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Radon Transform
Second Edition

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Preface to the Second Edition

The first edition of this book has been out of print for some time and I have decided to follow the publisher's kind suggestion to prepare a new edition. Many examples of the explicit inversion formulas and range theorems have been added, and the group-theoretic viewpoint emphasized. For example, the integral geometric viewpoint of the Poisson integral for the disk leads to interesting analogies with the X-ray transform in Euclidean 3-space. To preserve the introductory flavor of the book the short and self-contained Chapter V on Schwartz' distributions has been added. Here §5 provides proofs of the needed results about the Riesz potentials while §§3–4 develop the tools from Fourier analysis following closely the account in Hörmander's books [1963] and [1983]. There is some overlap with my books [1984] and [1994b] which, however, rely heavily on Lie group theory. The present book is much more elementary.

I am indebted to Sine Jensen for a critical reading of parts of the manuscript and to Hilgert and Schlichtkrull for concrete contributions mentioned at specific places in the text. Finally I thank Jan Wetzel and Bonnie Friedman for their patient and skillful preparation of the manuscript.

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Preface to the First Edition

The title of this booklet refers to a topic in geometric analysis which has its origins in results of Funk [1916] and Radon [1917] determining, respectively, a symmetric function on the two-sphere $\mathbf{S}^2$ from its great circle integrals and a function of the plane $\mathbf{R}^2$ from its line integrals. (See references.) Recent developments, in particular applications to partial differential equations, X-ray technology, and radio astronomy, have widened interest in the subject.

These notes consist of a revision of lectures given at MIT in the Fall of 1966, based mosty on my papers during 1959–1965 on the Radon transform and its generalizations. (The term "Radon Transform" is adopted from John [1955].)

The viewpoint for these generalizations is as follows.

The set of points on $\mathbf{S}^2$ and the set of great circles on $\mathbf{S}^2$ are both homogeneous spaces of the orthoginal group $O(3)$. Similarly, the set of points in $\mathbf{R}^2$ and the set of lines in $\mathbf{R}^2$ are both homogeneous spaces of the group $\mathbf{M}(2)$ of rigid motions of $\mathbf{R}^2$. This motivates our general Radon transform definition from [1965a, 1966a] which forms the framwork of Chapter II: Given two homogeneous spaces G/K and G/H of the same group G, the Radon transform $u \to \widehat{u}$ maps functions u on the first space to functions $\widehat{u}$ on the second space. For $\xi \in G/H$, $\widehat{u}(\xi)$ is defined as the (natural) integral of u over the set of points $x \in G/K$ which are incident to ξ in the sense of Chern [1942]. The problem of inverting $u \to \widehat{u}$ is worked out in a few cases.

It happens when G/K is a Euclidean space, and more generally when G/K is a Riemannian symmetric space, that the natural differential operators A on G/K are transferred by $u \to \widehat{u}$ into much more manageable differential operators $\widehat{A}$ on G/H; the connection is $(Au)\widehat{\ } = \widehat{A}\widehat{u}$. Then the theory of the transform $u \to \widehat{u}$ has significant applications to the study of properties of A.

On the other hand, the applications of the original Radon transform on $\mathbf{R}^2$ to X-ray technology and radio astronomy are based on the fact that for an unknown density u, X-ray attenuation measurements give $\widehat{u}$ directly and therefore yield u via Radon's inversion formula. More precisely, let B be a convex body, $u(x)$ its density at the point x, and suppose a thin beam of X-rays is directed at B along a line ξ. Then the line integral $\widehat{u}(\xi)$ of u along ξ equals $\log(I_o/I)$ where I_o and I, respectively, are the intensities of the beam before hitting B and after leaving B. Thus while the function u is at first unknown, the function $\widehat{u}$ is determined by the X-ray data.

The lecture notes indicated above have been updated a bit by including a short account of some applications (Chapter I, §7), by adding a few cprollaries (Corollaries 2.8 and 2.12, Theorem 6.3 in Chapter I, Corollaries 2.8

and 4.1 in Chapter III), and by giving indications in the bibliographical notes of some recent developments.

An effort has been made to keep the exposition rather elementary. The distribution theory and the theory of Riesz potentials, occasionally needed in Chapter I, is reviewed in some detail in §8 (now Chapter V). Apart from the general homogeneous space framework in Chapter II, the treatment is restricted to Euclidean and isotropic spaces (spaces which are "the same in all directions"). For more general symmetric spaces the treatment is postposed (except for §4 in Chapter III) to another occasion since more machinery from the theorem of semisimple Lie groups is required.

I am indebted to R. Melrose and R. Seeley for helpful suggestions and to F. Gonzalez and J. Orloff for critical reading of parts of the manuscript.

Cambridge, MA 1980

CHAPTER I

THE RADON TRANSFORM ON $\mathbf{R}^N$

§1 Introduction

It was proved by J. Radon in 1917 that a differentiable function on $\mathbf{R}^3$ can be determined explicitly by means of its integrals over the planes in $\mathbf{R}^3$. Let $J(\omega, p)$ denote the integral of f over the hyperplane $\langle x, \omega\rangle = p$, ω denoting a unit vector and $\langle\ ,\ \rangle$ the inner product. Then

$$f(x) = -\frac{1}{8\pi^2} L_x \left(\int_{\mathbf{S}^2} J(\omega, \langle \omega, x\rangle)\, d\omega \right) ,$$

where L is the Laplacian on $\mathbf{R}^3$ and $d\omega$ the area element on the sphere $\mathbf{S}^2$ (cf. Theorem 3.1).

We now observe that the formula above has built in a remarkable duality: first one integrates over the set of points in a hyperplane, then one integrates over the set of hyperplanes passing through a given point. This suggests considering the transforms $f \to \widehat{f}, \varphi \to \check{\varphi}$ defined below.

The formula has another interesting feature. For a fixed ω the integrand $x \to J(\omega, \langle\omega, x\rangle)$ is a *plane wave*, that is a function constant on each plane perpendicular to ω. Ignoring the Laplacian the formula gives a continuous decomposition of f into plane waves. Since a plane wave amounts to a function of just one variable (along the normal to the planes) this decomposition can sometimes reduce a problem for $\mathbf{R}^3$ to a similar problem for $\mathbf{R}$. This principle has been particularly useful in the theory of partial differential equations.

The analog of the formula above for the line integrals is of importance in radiography where the objective is the description of a density function by means of certain line integrals.

In this chapter we discuss relationships between a function on $\mathbf{R}^n$ and its integrals over k-dimensional planes in $\mathbf{R}^n$. The case $k = n-1$ will be the one of primary interest. We shall occasionally use some facts about Fourier transforms and distributions. This material will be developed in sufficient detail in Chapter V so the treatment should be self-contained.

Following Schwartz [1966] we denote by $\mathcal{E}(\mathbf{R}^n)$ and $\mathcal{D}(\mathbf{R}^n)$, respectively, the space of complex-valued $\mathcal{C}^\infty$ functions (respectively $\mathcal{C}^\infty$ functions of compact support) on $\mathbf{R}^n$. The space $\mathcal{S}(\mathbf{R}^n)$ of rapidly decreasing functions on $\mathbf{R}^n$ is defined in connection with (6) below. $C^m(\mathbf{R}^n)$ denotes the space of m times continuously differentiable functions. We write $C(\mathbf{R}^n)$ for $C^0(\mathbf{R}^n)$, the space of continuous function on $\mathbf{R}^n$.

For a manifold M, $C^m(M)$ (and $C(M)$) is defined similarly and we write $\mathcal{D}(M)$ for $\mathcal{C}_c^\infty(M)$ and $\mathcal{E}(M)$ for $\mathcal{C}^\infty(M)$.

§2 The Radon Transform of the Spaces $\mathcal{D}(\mathbf{R}^n)$ and $\mathcal{S}(\mathbf{R}^n)$. The Support Theorem

Let f be a function on $\mathbf{R}^n$, integrable on each hyperplane in $\mathbf{R}^n$. Let $\mathbf{P}^n$ denote the space of all hyperplanes in $\mathbf{R}^n$, $\mathbf{P}^n$ being furnished with the obvious topology. The *Radon transform* of f is defined as the function $\widehat{f}$ on $\mathbf{P}^n$ given by

$$\widehat{f}(\xi) = \int_\xi f(x)dm(x)\,,$$

where dm is the Euclidean measure on the hyperplane ξ. Along with the transformation $f \to \widehat{f}$ we consider also the *dual transform* $\varphi \to \check{\varphi}$ which to a continuous function φ on $\mathbf{P}^n$ associates the function $\check{\varphi}$ on $\mathbf{R}^n$ given by

$$\check{\varphi}(x) = \int_{x\in\xi} \varphi(\xi)\, d\mu(\xi)$$

where $d\mu$ is the measure on the compact set $\{\xi \in \mathbf{P}^n : x \in \xi\}$ which is invariant under the group of rotations around x and for which the measure of the whole set is 1 (see Fig. I.1). We shall relate certain function spaces on $\mathbf{R}^n$ and on $\mathbf{P}^n$ by means of the transforms $f \to \widehat{f}, \varphi \to \check{\varphi}$; later we obtain explicit inversion formulas.

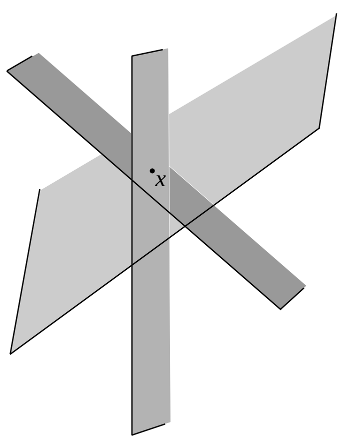

FIGURE I.1.

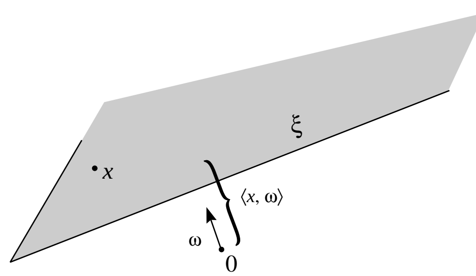

FIGURE I.2.

Each hyperplane $\xi \in \mathbf{P}^n$ can be written $\xi = \{x \in \mathbf{R}^n : \langle x, \omega\rangle = p\}$ where $\langle\ ,\ \rangle$ is the usual inner product, $\omega = (\omega_1, \ldots, \omega_n)$ a unit vector and $p \in \mathbf{R}$ (Fig. I.2). Note that the pairs (ω, p) and $(-\omega, -p)$ give the same ξ; the mapping $(\omega, p) \to \xi$ is a double covering of $\mathbf{S}^{n-1} \times \mathbf{R}$ onto $\mathbf{P}^n$. Thus $\mathbf{P}^n$ has a canonical manifold structure with respect to which this covering map is differentiable and regular. We thus identify continuous (differentiable) function on $\mathbf{P}^n$ with continuous (differentiable) functions φ on $\mathbf{S}^{n-1} \times \mathbf{R}$ satisfying the symmetry condition $\varphi(\omega, p) = \varphi(-\omega, -p)$. Writing $\widehat{f}(\omega, p)$ instead of $\widehat{f}(\xi)$ and f_t (with $t \in \mathbf{R}^n$) for the translated function $x \to f(t+x)$ we have

$$\widehat{f_t}(\omega, p) = \int_{\langle x,\omega\rangle = p} f(x+t)\, dm(x) = \int_{\langle y,\omega\rangle = p + \langle t,\omega\rangle} f(y)\, dm(y)$$

so

$$\widehat{f_t}(\omega, p) = \widehat{f}(\omega, p + \langle t, \omega\rangle)\,. \tag{1}$$

Taking limits we see that if $\partial_i = \partial/\partial x_i$

$$(\partial_i f)\widehat{}\,(\omega, p) = \omega_i \frac{\partial \widehat{f}}{\partial p}(\omega, p)\,. \tag{2}$$

Let L denote the Laplacian $\Sigma_i \partial_i^2$ on $\mathbf{R}^n$ and let $\square$ denote the operator

$$\varphi(\omega, p) \to \frac{\partial^2}{\partial p^2} \varphi(\omega, p)\,,$$

which is a well-defined operator on $\mathcal{E}(\mathbf{P}^n) = \mathcal{C}^\infty(\mathbf{P}^n)$. It can be proved that if $\mathbf{M}(n)$ is the group of isometries of $\mathbf{R}^n$, then L (respectively $\square$) generates the algebra of $\mathbf{M}(n)$-invariant differential operators on $\mathbf{R}^n$ (respectively $\mathbf{P}^n$).

Lemma 2.1. *The transforms $f \to \widehat{f}, \varphi \to \check{\varphi}$ intertwine L and $\square$, i.e.,*

$$(Lf)\widehat{} = \square(\widehat{f})\,, \quad (\square\varphi)^\vee = L\check{\varphi}\,.$$

Proof. The first relation follows from (2) by iteration. For the second we just note that for a certain constant c

$$\check{\varphi}(x) = c \int_{\mathbf{S}^{n-1}} \varphi(\omega, \langle x, \omega\rangle)\, d\omega\,, \tag{3}$$

where $d\omega$ is the usual measure on $\mathbf{S}^{n-1}$.

The Radon transform is closely connected with the Fourier transform

$$\widetilde{f}(u) = \int_{\mathbf{R}^n} f(x) e^{-i\langle x,\omega\rangle}\, dx \quad u \in \mathbf{R}^n.$$

In fact, if $s \in \mathbf{R}$, ω a unit vector,

$$\widetilde{f}(s\omega) = \int_{-\infty}^{\infty} dr \int_{\langle x,\omega\rangle = r} f(x) e^{-is\langle x,\omega\rangle}\, dm(x)$$

so

$$\widetilde{f}(s\omega) = \int_{-\infty}^{\infty} \widehat{f}(\omega, r) e^{-isr}\, dr\,. \tag{4}$$

This means that the n-dimensional Fourier transform is the 1-dimensional Fourier transform of the Radon transform. From (4), or directly, it follows that the Radon transform of the convolution

$$f(x) = \int_{\mathbf{R}^n} f_1(x-y) f_2(y)\, dy$$

is the convolution

$$\widehat{f}(\omega, p) = \int_{\mathbf{R}} \widehat{f_1}(\omega, p-q) \widehat{f_2}(\omega, q)\, dq\,. \tag{5}$$

We consider now the space $\mathcal{S}(\mathbf{R}^n)$ of complex-valued rapidly decreasing functions on $\mathbf{R}^n$. We recall that $f \in \mathcal{S}(\mathbf{R}^n)$ if and only if for each polynomial P and each integer $m \geq 0$,

$$\sup_x \big| |x|^m P(\partial_1, \ldots, \partial_n) f(x) \big| < \infty\,, \tag{6}$$

$|x|$ denoting the norm of x. We now formulate this in a more invariant fashion.

Lemma 2.2. *A function $f \in \mathcal{E}(\mathbf{R}^n)$ belongs to $\mathcal{S}(\mathbf{R}^n)$ if and only if for each pair $k, \ell \in \mathbb{Z}^+$*

$$\sup_{x \in \mathbf{R}^n} |(1+|x|)^k (L^\ell f)(x)| < \infty\,.$$

This is easily proved just by using the Fourier transforms.

In analogy with $\mathcal{S}(\mathbf{R}^n)$ we define $\mathcal{S}(\mathbf{S}^{n-1} \times \mathbf{R})$ as the space of $\mathcal{C}^\infty$ functions φ on $\mathbf{S}^{n-1} \times \mathbf{R}$ which for any integers $k, \ell \geq 0$ and any differential operator D on $\mathbf{S}^{n-1}$ satisfy

$$\sup_{\omega \in \mathbf{S}^{n-1}, r \in \mathbf{R}} \left| (1+|r|^k) \frac{d^\ell}{dr^\ell} (D\varphi)(\omega, r) \right| < \infty\,. \tag{7}$$

The space $\mathcal{S}(\mathbf{P}^n)$ is then defined as the set of $\varphi \in \mathcal{S}(\mathbf{S}^{n-1} \times \mathbf{R})$ satisfying $\varphi(\omega, p) = \varphi(-\omega, -p)$.

Lemma 2.3. *For each $f \in \mathcal{S}(\mathbf{R}^n)$ the Radon transform $\widehat{f}(\omega, p)$ satisfies the following condition: For $k \in \mathbb{Z}^+$ the integral*

$$\int_{\mathbf{R}} \widehat{f}(\omega, p) p^k\, dp$$

can be written as a k^{th} degree homogeneous polynomial in $\omega_1, \ldots, \omega_n$.

Proof. This is immediate from the relation

(8)
$$\int_{\mathbf{R}} \widehat{f}(\omega,p)p^k\,dp = \int_{\mathbf{R}} p^k\,dp \int_{\langle x,\omega\rangle=p} f(x)\,dm(x) = \int_{\mathbf{R}^n} f(x)\langle x,\omega\rangle^k\,dx\,.$$

In accordance with this lemma we define the space

$$\mathcal{S}_H(\mathbf{P}^n) = \left\{ F \in \mathcal{S}(\mathbf{P}^n) : \begin{array}{l} \text{For each } k \in \mathbb{Z}^+, \int_{\mathbf{R}} F(\omega,p)p^k\,dp \\ \text{is a homogeneous polynomial} \\ \text{in } \omega_1,\dots,\omega_n \text{ of degree } k \end{array} \right\}.$$

With the notation $\mathcal{D}(\mathbf{P}^n) = \mathcal{C}_c^\infty(\mathbf{P}^n)$ we write

$$\mathcal{D}_H(\mathbf{P}^n) = \mathcal{S}_H(\mathbf{P}^n) \cap \mathcal{D}(\mathbf{P}^n)\,.$$

According to Schwartz [1966], p. 249, the Fourier transform $f \to \widetilde{f}$ maps the space $\mathcal{S}(\mathbf{R}^n)$ onto itself. See Ch. V, Theorem 3.1. We shall now settle the analogous question for the Radon transform.

Theorem 2.4. *(The Schwartz theorem) The Radon transform $f \to \widehat{f}$ is a linear one-to-one mapping of $\mathcal{S}(\mathbf{R}^n)$ onto $\mathcal{S}_H(\mathbf{P}^n)$.*

Proof. Since

$$\frac{d}{ds}\widetilde{f}(s\omega) = \sum_{i=1}^n \omega_i(\partial_i\widetilde{f})$$

it is clear from (4) that for each fixed ω the function $r \to \widehat{f}(\omega,r)$ lies in $\mathcal{S}(\mathbf{R})$. For each $\omega_0 \in \mathbf{S}^{n-1}$ a subset of $(\omega_1,\dots,\omega_n)$ will serve as local coordinates on a neighborhood of ω_0 in $\mathbf{S}^{n-1}$. To see that $\widehat{f} \in \mathcal{S}(\mathbf{P}^n)$, it therefore suffices to verify (7) for $\varphi = \widehat{f}$ on an open subset $N \subset \mathbf{S}^{n-1}$ where ω_n is bounded away from 0 and $\omega_1,\dots,\omega_{n-1}$ serve as coordinates, in terms of which D is expressed. Since

(9) $$u_1 = s\omega_1, \dots, u_{n-1} = s\omega_{n-1}\,, \quad u_n = s(1-\omega_1^2-\cdots-\omega_{n-1}^2)^{1/2}$$

we have

$$\frac{\partial}{\partial\omega_i}(\widetilde{f}(s\omega)) = s\frac{\partial\widetilde{f}}{\partial u_i} - s\omega_i(1-\omega_1^2-\cdots-\omega_{n-1}^2)^{-1/2}\frac{\partial\widetilde{f}}{\partial u_n}\,.$$

It follows that if D is any differential operator on $\mathbf{S}^{n-1}$ and if $k,\ell \in \mathbb{Z}^+$ then

(10) $$\sup_{\omega\in N, s\in\mathbf{R}} \left|(1+s^{2k})\frac{d^\ell}{ds^\ell}(D\widetilde{f})(\omega,s)\right| < \infty\,.$$

We can therefore apply D under the integral sign in the inversion formula to (4),

$$\widehat{f}(\omega,r) = \frac{1}{2\pi}\int_{\mathbf{R}} \widetilde{f}(s\omega)e^{isr}\,ds$$

and obtain

$$(1+r^{2k})\frac{d^\ell}{dr^\ell}\Big(D_\omega(\widehat{f}(\omega,r))\Big)=\frac{1}{2\pi}\int\Big(1+(-1)^k\frac{d^{2k}}{ds^{2k}}\Big)\Big((is)^\ell D_\omega(\widetilde{f}(s\omega))\Big)e^{isr}ds\,.$$

Now (10) shows that $\widehat{f}\in\mathcal{S}(\mathbf{P}^n)$ so by Lemma 2.3, $\widehat{f}\in\mathcal{S}_H(\mathbf{P}^n)$.

Because of (4) and the fact that the Fourier transform is one-to-one it only remains to prove the surjectivity in Theorem 2.4. Let $\varphi\in\mathcal{S}_H(\mathbf{P}^n)$. In order to prove $\varphi=\widehat{f}$ for some $f\in\mathcal{S}(\mathbf{R}^n)$ we put

$$\psi(s,\omega)=\int_{-\infty}^{\infty}\varphi(\omega,r)e^{-irs}\,dr\,.$$

Then $\psi(s,\omega)=\psi(-s,-\omega)$ and $\psi(0,\omega)$ is a homogeneous polynomial of degree 0 in $\omega_1,\ldots,\omega_n$, hence constant. Thus there exists a function F on $\mathbf{R}^n$ such that

$$F(s\omega)=\int_{\mathbf{R}}\varphi(\omega,r)e^{-irs}\,dr\,.$$

While F is clearly smooth away from the origin we shall now prove it to be smooth at the origin too; this is where the homogeneity condition in the definition of $\mathcal{S}_H(\mathbf{P}^n)$ enters decisively. Consider the coordinate neighborhood $N\subset\mathbf{S}^{n-1}$ above and if $h\in\mathcal{C}^\infty(\mathbf{R}^n-0)$ let $h^*(\omega_1,\ldots,\omega_{n-1},s)$ be the function obtained from h by means of the substitution (9). Then

$$\frac{\partial h}{\partial u_i}=\sum_{j=1}^{n-1}\frac{\partial h^*}{\partial\omega_j}\frac{\partial\omega_j}{\partial u_i}+\frac{\partial h^*}{\partial s}\cdot\frac{\partial s}{\partial u_i}\quad(1\le i\le n)$$

and

$$\begin{aligned}\frac{\partial\omega_j}{\partial u_i}&=\frac{1}{s}(\delta_{ij}-\frac{u_iu_j}{s^2})\quad(1\le i\le n\,,\quad 1\le j\le n-1)\,,\\ \frac{\partial s}{\partial u_i}&=\omega_i\quad(1\le i\le n-1),\,\frac{\partial s}{\partial u_n}=(1-\omega_1^2-\cdots-\omega_{n-1}^2)^{1/2}\,.\end{aligned}$$

Hence

$$\begin{aligned}\frac{\partial h}{\partial u_i}&=\frac{1}{s}\frac{\partial h^*}{\partial\omega_i}+\omega_i\left(\frac{\partial h^*}{\partial s}-\frac{1}{s}\sum_{j=1}^{n-1}\omega_j\frac{\partial h^*}{\partial\omega_j}\right)\quad(1\le i\le n-1)\\ \frac{\partial h}{\partial u_n}&=(1-\omega_1^2-\cdots-\omega_{n-1}^2)^{1/2}\left(\frac{\partial h^*}{\partial s}-\frac{1}{s}\sum_{j=1}^{n-1}\omega_j\frac{\partial h^*}{\partial\omega_j}\right)\,.\end{aligned}$$

In order to use this for $h=F$ we write

$$F(s\omega)=\int_{-\infty}^{\infty}\varphi(\omega,r)\,dr+\int_{-\infty}^{\infty}\varphi(\omega,r)(e^{-irs}-1)\,dr\,.$$

By assumption the first integral is independent of ω. Thus using (7) we have for constant $K > 0$

$$\left|\frac{1}{s}\frac{\partial}{\partial\omega_i}(F(s\omega))\right| \leq K\int(1+r^4)^{-1}s^{-1}|e^{-isr}-1|\,dr \leq K\int\frac{|r|}{1+r^4}\,dr$$

and a similar estimate is obvious for $\partial F(s\omega)/\partial s$. The formulas above therefore imply that all the derivatives $\partial F/\partial u_i$ are bounded in a punctured ball $0 < |u| < \epsilon$ so F is certainly continuous at $u = 0$.

