for each $\varepsilon > 0$. At the same time, $F(z)\varphi(z)$ vanishes (with appropriate multiplicity) at each point of the given sequence Σ.

Proof of theorem. Fixing the number $\eta > 0$, we take the function $\delta(t)$ corresponding to it furnished by the lemma and $v(x)$, related to $\delta(t)$ as in the statement of the preceding theorem. The function $U(x)$ figuring in that result is, as we know, related to our F by the formula

$$U(x) \quad = \quad \log|F(x)| \;+\; Dx\log\left|\frac{x+1}{x-1}\right|.$$

Let us obtain a similar representation for $v(x)$.

Write

$$\nu(t) \quad = \quad \begin{cases} \eta t + \delta(t), & |t| \geqslant 1, \\ 0, & |t| < 1. \end{cases}$$

By property (ii) of the lemma, $\nu(t)$ is *increasing* on $(-\infty, \infty)$, and, by property (iii),

$$\frac{\nu(t)}{t} \longrightarrow \eta \qquad \text{as } t \longrightarrow \pm\infty.$$

$\nu(t)$ is in fact *piecewise constant*, with jump discontinuities at (and *only* at) the points x_k, $k = \pm 1, \pm 2, \ldots$ mentioned in the lemma's statement:

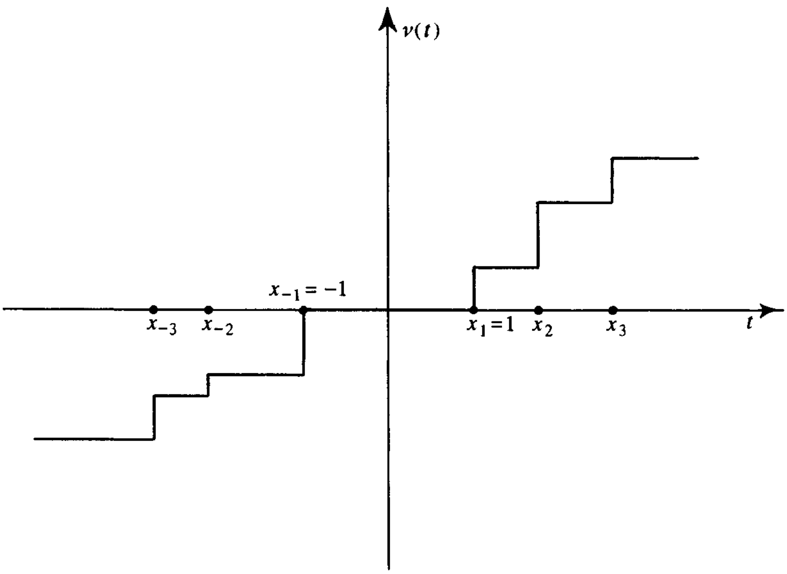

Figure 226

According to the lemma and discussion at the end of the last article, we have

$$v(x) \;=\; \int_{-\infty}^{\infty} \frac{x^2}{x-t}\frac{\delta(t)}{t^2}\,dt \;=\; \int_{|t|\geqslant 1} \frac{x^2}{x-t}\frac{\nu(t)}{t^2}\,dt \;+\; \eta x \log\left|\frac{x+1}{x-1}\right|$$

$$=\; \int_{-\infty}^{\infty} \left(\log\left|1-\frac{x}{t}\right| + \frac{x}{t}\right) d\nu(t) \;+\; \eta x \log\left|\frac{x+1}{x-1}\right|.$$

In terms of

$$H(z) \;=\; \int_{-\infty}^{\infty} \left(\log\left|1-\frac{z}{t}\right| + \frac{\Re z}{t}\right) d\nu(t),$$

we thus have our desired representation:

$$v(x) \;=\; H(x) \;+\; \eta x \log\left|\frac{x+1}{x-1}\right|.$$

Using this formula with the previous one for $U(x)$, we can reformulate the conclusion of the last theorem to get

$$\int_{-\infty}^{\infty} \frac{|\log|F(x)| + H(x)|}{1+x^2}\,dx \;<\; \infty.$$

The use of

$$\int_{-\infty}^{\infty} \left(\log\left|1-\frac{z}{t}\right| + \frac{\Re z}{t}\right) d[\nu(t)],$$

obviously the logarithm of the modulus of an entire function, in place of $H(z)$ comes now immediately to mind – one recalls the lemma of §A.1. (N.B. For $p \geqslant 0$, $[p]$ denotes, as usual, the greatest integer $\leqslant p$, *but when* $p < 0$, *we take* $[p]$ *as the least integer* $\geqslant p$, *so as to have* $[-p] = -[p]$.)

Here, one must be somewhat careful. The expression

$$\int_{-\infty}^{\infty} \left(\log\left|1-\frac{z}{t}\right| + \frac{\Re z}{t}\right) d\nu(t)$$

is very sensitive to small changes in ν because of the term $\Re z/t$ in the integrand; replacement of $\nu(t)$ by $[\nu(t)]$ usually produces a new term linear in $\Re z = x$ which spoils the convergence of the integral involving $\log|F|$

and H. What we do have is a relation

$$\int_{-\infty}^{\infty}\left(\log\left|1-\frac{x+\mathrm{i}}{t}\right|+\frac{x}{t}\right)(\mathrm{d}[\nu(t)]-\mathrm{d}\nu(t))$$

$$\leqslant \quad \gamma x \;+\; 2\log^{+}|x| \;+\; O(1), \qquad x\in\mathbb{R},$$

valid with a certain real constant γ.

To show this, a device from the proof of the theorem in article 1 is used.* Assuming, wlog, that $x \geqslant 0$, we observe that the *left-hand member* of the relation in question can be rewritten as

$$\int_{0}^{\infty}\log\left|1-\left(\frac{x+\mathrm{i}}{t}\right)^{2}\right|(\mathrm{d}[\nu(t)]-\mathrm{d}\nu(t))$$

$$+\quad\int_{0}^{\infty}\left(\log\left|1+\frac{x+\mathrm{i}}{t}\right|-\frac{x}{t}\right)(\mathrm{d}[-\nu(-t)]+\mathrm{d}\nu(-t)-\mathrm{d}[\nu(t)]+\mathrm{d}\nu(t)).$$

To estimate the *first* integral we fall back on the lemma from §A.1, according to which it is

$$\leqslant \quad \log^{+}|x| \;+\; O(1).$$

The *second* one we integrate by parts, remembering that $\nu(t) = 0$ for $|t| < 1$. When that is done, the integrated terms (involving the *differences* $(-\nu(-t))-[-\nu(-t)]$ and $\nu(t)-[\nu(t)]$) all vanish, leaving

$$x\int_{1}^{\infty}\frac{[\nu(t)]-\nu(t)+(-\nu(-t))-[-\nu(-t)]}{t^{2}}\,\mathrm{d}t$$

$$+\quad\int_{1}^{\infty}\left(\frac{\partial}{\partial t}\log\left|1+\frac{x+\mathrm{i}}{t}\right|\right)\{(-\nu(-t)-[-\nu(-t)]) \;-\; (\nu(t)-[\nu(t)])\}\,\mathrm{d}t.$$

The first term here is just γx, where

$$\gamma \quad = \quad \int_{1}^{\infty}\frac{[\nu(t)]-\nu(t)+(-\nu(-t))-[-\nu(-t)]}{t^{2}}\,\mathrm{d}t$$

(a quantity between -1 and 1). In the second term, the expression in $\{\ \}$ lies between -1 and 1, while

$$\frac{\partial}{\partial t}\log\left|1+\frac{x+\mathrm{i}}{t}\right| \quad < \quad 0$$

* One may also work directly with the expression $\int_{|t|\geqslant 1}\{\log|1-(x+\mathrm{i})/t| + x/t\}\mathrm{d}([\nu(t)]-\nu(t))$, adapting the proof of the lemma in §A.1 to the (improper) integral involving the first term in $\{\ \}$ and using partial integration on what remains.

for $1 \leqslant t < \infty$ since $x \geqslant 0$. The second term is therefore $\leqslant \log|1+x+\mathrm{i}|$ (cf. proof of the lemma in §A.1).

Putting these results together, we see that

$$\int_{-\infty}^{\infty}\left(\log\left|1-\frac{x+\mathrm{i}}{t}\right|+\frac{x}{t}\right)(\mathrm{d}[\nu(t)]-\mathrm{d}\nu(t))$$

$$\leqslant \quad \log^+|x| \ + \ O(1) \ + \ \gamma x \ + \ \log|1+x+\mathrm{i}|$$

for $x \geqslant 0$, proving the desired inequality when that is the case. A similar argument may be used when x is negative.

