Combining these inequalities we have

$$b-a < m_e(E)+m_e(CE)+4\epsilon.$$

Making $\epsilon \to 0$, it follows that

$$b-a \leqslant m_e(E)+m_e(CE),$$

which is equivalent to the result stated.

If $m_e(E) = 0$, it follows that $m_i(E) = 0$. Hence E is measurable, and its measure is zero.

10.23. We now come to the two fundamental theorems in the theory of measure.

First fundamental theorem. *If E_1, E_2,..., E_n,... are measurable sets, then the set $E = E_1+E_2+E_3+...$ is measurable, and*

$$m(E) \leqslant m(E_1)+m(E_2)+\ldots.$$

If E_1, E_2,... do not overlap, then the equality holds. (*Otherwise the series may diverge.*)

Second fundamental theorem. *If E_1, E_2,... are measurable sets, then the set $E_1 E_2 E_3$... is measurable.*

That is, the set of points belonging to any of the sets E_1, E_2,... is measurable, and so is the set of points belonging to all of them.

We shall begin by proving two lemmas on open sets, the first of which is the first fundamental theorem for open sets. We next prove a general theorem on exterior measure, and deduce from it the first fundamental theorem for the case where the sets do not overlap. Then we obtain the second theorem for two sets, and use it to complete the first theorem. Finally we use this result to complete the second theorem.

10.24. *If O_1, O_2,... are open sets (overlapping or not), and*

$$O = O_1+O_2+O_3+\ldots,$$

then

$$m(O) \leqslant m(O_1)+m(O_2)+\ldots. \tag{1}$$

We assume the convergence of the series on the right, since otherwise the theorem is meaningless.

Let the intervals of O_n be $(a_{m,n}, b_{m,n})$ $(m = 1, 2,...)$, and let those of O be (A_k, B_k) $(k = 1, 2,...)$. Let ϵ be a positive number less than $\frac{1}{2}(B_k-A_k)$. Then every point of the interval $(A_k+\epsilon, B_k-\epsilon)$ is an interior point of one of the intervals $(a_{m,n}, b_{m,n})$ which make up (A_k, B_k). If $\sum_k$ denotes a summation over these intervals, it follows from the Heine-Borel theorem, as in the previous proof, that

$$B_k - A_k - 2\epsilon \leqslant \sum_k (b_{m,n} - a_{m,n}).$$

Making $\epsilon \to 0$, we obtain

$$B_k - A_k \leqslant \sum_k (b_{m,n} - a_{m,n}), \tag{2}$$

and, summing with respect to k,

$$m(O) \leqslant \sum_{k=1}^{\infty} \sum_k (b_{m,n} - a_{m,n}). \tag{3}$$

Since a convergent double series of positive terms can be summed in any manner, the right-hand side of (3) can be rearranged in the form

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (b_{m,n} - a_{m,n}) = \sum_{n=1}^{\infty} m(O_n).$$

This proves the theorem.

If none of the sets overlap, each interval (A_k, B_k) coincides with one interval $(a_{m,n}, b_{m,n})$, and the inequalities (2) and (3), and so also (1), become equalities.

An enumerable set is measurable, and its measure is zero. For let the set be $x_1, x_2, \ldots$. Include x_1 in an open interval of length ϵ. If this does not include x_2, we can include x_2 in an interval of length $\frac{1}{2}\epsilon$; and so generally x_n in an interval of length $\epsilon/2^n$. Thus the given set can be included in an open set of measure not greater than 2ϵ. Since ϵ may be as small as we please, the exterior measure of the set is zero. Hence its measure is zero.

10.241. *If O and O' are open sets which together include all points of the interval (a, b), then*

$$m(OO') \leqslant m(O) + m(O') - (b - a).$$

By the Heine-Borel theorem we can select finite sets of the intervals of O and O', say Q from O and Q' from O', such that Q and Q' together include the whole interval $(a+\epsilon, b-\epsilon)$; and we may, by adding further intervals if necessary, suppose that

$$O = Q + R, \qquad O' = Q' + R',$$

where $m(R) < \epsilon$, $m(R') < \epsilon$. Now

$$OO' \prec QQ' + R + R',$$

so that by the previous lemma

$$m(OO') \leqslant m(QQ') + m(R) + m(R') < m(QQ') + 2\epsilon.$$

But $m(Q)+m(Q')-m(QQ') \geqslant b-a-2\epsilon$, from elementary considerations, and $m(O) \geqslant m(Q)$, $m(O') \geqslant m(Q')$. Making $\epsilon \to 0$, the result follows.*

10.25. *If $E_1, E_2,...$ are any sets, and*

$$E = E_1+E_2+...,$$

then $$m_e(E) \leqslant m_e(E_1)+m_e(E_2)+....$$

We can enclose E_n in an open set O_n such that

$$m(O_n) < m_e(E_n)+\frac{\epsilon}{2^n}.$$

Summing with respect to n, and using the result of § 10.24,

$$m(O) \leqslant m(O_1)+m(O_2)+... < m_e(E_1)+m_e(E_2)+...+\epsilon.$$

But O is an open set which includes E. Hence

$$m_e(E) \leqslant m(O).$$

Hence $$m_e(E) < m_e(E_1)+m_e(E_2)+...+\epsilon,$$

and, making $\epsilon \to 0$, the result follows.

10.26. *If $E_1, E_2,...$ are non-overlapping measurable sets, and*

$$E = E_1+E_2+...,$$

then E is measurable, and

$$m(E) = m(E_1)+m(E_2)+....$$

We may suppose that all the sets are included in (a, b).

(i) Consider first the case of two sets, $E = E_1+E_2$. We know already that

$$m_e(E) \leqslant m_e(E_1)+m_e(E_2) = m(E_1)+m(E_2).$$

Hence it is sufficient to prove that

$$m_i(E) \geqslant m(E_1)+m(E_2),$$

i.e. that $$m_e(CE) \leqslant m(CE_1)+m(CE_2)-(b-a).$$

Now we can include CE_1, CE_2, in open sets O_1, O_2, such that

$$m(O_1) < m(CE_1)+\epsilon, \qquad m(O_2) < m(CE_2)+\epsilon.$$

Since E_1 and E_2 have no common points, CE_1 and CE_2 together include the whole interval, and hence so do O_1 and O_2. Hence

$$m(O_1 O_2) \leqslant m(O_1)+m(O_2)-(b-a).$$

* Actually the two sides are equal. This follows in due course from the first fundamental theorem.

But O_1O_2 includes CE. Hence

$$m_e(CE) \leqslant m(O_1O_2) \leqslant m(O_1)+m(O_2)-(b-a)$$
$$< m(CE_1)+m(CE_2)+2\epsilon-(b-a),$$

and, making $\epsilon \to 0$, the result follows.

(ii) The theorem for any finite number of sets follows by repeated application of (1).

(iii) In the case of an infinity of sets, we have, for all values of n,

$$m(E_1)+m(E_2)+...+m(E_n) = m(E_1+...+E_n) \leqslant b-a.$$

Hence $\sum m(E_n)$ is convergent.

Let $S_n = E_1+...+E_n$. Then $CE < CS_n$, so that

$$m_e(CE) \leqslant m_e(CS_n) = m(CS_n) = b-a-m(E_1)-...-m(E_n).$$

Making $n \to \infty$ we obtain

$$m_e(CE) \leqslant b-a-\sum m(E_n),$$

i.e.
$$m_i(E) \geqslant \sum m(E_n).$$

Combining this with § 10.25, the result follows.

In particular, taking E_1, E_2,... to be open intervals, it follows that any open set is measurable in the general sense, and that the two definitions of the measure of an open set agree. Also any closed set, as the complement of an open set, is measurable.

If E_1 and E_2 are measurable sets, E_1 being included in E_2, then E_2-E_1 is measurable.

