The reader should draw a figure showing the regions of the z-plane which correspond to the upper half of the w-plane.

6.5. The function $w = \int\limits_0^z \frac{dt}{\sqrt{(1-t^2)}}$. We know that this function is equal to $\arcsin z$. Consider, however, what can be deduced from the integral about the representation of the z-plane on the w-plane.

Consider what part of the w-plane corresponds to the first quadrant in the z-plane. If $z = iy$ is purely imaginary, we have

$$w = \int\limits_0^y \frac{i\,ds}{\sqrt{1+s^2}},$$

which is also purely imaginary; and as $y \to \infty$, so does w. The two imaginary axes therefore correspond.

Again, as z increases along the real axis from 0, so does w, until z reaches 1, and w reaches the value

$$\int\limits_0^1 \frac{dt}{\sqrt{(1-t^2)}}.$$

Let us denote the value of this integral by I. Actually $I = \frac{1}{2}\pi$, but we need not assume that this is known.

We must now suppose that the path of integration passes above the point $z = 1$, say by a small semicircle. Then $\arg(1-t)$ decreases from 0 to $-\pi$, and so $\arg(1-t)^{-\frac{1}{2}}$ increases from 0 to $\frac{1}{2}\pi$; thus the integrand becomes purely imaginary, and we have

$$w = I + i \int\limits_1^z \frac{dt}{\sqrt{(t^2-1)}}.$$

Finally, as z tends to infinity along the real axis, w tends to infinity along the line $u = I$.

Hence the boundary of the first quadrant in the z-plane corresponds to the boundary of the half-strip $0 < u < I$, $v > 0$ in the w-plane.

Secondly, the function is simple in this region. We cannot deduce this from § 6.45 without some further argument, since both the regions extend to infinity. But it is easily seen directly. For, if t lies in the first quadrant, the imaginary part of t^2-1

is positive, and $\arg\sqrt{(1-t^2)}$ lies between $-\frac{1}{2}\pi$ and 0. Hence

$$f(z_2)-f(z_1)=\int_{z_1}^{z_2}\frac{dt}{\sqrt{(1-t^2)}},$$

and, taking the integral along the straight line, it is of the form

$$k\int\frac{d\lambda}{\rho e^{i\phi}},$$

where λ is a real variable, $\rho>0$, $-\frac{1}{2}\pi<\phi<0$. Such an integral plainly cannot vanish. Here the function cannot take any value twice.

Also the quadrant of the circle $|z|=R$, where R is large, corresponds to a curve which (by the previous remark) has no double point, and which connects the two sides of the strip and lies entirely at a great distance from the real axis. Hence, by the theorem of § 6.45, the quadrant is represented simply on the whole strip.

The next problem is to continue the function beyond this limited region. This can be done by the method of reflection of § 4.51. In fact all the boundaries in each figure are straight lines.

In the first place, the imaginary axes correspond. Hence, reflecting in these lines, we see that the second quadrant in the z-plane corresponds to the half-strip $-I<u<0$, $v>0$, in the w-plane. Hence the upper half of the z-plane corresponds to the half-strip $-I<u<I$, $v>0$.

Next, reflect with respect to the segment $(0, 1)$ of the real axis in the z-plane. We obtain the lower half of the z-plane. In the w-plane we obtain the half-strip $-I<u<I$, $v<0$.

Hence the whole strip $-I<u<I$ corresponds to the whole z-plane, but there are singularities at $z=\pm 1$ round which we must not pass; we may, for example, suppose the plane cut from $-\infty$ to -1 and from 1 to ∞.

Again, a reflection in the segment $(1,\infty)$ of the real z-axis corresponds to a reflection in the line $u=I$ in the w-plane. Hence the lower half of the z-plane (obtained by continuation to the right of $z=1$) corresponds to the half-strip $I<u<3I$, $v>0$.

It is plain that we can continue this process indefinitely. The whole w-plane is divided up into strips of breadth $2I$, each of which corresponds to the whole z-plane.

If we reflect a point w_0 of the strip $-I < u < I$, first in $u = I$ and then in $u = 3I$, we obtain the point w_0+4I. Meanwhile the corresponding z_0, being reflected twice in the real axis, has returned to its original value. Then w_0 and w_0+4I correspond to the same z_0; i.e. if $z = g(w)$, then $g(w) = g(w+4I)$. The inverse function $g(w)$ is therefore periodic, with period $4I$.

Example. Prove that the function

$$w = \int_0^z \frac{dt}{\sqrt{\{(1-t^2)(1-k^2t^2)\}}} \qquad (0 < k < 1)$$

represents the upper half of the z-plane on the rectangle in the w-plane bounded by the lines $u = -K$, $u = K$, $v = 0$, $v = K'$, where

$$K = \int_0^1 \frac{dt}{\sqrt{\{(1-t^2)(1-k^2t^2)\}}}, \qquad K' = \int_1^{1/k} \frac{dt}{\sqrt{\{(t^2-1)(1-k^2t^2)\}}}.$$

Prove that the inverse function $z = g(w)$ has the two periods $4K$ and $2iK'$. [Hurwitz-Courant, *Funktionentheorie*, pp. 302–3.]

6.6. Representation of a polygon on a half-plane. The functions of the previous section are examples of the representation of a polygon on a half-plane. It is possible to do this with *any* polygon. The complete proof would take us too far, but we can show in a general way how it is to be done.

Consider a polygon in the w-plane with n sides and angles $\alpha_1\pi, \alpha_2\pi, \dots, \alpha_n\pi$, where $\alpha_1+\dots+\alpha_n = n-2$. If $\alpha_m < 1$ $(m = 1, 2, \dots n)$ the polygon is convex. Some of the α's may be greater than 1, but the polygon must never cross itself. Suppose that the vertices of the polygon are to correspond to points $a_1, a_2, \dots, a_n$ on the real axis in the z-plane. So long as z remains on the real axis without passing any of the points $a_1, \dots$, w remains on the same side of the polygon; hence the angle between the z-curve and the w-curve is constant, i.e. $\arg(dw/dz)$ is constant (see § 6.1).

If $$\frac{dw}{dz} = C(z-a_1)^{\alpha_1-1}(z-a_2)^{\alpha_2-1}\dots(z-a_n)^{\alpha_n-1},$$

then dw/dz has this property. When z passes the point a_1 by a small circle above it, $\arg(z-a_1)$ decreases from π to 0, the amplitudes of the other factors returning to their original values. Hence $\arg(dw/dz)$ decreases by $\pi(\alpha_1-1)$. Hence the w-curve turns through $\pi(1-\alpha_1)$ in the positive direction. This corresponds to an angle $\pi\alpha_1$ of the polygon.

The required function is therefore of the form

$$w = C\int\limits_{z_0}^{z} (t-a_1)^{\alpha_1-1}(t-a_2)^{\alpha_2-1}...(t-a_n)^{\alpha_n-1}\,dt.$$

The integrand is $O(1/|t|^2)$ as $|t| \to \infty$; hence the integral converges as $z \to \pm\infty$, and to the same value in each case, since the integral along a large semicircle above the real axis tends to zero. Hence, as z describes the real axis, w describes a closed curve, and in fact, from the construction, a polygon with the prescribed angles. By first considering the real z-axis as closed by a large semicircle above it, we can apply the theorems of §§ 6.45–6, and we find that the interior of the polygon is represented simply on the upper half-plane.

To show that we can choose the constants so that a polygon with given sides, as well as given angles, can be represented, is more difficult. For a triangle, however, the result is easily obtained. Consider, for example, the triangle with vertices $w = i\surd 3$, 0, 1 (and angles $\frac{1}{6}\pi$, $\frac{1}{2}\pi$, $\frac{1}{3}\pi$), and let the vertices correspond to $z = -1$, 0, 1. The above theory gives

$$w = C\int\limits_{z_0}^{z} (t+1)^{-\frac{5}{6}}t^{-\frac{1}{2}}(t-1)^{-\frac{2}{3}}\,dt.$$

The origins correspond if $z_0 = 0$; and if we write the formula as

$$w = C'\int\limits_{0}^{z} (t+1)^{-\frac{5}{6}}t^{-\frac{1}{2}}(1-t)^{-\frac{2}{3}}\,dt,$$

where C' is real and positive, the directions of the real axes correspond. Finally, if

$$1 = C'\int\limits_{0}^{1} (t+1)^{-\frac{5}{6}}t^{-\frac{1}{2}}(1-t)^{-\frac{2}{3}}\,dt,$$

then $z = 1$ corresponds to $w = 1$, and the required representation is obtained.