Having established our relation, we proceed to construct the entire function $\varphi(z)$. Take S as the sequence of points where $[\nu(t)]$ *jumps*, each of those being *repeated* a number of times equal to the *magnitude* of the *jump* corresponding to it; S simply consists of some of the points x_k, $k = \pm 1, \pm 2, \ldots$, with certain repetitions. We then put

$$\varphi(z) \quad = \quad \mathrm{e}^{-\gamma z}\prod_{\lambda\in S}\left(1-\frac{z}{\lambda}\right)\mathrm{e}^{z/\lambda},$$

taking care to repeat each of the factors on the right as many times as the λ corresponding to it is repeated in S. This function $\varphi(z)$ is entire, and clearly

$$\log|\varphi(z)| \quad = \quad -\gamma\Re z \ + \ \int_{-\infty}^{\infty}\left(\log\left|1-\frac{z}{t}\right|+\frac{\Re z}{t}\right)\mathrm{d}[\nu(t)].$$

The inequality proved above now yields

$$\log|\varphi(x+\mathrm{i})| \quad \leqslant \quad H(x+\mathrm{i}) \ + \ 2\log^+|x| \ + \ O(1), \qquad x\in\mathbb{R}.$$

Obviously, $n_S(t) \ = \ [\nu(t)]$, so

$$\frac{n_S(t)}{t} \longrightarrow \eta \quad \text{as } t \longrightarrow \pm\infty.$$

Also,

$$\int_{-\infty}^{\infty}\frac{|n_S(t)-\eta t|}{1+t^2}\mathrm{d}t \ \leqslant \ \int_{-\infty}^{\infty}\frac{|\nu(t)-\eta t|+1}{1+t^2}\mathrm{d}t \quad = \quad \int_{-\infty}^{\infty}\frac{|\delta(t)|+1}{1+t^2}\mathrm{d}t \ < \ \infty$$

by property (iv) of the lemma. The hypothesis of the theorem in article 1 therefore holds for the function $\mathrm{e}^{\gamma z}\varphi(z)$, so it – and hence $\varphi(z)$ – is of *exponential type*. That result (as well as the second Lindelöf theorem of §B, Chapter III, on which it depends) is also easily adapted so as to apply to functions like $H(z)$, and we thus find, reasoning as for $\mathrm{e}^{\gamma z}\varphi(z)$, that

$$H(z) \ \leqslant \ \text{const.}|z| \ + \ O(1).$$

$H(z)$ is, of course, *harmonic* in $\Im z > 0$. These properties of H are used to get a grip on $\log|F(x+i)| + H(x+i)$, the idea being to then make use of our relation between $\log|\varphi(x+i)|$ and $H(x+i)$.

Because $F(z)$ is of exponential type, we certainly have

$$\log|F(z)| \;+\; H(z) \;\leqslant\; \text{const.}|z| \;+\; O(1),$$

with the left side harmonic in $\Im z > 0$. The functions $|F(z)|$ and $e^{H(z)}$ are actually continuous right up to the real axis, as long as we take the value of the latter one to be *zero* at the points of S; moreover,

$$\int_{-\infty}^{\infty} \frac{(\log|F(x)| + H(x))_+}{1+x^2}\,dx \;<\; \infty$$

by the observations made at the beginning of this proof ($(a)_+$ denotes $\max(a,0)$ for real a). We can therefore use the theorem of §E, Chapter III (actually, a variant of it having, however, exactly the same proof) so as to conclude that (with an appropriate constant A)

$$\log|F(x+i)| \;+\; H(x+i) \;\leqslant\; A \;+\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(\log|F(t)| + H(t))_+}{(x-t)^2+1}\,dt.$$

Now we bring in the above relation involving $|\varphi(x+i)|$ and $H(x+i)$, and get

$$\log|F(x+i)| \;+\; \log|\varphi(x+i)| \;\leqslant\; O(1) \;+\; 2\log^+|x| \;+\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\log|F(t)| + H(t)|}{(x-t)^2+1}\,dt,$$

from which we easily see that

$$\int_{-\infty}^{\infty} \frac{\log^+|F(x+i)\varphi(x+i)|}{x^2+1}\,dx \;<\; \infty,$$

following the procedure so often used in Chapter VI and elsewhere. Having arrived at this point, we may use the theorem from §E, Chapter III once more, this time in the half plane $\{\Im z \leqslant 1\}$ – the function $F(z)\varphi(z)$ *is* entire, and of exponential type. Doing that and then repeating the argument just referred to, we find that

$$\int_{-\infty}^{\infty} \frac{\log^+|F(x)\varphi(x)|}{1+x^2}\,dx \;<\; \infty,$$

which is what we wanted to prove.

We are done.

Remark. If the function $v(t) = \eta t + \delta(t)$ were known to be *integral valued*, $\log|\varphi(z)|$ could have been taken *equal* to $H(z)$ in the above proof, and the discussion about the effect of replacing $v(t)$ by $[v(t)]$ avoided. To realize this simplification, we would have had to modify the lemma's construction so as to make it yield a function $\delta(t)$ with $\delta(t) + \eta t$ integral valued. As a matter of fact, that can be done without too much difficulty, and one thus arrives at an alternative derivation of the preceding result. Such is the procedure followed by Redheffer in his survey article.

3. Determination of the completeness radius for real and complex sequences Λ

We are finally ready to apply the result stated in §A.2.

Theorem (Beurling and Malliavin, 1961). *Let Λ be a sequence of distinct real numbers having effective density $\tilde{D}_\Lambda < \infty$. Then the completeness radius associated with Λ is equal to $\pi\tilde{D}_\Lambda$.*

Proof. According to the discussion at the beginning of this §, it is enough to show that the $e^{i\lambda t}$, $\lambda \in \Lambda$, are not complete on any interval of length $> 2\pi\tilde{D}_\Lambda$, and for that purpose it suffices, as explained there, to establish, for arbitrary $\eta > 0$, the existence of a non-zero entire function $G(z)$ of exponential type $\leqslant \pi(\tilde{D}_\Lambda + 3\eta)$ with

$$\int_{-\infty}^{\infty} |G(x)|\,dx \quad < \quad \infty$$

and

$$G(\lambda) \;=\; 0 \quad \text{for } \lambda \in \Lambda.$$

Writing $D = \tilde{D}_\Lambda + \eta$, we take the real sequence $\Sigma \supseteq \Lambda$ such that

$$\int_{-\infty}^{\infty} \frac{|n_\Sigma(t) - Dt|}{1+t^2}\,dt \quad < \quad \infty,$$

used in article 1 at the start of our constructions. We then *throw away any points that Σ may have in $(-1,\ 1)$, so as to ensure that $n_\Sigma(t) = 0$ there.** This perhaps leaves us with a certain finite number of $\mu \in \Lambda$ *not belonging to* Σ (the points of Λ in $(-1,\ 1)$); those will be taken care of in a moment.

We next form

$$F(z) \;=\; \prod_{\substack{\lambda\in\Sigma \\ \lambda\notin(-1,1)}} \left(1 - \frac{z}{\lambda}\right) e^{z/\lambda}$$

as in article 1, and use the little multiplier theorem from article 2 to get

* That does not affect the preceding relation!

the non-zero entire function $\varphi(z)$ of exponential type described there, such that

$$\int_{-\infty}^{\infty} \frac{\log^+ |F(x)\varphi(x)|}{1+x^2} \mathrm{d}x \quad < \quad \infty.$$

As remarked just after the statement of the little multiplier theorem (and as one checks immediately), when $y \longrightarrow \pm\infty$,

$$\frac{\log |F(\mathrm{i}y)\varphi(\mathrm{i}y)|}{|y|} \quad \longrightarrow \quad \pi(D+\eta) \quad = \quad \pi(\tilde{D}_\Lambda + 2\eta).$$

Put now

$$F_0(z) \quad = \quad F(z) \prod_{\substack{\mu\in\Lambda \\ -1<\mu<1}} (z-\mu).$$

The relations just written obviously still hold with F_0 standing in place of F. The *theorem on the multiplier* enunciated in §A.2 therefore gives us a non-zero entire function $\psi(z)$ of exponential type $\leqslant \pi\eta$ with

$$|F_0(x)\varphi(x)\psi(x)| \leqslant \text{const.}, \qquad x \in \mathbb{R}.$$

In view of the previous relation, we clearly have

$$\limsup_{y\to\pm\infty} \frac{\log|F_0(\mathrm{i}y)\varphi(\mathrm{i}y)\psi(\mathrm{i}y)|}{|y|} \quad \leqslant \quad \pi(\tilde{D}_\Lambda + 3\eta),$$

so the *boundedness* of $F_0\varphi\psi$ on the real axis implies that that product is *of exponential type* $\leqslant \pi(\tilde{D}_\Lambda + 3\eta)$ by the *third* Phragmén–Lindelöf theorem from §C of Chapter III.

The function $\varphi(z)$ furnished by the little multiplier theorem has a zero at each point of the real sequence S, with

$$\frac{n_S(t)}{t} \longrightarrow \eta \quad \text{for } t \longrightarrow \pm\infty.$$

We can thus certainly take *two different points* $s_1, s_2 \in S$ (!), and $\varphi(z)/(z-s_1)(z-s_2)$ will still be entire. Let, finally,

$$G(z) \quad = \quad \frac{F_0(z)\varphi(z)\psi(z)}{(z-s_1)(z-s_2)}.$$

This function is entire and of exponential type $\leqslant \pi(\tilde{D}_\Lambda + 3\eta)$, and

$$\int_{-\infty}^{\infty} |G(x)| \mathrm{d}x \quad < \quad \infty.$$

Also, $G(z)$ vanishes at each point of Λ since $F_0(z)$ does. We are done.