For
$$C(E_2-E_1) = E_1+CE_2.$$

10.27. *If E and F are measurable sets, so is EF.*

Let both sets be included in (a,b), and suppose first that F is an interval (α,β). Let E_1 be the part of E in (α,β), E_2 the remainder. Similarly, if O is an open set containing E, let $O = O_1+O_2$. O_1 and O_2 are open sets containing respectively E_1 and E_2, if we neglect the points α and β, as we obviously may; and clearly

$$m(O) = m(O_1)+m(O_2).$$

Taking lower bounds,

$$m_e(E) = m_e(E_1)+m_e(E_2). \tag{1}$$

Similarly, if $e = CE = e_1+e_2$,

$$m_e(e) = m_e(e_1)+m_c(e_2). \tag{2}$$

But, since E is measurable,

$$m_e(E)+m_e(e) = b-a, \tag{3}$$

and by § 10.25

$$m_e(E_2)+m_e(e_2) \geqslant m_e(E_2+e_2) = b-a-(\beta-\alpha). \tag{4}$$

From (1), (2), (3) and (4) it follows that

$$m_e(E_1)+m_e(e_1) \leqslant \beta-\alpha,$$

and hence E_1 is measurable.

The result is therefore proved if F is an interval, and so, by the previous theorem, if F is an open set. In the general case we can include F in an open set O, and CF in O', so that $m(O)+m(O') < b-a+\epsilon$. Then

$$EF < EO, \qquad C(EF) = CF+F.CE < O'+O.CE,$$

so that

$$m_e(EF)+m_e\{C(EF)\} \leqslant m(EO)+m(O')+m(O.CE)$$
$$= m(O)+m(O') < b-a+\epsilon.$$

Making $\epsilon \to 0$, $m_e(EF)+m_e\{C(EF)\} \leqslant b-a$, whence the result.

If E_1 and E_2 are measurable, the set E of points belonging to E_2 but not to E_1 is measurable.

For $E = E_2.CE_1$.

10.28. We can now complete the proofs of the fundamental theorems. Let E_1, E_2,... be any sets, overlapping or not, and let E be their sum. Let

$$E_2' = E_2.CE_1, \qquad E_3' = E_3.C(E_1+E_2'),$$
$$E_4' = E_4.C(E_1+E_2'+E_3'),$$

and so on. Then E_1, E_2', E_3',... are non-overlapping measurable sets, and $E = E_1+E_2'+E_3'+\ldots$. Hence E is measurable, by § 10.26, and the proof of the first fundamental theorem is completed, the inequality then stated following from § 10.25.

Again, if $F = E_1E_2E_3\ldots$, then

$$CF = CE_1+CE_2+\ldots.$$

Hence, by what has just been proved, CF is measurable, and so F is measurable. This proves the second fundamental theorem.

10.29. Limiting sets. *If E_1, E_2,... are measurable sets, each contained in the following one, and E is their sum, then*

$$\lim_{n\to\infty} m(E_n) = m(E).$$

For the sets E_2-E_1, E_3-E_2,... are measurable and non-overlapping, and

$$E = E_1+(E_2-E_1)+(E_3-E_2)+...,$$

so that

$$m(E) = m(E_1)+m(E_2-E_1)+...$$
$$= \lim\{m(E_1)+m(E_2-E_1)+...+m(E_n-E_{n-1})\} = \lim m(E_n).$$

The set E is called the *outer limiting set* of the sets E_1, E_2,... .

If each of the sets E_1, E_2,... contains the next, and $E = E_1 E_2...$, then

$$\lim_{n\to\infty} m(E_n) = m(E).$$

This follows by complementary sets from the previous theorem. In this case the set E is called the *inner limiting set.*

Unlike most of the theorems on the measure of sets, the first of these results holds if 'measure' is replaced by 'exterior measure', whether the sets are measurable or not. This remark will be useful in the next chapter, where it happens to be inconvenient to verify that certain sets are measurable.

If E is the outer limiting set of a sequence E_n, then

$$\lim_{n\to\infty} m_e(E_n) = m_e(E).$$

Let E_n be included in an open set O_n such that

$$m(O_n) < m_e(E_n)+\epsilon.$$

Let $S_n = O_n O_{n+1} O_{n+2}...$, and let $S = S_1+S_2+...$. Then $E_n < S_n < O_n$, $E < S$, and $S_n < S_{n+1}$, so that S is the outer limiting set of the sets S_n (this is not necessarily true for O_n, which is why we introduce S_n). Hence

$$m_e(E) \leqslant m(S) = \lim m(S_n) < \lim m_e(E_n)+\epsilon,$$

and, making $\epsilon \to 0$, $m_e(E) \leqslant \lim m_e(E_n)$. But since a set which includes E also includes E_n, $m_e(E) \geqslant m_e(E_n)$ for every n. This proves the theorem.

10.291. Cantor's ternary set. The following set of points, defined by Cantor, has many interesting properties.

Divide the interval $(0, 1)$ into three equal parts, and remove the interior of the middle part. Next subdivide each of the two remaining parts into three equal parts, and remove the interiors of the middle parts of each of them; and repeat this process indefinitely. Thus at the pth step we remove 2^{p-1} intervals.

We denote these intervals, from left to right, by $\delta_{p,k}$, where k runs from 1 to 2^{p-1}. For each k the length of $\delta_{p,k}$ is 3^{-p}.

Let E be the set of points which remain. Then E is the set of points represented by the infinite decimals

$$\cdot a_1 a_2 \ldots a_n \ldots (3)$$

in the scale of 3 (indicated by the final figure), where the numbers $a_1, a_2, \ldots$ take the values 0 or 2 only, never the value 1; for example, E includes $\frac{2}{3} = \cdot 200\ldots$, and also $\frac{1}{3}$, which can be represented as $\cdot 0222\ldots$. In fact the first step described above removes from the interval all points for which the first figure is a 1 (except $\cdot 100\ldots = \cdot 022\ldots$); the second step removes all remaining points for which the second figure is a 1 (except $\cdot 010\ldots = \cdot 0022\ldots$, and $\cdot 210\ldots = \cdot 2022\ldots$); and so on. Notice also that the end-points of the intervals $\delta_{p,k}$ consist of all decimals $\cdot a_1 a_2 \ldots (3)$, where the digits after a certain point are all 0's or all 2's. This is obviously true for $\delta_{1,1}$; then $\delta_{2,1}$, $\delta_{2,2}$ are obtained by taking the first decimal as 0 or 2 and then the rest as the decimals corresponding to the ends of $\delta_{1,1}$; and so on. Thus the general form of the end-points of a $\delta_{p,k}$ is

$$\cdot a_1 \ldots a_m 0222\ldots(3), \qquad \cdot a_1 \ldots a_m 2000\ldots(3).$$

The set E is not enumerable; this may be proved in the same way that it was proved that the continuum was not enumerable. On the other hand, the measure of E is zero; for

$$m(E) = 1 - \sum m(\delta_{p,k}) = 1 - \sum_{p=1}^{\infty} \frac{2^{p-1}}{3^p} = 0.$$

We shall refer to this set again in § 11.72.

Example. Prove that the measure of the set of points in the interval (0, 1) representing numbers whose expressions as infinite decimals do not contain some particular digit (say 7) is zero.

10.3. Measurable functions. Let $f(x)$ be a bounded function of x in the interval $a \leqslant x \leqslant b$. We denote by $E(f > c)$ the set of points in (a, b) where $f(x) > c$; and similarly with other inequalities.

The function $f(x)$ is said to be measurable if any one of the sets

$$E(f \geqslant c), \qquad E(f < c), \qquad E(f > c), \qquad E(f \leqslant c)$$

is measurable for all values of c.

Any one of these four conditions implies the other three.

Suppose, for example, that the first holds. The second follows by complementary sets. Hence also the sets

$$E_n = E\left(f < c + \frac{1}{n}\right) \qquad (n = 1, 2, \ldots)$$

are all measurable. Hence the set

$$(E_1 - E_2) + (E_2 - E_3) + \ldots = E(c < f < c+1)$$

is measurable. Hence

$$E(f = c) = E(f \geqslant c) - E(f \geqslant c+1) - E(c < f < c+1)$$

is measurable, and the result clearly follows from this.

10.31. General properties of measurable functions.

(i) *Let f be a measurable function, k a constant. Then $k+f$, kf, and in particular $-f$, are measurable.*

This is obvious.

(ii) *If f and ϕ are measurable functions, the set $E(f > \phi)$ is measurable.*

If $f > \phi$, there is a rational number r such that $f > r > \phi$. Hence

$$E(f > \phi) = \sum_r E(f > r) \,.\, E(\phi < r)$$

where r runs through all rational numbers. Hence the result.