Examples. (i) Prove that the function

$$w = \int\limits_{0}^{z} \frac{dt}{(1-t^2)^{\frac{2}{3}}}$$

represents a half-plane on an equilateral triangle.

(ii) Prove that the function

$$w = \int_0^z \frac{dt}{\sqrt{(1-t^4)}}$$

represents the unit circle in the z-plane on a square in the w-plane. [Hurwitz-Courant. Put $z = (\zeta - i)/(\zeta + i)$.]

6.7. Representation of any region on a circle. A fundamental theorem of Riemann states that *any region with a suitable boundary can be represented on a circle by a simple analytic function.* It is beyond our scope to inquire exactly what forms of region are suitable. The region may be the interior of a closed curve; or one side of a curve which goes to infinity in both directions (e.g. a half-plane); or any form of strip between two such curves; or even the whole plane cut along a curve (e.g. along the real axis from 0 to infinity).

Let D be a region of one of the above types.

The function which represents any region simply on a bounded region must be simple and bounded. Let us first verify that there are such functions for D. Let a and b be two points on the boundary of D, and let

$$w = \sqrt{\left(\frac{z-a}{z-b}\right)}.$$

In D we can restrict ourselves to one branch of this function; this branch is simple, and the values taken by it cover a part only of the w-plane (since both branches together cover the whole w-plane once). Let w_0 be a point of the region not covered. Then $1/(w-w_0)$ is simple and bounded in D. Also

$$f(z) = \frac{p}{w-w_0} + q$$

is simple and bounded, and we can choose p and q so that, at a given point of D, $f(z) = 0$ and $f'(z) = 1$.

Consider all functions $f(z)$ which are simple and bounded in D, and such that $f(z) = 0$ and $f'(z) = 1$ at a given point P of D. Let $M(f)$ denote the maximum modulus of $f(z)$. Let ρ be the lower bound of $M(f)$ for all such functions.

There is then either a function $\phi(z)$ of the set such that $M(\phi) = \rho$; or a sequence $f_1, f_2, \ldots$ of functions of the set such that

$$\lim M(f_n) = \rho.$$

We shall show that the second alternative reduces to the first. Since the sequence $f_n(z)$ is bounded in D, we can, by § 5.22, select from it a partial sequence which tends uniformly to a limit in any region interior to D. Let $f_{n_1}(z)$, $f_{n_2}(z)$,... be such a sequence, and $\phi(z)$ its limit. Then $\phi(z)$ is also a function of the set; for it is bounded, and $\phi(z)=0$, $\phi'(z)=1$ at P, and $\phi(z)$ is simple (§ 6.44), being not constant since $\phi'(z)=1$. Also

$$M(\phi)\geqslant\rho$$

by definition of ρ; and

$$M(f_{n_\nu})<\rho+\epsilon \qquad (\nu>\nu_0),$$

i.e.

$$|f_{n_\nu}(z)|<\rho+\epsilon \qquad (\nu>\nu_0).$$

Making $\nu\to\infty$, it follows that

$$|\phi(z)|<\rho+\epsilon,$$

i.e. $M(\phi)\leqslant\rho$.

This proves the existence of a function $\phi(z)$ of the set with $M(\phi)=\rho$; and since $\phi(z)$ is not constant, $\rho>0$.

We shall show that the function $w=\phi(z)$ represents D simply on the circle $|w|<\rho$. In the proof, we may suppose that $\rho=1$. Let Δ be the region of the w-plane on which $w=\phi(z)$ represents D. Since $M(\phi)=1$, Δ is included in $|w|\leqslant 1$, and reaches its circumference at one point at least.

If the theorem is not true, Δ has a boundary point α inside the circle ($|\alpha|<1$). Then each branch of

$$w_1=\sqrt{\left(\frac{w-\alpha}{\bar{\alpha}w-1}\right)}$$

is regular for w in Δ. Also $|w_1|\leqslant 1$ if $|w|\leqslant 1$ (§ 6.24), and $w_1(0)=\sqrt{\alpha}$. Let

$$w_2=\frac{w_1-\sqrt{\alpha}}{\overline{\sqrt{\alpha}}w_1-1}.$$

Then $|w_2|\leqslant 1$ if $|w_1|\leqslant 1$. Also

$$\frac{dw_2}{dw}=\frac{dw_2}{dw_1}\cdot\frac{dw_1^2}{dw}\cdot\frac{1}{2w_1}=\frac{|\alpha|-1}{(\overline{\sqrt{\alpha}}w_1-1)^2}\cdot\frac{|\alpha|^2-1}{(\bar{\alpha}w-1)^2}\cdot\frac{1}{2w_1}$$

$$=\frac{|\alpha|-1}{(|\alpha|-1)^2}\cdot\frac{|\alpha|^2-1}{2\sqrt{\alpha}}=\frac{|\alpha|+1}{2\sqrt{\alpha}}$$

at $w=0$. The modulus of this is greater than unity. Hence

$$w_3=\frac{2\sqrt{\alpha}}{|\alpha|+1}w_2$$

is a function of the set considered, and

$$M(w_3) = \frac{2|\sqrt{\alpha}|}{|\alpha|+1} < 1.$$

This gives a contradiction, and the theorem follows.

6.71. Uniqueness theorem. Let D be a region in the z-plane which is the interior of a simple closed contour, or which is of one of the other types considered in § 6.7. *Then there is a uniquely determined function $w = f(z)$ which represents D simply on the interior of the unit circle in the w-plane, and is such that, if z_0 is a given point in D, $f(z_0) = 0$ and $f'(z_0)$ is real and positive.*

It follows from § 6.7 that there is one such function, say $w = f(z)$. Let $z = F(w)$ be the inverse function. Suppose that there is another function $w = g(z)$ with the same properties. Then the function $W = g\{F(w)\}$ represents the unit circle simply on itself, the centre and the direction of the real axis through it remaining unaltered. Hence, by § 6.42, $g\{F(w)\} = w$, i.e. $g(z) = f(z)$.

6.8. Further properties of simple functions. The class of functions $f(z)$ which are simple for $|z| < 1$, and such that $f(0) = 0$, $f'(0) = 1$, has been studied in great detail. The function $w = z$ belongs to the class, and represents the unit circle on itself. For all functions of the class the 'map' of the unit circle is subject to certain limitations. For the details we may refer to Bieberbach, *Funktionentheorie*, ii. 82–94, Landau, *Ergebnisse* (ed. 2), pp. 107–14, or Dienes, *The Taylor Series*, Ch. VIII. We shall, however, obtain the simplest property of the map. *For any function of the class, no boundary point of the map of the unit circle is nearer to the origin than the point $\frac{1}{4}$.*

We deduce this from the two following theorems.

Let

$$w = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots$$

be simple for $|z| > 1$, and regular except for the pole at infinity. Then

$$\sum_{n=1}^{\infty} n|a_n|^2 \leqslant 1.$$

Since the function is simple, any circle $|z| = r > 1$ corresponds to a simple closed curve in the w-plane, which encloses a positive area. If $w = u+iv$, $u = u(\theta)$, $v = v(\theta)$, on the curve, the area enclosed is

$$\int_0^{2\pi} u(\theta)v'(\theta)\,d\theta = \int_0^{2\pi} \frac{w(\theta)+\bar{w}(\theta)}{2}.\frac{w'(\theta)-\bar{w}'(\theta)}{2i}\,d\theta$$

$$= \tfrac{1}{4}\int_0^{2\pi}\left\{re^{i\theta}+re^{-i\theta}+\sum_{n=1}^{\infty}\frac{a_n e^{-in\theta}+\bar{a}_n e^{in\theta}}{r^n}\right\}\times$$

$$\times\left\{re^{i\theta}+re^{-i\theta}-\sum_{n=1}^{\infty}\frac{na_n e^{-in\theta}+n\bar{a}_n e^{in\theta}}{r^n}\right\}d\theta$$

$$= \tfrac{1}{2}\pi\left\{\left(r+\frac{\bar{a}_1}{r}\right)\left(r-\frac{a_1}{r}\right)+\left(r+\frac{a_1}{r}\right)\left(r-\frac{\bar{a}_1}{r}\right)-\sum_{n=2}^{\infty}\frac{2na_n\bar{a}_n}{r^{2n}}\right\}$$

$$= \pi r^2-\pi\sum_{n=1}^{\infty} n|a_n|^2 r^{-2n}.$$

Since this is positive

$$\sum_{n=1}^{\infty} n|a_n|^2 r^{-2n} \leqslant r^2,$$

and making $r \to 1$ the result follows.