More generally, we prove by induction that

(11)
$$\frac{\partial^q h}{\partial u_{i_1}\dots\partial u_{i_q}} = \sum_{1\leq i+j\leq q, 1\leq k_1,\cdots,k_i\leq n-1} A_{j,k_1\dots k_i}(\omega,s)\frac{\partial^{i+j}h^*}{\partial\omega_{k_1}\dots\partial\omega_{k_i}\partial s^j}$$

where the coefficients A have the form

$$A_{j,k_1\dots k_i}(\omega,s) = a_{j,k_1\dots k_i}(\omega)s^{j-q}\,. \tag{12}$$

For $q = 1$ this is in fact proved above. Assuming (11) for q we calculate

$$\frac{\partial^{q+1}h}{\partial u_{i_1}\dots\partial u_{i_{q+1}}}$$

using the above formulas for $\partial/\partial u_i$. If $A_{j,k_1\dots k_i}(\omega,s)$ is differentiated with respect to $u_{i_{q+1}}$ we get a formula like (12) with q replaced by $q+1$. If on the other hand the $(i+j)^{\text{th}}$ derivative of h^* in (11) is differentiated with respect to $u_{i_{q+1}}$ we get a combination of terms

$$s^{-1}\frac{\partial^{i+j+1}h^*}{\partial\omega_{k_1}\dots\partial\omega_{k_{i+1}}\partial s^j}\ ,\ \frac{\partial^{i+j+1}h^*}{\partial\omega_{k_1}\dots\partial\omega_{k_i}\partial s^{j+1}}$$

and in both cases we get coefficients satisfying (12) with q replaced by $q+1$. This proves (11)–(12) in general. Now

$$F(s\omega) = \int_{-\infty}^{\infty}\varphi(\omega,r)\sum_0^{q-1}\frac{(-isr)^k}{k!}\,dr + \int_{-\infty}^{\infty}\varphi(\omega,r)e_q(-irs)\,dr\,, \tag{13}$$

where

$$e_q(t) = \frac{t^q}{q!} + \frac{t^{q+1}}{(q+1)!} + \cdots$$

Our assumption on φ implies that the first integral in (13) is a polynomial in $u_1,\dots,u_n$ of degree $\leq q-1$ and is therefore annihilated by the differential operator (11). If $0 \leq j \leq q$, we have

$$|s^{j-q}\frac{\partial^j}{\partial s^j}(e_q(-irs))| = |(-ir)^q(-irs)^{j-q}e_{q-j}(-irs)| \leq k_j r^q\,, \tag{14}$$

where k_j is a constant because the function $t \to (it)^{-p} e_p(it)$ is obviously bounded on $\mathbf{R}\,(p \geq 0)$. Since $\varphi \in \mathcal{S}(\mathbf{P}^n)$ it follows from (11)–(14) that each q^{th} order derivative of F with respect to $u_1, \ldots, u_n$ is bounded in a punctured ball $0 < |u| < \epsilon$. Thus we have proved $F \in \mathcal{E}(\mathbf{R}^n)$. That F is rapidly decreasing is now clear from (7), Lemma 2.2 and (11). Finally, if f is the function in $\mathcal{S}(\mathbf{R}^n)$ whose Fourier transform is F then

$$\widetilde{f}(s\omega) = F(s\omega) = \int_{-\infty}^{\infty} \varphi(\omega, r) e^{-irs}\, dr\,;$$

hence by (4), $\widehat{f} = \varphi$ and the theorem is proved.

To make further progress we introduce some useful notation. Let $S_r(x)$ denote the sphere $\{y : |y - x| = r\}$ in $\mathbf{R}^n$ and $A(r)$ its area. Let $B_r(x)$ denote the open ball $\{y : |y - x| < r\}$. For a continuous function f on $S_r(x)$ let $(M^r f)(x)$ denote the mean value

$$(M^r f)(x) = \frac{1}{A(r)} \int_{S_r(x)} f(\omega)\, d\omega\,,$$

where $d\omega$ is the Euclidean measure. Let K denote the orthogonal group $\mathbf{O}(n)$, dk its Haar measure, normalized by $\int dk = 1$. If $y \in \mathbf{R}^n, r = |y|$ then

$$(15) \quad (M^r f)(x) = \int_K f(x + k \cdot y)\, dk\,.$$

(Fig. I.3) In fact, for x, y fixed both sides represent rotation-invariant functionals on $C(S_r(x))$, having the same value for the function $f \equiv 1$. The rotations being transitive on $S_r(x)$, (15) follows from the uniqueness of such invariant functionals. Formula (3) can similarly be written

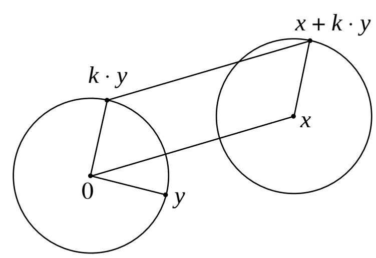

FIGURE I.3.

$$(16) \qquad \check{\varphi}(x) = \int_K \varphi(x + k \cdot \xi_0)\, dk$$

if ξ_0 is some fixed hyperplane through the origin. We see then that if $f(x) = 0(|x|^{-n}), \Omega_k$ the area of the unit sphere in $\mathbf{R}^k$, i.e., $\Omega_k = 2\frac{\pi^{k/2}}{\Gamma(k/2)}$,

$$\begin{aligned}(\widehat{f})^{\vee}(x) &= \int_K \widehat{f}(x + k \cdot \xi_0)\, dk = \int_K \left(\int_{\xi_0} f(x + k \cdot y)\, dm(y) \right) dk \\ &= \int_{\xi_0} (M^{|y|} f)(x)\, dm(y) = \Omega_{n-1} \int_0^{\infty} r^{n-2} \left(\frac{1}{\Omega_n} \int_{\mathcal{S}^{n-1}} f(x + r\omega)\, d\omega \right) dr\end{aligned}$$

so

$$(\widehat{f})^\vee(x) = \frac{\Omega_{n-1}}{\Omega_n} \int_{\mathbf{R}^n} |x-y|^{-1} f(y)\, dy\,. \tag{17}$$

We consider now the analog of Theorem 2.4 for the transform $\varphi \to \check{\varphi}$. But $\varphi \in \mathcal{S}_H(\mathbf{P}^n)$ does not imply $\check{\varphi} \in \mathcal{S}(\mathbf{R}^n)$. (If this were so and we by Theorem 2.4 write $\varphi = \widehat{f}, f \in \mathcal{S}(\mathbf{R}^n)$ then the inversion formula in Theorem 3.1 for $n = 3$ would imply $\int f(x)\, dx = 0$.) On a smaller space we shall obtain a more satisfactory result.

Let $\mathcal{S}^*(\mathbf{R}^n)$ denote the space of all functions $f \in \mathcal{S}(\mathbf{R}^n)$ which are orthogonal to all polynomials, i.e.,

$$\int_{\mathbf{R}^n} f(x)P(x)\, dx = 0 \quad \text{for all polynomials } P\,.$$

Similarly, let $\mathcal{S}^*(\mathbf{P}^n) \subset \mathcal{S}(\mathbf{P}^n)$ be the space of φ satisfying

$$\int_{\mathbf{R}} \varphi(\omega, r)p(r)\, dr = 0 \quad \text{for all polynomials } p\,.$$

Note that under the Fourier transform the space $\mathcal{S}^*(\mathbf{R}^n)$ corresponds to the subspace $\mathcal{S}_0(\mathbf{R}^n) \subset \mathcal{S}(\mathbf{R}^n)$ of functions all of whose derivatives vanish at 0.

Corollary 2.5. *The transforms $f \to \widehat{f}, \varphi \to \check{\varphi}$ are bijections of $\mathcal{S}^*(\mathbf{R}^n)$ onto $\mathcal{S}^*(\mathbf{P}^n)$ and of $\mathcal{S}^*(\mathbf{P}^n)$ onto $\mathcal{S}^*(\mathbf{R}^n)$, respectively.*

The first statement is clear from (8) if we take into account the elementary fact that the polynomials $x \to \langle x, \omega\rangle^k$ span the space of homogeneous polynomials of degree k. To see that $\varphi \to \check{\varphi}$ is a bijection of $\mathcal{S}^*(\mathbf{P}^n)$ onto $\mathcal{S}^*(\mathbf{R}^n)$ we use (17), knowing that $\varphi = \widehat{f}$ for some $f \in \mathcal{S}^*(\mathbf{R}^n)$. The right hand side of (17) is the convolution of f with the tempered distribution $|x|^{-1}$ whose Fourier transform is by Chapter V, §5 a constant multiple of $|u|^{1-n}$. (Here we leave out the trivial case $n = 1$.) By Chapter V, (12) this convolution is a tempered distribution whose Fourier transform is a constant multiple of $|u|^{1-n}\widetilde{f}(u)$. But, by Lemma 5.6, Chapter V this lies in the space $\mathcal{S}_0(\mathbf{R}^n)$ since $\widetilde{f}$ does. Now (17) implies that $\check{\varphi} = (\widehat{f})^\vee \in \mathcal{S}^*(\mathbf{R}^n)$ and that $\check{\varphi} \not\equiv 0$ if $\varphi \not\equiv 0$. Finally we see that the mapping $\varphi \to \check{\varphi}$ is surjective because the function

$$((\widehat{f})^\vee)\widetilde{\ }(u) = c|u|^{1-n}\widetilde{f}(u)$$

(where c is a constant) runs through $\mathcal{S}_0(\mathbf{R}^n)$ as f runs through $\mathcal{S}^*(\mathbf{R}^n)$.

We now turn to the space $\mathcal{D}(\mathbf{R}^n)$ and its image under the Radon transform. We begin with a preliminary result. (See Fig. I.4.)

Theorem 2.6. *(The support theorem.) Let $f \in C(\mathbf{R}^n)$ satisfy the following conditions:*

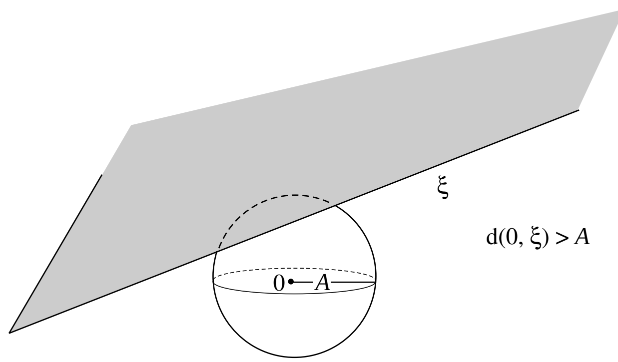

FIGURE I.4.

(i) *For each integer $k > 0, |x|^k f(x)$ is bounded.*

(ii) *There exists a constant $A > 0$ such that*

$$\widehat{f}(\xi) = 0 \ \textit{for} \ d(0,\xi) > A\,,$$

d denoting distance.

Then

$$f(x) = 0 \ \textit{for} \ |x| > A\,.$$

Proof. Replacing f by the convolution $\varphi * f$ where φ is a radial $\mathcal{C}^\infty$ function with support in a small ball $B_\epsilon(0)$ we see that it suffices to prove the theorem for $f \in \mathcal{E}(\mathbf{R}^n)$. In fact, $\varphi * f$ is smooth, it satisfies (i) and by (5) it satisfies (ii) with A replaced by $A+\epsilon$. Assuming the theorem for the smooth case we deduce that support $(\varphi * f) \subset B_{A+\epsilon}(0)$ so letting $\epsilon \to 0$ we obtain support $(f) \subset$ Closure $B_A(0)$.

To begin with we assume f is a radial function. Then $f(x) = F(|x|)$ where $F \in \mathcal{E}(\mathbf{R})$ and even. Then $\widehat{f}$ has the form $\widehat{f}(\xi) = \widehat{F}(d(0,\xi))$ where $\widehat{F}$ is given by

$$\widehat{F}(p) = \int_{\mathbf{R}^{n-1}} F((p^2 + |y|^2)^{1/2})\, dm(y)\,, \quad (p \geq 0)$$

because of the definition of the Radon transform. Using polar coordinates in $\mathbf{R}^{n-1}$ we obtain

$$\widehat{F}(p) = \Omega_{n-1} \int_0^\infty F((p^2 + t^2)^{1/2}) t^{n-2}\, dt\,. \tag{18}$$

Here we substitute $s = (p^2 + t^2)^{-1/2}$ and then put $u = p^{-1}$. Then (18) becomes

$$u^{n-3}\widehat{F}(u^{-1}) = \Omega_{n-1} \int_0^u (F(s^{-1})s^{-n})(u^2 - s^2)^{(n-3)/2}\, ds\,.$$

We write this equation for simplicity

$$h(u) = \int_0^u g(s)(u^2 - s^2)^{(n-3)/2}\, ds\,. \tag{19}$$

This integral equation is very close to Abel's integral equation (Whittaker-Watson [1927], Ch. IX) and can be inverted as follows. Multiplying both sides by $u(t^2 - u^2)^{(n-3)/2}$ and integrating over $0 \leq u \leq t$ we obtain

$$\begin{aligned}
&\int_0^t h(u)(t^2 - u^2)^{(n-3)/2} u\, du \\
&\qquad = \int_0^t \left[\int_0^u g(s)[(u^2 - s^2)(t^2 - u^2)]^{(n-3)/2}\, ds\right] u\, du \\
&\qquad = \int_0^t g(s) \left[\int_{u=s}^t u[(t^2 - u^2)(u^2 - s^2)]^{(n-3)/2}\, du\right] ds\,.
\end{aligned}$$

The substitution $(t^2 - s^2)V = (t^2 + s^2) - 2u^2$ gives an explicit evaluation of the inner integral and we obtain

$$\int_0^t h(u)(t^2 - u^2)^{(n-3)/2} u\, du = C \int_0^t g(s)(t^2 - s^2)^{n-2}\, ds\,,$$

where $C = 2^{1-n}\pi^{\frac{1}{2}}\Gamma((n-1)/2)/\Gamma(n/2)$. Here we apply the operator $\frac{d}{d(t^2)} = \frac{1}{2t}\frac{d}{dt}$ $(n-1)$ times whereby the right hand side gives a constant multiple of $t^{-1}g(t)$. Hence we obtain

$$F(t^{-1})t^{-n} = ct\left[\frac{d}{d(t^2)}\right]^{n-1} \int_0^t (t^2 - u^2)^{(n-3)/2} u^{n-2} \widehat{F}(u^{-1})\, du \tag{20}$$

where $c^{-1} = (n-2)!\Omega_n/2^n$. By assumption (ii) we have $\widehat{F}(u^{-1}) = 0$ if $u^{-1} \geq A$, that is if $u \leq A^{-1}$. But then (20) implies $F(t^{-1}) = 0$ if $t \leq A^{-1}$, that is if $t^{-1} \geq A$. This proves the theorem for the case when f is radial.

We consider next the case of a general f. Fix $x \in \mathbf{R}^n$ and consider the function

$$g_x(y) = \int_K f(x + k \cdot y)\, dk$$

as in (15). Then g_x satisfies (i) and

$$\widehat{g}_x(\xi) = \int_K \widehat{f}(x + k \cdot \xi)\, dk\,, \tag{21}$$

$x + k \cdot \xi$ denoting the translate of the hyperplane $k \cdot \xi$ by x. The triangle inequality shows that

$$d(0, x + k \cdot \xi) \geq d(0, \xi) - |x|\,, \quad x \in \mathbf{R}^n, k \in K\,.$$

Hence we conclude from assumption (ii) and (21) that

$$\widehat{g}_x(\xi) = 0 \quad \text{if} \quad d(0,\xi) > A + |x|\,. \tag{22}$$

But g_x is a radial function so (22) implies by the first part of the proof that

$$\int_K f(x + k \cdot y)\, dk = 0 \quad \text{if} \quad |y| > A + |x|\,. \tag{23}$$

Geometrically, this formula reads: The surface integral of f over $S_{|y|}(x)$ is 0 if the ball $B_{|y|}(x)$ contains the ball $B_A(0)$. The theorem is therefore a consequence of the following lemma.

Lemma 2.7. *Let $f \in C(\mathbf{R}^n)$ be such that for each integer $k > 0$,*

$$\sup_{x \in \mathbf{R}^n} |x|^k |f(x)| < \infty\,.$$

Suppose f has surface integral 0 over every sphere S which encloses the unit ball. Then $f(x) \equiv 0$ for $|x| > 1$.

Proof. The idea is to perturb S in the relation

$$\int_S f(s)\, d\omega(s) = 0 \tag{24}$$

slightly, and differentiate with respect to the parameter of the perturbations, thereby obtaining additional relations. (See Fig. I.5.) Replacing, as above, f with a suitable convolution $\varphi * f$ we see that it suffices to prove the lemma for f in $\mathcal{E}(\mathbf{R}^n)$. Writing $S = S_R(x)$ and viewing the exterior of the ball $B_R(x)$ as a union of spheres with center x we have by the assumptions,

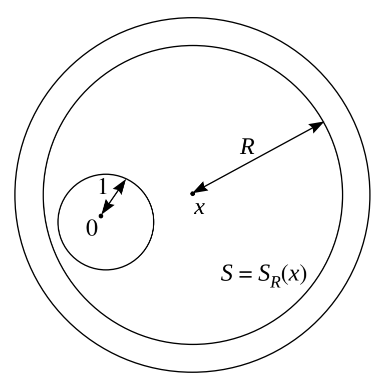

FIGURE I.5.

$$\int_{B_R(x)} f(y)\, dy = \int_{\mathbf{R}^n} f(y)\, dy\,,$$

which is a constant. Differentiating with respect to x_i we obtain

$$\int_{B_R(0)} (\partial_i f)(x+y)\,dy = 0\,. \tag{25}$$

We use now the divergence theorem

$$\int_{B_R(0)} (\mathrm{div} F)(y)\,dy = \int_{S_R(0)} \langle F, \mathbf{n}\rangle(s)\,d\omega(s) \tag{26}$$

for a vector field F on $\mathbf{R}^n$, $\mathbf{n}$ denoting the outgoing unit normal and $d\omega$ the surface element on $S_R(0)$. For the vector field $F(y) = f(x+y)\frac{\partial}{\partial y_i}$ we obtain from (25) and (26), since $\mathbf{n} = R^{-1}(s_1, \ldots, s_n)$,

$$\int_{S_R(0)} f(x+s)s_i\,d\omega(s) = 0\,. \tag{27}$$

But by (24)

$$\int_{S_R(0)} f(x+s)x_i\,d\omega(s) = 0$$

so by adding

$$\int_S f(s)s_i\,d\omega(s) = 0\,.$$

This means that the hypotheses of the lemma hold for $f(x)$ replaced by the function $x_i f(x)$. By iteration

$$\int_S f(s)P(s)\,d\omega(s) = 0$$

for any polynomial P, so $f \equiv 0$ on S. This proves the lemma as well as Theorem 2.6.

Corollary 2.8. *Let $f \in C(\mathbf{R}^n)$ satisfy (i) in Theorem 2.6 and assume*

$$\widehat{f}(\xi) = 0$$

for all hyperplanes ξ disjoint from a certain compact convex set C. Then

$$f(x) = 0 \quad \textit{for} \quad x \notin C\,. \tag{28}$$

In fact, if B is a closed ball containing C we have by Theorem 2.6, $f(x) = 0$ for $x \notin B$. But C is the intersection of such balls so (28) follows.

Remark 2.9. While condition (i) of rapid decrease entered in the proof of Lemma 2.7 (we used $|x|^k f(x) \in L^1(\mathbf{R}^n)$ for each $k > 0$) one may wonder whether it could not be weakened in Theorem 2.6 and perhaps even dropped in Lemma 2.7.

As an example, showing that the condition of rapid decrease can not be dropped in either result consider for $n = 2$ the function

$$f(x, y) = (x + iy)^{-5}$$

made smooth in $\mathbf{R}^2$ by changing it in a small disk around 0. Using Cauchy's theorem for a large semicircle we have $\int_\ell f(x)\,dm(x) = 0$ for every line ℓ outside the unit circle. Thus (ii) is satisfied in Theorem 2.6. Hence (i) cannot be dropped or weakened substantially.

This same example works for Lemma 2.7. In fact, let S be a circle $|z - z_0| = r$ enclosing the unit disk. Then $d\omega(s) = -ir\frac{dz}{z-z_0}$ so, by expanding the contour or by residue calculus,

$$\int_S z^{-5}(z - z_0)^{-1}\,dz = 0\,,$$

(the residue at $z = 0$ and $z = z_0$ cancel) so we have in fact

$$\int_S f(s)\,d\omega(s) = 0\,.$$

We recall now that $\mathcal{D}_H(\mathbf{P}^n)$ is the space of symmetric $\mathcal{C}^\infty$ functions $\varphi(\xi) = \varphi(\omega, p)$ on $\mathbf{P}^n$ of compact support such that for each $k \in \mathbb{Z}^+$, $\int_{\mathbf{R}} \varphi(\omega, p)p^k\,dp$ is a homogeneous kth degree polynomial in $\omega_1, \ldots, \omega_n$. Combining Theorems 2.4, 2.6 we obtain the following characterization of the Radon transform of the space $\mathcal{D}(\mathbf{R}^n)$. This can be regarded as the analog for the Radon transform of the Paley-Wiener theorem for the Fourier transform (see Chapter V).

Theorem 2.10. *(The Paley-Wiener theorem.) The Radon transform is a bijection of $\mathcal{D}(\mathbf{R}^n)$ onto $\mathcal{D}_H(\mathbf{P}^n)$.*

We conclude this section with a variation and a consequence of Theorem 2.6.

Lemma 2.11. *Let $f \in C_c(\mathbf{R}^n)$, $A > 0$, ω_0 a fixed unit vector and $N \subset S$ a neighborhood of ω_0 in the unit sphere $S \subset \mathbf{R}^n$. Assume*

$$\widehat{f}(\omega, p) = 0 \quad \textit{for} \quad \omega \in N,\, p > A\,.$$

Then

$$f(x) = 0 \ \textit{ in the half-space } \langle x, \omega_0 \rangle > A\,. \tag{29}$$

Proof. Let B be a closed ball around the origin containing the support of f. Let $\epsilon > 0$ and let H_ϵ be the union of the half spaces $\langle x, \omega \rangle > A + \epsilon$ as ω runs through N. Then by our assumption

$$\widehat{f}(\xi) = 0 \quad \text{if} \quad \xi \subset H_\epsilon\,. \tag{30}$$

Now choose a ball B_ϵ with a center on the ray from 0 through $-\omega_0$, with the point $(A+2\epsilon)\omega_0$ on the boundary, and with radius so large that any hyperplane ξ intersecting B but not B_ϵ must be in H_ϵ. Then by (30)

$$\widehat{f}(\xi)=0 \quad \text{whenever} \quad \xi\in\mathbf{P}^n,\ \xi\cap B_\epsilon=\emptyset\,.$$

Hence by Theorem 2.6, $f(x)=0$ for $x\notin B_\epsilon$. In particular, $f(x)=0$ for $\langle x,\omega_0\rangle > A+2\epsilon$; since $\epsilon>0$ is arbitrary, the lemma follows.

Corollary 2.12. *Let N be any open subset of the unit sphere $\mathbf{S}^{n-1}$. If $f\in C_c(\mathbf{R}^n)$ and*

$$\widehat{f}(\omega,p)=0 \quad \textit{for } p\in\mathbf{R},\omega\in N$$

then

$$f\equiv 0\,.$$

Since $\widehat{f}(-\omega,-p)=\widehat{f}(\omega,p)$ this is obvious from Lemma 2.11.

§3 The Inversion Formula

We shall now establish explicit inversion formulas for the Radon transform $f\to\widehat{f}$ and its dual $\varphi\to\check{\varphi}$.