Remark. If Λ has repeated points, the argument just made goes through without change. The entire function $G(z)$ thus obtained then vanishes with appropriate multiplicitly at each point of Λ.

In order to now obtain the completeness radius for sequences of *complex* numbers Λ, we proceed somewhat as in §H.3 of Chapter III.

Notation. For complex λ *with non-zero real part,* we write

$$\lambda' = 1/\Re(1/\lambda).$$

If Λ is any sequence of complex numbers, we let Λ' be the real sequence consisting of the λ' corresponding to the $\lambda \in \Lambda$ having non-zero real part. Should several members of Λ correspond to the *same* value for λ', we look on that value as *repeated an appropriate number of times* in Λ'.

We then have the

Theorem (Beurling and Malliavin, 1967). *Let Λ be any sequence of distinct complex numbers. If*

$$\sum_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \frac{|\Im \lambda|}{|\lambda|^2} = \infty,$$

the exponentials $e^{i\lambda t}$, $\lambda \in \Lambda$, are complete on any interval of finite length.

Otherwise, they are complete on any interval of length $< 2\pi \tilde{D}_{\Lambda'}$, and, if that quantity is finite, incomplete on any interval of length $> 2\pi \tilde{D}_{\Lambda'}$.

Proof. If the $e^{i\lambda t}$ are incomplete on (say) the interval $[-L, L]$, there is (as at the beginning of §D, Chapter IX) a non-zero measure μ on $[-L, L]$ with

$$\int_{-L}^{L} e^{i\lambda t}\, d\mu(t) = 0, \qquad \lambda \in \Lambda.$$

The non-zero entire function

$$G(z) = \int_{-L}^{L} e^{izt}\, d\mu(t)$$

of exponential type $\leqslant L$ thus vanishes at each point of Λ, so, since G is bounded on the real axis, we have

$$\sum_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \frac{|\Im \lambda|}{|\lambda|^2} < \infty$$

by §G.3 of Chapter III.

Let Σ denote the complete sequence of zeros of G (with repetitions according to multiplicities, as usual). The Hadamard representation for G is then

$$G(z) \quad = \quad Az^p e^{cz} \prod_{\substack{\mu\in\Sigma \\ \mu\neq 0}} \left(1 - \frac{z}{\mu}\right) e^{z/\mu}.$$

Denote by S the difference set

$$\Sigma \sim \{\lambda\in\Lambda:\ \Re\lambda \neq 0\};$$

then

$$G(z) \quad = \quad Az^p e^{cz} \prod_{\substack{\mu\in S \\ \mu\neq 0}} \left(1 - \frac{z}{\mu}\right) e^{z/\mu} \cdot \prod_{\substack{\lambda\in\Lambda \\ \Re\lambda\neq 0}} \left(1 - \frac{z}{\lambda}\right) e^{z/\lambda}.$$

Take now the function

$$G_0(z) \quad = \quad Az^p e^{cz} \prod_{\substack{\mu\in S \\ \mu\neq 0}} \left(1 - \frac{z}{\mu}\right) e^{z/\mu} \cdot \prod_{\lambda'\in\Lambda'} \left(1 - \frac{z}{\lambda'}\right) e^{z/\lambda}$$

(with exponentials $e^{z/\lambda}$ and *not* $e^{z/\lambda'}$ in the second product!). By work done in §H.3 of Chapter III, we see that $G_0(z)$ is of exponential type, and that

$$|G_0(x)| \leqslant |G(x)|, \qquad x\in\mathbb{R},$$

so that $G_0(x)$ is bounded on the real axis (like $G(x)$).

Write

$$B \quad = \quad \limsup_{y\to\infty} \frac{\log|G_0(iy)|}{y}$$

and

$$B' \quad = \quad \limsup_{y\to -\infty} \frac{\log|G_0(iy)|}{|y|}.$$

Observe also that

$$\log|G(z)| \quad \leqslant \quad L|\Im z| + O(1)$$

by the third Phragmén–Lindelöf theorem of §C, Chapter III, since G is of exponential type $\leqslant L$ and bounded on the real axis.

Apply now Levinson's theorem (Chapter III, §H.3) to the zero distribution for $G_0(z)$, and then use Jensen's formula on the one for $G(z)$ together with the estimate just written for the latter function. The two zero distributions are the same asymptotically,* so, by an argument just

* Refer to volume I, pp. 74–5.

like the one at the end of §H.3, Chapter III, it is found that

$$\frac{B+B'}{2} \leqslant L.$$

Once this is known, we have by §D of Chapter IX,

$$\pi\tilde{D}_{\Lambda'} \leqslant \frac{B+B'}{2} \leqslant L,$$

since $G_0(z)$ vanishes* at the points of $\Lambda' \subseteq \mathbb{R}$. *Incompleteness* of the $e^{i\lambda t}$, $\lambda \in \Lambda$, on $[-L, L]$ thus implies that $L \geqslant \pi\tilde{D}_{\Lambda'}$, and those exponentials must therefore be complete on any interval of length $< 2\pi\tilde{D}_{\Lambda'}$.

We must now show that if

$$\sum_{\substack{\lambda\in\Lambda\\ \lambda\neq 0}} \frac{|\Im\lambda|}{|\lambda|^2} < \infty$$

and $\tilde{D}_{\Lambda'} < \infty$, the $e^{i\lambda x}$, $\lambda \in \Lambda$, are *incomplete* on any interval of length $> 2\pi\tilde{D}_{\Lambda'}$. Fix any $\eta > 0$. The previous theorem and remark then give us an entire function $f(z) \not\equiv 0$ of exponential type $\leqslant \pi(\tilde{D}_{\Lambda'} + 3\eta)$, vanishing† at each point of Λ', and such that (wlog)

$$|f(x)| \leqslant 1, \qquad x \in \mathbb{R}.$$

Denote by Ξ the set of *non-zero, purely imaginary* $\mu \in \Lambda$, and then put

$$g(z) = f(z)\cdot\prod_{\lambda'\in\Lambda'}\left(\frac{1-z/\lambda}{1-z/\lambda'}\right)\cdot\prod_{\mu\in\Xi}\left(1-\frac{z}{\mu}\right).$$

Using the two Lindelöf theorems of §B, Chapter III we easily see that $g(z)$ is of exponential type, thanks to the convergence of the above sum of the $|\Im\lambda|/|\lambda|^2$. $g(z)$ vanishes at each $\lambda \in \Lambda$ (save that at the origin, in case $0 \in \Lambda$). By calculations like one made in §H.3, Chapter III, we also verify without difficulty that

$$\log\prod_{\lambda'\in\Lambda'}\left|\frac{1-iy/\lambda}{1-iy/\lambda'}\right| \leqslant o(|y|)$$

and

$$\log\prod_{\mu\in\Xi}\left|1-\frac{iy}{\mu}\right| \leqslant o(|y|)$$

* with the appropriate multiplicity at repeated points of Λ', which may well *have* some, although Λ does not

† with the appropriate multiplicity

as $y \longrightarrow \pm\infty$, the second on account of the convergence of $\sum_{\mu\in\Xi} 1/|\mu|$. Therefore, since $f(z)$ is of exponential type $\leqslant \pi(\tilde{D}_{\Lambda'} + 3\eta)$, we have

$$\limsup_{y\to\pm\infty} \frac{\log|g(\mathrm{i}y)|}{|y|} \quad \leqslant \quad \pi(\tilde{D}_{\Lambda'} + 3\eta).$$

It is now claimed that

$$\int_{-\infty}^{\infty} \frac{\log^+|g(x)|}{1+x^2}\mathrm{d}x \quad < \quad \infty.$$

Because $|f(x)| \leqslant 1$, this will follow from the convergence of

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\log^+\left|\prod_{\mu\in\Xi}\left(1 - \frac{x}{\mu}\right)\right|\mathrm{d}x$$

and of

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\log^+\left|\prod_{\lambda'\in\Lambda'}\left(\frac{1-x/\lambda}{1-x/\lambda'}\right)\right|\mathrm{d}x.$$

Using the pure imaginary character of the $\mu\in\Xi$ and the relation $1/\lambda' = \Re(1/\lambda)$, we see at once that the factors $1 - x/\mu$, $\mu\in\Xi$, $(1-x/\lambda)/(1-x/\lambda')$, $\lambda'\in\Lambda'$, are all in modulus $\geqslant 1$ for real x. *The* $\log^+$ *may therefore be replaced by* log *in these integrals.*

Once this is done, the resulting expressions are easily worked out explicitly using Poisson's formula for a half plane. In that way we find the *first* integral to be equal to

$$\pi \sum_{\mu\in\Xi} \log\left(1 + \frac{1}{|\mu|}\right)$$

and the *second* to be

$$\leqslant \quad \pi \sum_{\lambda'\in\Lambda'} \log\left(1 + \frac{|\Im\lambda|}{|\lambda|^2}\right).$$

Both of these sums, however, are *finite* since $\sum_{\substack{\lambda\in\Lambda \\ \lambda\neq 0}} |\Im\lambda|/|\lambda|^2 < \infty$. Convergence of the logarithmic integral involving g is thus established.

