi n}{2T}\right)^{\frac{1}{2}}\right\}^{-1}\sin f(n)$$

with (7.20.6)

$$f(n) = \tfrac{1}{4}\pi+2T\sinh^{-1}\left(\frac{\pi n}{2T}\right)^{\frac{1}{2}}+(\pi^2 n^2+2\pi nT)^{\frac{1}{2}}, \tag{7.20.7}$$

and

$$\Sigma_2 = 2\sum_{n\leqslant N'} d(n)n^{-\frac{1}{2}}\left(\log\frac{T}{2\pi n}\right)^{-1}\sin g(n)$$

where

$$g(n) = T\log\frac{T}{2\pi n}-T-\tfrac{1}{4}\pi.$$

Atkinson loses a minus sign on [1; p 375]. This is corrected above. In applications of the above formula one can usually show that Σ_2 may be ignored. On the Lindelöf hypothesis, for example, one has

$$\sum_{n\leqslant x} d(n)n^{-\frac{1}{2}-iT} \ll T^{\varepsilon}$$

for $x \ll T$, so that $\Sigma_2 \ll T^{\varepsilon}$ by partial summation; and in general one finds $\Sigma_2 \ll T^{2\mu(\frac{1}{2})+\varepsilon}$. The sum Σ_1 is closely analogous to that occuring in the explicit formula (12.4.4) for $\Delta(x)$ in Dirichlet's divisor problem. Indeed, if $n = o(T^{\frac{1}{3}})$ then the summands of (7.20.6) are

$$(-1)^n \left(\frac{2T}{\pi}\right)^{\frac{1}{4}} \frac{d(n)}{n^{\frac{3}{4}}} \cos\left(2\sqrt{(2\pi nT)} - \frac{\pi}{4}\right) + o\left(T^{\frac{1}{4}} \frac{d(n)}{n^{\frac{3}{4}}}\right).$$

7.21. Ingham's result has been improved by Heath-Brown [4] to give

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^4\,dt = \sum_{n=0}^{4} c_n T(\log T)^n + O(T^{\frac{7}{8}+\varepsilon}) \tag{7.21.1}$$

where $c_4 = (2\pi^2)^{-1}$ and

$$c_3 = 2\{4\gamma - 1 - \log(2\pi) - 12\zeta'(2)\pi^{-2}\}\pi^{-2}.$$

The proof requires an asymptotic formula for

$$\sum_{n \leqslant N} d(n)\,d(n+r)$$

with a good error term, uniform in r. Such estimates are obtained in Heath-Brown [4] by applying Weil's bound for the Kloosterman sum (see §7.24).

7.22. Better estimates for σ_k are now available. In particular we have $\sigma_3 \leqslant \frac{7}{12}$ and $\sigma_4 \leqslant \frac{5}{8}$. The result on σ_4 is due to Heath-Brown [8]. To deduce the estimate for σ_3 one merely uses Gabriel's convexity theorem (see § 9.19), taking $\alpha = \frac{1}{2}$, $\beta = \frac{5}{8}$, $\lambda = \frac{1}{4}$, $\mu = \frac{1}{6}$, and $\sigma = \frac{7}{12}$.

The key ingredient required to obtain $\sigma_4 \leqslant \frac{5}{8}$ is the estimate

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^{12}\,dt \ll T^2(\log T)^{17} \tag{7.22.1}$$

of Heath-Brown [2]. According to (7.20.2) this implies the bound $\mu(\frac{1}{2}) \leqslant \frac{1}{6}$. In fact, in establishing (7.22.1) it is shown that, if $|\zeta(\frac{1}{2}+it_r)| \geqslant V\,(>0)$ for $1 \leqslant r \leqslant R$, where $0 < t_r \leqslant T$ and $t_{r+1} - t_r \geqslant 1$, then

$$R \ll T^2 V^{-12}(\log T)^{16},$$

and, if $V \geqslant T^{\frac{2}{13}}(\log T)^2$, then

$$R \ll TV^{-6}(\log T)^8.$$

Thus one sees not only that $\zeta(\frac{1}{2}+it) \ll t^{\frac{1}{6}}(\log t)^{\frac{4}{3}}$, but also that the number

of points at which this bound is close to being attained is very small. Moreover, for $V \geqslant T^{\frac{2}{13}}(\log T)^2$, the behaviour corresponds to the, as yet unproven, estimate

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^6\,dt \ll T^{1+\varepsilon}.$$

To prove (7.22.1) one uses Atkinson's formula for $E(T)$ (see §7.20) to show that

$$\int_{T-G}^{T+G} |\zeta(\tfrac{1}{2}+it)|^2\,dt \ll G\log T +$$

$$G\sum_K (TK)^{-\frac{1}{4}}\left(|S(K)| + K^{-1}\int_0^K |S(x)|\,dx\right)e^{-G^2K/T}, \tag{7.22.2}$$

where K runs over powers of 2 in the range $T^{\frac{1}{3}} \leqslant K \leqslant TG^{-2}\log^3 T$, and

$$S(x) = S(x, K, T) = \sum_{K < n \leqslant K+x} (-1)^n d(n) e^{if(n)}$$

with $f(n)$ as in (7.20.7). The bound (7.22.2) holds uniformly for $\log^2 T \leqslant G \leqslant T^{\frac{5}{12}}$. In order to obtain the estimate (7.22.1) one proceeds to estimate how often the sum $S(x, K, T)$ can be large, for varying T. This is done by using a variant of Halász's method, as described in §9.28.

By following similar ideas, Graham, in work in the process of publication, has obtained

$$\int_0^T |\zeta(\tfrac{5}{7}+it)|^{196}\,dt \ll T^{14}(\log T)^{425}. \tag{7.22.3}$$

Of course there is no analogue of Atkinson's formula available here, and so the proof is considerably more involved. The result (7.22.3) contains the estimate $\mu(\frac{5}{7}) \leqslant \frac{1}{14}$ (which is the case $l = 4$ of Theorem 5.14) in the same way that (7.22.1) implies $\mu(\frac{1}{2}) \leqslant \frac{1}{6}$.

7.23. As in §7.9, one may define σ_k, for all positive real k, as the infimum of those σ for which (7.9.1) holds, and σ'_k similarly, for (7.9.2).

Then it is still true that $\sigma_k = \sigma'_k$, and that

$$\int_1^T |\zeta(\sigma+it)|^{2k}\,dt = T\sum_1^\infty d_k(n)^2 n^{-2\sigma} + O(T^{1-\delta})$$

for $\sigma > \sigma_k$, where $\delta = (\sigma, k) > 0$ may be explicitly determined. This may be proved by the method of Haselgrove [1]; see also Turganaliev [1]. In particular one may take $\delta(\sigma, \frac{1}{2}) = \frac{1}{2}(\sigma - \frac{1}{2})$ for $\frac{1}{2} < \sigma < 1$ (Ivic [3; (8.111)] or Turganaliev [1]). For some quite general approaches to these fractional moments the reader should consult Ingham (4) and Bohr and Jessen (4).

Mean values for $\sigma = \frac{1}{2}$ are far more difficult, and in no case other than $k = 1$ or 2 is an asymptotic formula for

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt = I_k(T),$$

say, known, even assuming the Riemann hypothesis. However Heath-Brown [7] has shown that

$$T(\log T)^{k^2} \ll I_k(T) \ll T(\log T)^{k^2} \quad \left(k = \frac{1}{n}\right),$$

Ramachandra [3], [4] having previously dealt with the case $k = \frac{1}{2}$. Jutila [4] observed that the implied constants may be taken to be independent of k. We also have

$$I_k(T) \gg T(\log T)^{k^2}$$

for any positive rational k. This is due to Ramachandra [4] when k is half an integer, and to Heath-Brown [7] in the remaining cases. (Titchmarsh [1; Theorem 29] states such a result for positive integral k, but the reference given there seems to yield only Theorem 7.19, which is weaker.) When k is irrational the best result known is Ramachandra's estimate [5]

$$I_k(T) \gg T(\log T)^{k^2}(\log\log T)^{-k^2}.$$

If one assumes the Riemann hypothesis one can obtain the better results

$$I_k(T) \ll T(\log T)^{k^2} \quad (0 \leqslant k \leqslant 2)$$

and

$$I_k(T) \gg T(\log T)^{k^2} \quad (k \geqslant 0), \tag{7.23.1}$$

for which see Ramachandra [4] or Heath-Brown [7]. Conrey and Ghosh [1] have given a particularly simple proof of (7.23.1) in the form

$$I_k(T) \geqslant \{C_k + o(1)\}\, T(\log T)^{k^2},$$

with

$$C_k = \{\Gamma(k^2+1)\}^{-1}\prod_p\left\{\left(1-\frac{1}{p}\right)^{k^2}\sum_{m=0}^\infty\left(\frac{\Gamma(k+m)}{m!\,\Gamma(k)}\right)^2 p^{-m}\right\}.$$

They suggest that this relation may even hold with equality (as it does when $k = 1$ or 2).

7.24. The work of Atkinson (2) alluded to at the end of §7.16 is of special historical interest, since it contains the first occurence of Kloosterman sums in the subject. These sums are defined by

$$S(q; a, b) = \sum_{\substack{n=1\\(n,q)=1}}^{q} \exp\left(\frac{2\pi i}{q}(an + b\bar{n})\right), \tag{7.24.1}$$

where $n\bar{n} \equiv 1 (\mathrm{mod}\ q)$. Such sums have been of great importance in recent work, notably that of Heath-Brown [4] mentioned in §7.21, and of Iwaniec [1] and Deshouillers and Iwaniec [2], [3] referred to later in this section. The key fact about these sums is the estimate

$$|S(q; a, b)| \leqslant d(q)q^{\frac{1}{2}}(q, a, b)^{\frac{1}{2}}, \tag{7.24.2}$$

which indicates a very considerable amount of cancellation in (7.24.1). This result is due to Weil [1] when q is prime (the most important case) and to Estermann [2] in general. Weil's proof uses deep methods from algebraic geometry. It is possible to obtain further cancellations by averaging $S(q; a, b)$ over q, a and b. In order to do this one employs the theory of non-holomorphic modular forms, as in the work of Deshouillers and Iwaniec [1]. This is perhaps the most profound area of current research in the subject.

One way to see how Kloosterman sums arise is to use (7.15.2). Suppose for example one considers

$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^2\left|\sum_{u\leqslant U} u^{-it}\right|^2 e^{-t/T}\,dt. \tag{7.24.3}$$

Applying (7.15.2) with $2\delta = 1/T + i\log(v/u)$ one is led to examine

$$\sum_{n=1}^\infty d(n)\exp\left(\frac{2\pi i n u}{v}e^{i/T}\right).$$

One may now replace $e^{i/T}$ by $1+(i/T)$ with negligible error, producing

$$\sum_{n=1}^{\infty} d(n)\exp\left(\frac{2\pi inu}{v}\right)\exp\left(-\frac{2\pi nu}{vT}\right)=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Gamma(s)\left(\frac{Tv}{2\pi u}\right)^{s}D\left(s,\frac{u}{v}\right)ds$$

where

$$D\left(s,\frac{u}{v}\right)=\sum_{n=1}^{\infty} d(n)\exp\left(\frac{2\pi inu}{v}\right)n^{-s}.$$

This Dirichlet series was investigated by Estermann [1], using the function $\zeta(s,a)$ of §2.17. It has an analytic continuation to the whole complex plane, and satisfies the functional equation

$$D\left(s,\frac{u}{v}\right)=2v^{1-2s}\frac{\Gamma(1-s)^2}{(2\pi)^{2-2s}}\left\{D\left(1-s,\frac{\bar{u}}{v}\right)-\cos(\pi s)\,D\left(1-s,-\frac{\bar{u}}{v}\right)\right\}$$

providing that $(u,v)=1$. To evaluate our original integral (7.24.3) it is necessary to average over u and v, so that one is led to consider

$$\sum_{\substack{u,v\leqslant U\\(u,v)=1}} D\left(1-s,\frac{\bar{u}}{v}\right)=\sum_{v\leqslant U}\sum_{n=1}^{\infty} d(n)n^{s-1}\sum_{\substack{u\leqslant U\\(u,v)=1}}\exp\left(\frac{2\pi in\bar{u}}{v}\right),$$

for example. In order to get a sharp bound for the innermost sum on the right one introduces the Kloosterman sum:

$$\sum_{\substack{u\leqslant U\\(u,v)=1}}\exp\left(\frac{2\pi in\bar{u}}{v}\right)=\sum_{\substack{m=1\\(m,v)=1}}^{v}\exp\left(\frac{2\pi in\bar{m}}{v}\right)\sum_{\substack{u\leqslant U\\u\equiv m\,(\mathrm{mod}\,v)}}1$$

$$=\sum_{\substack{m=1\\(m,v)=1}}^{v}\exp\left(\frac{2\pi in\bar{m}}{v}\right)\sum_{u\leqslant U}\left\{\frac{1}{v}\sum_{a=1}^{v}\exp\left(\frac{2\pi ia(m-u)}{v}\right)\right\}$$

$$=\frac{1}{v}\sum_{a=1}^{v}S(v;a,n)\sum_{u\leqslant U}\exp\left(-\frac{2\pi iau}{v}\right),$$

and one can now get a significant saving by using (7.24.2). Notice also that $S(v;a,n)$ is averaged over v, a and n, so that estimates for averages of Kloosterman sums are potentially applicable.

By pursuing such ideas and exploiting the connection with non-holomorphic modular forms, Iwaniec [1] showed that

$$\sum_{1}^{R}\int_{t_r}^{t_r+\Delta}|\zeta(\tfrac{1}{2}+it)|^4\,dt \ll (R\Delta+TR^{\frac{1}{2}}\Delta^{-\frac{1}{2}})\,T^{\varepsilon}$$

for $0\leqslant t_r\leqslant T$, $t_{r+1}-t_r\geqslant\Delta\geqslant T^{\frac{1}{2}}$. In particular, taking $R=1$, one has

$$\int_{T}^{T+T^{\frac{2}{3}}}|\zeta(\tfrac{1}{2}+it)|^4\,dt \ll T^{\frac{2}{3}+\varepsilon} \tag{7.24.4}$$

which again implies $\mu(\frac{1}{2})\leqslant\frac{1}{6}$, by (7.20.2). Moreover, by a suitable choice of the points t_r one can deduce (7.22.1), with $T^{2+\varepsilon}$ on the right.

Mean-value theorems involving general Dirichlet polynomials and partial sums of the zeta function are of interest, particularly in connection with the problems considered in Chapters 9 and 10. Such results may be proved by the methods of this chapter, but sharper estimates can be obtained by using Kloosterman sums and their connection with modular forms. Thus Deshouillers and Iwaniec [2], [3] established the bounds

$$\int_0^T|\zeta(\tfrac{1}{2}+it)|^4\left|\sum_{n\leqslant N}a_nn^{it}\right|^2dt \ll T^{\varepsilon}(T+T^{\frac{1}{2}}N^2+T^{\frac{3}{4}}N^{\frac{5}{4}})\sum_{n\leqslant N}|a_n|^2 \tag{7.24.5}$$

and

$$\int_0^T|\zeta(\tfrac{1}{2}+it)|^2\left|\sum_{m\leqslant M}a_mm^{it}\right|^2\left|\sum_{n\leqslant N}b_nn^{it}\right|^2dt$$

$$\ll T^{\varepsilon}(T+T^{\frac{1}{2}}M^{\frac{3}{2}}N+T^{\frac{1}{2}}MN^{\frac{1}{2}}+M^{\frac{7}{4}}N^{\frac{3}{2}})\left(\sum_{m\leqslant M}|a_m|^2\right)\left(\sum_{n\leqslant N}|b_n|^2\right) \tag{7.24.6}$$

for $N\leqslant M$. In a similar vein Balasubramanian, Conrey, and Heath-Brown [1] showed that

$$\int_0^T|\zeta(\tfrac{1}{2}+it)|^2\left|\sum_{m\leqslant M}\mu(m)F(m)m^{-\frac{1}{2}-it}\right|^2dt=CT+O_A\{T(\log T)^{-A}\}, \tag{7.24.7}$$

$$C=\sum_{m,n\leqslant M}\frac{\mu(m)\mu(n)}{mn}F(m)\overline{F(n)}\,(m,n)\left(\log\frac{T(m,n)^2}{2\pi mn}+2\gamma-1\right)$$

for $M\leqslant T^{\frac{9}{17}-\varepsilon}$, where A is any positive constant, and the function F satisfies $F(x)\ll 1$, $F'(x)\ll x^{-1}$. The proof requires Weil's estimate for the Kloosterman sum, if $T^{\frac{1}{2}}\leqslant M\leqslant T^{\frac{9}{17}-\varepsilon}$.

VIII
Ω-THEOREMS

8.1. Introduction. The previous chapters have been largely concerned with what we may call O-theorems, i.e. results of the form

$$\zeta(s) = O\{f(t)\}, \qquad 1/\zeta(s) = O\{g(t)\}$$

for certain values of σ.

In this chapter we prove a corresponding set of Ω-theorems, i.e. results of the form

$$\zeta(s) = \Omega\{\phi(t)\}, \qquad 1/\zeta(s) = \Omega\{\psi(t)\},$$

the Ω symbol being defined as the negation of o, so that $F(t) = \Omega\{\phi(t)\}$ means that the inequality $|F(t)| > A\phi(t)$ is satisfied for some arbitrarily large values of t.

If, for a given function $F(t)$, we have both

$$F(t) = O\{f(t)\}, \qquad F(t) = \Omega\{f(t)\},$$

we may say that the order of $F(t)$ is determined, and the only remaining question is that of the actual constants involved.

For $\sigma > 1$, the problems of $\zeta(\sigma+it)$ and $1/\zeta(\sigma+it)$ are both solved. For $\frac{1}{2} \leqslant \sigma \leqslant 1$ there remains a considerable gap between the O-results of Chapters V–VI and the Ω-results of the present chapter. We shall see later that, on the Riemann hypothesis, it is the Ω-results which represent the real truth, and the O-results which fall short of it. We are always more successful with Ω-theorems. This is perhaps not surprising, since an O-result is a statement about all large values of t, an Ω-result about some indefinitely large values only.

8.2. The first Ω results were obtained by means of Diophantine approximation, i.e. the approximate solution in integers of given equations. The following two theorems are used.

DIRICHLET'S THEOREM. *Given N real numbers $a_1, a_2,..., a_N$, a positive integer q, and a positive number t_0, we can find a number t in the range*

$$t_0 \leqslant t \leqslant t_0 q^N, \tag{8.2.1}$$

and integers $x_1, x_2,..., x_N$, such that

$$|ta_n - x_n| \leqslant 1/q \quad (n = 1, 2,..., N). \tag{8.2.2}$$

The proof is based on an argument which was introduced and employed extensively by Dirichlet. This argument, in its simplest form, is that, if there are $m+1$ points in m regions, there must be at least one region which contains at least two points.

Consider the N-dimensional unit cube with a vertex at the origin and edges along the coordinate axes. Divide each edge into q equal parts, and thus the cube into q^N equal compartments. Consider the q^N+1 points, in the cube, congruent (mod 1) to the points $(ua_1, ua_2,..., ua_N)$, where $u = 0, t_0, 2t_0,..., q^N t_0$. At least two of these points must lie in the same compartment. If these two points correspond to $u = u_1$, $u = u_2$ $(u_1 < u_2)$, then $t = u_2 - u_1$ clearly satisfies the requirements of the theorem.