Theorem 3.1. *The function f can be recovered from the Radon transform by means of the following inversion formula*

$$cf=(-L)^{(n-1)/2}((\widehat{f})^\vee) \quad f\in\mathcal{E}(\mathbf{R}^n)\,, \tag{31}$$

provided $f(x)=0(|x|^{-N})$ for some $N>n$. Here c is the constant

$$c=(4\pi)^{(n-1)/2}\Gamma(n/2)/\Gamma(1/2)\,.$$

Proof. We use the connection between the powers of L and the Riesz potentials in Chapter V, §5. Using (17) we in fact have

$$(\widehat{f})^\vee=2^{n-1}\pi^{\frac{n}{2}-1}\Gamma(n/2)I^{n-1}f\,. \tag{32}$$

By Chapter V, Proposition 5.7, we thus obtain the desired formula (31).

For n odd one can give a more geometric proof of (31). We start with some general useful facts about the mean value operator M^r. It is a familiar fact that if $f \in C^2(\mathbf{R}^n)$ is a radial function, i.e., $f(x) = F(r), r = |x|$, then

$$(33) \qquad (Lf)(x) = \frac{d^2F}{dr^2} + \frac{n-1}{r}\frac{dF}{dr}\,.$$

This is immediate from the relations

$$\frac{\partial^2 f}{\partial x_i^2} = \frac{\partial^2 f}{\partial r^2}\left(\frac{\partial r}{\partial x_i}\right)^2 + \frac{\partial f}{\partial r}\frac{\partial^2 r}{\partial x_i^2}\,.$$

Lemma 3.2. *(i)* $LM^r = M^rL$ *for each* $r > 0$.

(ii) For $f \in C^2(\mathbf{R}^n)$ *the mean value* $(M^rf)(x)$ *satisfies the "Darboux equation"*

$$L_x\big((M^rf)(x)\big) = \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right)(M^rf(x))\,,$$

that is, the function $F(x,y) = (M^{|y|}f)(x)$ *satisfies*

$$L_x(F(x,y)) = L_y(F(x,y))\,.$$

Proof. We prove this group theoretically, using expression (15) for the mean value. For $z \in \mathbf{R}^n, k \in K$ let T_z denote the translation $x \to x+z$ and R_k the rotation $x \to k \cdot x$. Since L is invariant under these transformations, we have if $r = |y|$,

$$\begin{aligned}
(LM^rf)(x) = \int_K L_x(f(x+k\cdot y))\,dk &= \int_K (Lf)(x+k\cdot y)\,dk = (M^rLf)(x)\\
= \int_K [(Lf)\circ T_x \circ R_k](y)\,dk &= \int_K [L(f\circ T_x\circ R_k)](y)\,dk\\
= L_y\left(\int_K f(x+k\cdot y)\right)\,dk &
\end{aligned}$$

which proves the lemma.

Now suppose $f \in \mathcal{S}(\mathbf{R}^n)$. Fix a hyperplane ξ_0 through 0, and an isometry $g \in \mathbf{M}(n)$. As k runs through $\mathbf{O}(n)$, $gk\cdot\xi_0$ runs through the set of hyperplanes through $g\cdot 0$, and we have

$$\check{\varphi}(g\cdot 0) = \int_K \varphi(gk\cdot\xi_0)\,dk$$

and therefore

$$\begin{aligned}
(\widehat{f})^\vee(g\cdot 0) &= \int_K\left(\int_{\xi_0} f(gk\cdot y)\,dm(y)\right)dk\\
&= \int_{\xi_0} dm(y)\int_K f(gk\cdot y)\,dk = \int_{\xi_0}(M^{|y|}f)(g\cdot 0)\,dm(y)\,.
\end{aligned}$$

Hence

$$((\widehat{f}))^{\vee}(x) = \Omega_{n-1} \int_0^{\infty} (M^r f)(x) r^{n-2}\, dr\,, \tag{34}$$

where Ω_{n-1} is the area of the unit sphere in $\mathbf{R}^{n-1}$. Applying L to (34), using (33) and Lemma 3.2, we obtain

$$L((\widehat{f})^{\vee}) = \Omega_{n-1} \int_0^{\infty} \left(\frac{d^2F}{dr^2} + \frac{n-1}{r}\frac{dF}{dr} \right) r^{n-2}\, dr \tag{35}$$

where $F(r) = (M^r f)(x)$. Integrating by parts and using

$$F(0) = f(x), \lim_{r\to\infty} r^k F(r) = 0\,,$$

we get

$$L((\widehat{f})^{\vee}) = \begin{cases} -\Omega_{n-1} f(x) & \text{if} \quad n = 3\,, \\ -\Omega_{n-1}(n-3) \int_0^{\infty} F(r) r^{n-4}\, dr & \quad (n > 3)\,. \end{cases}$$

More generally,

$$L_x \left(\int_0^{\infty} (M^r f)(x) r^k\, dr \right) = \begin{cases} -(n-2) f(x) & \text{if } k = 1\,, \\ -(n-1-k)(k-1) \int_0^{\infty} F(r) r^{k-2} dr, & (k > 1)\,. \end{cases}$$

If n is odd the formula in Theorem 3.1 follows by iteration. Although we assumed $f \in \mathcal{S}(\mathbf{R}^n)$ the proof is valid under much weaker assumptions.

Remark 3.3. The condition $f(x) = 0(|x|^{-N})$ for some $N > n$ cannot in general be dropped. In [1982] Zalcman has given an example of a smooth function f on $\mathbf{R}^2$ satisfying $f(x) = 0(|x|^{-2})$ on all lines with $\widehat{f}(\xi) = 0$ for all lines ξ and yet $f \neq 0$. The function is even $f(x) = 0(|x|^{-3})$ on each line which is not the x-axis. See also Armitage and Goldstein [1993].

Remark 3.4. It is interesting to observe that while the inversion formula requires $f(x) = 0(|x|^{-N})$ for *one* $N > n$ the support theorem requires $f(x) = 0(|x|^{-N})$ for *all* N as mentioned in Remark 2.9.

We shall now prove a similar inversion formula for the dual transform $\varphi \to \check{\varphi}$ on the subspace $\mathcal{S}^*(\mathbf{P}^n)$.

Theorem 3.5. *We have*

$$c\varphi = (-\Box)^{(n-1)/2} (\check{\varphi})\widehat{\ }, \quad \varphi \in \mathcal{S}^*(\mathbf{P}^n)\,,$$

where c is the constant $(4\pi)^{(n-1)/2}\Gamma(n/2)/\Gamma(1/2)$.

Here $\Box$ denotes as before the operator $\frac{d^2}{dp^2}$ and its fractional powers are again defined in terms of the Riesz' potentials on the 1-dimensional p-space.

If n is odd our inversion formula follows from the odd-dimensional case in Theorem 3.1 if we put $f = \check{\varphi}$ and take Lemma 2.1 and Corollary 2.5 into account. Suppose now n is even. We claim that

$$((-L)^{\frac{n-1}{2}} f)\widehat{\ } = (-\Box)^{\frac{n-1}{2}} \widehat{f} \qquad f \in \mathcal{S}^*(\mathbf{R}^n)\,. \tag{36}$$

By Lemma 5.6 in Chapter V, $(-L)^{(n-1)/2} f$ belongs to $\mathcal{S}^*(\mathbf{R}^n)$. Taking the 1-dimensional Fourier transform of $((-L)^{(n-1)/2} f)\widehat{\ }$ we obtain

$$\big((-L)^{(n-1)/2} f\big)\widetilde{\ }(s\omega) = |s|^{n-1} \widetilde{f}(s\omega)\,.$$

On the other hand, for a fixed ω, $p \to \widehat{f}(\omega, p)$ is in $\mathcal{S}^*(\mathbf{R})$. By the lemma quoted, the function $p \to ((-\Box)^{(n-1)/2} \widehat{f})(\omega, p)$ also belongs to $\mathcal{S}^*(\mathbf{R})$ and its Fourier transform equals $|s|^{n-1} \widetilde{f}(s\omega)$. This proves (36). Now Theorem 3.5 follows from (36) if we put in (36)

$$\varphi = \widehat{g}\,, \quad f = (\widehat{g})^\vee\,, \quad g \in \mathcal{S}^*(\mathbf{R}^n)\,,$$

because, by Corollary 2.5, $\widehat{g}$ belongs to $\mathcal{S}^*(\mathbf{P}^n)$.

Because of its theoretical importance we now prove the inversion theorem (3.1) in a different form. The proof is less geometric and involves just the one variable Fourier transform.

Let $\mathcal{H}$ denote the Hilbert transform

$$(\mathcal{H}F)(t) = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{F(p)}{t-p}\, dp \qquad F \in \mathcal{S}(\mathbf{R})$$

the integral being considered as the Cauchy principal value (see Lemma 3.7 below). For $\varphi \in \mathcal{S}(\mathbf{P}^n)$ let $\Lambda\varphi$ be defined by

$$(\Lambda\varphi)(\omega, p) = \begin{cases} \frac{d^{n-1}}{dp^{n-1}} \varphi(\omega, p) & n \text{ odd,} \\ \mathcal{H}_p \frac{d^{n-1}}{dp^{n-1}} \varphi(\omega, p) & n \text{ even.} \end{cases} \tag{37}$$

Note that in both cases $(\Lambda\varphi)(-\omega, -p) = (\Lambda\varphi)(\omega, p)$ so $\Lambda\varphi$ is a function on $\mathbf{P}^n$.

Theorem 3.6. *Let Λ be as defined by (37). Then*

$$cf = (\Lambda \widehat{f})^\vee\,, \quad f \in \mathcal{S}(\mathbf{R}^n)\,,$$

where as before

$$c = (-4\pi)^{(n-1)/2} \Gamma(n/2)/\Gamma(1/2)\,.$$

Proof. By the inversion formula for the Fourier transform and by (4)

$$f(x) = (2\pi)^{-n} \int_{\mathbf{S}^{n-1}} d\omega \int_0^\infty \left(\int_{-\infty}^{\infty} e^{-isp} \widehat{f}(\omega, p)\, dp \right) e^{is\langle x, \omega\rangle} s^{n-1}\, ds$$

which we write as

$$f(x) = (2\pi)^{-n}\int_{\mathbf{S}^{n-1}} F(\omega, x)\, d\omega = (2\pi)^{-n}\int_{\mathbf{S}^{n-1}} \tfrac{1}{2}(F(\omega,x)+F(-\omega,x))\, d\omega\,.$$

Using $\widehat{f}(-\omega,p) = \widehat{f}(\omega,-p)$ this gives the formula

$$f(x) = \tfrac{1}{2}(2\pi)^{-n}\int_{\mathbf{S}^{n-1}} d\omega \int_{-\infty}^{\infty} |s|^{n-1} e^{is\langle x,\omega\rangle}\, ds \int_{-\infty}^{\infty} e^{-isp}\widehat{f}(\omega,p)\, dp\,. \tag{38}$$

If n is odd the absolute value on s can be dropped. The factor s^{n-1} can be removed by replacing $\widehat{f}(\omega,p)$ by $(-i)^{n-1}\frac{d^{n-1}}{dp^{n-1}}\widehat{f}(\omega,p)$. The inversion formula for the Fourier transform on $\mathbf{R}$ then gives

$$f(x) = \tfrac{1}{2}(2\pi)^{-n}(2\pi)^{+1}(-i)^{n-1}\int_{\mathbf{S}^{n-1}} \left\{\frac{d^{n-1}}{dp^{n-1}}\widehat{f}(\omega,p)\right\}_{p=\langle x,\omega\rangle} d\omega$$

as desired.

In order to deal with the case n even we recall some general facts.

Lemma 3.7. *Let S denote the Cauchy principal value*

$$S : \psi \to \lim_{\epsilon\to 0}\int_{|x|\geq\epsilon} \frac{\psi(x)}{x}\, dx\,.$$

Then S is a tempered distribution and $\widetilde{S}$ is the function

$$\widetilde{S}(s) = -\pi i\,\mathrm{sgn}(s) = \begin{cases} -\pi i & s \geq 0 \\ \pi i & s < 0 \end{cases}\,.$$

Proof. It is clear that S is tempered. Also $xS = 1$ so

$$2\pi\delta = \widetilde{1} = (xS)^{\sim} = i(\widetilde{S})'\,.$$

But $\mathrm{sgn}' = 2\delta$ so $\widetilde{S} = -\pi i\,\mathrm{sgn} + C$. But $\widetilde{S}$ and sgn are odd so $C = 0$.

This implies

$$(\mathcal{H}F)^{\sim}(s) = \mathrm{sgn}(s)\widetilde{F}(s)\,. \tag{39}$$

For n even we write in (38), $|s|^{n-1} = \mathrm{sgn}(s)s^{n-1}$ and then (38) implies

$$f(x) = c_0\int_{\mathbf{S}^{n-1}} d\omega \int_{\mathbf{R}} \mathrm{sgn}(s) e^{is\langle x,\omega\rangle}\, ds \int_{\mathbf{R}} \frac{d^{n-1}}{dp^{n-1}}\widehat{f}(\omega,p)e^{-isp}\, dp\,, \tag{40}$$

where $c_0 = \frac{1}{2}(-i)^{n-1}(2\pi)^{-n}$. Now we have for each $F \in \mathcal{S}(\mathbf{R})$ the identity

$$\int_{\mathbf{R}} \mathrm{sgn}(s)e^{ist}\left(\int_{\mathbf{R}} F(p)e^{-ips}\, dp\right) ds = 2\pi(\mathcal{H}F)(t)\,.$$

In fact, if we apply both sides to $\widetilde{\psi}$ with $\psi \in \mathcal{S}(\mathbf{R})$, the left hand side is by (39)

$$\begin{aligned}&\int_{\mathbf{R}}\bigg(\int_{\mathbf{R}} \operatorname{sgn}(s)e^{ist}\widetilde{F}(s)\,ds\bigg)\widetilde{\psi}(t)\,dt\\ &\quad = \int_{\mathbf{R}} \operatorname{sgn}(s)\widetilde{F}(s)2\pi\psi(s)\,ds = 2\pi(\mathcal{H}F)\widetilde{\ }(\psi) = 2\pi(\mathcal{H}F)(\widetilde{\psi})\,.\end{aligned}$$

Putting $F(p) = \frac{d^{n-1}}{dp^{n-1}}\widehat{f}(\omega,p)$ in (40) Theorem 3.6 follows also for n even.

For later use we add here a few remarks concerning $\mathcal{H}$. Let $F \in \mathcal{D}$ have support contained in $(-R,R)$. Then

$$-i\pi(\mathcal{H}F)(t) = \lim_{\epsilon\to 0}\int_{\epsilon<|t-p|}\frac{F(p)}{t-p}\,dp = \lim_{\epsilon\to 0}\int_I \frac{F(p)}{t-p}\,dp$$

where $I = \{p : |p| < R, \epsilon < |t-p|\}$. We decompose this last integral

$$\int_I \frac{F(p)}{t-p}\,dp = \int_I \frac{F(p)-F(t)}{t-p}\,dp + F(t)\int_I \frac{dp}{t-p}\,.$$

The last term vanishes for $|t| > R$ and all $\epsilon > 0$. The first term on the right is majorized by

$$\int_{|p|<R} \Big|\frac{F(t)-F(p)}{t-p}\Big|\,dp \leq 2R\sup|F'|\,.$$

Thus by the dominated convergence theorem

$$\lim_{|t|\to\infty}(\mathcal{H}F)(t) = 0\,.$$

Also if $J \subset (-R,R)$ is a compact subset the mapping $F \to \mathcal{H}F$ is continuous from $\mathcal{D}_J$ into $\mathcal{E}(\mathbf{R})$ (with the topologies in Chapter V, §1).

§4 The Plancherel Formula

We recall that the functions on $\mathbf{P}^n$ have been identified with the functions φ on $\mathbf{S}^{n-1}\times\mathbf{R}$ which are even: $\varphi(-\omega,-p) = \varphi(\omega,p)$. The functional

(41) $$\varphi \to \int_{\mathbf{S}^{n-1}}\int_{\mathbf{R}} \varphi(\omega,p)\,d\omega\,dp \qquad \varphi \in C_c(\mathbf{P}^n)\,,$$

is therefore a well defined measure on $\mathbf{P}^n$, denoted $d\omega\,dp$. The group $\mathbf{M}(n)$ of rigid motions of $\mathbf{R}^n$ acts transitively on $\mathbf{P}^n$: it also leaves the measure $d\omega\,dp$ invariant. It suffices to verify this latter statement for the translations

T in $\mathbf{M}(n)$ because $\mathbf{M}(n)$ is generated by them together with the rotations around 0, and these rotations clearly leave $d\omega\, dp$ invariant. But

$$(\varphi \circ T)(\omega, p) = \varphi(\omega, p + q(\omega, T))$$

where $q(\omega, T) \in \mathbf{R}$ is independent of p so

$$\iint (\varphi \circ T)(\omega, p)\, d\omega\, dp = \iint \varphi(\omega, p + q(\omega, T))\, d\omega\, dp = \iint \varphi(\omega, p)\, dp\, d\omega\,,$$

proving the invariance.

In accordance with (49)–(50) in Ch. V the fractional power $\square^k$ is defined on $\mathcal{S}(\mathbf{P}^n)$ by

$$(-\square^k)\varphi(\omega, p) = \frac{1}{H_1(-2k)} \int_{\mathbf{R}} \varphi(\omega, q)|p - q|^{-2k-1}\, dq \tag{42}$$

and then the 1-dimensional Fourier transform satisfies

$$((-\square)^k \varphi)\widetilde{\ }(\omega, s) = |s|^{2k} \widetilde{\varphi}(\omega, s)\,. \tag{43}$$

Now, if $f \in \mathcal{S}(\mathbf{R}^n)$ we have by (4)

$$\widehat{f}(\omega, p) = (2\pi)^{-1} \int \widetilde{f}(s\omega) e^{isp}\, ds$$

and

$$(-\square)^{\frac{n-1}{4}} \widehat{f}(\omega, p) = (2\pi)^{-1} \int_{\mathbf{R}} |s|^{\frac{n-1}{2}} \widetilde{f}(s\omega) e^{isp}\, ds\,. \tag{44}$$

Theorem 4.1. *The mapping* $f \to \square^{\frac{n-1}{4}} \widehat{f}$ *extends to an isometry of* $L^2(\mathbf{R}^n)$ *onto the space* $L^2_e(\mathbf{S}^{n-1} \times \mathbf{R})$ *of even functions in* $L^2(\mathbf{S}^{n-1} \times \mathbf{R})$*, the measure on* $\mathbf{S}^{n-1} \times \mathbf{R}$ *being*

$$\tfrac{1}{2}(2\pi)^{1-n}\, d\omega\, dp\,.$$

Proof. By (44) we have from the Plancherel formula on $\mathbf{R}$

$$(2\pi) \int_{\mathbf{R}} |(-\square)^{\frac{n-1}{4}} \widehat{f}(\omega, p)|^2\, dp = \int_{\mathbf{R}} |s|^{n-1} |\widetilde{f}(s\omega)|^2\, ds$$

so by integration over $\mathbf{S}^{n-1}$ and using the Plancherel formula for $f(x) \to \widetilde{f}(s\omega)$ we obtain

$$\int_{\mathbf{R}^n} |f(x)|^2\, dx = \tfrac{1}{2}(2\pi)^{1-n} \int_{\mathbf{S}^{n-1} \times \mathbf{R}} |\square^{\frac{n-1}{4}} \widehat{f}(\omega, p)|^2\, d\omega\, dp\,.$$

It remains to prove that the mapping is surjective. For this it would suffice to prove that if $\varphi \in L^2(\mathbf{S}^{n-1} \times \mathbf{R})$ is even and satisfies

$$\int_{\mathbf{S}^{n-1}} \int_{\mathbf{R}} \varphi(\omega, p)(-\square)^{\frac{n-1}{4}} \widehat{f}(\omega, p)\, d\omega\, dp = 0$$

for all $f \in \mathcal{S}(\mathbf{R}^n)$ then $\varphi = 0$. Taking Fourier transforms we must prove that if $\psi \in L^2(\mathbf{S}^{n-1} \times \mathbf{R})$ is even and satisfies

$$\int_{\mathbf{S}^{n-1}} \int_{\mathbf{R}} \psi(\omega, s)|s|^{\frac{n-1}{2}} \widetilde{f}(s\omega)\, ds\, d\omega = 0 \tag{45}$$

for all $f \in \mathcal{S}(\mathbf{R}^n)$ then $\psi = 0$. Using the condition $\psi(-\omega, -s) = \psi(\omega, s)$ we see that

$$\begin{aligned} &\int_{\mathbf{S}^{n-1}} \int_{-\infty}^{0} \psi(\omega, s)|s|^{\frac{1}{2}(n-1)} \widetilde{f}(s\omega)\, ds\, d\omega \\ = &\int_{\mathbf{S}^{n-1}} \int_{0}^{\infty} \psi(\omega, t)|t|^{\frac{1}{2}(n-1)} \widetilde{f}(t\omega)\, dt\, d\omega \end{aligned}$$

so (45) holds with $\mathbf{R}$ replaced with the positive axis $\mathbf{R}^+$. But then the function

$$\Psi(u) = \psi\left(\frac{u}{|u|}, |u|\right)|u|^{-\frac{1}{2}(n-1)}\,, \quad u \in \mathbf{R}^n - \{0\}$$

satisfies

$$\int_{\mathbf{R}^n} \Psi(u)\widetilde{f}(u)\, du = 0\,, \quad f \in \mathcal{S}(\mathbf{R}^n)$$

so $\Psi = 0$ almost everywhere, whence $\psi = 0$.

If we combine the inversion formula in Theorem 3.6 with (46) below we obtain the following version of the Plancherel formula

$$c\int_{\mathbf{R}^n} f(x)g(x)\, dx = \int_{\mathbf{P}^n} (\Lambda\widehat{f})(\xi)\widehat{g}(\xi)\, d\xi\,.$$

§5 Radon Transform of Distributions

It will be proved in a general context in Chapter II (Proposition 2.2) that

$$\int_{\mathbf{P}^n} \widehat{f}(\xi)\varphi(\xi)\, d\xi = \int_{\mathbf{R}^n} f(x)\check{\varphi}(x)\, dx \tag{46}$$

for $f \in C_c(\mathbf{R}^n), \varphi \in C(\mathbf{P}^n)$ if $d\xi$ is a suitable fixed $\mathbf{M}(n)$-invariant measure on $\mathbf{P}^n$. Thus $d\xi = \gamma\, d\omega\, dp$ where γ is a constant, independent of f and φ. With applications to distributions in mind we shall prove (46) in a somewhat stronger form.

Lemma 5.1. *Formula (46) holds (with $\widehat{f}$ and $\check{\varphi}$ existing almost anywhere) in the following two situations:*

(a) $f \in L^1(\mathbf{R}^n)$ *vanishing outside a compact set;* $\varphi \in C(\mathbf{P}^n)$.

(b) $f \in C_c(\mathbf{R}^n)$, φ *locally integrable.*

Also $d\xi = \Omega_n^{-1}\, d\omega\, dp$.

Proof. We shall use the Fubini theorem repeatedly both on the product $\mathbf{R}^n \times \mathbf{S}^{n-1}$ and on the product $\mathbf{R}^n = \mathbf{R} \times \mathbf{R}^{n-1}$. Since $f \in L^1(\mathbf{R}^n)$ we have for each $\omega \in \mathbf{S}^{n-1}$ that $\widehat{f}(\omega, p)$ exists for almost all p and

$$\int_{\mathbf{R}^n} f(x)\, dx = \int_{\mathbf{R}} \widehat{f}(\omega, p)\, dp\,.$$

We also conclude that $\widehat{f}(\omega, p)$ exists for almost all $(\omega, p) \in \mathbf{S}^{n-1} \times \mathbf{R}$. Next we consider the measurable function

$$(x, \omega) \to f(x)\varphi(\omega, \langle \omega, x\rangle) \ \text{ on } \ \mathbf{R}^n \times \mathbf{S}^{n-1}\,.$$

We have

$$\begin{aligned}
&\int_{\mathbf{S}^{n-1}\times\mathbf{R}^n} |f(x)\varphi(\omega, \langle\omega, x\rangle)|\, d\omega\, dx \\
&\quad = \int_{\mathbf{S}^{n-1}} \left(\int_{\mathbf{R}^n} |f(x)\varphi(\omega, \langle\omega, x\rangle)|\, dx \right) d\omega \\
&\quad = \int_{\mathbf{S}^{n-1}} \left(\int_{\mathbf{R}} |f|\widehat{\ }(\omega, p)|\varphi(\omega, p)|\, dp \right) d\omega\,,
\end{aligned}$$

which in both cases is finite. Thus $f(x) \cdot \varphi(\omega, \langle\omega, x\rangle)$ is integrable on $\mathbf{R}^n \times \mathbf{S}^{n-1}$ and its integral can be calculated by removing the absolute values above. This gives the left hand side of (46). Reversing the integrations we conclude that $\check{\varphi}(x)$ exists for almost all x and that the double integral reduces to the right hand side of (46).