(iii) *If f and ϕ are measurable, so are $f+\phi$ and $f-\phi$.*

For

$$E(f+\phi > c) = E(f > c - \phi)$$

and the result follows from (ii). Similarly for $f-\phi$.

(iv) *If f and ϕ are measurable, so is $f\phi$.*

The function $\{f(x)\}^2$ is measurable, for, if $c > 0$,

$$E(f^2 > c) = E(f > \sqrt{c}) + E(f < -\sqrt{c}).$$

The general theorem then follows from the fact that

$$f\phi = \tfrac{1}{4}(f+\phi)^2 - \tfrac{1}{4}(f-\phi)^2.$$

(v) *If $f_n(x)$ is a sequence of measurable functions, then*

$$\varliminf_{n\to\infty} f_n(x), \qquad \varlimsup_{n\to\infty} f_n(x),$$

supposed finite, are measurable. In particular, if the sequence tends to a limit, the limit is measurable.

Let $f(x) = \varlimsup f_n(x)$. Let c be any real number, let

$$E_{m,n} = E\left(f_n > c + \frac{1}{m}\right) + E\left(f_{n+1} > c + \frac{1}{m}\right) + \ldots,$$

and let $E_m = E_{m,1} E_{m,2} E_{m,3} \ldots$. By the fundamental theorems

$E_{m,n}$ and E_m are measurable. Now E_m, the set of points common to all the sets $E_{m,n}$, is the set where $f_\nu > c+1/m$ for arbitrarily large values of ν. Hence

$$f = \overline{\lim} f_\nu \geqslant c+\frac{1}{m} > c$$

in E_m. Let $E = E_1+E_2+E_3+\dots$. Then E is measurable, and $f > c$ at all points of E. Conversely, if $f(x) > c$, then there is an integer m such that $f_\nu(x) > c+1/m$ for arbitrarily large values of ν, and so x belongs to one of the sets E_m. Hence $E = E(f > c)$, which proves the theorem.

(vi) *A continuous function is measurable.* For if $f(x)$ is continuous, it is easily seen that $E(f \leqslant c)$ is closed. Hence $E(f > c)$ is open, and so measurable.

All the ordinary functions of analysis may be obtained by limiting processes from continuous functions, and so are measurable. The same thing is true of some of the more artificial functions. For example,

$$\lim_{n\to\infty}\{\cos m!\pi x\}^{2n}$$

is the limit of a continuous function, and is equal to 1 if $m!x$ is an integer, and otherwise is zero. If x is rational, $m!x$ is an integer if m is large enough. Hence

$$f(x) = \lim_{m\to\infty}\lim_{n\to\infty}\{\cos m!\pi x\}^{2n}$$

is equal to 1 if x is rational, and to 0 otherwise. The fact that this function is measurable has, of course, been proved more directly (§ 10.22).

10.4. The Lebesgue integral of a bounded function. We are now in a position to define the Lebesgue integral of any bounded measurable function.

If $f(x)$ is the characteristic function of a set E, i.e. $f(x) = 1$ in E and 0 elsewhere, a natural definition of the integral is

$$\int_a^b f(x)\,dx = m(E).$$

If $f(x) = k$ in E and 0 elsewhere, then we take

$$\int_a^b f(x)\,dx = km(E).$$

In the general case, let α and β be the lower and upper bounds of $f(x)$. As in the case of Riemann integration, the integral is defined as the limit of the sum; but this time the sum is obtained by dividing up the interval of variation of $f(x)$. We take numbers $y_0, y_1, \ldots, y_{n+1}$ such that

$$\alpha = y_0 < y_1 < y_2 < \ldots < y_{n-1} < y_n = \beta.$$

Let e_ν be the set where $y_\nu \leqslant f(x) < y_{\nu+1}$ $(\nu = 0, \ldots, n-1)$, and e_n the set where $f(x) = \beta$. Since $f(x)$ is measurable, all the sets e_ν are measurable. Putting $y_{n+1} = \beta$, let

$$s = \sum_{\nu=0}^{n} y_\nu m(e_\nu), \qquad S = \sum_{\nu=0}^{n} y_{\nu+1} m(e_\nu).$$

The Lebesgue integral of $f(x)$ over (a, b) is the common limit of the sums s and S when the number of division-points y_ν is increased indefinitely, so that the greatest value of $y_{\nu+1}-y_\nu$ tends to zero.

To justify the definition we have to prove that the two limits exist and are equal.

Suppose the interval (α, β) divided up in two different ways, each difference $y_{\nu+1}-y_\nu$ in each way being less than ϵ. Let the sums formed in these two ways be s, S and s', S'. Then

$$S-s = \sum_{\nu=0}^{n} (y_{\nu+1}-y_\nu) m(e_\nu) \leqslant \epsilon \sum_{\nu=0}^{n} m(e_\nu) = \epsilon(b-a),$$

and similarly $S'-s' \leqslant \epsilon(b-a)$.

We now divide up the interval (α, β) by taking all the division-points of the first two ways at once. This gives two more sums, s'' and S''. Now the insertion of a new division-point does not decrease a lower sum or increase an upper sum; for example, if we insert a point η between y_ν and $y_{\nu+1}$ we have

$$y_\nu m(e_\nu) \leqslant y_\nu m\{E(y_\nu \leqslant f < \eta)\} + \eta m\{E(\eta \leqslant f < y_{\nu+1})\},$$

so that the lower sum is not decreased. Applying this principle repeatedly, we obtain

$$s \leqslant s'', \qquad s' \leqslant s'',$$

and similarly $$S'' \leqslant S, \qquad S'' \leqslant S'.$$

It follows that the intervals (s, S) and (s', S') have points in common, e.g. all points of the interval (s'', S''). Hence the numbers s, s', S, S' all lie within an interval of length $2\epsilon(b-a)$. The existence and equality of the limits then follow from the general principle of convergence.

10.41. Comparison with Riemann's definition. Perhaps the most obvious difference to the beginner is that, in Lebesgue's definition, we divide up the interval of variation of the function instead of the interval of integration. This, however, is comparatively unimportant. What is essential is that we use the general theory of 'measure' of sets instead of the more limited theory of 'extent'. It would be possible to build up an integral from integrals of characteristic functions, but using extent instead of measure. This would be substantially equivalent to Riemann's definition. On the other hand, it is possible to define an integral equivalent to Lebesgue's by dividing up the interval of integration in a suitable way.

In both Riemann's and Lebesgue's definitions we have upper and lower sums which tend to limits. In the Riemann case the two limits are not necessarily the same, and the function is only integrable if they are the same. In the Lebesgue case the two limits are necessarily the same, their equality being a consequence of the assumption that the function is measurable.

Lebesgue's definition is more general than Riemann's. For the characteristic function of the set of rational points has a Lebesgue integral, but not a Riemann integral; and we shall see later that, if a function has a Riemann integral, then it also has a Lebesgue integral, and the two are equal.

We use the same notation

$$\int_a^b f(x)\,dx$$

for a Lebesgue integral as we have done for a Riemann integral. When it is necessary to distinguish a Riemann integral from a Lebesgue integral, we shall denote the former by

$$R\int_a^b f(x)\,dx.$$

10.42. Integral over any measurable set. Let E be any measurable set contained in an interval (a, b). The integral of $f(x)$ over E may be defined in the same way as the integral over an interval. The sets e_ν of § 10.4 are now the sub-sets of E where $y_\nu \leqslant f(x) < y_{\nu+1}$; the proof of the existence of the integral is practically unchanged. The integral is written

$$\int_E f(x)\,dx.$$

Any integral over a set of measure zero is zero. For all the sets e_ν are of measure zero, and so the sums s and S are always 0.

We might also define the integral by putting $f(x)=0$ in CE, and then using the definition of the integral over an interval. It is easily seen that the two definitions are equivalent.

10.43. Henceforward we shall assume that all sets and functions introduced are measurable, without always saying so explicitly.

10.44. Elementary properties of the integral of a bounded function.

(i) **The mean-value theorem.** *If* $\alpha \leqslant f(x) \leqslant \beta$, *then*

$$\alpha m(E) \leqslant \int_E f(x)\,dx \leqslant \beta m(E).$$

For it is easily seen that $\alpha m(E) \leqslant s \leqslant \beta m(E)$, and the result follows in the limit.