If $$w = f(z) = z+a_2z^2+\ldots$$ *is simple in* $|z|<1$, *then* $|a_2| \leqslant 2$.

The function

$$F(z) = \sqrt{\{f(z^2)\}} = z+\tfrac{1}{2}a_2z^3+\ldots$$

is also simple in $|z|<1$; for it is regular, since $f(z^2)$ does not vanish except at $z=0$, where it has a double zero; and if $F(z_1) = F(z_2)$, then $f(z_1^2) = f(z_2^2)$, and hence, since $f(z)$ is simple, $z_1^2 = z_2^2$, i.e. $z_1 = \pm z_2$. But $F(z)$ is an odd function, so that $z_1 = -z_2$ gives $F(z_1) = -F(z_2)$. Hence the only solution of $F(z_1) = F(z_2)$ is $z_1 = z_2$, i.e. $F(z)$ is simple.

It follows that

$$\left\{F\left(\frac{1}{z}\right)\right\}^{-1} = z-\frac{\frac{1}{2}a_2}{z}+\ldots$$

is simple for $|z|>1$. Hence by the previous theorem

$$\tfrac{1}{4}|a_2|^2+\ldots \leqslant 1,$$

and the result follows.

Now let $$w = f(z) = z+a_2z^2+\ldots$$

be a function of the class considered in the main theorem. Let c be a value which it does not take in the unit circle, i.e. a point outside the 'map' of the unit circle. Then

$$\frac{c\,f(z)}{c-f(z)} = z+\left(a_2+\frac{1}{c}\right)z^2+\ldots$$

is regular and simple for $|z| < 1$. Hence

$$\left|a_2+\frac{1}{c}\right| \leqslant 2,$$

$$\left|\frac{1}{c}\right| \leqslant 2+|a_2| \leqslant 4,$$

$$|c| \geqslant \tfrac{1}{4},$$

and the result follows.

Example. The function $z/(1-z)^2$ belongs to the above class. It has $a_2 = 2$, and it gives a map passing through $w = -\frac{1}{4}$.

[The only solution of

$$\frac{z}{(1-z)^2} = \frac{z'}{(1-z')^2}, \qquad |z| < 1, \qquad |z'| < 1,$$

is $z = z'$.]

MISCELLANEOUS EXAMPLES

1. In a given linear transformation, the point z_0 is such that there is some circle $|z-z_0| = R$ which transforms into a concentric circle $|w-z_0| = R'$. Show that the locus of z_0 is a rectangular hyperbola; and that to each point z_0 on the locus corresponds just one circle (real or imaginary) which transforms into a concentric circle.

2. Show that, if $$\frac{dz}{dw} = -2i\left(w-\frac{1}{w}\right),$$ and the constant of integration is properly chosen, the whole z-plane cut along the semi-infinite lines $x = \pm\pi$, $y \leqslant 0$, corresponds to the upper half of the w-plane.

3. Show that, if $$\frac{dz}{dw} = \frac{w}{\sqrt{(w^2-a^2)}},$$ and a and the constant of integration and the value of the square root are properly chosen, the upper half of the w-plane corresponds to the upper half of the z-plane, cut along the imaginary axis from $z = 0$ to a point $z = ik$.

4. If $f(z)$ is regular inside and on the unit circle, $|f(z)| \leqslant M$ on the circle, and $f(a) = 0$, where $|a| < 1$, then $$|f(z)| \leqslant M\left|\frac{z-a}{\bar{a}z-1}\right|$$ inside the circle.

5. If $f(z)$ is regular inside and on the unit circle, $|f(z)| \leqslant M$ on the circle, and $f(0) = a$, where $0 < a < M$, then

$$|f(z)| \leqslant M \frac{M|z|+a}{a|z|+M}$$

inside the circle.

[Consider $F(z) = M\{f(z)-a\}/\{af(z)-M^2\}$.]

6. Either branch of the function

$$\frac{z}{\sqrt{(1-z)}}$$

is simple for $|z| < 1$.

7. Show that the function

$$\frac{z}{(1-z)^3}$$

is simple for $|z| < \frac{1}{2}$, but not in any larger circle with centre at the origin.

8. Show that the function

$$f(z) = z+a_2z^2+a_3z^3+\ldots$$

is simple for $|z| < 1$ if

$$\sum_{n=2}^{\infty} n|a_n| \leqslant 1.$$

CHAPTER VII

POWER SERIES WITH A FINITE RADIUS OF CONVERGENCE

7.1. The circle of convergence. We know that every power series has a circle of convergence, within which it converges, and outside which it diverges. The radius of this circle may, however, be infinite, so that the circle includes the whole plane. In this chapter we shall consider power series which have a finite radius of convergence.

The radius of convergence of a power series is determined by the moduli of the coefficients in the series.

The power series

$$\sum_{n=0}^{\infty} a_n z^n \tag{1}$$

has the radius of convergence

$$R = \varliminf_{n\to\infty} |a_n|^{-1/n}. \tag{2}$$

Suppose that R is defined by (2). If z is a point where the series (1) converges, $a_n z^n \to 0$ as $n \to \infty$. Hence, if n is sufficiently large,

$$|a_n z^n| < 1,$$

i.e.

$$|z| < |a_n|^{-1/n}.$$

Making $n \to \infty$, it follows that $|z| \leqslant R$. Hence the radius of convergence does not exceed R.

On the other hand, for sufficiently large values of n,

$$|a_n|^{-1/n} > R-\epsilon,$$

i.e.

$$|a_n| < (R-\epsilon)^{-n}.$$

Hence the series (1) is convergent if $\sum (R-\epsilon)^{-n}|z|^n$ is convergent, i.e. if $|z| < R-\epsilon$. Since ϵ is arbitrarily small, the series (1) is convergent if $|z| < R$. Thus the radius of convergence is at least equal to R. Putting together the two results, the theorem follows.

Examples. (i) Find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{2n!}{(n!)^2} z^n, \quad \sum_{n=1}^{\infty} \frac{n!}{n^n} z^n, \quad \sum_{n=0}^{\infty} n!\, z^{n^2}.$$

(ii) If $R = 1$, and the only singularities on the unit circle are simple poles, then a_n is bounded. [For

$$f(z) = \frac{a}{1-ze^{-i\alpha}} + \frac{b}{1-ze^{-i\beta}} + \dots + \frac{k}{1-ze^{-i\kappa}} + g(z),$$

where $g(z)$ is regular for $|z| < 1+\delta$ $(\delta > 0)$. Hence $g(z) = \sum b_n z^n$, where $b_n = o(1)$.]

(iii) If $R = 1$, and the only singularities on the unit circle are poles of order p, then $a_n = O(n^{p-1})$.

7.11. We also know from the Cauchy-Taylor theorem that the circle of convergence of the series passes through the singularity or singularities of the function which are nearest to the origin. *Hence the modulus of the nearest singularity can be determined from the moduli of the coefficients in the series.*

7.2. Position of the singularities. While the modulus of the nearest singularities is determined in quite a simple way, their exact position is not usually so easy to find. There are, however, some special cases in which we can identify a particular point as a singularity.

In the following theorems we shall take the radius of convergence to be unity; we can, of course, pass from this to the general case by a simple transformation.

7.21. *If $a_n \geqslant 0$ for all values of n, then $z = 1$ is a singular point.*

Suppose, on the contrary, that $z = 1$ is regular. Then, if we take a point ρ on the real axis between 0 and 1, there is a circle with centre ρ which includes the point 1, and in which the function is regular. If $f(z)$ is the function, the Taylor's series about ρ is

$$\sum_{\nu=0}^{\infty} \frac{f^{(\nu)}(\rho)}{\nu!}(z-\rho)^{\nu}, \tag{1}$$

and this converges at a point $z = 1+\delta$ $(\delta > 0)$. Now

$$f^{(\nu)}(\rho) = \sum_{n=\nu}^{\infty} n(n-1)\dots(n-\nu+1)a_n \rho^{n-\nu}, \tag{2}$$

and so the above series is

$$\sum_{\nu=0}^{\infty} \frac{(z-\rho)^{\nu}}{\nu!} \sum_{n=\nu}^{\infty} n(n-1)\dots(n-\nu+1)a_n \rho^{n-\nu}.$$

This is a double series of positive terms, convergent for $z = 1+\delta$.