The formula (46) dictates how to define the Radon transform and its dual for distributions (see Chapter V). In order to make the definitions formally consistent with those for functions we would require $\widehat{S}(\varphi) = S(\check{\varphi})$, $\check{\Sigma}(f) = \Sigma(\widehat{f})$ if S and Σ are distributions on $\mathbf{R}^n$ and $\mathbf{P}^n$, respectively. But while $f \in \mathcal{D}(\mathbf{R}^n)$ implies $\widehat{f} \in \mathcal{D}(\mathbf{P}^n)$ a similar implication does not hold for φ; we do not even have $\check{\varphi} \in \mathcal{S}(\mathbf{R}^n)$ for $\varphi \in \mathcal{D}(\mathbf{P}^n)$ so $\widehat{S}$ cannot be defined as above even if S is assumed to be tempered. Using the notation $\mathcal{E}$ (resp. $\mathcal{D}$) for the space of $\mathcal{C}^\infty$ functions (resp. of compact support) and $\mathcal{D}'$ (resp. $\mathcal{E}'$) for the space of distributions (resp. of compact support) we make the following definition.

Definition. For $S \in \mathcal{E}'(\mathbf{R}^n)$ we define the functional $\widehat{S}$ by

$$\widehat{S}(\varphi) = S(\check{\varphi}) \qquad \text{for } \varphi \in \mathcal{E}(\mathbf{P}^n)\,;$$

for $\Sigma \in \mathcal{D}'(\mathbf{P}^n)$ we define the functional $\check{\Sigma}$ by

$$\check{\Sigma}(f) = \Sigma(\widehat{f}) \qquad \text{for } f \in \mathcal{D}(\mathbf{R}^n)\,.$$

Lemma 5.2. *(i) For each $\Sigma \in \mathcal{D}'(\mathbf{P}^n)$ we have $\check{\Sigma} \in \mathcal{D}'(\mathbf{R}^n)$.*

(ii) For each $S \in \mathcal{E}'(\mathbf{R}^n)$ we have $\widehat{S} \in \mathcal{E}'(\mathbf{P}^n)$.

Proof. For $A > 0$ let $\mathcal{D}_A(\mathbf{R}^n)$ denote the set of functions $f \in \mathcal{D}(\mathbf{R}^n)$ with support in the closure of $B_A(0)$. Similarly let $\mathcal{D}_A(\mathbf{P}^n)$ denote the set of functions $\varphi \in \mathcal{D}(\mathbf{P}^n)$ with support in the closure of the "ball"

$$\beta_A(0) = \{\xi \in \mathbf{P}^n : d(0, \xi) < A\} .$$

The mapping of $f \to \widehat{f}$ from $\mathcal{D}_A(\mathbf{R}^n)$ to $\mathcal{D}_A(\mathbf{P}^n)$ being continuous (with the topologies defined in Chapter V, §1) the restriction of $\check{\Sigma}$ to each $\mathcal{D}_A(\mathbf{R}^n)$ is continuous so (i) follows. That $\widehat{S}$ is a distribution is clear from (3). Concerning its support select $R > 0$ such that S has support inside $B_R(0)$. Then if $\varphi(\omega, p) = 0$ for $|p| \leq R$ we have $\check{\varphi}(x) = 0$ for $|x| \leq R$ whence $\widehat{S}(\varphi) = S(\check{\varphi}) = 0$.

Lemma 5.3. *For $S \in \mathcal{E}'(\mathbf{R}^n), \Sigma \in \mathcal{D}'(\mathbf{P}^n)$ we have*

$$(LS)\widehat{\ } = \Box\widehat{S} , \quad (\Box\Sigma)^{\vee} = L\check{\Sigma} .$$

Proof. In fact by Lemma 2.1,

$$(LS)\widehat{\ }(\varphi) = (LS)(\check{\varphi}) = S(L\check{\varphi}) = S((\Box\varphi)^{\vee}) = \widehat{S}(\Box\varphi) = (\Box\widehat{S})(\varphi) .$$

The other relation is proved in the same manner.

We shall now prove an analog of the support theorem (Theorem 2.6) for distributions. For $A > 0$ let $\beta_A(0)$ be defined as above and let supp denote support.

Theorem 5.4. *Let $T \in \mathcal{E}'(\mathbf{R}^n)$ satisfy the condition*

$$\operatorname{supp} \widehat{T} \subset C\ell(\beta_A(0)) , \quad (C\ell = \text{ closure}) .$$

Then

$$\operatorname{supp}(T) \subset C\ell(B_A(0)) .$$

Proof. For $f \in \mathcal{D}(\mathbf{R}^n)$, $\varphi \in \mathcal{D}(\mathbf{P}^n)$ we can consider the "convolution"

$$(f \times \varphi)(\xi) = \int_{\mathbf{R}^n} f(y)\varphi(\xi - y)\, dy ,$$

where for $\xi \in \mathbf{P}^n$, $\xi - y$ denotes the translate of the hyperplane ξ by $-y$. Then

$$(f \times \varphi)^\vee = f * \check{\varphi}\,.$$

In fact, if ξ_0 is any hyperplane through 0,

$$\begin{aligned}(f \times \varphi)^\vee(x) &= \int_K dk \int_{\mathbf{R}^n} f(y)\varphi(x + k \cdot \xi_0 - y)\,dy \\ &= \int_K dk \int_{\mathbf{R}^n} f(x-y)\varphi(y + k \cdot \xi_0)\,dy = (f * \check{\varphi})(x)\,.\end{aligned}$$

By the definition of $\widehat{T}$, the support assumption on $\widehat{T}$ is equivalent to

$$T(\check{\varphi}) = 0$$

for all $\varphi \in \mathcal{D}(\mathbf{P}^n)$ with support in $\mathbf{P}^n - C\ell(\beta_A(0))$. Let $\epsilon > 0$, let $f \in \mathcal{D}(\mathbf{R}^n)$ be a symmetric function with support in $C\ell(B_\epsilon(0))$ and let $\varphi = \mathcal{D}(\mathbf{P}^n)$ have support contained in $\mathbf{P}^n - C\ell(\beta_{A+\epsilon}(0))$. Since $d(0, \xi - y) \leq d(0, \xi) + |y|$ it follows that $f \times \varphi$ has support in $\mathbf{P}^n - C\ell(\beta_A(0))$; thus by the formulas above, and the symmetry of f,

$$(f * T)(\check{\varphi}) = T(f * \check{\varphi}) = T((f \times \varphi)^\vee) = 0\,.$$

But then

$$(f * T)\widehat{\ }(\varphi) = (f * T)(\check{\varphi}) = 0\,,$$

which means that $(f * T)\widehat{\ }$ has support in $C\ell(\beta_{A+\epsilon}(0))$. But now Theorem 2.6 implies that $f * T$ has support in $C\ell(B_{A+\epsilon})(0)$. Letting $\epsilon \to 0$ we obtain the desired conclusion, $\operatorname{supp}(T) \subset C\ell(B_A(0))$.

We can now extend the inversion formulas for the Radon transform to distributions. First we observe that the Hilbert transform $\mathcal{H}$ can be extended to distributions T on $\mathbf{R}$ of compact support. It suffices to put

$$\mathcal{H}(T)(F) = T(-\mathcal{H}F)\,, \quad F \in \mathcal{D}(\mathbf{R})\,.$$

In fact, as remarked at the end of §3, the mapping $F \longrightarrow \mathcal{H}F$ is a continuous mapping of $\mathcal{D}(\mathbf{R})$ into $\mathcal{E}(\mathbf{R})$. In particular $\mathcal{H}(T) \in \mathcal{D}'(\mathbf{R})$.

Theorem 5.5. *The Radon transform* $S \longrightarrow \widehat{S}$ $\quad (S \in \mathcal{E}'(\mathbf{R}^n))$ *is inverted by the following formula*

$$cS = (\Lambda \widehat{S})^\vee\,, \quad S \in \mathcal{E}'(\mathbf{R}^n)\,,$$

where the constant $c = (-4\pi)^{(n-1)/2}\Gamma(n/2)/\Gamma(1/2)$.

In the case when n *is odd we have also*

$$c\,S = L^{(n-1)/2}((\widehat{S})^\vee)\,.$$

Remark 5.6. Since $\widehat{S}$ has compact support and since Λ is defined by means of the Hilbert transform the remarks above show that $\Lambda\widehat{S} \in \mathcal{D}'(\mathbf{P}^n)$ so the right hand side is well defined.

Proof. Using Theorem 3.6 we have

$$(\Lambda\widehat{S})^\vee(f) = (\Lambda\widehat{S})(\widehat{f}) = \widehat{S}(\Lambda\widehat{f}) = S((\Lambda\widehat{f})^\vee) = cS(f)\,.$$

The other inversion formula then follows, using the lemma.

In analogy with β_A we define the "sphere" σ_A in $\mathbf{P}^n$ as

$$\sigma_A = \{\xi \in \mathbf{P}^n : d(0,\xi) = A\}\,.$$

From Theorem 5.5 we can then deduce the following complement to Theorem 5.4.

Corollary 5.7. *Suppose n is odd. Then if $S \in \mathcal{E}'(\mathbf{R}^n)$,*

$$\mathrm{supp}(\widehat{S}) \in \sigma_R \Rightarrow \mathrm{supp}(S) \subset S_R(0)\,.$$

To see this let $\epsilon > 0$ and let $f \in \mathcal{D}(\mathbf{R}^n)$ have $\mathrm{supp}(f) \subset B_{R-\epsilon}(0)$. Then $\mathrm{supp}\,\widehat{f} \in \beta_{R-\epsilon}$ and since Λ is a differential operator, $\mathrm{supp}(\Lambda\widehat{f}) \subset \beta_{R-\epsilon}$. Hence

$$cS(f) = S((\Lambda\widehat{f})^\vee) = \widehat{S}(\Lambda\widehat{f}) = 0$$

so $\mathrm{supp}(S) \cap B_{R-\epsilon}(0) = \emptyset$. Since $\epsilon > 0$ is arbitrary,

$$\mathrm{supp}(S) \cap B_R(0) = \emptyset\,.$$

On the other hand by Theorem 5.4, $\mathrm{supp}(S) \subset \overline{B_R(0)}$. This proves the corollary.

Let M be a manifold and $d\mu$ a measure such that on each local coordinate patch with coordinates $(t_1,\ldots,t_n)$ the Lebesque measure $dt_1,\ldots,dt_n$ and $d\mu$ are absolutely continuous with respect to each other. If h is a function on M locally integrable with respect to $d\mu$ the distribution $\varphi \to \int \varphi h\, d\mu$ will be denoted T_h.

Proposition 5.8. *(a) Let $f \in L^1(\mathbf{R}^n)$ vanish outside a compact set. Then the distribution T_f has Radon transform given by*

$$\widehat{T}_f = T_{\widehat{f}}\,. \tag{47}$$

(b) Let φ be a locally integrable function on $\mathbf{P}^n$. Then

$$(T_\varphi)^\vee = T_{\check{\varphi}}\,. \tag{48}$$

Proof. The existence and local integrability of $\widehat{f}$ and $\check{\varphi}$ was established during the proof of Lemma 5.1. The two formulas now follow directly from Lemma 5.1.

As a result of this proposition the smoothness assumption can be dropped in the inversion formula. In particular, we can state the following result.

Corollary 5.9. *(n odd.) The inversion formula*

$$cf = L^{(n-1)/2}((\widehat{f})^{\vee}),$$

$c = (-4\pi)^{(n-1)/2}\Gamma(n/2)/\Gamma(1/2)$, *holds for all* $f \in L^1(\mathbf{R}^n)$ *vanishing outside a compact set, the derivative interpreted in the sense of distributions.*

Examples. If μ is a measure (or a distribution) on a closed submanifold S of a manifold M the distribution on M given by $\varphi \to \mu(\varphi|S)$ will also be denoted by μ.

(a) Let δ_0 be the delta distribution $f \to f(0)$ on $\mathbf{R}^n$. Then

$$\widehat{\delta_0}(\varphi) = \delta_0(\check{\varphi}) = \Omega_n^{-1} \int_{S^{n-1}} \varphi(\omega, 0)\, d\omega$$

so

$$\widehat{\delta_0} = \Omega_n^{-1} m_{\mathbf{S}^{n-1}} \tag{49}$$

the normalized measure on $\mathbf{S}^{n-1}$ considered as a distribution on $\mathbf{S}^{n-1} \times \mathbf{R}$.

(b) Let ξ_0 denote the hyperplane $x_n = 0$ in $\mathbf{R}^n$, and δ_{ξ_0} the delta distribution $\varphi \to \varphi(\xi_0)$ on $\mathbf{P}^n$. Then

$$(\delta_{\xi_0})^{\vee}(f) = \int_{\xi_0} f(x)\, dm(x)$$

so

$$(\delta_{\xi_0})^{\vee} = m_{\xi_0}, \tag{50}$$

the Euclidean measure of ξ_0.

(c) Let χ_B be the characteristic function of the unit ball $B \subset \mathbf{R}^n$. Then by (47),

$$\widehat{\chi}_B(\omega, p) = \begin{cases} \frac{\Omega_{n-1}}{n-1}(1-p^2)^{(n-1)/2} & , \ |p| \leq 1 \\ 0 & , \ |p| > 1 \end{cases}.$$

(d) Let Ω be a bounded convex region in $\mathbf{R}^n$ whose boundary is a smooth surface. We shall obtain a formula for the volume of Ω in terms of the areas of its hyperplane sections. For simplicity we assume n odd. The characteristic function χ_Ω is a distribution of compact support and $(\chi_\Omega)\widehat{}$ is thus well defined. Approximating χ_Ω in the L^2-norm by a sequence $(\psi_n) \subset \mathcal{D}(\Omega)$ we see from Theorem 4.1 that $\partial_p^{(n-1)/2}\widehat{\psi}_n(\omega, p)$ converges in the L^2-norm on $\mathbf{P}^n$. Since

$$\int \widehat{\psi}(\xi)\varphi(\xi)\, d\xi = \int \psi(x)\check{\varphi}(x)\, dx$$

it follows from Schwarz' inequality that $\widehat{\psi}_n \longrightarrow (\chi_\Omega)\widehat{}$ in the sense of distributions and accordingly $\partial^{(n-1)/2}\widehat{\psi}_n$ converges as a distribution to $\partial^{(n-1)/2}((\chi_\Omega)\widehat{}\,)$. Since the L^2 limit is also a limit in the sense of distributions this last function equals the L^2 limit of the sequence $\partial^{(n-1)/2}\widehat{\psi}_n$. From Theorem 4.1 we can thus conclude the following result:

Theorem 5.10. *Let $\Omega \subset \mathbf{R}^n$ (n odd) be a convex region as above and $V(\Omega)$ its volume. Let $A(\omega, p)$ denote the $(n-1)$-dimensional area of the intersection of Ω with the hyperplane $\langle x, \omega\rangle = p$. Then*

$$V(\Omega) = \tfrac{1}{2}(2\pi)^{1-n} \int_{\mathbf{S}^{n-1}} \int_{\mathbf{R}} \left| \frac{\partial^{(n-1)/2} A(\omega, p)}{\partial p^{(n-1)/2}} \right|^2 dp\, d\omega\,. \tag{51}$$

§6 Integration over d-planes. X-ray Transforms. The Range of the d-plane Transform

Let d be a fixed integer in the range $0 < d < n$. We define the *d-dimensional Radon transform* $f \to \widehat{f}$ by

$$\widehat{f}(\xi) = \int_\xi f(x)\, dm(x) \qquad \xi \quad \text{a } d\text{-plane}\,. \tag{52}$$

Because of the applications to radiology indicated in § 7,b) the 1-dimensional Radon transform is often called the *X-ray transform*. Since a hyperplane can be viewed as a disjoint union of parallel d-planes parameterized by $\mathbf{R}^{n-1-d}$ it is obvious from (4) that the transform $f \to \widehat{f}$ is injective. Similarly we deduce the following consequence of Theorem 2.6.

Corollary 6.1. *Let $f, g \in C(\mathbf{R}^n)$ satisfy the rapid decrease condition: For each $m > 0$, $|x|^m f(x)$ and $|x|^m g(x)$ are bounded on $\mathbf{R}^n$. Assume for the d-dimensional Radon transforms*

$$\widehat{f}(\xi) = \widehat{g}(\xi)$$

whenever the d-plane ξ lies outside the unit ball. Then

$$f(x) = g(x) \quad \text{for} \quad |x| > 1\,.$$

We shall now generalize the inversion formula in Theorem 3.1. If φ is a continuous function on the space of d-planes in $\mathbf{R}^n$ we denote by $\check{\varphi}$ the point function

$$\check{\varphi}(x) = \int_{x\in\xi} \varphi(\xi)\, d\mu(\xi)\,,$$

where μ is the unique measure on the (compact) space of d-planes passing through x, invariant under all rotations around x and with total measure 1. If σ is a fixed d-plane through the origin we have in analogy with (16),

$$\check{\varphi}(x) = \int_K \varphi(x + k\cdot\sigma)\, dk\,. \tag{53}$$

Theorem 6.2. *The d-dimensional Radon transform in $\mathbf{R}^n$ is inverted by the formula*

$$cf = (-L)^{d/2}((\widehat{f})^\vee)\,, \tag{54}$$

where $c = (4\pi)^{d/2}\Gamma(n/2)/\Gamma((n-d)/2)$. Here it is assumed that $f(x) = 0(|x|^{-N})$ for some $N > n$.

Proof. We have in analogy with (34)

$$\begin{aligned} (\widehat{f})^\vee(x) &= \int_K \left(\int_\sigma f(x+k\cdot y)\, dm(y)\right) dk \\ &= \int_\sigma dm(y) \int_K f(x+k\cdot y)\, dk = \int_\sigma (M^{|y|}f)(x)\, dm(y)\,. \end{aligned}$$

Hence

$$(\widehat{f})^\vee(x) = \Omega_d \int_0^\infty (M^r f)(x) r^{d-1}\, dr$$

so using polar coordinates around x,

$$(\widehat{f})^\vee(x) = \frac{\Omega_d}{\Omega_n} \int_{\mathbf{R}^n} |x-y|^{d-n} f(y)\, dy\,. \tag{55}$$

The theorem now follows from Proposition 5.7 in Chapter V.

As a consequence of Theorem 2.10 we now obtain a generalization, characterizing the image of the space $\mathcal{D}(\mathbf{R}^n)$ under the d-dimensional Radon transform.

The set $\mathbf{G}(d,n)$ of d-planes in $\mathbf{R}^n$ is a manifold, in fact a homogeneous space of the group $\mathbf{M}(n)$ of all isometries of $\mathbf{R}^n$. Let $\mathbf{G}_{d,n}$ denote the manifold of all d-dimensional subspaces (d-planes through 0) of $\mathbf{R}^n$. The parallel translation of a d-plane to one through 0 gives a mapping π of $\mathbf{G}(d,n)$ onto $\mathbf{G}_{d,n}$. The inverse image $\pi^{-1}(\sigma)$ of a member $\sigma \in \mathbf{G}_{d,n}$ is naturally identified with the orthogonal complement $\sigma^\perp$. Let us write

$$\xi = (\sigma, x'') = x'' + \sigma \ \text{ if } \ \sigma = \pi(\xi) \ \text{ and } \ x'' = \sigma^\perp \cap \xi\,.$$

(See Fig. I.6.) Then (52) can be written

(56)

$$\widehat{f}(x'' + \sigma) = \int_\sigma f(x' + x'')\, dx' .$$

For $k \in \mathbb{Z}^+$ we consider the polynomial

(57)

$$P_k(u) = \int_{\mathbf{R}^n} f(x)\langle x, u\rangle^k\, dx .$$

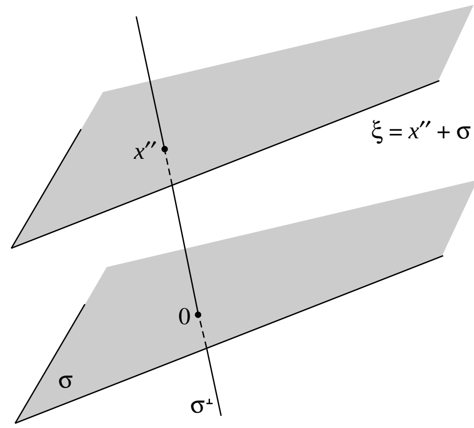

FIGURE I.6.

If $u = u'' \in \sigma^\perp$ this can be written

$$\int_{\mathbf{R}^n} f(x)\langle x, u''\rangle^k\, dx = \int_{\sigma^\perp}\int_\sigma f(x' + x'')\langle x'', u''\rangle^k\, dx'\, dx''$$

so the polynomial

$$P_{\sigma,k}(u'') = \int_{\sigma^\perp} \widehat{f}(x'' + \sigma)\langle x'', u''\rangle^k\, dx''$$

is the restriction to $\sigma^\perp$ of the polynomial P_k.

In analogy with the space $\mathcal{D}_H(\mathbf{P}^n)$ in No. 2 we define the space $\mathcal{D}_H(\mathbf{G}(d, n))$ as the set of $\mathcal{C}^\infty$ functions

$$\varphi(\xi) = \varphi_\sigma(x'') = \varphi(x'' + \sigma) \quad (\text{if } \ \xi = (\sigma, x''))$$

on $\mathbf{G}(d, n)$ of compact support satisfying the following condition.

(H) : *For each $k \in \mathbb{Z}^+$ there exists a homogeneous k^{th} degree polynomial P_k on $\mathbf{R}^n$ such that for each $\sigma \in \mathbf{G}_{d,n}$ the polynomial*

$$P_{\sigma,k}(u'') = \int_{\sigma^\perp} \varphi(x'' + \sigma)\langle x'', u''\rangle^k\, dx'' , \quad u'' \in \sigma^\perp ,$$

coincides with the restriction $P_k|\sigma^\perp$.

Theorem 6.3. *The d-dimensional Radon transform is a bijection of $\mathcal{D}(\mathbf{R}^n)$ onto $\mathcal{D}_H(\mathbf{G}(d, n))$.*

Proof. For $d = n - 1$ this is Theorem 2.10. We shall now reduce the case of general $d \leq n - 2$ to the case $d = n - 1$. It remains just to prove the surjectivity in Theorem 6.3.

We shall actually prove a stronger statement.

Theorem 6.4. *Let $\varphi \in \mathcal{D}(\mathbf{G}(d,n))$ have the property: For each pair $\sigma, \tau \in \mathbf{G}_{d,n}$ and each $k \in \mathbb{Z}^+$ the polynomials*

$$\begin{aligned} P_{\sigma,k}(u) &= \int_{\sigma^\perp} \varphi(x'' + \sigma)\langle x'', u\rangle^k \, dx'' \qquad u \in \mathbf{R}^n \\ P_{\tau,k}(u) &= \int_{\tau^\perp} \varphi(y'' + \tau)\langle y'', u\rangle^k \, dy'' \qquad u \in \mathbf{R}^n \end{aligned}$$

agree for $u \in \sigma^\perp \cap \tau^\perp$. Then $\varphi = \widehat{f}$ for some $f \in \mathcal{D}(\mathbf{R}^n)$.

