Dirichlet Series

John E. McCarthy

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Preface

In 2005, I taught a graduate course on Dirichlet series at Washington University. One of the students in the course, David Opěla, took notes and TeX'ed them up. We planned to turn these notes into a book, but the project stalled.

In 2015, I taught the course again, and revised the notes. I still intend to write a proper book, eventually, but until then I decided to make the notes available to anybody who is interested. The notes are not complete, and in particular lack a lot of references to recent papers.

Dirichlet series have been studied since the 19^{th} century, but as individual functions. Henry Helson in 1969 [**Hel69**] had the idea of studying function spaces of Dirichlet series, but this idea did not really take off until the landmark paper [**HLS97**] of Hedenmalm, Lindqvist and Seip that introduced a Hilbert space of Dirichlet series that is analogous to the Hardy space on the unit disk. This space, and variations of it, has been intensively studied, and the results are of great interest.

I would like to thank all the students who took part in the courses, and my two Ph.D. students, Brian Maurizi and Meredith Sargent, who did research on Dirichlet series. I would especially like to thank David Opěla for his work in rendering the original course notes into a legible draft. I would also like to thank the National Science Foundation, that partially supported me during the entire long genesis of this project, with grants DMS 0501079, DMS 0966845, DMS 1300280, DMS 1565243.

Notation

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$
$\mathbb{N}^+ = \{1, 2, 3, 4, \dots\}$
$\mathbb{Z}$ = integers
$\mathbb{Q}$ = rationals
$\mathbb{R}$ = reals
$\mathbb{C}$ = complex numbers
$\mathbb{P} = \{2, 3, 5, 7, \dots\} = \{p_1, p_2, p_3, p_4, \dots\}$
$\mathbb{P}_k = \{p_1, p_2, \dots, p_k\}$
$\mathbb{N}_k = \{n \in \mathbb{N}^+ :\ \text{all prime factors of } n \text{ lie in } \mathbb{P}_k\}$
$s = \sigma + it,\ s \in \mathbb{C},\ \sigma, t \in \mathbb{R}$
$\Omega_\rho = \{s \in \mathbb{C};\ \mathrm{Re}\ s > \rho\}$
$\pi(x) = \#$ of primes $\leq x$
$\mu(n)$ = Möbius function
$d(k)$ = number of divisors of k
$d_j(k)$ = number of ways to factor k into exactly j factors
$\phi(n)$ = Euler totient function
$\Phi(s) = \sum_{p \in \mathbb{P}} \frac{\log p}{p^s}$
$\Theta(x) = \sum_{p \leq x} \log p$
σ_c = abscissa of convergence
σ_a = abscissa of absolute convergence
$\sigma_1 = \max(0, \sigma_c)$
σ_u = abscissa of uniform convergence
σ_b = abscissa of bounded convergence
$F(x)$ summatory function
$\displaystyle ⨍_{-T}^{T}$ normalized integral
ε_n = Rademacher sequence
$\mathbb{E}$ = Expectation
$\mathbb{T}$ = torus
$z^{r(n)} := z_1^{t_1} \dots z_l^{t_l}$, where $n = p_1^{t_1} \cdots p_l^{t_l}$
$\mathcal{B} : \sum a_n z^{r(n)} \mapsto \sum a_n n^{-s}$
$\mathcal{Q} : \sum a_n n^{-s} \mapsto \sum a_n z^{r(n)}$
$\mathbb{T}^\infty$ = infinite torus

$\beta(x) = \sqrt{2}\sin(\pi x)$
$\mathcal{H}^2 = \{\sum_{n=1}^{\infty} a_n n^{-s} : \ \sum_n |a_n|^2 < \infty\}$
$\mathrm{Mult}\,(\mathcal{X}) = \{\varphi : \ \varphi f \in \mathcal{X}, \ \forall f \in \mathcal{X}\}$
$M_\varphi : \ f \mapsto \varphi f$
$\mathbb{D}^\infty =$ infinite polydisk
$E(\varepsilon, f)$ ε-translation numbers of f
$\mathcal{H}^2_w =$ weighted space of Dirichlet series
$H^2_w =$ weighted space of power series
$Q_K : \left(\sum_{n=1}^{\infty} a_n n^{-s} \right) \mapsto \sum_{n \in \mathbb{N}_K} a_n n^{-s}$
$\rho =$ Haar measure on $\mathbb{T}^\infty$
$\ell^2(G) =$ Hilbert space of square-summable functions on the group G
$\mathcal{X}_q =$ Dirichlet characters modulo q
$L(s, \chi) =$ Dirichlet L series
$H^p_\infty(\Omega_{1/2}) = \{g \in \mathrm{Hol}\ (\Omega_{1/2}) : \ [\sup_{\theta \in \mathbb{R}} \ \sup_{\sigma > 1/2} \int_\theta^{\theta+1} |g(\sigma + it)|^p \ dt]^{\frac{1}{p}} < \infty\}$
$\|f\|_{\mathcal{H}^p} = \left[\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(it)|^p \ dt \right]^{1/p}$
$\preccurlyeq$ The left-hand side is less than or equal to a constant times the right-hand side
$\approx$ Each side is $\preccurlyeq$ the other side
$\rho_{\mathcal{A}}(x, y) = \sup\{|\phi(y)\| \ : \ \phi(x) = 0, \ \|\phi\| \leq 1\}$
$\mathcal{H}^\infty \ = \ H^\infty(\Omega_0) \cap \mathbb{D}$
$\mathcal{E} \ : \ f \mapsto \langle f, g_i \rangle$
$g_i = k_{\lambda_i} / \|k_{\lambda_i}\|$

CHAPTER 1

Introduction

A *Dirichlet series* is a series of the form

$$\sum_{n=1}^{\infty} a_n n^{-s} =: f(s), \ \ s \in \mathbb{C}.$$

The most famous example is the *Riemann zeta function*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

NOTATION 1.1. By long-standing tradition, the complex variable in a Dirichlet series is denoted by s, and it is written as

$$s \ = \ \sigma + it.$$

We shall always use σ for $\Re(s)$ and t for $\Im(s)$.

NOTE 1.2. The Dirichlet series for $\zeta(s)$ converges if $\sigma > 1$; in fact, it converges absolutely for such s, since

$$|n^{-s}| = |e^{-(\sigma+it)\log n}| = |e^{-(\sigma+it)\log n}| = n^{-\sigma}.$$

Also, if $\sigma \leq 0$ or $0 < s \leq 1$, the series diverges, in the first case because the terms do not tend to zero, in the second by comparison with the harmonic series.

REMARK 1.3. Consider the power series $\sum_{n=1}^{\infty} z^n$; it converges to $\frac{1}{1-z}$, but only in the open unit disk. Nonetheless, it determines the analytic function $f(z) = \frac{1}{1-z}$ everywhere, since it has a unique analytic continuation to $\mathbb{C} \setminus \{1\}$. The Riemann zeta function can also be analytically continued outside of the region where it is defined by the series.

For this continuation, it can be shown that $\zeta(-2n) = 0$, for all $n \in \mathbb{N}^+$ and that there are no other zeros outside of the strip $0 \leq \mathrm{Re}\ s \leq 1$. The *Riemann hypothesis*, proposed by Bernhard Riemann in 1859, is one of the most famous unanswered conjectures in mathematics. It states that all the zeros other than the even negative integers have real part equal to $\frac{1}{2}$.

We shall prove in Theorem 2.19 that the zeta function has no zeroes on the line $\{\Re s = 1\}$.

The importance of the Riemann zeta function and the Riemann hypothesis lies in their intimate connection with prime numbers and their distribution. On the simplest level, this can be explained by the Euler Product formula below.

Recall that an infinite product $\prod_{n=1}^{\infty} a_n$ is said to *converge*, if the partial products tend to a non-zero finite number (or if one of the a_n's is zero). This is equivalent to the requirement that $\sum_{n=1}^{\infty} \log a_n$ converges (or $a_n = 0$, for some $n \in \mathbb{N}^+$). See *e.g.* [**Gam01**, XIII.3].

Notation 1.4. We shall let $\mathbb{P}$ denote the set of primes, and when convenient we shall write

$$\mathbb{P} \; = \; \{p_1, p_2, p_3, p_4, \dots\} \; = \; \{2, 3, 5, 7, \dots\}$$

to label the primes in increasing order. We shall let $\mathbb{P}_k$ denote the first k primes.

Theorem 1.5. **(Euler Product formula)** *For $\sigma > 1$,*

$$\prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1} = \zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

Formal proof:

$$\begin{aligned}
\prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1} &= \left(1 - \frac{1}{2^s}\right)^{-1} \left(1 - \frac{1}{3^s}\right)^{-1} \left(1 - \frac{1}{5^s}\right)^{-1} \dots \\
&= \left(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \frac{1}{2^{3s}} + \dots\right) \times \\
&\quad \left(1 + \frac{1}{3^s} + \frac{1}{3^{2s}} + \frac{1}{3^{3s}} + \dots\right) \dots
\end{aligned}$$

If we formally multiply out this infinite product, we can only obtain a non-zero product by choosing 1 from all but finitely many brackets. This product will be $\frac{1}{q_1^{r_1 s} q_2^{r_2 s} \dots q_k^{r_k s}} = \frac{1}{n^s}$. For each $n \in \mathbb{N}^+$, the term $\frac{1}{n^s}$ will appear exactly once, by the existence and uniqueness of prime factoring.

For a *rigorous proof* assume that Re $s > 1$, and fix $k \in \mathbb{N}^+$. Then

$$\begin{aligned} \prod_{p \in \mathbb{P}_k} \left(1 - \frac{1}{p^s}\right)^{-1} &= \prod_{p \in \mathbb{P}_k} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \dots\right) \\ &= \sum_{n = p_1^{r_1} \dots p_k^{r_k}} \frac{1}{n^s}, \end{aligned} \tag{1.6}$$

where the last equality holds by a variation of the formal argument above and convergence is not a problem, since we are multiplying finitely many absolutely convergent series.

Using (1.6), we have, for Re $s > 1$,

$$\left|\zeta(s) - \prod_{p \in \mathbb{P}_k} \left(1 - \frac{1}{p^s}\right)^{-1}\right| = \left|\sum_{\{n;\ p_l | n,\ l > k\}} \frac{1}{n^s}\right| \leq \sum_{n \geq p_{k+1}} \frac{1}{n^\sigma} \to 0, \text{ as } k \to \infty.$$

Thus the product converges to $\zeta(s)$.

To see that the limit is non-zero, we have

$$\begin{aligned} \left|1 - (1 - \frac{1}{p^s})^{-1}\right| &\leq \frac{1}{p^\sigma} \frac{1}{p^\sigma - 1} \\ &\leq \frac{2}{p^\sigma} \quad \text{for } p \text{ large.} \end{aligned}$$

Since $\sigma > 1$, this means that the infinite product converges absolutely, and therefore $\sum \log(1 - \frac{1}{p^s})^{-1}$ converges absolutely. □

NOTATION 1.7. We shall let Ω_ρ denote the open half-plane

$$\Omega_\rho = \{s : \Re(s) > \rho\}.$$

COROLLARY 1.8. *$\zeta(s)$ has no zeros in Ω_1.*

Proof: For $s \in \Omega_1$, $\zeta(s)$ is given by an absolutely convergent product. Thus, it can only be zero if one of the terms is zero. But $\left(1 - \frac{1}{p^s}\right)^{-1} = 0$ if and only if $p^s = 0$, which never happens. □

THEOREM 1.9. $\sum_{p \in \mathbb{P}} \frac{1}{p} = \infty$.

Proof: Suppose not, then $\sum_{p \in \mathbb{P}} \frac{1}{p}$ converges. By the Taylor expansion of $\log(1 - x)$, for x close enough to 0,

$$-x \leq \log(1 - x) \leq -\frac{x}{2},$$

so we conclude that $\sum_{p\in\mathbb{P}} \log\left(1-\frac{1}{p}\right)$ also converges. Since

$$\log\left(1-\frac{1}{p}\right) \; < \; \log\left(1-\frac{1}{p^\sigma}\right),$$

for all $\sigma > 1$ and $p \in \mathbb{P}$, we get

$$\begin{aligned} -\infty \quad &< \quad \sum_{p\in\mathbb{P}} \log\left(1-\frac{1}{p}\right) \\ &< \quad \lim_{\sigma\to 1+} \sum_{p\in\mathbb{P}} \log\left(1-\frac{1}{p^\sigma}\right) \\ &= \quad -\lim_{\sigma\to 1+} \log\frac{1}{\zeta(\sigma)} \\ &= \quad -\infty, \end{aligned}$$

a contradiction. □

The following discrete version of integration by parts is often useful when working with Dirichlet series. In it, integrals are replaced by sums, and derivatives by differences. (In the familiar formula $\int_m^n u dv = u(n)v(n) - u(m)v(m) - \int_m^n v du$, we let u correspond to b, v to A and thus dv to a.)

In fact, one can prove integration by parts for Riemann integrals using the definition (via Riemann sums) and Lemma 1.10.

Lemma 1.10. **(Abel's Summation by parts formula)** *Let* $A_n = \sum_{k=1}^n a_k$, *then*

$$\sum_{k=m}^{n} a_k b_k \;=\; A_n b_n - A_{m-1} b_m + \sum_{k=m}^{n-1} A_k (b_k - b_{k+1}).$$

Proof: Since $a_k = A_k - A_{k-1}$, we have

$$\begin{aligned} \sum_{k=m}^{n} a_k b_k \quad &= \quad \sum_{k=m}^{n} [A_k - A_{k-1}] b_k \\ &= \quad \sum_{k=m}^{n} A_k b_k - \sum_{k=m}^{n} A_{k-1} b_k \\ &= \quad \sum_{k=m}^{n-1} A_k [b_k - b_{k+1}] - A_{m-1} b_m + A_n b_n. \end{aligned}$$

□

NOTATION 1.11. For $x > 0$, we let $\pi(x)$ denote the number of primes less than or equal to x.

The prime number theorem (see Chapter 2) is an estimate of how big $\pi(n)$ is for large n. We can use the Euler product formula to relate π and the Riemann zeta function.

THEOREM 1.12. *For $\sigma > 1$,*

$$\log \zeta(s) = s \int_2^\infty \frac{\pi(x)}{x(x^s - 1)}\, dx \ .$$

Proof: In the following calculation we use the fact that $[\pi(k) - \pi(k-1)]$ is equal to 1 if k is a prime, and 0 if k is composite; the equality $\sum_{k=1}^n [\pi(k) - \pi(k-1)] = \pi(n)$; and summation by parts.

$$\begin{aligned}
\log \zeta(s) &= -\sum_{p \in \mathbb{P}} \log\left(1 - \frac{1}{p^s}\right) \\
&= -\sum_{k=2}^{\infty} [\pi(k) - \pi(k-1)] \log\left(1 - \frac{1}{k^s}\right) \\
&= -\lim_{L \to \infty} \sum_{k=2}^{L} [\pi(k) - \pi(k-1)] \log\left(1 - \frac{1}{k^s}\right) \\
&= -\lim_{L \to \infty} \left\{ \sum_{k=2}^{L-1} \pi(k) \left[\log\left(1 - \frac{1}{k^s}\right) - \log\left(1 - \frac{1}{(k+1)^s}\right)\right] \right. \\
&\quad \left. + \ \pi(1) \log\left(1 - \frac{1}{2^s}\right) - \pi(L) \log\left(1 - \frac{1}{L^s}\right) \right\}
\end{aligned}$$

The penultimate term vanishes, since $\pi(1) = 0$. As for the last term, the trivial bound $\pi(L) \leq L$ gives

$$\left| \pi(L) \log\left(1 - \frac{1}{L^s}\right) \right| \leq L \cdot L^{-\sigma} \to 0 \text{ as } L \to \infty.$$

We let $L \to \infty$, and use the fact that $\frac{d}{dx} \log(1 - \frac{1}{x^s}) = \frac{s}{x^{s+1} - x}$, to get:

$$\begin{aligned}
\log \zeta(s) &= -\sum_{k=2}^{\infty} \pi(k) \left[\log\left(1 - \frac{1}{k^s}\right) - \log\left(1 - \frac{1}{(k+1)^s}\right)\right] \\
&= -\sum_{k=2}^{\infty} \pi(k) \int_k^{k+1} \frac{-s}{x^{s+1} - x}\, dx \\
&= s \int_2^\infty \frac{\pi(x)}{x^{s+1} - x}\, dx \ .
\end{aligned}$$

$\square$

Notation 1.13. The *Möbius function* is helpful when working with the Riemann zeta function. It is given as follows:

$$\mu(n) = \begin{cases} 1, & n = 1, \\ (-1)^k, & \text{if } n \text{ is the product of } k \text{ distinct primes,} \\ 0, & \text{otherwise.} \end{cases}$$

Its values for the first few positive integer are in the table below:

n	1	2	3	4	5	6	7	8	9	10	11	12
$\mu(n)$	1	-1	-1	0	-1	1	-1	0	0	1	-1	0

Theorem 1.14. *For $\sigma > 1$,*

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \mu(n) n^{-s}.$$

Proof: We only present a formal proof — convergence can be checked in the same way as was done for the Euler product formula.

$$\begin{aligned} \frac{1}{\zeta(s)} &= \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right) \\ &= \left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{5^s}\right)\cdots \\ &= 1 - \sum_{p \in \mathbb{P}} p^{-s} + \sum_{p,q \in \mathbb{P},\ p \neq q} p^{-s} q^{-s} - \ldots \\ &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \end{aligned}$$

□

It is obvious that the Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ converges (converges absolutely, respectively) for all $s \in \Omega_\rho$ if and only if the series $\sum_{n=1}^{\infty} (a_n n^{-\rho}) n^{-s}$ converges (conv. abs., resp.) for all $s \in \Omega_0$. This ability to translate the Dirichlet series horizontally often allows one to simplify calculations. (It is analogous to working with power series and assuming the center is at 0). The proof of the proposition below is a typical example of this.

The following "uniqueness-of-coefficients" theorem will be used frequently.

Proposition 1.15. *Suppose that $\sum_{n=1}^{\infty} a_n n^{-s}$ converges absolutely to $f(s)$ in some half-plane Ω_ρ and $f(s) \equiv 0$ in Ω_ρ. Then $a_n = 0$ for all $n \in \mathbb{N}^+$.*

Proof: As remarked, we may assume that $\rho < 0$, so in particular, $\sum |a_n| < \infty$. Suppose all the a_n's are not 0, and let n_0 be the smallest natural number such that $a_{n_0} \neq 0$.
Claim: $\lim_{\sigma\to\infty} f(\sigma)n_0^\sigma = a_{n_0}$.

To prove the claim note that

$$\begin{aligned} 0 &\leq n_0^\sigma \left| \sum_{n>n_0} a_n n^{-\sigma} \right| \\ &\leq \sum_{n>n_0} |a_n| \left(\frac{n_0}{n}\right)^\sigma \\ &\leq \left(\frac{n_0}{n_0+1}\right)^\sigma \sum_{n>n_0} |a_n|, \end{aligned}$$

and the last term tends to 0 as $\sigma \to \infty$, since $\sum |a_n|$ converges. As

$$f(\sigma)n_0^\sigma = a_{n_0} + n_0^\sigma \sum_{n>n_0} a_n n^{-\sigma},$$

the claim is proved.

The proof is also finished, because the limit in the claim is obviously 0, a contradiction. □

Recall that the Cauchy product formula for the product of power series states that

$$\left(\sum a_n z^n\right)\left(\sum b_m z^m\right) = \sum_{k=0}^{\infty} \left(\sum_{0\leq n\leq k} a_n b_{k-n}\right) z^k,$$

if at least one of the sums on the left-hand side converges absolutely. The Dirichlet series analogue below involves the sum over all divisors of a given integer. The multiplicative structure of the natural numbers is far more complex than their additive structure. Indeed, as an additive semigroup $\mathbb{N}^+$ is singly generated, while as a multiplicative semigroup it is not finitely generated — the smallest set of generators is $\mathbb{P}$. This is one of the reasons why the theory of Dirichlet series is more complicated than the theory of power series. Now, we state the Dirichlet series analogue of the Cauchy product formula. The proof is immediate.

THEOREM 1.16. *Assume that $\sum_{n=1}^\infty a_n n^{-s}$ and $\sum_{m=1}^\infty b_m m^{-s}$ converge absolutely. Then*

$$\left(\sum_{n=1}^{\infty} a_n n^{-s}\right)\left(\sum_{m=1}^{\infty} b_m m^{-s}\right) = \sum_{k=1}^{\infty}\left(\sum_{n|k} a_n b_{k/n}\right) k^{-s},$$

with absolute convergence.

COROLLARY 1.17. *For $\sigma > 1$,*

$$\zeta^2(s) = \sum_{k=1}^{\infty} d(k) k^{-s},$$

where $d(k)$ denotes the number of divisors of k. More generally,

$$\zeta^j(s) = \sum_{k=1}^{\infty} d_j(k) k^{-s},$$

where $d_j(k)$ denotes the number of ways to factor k into exactly j factors. Here, 1 is allowed to be a factor and two factorings that differ only by the order of the factors are considered to be distinct.

Proof: We shall prove the first formula. Using Theorem 1.16, we have, for $\sigma > 1$,

$$\begin{aligned}
\zeta^2(s) &= \left(\sum_{n=1}^{\infty} n^{-s}\right)\left(\sum_{m=1}^{\infty} m^{-s}\right) \\
&= \sum_{k=1}^{\infty}\left(\sum_{n|k} 1\right) k^{-s} \\
&= \sum_{k=1}^{\infty} d(k) k^{-s}.
\end{aligned}$$

The proof of the second formula is analogous. □

The formula for $\frac{1}{\zeta(s)}$ implies the following identity for the Möbius function. (It can also be proved directly.)

COROLLARY 1.18. $\sum_{n|k} \mu(n) = 0$, *for all $k \geq 2$.*

Proof: For $\sigma > 1$, write

$$\begin{aligned}
1 &= \zeta(s)\zeta^{-1}(s) \\
&= \left(\sum_{m=1}^{\infty} \frac{1}{m^s}\right)\left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}\right) \\
&= \sum_{k=1}^{\infty}\left(\sum_{n|k} \mu(n)\right) \frac{1}{k^s}.
\end{aligned}$$

Comparing the coefficients of the outer-most Dirichlet series completes the proof. □

PROPOSITION 1.19. **(Möbius inversion formula)** *Let f, g be functions on $\mathbb{N}^+$. If*

$$g(q) = \sum_{n|q} f(n), \qquad \textit{then} \qquad f(q) = \sum_{d|q} \mu(q/p) g(d).$$

Proof:

$$\begin{aligned}
\sum_{d|q} \mu(q/p) g(d) &= \sum_{d|q} \mu(q/p) \sum_{n|d} f(n) \\
&= \sum_{n|q} \left(\sum_{d|q,\ \frac{q}{d} | \frac{q}{n}} \mu(q/d) \right) f(n) \\
&= \sum_{n|q} \left(\sum_{s | \frac{q}{n}} \mu(s) \right) f(n) \\
&= f(q),
\end{aligned}$$

since, by the preceding corollary, the bracket is non-zero only when $q/n = 1$. □

DEFINITION 1.20. The *Euler totient function* $\phi(n)$ is defined as $\#\{1 \leq k \leq n;\ \gcd(n, k) = 1\}$.

Clearly, $\phi(p) = p - 1$, iff p is a prime. In fact, one can express $\phi(n)$ in terms of the prime factors of n.

LEMMA 1.21. *If $n = q_1^{r_1} \ldots q_k^{r_k}$ with $r_j > 0$, then*

$$\phi(n) = n \prod_{j=1}^{k} \left(1 - \frac{1}{q_j} \right).$$

Proof: First, note that $\gcd(n, m) \neq 1$, if and only if, $q_j \big| m$, for some $1 \leq j \leq k$. Consider the uniform probability distribution on $\{1, \ldots, n\}$. Let E_j be the event that q_j divides a randomly chosen number in $\{1, \ldots, n\}$. For any l that divides n, there are exactly n/l numbers in $\{1, \ldots, n\}$ divisible by l. Thus, the events $\{E_j\}_{j=1}^k$ are independent and hence so are their complements. Hence, $\phi(n)/n$, the probability that a randomly chosen number is not divisible by any q_j, is equal to the product of the probabilities that it is not divisible by q_j, that is $\prod_j (1 - 1/q_j)$. □

THEOREM 1.22. *For $\sigma > 2$,*

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\phi(n)}{n^s}.$$

Proof: Again, we will only prove it formally, since turning it into a rigorous proof is routine, but renders the proof harder to read. By the Euler product formula, we have

$$\begin{aligned}
\frac{\zeta(s-1)}{\zeta(s)} &= \prod_{p\in\mathbb{P}} \frac{\left(1-\frac{1}{p^s}\right)}{\left(1-\frac{1}{p^{s-1}}\right)} \\
&= \prod_{p\in\mathbb{P}} \left(1-\frac{1}{p^s}\right)\left[1+\frac{p}{p^s}+\frac{p^2}{p^{2s}}+\dots\right] \\
&= \prod_{p\in\mathbb{P}} \left(\left[1+\frac{p}{p^s}+\frac{p^2}{p^{2s}}+\dots\right]-\left[\frac{1}{p^s}+\frac{p}{p^{2s}}+\frac{p^2}{p^{3s}}+\dots\right]\right) \\
&= \prod_{p\in\mathbb{P}} \left[1+\left(1-\frac{1}{p}\right)\left(\frac{p}{p^s}+\frac{p^2}{p^{2s}}+\dots\right)\right] \\
&= \sum_{n=1}^{\infty} a_n n^{-s},
\end{aligned}$$

where

$$a_n = \prod_{j=1}^{k}\left(1-\frac{1}{q_j}\right) q_j^{r_j} = n\prod_{j=1}^{k}\left(1-\frac{1}{q_j}\right) = \phi(n),$$

for $n = q_1^{r_1}\dots q_k^{r_k}$. □

1.1. Exercises

1. Prove that if $\chi : \mathbb{N}^+ \to \mathbb{T}\cup\{0\}$ is a quasi-character, which means $\chi(mn) = \chi(m)\chi(n)$, the same argument that proved the Euler product formula (Theorem 1.5) shows that

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^{-s}} = \prod_{p\in\mathbb{P}} \left(\frac{1}{1-\chi(p)p^{-s}}\right).$$

1.2. Notes

For a thorough treatment of the Riemann zeta function, see [**Tit86**]. The material in this chapter comes from the first few pages of Titchmarsh's magisterial book.