The theorem may be extended as follows. Suppose that we give u the values $0, t_0, 2t_0,..., mq^N t_0$. We obtain mq^N+1 points, of which one compartment must contain at least $m+1$. Let these points correspond to $u = u_1,..., u_{m+1}$. Then $t = u_2-u_1,..., u_m - u_1$, all satisfy the requirements of the theorem.

We conclude that the interval $(t_0, mq^N t_0)$ contains at least m solutions of the inequalities (8.2.2), any two solutions differing by at least t_0.

8.3. KRONECKER'S THEOREM. *Let $a_1, a_2,..., a_N$ be linearly independent real numbers, i.e. numbers such that there is no linear relation*

$$\lambda_1 a_1 + ... + \lambda_N a_N = 0$$

in which the coefficients $\lambda_1,...$ are integers not all zero. Let $b_1,..., b_N$ be any real numbers, and q a given positive number. Then we can find a number t and integers $x_1,..., x_N$, such that

$$|ta_n - b_n - x_n| \leqslant 1/q \quad (n = 1, 2,..., N). \tag{8.3.1}$$

If all the numbers b_n are 0, the result is included in Dirichlet's theorem. In the general case, we have to suppose the a_n linearly independent; for example, if the a_n are all zero, and the b_n are not all integers, there is in general no t satisfying (8.3.1). Also the theorem assigns no upper bound for the number t such as the q^N of Dirichlet's theorem. This makes a considerable difference to the results which can be deduced from the two theorems.

Many proofs of Kronecker's theorem are known.† The following is due to Bohr (15).

We require the following lemma

LEMMA. *If $\phi(x)$ is positive and continuous for $a \leqslant x \leqslant b$, then*

$$\lim_{n\to\infty} \left\{ \int_a^b \{\phi(x)\}^n \, dx \right\}^{1/n} = \max_{a \leqslant x \leqslant b} \phi(x).$$

A similar result holds for an integral in any number of dimensions.

† Bohr (15), (16), Bohr and Jessen (3), Estermann (3), Lettenmeyer (1).

Let $M = \max \phi(x)$. Then

$$\left\{\int_a^b \{\phi(x)\}^n\,dx\right\}^{1/n} \leqslant \{(b-a)M^n\}^{1/n} = (b-a)^{1/n}M.$$

Also, given ϵ, there is an interval, (α,β) say, throughout which

$$\phi(x) \geqslant M-\epsilon.$$

Hence

$$\left\{\int_a^b \{\phi(x)\}^n\,dx\right\}^{1/n} \geqslant \{(\beta-\alpha)(M-\epsilon)^n\}^{1/n} = (\beta-\alpha)^{1/n}(M-\epsilon),$$

and the result is clear. A similar proof holds in the general case.

Proof of Kronecker's theorem. It is sufficient to prove that we can find a number t such that each of the numbers

$$e^{2\pi i(a_n t-b_n)} \quad (n = 1, 2,\ldots, N)$$

differs from 1 by less than a given ϵ; or, if

$$F(t) = 1+\sum_{n=1}^{N} e^{2\pi i(a_n t-b_n)},$$

that the upper bound of $|F(t)|$ for real values of t is $N+1$. Let us denote this upper bound by L. Clearly $L \leqslant N+1$.

Let
$$G(\phi_1,\phi_2,\ldots,\phi_N) = 1+\sum_{n=1}^{N} e^{2\pi i\phi_n},$$

where the numbers ϕ_1, $\phi_2,\ldots$, ϕ_N are independent real variables, each lying in the interval $(0,1)$. Then the upper bound of $|G|$ is $N+1$, this being the value of $|G|$ when $\phi_1 = \phi_2 = \ldots = \phi_N = 0$.

We consider the polynomial expansions of $\{F(t)\}^k$ and $\{G(\phi_1,\ldots,\phi_N)\}^k$, where k is an arbitrary positive integer; and we observe that each of these expansions contains the same number of terms. For, the numbers a_1, $a_2,\ldots$, a_N being linearly independent, no two terms in the expansion of $\{F(t)\}^k$ fall together. Also the moduli of corresponding terms are equal. Thus if

$$\{G(\phi_1,\ldots,\phi_N)\}^k = 1+\sum C_q\, e^{2\pi i(\lambda_{q,1}\phi_1+\ldots+\lambda_{q,N}\phi_N)},$$

then
$$\{F(t)\}^k = 1+\sum C_q\, e^{2\pi i\{\lambda_{q,1}(a_1t-b_1)+\ldots+\lambda_{q,N}(a_Nt-b_N)\}}$$
$$= 1+\sum C_q\, e^{2\pi i(\alpha_q t-\beta_q)},$$

say. Now the mean values

$$F_k = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} |F(t)|^{2k}\,dt$$

and
$$G_k = \int_0^1\int_0^1\ldots\int_0^1 |G(\phi_1,\ldots,\phi_N)|^{2k}\,d\phi_1\ldots d\phi_N$$

are equal, each being equal to

$$1+\sum C_q^2.$$

This is easily seen in each case on expressing the squared modulus as a product of conjugates and integrating term by term.

Since $N+1$ is the upper bound of $|G|$, the lemma gives

$$\lim_{k\to\infty} G_k^{1/2k} = N+1.$$

Hence also
$$\lim_{k\to\infty} F_k^{1/2k} = N+1.$$

But plainly
$$F_k^{1/2k} \leqslant L$$

for all values of k. Hence $L \geqslant N+1$, and so in fact $L = N+1$. This proves the theorem.

8.4. THEOREM 8.4. *If $\sigma > 1$, then*

$$|\zeta(s)| \leqslant \zeta(\sigma) \tag{8.4.1}$$

for all values of t, while

$$|\zeta(s)| \geqslant (1-\epsilon)\zeta(\sigma) \tag{8.4.2}$$

for some indefinitely large values of t.

We have
$$|\zeta(s)| = \left|\sum_{n=1}^{\infty} n^{-s}\right| \leqslant \sum_{n=1}^{\infty} n^{-\sigma} = \zeta(\sigma),$$

so that the whole difficulty lies in the second part. To prove this we use Dirichlet's theorem. For all values of N

$$\zeta(s) = \sum_{n=1}^{N} n^{-\sigma}e^{-it\log n}+\sum_{n=N+1}^{\infty} n^{-\sigma-it},$$

and hence (the modulus of the first sum being not less than its real part)

$$|\zeta(s)| \geqslant \sum_{n=1}^{N} n^{-\sigma}\cos(t\log n)-\sum_{n=N+1}^{\infty} n^{-\sigma}. \tag{8.4.3}$$

By Dirichlet's theorem there is a number t ($t_0 \leqslant t \leqslant t_0 q^N$) and integers $x_1,\ldots, x_N$, such that, for given N and q ($q \geqslant 4$),

$$\left|\frac{t\log n}{2\pi}-x_n\right| \leqslant \frac{1}{q} \quad (n = 1, 2,\ldots, N).$$

Hence $\cos(t\log n) \geqslant \cos(2\pi/q)$ for these values of n, and so

$$\sum_{n=1}^{N} n^{-\sigma}\cos(t\log n) \geqslant \cos(2\pi/q)\sum_{n=1}^{N} n^{-\sigma} > \cos(2\pi/q)\zeta(\sigma)-\sum_{n=N+1}^{\infty} n^{-\sigma}.$$

Hence by (8.4.3)

$$|\zeta(s)| \geqslant \cos(2\pi/q)\zeta(\sigma)-2\sum_{N+1}^{\infty} n^{-\sigma}.$$

Now
$$\zeta(\sigma) = \sum_{n=1}^{\infty} n^{-\sigma} > \int_1^{\infty} u^{-\sigma}\,du = \frac{1}{\sigma-1},$$

and
$$\sum_{N+1}^{\infty} n^{-\sigma} < \int_N^{\infty} u^{-\sigma}\,du = \frac{N^{1-\sigma}}{\sigma-1}.$$

Hence
$$|\zeta(s)| \geqslant \{\cos(2\pi/q)-2N^{1-\sigma}\}\zeta(\sigma), \tag{8.4.4}$$

and the result follows if q and N are large enough.

THEOREM 8.4 (A). *The function $\zeta(s)$ is unbounded in the open region*

$$\sigma > 1,\ t > \delta > 0.$$

This follows at once from the previous theorem, since the upper bound $\zeta(\sigma)$ of $\zeta(s)$ itself tends to infinity as $\sigma \to 1$.

THEOREM 8.4 (B). *The function $\zeta(1+it)$ is unbounded as $t \to \infty$.*

This follows from the previous theorem and the theorem of Phragmén and Lindelöf. Since $\zeta(2+it)$ is bounded, if $\zeta(1+it)$ were also bounded $\zeta(s)$ would be bounded throughout the half-strip $1 \leqslant \sigma \leqslant 2$, $t > \delta$; and this is false, by the previous theorem.

8.5. Dirichlet's theorem also gives the following more precise result.†

THEOREM 8.5. *However large t_1 may be, there are values of s in the region $\sigma > 1$, $t > t_1$, for which*

$$|\zeta(s)| > A \log\log t. \tag{8.5.1}$$

Also
$$\zeta(1+it) = \Omega(\log\log t). \tag{8.5.2}$$

Take $t_0 = 1$ and $q = 6$ in the proof of Theorem 8.4. Then (8.4.4) gives

$$|\zeta(s)| \geqslant (\tfrac{1}{2}-2N^{1-\sigma})/(\sigma-1) \tag{8.5.3}$$

for a value of t between 1 and 6^N. We choose N to be the integer next above $8^{1/(\sigma-1)}$. Then

$$|\zeta(s)| \geqslant \frac{1}{4(\sigma-1)} \geqslant \frac{\log(N-1)}{4\log 8} > A\log N \tag{8.5.4}$$

for a value of t such that $N > A\log t$. The required inequality (8.5.1) then follows from (8.5.4). It remains only to observe that the value of t in question must be greater than any assigned t_1, if $\sigma-1$ is sufficiently small; otherwise it would follow from (8.5.3) that $\zeta(s)$ was unbounded

† Bohr and Landau (1).

in the region $\sigma > 1$, $1 < t \leqslant t_1$; and we know that $\zeta(s)$ is bounded in any such region.

The second part of the theorem now follows from the first by the Phragmén–Lindelöf method. Consider the function

$$f(s) = \frac{\zeta(s)}{\log\log s},$$

the branch of $\log\log s$ which is real for $s > 1$, and is restricted to $|s| > 1$, $\sigma > 0$, $t > 0$ being taken. Then $f(s)$ is regular for $1 \leqslant \sigma \leqslant 2$, $t > \delta$. Also $|\log\log s| \sim \log\log t$ as $t \to \infty$, uniformly with respect to σ in the strip. Hence $f(2+it) \to 0$ as $t \to \infty$, and so, if $f(1+it) \to 0$, $f(s) \to 0$ uniformly in the strip.† This contradicts (8.5.1), and so (8.5.2) follows.

It is plain that arguments similar to the above may be applied to all Dirichlet series, with coefficients of fixed sign, which are not absolutely convergent on their line of convergence. For example, the series for $\log\zeta(s)$ and its differential coefficients are of this type. The result for $\log\zeta(s)$ is, however, a corollary of that for $\zeta(s)$, which gives at once

$$|\log\zeta(s)| > \log\log\log t - A$$

for some indefinitely large values of t in $\sigma > 1$. For the nth differential coefficient of $\log\zeta(s)$ the result is that

$$\left|\left(\frac{d}{ds}\right)^n \log\zeta(s)\right| > A_n(\log\log t)^n$$

for some indefinitely large values of t in $\sigma > 1$.

8.6. We now turn to the corresponding problems‡ for $1/\zeta(s)$. We cannot apply the argument depending on Dirichlet's theorem to this function, since the coefficients in the series

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

are not all of the same sign; nor can we argue similarly with Kronecker's theorem, since the numbers $(\log n)/2\pi$ are not linearly independent. Actually we consider $\log\zeta(s)$, which depends on the series $\sum p^{-s}$, to which Kronecker's theorem can be applied.

THEOREM 8.6. *The function $1/\zeta(s)$ is unbounded in the open region $\sigma > 1$, $t > \delta > 0$.*

We have for $\sigma \geqslant 1$

$$\left|\log\zeta(s) - \sum_p \frac{1}{p^s}\right| = \left|\sum_p \sum_{m=2}^{\infty} \frac{1}{mp^{ms}}\right| \leqslant \sum_p \sum_{m=2}^{\infty} \frac{1}{p^m} = \sum_p \frac{1}{p(p-1)} = A.$$

† See e.g. my *Theory of Functions*, § 5.63, with the angle transformed into a strip.
‡ Bohr and Landau (7).

Now
$$\mathbf{R}\Big(\sum_p \frac{1}{p^s}\Big) = \sum_{n=1}^{\infty} \frac{\cos(t\log p_n)}{p_n^\sigma} \leqslant \sum_{n=1}^{N} \frac{\cos(t\log p_n)}{p_n^\sigma} + \sum_{n=N+1}^{\infty} \frac{1}{p_n^\sigma}.$$
Also *the numbers* $\log p_n$ *are linearly independent.* For it follows from the theorem that an integer can be expressed as a product of prime factors in one way only, that there can be no relation of the form
$$p_1^{\lambda_1} p_2^{\lambda_2} \dots p_N^{\lambda_N} = 1,$$
where the λ's are integers, and therefore no relation of the form
$$\lambda_1 \log p_1 + \dots + \lambda_N \log p_N = 0.$$
Hence also the numbers $(\log p_n)/2\pi$ are linearly independent. It follows therefore from Kronecker's theorem that we can find a number t and integers $x_1,\dots, x_N$ such that
$$\left| t\frac{\log p_n}{2\pi} - \tfrac{1}{2} - x_n \right| \leqslant \tfrac{1}{6} \quad (n = 1, 2,\dots, N),$$
or
$$|t\log p_n - \pi - 2\pi x_n| \leqslant \tfrac{1}{3}\pi \quad (n = 1, 2,\dots, N).$$
Hence for these values of n
$$\cos(t\log p_n) = -\cos(t\log p_n - \pi - 2\pi x_n) \leqslant -\cos\tfrac{1}{3}\pi = -\tfrac{1}{2},$$
and hence
$$\mathbf{R}\Big(\sum_p \frac{1}{p^s}\Big) \leqslant -\frac{1}{2}\sum_{n=1}^{N} \frac{1}{p_n^\sigma} + \sum_{n=N+1}^{\infty} \frac{1}{p_n^\sigma}.$$
Since $\sum p_n^{-1}$ is divergent, we can, if H is any assigned positive number, choose σ so near to 1 that $\sum p_n^{-\sigma} > H$. Having fixed σ, we can choose N so large that
$$\sum_{n=1}^{N} p_n^{-\sigma} > \tfrac{3}{4}H, \qquad \sum_{n=N+1}^{\infty} p_n^{-\sigma} < \tfrac{1}{4}H.$$
Then
$$\mathbf{R}\Big(\sum_p p^{-s}\Big) < -\tfrac{3}{8}H + \tfrac{1}{4}H = -\tfrac{1}{8}H.$$
Since H may be as large as we please, it follows that $\mathbf{R}(\sum p^{-s})$, and so $\log|\zeta(s)|$, takes arbitrarily large negative values. This proves the theorem.

THEOREM 8.6 (A). *The function* $1/\zeta(1+it)$ *is unbounded as* $t \to \infty$.

This follows from the previous theorem in the same way as Theorem 8.4 (B) from Theorem 8.4 (A).

We cannot, however, proceed to deduce an analogue of Theorem 8.5 for $1/\zeta(s)$. In proving Theorem 8.5, each of the numbers $\cos(t\log n)$ has to be made as near as possible to 1, and this can be done by Dirichlet's theorem. In Theorem 8.6, each of the numbers $\cos(t\log p_n)$ has to be made as near as possible to -1, and this requires Kronecker's theorem.

Now Theorem 8.5 depends on the fact that we can assign an upper limit to the number t which satisfies the conditions of Dirichlet's theorem. Since there is no such upper limit in Kronecker's theorem, the corresponding argument for $1/\zeta(s)$ fails. We shall see later that the analogue of Theorem 8.5 is in fact true, but it requires a much more elaborate proof.

8.7. Before proceeding to these deeper theorems, we shall give another method of proving some of the above results.† This method deals directly with integrals of high powers of the functions in question, and so might be described as a short cut which avoids explicit use of Diophantine approximation.

We write
$$M\{|f(s)|^2\} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(\sigma+it)|^2\,dt,$$
and prove the following lemma.

LEMMA. *Let*
$$g(s) = \sum_{m=1}^{\infty} \frac{b_m}{m^s}, \qquad h(s) = \sum_{n=1}^{\infty} \frac{c_n}{n^s}$$
be absolutely convergent for a given value of σ, *and let every* m *with* $b_m \neq 0$ *be prime to every* n *with* $c_n \neq 0$. *Then for such* σ
$$M\{|g(s)h(s)|^2\} = M\{|g(s)|^2\}M\{|h(s)|^2\}.$$
By Theorem 7.1
$$M\{|g(s)|^2\} = \sum_{m=1}^{\infty} \frac{|b_m|^2}{m^{2\sigma}}, \qquad M\{|h(s)|^2\} = \sum_{n=1}^{\infty} \frac{|c_n|^2}{n^{2\sigma}}.$$
Now
$$g(s)h(s) = \sum_{r=1}^{\infty} \frac{d_r}{r^s},$$
where each term $d_r r^{-s}$ is the product of two terms $b_m m^{-s}$ and $c_n n^{-s}$. Hence
$$M\{|g(s)h(s)|^2\} = \sum_{r=1}^{\infty} \frac{|d_r|^2}{r^{2\sigma}} = \sum\sum \frac{|b_m c_n|^2}{(mn)^{2\sigma}} = M\{|g(s)|^2\}M\{|h(s)|^2\}.$$

We can now prove the analogue for $1/\zeta(s)$ of Theorem 8.4.

THEOREM 8.7. *If* $\sigma > 1$, *then*
$$\left|\frac{1}{\zeta(s)}\right| \leqslant \frac{\zeta(\sigma)}{\zeta(2\sigma)} \tag{8.7.1}$$
for all values of t, *while*
$$\left|\frac{1}{\zeta(s)}\right| \geqslant (1-\epsilon)\frac{\zeta(\sigma)}{\zeta(2\sigma)} \tag{8.7.2}$$
for some indefinitely large values of t.

† Bohr and Landau (7).

We have, for $\sigma > 1$,

$$\left|\frac{1}{\zeta(s)}\right| = \left|\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}\right| \leqslant \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^\sigma}.$$

Since
$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p \left(1-\frac{1}{p^s}\right)$$
we have also

$$\sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^\sigma} = \prod_p \left(1+\frac{1}{p^\sigma}\right) = \prod_p \left(\frac{1-p^{-2\sigma}}{1-p^{-\sigma}}\right) = \frac{\zeta(\sigma)}{\zeta(2\sigma)},$$

and the first part follows.