Proof. Let $\varphi = \mathcal{D}(\mathbf{G}(d,n))$ have the property above. Let $\omega \in \mathbf{R}^n$ be a unit vector. Let $\sigma, \tau \in \mathbf{G}_{d,n}$ be perpendicular to ω. Consider the $(n-d-1)$-dimensional integral

$$\Psi_\sigma(\omega, p) = \int_{\langle \omega, x''\rangle = p,\, x'' \in \sigma^\perp} \varphi(x'' + \sigma) d_{n-d-1}(x'')\,, \quad p \in \mathbf{R}\,. \tag{58}$$

We claim that

$$\Psi_\sigma(\omega, p) = \Psi_\tau(\omega, p)\,.$$

To see this consider the moment

$$\begin{aligned} &\int_{\mathbf{R}} \Psi_\sigma(\omega, p) p^k \, dp \\ &= \int_{\mathbf{R}} p^k \left(\int \varphi(x'' + \sigma) d_{n-d-1}(x'') \right) dp = \int_{\sigma^\perp} \varphi(x'' + \sigma)\langle x'', \omega\rangle^k \, dx'' \\ &= \int_{\tau^\perp} \varphi(y'' + \tau)\langle y'', \omega\rangle^k \, dy'' = \int_{\mathbf{R}} \Psi_\tau(\omega, p) p^k \, dp\,. \end{aligned}$$

Thus $\Psi_\sigma(\omega, p) - \Psi_\tau(\omega, p)$ is perpendicular to all polynomials in p; having compact support it would be identically 0. We therefore put $\Psi(\omega, p) = \Psi_\sigma(\omega, p)$. Observe that Ψ is smooth; in fact for ω in a neighborhood of a fixed ω_0 we can let σ depend smoothly on ω so by (58), $\Psi_\sigma(\omega, p)$ is smooth.

Writing

$$\langle x'', \omega\rangle^k = \sum_{|\alpha| = k} p_\alpha(x'')\omega^\alpha\,, \qquad \omega^\alpha = \omega_1^{\alpha_1} \dots \omega_n^{\alpha_n}$$

we have

$$\int_{\mathbf{R}} \Psi(\omega, p) p^k \, dp = \sum_{|\alpha| = k} A_\alpha \omega^\alpha\,,$$

where

$$A_\alpha = \int_{\sigma^\perp} \varphi(x'' + \sigma) p_\alpha(x'') \, dx''\,.$$

Here A_α is independent of σ if $\omega \in \sigma^\perp$; in other words, viewed as a function of ω, A_α has for each σ a constant value as ω varies in $\sigma^\perp \cap S_1(0)$. To see that this value is the same as the value on $\tau^\perp \cap S_1(0)$ we observe that there exists a $\rho \in \mathbf{G}_{d,n}$ such that $\rho^\perp \cap \sigma^\perp \neq 0$ and $\rho^\perp \cap \tau^\perp \neq 0$. (Extend the 2-plane spanned by a vector in $\sigma^\perp$ and a vector in $\tau^\perp$ to an $(n-d)$-plane.) This shows that A_α is constant on $S_1(0)$ so $\Psi \in \mathcal{D}_H(\mathbf{P}^n)$. Thus by Theorem 2.10,

$$\Psi(\omega, p) = \int_{\langle x,\omega\rangle = p} f(x)\, dm(x) \tag{59}$$

for some $f \in \mathcal{D}(\mathbf{R}^n)$. It remains to prove that

$$\varphi(x'' + \sigma) = \int_\sigma f(x' + x'')\, dx' . \tag{60}$$

But as x'' runs through an arbitrary hyperplane in $\sigma^\perp$ it follows from (58) and (59) that both sides of (60) have the same integral. By the injectivity of the $(n-d-1)$-dimensional Radon transform on $\sigma^\perp$ equation (60) follows. This proves Theorem 6.4.

Theorem 6.4 raises the following elementary question: If a function f on $\mathbf{R}^n$ is a polynomial on each k-dimensional subspace, is f itself a polynomial? The answer is no for $k = 1$ but yes if $k > 1$. See Proposition 6.13 below, kindly communicated by Schlichtkrull.

We shall now prove another characterization of the range of $\mathcal{D}(\mathbf{R}^n)$ under the d-plane transform (for $d \leq n-2$). The proof will be based on Theorem 6.4.

Given any $d+1$ points $(x_0, \ldots, x_d)$ in general position let $\xi(x_0, \ldots, x_d)$ denote the d-plane passing through them. If $\varphi \in \mathcal{E}(\mathbf{G}(d,n))$ we shall write $\varphi(x_0, \ldots, x_d)$ for the value $\varphi(\xi(x_0, \ldots, x_d))$. We also write $V(\{x_i - x_0\}_{i=1,d})$ for the volume of the parallelepiped spanned by vectors $(x_i - x_0), (1 \leq i \leq d)$. The mapping

$$(\lambda_1, \ldots, \lambda_d) \to x_0 + \Sigma_{i=1}^d \lambda_i (x_i - x_0)$$

is a bijection of $\mathbf{R}^d$ onto $\xi(x_0, \ldots, x_d)$ and

$$\widehat{f}(x_0, \ldots, x_d) = V(\{x_i - x_0\}_{i=1,d}) \int_{\mathbf{R}^d} f(x_0 + \Sigma_i \lambda_i (x_i - x_0))\, d\lambda . \tag{61}$$

The range $\mathcal{D}(\mathbf{R}^n)$ can now be described by the following alternative to Theorem 6.4. Let x_i^k denote the k^{th} coordinate of x_i.

Theorem 6.5. *If $f \in \mathcal{D}(\mathbf{R}^n)$ then $\varphi = \widehat{f}$ satisfies the system*

$$(\partial_{i,k}\partial_{j,\ell} - \partial_{j,k}\partial_{i,\ell})\big(\varphi(x_0, \ldots, x_d)/V(\{x_i - x_0\}_{i=1,d})\big) = 0 , \tag{62}$$

where

$$0 \leq i, j \leq d\,,\ 1 \leq k, \ell \leq n\,, \partial_{i,k} = \partial/\partial x_i^k .$$

Conversely, if $\varphi \in \mathcal{D}(\mathbf{G}(d,n))$ satisfies (62) then $\varphi = \widehat{f}$ for some $f \in \mathcal{D}(\mathbf{R}^n)$.

The validity of (62) for $\varphi = \widehat{f}$ is obvious from (61) just by differentiation under the integral sign. For the converse we first prove a simple lemma.

Lemma 6.6. *Let* $\varphi \in \mathcal{E}(\mathbf{G}(d,n))$ *and* $A \in \mathbf{O}(n)$. *Let* $\psi = \varphi \circ A$. *Then if* $\varphi(x_0, \dots, x_d)$ *satisfies (62) so does the function*

$$\psi(x_0, \dots, x_d) = \varphi(Ax_0, \dots, Ax_d)\,.$$

Proof. Let $y_i = Ax_i$ so $y_i^\ell = \Sigma_p a_{\ell p} x_i^p$. Then, if $D_{i,k} = \partial/\partial y_i^k$,

$$(\partial_{i,k}\partial_{j,\ell} - \partial_{j,k}\partial_{i,\ell}) = \Sigma_{p,q=1}^n a_{pk} a_{q\ell}(D_{i,p}D_{j,q} - D_{i,q}D_{j,p})\,. \tag{63}$$

Since A preserves volumes, the lemma follows.

Suppose now φ satisfies (62). We write $\sigma = (\sigma_1, \dots, \sigma_d)$ if (σ_j) is an orthonormal basis of σ. If $x'' \in \sigma^\perp$, the $(d+1)$-tuple

$$(x'', x'' + \sigma_1, \dots, x'' + \sigma_d)$$

represents the d-plane $x'' + \sigma$ and the polynomial

$$\begin{aligned}(64)\ P_{\sigma,k}(u'') &= \int_{\sigma^\perp} \varphi(x''+\sigma)\langle x'', u''\rangle^k \, dx'' \\ &= \int_{\sigma^\perp} \varphi(x'', x''+\sigma_1, \dots, x''+\sigma_d)\langle x'', u''\rangle^k \, dx''\,, \quad u'' \in \sigma^\perp\,,\end{aligned}$$

depends only on σ. In particular, it is invariant under orthogonal transformations of $(\sigma_1, \dots, \sigma_d)$. In order to use Theorem 6.4 we must show that for any $\sigma, \tau \in \mathbf{G}_{d,n}$ and any $k \in \mathbb{Z}^+$,

$$P_{\sigma,k}(u) = P_{\tau,k}(u) \ \text{ for } \ u \in \sigma^\perp \cap \tau^\perp\,, \quad |u| = 1\,. \tag{65}$$

The following lemma is a basic step towards (65).

Lemma 6.7. *Assume* $\varphi \in \mathbf{G}(d,n)$ *satisfies (62). Let*

$$\sigma = (\sigma_1, \dots, \sigma_d), \tau = (\tau_1, \dots, \tau_d)$$

be two members of $\mathbf{G}_{d,n}$. *Assume*

$$\sigma_j = \tau_j \quad \text{for } 2 \le j \le d\,.$$

Then

$$P_{\sigma,k}(u) = P_{\tau,k}(u) \quad \text{for } u \in \sigma^\perp \cap \tau^\perp\,, \quad |u| = 1\,.$$

Proof. Let $e_i (1 \le i \le n)$ be the natural basis of $\mathbf{R}^n$ and $\epsilon = (e_1, \dots, e_d)$. Select $A \in \mathbf{O}(n)$ such that

$$\sigma = A\epsilon\,, \quad u = Ae_n\,.$$

Let

$$\eta = A^{-1}\tau = (A^{-1}\tau_1, \dots, A^{-1}\tau_d) = (A^{-1}\tau_1, e_2, \dots, e_d)\,.$$

The vector $E = A^{-1}\tau_1$ is perpendicular to e_j $(2 \leq j \leq d)$ and to e_n (since $u \in \tau^{\perp}$). Thus

$$E = a_1 e_1 + \sum_{d+1}^{n-1} a_i e_i \qquad (a_1^2 + \sum_i a_i^2 = 1)\,.$$

In (64) we write $P^{\varphi}_{\sigma,k}$ for $P_{\sigma,k}$. Putting $x'' = Ay$ and $\psi = \varphi \circ A$ we have

$$P^{\varphi}_{\sigma,k}(u) = \int_{\epsilon^{\perp}} \varphi(Ay, A(y+e_1), \dots, A(y+e_d))\langle y, e_n\rangle^k \; dy = P^{\psi}_{\epsilon,k}(e_n)$$

and similarly

$$P^{\varphi}_{\tau,k}(u) = P^{\psi}_{\eta,k}(e_n)\,.$$

Thus, taking Lemma 6.6 into account, we have to prove the statement:

$$P_{\epsilon,k}(e_n) = P_{\eta,k}(e_n)\,, \tag{66}$$

where $\epsilon = (e_1, \dots, e_d), \eta = (E, e_2, \dots, e_d)$, E being any unit vector perpendicular to e_j $(2 \leq j \leq d)$ and to e_n. First we take

$$E = E_t = \sin t\, e_1 + \cos t\, e_i \quad (d < i < n)$$

and put $\epsilon_t = (E_t, e_2, \dots, e_d)$. We shall prove

$$P_{\epsilon_t,k}(e_n) = P_{\epsilon,k}(e_n)\,. \tag{67}$$

Without restriction of generality we can take $i = d+1$. The space $\epsilon_t^{\perp}$ consists of the vectors

$$x_t = (-\cos t\, e_1 + \sin t\, e_{d+1})\lambda_{d+1} + \sum_{i=d+2}^{n} \lambda_i e_i\,, \quad \lambda_i \in \mathbf{R}\,. \tag{68}$$

Putting $P(t) = P_{\epsilon_t,k}(e_n)$ we have

$$P(t) = \int_{\mathbf{R}^{n-d}} \varphi(x_t, x_t + E_t, x_t + e_2, \dots, x_t + e_d)\lambda_n^k \, d\lambda_n \dots d\lambda_{d+1}\,. \tag{69}$$

In order to use (62) we replace φ by the function

$$\psi(x_0,\dots,x_d)=\varphi(x_0,\dots,x_d)/V(\{x_i-x_0\}_{i=1,d})\,.$$

Since the vectors in (69) span volume 1 replacing φ by ψ in (69) does not change $P(t)$. Applying $\partial/\partial t$ we get (with $d\lambda=d\lambda_n\dots d\lambda_{d+1}$),

$$\begin{aligned} P'(t) &= \int_{\mathbf{R}^{n-d}}\Big[\sum_{j=0}^{d}\lambda_{d+1}(\sin t\,\partial_{j,1}\psi+\cos t\,\partial_{j,d+1}\psi) \\ &+ \cos t\,\partial_{1,1}\psi-\sin t\,\partial_{1,d+1}\psi\Big]\lambda_n^k\,d\lambda\,. \end{aligned} \tag{70}$$

Now φ is a function on $\mathbf{G}(d,n)$. Thus for each $i\neq j$ it is invariant under the substitution

$$y_k=x_k\ (k\neq i),\ y_i=sx_i+(1-s)x_j=x_j+s(x_i-x_j)\,,\quad s>0$$

whereas the volume changes by the factor s. Thus

$$\psi(y_0,\dots,y_d)=s^{-1}\psi(x_0,\dots,x_d)\,.$$

Taking $\partial/\partial s$ at $s=1$ we obtain

$$\psi(x_0,\dots,x_d)+\sum_{k=1}^{n}(x_i^k-x_j^k)(\partial_{i,k}\psi)(x_0,\dots,x_d)=0\,. \tag{71}$$

Note that in (70) the derivatives are evaluated at

$$(x_0,\dots,x_d)=(x_t,x_t+E_t,x_t+e_2,\dots,x_t+e_d)\,. \tag{72}$$

Using (71) for $(i,j)=(1,0)$ and $(i,j)=(0,1)$ and adding we obtain

$$\sin t\,(\partial_{0,1}\psi+\partial_{1,1}\psi)+\cos t\,(\partial_{0,d+1}\psi+\partial_{1,d+1}\psi)=0\,. \tag{73}$$

For $i\geq 2$ we have

$$x_i-x_0=e_i,\ x_i-x_1=-\sin t\,e_1-\cos t\,e_{d+1}+e_i\,,$$

and this gives the relations (for $j=0$ and $j=1$)

$$\psi(x_0,\dots,x_d)+(\partial_{i,i}\psi)(x_0,\dots,x_d)=0\,, \tag{74}$$

$$\psi-\sin t\ (\partial_{i,1}\psi)-\cos t\ (\partial_{i,d+1}\psi)+\partial_{i,i}\psi=0\,. \tag{75}$$

Thus by (73)–(75) formula (70) simplifies to

$$P'(t)=\int_{\mathbf{R}^{n-d}}[\cos t\ (\partial_{1,1}\psi)-\sin t\ (\partial_{1,d+1}\psi)]\,\lambda_n^k\,d\lambda\,.$$

In order to bring in 2^{nd} derivatives of ψ we integrate by parts in λ_n,

$$(76)\quad (k+1)P'(t) = \int_{\mathbf{R}^{n-d}} -\frac{\partial}{\partial \lambda_n}\left[\cos t\ (\partial_{1,1}\psi) - \sin t\ (\partial_{1,d+1}\psi)\right]\lambda_n^{k+1}\,d\lambda\,.$$

Since the derivatives $\partial_{j,k}\psi$ are evaluated at the point (72) we have in (76)

$$(77)\qquad \frac{\partial}{\partial \lambda_n}(\partial_{j,k}\psi) = \sum_{i=0}^{d} \partial_{i,n}(\partial_{j,k}\psi)$$

and also, by (68) and (72),

$$(78)\qquad \frac{\partial}{\partial \lambda_{d+1}}(\partial_{j,k}\psi) = -\cos t\sum_{0}^{d}\partial_{i,1}(\partial_{j,k}\psi) + \sin t\sum_{0}^{d}\partial_{i,d+1}(\partial_{j,k}\psi)\,.$$

We now plug (77) into (76) and then invoke equations (62) for ψ which give

$$(79)\quad \sum_{0}^{d}\partial_{i,n}\partial_{1,1}\psi = \partial_{1,n}\sum_{0}^{d}\partial_{i,1}\psi\,,\qquad \sum_{0}^{d}\partial_{i,n}\partial_{1,d+1}\psi = \partial_{1,n}\sum_{0}^{d}\partial_{i,d+1}\psi\,.$$

Using (77) and (79) we see that (76) becomes

$$-(k+1)P'(t) = \int_{\mathbf{R}^{n-d}}\left[\partial_{1,n}(\cos t\,\Sigma_i\partial_{i,1}\psi - \sin t\,\Sigma_i\partial_{i,d+1}\psi)\right](x_t, x_t+E_t, \dots, x_t+e_d)\lambda_n^{k+1}\,d\lambda$$

so by (78)

$$(k+1)P'(t) = \int_{\mathbf{R}^{n-d}}\frac{\partial}{\partial\lambda_{d+1}}(\partial_{1,n}\psi)\lambda_n^{k+1}\,d\lambda\,.$$

Since $d+1<n$, the integration in λ_{d+1} shows that $P'(t)=0$, proving (67).

This shows that without changing $P_{\epsilon,k}(e_n)$ we can pass from $\epsilon = (e_1,\dots,e_d)$ to

$$\epsilon_t = (\sin\ t\,e_1 + \cos\ t\,e_{d+1}, e_2, \dots, e_d)\,.$$

By iteration we can replace e_1 by

$$\sin t_{n-d-1}\dots\sin t_1 e_1 + \sin t_{n-d-1}\dots\sin t_2\cos t_1 e_{d+1} + \dots + \cos t_{n-d-1}e_{n-1},$$

but keeping $e_2,\dots,e_d$ unchanged. This will reach an arbitrary E so (66) is proved.

We shall now prove (65) in general. We write σ and τ in orthonormal bases, $\sigma = (\sigma_1, \ldots, \sigma_d), \tau = (\tau_1, \ldots, \tau_d)$. Using Lemma 6.7 we shall pass from σ to τ without changing $P_{\sigma,k}(u)$, u being fixed.

Consider τ_1. If two members of σ, say σ_j and σ_k, are both not orthogonal to τ_1 that is ($\langle \sigma_j, \tau_1\rangle \neq 0, \langle \sigma_k, \tau_1\rangle \neq 0$) we rotate them in the (σ_j, σ_k)-plane so that one of them becomes orthogonal to τ_1. As remarked after (63) this has no effect on $P_{\sigma,k}(u)$. We iterate this process (with the same τ_1) and end up with an orthogonal frame $(\sigma_1^*, \ldots, \sigma_d^*)$ of σ in which at most one entry σ_i^* is not orthogonal to τ_1. In this frame we replace this σ_i^* by τ_1. By Lemma 6.7 this change of σ does not alter $P_{\sigma,k}(u)$.

We now repeat this process with $\tau_2, \tau_3 \ldots$, etc. Each step leaves $P_{\sigma,k}(u)$ unchanged (and u remains fixed) so this proves (65) and the theorem.

We consider now the case $d = 1, n = 3$ in more detail. Here $f \to \widehat{f}$ is the X-ray transform in $\mathbf{R}^3$. We also change the notation and write ξ for x_0, η for x_1 so $V(\{x_1 - x_0\})$ equals $|\xi - \eta|$. Then Theorem 6.5 reads as follows.

Theorem 6.8. *The X-ray transform $f \to \widehat{f}$ in $\mathbf{R}^3$ is a bijection of $\mathcal{D}(\mathbf{R}^3)$ onto the space of $\varphi \in \mathcal{D}(\mathbf{G}(1,3))$ satisfying*

$$\left(\frac{\partial}{\partial \xi_k}\frac{\partial}{\partial \eta_\ell} - \frac{\partial}{\partial \xi_\ell}\frac{\partial}{\partial \eta_k}\right)\left(\frac{\varphi(\xi,\eta)}{|\xi-\eta|}\right) = 0\,, \quad 1 \leq k, \ell \leq 3\,. \tag{80}$$

Now let $\mathbf{G}'(1,3) \subset \mathbf{G}(1,3)$ denote the open subset consisting of the *non-horizontal* lines. We shall now show that for $\varphi \in \mathcal{D}(\mathbf{G}(1,n))$ (and even for $\varphi \in \mathcal{E}(\mathbf{G}'(1,n))$) the validity of (80) for $(k,\ell) = (1,2)$ implies (80) for general (k,ℓ). Note that (71) (which is also valid for $\varphi \in \mathcal{E}(\mathbf{G}'(1,n))$) implies

$$\frac{\varphi(\xi,\eta)}{|\xi-\eta|} + \sum_1^3 (\xi_i - \eta_i)\frac{\partial}{\partial \xi_i}\left(\frac{\varphi(\xi,\eta)}{|\xi-\eta|}\right) = 0\,.$$

Here we apply $\partial/\partial\eta_k$ and obtain

$$\left(\sum_{i=1}^3 (\xi_i - \eta_i)\frac{\partial^2}{\partial\xi_i\partial\eta_k} - \frac{\partial}{\partial\xi_k} + \frac{\partial}{\partial\eta_k}\right)\left(\frac{\varphi(\xi,\eta)}{|\xi-\eta|}\right) = 0\,.$$

Exchanging ξ and η and adding we derive

$$\sum_{i=1}^3 (\xi_i - \eta_i)\left(\frac{\partial^2}{\partial\xi_i\partial\eta_k} - \frac{\partial^2}{\partial\xi_k\partial\eta_i}\right)\left(\frac{\varphi(\xi,\eta)}{|\xi-\eta|}\right) = 0 \tag{81}$$

for $k = 1,2,3$. Now assume (80) for $(k,\ell) = (1,2)$. Taking $k = 1$ in (81) we derive (80) for $(k,\ell) = (1,3)$. Then taking $k = 3$ in (81) we deduce (80) for $(k,\ell) = (3,2)$. This verifies the claim above.

We can now put this in a simpler form. Let $\ell(\xi,\eta)$ denote the line through the points $\xi \neq \eta$. Then the mapping

$$(\xi_1, \xi_2, \eta_1, \eta_2) \to \ell((\xi_1,\xi_2,0),(\eta_1,\eta_2,-1))$$

is a bijection of $\mathbf{R}^4$ onto $\mathbf{G}'(1,3)$. The operator

$$\Lambda = \frac{\partial^2}{\partial\xi_1\partial\eta_2} - \frac{\partial^2}{\partial\xi_2\partial\eta_1} \tag{82}$$

is a well defined differential operator on the dense open set $\mathbf{G}'(1,3)$. If $\varphi \in \mathcal{E}(\mathbf{G}(1,3))$ we denote by ψ the restriction of the function $(\xi,\eta) \to \varphi(\xi,\eta)/|\xi-\eta|$ to $\mathbf{G}'(1,3)$. Then we have proved the following result.

Theorem 6.9. *The X-ray transform $f \to \widehat{f}$ is a bijection of $\mathcal{D}(\mathbf{R}^3)$ onto the space*

$$\{\varphi \in \mathcal{D}(\mathbf{G}(1,3)) : \Lambda\psi = 0\}\,. \tag{83}$$

We shall now rewrite the differential equation (83) in Plücker coordinates. The line joining ξ and η has Plücker coordinates $(p_1,p_2,p_3,q_1,q_2,q_3)$ given by

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \xi_1 & \xi_2 & \xi_3 \\ \eta_1 & \eta_2 & \eta_3 \end{vmatrix} = p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}\,, \quad q_i = \begin{vmatrix} \xi_i & 1 \\ \eta_i & 1 \end{vmatrix}$$

which satisfy

$$p_1q_1 + p_2q_2 + p_3q_3 = 0\,. \tag{84}$$

Conversely, each ratio $(p_1 : p_2 : p_3 : q_1 : q_2 : q_3)$ determines uniquely a line provided (84) is satisfied. The set $\mathbf{G}'(1,3)$ is determined by $q_3 \neq 0$. Since the common factor can be chosen freely we fix q_3 as 1. Then we have a bijection $\tau : \mathbf{G}'(1,3) \to \mathbf{R}^4$ given by

$$x_1 = p_2 + q_2,\ x_2 = -p_1 - q_1,\ x_3 = p_2 - q_2,\ x_4 = -p_1 + q_1 \tag{85}$$

with inverse

$$(p_1,p_2,p_3,q_1,q_2) = \\ \big(\tfrac{1}{2}(-x_2-x_4), \tfrac{1}{2}(x_1+x_3), \tfrac{1}{4}(-x_1^2-x_2^2+x_3^2+x_4^2), \tfrac{1}{2}(-x_2+x_4), \tfrac{1}{2}(x_1-x_3)\big).$$

Theorem 6.10. *If $\varphi \in \mathcal{D}(\mathbf{G}(1,3))$ satisfies (83) then the restriction $\varphi|\mathbf{G}'(1,3)$ (with $q_3 = 1$) has the form*

$$\varphi(\xi,\eta) = |\xi-\eta|\ u(p_2+q_2, -p_1-q_1, p_2-q_2, -p_1+q_1) \tag{86}$$

where u satisfies

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2} - \frac{\partial^2 u}{\partial x_4^2} = 0\,. \tag{87}$$

On the other hand, if u satisfies (87) then (86) defines a function φ on $\mathbf{G}'(1,3)$ which satisfies (80).