(ii) *The integral is additive for a finite number or for an enumerable infinity of non-overlapping sets included in a finite interval. That is, if*

$$E = E_1 + E_2 + \dots,$$

then
$$\int_E f(x)\,dx = \int_{E_1} f(x)\,dx + \int_{E_2} f(x)\,dx + \dots.$$

Suppose first that there are two sets, E_1 and E_2. Inserting division-points y_ν, the sets E, E_1, E_2 are divided into sub-sets e_ν, e_ν^1, e_ν^2, such that

$$m(e_\nu) = m(e_\nu^1) + m(e_\nu^2).$$

Hence
$$\int_{E_1} + \int_{E_2} = \lim \sum y_\nu m(e_\nu^1) + \lim \sum y_\nu m(e_\nu^2)$$
$$= \lim \sum y_\nu m(e_\nu) = \int_E.$$

Similarly for any finite number of sets.

If there are an infinity of sets, let S_n be the sum of the first n, R_n the remainder. Then

$$\int_E = \int_{S_n} + \int_{R_n}.$$

But, by the mean-value theorem, if $|f(x)| \leqslant M$, then

$$\left|\int_{R_n} f(x)\,dx\right| \leqslant Mm(R_n),$$

and this tends to zero as $n \to \infty$, since the series $\sum m(E_n)$ is convergent. Hence

$$\int_E = \lim \int_{S_n} = \int_{E_1} + \int_{E_2} + \dots .$$

(iii) *If, in a set E, $f(x) \leqslant \phi(x)$, then*

$$\int_E f(x)\,dx \leqslant \int_E \phi(x)\,dx.$$

Take division-points y_ν, and define the sets e_ν by means of $f(x)$. Then, in e_ν, $\phi(x) \geqslant f(x) \geqslant y_\nu$. Hence

$$\int_E \phi(x)\,dx = \sum \int_{e_\nu} \phi(x)\,dx \geqslant \sum y_\nu m(e_\nu).$$

The right-hand side tends to $\int_E f(x)\,dx$, whence the result follows.

(iv) *The integral of the sum of a finite number of bounded measurable functions is the sum of the integrals of the separate functions.*

In the first place, if k is a constant,

$$\int_E (f+k)\,dx = \int_E f\,dx + \int_E k\,dx = \int_E f\,dx + km(E).$$

For calculate the sum s relative to $f(x)$ with the scale $y_0, y_1, \dots$, and the sum s' relative to $f(x)+k$ with the scale $y_0+k, y_1+k, \dots$. Then

$$s' = \sum (y_\nu+k)m(e_\nu) = s+km(E),$$

and the result follows in the limit.

Now consider any two functions $f(x)$ and $\phi(x)$. We have

$$\begin{aligned}\int_E \{f(x)+\phi(x)\}\,dx &= \sum \int_{e_\nu} (f+\phi)\,dx \\ &\geqslant \sum \int_{e_\nu} (y_\nu+\phi)\,dx \\ &= s + \int_E \phi\,dx\end{aligned}$$

by what has just been proved. Similarly, replacing y_ν by $y_{\nu+1}$, we obtain

$$\int_E (f+\phi)\,dx \leqslant S + \int_E \phi\,dx.$$

The result now follows in the limit.

The result for any finite number of functions is obtained by repeated application of the result for two functions.

(v) *If k is a constant,*

$$\int_E k f(x)\,dx = k \int_E f(x)\,dx.$$

This is obvious if $k = 0$. If $k > 0$, calculate the second integral with the scale y_ν, and the first with the scale ky_ν. Then the sets e_ν are the same in each case, and $s = ks'$, whence the result.

(vi) *We have*

$$\left|\int_E f(x)\,dx\right| \leqslant \int_E |f(x)|\,dx.$$

Let E_1 be the set where $f(x) \geqslant 0$, E_2 the set where $f(x) < 0$. Then

$$\int_E f\,dx = \int_{E_1} f\,dx - \int_{E_2} |f|\,dx,$$

$$\int_E |f|\,dx = \int_{E_1} f\,dx + \int_{E_2} |f|\,dx,$$

and the result is obvious.

(vii) A relation which holds except in a set of measure zero is said to hold *almost everywhere.*

Two functions which are equal almost everywhere have the same integral.

Let $f(x) = \phi(x)$ at all points of E, except in a set e of measure zero. Then

$$\int_E (f-\phi)\,dx = \int_e (f-\phi)\,dx + \int_{E.Ce} (f-\phi)\,dx.$$

The first term is zero because $m(e) = 0$, and the second because the integrand is everywhere zero. Hence

$$\int_E f\,dx = \int_E \phi\,dx.$$

(viii) *If* $f(x) \geqslant 0$ *and* $\int_E f(x)\,dx = 0$, *then* $f(x) = 0$ *almost everywhere in* E.

Let $E_0 = E(f=0)$, and

$$E_n = E\big(M/(n+1) < f \leqslant M/n\big), \qquad n = 1, 2, \ldots,$$

where M is the upper bound of f. Then $E = E_0 + E_1 + E_2 + \ldots$; and

$$m(E_n) \leqslant \frac{n+1}{M} \int_{E_n} f\,dx \leqslant \frac{n+1}{M} \int_E f\,dx = 0.$$

Thus $m(E_n) = 0$ for $n = 1, 2, \ldots$, and the result follows.

10.5. Lebesgue's convergence theorem (theorem of bounded convergence). *Let $f_n(x)$ be a sequence of measurable functions such that $|f_n(x)| \leqslant M$ for all values of n, when x is in a set E, and let*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for all values of x in E. Then

$$\lim_{n\to\infty} \int_E f_n(x)\,dx = \int_E f(x)\,dx.$$

Since sets of measure zero can be omitted from the integrals, it is sufficient that the conditions should hold *almost everywhere*.

Since $|f_n(x)| \leqslant M$ for each n, $|f(x)| \leqslant M$. Hence $f(x)$ is integrable, and we have to prove that

$$\lim \int_E \{f(x)-f_n(x)\}\,dx = 0.$$

Let $g_n = |f-f_n|$, let ϵ be any positive number, and let

$$E_1 = E(\epsilon > g_1, g_2,\ldots), \qquad E_2 = E(g_1 \geqslant \epsilon > g_2, g_3,\ldots),$$
$$E_3 = E(g_2 \geqslant \epsilon > g_3, g_4,\ldots),$$

and so on. Then the sets E_k are measurable; they are non-overlapping, since $g_k \geqslant \epsilon$ in E_{k+1}, but not in $E_1,\ldots, E_k$, so that E_{k+1} has no point in common with $E_1,\ldots, E_k$; and every point of E belongs to some E_k; for $g_n(x) \to 0$ for every x, so that to every x corresponds a first number k such that $g_k, g_{k+1},\ldots$ are all less than ϵ, and then x belongs to E_k.

It follows that

$$\int_E g_n\,dx = \int_{E_1} g_n\,dx + \int_{E_2} g_n\,dx + \ldots.$$

Now $g_n < \epsilon$ in $E_1,\ldots, E_n$, and $g_n \leqslant 2M$ everywhere. Hence

$$\int_E g_n\,dx \leqslant \epsilon\{m(E_1)+\ldots+m(E_n)\}+2M\{m(E_{n+1})+\ldots\}.$$

Making $n \to \infty$, it follows that

$$\overline{\lim} \int_E g_n\,dx \leqslant \epsilon m(E).$$

Hence, making $\epsilon \to 0$, it follows that

$$\lim \int_E g_n\,dx = 0,$$

and the theorem follows.

The theorem is not true for Riemann integrals, because the function $f(x)$ is not necessarily integrable in Riemann's sense, even if each $f_n(x)$ is. For example, let r_1, $r_2,\ldots$ be the rational points in $(0, 1)$, and let $f_n(x) = 1$ if $x = r_1, r_2,\ldots$ or r_n, and $f_n(x) = 0$ elsewhere. Then

$$R\int_0^1 f_n(x)\,dx = 0$$

for every n; but $f(x) = 1$ for every rational x, and $f(x) = 0$ for irrational x, so that $f(x)$ is not integrable in Riemann's sense.