Hence we may invert the order of the summations, and we obtain

$$\sum_{n=0}^{\infty} a_n \sum_{\nu=0}^{n} \frac{n(n-1)\ldots(n-\nu+1)}{\nu!}(z-\rho)^\nu \rho^{n-\nu}$$
$$= \sum_{n=0}^{\infty} a_n\{(z-\rho)+\rho\}^n = \sum_{n=0}^{\infty} a_n z^n.$$

Hence the original series is convergent for $z = 1+\delta$, contrary to the hypothesis that the radius of convergence is 1. This proves the theorem.

Another proof, due to Pringsheim, is as follows. There is at least one singularity, say $e^{i\alpha}$, on the unit circle. The Taylor's series about $\rho e^{i\alpha}$, where $0 < \rho < 1$, is

$$\sum_{\nu=0}^{\infty} \frac{f^{(\nu)}(\rho e^{i\alpha})}{\nu!}(z-\rho e^{i\alpha})^\nu,$$

and, since $e^{i\alpha}$ is a singularity, this has the radius of convergence $1-\rho$. But it is clear from (2) that, if $a_n \geqslant 0$ for all values of n,

$$|f^{(\nu)}(\rho e^{i\alpha})| \leqslant f^{(\nu)}(\rho).$$

Hence the radius of convergence of (1) does not exceed $1-\rho$. Hence $z = 1$ is a singularity.

7.22. *If a_n is real for all values of n, and $\sum a_n$ is properly divergent, i.e.*

$$s_n = a_0+a_1+\ldots+a_n \to \infty \text{ (or } \to -\infty),$$

then $z = 1$ is a singular point.

We have, for $|z| < 1$,

$$\frac{f(z)}{1-z} = \sum_{n=0}^{\infty} a_n z^n \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} s_n z^n. \tag{1}$$

by § 1.65, the series being absolutely convergent. Hence

$$f(z) = (1-z)\sum_{n=0}^{\infty} s_n z^n$$
$$= (1-z)\sum_{n=0}^{N} s_n z^n + (1-z)\sum_{n=N+1}^{\infty} s_n z^n = f_1(z)+f_2(z),$$

say. Suppose that $s_n \to \infty$. Then, given any positive number G, however large, we can choose N so that $s_n > G$ $(n > N)$. Then, if $0 < z < 1$,

$$f_2(z) > (1-z)\sum_{n=N+1}^{\infty} Gz^n = Gz^{N+1}.$$

Having fixed N, we can choose z_0 so near to 1 that

$$z^{N+1} > \tfrac{1}{2}, \qquad |f_1(z)| < \tfrac{1}{4}G \qquad (z > z_0),$$

since $z^{N+1} \to 1$ and $f_1(z) \to 0$. Hence

$$f(z) > \tfrac{1}{4}G \qquad (z > z_0),$$

i.e. $f(z) \to \infty$ as $z \to 1$. This proves the theorem.

If we merely know that $|s_n| \to \infty$, we cannot deduce that $z = 1$ is a singularity. For example,

$$\frac{1}{(1+z)^3} = 1-3z+...+(-1)^n \tfrac{1}{2}(n+1)(n+2)z^n+...,$$

and here $|s_n| \sim \frac{1}{4}n^2$, though the function is regular at $z = 1$.

7.23. General tests for singular points. If we consider any particular point on the circle of convergence, we can devise a test to determine whether it is a singularity or not; but it is not one which lends itself to simple calculations.

We may suppose that the radius of convergence is 1, and, by a preliminary transformation, we may bring the point to be considered to $z = 1$.

The principle to be used is that, if we expand $f(z)$ about a point on the real axis between 0 and 1, the circle of convergence includes $z = 1$ if $f(z)$ is regular at this point, and not otherwise. But we can make a transformation which brings the formula into a simpler form than the direct application of the principle would give.

Let
$$F(w) = \frac{1}{1-w} f\left(\frac{w}{1-w}\right).$$

Then $F(w)$ is regular for $\mathbf{R}(w) < \frac{1}{2}$, since $\mathbf{R}(w) < \frac{1}{2}$ gives $|w| < |1-w|$. Now

$$F(w) = \sum_{m=0}^{\infty} \frac{a_m w^m}{(1-w)^{m+1}} = \sum_{m=0}^{\infty} a_m w^m \sum_{r=0}^{\infty} \frac{(m+r)!}{m!\,r!} w^r$$
$$= \sum_{n=0}^{\infty} w^n \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} a_m.$$

Let
$$b_n = \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} a_m.$$

Then a necessary and sufficient condition that $z = 1$ should be a singularity of $f(z)$, i.e. that $w=\frac{1}{2}$ should be a singularity of $F(w)$, is that

$$\varlimsup_{n\to\infty} |b_n|^{-\frac{1}{n}} = \tfrac{1}{2}.$$

For then $F(w)$ has a singularity on $|w|=\frac{1}{2}$, and every point other than $w=\frac{1}{2}$ is known to be regular.

By using other transformations we obtain a variety of other equivalent conditions.

Example. Prove that every point on the unit circle is a singularity of

$$f(z) = \sum_{n=0}^{\infty} z^{2^n}.$$

[For the point $z = e^{i\theta}$, we have to consider

$$b_n = \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} a_m,$$

where $a_m = e^{i2^r\theta}$ if $m = 2^r$, and $a_m = 0$ otherwise. Clearly

$$|b_n| \leqslant \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} = 2^n.$$

Also
$$b_{2^n} = \sum_{m=0}^{n} \frac{2^n!}{2^m!(2^n-2^m)!} e^{i2^m\theta}.$$

The modulus of the term $m=n-1$ is asymptotic to $A2^{2^n-\frac{1}{2}n}$, by Stirling's theorem. Also, if u_m denotes the general term, and $0<m\leqslant n-2$,

$$\left|\frac{u_m}{u_{m+1}}\right| = \frac{(2^m+1)\ldots 2^{m+1}}{(2^n-2^{m+1}+1)\ldots(2^n-2^m)} < \left(\frac{2^{m+1}}{2^n-2^m}\right)^{2^m} < \left(\frac{2}{3}\right)^{2^m} < \frac{4}{9},$$

and the remainder is easily seen to be negligible. Hence

$$\lim |b_{2^n}|^{\frac{1}{2^n}} = 2.]$$

7.3. Convergence of the series and regularity of the function. It will be noticed that we have not used the convergence or divergence of the original series as a test for regularity or singularity of the function. In general no such test is possible, for all possible relations can occur. If

$$f(z) = \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n} = \log\frac{1}{1+z},$$

the series is convergent, and the function regular, at $z=1$; and the series is divergent, and the function singular, at $z=-1$.

On the other hand, if

$$f(z) = \sum_{n=0}^{\infty} (-1)^n z^n = \frac{1}{1+z}$$

the series is divergent, but the function is regular, at $z = 1$; while if

$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} = \int_0^z \frac{1}{w} \log \frac{1}{1-w}\, dw$$

the series is convergent at $z = 1$, but $f(z)$ has a singularity.

7.31. There is, however, one case in which divergence of the series indicates a singularity of the function; the case where $a_n \to 0$. This follows from the following theorem.

If

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

and $a_n \to 0$, *the series is convergent at every point of the unit circle where the function is regular.*

Two proofs of this theorem have been given. One, due to M. Riesz, is essentially a 'complex variable' method, and is given by Landau, *Ergebnisse*, § 18. The following proof, due to W. H. Young (7), is of Fourier-series type. In some respects it is not so simple as Riesz's, but it can easily be adapted to give more general results.

We may without loss of generality take the point in question to be $z = 1$; and we may suppose that $f(1) = 0$. We have then to prove that $s_n \to 0$.