CHAPTER 2

The Prime Number Theorem

2.1. Statement of the Prime number theorem

We have defined $\pi(n)$ to be the number of primes less than or equal to n. Euclid's proof that there are an infinite number of primes says that $\lim_{n\to\infty} \pi(n) = \infty$; but how fast does it grow? By Theorem 1.9 and Abel's summation by parts formula we know

$$\begin{aligned}\infty &= \sum_{p\in\mathbb{P}} \frac{1}{p} \\ &= \sum_{n=2}^{\infty} [\pi(n) - \pi(n-1)] \frac{1}{n} \\ &\approx \sum_{n=2}^{\infty} \pi(n) \frac{1}{n^2},\end{aligned}$$

so $\pi(n)$ cannot be $O(n^\alpha)$ for any $\alpha < 1$.

Gauss conjectured that

$$\pi(x) \sim \frac{x}{\log x}, \tag{2.1}$$

where the asymptotic symbol $\sim$ in (2.1) means that the ratio of the quantities on either side tends to 1 as $x \to \infty$. Tchebyshev proved that

$$.93 \frac{x}{\log x} \leq \pi(x) \leq 1.1 \frac{x}{\log x}$$

for x large, and also showed that if

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x}$$

exists, it must be 1. The full prime number theorem was proved in 1896 by de la Vallée Poussin and Hadamard.

Theorem 2.2. **[Prime Number Theorem]**

$$\pi(x) \sim \frac{x}{\log x}.$$

Looking at some examples, we see that

$$\left.\begin{aligned} \pi(10^6) &= 78,498 \\ \frac{10^6}{\log(10^6)} &\approx 72,382 \end{aligned}\right\} \implies \frac{\pi(10^6)}{\frac{10^6}{\log 10^6}} \approx 1.08$$

and

$$\left.\begin{aligned} \pi(10^9) &= 50,847,478 \\ \frac{10^9}{\log(10^9)} &\approx 48,254,942 \end{aligned}\right\} \implies \frac{\pi(10^9)}{\frac{10^9}{\log 10^9}} \approx 1.05$$

Definition 2.3. For $s \in \Omega_1$, we define

$$\Phi(s) := \sum_{p\in\mathbb{P}} \frac{\log p}{p^s}.$$

It is easy to see that this Dirichlet series converges absolutely in Ω_1.

Definition 2.4. For $x \in \mathbb{R}$, define

$$\Theta(x) := \sum_{p\in\mathbb{P},\ p\le x} \log p.$$

The key to proving the Prime number theorem is establishing the estimate $\Theta(x) \sim x$ (Proposition 2.27).

Say more here?

2.2. Proof of the Prime number theorem

We will now prove the Prime number theorem in a series of steps.

Lemma 2.5.

$$\Theta(x) = O(x) \text{ as } x \to \infty, \text{i.e., } \limsup_{x\to\infty} \frac{\Theta(x)}{x} < \infty.$$

Proof: Note that

$$\binom{2n}{n} \ge \prod_{n<p\le 2n,\ p\in\mathbb{P}} p.$$

Indeed, the LHS is a positive integer that is divisible by the RHS. Note that we have not yet proved that there are any primes between n and $2n$, so the RHS may be an empty product (we interpret empty products as having the value 1).

Thus,

$$\binom{2n}{n} \ge \prod_{n<p\le 2n} p = e^{\Theta(2n)-\Theta(n)}.$$

Now, by the binomial theorem,

$$2^{2n} = (1+1)^{2n} = \binom{2n}{0} + \cdots + \binom{2n}{n} + \cdots + \binom{2n}{2n},$$

Thus

$$2^{2n} \geq \binom{2n}{n} \implies e^{n\log 4} \geq \binom{2n}{n} \geq e^{\Theta(2n)-\Theta(n)},$$

and consequently

$$\Theta(2n) - \Theta(n) \leq n\log 4.$$

For $x \in \mathbb{R}, x \geq 1$, we have

$$\begin{aligned} \Theta(2x) - \Theta(x) &\leq \Theta(\lfloor 2x \rfloor) - \Theta(\lfloor x \rfloor) \\ &\leq \Theta(2\lfloor x \rfloor + 1) - \Theta(\lfloor x \rfloor) \\ &\leq \log(\lfloor 2x \rfloor + 1) + \Theta(\lfloor 2x \rfloor) - \Theta(\lfloor x \rfloor) \\ &\leq cx. \end{aligned}$$

Now fix x and choose $n \in \mathbb{N}$ such that $\frac{x}{2^{n+1}} \leq 1 \leq \frac{x}{2^n}$. Then, by telescoping,

$$\begin{aligned} \Theta(x) - \Theta(1) &= \sum_{j=0}^{n} \Theta(\frac{x}{2^j}) - \Theta(\frac{x}{2^{j+1}}) \\ &\leq \sum_{j=0}^{n} c\frac{x}{2^{j+1}} \\ &= cx. \end{aligned}$$

Since $\Theta(1) = 0$, we conclude that

$$\Theta(x) = O(x) \tag{2.6}$$

□

Recall that $\Phi(s) = \sum_{p\in\mathbb{P}} \frac{\log p}{p^s}$. Since $\sum_{p\in\mathbb{P}} \frac{1}{p} = \infty$ (Theorem 1.9) we conclude that $\Phi(s)$ has a pole at 1.

LEMMA 2.7. *The function $\Phi(s) - \frac{1}{s-1}$ is holomorphic in $\overline{\Omega_1}$.*

PROOF: In Ω_1,

$$\zeta(s) = \prod_{p\in\mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1} = \prod_{p\in\mathbb{P}} \frac{1}{1-p^s}$$

By logarithmic differentiation we obtain

$$\begin{aligned}\frac{\zeta(s)}{\zeta'(s)} &= -\sum_{p\in\mathbb{P}} \frac{\frac{\partial}{\partial s}(1-p^{-s})}{1-p^{-s}} \\ &= -\sum_{p\in\mathbb{P}} (p^{-s}\log p)\frac{1}{1-p^{-s}} \\ &= -\sum_{p\in\mathbb{P}} \frac{\log p}{p^s-1} \qquad (2.8)\end{aligned}$$

Now

$$\frac{1}{p^s-1} = \frac{1}{p^s} + \frac{1}{p^s(p^s-1)} \qquad (2.9)$$

Combining (2.8) and (2.9), we obtain, for $s \in \Omega_1$,

$$\frac{-\zeta'(s)}{\zeta(s)} = \sum_{p\in\mathbb{P}} \frac{\log p}{p^s} + \sum_{p\in\mathbb{P}} \frac{\log p}{p^s(p^s-1)} \qquad (2.10)$$

Note that we can rearrange the terms since the series converge absolutely in Ω_1. Thus, for $s \in \Omega_1$,

$$\frac{-\zeta'(s)}{\zeta(s)} = \Phi(s) + \sum_{p\in\mathbb{P}} \frac{\log p}{p^s(p^s-1)} \qquad (2.11)$$

The second term on the RHS defines an analytic function in $\Omega_{1/2}$ as the series converges there absolutely. Thus in $\Omega_{1/2}$, any information about the analyticity of $\frac{-\zeta'(s)}{\zeta(s)}$ translates into the analyticity of $\Phi(s)$.

The function $\zeta(s)$ has a pole at 1 with residue 1 and so $\zeta(s) - \frac{1}{s-1}$ is analytic near 1, and consequently, $\zeta'(s) + \frac{1}{(s-1)^2}$ is analytic near 1. Thus $\frac{\zeta'(s)}{\zeta(s)} - \frac{-\frac{1}{(s-1)^2}}{\frac{1}{s-1}} = \frac{\zeta'(s)}{\zeta(s)} - \frac{1}{(s-1)}$ is analytic near 1.

Thus

$$\Phi(s) = \frac{\zeta'(s)}{\zeta(s)} - \sum_{p\in\mathbb{P}} \frac{\log p}{p^s(p^s-1)} \qquad (2.12)$$

is holomorphic in $\Omega_{1/2} \cap \{s :\ \zeta(s) \neq 0\}$.

It remains to prove that $\zeta(s) \neq 0$ if Re $s \geq 1$.

Definition 2.13. The *von Mangoldt function* $\Lambda : \mathbb{N}_0 \to \mathbb{R}$, is defined as

$$\Lambda(m) = \begin{cases} \log p, \text{ if } m = p^k, \\ 0, \text{ else.} \end{cases} \qquad (2.14)$$

PROPOSITION 2.15. *For $s \in \Omega_1$*

$$\frac{-\zeta'(s)}{\zeta(s)} = \sum_{n \geq 2} \frac{\Lambda(n)}{n^s} = \sum_{p \in \mathbb{P}} \frac{\log p}{p^s - 1} \tag{2.16}$$

holds.

PROOF: We have $\zeta(s) = \prod_{p \in \mathbb{P}} (1 - \frac{1}{p^s})^{-1}$ and thus

$$\begin{aligned} \frac{\zeta'(s)}{\zeta(s)} &= -\sum_{p \in \mathbb{P}} \log p \frac{p^{-s}}{1 - \frac{1}{p^s}} \\ &= -\sum_{p \in \mathbb{P}} \frac{\log p}{p^s - 1} \end{aligned}$$

which proves that the first and last term in the statement of the proposition are equal. For Re $s > 1$, $\|1/p^s\| < 1$, so the first equality above yields

$$\begin{aligned} \frac{\zeta'(s)}{\zeta(s)} &= -\sum_{p \in \mathbb{P}} (\log p) p^{-s} (1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots) \\ &= -\sum_{p \in \mathbb{P}} \sum_{k=1}^{\infty} \log p (p^k)^{-s}. \end{aligned}$$

This double summation goes over exactly those numbers $n = p^k$ for which $\Lambda(n)$ does not vanish and thus, for $s \in \Omega_1$,

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{p \in \mathbb{P}} \sum_{k=1}^{\infty} \log p (p^k)^{-s} = \sum_{n \geq 2} \frac{\Lambda(n)}{n^s} \tag{2.17}$$

□

LEMMA 2.18. *Let $x_0 \in \mathbb{R}$ and assume F is holomorphic in a neighborhood of x_0, $F(x_0) = 0$ and $F \neq 0$. Then there exists $\varepsilon > 0$ such that*

$$Re \left(\frac{F'(x)}{F(x)} \right) > 0$$

for $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$.

PROOF: Write $F(x) = a_k(s-x_0)^k + a_{k+1}(s-x_0)^{k+1} + \dots$ where $k > 0$. Then

$$\begin{aligned}\frac{F'(x)}{F(x)} &= \frac{ka_k(x-x_0)^{k-1} + (k+1)a_{k+1}(x-x_0)^k + \dots}{a_k(x-x_0)^k + a_{k+1}(x-x_0)^{k+1} + \dots} \\ &= \frac{k + \frac{(k+1)a_{k+1}}{a_k}(x-x_0) + \dots}{(x-x_0) + \frac{a_{k+1}}{a_k}(x-x_0)^2 + \dots} \\ &\approx \frac{k}{x-x_0} > 0,\end{aligned}$$

for $x \in (x_0, x_0+\varepsilon)$. □

THEOREM 2.19. *The Riemann ζ function does not vanish on the line $\{\Re(s) = 1\}$.*

PROOF: Suppose that $\zeta(1+it_0) = 0$, for $t_0 \in \mathbb{R} \setminus \{0\}$. Define

$$F(s) := \zeta^3(s)\zeta^4(s+it_0)\zeta(s+2it_0). \tag{2.20}$$

At $s = 1$, we see that ζ^3 has a pole of order 3, and $\zeta^4(s+it_0)$ vanishes to order 4, so $F(1) = 0$. Thus, in a neighborhood of 1, F is holomorphic. Using Lemma 2.18, Re $\left(\frac{F'(x)}{F(x)}\right) > 0$ for $x \in (1, 1+\varepsilon)$. Computing

$$\begin{aligned}\frac{F'(x)}{F(x)} &= 3\frac{\zeta'(x)}{\zeta(x)} + 4\frac{\zeta'(x+it_0)}{\zeta(x+it_0)} + \frac{\zeta'(x+2it_0)}{\zeta(x+2it_0)} \\ &= \sum_{n\geq 2} \Lambda(n)\left[-3n^{-x} - 4n^{-x}e^{-it_0\log n} - n^{-x}e^{-2it_0\log n}\right],\end{aligned}$$

thus,

$$\begin{aligned}\text{Re}\,\frac{F'(x)}{F(x)} &= \sum_{n\geq 2} -\Lambda(n)n^{-x}\left[3 + 4\cos(t_0\log n) + \cos(2t_0\log n)\right] \\ &= \sum_{n\geq 2} -\Lambda(n)n^{-x}\left[2 + 4\cos(t_0\log n) + 2\cos(t_0\log n)\right]\end{aligned}$$

We observe that $-\Lambda(n)n^{-x} \leq 0$ for every $n \geq 2$ while the term in the square bracket is always non-negative, since it the square of

$$\sqrt{2}[1 + \cos(t_0\log n)],$$

a contradiction with Lemma 2.18. □

LEMMA 2.21. *Let $f(t) : [0,\infty) \to \mathbb{C}$ be bounded and suppose that*

$$g(s) = \int_0^\infty f(t)e^{-st}\,dt \tag{2.22}$$

extends to a holomorphic function in $\overline{\Omega_0}$. *Then* $\int_0^\infty f(t)\, dt$ *exists and equals* $g(0)$.

PROOF: Let

$$g_T(s) = \int_0^T f(t) e^{-st}\, dt\ . \tag{2.23}$$

Then g_T is an entire function by Morera's theorem, and $g_T(0) = \int_0^T f(t)\, dt$. We want to show that $\lim_{t\to\infty} g_T(0) = g(0)$.

insert image around here

For $R,\ \delta > 0$ let $U_{R,\delta} := \mathbb{D}(0,R) \cap \Omega_{-\delta}$. For any $R > 0$ there is $\delta > 0$ such that g is holomorphic in $\overline{U_{R,\delta}}$, since by hypothesis g is holomorphic in a neighborhood of $\overline{\Omega_0}$. Let $C := \partial U_{R,\delta}$ and $C_+ = C \cap \Omega_0$ and $C_- = C \setminus \overline{\Omega_0}$. By Cauchy's theorem:

$$g(0) - g_T(0) = \frac{1}{2\pi i} \int_C [g(s) - g_t(s)]\, e^{sT} \left(1 + \frac{s^2}{R^2}\right) \frac{ds}{s}, \tag{2.24}$$

since $e^{st}(1 + \frac{s^2}{R^2})$ has value 1 at 0 and is holomophic everywhere in our contour. Let $h(s) := [g(s) - g_t(s)]\, e^{sT} \left(1 + \frac{s^2}{R^2}\right)$. In Ω_0, we have

$$\begin{aligned} |g(s) - g_T(s)| &= \left| \int_T^\infty f(t) e^{-st}\, dt \right| \\ &\leq M \left| \int_T^\infty e^{-st}\, dt \right| \\ &= M \left| \int_T^\infty e^{-(\mathrm{Re}\ s)t}\, dt \right| \\ &= M \frac{1}{\mathrm{Re}\ s} e^{-\mathrm{Re}\ s\ T} \end{aligned}$$

Thus

$$\begin{aligned} \left| \int_{C_+} h(s) \frac{ds}{s} \right| &\leq \int_{C_+} |f(t) e^{-st}\, dt\ | \\ &\leq M \int_{C_+} \frac{e^{-\mathrm{Re}\ sT}}{\mathrm{Re}\ s} \left| \frac{e^{sT}}{s} \left(1 + \frac{s^2}{R^2}\right) \right| |\ ds\ | \end{aligned}$$

For $s \in C_+$, we have $|s| = R$ and so

$$\left(1 + \frac{s^2}{R^2}\right) \frac{1}{s} = \frac{R^2 + s^2}{R^2 s} = \frac{|s|^2 + s}{R^2 s} = \frac{\overline{s} + s}{R^2}$$

Thus,

$$\begin{aligned}\left|\int_{C+} h(s)\frac{ds}{s}\right| &\leq \frac{M}{2\pi}\int_{C_+}\frac{e^{-\text{Re } sT}}{\text{Re } s}\frac{e^{\text{Re } sT}}{s}\frac{2\text{Re } s}{R^2}|\ ds\ | \\ &\leq \frac{M}{\pi R^2}\pi R \\ &= \frac{M}{R}\end{aligned}$$

We conclude $\int_{C_+} h(s)\frac{ds}{s} \to 0$ as $R \to 0$.

For C_-, we will show that both

$$I_1(T) \ := \int_{C_-} |g(s)|\, e^{sT}\left(1+\frac{s^2}{R^2}\right)\frac{ds}{s}$$

and

$$I_2(T) \ := \int_{C_-} |g_T(s)|\, e^{sT}\left(1+\frac{s^2}{R^2}\right)\frac{ds}{s}$$

tend to 0 as R tends to 0.

We start with I_1:

$$\begin{aligned}|g_T(s)| &= \left|\int_0^T f(t)e^{-st}\ dt\ \right| \\ &\leq M\int_0^T e^{-(\text{Re } s)t}\ dt \\ &\leq M\int_{-\infty}^T e^{-(\text{Re } s)t}\ dt \\ &= \frac{M}{\text{Re } s}e^{-\text{Re } st}\end{aligned}$$

Therefore,

$$I_1(T) \leq \int_{C_-}\frac{Me^{-(\text{Re } s)t}}{|\text{Re } s|}\left|e^{sT}\left(1+\frac{s^2}{R^2}\right)\frac{1}{s}\right|\,|\ ds\ |$$

and since g_T is an entire function, we can integrate over the semicircle Γ_- instead of C_- and use the same estimates as in Ω_0 to get

$$I_1(T) \leq \frac{M}{R}.$$

Now

$$I_2(T) = \int_{C_-}\left[g(s)\left(1+\frac{s^2}{R^2}\right)\frac{1}{s}\right]e^{sT}\ ds$$

and the expression in square bracket is independent of T and holomorphic in a neighborhood of C_- while $e^{sT} \to 0$ as $T \to \infty$. Using dominated convergence theorem, we conclude that $I_2 \to 0$ as $T \to \infty$.

Thus,

$$\begin{aligned} |g(0) - g(T)| &\leq \left| \int_{C_+} h(s) \frac{ds}{s} \right| + |I_1(T)| + |I_2(T) \\ &\leq \frac{M}{R} + \frac{M}{R} + I_2(T) \to \frac{2M}{R} \end{aligned}$$

Taking the limit as $R \to \infty$ implies that $g(0) = \lim_{T\to\infty} g_T(0)$ □

LEMMA 2.25. *The integral* $\int_1^\infty \frac{\Theta(x)-x}{x^2}\, dx$ *converges.*

PROOF: For Re $s > 1$,

$$\Phi(s) = \sum_{p\in pri} \frac{\log p}{p^s} = \int_1^\infty \frac{d\Theta(x)}{dx}$$

We are using the Stiltjes integral in the last expression because $\Theta(x)$ is a step function.

We use integration by parts with $u := x^{-s}$ and $dv := d\Theta(x)$. Then $du = -sx^{-(s+1)}\, dx$ and $v(x) = \Theta(x)$, giving

$$\Phi(x) = x^{-s}\Theta(x)\Big|_1^\infty + s\int_1^\infty \frac{\Theta(x)}{x^{s+1}}\, dx\ .$$

The first term vanishes since $\Theta(x) = O(x)$ as $x \to \infty$. We conclude that

$$\Phi(s) = s\int_1^\infty \frac{\Theta(x)}{x^{s+1}}\, dx\ .$$

Now let us use the substitution, $x = e^t$ to get

$$\Phi(s) = s\int_0^\infty \Theta(e^t)e^{-ts}\, dt\ .$$

We want apply Lemma 2.21 to $f(t) := \Theta(e^t)e^{-t} - 1$ and $g(s) = \frac{\Theta(s+1)}{s+1} - \frac{1}{s}$. By Lemma 2.5, we get that $f(t)$ is bounded and by Lemma 2.7, we know that $\frac{\Theta(s+1)}{s+1} - \frac{1}{s}$ is holomorphic in $\overline{\Omega_0}$. In order to apply Lemma 2.21, we need to check that $g(s)$ is the Laplace transform of $f(t)$.

We have

$$\int_0^\infty \Theta(e^t)e^{-t}e^{-ts}\, dt\ = \int_0^\infty \Theta(e^t)e^{-t(s+1)}\, dt$$

and

$$\int_0^\infty 1e^{-ts}\, dt\ = \frac{1}{s}$$

and thus $g(s)$ is the Laplace transform of $f(t)$, and we can apply Lemma 2.21 to conclude that $\int_0^\infty f(t)\ dt$ exists.

$$\begin{aligned}
\int_0^\infty f(t)dt &= \int_0^\infty \left[\Theta(e^t)e^{-t} - 1\right]\ dt \\
&= \int_1^\infty \left[\Theta(x)\frac{1}{x} - 1\right] \frac{dx}{x} \\
&= \int_1^\infty \left[\frac{\Theta(x) - x}{x^2}\right]\ dx
\end{aligned}$$

which concludes the proof. □

NOTE 2.26. See [**Fol99**, p. 107] for information on integration by parts in the context of the Stieltjes integrals.

PROPOSITION 2.27. $\lim_{x\to\infty} \frac{\Theta(x)}{x} = 1$, *that is,* $\Theta(x) \sim x$.

PROOF: We will proceed by contraction. There are two cases.

First assume that $\limsup_{x\to\infty} \frac{\Theta(x)}{x} > 1$. Thus, there exists $\lambda > 1$ and a sequence $\{x_n\}$ with $x_n \to \infty$ such that $\Theta(x_n) > \lambda x_n$. Then, since Θ is non-decreasing,

$$\int_{x_n}^{\lambda x_n} \frac{\Theta(t) - t}{t^2}\ dt \ \geq \int_{x_n}^{\lambda x_n} \frac{\lambda x_n - t}{t^2}\ dt \ =: c_\lambda.$$

We integrate the two pieces,

$$\int_{x_n}^{\lambda x_n} \frac{\lambda x_n}{t^2}\ dt \ = \lambda x_n \left(-\frac{1}{t}\Big|_{x_n}^{\lambda x_n}\right) = \lambda - 1$$

and

$$\int_{x_n}^{\lambda x_n} \frac{dt}{t} = \log(\lambda x_n) - \log x_n = \log \lambda$$

to conclude that $c_\lambda = \lambda - 1 - \log\lambda > 0$ by a well-known inequality for log. This implies that $\int_1^\infty \frac{\Theta(x)-x}{x^2}\ dx$ does not converge, a contradiction.

The second case is that $\liminf_{x\to\infty} \frac{\Theta(x)}{x} < 1$, so there is $\lambda < 1$ and a sequence $\{x_n\}$ with $x_n \to \infty$ and $\frac{\Theta(x_n)}{x_n} < \lambda$. As before,

$$\int_{\lambda x_n}^{x_n} \frac{\Theta(t) - t}{t^2}\ dt \ \leq \int_{x_n}^{\lambda x_n} \frac{\lambda x_n - t}{t^2}\ dt \ = -c_\lambda = -(\lambda - 1 - \log\lambda) < 0$$

and we reach a contraction as in the first case. □

Proof of Theorem 2.2. We can estimate

$$\begin{aligned}\Theta(x) &= \sum_{p\le x}\log p\\ &\le \sum_{p\le x}\log x\\ &= \pi(x)\log x.\end{aligned}$$

By Proposition 2.27, $\Theta(x)\sim x$ and thus we have the bound

$$\limsup_{x\to\infty}\ \pi(x)\frac{\log x}{x}\ \ge\ 1.$$

For the other bound, let $\varepsilon>0$, and write

$$\begin{aligned}\Theta(x) &\ge \sum_{x^{1-\varepsilon}\le p\le x}\log p\\ &\ge \sum_{x^{1-\varepsilon}\le p\le x}\log x^{1-\varepsilon}\\ &= [\pi(x)-\pi(x^{1-\varepsilon})](1-\varepsilon)\log x\\ &= (1-\varepsilon)\log x\left[\pi(x)+O(x^{1-\varepsilon})\right]\end{aligned}$$

where the last equality come from Lemma 2.5.

We have

$$\pi(x)\log x\le\frac{1}{1-\varepsilon}\Theta(x)+O(x^{1-\varepsilon}\log x)$$

and hence

$$\frac{\pi(x)\log x}{x}\le\frac{1}{1-\varepsilon}\frac{\Theta(x)}{x}+O(x^{-\varepsilon}\log x).$$

Using Proposition 2.27 again, we get

$$\limsup_{x\to\infty}\ \pi(x)\frac{\log x}{x}\ \le\ \frac{1}{1-\varepsilon}$$

for every $\varepsilon>0$. Taking $\varepsilon\to 0$ yields

$$\limsup_{x\to\infty}\ \pi(x)\frac{\log x}{x}\ \le\ 1$$

which concludes the proof. □

2.3. Historical Notes

The *offset logarithmic integral function*, $Li(x):=\int_2^x\frac{dt}{\log t}$ satisfies $Li(x)\approx\frac{x}{\log x}\approx\pi(x)$ but is a better approximation to $\pi(x)$.