Thanks to that convergence, *the theorem from* §A.2 *applies to our entire function* g. There is, in other words, a non-zero entire function $\psi(z)$ of exponential type $\leqslant \pi\eta$ with $g(x)\psi(x)$ bounded on $\mathbb{R}$; reasoning as at the end of the preceding theorem's proof we can even choose ψ so as to have

$$\int_{-\infty}^{\infty} |xg(x)\psi(x)|\,\mathrm{d}x \quad < \quad \infty.$$

Referring to the above estimate on $\log|g(\mathrm{i}y)|/|y|$ and applying the usual Phragmén–Lindelöf theorem, we see that $zg(z)\psi(z)$ is of exponential type

$\leqslant \pi(\tilde{D}_{\Lambda'}+4\eta)$. This function certainly vanishes at each $\lambda\in\Lambda$, so the $e^{i\lambda t}$, $\lambda\in\Lambda$, are *not* complete on $[-\pi(\tilde{D}_{\Lambda'}+4\eta),\ \pi(\tilde{D}_{\Lambda'}+4\eta)]$ according to the discussion at the beginning of this §.

The theorem is completely proved.

Problem 39

Let Λ be any sequence of distinct complex numbers. Show that the completeness radius associated with Λ is equal to π times the *infimum* of the numbers $c > 0$ with the following property: there exist *distinct integers* n_λ corresponding to the different non-zero λ in Λ such that

$$\sum_{\substack{\lambda\in\Lambda\\ \lambda\neq 0}}\left|\frac{1}{\lambda}-\frac{c}{n_\lambda}\right| \leqslant \infty.$$

This criterion is due to Redheffer. (Hint: Look again at the constructions in §D.2 of Chapter IX.)

C. The multiplier theorem for weights with uniformly continuous logarithms

The result of Beurling and Malliavin enunciated in §A.2 is broader in scope than may appear at first sight. One can, for instance, deduce from it another multiplier theorem for weights fulfilling a simple descriptive regularity condition. This is done in article 1 below; the work depends on some elementary material from Chapter VI and the first part of Chapter VII.

In article 2, the theorem of article 1 is used to extend a result obtained in problem 11 (Chapter VII, §A.2) to certain unbounded measures on $\mathbb{R}$.

1 The multiplier theorem

Theorem (Beurling and Malliavin, 1961). Let $W(x) \geqslant 1$, *and let* $\log W(x)$ *be uniformly continuous* on* $\mathbb{R}$. *Then* W *admits multipliers iff*

* Beurling and Malliavin require only that $\omega(s) = \operatorname{ess\,sup}_{x\in\mathbb{R}}|\log W(x+s) - \log W(x)|$ be *finite* for a set of $s\in\mathbb{R}$ having positive Lebesgue measure. To reduce the treatment under this less stringent assumption to that of the uniform Lip 1 case handled below, they observe that there must be some $M < \infty$ with $\omega(s) \leqslant M$ on a Lebesgue measurable set E with $|E| > 0$. But then $E - E$ includes a whole interval $(-h, h)$, $h > 0$, so $\omega(s) \leqslant 2M$ for $|s| < h$, ω being clearly *even* and *subadditive*. From this point, one proceeds as in the text, passing from W to W_h; the only changes are in the constants.

This argument is valid as long as $W(x) \geqslant 1$ is Lebesgue measurable, for then

$$\omega(s) = \lim_{p\to\infty}\left(\int_{-\infty}^{\infty}|\log W(x+s) - \log W(x)|^p e^{-2|x|}\,dx\right)^{1/p}$$

is also Lebesgue measurable (the *integrals* are by Tonelli's theorem, $\log W(x+s) - \log W(x)$ being Lebesgue measurable on $\mathbb{R}^2$).

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2}\,dx \quad < \quad \infty.$$

Remark. The weaker assumption that $\log\log W(x)$ is uniformly continuous on $\mathbb{R}$ *does not imply* that W admits multipliers when the above integral is convergent. For an example, see the following chapter.

Proof of theorem. As explained at the beginning of §A, convergence of the integral in question is certainly *necessary if* W is to admit multipliers; we therefore need only concern ourselves with the *sufficiency* of that convergence in the present circumstances.

We may, to begin with, replace the hypothesis of *uniform continuity* for $\log W(x)$ by the stronger one that

$$|\log W(x) - \log W(x')| \;\leqslant\; C|x-x'| \qquad \text{for } x, x' \in \mathbb{R},$$

i.e., that $\log W$ be *uniformly* Lip 1 on $\mathbb{R}$. Indeed, the former property gives us a fixed $h > 0$ such that

$$|\log W(x) - \log W(x')| \;\leqslant\; 1 \quad \text{whenever } |x-x'| \;\leqslant\; h.$$

Take any smooth positive function φ supported on $[-h, h]$ with

$$\int_{-h}^{h} \varphi(t)\,dt \;=\; 1,$$

and define a new weight $W_h(x)$ by putting

$$\log W_h(x) \;=\; \int_{-h}^{h} (\log W(x-t))\varphi(t)\,dt \;=\; \int_{-\infty}^{\infty} \varphi(x-s)\log W(s)\,ds.$$

Then, by our choice of h,

$$\log W(x) \;-\; 1 \;\leqslant\; \log W_h(x) \;\leqslant\; \log W(x) \;+\; 1.$$

Again,

$$\frac{d\log W_h(x)}{dx} \;=\; \int_{-\infty}^{\infty} \varphi'(x-s)\log W(s)\,ds \;=\; \int_{-h}^{h} \varphi'(t)\log W(x-t)\,dt.$$

Here, since $\varphi(-h) \;=\; \varphi(h) \;=\; 0$,

$$\int_{-h}^{h} \varphi'(t)\log W(x)\,dt \;=\; 0,$$

so

$$\frac{d\log W_h(x)}{dx} \;=\; \int_{-h}^{h} \varphi'(t)\big(\log W(x-t) - \log W(x)\big)\,dt.$$

By the choice of h, the integral on the right is in absolute value

$$\leqslant \int_{-h}^{h} |\varphi'(t)|\,dt = C, \quad \text{say,}$$

so

$$\left|\frac{d \log W_h(x)}{dx}\right| \leqslant C, \qquad x \in \mathbb{R},$$

and $\log W_h(x)$ satisfies the Lipschitz condition written above.

We also have

$$e^{-1} W(x) \leqslant W_h(x) \leqslant e\, W(x)$$

by the previous estimate, so

$$\int_{-\infty}^{\infty} \frac{\log W_h(x)}{1+x^2}\,dx < \infty$$

provided that the corresponding integral with W is finite. If, then, we can conclude that $W_h(x)$ admits multipliers, the left-hand half of the preceding double inequality shows that $W(x)$ also does so, and it is enough to establish the theorem for weights W with $\log W$ uniformly Lip 1 on $\mathbb{R}$.

Assuming henceforth this Lipschitz condition on $\log W(x)$ and the convergence of the corresponding logarithmic integral, we set out to show that W admits multipliers. Our idea is to produce an entire function $K(z)$ of exponential type such that

$$\int_{-\infty}^{\infty} \frac{\log^+ |K(x)|}{1+x^2}\,dx < \infty,$$

while

$$4K(x) \geqslant (W(x))^{\alpha} \qquad \text{for } x \in \mathbb{R}$$

with a certain constant $\alpha > 0$. Application of the theorem from §A.2 to $K(z)$ will then yield multipliers for W.

Following a procedure of Akhiezer used in Chapters VI and VII, we form the new weight

$$\begin{aligned} W_1(x) = \sup\{|f(x)| : \ & f \text{ entire, of exponential type} \leqslant 1, \\ & \text{bounded on } \mathbb{R}, \text{ and } |f(t)/W(t)| \leqslant 1 \text{ on } \mathbb{R}\}. \end{aligned}$$

If $\log W(x)$ satisfies the Lipschitz condition written above (with Lipschitz constant C) we have, by the *first* theorem of §A.1, Chapter VII,

$$W_1(x) \geqslant \tfrac{1}{2}(W(x))^{1/\sqrt{(C^2+1)}} \qquad \text{for } x \in \mathbb{R}.$$

What we want, then, is an entire function $K(z)$ of exponential type with convergent logarithmic integral, such that

$$K(x) \geqslant (W_1(x))^2,$$

say, for $x \in \mathbb{R}$.