It follows from § 7.22 (1) that

$$s_n = \frac{1}{2\pi i} \int \frac{f(z)}{1-z} \frac{dz}{z^{n+1}}.$$

Taking the contour to be the circle $|z| = r < 1$, we have

$$s_n = \frac{1}{2\pi r^n} \int_{-\pi}^{\pi} \frac{f(re^{i\theta})}{1-re^{i\theta}} e^{-in\theta}\, d\theta.$$

Let $0 < \delta < \pi$, and let $\phi(\theta) = \phi(\theta, \delta, r)$ be such that

(i) $\phi(\theta) = 1/(1-re^{i\theta})$ for $-\pi < \theta < -\delta$ and $\delta < \theta < \pi$;

(ii) $\phi(\theta)$ and $\phi'(\theta)$ are continuous for $-\pi < \theta < \pi$;

(iii) $|\phi(\theta)| < K$, $|\phi'(\theta)| < K$, $|\phi''(\theta)| < K$, for $-\pi < \theta < \pi$, where K depends on δ but not on r. For example, if

$$\phi(\theta) = a\theta^3+b\theta^2+c\theta+d \qquad (-\delta \leqslant \theta \leqslant \delta),$$

we can determine the coefficients so that

$$\phi(\pm\delta) = \frac{1}{1-re^{\pm i\delta}}, \quad \phi'(\pm\delta) = \frac{ire^{\pm i\delta}}{(1-re^{\pm i\delta})^2}.$$

Then (ii) is satisfied; and a, b, c, and d are linear functions of $\phi(\pm\delta)$ and $\phi'(\pm\delta)$, the moduli of these not exceeding $\frac{1}{2}\operatorname{cosec}\frac{1}{2}\delta$ and $\frac{1}{4}\operatorname{cosec}^2\frac{1}{2}\delta$ respectively. Hence (iii) is satisfied.

We can then write

$$2\pi r^n s_n = \int_{-\delta}^{\delta} \frac{f(re^{i\theta})}{1-re^{i\theta}} e^{-in\theta}\,d\theta + \int_{-\pi}^{\pi} f(re^{i\theta})\phi(\theta)e^{-in\theta}\,d\theta - \int_{-\delta}^{\delta} f(re^{i\theta})\phi(\theta)e^{-in\theta}\,d\theta$$

$$= I_1+I_2-I_3.$$

Since $f(z)$ is regular at $z=1$, and $f(1)=0$, we have

$$f(re^{i\theta}) = O(|1-re^{i\theta}|)$$

in an interval $|\theta| \leqslant \theta_0$, uniformly for $r_0 \leqslant r \leqslant 1$. Hence

$$I_1 = \int_{-\delta}^{\delta} O(1)\,d\theta = O(\delta).$$

Suppose now that δ is fixed. We have

$$I_2 = \sum_{m=1}^{\infty} a_m r^m \int_{-\pi}^{\pi} e^{i(m-n)\theta}\phi(\theta)\,d\theta,$$

by uniform convergence; and integrating by parts twice each integral except the nth,

$$I_2 = a_n r^n \int_{-\pi}^{\pi} \phi(\theta)\,d\theta - \sum_{m\neq n} \frac{a_m r^m}{(m-n)^2} \int_{-\pi}^{\pi} e^{i(m-n)\theta}\phi''(\theta)\,d\theta,$$

all the integrated terms cancelling. Let $\epsilon_\nu = \max\limits_{m>\nu}(|a_m|)$, so that $\epsilon_\nu \to 0$. Then

$$|I_2| \leqslant 2\pi K\left\{\epsilon_n+\epsilon_0 \sum_{m\leqslant\frac{1}{2}n} \frac{1}{(m-n)^2}+\epsilon_{\frac{1}{2}n} \sum_{m>\frac{1}{2}n} \frac{1}{(m-n)^2}\right\}$$

$$= O(\epsilon_n)+O(1/n)+O(\epsilon_{\frac{1}{2}n}).$$

Finally

$$I_3 = \left[\frac{f\phi e^{-in\theta}}{-in}\right]_{\delta}^{-\delta} + \frac{1}{in}\int_{-\delta}^{\delta} (f'\phi + f\phi')e^{-in\theta}\,d\theta = O\left(\frac{1}{n}\right).$$

Given ϵ, we can choose δ so that $|I_1| < \frac{1}{3}\epsilon$ for all values of n; and, having fixed δ, we can choose $n_0 = n_0(\epsilon)$ so large that $|I_2| < \frac{1}{3}\epsilon$ and $|I_3| < \frac{1}{3}\epsilon$ for $n > n_0$. Hence

$$2\pi r^n |s_n| < \epsilon \qquad (n > n_0).$$

Making $r \to 1$, it follows that $2\pi |s_n| < \epsilon$ $(n > n_0)$, i.e. $s_n \to 0$.

The reader will notice that we have not used the full force of the hypothesis '$f(z)$ is regular at $z = 1$'; and the proof would hold with little change if e.g. $f(z) = O(|1-z|^{\alpha})$, where $\alpha > 0$. For the more general form of the theorem we must refer the reader to Young's paper.

7.4. Over-convergence.* We know that, at every point outside the circle of convergence of a power series, the series is divergent. But if, instead of considering the whole sequence of partial sums of the series, we consider particular sequences of these sums, it is sometimes possible to obtain a convergent sequence. This is shown by the following example.

Let
$$f(z) = \sum_{n=1}^{\infty} \frac{\{z(1-z)\}^{4^n}}{p_n},$$

where p_n is the maximum coefficient in the polynomial $\{z(1-z)\}^{4^n}$. Then in each of the polynomials

$$\frac{\{z(1-z)\}^{4^n}}{p_n}$$

the moduli of the coefficients do not exceed 1, and one of them is actually equal to 1. Also the highest term in this polynomial is of degree 2.4^n, whereas the lowest term in the next polynomial is of degree 4^{n+1}. Hence, if we expand $f(z)$ in powers of z, each term is a single term of one of the above polynomials. The radius of convergence of this series is 1, since $|a_n| \leqslant 1$ for all n, while $a_n = 1$ for an infinity of values of n.

In particular, the above series of polynomials is convergent for $|z| < 1$. But, since it is formally unchanged by the substitution $z = 1-w$, it is also convergent for $|w| < 1$, i.e. for $|1-z| < 1$.

* Ostrowski (1), Zygmund (1), Estermann (2).

The special sequence of partial sums obtained by taking each polynomial as a whole is therefore convergent in a region which lies partly outside the unit circle.

A power series which has a sequence of partial sums convergent outside the circle of convergence of the series is said to be 'over-convergent'. Of course a power series can only be over-convergent in the neighbourhood of a point of the circle where the function is regular. We shall next define a class of functions which have this property of over-convergence in the neighbourhood of every point of the circle where the function is regular.

7.41. *Suppose that the power series*

$$f(z)=\sum_{n=0}^{\infty} a_n z^n$$

has the radius of convergence 1, *and that there are an infinite number of gaps in the sequence of coefficients, i.e. there are sequences of suffixes* p_k, q_k, *such that* $a_n=0$ *for* $p_k<n<q_k$; *and* $q_k \geqslant (1+\vartheta)p_k$, *with a fixed positive* ϑ.

Then the sequence of the corresponding partial sums

$$s_{p_k}(z)=\sum_{n=0}^{p_k} a_n z^n$$

is convergent in a region of which every regular point of $f(z)$ *on the circle of convergence is an interior point.*

To prove this it is sufficient to consider the point $z=1$. Suppose that $f(z)$ is regular at $z=1$. Then, if δ is small enough, it is regular in and on the circle with centre $\frac{1}{2}$ and radius $\frac{1}{2}+\delta$.

We apply Hadamard's three-circles theorem to the function

$$\phi(z)=f(z)-s_{p_k}(z),$$

and the circles with centre $\frac{1}{2}$ and radii $\frac{1}{2}-\delta$, $\frac{1}{2}+\epsilon$, $\frac{1}{2}+\delta$, where $0<\epsilon<\delta$. If M_1, M_2, M_3 are the maximum moduli of $\phi(z)$ on these circles, then

$$M_2^{\log\frac{1+2\delta}{1-2\delta}} \leqslant M_1^{\log\frac{1+2\delta}{1+2\epsilon}} M_3^{\log\frac{1+2\epsilon}{1-2\delta}}. \tag{1}$$

In order to prove that $s_{p_k}(z)\to f(z)$ in a region including $z=1$, it is sufficient to show that we can take ϵ so small that $M_2\to 0$ when $p_k\to\infty$. The idea of the proof is that, while M_3 is substantially of the order $(1+\delta)^{p_k}$, M_1 behaves like $(1-\delta)^{q_k}$, and so, since q_k is greater than p_k, the right-hand side of (1) is small when p_k is large.