Gauss conjectured that $\pi(n)\le Li(n)$. This was disproved by E. Littlewood in 1914.

During the proof of the Prime Number Theorem, we used the fact that $\zeta(s)$ does not vanish for Re $s \geq 1$. More precise estimates showing that the zeros of $\zeta(s)$ must lie "close to" the critical line $\{\text{Re } s = 1/2\}$ yield estimates on the error $|\pi(x) - Li(x)|$.

The Riemann hypothesis is equivalent to the error estimate

$$\pi(x) = Li(x) + O(\sqrt{x}\log x).$$

The best known estimate is of the error is

$$\pi(x) = Li(x) + O(xe^{-\frac{A(\log x)^{3/5}}{(\log\log x)^{1/5}}}).$$

CHAPTER 3

Convergence of Dirichlet Series

We will now investigate convergence of Dirichlet series. Much of the general theory holds for *generalized Dirichlet series*, that is, series of the form

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}.$$

An ordinary Dirichlet series corresponds, of course, to the case $\lambda_n = \log n$.

When dealing with a generalized Dirichlet series, we shall always assume that λ_n is a strictly increasing sequence tending to infinity, and that $\lambda_1 \geq 0$. Sometimes, an additional assumption is needed, such as the Bohr condition, namely $\lambda_{n+1} - \lambda_n \geq c/n$, for some $c > 0$.

Recall that for a power series $\sum_{n=1}^{\infty} a_n z^n$ there exists a (unique) value $R \in [0, \infty]$, called the *radius of convergence*, such that

(1) if $|z| < R$, then $\sum_{n=1}^{\infty} a_n z^n$ converges,
(2) if $|z| > R$, then $\sum_{n=1}^{\infty} a_n z^n$ diverges,
(3) for any $r < R$, the series $\sum_{n=1}^{\infty} a_n z^n$ converges uniformly and absolutely in $\{|z| \leq r\}$ and the sum is bounded on this set,
(4) on the circle $\{|z| = R\}$, the behavior is more delicate.

As we shall see, the situation for Dirichlet series is more complicated. In particular, compare the third point above with Proposition 3.10.

We start with a basic result.

THEOREM 3.1. *If the series $\sum_{n=1}^{\infty} a_n n^{-s}$ converges at some $s_0 \in \mathbb{C}$, then, for every $\delta > 0$, it converges uniformly in the sector $\{s : -\frac{\pi}{2} + \delta < \arg(s - s_0) < \frac{\pi}{2} - \delta\}$.*

Proof: As usual, we may assume $s_0 = 0$, that is, $\sum_n a_n$ converges. Let $r_n := \sum_{k=n+1}^{\infty} a_k$, and fix $\varepsilon > 0$. Then there exist $n_0 \in \mathbb{N}$ such that $|r_n| < \varepsilon$ for all $n \geq n_0$. Using summation by parts, for s in the sector and $M, N > n_0$

$$\begin{aligned} \sum_{n=M}^{N} a_n n^{-s} &= \sum_{n=M}^{N} (r_{n-1} - r_n) n^{-s} \\ &= \sum_{n=M}^{N-1} r_n \left[\frac{1}{(n+1)^s} - \frac{1}{n^s} \right] \qquad (3.2) \\ &\quad + \frac{r_{M-1}}{M^s} - \frac{r_N}{N^s}. \end{aligned}$$

The absolute values of the last two terms are bounded by ε, since their numerators are bounded by ε while the denominators have absolute value at least 1. To estimate (3.2), note that

$$\frac{1}{(n+1)^s} - \frac{1}{n^s} = \int_n^{n+1} \frac{-s}{x^{s+1}}\, dx\ ,$$

so that

$$\left| \frac{1}{(n+1)^s} - \frac{1}{n^s} \right| \le |s| \int_n^{n+1} \frac{dx}{|x^{s+1}|} = \frac{|s|}{\sigma} \left[\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma} \right]. \qquad (3.3)$$

Thus the absolute value of (3.2) satisfies, for $M, N > n_0$,

$$\begin{aligned} \left| \sum_{n=M}^{N-1} r_n \left[\frac{1}{(n+1)^s} - \frac{1}{n^s} \right] \right| &\le \sum_{n=M}^{N-1} |r_n| \frac{|s|}{\sigma} \left[\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma} \right] \\ &\le \varepsilon \frac{|s|}{\sigma} \sum_{n=M}^{N-1} \left[\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma} \right] \\ &\le \varepsilon \frac{|s|}{\sigma} \left[\frac{1}{M^\sigma} - \frac{1}{N^\sigma} \right] \\ &\le c(\delta)\varepsilon, \qquad (3.4) \end{aligned}$$

since $\frac{|s|}{\sigma} = |1/\cos(\arg s)| \le 1/\cos\left(\frac{\pi}{2} - \delta\right) =: c(\delta)$. This proves that the series is uniformly Cauchy, and hence uniformly convergent. $\square$

Corollary 3.5. *If $\sum_{n=1}^{\infty} a_n n^{-s}$ converges at $s_0 \in \mathbb{C}$, then it converges in Ω_{σ_0}.*

Proof: This follows from the inclusion $\Omega_{\sigma_0} \subset \bigcup_{\delta>0} \{s : \arg|s - s_0| < \frac{\pi}{2} - \delta\}$. $\square$

This implies that there exists a unique value $\sigma_c \in [-\infty, \infty]$ such that the Dirichlet series converges to the right of it, and diverges to the left of it.

DEFINITION 3.6. The *abscissa of convergence* of the Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ is the extended real number $\sigma_c \in [-\infty, \infty]$ with the following properties

(1) if Re $s > \sigma_c$, then $\sum_{n=1}^{\infty} a_n n^{-s}$ converges,
(2) if Re $s < \sigma_c$, then $\sum_{n=1}^{\infty} a_n n^{-s}$ diverges.

NOTE 3.7. To determine the abscissa of convergence, it is enough to look at convergence of the series for $s \in \mathbb{R}$.

EXAMPLE 3.8. It may not be true that the series $\sum_{n=1}^{\infty} a_n n^{-s}$ converges absolutely in $\Omega_{\sigma_c+\delta}$ for every $\delta > 0$, in contrast with the behavior of power series. An example of this phenomenon is the *alternating zeta function* defined as

$$\tilde{\zeta}(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}.$$

First note that $\sigma_c = 0$ for this series. Indeed, the alternating series test implies convergence for all $\sigma > 0$, and the series clearly diverges if $\sigma \leq 0$. Absolute convergence of the series is convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^\sigma}$, so occurs if and only if $\Re(s) > 1$.

DEFINITION 3.9. Given a Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$, the *abscissa of absolute convergence* is defined as

$$\begin{aligned}\sigma_a &= \inf\left\{\rho : \sum_{n=1}^{\infty} a_n n^{-s} \text{ converges absolutely for some } s \text{ with Re } s = \rho\right\} \\ &= \inf\left\{\rho : \sum_{n=1}^{\infty} a_n n^{-s} \text{ converges absolutely for all } s \text{ with Re } s \geq \rho\right\}.\end{aligned}$$

PROPOSITION 3.10. *For any Dirichlet series, we have*

$$\sigma_c \leq \sigma_a \leq \sigma_c + 1.$$

Proof: The first inequality is obvious. For the second, assume, by the usual trick, that $\sigma_c = 0$. We need to show that for $\sigma > 1$, $\sum_{n=1}^{\infty} |a_n n^{-s}|$ converges. Take $\varepsilon > 0$ such that $\sigma - \varepsilon > 1$. Then,

$$\sum_{n=1}^{\infty} \left|\frac{a_n}{n^s}\right| = \sum_{n=1}^{\infty} \frac{|a_n|}{n^\sigma} = \sum_{n=1}^{\infty} \frac{|a_n|}{n^\varepsilon} \cdot \frac{1}{n^{\sigma-\varepsilon}}, \leq C \sum_{n=1}^{\infty} \frac{1}{n^{\sigma-\varepsilon}} < \infty,$$

where $C := \sup_n \left|\frac{a_n}{n^\varepsilon}\right|$ is finite, since $\sigma_c = 0$. $\square$

REMARK 3.11. If $a_n > 0$ for all $n \in \mathbb{N}^+$, then $\sigma_c = \sigma_a$. This follows immediately by considering $s \in \mathbb{R}$.

Recall that for the radius of convergence of a power series, we have the following formula

$$1/R \;=\; \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}.$$

The following is an analogous formula for the abscissa of convergence of a Dirichlet series.

THEOREM 3.12. *Let $\sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series, and let σ_c be its abscissa of convergence. Let $s_n = a_1 + \cdots + a_n$ and $r_n = a_{n+1} + a_{n+2} + \ldots$.*

(1) *If $\sum a_n$ diverges, then $0 \le \sigma_c = \limsup_{n\to\infty} \frac{\log|s_n|}{\log n}$.*
(2) *If $\sum a_n$ converges, then $0 \ge \sigma_c = \limsup_{n\to\infty} \frac{\log|r_n|}{\log n}$.*

Proof: We will show (1); the second part has a similar proof. Hence we assume that $\sum_{n=1}^{\infty} a_n$ diverges and define

$$\alpha := \limsup_{n\to\infty} \frac{\log|s_n|}{\log n}.$$

We will first prove the inequality $\alpha \le \sigma_c$. Assume that $\sum_{n=1}^{\infty} a_n n^{-\sigma}$ converges. Thus $\sigma > 0$ and we need to show that $\sigma \ge \alpha$. Let $b_n = a_n n^{-\sigma}$ and $B_n = \sum_{k=1}^{n} b_k$ (so that $B_0 = 0$). By assumption, the sequence $\{B_n\}$ is bounded, say by M, and we can use summation by parts as follows:

$$\begin{aligned} s_N &= \sum_{n=1}^{N} a_n \\ &= \sum_{n=1}^{N} b_n n^{\sigma} \\ &= \sum_{n=1}^{N-1} B_n[n^{\sigma} - (n+1)^{\sigma}] + B_N N^{\sigma} \end{aligned}$$

so that

$$\begin{aligned} |s_n| &\le M\sum_{n=1}^{N-1}[(n+1)^{\sigma} - n^{\sigma}] + MN^{\sigma} \\ &\le 2MN^{\sigma}. \end{aligned}$$

Applying the natural logarithm to both sides yields

$$\log|s_n| \;\le\; \sigma\log N + \log 2M,$$

so

$$\frac{\log|s_n|}{\log N} \leq \sigma + \frac{\log 2M}{\log N},$$

and this tends to σ as $N \to \infty$, giving the desired upper bound for α.

We need to show the other inequality: $\sigma_c \leq \alpha$. Suppose that $\sigma > \alpha$; we need to show that $\sum_{n=1}^{\infty} a_n n^{-\sigma}$ converges. Choose an $\varepsilon > 0$ such that $\alpha + \varepsilon < \sigma$. By definition, there exist $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\frac{\log|s_n|}{\log n} \leq \alpha + \varepsilon.$$

This implies that

$$\log|s_n| \;\leq\; (\alpha+\varepsilon)\log n \;=\; \log(n^{\alpha+\varepsilon}).$$

Thus, $|s_n| \leq n^{\alpha+\varepsilon}$, for all $n \geq n_0$. Observe that

$$\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma} \;=\; \sigma\int_n^{n+1} \frac{du}{u^{\sigma+1}} \;\leq\; \sigma n^{-(\sigma+1)}.$$

Using summation by parts, we can compute

$$\begin{aligned}
\sum_{n=M+1}^{N} \frac{a_n}{n^\sigma} \;&=\; \sum_{n=M}^{N} s_n\left[n^{-\sigma} - (n+1)^{-\sigma}\right] + s_N(N+1)^{-\sigma} - s_M M^{-\sigma} \\
&\leq\; \sum_{n=M}^{N} n^{\alpha+\varepsilon}\left[\sigma n^{-\sigma-1}\right] + N^{\alpha+\varepsilon}N^{-\sigma} + M^{\alpha+\varepsilon}M^{-\sigma} \\
&\lesssim\; (M-1)^{\alpha+\varepsilon-\sigma},
\end{aligned}$$

and the last quantity tends to zero as M tends to ∞.

We estimated $\sum_{n=M}^{N} n^{\alpha+\varepsilon-\sigma-1}$ by the integral $\int_{M-1}^{N-1} x^{\alpha+\varepsilon-\sigma-1}\,dx \lesssim (M-1)^{\alpha+\varepsilon-\sigma}$, and the symbol $\lesssim$ means less than or equal to a constant times the right hand-side (where the constant depends on $\alpha+\varepsilon-\sigma$, but, critically, not on M). $\square$

Exercise 3.13. Prove (2) of Theorem 3.12.

From the formulae above we can simply deduce formulae for the abscissa of absolute convergence, although these can be derived easily on their own.

Corollary 3.14. *For a Dirichlet series* $\sum_{n=1}^{\infty} a_n n^{-s}$, *we have*

(1) *if* $\sum|a_n|$ *diverges, then* $\sigma_a = \limsup_{n\to\infty} \frac{\log(|a_1|+\cdots+|a_n|)}{\log n} \geq 0$,
(2) *if* $\sum|a_n|$ *converges, then* $\sigma_a = \limsup_{n\to\infty} \frac{\log(|a_{n+1}|+|a_{n+2}|+\dots)}{\log n} \leq 0$.

Proof: Recall that to determine the abscissae, one only needs to consider $s \in \mathbb{R}$ and then absolute convergence of the series is exactly convergence of the Dirichlet series whose coefficient are the absolute values of the original coefficients. $\square$

EXAMPLE 3.15. *The series*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{p_n^s}$$

has $\sigma_c = 0$ *and* $\sigma_a = 1$*.*

Proof: The series of coefficients diverges and so we use the first of the pair of formulae for each abscissae:

$$\sigma_c \;=\; \limsup_{n\to\infty} \frac{\log 1}{\log n} \;=\; 0,$$

and, using the prime number theorem,

$$\sigma_a \;=\; \limsup_{n\to\infty} \frac{\log(\pi(n))}{\log n} \;=\; \limsup_{n\to\infty} \frac{\log n - \log(\log n)}{\log n} \;=\; 1.$$

EXERCISE 3.16. Show that Theorem 3.1 holds for the generalized Dirichlet series $\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$, (assuming, as we always do, that λ_n is an increasing sequence tending to infinity).

(Hint: Find a substitute for (3.3), by considering the integral $\int se^{-sx} dx$.)

Therefore generalized Dirichlet series also have an abscissa of convergence.

EXERCISE 3.17. Show that Theorem3.12 implies that if the abscissa of convergence $\sigma_c \geq 0$, then

$$\forall\, \varepsilon > 0, \quad s_n \;=\; O(n^{\sigma_c + \varepsilon}). \tag{3.18}$$

CHAPTER 4

Perron's and Schnee's formulae

Suppose you know the function values $f(s)$ of some function f that, at least in some half-plane, can be represented by the Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$. How do you determine the coefficients a_n? We have seen one way already in Proposition 1.15:

$$\begin{aligned} a_1 &= \lim_{s\to\infty} f(s) \\ a_2 &= \lim_{s\to\infty} 2^s[f(s)-a_1] \\ a_3 &= \lim_{s\to\infty} 3^s[f(s)-a_1-a_2 2^{-s}] \end{aligned}$$

and so on. The disadvantage is that these formulae are inductive. Schnee's theorem (Theorem 4.11) gives an integral formula for a_n, and Perron's formula (Theorem 4.5) gives a formula for the partial sums.

First, we need to recall the Mellin transform.

Definition 4.1. Suppose that $g(x)x^{\sigma-1} \in L^1(0,\infty)$, then

$$(\mathcal{M}g)(s) \ := \int_0^\infty g(x)x^{s-1}\ ds$$

is the *Mellin transform* of g at $s = \sigma + it$.

Remark 4.2. The Mellin transform is closely related to the Fourier transform and the Laplace transform. From one point of view, the Fourier transform is the Gelfand transform for the group $(\mathbb{R},+)$, while the Mellin transform is the Gelfand transform for the group $(\mathbb{R}^+,\times)$. The two groups are isomorphic and homeomorphic via the exponential map, and we can use this to derive the formula for the inverse of the Mellin transform.

Here is an inverse transform theorem for the Fourier transform. BV_{loc} means locally of bounded variation, *i.e.* every point has a neighborhood on which the total variation of the function is finite.

Theorem 4.3. *If* $h \in BV_{loc}(-\infty,\infty) \cap L^1(-\infty,\infty)$, *then*

$$\frac{1}{2}\left[h(\lambda^+)+h(\lambda^-)\right] = \frac{1}{2\pi}\lim_{T\to\infty}\int_{-T}^{T}(\mathcal{F}h)(\xi)e^{i\lambda\xi}\ d\xi,$$

for all $\lambda \in \mathbb{R}$.

Proof: See [**Tit37**, Thm. 24]. $\square$

This gives us the following formula for the inverse of the Mellin transform.

THEOREM 4.4. *Suppose that* $g \in BV_{loc}(0, \infty)$. *Let* $\sigma \in \mathbb{R}$, *and assume that* $g(x)x^{\sigma-1} \in L^1(0, \infty)$. *Then*

$$\frac{1}{2}\left[g(x^+) + g(x^-)\right] = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\sigma-iT}^{\sigma+iT} (\mathcal{M}g)(s)x^{-s}\, ds,$$

for all $x > 0$.

Proof: Let $\lambda = \log x$, then $G(\lambda) := g(e^\lambda)$ belongs to $BV_{loc}(-\infty, \infty)$. Let $h(\lambda) := G(\lambda)e^{\lambda\sigma}$. Then

$$\begin{aligned}\int_{-\infty}^{\infty} |h(\lambda)|\, d\lambda &= \int_{-\infty}^{\infty} |g(e^\lambda)| e^{\lambda(\sigma-1)} e^\lambda\, d\lambda \\ &= \int_0^\infty |g(x)| x^{\sigma-1}\, dx \\ &< \infty,\end{aligned}$$

so h belongs to $L^1(-\infty, \infty)$. It also belong to $BV_{loc}(-\infty, \infty)$, because, locally, it is the product of a function of bounded variation and a bounded increasing function. We have

$$\begin{aligned}(\mathcal{M}g)(s) &= \int_0^\infty g(x)x^{s-1}\, dx \\ &= \int_0^\infty G(\lambda)e^{\lambda s}\, d\lambda \\ &= \int_{-\infty}^{\infty} \left(G(\lambda)e^{\lambda\sigma}\right) e^{i\lambda t}\, dt \\ &= \mathcal{F}\left(G(\lambda)e^{\lambda\sigma}\right)(-t) \\ &= (\mathcal{F}h)(-t).\end{aligned}$$

Now, we apply Theorem 4.3 to h.

$$\begin{aligned}\frac{1}{2}\left[g(x^+) + g(x^-)\right] &= e^{-\lambda\sigma}\frac{1}{2}\left[h(\lambda^+) + h(\lambda^-)\right] \\ &= e^{-\lambda\sigma}\frac{1}{2\pi} \lim_{T\to\infty} \int_{-T}^{T} (\mathcal{F}h)(\xi)e^{i\lambda\xi}\, d\xi \\ &= e^{-\lambda\sigma}\frac{1}{2\pi} \lim_{T\to\infty} \int_{-T}^{T} (\mathcal{M}g)(\sigma - it)e^{i\lambda t}\, dt \\ &= e^{-\lambda\sigma}\frac{1}{2\pi} \lim_{T\to\infty} \int_{-T}^{T} (\mathcal{M}g)(\sigma + it)e^{-i\lambda t}\, dt\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi} \lim_{T\to\infty} \int_{-T}^{T} (\mathcal{M}g)(\sigma+it)e^{-\lambda(\sigma+it)}\, dt \\
&= \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\sigma-iT}^{\sigma+iT} (\mathcal{M}g)(s)e^{-\lambda s}\, ds \\
&= \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\sigma-iT}^{\sigma+iT} (\mathcal{M}g)(s)x^{-s}\, ds,
\end{aligned}
$$

and we are done. □

Given a Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$, let $F(x) = \sum_{n\le x}' a_n$, where $\sum'$ means that for $x = m \in \mathbb{N}^+$, the last term of the sum is replaced by $\frac{a_m}{2}$ so that the function $F(x)$ satisfies

$$F(x) = \frac{1}{2}\left[F(x^+) + F(x^-)\right]$$

for all x. This function $F(x)$ is called the *summatory function* of the Dirichlet series.

THEOREM 4.5. **(Perron's formula)** *For a Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, the summatory function satisfies*

$$F(x) \;=\; \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\sigma-iT}^{\sigma+iT} \frac{f(w)}{w} x^w dw, \tag{4.6}$$

for all $\sigma > \max(0, \sigma_c)$.

Before we prove Perron's formula, we need the following two propositions.

PROPOSITION 4.7. *Let $F_\sigma(x) = \sum_{n\le x}' a_n n^{-\sigma}$, then*

(1) $F_\sigma(x) = x^{-\sigma}F(x) + \sigma\int_0^x F(y)y^{-\sigma-1}\, dy$,
(2) $F(x) = x^{\sigma}F_\sigma(x) - \sigma\int_0^x F_\sigma(y)y^{\sigma-1}\, dy$.

Proof: First note that if $\sigma = 0$, the formulae hold trivially.

To prove (1), evaluate the integral on the RHS by parts, assuming that $x \notin \mathbb{N}$:

$$
\begin{aligned}
\text{RHS} \;&=\; x^{-\sigma}F(x) + \Big[-F(y)y^{-\sigma}\Big]_0^x + \int_0^x y^{-\sigma} dF(y) \\
&=\; \sum_{n\le x} a_n n^{-\sigma} = F_\sigma(x).
\end{aligned}
$$

If $x_0 \in \mathbb{N}^+$, note that the difference between the limit of the LHS as $x \to x_0-$ and the value of the LHS at x_0 is $\frac{1}{2}a_n n^{-\sigma}$ and the same is true for the RHS, since the integral on the RHS depends continuously on x. Since the two sides were equal for all $x \in (x_0-1, x_0)$ and they jump by the same amount at x_0, they are equal at x_0 as well.

To prove (2) one can either do an analogous calculation, or set $b_n = a_n n^{-\sigma}$, let $G_\sigma(x) = \sum'_{n\le x} b_n n^{-\sigma}$, and let $G(x) = G_0(x)$. Now we apply (1) with G in place of F and $\tilde\sigma = -\sigma$ instead of σ to get

$$\begin{aligned} F(x) &= G_{\tilde\sigma}(x) \\ &= x^{-\tilde\sigma} G(x) \ + \ \tilde\sigma \int_0^x G(y) y^{-\tilde\sigma - 1}\, dy \\ &= x^{\sigma} F_\sigma(x) \ - \ \sigma \int_0^x F_\sigma(y) y^{\sigma-1}\, dy, \end{aligned}$$

since $F(x) = G_{-\sigma}(x)$ and $F_\sigma(x) = G(x)$. $\square$

The following is a necessary condition for a function to be representable by a Dirichlet series.

Proposition 4.8. *Consider the Dirichlet series $f(s) \sim \sum_{n=1}^\infty a_n n^{-s}$ and take a positive σ satisfying $\sigma > \sigma_1 := max(0, \sigma_c)$. Then*

$$f(\sigma + it) = o(|t|), \ \text{as } |t| \to \infty. \tag{4.9}$$

Proof: By Theorem 3.12, we know that $F(x)x^{-\sigma} \to 0$ as $x \to \infty$ (see Exercise 3.17). Since $F(x)$ is 0 if $x < 1$, we have that $F(x)x^{-\sigma-1} \in L^1(0,\infty)$. By Proposition 4.7,

$$f(\sigma) \ = \ \lim_{x\to\infty} F_\sigma(x) \ = \ \lim_{x\to\infty} x^{-\sigma}F(x) \ + \ \sigma \int_0^\infty F(y) y^{-\sigma-1}\, dy.$$

Since the first term tends to 0, we obtain

$$\frac{f(\sigma)}{\sigma} \ = \ (\mathcal{M}F)(-\sigma), \ \text{for all } \sigma > \sigma_1. \tag{4.10}$$

In fact, (4.10) holds for all $s \in \Omega_{\sigma_1}$, since both sides are analytic there. As The function $H(\lambda) := F(e^\lambda)e^{-\lambda s}$ is integrable and $\mathcal{F}(H)(t) = (\mathcal{M}F)(-s)$, by a similar change of variables argument to the one we used in the proof of Theorem 4.4. So by the Riemann-Lebesgue lemma we have

$$\begin{aligned} \lim_{t\to\pm\infty} \frac{f(s)}{s} &= \lim_{t\to\pm\infty} (\mathcal{M}F)(-s) \\ &= \lim_{t\to\pm\infty} \mathcal{F}(H)(t) \\ &= 0. \end{aligned}$$

Therefore we get (4.9), since $|s| \approx |t|$ as $t \to \pm\infty$. $\square$

We will now prove Perron's formula (4.6).

Proof: The function F is in $BV_{loc}(0,\infty)$, and $F(x)x^{-\sigma-1} \in L^1(0,\infty)$. So we can apply the Mellin inversion formula to $\mathcal{M}F(-s) = \frac{f(s)}{s}$ and use the substitution $u = -w$ as follows:

$$\begin{aligned} F(x) = \frac{1}{2}[F(x^+) + F(x^-)] &= \frac{1}{2\pi i}\lim_{T\to\infty}\int_{-\sigma-iT}^{-\sigma+iT} (\mathcal{M}F)(u)x^{-u}\, du \\ &= -\frac{1}{2\pi i}\lim_{T\to\infty}\int_{\sigma+iT}^{\sigma-iT} \frac{f(w)}{w}x^{w}\, dw \\ &= \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\sigma-iT}^{\sigma+iT} \frac{f(w)}{w}x^{w}\, dw, \end{aligned}$$

and we are done. □

One can use this formula to estimate the growth of a_n from estimates of the growth of $f(w)$. Also, note that the formula might hold for smaller σ's, provided that f extends holomorphically to larger half-planes. This follows from the Cauchy integral formula applied to integrals along long vertical rectangles.