To prove the second part, write

$$\frac{1}{\zeta(s)} = \prod_{n=1}^{N} \left(1-\frac{1}{p_n^s}\right)\eta_N(s),$$

$$\frac{1}{\{\zeta(s)\}^k} = \prod_{n=1}^{N} \left(1-\frac{1}{p_n^s}\right)^k \{\eta_N(s)\}^k.$$

By repeated application of the lemma it follows that

$$M\left\{\frac{1}{|\zeta(s)|^{2k}}\right\} = \prod_{n=1}^{N} M\left\{\left|\left(1-\frac{1}{p_n^s}\right)\right|^{2k}\right\} M\{|\eta_N(s)|^{2k}\}.$$

Now, for every p,

$$M\left\{\left|1-\frac{1}{p^s}\right|^{2k}\right\} = \frac{\log p}{2\pi} \int_0^{2\pi/\log p} \left|1-\frac{1}{p^s}\right|^{2k} dt,$$

since the integrand is periodic with period $2\pi/\log p$; and

$$M\{|\eta_N(s)|^{2k}\} \geqslant 1,$$

since the Dirichlet series for $\{\eta_N(s)\}^k$ begins with $1+\ldots$. Hence

$$M\left\{\frac{1}{|\zeta(s)|^{2k}}\right\} \geqslant \prod_{n=1}^{N} \frac{\log p_n}{2\pi} \int_0^{2\pi/\log p_n} \left|1-\frac{1}{p_n^s}\right|^{2k} dt.$$

Now
$$\lim_{k\to\infty} \left(\int_0^{2\pi/\log p} \left|1-\frac{1}{p^s}\right|^{2k} dt\right)^{1/2k} = \max_{0\leqslant t\leqslant 2\pi/\log p} \left|1-\frac{1}{p^s}\right| = 1+\frac{1}{p^\sigma}.$$

Hence
$$\varliminf_{k\to\infty}\left[M\left\{\frac{1}{|\zeta(s)|^{2k}}\right\}\right]^{1/2k} \geqslant \prod_{n=1}^{N}\left(1+\frac{1}{p_n^\sigma}\right).$$

Since the left-hand side is independent of N, we can make $N \to \infty$ on the right, and obtain

$$\varliminf_{k\to\infty}\left[M\left\{\frac{1}{|\zeta(s)|^{2k}}\right\}\right]^{1/2k} \geqslant \frac{\zeta(\sigma)}{\zeta(2\sigma)}.$$

Hence to any ϵ corresponds a k such that

$$\left[M\left\{\frac{1}{|\zeta(s)|^{2k}}\right\}\right]^{1/2k} > (1-\epsilon)\frac{\zeta(\sigma)}{\zeta(2\sigma)},$$

and (8.7.2) now follows.

Since $\zeta(\sigma)/\zeta(2\sigma) \to \infty$ as $\sigma \to 1$, this also gives an alternative proof of Theorem 8.6

It is easy to see that a similar method can be used to prove Theorem 8.4 (A). It is also possible to prove Theorems 8.4 (B) and 8.6 (A) directly by this method without using the Phragmén–Lindelöf theorem. This, however, requires an extension of the general mean-value theorem for Dirichlet series.

8.8. THEOREM 8.8.† *However large t_0 may be, there are values of s in the region $\sigma > 1$, $t > t_0$ for which*

$$\left|\frac{1}{\zeta(s)}\right| > A \log\log t.$$

Also
$$\frac{1}{\zeta(1+it)} = \Omega(\log\log t).$$

As in the case of Theorem 8.5, it is enough to prove the first part. We first prove some lemmas. The object of these lemmas is to supply, for the particular case in hand, what Kronecker's theorem lacks in the general case, viz. an upper bound for the number t which satisfies the conditions (8.3.1).

LEMMA α. *If m and n are different positive integers,*

$$\left|\log\frac{m}{n}\right| > \frac{1}{\max(m,n)}.$$

For if $m < n$

$$\log\frac{n}{m} \geqslant \log\frac{n}{n-1} = \frac{1}{n}+\frac{1}{2}\frac{1}{n^2}+\ldots > \frac{1}{n}.$$

LEMMA β. *If $p_1,\ldots, p_N$ are the first N primes, and $\mu_1,\ldots, \mu_N$ are integers, not all 0 (not necessarily positive), then*

$$\left|\log \prod_{n=1}^{N} p_n^{\mu_n}\right| > p_N^{-\mu N} \quad (\mu = \max|\mu_n|).$$

For $\prod_{n=1}^{N} p_n^{\mu_n} = u/v$, where

$$u = \prod_{\mu_n>0} p_n^{\mu_n}, \qquad v = \prod_{\mu_n<0} p_n^{-\mu_n},$$

† Bohr and Landau (7).

and u and v, being mutually prime, are different. Also

$$\max(u,v) \leqslant \prod_{n=1}^{N} p_n^{\mu} \leqslant p_N^{N\mu},$$

and the result follows from Lemma α.

LEMMA γ. *The number of solutions in positive or zero integers of the equation*

$$\nu_0+\nu_1+\ldots+\nu_N = k$$

does not exceed $(k+1)^N$.

For $N = 1$ the number of solutions is $k+1$, so that the theorem holds. Suppose that it holds for any given N. Then for given ν_{N+1} the number of solutions of

$$\nu_0+\nu_1+\ldots+\nu_N = k-\nu_{N+1}$$

does not exceed $(k-\nu_{N+1}+1)^N \leqslant (k+1)^N$; and ν_{N+1} can take $k+1$ values. Hence the total number of solutions is $\leqslant (k+1)^{N+1}$, whence the result.

LEMMA δ. *For $N > A$, there exits a t satisfying $0 \leqslant t \leqslant \exp(N^6)$ for which*

$$\cos(t\log p_n) < -1+\frac{1}{N} \quad (n \leqslant N).$$

Let $N > 1$, $k > 1$. Then

$$\Big(\sum_{n=0}^{N} x_n\Big)^k = \sum c(\nu_0,\ldots,\nu_N)x_0^{\nu_0}\ldots x_N^{\nu_N},$$

where
$$c(\nu_0,\ldots,\nu_N) = \frac{k!}{\nu_0!\ldots\nu_N!}, \qquad \sum \nu_n = k.$$

The number of distinct terms in the expansion is at most $(k+1)^N < k^{2N}$, by Lemma γ. Hence

$$(\sum c)^2 \leqslant \sum c^2 \sum 1 < k^{2N} \sum c^2,$$

so that
$$\sum c^2 > k^{-2N}(\sum c)^2 = k^{-2N}(N+1)^{2k}.$$

Let
$$F(t) = 1-\sum_{n=1}^{N} e^{it\log p_n},$$

so that

$$\{F(t)\}^k = \sum c(\nu_0,\ldots,\nu_N)(-1)^{\nu_1+\ldots+\nu_N}\exp\Big(it\sum_{1}^{N}\nu_n\log p_n\Big),$$

$$|F(t)|^{2k} = \sum_{\nu,\nu'}\sum cc'(-1)^{\Sigma\nu_n+\Sigma\nu'_n}\exp\Big\{it\sum_n(\nu_n-\nu'_n)\log p_n\Big\}$$

$$= \Sigma_1+\Sigma_2,$$

where Σ_1 is taken over values of (ν,ν') for which $\nu_1 = \nu'_1$, $\nu_2 = \nu'_2,\ldots,$ and

Σ_2 over the remainder. Now

$$\frac{1}{T}\int_0^T e^{i\alpha t}\,dt = 1 \quad (\alpha = 0),$$

$$\left|\frac{1}{T}\int_0^T e^{i\alpha t}\,dt\right| = \left|\frac{e^{i\alpha T}-1}{i\alpha T}\right| \leqslant \frac{2}{|\alpha|T} \quad (\alpha \neq 0).$$

Hence
$$\frac{1}{T}\int_0^T |F(t)|^{2k}\,dt \geqslant \Sigma_1 c^2 - \Sigma_2\frac{2cc'}{|\sum(\nu_n-\nu'_n)\log p_n|T}.$$

By Lemma β, since the numbers $\nu_n-\nu'_n$ are not all 0,

$$|\sum(\nu_n-\nu'_n)\log p_n| = \left|\log\prod_1^N p_n^{(\nu_n-\nu'_n)}\right| > p_N^{-N\max|\nu_n-\nu'_n|} \geqslant p_N^{-kN}.$$

Hence

$$\frac{1}{T}\int_0^T |F(t)|^{2k}\,dt \geqslant \sum c^2-\frac{2p_N^{kN}}{T}\sum\sum cc'$$

$$\geqslant k^{-2N}(\sum c)^2-\frac{2p_N^{kN}}{T}(\sum c)^2$$

$$= \left(k^{-2N}-\frac{2p_N^{kN}}{T}\right)(N+1)^{2k}.$$

In this we take $k = N^4$, $T = e^{N^6}$, and obtain, for $N > A$,

$$k^{-2N}-\frac{2p_N^{kN}}{T} = N^{-8N}-2\left(\frac{p_N}{e^N}\right)^{kN} > e^{-N^3/(N+1)}.$$

Hence

$$\left\{\frac{1}{T}\int_0^T |F(t)|^{2k}\,dt\right\}^{1/2k} \geqslant (N+1)e^{-1/(2N(N+1))} > N+1-\frac{1}{2N}.$$

Hence there is a t in $(0, e^{N^6})$ such that

$$|F(t)| > N+1-\frac{1}{2N}.$$

Suppose, however, that $\cos(t\log p_n) \geqslant -1+1/N$ for some value of n. Then

$$|F(t)| \leqslant N-1+|1-e^{it\log p_n}| = N-1+\sqrt{2}(1-\cos t\log p_n)^{\frac{1}{2}}$$

$$\leqslant N-1+\sqrt{2}\left(2-\frac{1}{N}\right)^{\frac{1}{2}} \leqslant N+1-\frac{1}{2N},$$

a contradiction. Hence the result.

We can now prove Theorem 8.8. As in § 8.6, for $\sigma > 1$

$$\log\frac{1}{|\zeta(s)|} = -\sum\frac{\cos(t\log p_n)}{p_n^\sigma}+O(1).$$

Let now N be large, $t = t(N)$ the number of Lemma δ, $\delta = 1/\log N$, and $\sigma = 1+\delta$. Then

$$\log\frac{A}{|\zeta(s)|} \geqslant -\sum\frac{\cos(t\log p_n)}{p_n^\sigma} \geqslant \left(1-\frac{1}{N}\right)\sum_1^N\frac{1}{p_n^\sigma}-\sum_{N+1}^\infty\frac{1}{p_n^\sigma}$$

$$> \left(1-\frac{1}{N}\right)\sum\frac{1}{p^\sigma}-2\sum_{N+1}^\infty\frac{1}{p_n^\sigma} > \left(1-\frac{1}{N}\right)\{\log\zeta(\sigma)-A\}-2\sum_{N+1}^\infty\frac{1}{(An\log n)^\sigma}$$

$$> \left(1-\frac{1}{N}\right)\left(\log\frac{1}{\delta}-A\right)-\frac{A}{\log N}\sum_{N+1}^\infty\frac{1}{n^\sigma},$$

$$\log\frac{A\delta}{|\zeta(s)|} > -A-\frac{1}{N}\log\frac{1}{\delta}-\frac{A}{\log N}\,\frac{N^{1-\sigma}}{\sigma-1} > -A,$$

$$\frac{1}{|\zeta(s)|} > \frac{A}{\delta} = A\log N > A\log\log t.$$

The number $t = t(N)$ evidently tends to infinity with N, since $1/\zeta(s)$ is bounded in $|t| \leqslant A$, $\sigma \geqslant 1$, and the proof is completed.

8.9. In Theorems 8.5 and 8.8 we have proved that each of the inequalities

$$|\zeta(1+it)| > A\log\log t, \qquad 1/|\zeta(1+it)| > A\log\log t$$

is satisfied for some arbitrarily large values of t, if A is a suitable constant. We now consider the question how large the constant can be in the two cases.

Since neither $|\zeta(1+it)|/\log\log t$ nor $|\zeta(1+it)|^{-1}/\log\log t$ is known to be bounded, the question of the constants might not seem to be of much interest. But we shall see later that on the Riemann hypothesis they are both bounded; in fact if

$$\lambda = \overline{\lim_{t\to\infty}}\frac{|\zeta(1+it)|}{\log\log t}, \qquad \mu = \overline{\lim_{t\to\infty}}\frac{1/|\zeta(1+it)|}{\log\log t}, \tag{8.9.1}$$

then, on the Riemann hypothesis,

$$\lambda \leqslant 2e^\gamma, \qquad \mu \leqslant \frac{12}{\pi^2}e^\gamma, \tag{8.9.2}$$

where γ is Euler's constant.

There is therefore a certain interest in proving the following results.†

THEOREM 8.9 (A). $\displaystyle\overline{\lim_{t\to\infty}}\frac{|\zeta(1+it)|}{\log\log t} \geqslant e^\gamma.$

THEOREM 8.9 (B). $\displaystyle\overline{\lim_{t\to\infty}}\frac{1/|\zeta(1+it)|}{\log\log t} \geqslant \frac{6}{\pi^2}e^\gamma.$

Thus on the Riemann hypothesis it is only a factor 2 which remains in doubt in each case.

We first prove some identities and inequalities. As in § 7.19, if

$$F_k(s) = \sum_{n=1}^\infty\frac{d_k^2(n)}{n^s} \quad (\sigma > 1) \tag{8.9.3}$$

and

$$f_k(x) = \sum_{m=0}^\infty\left\{\frac{(k+m-1)!}{(k-1)!\,m!}\right\}^2 x^m, \tag{8.9.4}$$

then

$$F_k(s) = \prod_p f_k(p^{-s}). \tag{8.9.5}$$

Now for real x

$$f_k(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{m=0}^\infty\frac{(k+m-1)!}{(k-1)!\,m!}x^{\frac{1}{2}m}e^{im\phi}\right|^2 d\phi$$

$$= \frac{1}{\pi}\int_0^\pi\frac{d\phi}{|1-x^{\frac{1}{2}}e^{i\phi}|^{2k}} = \frac{1}{\pi}\int_0^\pi\frac{d\phi}{(1-2\sqrt{x}\cos\phi+x)^k}. \tag{8.9.6}$$

Using the familiar formula

$$P_n(z) = \frac{1}{\pi}\int_0^\pi\frac{d\phi}{\{z-\sqrt{(z^2-1)}\cos\phi\}^{n+1}} \tag{8.9.7}$$

for the Legendre polynomial of degree n, we see that

$$f_k(x) = \frac{1}{(1-x)^k}P_{k-1}\left(\frac{1+x}{1-x}\right). \tag{8.9.8}$$

Naturally this identity holds also for complex x; it gives

$$F_k(s) = \prod_p\frac{1}{(1-p^{-s})^k}P_{k-1}\left(\frac{1+p^{-s}}{1-p^{-s}}\right) = \zeta^k(s)\prod_p P_{k-1}\left(\frac{1+p^{-s}}{1-p^{-s}}\right). \tag{8.9.9}$$

A similar set of formulae holds for $1/\zeta(s)$. We have

$$\frac{1}{\{\zeta(s)\}^k} = \prod_p\left(1-\frac{1}{p^s}\right)^k = \prod_p\left(1-\frac{k}{p^s}+\frac{k(k-1)}{1.2}\frac{1}{p^{2s}}-\ldots+\frac{(-1)^k}{p^{ks}}\right).$$

† Littlewood (5), (6), Titchmarsh (4), (14).

Hence
$$\frac{1}{\zeta^k(s)} = \sum_{n=1}^{\infty} \frac{b_k(n)}{n^s}, \tag{8.9.10}$$
where the coefficients $b_k(n)$ are determined in an obvious way from the above product. They are integers, but are not all positive.

The form of these coefficients shows that
$$\sum_{n=1}^{\infty} \frac{|b_k(n)|}{n^s} = \prod_p \left(1+\frac{k}{p^s}+\dots+\frac{1}{p^{ks}}\right) = \prod_p \left(1+\frac{1}{p^s}\right)^k$$
$$= \prod_p \left(1-\frac{1}{p^{2s}}\right)^k \left(1-\frac{1}{p^s}\right)^{-k} = \left\{\frac{\zeta(s)}{\zeta(2s)}\right\}^k. \tag{8.9.11}$$

Again, let
$$G_k(s) = \sum_{n=1}^{\infty} \frac{b_k^2(n)}{n^s}. \tag{8.9.12}$$
As in the case of $F_k(s)$,
$$G_k(s) = \prod_p \left(1+\frac{k^2}{p^s}+\frac{k^2(k-1)^2}{1^2 . 2^2}\frac{1}{p^{2s}}+\dots+\frac{1}{p^{ks}}\right) = \prod_p g_k(p^{-s}), \tag{8.9.13}$$
say. Now, for real x,
$$g_k(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\sum_{m=0}^{k} \frac{k!}{m!(k-m)!} x^{\frac{1}{2}m} e^{im\phi}\right|^2 d\phi$$
$$= \frac{1}{\pi}\int_0^{\pi} |1+x^{\frac{1}{2}}e^{i\phi}|^{2k}\, d\phi = \frac{1}{\pi}\int_0^{\pi} (1+2x^{\frac{1}{2}}\cos\phi+x)^k\, d\phi.$$
Comparing this with the formula
$$P_n(z) = \frac{1}{\pi}\int_0^{\pi} \{z+\sqrt{(z^2-1)}\cos\phi\}^n\, d\phi$$
we see that†
$$g_k(x) = (1-x)^k P_k\left(\frac{1+x}{1-x}\right). \tag{8.9.14}$$
Hence
$$G_k(s) = \prod_p (1-p^{-s})^k P_k\left(\frac{1+p^{-s}}{1-p^{-s}}\right) = \frac{1}{\zeta^k(s)} \prod_p P_k\left(\frac{1+p^{-s}}{1-p^{-s}}\right).$$
We have also the identity
$$F_{k+1}(s) = \zeta^{2k+1}(s) G_k(s). \tag{8.9.15}$$

† This formula is, essentially, Murphy's well-known formula
$$P_k(\cos\theta) = \cos^{2k}\tfrac{1}{2}\theta\, F(-k, -k; 1; -\tan^2\tfrac{1}{2}\theta)$$
with $x = -\tan^2\frac{1}{2}\theta$; cf. Hobson, *Spherical and Ellipsoidal Harmonics*, pp. 22, 31.

Again for $0 < x < \frac{1}{2}$
$$f_k(x) > \frac{1}{\pi}\int_0^{\pi/k} \frac{d\phi}{(1-2\sqrt{x}\cos\phi+x)^k}$$
$$= \frac{1}{\pi(1-\sqrt{x})^{2k}} \int_0^{\pi/k} \left\{1-\frac{2\sqrt{x}(1-\cos\phi)}{1-2\sqrt{x}\cos\phi+x}\right\}^k d\phi$$
$$= \frac{1}{\pi(1-\sqrt{x})^{2k}} \int_0^{\pi/k} \left\{1+O\left(\frac{1}{k^2}\right)\right\}^k d\phi > \frac{1}{2k(1-\sqrt{x})^{2k}} \tag{8.9.16}$$
if k is large enough. Hence also
$$g_k(x) = (1-x)^{2k+1} f_{k+1}(x) > \frac{(1+\sqrt{x})^{2k+1}}{2k+2} > \frac{(1+\sqrt{x})^{2k}}{3k} \tag{8.9.17}$$
for k large enough; and
$$g_k(x) \leqslant \frac{1}{\pi}\int_0^{\pi} (1+\sqrt{x})^{2k}\, d\phi = (1+\sqrt{x})^{2k} \tag{8.9.18}$$
for all values of x and k.