To every positive η (say with $\eta < \frac{1}{2}\delta$) corresponds a K such that

$$|a_n| < K(1-\eta)^{-n}.$$

Hence, as $k \to \infty$,

$$M_1 \leqslant |a_{q_k} z^{q_k}| + |a_{q_k+1} z^{q_k+1}| + \dots$$

$$< K \frac{\left(\dfrac{1-\delta}{1-\eta}\right)^{q_k}}{1-\dfrac{1-\delta}{1-\eta}} = O\left\{\left(\frac{1-\delta}{1-\eta}\right)^{q_k}\right\} = O\left\{\left(\frac{1-\delta}{1-\eta}\right)^{(1+\vartheta)p_k}\right\}.$$

Also, if M is the maximum modulus of $f(z)$ on the outer circle,

$$M_3 \leqslant M + |a_0| + \dots + |a_{p_k} z^{p_k}|$$

$$\leqslant M + K\left\{1 + \frac{1+\delta}{1-\eta} + \dots + \left(\frac{1+\delta}{1-\eta}\right)^{p_k}\right\} = O\left\{\left(\frac{1+\delta}{1-\eta}\right)^{p_k}\right\}.$$

Hence the right-hand side of (1) is

$$O\left[\left\{\left(\frac{1-\delta}{1-\eta}\right)^{(1+\vartheta)\log\frac{1+2\delta}{1+2\epsilon}} \left(\frac{1+\delta}{1-\eta}\right)^{\log\frac{1+2\epsilon}{1-2\delta}}\right\}^{p_k}\right].$$

When $\epsilon \to 0$, $\eta \to 0$, the expression in brackets tends to

$$(1-\delta)^{(1+\vartheta)\log(1+2\delta)}(1+\delta)^{-\log(1-2\delta)},$$

which is less than 1 if δ is small enough; for its logarithm $\sim -2\vartheta\delta^2$ as $\delta \to 0$, and so is negative for small δ. Hence we may take ϵ and η so small that the original expression is less than 1; and the result then follows.

7.42. The occurrence of gaps in the series is not merely a useful device for producing over-convergence. It has an essential connexion with it. This is shown by the following theorem, which is a sort of converse of the preceding one.

If a sequence $s_{p_k}(z)$ of partial sums of the series $f(z) = \sum a_n z^n$, with radius of convergence 1, *is uniformly convergent in the neighbourhood of a point on the unit circle, then*

$$f(z) = g(z) + r(z),$$

where the power series $g(z)$ has an infinite number of gaps p_k, q_k, where $q_k > (1+\vartheta)p_k$, and the radius of convergence of the power series $r(z)$ is greater than 1.

We shall not give the proof, which is more difficult than that of the direct theorem.

7.43. Hadamard's gap theorem. *If, in the power series*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

$a_n = 0$ *except when* n *belongs to a sequence* n_k *such that* $n_{k+1} > (1+\vartheta)n_k$, *where* $\vartheta > 0$, *then the circle of convergence of the series is a natural boundary of the function.*

This is an almost immediate corollary of the theorem on over-convergence. For, if $f(z)$ were regular at any point of the circle, the series would be over-convergent at that point, i.e. the sequence

$$s_{n_k}(z) = \sum_{n=1}^{n_k} a_n z^n$$

would be convergent at a point outside the circle. But for a series of the given form this sequence of partial sums is the same as the whole sequence of partial sums. Hence over-convergence is impossible, and consequently every point of the circle of convergence is a singularity of $f(z)$.

7.44. Mordell's proof of the theorem.* This is a very simple direct proof. Suppose that the radius of convergence is 1. Let $z = aw^p + bw^{p+1}$, where $0 < a < 1$, $a+b = 1$, and p is a positive integer. Clearly $|z| \leqslant 1$ if $|w| \leqslant 1$; and it is easily seen that $|z| < 1$ if $|w| \leqslant 1$, except that $z = 1$ if $w = 1$. Let

$$\begin{aligned}\phi(w) = f(z) &= \sum a_n(aw^p + bw^{p+1})^n \\ &= \sum a_n(a^n w^{pn} + \ldots + b^n w^{(p+1)n}) = \sum b_n w^n.\end{aligned}$$

Then $\phi(w)$ is regular for $|w| \leqslant 1$, except possibly at $w = 1$. We shall show that the radius of convergence of the power series for $\phi(w)$ is 1, and hence that $w = 1$ is a singularity of $\phi(w)$.

We observe that, in the last expression but one for $\phi(w)$, no power of w occurs twice if

$$(p+1)n_k < pn_{k+1},$$

i.e.

$$p\left(\frac{n_{k+1}}{n_k} - 1\right) > 1$$

throughout the series; and this is true if $p > 1/\vartheta$. The expression $\sum b_n w^n$ is then obtained by simply omitting the brackets in the previous expression.

* Mordell (1).

If the series for $\phi(w)$ had a radius of convergence greater than 1, it would be convergent for a real $w > 1$, and therefore the series for $f(z)$ would be convergent for a real $z > 1$, which is false. This proves the theorem.

There is still another proof,* depending on the criterion of § 7.23.

The theorem of § 7.41 can be proved in a similar way.† Let the series for $f(z)$ satisfy the condition of § 7.41. Then $\phi(w)$ can have no singularity for $|w| \leqslant 1$ except possibly at $w = 1$. Hence if $f(z)$ is regular at $z = 1$, $\phi(w)$ is regular at $w = 1$, and so in $|w| < 1+\delta$ for some positive δ. Hence $\sum b_n w^n$ converges for $|w| < 1+\delta$, and in particular $\sum_{n=0}^{(p+1)p_k} b_n w^n$ converges for $|w| < 1+\delta$. Hence $\sum_{n=0}^{p_k} a_n z^n$ converges in a region of which $z = 1$ is an interior point.

7.5. Asymptotic behaviour near the circle of convergence.

If the coefficients in the power series satisfy a sufficiently simple law as $n \to \infty$, we can deduce an asymptotic expression for the function $f(z)$ as z approaches the circle of convergence along a radius vector. The simplest case of this process is given by the following theorem.

Let
$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad g(x) = \sum_{n=0}^{\infty} b_n x^n,$$
where $a_n \geqslant 0$, $b_n \geqslant 0$, and the series converge for $0 < x < 1$ and diverge for $x = 1$. If, as $n \to \infty$,
$$a_n \sim C b_n, \tag{1}$$
then as $x \to 1$
$$f(x) \sim C g(x). \tag{2}$$

Given ϵ, we can find N such that
$$|a_n - C b_n| < \epsilon b_n \qquad (n > N).$$

* See Landau, *Ergebnisse*, § 19.

† Pointed out by Mr. M. M. Crum.

Then

$$\begin{aligned}|f(x)-Cg(x)| &= \left|\sum_{n=0}^{\infty}(a_n-Cb_n)x^n\right| \\ &\leqslant \left|\sum_{n=0}^{N}(a_n-Cb_n)x^n\right| + \left|\sum_{n=N+1}^{\infty}(a_n-Cb_n)x^n\right| \\ &\leqslant \sum_{n=0}^{N}|a_n-Cb_n| + \epsilon\sum_{n=N+1}^{\infty}b_n x^n \\ &\leqslant \sum_{n=0}^{N}|a_n-Cb_n| + \epsilon g(x).\end{aligned}$$

Having fixed N, we can, since $g(x)\to\infty$, choose δ so that

$$\sum_{n=0}^{N}|a_n-Cb_n| < \epsilon g(x) \qquad (x>1-\delta).$$

Then $$|f(x)-Cg(x)| < 2\epsilon g(x) \qquad (x>1-\delta),$$

which proves the theorem.

The same result, however, holds under more general conditions. *Let the series converge for* $0<x<1$; *let*

$$s_n = a_0+a_1+...+a_n, \qquad t_n = b_0+b_1+...+b_n,$$

and let s_n *and* t_n *be positive, and* $\sum s_n$ *and* $\sum t_n$ *divergent, and let*

$$s_n \sim Ct_n. \tag{3}$$

Then (2) *is still true.*

For as in § 7.22, for $0<x<1$

$$f(x) = (1-x)\sum_{n=0}^{\infty}s_n x^n, \qquad g(x) = (1-x)\sum_{n=0}^{\infty}t_n x^n,$$

and by the previous theorem

$$\sum_{n=0}^{\infty}s_n x^n \sim C\sum_{n=0}^{\infty}t_n x^n.$$

Hence the result.