Recall that one can use the Cauchy integral formula to obtain the coefficients of a power series from the values of the function it represents, namely

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}\, d\theta.$$

The following theorem is a Dirichlet series analogue.

THEOREM 4.11. **(Schnee)** *Consider the Dirichlet series $f(s) \sim \sum_{n=1}^{\infty} a_n n^{-s}$. One has, for $\sigma > \sigma_c$,*

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(\sigma+it)e^{i\lambda t}\, dt = \begin{cases} a_n n^{-\sigma}, \text{if } \lambda = \log n, \\ 0, \text{ otherwise.} \end{cases} \tag{4.12}$$

Proof: Formally, exchanging the order of summation and integration, one gets

$$\begin{aligned} \frac{1}{2T}\int_{-T}^{T} f(\sigma+it)e^{i\lambda t}\, dt &= \fint_{-T}^{T}\sum a_n e^{(-\sigma-it)\log n + i\lambda t}\, dt \\ &= \sum a_n n^{-\sigma}\fint_{-T}^{T} e^{i(\lambda-\log n)t}\, dt. \end{aligned} \tag{4.13}$$

We write $\fint_{-T}^{T}$ to denote the normalized integral, obtained by dividing by the size of the set over which we are integrating. The integral in

(4.13) is 1 if $\lambda = \log n$, and tends to 0 as $T \to \infty$ otherwise, since for $\alpha \neq 0$, one has

$$\fint_{-T}^{T} e^{i\alpha t}\, dt \;=\; \frac{1}{2Ti\alpha}\left[e^{i\alpha T} - e^{-i\alpha T}\right] \;=\; \frac{\sin(\alpha T)}{\alpha T}.$$

This computation works fine for finite sums, and hence we can change finitely many coefficients of the series. So we may assume that $a_n = 0$, if $\log n \leq \lambda + 1$, and then we must show that the LHS of (4.12) is zero.

Case (i): $\sigma > 0$.

Consider the integral inside the limit. Then t lies in the finite interval $[-T, T]$ and on this interval the series converges uniformly (since it is contained in an appropriate sector, and we can apply Theorem 3.1). Thus we may interchange the order of summation and integration and then use integration by parts as follows

$$\begin{aligned}
\frac{1}{2T}\int_{-T}^{T} \sum_{n \geq e^{\lambda+1}} a_n n^{-\sigma} e^{i(\lambda - \log n)t}\, dt &= \sum a_n n^{-\sigma} \fint_{-T}^{T} e^{i(\lambda-\log n)t}\, dt \\
&= \int_0^\infty x^{-\sigma} \fint_{-T}^{T} e^{i(\lambda-\log x)t}\, dt\; dF(x) \\
&= \int_0^\infty x^{-\sigma} \frac{\sin[(\lambda - \log x)T]}{(\lambda - \log x)T}\, dF(x) \\
&= \left[\frac{x^{-\sigma}\sin[(\lambda-\log x)T]}{(\lambda-\log x)T} F(x)\right]_0^\infty \qquad (4.14) \\
&- \int_0^\infty F(x) \frac{d}{dx}\left[\frac{x^{-\sigma}\sin[(\lambda-\log x)T]}{(\lambda-\log x)T}\right] \qquad (4.15)
\end{aligned}$$

Since $F(x) = 0$ for $x < 1$, the term in brackets in (4.14) vanishes at 0. At infinity, $F(x) = O(x^{\sigma_1+\varepsilon}) = o(x^\sigma)$, by choosing ε small enough. (Again we let σ_1 denote $\max(0, \sigma_c)$). Hence the expression is $o((\log x)^{-1})$ and so the whole term (4.14) vanishes.

We will show that the limit of (4.15) as $T \to \infty$ vanishes as well. We need to differentiate the square bracket. We obtain three terms:

$$\begin{aligned}
\frac{d}{dx}\left[\frac{x^{-\sigma}\sin[(\lambda-\log x)T]}{(\lambda-\log x)T}\right] &= \frac{-\sigma x^{-\sigma-1}\sin[(\lambda-\log x)T]}{(\lambda-\log x)T} \\
&+ \frac{-x^{-\sigma-1}T\cos[(\lambda-\log x)T]}{(\lambda-\log x)T} \\
&+ \frac{x^{-\sigma-1}\sin[(\lambda-\log x)T]}{(\lambda-\log x)^2 T}.
\end{aligned}$$

For each of these terms, we estimate the corresponding integral. Recall that $F(x) = O(x^{\sigma_1+\varepsilon})$ for any positive ε. In particular, $x^{-\sigma}F(x) = O(x^{-\delta})$, for any $\delta < \sigma - \sigma_1$. The first and third terms are similar and we get, as $T \to \infty$

$$\begin{aligned} I: &\quad \left|\int_{e^{\lambda+1}}^{\infty} F(x)\frac{-\sigma x^{-\sigma-1}\sin[(\lambda-\log x)T]}{(\lambda-\log x)T}\,dx\right| = \frac{1}{T}\int_{e^{\lambda+1}}^{\infty} O(x^{-1-\delta}) \to 0, \\ III: &\quad \left|\int_{e^{\lambda+1}}^{\infty} F(x)\frac{x^{-\sigma-1}\sin[(\lambda-\log x)T]}{(\lambda-\log x)^2T}\,dx\right| = \frac{1}{T}\int_{e^{\lambda+1}}^{\infty} O(x^{-1-\delta}) \to 0. \end{aligned}$$

The remaining term is more delicate. We will use the change of variables $u = (\log x - \lambda)$, so that $dx = e^{u+\lambda}du$. We have

$$\begin{aligned} II: \int_{e^{\lambda+1}}^{\infty} F(x)\frac{-x^{-\sigma-1}T\cos[(\lambda-\log x)T]}{(\lambda-\log x)T}\,dx &= -\int_{1}^{\infty} F(e^{u+\lambda})e^{-(1+\sigma)(u+\lambda)}\frac{\cos Tu}{u}e^{\lambda+u}\,du \\ &= -\int_{1}^{\infty}\frac{F(e^{u+\lambda})e^{-\sigma(u+\lambda)}}{u}\cos Tu\,du, \end{aligned}$$

and the last integral tends to 0 as $T \to \infty$ by the Riemann-Lebesgue lemma. Indeed,

$$g(u) := F(e^{u+\lambda})\frac{e^{-\sigma(u+\lambda)}}{u} = O(e^{-\delta(u+\lambda)}), \text{ as } u \to \infty,$$

and thus belongs to L^1.

Case (ii): $\sigma \leq 0$.

Choose some a such that $\sigma + a > 0$, and define $g(s) = f(s-a)$. Then

$$\frac{1}{2T}\int_{-T}^{T} f(\sigma+it)e^{i\lambda t}\,dt = \frac{1}{2T}\int_{-T}^{T} g((\sigma+a)+it)e^{i\lambda t}\,dt,$$

and we can reduce to Case (i). □

EXERCISE 4.16. Check that the same proof yields Schnee's theorem for generalized Dirichlet series. Let λ_n be a strictly increasing sequence with $\lim_{n\to\infty}\lambda_n = \infty$. Define the abscissa of convergence for $f(s) = \sum a_n e^{-\lambda_n s}$ just as for an ordinary Dirichlet series. Then, for $\sigma > \sigma_c$,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(\sigma+it)e^{i\mu t}\,dt = \begin{cases} a_n e^{-\lambda_n\sigma}, \text{if } \mu = \lambda_n, \\ 0, \text{ otherwise.}\end{cases} \qquad (4.17)$$

4.1. Notes

Perron's formula, like Schnee's theorem, also holds for generalized Dirichlet series. For further results in this vein, see [**Hel05**, Ch. 1].

CHAPTER 5

Abscissae of uniform and bounded convergence

5.1. Uniform Convergence

We introduced the alternating zeta function $\tilde{\zeta}$ in Example 3.8, and showed its abscissa of convergence was 0, whilst its abscissa of absolute convergence was 1. In the strip $\{0 < \Re s < 1\}$, one can ask whether there is another form of convergence, intermediate between absolute and pointwise conditional convergence. For example, in what half-planes does the series converge uniformly or to a bounded function?

The values of the alternating zeta function are closely related to the values of the Riemann zeta function; more precisely,

$$\tilde{\zeta}(s) = (2^{1-s} - 1)\zeta(s). \tag{5.1}$$

Indeed, for $\sigma > 1$, both series converge absolutely, so we can reorder the terms freely, and hence

$$\begin{aligned}
\tilde{\zeta}(s) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} \\
&= \sum_{n=1}^{\infty} \frac{-1}{n^s} + 2\sum_{n=1}^{\infty} \frac{1}{(2n)^s} \\
&= (-1 + 2^{1-s})\zeta(s).
\end{aligned}$$

We will see later [**?**] that $\zeta(s)$ can be analytically continued to $\mathbb{C} \setminus \{1\}$, and that this continuation is unbounded on any of the lines $\{s : \text{Re } s = \alpha\}$ with $\alpha \in (0, 1)$. The relationship (5.1) will hold for the continuation as well, since both sides are analytic, and shows that $\tilde{\zeta}(s)$ must also be unbounded on $\{\text{Re } s = \alpha\}$. Hence the convergence cannot be uniform on this line either. We can get uniform convergence, however, provided we divide by s, as the following proposition shows.

Proposition 5.2. *If $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ converges at $s_0 = 0$, then, for any $\delta > 0$,*

$$\frac{1}{s} \sum_{n=1}^{\infty} a_n n^{-s}$$

converges uniformly to $\frac{f(s)}{s}$ in Ω_δ.

Proof: We use the same estimates as when we proved uniform convergence in the sector, but we replace the inequality (3.4) by

$$\varepsilon\frac{|s|}{\sigma}\left[\frac{1}{M^\sigma}-\frac{1}{(N+1)^\sigma}\right] \;\leq\; \frac{|s|}{\delta}\varepsilon.$$

This is an estimate for the main term of $\sum_{n=M}^N a_n n^{-s}$. Each of the two other terms was estimated by ε, so with the extra $1/s$, we obtain, using $1/|s|<1/\delta$,

$$\left|\frac{1}{s}\sum_{n=M}^{N} a_n n^{-s}\right| \;\leq\; 3\frac{\varepsilon}{\delta},$$

for $M,N\geq n_0$ and $s\in\Omega_\delta$. Thus we are done. □

Definition 5.3. For a Dirichlet series $f(s)\sim\sum_{n=1}^\infty a_n n^{-s}$ we define the *abscissa of uniform convergence* σ_u as

$$\sigma_u := \inf\left\{\rho:\ \sum_{n=1}^{\infty} a_n n^{-s} \text{ converges uniformly in } \Omega_\rho\right\},$$

and the *abscissa of bounded convergence* σ_b as

$$\sigma_b := \inf\left\{\rho:\ \sum_{n=1}^{\infty} a_n n^{-s} \text{ converges to a bounded function in } \Omega_\rho\right\}.$$

If a Dirichlet series converges absolutely at some $s_0\in\mathbb{C}$, then it converges uniformly in the closed half-plane $\overline{\Omega_{\sigma_0}}$ by the comparison criterion. Also, if a Dirichlet series converges uniformly in some half-plane Ω_{σ_0}, for N large enough, the sum differs by at most 1 from the partial sum $\sum_{n=1}^N a_n n^{-s}$, for all $s\in\Omega_{\sigma_0}$. But (the absolute value of) this partial sum is bounded by $\sum_{n=1}^N |a_n| n^{-\sigma_0}<\infty$, and so the Dirichlet series converges to a bounded function in Ω_{σ_0}. Combining these two observations with the previously known inequalities between σ_c and σ_a and the obvious inequality $\sigma_c\leq\sigma_b$ we obtain

$$\sigma_c \;\leq\; \sigma_b \;\leq\; \sigma_u \;\leq\; \sigma_a \;\leq\; \sigma_c+1.$$

In fact, $\sigma_b=\sigma_u$, a result due to Bohr in 1913 [**Boh13b**].

Theorem 5.4. **(H. Bohr)** *Suppose that a Dirichlet series converges somewhere and extends analytically to a bounded function in Ω_ρ. Then for all $\delta>0$, the Dirichlet series converges uniformly in $\Omega_{\rho+\delta}$.*

Proof: Suppose that $|f|\leq K$ in $\overline{\Omega_\rho}$ and fix $0<\delta<1$. If $\rho\geq\sigma_a$, we are done by the chain of inequalities above. Thus, we may assume that $\rho < \sigma_a$. Observe, that it is enough to prove the following estimate for $\sigma \geq \rho + \delta$:

$$\left| f(s) - \sum_{n=1}^{N} a_n n^{-s} \right| \;\leq\; C(K,\delta) N^{-\delta} \log N, \tag{5.2}$$

since the right-hand side is $o(1)$ as $N \to \infty$.

To prove 5.2, we fix s and N and define

$$g(z) \;:=\; \frac{f(z)}{z-s} \left(N + \frac{1}{2}\right)^{z-s}.$$

Let d denote $\sigma_a - \rho + 2$, and integrate g around the rectangle with vertices $s - \delta \pm iN^d$ and $s + (\sigma_a - \rho) \pm iN^d$.

It would be nice to put a picture in here

By the residue theorem, we obtain

$$\int_{\square} g(z)\; dz \;=\; 2\pi i f(s).$$

Consider the left-hand edge of the rectangle (LHE), on it we can estimate

$$|g(z)| \;\leq\; \frac{K}{\sqrt{\delta^2 + \operatorname{Im}^2(z-s)}} \left(N + \frac{1}{2}\right)^{-\delta}$$

so that

$$\begin{aligned}
\left| \int_{LHE} g(z)\; dz \right| \;&\lesssim\; KN^{-\delta} \int_{-N^d}^{N^d} \frac{1}{\sqrt{\delta^2 + y^2}}\; dy \\
&=\; KN^{-\delta} \Big[\log\left(y + \sqrt{\delta^2 + y^2} \right) \Big]_{-N^d}^{N^d} \\
&\leq\; CKN^{-\delta} [\log N + \log \delta] \\
&=\; C(K,\delta) N^{-\delta} \log N.
\end{aligned}$$

As for the integration over both of the horizontal edges (HE), we can use the same estimate

$$\begin{aligned}
\left| \int_{HE} g(z)\; dz \right| \;&\leq\; KN^{-d} \int_{\sigma-\delta}^{\sigma+d-2} \left(N + \frac{1}{2}\right)^{x-\sigma} dx \\
&\lesssim\; KN^{-d} \left[\frac{1}{\log N} N^{x-\sigma} \right]_{x=\sigma-\delta}^{x=\sigma+d-2} \\
&\lesssim\; \frac{KN^{-2}}{\log N}.
\end{aligned}$$

Hence, we can conclude that

$$2\pi i f(s) \;=\; \int_{RHE} g(z)\; dz + O(N^{-\delta}\log N).$$

Since the series converges absolutely on RHE, we can interchange the order of integration and summation

$$\begin{aligned}
\int_{RHE} g(z)\; dz &= \int_{RHE} \sum_{n=1}^{\infty} a_n n^{-z} \left(N+\frac{1}{2}\right)^{z-s} \frac{1}{z-s}\; dz \\
&= \sum_{n=1}^{\infty} a_n \int_{RHE} n^{-z} \left(N+\frac{1}{2}\right)^{z-s} \frac{1}{z-s}\; dz \\
&= \sum_{n=1}^{\infty} a_n n^{-s} \int_{RHE} \left(\frac{N+\frac{1}{2}}{n}\right)^{z-s} \frac{1}{z-s}\; dz
\end{aligned}$$

We will show that the contribution of the tail of the series above — the sum for $n > N$ — is small, while the sum over $n \leq N$ is approximately the partial sum of the Dirichlet series.

First, assume that $n > N$, i.e., $n \geq N+1$. Apply Cauchy's theorem to the rectangular path whose left-hand edge is RHE and whose horizontal sides have length L, and let L tend to infinity. Since the integrand has no poles in the region encompassed by this rectangle, the integral over the closed path vanishes. On the new right-hand edge, the integrand decays exponentially with L and so the limit of the integral over this edge tends to 0. On the top edge (and similarly, on the bottom one), we estimate as follows,

$$\begin{aligned}
\left| \int_{s+(\sigma_a-\rho)\pm iN^d}^{\infty+it\pm iN^d} \left(\frac{N+\frac{1}{2}}{n}\right)^{z-s} \frac{dz}{z-s}\; dz \right| &\leq \frac{1}{N^d} \int_{\sigma+(\sigma_a-\rho)}^{\infty} \left(\frac{N+\frac{1}{2}}{n}\right)^{x-\sigma} dx \\
&= \frac{1}{N^d} \int_{\sigma+(\sigma_a-\rho)}^{\infty} e^{(x-\sigma)\log\left(\frac{N+\frac{1}{2}}{n}\right)}\; dx \\
&= \frac{1}{N^d} \frac{1}{-\log\left(\frac{N+\frac{1}{2}}{n}\right)} e^{(\sigma_a-\rho)\log\left(\frac{N+\frac{1}{2}}{n}\right)}.
\end{aligned}$$

The expression $\log\left(\frac{N+\frac{1}{2}}{n}\right)$ is minimized when $n = N+1$. So

$$\begin{aligned}
\left|\log\left(\frac{N+\frac{1}{2}}{n}\right)\right| &\geq -\log\left(1-\frac{1}{2N+2}\right) \\
&> \frac{1}{2(N+1)}.
\end{aligned}$$

Hence, we can estimate the tail of the series by

$$\begin{aligned}\left|\sum_{n=N+1}^{\infty} a_n n^{-s} \int_{RHE} \left(\frac{N+\frac{1}{2}}{n}\right)^{z-s} \frac{1}{z-s}\, dz\right| &\lesssim \sum_{n=N+1}^{\infty} \frac{|a_n|}{n^{\sigma}} N^{1-d} \left(\frac{N+\frac{1}{2}}{n}\right)^{\sigma_a-\rho} \\ &= N^{1-d}\left(N+\frac{1}{2}\right)^{\sigma_a-\rho} \sum_{n=N+1}^{\infty} \frac{|a_n|}{n^{\sigma+\sigma_a-\rho}} \\ &\lesssim N^{-1},\end{aligned}$$

since $\sum \frac{|a_n|}{n^{\sigma+\sigma_a-\rho}}$ converges.

If $n \leq N$, we use Cauchy's theorem again, but now with a rectangular path whose right-hand edge is RHE and whose width L tends to infinity. The residue theorem now implies that

$$\int_{\Box} \left(\frac{N+\frac{1}{2}}{n}\right)^{z-s} \frac{1}{z-s}\, dz \;=\; 2\pi i.$$

The integrand decays exponentially on the left-hand edge, and so the integral over that edge tends to zero. As for the top edge (and also the bottom one)

$$\begin{aligned}\left|\int_{-\infty+it\pm iN^d}^{s+(\sigma_a-\rho)\pm iN^d} \left(\frac{N+\frac{1}{2}}{n}\right)^{z-s} \frac{dz}{z-s}\, dz\right| &\leq \frac{1}{N^d}\int_{-\infty}^{\sigma+(\sigma_a-\rho)} \left(\frac{N+\frac{1}{2}}{n}\right)^{x-\sigma} dx \\ &= \frac{1}{N^d}\int_{-\infty}^{\sigma+(\sigma_a-\rho)} e^{(x-\sigma)\log\left(\frac{N+\frac{1}{2}}{n}\right)} dx \\ &= \frac{1}{N^d}\frac{1}{\log\left(\frac{N+\frac{1}{2}}{n}\right)} e^{(\sigma_a-\rho)\log\left(\frac{N+\frac{1}{2}}{n}\right)} \\ &\leq N^{-d}\frac{1}{\log\left(\frac{N+\frac{1}{2}}{N}\right)}\left(\frac{N+\frac{1}{2}}{n}\right)^{\sigma_a-\rho} \\ &\lesssim N^{1-d}\left(\frac{N+\frac{1}{2}}{n}\right)^{\sigma_a-\rho} \\ &\lesssim N^{-1}n^{-\sigma_a+\rho}\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{n=1}^{N} a_n n^{-s} \int_{RHE} \left(\frac{N+\frac{1}{2}}{n}\right)^{z-s} \frac{1}{z-s}\, dz &= 2\pi i \sum_{n=1}^{N} \frac{a_n}{n^s} + O(N^{-1}n^{-\sigma_a+\rho}) \sum_{n=1}^{N} \frac{|a_n|}{n^{\sigma}} \\ &= 2\pi i \sum_{n=1}^{N} \frac{a_n}{n^s} + O(N^{-1}),\end{aligned}$$

where we used boundedness of the partial sums of the convergent series $\sum_n \frac{|a_n|}{n^{\sigma+\sigma_a-\rho}}$. We have shown that $\frac{1}{2\pi i}\int_{RHE} g(z)\,dz$ is close to both the partial sum of the Dirichlet series and $f(s)$ (and the error is as in (5.2), and does not depend on s). □

The promised equality of the two new abscissae is now an immediate corollary.

COROLLARY 5.5. *The equality $\sigma_b = \sigma_u$ holds for any Dirichlet series.*

Note, however, that the above corollary does not imply that if a Dirichlet series converges to a bounded function in some half-plane, it will converge uniformly in that half-plane. We only know that it will converge uniformly in every strictly smaller half-plane.

REMARK 5.6. The function $g(z)$ used in the proof of the theorem above comes from Perron's formula which can be restated as (in the special case of $x = N + \frac{1}{2}$)

$$\sum_{n\le N} a_n n^{-s} \;=\; \frac{1}{2\pi i}\int_{\sigma+it-iT}^{\sigma+it+iT} f(z)\frac{(N+\frac{1}{2})^{z-s}}{z-s}\,dz \;+\; e_{N,T},$$

where $e_{N,T}$ is an error term that comes from not taking the limit in T. One can also prove this formula using the estimates above.

5.2. The Bohr correspondence

Bohr's idea was to use the following correspondence between Dirichlet series and power series in infinitely many variables. For a positive integer with prime factorization $n = p_1^{k_1}\ldots p_l^{k_l}$, we define

$$z^{r(n)} \;:=\; z_1^{k_1}\ldots z_l^{k_l}.$$

We have an isomorphism between formal power series in infinitely many variables $z_1, z_2, \ldots$ and Dirichlet series, given by

$$\mathcal{B} : \sum_n a_n z^{r(n)} \;\mapsto\; \sum_n a_n n^{-s}. \tag{5.7}$$

We shall write $\mathcal{Q}$ for the inverse of $\mathcal{B}$:

$$\mathcal{Q} : \sum_n a_n n^{-s} \;\mapsto\; \sum_n a_n z^{r(n)}. \tag{5.8}$$

The map $\mathcal{B}$ is an evaluation homomorphism — indeed, we evaluate the power series on the one-dimensional set $\{(z_i) : \; z_i = p_i^{-s}\}$. It is clearly onto, and it has a trivial kernel because the right-hand side is 0 iff all the coefficients vanish.

For finite series, we can norm both spaces so that $\mathcal{B}$ will be isometric. Indeed, we have

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left|\sum_{n=1}^{N}a_n n^{-it}\right|^2 dt &= \sum_{n=1}^{N}|a_n|^2\\ &= \int_{\mathbb{T}^\infty}\left|\sum_{n=1}^{N}a_n e^{2\pi i t\cdot r(n)}\right|^2 dt. \quad (5.9)\end{aligned}$$

By $\mathbb{T}^\infty$ we mean the infinite torus

$$\mathbb{T}^\infty \;=\; \{(e^{2\pi i t_1}, e^{2\pi i t_2}, \dots) :\; 0 \le t_j < 1 \;\forall j \in \mathbb{N}^+\}$$

which we identify with the infinite product

$$[0,1)\times[0,1)\times\cdots$$

on which we put the product probability measure of Lebesgue measure on each interval.

We shall investigate when (5.9) holds for infinite sums in Theorem 6.39.

Flesh this section out.

5.3. Bohnenblust-Hille Theorem

We will now proceed to show that $\sigma_a - \sigma_b \le \frac{1}{2}$, and that this bound is sharp. Originally, Hille and Bohnenblust exhibited an example of a Dirichlet series for which equality holds in the above inequality. Their construction was extremely complicated.

Instead of going through their construction, we shall show that such an example exists using a probabilistic method. This is a non-constructive method, used in other fields, in particular in combinatorics/graph theory.

Before describing the probabilistic method we mention two analogous methods: the "cardinality method" and the "Baire category method". Recall that one can prove the existence of transcendental numbers by showing that there are only countably many algebraic numbers (and uncountably many real numbers). This is much easier than proving that a concrete number is transcendental. Similarly, the existence of a nowhere differentiable continuous function on an interval I can be proved by showing that the set of all continuous functions with a derivative at at least one point is of the first category (and thus cannot equal the complete metric space of all continuous function on I). The construction of a particular example is again fairly technical.

The probabilistic method is similar in spirit. Instead of exhibiting a concrete example of an object with some given property, we consider some set S of objects and equip it with a convenient probability measure. We strive to show that a randomly chosen object will have the desired property with a non-zero probability. Although it might seem that this will rarely work, the probability method has been very successful, especially when examples with the given property have a complicated structure or description.

In our case, we will need to consider random series of functions of the form

$$f_\varepsilon(s) \;=\; \sum_{n=1}^{\infty} \varepsilon_n a_n n^{-s},$$

where $\{\varepsilon_n\}$ is a *Rademacher sequence*, that is, a sequence of independent random variables, such that each $\varepsilon_n \in \{\pm 1\}$ and $\mathrm{Prob}\,(\varepsilon_n = 1) = \mathrm{Prob}\,(\varepsilon_n = -1) = 1/2$. One can also consider the random series

$$f_\omega(s) \;=\; \sum_{n=1}^{\infty} a_n e^{in\omega_n} n^{-s},$$

where $\omega = \{\omega_n\}_{n=1}^{\infty}$ is a sequence of random variables that are independent and such that each ω_n is uniformly distributed on $[0, 2\pi]$. Both $\{\varepsilon_n\}$ and $\{\omega_n\}$ are i.i.d.'s, that is, independent and identically distributed.