10.51. The theorem of bounded convergence may be stated as a theorem on term-by-term integration of series. *If the series*

$$u_1(x)+u_2(x)+\ldots$$

converges in a set E to $s(x)$, and its partial sums

$$s_n(x) = u_1(x)+\ldots+u_n(x)$$

are bounded for all values of n, when x is in E, then

$$\int_E s(x)\,dx = \int_E u_1(x)\,dx + \int_E u_2(x)\,dx + \ldots.$$

This is the final form of the theorem of bounded convergence proved for Riemann integrals in § 1.76.

10.52. Egoroff's theorem.* *If a sequence of functions converges to a finite limit almost everywhere in a set E, then, given δ, we can find a set of measure greater than $m(E)-\delta$ in which the sequence converges uniformly.*

Let $f_n(x)$ be the sequence, let E' be the set where $f_n(x)$ converges, say to $f(x)$, and let $g_n = |f-f_n|$.

Let $\epsilon_1,\ldots, \epsilon_r,\ldots$ be a sequence of positive numbers tending to zero. Let $S_{n,r}$ be the sub-set of E' where $g_\nu < \epsilon_r$ for $\nu \geqslant n$. Then each of the sets $S_{1,r}$, $S_{2,r},\ldots$ is contained in the next, and their outer limiting set (§ 10.29) is E', since $g_\nu \to 0$ everywhere in E'. Hence we can determine $n(r)$ so that

$$m(E'-S_{n(r),r}) < \frac{\delta}{2^r}.$$

Let $$S = S_{n(1),1}S_{n(2),2}\ldots S_{n(r),r}\ldots.$$

Then, in S, $g_n < \epsilon_r$ $(n \geqslant n(r))$ for all values of r, i.e. $g_n \to 0$ uniformly in S; and

$$m(E-S) = m(E'-S) \leqslant \sum_{r=1}^{\infty} m(E'-S_{n(r),r}) < \sum_{r=1}^{\infty} \frac{\delta}{2^r} = \delta.$$

This proves the theorem.

Example. Use Egoroff's theorem to prove Lebesgue's convergence theorem.

10.6. *If $f(x)$ has a Riemann integral over (a,b), then it has a Lebesgue integral over the same interval, and the two are equal.*

The result is easily proved if we assume that $f(x)$ is measurable;

* Egoroff (1).

for then it certainly has a Lebesgue integral. Dividing up the interval (a,b) by the points $x_0, x_1,..., x_n$, and denoting by m_ν, M_ν the lower and upper bounds of $f(x)$ in $x_\nu < x \leqslant x_{\nu+1}$, we have

$$\sum_{\nu=0}^{n-1} m_\nu(x_{\nu+1}-x_\nu) \leqslant \sum_{\nu=0}^{n-1} \int_{x_\nu}^{x_{\nu+1}} f(x)\,dx \leqslant \sum_{\nu=0}^{n-1} M_\nu(x_{\nu+1}-x_\nu).$$

The middle term is the Lebesgue integral, while each of the extreme terms tends to the Riemann integral. Hence they are equal.

To prove that $f(x)$ is necessarily measurable if it has a Riemann integral, let

$$\phi(x) = m_\nu \quad (x_\nu < x \leqslant x_{\nu+1}), \qquad \Phi(x) = M_\nu \quad (x_\nu < x \leqslant x_{\nu+1}).$$

Then

$$\sum_{\nu=0}^{n-1} m_\nu(x_{\nu+1}-x_\nu) = \int_a^b \phi(x)\,dx, \qquad \sum_{\nu=0}^{n-1} M_\nu(x_{\nu+1}-x_\nu) = \int_a^b \Phi(x)\,dx.$$

Consider now an enumerable infinity of modes of division of the interval (a,b) such that $\max(x_{\nu+1}-x_\nu) \to 0$; and let each set of division-points contain the previous set. Let E be the set of all the division-points. E is enumerable and so of measure zero, and so may be neglected in integration. At any point x not in E, $\phi(x)$ does not decrease, and $\Phi(x)$ does not increase, as we insert division points. Hence $\phi(x) \to m(x)$, $\Phi(x) \to M(x)$, where $m(x)$ and $M(x)$ are the 'lower and upper bounds of $f(x)$ at x', i.e. the limits of the lower and upper bounds in indefinitely small intervals containing x. Also $\phi(x)$ and $\Phi(x)$ are measurable, and hence so are $m(x)$ and $M(x)$; and, by Lebesgue's convergence theorem,

$$\lim \int_a^b \phi(x)\,dx = \int_a^b m(x)\,dx, \qquad \lim \int_a^b \Phi(x)\,dx = \int_a^b M(x)\,dx.$$

But if $f(x)$ has a Riemann integral, each of these limits is equal to it. Hence

$$\int_a^b \{M(x)-m(x)\}\,dx = 0.$$

Since $M(x) \geqslant m(x)$ it follows by § 10.44 (viii) that $M(x) = m(x)$ almost everywhere; and since $M(x) \geqslant f(x) \geqslant m(x)$ it follows that $f(x) = m(x)$ almost everywhere. Hence $f(x)$ is measurable.

10.7. The Lebesgue integral of an unbounded function. Let $f(x)$ be an unbounded measurable function, and suppose first that $f(x) \geqslant 0$. Let $\{f(x)\}_n$, or simply $(f)_n$, denote $f(x)$ at points where $f(x) \leqslant n$, but n where $f(x) > n$. Then $\{f(x)\}_n$ is bounded and measurable, and so integrable. We define the integral of $f(x)$ over the set E to be the limit, if it exists, of the integral of $\{f(x)\}_n$,

$$\int_E f(x)\,dx = \lim_{n\to\infty} \int_E \{f(x)\}_n\,dx.$$

For a positive function $f(x)$ to be integrable over E, it is clearly necessary and sufficient that

$$\int_E \{f(x)\}_n\,dx$$

should be bounded.

The integral of a negative function may be defined in a similar way. In the general case, let $f(x) \geqslant 0$ in E_1, $f(x) < 0$ in E_2. Then we define the integral of $f(x)$ by the equation

$$\int_E f(x)\,dx = \int_{E_1} f(x)\,dx + \int_{E_2} f(x)\,dx.$$

A function which is integrable in this sense is 'absolutely integrable', i.e. $|f(x)|$ is also integrable. In fact it is clear that

$$\int_E |f(x)|\,dx = \int_{E_1} f(x)\,dx - \int_{E_2} f(x)\,dx.$$

It would of course be possible to define integrals which are not absolutely convergent; but we shall see that integrals of the above kind preserve all the characteristic properties of integrals of bounded functions, whereas this would not be true of non-absolutely convergent integrals.

We shall henceforth use the word 'integrable' to describe any function, bounded or unbounded, which has an integral in the above sense.

The use of the expression 'infinity', introduced in § 5.701, is also very convenient here. For example, if

$$\int_E \{f(x)\}_n\,dx$$

tends to infinity with n, we write

$$\int_E f(x)\,dx = \infty.$$

Examples. (i) Show that $\int_0^1 x^{-a}\,dx$ exists as a Lebesgue integral, and is equal to $1/(1-a)$, if $0 < a < 1$; but is infinite if $a \geqslant 1$.

[The Lebesgue definition of the integral is

$$\lim_{n\to\infty}\left\{\int_0^{n^{-1/a}} n\,dx + \int_{n^{-1/a}}^1 x^{-a}\,dx\right\},$$

and the results are the same as in the elementary theory.]

(ii) More generally, let $f(x)$ be positive, and bounded in $(\epsilon, 1)$ for every positive ϵ. Then

$$\int_0^1 f(x)\,dx = \lim_{\epsilon\to 0}\int_\epsilon^1 f(x)\,dx$$

in the sense that both sides are finite and equal, or both infinite.

(iii) The function

$$f(x) = \frac{d}{dx}\left(x^2\sin\frac{1}{x^2}\right) = 2x\sin\frac{1}{x^2} - \frac{2}{x}\cos\frac{1}{x^2}$$

is not integrable in Lebesgue's sense over $(0, 1)$.