In particular, *if* $s_n \sim Cn$, *then*

$$f(x) \sim \frac{C}{1-x}.$$

Examples. (i) If $p<1$, as $x\to 1$

$$\sum_{n=1}^{\infty}\frac{x^n}{n^p} \sim \frac{\Gamma(1-p)}{(1-x)^{1-p}}.$$

[We have $$(1-x)^{p-1} = \frac{1}{\Gamma(1-p)}\sum_{n=0}^{\infty}\frac{\Gamma(n-p+1)}{\Gamma(n+1)}x^n,$$

and we can use the lemma of § 1.87.]

(ii) Show that if

$$F(\alpha, \beta, \gamma, x) = 1 + \frac{\alpha . \beta}{1 . \gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 . 2 . \gamma(\gamma+1)} x^2 + \dots,$$

then, as $x \to 1$,

$$F(\alpha, \beta, \gamma, x) \sim \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \frac{1}{(1-x)^{\alpha+\beta-\gamma}}$$

if $\alpha+\beta > \gamma$; and that

$$F(\alpha, \beta, \alpha+\beta, x) \sim \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \log \frac{1}{1-x}.$$

7.51. The converse problem. It is easily seen that there is no general converse of the above theorems; from the asymptotic behaviour of $f(x)$ we cannot deduce that of a_n, or even of s_n. Consider, for example, the function

$$f(x) = \frac{1}{(1+x)^2(1-x)} = (1-x) \sum_{n=0}^{\infty} (n+1)x^{2n}$$

$$= \sum_{n=0}^{\infty} (n+1)(x^{2n} - x^{2n+1}).$$

Here $s_{2m+1} = 0$, while $s_{2m} = m+1$; hence s_n oscillates infinitely, though $f(x) \sim \frac{1}{4}/(1-x)$.

The coefficients in this example are, of course, not all positive; and this is, in a sense, the cause of the failure of the converse theorem. If we assume that all the coefficients are positive, we can state a precise converse of the last result of the previous section.

If $a_n \geqslant 0$ for all values of n, and as $x \to 1$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \sim \frac{1}{1-x},$$

then as $n \to \infty$

$$s_n = \sum_{\nu=0}^{n} a_\nu \sim n.$$

This theorem is due to Hardy and Littlewood.* We shall give an extremely elegant proof which has recently been obtained by Karamata.†

7.52. In order to appreciate the point of the proof, it may be well to see what can be proved by fairly obvious arguments. In the first place

$$f(x) \geqslant \sum_{\nu=0}^{n} a_\nu x^\nu \geqslant x^n s_n$$

* Hardy and Littlewood (2). † Karamata (1).

for all values of x and n. Taking $x = e^{-1/n}$, we obtain, since $f(x) < A/(1-x)$,

$$e^{-1}s_n \leqslant \frac{A}{1-x} = \frac{A}{1-e^{-1/n}} < An,$$

say
$$s_n < A_1 n. \tag{1}$$

On the other hand, using (1), we have

$$\begin{aligned} f(x) &= (1-x)\sum_{m=0}^{\infty} s_m x^m \\ &< (1-x)s_n \sum_{m=0}^{n-1} x^m + A_1(1-x)\sum_{m=n}^{\infty} mx^m \\ &< s_n + A_1 n x^n + \frac{A_1 x^{n+1}}{1-x}. \end{aligned}$$

Taking $x = e^{-\lambda/n}$, we obtain, since $f(x) > A/(1-x) > An/\lambda$ if $n > 2\lambda$,

$$\frac{An}{\lambda} < s_n + Ane^{-\lambda} + \frac{Ane^{-\lambda}}{\lambda}.$$

Hence, if λ is sufficiently large,

$$s_n > A_2 n. \tag{2}$$

What we have to show is that A_1 and A_2 can be replaced by $1+\epsilon$ and $1-\epsilon$ respectively. The above argument is too crude to do this, and the method actually used is far from being an obvious one.

7.53. Karamata's proof. The proof depends on the well-known theorem of Weierstrass, that we can approximate uniformly to any continuous function by a sequence of polynomials.* Let $g(x)$ be continuous in $(0, 1)$, and ϵ a given positive number. Then there are polynomials $p(x)$, $P(x)$, such that

$$p(x) \leqslant g(x) \leqslant P(x), \tag{1}$$

and
$$\int_0^1 \{g(x)-p(x)\}\,dx \leqslant \epsilon, \qquad \int_0^1 \{P(x)-g(x)\}\,dx \leqslant \epsilon. \tag{2}$$

This is obviously true if $p(x)$ and $P(x)$ differ by at most $\frac{1}{2}\epsilon$ from $g(x)-\frac{1}{2}\epsilon$ and $g(x)+\frac{1}{2}\epsilon$ respectively.

If $g(x)$ has a discontinuity of the first kind in the interval, say at $x = c$, we can still construct polynomials satisfying (1) and (2). Suppose, for example, that $g(c-0) < g(c+0)$. Let

* A proof is given in § 13.33. For another proof see Goursat, *Cours d'Analyse*, t. 1, § 206.

$\phi(x) = g(x)+\frac{1}{2}\epsilon$ for $x < c-\delta$ and for $x > c$; and, for $c-\delta \leqslant x \leqslant c$, let $\phi(x) = \max\{l(x), g(x)+\frac{1}{4}\epsilon\}$, where $l(x)$ is the linear function of x such that $l(c-\delta) = g(c-\delta)+\frac{1}{2}\epsilon$, $l(c) = g(c+0)+\frac{1}{2}\epsilon$. Then $\phi(x)$ is continuous, and $\phi(x) > g(x)$. It is easily seen that, if δ is small enough, a polynomial $P(x)$ which approximates sufficiently closely to $\phi(x)$ has the required properties. Similarly we may construct $p(x)$.

To prove the theorem of Hardy and Littlewood, we first prove that

$$\lim_{x\to 1}(1-x)\sum_{n=0}^{\infty} a_n x^n P(x^n) = \int_0^1 P(t)\,dt \tag{3}$$

for any polynomial $P(x)$. It is clearly sufficient to consider the case $P(x) = x^k$. Then the left-hand side is

$$(1-x)\sum_{n=0}^{\infty} a_n x^{n+kn} = \frac{1-x}{1-x^{k+1}}\left\{(1-x^{k+1})\sum_{n=0}^{\infty} a_n (x^{k+1})^n\right\}$$

$$\to \frac{1}{k+1} = \int_0^1 x^k\,dx,$$

and the result follows.

Next, we have

$$\lim_{x\to 1}(1-x)\sum_{n=0}^{\infty} a_n x^n g(x^n) = \int_0^1 g(t)\,dt \tag{4}$$

if $g(t)$ is continuous, or has a discontinuity of the first kind. For let $p(x)$ and $P(x)$ be polynomials satisfying (1) and (2). Then, since $g(x) \leqslant P(x)$, and the coefficients are positive,

$$\overline{\lim}(1-x)\sum_{n=0}^{\infty} a_n x^n g(x^n) \leqslant \overline{\lim}(1-x)\sum_{n=0}^{\infty} a_n x^n P(x^n)$$

$$= \int_0^1 P(t)\,dt < \int_0^1 g(t)\,dt + \epsilon.$$

Making $\epsilon \to 0$, it follows that

$$\overline{\lim}(1-x)\sum_{n=0}^{\infty} a_n x^n g(x^n) \leqslant \int_0^1 g(t)\,dt.$$

Similarly, arguing with $p(x)$, we obtain

$$\underline{\lim}(1-x)\sum_{n=0}^{\infty} a_n x^n g(x^n) \geqslant \int_0^1 g(t)\,dt,$$

and (4) follows.

Now let

$$g(t) = 0 \ (0 \leqslant t < e^{-1}), \qquad = 1/t \ (e^{-1} \leqslant t \leqslant 1).$$

Then

$$\int_0^1 g(t)\,dt = \int_{1/e}^1 \frac{dt}{t} = 1. \tag{5}$$

Let $x = e^{-1/N}$. Then

$$\sum_{n=0}^{\infty} a_n x^n g(x^n) = \sum_{n \leqslant 1/\log(1/x)} a_n = \sum_{n=0}^{N} a_n = s_N,$$

and so, by (4) and (5), $s_N \sim 1/(1-x) \sim N$. This proves the theorem.