8.10. Proof of Theorem 8.9 (A). Let $\sigma > 1$. Then
$$\int_{-T}^{T} \left(1-\frac{|t|}{T}\right) |\zeta(\sigma+it)|^{2k}\, dt = \int_{-T}^{T} \left(1-\frac{|t|}{T}\right) \sum_{m=1}^{\infty} \frac{d_k(m)}{m^{\sigma+it}} \sum_{n=1}^{\infty} \frac{d_k(n)}{n^{\sigma-it}}\, dt$$
$$= \sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^{2\sigma}} \int_{-T}^{T} \left(1-\frac{|t|}{T}\right) dt + \sum_{m\neq n}\sum \frac{d_k(m)d_k(n)}{(mn)^\sigma} \int_{-T}^{T} \left(1-\frac{|t|}{T}\right)\left(\frac{n}{m}\right)^{it} dt$$
$$= T\sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^{2\sigma}} + \sum_{m\neq n}\sum \frac{d_k(m)d_k(n)}{(mn)^\sigma} \frac{4\sin^2\{\frac{1}{2}T\log(n/m)\}}{T\log^2(n/m)}$$
$$\geqslant T\sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^{2\sigma}} = TF_k(2\sigma). \tag{8.10.1}$$
Since (from its original definition) $f_k(p^{-2\sigma}) \geqslant 1$ for all values of p,
$$F_k(2\sigma) \geqslant \prod_{p\leqslant x} f_k(p^{-2\sigma}) \geqslant \prod_{p\leqslant x} \left\{\frac{1}{2k}\left(1-\frac{1}{p^\sigma}\right)^{-2k}\right\} \tag{8.10.2}$$
for any positive x and k large enough. Here the number of factors is $\pi(x) < Ax/\log x$. Hence if $x > \sqrt{k}$
$$\prod_{p\leqslant x} \frac{1}{2k} \geqslant \left(\frac{1}{2k}\right)^{Ax/\log x} = \exp\left(-\frac{Ax\log 2k}{\log x}\right) > e^{-Ax}. \tag{8.10.3}$$

Also

$$\log\prod_{p\leqslant x}\frac{1-p^{-\sigma}}{1-p^{-1}}=\sum_{p\leqslant x}\log\frac{1-p^{-\sigma}}{1-p^{-1}}=\sum_{p\leqslant x}O\left(\frac{1}{p^{\sigma}}-\frac{1}{p}\right)$$

$$=\sum_{p\leqslant x}O\left(\log p\int_1^{\sigma}\frac{du}{p^u}\right)=O\left\{(\sigma-1)\sum_{p\leqslant x}\frac{\log p}{p}\right\}=O\{(\sigma-1)\log x\}. \tag{8.10.4}$$

Hence
$$F_k(2\sigma)>e^{-Ax-Ak(\sigma-1)\log x}\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)^{-2k},$$
and
$$\left\{\frac{1}{T}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)|\zeta(\sigma+it)|^{2k}\,dt\right\}^{1/2k}>e^{-Ax/k-A(\sigma-1)\log x}\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)^{-1}$$
$$>\{e^{\gamma}+o(1)\}e^{-Ax/k-A(\sigma-1)\log x}\log x$$
as $x\to\infty$, by (3.15.2).

Let $x=\delta k$, where $k^{-\frac{1}{2}}<\delta<1$, and $\sigma=1+\eta/\log k$, where $0<\eta<1$. Then the right-hand side is greater than
$$\{e^{\gamma}+o(1)\}e^{-A\delta-A\eta}\left(\log k-\log\frac{1}{\delta}\right).$$
Also, if $m_{\sigma,T}=\max\limits_{1\leqslant|t|\leqslant T}|\zeta(\sigma+it)|$, the left-hand side does not exceed
$$\left\{\frac{2}{T}\int_0^1\left(1-\frac{|t|}{T}\right)\left(\frac{2}{\sigma-1}\right)^{2k}dt\right\}^{1/2k}+\left\{\frac{2}{T}\int_1^T\left(1-\frac{|t|}{T}\right)m_{\sigma,T}^{2k}\,dt\right\}^{1/2k}$$
$$<\left(\frac{2}{T}\right)^{1/2k}\frac{2}{\sigma-1}+2^{1/2k}m_{\sigma,T}.$$
Hence
$$m_{\sigma,T}>2^{-1/2k}\{e^{\gamma}+o(1)\}e^{-A\delta-A\eta}\left(\log k-\log\frac{1}{\delta}\right)-\frac{2\log k}{T^{1/2k}\eta}.$$
Let $T=\eta^{-4k}$, so that
$$\log\log T=\log k+\log\left(4\log\frac{1}{\eta}\right).$$
Then
$$m_{\sigma,T}>2^{-1/2k}\{e^{\gamma}+o(1)\}e^{-A\delta-A\eta}\left\{\log\log T-\log\left(4\log\frac{1}{\eta}\right)-\log\frac{1}{\delta}\right\}-$$
$$-2\eta\left\{\log\log T-\log\left(4\log\frac{1}{\eta}\right)\right\}.$$
Giving δ and η arbitrarily small values, and then making $k\to\infty$, i.e. $T\to\infty$, we obtain
$$\overline{\lim}\frac{m_{\sigma,T}}{\log\log T}\geqslant e^{\gamma},$$
where, of course, σ is a function of T.

The result now follows by the Phragmén–Lindelöf method. Let
$$f(s)=\frac{\zeta(s)}{\log\log(s+hi)}$$
where $h>4$, and let
$$\lambda=\overline{\lim}\frac{|\zeta(1+it)|}{\log\log t}.$$
We may suppose λ finite, or there is nothing to prove. On $\sigma=1$, $t\geqslant 0$, we have
$$|f(s)|\leqslant\frac{|\zeta(s)|}{\log\log t}<\lambda+\epsilon\quad(t>t_0).$$
Also, on $\sigma=2$, $\quad |f(s)|=o(1)<\lambda+\epsilon\quad(t>t_1).$

We can choose h so that $|f(s)|<\lambda+\epsilon$ also on the remainder of the boundary of the strip bounded by $\sigma=1$, $\sigma=2$, and $t=1$. Then, by the Phragmén–Lindelöf theorem, $|f(s)|<\lambda+\epsilon$ in the interior, and so
$$\overline{\lim}\frac{|\zeta(s)|}{\log\log t}=\overline{\lim}\frac{|\zeta(s)|}{\log\log(t+h)}\leqslant\lambda.$$
Hence $\lambda\geqslant e^{\gamma}$, the required result.

8.11. Proof of Theorem 8.9 (B). The above method depends on the fact that $d_k(n)$ is positive. Since $b_k(n)$ is not always positive, a different method is required in this case.

Let $\sigma>1$, and let N be any positive number. Then
$$\frac{1}{T}\int_0^T\left|\sum_{n\leqslant N}\frac{b_k(n)}{n^s}\right|^2dt=\frac{1}{T}\int_0^T\sum_{m\leqslant N}\frac{b_k(m)}{m^{\sigma+it}}\sum_{n\leqslant N}\frac{b_k(n)}{n^{\sigma-it}}\,dt$$
$$=\sum_{n\leqslant N}\frac{b_k^2(n)}{n^{2\sigma}}+\frac{1}{T}\sum_{m\neq n}\sum\frac{b_k(m)b_k(n)}{m^{\sigma}n^{\sigma}}\int_0^T\left(\frac{n}{m}\right)^{it}dt$$
$$\geqslant\sum_{n\leqslant N}\frac{b_k^2(n)}{n^{2\sigma}}-\frac{1}{T}\sum_{m\neq n}\sum\frac{|b_k(m)b_k(n)|}{m^{\sigma}n^{\sigma}}\frac{2}{|\log n/m|}.$$
Now
$$\left|\log\frac{n}{m}\right|\geqslant\log\frac{n+1}{n}\geqslant\frac{1}{2n}\geqslant\frac{1}{2N},$$
so that the last sum does not exceed
$$\frac{4N}{T}\sum_{m\neq n}\sum\frac{|b_k(m)b_k(n)|}{m^{\sigma}n^{\sigma}}<\frac{4N}{T}\left(\sum_{n=1}^{\infty}\frac{|b_k(n)|}{n^{\sigma}}\right)^2=\frac{4N}{T}\left\{\frac{\zeta(\sigma)}{\zeta(2\sigma)}\right\}^{2k}.$$
Since $\zeta(\sigma)\sim 1/(\sigma-1)$ as $\sigma\to 1$, and $\zeta(2)>1$, we have, if σ is sufficiently near to 1,
$$\frac{\zeta(\sigma)}{\zeta(2\sigma)}<\frac{1}{\sigma-1}.$$

Hence the above last sum is less than

$$\frac{4N}{T(\sigma-1)^{2k}}.$$

Also

$$\left|\frac{1}{\zeta^k(s)}-\sum_{n\leqslant N}\frac{b_k(n)}{n^s}\right| \leqslant \sum_{n>N}\frac{|b_k(n)|}{n^\sigma} < \frac{1}{N^{\frac{1}{2}\sigma-\frac{1}{2}}}\sum_{n>N}\frac{|b_k(n)|}{n^{\frac{1}{2}\sigma+\frac{1}{2}}}$$

$$< \frac{1}{N^{\frac{1}{2}\sigma-\frac{1}{2}}}\left\{\frac{\zeta(\frac{1}{2}\sigma+\frac{1}{2})}{\zeta(\sigma+1)}\right\}^k < \frac{1}{N^{\frac{1}{2}\sigma-\frac{1}{2}}}\left(\frac{2}{\sigma-1}\right)^k$$

for σ sufficiently near to 1. Since for $\sigma > 2$

$$G_k(\sigma) \leqslant \prod_p\left(1+\frac{1}{p^{\frac{1}{2}\sigma}}\right)^{2k} = \prod_p\left(\frac{1-p^{-\sigma}}{1-p^{-\frac{1}{2}\sigma}}\right)^{2k} = \left\{\frac{\zeta(\frac{1}{2}\sigma)}{\zeta(\sigma)}\right\}^{2k},$$

we have similarly

$$G_k(2\sigma)-\sum_{n\leqslant N}\frac{b_k^2(n)}{n^{2\sigma}} = \sum_{n>N}\frac{b_k^2(n)}{n^{2\sigma}} < \frac{1}{N^{\sigma-1}}\sum_{n>N}\frac{b_k^2(n)}{n^{\sigma+1}}$$

$$< \frac{G_k(\sigma+1)}{N^{\sigma-1}} < \frac{1}{N^{\sigma-1}}\left\{\frac{\zeta(\frac{1}{2}\sigma+\frac{1}{2})}{\zeta(\sigma+1)}\right\}^{2k} < \frac{1}{N^{\sigma-1}}\left(\frac{2}{\sigma-1}\right)^{2k}.$$

These two differences are therefore both bounded if

$$N = \left(\frac{2}{\sigma-1}\right)^{2k/(\sigma-1)}$$

With this value of N we have

$$\frac{1}{T}\int_0^T\left|\frac{1}{\zeta^k(s)}+O(1)\right|^2 dt = \frac{1}{T}\int_0^T\left|\sum_{n\leqslant N}\frac{b_k(n)}{n^s}\right|^2 dt$$

$$> G_k(2\sigma)-\frac{4N}{T(\sigma-1)^{2k}}+O(1)$$

$$> \prod_{p\leqslant x}\left\{\frac{1}{3k}\left(1+\frac{1}{p^\sigma}\right)^{2k}\right\}-\frac{4N}{T(\sigma-1)^{2k}}+O(1)$$

by (8.9.17). Now

$$\log\prod_{p\leqslant x}\frac{1+p^{-1}}{1+p^{-\sigma}} = O\{(\sigma-1)\log x\}$$

as in (8.10.4). Hence, as in (8.10.3) and (3.15.3),

$$\prod_{p\leqslant x}\left\{\frac{1}{3k}\left(1+\frac{1}{p^\sigma}\right)^{2k}\right\} > e^{-Ax-Ak(\sigma-1)\log x}\{b+o(1)\}^{2k}\log^{2k}x$$

where $b = 6e^\gamma/\pi^2$.

Choosing x and σ as in the last proof,

$$\frac{N}{(\sigma-1)^{2k}} < \left(\frac{2\log k}{\eta}\right)^{2k\log k/\eta+2k},$$

and we obtain

$$\frac{1}{T}\int_0^T\left|\frac{1}{\zeta^k(s)}+O(1)\right|^2 dt > e^{-A\delta k-A\eta k}\{b+o(1)\}^{2k}\log^{2k}\delta k-$$

$$-\frac{4}{T}\left(\frac{2\log k}{\eta}\right)^{2k\log k/\eta+2k}+O(1).$$

Finally, let

$$T = \left(\frac{2\log k}{\eta}\right)^{2k\log k/\eta+2k}.$$

Then

$$\log\log T = \log k+\log\left(\frac{2\log k}{\eta}+2\right)+\log\log\frac{2\log k}{\eta} < (1+\epsilon)\log k$$

for $k > k_1 = k_1(\epsilon,\eta)$. Hence

$$\frac{1}{T}\int_0^T\left|\frac{1}{\zeta^k(s)}+O(1)\right|^2 dt > e^{-A\delta k-A\eta k}\{b+o(1)\}^{2k}\left(\frac{\log\log T}{1+\epsilon}-\log\frac{1}{\delta}\right)^{2k}+O(1).$$

Let

$$M_{\sigma,T} = \max_{0\leqslant t\leqslant T}\frac{1}{|\zeta(\sigma+it)|}.$$

Since the first term on the right of the above inequality tends to infinity with k (for fixed δ, η, and ϵ) it is clear that $M_{\sigma,T}^k$ tends to infinity. Hence

$$\left|\frac{1}{\zeta^k(s)}+O(1)\right| < 2M_{\sigma,T}^k$$

if k is large enough, and we deduce that

$$4M_{\sigma,T}^{2k} > \tfrac{1}{2}e^{-A\delta k-A\eta k}\{b+o(1)\}^{2k}\left(\frac{\log\log T}{1+\epsilon}-\log\frac{1}{\delta}\right)^{2k}$$

for k large enough. Hence

$$M_{\sigma,T} > \frac{1}{8^{1/2k}}e^{-A\delta-A\eta}\{b+o(1)\}\left(\frac{\log\log T}{1+\epsilon}-\log\frac{1}{\delta}\right).$$

Giving δ, ϵ, and η arbitrarily small values, and then varying T, we obtain

$$\overline{\lim}\frac{M_{\sigma,T}}{\log\log T} \geqslant b.$$

The theorem now follows as in the previous case.

8.12. The above theorems are mainly concerned with the neighbourhood of the line $\sigma = 1$. We now penetrate further into the critical strip, and prove†

THEOREM 8.12. *Let σ be a fixed number in the range $\frac{1}{2} \leqslant \sigma < 1$. Then the inequality*

$$|\zeta(\sigma+it)| > \exp(\log^\alpha t)$$

is satisfied for some indefinitely large values of t, provided that

$$\alpha < 1-\sigma.$$

Throughout the proof k is supposed large enough, and δ small enough, for any purpose that may be required. We take $\frac{1}{2} < \sigma < 1$, and the constants $C_1, C_2, \ldots$, and those implied by the symbol O, are independent of k and δ, but may depend on σ, and on ϵ when it occurs. The case $\sigma = \frac{1}{2}$ is deduced finally from the case $\sigma > \frac{1}{2}$.

We first prove some lemmas.

LEMMA α. *Let*

$$\Gamma(s)\zeta^k(s) = \sum_{m=0}^{k-1} \frac{(-1)^m m!\, a_m^{(k)}}{(s-1)^{m+1}} + \ldots$$

in the neighbourhood of $s = 1$. Then

$$|a_m^{(k)}| < e^{C_1 k} \quad (1 \leqslant m \leqslant k).$$

The $a_m^{(k)}$ are the same as those of § 7.13. We have

$$\Gamma(s) = \sum_{n=0}^{\infty} c_n (s-1)^n, \qquad \zeta^k(s) = (s-1)^{-k} \sum_{n=0}^{\infty} e_n^{(k)} (s-1)^n,$$

where $\quad |c_n| \leqslant C_2^n, \quad |e_n^{(1)}| \leqslant C_3^n \quad (C_2 > 1,\ C_3 > 1).$

Hence $e_n^{(k)}$ is less than the coefficient of $(s-1)^n$ in

$$\Big\{\sum_{n=0}^{\infty} C_3^n (s-1)^n\Big\}^k = \{1 - C_3(s-1)\}^{-k} = \sum_{n=0}^{\infty} \frac{(k+n-1)!}{(k-1)!\, n!} C_3^n (s-1)^n.$$

Hence

$$m!\,|a_m^{(k)}| = \left|\sum_{n=0}^{k-m-1} c_{k-m-n-1}\, e_n^{(k)}\right| < \sum_{n=0}^{k-m-1} C_2^{k-m-n-1} \frac{(k+n-1)!}{(k-1)!\, n!} C_3^n$$

$$< k C_2^k C_3^k \frac{(2k-2)!}{\{(k-1)!\}^2} < e^{C_1 k}.$$

LEMMA β.

$$\frac{1}{\pi} \int_{-\infty}^{\infty} |\Gamma(\sigma+it)\zeta^k(\sigma+it) e^{(\frac{1}{2}\pi - \delta)t}|^2\, dt$$

$$> \int_1^\infty \left|\sum_{n=1}^\infty d_k(n) \exp(-inxe^{-i\delta})\right|^2 x^{2\sigma-1}\, dx - \exp(C_4\, k \log k).$$

† Titchmarsh (4).

By (7.13.3) the left-hand side is greater than

$$2\int_1^\infty |\phi_k(ixe^{-i\delta})|^2 x^{2\sigma-1}\, dx \geqslant \int_1^\infty \left|\sum_{n=1}^\infty d_k(n)\exp(-inxe^{-i\delta})\right|^2 x^{2\sigma-1}\, dx -$$

$$-2\int_1^\infty |R_k(ixe^{-i\delta})|^2 x^{2\sigma-1}\, dx.$$

Since $|\log(ixe^{-i\delta})| \leqslant \log x + \frac{1}{2}\pi$,

$$|R_k(ixe^{-i\delta})| \leqslant \frac{1}{x}\{|a_0^{(k)}| + |a_1^{(k)}|(\log x + \tfrac{1}{2}\pi) + \ldots + a_{k-1}^{(k)}(\log x + \tfrac{1}{2}\pi)^{k-1}\}$$

$$\leqslant \frac{ke^{C_1 k}(\log x + \frac{1}{2}\pi)^{k-1}}{x},$$

and

$$\int_1^\infty (\log x + \tfrac{1}{2}\pi)^{2k-2} x^{2\sigma-3}\, dx < \int_1^\infty (2\log x)^{2k-2} x^{2\sigma-3}\, dx + \int_1^\infty \pi^{2k-2} x^{2\sigma-3}\, dx$$

$$= \frac{\Gamma(2k-1)}{2(1-\sigma)^{2k-1}} + \frac{\pi^{2k-2}}{2-2\sigma}.$$

The result now clearly follows.