In order to obtain $K(z)$, we use Akhiezer's theory of weighted approximation by sums of exponentials, presented in Chapter VI. We work, however, with a weighted L_2 norm instead of the weighted uniform one used there. Taking

$$\Omega(x) \quad = \quad (1+x^2)^{1/2} W(x),$$

let us consider approximation by finite linear combinations of the $e^{i\lambda x}$, $-1 \leqslant \lambda \leqslant 1$, in the norm $\| \quad \|_{\Omega,2}$ defined by

$$\|g\|_{\Omega,2} \quad = \quad \sqrt{\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\left|\frac{g(t)}{\Omega(t)}\right|^2 dt\right)}.$$

According to our assumed convergence of the logarithmic integral involving W, we have

$$\int_{-\infty}^{\infty} \frac{\log \Omega(x)}{1+x^2} dx \quad < \quad \infty.$$

Hence, by a version of T. Hall's theorem (the *first* one of §D, Chapter VI) appropriate to approximation in the norm $\| \quad \|_{\Omega,2}$ (see §§E.2 and G of Chapter VI), linear combinations of the $e^{i\lambda x}$, $-1 \leqslant \lambda \leqslant 1$, are *not* $\| \quad \|_{\Omega,2}$ dense in the space of functions for which that norm is finite. This, and the Akhiezer theorem (Chapter VI, §E.2) corresponding to the norm $\| \quad \|_{\Omega,2}$ (see again §G, Chapter VI) imply that

$$\int_{-\infty}^{\infty} \frac{\log \Omega_1(x)}{1+x^2} dx \quad < \quad \infty,$$

where

$$\Omega_1(x) \;=\; \sup\{\,|f(x)| : f \text{ entire, of exponential type} \leqslant 1, \text{ bounded on } \mathbb{R}, \text{ and } \|f\|_{\Omega,2} \leqslant 1\}.$$

Observe now that for any function f with $|f(t)/W(t)| \leqslant 1$ on $\mathbb{R}$, we certainly have $\|f\|_{\Omega,2} \leqslant 1$. Clearly, then,

$$\Omega_1(x) \geqslant W_1(x),$$

and, if we can show that $(\Omega_1(x))^2$ *coincides with an entire function of exponential type on the real axis*, we can simply take the latter as the function K we are seeking.

For that purpose we resort to a simple general argument. The space of Lebesgue measurable functions with finite $\| \quad \|_{\Omega,2}$ norm is certainly *separable*, so, since the entire functions of exponential type $\leqslant 1$ bounded on $\mathbb{R}$ *belong* to that space, we may choose a (countable!) *sequence* of those which is $\| \quad \|_{\Omega,2}$ *dense in the collection of all of them.* Using the inner product

$$\langle f, g \rangle_\Omega \quad = \quad \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x)\overline{g(x)}}{(\Omega(x))^2} \, dx,$$

one then applies *Schmidt's orthogonalization procedure* to that dense sequence, obtaining, after normalization, a sequence of entire functions $\varphi_n(z)$ of exponential type $\leqslant 1$, bounded on $\mathbb{R}$, with $\| \varphi_n \|_{\Omega,2} = 1$ and

$$\langle \varphi_n, \varphi_m \rangle_\Omega \ = \ 0 \quad \text{if } n \neq m.$$

Finite linear combinations of these φ_n are also $\| \quad \|_{\Omega,2}$ dense in the collection of all such entire functions.

Fixing any $x_0 \in \mathbb{R}$ and any N, we look at the finite linear combinations

$$S(x) \quad = \quad \sum_{k=1}^{N} a_k \varphi_k(x)$$

such that $\| S \|_{\Omega,2} \leqslant 1$, seeking the one which makes $|S(x_0)|$ *a maximum.* Since the φ_k are orthonormal with respect to $\langle \ , \ \rangle_\Omega$, the condition on $\| S \|_{\Omega,2}$ is equivalent to

$$\sum_{k=1}^{N} |a_k|^2 \quad \leqslant \quad 1,$$

so, by Schwarz' inequality,

$$|S(x_0)| \quad \leqslant \quad \sqrt{\left(\sum_{k=1}^{N} |\varphi_k(x_0)|^2 \right)}.$$

For proper choice of the coefficients a_k, the two sides are the same; the *maximum value* of $|S(x_0)|$ for the sums S is *thus equal to the quantity on the right.*

Put

$$K_N(z) \quad = \quad \sum_{k=1}^{N} \varphi_k(z) \overline{\varphi_k(\bar{z})};$$

this function is entire, of exponential type $\leqslant 2$ (*sic* !), and bounded on the real axis, where it is also $\geqslant 0$. As we have just seen, for each given $x \in \mathbb{R}$, the maximum value of $|S(x)|$ for sums S of the kind just specified is equal to $\sqrt{(K_N(x))}$. It is now claimed that when $N \longrightarrow \infty$, the $K_N(z)$ converge u.c.c to a certain entire function $K(z)$ of exponential type $\leqslant 2$, and that

$$K(x) = (\Omega_1(x))^2, \qquad x \in \mathbb{R}.$$

Given any particular N, we have, for each of the sums S,

$$|S(x)| \leqslant \Omega_1(x)$$

by *definition* of the function on the right, so

$$0 \leqslant K_N(x) \leqslant (\Omega_1(x))^2.$$

As we already know,

$$\int_{-\infty}^{\infty} \frac{\log(\Omega_1(x))^2}{1+x^2} \mathrm{d}x < \infty.$$

Therefore, since the $K_N(z)$ are of exponential type $\leqslant 2$, the last relation implies that they all satisfy a *uniform estimate* of the form

$$|K_N(z)| \leqslant C_\varepsilon \exp(2|\Im z| + \varepsilon|z|), \qquad z \in \mathbb{C}.$$

Here $\varepsilon > 0$ is arbitrary, and C_ε depends on it, but is *completely independent* of N. The statement just made is nothing other than an *adaptation*, to approximation in the norm $\| \quad \|_{\Omega,2}$, of the *fourth* theorem in §E.2, Chapter VI, proved by the familiar Akhiezer argument of §B.2 in that chapter.

By the estimate just found, the K_N form a normal family in the complex plane, and any convergent sequence of them tends to an entire function for which the same estimate holds. However, $K_{N+1}(x) \geqslant K_N(x)$ on $\mathbb{R}$, so the entire sequence of the K_N is *already* convergent, and

$$K(z) = \lim_{N \to \infty} K_N(z)$$

is an entire function, obviously of exponential type $\leqslant 2$.

We still have to prove that

$$K(x) = (\Omega_1(x))^2$$

on $\mathbb{R}$. Of course, $0 \leqslant K(x) \leqslant (\Omega_1(x))^2$ since each $K_N(x)$ has that property, and it suffices to show the reverse inequality. Take any $x_0 \in \mathbb{R}$, and choose an entire function $f(z)$ of exponential type $\leqslant 1$, bounded on $\mathbb{R}$, with $\|f\|_{\Omega,2} \leqslant 1$ and at the same time $|f(x_0)|$ *close to* $\Omega_1(x_0)$. By our choice of the φ_n, the orthogonal series development

$$\sum_k \langle f, \varphi_k \rangle_\Omega \, \varphi_k(x)$$

converges in norm $\| \quad \|_{\Omega,2}$ to $f(x)$. For the partial sums

$$P_N(x) \;=\; \sum_{k=1}^{N} \langle f, \varphi_k \rangle_\Omega \, \varphi_k(x)$$

we have, however,

$$\|P_N\|_{\Omega,2} \;\leqslant\; \|f\|_{\Omega,2} \;\leqslant\; 1,$$

so by definition,

$$|P_N(x)| \;\leqslant\; \Omega_1(x), \quad x \in \mathbb{R}.$$

Hence, since the P_N are of exponential type $\leqslant 1$, another application of our version of the fourth theorem from §E.2, Chapter VI, gives us the uniform estimate

$$|P_N(z)| \;\leqslant\; \tilde{C}_\varepsilon \exp(|\Im z| + \varepsilon |z|), \qquad z \in \mathbb{C},$$

on them. (Again, $\varepsilon > 0$ is arbitrary and $\tilde{C}_\varepsilon$ depends on it, but is independent of N.) The function $f(z)$ of course satisfies the same kind of estimate, and u.c.c convergence of the $P_N(z)$ to $f(z)$ now follows from the relation

$$\|f - P_N\|_{\Omega,2} \;\xrightarrow[N]{}\; 0$$

by a simple normal family argument. We see in particular that

$$P_N(x_0) \;\xrightarrow[N]{}\; f(x_0).$$

Since $\|f\|_{\Omega,2} \leqslant 1$, however,

$$|P_N(x_0)| \;\leqslant\; \sqrt{\left(\sum_{k=1}^{N} |\langle f, \varphi_k \rangle_\Omega|^2\right)} \sqrt{(K_N(x_0))} \;\leqslant\; \sqrt{(K_N(x_0))},$$

so

$$\sqrt{(K_N(x_0))} \;\geqslant\; |f(x_0)| \;-\; \varepsilon$$

with arbitrary $\varepsilon > 0$ for large enough N. Thence,

$$\sqrt{(K(x_0))} \;\geqslant\; |f(x_0)|.$$

But we chose f with $|f(x_0)|$ *close* to $\Omega_1(x_0)$ – indeed, *as close as we like.* Finally, then,

$$\sqrt{(K(x_0))} \;\geqslant\; \Omega_1(x_0),$$

whence

$$K(x) \;=\; (\Omega_1(x))^2, \qquad x \in \mathbb{R},$$

the reverse inequality having already been noted.