Proof. First assume $\varphi \in \mathcal{D}(\mathbf{G}(1,3))$ satisfies (83) and define $u \in \mathcal{E}(\mathbf{R}^4)$ by

$$u(\tau(\ell)) = \varphi(\ell)(1+q_1^2+q_2^2)^{-\frac{1}{2}}\,, \tag{88}$$

where $\ell \in \mathbf{G}'(1,3)$ has Plücker coordinates $(p_1,p_2,p_3,q_1,q_2,1)$. On the line ℓ consider the points ξ,η for which $\xi_3=0, \eta_3=-1$ (so $q_3=1$). Then since

$$p_1=-\xi_2,\, p_2=\xi_1,\, q_1=\xi_1-\eta_1,\, q_2=\xi_2-\eta_2$$

we have

$$\frac{\varphi(\xi,\eta)}{|\xi-\eta|} = u(\xi_1+\xi_2-\eta_2,\, -\xi_1+\xi_2+\eta_1,\, \xi_1-\xi_2+\eta_2,\, \xi_1+\xi_2-\eta_1)\,. \tag{89}$$

Now (83) implies (87) by use of the chain rule.

On the other hand, suppose $u \in \mathcal{E}(\mathbf{R}^4)$ satisfies (87). Define φ by (88). Then $\varphi \in \mathcal{E}(\mathbf{G}'(1,3))$ and by (89),

$$\Lambda\left(\frac{\varphi(\xi,\eta)}{|\xi-\eta|}\right) = 0\,.$$

As shown before the proof of Theorem 6.9 this implies that the whole system (80) is verified.

We shall now see what implications Ásgeirsson's mean-value theorem (Theorem 4.5, Chapter V) has for the range of the X-ray transform. We have from (85),

$$\int_0^{2\pi} u(r\cos\varphi, r\sin\varphi,0,0)\,d\varphi = \int_0^{2\pi} u(0,0,r\cos\varphi,r\sin\varphi)\,d\varphi\,. \tag{90}$$

The first points $(r\cos\varphi, r\sin\varphi,0,0)$ correspond via (85) to the lines with

$$(p_1,p_2,p_3,q_1,q_2,q_3) \;=\; (-\tfrac{r}{2}\sin\varphi, \tfrac{r}{2}\cos\varphi, -\tfrac{r^2}{4}, -\tfrac{r}{2}\sin\varphi, \tfrac{r}{2}\cos\varphi, 1)$$

containing the points

$$\begin{aligned}(\xi_1,\xi_2,\xi_3) &= (\tfrac{r}{2}\cos\varphi, \tfrac{r}{2}\sin\varphi, 0)\\ (\eta_1,\eta_2,\eta_3) &= (\tfrac{r}{2}(\sin\varphi+\cos\varphi), +\tfrac{r}{2}(\sin\varphi-\cos\varphi), -1)\end{aligned}$$

with $|\xi-\eta|^2 = 1+\frac{r^2}{4}$. The points $(0,0,r\cos\varphi,r\sin\varphi)$ correspond via (85) to the lines with

$$(p_1,p_2,p_3,q_1,q_2,q_3) = (-\tfrac{r}{2}\sin\varphi, \tfrac{r}{2}\cos\varphi, \tfrac{r^2}{4}, \tfrac{r}{2}\sin\varphi, -\tfrac{r}{2}\cos\varphi, 1)$$

containing the points

$$\begin{aligned}(\xi_1,\xi_2,\xi_3) &= (\tfrac{r}{2}\cos\varphi, \tfrac{r}{2}\sin\varphi, 0)\\ (\eta_1,\eta_2,\eta_3) &= (\tfrac{r}{2}(\cos\varphi-\sin\varphi), \tfrac{r}{2}(\cos\varphi+\sin\varphi), -1)\end{aligned}$$

with $|\xi - \eta|^2 = 1 + \frac{r^2}{4}$. Thus (90) takes the form

$$(91)\quad \int_0^{2\pi} \varphi(\tfrac{r}{2}\cos\theta, \tfrac{r}{2}\sin\theta, 0, \tfrac{r}{2}(\sin\theta + \cos\theta), \tfrac{r}{2}(\sin\theta - \cos\theta), -1)\, d\theta$$

$$= \int_0^{2\pi} \varphi(\tfrac{r}{2}\cos\theta, \tfrac{r}{2}\sin\theta, 0, \tfrac{r}{2}(\cos\theta - \sin\theta), \tfrac{r}{2}(\cos\theta + \sin\theta), -1)\, d\theta\,.$$

The lines forming the arguments of φ in these integrals are the two families of generating lines for the hyperboloid (see Fig. I.7)

$$x^2 + y^2 = \tfrac{r^2}{4}(z^2 + 1)\,.$$

Definition. A function $\varphi \in \mathcal{E}(\mathbf{G}'(1,3))$ is said to be a *harmonic line function* if

$$\Lambda\left(\frac{\varphi(\xi,\eta)}{|\xi - \eta|}\right) = 0\,.$$

Theorem 6.11. *A function $\varphi \in \mathcal{E}(\mathbf{G}'(1,3))$ is a harmonic line function if and only if for each hyperboloid of revolution H of one sheet and vertical axis the mean values of φ over the two families of generating lines of H are equal. (The variable of integration is the polar angle in the equatorial plane of H.).*

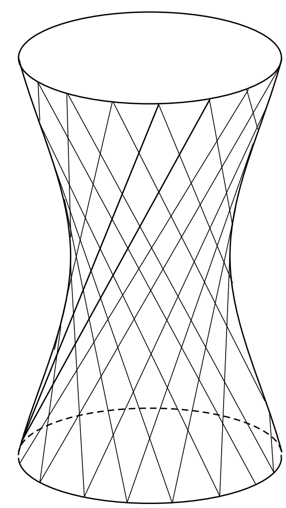

FIGURE I.7.

The proof of (91) shows that φ harmonic implies the mean value property for φ. The converse follows since (90) (with $(0,0)$ replaced by an arbitrary point in $\mathbf{R}^2$) is equivalent to (87) (Chapter V, Theorem 4.5).

Corollary 6.12. *Let $\varphi \in \mathcal{D}(\mathbf{G}(1,3))$. Then φ is in the range of the X-ray transform if and only if φ has the mean value property for arbitrary hyperboloid of revolution of one sheet (and arbitrary axis).*

We conclude this section with the following result due to Schlichtkrull mentioned in connection with Theorem 6.4.

Proposition 6.13. *Let f be a function on $\mathbf{R}^n$ and $k \in \mathbb{Z}^+$, $1 < k < n$. Assume that for each k-dimensional subspace $E_k \subset \mathbf{R}^n$ the restriction $f|E_k$ is a polynomial on E_k. Then f is a polynomial on $\mathbf{R}^n$.*

For $k = 1$ the result is false as the example $f(x,y) = xy^2/(x^2 + y^2)$, $f(0,0) = 0$ shows. We recall now the Lagrange interpolation formula. Let $a_0, \ldots, a_m$ be distinct numbers in $\mathbf{C}$. Then each polynomial $P(x)$ $(x \in \mathbf{R})$

of degree $\leq m$ can be written

$$P(x) = P(a_0)Q_0(x) + \cdots + P(a_m)Q_m(x)\,,$$

where

$$Q_i(x) = \prod_{j=0}^{m}(x-a_j)/(x-a_i)\prod_{j\neq i}(a_i-a_j)\,.$$

In fact, the two sides agree at $m+1$ distinct points. This implies the following result.

Lemma 6.14. *Let $f(x_1,\ldots,x_n)$ be a function on $\mathbf{R}^n$ such that for each i with $x_j (j\neq i)$ fixed the function $x_i \to f(x_1,\ldots,x_n)$ is a polynomial. Then f is a polynomial.*

For this we use Lagrange's formula on the polynomial $x_1 \longrightarrow f(x_1,x_2,\ldots,x_n)$ and get

$$f(x_1,\ldots,x_n) = \sum_{j=0}^{m} f(a_j,x_2,\ldots,x_m)Q_j(x_1)\,.$$

The lemma follows by iteration.

For the proposition we observe that the assumption implies that f restricted to each 2-plane E_2 is a polynomial on E_2. For a fixed $(x_2,\ldots,x_n)$ the point $(x_1,\ldots,x_n)$ is in the span of $(1,0,\ldots,0)$ and $(0,x_2,\ldots,x_n)$ so $f(x_1,\ldots,x_n)$ is a polynomial in x_1. Now the lemma implies the result.

§7 Applications

A. Partial differential equations.

The inversion formula in Theorem 3.1 is very well suited for applications to partial differential equations. To explain the underlying principle we write the inversion formula in the form

$$f(x) = \gamma\, L_x^{\frac{n-1}{2}}\left(\int_{\mathbf{S}^{n-1}} \widehat{f}(\omega,\langle x,\omega\rangle)\,d\omega\right). \tag{92}$$

where the constant γ equals $\frac{1}{2}(2\pi i)^{1-n}$. Note that the function $f_\omega(x) = \widehat{f}(\omega,\langle x,\omega\rangle)$ is a *plane wave with normal* ω, that is, it is constant on each hyperplane perpendicular to ω.

Consider now a differential operator

$$D = \sum_{(k)} a_{k_1\ldots k_n}\partial_1^{k_1}\ldots\partial_n^{k_n}$$

with constant coefficients $a_{k_1,\ldots,k_n}$, and suppose we want to solve the differential equation

$$Du = f \tag{93}$$

where f is a given function in $\mathcal{S}(\mathbf{R}^n)$. To simplify the use of (92) we assume n to be odd. We begin by considering the differential equation

$$Dv = f_\omega\,, \tag{94}$$

where f_ω is the plane wave defined above and we look for a solution v which is also a plane wave with normal ω. But a plane wave with normal ω is just a function of one variable; also if v is a plane wave with normal ω so is the function Dv. The differential equation (94) (with v a plane wave) is therefore an *ordinary* differential equation with constant coefficients. Suppose $v = u_\omega$ is a solution and assume that this choice can be made smoothly in ω. Then the function

$$u = \gamma\, L^{\frac{n-1}{2}} \int_{\mathbf{S}^{n-1}} u_\omega\, d\omega \tag{95}$$

is a solution to the differential equation (93). In fact, since D and $L^{\frac{n-1}{2}}$ commute we have

$$Du = \gamma L^{\frac{n-1}{2}} \int_{\mathbf{S}^{n-1}} Du_\omega\, d\omega = \gamma L^{\frac{n-1}{2}} \int_{\mathbf{S}^{n-1}} f_\omega\, d\omega = f\,.$$

This method only assumes that the plane wave solution u_ω to the ordinary differential equation $Dv = f_\omega$ exists and can be chosen so as to depend smoothly on ω. This cannot always be done because D might annihilate all plane waves with normal ω. (For example, take $D = \partial^2/\partial x_1 \partial x_2$ and $\omega = (1, 0)$.) However, if this restriction to plane waves is never 0 it follows from a theorem of Trèves [1963] that the solution u_ω can be chosen depending smoothly on ω. Thus we can state

Theorem 7.1. *Assuming the restriction D_ω of D to the space of plane waves with normal ω is $\neq 0$ for each ω formula (95) gives a solution to the differential equation $Du = f \quad (f \in \mathcal{S}(\mathbf{R}^n))$.*

The method of plane waves can also be used to solve the Cauchy problem for hyperbolic differential equations with constant coefficients. We illustrate this method by means of the wave equation $\mathbf{R}^n$,

$$Lu = \frac{\partial^2 u}{\partial t^2},\ u(x,0) = f_0(x),\ u_t(x,0) = f_1(x)\,, \tag{96}$$

f_0, f_1 being given functions in $\mathcal{D}(\mathbf{R}^n)$.

Lemma 7.2. *Let $h \in C^2(\mathbf{R})$ and $\omega \in \mathbf{S}^{n-1}$. Then the function*

$$v(x,t) = h(\langle x, \omega\rangle + t)$$

satisfies $Lv = (\partial^2/\partial t^2)v$.

The proof is obvious. It is now easy, on the basis of Theorem 3.6, to write down the unique solution of the Cauchy problem (96).

Theorem 7.3. *The solution to (96) is given by*

$$u(x,t) = \int_{\mathbf{S}^{n-1}} (Sf)(\omega, \langle x, \omega\rangle + t)\, d\omega \tag{97}$$

where

$$Sf = \begin{cases} c(\partial^{n-1}\widehat{f}_0 + \partial^{n-2}\widehat{f}_1)\,, & n \text{ odd} \\ c\,\mathcal{H}(\partial^{n-1}\widehat{f}_0 + \partial^{n-2}\widehat{f}_1)\,, & n \text{ even}\,. \end{cases}$$

Here $\partial = \partial/\partial p$ and the constant c equals

$$c = \tfrac{1}{2}(2\pi i)^{1-n}\,.$$

Lemma 7.2 shows that (97) is annihilated by the operator $L - \partial^2/\partial t^2$ so we just have to check the initial conditions in (96).

(a) If $n > 1$ is odd then $\omega \to (\partial^{n-1}\widehat{f}_0)(\omega, \langle x, \omega\rangle)$ is an even function on $\mathbf{S}^{n-1}$ but the other term in Sf, that is the function $\omega \to (\partial^{n-2}\widehat{f}_1)(\omega, \langle x, \omega\rangle)$, is odd. Thus by Theorem 3.6, $u(x,0) = f_0(x)$. Applying $\partial/\partial t$ to (97) and putting $t = 0$ gives $u_t(x,0) = f_1(x)$, this time because the function $\omega \to (\partial^n \widehat{f}_0)(\omega, \langle x, \omega\rangle)$ is odd and the function $\omega \to (\partial^{n-1}\widehat{f}_1)(\omega, \langle x, \omega\rangle)$ is even.

(b) If n is even the same proof works if we take into account the fact that $\mathcal{H}$ interchanges odd and even functions on $\mathbf{R}$.

Definition. For the pair $f = \{f_0, f_1\}$ we refer to the function Sf in (97) as the *source*.

In the terminology of Lax-Philips [1967] the wave $u(x,t)$ is said to be

(a) *outgoing* if $u(x,t) = 0$ in the *forward cone* $|x| < t$;

(b) *incoming* if $u(x,t) = 0$ in the *backward cone* $|x| < -t$.

The notation is suggestive because "outgoing" means that the function $x \to u(x,t)$ vanishes in larger balls around the origin as t increases.

Corollary 7.4. *The solution $u(x,t)$ to (96) is*

(i) outgoing if and only if $(Sf)(\omega, s) = 0$ for $s > 0$, all ω.

(ii) incoming if and only if $(Sf)(\omega, s) = 0$ for $s < 0$, all ω.

Proof. For (i) suppose $(Sf)(\omega, s) = 0$ for $s > 0$. For $|x| < t$ we have $\langle x, \omega\rangle + t \geq -|x| + t > 0$ so by (97) $u(x,t) = 0$ so u is outgoing. Conversely, suppose $u(x,t) = 0$ for $|x| < t$. Let $t_0 > 0$ be arbitrary and let $\varphi(t)$ be a smooth function with compact support contained in (t_0, ∞).

Then if $|x| < t_0$ we have

$$\begin{aligned} 0 &= \int_{\mathbf{R}} u(x,t)\varphi(t)\,dt = \int_{\mathbf{S}^{n-1}} d\omega \int_{\mathbf{R}} (Sf)(\omega, \langle x, \omega\rangle + t)\varphi(t)\,dt \\ &= \int_{\mathbf{S}^{n-1}} d\omega \int_{\mathbf{R}} (Sf)(\omega, p)\varphi(p - \langle x, \omega\rangle)\,dp\,. \end{aligned}$$

Taking arbitrary derivative $\partial^k/\partial x_{i_1}\dots\partial x_{i_k}$ at $x = 0$ we deduce

$$\int_{\mathbf{R}} \left(\int_{\mathbf{S}^{n-1}} (Sf)(\omega, p)\omega_{i_1}\dots\omega_{i_k}\,d\omega \right) (\partial^k\varphi)(p)\,dp = 0$$

for each k and each $\varphi \in \mathcal{D}(t_0, \infty)$. Integrating by parts in the p variable we conclude that the function

$$p \to \int_{\mathbf{S}^{n-1}} (Sf)(\omega, p)\omega_{i_1}\dots\omega_{i_k}\,d\omega\,, \quad p \in \mathbf{R} \tag{98}$$

has its k^{th} derivative $\equiv 0$ for $p > t_0$. Thus it equals a polynomial for $p > t_0$. However, if n is odd the function (98) has compact support so it must vanish identically for $p > t_0$.

On the other hand, if n is even and $F \in \mathcal{D}(\mathbf{R})$ then as remarked at the end of §3, $\lim_{|t|\to\infty}(\mathcal{H}F)(t) = 0$. Thus we conclude again that expression (98) vanishes identically for $p > t_0$.

Thus in both cases, if $p > t_0$, the function $\omega \to (Sf)(\omega, p)$ is orthogonal to all polynomials on $\mathbf{S}^{n-1}$, hence must vanish identically.

One can also solve (96) by means of the *Fourier transform*

$$\widetilde{f}(\zeta) = \int_{\mathbf{R}^n} f(x)e^{-i\langle x, \zeta\rangle}\,dx\,.$$

Assuming the function $x \to u(x,t)$ in $\mathcal{S}(\mathbf{R}^n)$ for a given t we obtain

$$\widetilde{u}_{tt}(\zeta, t) + \langle \zeta, \zeta\rangle \widetilde{u}(\zeta, t) = 0\,.$$

Solving this ordinary differential equation with initial data given in (96) we get

$$\widetilde{u}(\zeta, t) = \widetilde{f}_0(\zeta)\cos(|\zeta|t) + \widetilde{f}_1(\zeta)\frac{\sin(|\zeta|t)}{|\zeta|}\,. \tag{99}$$

The function $\zeta \to \sin(|\zeta|t)/|\zeta|$ is entire of exponential type $|t|$ on $\mathbf{C}^n$ (of at most polynomial growth on $\mathbf{R}^n$). In fact, if $\varphi(\lambda)$ is even, holomorphic on $\mathbf{C}$ and satisfies the exponential type estimate (13) in Theorem 3.3, Ch. V, then the same holds for the function Φ on $\mathbf{C}^n$ given by $\Phi(\zeta) = \Phi(\zeta_1, \dots, \zeta_n) = \varphi(\lambda)$ where $\lambda^2 = \zeta_1^2 + \cdots + \zeta_n^2$. To see this put

$$\lambda = \mu + iv\,, \quad \zeta = \xi + i\eta \qquad \mu, \nu \in \mathbf{R}\,, \quad \xi\,, \eta \in \mathbf{R}^n\,.$$

Then

$$\mu^2 - \nu^2 = |\xi|^2 - |\eta|^2\,, \quad \mu^2\nu^2 = (\xi \cdot \eta)^2\,,$$

so

$$|\lambda|^4 = (|\xi|^2 - |\eta|^2)^2 + 4(\xi \cdot \eta)^2$$

and

$$2|\operatorname{Im}\lambda|^2 = |\eta|^2 - |\xi|^2 + \left[(|\xi|^2 - |\eta|^2)^2 + 4(\xi \cdot \eta)^2\right]^{1/2}\,.$$

Since $|(\xi \cdot \eta)| \leq |\xi|\,|\eta|$ this implies $|\operatorname{Im}\lambda| \leq |\eta|$ so the estimate (13) follows for Φ. Thus by Theorem 3.3, Chapter V there exists a $T_t \in \mathcal{E}'(\mathbf{R}^n)$ with support in $\overline{B_{|t|}(0)}$ such that

$$\frac{\sin(|\zeta|t)}{|\zeta|} = \int_{\mathbf{R}^n} e^{-i\langle \zeta, x \rangle}\, dT_t(x)\,.$$

Theorem 7.5. *Given $f_0, f_1 \in \mathcal{E}(\mathbf{R}^n)$ the function*

$$u(x,t) = (f_0 * T_t')(x) + (f_1 * T_t)(x) \tag{100}$$

satisfies (96). Here T_t' stands for $\partial_t(T_t)$.

Note that (96) implies (100) if f_0 and f_1 have compact support. The converse holds without this support condition.

Corollary 7.6. *If f_0 and f_1 have support in $B_R(0)$ then u has support in the region*

$$|x| \leq |t| + R\,.$$

In fact, by (100) and support property of convolutions (Ch. V, §2), the function $x \to u(x,t)$ has support in $B_{R+|t|}(0)^-$. While Corollary 7.6 implies that for $f_0, f_1 \in \mathcal{D}(\mathbf{R}^n)$ u has support in a suitable solid cone we shall now see that Theorem 7.3 implies that if n is odd u has support in a conical shell (see Fig. I.8).

Corollary 7.7. *Let n be odd. Assume f_0 and f_1 have support in the ball $B_R(0)$.*

(i) Huygens' Principle. The solution u to (96) has support in the conical shell

$$|t| - R \leq |x| \leq |t| + R\,, \tag{101}$$

which is the union for $|y| \leq R$ of the light cones,

$$C_y = \{(x,t) : |x - y| = |t|\}\,.$$

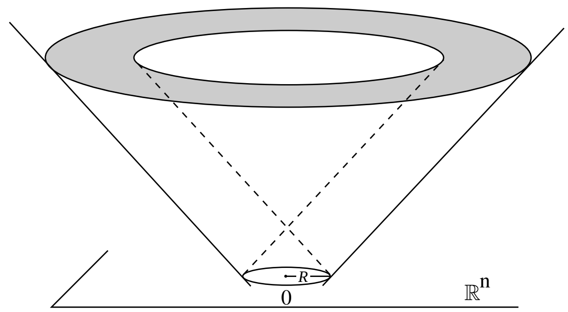

FIGURE I.8.

(ii) The solution to (96) is outgoing if and only if

$$\widehat{f}_0(\omega,p) = \int_p^\infty \widehat{f}_1(\omega,s)\,ds\,, \quad p>0\,,\ \textit{all}\ \omega \tag{102}$$

and incoming if and only if

$$\widehat{f}_0(\omega,p) = -\int_{-\infty}^p \widehat{f}_1(\omega,s)\,ds\,, \quad p<0\,,\ \textit{all}\ \omega\,.$$

Note that Part (ii) can also be stated: The solution is outgoing (incoming) if and only if

$$\int_\pi f_0 = \int_{H_\pi} f_1 \qquad \left(\int_\pi f_0 = -\int_{H_\pi} f_1\right)$$

for an arbitrary hyperplane $\pi (0 \notin \pi)$ H_π being the halfspace with boundary π which does not contain 0.

To verify (i) note that since n is odd, Theorem 7.3 implies

$$u(0,t) = 0 \quad \text{for} \quad |t| \geq R\,. \tag{103}$$

If $z \in \mathbf{R}^n$, $F \in \mathcal{E}(\mathbf{R}^n)$ we denote by F^z the translated function $y \to F(y+z)$. Then u^z satisfies (96) with initial data f_0^z, f_1^z which have support contained in $B_{R+|z|}(0)$. Hence by (103)

$$u(z,t) = 0 \quad \text{for} \quad |t| > R + |z|\,. \tag{104}$$

The other inequality in (101) follows from Corollary 7.6.

For the final statement in (i) we note that if $|y| \leq R$ and $(x,t) \in C_y$ then $|x-y| = t$ so $|x| \leq |x-y| + |y| \leq |t| + R$ and $|t| = |x-y| \leq |x| + R$ proving (101). Conversely, if (x,t) satisfies (101) then $(x,t) \in C_y$ with $y = x - |t|\frac{x}{|x|} = \frac{x}{|x|}(|x|-t)$ which has norm $\leq R$.

For (ii) we just observe that since $\widehat{f}_i(\omega,p)$ has compact support in p, (102) is equivalent to (i) in Corollary 7.4.