LEMMA γ.

$$\int_1^\infty \left|\sum_{n=1}^\infty d_k(n)\exp(-inxe^{-i\delta})\right|^2 x^{2\sigma-1}\, dx$$

$$> \frac{C_5}{\delta^{2\sigma}} \sum_{n=1}^\infty \frac{d_k^2(n)}{n^{2\sigma}} e^{-2n\sin\delta} - C_6 \log\frac{1}{\delta} \sum_{n=1}^\infty d_k^2(n) e^{-n\sin\delta}.$$

The left-hand side is equal to

$$\sum_{m=1}^\infty \sum_{n=1}^\infty d_k(m) d_k(n) \int_1^\infty \exp(imxe^{i\delta} - inxe^{-i\delta}) x^{2\sigma-1}\, dx$$

$$= \sum_{m=n} + \sum_{m \neq n} = \Sigma_1 + \Sigma_2.$$

Now $\qquad \displaystyle\int_1^\infty e^{-2nx\sin\delta} x^{2\sigma-1}\, dx = (2n\sin\delta)^{-2\sigma} \int_{2n\sin\delta}^\infty e^{-y} y^{2\sigma-1}\, dy,$

and for $2n\sin\delta \leqslant 1$

$$\int_{2n\sin\delta}^\infty e^{-y} y^{2\sigma-1}\, dy \geqslant \int_1^\infty e^{-y} y^{2\sigma-1}\, dy = C_7 > C_7 e^{-2n\sin\delta},$$

while for $2n\sin\delta > 1$

$$\int_{2n\sin\delta}^\infty e^{-y} y^{2\sigma-1}\, dy > \int_{2n\sin\delta}^\infty e^{-y}\, dy = e^{-2n\sin\delta}.$$

Hence

$$\Sigma_1 = \sum_{n=1}^{\infty} d_k^2(n) \int_1^{\infty} e^{-2nx\sin\delta} x^{2\sigma-1}\,dx > \frac{C_5}{\delta^{2\sigma}} \sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^{2\sigma}} e^{-2n\sin\delta}.$$

Also, using (7.14.4),

$$|\Sigma_2| < C_8 \sum_{m=2}^{\infty} \sum_{n=1}^{m-1} d_k(m) d_k(n) \frac{e^{-m\sin\delta}}{m-n}$$

$$= C_8 \sum_{r=1}^{\infty} \sum_{m=r+1}^{\infty} d_k(m) d_k(m-r) \frac{e^{-m\sin\delta}}{r}$$

$$= C_8 \sum_{r=1}^{\infty} \frac{e^{-\frac{1}{2}r\sin\delta}}{r} \sum_{m=r+1}^{\infty} d_k(m) e^{-\frac{1}{2}m\sin\delta} d_k(m-r) e^{-\frac{1}{2}(m-r)\sin\delta}$$

$$\leqslant C_8 \sum_{r=1}^{\infty} \frac{e^{-\frac{1}{2}r\sin\delta}}{r} \left\{ \sum_{m=r+1}^{\infty} d_k^2(m) e^{-m\sin\delta} \sum_{m=r+1}^{\infty} d_k^2(m-r) e^{-(m-r)\sin\delta} \right\}^{\frac{1}{2}}$$

$$< C_8 \sum_{r=1}^{\infty} \frac{e^{-\frac{1}{2}r\sin\delta}}{r} \sum_{m=1}^{\infty} d_k^2(m) e^{-m\sin\delta} < C_8 \log\frac{1}{\delta} \sum_{m=1}^{\infty} d_k^2(m) e^{-m\sin\delta}.$$

This proves the lemma.

LEMMA δ. *For* $\sigma > 1$

$$\exp\left\{C_9\left(\frac{k}{\log k}\right)^{2/\sigma}\right\} < F_k(\sigma) < \exp(C_{10} k^{2/\sigma}).$$

It is clear from (8.9.6) that

$$f_k(x) \leqslant (1-\sqrt{x})^{-2k} \quad (0 < x < 1).$$

Also it is easily verified that

$$\left\{\frac{(k+m-1)!}{(k-1)!\,m!}\right\}^2 \leqslant \frac{(k^2+m-1)!}{(k^2-1)!\,m!}.$$

Hence, for $0 < x < 1$,

$$f_k(x) \leqslant \sum_{m=0}^{\infty} \frac{(k^2+m-1)!}{(k^2-1)!\,m!} x^m = (1-x)^{-k^2}.$$

Hence

$$\log F_k(\sigma) = \sum_{p^\sigma \leqslant k^2} \log f_k(p^{-\sigma}) + \sum_{p^\sigma > k^2} \log f_k(p^{-\sigma})$$

$$\leqslant 2k \sum_{p^\sigma \leqslant k^2} \log(1-p^{-\frac{1}{2}\sigma})^{-1} + k^2 \sum_{p^\sigma > k^2} \log(1-p^{-\sigma})^{-1}$$

$$= O\Big(k \sum_{p^\sigma \leqslant k^2} p^{-\frac{1}{2}\sigma}\Big) + O\Big(k^2 \sum_{p^\sigma > k^2} p^{-\sigma}\Big)$$

$$= O\{k(k^{2/\sigma})^{1-\frac{1}{2}\sigma}\} + O\{k^2(k^{2/\sigma})^{1-\sigma}\} = O(k^{2/\sigma}).$$

On the other hand, (8.10.2) gives

$$\log F_k(\sigma) > 2k \sum_{p<x} \log(1-p^{-\frac{1}{2}\sigma})^{-1} - \sum_{p<x} \log 2k$$

$$> 2k \sum_{p<x} p^{-\frac{1}{2}\sigma} - C_{11} \frac{x}{\log x} \log 2k$$

$$> C_{12} k \frac{x^{1-\frac{1}{2}\sigma}}{\log x} - C_{11} \frac{x}{\log x} \log 2k.$$

Taking

$$x = \left(\frac{C_{12}}{2C_{11}} \frac{k}{\log k}\right)^{2/\sigma}$$

the other result follows.

Proof of Theorem 8.12 *for* $\frac{1}{2} < \sigma < 1$. It follows from Lemmas β and γ and Stirling's theorem that

$$\int_0^{\infty} |\zeta(\sigma+it)|^{2k} e^{-2\delta t} t^{2\sigma-1}\,dt > \frac{C_{13}}{\delta^{2\sigma}} \sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^{2\sigma}} e^{-2n\sin\delta} -$$

$$- C_{14} \log\frac{1}{\delta} \sum_{n=1}^{\infty} d_k^2(n) e^{-n\sin\delta} - C_{15} e^{C_4 k \log k}.$$

Now, if $0 < \epsilon < 2\sigma-1$,

$$\sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^{2\sigma}} e^{-2n\sin\delta} = F_k(2\sigma) - \sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^{2\sigma}} (1-e^{-2n\sin\delta})$$

$$> F_k(2\sigma) - C_{16} \sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^{2\sigma}} (n\delta)^{\epsilon}$$

$$= F_k(2\sigma) - C_{16}\, \delta^{\epsilon} F_k(2\sigma-\epsilon)$$

$$> \exp\left\{C_9\left(\frac{k}{\log k}\right)^{1/\sigma}\right\} - C_{16}\, \delta^{\epsilon} \exp\{C_{10} k^{2/(2\sigma-\epsilon)}\},$$

and

$$\sum_{n=1}^{\infty} d_k^2(n) e^{-n\sin\delta} < C_{17} \sum_{n=1}^{\infty} d_k^2(n) (n\delta)^{\epsilon-2\sigma} = C_{17}\, \delta^{\epsilon-2\sigma} F_k(2\sigma-\epsilon)$$

$$< C_{17}\, \delta^{\epsilon-2\sigma} \exp\{C_{10} k^{2/(2\sigma-\epsilon)}\}.$$

Let

$$\delta = \exp\left\{-\frac{C_{10}}{\epsilon} k^{2/(2\sigma-\epsilon)}\right\}.$$

Then

$$\int_0^{\infty} |\zeta(\sigma+it)|^{2k} e^{-2\delta t} t^{2\sigma-1}\,dt > \frac{1}{\delta^{2\sigma}} \left[C_{13} \exp\left\{C_9\left(\frac{k}{\log k}\right)^{1/\sigma}\right\} - C_{13} C_{16} - \right.$$

$$\left. - C_{14} C_{17} \frac{C_{10}}{\epsilon} k^{2/(2\sigma-\epsilon)} \right] - C_{15} e^{C_4 k \log k}$$

$$> \frac{C_{18}}{\delta^{2\sigma}} \exp\left\{C_9\left(\frac{k}{\log k}\right)^{1/\sigma}\right\}.$$

Suppose now that

$$|\zeta(\sigma+it)| \leqslant \exp(\log^\alpha t) \quad (t \geqslant t_0)$$

where $0 < \alpha < 1$. Then

$$\int_0^\infty |\zeta(\sigma+it)|^{2k} e^{-2\delta t} t^{2\sigma-1}\,dt \leqslant C_{19}^{2k} + \int_1^\infty e^{2k\log^\alpha t - 2\delta t} t^{2\sigma-1}\,dt.$$

If $t > k^2/\delta^2$, $k > k_0$, then

$$\frac{k}{\delta} < \sqrt{t} < \frac{1}{2}\frac{t}{\log^\alpha t}.$$

Hence

$$\int_1^\infty e^{2k\log^\alpha t - 2\delta t} t^{2\sigma-1}\,dt \leqslant e^{2k\log^\alpha(k^2/\delta^2)} \int_1^{k^2/\delta^2} e^{-2\delta t} t^{2\sigma-1}\,dt + \int_{k^2/\delta^2}^\infty e^{-\delta t} t^{2\sigma-1}\,dt$$

$$< e^{2k\log^\alpha(k^2/\delta^2)} \frac{C_{20}}{\delta^{2\sigma}}.$$

Hence $$\left(\frac{k}{\log k}\right)^{1/\sigma} = O\left(k \log^\alpha \frac{k}{\delta}\right) = O(k^{1+(2\alpha)/(2\sigma-\epsilon)}).$$

Hence $$\frac{1}{\sigma} \leqslant 1 + \frac{2\alpha}{2\sigma-\epsilon},$$

and since ϵ may be as small as we please

$$\frac{1}{\sigma} \leqslant 1 + \frac{\alpha}{\sigma}, \qquad \alpha \geqslant 1-\sigma.$$

The case $\sigma = \frac{1}{2}$. Suppose that

$$\zeta(\tfrac{1}{2}+it) = O\{\exp(\log^\beta t)\} \quad (0 < \beta < \tfrac{1}{2}).$$

Then the function $$f(s) = \zeta(s)\exp(-\log^\beta s)$$

is bounded on the lines $\sigma = \frac{1}{2}$, $\sigma = 2$, $t > t_0$, and it is $O(t)$ uniformly in this strip. Hence by the Phragmén–Lindelöf theorem $f(s)$ is bounded in the strip, i.e.

$$\zeta(\sigma+it) = O\{\exp(\log^\beta t)\}$$

for $\frac{1}{2} < \sigma < 2$. Since this is not true for $\frac{1}{2} < \sigma < 1-\beta$, it follows that $\beta \geqslant \frac{1}{2}$.

NOTES FOR CHAPTER 8

8.13. Levinson [1] has sharpened Theorems 8.9(A) and 8.9(B) to show that the inequalities

$$|\zeta(1+it)| \geqslant e^\gamma \log\log t + O(1)$$

and

$$\frac{1}{|\zeta(1+it)|} \geqslant \frac{6e^\gamma}{\pi^2}(\log\log t - \log\log\log t) + O(1)$$

each hold for arbitrarily large t. Theorem 8.12 has also been improved, by Montgomery [3]. He showed that for any σ in the range $\frac{1}{2} < \sigma < 1$, and for any real ϑ, there are arbitrarily large t such that

$$\mathbf{R}\{e^{i\vartheta}\log\zeta(\sigma+it)\} \geqslant \tfrac{1}{20}(\sigma-\tfrac{1}{2})^{-1}(\log t)^{1-\sigma}(\log\log t)^{-\sigma}.$$

Here $\log\zeta(s)$ is, as usual, defined by continuous variation along lines parallel to the real axis, using the Dirichlet series (1.1.9) for $\sigma > 1$. It follows in particular that

$$\zeta(\sigma+it) = \Omega\left\{\exp\left(\frac{\frac{1}{20}}{\sigma-\frac{1}{2}}\frac{(\log t)^{1-\sigma}}{(\log\log t)^\sigma}\right)\right\} \quad (\tfrac{1}{2} < \sigma < 1),$$

and the same for $\zeta(\sigma+it)^{-1}$. For $\sigma = \frac{1}{2}$ the best result is due to Balasubramanian and Ramachandra [2], who showed that

$$\max_{T\leqslant t\leqslant T+H} |\zeta(\tfrac{1}{2}+it)| \geqslant \exp\left(\tfrac{3}{4}\frac{(\log H)^{\frac{1}{2}}}{(\log\log H)^{\frac{1}{2}}}\right)$$

if $(\log T)^\delta \leqslant H \leqslant T$ and $T \geqslant T(\delta)$, where δ is any positive constant. Their method is akin to that of §8.12, in that it depends on a lower bound for a mean value of $|\zeta(\frac{1}{2}+it)|^{2k}$, uniform in k. By constrast the method of Montgomery [3] uses the formula

$$\frac{4}{\pi}\int_{-(\log t)^2}^{(\log t)^2} e^{-i\vartheta}\log\zeta(\sigma+it+iy)\left(\frac{\sin\frac{1}{2}y}{y}\right)^2\{1+\cos(\vartheta+y\log x)\}dy$$

$$= \sum_{|\log n/x|\leqslant\frac{1}{2}} \frac{\Lambda(n)}{\log n} n^{-\sigma-it}\left(\tfrac{1}{2} - \left|\log\frac{n}{x}\right|\right) + O\{x(\log t)^{-2}\}. \quad (8.13.1)$$

This is valid for any real x and ϑ, providing that $\zeta(s) \neq 0$ for $\mathbf{R}(s) \geqslant \sigma$ and $|\mathbf{I}(s) - t| \leqslant 2(\log t)^2$. After choosing x suitably one may use the extended version of Dirichlet's theorem given in §8.2 to show that the real part of the sum on the right of (8.13.1) is large at points $t_1 < \ldots < t_N \leqslant T$, spaced at least $4(\log T)^2$ apart. One can arrange that N exceeds $N(\sigma, T)$, whence at least one such t_n will satisfy the condition that $\zeta(s) \neq 0$ in the corresponding rectangle.

IX

THE GENERAL DISTRIBUTION OF THE ZEROS

9.1. In § 2.12 we deduced from the general theory of integral functions that $\zeta(s)$ *has an infinity of complex zeros*. This may be proved directly as follows.

We have

$$\frac{1}{2^2}+\frac{1}{3^2}+\ldots<\frac{1}{2^2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots=\frac{1}{4}+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots=\frac{3}{4}.$$

Hence for $\sigma \geqslant 2$

$$|\zeta(s)| \leqslant 1+\frac{1}{2^\sigma}+\frac{1}{3^\sigma}+\ldots \leqslant 1+\frac{1}{2^2}+\ldots<\frac{7}{4}, \tag{9.1.1}$$

and

$$|\zeta(s)| \geqslant 1-\frac{1}{2^\sigma}-\ldots \geqslant 1-\frac{1}{2^2}-\ldots>\frac{1}{4}. \tag{9.1.2}$$

Also

$$\mathbf{R}\{\zeta(s)\}=1+\frac{\cos(t\log 2)}{2^\sigma}+\ldots \geqslant 1-\frac{1}{2^2}-\ldots>\frac{1}{4}. \tag{9.1.3}$$

Hence for $\sigma \geqslant 2$ we may write

$$\log\zeta(s)=\log|\zeta(s)|+i\arg\zeta(s),$$

where $\arg\zeta(s)$ is that value of $\arctan\{\mathbf{I}\zeta(s)/\mathbf{R}\zeta(s)\}$ which lies between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. It is clear that

$$|\log\zeta(s)|<A \quad (\sigma \geqslant 2). \tag{9.1.4}$$

For $\sigma<2$, $t \neq 0$, we define $\log\zeta(s)$ as the analytic continuation of the above function along the straight line $(\sigma+it, 2+it)$, provided that $\zeta(s) \neq 0$ on this segment of line.

Now consider a system of four concentric circles C_1, C_2, C_3, C_4, with centre $3+iT$ and radii 1, 4, 5, and 6 respectively. Suppose that $\zeta(s) \neq 0$ in or on C_4. Then $\log\zeta(s)$, defined as above, is regular in C_4. Let M_1, M_2, M_3 be its maximum modulus on C_1, C_2, and C_3 respectively.

Since $\zeta(s)=O(t^A)$, $\mathbf{R}\{\log\zeta(s)\}<A\log T$ in C_4, and the Borel–Carathéodory theorem gives

$$M_3 \leqslant \frac{2.5}{6-5}A\log T+\frac{6+5}{6-5}\log|\zeta(3+iT)|<A\log T.$$

Also $M_1<A$, by (9.1.4). Hence Hadamard's three-circles theorem, applied to the circles C_1, C_2, C_3, gives

$$M_2 \leqslant M_1^\alpha M_3^\beta<A\log^\beta T,$$

where

$$1-\alpha=\beta=\log 4/\log 5<1.$$

Hence

$$\zeta(-1+iT)=O\{\exp(\log^\beta T)\}=O(T^\epsilon).$$

But by (9.1.2), and the functional equation (2.1.1) with $\sigma=2$,

$$|\zeta(-1+iT)|>AT^{\frac{3}{2}}.$$

We have thus obtained a contradiction. Hence every such circle C_4 contains at least one zero of $\zeta(s)$, and so there are an infinity of zeros. The argument also shows that the gaps between the ordinates of successive zeros are bounded.

9.2. The function $N(T)$. Let $T>0$, and let $N(T)$ denote the number of zeros of the function $\zeta(s)$ in the region $0 \leqslant \sigma \leqslant 1$, $0<t \leqslant T$. The distribution of the ordinates of the zeros can then be studied by means of formulae involving $N(T)$.

The most easily proved result is

THEOREM 9.2. *As* $T \to \infty$

$$N(T+1)-N(T)=O(\log T). \tag{9.2.1}$$

For it is easily seen that

$$N(T+1)-N(T) \leqslant n(\surd 5),$$

where $n(r)$ is the number of zeros of $\zeta(s)$ in the circle with centre $2+iT$ and radius r. Now, by Jensen's theorem,

$$\int\limits_0^3 \frac{n(r)}{r}\,dr=\frac{1}{2\pi}\int\limits_0^{2\pi}\log|\zeta(2+iT+3e^{i\theta})|\,d\theta-\log|\zeta(2+iT)|.$$

Since $|\zeta(s)|<t^A$ for $-1 \leqslant \sigma \leqslant 5$, we have

$$\log|\zeta(2+iT+3e^{i\theta})|<A\log T.$$

Hence

$$\int\limits_0^3 \frac{n(r)}{r}\,dr<A\log T+A<A\log T.$$

Since

$$\int\limits_0^3 \frac{n(r)}{r}\,dr \geqslant \int\limits_{\surd 5}^3 \frac{n(r)}{r}\,dr \geqslant n(\surd 5)\int\limits_{\surd 5}^3 \frac{dr}{r}=An(\surd 5),$$

the result (9.2.1) follows.