We are at this point essentially done. The entire function $K(z)$ of exponential type $\leqslant 2$ satisfies the relation just written. We have

$$\Omega_1(x) \geqslant W_1(x) \geqslant \tfrac{1}{2}(W(x))^{1/\sqrt{(C^2+1)}}$$

on $\mathbb{R}$, where $W(x) \geqslant 1$, so $4K(x) \geqslant 1$, $x \in \mathbb{R}$, and it follows from

$$\int_{-\infty}^{\infty} \frac{\log \Omega_1(x)}{1+x^2} \mathrm{d}x < \infty$$

that

$$\int_{-\infty}^{\infty} \frac{\log^+ K(x)}{1+x^2} \mathrm{d}x < \infty.$$

The theorem of Beurling and Malliavin from §A.2 now gives us, for any $\eta > 0$, an entire function $\psi(z) \not\equiv 0$ of exponential type $\leqslant \eta$ with

$$4K(x)|\psi(x)| \leqslant 1, \qquad x \in \mathbb{R},$$

i.e.,

$$|\psi(x)|(W(x))^{2/\sqrt{(C^2+1)}} \leqslant 1, \qquad x \in \mathbb{R}.$$

Taking any fixed integer m with

$$\frac{1}{m} < \frac{2}{\sqrt{(C^2+1)}},$$

we get

$$W(x)|(\psi(x))^m| \leqslant 1 \quad \text{on } \mathbb{R}.$$

Here $(\psi(z))^m$ is entire, of exponential type $\leqslant m\eta$, and not identically zero. Hence $W(x)$ admits multipliers, $\eta > 0$ being arbitrary. The theorem is proved.

Remark. Beurling and Malliavin did not derive this result from their theorem stated in §A.2. Instead, they gave an independent proof similar to the one furnished by them for the latter result. See the end of §C.5 in Chapter XI.

2. A theorem of Beurling

In order to indicate the location and extent of the intervals on $\mathbb{R}$ where a complex measure μ has little or no mass, Beurling, in his Stanford lectures, used a function $\sigma(x)$ related to μ by the formula

$$\mathrm{e}^{-\sigma(x)} = \int_{-\infty}^{\infty} \mathrm{e}^{-|x-t|}\,|\mathrm{d}\mu(t)|.$$

(Truth to tell, Beurling wrote $\sigma(x)$ where we write $-\sigma(x)$. Some of the formulas used in working with this function look a little simpler if the minus sign is taken in the exponent as we do here.)

For *finite* measures μ, $\sigma(x)$ is bounded below – $\sigma(x)$ is *positive* if $\int_{\mathbb{R}} |d\mu(t)| \leqslant 1$. *Large* values of $\sigma(x)$ then correspond to the abscissae near which μ has *very little mass.* In problem 11 (§A.2, Chapter VII) the reader was asked to show that if the function $\sigma(x)$ associated with a finite complex measure μ is *so large that*

$$\int_{-\infty}^{\infty} \frac{\sigma(x)}{1+x^2} dx = \infty,$$

then the Fourier–Stieltjes transform

$$\hat{\mu}(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda t} d\mu(t)$$

cannot vanish over any interval of positive length without μ's vanishing identically. Beurling originally established the gap theorem in §A.2, Chapter VII, with the help of this result, which is also due to him.

The multiplier theorem from the preceding article may be used to show that the result quoted is, in a certain sense, *best possible.* This application, set as problem 40, may be found at the end of the present article. Right now, we have in mind another application of that multiplier theorem, namely, *Beurling's extension of his result to certain unbounded complex measures μ.* This is also from his Stanford lectures.

When μ is *unbounded*, we can still define $\sigma(x)$ by means of the formula

$$e^{-\sigma(x)} = \int_{-\infty}^{\infty} e^{-|x-t|} |d\mu(t)|$$

as long as we admit the possibility that $\sigma(x) = -\infty$. In the case, however, that $\sigma(x) > -\infty$ for *any value* of x, *it is* $> -\infty$ *for all.* The reason for this is that $\sigma(x)$, if it is $> -\infty$ anywhere on $\mathbb{R}$, *is uniformly* Lip 1 there. To see that, we need only note that

$$|x'-t| \geqslant |x-t| - |x-x'|, \qquad t \in \mathbb{R},$$

whence

$$\int_{-\infty}^{\infty} e^{-|x'-t|} |d\mu(t)| \leqslant e^{|x'-x|} \int_{-\infty}^{\infty} e^{-|x-t|} |d\mu(t)|,$$

and

$$\sigma(x') \geqslant \sigma(x) - |x'-x|.$$

Interchanging x and x', we find that

$$\boxed{|\sigma(x') - \sigma(x)| \leqslant |x' - x|,}$$

a relation used several times in the following discussion.

Let us consider an unbounded μ for which $\sigma(x) > -\infty$. In this more general situation, $\sigma(x)$ is usually *not* bounded below (as it was for finite μ), and we need to look separately at

$$\sigma^+(x) = \max(\sigma(x), 0)$$

and

$$\sigma^-(x) = -\min(\sigma(x), 0) \qquad (sic!).$$

Since $\sigma(x)$ is uniformly Lip 1, *so are* $\sigma^+(x)$ *and* $\sigma^-(x)$.

The functions σ^+ and σ^- serve different purposes. *Large values of* $\sigma^+(x)$ correspond (as in the case of $\sigma(x)$ when dealing with finite measures) to the abscissae near which μ *has very little mass.* $\sigma^-(x)$, on the other hand, is *large near the places where* μ *has a great deal of mass.* With unbounded μ, one expects to come upon more and more such places (where $\sigma^-(x)$ assumes ever larger values) as x goes out to $+\infty$ or $-\infty$ along the real axis.

Beurling considered unbounded measures μ having *growth limited in such a way as to make*

$$\boxed{\int_{-\infty}^{\infty} \frac{\sigma^-(x)}{1+x^2}\,dx < \infty.}$$

Lemma. *Under the boxed condition on* σ^-, $\sigma^-(x)$ *is* $o(|x|)$ *for* $x \longrightarrow \pm\infty$.

Proof. Let $0 < c < 1$, and suppose that for any large x_0, we have

$$\sigma^-(x_0) \geqslant 2cx_0.$$

Then, by the Lip 1 property of σ^-,

$$\sigma^-(x) \geqslant cx_0 \quad \text{for } (1-c)x_0 \leqslant x \leqslant (1+c)x_0,$$

so

$$\int_{(1-c)x_0}^{(1+c)x_0} \frac{\sigma^-(x)}{x^2}\,dx \geqslant \frac{2c^2}{(1+c)^2}.$$

If the boxed relation holds, this cannot happen for arbitrarily large values of x_0, and $\sigma^-(x)$ must be $o(|x|)$ for $x \longrightarrow \infty$. Similarly for $x \longrightarrow -\infty$.

Lemma

$$\int_{x-1}^{x+1} e^{-\sigma^-(t)}|d\mu(t)| \quad \leqslant \quad e^2.$$

Proof. By definition,

$$e^{-\sigma(x)} \quad \geqslant \quad \int_{x-1}^{x+1} e^{-|x-t|}|d\mu(t)| \quad \geqslant \quad e^{-1}\int_{x-1}^{x+1} |d\mu(t)|.$$

This holds *a fortiori* if $\sigma(x)$ is replaced by $-\sigma^-(x) \leqslant \sigma(x)$. The Lip 1 property of σ^- now makes

$$\sigma^-(t) \quad \geqslant \quad \sigma^-(x) - 1, \qquad x-1 \leqslant t \leqslant x+1,$$

so we have

$$\int_{x-1}^{x+1} e^{-\sigma^-(t)}|d\mu(t)| \quad \leqslant \quad e\int_{x-1}^{x+1} e^{-\sigma^-(x)}|d\mu(t)| \quad \leqslant \quad e^2.$$

Done.

From these two lemmas we see that if the function σ^- corresponding to an unbounded complex measure μ fulfills the above boxed condition,

$$\int_{-\infty}^{\infty} e^{-\delta|t|}|d\mu(t)|$$

is convergent for every $\delta > 0$. In that circumstance, *the Fourier–Stieltjes transforms*

$$\hat{\mu}_\delta(\lambda) \quad = \quad \int_{-\infty}^{\infty} e^{i\lambda t}e^{-\delta|t|}\,d\mu(t)$$

are available. The $\hat{\mu}_\delta$ are nothing but the Abel means frequently used in harmonic analysis to try to give meaning to the expression

$$\int_{-\infty}^{\infty} e^{i\lambda t}\,d\mu(t)$$

when the integral is not absolutely convergent. It is often possible to interpret the latter as a limit (in some sense) of the $\hat{\mu}_\delta(\lambda)$ as $\delta \longrightarrow 0$ for certain (or sometimes even all) values of $\lambda \in \mathbb{R}$. We have examples of such treatment in the first lemma and theorem of §H.1, Chapter VI.