When we have a Rademacher sequence, we use $\mathbb{E}$ to denote the expectation, that is the average over all choices of sign, of some function that depends on the sequence:

$$\mathbb{E}[\sum \varepsilon_n g_n].$$

If the sequence is finite of length K, this just means adding up all 2^K choices and dividing by 2^K. If the sequence is infinite, one must replace this by integrating over the space $\{-1, 1\}^\infty$ with the product probability measure.

Note that a sequence of i.i.d.'s has a canonical probability distribution associated to it, namely the product probability. Heuristically, to choose a random sequence, we can choose it element by element, and since these elements should be independent, we arrive at the product probability.

As an example of a theorem about random Dirichlet series we prove the following proposition.

PROPOSITION 5.10. *Let $\{a_n\}$ be a sequence of complex numbers and let $\{\omega_n\}_{n=1}^{\infty}$ be sequence of i.i.d.'s which are uniformly distributed on*

$[0; 2\pi]$. *Denote* $f_\omega(s) := \sum_{n=1}^{\infty} a_n e^{in\omega_n} n^{-s}$, *as above. Then there exists some* $\tilde{\sigma} = \tilde{\sigma}(\{a_n\})$ *such that* $\sigma_c(f_\omega) = \tilde{\sigma}$ *almost surely.*

Proof: Given a sequence of random variables, a *tail event* is an event whose incidence is not changed by changing the values assumed by any finitely many elements of the sequence. The zero-one law of probability asserts that any tail event associated to a sequence of i.i.d.'s happens with probability either 0 or 1 [**Kah85**, p.7]. Consider the events $B_a = \{f_\omega : \sigma_c(f_\omega) \le a\}$ for $a \in \mathbb{R}$. These are clearly tail events. Let

$$\tilde{\sigma} \ := \ \inf \{a : \ \mathrm{Prob}\,(B_a) = 1\},$$

where we agree that $\inf \emptyset = \infty$. Since the events B_a are nested, we have

$$\begin{aligned} \mathrm{Prob}\,(B_a) = 0, &\ \text{ for all } a < \tilde{\sigma}, \\ \mathrm{Prob}\,(B_a) = 1, &\ \text{ for all } a > \tilde{\sigma}, \end{aligned}$$
$$\text{and} \quad \{f_\omega : \ \sigma_c(f_\omega) = \tilde{\sigma}\} \ = \ \left(\bigcap\nolimits_n B_{\tilde{\sigma}+\frac{1}{n}}\right) \setminus \left(\bigcup\nolimits_n B_{\tilde{\sigma}-\frac{1}{n}}\right),$$

which easily implies that $\tilde{\sigma}$ has the desired property. □

Let f be a holomorphic function on a domain $\Omega \subset \mathbb{C}$. We say that $\partial\Omega$ is a *natural boundary* for f, if no point $z_0 \in \partial\Omega$ has a neighborhood to which it can be holomorphically continued. Proposition 5.10 can be strengthened in the following way [**Kah85**, p. 44].

THEOREM 5.11. *Let* ω_n *and* $\overline{\sigma}$ *be as above. Then, with probability* 1*, the line* $\{Re\ s = \overline{\sigma}\}$ *is the natural boundary for the Dirichlet series* $\sum a_n e^{i\omega_n} n^{-s}$.

We will need the following theorem, which we shall prove as Corollary 5.23 below. We shall use multi-index notation, where $\alpha \in \mathbb{Z}^r$ — see Appendix 11.1. We shall use $\mathbb{T}^r$ to denote the r-torus, which by an abuse of notation we shall identify with both $\{(e^{2\pi i t_1}, \cdots, e^{2\pi i t_r}) : 0 \le t_j \le 1\ \forall\ j\}$ and $\{(t_1, \cdots, t_r) \ : \ 0 \le t_j \le 1\ \forall\ j\}$.

THEOREM 5.12. *There exists a universal constant* $C > 0$ *such that for every* $r \in \mathbb{N}^+$, *every* $N \ge 2$, *and every choice of coefficients* $c_\alpha \in \mathbb{C}$, *with* $|\alpha| = |\alpha_1| + \cdots + |\alpha_r| \le N$, *there exists some choice of signs such that*

$$\sup_{t \in \mathbb{T}^r} \left| \sum_{|\alpha| \le N} \pm c_\alpha e^{2\pi i(\alpha_1 t_1 + \cdots + \alpha_r t_r)} \right| \ \le \ C \left[r \log N \sum |c_\alpha|^2 \right]^{\frac{1}{2}}. \qquad (5.13)$$

By Fubini's theorem and the orthogonality of $\{e^{2\pi i\alpha\cdot t}\}$, for any choice of signs

$$\int_{\mathbb{T}^r}\left|\sum_{\alpha} \pm c_n e^{2\pi i\alpha\cdot t}\right|^2 dt \;=\; \sum_{\alpha}|c_\alpha|^2,$$

so the left-hand side of (5.13) is at least $\sum_\alpha |c_\alpha|^2$. The theorem says that for some choice of signs, this estimate is only off by a factor of $\sqrt{r\log N}$.

Note that choosing all c_α positive and using the Cauchy-Schwarz inequality yields the following much cruder estimate:

$$\begin{aligned}\sup_{t\in\mathbb{T}^r}\left|\sum_{|\alpha|\le N} c_\alpha e^{i(\alpha_1 t_1+\cdots+\alpha_r t_r)}\right| \;&=\; \sum_{\alpha} c_\alpha \\ &\le\; \sqrt{C_N}\left(\sum_{\alpha}|c_\alpha|^2\right)^{\frac{1}{2}},\end{aligned}$$

where C_N is the number of terms, roughly N^r, if $N \gg r$.

We will need the following lemma.

Lemma 5.14. *Let*

$$P(t) \;=\; \sum_{|\alpha|\le N} c_\alpha e^{2\pi i(\alpha_1 t_1+\cdots+\alpha_r t_r)}$$

be a trigonometric polynomial on $\mathbb{T}^r$. If P is real, then there exists an r-dimensional cube $I \subset \mathbb{T}^r$ of volume $(N+1)^{-2r}$ on which $|P(t_1,\ldots,t_r)| \ge \frac{1}{2}\|P\|_\infty$.

Proof: By multiplying P by (-1), if necessary, we may assume that there exists $\theta = (\theta_1,\ldots,\theta_r) \in \mathbb{T}^r$ such that

$$P(\theta) = \|P\|_\infty.$$

By the mean value theorem, we conclude that for any $t = (t_1,\ldots,t_r) \in \mathbb{T}^r$, there exists $\tilde{\theta}$ belonging to the segment connecting t and θ such that

$$P(t) - P(\theta) \;=\; \sum_{j=1}^{r}(t_j-\theta_j)\frac{\partial P}{\partial t_j}(\tilde{\theta}),$$

Thus,

$$|P(t)-P(\theta)| \;\le\; \max_j |t_j-\theta_j| \sum_{j=1}^{r}\left|\frac{\partial P}{\partial t_j}(\tilde{\theta})\right| \tag{5.15}$$

There exists a choice of signs $s_j \in \{\pm 1\}$ so that

$$\frac{d}{dx}_{|x=0} P(\tilde{\theta}_1 + s_1 x, \ldots, \tilde{\theta}_r + s_r x) \;=\; \sum_j \left| \frac{\partial P}{\partial t_j}(\tilde{\theta}) \right|.$$

We fix this choice, and define a trigonometric polynomial of degree at most N

$$Q(x) = P(\tilde{\theta}_1 + s_1 x, \ldots, \tilde{\theta}_r + s_r x).$$

Then $Q(x) = \sum_k b_k e^{ikx}$ and $Q'(x) = \sum_k ikb_k e^{ikx}$. Note that by integrating against e^{-ikx} we obtain $|b_k| \le \|Q\|_\infty$, and hence

$$\begin{aligned} |Q'(0)| &\le \sum_k |kb_k| \\ &\le \max_k |b_k| \sum_{k=-N}^{N} k \\ &\le \|Q\|_\infty N(N+1) \\ &\le \|P\|_\infty N(N+1). \end{aligned}$$

Thus, we can continue our estimate from (5.15)

$$|P(t) - P(\theta)| \;\le\; \|P\|_\infty N(N+1) \sup_j |t_j - \theta_j|. \tag{5.16}$$

Since $|P(\theta)| = \|P\|_\infty$, whenever the right-hand side of (5.16) is bounded by $\frac{1}{2}\|P\|_\infty$, we have $P(t) \ge \frac{\|P\|_\infty}{2}$. This will occur if

$$\sup_j |t_j - \theta_j| \le \frac{1}{2N(N+1)}.$$

The set of such t's is a cube of volume $[N(N+1)]^{-r} \ge (N+1)^{-2r}$. $\Box$

THEOREM 5.17. *Let $\{P_n\}_{n=1}^K$ be a finite set of complex trigonometric polynomials in r variables of degree less than or equal to N, with $N \ge 1$. Let $Q(t_1, \ldots, t_r) = \sum_n \varepsilon_n P_n(t_1, \ldots, t_r)$, where ε_n is a Rademacher sequence. Then*

$$\text{Prob}\left(\|Q\|_\infty \ge \left[32 r \log \gamma N \sum_n \|P_n\|_\infty^2 \right]^{\frac{1}{2}} \right) \le \frac{2}{\gamma},$$

for all real $\gamma \ge 8$.

Proof: First suppose that all P_n's are real, let $\tau = \sum_n \|P_n\|_\infty^2$ and $M = \|Q\|_\infty$ (here $M = M(\varepsilon)$ is a random variable). Let λ be an arbitrary real number. Then, using the inequality $\frac{1}{2}(e^x+e^{-x}) \leq e^{\frac{x^2}{2}}$ yields

$$
\begin{aligned}
\mathbb{E}\left(e^{\lambda Q(t)}\right) &= \mathbb{E}\left(e^{\lambda \sum_n \varepsilon_n P_n(t)}\right) \\
&= \mathbb{E}\left(\prod_n e^{\lambda \varepsilon_n P_n(t)}\right) \\
&= \prod_n \mathbb{E}\left(e^{\lambda \varepsilon_n P_n(t)}\right) \\
&= \prod_n \left(\frac{1}{2}[e^{\lambda P_n(t)} + e^{-\lambda P_n(t)}]\right) \\
&\leq \prod_n e^{\lambda^2 \frac{P_n^2(t)}{2}} \\
&\leq \prod_n e^{\frac{\lambda^2}{2}\|P_n\|_\infty^2} \\
&= e^{\frac{\lambda^2}{2}\sum_n \|P_n\|_\infty^2} \\
&= e^{\frac{\tau\lambda^2}{2}}. \qquad (5.18)
\end{aligned}
$$

By Lemma 5.14, there exists an interval $I = I(\varepsilon) \subset \mathbb{T}^r$ of volume at least $(N+1)^{-2r}$ such that $|Q| \geq \frac{1}{2}\|Q\|_\infty$ of I. For fixed $\varepsilon = \{\varepsilon_n\}$ we thus have

$$
\begin{aligned}
e^{\frac{\lambda M(\varepsilon)}{2}} &\leq \frac{1}{vol(I(\varepsilon))}\int_{I(\varepsilon)} e^{\lambda Q(t)} + e^{-\lambda Q(t)}\; dt \\
&\leq (N+1)^{2r}\int_{\mathbb{T}^r} e^{\lambda Q(t)} + e^{-\lambda Q(t)}\; dt
\end{aligned}
$$

Taking the expected value and using estimate (5.18) yields

$$
\begin{aligned}
\mathbb{E}\left(e^{\frac{\lambda M}{2}}\right) &\leq (N+1)^{2r}\mathbb{E}\left(\int_{\mathbb{T}^r} e^{\lambda Q(t)} + e^{-\lambda Q(t)}\; dt\right) \\
&= (N+1)^{2r}\int_{\mathbb{T}^r} \mathbb{E}\left(e^{\lambda Q(t)} + e^{-\lambda Q(t)}\right)\; dt \\
&\leq (N+1)^{2r}\int_{\mathbb{T}^r} 2e^{\frac{\tau\lambda^2}{2}}\; dt \\
&= 2(N+1)^{2r}e^{\frac{\tau\lambda^2}{2}} \\
&= e^{\frac{\tau\lambda^2}{2}+\log 2+2r\log(N+1)}
\end{aligned}
$$

Thus,

$$
\mathbb{E}\left(e^{\frac{\lambda M}{2}-\frac{\lambda^2\tau}{2}-\log 2-2r\log(N+1)}\right) \leq 1,
$$

and hence, by Chebyshev's inequality,

$$\text{Prob}\left(e^{\frac{\lambda M}{2}-\frac{\lambda^2\tau}{2}-\log 2-2r\log(N+1)} \geq \gamma\right) \;\leq\; \frac{1}{\gamma}. \tag{5.19}$$

The event on the left-hand side of (5.19) is equivalent to

$$\frac{\lambda M-\lambda^2\tau}{2} - \log 2 - 2r\log(N+1) \;\geq\; \log\gamma. \tag{5.20}$$

Choose $\lambda = \sqrt{\frac{2}{\tau}\log[2\gamma(N+1)^{2r}]}$, then, after algebraic manipulations, (5.20) becomes

$$M\sqrt{\frac{2}{\tau}\log[2\gamma(N+1)^{2r}]} \;\geq\; 4\log[2\gamma(N+1)^{2r}],$$

which is the same as

$$M \;\geq\; 2\sqrt{2\tau}\sqrt{\log[2\gamma(N+1)^{2r}]}. \tag{5.21}$$

For $\gamma \geq 8$ we have

$$2\gamma(N+1)^{2r} \leq (\gamma N)^{2r},$$

so (5.21) will hold if

$$\begin{aligned} M &\geq 2\sqrt{2\tau}\sqrt{\log[\gamma N]^{2r}} \\ &= 4\sqrt{r\tau\log[\gamma N]}. \end{aligned}$$

Recalling that $M = \|Q\|_\infty$ and $\tau = \sum_n \|P_n\|^2$ we obtain

$$\text{Prob}\left(\|Q\|_\infty \;\geq\; 4\left[r\log[\gamma N]\sum_n \|P_n\|^2\right]^{\frac{1}{2}}\right) \;\leq\; \frac{1}{\gamma},$$

when Q is real.

If Q is complex and

$$\|Q\|_\infty \;\geq\; 4\big[2r\log[\gamma N]\sum_n \|P_n\|_\infty^2\big]^{\frac{1}{2}},$$

then one of the two following inequalities must hold:

$$\begin{aligned} \|\text{Re } Q\|_\infty &\geq 4\left[r\log[\gamma N]\sum_n \|\text{Re } P_n\|_\infty^2\right]^{\frac{1}{2}}, \\ \|\text{Im } Q\|_\infty &\geq 4\left[r\log[\gamma N]\sum_n \|\text{Im } P_n\|_\infty^2\right]^{\frac{1}{2}}. \end{aligned}$$

But since these inequalities involve real polynomials, either of them happens with probability at most $\frac{1}{\gamma}$, by the real case. The probability that at least one of them happens is thus at most $\frac{2}{\gamma}$. $\square$

COROLLARY 5.22. *Let $N \geq 2$, and let $c_\alpha \in \mathbb{C}$ be given for every $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}^r$ with $|\alpha| \leq N$. Then, for any $\gamma \geq 8$, there exists $C > 0$ such that*

$$\text{Prob}\left(\left\| \sum_{|\alpha|\leq N} \varepsilon_\alpha c_\alpha e^{i\alpha\cdot t} \right\|_\infty \geq C(r\log N)^{\frac{1}{2}} \left[\sum_{|\alpha|\leq N} |c_\alpha|^2 \right]^{\frac{1}{2}} \right) \;\leq\; \frac{2}{\gamma}.$$

Proof: Fix $\gamma \geq 8$, and choose $C > 0$ such that $C^2 \geq 32\left(1 + \frac{\log\gamma}{\log N}\right)$. Let $P_\alpha(t) = c_\alpha e^{i\alpha\cdot t}$ and use Theorem 5.17. $\square$

COROLLARY 5.23. *There exist a choice of signs $\{\varepsilon_\alpha\}$ such that*

$$\left\| \sum_{|\alpha|\leq N} \varepsilon_\alpha c_\alpha e^{i\alpha\cdot t} \right\|_\infty \;\leq\; C(r\log N)^{\frac{1}{2}} \left[\sum_{|\alpha|\leq N} |c_\alpha|^2 \right]^{\frac{1}{2}}.$$

Proof: For any $\gamma \geq 8$, the probability that a random series will not have the property is at most $\frac{2}{\gamma} < 1$. $\square$

THEOREM 5.24. **(H. Bohr)** *For any Dirichlet series $\sigma_a - \sigma_u \leq \frac{1}{2}$.*

Proof: Let $\rho > \sigma_u$, then $\sum_{n=1}^\infty a_n n^{-s}$ converges uniformly in $\overline{\Omega_\rho}$. Fix $s \in \mathbb{C}$ with $\operatorname{Re} s = \rho + \frac{1}{2} + \varepsilon$. By the Cauchy-Schwarz inequality,

$$\begin{aligned} \sum_n |a_n n^{-s}| &= \sum_n |a_n| n^{-(\rho+\frac{1}{2}+\varepsilon)} \\ &\leq \left(\sum_n |a_n|^{-2\rho}\right)^{\frac{1}{2}} \left(\sum_n n^{-(1+2\varepsilon)}\right)^{\frac{1}{2}}, \qquad (5.25) \end{aligned}$$

where the second sum converges. By uniform convergence, there exists $K > 0$ such that for every $t \in \mathbb{R}$ and $N \in \mathbb{N}^+$

$$\left| \sum_{n=1}^N a_n n^{-(\rho+it)} \right| \;\leq\; K.$$

Consequently,

$$\begin{aligned} K^2 &\geq \left| \sum_{n=1}^N a_n n^{-(\rho+it)} \right|^2 \\ &= \sum_{n=1}^N |a_n|^2 n^{-2\rho} + 2\operatorname{Re} \sum_{1\leq n<m\leq N} a_n \overline{a}_m (nm)^{-\rho} e^{it\log\frac{m}{n}}. \end{aligned}$$

Taking the normalized integral yields

$$K^2 \;\geq\; \sum_{n=1}^{N} |a_n|^2 n^{-2\rho} + 2\,\mathrm{Re} \sum_{1\leq n<m\leq N} a_n \overline{a}_m (nm)^{-\rho} ⨍_{-T}^{T} e^{it\log\frac{m}{n}}\; dt.$$

Taking the limit as T tends to ∞, the mixed terms tend to 0 and so we conclude that

$$\sum_{n=1}^{N} |a_n|^2 n^{-2\rho} \;\leq K^2,$$

for all $N \in \mathbb{N}^+$. Thus the first sum on the right-hand side of (5.25) is bounded, and so $\sum_n |a_n n^{-s}|$ converges. Thus, $\sigma_a \leq \frac{1}{2} + \rho + \varepsilon$. Since this is true for every $\rho > \sigma_u$ and $\varepsilon > 0$, we get $\sigma_a \leq \frac{1}{2} + \sigma_u$. ☐

THEOREM 5.26. **(Bohnenblust-Hille, 1931)** *There exist a Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ for which $\sigma_u = \frac{1}{2}$ and $\sigma_a = 1$.*

We shall present a probabilistic proof, due to H. Boas [**Boa97**].

Proof: Each a_n will be an element of $\{\pm 1, 0\}$ and the coefficients will be constructed in groups, starting with $k = 2$. To construct the k^{th} group, choose a homogeneous polynomial Q_k of degree k in 2^k variables with coefficients $\varepsilon_j \in \{\pm 1\}$, with $j = (j_1, \ldots, j_{2^k})$,

$$Q_k(z_1, z_2, \ldots, z_{2^k}) \;=\; \sum_{|j|=k} \varepsilon_j z_1^{j_1} \ldots z_{2^k}^{j_{2^k}}$$

so that

$$\|Q_k\|_\infty \;\leq\; C\left[2^k \log k \sum_{|j|=k} |\varepsilon_j|^2\right]^{\frac{1}{2}}.$$

This is possible, by Corollary 5.23. By Lemma 5.29, the number of (monic) monomials of degree k in 2^k variables is $\binom{2^k+k-1}{k}$. We conclude that

$$\|Q_k\|_\infty \;\leq\; C\left[2^k \log k \binom{2^k+k-1}{k}\right]^{\frac{1}{2}}.$$

We convert the Q_k's into Dirichlet series as in (5.7)

$$f_k(s) \;:=\; (\mathcal{B}Q_k)(s) \;=\; \sum_{|j|=k} \varepsilon_j \left(p_{2^k}^{j_1} \ldots p_{2^k+2^k-1}^{j_{2^k}}\right)^{-s},$$

and let $f = \sum_{k=2}^{\infty} f_k$, thought of as a Dirichlet series. Then the coefficients of f lie in $\{\pm 1, 0\}$, since each n can appear in at most one f_k.

Claim 1: $\sigma_a(f) = 1$.

Proof: In f_k, the number of non-zero coefficients is

$$\binom{2^k+k-1}{k} \;\geq\; \frac{(2^k)^k}{k!} \;\geq\; \frac{2^{k^2}}{k^k}.$$

By the prime number theorem, $p_k \approx k \log k$, so that $p_{2^{k+1}} \leq M 2^k k$, for some $M > 1$. Hence any n that has a non-zero coefficient in f_k must satisfy

$$n \;\leq\; \left(M 2^k k\right)^k.$$

Thus, we can estimate for $\sigma < 1$,

$$\begin{aligned} \sum_n |a_n| n^{-\sigma} &\geq \sum_k \frac{2^{k^2}}{k^k} \left(M 2^k k\right)^{-k\sigma} \\ &= \sum_k \frac{2^{k^2(1-\sigma)}}{k^{k(1+\sigma)} M^{k\sigma}} \end{aligned} \tag{5.27}$$

By the root test (or ratio test), (5.27) diverges for $\sigma < 1$.

Since for $\sigma > 1$ the series converges absolutely (by comparison to $\sum_n n^{-\sigma}$), we conclude that $\sigma_a = 1$.

Claim 2: $\sigma_u(f) = \frac{1}{2}$.

Proof: Fix $\varepsilon > 0$, let $\sigma = \frac{1}{2} + \varepsilon$, and note that

$$\begin{aligned} |f_k(\sigma + it)| &= |Q_k(p_{2^k}^{-s}, \dots, p_{2^{k+1}-1}^{-s})| \\ &= \left| \sum_{|j|=k} \varepsilon_j \left(p_{2^k}^{j_1} \dots p_{2^{k+1}-1}^{j_{2^k}}\right)^{\sigma} \left(p_{2^k}^{j_1} \dots p_{2^{k+1}-1}^{j_{2^k}}\right)^{it} \right|. \end{aligned}$$

Thus,

$$\begin{aligned} \sup_t |f_k(\sigma + it)| &\leq \sup_{|z_i| = p_{2^k-1+i}^{-\sigma},\ i=1,\dots,2^k} |Q_k(z_1, \dots, z_{2^k})| \\ &\leq p_{2^k}^{-k\sigma} \sup_{|z_i|=1} |Q_k| \\ &\leq C p_{2^k}^{-k\sigma} \left[2^k \log k \binom{2^k+k-1}{k}\right]^{\frac{1}{2}} \qquad (5.28) \\ &\lesssim (2^k k \log 2)^{-k\sigma} \left[2^k \log k 2^{k^2}\right]^{\frac{1}{2}} \\ &= (k \log 2)^{-k\sigma} 2^{k^2(-\sigma + \frac{1}{2} + \frac{1}{2k})} \sqrt{\log k} \\ &= (k \log 2)^{-k\sigma} 2^{k^2(-\varepsilon + \frac{1}{2k})} \sqrt{\log k}. \end{aligned}$$

The series $\sum_k |f_k|$ is thus estimated by a summable series. Hence, $\sum_k f_k$ converges to a holomorphic function which is bounded in $\Omega_{1/2+\varepsilon}$

and equal to f in Ω_1. Letting $\varepsilon \to 0+$ yields, by Theorem 5.4, $\sigma_u = \sigma_b \leq \frac{1}{2}$. Thus, by Theorem 5.24 and Claim 1, $\sigma_u = \frac{1}{2}$. ☐

LEMMA 5.29. *The number of monomials of degree m in n variables is $\binom{n+m-1}{m}$.*

PROOF: From a linear array of $n+m-1$ objects, choose $n-1$ and color them black. Let the power of z_i be the number of non-colored objects between the $(i-1)^{\text{st}}$ black one and the i^{th} one. ☐

EXERCISE 5.30. Fill in the details that the series in (5.28) converges.

EXERCISE 5.31. Show that for all $x \in [0, \frac{1}{2}]$, there is a Dirichlet series such that $\sigma_a - \sigma_u$ is exactly x.

(Hint: Although Bohnenblust and Hille did not spot it, this result is a one-line consequence of Theorem 5.26. If you find the right line!)

5.4. Notes

The proofs of the Bohnenblust-Hille theorem in Section 5.3 and Bohr's Theorem 5.4 are based on H. Boas's article [**Boa97**]. The original proofs are in [**BH31**] and [**Boh13b**], respectively. Theorem 5.24 was proved in [**Boh13a**].

Talk about recent advances, in particular [**DFOC$^+$11**].