Thus (102) implies that for $t>0$, $u(x,t)$ has support in the thinner shell $|t| \leq |x| \leq |t| + R$.

B. X-ray Reconstruction.

The classical interpretation of an X-ray picture is an attempt at reconstructing properties of a 3-dimensional body by means of the X-ray projection on a plane.

In modern X-ray technology the picture is given a more refined mathematical interpretation. Let $B \subset \mathbf{R}^3$ be a body (for example a part of a human body) and let $f(x)$ denote its density at a point x. Let ξ be a line in $\mathbf{R}^3$ and suppose a thin beam of X-rays is directed at B along ξ. Let I_0 and I respectively, denote the intensity of the beam before entering B and after leaving B (see Fig. I.9). As the X-ray traverses the distance Δx along ξ it will undergo the relative intensity loss $\Delta I/I = f(x)\,\Delta x$. Thus $dI/I = -f(x)\,dx$ whence

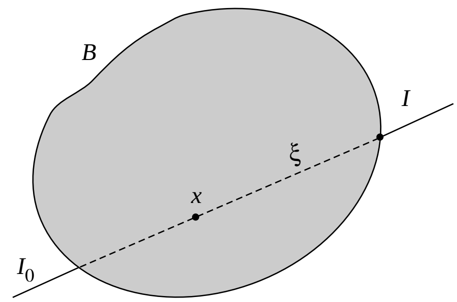

FIGURE I.9.

$$\log(I_0/I) = \int_\xi f(x)\,dx\,, \tag{105}$$

the integral $\widehat{f}(\xi)$ of f along ξ. Since the left hand side is determined by the X-ray picture, the *X-ray reconstruction problem* amounts to the determination of the function f by means of its line integrals $\widehat{f}(\xi)$. *The inversion formula in Theorem 3.1 gives an explicit solution of this problem.*

If $B_0 \subset B$ is a convex subset (for example the heart) it may be of interest to determine the density of f outside B_0 using only X-rays which do not intersect B_0. *The support theorem (Theorem 2.6, Cor. 2.8 and Cor. 6.1) implies that f is determined outside B_0 on the basis of the integrals $\widehat{f}(\xi)$ for which ξ does not intersect B_0.* **Thus the density outside the heart can be determined by means of X-rays which bypass the heart.**

In practice one can of course only determine the integrals $\widehat{f}(\xi)$ in (105) for *finitely* many directions. A compensation for this is the fact that only an approximation to the density f is required. One then encounters the mathematical problem of selecting the directions so as to optimize the approximation.

As before we represent the line ξ as the pair $\xi = (\omega, z)$ where $\omega \in \mathbf{R}^n$ is a unit vector in the direction of ξ and $z = \xi \cap \omega^\perp$ ($\perp$ denoting orthogonal complement). We then write

$$\widehat{f}(\xi) = \widehat{f}(\omega, z) = (P_\omega f)(z)\,. \tag{106}$$

The function $P_\omega f$ is the X-ray picture or the *radiograph* in the direction ω. Here f is a function on $\mathbf{R}^n$ vanishing outside a ball B around the origin and for the sake of Hilbert space methods to be used it is convenient to assume in addition that $f \in L^2(B)$. Then $f \in L^1(\mathbf{R}^n)$ so by the Fubini theorem we have: for each $\omega \in \mathbf{S}^{n-1}$, $P_\omega f(z)$ is defined for almost all $z \in \omega^\perp$. Moreover, we have in analogy with (4),

$$\widetilde{f}(\zeta) = \int_{\omega^\perp} (P_\omega f)(z) e^{-i\langle z,\zeta\rangle}\, dz \quad (\zeta \in \omega^\perp)\,. \tag{107}$$

Proposition 7.8. *An object is determined by any infinite set of radiographs.*

In other words, a compactly supported function f is determined by the functions $P_\omega f$ for any infinite set of ω.

Proof. Since f has compact support $\widetilde{f}$ is an analytic function on $\mathbf{R}^n$. But if $\widetilde{f}(\zeta) = 0$ for $\zeta \in \omega^\perp$ we have $\widetilde{f}(\eta) = \langle\omega,\eta\rangle g(\eta) \quad (\eta \in \mathbf{R}^n)$ where g is also analytic. If $P_{\omega_1} f, \ldots, P_{\omega_k} f \ldots$ all vanish identically for an infinite set $\omega_1, \ldots, \omega_k \ldots$ we see that for each k

$$\widetilde{f}(\eta) = \prod_{i=1}^{k} \langle\omega_i,\eta\rangle g_k(\eta)\,,$$

where g_k is analytic. But this would contradict the power series expansion of $\widetilde{f}$ which shows that for a suitable $\omega \in \mathbf{S}^{n-1}$ and integer $r \geq 0$, $\lim_{t\to 0} \widetilde{f}(t\omega)t^{-r} \neq 0$.

If only finitely many radiographs are used we get the opposite result.

Proposition 7.9. *Let $\omega_1, \ldots, \omega_k \in \mathbf{S}^{n-1}$ be an arbitrary finite set. Then there exists a function $f \in \mathcal{D}(\mathbf{R}^n), f \not\equiv 0$ such that*

$$P_{\omega_i} f \equiv 0 \quad \textit{for all} \quad 1 \leq i \leq k\,.$$

Proof. We have to find $f \in \mathcal{D}(\mathbf{R}^n), f \not\equiv 0$, such that $\widetilde{f}(\zeta) = 0$ for $\zeta \in \omega_i^\perp (1 \leq i \leq k)$. For this let D be the constant coefficient differential operator such that

$$(Du)\widetilde{\ }(\eta) = \prod_{1}^{k} \langle\omega_i,\eta\rangle \widetilde{u}(\eta) \quad \eta \in \mathbf{R}^n\,.$$

If $u \not\equiv 0$ is any function in $\mathcal{D}(\mathbf{R}^n)$ then $f = Du$ has the desired property.

We next consider the problem of *approximate reconstruction* of the function f from a finite set of radiographs $P_{\omega_1} f, \ldots, P_{\omega_k} f$.

Let N_j denote the null space of P_{ω_j} and let P_j the orthogonal projection of $L^2(B)$ on the plane $f + N_j$; in other words

$$P_j g = Q_j(g - f) + f\,, \tag{108}$$

where Q_j is the (linear) projection onto the subspace $N_j \subset L^2(B)$. Put $P = P_k \dots P_1$. Let $g \in L^2(B)$ be arbitrary (the initial guess for f) and form the sequence $P^m g, m = 1, 2, \dots$. Let $N_0 = \cap_1^k N_j$ and let P_0 (resp. Q_0) denote the orthogonal projection of $L^2(B)$ on the plane $f + N_0$ (subspace N_0). We shall prove that the sequence $P^m g$ converges to the projection $P_0 g$. This is natural since by $P_0 g - f \in N_0$, $P_0 g$ and f have the same radiographs in the directions $\omega_1, \dots, \omega_k$.

Theorem 7.10. *With the notations above,*

$$P^m g \longrightarrow P_0 g \quad as \quad m \longrightarrow \infty$$

for each $g \in L^2(B)$.

Proof. We have, by iteration of (108)

$$(P_k \dots P_1)g - f = (Q_k \dots Q_1)(g - f)$$

and, putting $Q = Q_k \dots Q_1$ we obtain

$$P^m g - f = Q^m (g - f)\,.$$

We shall now prove that $Q^m g \longrightarrow Q_0 g$ for each g; since

$$P_0 g = Q_0 (g - f) + f$$

this would prove the result. But the statement about Q^m comes from the following general result about abstract Hilbert space.

Theorem 7.11. *Let $\mathcal{H}$ be a Hilbert space and Q_i the projection of $\mathcal{H}$ onto a subspace $N_i \subset \mathcal{H}(1 \le i \le k)$. Let $N_0 = \cap_1^k N_i$ and $Q_0 : \mathcal{H} \longrightarrow N_0$ the projection. Then if $Q = Q_k \dots Q_1$*

$$Q^m g \longrightarrow Q_0 g \quad for\ each \quad g \in \mathcal{H},.$$

Since Q is a contraction ($\|Q\| \le 1$) we begin by proving a simple lemma about such operators.

Lemma 7.12. *Let $T : \mathcal{H} \longrightarrow \mathcal{H}$ be a linear operator of norm ≤ 1. Then*

$$\mathcal{H} = C\ell((I - T)\mathcal{H}) \oplus \ Null\ space\ (I - T)$$

is an orthogonal decomposition, $C\ell$ denoting closure, and I the identity.

Proof. If $Tg = g$ then since $\|T^*\| = \|T\| \le 1$ we have

$$\|g\|^2 = (g, g) = (Tg, g) = (g, T^* g) \le \|g\| \|T^* g\| \le \|g\|^2$$

so all terms in the inequalities are equal. Hence

$$\|g - T^* g\|^2 = \|g\|^2 - (g, T^* g) - (T^* g, g) + \|T^* g\|^2 = 0$$

so $T^* g = g$. Thus $I - T$ and $I - T^*$ have the same null space. But $(I - T^*)g = 0$ is equivalent to $(g, (I - T)\mathcal{H}) = 0$ so the lemma follows.

Definition. An operator T on a Hilbert space $\mathcal{H}$ is said to have *property* S if

$$\|f_n\| \leq 1, \|Tf_n\| \longrightarrow 1 \ \text{ implies } \|(I-T)f_n\| \longrightarrow 0\,. \tag{109}$$

Lemma 7.13. *A projection, and more generally a finite product of projections, has property* (S).

Proof. If T is a projection then

$$\|(I-T)f_n\|^2 = \|f_n\|^2 - \|Tf_n\|^2 \leq 1 - \|Tf_n\|^2 \longrightarrow 0$$

whenever

$$\|f_n\| \leq 1 \text{ and } \|Tf_n\| \longrightarrow 1\,.$$

Let T_2 be a projection and suppose T_1 has property (S) and $\|T_1\| \leq 1$. Suppose $f_n \in \mathcal{H}$ and $\|f_n\| \leq 1, \|T_2T_1f_n\| \longrightarrow 1$. The inequality implies $\|T_1f_n\| \leq 1$ and since

$$\|T_1f_n\|^2 = \|T_2T_1f_n\|^2 + \|(I-T_2)(T_1f_n)\|^2$$

we also deduce $\|T_1f_n\| \longrightarrow 1$. Writing

$$(I-T_2T_1)f_n = (I-T_1)f_n + (I-T_2)T_1f_n$$

we conclude that T_2T_1 has property (S). The lemma now follows by induction.

Lemma 7.14. *Suppose T has property (S) and $\|T\| \leq 1$. Then for each $f \in \mathcal{H}$*

$$T^n f \longrightarrow \pi f \quad \textit{as} \quad n \longrightarrow \infty\,,$$

where π is the projection onto the fixed point space of T.

Proof. Let $f \in \mathcal{H}$. Since $\|T\| \leq 1$, $\|T^nf\|$ decreases monotonically to a limit $\alpha \geq 0$. If $\alpha = 0$ we have $T^nf \longrightarrow 0$. By Lemma 7.12 $\pi T = T\pi$ so $\pi f = T^n\pi f = \pi T^n f$ so $\pi f = 0$ in this case. If $\alpha > 0$ we put $g_n = \|T^nf\|^{-1}(T^nf)$. Then $\|g_n\| = 1$ and $\|Tg_n\| \to 1$. Since T has property (S) we deduce

$$T^n(I-T)f = \|T^nf\|(I-T)g_n \longrightarrow 0,.$$

Thus $T^nh \longrightarrow 0$ for all h in the range of $I-T$. If g is in the closure of this range then given $\epsilon > 0$ there exist $h \in (I-T)\mathcal{H}$ such that $\|g-h\| < \epsilon$. Then

$$\|T^ng\| \leq \|T^n(g-h)\| + T^nh\| < \epsilon + \|T^nh\|$$

whence $T^ng \longrightarrow 0$. On the other hand, if h is in the null space of $I-T$ then $Th = h$ so $T^nh \longrightarrow h$. Now the lemma follows from Lemma 7.12.

In order to deduce Theorem 7.11 from Lemmas 7.13 and 7.14 we just have to verify that N_0 is the fixed point space of Q. But if $Qg = g$ then

$$\|g\| = \|Q_k \dots Q_1 g\| \leq \|Q_{k-1} \dots Q_1 g\| \leq \dots \leq \|Q_1 g\| \leq \|g\|$$

so equality signs hold everywhere. But the Q_i are projections so the norm identities imply

$$g = Q_1 g = Q_2 Q_1 g = \dots = Q_k \dots Q_1 g$$

which shows $g \in N_0$. This proves Theorem 7.11.

Bibliographical Notes

§§*1-2.* The inversion formulas

(i) $f(x) = \frac{1}{2}(2\pi i)^{1-n} L_x^{(n-1)/2} \int_{\mathbf{S}^{n-1}} J(\omega, \langle \omega, x \rangle)\, , d\omega$ (n odd)

(ii) $f(x) = \frac{1}{2}(2\pi i)^{-n} L_x^{(n-2)/2} \int_{\mathbf{S}^{n-1}} d\omega \int_{-\infty}^{\infty} \frac{dJ(\omega,p)}{p-(\omega,x)}$ (n even)

for a function $f \in \mathcal{D}(X)$ in terms of its plane integrals $J(\omega, p)$ go back to Radon [1917] and John [1934], [1955]. According to Bockwinkel [1906] the case $n = 3$ had been proved before 1906 by H.A. Lorentz, but fortunately, both for Lorentz and Radon, the transformation $f(x) \to J(\omega, p)$ was not baptized "Lorentz transformation". In John [1955] the proofs are based on the Poisson equation $Lu = f$. Other proofs, using distributions, are given in Gelfand-Shilov [1960]. See also Nievergelt [1986]. The dual transforms, $f \to \widehat{f}, \varphi \to \check{\varphi}$, the unified inversion formula and its dual,

$$cf = L^{(n-1)/2}((\widehat{f})^{\vee})\, , \quad c\varphi = \square^{(n-1)/2}((\check{\varphi})\widehat{)}$$

were given by the author in [1964]. The second proof of Theorem 3.1 is from the author's paper [1959]. It is valid for constant curvature spaces as well. The version in Theorem 3.6 is also proved in Ludwig [1966].

The support theorem, the Paley-Wiener theorem and the Schwartz theorem (Theorems 2.4,2.6, 2.10) are from Helgason [1964], [1965a]. The example in Remark 2.9 was also found by D.J. Newman, cf. Weiss' paper [1967], which gives another proof of the support theorem. See also Droste [1983]. The local result in Corollary 2.12 goes back to John [1935]; our derivation is suggested by the proof of a similar lemma in Flensted-Jensen [1977], p. 81. Another proof is in Ludwig [1966].See Palamodov and Denisjuk [1988] for a related inversion formula.

The simple geometric Lemma 2.7 is from the authors paper [1965a] and is extended to hyperbolic spaces in [1980b]. In the Proceedings containing

[1966a] the author raised the problem (p. 174) to extend Lemma 2.7 to each complete simply connected Riemannian manifold M of negative curvature. If in addition M is analytic this was proved by Quinto [1993b] and Grinberg and Quinto [1998]. This is an example of injectivity and support results obtained by use of the techniques of microlocal analysis and wave front sets. As further samples involving very general Radon transforms we mention Boman [1990], [1992], [1993], Boman and Quinto [1987], [1993], Quinto [1983], [1992], [1993b], [1994a], [1994b], Agranovsky and Quinto [1996], Gelfand, Gindikin and Shapiro [1979].

Corollary 2.8 is derived by Ludwig [1966] in a different way. There he proposes alternative proofs of the Schwartz- and Paley-Wiener theorems by expanding $\widehat{f}(\omega, p)$ in spherical harmonics in ω. However, the principal point—the smoothness of the function F in the proof of Theorem 2.4—is overlooked. Theorem 2.4 is from the author's papers [1964] [1965a].

Since the inversion formula is rather easy to prove for odd n it is natural to try to prove Theorem 2.4 for this case by showing directly that if $\varphi \in \mathcal{S}_H(\mathbf{P}^n)$ then the function $f = cL^{(n-1)/2}(\check{\varphi})$ for n odd belongs to $\mathcal{S}(\mathbf{R}^n)$ (in general $\check{\varphi} \notin \mathcal{S}(\mathbf{R}^n)$). This approach is taken in Gelfand-Graev-Vilenkin [1966], pp. 16-17. However, this method seems to offer some unresolved technical difficulties. For some generalizations see Kuchment and Lvin [1990], Aguilar, Ehrenpreis and Kuchment [1996] and Katsevich [1997]. Cor. 2.5 is stated in Semyanisty [1960].

§5. The approach to Radon transforms of distributions adopted in the text is from the author's paper [1966a]. Other methods are proposed in Gelfand-Graev-Vilenkin [1966] and in Ludwig [1966]. See also Ramm [1995].

§6. The d-plane transform and Theorem 6.2 are from Helgason [1959], p. 284. Formula (55) was already proved by Fuglede [1958]. The range characterization for the d-plane transform in Theorem 6.3 is from the 1980-edition of this book and was used by Kurusa [1991] to prove Theorem 6.5, which generalizes John's range theorem for the X-ray transform in $\mathbf{R}^3$ [1938]. The geometric range characterization (Corollary 6.12) is also due to John [1938]. Papers devoted to the d-plane range question for $\mathcal{S}(\mathbf{R}^n)$ are Gelfand-Gindikin and Graev [1982], Grinberg [1987], Richter [1986] and Gonzalez [1991]. This last paper gives the range as the kernel of a single 4^{th} order differential operator on the space of d-planes. As shown by Gonzalez, the analog of Theorem 6.3 fails to hold for $\mathcal{S}(\mathbf{R}^n)$. An L^2-version of Theorem 6.3 was given by Solmon [1976], p. 77. Proposition 6.13 was communicated to me by Schlichtkrull.

Some difficulties with the d-plane transform on $L^2(\mathbf{R}^n)$ are pointed out by Smith and Solmon [1975] and Solmon [1976], p. 68. In fact, the function $f(x) = |x|^{-\frac{1}{2}n}(\log|x|)^{-1} \quad (|x| \geq 2)$, 0 otherwise, is square integrable on $\mathbf{R}^n$ but is not integrable over any plane of dimension $\geq \frac{n}{2}$. Nevertheless, see for example Rubin [1998a], Strichartz [1981] for L^p-extensions of the d-plane transform.

§7. The applications to partial differential equations go in part back to Herglotz [1931]; see John [1955]. Other applications of the Radon transform to partial differential equations with constant coefficients can be found in Courant-Lax [1955], Gelfand-Shapiro [1955], John [1955], Borovikov [1959], Gårding [1961] and Ludwig [1966]. Our discussion of the wave equation (Theorem 7.3 and Corollary 7.4) is closely related to the treatment in Lax-Phillips [1967], Ch. IV, where however, the dimension is assumed to be odd. Applications to general elliptic equations were given by John [1955].

While the Radon transform on $\mathbf{R}^n$ can be used to "reduce" partial differential equations to ordinary differential equations one can use a Radon type transform on a symmetric space X to "reduce" an invariant differential operator D on X to a partial differential operator with constant coefficients. For an account of these applications see the author's monograph [1994b], Chapter V.

While the applications to differential equations are perhaps the most interesting to mathematicians, the tomographic applications of the X-ray transform have revolutionized medicine. These applications originated with Cormack [1963], [1964] and Hounsfield [1973]. For the approximate reconstruction problem, including Propositions 7.8 and 7.9 and refinements of Theorems 7.10, 7.11 see Smith, Solmon and Wagner [1977], Solmon [1976] and Hamaker and Solmon [1978]. Theorem 7.11 is due to Halperin [1962], the proof in the text to Amemiya and Ando [1965]. For an account of some of those applications see e.g. Deans [1983], Natterer [1986] and Ramm and Katsevich [1996]. Applications in radio astronomy appear in Bracewell and Riddle [1967].

CHAPTER II

A DUALITY IN INTEGRAL GEOMETRY. GENERALIZED RADON TRANSFORMS AND ORBITAL INTEGRALS

§1 Homogeneous Spaces in Duality

The inversion formulas in Theorems 3.1, 3.5, 3.6 and 6.2, Ch. I suggest the general problem of determining a function on a manifold by means of its integrals over certain submanifolds. In order to provide a natural framework for such problems we consider the Radon transform $f \to \widehat{f}$ on $\mathbf{R}^n$ and its dual $\varphi \to \check{\varphi}$ from a group-theoretic point of view, motivated by the fact that the isometry group $\mathbf{M}(n)$ acts transitively both on $\mathbf{R}^n$ and on the hyperplane space $\mathbf{P}^n$. Thus

$$\text{(1)} \qquad \mathbf{R}^n = \mathbf{M}(n)/\mathbf{O}(n)\,, \quad \mathbf{P}^n = \mathbf{M}(n)/\mathbb{Z}_2 \times \mathbf{M}(n-1)\,,$$

where $\mathbf{O}(n)$ is the orthogonal group fixing the origin $0 \in \mathbf{R}^n$ and $\mathbb{Z}_2 \times \mathbf{M}(n-1)$ is the subgroup of $\mathbf{M}(n)$ leaving a certain hyperplane ξ_0 through 0 stable. ($\mathbb{Z}_2$ consists of the identity and the reflection in this hyperplane.)

We observe now that a point $g_1\mathbf{O}(n)$ in the first coset space above lies on a plane $g_2(\mathbb{Z}_2 \times \mathbf{M}(n-1))$ in the second if and only if these cosets, considered as subsets of $\mathbf{M}(n)$, have a point in common. In fact

$$\begin{aligned} g_1 \cdot 0 \subset g_2 \cdot \xi_0 \quad &\Leftrightarrow \quad g_1 \cdot 0 = g_2 h \cdot 0 \text{ for some } h \in \mathbb{Z}_2 \times \mathbf{M}(n-1) \\ &\Leftrightarrow \quad g_1 k = g_2 h \text{ for some } k \in \mathbf{O}(n)\,. \end{aligned}$$

This leads to the following general setup.

Let G be a locally compact group, X and Ξ two left coset spaces of G,

$$\text{(2)} \qquad X = G/K\,, \quad \Xi = G/H\,,$$

where K and H are closed subgroups of G. Let $L = K \cap H$. We assume that the subset $KH \subset G$ is *closed*. This is automatic if one of the groups K or H is compact.

Two elements $x \in X$, $\xi \in \Xi$ are said to be *incident* if as cosets in G they intersect. We put (see Fig. II.1)

$$\begin{aligned} \check{x} \quad &= \quad \{\xi \in \Xi : x \text{ and } \xi \text{ incident}\} \\ \widehat{\xi} \quad &= \quad \{x \in X : x \text{ and } \xi \text{ incident}\}\,. \end{aligned}$$

Let $x_0 = \{K\}$ and $\xi_0 = \{H\}$ denote the origins in X and Ξ, respectively. If $\Pi : G \to G/H$ denotes the natural mapping then since $\check{x}_0 = K \cdot \xi_0$ we

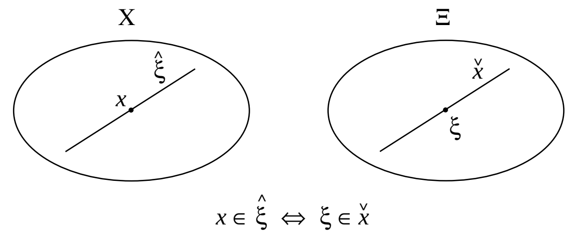

FIGURE II.1.

have

$$\Pi^{-1}(\Xi - \check{x}_0) = \{g \in G : gH \notin KH\} = G - KH\,.$$

In particular $\Pi(G - KH) = \Xi - \check{x}_0$ so since Π is an open mapping, $\check{x}_0$ is a closed subset of Ξ. This proves

Lemma 1.1. *Each $\check{x}$ and each $\widehat{\xi}$ is closed.*

Using the notation $A^g = gAg^{-1}$ $(g \in G, A \subset G)$ we have the following lemma.