Definition. If $\int_{-\infty}^{\infty} e^{-\delta|t|}|d\mu(t)| < \infty$ for each $\delta > 0$, we say that "$\hat{\mu}(\lambda)$" ($\hat{\mu}(\lambda)$ itself is in general not defined!) vanishes on a closed interval $[a, b]$ of $\mathbb{R}$ provided that

$$\hat{\mu}_\delta(\lambda) \longrightarrow 0 \textit{ uniformly } \text{for } a \leqslant \lambda \leqslant b$$

as $\delta \longrightarrow 0$.

In terms of this notion, Beurling's extension of the result established in problem 11 takes the following form:

Theorem (Beurling) *Let μ be a complex-valued Radon measure on $\mathbb{R}$ for which*

$$\int_{-\infty}^{\infty} \frac{\sigma^-(x)}{1+x^2}dx \quad < \quad \infty,$$

but at the same time,

$$\int_{-\infty}^{\infty} \frac{\sigma^+(x)}{1+x^2}dx \quad = \quad \infty,$$

σ^- and σ^+ being the functions related to μ in the manner described above. If also "$\hat{\mu}(\lambda)$" vanishes on an interval of positive length, μ is identically zero.

Remark. According to the previous discussion, the condition on $\sigma^-(x)$ means that μ, although (perhaps) unbounded, *does not accumulate too much mass* anywhere. The one involving $\sigma^+(x)$ means that there are also *large parts* of $\mathbb{R}$ where μ has *very little mass.*

Proof of theorem. Let, wlog, "$\hat{\mu}(\lambda)$" vanish on $[-A, A]$, where $A > 0$. The function

$$\sigma^-(x) \; + \; \log(1+x^2)$$

is uniformly Lip 1 on $\mathbb{R}$, so, *by the theorem of the preceding article*, the integral condition on σ^- makes it possible for us to get a non-zero entire function $f(z)$ of exponential type $a < A$ such that

$$|f(x)| \quad \leqslant \quad \frac{e^{-\sigma^-(x)}}{1+x^2}.$$

From this, the second of the above lemmas yields

$$\int_{x-1}^{x+1} |f(t)|\,|d\mu(t)| \quad \leqslant \quad \frac{1}{1+(|x|-1)^2}\int_{x-1}^{x+1} e^{-\sigma^-(t)}|d\mu(t)|$$

$$\leqslant \quad \frac{e^2}{1+(|x|-1)^2} \qquad \text{for } |x| \geqslant 1,$$

so by summation over integer values of x, we get

$$\int_{-\infty}^{\infty} |f(t)||d\mu(t)| \quad < \quad \infty,$$

making

$$dv(t) \;=\; f(t)d\mu(t)$$

a totally finite measure on $\mathbb{R}$.

To ν we now apply the result from problem 11. Write

$$e^{-\tau(x)} \;=\; \int_{-\infty}^{\infty} e^{-|x-t|}|d\nu(t)|;$$

$\tau(x)$ is just the analogue of the function $\sigma(x)$ corresponding to the finite measure ν. Without loss of generality, $\int_{-\infty}^{\infty}|d\nu(t)| \leqslant 1$, so $\tau(x) \geqslant 0$. Also, $|f(x)| \leqslant 1$, so $|d\nu(t)| \leqslant |d\mu(t)|$ and

$$e^{-\tau(x)} \;\leqslant\; e^{-\sigma(x)},$$

i.e., $\tau(x) \geqslant \sigma(x)$. Combining this with the previous inequality, we get

$$\tau(x) \;\geqslant\; \sigma^{+}(x),$$

whence

$$\int_{-\infty}^{\infty} \frac{\tau(x)}{1+x^2}dx \;=\; \infty$$

by hypothesis.

It is now claimed that

$$\int_{-\infty}^{\infty} e^{i\lambda_0 t}d\nu(t) \;=\; 0 \quad \text{for } |\lambda_0| \leqslant A-a.$$

By the Paley–Wiener theorem, we have

$$f(t) \;=\; \int_{-a}^{a} e^{it\lambda}\varphi(\lambda)d\lambda,$$

where φ is (under the present circumstances) a continuous function on $[-a, a]$. Therefore,

$$\begin{aligned}\int_{-\infty}^{\infty} e^{i\lambda_0 t}d\nu(t) \;&=\; \lim_{\delta\to 0}\int_{-\infty}^{\infty} e^{-\delta|t|}e^{i\lambda_0 t}f(t)d\mu(t)\\ &=\; \lim_{\delta\to 0}\int_{-\infty}^{\infty}\int_{-a}^{a} e^{i\lambda_0 t}e^{it\lambda}\varphi(\lambda)e^{-\delta|t|}d\lambda\, d\mu(t)\\ &=\; \lim_{\delta\to 0}\int_{-a}^{a}\int_{-\infty}^{\infty} e^{-\delta|t|}e^{i(\lambda_0+\lambda)t}d\mu(t)\varphi(\lambda)\,d\lambda;\end{aligned}$$

here, for each $\delta > 0$, absolute convergence holds throughout. The last limit is just

$$\lim_{\delta\to 0}\int_{-a}^{a} \varphi(\lambda)\hat{\mu}_\delta(\lambda+\lambda_0)\mathrm{d}x$$

which is, however, *zero* when $|\lambda_0| \leqslant A-a$ since then $\hat{\mu}_\delta(\lambda+\lambda_0)\longrightarrow 0$ uniformly for $|\lambda| \leqslant a$ as $\delta \longrightarrow 0$.

The claim just established and the integral condition on $\tau(x)$ *now make* $\nu \equiv 0$ *by problem* 11. That is,

$$f(x)\mathrm{d}\mu(x) \equiv 0.$$

If the function f vanishes at all on $\mathbb{R}$, it does so only at certain points x_n isolated from each other, for f is *entire* and *not identically zero.* What we have just proved is that μ, *if not identically zero,* has all its mass distributed on the points x_n. Then there must be *one* of those points, say x_0, for which

$$\mu(\{x_0\}) \neq 0.$$

That, however, *cannot happen.* If, for instance, x_0 is a k-fold zero of $f(z)$, we may *repeat the above argument using the entire function*

$$f_0(z) = \frac{f(z)}{(z-x_0)^k}$$

instead of f; doing so, we then find that

$$f_0(t)\mathrm{d}\mu(t) \equiv 0.$$

Since $f_0(x_0) \neq 0$, we thus have

$$\mu(\{x_0\}) = 0,$$

a contradiction.

The measure μ must hence vanish identically. We are done.

Problem 40

Show that the result from problem 11 is best possible in the following sense:

If μ is a finite complex measure and

$$e^{-\sigma(x)} = \int_{-\infty}^{\infty} e^{-|x-t|}|\mathrm{d}\mu(t)|,$$

then, in the case that

$$\int_{-\infty}^{\infty} \frac{\sigma(x)}{1+x^2}\,dx \quad < \quad \infty,$$

there *is* a finite non-zero complex measure ν on $\mathbb{R}$ such that

$$\int_{-\infty}^{\infty} e^{-|x-t|}|d\nu(t)| \quad \leqslant \quad e^{-\sigma(x)}$$

but $\hat{\nu}(\lambda) \equiv 0$ outside some finite interval.

(Hint. One takes $d\nu(t) = f(t)dt$ where $f(t)$ is an entire function of exponential type chosen to satisfy $|f(x)| \leqslant e^{-\sigma(x)}/\pi(1+x^2)$.)

Yet another application of the result from article 1 is found near the end of Louis de Branges' book.

D. Poisson integrals of certain functions having given weighted quadratic norms

A condition involving the existence of multipliers is encountered when one desires to estimate certain harmonic functions whose boundary data are controlled by weighted norms. As a very simple example, let us consider the problem of estimating

$$U(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\,U(t)\,dt, \qquad \Im z > 0,$$

when it is known that

$$\int_{-\infty}^{\infty} |U(t)|^2 w(t)\,dt \quad \leqslant \quad 1$$

with some given function $w(t) \geqslant 0$ belonging to $L_1(\mathbb{R})$. We may, if we like, require that

$$|U(t)| \leqslant \text{ some } M \quad \text{for } t \in \mathbb{R},$$

where M is *unknown and beyond our control.* Is it possible, in these circumstances, to say anything about the magnitude of $|U(z)|$?

If the bounded function $U(t)$ is permitted to be *arbitrary*, a simple condition on w is both necessary and sufficient for the existence of an estimate on $U(z)$. Then, we may wlog take $z = i$, and rewrite $\pi U(i)$ as

$$\int_{-\infty}^{\infty} \frac{1}{w(t)(t^2+1)}\,U(t)w(t)\,dt.$$

The very rudiments of analysis now tell us that this integral is bounded for

$$\int_{-\infty}^{\infty} |U(t)|^2 w(t)\mathrm{d}t \quad \leqslant \quad 1$$

if and only if

$$\int_{-\infty}^{\infty} \frac{1}{w(t)(t^2+1)^2}\mathrm{d}t \quad < \quad \infty.$$

It is for such w, then, and only for them, that the estimate in question (with arbitrary z having $\Im z > 0$) is available.