7.6. Abel's theorem and its converse. In this section we return to a subject already discussed in Chapter I. In § 1.22 we proved Abel's theorem for real power series: *if the series*

$$\sum_{n=0}^{\infty} a_n$$

converges to the sum s, then

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \to s$$

as $x \to 1$ through real values. In § 1.23 we proved Tauber's theorem, that *the converse deduction holds, provided that* $a_n = o(1/n)$. We shall now consider a number of generalizations of these theorems.*

7.61. *If*

$$\sum_{n=0}^{\infty} a_n = s, \tag{1}$$

then

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \to s \tag{2}$$

as $z \to 1$ along any path lying between two chords of the unit circle which pass through $z = 1$.

As in § 1.22, it is sufficient to show that the power series is uniformly convergent, but now we must prove uniform convergence in a region included between two chords through $z = 1$, and a sufficiently small circle with centre at $z = 1$.

We have to adapt the argument used to prove Abel's lemma (§ 1.131) to the present conditions. Let

$$s_{n,p} = a_n + a_{n+1} + \ldots + a_p,$$

* Landau, *Ergebnisse*, Ch. III, and Hardy and Littlewood (1), (2), (3), (4).

so that $|s_{n,p}| < \epsilon$ $(n_0 \leqslant n < p)$. Then

$$\sum_{\nu=n}^{m} a_\nu z^\nu = s_{n,n} z^n + (s_{n,n+1} - s_{n,n}) z^{n+1} + \ldots + (s_{n,m} - s_{n,m-1}) z^m$$
$$= s_{n,n}(z^n - z^{n+1}) + \ldots + s_{n,m-1}(z^{m-1} - z^m) + s_{n,m} z^m.$$

Hence for $n \geqslant n_0$

$$\left|\sum_{\nu=n}^{m} a_\nu z^\nu\right| \leqslant \epsilon \left\{\sum_{\nu=n}^{m-1} |z^\nu - z^{\nu+1}| + |z|^m\right\}$$
$$< \epsilon\{|1-z| \sum_{\nu=0}^{\infty} |z|^\nu + 1\}$$
$$= \epsilon \left\{\frac{|1-z|}{1-|z|} + 1\right\}.$$

The result now follows as in the previous case, provided that

$$\frac{|1-z|}{1-|z|}$$

is bounded as $z \to 1$ on the path considered. It is this that makes it necessary to restrict the path, for this function can be made large by taking z near to 1, but still nearer to the circumference.

Suppose, then, that

$$|1-z| \leqslant k(1-|z|) \qquad (k > 1). \tag{3}$$

This inequality is satisfied in a region bounded by the curve

$$|1-z| = k(1-|z|).$$

Putting $1-z = \rho e^{i\phi}$, the equation becomes

$$\rho = k - k|1-\rho e^{i\phi}|,$$

i.e.
$$(\rho - k)^2 = k^2(1 - 2\rho\cos\phi + \rho^2),$$

i.e.
$$\rho = 2\frac{k^2\cos\phi - k}{k^2 - 1}.$$

This represents a curve with two branches through $z = 1$, each making an angle $\arccos(1/k)$ with the real axis. By choosing k sufficiently large we can make the curve include any region of the required type. Since (3) is satisfied inside the curve, the theorem now follows.

7.62. We can also obtain a similar extension of Tauber's theorem.

If $f(z) \to s$ as $z \to 1$ along a path satisfying the same conditions as before, and $a_n = o(1/n)$, then $\sum a_n$ converges to the sum s.

In view of the above analysis, the proof given in § 1.23 now requires little modification. We have to prove that

$$S_1-S_2=\sum_{n=N+1}^{\infty} a_n z^n-\sum_{n=0}^{N} a_n(1-z^n)\to 0,$$

where $N=[1/(1-|z|)]$. As before, if $|na_n|<\epsilon$ for $n>N$,

$$|S_1|=\left|\sum_{N+1}^{\infty} na_n.\frac{z^n}{n}\right|<\frac{\epsilon}{N+1}\sum_{N+1}^{\infty}|z|^n<\frac{\epsilon}{(N+1)(1-|z|)}<\epsilon.$$

Now $\qquad |1-z^n|=|(1-z)(1+z+...+z^{n-1})|\leqslant|1-z|n.$

Hence, if 7.61 (3) is satisfied,

$$|S_2|\leqslant\sum_{n=1}^{N}|na_n(1-z)|\leqslant k(1-|z|)\sum_{n=1}^{N}n|a_n|\leqslant\frac{k}{N}\sum_{n=1}^{N}n|a_n|,$$

and this tends to zero, by the lemma of § 1.23. This proves the theorem.

7.63. Tauber's theorem for regular paths. It is not possible to extend Abel's theorem, at any rate in its obvious form, to paths which touch the unit circle; for example, it is known* that the series

$$\sum_{n=1}^{\infty} n^{-b}e^{in^a} \qquad (0<a<1)$$

is convergent if $b>1-a$; but, if $b<1-\frac{1}{2}a$, the function

$$f(z)=\sum_{n=1}^{\infty} n^{-b}e^{in^a}z^n$$

does not tend to a limit as $z\to 1$ along an arc of a circle touching the unit circle at $z=1$.

On the other hand, we can obtain an extension of Tauber's theorem to paths which touch the circle, provided that they are sufficiently regular.

A path will be called 'regular' if it is defined by equations $x=x(t)$, $y=y(t)$, where $x'(t)$ and $y'(t)$ are continuous and never both 0, so that there is a definite tangent at each point.

If $f(z)\to s$ as $z\to 1$ along a regular path inside the circle, and $a_n=o(1/n)$, then $\sum a_n$ converges to the sum s.

We may suppose without loss of generality that $s=0$ Let

* See Hardy and Littlewood (3), p. 207.

C be the path in question. Then the integral

$$\int_z^1 f(w)\,dw,$$

taken along C, exists; and it is $o(|1-z|)$ as $z \to 1$. For, given ϵ, we have $|f(w)| \leqslant \epsilon$ for w sufficiently near to 1 on C. Hence

$$\left|\int_z^1 f(w)\,dw\right| \leqslant \epsilon\, l(z),$$

where $l(z)$ is the length of C from z to 1. But $l(z) \sim |1-z|$ as $z \to 1$; for if $t = 0$ corresponds to $z = 1$,

$$\frac{l(z)}{t} = \frac{1}{t}\int_0^t [\{x'(u)\}^2+\{y'(u)\}^2]^{\frac{1}{2}}\,du \to [\{x'(0)\}^2+\{y'(0)\}^2]^{\frac{1}{2}},$$

$x'(u)$ and $y'(u)$ being continuous; and

$$\frac{x-1}{t} \to x'(0), \qquad \frac{y}{t} \to y'(0).$$

Hence
$$\int_z^1 f(w)\,dw = o(|1-z|). \tag{1}$$

Now if z and z' are points on C,

$$\int_z^{z'} f(w)\,dw = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z'^{n+1}-z^{n+1}).$$

This series converges uniformly with respect to z' for $|z'| \leqslant 1$ (since $a_n = o(1/n)$). Hence, making $z' \to 1$,

$$\int_z^1 f(w)\,dw = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(1-z^{n+1}). \tag{2}$$

Let $N = [1/|1-z|]$. Then

$$\int_z^1 f(w)\,dw = \sum_{n=0}^{N} + \sum_{N+1}^{\infty} \frac{a_n}{n+1}(1-z^{n+1}) = \sum\nolimits_1 + \sum\nolimits_2,$$

and
$$\sum\nolimits_2 = \sum_{N+1}^{\infty} o\left(\frac{1}{n^2}\right) = o\left(\frac{1}{N}\right) = o(|1-z|). \tag{3}$$

Also

$$\begin{aligned} 1-z^{n+1} &= (1-z)(1+z+\dots+z^n) \\ &= (1-z)(n+1)-(1-z)^2\{n+(n-1)z+\dots+z^{n-1}\} \\ &= (1-z)(n+1)+O(|1-z|^2 n^2). \end{aligned}$$