[The function is continuous over $(\epsilon, 1)$, and $\lim_{\epsilon\to 0}\int_\epsilon^1 f(x)\,dx$ exists. But

$$\int_0^1 |f(x)|\,dx = \infty;$$

for

$$|f(x)| \geqslant \frac{2}{x}\left|\cos\frac{1}{x^2}\right| - 2x \geqslant \frac{1}{x} - 2x$$

in each of the intervals $\{(2n+\frac{1}{3})\pi\}^{-\frac{1}{2}} \leqslant x \leqslant \{(2n-\frac{1}{3})\pi\}^{-\frac{1}{2}}$, and it is easily seen from this that

$$\int_0^1 \{|f(x)|\}_n\,dx > A\log n.]$$

(iv) Let $f(x)$ be any measurable function in E, and let e_n be the sub-set of E where $n-1 \leqslant f(x) < n$. Then the necessary and sufficient condition that $f(x)$ should be integrable over E is that $\sum_{n=-\infty}^{\infty} |n| m(e_n)$ should be convergent.

(v) We might define the integral of a positive unbounded function $f(x)$ by taking $\{f(x)\}^n = f(x)$ if $f(x) \leqslant n$, and otherwise $\{f(x)\}^n = 0$, and substituting $\{f(x)\}^n$ for $\{f(x)\}_n$ in Lebesgue's definition. Show that this definition is equivalent to that of Lebesgue.

(vi) If $|f(x)| \leqslant \phi(x)$, and $\phi(x)$ is integrable over E, then $f(x)$ is integrable over E.

(vii) If $f(x)$ is integrable over E, and E_n is the part of E where $|f(x)| \geqslant n$, then $m(E_n) = o(1/n)$.

(viii) If $f(x) = 0$ at every point of Cantor's ternary set, and $f(x) = p$ in each of the complementary intervals of length 3^{-p}, then

$$\int_0^1 f(x)\,dx$$

exists in Lebesgue's sense and is equal to 3.

10.71. Elementary properties of integrals. *The integral is additive, i.e. if E_1, E_2,... are non-overlapping sets, and $E = E_1+E_2+\ldots$, then*

$$\int_E f\,dx = \int_{E_1} f\,dx + \int_{E_2} f\,dx + \ldots.$$

We may suppose without loss of generality that $f \geqslant 0$; for if the result is true for positive functions it is true similarly for negative functions, and so by addition in the general case. This remark simplifies many of our proofs.

We define $(f)_n$ as before. The integral of $(f)_n$ is additive, so that

$$\int_E (f)_n\,dx = \sum_{k=1}^{\infty} \int_{E_k} (f)_n\,dx \leqslant \sum_{k=1}^{\infty} \int_{E_k} f\,dx.$$

Now make $n \to \infty$. If there are only a finite number of sets, the result follows (from the equality). If there are an infinity of sets we obtain (from the inequality)

$$\int_E f\,dx \leqslant \sum \int_{E_k} f\,dx.$$

But for any value of K

$$\int_E (f)_n\,dx \geqslant \sum_{k=1}^{K} \int_{E_k} (f)_n\,dx.$$

Making $n \to \infty$ first, and then $K \to \infty$, we obtain

$$\int_E f\,dx \geqslant \sum \int_{E_k} f\,dx.$$

Hence the result. (Notice the analogy with the proof given in § 1.62 that a double series of positive terms may be summed by rows or by columns to the same sum.)

10.72. *The sum of a finite number of integrable functions is integrable, and the integral of the sum is the sum of the integrals of the separate functions.*

It is sufficient to consider two functions, say $f(x)$ and $g(x)$. Suppose first that they are both positive, and let $\phi = f+g$. Then

$$(\phi)_n \leqslant (f)_n+(g)_n \leqslant (\phi)_{2n}.$$

Hence

$$\int_E (\phi)_n \, dx \leqslant \int_E (f)_n \, dx + \int_E (g)_n \, dx \leqslant \int_E (\phi)_{2n} \, dx,$$

and making $n \to \infty$

$$\int_E \phi \, dx \leqslant \int_E f \, dx + \int_E g \, dx \leqslant \int_E \phi \, dx.$$

which gives the required result.

If $f \geqslant 0$, $g < 0$, consider the set where $\phi \geqslant 0$. Here

$$f = \phi + (-g),$$

and the result follows from the previous case. Similarly where $\phi < 0$ we consider $-g = f + (-\phi)$.

Having proved the result for the sum and difference of positive functions, the general result now follows.

10.73. The following results can easily be deduced from the corresponding results for bounded functions:

(i) *If k is a constant,*

$$\int_E kf \, dx = k \int_E f \, dx.$$

(ii)
$$\left| \int_E f \, dx \right| \leqslant \int_E |f| \, dx.$$

(iii) *Two functions which are equal almost everywhere have the same integral.*

(iv) *If $f(x) \geqslant 0$, $\int_E f(x) = 0$, then $f(x) = 0$ almost everywhere in E.*

(v) *If $f(x)$ is integrable over E, and $E_1, E_2, \ldots$ is a sequence of sets contained in E such that $m(E_k) \to 0$, then $\int_{E_k} f(x) \, dx \to 0$, and indeed uniformly for all such sequences of sets.*

For, supposing, as we may, that $f(x) \geqslant 0$, choose n so that

$$\int_E [f(x) - \{f(x)\}_n] \, dx < \epsilon.$$

Having fixed n, we have

$$\int_{E_k} \{f(x)\}_n \, dx \leqslant nm(E_k) < \epsilon \qquad (k > k_0).$$

Hence

$$\begin{aligned} \int_{E_k} f(x) \, dx &= \int_{E_k} \{f(x)\}_n \, dx + \int_{E_k} [f(x) - \{f(x)\}_n] \, dx \\ &\leqslant \int_{E_k} \{f(x)\}_n \, dx + \int_E [f(x) - \{f(x)\}_n] \, dx \\ &< 2\epsilon \qquad (k > k_0), \end{aligned}$$

and the result follows.

Example. Let $f(x)$ be integrable, and $\phi(x)$ integrable in Riemann's sense, over (a, b). Dividing up the interval (a, b) by points x_ν as in § 10.1, prove that, as $\max(x_{\nu+1}-x_\nu) \to 0$,

$$\lim \sum_{\nu=0}^{n-1} \phi(x_\nu) \int_{x_\nu}^{x_{\nu+1}} f(x)\, dx = \int_a^b \phi(x)f(x)\, dx.$$

[Titchmarsh (1).]

10.8. The general convergence theorem of Lebesgue. *If $f_n(x)$ is a sequence of functions such that $|f_n(x)| \leqslant F(x)$, where $F(x)$ is integrable over E, for all values of n and all values of x in E, and*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for all values of x in E, then

$$\lim_{n\to\infty} \int_E f_n(x)\, dx = \int_E f(x)\, dx.$$

As usual, it is sufficient that the conditions should hold almost everywhere. The proof is almost the same as that of the theorem of bounded convergence. We define the sets E_n as before; by § 10.71 the series

$$\sum \int_{E_n} F(x)\, dx$$

is convergent, and we have

$$\int_E g_n\, dx \leqslant \epsilon\{m(E_1)+...+m(E_n)\}+$$
$$+2\int_{E_{n+1}} F(x)\, dx + 2\int_{E_{n+2}} F(x)\, dx + ...$$

Making $n \to \infty$ it follows that

$$\overline{\lim} \int_E g_n\, dx \leqslant \epsilon m(E),$$

and the result now follows as before.

The above theorem enables us to prove a new theorem on term-by-term integration of series. *We may multiply a boundedly convergent series by any integrable function, and integrate term by term.* For if $s_n(x)$ is the nth partial sum of the series, and $|s_n(x)| \leqslant M$, and $\phi(x)$ is the integrable function, we have

$$|\phi(x)s_n(x)| \leqslant M|\phi(x)|,$$

which is integrable, and may be taken as the $F(x)$ of the above proof.