CHAPTER 6

Hilbert Spaces of Dirichlet Series

6.1. Beurling's problem: The statement

We will motivate our discussion by considering a problem posed by A. Beurling in 1945. If we set $\beta(x) = \sqrt{2}\sin(\pi x)$, the set

$$\{\beta(nx) : \ n \in \mathbb{N}^+\}$$

forms an orthonormal basis of $L^2([0;1])$.

PROPOSITION 6.1. *If $\psi : \mathbb{R}^+ \to \mathbb{C}$ is 2-periodic, and $\{\psi(nx)\}_{n\in\mathbb{N}^+}$ is an orthonormal basis for $L^2([0;1])$, then $\psi = e^{i\theta}\beta$, for some $\theta \in \mathbb{R}$.*

Proof: Extend ψ to an odd function on $\mathbb{R}$. Then ψ is odd and 2-periodic, so we can expand it into a sine series $\psi(x) = \sum_{k=1}^{\infty} c_k\beta(kx)$. Since $\{\psi(nx)\}_{n\in\mathbb{N}^+}$ is an orthonormal basis, we have

$$\begin{aligned}
1 \ = \ \|\beta(mx)\|^2 \quad &= \quad \sum_{n=1}^{\infty} |\langle \beta(mx), \psi(nx)\rangle|^2 \\
&= \quad \sum_{n=1}^{\infty} |\langle \beta(mx), \sum_{k=1}^{\infty} c_k\beta(nkx)\rangle|^2 \\
&= \quad \sum_{n=1}^{\infty} \left| \sum_{k;\ kn=m} c_k \right|^2 \\
&= \quad \sum_{k|m} |c_k|^2.
\end{aligned}$$

Letting $m = 1$, we obtain $|c_1|^2 = 1$. Thus, for $m \geq 2$, we have $1 + \cdots + |c_m|^2 = 1$ (where the middle terms are non-negative) and so $|c_m| = 0$. $\square$

DEFINITION 6.2. Let $\{v_n\}$ be a set of vectors in a Hilbert space $\mathcal{H}$. We say that $\{v_n\}$ is a *Riesz basis*, if $\overline{\text{span}}\ \{v_n\} = \mathcal{H}$ and the *Gram matrix* G given by

$$G_{ij} := \langle v_j, v_i\rangle$$

is bounded and bounded below, that is, for all $\{a_n\}_{n=1}^\infty \in \ell^2$:

$$c_1 \sum_{j=1}^\infty |a_j|^2 \;\le\; \sum_{i,j=1}^\infty a_i \overline{a}_j G_{ij} \;\le\; c_2 \sum_{j=1}^\infty |a_j|^2. \tag{6.3}$$

PROPOSITION 6.4. *The set* $\{v_n\}_{n=1}^\infty$ *is a Riesz basis if and only if the map*

$$T : \sum_{n=1}^\infty a_n e_n \;\mapsto\; \sum_{n=1}^\infty a_n v_n$$

is bounded and invertible, where $\{e_n\}$ *is an orthonormal basis for* $\mathcal{H}$.

Proof: We have

$$\left\| T \sum_n a_n e_n \right\|^2 \;=\; \left\| \sum_n a_n v_n \right\|^2 \;=\; \sum_{m,n} a_n \overline{a}_m G_{mn}$$

and

$$\left\| \sum_n a_n e_n \right\|^2 = \sum_n |a_n|^2.$$

Thus condition (6.3) is equivalent to boundedness of T from below and above. Moreover, T is onto if and only if the span of $\{v_n\}$ is dense in $\mathcal{H}$. The claim follows by recalling that a map is invertible if and only if it is bounded, bounded from below, and onto. □

Here is Beurling's question.

QUESTION 6.5. **(Beurling)** For which odd 2-periodic functions $\psi : \mathbb{R} \to \mathbb{C}$ does the sequence $\{\psi(nx)\}_{n=1}^\infty$ form a Riesz basis for $L^2([0;1])$?

REMARK 6.6. A frame is a set of vectors $\{v_n\}_{n=1}^\infty$ in $\mathcal{H}$ such that for some $c_1, c_2 > 0$

$$c_1 \|v\|^2 \;\le\; \sum_{n=1}^\infty |\langle v, v_n \rangle|^2 \;\le\; c_2 \|v\|^2$$

holds for every $v \in \mathcal{H}$. (Unlike a Riesz basis, they do not need to be linearly independent).

The following problem attracted a lot of attention; it has many equivalent reformulations.

CONJECTURE 6.7. **(Feichtinger)** Suppose that $\{v_n\}_{n=1}^\infty$ is a set of unit vectors in $\mathcal{H}$ that form a frame. Does it follow that $\{v_n\}_{n=1}^\infty$ is a finite union of Riesz bases?

The conjecture was proved, in the affirmative, by A. Marcus, D. Spielman and N. Srivastava [**MSS15**].

Beurling's idea was to consider the Hilbert space of Dirichlet series

$$\mathcal{H}^2 := \left\{ \sum_{n=1}^{\infty} a_n n^{-s} : \sum_n |a_n|^2 < \infty \right\}. \tag{6.8}$$

Let us first observe that for any $f \in \mathcal{H}^2$ we have $\sigma_a \leq \frac{1}{2}$. Indeed, by the Cauchy-Schwarz inequality,

$$\left| \sum_n a_n n^{-s} \right| \leq \left(\sum_n |a_n|^2 \right)^{\frac{1}{2}} \left(\sum_n n^{-2\sigma} \right)^{\frac{1}{2}} < \infty,$$

whenever $2\sigma > 1$. In fact, the above estimate shows that for any $s_0 \in \Omega_{1/2}$, the map $\mathcal{H}^2 \ni f \mapsto f(s_0)$ is a bounded linear functional. Therefore it is given by the inner product with a function $k_{s_0} \in \mathcal{H}^2$, the so-called *reproducing kernel* at s_0, i.e.,

$$f(s_0) = \langle f, k_{s_0} \rangle \qquad \text{for all } f \in \mathcal{H}^2.$$

For any Hilbert (or Banach) space of analytic functions $\mathcal{X}$, we define its *multiplier algebra* by

$$\mathrm{Mult}\,(\mathcal{X}) = \big\{ \varphi;\ \varphi f \in \mathcal{X},\ \forall f \in \mathcal{X} \big\}.$$

It is easy to check that the following hold

- $1 \in \mathcal{X} \implies \mathrm{Mult}\,(\mathcal{X}) \subset \mathcal{X}$,
- $\mathrm{Mult}\,(\mathcal{X})$ is an algebra.

Clearly, multiplication by k^{-s} is isometric on $\mathcal{H}^2$, for all $k \in \mathbb{N}$. Consequently, every finite Dirichlet series lies in $\mathrm{Mult}\,(\mathcal{H}^2)$.

Also, note that $\sup_{s \in \Omega_{1/2}} |k^{-s}| = k^{-\frac{1}{2}} \to 0$ as k tends to ∞. Thus, $\|f\|_{\mathrm{Mult}\,(\mathcal{H}^2)} \not\lesssim \|f\|_{H^\infty(\Omega_{1/2})}$.

The following result — multiplication operators are bounded if they are everywhere defined — is true in great generality (see Section 11.4).

Proposition 6.9. *The multiplication operator M_φ is bounded on $\mathcal{H}^2$ for every $\varphi \in \mathrm{Mult}\,(\mathcal{H}^2)$.*

Proof: Multiplication operators on a Banach space of functions in which norm convergence implies pointwise convergence (or at least a.e. convergence) are easily seen to be closed. Indeed, suppose that $f_n \to f$ and $M_\varphi f_n \to g$. Then, for every $s \in \Omega_{1/2}$, $f_n(s) \to f(s)$ and so $(M_\varphi f_n)(s) = \varphi(s) f_n(s) \to \varphi(s) f(s) = (M_\varphi f)(s)$. On the other hand, $M_\varphi f_n(s) \to g(s)$, for all $s \in \Omega_{1/2}$. We conclude that $(M_\varphi f)(s) = g(s)$ for all $s \in \Omega_{1/2}$ and hence $M_\varphi f = g$. Thus, M_φ is closed. Hence, M_φ is an everywhere defined closed linear operator on a Banach space, and the closed graph theorem states that such operators are necessarily bounded. $\square$

Now let ψ be an odd 2-periodic function on $\mathbb{R}$. We can expand it into a Fourier series $\psi(x) = \sum_{n=1}^{\infty} c_n \beta(nx)$. The sequence $\{\psi(kx)\}_{k\in\mathbb{N}^+}$ is a Riesz basis, if and only if it spans L^2 and the operator $T : \sum_k a_k \beta(kx) \mapsto \sum_k a_k \psi(kx)$ is bounded and bounded below. Denote $\psi_k(x) := \psi(kx)$ and analyze the condition on T:

$$\begin{aligned}\Big\| \sum_k a_k \psi_k \Big\|^2 &= \Big\| \sum_k a_k \sum_n c_n \beta(nkx) \Big\|^2 \\ &= \Big\langle \sum_{k,n} a_k c_n \beta(nkx), \sum_{j,m} a_j c_m \beta(mjx) \Big\rangle \\ &= \sum_{k,n,j,m;\ kn=jm} a_k c_n \overline{a}_j \overline{c}_m,\end{aligned}$$

and thus we want

$$\sum_{k,n,j,m;\ kn=jm} a_k c_n \overline{a}_j \overline{c}_m \approx \sum_k |a_k|^2. \tag{6.9}$$

Let us define auxilliary functions in $\mathcal{H}^2$:

$$g(s) := \sum_n c_n n^{-s}, \qquad f(s) := \sum_k a_k k^{-s}.$$

We have

$$\|gf\|^2_{\mathcal{H}^2} = \Big\langle \sum_{n,k} c_n a_k (nk)^{-s}, \sum_{m,j} c_m a_j (mj)^{-s} \Big\rangle = \sum_{k,n,j,m;\ kn=jm} a_k c_n \overline{a}_j \overline{c}_m,$$

and so the condition (6.9) holds, if and only if $\|gf\|_{\mathcal{H}^2} \approx \|f\|_{\mathcal{H}^2}$, i.e., when M_g is bounded and bounded below.

Let us also look the density of the span of $\{\psi_n\}_n$. It is equivalent to

$$\begin{aligned}\overline{\text{span}} \Big\{ \sum_n c_n \beta(nkx) \Big\}_{k\in\mathbb{N}^+} = L^2([0;1]) \quad &\Longleftrightarrow \quad \overline{\text{span}} \Big\{ \sum_n c_n e_{nk} \Big\}_{k\in\mathbb{N}^+} = \ell^2(\mathbb{N}) \\ &\Longleftrightarrow \quad \overline{\text{span}} \Big\{ \sum_n c_n (nk)^{-s} \Big\}_{k\in\mathbb{N}^+} = \mathcal{H}^2 \\ &\Longleftrightarrow \quad \overline{\text{span}} \Big\{ k^{-s} g(s) \Big\}_{k\in\mathbb{N}^+} = \mathcal{H}^2.\end{aligned}$$

The last condition implies that range of M_g is dense. But since M_g is bounded below, it has a closed range and thus is onto. Therefore M_g is invertible, or $M_{1/g}$ is bounded. Conversely, if M_g is invertible, the image of the dense set span $\{k^{-s}\}_{k\in\mathbb{N}^+}$ is dense, and so the density of span $\{\psi_k\}_k$ follows by the above equivalences. We have proved:

PROPOSITION 6.10. *Let $\psi(x) = \sum_{n=1}^{\infty} c_n \beta(nx)$ be a odd 2-periodic function on $\mathbb{R}$. Then $\{\psi(kx)\}_{k \in \mathbb{N}^+}$ is a Riesz basis, if and only if both g and $1/g$ are multipliers of $\mathcal{H}^2$, where $g(s) = \sum_{n=1}^{\infty} c_n n^{-s}$.*

In view of Proposition 6.10, Beurling's question 6.5 would be answered if we could answer the following question:

QUESTION 6.11. What are the multipliers of $\mathcal{H}^2$?

6.2. Reciprocals of Dirichlet Series

PROPOSITION 6.12. *If $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is a Dirichlet series that converges somewhere and satisfies $a_1 \neq 0$, then $g(s) = \frac{1}{f(s)}$ is also given by the sum of a somewhere-convergent Dirichlet series. Moreover, $\sigma_b(g) = \inf\{\rho : \inf |f| \big|_{\Omega_\rho} > 0\}$.*

Proof: By rescaling, we may assume that $a_1 = 1$, and by shifting the series so that $\sigma_a < 0$, we have $\sup |a_n| \leq M$. We will construct the coefficients b_k of g inductively. Clearly, $b_1 = 1$. For $n \geq 2$, we have

$$0 \;=\; \widehat{fg}(n) \;=\; \sum_{k|n} a_{n/k} b_k. \tag{6.13}$$

Equations (6.13) can be solved for b_k, first when k is a prime, then a power of a prime, then when k has two distinct prime factors, and so on.

Claim: If $n = p_1^{i_1} \dots p_r^{i_r}$, then $|b_n| \leq n^2 M^{|i|}$.

Proof: For $n = 1$, $b_n = 1$ and so the claim holds. Assume inductively that the claim holds for all $m < n$. By (6.13), we have

$$\begin{aligned}
|b_n| &\leq \sum_{k|n,\ k\geq 2} |a_k b_{n/k}| \\
&\leq M \sum_{k \geq 2} b_{n/k} \\
&\leq M \sum_{k\geq 2} \left(\frac{n}{k}\right)^2 M^{|i|-1} \\
&\leq M^{|i|} n^2 \sum_{k \geq 2} \frac{1}{k^2} \\
&= M^{|i|} n^2 \left(\frac{\pi^2}{6} - 1\right),
\end{aligned}$$

and the claim follows, since $\frac{\pi^2}{6} < 2$.

Since $|i| \leq \log_2 n$, we obtain

$$\begin{aligned} |b_n| &\leq M^{|i|}n^2 \\ &\leq M^{\log_2 n}n^2 \\ &= n^{\log_2 M}n^2 \\ &= n^{2+\log_2 M}. \end{aligned}$$

Hence, for Re $s > 3 + \log_2 M$, the Dirichlet series $\sum_n b_n n^{-s}$ converges absolutely.

Now, g is bounded in Ω_ρ, if and only if $\inf |f|\big|_{\Omega_\rho} > 0$. As g is given by a convergent Dirichlet series in $\Omega_{3+\log_2 M}$, by Theorem 5.4,

$$\sigma_b(g) \leq \inf\{\rho : \ \inf |f|\big|_{\Omega_\rho} > 0\}.$$

The reverse inequality is obvious. $\square$

Note that the condition $a_1 \neq 0$ is necessary, since $a_1 = \lim_{\sigma\to\infty} f(\sigma)$.

6.3. Kronecker's Theorem

Theorem 6.14. **(Kronecker)**

(1) *Let* $\theta_1, \ldots, \theta_k \in \mathbb{R}$ *be linearly independent over* $\mathbb{Q}$*, and let* $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$*,* $T, \varepsilon > 0$ *be given. Then there exist* $t > T$ *and* $q_1, \ldots, q_k \in \mathbb{Z}$ *such that*

$$|t\theta_j - \alpha_j - q_j| < \varepsilon, \ 1 \leq j \leq k.$$

(2) *Let* $1, \theta_1, \ldots, \theta_k \in \mathbb{R}$ *be linearly independent over* $\mathbb{Q}$*, and let* $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$*,* $T, \varepsilon > 0$ *be given. Then there exist* $\mathbb{N} \ni n > T$ *and* $q_1, \ldots, q_k \in \mathbb{Z}$ *such that*

$$|n\theta_j - \alpha_j - q_j| < \varepsilon, \ 1 \leq j \leq k.$$

Proof: (1) $\Longrightarrow$ (2): Assume that all θ_j's lie in $(-M, M)$. Fix $0 < \varepsilon < 1$, and apply (1) to the $(k+1)$-tuples $\theta_1, \ldots, \theta_k, 1$ and $\alpha_1, \ldots, \alpha_k, 0$, $T = N + 1$ and $\varepsilon/(M+1)$. Let $n = q_{k+1}$, then $|t - n| < \varepsilon/(M+1)$. Thus, for $1 \leq j \leq k$, we have

$$\begin{aligned} |n\theta_j - \alpha_j - q_j| &\leq |n-t|\theta_j + |t\theta_j - \alpha_j - q_j| \\ &< \frac{M\varepsilon}{M+1} + \frac{\varepsilon}{M+1}. \end{aligned}$$

To prove (1), define $F(t) := 1 + \sum_{j=1}^k e^{2\pi i[\theta_j t - \alpha_j]}$. We need to show that $\limsup_{t\to\infty} |F(t)| = k + 1$. Fix $m \in \mathbb{N}$, and define $\alpha = (0, \alpha_1, \ldots, \alpha_k)$, $\theta = (0, \theta_1, \ldots, \theta_k)$ and $j = (j_0, \ldots, j_k)$. Then

$$[F(t)]^m = \sum_{|j|=j_0+\cdots+j_k=m} a_j e^{2\pi i t\gamma_j},$$

where $a_j = \frac{m!}{j!} e^{-2\pi i j\cdot\alpha}$ and $\gamma_j = j\cdot\theta$. Indeed, there are $\frac{m!}{j!}$ ways to get $\prod_l e^{2\pi i t j_l \theta_l}$ in the product, and, by independence of θ_j's over $\mathbb{Q}$, distinct j's yield distinct γ_j's. Also, $\sum_{|j|=m} |a_j| = (k+1)^m$, since there are $(k+1)$ terms, each with a coefficient of modulus 1.

Suppose that $\limsup_{t\to\infty} F(t) < k+1$. Then there exist $M > 0$ and $\lambda < k+1$ such that $|F(t)| \le \lambda$ for all $t > M$. Consequently,

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T |F(t)|^m \, dt \;\le\; \lambda^m.$$

Since $[F(t)]^m$ is a finite combination of exponentials,

$$\begin{aligned} |a_j| &= \left| \lim_{T\to\infty} \frac{1}{T}\int_0^T [F(t)]^m e^{-2\pi i t\gamma_j} \, dt \right| \\ &\le \limsup_{T\to\infty} \frac{1}{T}\int_0^T |F(t)|^m \, dt \\ &\le \lambda^m. \end{aligned} \tag{6.15}$$

Note that there are $\binom{m+k}{k} \le (m+1)^k$ possible j's. Thus, summing the inequality (6.15) over all j's yields

$$\begin{aligned} (k+1)^m &= \sum_{|j|=m} |a_j| \\ &\le (m+1)^k \lambda^m, \end{aligned}$$

a contradiction for large m. $\square$

REMARK 6.16. Let $q_1, \ldots, q_k$ be distinct primes. Then $\log q_1, \ldots, \log q_k$ are linearly independent over $\mathbb{Q}$.

Proof: If not, then for some rational numbers $r_1, \ldots, r_k$ we have $\sum_j r_j \log q_k = 0$ and by clearing the denominators, there exists integers $n_1, \ldots, n_k$ so that

$$\sum_k n_k \log q_k = 0 \implies \prod_k q_k^{n_k} = 1.$$

Thus all n_k's must be zero by the uniqueness of prime factorization. $\square$

6.4. Power series in infinitely many variables

Recall from (5.8) that given $f \in \mathcal{H}^2$, $f = \sum_{n=1}^\infty a_n n^{-s}$, we have a formal power series in infinitely many variables

$$(\mathcal{Q}f)(z) = \sum_{n=1}^\infty a_n z^{r(n)}.$$

Let $\mathbb{D}^\infty$ denote $\{(z_i)_{i=1}^\infty;\ |z_i| < 1\}$ — the *infinite polydisk*.

PROPOSITION 6.17. *If $f \in \mathcal{H}^2$ and $z \in \mathbb{D}^\infty \cap \ell^2$, then $(\mathcal{Q}f)(z)$ is well-defined.*

Proof: Using the Cauchy-Schwarz inequality, we obtain

$$|\mathcal{Q}f(z)|^2 \ \le \ \left(\sum_{n=1}^\infty |a_n|^2\right)\left(\sum_{n=1}^\infty |z|^{2r(n)}\right).$$

For $z \in \mathbb{D}^\infty$, observe that the map $n \mapsto \psi_z(n) := z^{[n]}$ is multiplicative and satisfies $|\psi_z(n)| \le 1$, for all $n \in \mathbb{N}^+$. It follows that

$$\begin{aligned} \sum_{n=1}^\infty \left|z^{r(n)}\right|^2 &= \prod_{i=1}^\infty \frac{1}{1-|z_i|^2} \\ &= \prod_{p\in\mathbb{P}} \frac{1}{1-|\phi(p)|^2}. \end{aligned}$$

Therefore

$$|\mathcal{Q}f(z)| \ \le \ \|f\|_{\mathcal{H}^2}\left[\prod_{i=1}^\infty \frac{1}{1-|z_i|^2}\right]^{1/2}.$$

This is finite if $z \in \mathbb{D}^\infty \cap \ell^2$. □

REMARK 6.18. A character on $(\mathbb{N}^+,\cdot)$ is a multiplicative map from $\mathbb{N}^+$ to $\mathbb{T}$. A *quasi-character* on $(\mathbb{N}^+,\cdot)$ is a multiplicative map from $\mathbb{N}^+$ to $\overline{\mathbb{D}}$. So ψ_z is a quasi-character.

Hilbert, in 1909, asked:

QUESTION 6.19. Does $\mathcal{Q}f(z)$ make sense on a larger set than $\mathbb{D}^\infty \cap \ell^2$?

This was his answer. Let $z = (z_1, z_2, \dots)$, and let $z_{(m)}$ denote $(z_1, \dots, z_m, 0, 0, \dots)$. Consider the sequence $F_m(z) := F(z_{(m)})$; this is called the m^{te}-Abschnitt (or cut-off). If $f \in \mathcal{H}^2$ and $F = \mathcal{Q}f$, then the functions F_m are well-defined on $\mathbb{D}^\infty$ by Proposition 6.17.

PROPOSITION 6.20. **(Hilbert)** *Suppose that there exists $C > 0$ such that*

$$|F_m(z)| \le C \quad \forall\ z\ \in\ \mathbb{D}^\infty,\ \forall\ m \in \mathbb{N}^+.$$

Then, for every $z \in \mathbb{D}^\infty \cap c_0$, the limit

$$\lim_{m\to\infty} F_m(z) =: F(z)$$

exists.

Proof: Fix $z \in \mathbb{D}^\infty \cap c_0$ and an $\varepsilon > 0$. Then, there exists $K \in \mathbb{N}$ such that $|z_k| < \frac{\varepsilon}{2C}$ holds for all $k > K$. Fix $n > m > K$, and consider the function $f \in H^\infty(\mathbb{D}^{n-m})$ given by

$$f(w_{m+1}, \ldots, w_n) \;:=\; F(z_1, \ldots, z_m, w_{m+1}, \ldots, w_n, 0, 0, \ldots).$$

Now, we apply the polydisk version of Schwarz's lemma, Lemma 11.2, to $g(w) := \frac{f(w)-f(0)}{2C}$. Since $g : \mathbb{D}^{n-m} \to \mathbb{D}$, we conclude that

$$|g(z_{m+1}, \ldots, z_n)| \;\leq\; \max_{i=m+1,\ldots,n} |z_i| \;<\; \frac{\varepsilon}{2C},$$

so that

$$|f(z) - f(0)| \;\leq\; \frac{\varepsilon}{2C} \cdot 2C = \varepsilon.$$

Thus, the sequence $\{F_m(z)\}_m$ is Cauchy. □

Definition 6.21. We define $H^\infty(\mathbb{D}^\infty)$ by

$$H^\infty(\mathbb{D}^\infty) \;:=\; \left\{ F(z) = \sum_{n=1}^{\infty} a_n z^{r(n)} \;:\; |F_m(z)| \leq C,\ \forall\, m \in \mathbb{N}, z \in \mathbb{D}^\infty \right\}. \tag{6.22}$$

The norm of $F \in H^\infty(\mathbb{D}^\infty)$ is the smallest C that satisfies the inequality in (6.22).

6.5. Besicovitch's Theorem

Definition 6.23. (1) Let $f \in \mathrm{Hol}\,(\Omega_\rho)$, let $\varepsilon > 0$. We say that $\tau \in \mathbb{R}$ is an *ε-translation number* of f, if

$$\sup_{s \in \Omega_\rho} |f(s + i\tau) - f(s)| < \varepsilon.$$

We shall let $E(\varepsilon, f)$ denotes the set of ε-translation numbers of f.

(2) A set $S \subset \mathbb{R}$ is called *relatively dense*, if there exists $L < \infty$ such that each interval of length L contains at least one element of S.

(3) A function $f \in \mathrm{Hol}\,(\Omega_\rho)$ is *uniformly almost periodic* in Ω_ρ, if for all $\varepsilon > 0$, the set of ε-translation numbers of f is relatively dense.

Example 6.24. The function $f(s) = 2^{-s} + 3^{-s}$ is uniformly almost periodic in the half-plane Ω_ρ for every $\rho \in \mathbb{R}$.

It follows from Kronecker's theorem that for every $\varepsilon > 0$ there exists an arbitrarily large ε-translation number. Indeed, let $\theta_1 = \frac{\log 2}{2\pi}$,

$\theta_2 = \frac{\log 3}{2\pi}$ and $\alpha_1 = \alpha_2 = 0$. Then there exists an arbitrarily large $\tau \in \mathbb{R}$ so that

$$\text{dist}\left\{\frac{\tau \log 2}{2\pi}, \mathbb{Z}\right\} < \varepsilon \quad \text{and} \quad \text{dist}\left\{\frac{\tau \log 3}{2\pi}, \mathbb{Z}\right\} < \varepsilon.$$

Thus,

$$\begin{aligned} |2^{-(s+i\tau)} - 2^{-s}| &= |2^{-s}(e^{-i\tau \log 2} - 1)| \\ &\leq 2^{-\rho} 2\pi(\log 2) \text{ dist}\left\{\frac{\tau \log 2}{2\pi}, \mathbb{Z}\right\} \\ &< C\varepsilon. \end{aligned}$$

Similarly, one obtains

$$|3^{-(s+i\tau)} - 3^{-s}| \leq 3^{-\rho} 2\pi(\log 3) \text{ dist } \{\frac{\tau \log 3}{2\pi}, \mathbb{Z}\} < C\varepsilon.$$

There exists a refined version of Kronecker's theorem that implies that the ε-translation numbers of f are relatively dense, so f is uniformly almost periodic. However, the claim also follows from Corollary 6.28 below.