The situation alters when we restrict the *spectrum* of the functions $U(t)$ under consideration. In order not to get bogged down here in questions of harmonic analysis not really germane to the matter at hand, let us simply say that we look at *arbitrary finite sums*

$$S(t) \quad = \quad \sum_{\lambda\in\Sigma} A_\lambda e^{i\lambda t}$$

with some prescribed closed $\Sigma \subseteq \mathbb{R}$ — the *spectrum* for those sums. When $\Sigma = \mathbb{R}$, such sums are of course w^* dense in $L_\infty(\mathbb{R})$, because an L_1 function whose Fourier transform is everywhere zero must vanish identically. In that case we may think crudely of the collection of sums S as filling out the set of bounded functions U 'for all practical purposes', and our problem boils down to the simple one with the solution just described. It is thus natural to ask what happens when $\Sigma \neq \mathbb{R}$, and the simplest situation in which this occurs is the one where $\mathbb{R} \sim \Sigma$ *consists of one finite interval.* Then, we may take the complementary interval to be symmetric about 0, and it is possible to describe completely the functions w for which estimates of the above kind on the sums $S(t)$ exist. The description is in terms of multipliers.

Theorem. *Let $w \geqslant 0$ belong to $L_1(\mathbb{R})$ and let $a > 0$. A necessary and sufficient condition for the existence of a β, $\Im\beta > 0$, and corresponding constant C_β, such that*

$$\left|\int_{-\infty}^{\infty} \frac{\Im\beta}{|t-\beta|^2} S(t)\mathrm{d}t\right| \quad \leqslant \quad C_\beta\sqrt{\left(\int_{-\infty}^{\infty} |S(t)|^2 w(t)\mathrm{d}t\right)}$$

for all finite sums

$$S(t) \quad = \quad \sum_{|\lambda|\geqslant a} A_\lambda e^{i\lambda t},$$

is that there exist a non-zero entire function $\varphi(t)$ of exponential type $\leqslant a$

which makes

$$\int_{-\infty}^{\infty} \frac{|\varphi(t)|^2}{w(t)(t^2+1)^2}\,dt \quad < \quad \infty.$$

Proof*: *Necessity. It is convenient to work with the Hilbert space norm

$$\|f\| \quad = \quad \sqrt{\left(\int_{-\infty}^{\infty} |f(t)|^2 w(t)\,dt\right)}$$

and corresponding inner product

$$\langle f, g\rangle \quad = \quad \int_{-\infty}^{\infty} f(t)\overline{g(t)}w(t)\,dt.$$

Assuming, then, that for some β, $\Im\beta > 0$, we have

$$\left|\int_{-\infty}^{\infty} \frac{\Im\beta}{|\beta-t|^2} S(t)\,dt\right| \quad \leqslant \quad C_\beta\|S\|$$

for all sums S of the given form, there must, by the Hahn–Banach theorem*, be a measurable $k(t)$ for which

$$\|k\| \quad < \quad \infty$$

and

$$\int_{-\infty}^{\infty} \frac{\Im\beta}{|\beta-t|^2} S(t)\,dt \quad = \quad \langle S, k\rangle$$

for such S. Taking just $S(t) = e^{i\lambda t}$ with $|\lambda| \geqslant a$, we find that

$$\int_{-\infty}^{\infty} \left(w(t)\overline{k(t)} - \frac{\Im\beta}{|\beta-t|^2}\right) e^{i\lambda t}\,dt \quad = \quad 0$$

for such λ.

This shows, to begin with, that $w(t)\overline{k(t)}$ *is certainly not a.e. zero.* Since $\|k\| < \infty$ and $w \in L_1(\mathbb{R})$, we see by Schwarz' inequality that $w(t)\overline{k(t)} \in L_1(\mathbb{R})$. According to the last relation, then, the Fourier transform of the integrable function

$$w(t)\overline{k(t)} \quad - \quad \frac{\Im\beta}{|\beta-t|^2}$$

vanishes outside $[-a, a]$, so the latter must coincide a.e. on the real axis with $\psi(t)$, where ψ *is an entire function of exponential type* $\leqslant a$.

* or rather that theorem's special and elementary version for *Hilbert space*

We have

$$w(t)\overline{k(t)} \quad = \quad \psi(t) \;+\; \frac{\Im\beta}{(t-\beta)(t-\bar\beta)} \qquad \text{a.e., } t\in\mathbb{R}.$$

Here,

$$\varphi(t) \quad = \quad \psi(t)(t-\beta)(t-\bar\beta) \;+\; \Im\beta$$

is *also* entire and of exponential type $\leqslant a$, and

$$\varphi(t) \quad = \quad w(t)\overline{k(t)}\,|t-\beta|^2 \qquad \text{a.e., } t\in\mathbb{R},$$

so $\varphi \not\equiv 0$. Also,

$$\int_{-\infty}^{\infty} \frac{|\varphi(t)|^2}{w(t)|t-\beta|^4}\,dt \quad = \quad \|k\|^2 \quad < \quad \infty,$$

whence

$$\int_{-\infty}^{\infty} \frac{|\varphi(t)|^2}{w(t)(t^2+1)^2}\,dt \quad < \quad \infty.$$

Sufficiency. We continue to use the norm symbol $\|\quad\|$ introduced above with the same meaning as before. Suppose there *is* a non-zero entire function φ of exponential type $\leqslant a$ such that

$$\int_{-\infty}^{\infty} \frac{|\varphi(t)|^2}{w(t)(t^2+1)^2}\,dt \quad < \quad \infty.$$

One may, to begin with, *exclude* the case where $\varphi(t)$ is *constant*, for then we would have

$$\int_{-\infty}^{\infty} \frac{dt}{w(t)(t^2+1)^2} \quad < \quad \infty,$$

making

$$\left|\int_{-\infty}^{\infty} \frac{\Im\beta}{|t-\beta|^2}\,U(t)\,dt\right| \quad \leqslant \quad C_\beta\|U\|$$

for *any* β, $\Im\beta > 0$, and *all* bounded U, by the discussion at the beginning of this §.

One may also take $\varphi(t)$ to be *real-valued* on $\mathbb{R}$. Indeed,

$$\varphi(z) \quad = \quad \frac{\varphi(z)+\overline{\varphi(\bar z)}}{2} \;+\; \frac{\varphi(z)-\overline{\varphi(\bar z)}}{2},$$

and *one* of the two functions on the right must be $\not\equiv 0$. *Both* are entire and of exponential type $\leqslant a$, and, on the *real axis*, the *first* coincides with $\Re\varphi(t)$ and the *second* with $i\Im\varphi(t)$. The *first* one, or else the *second* one divided by i, will thus do the job.

Let $\Im\beta > 0$. Then

$$\frac{\varphi(t)}{(t-\beta)(t-\bar{\beta})} = \frac{\varphi(t) - \dfrac{\varphi(\beta)}{\beta-\bar{\beta}}(t-\bar{\beta}) - \dfrac{\varphi(\bar{\beta})}{\bar{\beta}-\beta}(t-\beta)}{(t-\beta)(t-\bar{\beta})}$$

$$+ \frac{\varphi(\beta)}{(\beta-\bar{\beta})(t-\beta)} + \frac{\varphi(\bar{\beta})}{(\bar{\beta}-\beta)(t-\bar{\beta})}.$$

The first term on the right, $\psi(t)$, is a certain *entire function of exponential type* $\leqslant a$, and, after multiplying by $\beta - \bar{\beta}$ and collecting terms, we get

$$\frac{(\Im\beta)\varphi(t)}{|t-\beta|^2} = (\Im\beta)\psi(t) + \frac{1}{2i}\left(\frac{\varphi(\beta)}{t-\beta} - \frac{\varphi(\bar{\beta})}{t-\bar{\beta}}\right),$$

that is,

$$\frac{(\Im\beta)\varphi(t)}{|t-\beta|^2} = (\Im\beta)\psi(t) + (\Im\varphi(\beta))\frac{t-\Re\beta}{|t-\beta|^2} + (\Re\varphi(\beta))\frac{\Im\beta}{|t-\beta|^2},$$

since $\varphi(\bar{\beta}) = \overline{\varphi(\beta)}$, φ being real on $\mathbb{R}$.

Suppose we can choose β *in such a way that* $\Im\varphi(\beta) = 0$ *but* $\Re\varphi(\beta) \neq 0$. *Then*

$$\left|\int_{-\infty}^{\infty} \frac{\Im\beta}{|t-\beta|^2} S(t)\,dt\right| \leqslant C_\beta \| S \|$$

for the sums

$$S(t) = \sum_{|\lambda| \geqslant a} A_\lambda e^{i\lambda t}.$$

Indeed,

$$\int_{-\infty}^{\infty} \frac{|\varphi(t)|}{|t-\beta|^2}\,dt \leqslant \sqrt{\left(\int_{-\infty}^{\infty} \frac{|\varphi(t)|^2}{w(t)|t-\beta|^4}\,dt \int_{-\infty}^{\infty} w(t)\,dt\right)},$$

a finite quantity, so the *left side* of the last identity is in $L_1(\mathbb{R})$. On the *right side*, the term

$$(\Im\varphi(\beta))\frac{t-\Re\beta}{|t-\beta|^2}$$