10.81. The following theorem is often useful. Its original form is due to Fatou.*

If $f_n(x) \geqslant 0$ for all values of n, and x in E, and $f_n(x) \to f(x)$ as $n \to \infty$, then

$$\int_E f(x)\,dx \leqslant \varliminf_{n\to\infty} \int_E f_n(x)\,dx.$$

The statement implies that, if the right-hand side is finite, then $f(x)$ is finite almost everywhere and integrable; while, if $f(x)$ is not integrable, or is infinite in a set of positive measure, then

$$\lim_{n\to\infty} \int_E f_n(x)\,dx = \infty.$$

It is easily seen that, with the usual notation,

$$\lim_{n\to\infty} \{f_n(x)\}_k = \{f(x)\}_k.$$

Hence, by the theorem of bounded convergence,

$$\lim_{n\to\infty} \int_E \{f_n(x)\}_k\,dx = \int_E \{f(x)\}_k\,dx.$$

But
$$\int_E \{f_n(x)\}_k\,dx \leqslant \int_E f_n(x)\,dx,$$

and hence
$$\varliminf_{n\to\infty} \int_E f_n(x)\,dx \geqslant \int_E \{f(x)\}_k\,dx.$$

Making $k \to \infty$, the result follows at once if $f(x)$ is finite almos everywhere, the set where $f(x)$ is infinite being omitted from th integral. If $f(x) = \infty$ in a set e of positive measure, then

$$\int_E \{f(x)\}_k\,dx \geqslant km(e)$$

for all values of k, and the result follows.

10.82. A convergence theorem for monotonic sequence

Let $f_1(x)$, $f_2(x)$,... be a sequence of positive integrable function non-decreasing for every value of x in E. Let $f(x)$ be the lim finite or infinite, of the sequence. Then

$$\lim_{n\to\infty} \int_E f_n(x)\,dx = \int_E f(x)\,dx$$

in the following sense:

(i) *if the left-hand side is finite, then $f(x)$ is finite almost eve where and integrable, and the equality holds;*

* Fatou (1), p. 375.

(ii) *if the right-hand side is finite, so is the left-hand side, and equality holds*;

(iii) *if the left-hand side is infinite, then $f(x)$ is not integrable or is infinite in a set of positive measure*;

(iv) *the converse of* (iii) *holds.*

If the left-hand side is finite, so is the right-hand side, by Fatou's theorem; and equality in cases (i) and (ii) follows from Lebesgue's convergence theorem, since $f_n(x) \leqslant f(x)$. Then (iii) follows from (ii) and (iv) from (i).

10.83. We can now put the theorem of § 1.77 on integration of series into a more satisfactory form.

If $u_n(x) \geqslant 0$ for all values of n and x, then

$$\int_a^b \{\sum u_n(x)\}\, dx = \sum \int_a^b u_n(x)\, dx,$$

provided that either side is convergent.

For the partial sum $s_n(x) = u_1(x)+...+u_n(x)$ is positive, and non-decreasing for every value of x.

In particular, the convergence of the right-hand side implies the convergence of $\sum u_n(x)$ for almost all values of x.

We have still to consider the case where the range of integration is infinite; but as we have not yet discussed infinite Lebesgue integrals of this kind, we must postpone the complete result until the end of the next section.

10.9. Integrals over an infinite range. Let $f(x)$ be a function which is integrable over the interval (a, b), for all finite values of b. Let $f_1(x) = f(x)$ where $f(x) \geqslant 0$, and $f_1(x) = 0$ elsewhere; and let $f_2(x) = -f(x)$ where $f(x) < 0$, and $f_2(x) = 0$ elsewhere. Then

$$\int_a^b f(x)\, dx = \int_a^b f_1(x)\, dx - \int_a^b f_2(x)\, dx.$$

Each integral on the right is a non-decreasing function of b, and so tends to a finite limit or to positive infinity as $b \to \infty$. We write

$$\int_a^\infty f_1(x)\, dx = \lim_{b\to\infty} \int_a^b f_1(x)\, dx, \qquad \int_a^\infty f_2(x)\, dx = \lim_{b\to\infty} \int_a^b f_2(x)\, dx,$$

CHAPTER XI

DIFFERENTIATION AND INTEGRATION

11.1. Introduction. The 'fundamental theorem of the integral calculus' is that differentiation and integration are inverse processes. This general principle may be interpreted in two different ways. If $f(x)$ is integrable, the function

$$F(x) = \int_a^x f(t)\,dt \tag{1}$$

is called the indefinite integral of $f(x)$; and the principle asserts that

$$F'(x) = f(x). \tag{2}$$

On the other hand, if $F(x)$ is a given function, and $f(x)$ is defined by (2), the principle asserts that

$$\int_a^x f(t)\,dt = F(x) - F(a). \tag{3}$$

The main object of this chapter is to consider in what sense these theorems are true.

As in elementary theory, (2) follows from (1) for every value of x for which $f(x)$ is continuous. For we can choose h_0 so small that $|f(t)-f(x)| < \epsilon$ for $|t-x| \leqslant h_0$; and then

$$\left|\frac{F(x+h)-F(x)}{h} - f(x)\right| = \left|\frac{1}{h}\int_x^{x+h} \{f(t)-f(x)\}\,dt\right| \leqslant \epsilon \quad (|h| < h_0),$$

by the mean-value theorem. This proves (2).

However, in the Lebesgue theory we consider functions which are in general discontinuous, so that the above argument does not apply to them. Actually the interesting question is, not whether (2) holds at particular points, but whether it is true in general; and to this we can give a satisfactory answer.

If $f(x)$ is any integrable function, its indefinite integral $F(x)$ has almost everywhere a finite differential coefficient equal to $f(x)$.

The problem of deducing (3) from (2) is much more difficult. We require in the first place that $F'(x)$ should exist at any rate almost everywhere, and, as we shall see in § 11.22, this is not necessarily so. Secondly, if $F'(x)$ exists we require that it should be integrable. If we were relying on the Riemann theory, we should find a fundamental difficulty here; for Volterra has shown by an example* that $F'(x)$ may exist everywhere and be bounded, and yet not be integrable in Riemann's sense. In the Lebesgue theory, a differential coefficient is measurable, and so integrable if it is bounded. But, if it is unbounded, it is not necessarily integrable in the Lebesgue sense. The problem has received a satisfactory answer, but it requires a more general process, known as totalization, or Denjoy integration, which we have not space to consider here. The result is that if $F'(x)$ is finite everywhere, then (3) follows from (2) if the integral is taken in the Denjoy sense.

11.2. Differentiation throughout an interval. The ordinary functions of analysis are differentiable in general, i.e. for most values of the variable, though there may be special points at which they are not differentiable. The exceptional points are usually isolated. This seems to have created the impression at one time that a continuous function necessarily has a differential coefficient in general. It was, however, shown by Weierstrass that this is quite untrue. *There is a continuous function which has no differential coefficient anywhere.*

Nevertheless, the idea that an 'ordinary function' has a differential coefficient in general is correct, if we attach this vague expression to a different class of functions. We shall see that it is true in the sense that *a monotonic function has a finite differential coefficient almost everywhere.*

We shall first consider non-differentiable functions, and then proceed to the constructive side of the theory.

11.21. Continuous non-differentiable functions. There are many simple examples of continuous functions which are not differentiable at particular points; for example, if $f(x) = |x|$, the ratio

$$\frac{f(h)-f(0)}{h}$$

tends to different limits, 1 and -1, as $h \to 0$ by positive or negative values; and if $f(x) = x \sin 1/x$ $(x \neq 0)$, $f(0) = 0$, the ratio does not tend to any definite limit.

We can next, by a method known as the condensation of

* Hobson, vol. i, p. 461.

singularities, construct continuous functions which are not differentiable in a set which is everywhere dense, for example in the set of rational points. Let r_1, r_2,... denote the rational numbers between 0 and 1, and let

$$F(x)=\sum_{n=1}^{\infty}a_n f(x-r_n),$$

where $f(x)$ has an assigned singularity at $x=0$, and the coefficients a_n tend to zero sufficiently rapidly. Then $F(x)$ will have the assigned singularity at every rational point. For example,

$$F(x)=\sum_{n=1}^{\infty}\frac{|x-r_n|}{3^n}$$

is continuous, since the series is uniformly convergent; but it is not differentiable at any rational point; for

$$\frac{F(r_k+h)-F(r_k)}{h}=\sum_{n=1}^{k-1}\frac{|r_k+h-r_n|-|r_k-r_n|}{h.3^n}+\frac{|h|}{h.3^k}+$$
$$+\sum_{k+1}^{\infty}\frac{|r_k+h-r_n|-|r_k-r_n|}{h.3^n};$$

and as $h\to 0$ the first term tends to a limit, the second term tends to $\pm 1/3^k$ according as $h>0$ or $h<0$, and, if $|h|<1$, the third term does not exceed

$$\sum_{k+1}^{\infty}\frac{1}{3^n}=\frac{1}{2\,.\,3^k}$$

in absolute value. Hence $F'(r_k)$ does not exist.