THEOREM 6.25. **(Besicovitch)** *Suppose $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ and the series converges uniformly in Ω_ρ. Then f is uniformly almost periodic.*

LEMMA 6.26. *Suppose f is uniformly almost periodic and uniformly continuous in Ω_ρ, and let $0 < \varepsilon_1 < \varepsilon_2$ be arbitrary. Then there exists a $\delta > 0$ such that for each $\tau \in E(\varepsilon_1, f)$, the inclusion $(\tau - \delta, \tau + \delta) \subset E(\varepsilon_2, f)$ holds.*

Proof: Let $\delta > 0$ be such that for every $0 < \delta' < \delta$ and $z \in \Omega_\rho$,

$$|f(z + i\delta') - f(z)| < \varepsilon_2 - \varepsilon_1.$$

For any $\tau' \in (\tau - \delta, \tau + \delta)$, write $\tau' = \tau + \delta'$ with $0 < |\delta'| < \delta$. Then the inequality

$$\begin{aligned} |f(z + i\tau') - f(z)| &\leq |f(z + i(\tau + \delta') - f(z + i\tau)| + |f(z + i\tau) - f(z)| \\ &< (\varepsilon_2 - \varepsilon_1) + \varepsilon_1 \;=\; \varepsilon_2 \end{aligned}$$

holds. □

LEMMA 6.27. *Let $\varepsilon, \delta > 0$ and let f_1, f_2 be uniformly almost periodic and uniformly continuous functions. Then the set*

$$P = \{\tau \in E(\varepsilon, f_1) : \text{ dist } (\tau, E(\varepsilon, f_2)) < \delta\}$$

is relatively dense.

Proof: For a uniformly almost periodic function f and $\varepsilon > 0$, let $L(\varepsilon, f)$ denote the infimum of those $L > 0$ such that any interval of length L contains an ε-translation number of f. Choose $K \in \mathbb{N}$ so that $L = \delta K$ is greater than $\max\{L(\frac{\varepsilon}{2}, f_1), L(\frac{\varepsilon}{2}, f_2)\}$. Write

$$\mathbb{R} \;=\; \bigcup_{n\in\mathbb{Z}} \big[(n-1)L, nL\big) \;=\; \bigcup_{n\in\mathbb{Z}} I_n.$$

In each I_n there exist $\tau_1^{(n)} \in E(\frac{\varepsilon}{2}, f_1)$ and $\tau_2^{(n)} \in E(\frac{\varepsilon}{2}, f_2)$ and clearly $-L < \tau_1^{(n)} - \tau_2^{(n)} \leq L$. Decompose $[-L, L)$ into $2K$ disjoint intervals J_l of length δ. Since this is a finite number, there exists $n_0 \in \mathbb{N}$ such that if any interval J_l contains some point in the set $\{\tau_1^{(n)} - \tau_2^{(n)}\}_{n\in\mathbb{Z}}$, then it contains a point in the set $\{\tau_1^{(n)} - \tau_2^{(n)}\}_{n=-n_0}^{n_0}$. Thus, for any $n \in \mathbb{Z}$, there exists $n' \in \{-n_0, \ldots, n_0\}$ such that

$$\left|(\tau_1^{(n)} - \tau_2^{(n)}) - (\tau_1^{(n')} - \tau_2^{(n')})\right| \;<\; \delta.$$

Equivalently,

$$\tau \;:=\; \left(\tau_1^{(n)} - \tau_1^{(n')}\right) \;=\; \left(\tau_2^{(n)} - \tau_2^{(n')}\right) + \theta\delta,$$

with $|\theta| < 1$. By the triangle inequality, this implies that τ lies in $E(\varepsilon, f_1)$, and is closer than δ to an element of $E(\varepsilon, f_2)$, namely $(\tau_2^{(n)} - \tau_2^{(n')})$. In other words, $\tau \in P$.

We will now show that P is relatively dense. Consider an arbitrary interval I of length $(2n_0 + 3)L$ and find the integer n for which $\tau_1^{(n)}$ is closest to the center of I. Then the distance of $\tau_1^{(n)}$ from the center of I is at most L. Find the corresponding n' and τ, and conclude that

$$|\tau - \tau_1^{(n)}| \;=\; |\tau_1^{(n')}| \;\leq\; n_0 L.$$

This means that τ lies in I, and so the set P intersects every interval of length $(2n_0 + 3)L$. □

Corollary 6.28. *Let f_1 and f_2 be both uniformly almost periodic and uniformly continuous. Then $f_1 + f_2$ is also uniformly almost periodic.*

Proof: Fix $\varepsilon > 0$, and apply Lemma 6.26 to $f = f_2$, $\varepsilon_1 = \frac{\varepsilon}{3}$ and $\varepsilon_2 = \frac{2\varepsilon}{3}$. We obtain $\delta > 0$ such that $\{\tau : \text{ dist } (\tau, E(\frac{\varepsilon}{3}, f_2) < \delta\} \subseteq E(\frac{2\varepsilon}{3}, f_2)$. Now apply Lemma 6.27 to conclude that

$$\{\tau \in E(\tfrac{\varepsilon}{3}, f_1) : \text{ dist } (\tau, E(\tfrac{\varepsilon}{3}, f_2)) < \delta\}$$

is relatively dense. But, by the triangle inequality, any τ in the above set is an ε-translation number for $f_1 + f_2$. □

Proof: (of Theorem 6.25) Since a finite Dirichlet series is uniformly continuous, it follows inductively from Corollary 6.28 that it is also uniformly almost periodic. Therefore it is sufficient to prove that the uniform limit of uniformly almost periodic functions is also uniformly almost periodic.

Fix $\varepsilon > 0$. Find N so that $\|f_n - f\|_\infty < \varepsilon/3$ holds for all $n \geq N$. Then any $\varepsilon/3$-translation number τ of f_N is an ε-translation number of f, since

$$\begin{aligned} |f(z+\tau) - f(z)| &\leq |f(z+\tau) - f_N(z+\tau)| + |f_N(z+\tau) - f_N(z)| + |f_N(z) - f(z)| \\ &< \varepsilon \qquad \forall\, z. \end{aligned}$$

□

6.6. The spaces $\mathcal{H}^2_w$

Definition 6.29. Let $w = \{w_n\}_{n=1}^\infty$ be a sequence of positive real numbers which are in this context called a *weight*. Define the Hilbert space $\mathcal{H}^2_w$ of Dirichlet series by

$$\mathcal{H}^2_w \ := \left\{ \sum_n a_n n^{-s} : \ \sum_n |a_n|^2 w_n < \infty \right\}.$$

Remark 6.30. Note that if $f \in \mathcal{H}^2_w$, then f' is in the space with weights $w_n(\log n)^2$.

One way to obtain interesting weights is from measures on the positive real axis. Let μ be a positive Radon measure on $[0,\infty)$ such that

$$0 \in \operatorname{supp} \mu \tag{6.31}$$

$$\textstyle\int_0^\infty 4^{-\sigma} \ d\mu(\sigma) < \infty. \tag{6.32}$$

We define the weight sequence by

$$w_n \ := \ \int_0^\infty n^{-2\sigma} \ d\mu(\sigma). \tag{6.33}$$

One example of course is when μ is the Dirac measure at 0 denoted by δ_0, and all the weights are 1, giving $\mathcal{H}^2$. Here is another class.

Example 6.33. For each $\alpha < 0$, define μ_α on $[0,\infty)$ by

$$d\mu_\alpha(\sigma) \ = \ \frac{2^{-\alpha}}{\Gamma(-\alpha)} \sigma^{-1-\alpha} d\sigma.$$

Then for each $n \geq 2$, we have from (6.33)

$$w_n \ = \ (\log n)^\alpha. \tag{6.34}$$

Since w_1 is infinite, it is convenient to assume that sums $\sum_n a_n n^{-s}$ start at $n = 2$ when dealing with these spaces.

REMARK 6.35. On the unit disk, one can define spaces H^2_w by

$$H^2_w \ := \left\{ \sum_n a_n z^n : \ \sum_n |a_n|^2 w_n < \infty \right\}. \tag{6.36}$$

A special case is when

$$w_n \ = \ (n+1)^\alpha.$$

Then $\alpha = 0$ corresponds to the Hardy space, $\alpha = -1$ to the Bergman space, and $\alpha = 1$ to the Dirichlet space, the space of functions whose derivatives are in the Bergman space. The theory of the Hardy space on the disk is fairly well-developed – see *e.g.* [**Koo80, Dur70**] for a first course, or [**Nik85**] for a second. The Bergman space (and the other spaces with $\alpha < 0$ in this scale, that all come from L^2-norms of radial measures) is more complicated — see *e.g.* [**DS04, HKZ00**]. The Dirichlet space on the disk is even more complicated analytically, though it does have the complete Pick property. See *e.g.* [**EFKMR14**].

This section should be seen as an attempt to continue the analogy of Remark 6.35. The case $\alpha = 0$ in (6.34) we think of as a Hardy-type space, and the case $\alpha = -1$ in (6.34) we think of as a Bergman-type space. When $\alpha > 0$, we can still define weights by (6.34), though they do not come from a measure as in (6.33). By Remark 6.30, we can think of $\alpha = 1$, for example, as the space of functions whose first derivatives lie in the space with $\alpha = -1$. This would render this space a "Dirichlet space" of Dirichlet series, which is perhaps a surfeit of Dirichlet.

If the weights are defined by (6.33), then, for every $\varepsilon > 0$,

$$\begin{aligned} w_n \quad &\geq \quad \int_0^\varepsilon n^{-2\sigma} \, d\mu \\ &\geq \quad \mu([0,\varepsilon]) n^{-2\varepsilon}, \end{aligned} \tag{6.37}$$

and, consequently, the weight sequence cannot decrease to 0 very fast.

PROPOSITION 6.38. *Suppose w_n is a weight sequence that is bounded below by $n^{-2\varepsilon}$ for every $\varepsilon > 0$. Then for any $f \in \mathcal{H}^2_w$, we have $\sigma_a(f) \leq \frac{1}{2}$.*

Proof: Take $\sigma > \frac{1}{2}$, and choose $\varepsilon > 0$ such that $\sigma - \varepsilon > \frac{1}{2}$. Then, by the Cauchy-Schwarz inequality,

$$\begin{aligned} \sum_n |a_n| n^{-\sigma} \quad &= \quad \sum_n |a_n n^{-\varepsilon}| n^{-(\sigma-\varepsilon)} \\ &\leq \quad \left(\sum_n |a_n|^2 n^{-2\varepsilon} \right)^{\frac{1}{2}} \left(\sum n^{-2(\sigma-\varepsilon)} \right)^{\frac{1}{2}}. \end{aligned}$$

The first term is finite by (6.37), and the second since $2(\sigma - \varepsilon) > 1$. $\square$

The following theorem, in the case that $\mu = \delta_0$, is due to F. Carlson [**Car22**]. If $w_1 < \infty$, we assume that the Dirichlet series for f starts at $n = 1$; if w_1 is infinite, we start the series at $n = 2$ (Condition (6.32) says that $w_2 < \infty$).

THEOREM 6.39. *Let μ satisfy (6.31) and (6.32), and define w_n by (6.33). Assume that $f = \sum_n a_n n^{-s}$ has $\sigma_b(f) \leq 0$. Then*

$$\sum_n |a_n|^2 w_n \; = \; \lim_{c \to 0+} \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \int_0^\infty |f(s+c)|^2 \, d\mu(\sigma) \, dt. \qquad (6.40)$$

Moreover, if $\mu(\{0\}) = 0$, then the right-hand side becomes

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \int_0^\infty |f(s)|^2 \, d\mu(\sigma) \, dt.$$

Proof: Fix $0 < c < 1$, and let $0 < \varepsilon < 1$. Define δ by

$$\delta \; = \; \frac{\varepsilon}{(1 + \mu[0, \frac{1}{c}])(1 + 2\|f\|_{\Omega_c})}.$$

Since the Dirichlet series of f converges uniformly in $\overline{\Omega_c}$, there exists N such that

$$|\sum_{n \leq N'} a_n n^{-s} - f(s)| \; < \; \delta, \quad \forall \, s \in \overline{\Omega_c}, \; \forall \, N' > N.$$

Then

$$\begin{aligned} \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \int_0^{1/c} |f(s+c)|^2 \, d\mu(\sigma) \, dt \;&=\; \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \int_0^{1/c} |\sum_{n \leq N'} a_n n^{-s-c}|^2 \, d\mu(\sigma) \, dt + O(\varepsilon) \\ &=\; \sum_{n \leq N'} |a_n|^2 \int_0^{1/c} n^{-2\sigma - 2c} \, d\mu(\sigma) + O(\varepsilon) \end{aligned}$$

Let N' tend to infinity, and c tend to 0, to get that the difference between the left and right sides of (6.40) are at most ε; since this is arbitrary, the two sides must be equal.

As $\lim_{T \to \infty} \fint_{-T}^{T} |f(s+c)|^2 dt$ is monotonically increasing as $c \to 0^+$, the monotone convergence theorem proves the second part of the theorem. $\square$

In particular, if $d\mu = d\mu_{-1} = 2dm$, we obtain

$$\sum_n |a_n|^2 \frac{1}{\log n} \; = 2 \; \lim_{T \to \infty} \fint_{-T}^{T} \int_0^\infty |f(s)|^2 \, dm(\sigma) \, dt,$$

and for $\mu = \delta_0$, we get

$$\sum_n |a_n|^2 \;=\; \lim_{c\to 0+} \lim_{T\to\infty} \fint_{-T}^{T} |f(c+it)|^2 \, dt.$$

6.7. Multiplier algebras of $\mathcal{H}^2$ and $\mathcal{H}^2_w$

NOTATION 6.41. Let us denote by $\mathcal{D}$ the set of functions expressible as Dirichlet series which converge somewhere, that is,

$$\mathcal{D} \;:=\; \Big\{ f : \; \exists \rho \text{ such that } f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \text{ in } \Omega_\rho \Big\}.$$

Since $\sigma_a \leq \sigma_c + 1$, $\mathcal{D}$ is also the set of Dirichlet series that converge absolutely in some half-plane.

The following theorem is due to H. Hedenmalm, P. Lindqvist and K. Seip, in their ground-breaking paper [**HLS97**].

THEOREM 6.42. *Let* μ *and* $\{w_n\}$ *satisfy* (6.31) – (6.33). *Then* $\mathrm{Mult}\,(\mathcal{H}^2_w)$ *is isometrically isomorphic to* $H^\infty(\Omega_0) \cap \mathcal{D}$.

REMARK 6.43. Before we prove the theorem, note that it implies that the multiplier algebra is independent of the weight w. The situation is analogous to a similar phenomenon on the disk. For any sequence $w = \{w_n\}_{n=0}^{\infty}$, one can define a Hilbert space of holomorphic functions H^2_w by (6.36). If the sequence w comes from a radial positive Radon measure μ on $\overline{\mathbb{D}}$ such that $\mathbb{T} \subset \operatorname{supp} \mu$ as

$$w_n \;=\; \int_{\overline{\mathbb{D}}} |z|^{2n} \, d\mu(z),$$

then $\{w_n\}_n$ is non-increasing and, since the measure is radial, the sequence $\{z^n\}_{n\in\mathbb{N}}$ is an orthogonal basis of H^2_w. (Saying the measure is radial means $d\mu = d\theta d\nu(r)$ for some measure ν on $[0,1]$). Thus, the norm on H^2_w is given by integration:

$$\|f\|^2 \;=\; \int_{\overline{\mathbb{D}}} |f(z)|^2 \, d\mu(z).$$

For all these spaces,

$$\mathrm{Mult}\,(H^2_w) \;=\; H^\infty(\mathbb{D}), \tag{6.44}$$

the bounded analytic functions on the disk. Indeed, if μ is carried by the open disk, this follows from Proposition 11.9. If μ puts weight on the circle, the theorem is still true, and can most easily be seen by writing

$$\int_{\overline{\mathbb{D}}} |f(z)|^2 \, d\mu(z) \;=\; \lim_{r \nearrow 1} \int_{\overline{\mathbb{D}}} |f(rz)|^2 \, d\mu(z).$$

In particular, (6.44) holds for all the spaces with $w_n = (n+1)^\alpha$ for $\alpha \leq 0$.

REMARK 6.45. There exist many functions in $H^\infty(\Omega_0) \setminus \mathcal{D}$, for example $f(s) = \left(\frac{3}{2}\right)^{-s}$ and $g(s) = \frac{s}{(s+1)^2}$.

Before embarking on the proof of the theorem, recall the following fact. It is a version of the Phragmén-Lindelöf principle — a maximum modulus principle for unbounded domains. This particular version is known as the three line lemma.

LEMMA 6.46. *Let f be a bounded holomorphic function in $\{z \in \mathbb{C};\ a < Re\ z < b\}$, let $N(\sigma) := \sup_{t \in \mathbb{R}} |f(\sigma + it)|$. Then the function N is logarithmically convex, that is,*

$$N(\sigma) \;\leq\; N(a)^{\frac{b-\sigma}{b-a}} N(b)^{\frac{\sigma-a}{b-a}}.$$

Proof: See Theorem 12.8, p. 274 in [**Rud86**]. ☐

REMARK 6.47. The lemma does not hold without the assumption that f is bounded in the strip. Indeed, consider the function $f(z) = e^{e^{iz}}$. It is holomorphic in the strip $\{-\frac{\pi}{2} < \text{Re } z < \frac{\pi}{2}\}$, bounded on its boundary $\{|\text{Re } z| = \frac{\pi}{2}\}$, but $\lim_{t \to -\infty} f(it) = \infty$. However, one can weaken the assumption of boundedness of f to an appropriate restriction on the growth of f.

The following lemma is trivial if $1 \in \mathcal{H}^2_w$.

LEMMA 6.48. *Any multiplier of $\mathcal{H}^2_w$ lies in $\mathcal{D}$.*

PROOF: If φ belongs to Mult $(\mathcal{H}^2_w)$, then both $\varphi(s)2^{-s}$ and $\varphi(s)3^{-s}$ are in $\mathcal{D}$. So

$$\begin{aligned}\varphi(s)2^{-s} &= \sum a_n n^{-s} \\ \varphi(s)3^{-s} &= \sum b_n n^{-s}.\end{aligned}$$

Multiplying the first equation by 3^{-s} and the second by 2^{-s}, we conclude that a_n is zero when n is odd (and b_n is zero when n is not divisible by 3), so φ itself can be represented by an ordinary Dirichlet series. ☐

PROPOSITION 6.49. *Let $\varphi(s) = \sum_{n=1}^\infty b_n n^{-s}$ with $\sigma_b \leq 0$. Then $\|M_\varphi\| = \|\varphi\|_{\Omega_0}$.*

Proof: Let $f(s) = \sum_{n\le N} a_n n^{-s}$, then $\sigma_b(\varphi f) \le 0$. By Theorem 6.39,

$$\begin{aligned} \|\varphi f\|^2_{\mathcal{H}^2_w} &= \lim_{c\to 0+}\lim_{T\to\infty} \fint_{-T}^{T}\int_0^\infty |\varphi(s+c)|^2|f(s+c)|^2\, d\mu(\sigma)\, dt \\ &\le \|\varphi\|^2_{\Omega_0}\cdot\|f\|^2_{\mathcal{H}^2_w}. \end{aligned}$$

Hence M_φ is bounded on a dense subset of $\mathcal{H}^2_w$, and therefore extends to a bounded operator on all of $\mathcal{H}^2_w$, which must be multiplication by ϕ. (Why?) Also, the estimate above shows that $\|M_\varphi\| \le \|\varphi\|_{\Omega_0}$.

Conversely, assume that $\|M_\varphi\| = 1$ and $1 < \|\varphi\|_{\Omega_0}$ (possibly infinite). Let

$$N(\sigma) := \sup_{t\in\mathbb{R}} |\varphi(\sigma+it)|.$$

Clearly, $N(\sigma)\to|b_1|$ as $\sigma\to\infty$, and for any $\sigma>0$, we have

$$N^2(\sigma) \ \ge\ \lim_{T\to\infty}\fint_{-T}^{T}|\varphi(\sigma+it)|^2\, dt \ =\ \sum_{n=1}^\infty |b_n|^2 n^{-2\sigma} \ >\ |b_1|^2,$$

unless φ is a constant (in which case the Proposition is obvious). For any $0<a<b$ one can apply the three line lemma, 6.46, to conclude that $\log N$ is convex, so it must be convex on the half-line $(0,\infty)$. Since

$$\lim_{\sigma\to\infty}\log N(\sigma) \ =\ \log|b_1| \ <\infty,$$

we must have that $\log N$, and hence N, is a decreasing function on $(0,\infty)$.

For each $c>0$, $\sum_n b_n n^{-s}$ converges uniformly in $\overline{\Omega_c}$, and hence by Theorem 6.25, φ is uniformly continuous and uniformly almost periodic in this half-plane. Thus, there exist $\varepsilon_1,\varepsilon_2,\varepsilon_3$ and ε_4 positive such that

$$\left|\{t:\ |\varphi(\sigma+it)| \ \ge\ 1+\varepsilon_1,\ -T<t<T\}\right| \ \ge\ \varepsilon_2(2T) \tag{6.50}$$

holds for every sufficiently large $T>0$, and $\sigma\in(\varepsilon_3,\varepsilon_3+\varepsilon_4)$. Indeed, choose ε_3 so that $N(\varepsilon_3)>1$. Then there is some $\varepsilon_1>0$ and some rectangle R with non-empty interior,

$$R \ =\ \{\sigma+it:\varepsilon_3\le\sigma\le\varepsilon_3+\varepsilon_4,\ t_1\le t\le t_1+h\},$$

such that $|\varphi| > 1+2\varepsilon_1$ on R. By the definition of uniform almost periodicity, there exists some L such that every interval of length L contains an ε_1 translation number of φ. For $T>L$, every interval of length $2T$ contains at least $\frac{T}{L}$ disjoint sub-intervals of length L, so for any $\sigma\in[\varepsilon_3,\varepsilon_3+\varepsilon_4]$ the left-hand side of (6.50) is at least $\frac{T}{L}h$. Setting $\varepsilon_2 = \frac{h}{2L}$ yields the inequality (6.50).

Now, on one hand, we have

$$\|M_\varphi^j 2^{-s}\|_{\mathcal{H}^2_w} \;\leq\; \|M_\varphi\|^j \cdot \|2^{-s}\|_{\mathcal{H}^2_w} \;=\; \|2^{-s}\|_{\mathcal{H}^2_w},$$

so that this sequence of norms is bounded by $\sqrt{w_2}$. On the other hand,

$$\begin{aligned}\|M_\varphi^j 2^{-s}\|^2_{\mathcal{H}^2_w} \quad &\geq \quad \lim_{T\to\infty} \int_0^{\varepsilon_4} \fint_{-T}^{T} |2^{-(s+\varepsilon_3)} \varphi^j(s+\varepsilon_3)|^2 \, dt \, d\mu(\sigma) \\ &\geq \quad \mu([0,\varepsilon_4]) 2^{-2(\varepsilon_3+\varepsilon_4)} \varepsilon_2 (1+\varepsilon_1)^{2j},\end{aligned}$$

and this tends to infinity as j tends to ∞, a contradiction. $\square$

For later use, note that the proof of Proposition 6.49 shows:

LEMMA 6.51. *If $\varphi = \sum_{n=1}^\infty b_n n^{-s}$ satisfies $\sigma_b(\varphi) \leq 0$, and $\|\varphi\|_{\Omega_0} > 1$, then*

$$\sup_{j\in\mathbb{N}^+} \|M_\varphi^j 2^{-s}\| \;=\; \infty.$$

For $K \in \mathbb{N}^+$, define

$$\mathbb{N}_K \;=\; \{n = p_1^{r_1} \cdot \ldots \cdot p_K^{r_K};\; r_j \in \mathbb{N}\},$$

where, as usual, p_l is the l-th prime. Clearly, $n \in \mathbb{N}_K$, if and only if $p_l \not| n$ for all $l > K$. Let $Q_K : \mathcal{D} \to \mathcal{D}$ be the map defined by

$$Q_K\left(\sum_{n=1}^\infty a_n n^{-s}\right) \;=\; \sum_{n\in\mathbb{N}_K} a_n n^{-s}.$$

The map Q_K is well-defined, since if $\sum_{n=1}^\infty a_n n^{-s}$ converges absolutely in Ω_ρ, then so does $\sum_{n\in\mathbb{N}_K} a_n n^{-s}$.

We need the following observations.

LEMMA 6.52. *For any $K \in \mathbb{N}^+$, the map Q_K has the following properties:*

1. *The restriction of Q_K to $\mathcal{H}^2_w$ is the orthogonal projection onto $\overline{\text{span}}\, \{n^{-s} :\; n \in \mathbb{N}_K\}$.*
2. *For any $\varphi, f \in \mathcal{D}$, $Q_K(\varphi f) = (Q_K\varphi)(Q_K f)$.*
3. *If $\varphi \in \text{Mult}\,(\mathcal{H}^2_w)$, then $Q_K M_\varphi Q_K = M_{Q_K\varphi} Q_K = Q_K M_\varphi$.*

Proof: (1) follows immediately from the orthogonality of the functions $\{n^{-s}\}_{n\in\mathbb{N}^+}$.