Naturally it also follows that

$$N(T+h)-N(T)=O(\log T)$$

for any fixed value of h. In particular, the multiplicity of a multiple zero of $\zeta(s)$ in the region considered is at most $O(\log T)$.

9.3. The closer study of $N(T)$ depends on the following theorem.† If T is not the ordinate of a zero, let $S(T)$ denote the value of

$$\pi^{-1}\arg\zeta(\tfrac{1}{2}+iT)$$

obtained by continuous variation along the straight lines joining 2, $2+iT$, $\frac{1}{2}+iT$, starting with the value 0. If T is the ordinate of a zero, let $S(T) = S(T+0)$. Let

$$L(T) = \frac{1}{2\pi}T\log T - \frac{1+\log 2\pi}{2\pi}T + \frac{7}{8}. \tag{9.3.1}$$

THEOREM 9.3. *As* $T \to \infty$

$$N(T) = L(T)+S(T)+O(1/T). \tag{9.3.2}$$

The number of zeros of the function $\Xi(z)$ (see § 2.1) in the rectangle with vertices at $z = \pm T \pm \frac{3}{2}i$ is $2N(T)$, so that

$$2N(T) = \frac{1}{2\pi i}\int \frac{\Xi'(z)}{\Xi(z)}\,dz$$

taken round the rectangle. Since $\Xi(z)$ is even and real for real z, this is equal to

$$\frac{2}{\pi i}\left(\int\limits_{T}^{T+\frac{3}{2}i} + \int\limits_{T+\frac{3}{2}i}^{\frac{3}{2}i}\right)\frac{\Xi'(z)}{\Xi(z)}\,dz = \frac{2}{\pi i}\left(\int\limits_{2}^{2+iT} + \int\limits_{2+iT}^{\frac{1}{2}+iT}\right)\frac{\xi'(s)}{\xi(s)}\,ds$$

$$= \frac{2}{\pi}\Delta\arg\xi(s),$$

where Δ denotes the variation from 2 to $2+iT$, and thence to $\frac{1}{2}+iT$, along straight lines. Recalling that

$$\xi(s) = \tfrac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s),$$

we obtain

$$\pi N(T) = \Delta\arg s(s-1)+\Delta\arg\pi^{-\frac{1}{2}s}+\Delta\arg\Gamma(\tfrac{1}{2}s)+\Delta\arg\zeta(s).$$

Now

$$\Delta\arg s(s-1) = \arg(-\tfrac{1}{4}-T^2) = \pi,$$

$$\Delta\arg\pi^{-\frac{1}{2}s} = \Delta\arg e^{-\frac{1}{2}s\log\pi} = -\tfrac{1}{2}T\log\pi,$$

and by (4.12.1)

$$\begin{aligned}\Delta\arg\Gamma(\tfrac{1}{2}s) &= \mathbf{I}\log\Gamma(\tfrac{1}{4}+\tfrac{1}{2}iT)\\ &= \mathbf{I}\{(-\tfrac{1}{4}+\tfrac{1}{2}iT)\log(\tfrac{1}{2}iT)-\tfrac{1}{2}iT+O(1/T)\}\\ &= \tfrac{1}{2}T\log\tfrac{1}{2}T-\tfrac{1}{8}\pi-\tfrac{1}{2}T+O(1/T).\end{aligned}$$

Adding these results, we obtain the theorem, provided that T is not the ordinate of a zero. If T is the ordinate of a zero, the result follows from

† Backlund (2), (3).

the definitions and what has already been proved, the term $O(1/T)$ being continuous.

The problem of the behaviour of $N(T)$ is thus reduced to that of $S(T)$.

9.4. We shall now prove the following lemma.

LEMMA. *Let* $0 \leqslant \alpha < \beta < 2$. *Let* $f(s)$ *be an analytic function, real for real* s, *regular for* $\sigma \geqslant \alpha$ *except at* $s = 1$; *let*

$$|\mathbf{R}f(2+it)| \geqslant m > 0$$

and

$$|f(\sigma'+it')| \leqslant M_{\sigma,t} \quad (\sigma' \geqslant \sigma,\ 1 \leqslant t' \leqslant t).$$

Then if T *is not the ordinate of a zero of* $f(s)$

$$|\arg f(\sigma+iT)| \leqslant \frac{\pi}{\log\{(2-\alpha)/(2-\beta)\}}\left(\log M_{\alpha,T+2}+\log\frac{1}{m}\right)+\frac{3\pi}{2} \tag{9.4.1}$$

for $\sigma \geqslant \beta$.

Since $\arg f(2) = 0$, and

$$\arg f(s) = \arctan\left\{\frac{\mathbf{I}f(s)}{\mathbf{R}f(s)}\right\},$$

where $\mathbf{R}f(s)$ does not vanish on $\sigma = 2$, we have

$$|\arg f(2+iT)| < \tfrac{1}{2}\pi.$$

Now if $\mathbf{R}f(s)$ vanishes q times between $2+iT$ and $\beta+iT$, this interval is divided into $q+1$ parts, throughout each of which $\mathbf{R}\{f(s)\} \geqslant 0$ or $\mathbf{R}\{f(s)\} \leqslant 0$. Hence in each part the variation of $\arg f(s)$ does not exceed π. Hence

$$|\arg f(s)| \leqslant (q+\tfrac{3}{2})\pi \quad (\sigma \geqslant \beta).$$

Now q is the number of zeros of the function

$$g(z) = \tfrac{1}{2}\{f(z+iT)+f(z-iT)\}$$

for $\mathbf{I}(z) = 0$, $\beta \leqslant \mathbf{R}(z) \leqslant 2$; hence $q \leqslant n(2-\beta)$, where $n(r)$ denotes the number of zeros of $g(z)$ for $|z-2| \leqslant r$. Also

$$\int\limits_{0}^{2-\alpha}\frac{n(r)}{r}\,dr \geqslant \int\limits_{2-\beta}^{2-\alpha}\frac{n(r)}{r}\,dr \geqslant n(2-\beta)\log\frac{2-\alpha}{2-\beta},$$

and by Jensen's theorem

$$\int\limits_{0}^{2-\alpha}\frac{n(r)}{r}\,dr = \frac{1}{2\pi}\int\limits_{0}^{2\pi}\log|g\{2+(2-\alpha)e^{i\theta}\}|\,d\theta - \log|g(2)|$$

$$\leqslant \log M_{\alpha,T+2}+\log 1/m.$$

This proves the lemma.

We deduce

THEOREM 9.4. *As* $T \to \infty$

$$S(T) = O(\log T), \tag{9.4.2}$$

i.e.

$$N(T) = \frac{1}{2\pi} T \log T - \frac{1+\log 2\pi}{2\pi} T + O(\log T). \tag{9.4.3}$$

We apply the lemma with $f(s) = \zeta(s)$, $\alpha = 0$, $\beta = \frac{1}{2}$, and (9.4.2) follows, since $\zeta(s) = O(t^A)$. Then (9.4.3) follows from (9.3.2).

Theorem 9.4 has a number of interesting consequences. It gives another proof of Theorem 9.2, since $(0 < \theta < 1)$

$$L(T+1) - L(T) = L'(T+\theta) = O(\log T).$$

We can also prove the following result.

If the zeros $\beta+i\gamma$ *of* $\zeta(s)$ *with* $\gamma > 0$ *are arranged in a sequence* $\rho_n = \beta_n + i\gamma_n$ *so that* $\gamma_{n+1} \geqslant \gamma_n$, *then as* $n \to \infty$

$$|\rho_n| \sim \gamma_n \sim \frac{2\pi n}{\log n}. \tag{9.4.4}$$

We have $$N(T) \sim \frac{1}{2\pi} T \log T.$$

Hence $$2\pi N(\gamma_n \pm 1) \sim (\gamma_n \pm 1)\log(\gamma_n \pm 1) \sim \gamma_n \log \gamma_n.$$

Also $$N(\gamma_n - 1) \leqslant n \leqslant N(\gamma_n + 1).$$

Hence $$2\pi n \sim \gamma_n \log \gamma_n.$$

Hence $$\log n \sim \log \gamma_n,$$

and so $$\gamma_n \sim \frac{2\pi n}{\log n}.$$

Also $|\rho_n| \sim \gamma_n$, since $\beta_n = O(1)$.

We can also deduce the result of § 9.1, that the gaps between the ordinates of successive zeros are bounded. For if $|S(t)| \leqslant C \log t$ $(t \geqslant 2)$,

$$N(T+H) - N(T) = \frac{1}{2\pi} \int_T^{T+H} \log \frac{t}{2\pi}\, dt + S(T+H) - S(T) + O\left(\frac{1}{T}\right)$$
$$\geqslant \frac{H}{2\pi} \log \frac{T}{2\pi} - C\{\log(T+H) + \log T\} + O\left(\frac{1}{T}\right),$$

which is ultimately positive if H is a constant greater than $4\pi C$.

The behaviour of the function $S(T)$ appears to be very complicated. It must have a discontinuity k where T passes through the ordinate of a zero of $\zeta(s)$ of order k (since the term $O(1/T)$ in the above theorem is in fact continuous). Between the zeros, $N(T)$ is constant, so that the

variation of $S(T)$ must just neutralize that of the other terms. In the formula (9.3.1), the term $\frac{7}{8}$ is presumably overwhelmed by the variations of $S(T)$. On the other hand, in the integrated formula

$$\int_0^T N(t)\, dt = \int_0^T L(t)\, dt + \int_0^T S(t)\, dt + O(\log T)$$

the term in $S(T)$ certainly plays a much smaller part, since, as we shall presently prove, the integral of $S(t)$ over $(0, T)$ is still only $O(\log T)$. Presumably this is due to frequent variations in the sign of $S(t)$. Actually we shall show that $S(t)$ changes sign an infinity of times.

9.5. A problem of analytic continuation.

The above theorems on the zeros of $\zeta(s)$ lead to the solution of a curious subsidiary problem of analytic continuation.† Consider the function

$$P(s) = \sum_p \frac{1}{p^s}. \tag{9.5.1}$$

This is an analytic function of s, regular for $\sigma > 1$. Now by (1.6.1)

$$P(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \zeta(ns). \tag{9.5.2}$$

As $n \to \infty$, $\log \zeta(ns) \sim 2^{-ns}$. Hence the right-hand side represents an analytic function of s, regular for $\sigma > 0$, except at the singularities of individual terms. These are branch-points arising from the poles and zeros of the functions $\zeta(ns)$; there are an infinity of such points, but they have no limit-point in the region $\sigma > 0$. Hence $P(s)$ is regular for $\sigma > 0$, except at certain branch-points.

Similarly, the function

$$Q(s) = -P'(s) = -\sum_{n=1}^{\infty} \mu(n) \frac{\zeta'(ns)}{\zeta(ns)} \tag{9.5.3}$$

is regular for $\sigma > 0$, except at certain simple poles.

We shall now prove that *the line* $\sigma = 0$ *is a natural boundary of the functions* $P(s)$ *and* $Q(s)$.

We shall in fact prove that every point of $\sigma = 0$ is a limit-point of poles of $Q(s)$. By symmetry, it is sufficient to consider the upper half-line. Thus it is sufficient to prove that for every $u > 0$, $\delta > 0$, the square

$$0 < \sigma < \delta, \qquad u < t \leqslant u+\delta \tag{9.5.4}$$

contains at least one pole of $Q(s)$.

† Landau and Walfisz (1).

As $p \to \infty$ through primes,

$$N\{p(u+\delta)\} \sim \frac{1}{2\pi}(u+\delta)p\log p, \qquad N(pu) \sim \frac{1}{2\pi}up\log p,$$

by Theorem 9.4. Hence for all $p \geqslant p_0(\delta, u)$

$$N\{p(u+\delta)\}-N(pu) > 0. \tag{9.5.5}$$

Also, by Theorem 9.2, the multiplicity $v(\rho)$ of each zero $\rho = \beta+i\gamma$ with ordinate $\gamma \geqslant 2$ is less than $A\log\gamma$, where A is an absolute constant.

Now choose $p = p(\delta, u)$ satisfying the conditions

$$p > \frac{1}{\delta}, \qquad p \geqslant \frac{2}{u}, \qquad p \geqslant p_0(\delta, u), \qquad p > A\log\{p(u+\delta)\}.$$

There is then, by (9.5.5), a zero ρ of $\zeta(s)$ in the rectangle

$$\tfrac{1}{2} \leqslant \sigma < 1, \qquad pu < t \leqslant p(u+\delta). \tag{9.5.6}$$

Since $\gamma > pu \geqslant 2$, its multiplicity $v(\rho)$ satisfies

$$v(\rho) < A\log\gamma \leqslant A\log\{p(u+\delta)\} < p,$$

and so is not divisible by p.

The point ρ/p belongs to the square (9.5.4). We shall show that this point is a pole of $Q(s)$. Let m run through the positive integers (finite in number) for which $\zeta(m\rho/p) = 0$. Then we have to prove that

$$\sum \frac{\mu(m)}{m} v\left(\frac{m\rho}{p}\right) \neq 0. \tag{9.5.7}$$

The term of this sum corresponding to $m = p$ is $-v(\rho)/p$. No other m occurring in the sum is divisible by p, since for $m \geqslant 2p$

$$\mathbf{R}\left(\frac{m\rho}{p}\right) = \frac{m\beta}{p} \geqslant \frac{2p.\frac{1}{2}}{p} = 1.$$

Hence

$$\sum \frac{\mu(m)}{m} v\left(\frac{m\rho}{p}\right) = \frac{a}{b} - \frac{v(\rho)}{p},$$

where a and b are integers, and p is not a factor of b. Since p is also not a factor of $v(\rho)$, $ap \neq bv(\rho)$, and (9.5.7) follows.

There are various other functions with similar properties. For example,† let

$$f_{l,k}(s) = \sum_{n=1}^{\infty} \frac{\{d_k(n)\}^l}{n^s},$$

where k and l are positive integers, $k \geqslant 2$. By (1.2.2) and (1.2.10), $f_{l,k}(s)$ is a meromorphic function of s if $l = 1$, or if $l = 2$ and $k = 2$. *For all other values of l and k, $f_{l,k}(s)$ has $\sigma = 0$ as a natural boundary, and it has no singularities other than poles in the half-plane $\sigma > 0$.*

† Estermann (1).

9.6. An approximate formula for $\zeta'(s)/\zeta(s)$. The following approximate formula for $\zeta'(s)/\zeta(s)$ in terms of the zeros near to s is often useful.

THEOREM 9.6 (A). *If $\rho = \beta+i\gamma$ runs through zeros of $\zeta(s)$,*

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{|t-\gamma|\leqslant 1} \frac{1}{s-\rho} + O(\log t), \tag{9.6.1}$$

uniformly for $-1 \leqslant \sigma \leqslant 2$.

Take $f(s) = \zeta(s)$, $s_0 = 2+iT$, $r = 12$ in Lemma α of § 3.9. Then $M = A\log T$, and we obtain

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{|\rho-s_0|\leqslant 6} \frac{1}{s-\rho} + O(\log T) \tag{9.6.2}$$

for $|s-s_0| \leqslant 3$, and so in particular for $-1 \leqslant \sigma \leqslant 2$, $t = T$. Replacing T by t in the particular case, we obtain (9.6.2) with error $O(\log t)$, and $-1 \leqslant \sigma \leqslant 2$. Finally any term occurring in (9.6.2) but not in (9.6.1) is bounded, and the number of such terms does not exceed

$$N(t+6)-N(t-6) = O(\log t)$$

by Theorem 9.2. This proves (9.6.1).

Another proof depends on (2.12.7), which, by a known property of the Γ-function, gives

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right) + O(\log t).$$

Replacing s by $2+it$ and subtracting,

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{\rho}\left(\frac{1}{s-\rho}-\frac{1}{2+it-\rho}\right) + O(\log t),$$

since $\zeta'(2+it)/\zeta(2+it) = O(1)$.

Now

$$\sum_{|t-\gamma|\leqslant 1} \frac{1}{2+it-\rho} = \sum_{|t-\gamma|\leqslant 1} O(1) = O(\log t)$$

by Theorem 9.2. Also

$$\sum_{t+n<\gamma\leqslant t+n+1}\left(\frac{1}{s-\rho}-\frac{1}{2+it-\rho}\right) = \sum_{t+n<\gamma\leqslant t+n+1} \frac{2-\sigma}{(s-\rho)(2+it-\rho)}$$
$$= \sum_{t+n<\gamma\leqslant t+n+1} O\left\{\frac{1}{(\gamma-t)^2}\right\} = \sum_{t+n<\gamma\leqslant t+n+1} O\left(\frac{1}{n^2}\right) = O\left\{\frac{\log(t+n)}{n^2}\right\},$$

again by Theorem 9.2. Since

$$\sum_{n=1}^{\infty} \frac{\log(t+n)}{n^2} < \sum_{n\leqslant t} \frac{\log 2t}{n^2} + \sum_{n>t} \frac{\log 2n}{n^2} = O(\log t),$$

it follows that $$\sum_{\gamma>t+1}\left(\frac{1}{s-\rho}-\frac{1}{2+it-\rho}\right) = O(\log t).$$

Similarly $$\sum_{\gamma<t-1}\left(\frac{1}{s-\rho}-\frac{1}{2+it-\rho}\right) = O(\log t)$$

and the result follows again.

The corresponding formula for $\log\zeta(s)$ is given by

THEOREM 9.6 (B). *We have*

$$\log\zeta(s) = \sum_{|t-\gamma|\leqslant 1}\log(s-\rho)+O(\log t) \tag{9.6.3}$$

uniformly for $-1 \leqslant \sigma \leqslant 2$, *where* $\log\zeta(s)$ *has its usual meaning, and* $-\pi < \mathbf{I}\log(s-\rho) \leqslant \pi$.

Integrating (9.6.1) from s to $2+it$, and supposing that t is not equal to the ordinate of any zero, we obtain

$$\log\zeta(s)-\log\zeta(2+it) = \sum_{|t-\gamma|\leqslant 1}\{\log(s-\rho)-\log(2+it-\rho)\}+O(\log t).$$

Now $\log\zeta(2+it)$ is bounded; also $\log(2+it-\rho)$ is bounded, and there are $O(\log t)$ such terms. Their sum is therefore $O(\log t)$. The result therefore follows for such values of t, and then by continuity for all values of s in the strip other than the zeros.

9.7. As an application of Theorem 9.6 (B) we shall prove the following theorem on the minimum value of $\zeta(s)$ in certain parts of the critical strip. We know from Theorem 8.12 that $|\zeta(s)|$ is sometimes large in the critical strip, but we can prove little about the distribution of the values of t for which it is large. The following result† states a much weaker inequality, but states it for many more values of t.

THEOREM 9.7. *There is a constant A such that each interval* $(T, T+1)$ *contains a value of t for which*

$$|\zeta(s)| > t^{-A} \quad (-1 \leqslant \sigma \leqslant 2). \tag{9.7.1}$$

Further, if H is any number greater than unity, then

$$|\zeta(s)| > T^{-AH} \tag{9.7.2}$$

for $-1 \leqslant \sigma \leqslant 2$, $T \leqslant t \leqslant T+1$, *except possibly for a set of values of t of measure* $1/H$.