To obtain functions which are *everywhere* non-differentiable we have to use quite different methods. The first example of such a function was given by Weierstrass.

11.22. Weierstrass's non-differentiable function. This function is defined by the series

$$f(x)=\sum_{n=0}^{\infty}b^n\cos(a^n\pi x),$$

where $0<b<1$, and a is an odd positive integer. The series is uniformly convergent in any interval, so that $f(x)$ is everywhere continuous. On the other hand, if $ab>1$, the series obtained by term-by-term differentiation is divergent. This in itself does not prove that $f(x)$ is not differentiable, but it suggests possibilities in this direction. We shall prove that *if* $ab > 1+\frac{3}{2}\pi$, *the function has no finite differential coefficient for any value of* x.

We have

$$\frac{f(x+h)-f(x)}{h} = \sum_{n=0}^{\infty} b^n \frac{\cos\{a^n\pi(x+h)\}-\cos(a^n\pi x)}{h}$$

$$= \sum_{n=0}^{m-1} + \sum_{m}^{\infty} = S_m + R_m,$$

say. Now

$$|\cos\{a^n\pi(x+h)\}-\cos(a^n\pi x)| = |a^n\pi h \sin\{a^n\pi(x+\theta h)\}| \leqslant a^n\pi|h|,$$

so that

$$|S_m| \leqslant \sum_{n=0}^{m-1} \pi a^n b^n = \pi \frac{a^m b^m - 1}{ab-1} < \pi \frac{a^m b^m}{ab-1}.$$

We next obtain a lower limit for R_m, giving h a particular value. We can write

$$a^m x = \alpha_m + \xi_m,$$

where α_m is an integer, and $-\frac{1}{2} \leqslant \xi_m < \frac{1}{2}$. Let

$$h = \frac{1-\xi_m}{a^m}.$$

Then

$$0 < h \leqslant \frac{3}{2a^m}.$$

Also $\quad a^n\pi(x+h) = a^{n-m} . a^m\pi(x+h) = a^{n-m}\pi(\alpha_m+1).$

Since a is odd, it follows that

$$\cos\{a^n\pi(x+h)\} = (-1)^{a^{n-m}(\alpha_m+1)} = (-1)^{\alpha_m+1}.$$

Again

$$\cos(a^n\pi x) = \cos\{a^{n-m}\pi(\alpha_m+\xi_m)\} = \cos(a^{n-m}\pi\alpha_m)\cos(a^{n-m}\pi\xi_m)$$
$$= (-1)^{\alpha_m}\cos(a^{n-m}\pi\xi_m).$$

Hence

$$R_m = \frac{(-1)^{\alpha_m+1}}{h} \sum_{n=m}^{\infty} b^n\{1+\cos(a^{n-m}\pi\xi_m)\}.$$

All the terms of this series are positive, and hence, taking the first term only,

$$|R_m| > \frac{b^m}{|h|} > \frac{2}{3}a^m b^m.$$

Hence

$$\left|\frac{f(x+h)-f(x)}{h}\right| \geqslant |R_m|-|S_m| > \left(\frac{2}{3}-\frac{\pi}{ab-1}\right)a^m b^m.$$

If $ab > 1+\frac{3}{2}\pi$, the factor in brackets is positive; and when $m \to \infty$, $h \to 0$, and the expression on the right tends to infinity. Hence $\{f(x+h)-f(x)\}/h$ takes arbitrarily large values, so that $f'(x)$ does not exist or is not finite.

The graph of the function may be said to consist of an infinity of infinitesimal crinkles; but it is almost impossible to form any definite picture of it which does not obscure its essential feature.*

11.23. The following example of a continuous non-differentiable function is due to van der Waerden.† The function is similar to Weierstrass's, but the result is obtained in quite a different way.

Let $f_n(x)$ denote the distance between x and the nearest number of the form $m/10^n$, where m is an integer. Then the function

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

is a continuous non-differentiable function.

Each $f_n(x)$ is continuous; and $|f_n(x)| < 10^{-n}$, so that the series is uniformly convergent. Hence $f(x)$ is continuous.

Let x be any number in the interval $(0, 1)$, and suppose it expressed as a decimal. If the qth figure is 4 or 9, let $x' = x-10^{-q}$; otherwise let $x' = x+10^{-q}$. Then if $n < q$, the nearest number $m/10^n$ is the same for x and x', and x and x' lie on the same side of it; while if $n \geqslant q$, the numbers $m/10^n$ and $m'/10^n$ corresponding to x and x' differ by $x-x'$. These rules may be verified by considering simple examples, such as $q = 2$, $x = \cdot 326$, $\cdot 346$, or $\cdot 396$.

It follows that

$$\begin{aligned} f_n(x')-f_n(x) &= \pm(x'-x) \qquad && (n < q) \\ &= 0 && (n \geqslant q). \end{aligned}$$

Hence
$$f(x')-f(x) = \sum_{n=1}^{q-1} \pm(x'-x) = p(x'-x),$$

* For further properties of this function see Hardy (7), where the same result is obtained for $ab > 1$. A general method of constructing continuous non-differentiable functions is given by Knopp (2).

† Van der Waerden (1).

where p is an integer, and is odd or even with $q-1$. Hence $\{f(x')-f(x)\}/(x'-x)$ cannot tend to a finite limit as $x' \to x$.

11.3. The four derivates of a function. Whether the differential coefficient

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

exists or not, the four expressions

$$\varlimsup_{h\to+0}\frac{f(x+h)-f(x)}{h}, \qquad \varliminf_{h\to+0}\frac{f(x+h)-f(x)}{h},$$

$$\varlimsup_{h\to-0}\frac{f(x+h)-f(x)}{h}, \qquad \varliminf_{h\to-0}\frac{f(x+h)-f(x)}{h}$$

always have a meaning, being either finite, or positive or negative infinity. They are called the upper and lower derivates on the right, and the upper and lower derivates on the left, respectively. We shall denote them by

$$D^+f(x), \qquad D_+f(x), \qquad D^-f(x), \qquad D_-f(x)$$

respectively, the sign referring to that of h in the above ratio, and its position corresponding to the 'lower' or 'upper' limit. If $D^+f = D_+f$, the function is said to have a right-hand derivative, if $D^-f = D_-f$, a left-hand derivative. The necessary and sufficient condition for the existence of the ordinary differential coefficient is that all the derivates should be equal.

We denote the left-hand and right-hand derivatives, when they exist, by $f'_-(x)$ and $f'_+(x)$.

Examples. (i) The function $\sqrt{x^2}$, where the positive value of the square root is always taken, has different left-hand and right-hand derivatives at $x = 0$.

(ii) Let $f(x) = x\sin 1/x$ $(x \neq 0)$, 0 $(x = 0)$. Then at $x = 0$

$$D_+f = -1, \qquad D^+f = 1, \qquad D_-f = -1, \qquad D^-f = 1.$$

(iii) Let
$$f(x) = \begin{cases} ax\sin^2 1/x + bx\cos^2 1/x & (x > 0) \\ 0 & (x = 0) \\ a'x\sin^2 1/x + b'x\cos^2 1/x & (x < 0), \end{cases}$$

where $a < b$, $a' < b'$. Then at $x = 0$

$$D_+f = a, \qquad D^+f = b, \qquad D_-f = a', \qquad D^-f = b'.$$

(iv) If $f(x)$ is continuous in (a, b), and one of its derivates is non-negative in the interval, then $f(a) \leqslant f(b)$.

[Let $D^+f \geqslant 0$, for example. Suppose that $f(b)-f(a) < -\epsilon(b-a)$, and let $\phi(x) = f(x)-f(a)+\epsilon(x-a)$. Then $\phi(b) < 0$. Also $\phi(x) > 0$ for some