(2) By linearity, we only need to check that $Q_K(n^{-s}m^{-s}) = Q_K(n^{-s})Q_K(m^{-s})$, for all $m, n \in \mathbb{N}^+$. This follows from the facts that if p is prime, then $p \not| nm$ if and only if $p \not| n$ and $p \not| m$, and

$$Q_K n^{-s} = \begin{cases} n^{-s}, \ p_l \not| n, \text{ for all } l > K, \\ 0, \text{ otherwise.} \end{cases}$$

(3) Let $f \in \mathcal{H}^2_w$, then, using (2), we get

$$\begin{aligned} Q_K M_\varphi Q_K f &= Q_K(\varphi Q_K f) \\ &= (Q_K\varphi)(Q_K^2 f) \\ &= Q_K(\varphi f). \end{aligned}$$

Also,

$$\begin{aligned} M_{Q_K\varphi} Q_K f &= (Q_K\varphi)(Q_K f) \\ &= Q_K(\varphi f). \end{aligned}$$

□

PROPOSITION 6.53. $\mathrm{Mult}\,(\mathcal{H}^2_w) \subset H^\infty(\Omega_0) \cap \mathcal{D}$.

Proof: Let $f = \sum a_n n^{-s} \in \mathcal{H}^2_w$, and fix $K \in \mathbb{N}^+$, $s \in \Omega_0$. Then

$$\begin{aligned} |Q_K f(s)| &= \left| \sum_{n \in \mathbb{N}_K} a_n n^{-s} \right| \\ &\leq \left[\sum_{n \in \mathbb{N}_K} n^{-\sigma} \right] \sup_{n \in \mathbb{N}_K} |a_n| \\ &= \left[\prod_{j=1}^{K} \frac{1}{1 - p_j^{-\sigma}} \right] \sup_{n \in \mathbb{N}_K} |a_n|. \end{aligned}$$

So, if $\sup_{n \in \mathbb{N}_K} |a_n|$ is finite, then $Q_K(f)$ is bounded in Ω_ρ for all $\rho > 0$. Since $\sum_n |a_n|^2 \omega_n$ converges, $\{|a_n|^2 \omega_n\}$ is bounded, and hence by (6.37), $|a_n| = O(n^\varepsilon)$ for all $\varepsilon > 0$. Thus, for any $\varepsilon > 0$, the Dirichlet series of $f_\varepsilon(s) := f(s + \varepsilon)$ has bounded coefficients. Consequently, $Q_K f_\varepsilon \in H^\infty(\Omega_\rho)$, which is the same as saying $Q_K f_{\varepsilon+\rho} \in H^\infty(\Omega_0)$. Since $\varepsilon > 0$ and $\rho > 0$ were arbitrary, we conclude that

$$Q_K f_\varepsilon \in H^\infty(\Omega_0), \quad \forall\, K \in \mathbb{N}^+,\ \varepsilon > 0,\ f \in \mathcal{H}^2_w.$$

Let φ be in $\mathrm{Mult}\,(\mathcal{H}^2_w)$. Then $\varphi 2^{-s} \in \mathcal{H}^2_w$, and so $2^{-s} Q_K(\varphi) = Q_K(\varphi 2^{-s}) \in H^\infty(\Omega_\varepsilon)$, for all $\varepsilon > 0$. Since we know $\varphi \in \mathcal{D}$ by Lemma 6.48 it follows that $\sigma_b(Q_K\varphi) \leq 0$.

By Lemma 6.51, applied to $Q_K\varphi$, we get

$$\|Q_K\varphi\|_{\Omega_0} \leq \|M_{Q_K\varphi}|_{Q_K\mathcal{H}^2_w}\|. \tag{6.54}$$

By Lemma 6.52,

$$\begin{aligned} \|M_{Q_K\varphi}|_{Q_K\mathcal{H}^2_w}\| &= \|Q_K M_\varphi Q_K\| \\ &\leq \|M_\varphi\|. \end{aligned}$$

So by (6.54),

$$\|Q_K\varphi\|_{\Omega_0} \leq \|M_\varphi\| \quad \forall\, K \in \mathbb{N}^+.$$

Using normal families, we conclude that some subsequence $Q_{K_l}\varphi$ converges to some function $\psi \in H^\infty(\Omega_0)$ uniformly on compact subsets of Ω_0. But $Q_K\varphi \to \varphi$ uniformly on compact subsets of $\Omega_{\sigma_u(\varphi)}$ and hence, $\varphi = \psi$ in $\Omega_{\sigma_u(\varphi)}$. By uniqueness of analytic functions, we conclude that $\varphi = \psi$ in Ω_0. $\square$

Combining Propositions 6.49 and 6.53, we complete the proof of Theorem 6.42. This also concludes the solution to Beurling's problem [**HLS97**].

COROLLARY 6.55. **(Hedenmalm, Lindqvist, Seip)** *Let $\psi(x) = \sqrt{2}\sum_{n=1}^\infty c_n \sin(n\pi x)$ be an odd, 2-periodic function on $\mathbb{R}$. Then $\{\psi(nx)\}_{n\in\mathbb{N}^+}$ forms a Riesz basis for $L^2([0,1])$ if and only if the function $\varphi(s) = \sum_{n=1}^\infty c_n n^{-s}$ is bounded and bounded from below in Ω_0.*

6.8. Cyclic Vectors

Consider the following variant of Beurling's question. Let $\psi : [0;1] \to \mathbb{C}$ be in L^2. When is the set $\{\psi(nx) : \ n \in \mathbb{N}^+\}$ complete, i.e., when do we have

$$\overline{\text{span}}\ \{\psi(nx) : \ n \in \mathbb{N}^+\} = L^2([0;1])?$$

As before, we can write $\psi(x) = \sum_{n=1}^\infty c_n\beta(x)$, and translate this problem to $\mathcal{H}^2$. Let $f(s) = \sum_{n=1}^\infty c_n n^{-s}$. When is

$$\overline{\text{span}}\ \{f(ns) : \ n \in \mathbb{N}^+\} \ = \ \mathcal{H}^2?$$

Since $f(ns) = (M_{n^{-s}}f)(s)$, it is equivalent to requiring that

$$\overline{\text{span}}\ \{f \cdot \mathcal{D}\} = \mathcal{H}^2,$$

i.e. that f is a *cyclic vector* for the collection of multipliers $\{M_{p^{-s}} : p \in \mathbb{P}\}$. An obvious necessary condition is that f does not vanish in $\Omega_{1/2}$. We record this open question.

QUESTION 6.56. Which Dirichlet series f satisfy $\overline{\text{span}}\ \{f\cdot\mathcal{D}\} = \mathcal{H}^2$?

6.9. Exercises

EXERCISE 6.57. Show that $\mathcal{H}^2$ contains a function f with $\sigma_a(f) = \frac{1}{2}$.

EXERCISE 6.58. Prove that the reproducing kernel for $\mathcal{H}^2_w$ is given by

$$k(s,u) \ = \ \sum_n \frac{1}{w_n} n^{-s-\bar{u}}. \tag{6.59}$$

EXERCISE 6.60. Prove (6.34).

EXERCISE 6.61. Show that if $\alpha \in \mathbb{Z}$, and $w_n = (\log n)^\alpha$, the reproducing kernel for $\mathcal{H}^2_w$ can be written in terms of the ζ function (if $\alpha = 0$), its derivatives (if $\alpha < 0$) or integrals (if $\alpha > 0$), after adjusting if necessary for the constant term.

EXERCISE 6.62. Prove that $\sum a_n n^{-s}$ is in $\mathcal{D}$ if and only if a_n is bounded by a polynomial in n.

6.10. Notes

The proof we give of Besicovitch's theorem 6.25 is from his book [**Bes32**, p. 144]. In the book he also develops the theory of functions that are almost periodic in the L^p-sense (where the L^p-norm of the difference between f and a vertical translate of it is less than ϵ).

The solution to Beurling's problem, and the proof of Theorem 6.42 (in the most important case, $\mathcal{H}^2_w = \mathcal{H}^2$) is due to Hedenmalm, Lindqvist and Seip [**HLS97**]. The spaces $\mathcal{H}^2_w$ were first studied in [**McCa04**].

In Carlson's theorem 6.39, if μ has a point mass at 0, then one cannot take the limit with respect to c inside the integral in (6.40). Indeed, E. Saksman and K. Seip prove the following theorem in [**SS09**]:

THEOREM 6.63. *(1) There exists a function f in $H^\infty(\Omega_0) \cap \mathcal{D}$ such that $\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(it)|^2\, dt$ does not exist.*

(2) For all $\varepsilon > 0$, there exists $g = \sum_{n=1}^\infty a_n n^{-s} \in H^\infty(\Omega_0) \cap \mathcal{D}$ that is a singular inner function and such that $\sum |a_n|^2 < \varepsilon$.

For a more refined version of Carlson's theorem, see [**QQ13**, Section 7.4].

CHAPTER 7

Characters

7.1. Vertical Limits

Let us return to the map $\mathcal{Q} : \mathcal{D} \to \text{Hol}\,(\mathbb{D}^\infty)$. Consider the group $(\mathbb{Q}^+, \cdot)$ equipped with discrete topology. Its *dual group* K — the group of all characters,

$$K = \{\chi : \mathbb{Q}^+ \to \mathbb{T};\ \chi(mn) = \chi(m)\chi(n), \text{for all } m, n \in \mathbb{Q}^+\}$$

is isomorphic (as a topological group) to $\mathbb{T}^\infty$ via the map $\chi \mapsto \{\chi(p_k)\}_{k\in\mathbb{N}^+} = (\chi(2), \chi(3), \chi(5), \dots)$. The topology on K is the topology of pointwise convergence. It corresponds to the product topology on $\mathbb{T}^\infty$. The group $\mathbb{T}^\infty$ is also equipped with a Haar measure, which is the infinite product of the Haar measures on $\mathbb{T}$. We shall use ρ to denote Haar measure on $\mathbb{T}^\infty$.

Given any set X, a *flow* on X is family of maps $T_t : X \to X$, where t is a real parameter, that satisfy T_0 is the identity, and $T_s \circ T_t = T_{s+t}$. If X is equipped with some structure (measure space, topological space, smooth manifold, ...), we usually assume that T_t is compatible with this structure (*i.e.* each T_t is measurable, continuous, smooth, ...).

Given a sequence of real number $\{\alpha_n\}_{n\in\mathbb{N}}$, we define a flow on $\mathbb{T}^\infty$ by

$$T_t(z_1, z_2, \dots) := (e^{-it\alpha_1} z_1, e^{-it\alpha_2} z_2, \dots),$$

the so-called *Kronecker flow*. Note that the Kronecker flow is continuous and measurable.

Definition 7.1. A measurable flow on a probability space is *ergodic*, if all invariant sets have measure 0 or 1.

Theorem 7.2. *The Kronecker flow is ergodic if and only if $\{\alpha_n\}$ are linearly independent over $\mathbb{Q}$.*

Proof: See [**CFS82**]. □

In particular, if $\alpha_n = \log p_n$, the Kronecker flow is ergodic. (See Theorem 6.14.) The ergodic theorem (of which there are many variants) says that for an ergodic flow, the time average (the left-hand side of (7.4)) equals the space average (the right-hand side).

THEOREM 7.3. **(Birkhoff-Khinchin)** *Let T_t be an ergodic flow on K. Then*

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} g(T_t\chi_0)\ dt \ = \ \int_K g(\chi)d\rho(\chi), \tag{7.4}$$

for all χ_0, if $g \in \mathcal{C}(K)$, and for a.e. χ_0, if $g \in L^1$.

Proof: See [**CFS82**]. □

LEMMA 7.5. *Let $f \sim \sum_{n=1}^{\infty} a_n n^{-s}$ safisfies $\sigma_u(f) < 0$. Then $\mathcal{Q}f \in \mathcal{C}(\mathbb{T}^\infty)$.*

Proof: It suffices to show that the series for $\mathcal{Q}f$ is uniformly Cauchy, since the partial sums are clearly continuous.

Let $L = \sup_n |a_n| < \infty$. Fix $0 < \varepsilon < 1$, and find $N \in \mathbb{N}$ such that for all $M_2 > M_1 > N$

$$\Big|\sum_{n=M_1}^{M_2} a_n n^{it}\Big| \ < \ \varepsilon.$$

Thus, for all $t \in \mathbb{R}$,

$$\Big|\sum_{n=M_1}^{M_2} a_n \Big[e^{it\log p_1}\Big]^{r_1(n)} \dots \Big[e^{it\log p_k}\Big]^{r_k(n)}\Big| \ < \ \varepsilon.$$

Note that, if $w_1, \dots, w_k, \zeta_1, \dots, \zeta_k \in \mathbb{T}$, then

$$|w_1 \dots w_k - \zeta_1 \dots \zeta_k| \ \leq \ |w_1 - \zeta_1| + \dots + |w_k - \zeta_k|. \tag{7.5}$$

This can be proven by induction on k using the inequality $|w_1w_2 - \zeta_1\zeta_2| \leq |w_1 - \zeta_1| + |w_2 - \zeta_2|$, which follows easily from the triangle inequality.

Fix $z \in \mathbb{T}^\infty$ and $M_2 > M_1 > N$ as above. By Kronecker's theorem 6.14, we can find $t \in \mathbb{R}$ such that $|e^{it\log p_j} - z_j| < \frac{\varepsilon}{M_2 L}$ holds for all j's such that $p_j \leq M_2$. Thus we have,

$$\begin{aligned} \Big|\sum_{n=M_1}^{M_2} a_n z^{r(n)}\Big| \ &\leq \ \Big|\sum_{n=M_1}^{M_2} a_n \Big[z^{r(n)} - \Big[e^{it\log p_1}\Big]^{r_1(n)} \dots \Big[e^{it\log p_k}\Big]^{r_k(n)}\Big]\Big| \\ &+ \ \Big|\sum_{n=M_1}^{M_2} a_n \Big[e^{it\log p_1}\Big]^{r_1(n)} \dots \Big[e^{it\log p_k}\Big]^{r_k(n)}\Big| \\ &< \ \varepsilon + \varepsilon = 2\varepsilon, \end{aligned}$$

where we used the inequality (7.5) to estimate the first term. □

This gives another proof of Carlson's theorem, 6.39.

THEOREM 7.6. *Let* $f \sim \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{H}^2$, *and let* $x > \frac{1}{2}$. *Then*

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(x+it)|^2 \, ds \;=\; \sum_{n=1}^{\infty} |a_n|^2 n^{-2x}. \tag{7.7}$$

Proof: Since $\sigma_u(f) \leq \sigma_a(f) \leq \frac{1}{2}$, we obtain $\sigma_u(f_x) < 0$, for $x > \frac{1}{2}$.

Since $\mathcal{Q}f_x$ is continuous on $\mathbb{T}^\infty$ by Lemma 7.5, we can apply the Birkhoff-Khinchin ergodic theorem 7.3 for any character $\chi_0 \in K$ to get

$$\begin{aligned}
\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\mathcal{Q}f_x(T_t\chi_0)|^2 \, dt \quad &= \quad \int_K |\mathcal{Q}f_x(\chi)|^2 \, d\rho(\chi) \\
&= \quad \sum_{q\in\mathbb{Q}_+} |\widehat{\mathcal{Q}f_x}(q)|^2 \qquad (7.8)\\
&= \quad \sum_{n=1}^{\infty} |a_n|^2 n^{-2x}. \qquad (7.9)
\end{aligned}$$

We used Plancherel's theorem to obtain (7.8), and the fact that $\mathcal{Q}f$ is a sum only over positive powers of z means the only non-zero terms in (7.8) are when $q \in \mathbb{N}^+$, giving (7.8). Choosing the trivial character $\chi_0(n) \equiv 1$ yields

$$\begin{aligned}
(\mathcal{Q}f_x)(T_t\chi_0) \quad &= \quad \sum_n a_n n^{-x} n^{-it} \chi_0(n) \\
&= \quad f(x+it),
\end{aligned}$$

giving (7.7). □

For every $\tau \in \mathbb{R}$, the map $f \mapsto f_{i\tau}$ is unitary on $\mathcal{H}^2$. Thus, by Corollary 11.8, for every sequence $\{\tau_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$, there is a subsequence τ_{k_l} such that $\{f_{i\tau_{k_l}}\}$ converges uniformly on compact subsets of $\Omega_{1/2}$.

DEFINITION 7.10. Let $f \in \mathcal{H}^2$, and let $\{\tau_k\}_{k\in\mathbb{N}}$ be a sequence of real numbers. If the sequence $f_{i\tau_k}$ converges uniformly on compact subsets of $\Omega_{1/2}$ to a function g, then g is called a *vertical limit function* of f.

PROPOSITION 7.11. *Let* $f \in \mathcal{H}^2$, *and let* χ *be a character. Then* $f_\chi(s) := \sum_{n=1}^{\infty} a_n \chi(n) n^{-s}$ *is a vertical limit function of* f. *Conversely, all vertical limit functions have this form for some character* χ.

Proof: Fix a character χ and let $k \in \mathbb{N}^+$. By Kronecker's theorem, we can find $\tau_k \in \mathbb{R}$ such that $|e^{i\tau_k \log p_j} - \chi(p_j)| \leq 1/k$ holds for $j = 1, \ldots, k$. Define $f_k := f_{i\tau_k}$. Then using inequality (7.5), we conclude that for any $n \in \mathbb{N}^+$, $n = p_1^{r_1(n)} \ldots p_l^{r_l(n)}$,

$$\left|\widehat{f_k}(n) - \widehat{f_\chi}(n)\right| \quad = \quad \left|\hat{f}(n) n^{i\tau_k} - \hat{f}(n)\chi(n)\right|$$

$$\begin{aligned} &= \left|\hat{f}(n)\right| \cdot \left| \prod_{j=1}^{l} \left[e^{i \log p_j \tau_k}\right]^{r_j} - \prod_{j=1}^{l} \chi(p_j)^{r_j} \right| \\ &\leq \|f\|_{\mathcal{H}^2} \sum_{j=1}^{l} r_j \left|e^{i \log p_j \tau_k} - \chi(p_j)\right| \\ &\leq \frac{1}{k} \, \|f\|_{\mathcal{H}^2} \sum_{j=1}^{l} r_j, \end{aligned}$$

and this last expression tends to 0 as $k \to \infty$. Proposition 11.7 now implies that f_χ is a vertical limit function of f.

Conversely, let g be a vertical limit function of f. Using Proposition 11.7 again, we conclude that there exists a sequence $\{\tau_k\}_{k \in \mathbb{N}} \subset \mathbb{R}$ such that $\widehat{f_{i\tau_k}}(n) \to \hat{g}(n)$ for all $n \in \mathbb{N}$. Equivalently,

$$n^{i\tau_k} \to \frac{\hat{g}(n)}{\hat{f}(n)}, \text{ as } k \to \infty.$$

Since $n \mapsto n^{i\tau_k}$ is a character for all $k \in \mathbb{N}$, so is the limit: $n \mapsto \hat{g}(n)/\hat{f}(n)$. $\square$

Let us now turn to the Lindelöf hypothesis, a conjecture weaker than the Riemann hypothesis, but one that could be possibly approached by the tools of functional analysis.

Recall that the alternating zeta function is given by $\tilde{\zeta}(s) = \sum_{n=1}^{\infty} (-1)^n n^{-s}$. We have seen that $\tilde{\zeta}(s) = (2^{1-s} - 1)\zeta(s)$. This implies that $\tilde{\zeta}(s)$ and $\zeta(s)$ are of comparable size in $\{s \in \mathbb{C} : \text{ Re } s > 0, |1 - \text{Re } s| > \varepsilon\}$, for any $\varepsilon > 0$.

Conjecture 7.12. **(Lindelöf hypothesis)** For every $\sigma > \frac{1}{2}$ and $k \in \mathbb{N}^+$

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\tilde{\zeta}^k(\sigma + it)|^2 \, dt < \infty$$

holds.

Recall that $d_k(n)$, defined in Corollary 1.17, is the number of ways n can be factored into exactly k factors, allowing 1 and where the order matters.

Lemma 7.13. *Let k be a natural number and let $\varepsilon > 0$. Then*

$$d_k(n) = O(n^\varepsilon) \text{ as } n \to \infty.$$

Proof: Note that $d_2(n)$ is the number of divisors of n. Also, $d_3(n) \le d_2(n)^2$, since

$$d_3(n) \;=\; \sum_{l|n} d_2\left(\frac{n}{l}\right) \;\le\; \sum_{l|n} d_2(n) \;=\; d_2(n)^2.$$

Applying this argument inductively, we obtain $d_k(n) \le d_2(n)^{k-1}$ and thus it is enough to show that $d_2(n) = O(n^\varepsilon)$ for all $\varepsilon > 0$.

Fix $\varepsilon > 0$. We need to show that there exist $C = C(\varepsilon)$ such that $d_2(n) \le Cn^\varepsilon$ holds for all $n \in \mathbb{N}^+$, or equivalently, that

$$\log d_2(n) \;\le\; \varepsilon \log n + \log C.$$

Write $n = \prod_{j=1}^l p_j^{t_j}$ with $t_j \ge 0$ and $t_l > 0$, then $d_2(n) = \prod_{j=1}^l (1 + t_j)$. We want to show that

$$\sum_{j=1}^{l} [\log(1 + t_j) - \varepsilon t_j \log p_j] \;\le\; \log C$$

for all $n \in \mathbb{N}$. Clearly, if $\log p_j \ge 1/\varepsilon$, then the j^{th} summand is non-positive, because $\log(1 + t_j) < t_j$. As $t_j \to \infty$, the j^{th} summand tends to $-\infty$. Hence each of the finitely many summands with $\log p_j < 1/\varepsilon$ is bounded. $\square$

Suppose that the Carlson theorem applied to $\tilde{\zeta}^k(s)$. Then

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\tilde{\zeta}^k(\sigma + it)|^2 \, dt \quad &= \quad \sum_{n=1}^{\infty} n^{-2\sigma} |\widehat{\tilde{\zeta}^k}(n)|^2 \\ &\le \quad \sum_{n=1}^{\infty} n^{-2\sigma} |\widehat{\zeta^k}(n)|^2 \\ &< \quad \infty,\end{aligned}$$

since $\widehat{\zeta^k}(n) = d_k(n) = O(n^\varepsilon)$ for all $\varepsilon > 0$ by Lemma 7.13. Thus we would have proved the Lindelöf hypothesis. Conversely, the following is known.

THEOREM 7.14. **(Titchmarsh)** *If*

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\tilde{\zeta}^k(\sigma + it)|^2 \, dt < \infty,$$

then it equals to $\sum_{n=1}^{\infty} n^{-2\sigma} |\widehat{\tilde{\zeta}^k}(n)|^2$.

7.2. Helson's Theorem

We will need some properties of Hardy spaces of the right half-plane Ω_0. There is more than one natural definition. We will consider two of them. Let $\psi : \Omega_0 \to \mathbb{D}$ be the standard conformal mapping of the right half-plane onto the disk, that is, $\psi(z) = \frac{1-z}{1+z}$. For $1 \leq p \leq \infty$, we define the conformally invariant Hardy space as

$$H_i^p(\Omega_0) = \{g \circ \psi;\ g \in H^p(\mathbb{D})\}.$$

For $1 \leq p < \infty$, writing $e^{i\theta} = \psi(-it) = \frac{1+it}{1-it}$, and changing variables yields

$$\begin{aligned} \|g\|^p_{H^p(\mathbb{D})} &= \int_{\mathbb{T}} |g(e^{i\theta})|^p \, \frac{d\theta}{2\pi} \\ &= \int_{\mathbb{R}} |(g \circ \psi)(-it)|^p \left| \frac{d\theta}{dt} \right| \frac{dt}{2\pi} \\ &= \int_{\mathbb{R}} |(g \circ \psi)(-it)|^p \frac{dt}{\pi(1+t^2)}. \end{aligned}$$

Any function $g \in H^p(\mathbb{D})$ extends to an L^p function on $\mathbb{T}$ satisfying

$$\int_{\mathbb{T}} g(e^{i\theta}) e^{in\theta} \frac{d\theta}{2\pi} \ = \ 0, \text{ for all } n \in \mathbb{N}^+. \tag{7.10}$$

Conversely, any L^p function on $\mathbb{T}$ satisfying (7.10) is the boundary value of function in $H^p(\mathbb{D})$.

Let μ be the measure on the real axis give by $d\mu(t) = \frac{dt}{\pi(1+t^2)}$.

We deduce that a Lebesgue measurable function $f : i\mathbb{R} \to \mathbb{C}$ belongs to $H_i^p(\Omega_0)$, if and only if

$$\|f\|^p_{H_i^p(\Omega_0)} \ := \ \int_{\mathbb{R}} |f(it)|^p \, d\mu(t) \ < \ \infty,$$

and

$$\int_{\mathbb{R}} f(it) \left(\frac{1-it}{1+it} \right)^n \, d\mu(t) \ = \ 0, \text{ for all } n \in \mathbb{N}^+. \tag{7.11}$$

Here is the second definition for the Hardy spaces of the half-plane. For $1 \leq p < \infty$, set

$$H^p(\Omega_0) := \left\{ f \in \text{Hol}\ (\Omega_0) \ \middle|\ \|f\|^p_{H^p(\Omega_0)} := \sup_{\sigma > 0} \int_{-\infty}^{\infty} |f(\sigma + it)|^p \, dt < \infty \right\}.$$

For any function $f \in H^p(\Omega_0)$ and almost every $t \in \mathbb{R}$, the limit $\tilde{f}(it) := \lim_{\sigma \to 0+} f(\sigma + it)$ exists and satisfies $\tilde{f} \in L^p(i\mathbb{R})$. One can recover f from $\tilde{f}$ by convolution with the Poisson kernel. For both $H_i^p(\Omega_0)$ and $H^p(\Omega_0)$ we identify the functions with their boundary values.