Taking real parts in (9.6.3),

$$\log|\zeta(s)| = \sum_{|t-\gamma|\leqslant 1}\log|s-\rho|+O(\log t)$$
$$\geqslant \sum_{|t-\gamma|\leqslant 1}\log|t-\gamma|+O(\log t). \tag{9.7.3}$$

† Valiron (1), Landau (8), (18), Hoheisel (3).

Now

$$\int_T^{T+1}\sum_{|t-\gamma|\leqslant 1}\log|t-\gamma|\,dt = \sum_{T-1\leqslant\gamma\leqslant T+2}\int_{\max(\gamma-1,T)}^{\min(\gamma+1,T+1)}\log|t-\gamma|\,dt$$
$$\geqslant \sum_{T-1\leqslant\gamma\leqslant T+2}\int_{\gamma-1}^{\gamma+1}\log|t-\gamma|\,dt$$
$$= \sum_{T-1\leqslant\gamma\leqslant T+2}(-2) > -A\log T.$$

Hence $$\sum_{|t-\gamma|\leqslant 1}\log|t-\gamma| > -A\log T$$

for some t in $(T, T+1)$.

Hence $\log|\zeta(s)| > -A\log T$ for some t in $(T, T+1)$ and all σ in $-1 \leqslant \sigma \leqslant 2$; and

$$\log|\zeta(s)| > -AH\log T$$

except in a set of measure $1/H$. This proves the theorem.

The exceptional values of t are, of course, those in the neighbourhood of ordinates of zeros of $\zeta(s)$.

9.8. Application to a formula of Ramanujan.† Let a and b be positive numbers such that $ab = \pi$, and consider the integral

$$\frac{1}{2\pi i}\int a^{-2s}\frac{\Gamma(s)}{\zeta(1-2s)}\,ds = \frac{1}{2\pi i}\int\frac{b^{2s}}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{2}-s)}{\zeta(2s)}\,ds$$

taken round the rectangle $(1\pm iT, -\frac{1}{2}\pm iT)$. The two forms are equivalent on account of the functional equation.

Let $T\to\infty$ through values such that $|T-\gamma| > \exp(-A_1\gamma/\log\gamma)$ for every ordinate γ of a zero of $\zeta(s)$. Then by (9.7.3)

$$\log|\zeta(\sigma+iT)| \geqslant -\sum_{|T-\gamma|\leqslant 1}A_1\gamma/\log\gamma+O(\log T) > -A_2 T$$

where $A_2 < \frac{1}{4}\pi$ if A_1 is small enough, and $T > T_0$. It now follows from the asymptotic formula for the Γ-function that the integrals along the horizontal sides of the contour tend to zero as $T\to\infty$ through the above values. Hence by the theorem of residues‡

$$\frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}a^{-2s}\frac{\Gamma(s)}{\zeta(1-2s)}\,ds-\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\frac{b^{2s}}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{2}-s)}{\zeta(2s)}\,ds$$
$$= -\frac{1}{2\sqrt{\pi}}\sum_\rho b^\rho\frac{\Gamma(\frac{1}{2}-\frac{1}{2}\rho)}{\zeta'(\rho)}.$$

† Hardy and Littlewood (2), 156–9.

‡ In forming the series of residues we have supposed for simplicity that the zeros of $\zeta(s)$ are all simple.

The first term on the left is equal to

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\frac{1}{2\pi i}\int\limits_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\left(\frac{n}{a}\right)^{2s}\Gamma(s)\,ds = -\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\{1-e^{-(a/n)^2}\}$$

$$= \sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-(a/n)^2}.$$

Evaluating the other integral in the same way, and multiplying through by $\sqrt{a}$, we obtain Ramanujan's result

$$\sqrt{a}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-(a/n)^2}-\sqrt{b}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-(b/n)^2} = -\frac{1}{2\sqrt{b}}\sum b^{\rho}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}\rho)}{\zeta'(\rho)}. \tag{9.8.1}$$

We have, of course, not proved that the series on the right is convergent in the ordinary sense. We have merely proved that it is convergent if the terms are bracketed in such a way that two terms for which

$$|\gamma-\gamma'| < \exp(-A_1\gamma/\log\gamma)+\exp(-A_1\gamma'/\log\gamma')$$

are included in the same bracket. Of course the zeros are, on the average, much farther apart than this, and it is quite possible that the series may converge without any bracketing. But we are unable to prove this, even on the Riemann hypothesis.

9.9. We next prove a general formula concerning the zeros of an analytic function in a rectangle.† Suppose that $\phi(s)$ is meromorphic in and upon the boundary of a rectangle bounded by the lines $t = 0$, $t = T$, $\sigma = \alpha$, $\sigma = \beta$ $(\beta > \alpha)$, and regular and not zero on $\sigma = \beta$. The function $\log\phi(s)$ is regular in the neighbourhood of $\sigma = \beta$, and here, starting with any one value of the logarithm, we define $F(s) = \log\phi(s)$. For other points s of the rectangle, we define $F(s)$ to be the value obtained from $\log\phi(\beta+it)$ by continuous variation along $t =$ constant from $\beta+it$ to $\sigma+it$, provided that the path does not cross a zero or pole of $\phi(s)$; if it does, we put

$$F(s) = \lim_{\epsilon\to+0}F(\sigma+it+i\epsilon).$$

Let $\nu(\sigma', T)$ denote the excess of the number of zeros over the number of poles in the part of the rectangle for which $\sigma > \sigma'$, including zeros or poles on $t = T$, but not those on $t = 0$.

Then

$$\int F(s)\,ds = -2\pi i\int\limits_{\alpha}^{\beta}\nu(\sigma, T)\,d\sigma, \tag{9.9.1}$$

the integral on the left being taken round the rectangle in the positive direction.

† Littlewood (4).

We may suppose $t = 0$ and $t = T$ to be free from zeros and poles of $\phi(s)$; it is easily verified that our conventions then ensure the truth of the theorem in the general case.

We have

$$\int F(s)\,ds = \int\limits_{\alpha}^{\beta}F(\sigma)\,d\sigma-\int\limits_{\alpha}^{\beta}F(\sigma+iT)\,d\sigma+\int\limits_{0}^{T}\{F(\beta+it)-F(\alpha+it)\}\,i\,dt. \tag{9.9.2}$$

The last term is equal to

$$\int\limits_{0}^{T}i\,dt\int\limits_{\alpha}^{\beta}\frac{\phi'(\sigma+it)}{\phi(\sigma+it)}\,d\sigma = \int\limits_{\alpha}^{\beta}d\sigma\int\limits_{\sigma}^{\sigma+iT}\frac{\phi'(s)}{\phi(s)}\,ds,$$

and by the theorem of residues

$$\int\limits_{\sigma}^{\sigma+iT}\frac{\phi'(s)}{\phi(s)}\,ds = \left(\int\limits_{\sigma}^{\beta}+\int\limits_{\beta}^{\beta+iT}-\int\limits_{\sigma+iT}^{\beta+iT}\right)\frac{\phi'(s)}{\phi(s)}\,ds-2\pi i\nu(\sigma, T)$$

$$= F(\sigma+iT)-F(\sigma)-2\pi i\nu(\sigma, T).$$

Substituting this in (9.9.2), we obtain (9.9.1).

We deduce

THEOREM 9.9. *If*
$$S_1(T) = \int\limits_{0}^{T}S(t)\,dt,$$

then
$$S_1(T) = \frac{1}{\pi}\int\limits_{\frac{1}{2}}^{2}\log|\zeta(\sigma+iT)|\,d\sigma+O(1). \tag{9.9.3}$$

Take $\phi(s) = \zeta(s)$, $\alpha = \frac{1}{2}$, in the above formula, and take the real part. We obtain

$$\int\limits_{\frac{1}{2}}^{\beta}\log|\zeta(\sigma)|\,d\sigma-\int\limits_{0}^{T}\arg\zeta(\beta+it)\,dt-\int\limits_{\frac{1}{2}}^{\beta}\log|\zeta(\sigma+iT)|\,d\sigma+$$
$$+\int\limits_{0}^{T}\arg\zeta(\tfrac{1}{2}+it)\,dt = 0, \tag{9.9.4}$$

the term in $\nu(\sigma, T)$, being purely imaginary, disappearing. Now make $\beta\to\infty$. We have

$$\log\zeta(s) = \log\left(1+\frac{1}{2^s}+\dots\right) = O(2^{-\sigma})$$

as $\sigma\to\infty$, uniformly with respect to t. Hence $\arg\zeta(s) = O(2^{-\sigma})$, so that the second integral tends to 0 as $\beta\to\infty$. Also the first integral is a constant, and

$$\int\limits_{2}^{\beta}\log|\zeta(\sigma+iT)|\,d\sigma = \int\limits_{2}^{\beta}O(2^{-\sigma})\,d\sigma = O(1).$$

Hence the result.

THEOREM 9.9 (A). $S_1(T) = O(\log T)$.

By Theorem 9.6 (B)

$$\int_{\frac12}^{2} \log|\zeta(s)|\, d\sigma = \sum_{|t-\gamma|\leqslant 1} \int_{\frac12}^{2} \log|s-\rho|\, d\sigma + O(\log t).$$

The terms of the last sum are bounded, since

$$\tfrac{3}{2}\log(\tfrac{9}{4}+1) \geqslant \int_{\frac12}^{2} \log\{(\sigma-\beta)^2+(\gamma-t)^2\}\, d\sigma \geqslant 2\int_{\frac12}^{2} \log|\sigma-\beta|\, d\sigma > -A.$$

Hence
$$\int_{\frac12}^{2} \log|\zeta(s)|\, d\sigma = O(\log t), \tag{9.9.5}$$

and the result follows from the previous theorem.

It was proved by F. and R. Nevanlinna (1), (2) that

$$\int_0^T \frac{S(t)}{t}\, dt = A + O\left(\frac{\log T}{T}\right). \tag{9.9.6}$$

This follows from the previous result by integration by parts; for

$$\int_1^T \frac{S(t)}{t}\, dt = \left[\frac{S_1(t)}{t}\right]_1^T + \int_1^T \frac{S_1(t)}{t^2}\, dt = A + \frac{S_1(T)}{T} - \int_T^\infty \frac{S_1(t)}{t^2}\, dt.$$

Since $S_1(T) = O(\log T)$, the middle term is $O(T^{-1}\log T)$, and the last term is

$$O\left(\int_T^\infty \frac{\log t}{t^2}\, dt\right) = O\left(-\left[\frac{\log t}{t}\right]_T^\infty + \int_T^\infty \frac{dt}{t^2}\right) = O\left(\frac{\log T}{T}\right).$$

Hence the result follows. A similar result clearly holds for

$$\int_1^T \frac{S(t)}{t^\alpha}\, dt \quad (0 < \alpha < 1).$$

It has recently been proved by A. Selberg (5) that

$$S(t) = \Omega_{\pm}\{(\log t)^{\frac13}(\log\log t)^{-\frac73}\} \tag{9.9.7}$$

with a similar result for $S_1(t)$; and that

$$S_1(t) = \Omega_{+}\{(\log t)^{\frac13}(\log\log t)^{-4}\}. \tag{9.9.8}$$

9.10. THEOREM 9.10.† *$S(t)$ has an infinity of changes of sign.*

Consider the interval (γ_n, γ_{n+1}) in which $N(t) = n$. Let $l(t)$ be the

† Titchmarsh (17).

linear function of t such that $l(\gamma_n) = S(\gamma_n)$, $l(\gamma_{n+1}) = S(\gamma_{n+1}-0)$. Then for $\gamma_n < t < \gamma_{n+1}$

$$\begin{aligned} l(t)-S(t) &= \{S(\gamma_{n+1}-0)-S(\gamma_n)\}\frac{t-\gamma_n}{\gamma_{n+1}-\gamma_n} - \{S(t)-S(\gamma_n)\} \\ &= -\{L(\gamma_{n+1})-L(\gamma_n)\}\frac{t-\gamma_n}{\gamma_{n+1}-\gamma_n} + \{L(t)-L(\gamma_n)\} + O\left(\frac{1}{\gamma_n}\right), \end{aligned}$$

using (9.3.2) and the fact that $N(t)$ is constant in the interval. The first two terms on the right give

$$\begin{aligned} &-L'(\xi)(t-\gamma_n)+L'(\eta)(t-\gamma_n) \quad (\gamma_n < \eta < t,\ \gamma_n < \xi < \gamma_{n+1}) \\ &= L''(\xi_1)(\eta-\xi)(t-\gamma_n) \quad (\xi_1 \text{ between } \xi \text{ and } \eta) \\ &= O(1/\gamma_n) \end{aligned}$$

since $\gamma_{n+1}-\gamma_n = O(1)$. Hence

$$\begin{aligned} \int_{\gamma_n}^{\gamma_{n+1}} S(t)\, dt &= \int_{\gamma_n}^{\gamma_{n+1}} l(t)\, dt + O\left(\frac{\gamma_{n+1}-\gamma_n}{\gamma_n}\right) \\ &= \tfrac12(\gamma_{n+1}-\gamma_n)\{S(\gamma_n)+S(\gamma_{n+1}-0)\} + O\left(\frac{\gamma_{n+1}-\gamma_n}{\gamma_n}\right). \end{aligned}$$

Suppose that $S(t) \geqslant 0$ for $t > t_0$. Then

$$N(\gamma_n) \geqslant N(\gamma_n - 0)+1$$

gives
$$S(\gamma_n) \geqslant S(\gamma_n - 0)+1 \geqslant 1.$$

Hence
$$\begin{aligned} \int_{\gamma_n}^{\gamma_{n+1}} S(t)\, dt &\geqslant \tfrac12(\gamma_{n+1}-\gamma_n) + O\left(\frac{\gamma_{n+1}-\gamma_n}{\gamma_n}\right) \\ &\geqslant \tfrac14(\gamma_{n+1}-\gamma_n) \quad (n \geqslant n_0). \end{aligned}$$

Hence
$$\int_{\gamma_{n_0}}^{\gamma_N} S(t)\, dt \geqslant \tfrac14(\gamma_N - \gamma_{n_0}),$$

contrary to Theorem 9.9 (A). Similarly the hypothesis $S(t) \leqslant 0$ for $t > t_0$ can be shown to lead to a contradiction.

It has been proved by A. Selberg (5) that $S(t)$ changes sign at least

$$T(\log T)^{\frac13} e^{-A\sqrt{\log\log T}}$$

times in the interval $(0, T)$.

9.11. At the present time no improvement on the result

$$S(T) = O(\log T)$$

is known. But it is possible to prove directly some of the results which would follow from such an improvement. We shall first prove†

THEOREM 9.11. *The gaps between the ordinates of successive zeros of $\zeta(s)$ tend to 0.*

† Littlewood (3).

This would follow at once from (9.3.2) if it were possible to prove that $S(t) = o(\log t)$.

The argument given in § 9.1 shows that the gaps are bounded. Here we have to apply a similar argument to the strip $T-\delta \leqslant t \leqslant T+\delta$, where δ is arbitrarily small, and it is clear that we cannot use four concentric circles. But the ideas of the theorems of Borel–Carathéodory and Hadamard are in no way essentially bound up with sets of concentric circles, and the difficulty can be surmounted by using suitable elongated curves instead.

Let D_4 be the rectangle with centre $3+iT$ and a corner at $-3+i(T+\delta)$, the sides being parallel to the axes. We represent D_4 conformally on the unit circle D_4' in the z-plane, so that its centre $3+iT$ corresponds to $z = 0$. By this representation a set of concentric circles $|z| = r$ inside D_4' will correspond to a set of convex curves inside D_4, such that as $r \to 0$ the curve shrinks upon the point $3+iT$, while as $r \to 1$ it tends to coincidence with D_4. Let D_1', D_2', D_3' be circles (independent, of course, of T) for which the corresponding curves D_1, D_2, D_3 in the s-plane pass through the points $2+iT$, $-1+iT$, $-2+iT$ respectively.

The proof now proceeds as before. We consider the function

$$f(z) = \log \zeta\{s(z)\},$$

where $s = s(z)$ is the analytic function corresponding to the conformal representation; and we apply the theorems of Borel–Carathéodory and Hadamard in the same way as before.

9.12. We shall now obtain a more precise result of the same kind.†

THEOREM 9.12. *For every large positive T, $\zeta(s)$ has a zero $\beta+i\gamma$ satisfying*

$$|\gamma-T| < \frac{A}{\log\log\log T}.$$

This was first proved by Littlewood by a detailed study of the conformal representation used in the previous proof. This involves rather complicated calculations with elliptic functions. We shall give here two proofs which avoid these calculations.

In the first, we replace the rectangles by a succession of circles. Let T be a large positive number, and suppose that $\zeta(s)$ has no zero $\beta+i\gamma$ such that $T-\delta \leqslant \gamma \leqslant T+\delta$, where $\delta < \frac{1}{2}$. Then the function

$$f(s) = \log \zeta(s),$$

where the logarithm has its principal value for $\sigma > 2$, is regular in the rectangle

$$-2 \leqslant \sigma \leqslant 3, \qquad T-\delta \leqslant t \leqslant T+\delta.$$

† Littlewood (3); proofs given here by Titchmarsh (13), Kramaschke (1).

Let c_ν, C_ν, $\mathbf{C}_\nu$, Γ_ν be four concentric circles, with centre $2-\frac{1}{4}\nu\delta+iT$, and radii $\frac{1}{8}\delta$, $\frac{1}{2}\delta$, $\frac{3}{4}\delta$, and δ respectively. Consider these sets of circles for $\nu = 0, 1,\ldots, n$, where $n = [12/\delta]+1$, so that $2-\frac{1}{4}n\delta \leqslant -1$, i.e. the centre of the last circle lies on, or to the left of, $\sigma = -1$. Let m_ν, M_ν, and $\mathbf{M}_\nu$ denote the maxima of $|f(s)|$ on c_ν, C_ν, and $\mathbf{C}_\nu$ respectively.

Let A_1, A_2,... denote absolute constants (it is convenient to preserve their identity throughout the proof). We have $\mathbf{R}\{f(s)\} < A_1 \log T$ on all the circles, and $|f(2+iT)| < A_2$. Hence the Borel–Carathéodory theorem for the circles $\mathbf{C}_0$ and Γ_0 gives

$$\mathbf{M}_0 < \frac{\delta+\frac{3}{4}\delta}{\delta-\frac{3}{4}\delta}(A_1 \log T+A_2) = 7(A_1 \log T+A_2),$$

and in particular

$$|f(2-\tfrac{1}{4}\delta+iT)| < 7(A_1 \log T+A_2).$$

Hence, applying the Borel–Carathéodory theorem to $\mathbf{C}_1$ and Γ_1,

$$\mathbf{M}_1 < 7\{A_1 \log T+|f(2-\tfrac{1}{4}\delta+iT)|\} < (7+7^2)A_1 \log T+7^2A_2.$$

So generally $\quad \mathbf{M}_\nu < (7+\ldots+7^{\nu+1})A_1 \log T+7^{\nu+1}A_2,$

or, say,
$$\mathbf{M}_\nu < 7^\nu A_3 \log T. \tag{9.12.1}$$

Now by Hadamard's three-circles theorem

$$M_\nu \leqslant m_\nu^a \mathbf{M}_\nu^b,$$

where a and b are positive constants such that $a+b = 1$; in fact $a = \log\frac{3}{2}/\log 3$, $b = \log 2/\log 3$. Also, since the circle $C_{\nu-1}$ includes the circle c_ν, $m_\nu \leqslant M_{\nu-1}$. Hence

$$M_\nu \leqslant M_{\nu-1}^a \mathbf{M}_\nu^b \quad (\nu = 1, 2,\ldots, n).$$

Thus $\quad M_1 \leqslant M_0^a \mathbf{M}_1^b, \qquad M_2 \leqslant M_1^a \mathbf{M}_2^b \leqslant M_0^{a^2} \mathbf{M}_1^{ab} \mathbf{M}_2^b,$

and so on, giving finally

$$M_n \leqslant M_0^{a^n} \mathbf{M}_1^{a^{n-1}b} \mathbf{M}_2^{a^{n-2}b} \ldots \mathbf{M}_n^b.$$

Hence, by (9.12.1),

$$M_n \leqslant M_0^{a^n} 7^{a^{n-1}b+2a^{n-2}b+\ldots+nb}(A_3 \log T)^{a^{n-1}b+a^{n-2}b+\ldots+b}.$$

Now

$$a^{n-1}b+2a^{n-2}b+\ldots+nb < n^2,$$

$$a^{n-1}b+a^{n-2}b+\ldots+b = b(1-a^n)/(1-a) = 1-a^n.$$

Hence $\quad M_n \leqslant M_0^{a^n} 7^{n^2}(A_3 \log T)^{1-a^n} < A_4\, 7^{n^2}(\log T)^{1-a^n},$

since M_0 is bounded as $T \to \infty$.

But $|\zeta(s)| > t^{A_5}$ for $\sigma \leqslant -1$, $t > t_0$, so that $M_n > A_5 \log T$. Hence

$$A_5 < A_4\, 7^{n^2}(\log T)^{-a^n},$$

$$\log\log T < \left(\frac{1}{a}\right)^n \left(n^2 \log 7-\log\frac{A_5}{A_4}\right),$$

$$\log\log\log T < n \log\frac{1}{a}+A_6 \log n,$$

so that
$$\delta < \frac{12}{n-1} < \frac{A}{\text{logloglog}\, T},$$
and the result follows.

9.13. Second Proof. Consider the angular region in the s-plane with vertex at $s = -3+iT$, bounded by straight lines making angles $\pm\frac{1}{2}\alpha\,(0 < \alpha < \pi)$ with the real axis.

Let
$$w = (s+3-iT)^{\pi/\alpha}.$$
Then the angular region is mapped on the half-plane $\mathbf{R}(w) \geqslant 0$. The point $s = 2+iT$ corresponds to
$$w = 5^{\pi/\alpha}.$$
Let
$$z = \frac{w-5^{\pi/\alpha}}{w+5^{\pi/\alpha}}.$$
Then the angular region corresponds to the unit circle in the z-plane, and $s = 2+iT$ corresponds to its centre $z = 0$. If $s = \sigma+iT$ corresponds to $z = -r$, then
$$(\sigma+3)^{\pi/\alpha} = w = 5^{\pi/\alpha}\frac{1-r}{1+r},$$
i.e.
$$r = \left\{1-\left(\frac{\sigma+3}{5}\right)^{\pi/\alpha}\right\}\Big/\left\{1+\left(\frac{\sigma+3}{5}\right)^{\pi/\alpha}\right\}.$$

Suppose that $\zeta(s)$ has no zeros in the angular region, so that $\log\zeta(s)$ is regular in it.

Let $s = \frac{3}{2}+iT$, $-1+iT$, $-2+iT$ correspond to $z = -r_1, -r_2, -r_3$ respectively. Let M_1, M_2, M_3 be the maxima of $|\log\zeta(s)|$ on the s-curves corresponding to $|z| = r_1, r_2, r_3$. Then Hadamard's three-circles theorem gives
$$\log M_2 \leqslant \frac{\log r_3/r_2}{\log r_3/r_1}\log M_1 + \frac{\log r_2/r_1}{\log r_3/r_1}\log M_3.$$

It is easily verified that, on the curve corresponding to $|z| = r_1$, $\sigma \geqslant \frac{3}{2}$. For if $w = \xi+i\eta$, then
$$\sigma = -3+(\xi^2+\eta^2)^{\alpha/2\pi}\cos\left(\frac{\alpha}{\pi}\arctan\frac{\eta}{\xi}\right),$$
which is a minimum at $\eta = 0$, for given ξ, if $0 < \alpha < \frac{1}{2}\pi$; and the minimum is $-3+\xi^{\alpha/\pi}$, which, as a function of ξ, is a minimum when ξ is a minimum, i.e. when $z = -r_1$. It therefore follows that $\log M_1 < A$.

Since $\mathbf{R}\{\log\zeta(s)\} < A\log T$ in the angle, it follows from the Borel–Carathéodory theorem that
$$M_3 < \frac{2}{1-r_3}(A\log T+A) < \frac{A\log T}{1-r_3}.$$

Hence
$$\log M_2 \leqslant A+\frac{\log r_2/r_1}{\log r_3/r_1}\log\left(\frac{A\log T}{1-r_3}\right).$$
Now if r_1, r_2, and r_3 are sufficiently near to 1, i.e. if α is sufficiently small,
$$\frac{\log r_2/r_1}{\log r_3/r_1} = \frac{\log\left(1+\dfrac{r_2-r_1}{r_1}\right)}{\log\left(1+\dfrac{r_3-r_1}{r_1}\right)} \leqslant \left(\frac{r_2-r_1}{r_3-r_1}\right)^{\frac{1}{2}},$$
and
$$\frac{r_2-r_1}{r_3-r_1} = \frac{\dfrac{1-r_1}{1+r_1}-\dfrac{1-r_2}{1+r_2}}{\dfrac{1-r_1}{1+r_1}-\dfrac{1-r_3}{1+r_3}}\,\frac{1+r_2}{1+r_3} < \frac{(\frac{9}{10})^{\pi/\alpha}-(\frac{2}{5})^{\pi/\alpha}}{(\frac{9}{10})^{\pi/\alpha}-(\frac{1}{5})^{\pi/\alpha}}$$
$$< 1-A(\tfrac{4}{9})^{\pi/\alpha}.$$
Hence
$$\frac{\log r_2/r_1}{\log r_3/r_1} < 1-A(\tfrac{4}{9})^{\pi/\alpha}.$$
Also
$$1/(1-r_3) < A5^{\pi/\alpha}.$$
Hence
$$\log M_2 < A+\{1-A(\tfrac{4}{9})^{\pi/\alpha}\}\left\{\text{loglog}\,T+\frac{\pi}{\alpha}\log 5+A\right\}.$$

Let $\alpha = \pi/(c\,\text{logloglog}\,T)$. Then
$$\log M_2 < A+\{1-A(\text{loglog}\,T)^{-c\log\frac{9}{4}}\}\{\text{loglog}\,T+c\log 5\,\text{logloglog}\,T+A\}$$
$$< \text{loglog}\,T-(\text{loglog}\,T)^{\frac{1}{2}}$$
if $c\log\frac{9}{4} < \frac{1}{2}$ and T is large enough. Hence
$$M_2 < \log T\, e^{-(\text{loglog}\,T)^{\frac{1}{2}}} < \epsilon\log T \quad (T > T_0(\epsilon)).$$
In particular
$$\log|\zeta(-1+iT)| < \epsilon\log T,$$
$$|\zeta(-1+iT)| < T^{\epsilon}.$$
But
$$|\zeta(-1+iT)| = |\chi(-1+iT)\zeta(2-iT)| > KT^{\frac{3}{2}}.$$
We thus obtain a contradiction, and the result follows.

9.14. Another result† in the same order of ideas is

THEOREM 9.14. *For any fixed h, however small,*
$$N(T+h)-N(T) > K\log T$$
for $K = K(h)$, $T > T_0$.

This result is not a consequence of Theorem 9.4 if h is less than a certain value.

Consider the same angular region as before, with a new α such that

† Not previously published.

$\tan\alpha \leqslant \frac{1}{4}$, and suppose now that $\zeta(s)$ has zeros $\rho_1, \rho_2,\ldots, \rho_n$ in the angular region. Let

$$F(s) = \frac{\zeta(s)}{(s-\rho_1)\ldots(s-\rho_n)}.$$

Let C be the circle with centre $\frac{1}{2}+iT$ and radius 3. Then $|s-\rho_\nu| \geqslant 1$ on C. Hence

$$|F(s)| \leqslant |\zeta(s)| < T^A$$

on C, and so also inside C.

Let $f(s) = \log F(s)$. Then $f(s)$ is regular in the angle, and

$$\mathbf{R} f(s) < A \log T.$$

Also

$$f(2+iT) = \log\zeta(2+iT) - \sum_{\nu=1}^{n} \log(2+iT-\rho_\nu)$$
$$= O(1) + \sum_{\nu=1}^{n} O(1) = O(n).$$

Let M_1, M_2, and M_3 now denote the maxima of $|f(s)|$ on the three s-curves. Then

$$M_3 < \frac{A}{1-r_3}(\log T+n).$$

Also $M_1 < An$, as for $f(2+iT)$. Hence

$$\log|f(-1+iT)| \leqslant \log M_2$$
$$< \frac{\log r_3/r_2}{\log r_3/r_1}(A+\log n) + \frac{\log r_2/r_1}{\log r_3/r_1}\log\left\{\frac{A(n+\log T)}{1-r_3}\right\}$$
$$< A+\log n + \frac{\log r_2/r_1}{\log r_3/r_1}\left\{\log\frac{1}{1-r_3} + \log\left(\frac{\log T}{n}\right)\right\}$$
$$< A+\log n + \{1-A(\tfrac{4}{9})^{\pi/\alpha}\}\left\{\frac{\pi}{\alpha}\log 5 + \log\left(\frac{\log T}{n}\right)\right\}$$

as before. But

$$|f(-1+iT)| = \left|\log\zeta(-1+iT) - \sum_{\nu=1}^{n}\log(-1+iT-\rho_\nu)\right|$$
$$\geqslant \log|\zeta(-1+iT)| - \sum_{\nu=1}^{n} O(1)$$
$$> A_1 \log T - A_2 n,$$

say. If $n > \frac{1}{2}(A_1/A_2)\log T$ the theorem follows at once. Otherwise

$$|f(-1+iT)| > \tfrac{1}{2}A_1 \log T,$$

and we obtain

$$\log\left(\frac{\log T}{n}\right) < A + \{1-A(\tfrac{4}{9})^{\pi/\alpha}\}\left\{\frac{\pi}{\alpha}\log 5 + \log\left(\frac{\log T}{n}\right)\right\},$$

$$A(\tfrac{4}{9})^{\pi/\alpha}\log\left(\frac{\log T}{n}\right) < A + \{1-A(\tfrac{4}{9})^{\pi/\alpha}\}\frac{\pi}{\alpha}\log 5,$$

and hence
$$\log\log\left(\frac{\log T}{n}\right) < \frac{\pi}{\alpha}\log\frac{9}{4} + \log\frac{1}{\alpha} + A < \frac{A}{\alpha},$$

$$n > e^{-e^{A/\alpha}}\log T.$$

This proves the theorem.

9.15. The function $N(\sigma, T)$. We define $N(\sigma, T)$ to be the number of zeros $\beta+i\gamma$ of the zeta-function such that $\beta > \sigma$, $0 < t \leqslant T$. For each T, $N(\sigma, T)$ is a non-increasing function of σ, and is 0 for $\sigma \geqslant 1$. On the Riemann hypothesis, $N(\sigma, T) = 0$ for $\sigma > \frac{1}{2}$. Without any hypothesis, all that we can say so far is that

$$N(\sigma, T) \leqslant N(T) < AT\log T$$

for $\frac{1}{2} < \sigma < 1$.

The object of the next few sections is to improve upon this inequality for values of σ between $\frac{1}{2}$ and 1.

We return to the formula (9.9.1). Let $\phi(s) = \zeta(s)$, $\alpha = \sigma_0$, $\beta = 2$, and this time take the imaginary part. We have

$$\nu(\sigma, T) = N(\sigma, T) \quad (\sigma < 1), \qquad \nu(\sigma, T) = 0 \quad (\sigma \geqslant 1).$$

We obtain, if T is not the ordinate of a zero,

$$2\pi\int_{\sigma_0}^{1} N(\sigma, T)\,d\sigma = \int_0^T \log|\zeta(\sigma_0+it)|\,dt - \int_0^T \log|\zeta(2+it)|\,dt +$$
$$+ \int_{\sigma_0}^{2} \arg\zeta(\sigma+iT)\,d\sigma + K(\sigma_0),$$

where $K(\sigma_0)$ is independent of T. We deduce†

THEOREM 9.15. *If $\frac{1}{2} \leqslant \sigma_0 \leqslant 1$, and $T \to \infty$,*

$$2\pi\int_{\sigma_0}^{1} N(\sigma, T)\,d\sigma = \int_0^T \log|\zeta(\sigma_0+it)|\,dt + O(\log T).$$

We have

$$\int_0^T \log|\zeta(2+it)|\,dt = \mathbf{R}\sum_{n=2}^{\infty}\frac{\Lambda_1(n)}{n^2}\frac{n^{-iT}-1}{-i\log n} = O(1).$$

Also, by § 9.4, $\arg\zeta(\sigma+iT) = O(\log T)$ uniformly for $\sigma \geqslant \frac{1}{2}$, if T is not the ordinate of a zero. Hence the integral involving $\arg\zeta(\sigma+iT)$ is $O(\log T)$. The result follows if T is not the ordinate of a zero, and this restriction can then be removed from considerations of continuity.

† Littlewood (4).

THEOREM 9.15 (A).† *For any fixed σ greater than $\frac{1}{2}$,*

$$N(\sigma, T) = O(T).$$

For any non-negative continuous $f(t)$

$$\frac{1}{b-a}\int_a^b \log f(t)\,dt \leqslant \log\left\{\frac{1}{b-a}\int_a^b f(t)\,dt\right\}.$$

Thus, for $\frac{1}{2} < \sigma < 1$,

$$\int_0^T \log|\zeta(\sigma+it)|\,dt = \tfrac{1}{2}\int_0^T \log|\zeta(\sigma+it)|^2\,dt$$

$$\leqslant \tfrac{1}{2}T\log\left\{\frac{1}{T}\int_0^T |\zeta(\sigma+it)|^2\,dt\right\} = O(T)$$

by Theorem 7.2. Hence, by Theorem 9.15,

$$\int_{\sigma_0}^1 N(\sigma, T)\,d\sigma = O(T)$$

for $\sigma_0 > \frac{1}{2}$. Hence, if $\sigma_1 = \frac{1}{2}+\frac{1}{2}(\sigma_0-\frac{1}{2})$,

$$N(\sigma_0, T) \leqslant \frac{1}{\sigma_0-\sigma_1}\int_{\sigma_1}^{\sigma_0} N(\sigma, T)\,d\sigma \leqslant \frac{2}{\sigma_0-\frac{1}{2}}\int_{\sigma_1}^1 N(\sigma, T)\,d\sigma = O(T),$$

the required result.

From this theorem, and the fact that $N(T) \sim AT\log T$, it follows that *all but an infinitesimal proportion of the zeros of $\zeta(s)$ lie in the strip $\frac{1}{2}-\delta < \sigma < \frac{1}{2}+\delta$, however small δ may be.*

9.16. We shall next prove a number of theorems in which the $O(T)$ of Theorem 9.15 (A) is replaced by $O(T^{\theta})$, where $\theta < 1$.‡ We do this by applying the above methods, not to $\zeta(s)$ itself, but to the function

$$\zeta(s)M_X(s) = \zeta(s)\sum_{n<X}\frac{\mu(n)}{n^s}.$$

The zeros of $\zeta(s)$ are zeros of $\zeta(s)M_X(s)$. If $\sigma > 1$, $M_X(s) \to 1/\zeta(s)$ as $X \to \infty$, so that $\zeta(s)M_X(s) \to 1$. On the Riemann hypothesis this is also true for $\frac{1}{2} < \sigma \leqslant 1$. Of course we cannot prove this without any hypothesis; but we can choose X so that the additional factor neutralizes to a certain extent the peculiarities of $\zeta(s)$, even for values of σ less than 1.

Let
$$f_X(s) = \zeta(s)M_X(s)-1.$$

† Bohr and Landau (4), Littlewood (4).
‡ Bohr and Landau (5), Carlson (1), Landau (12), Titchmarsh (5), Ingham (5).

We shall first prove

THEOREM 9.16. *If for some $X = X(\sigma, T)$, $T^{1-l(\sigma)} \leqslant X < T^A$,*

$$\int_{\frac{1}{2}T}^T |f_X(s)|^2\,dt = O(T^{l(\sigma)}\log^m T)$$

as $T \to \infty$, uniformly for $\sigma \geqslant \alpha$, where $l(\sigma)$ is a positive non-increasing function with a bounded derivative, and m is a constant $\geqslant 0$, then

$$N(\sigma, T) = O(T^{l(\sigma)}\log^{m+1} T)$$

uniformly for $\sigma \geqslant \alpha+1/\log T$.

We have
$$f_X(s) = \zeta(s)\sum_{n<X}\frac{\mu(n)}{n^s}-1 = \sum\frac{a_n(X)}{n^s},$$

where $a_1(X) = 0$,

$$a_n(X) = \sum_{d|n}\mu(d) = 0 \quad (n < X),$$

and
$$|a_n(X)| = \left|\sum_{\substack{d|n\\ d<X}}\mu(d)\right| \leqslant d(n)$$

for all n and X.

Let
$$1-f_X^2 = \zeta M_X(2-\zeta M_X) = \zeta(s)g(s) = h(s)$$

say, where $g(s) = g_X(s)$ and $h(s) = h_X(s)$ are regular except at $s = 1$. Now for $\sigma \geqslant 2$, $X > X_0$,

$$|f_X(s)|^2 \leqslant \left(\sum_{n\geqslant X}\frac{d(n)}{n^2}\right)^2 = O(X^{2\epsilon-2}) < \frac{1}{2X} < \frac{1}{2},$$

so that $h(s) \neq 0$. Applying (9.9.1) to $h(s)$, and writing

$$\nu(\sigma, T_1, T_2) = \nu(\sigma, T_2)-\nu(\sigma, T_1),$$

we obtain

$$2\pi\int_{\sigma_0}^2 \nu(\sigma, \tfrac{1}{2}T, T)\,d\sigma = \int_{\frac{1}{2}T}^T \{\log|h(\sigma_0+it)|-\log|h(2+it)|\}\,dt +$$

$$+\int_{\sigma_0}^2 \{\arg h(\sigma+iT)-\arg h(\sigma+\tfrac{1}{2}iT)\}\,d\sigma.$$

Now
$$\log|h(s)| \leqslant \log\{1+|f_X(s)|^2\} \leqslant |f_X(s)|^2,$$

so that, if $\sigma_0 \geqslant \alpha$,

$$\int_{\frac{1}{2}T}^T \log|h(\sigma_0+it)|\,dt \leqslant \int_{\frac{1}{2}T}^T |f_X(\sigma_0+it)|^2\,dt = O(T^{l(\sigma_0)}\log^m T).$$

Next

$$-\log|h(2+it)| \leqslant -\log\{1-|f_X(2+it)|^2\} \leqslant 2|f_X(2+it)|^2 < X^{-1}$$