Lemma 1.2. *Let $g, \gamma \in G$, $x \in gK$, $\xi = \gamma H$. Then*

$$\check{x} \textit{ is an orbit of } K^g,\ \widehat{\xi} \textit{ is an orbit of } H^\gamma\,,$$

and

$$\check{x} = K^g/L^g\,, \quad \widehat{\xi} = H^\gamma/L^\gamma\,.$$

Proof. By definition

$$\check{x} = \{\delta H \,:\, \delta H \cap gK \neq \emptyset\} = \{gkH \,:\, k \in K\} \tag{3}$$

which is the orbit of the point gH under gKg^{-1}. The subgroup fixing gH is $gKg^{-1} \cap gHg^{-1} = L^g$. Also (3) implies

$$\check{x} = g \cdot \check{x}_0 \qquad \widehat{\xi} = \gamma \cdot \widehat{\xi}_0\,,$$

where the dot $\cdot$ denotes the action of G on X and Ξ.

Lemma 1.3. *Consider the subgroups*

$$\begin{aligned} K_H &= \{k \in K \,:\, kH \cup k^{-1}H \subset HK\} \\ H_K &= \{h \in H \,:\, hK \cup h^{-1}K \subset KH\}\,. \end{aligned}$$

The following properties are equivalent:

(a) $K \cap H = K_H = H_K$.

(b) *The maps $x \to \check{x}\,(x \in X)$ and $\xi \to \widehat{\xi}\,(\xi \in \Xi)$ are injective.*

We think of property (a) as a kind of *transversality* of K and H.

Proof. Suppose $x_1 = g_1K$, $x_2 = g_2K$ and $\check{x}_1 = \check{x}_2$. Then by (3) $g_1 \cdot \check{x}_0 = g_1 \cdot \check{x}_0$ so $g \cdot \check{x}_0 = \check{x}_0$ if $g = g_1^{-1}g_2$. In particular $g \cdot \xi_0 \subset \check{x}_0$ so $g \cdot \xi_0 = k \cdot \xi_0$ for some $k \in K$. Hence $k^{-1}g = h \in H$ so $h \cdot \check{x}_0 = \check{x}_0$, that is $hK \cdot \xi_0 = K \cdot \xi_0$. As a relation in G, this means $hKH = KH$. In particular $hK \subset KH$. Since $h \cdot \check{x}_0 = \check{x}_0$ implies $h^{-1} \cdot \check{x}_0 = \check{x}_0$ we have also $h^{-1}K \subset KH$ so by (b) $h \in K$ which gives $x_1 = x_2$.

On the other hand, suppose the map $x \to \check{x}$ is injective and suppose $h \in H$ satisfies $h^{-1}K \cup hK \subset KH$. Then

$$hK \cdot \xi_0 \subset K \cdot \xi_0 \text{ and } h^{-1}K \cdot \xi_0 \subset K \cdot \xi_0\,.$$

By Lemma 1.2, $h \cdot \check{x}_0 \subset \check{x}_0$ and $h^{-1} \cdot \check{x}_0 \subset \check{x}_0$. Thus $h \cdot \check{x}_0 = \check{x}_0$ whence by the assumption, $h \cdot x_0 = x_0$ so $h \in K$.

Thus we see that under the transversality assumption a) the elements ξ can be viewed as the subsets $\widehat{\xi}$ of X and the elements x as the subsets $\check{x}$ of Ξ. We say X and Ξ are *homogeneous spaces in duality*.

The maps are also conveniently described by means of the following *double fibration*

(4)
$$\begin{tikzcd} & G/L \arrow[dl, "p"'] \arrow[dr, "\pi"] & \\ G/K & & G/H \end{tikzcd}$$

where $p(gL) = gK$, $\pi(\gamma L) = \gamma H$. In fact, by (3) we have

$$\check{x} = \pi(p^{-1}(x)) \qquad \widehat{\xi} = p(\pi^{-1}(\xi))\,.$$

We now prove some group-theoretic properties of the incidence, supplementing Lemma 1.3.

Theorem 1.4. *(i) We have the identification*

$$G/L = \{(x, \xi) \in X \times \Xi \,:\, x \textit{ and } \xi \textit{ incident}\}$$

via the bijection $\tau \,:\, gL \to (gK, gH)$.

(ii) The property

$$KHK = G$$

is equivariant to the property:

Any two $x_1, x_2 \in X$ are incident to some $\xi \in \Xi$. A similar statement holds for $HKH = G$.

(iii) The property

$$HK \cap KH = K \cup H$$

is equivalent to the property:

For any two $x_1 \neq x_2$ in X there is at most one $\xi \in \Xi$ incident to both. By symmetry, this is equivalent to the property:

For any $\xi_1 \neq \xi_2$ in Ξ there is at most one $x \in X$ incident to both.

Proof. (i) The map is well-defined and injective. The surjectivity is clear because if $gK \cap \gamma H \neq \emptyset$ then $gk = \gamma h$ and $\tau(gkL) = (gK, \gamma H)$.

(ii) We can take $x_2 = x_0$. Writing $x_1 = gK$, $\xi = \gamma H$ we have

$$\begin{aligned} x_0, \xi \text{ incident } &\Leftrightarrow \gamma h &=& \quad k \quad (\text{some } h \in H, k \in K) \\ x_1, \xi \text{ incident } &\Leftrightarrow \gamma h_1 &=& \quad g_1 k_1 \quad (\text{some } h_1 \in H, k_1 \in K) \end{aligned}$$

Thus if x_0, x_1 are incident to ξ we have $g_1 = kh^{-1}h_1k_1^{-1}$. Conversely if $g_1 = k'h'k''$ we put $\gamma = k'h'$ and then x_0, x_1 are incident to $\xi = \gamma H$.

(iii) Suppose first $KH \cap HK = K \cup H$. Let $x_1 \neq x_2$ in X. Suppose $\xi_1 \neq \xi_2$ in Ξ are both incident to x_1 and x_2. Let $x_i = g_iK$, $\xi_j = \gamma_j H$. Since x_i is incident to ξ_j there exist $k_{ij} \in K$, $h_{ij} \in H$ such that

$$g_i k_{ij} = \gamma_j h_{ij} \qquad i = 1, 2\,; \; j = 1, 2\,. \tag{5}$$

Eliminating g_i and γ_j we obtain

$$k_{2\,2}^{-1} k_{2\,1} h_{2\,1}^{-1} h_{1\,1} = h_{2\,2}^{-1} h_{1\,2} k_{1\,2}^{-1} k_{1\,1}\,. \tag{6}$$

This being in $KH \cap HK$ it lies in $K \cup H$. If the left hand side is in K then $h_{2\,1}^{-1}h_{1\,1} \in K$ so

$$g_2K = \gamma_1 h_{2\,1}K = \gamma_1 h_{1\,1}K = g_1K\,,$$

contradicting $x_2 \neq x_1$. Similarly if expression (6) is in H then $k_{1\,2}^{-1}k_{1\,1} \in H$ so by (5) we get the contradiction

$$\gamma_2 H = g_1 k_{1\,2} H = g_1 k_{1\,1} H = \gamma_1 H\,.$$

Conversely, suppose $KH \cap HK \neq K \cup H$. Then there exist h_1, h_2, k_1, k_2 such that $h_1k_1 = k_2h_2$ and $h_1k_1 \notin K \cup H$. Put $x_1 = h_1K, \xi_2 = k_2H$. Then $x_1 \neq x_0, \xi_0 \neq \xi_2$, yet both ξ_0 and ξ_2 are incident to both x_0 and x_1.

Examples

(i) *Points outside hyperplanes.* We saw before that if in the coset space representation (1) $\mathbf{O}(n)$ is viewed as the isotropy group of 0 and $\mathbb{Z}_2\mathbf{M}(n-1)$ is viewed as the isotropy group of a hyperplane *through* 0 then the abstract incidence notion is equivalent to the naive one: $x \in \mathbf{R}^n$ is incident to $\xi \in \mathbf{P}^n$ if and only if $x \in \xi$.

On the other hand we can also view $\mathbb{Z}_2\mathbf{M}(n-1)$ as the isotropy group of a hyperplane ξ_δ at a distance $\delta > 0$ from 0. (This amounts to a different embedding of the group $\mathbb{Z}_2\mathbf{M}(n-1)$ into $\mathbf{M}(n)$.) Then we have the following generalization.

Proposition 1.5. *The point $x \in \mathbf{R}^n$ and the hyperplane $\xi \in \mathbf{P}^n$ are incident if and only if distance $(x, \xi) = \delta$.*

Proof. Let $x = gK$, $\xi = \gamma H$ where $K = \mathbf{O}(n), H = \mathbb{Z}_2\mathbf{M}(n-1)$. Then if $gK \cap \gamma H \neq \emptyset$, we have $gk = \gamma h$ for some $k \in K, h \in H$. Now the orbit $H \cdot 0$ consists of the two planes ξ'_δ and ξ''_δ parallel to ξ_δ at a distance δ from ξ_δ. The relation

$$g \cdot 0 = \gamma h \cdot 0 \in \gamma \cdot (\xi'_\delta \cup \xi''_\delta)$$

together with the fact that g and γ are isometries shows that x has distance δ from $\gamma \cdot \xi_\delta = \xi$.

On the other hand if distance $(x, \xi) = \delta$ we have $g \cdot 0 \in \gamma \cdot (\xi'_\delta \cup \xi''_\delta) = \gamma H \cdot 0$ which means $gK \cap \gamma H \neq \emptyset$.

(ii) *Unit spheres.* Let σ_0 be a sphere in $\mathbf{R}^n$ of radius one passing through the origin. Denoting by Σ the set of all *unit* spheres in $\mathbf{R}^n$ we have the dual homogeneous spaces

$$\mathbf{R}^n = \mathbf{M}(n)/\mathbf{O}(n)\,; \quad \Sigma = \mathbf{M}(n)/\mathbf{O}^*(n) \tag{7}$$

where $\mathbf{O}^*(n)$ is the set of rotations around the center of σ_0. Here a point $x = g\mathbf{O}(n)$ is incident to $\sigma_0 = \gamma\mathbf{O}^*(n)$ if and only if $x \in \sigma$.

§2 The Radon Transform for the Double Fibration

With K, H and L as in §1 we assume now that K/L and H/L have positive measures $d\mu_0 = dk_L$ and $dm_0 = dh_L$ invariant under K and H, respectively. This is for example guaranteed if L is compact.

Lemma 2.1. *Assume the transversality condition (a). Then there exists a measure on each $\check{x}$ coinciding with $d\mu_0$ on $K/L = \check{x}_0$ such that whenever $g \cdot \check{x}_1 = \check{x}_2$ the measures on $\check{x}_1$ and $\check{x}_2$ correspond under g. A similar statement holds for dm on $\widehat{\xi}$.*

Proof. If $\check{x} = g \cdot \check{x}_0$ we transfer the measure $d\mu_0 = dk_L$ over on $\check{x}$ by the map $\xi \to g \cdot \xi$. If $g \cdot \check{x}_0 = g_1 \cdot \check{x}_0$ then $(g \cdot x_0)^\vee = (g_1 \cdot x_0)^\vee$ so by Lemma 1.3, $g \cdot x_0 = g_1 \cdot x_0$ so $g = g_1 k$ with $k \in K$. Since $d\mu_0$ is K-invariant the lemma follows.

The measures defined on each $\check{x}$ and $\widehat{\xi}$ under condition (a) are denoted by $d\mu$ and dm, respectively.

Definition. The Radon transform $f \to \widehat{f}$ and its dual $\varphi \to \check{\varphi}$ are defined by

$$\widehat{f}(\xi) = \int_{\widehat{\xi}} f(x)\, dm(x)\,, \qquad \check{\varphi}(x) = \int_{\check{x}} \varphi(\xi)\, d\mu(\xi)\,. \tag{8}$$

whenever the integrals converge. Because of Lemma 1.1, this will always happen for $f \in C_c(X)$, $\varphi \in C_c(\Xi)$.

In the setup of Proposition 1.5, $\widehat{f}(\xi)$ is the integral of f over the two hyperplanes at distance δ from ξ and $\check{\varphi}(x)$ is the average of φ over the set of hyperplanes at distance δ from x. For $\delta = 0$ we recover the transforms of Ch. I, §1.

Formula (8) can also be written in the group-theoretic terms,

$$\widehat{f}(\gamma H) = \int_{H/L} f(\gamma h K)\, dh_L\,, \quad \check{\varphi}(gK) = \int_{K/L} \varphi(gkH)\, dk_L\,. \tag{9}$$

Note that (9) serves as a definition even if condition (a) in Lemma 1.3 is not satisfied. In this abstract setup the spaces X and Ξ have equal status. The theory in Ch. I, in particular Lemma 2.1, Theorems 2.4, 2.10, 3.1 thus raises the following problems:

Principal Problems:

A. Relate function spaces on X and on Ξ by means of the transforms $f \to \widehat{f}$, $\varphi \to \check{\varphi}$. In particular, determine their ranges and kernels.

B. Invert the transforms $f \to \widehat{f}$, $\varphi \to \check{\varphi}$ on suitable function spaces.

C. In the case when G is a Lie group so X and Ξ are manifolds let $\mathbf{D}(X)$ and $\mathbf{D}(\Xi)$ denote the algebras of G-invariant differential operators on X and Ξ, respectively. Is there a map $D \to \widehat{D}$ of $\mathbf{D}(X)$ into $\mathbf{D}(\Xi)$ and a map $E \to \check{E}$ of $\mathbf{D}(\Xi)$ into $\mathbf{D}(X)$ such that

$$(Df)\widehat{\ } = \widehat{D}\widehat{f}\,, \quad (E\varphi)^\vee = \check{E}\check{\varphi}\,?$$

Although weaker assumptions would be sufficient, we assume now that the groups G, K, H and L all have bi-invariant Haar measures dg, dk, dh and $d\ell$. These will then generate invariant measures dg_K, dg_H, dg_L, dk_L, dh_L on G/K, G/H, G/L, K/L, H/L, respectively. This means that

$$\int_G F(g)\, dg = \int_{G/K} \left(\int_K F(gk)\, dk \right) dg_K \tag{10}$$

and similarly dg and dh determine dg_H, etc. Then

$$\int_{G/L} Q(gL)\, dg_L = c \int_{G/K} dg_K \int_{K/L} Q(gkL)\, dk_L \tag{11}$$

for $Q \in C_c(G/L)$ where c is a constant. In fact, the integrals on both sides of (11) constitute invariant measures on G/L and thus must be proportional. However,

$$\int_G F(g)\, dg = \int_{G/L} \left(\int_L F(g\ell)\, d\ell \right) dg_L \tag{12}$$

and

$$\int_K F(k)\, dk = \int_{K/L} \left(\int_L F(k\ell)\, d\ell \right) dk_L\, . \tag{13}$$

Using (13) on (10) and combining with (11) we see that the constant c equals 1.

We shall now prove that $f \to \widehat{f}$ and $\varphi \to \check{\varphi}$ are adjoint operators. We write dx for dg_K and $d\xi$ for dg_H.

Proposition 2.2. *Let $f \in C_c(X)$, $\varphi \in C_c(\Xi)$. Then $\widehat{f}$ and $\check{\varphi}$ are continuous and*

$$\int_X f(x)\check{\varphi}(x)\, dx = \int_\Xi \widehat{f}(\xi)\varphi(\xi)\, d\xi\, .$$

Proof. The continuity statement is immediate from (9). We consider the function

$$P = (f \circ p)(\varphi \circ \pi)$$

on G/L. We integrate it over G/L in two ways using the double fibration (4). This amounts to using (11) and its analog with G/K replaced by G/H with $Q = P$. Since $P(gk\,L) = f(gK)\varphi(gkH)$ the right hand side of (11) becomes

$$\int_{G/K} f(gK)\check{\varphi}(gK)\, dg_K\, .$$

If we treat G/H similarly, the lemma follows.

The result shows how to define the Radon transform and its dual for measures and, in case G is a Lie group, for distributions.

Definition. Let s be a measure on X of compact support. Its Radon transform is the functional $\widehat{s}$ on $C_c(\Xi)$ defined by

$$\widehat{s}(\varphi) = s(\check{\varphi})\, . \tag{14}$$

Similarly $\check{\sigma}$ is defined by

$$\check{\sigma}(f) = \sigma(\widehat{f})\, , \qquad f \in C_c(X) \tag{15}$$

if σ is a compactly supported measure on Ξ.

Lemma 2.3. *(i) If s is a compactly supported measure on X, $\widehat{s}$ is a measure on Ξ.*

(ii) If s is a bounded measure on X and if $\check{x}_0$ has finite measure then $\widehat{s}$ as defined by (14) is a bounded measure.

Proof. (i) The measure s can be written as a difference $s = s^+ - s^-$ of two positive measures, each of compact support. Then $\widehat{s} = \widehat{s^+} - \widehat{s^-}$ is a difference of two positive *functionals* on $C_c(\Xi)$.

Since a positive functional is necessarily a measure, $\widehat{s}$ is a measure.

(ii) We have

$$\sup_x |\check{\varphi}(x)| \leq \sup_\xi |\varphi(\xi)| \, \mu_0(\check{x}_0)$$

so for a constant K,

$$|\widehat{s}(\varphi)| = |s(\check{\varphi})| \leq K \sup |\check{\varphi}| \leq K \mu_0(\check{x}_0) \sup |\varphi| \, ,$$

and the boundedness of $\widehat{s}$ follows.

If G is a Lie group then (14), (15) with $f \in \mathcal{D}(X)\,, \quad \varphi \in \mathcal{D}(\Xi)$ serve to define the Radon transform $s \to \widehat{s}$ and the dual $\sigma \to \check{\sigma}$ for distributions s and σ of compact support. We consider the spaces $\mathcal{D}(X)$ and $\mathcal{E}(X)$ $(= \mathcal{C}^\infty(X))$ with their customary topologies (Chapter V,§1). The duals $\mathcal{D}'(X)$ and $\mathcal{E}'(X)$ then consist of the distributions on X and the distributions on X of compact support, respectively.

Proposition 2.4. *The mappings*

$$\begin{aligned} f \in \mathcal{D}(X) \quad &\to \quad \widehat{f} \in \mathcal{E}(\Xi) \\ \varphi \in \mathcal{D}(\Xi) \quad &\to \quad \check{\varphi} \in \mathcal{E}(X) \end{aligned}$$

are continuous. In particular,

$$\begin{aligned} s \in \mathcal{E}'(X) \quad &\Rightarrow \quad \widehat{s} \in \mathcal{D}'(\Xi) \\ \sigma \in \mathcal{E}'(\Xi) \quad &\Rightarrow \quad \check{\sigma} \in \mathcal{D}'(X) \, . \end{aligned}$$

Proof. We have

$$\widehat{f}(g \cdot \xi_0) = \int_{\widehat{\xi}_0} f(g \cdot x) \, dm_0(x) \, . \tag{16}$$

Let g run through a local cross section through e in G over a neighborhood of ξ_0 in Ξ. If $(t_1, \ldots, t_n)$ are coordinates of g and $(x_1, \ldots, x_m)$ the coordinates of $x \in \widehat{\xi}_0$ then (16) can be written in the form

$$\widehat{F}(t_1, \ldots, t_n) = \int F(t_1, \ldots, t_n \, ; \, x_1, \ldots, x_m) \, dx_1 \ldots dx_m \, .$$

Now it is clear that $\widehat{f} \in \mathcal{E}(\Xi)$ and that $f \to \widehat{f}$ is continuous, proving the proposition.

The result has the following refinement.

Proposition 2.5. *Assume K compact. Then*

(i) $f \to \widehat{f}$ *is a continuous mapping of* $\mathcal{D}(X)$ *into* $\mathcal{D}(\Xi)$.

(ii) $\varphi \to \check{\varphi}$ *is a continuous mapping of* $\mathcal{E}(\Xi)$ *into* $\mathcal{E}(X)$.

A self-contained proof is given in the author's book [1994b], Ch. I, § 3. The result has the following consequence.

Corollary 2.6. *Assume K compact. Then* $\mathcal{E}'(X)\widehat{\ } \subset \mathcal{E}'(\Xi)$, $\mathcal{D}'(\Xi)^{\vee} \subset \mathcal{D}'(X)$.

In Chapter I we have given solutions to Problems A, B, C in some cases. Further examples will be given in § 4 of this chapter and Chapter III will include their solution for the antipodal manifolds for compact two-point homogeneous spaces.

The variety of the results for these examples make it doubtful that the individual results could be captured by a general theory. Our abstract setup in terms of homogeneous spaces in duality is therefore to be regarded as a framework for examples rather than as axioms for a general theory.

Nevertheless, certain general features emerge from the study of these examples. If $\dim X = \dim \Xi$ and $f \to \widehat{f}$ is injective the range consists of functions which are either arbitrary or at least subjected to rather weak conditions. As the difference $\dim \Xi - \dim X$ increases more conditions are imposed on the functions in the range. (See the example of the d-plane transform in $\mathbf{R}^n$.) In the case when G is a Lie group there is a group-theoretic explanation for this. Let $\mathbf{D}(G)$ denote the algebra of left-invariant differential operators on G. Since $\mathbf{D}(G)$ is generated by the left invariant vector fields on G, the action of G on X and on Ξ induces homomorphisms

$$\lambda : \mathbf{D}(G) \longrightarrow E(X)\,, \tag{17}$$

$$\Lambda : \mathbf{D}(G) \longrightarrow E(\Xi)\,, \tag{18}$$

where for a manifold M, $E(M)$ denotes the algebra of all differential operators on M. Since $f \to \widehat{f}$ and $\varphi \to \check{\varphi}$ commute with the action of G we have for $D \in \mathbf{D}(G)$,

$$(\lambda(D)f)\widehat{\ } = \Lambda(D)\widehat{f}\,, \quad (\Lambda(D)\varphi)^{\vee} = \lambda(D)\check{\varphi}\,. \tag{19}$$

Therefore $\Lambda(D)$ annihilates the range of $f \to \widehat{f}$ if $\lambda(D) = 0$. In some cases, including the case of the d-plane transform in $\mathbf{R}^n$, the range is characterized as the null space of these operators $\Lambda(D)$ (with $\lambda(D) = 0$).

§3 Orbital Integrals

As before let $X = G/K$ be a homogeneous space with origin $o = (K)$. Given $x_0 \in X$ let G_{x_0} denote the subgroup of G leaving x_0 fixed, i.e., the isotropy subgroup of G at x_0.

Definition. A *generalized sphere* is an orbit $G_{x_0} \cdot x$ in X of some point $x \in X$ under the isotropy subgroup at some point $x_0 \in X$.

Examples. (i) If $X = \mathbf{R}^n$, $G = \mathbf{M}(n)$ then the generalized spheres are just the spheres.

(ii) Let X be a locally compact subgroup L and G the product group $L \times L$ acting on L on the right and left, the element $(\ell_1, \ell_2) \in L \times L$ inducing action $\ell \to \ell_1 \ell \, \ell_2^{-1}$ on L. Let ΔL denote the diagonal in $L \times L$. If $\ell_0 \in L$ then the isotropy subgroup of ℓ_0 is given by

$$(L \times L)_{\ell_0} = (\ell_0, e)\Delta L(\ell_0^{-1}, e) \tag{20}$$

and the orbit of ℓ under it by

$$(L \times L)_{\ell_0} \cdot \ell = \ell_0(\ell_0^{-1}\ell)^L .$$

that is the left translate by ℓ_0 of the conjugacy class of the element $\ell_0^{-1}\ell$. Thus the *generalized spheres in the group L are the left (or right) translates of its conjugacy classes*.

Coming back to the general case $X = G/K = G/G_0$ we assume that G_0, and therefore each G_{x_0}, is unimodular. But $G_{x_0} \cdot x = G_{x_0}/(G_{x_0})_x$ so $(G_{x_0})_x$ unimodular implies the orbit $G_{x_0} \cdot x$ has an invariant measure determined up to a constant factor. We can now consider the following general problem (following Problems A, B, C above).

D. *Determine a function f on X in terms of its integrals over generalized spheres.*

Remark 3.1. In this problem it is of course significant how the invariant measures on the various orbits are normalized.

(a) If G_0 is compact the problem above is rather trivial because each orbit $G_{x_0} \cdot x$ has finite invariant measure so $f(x_0)$ is given as the limit as $x \to x_0$ of the average of f over $G_{x_0} \cdot x$.

(b) Suppose that for each $x_0 \in X$ there is a G_{x_0}-invariant open set $C_{x_0} \subset X$ containing x_0 in its closure such that for each $x \in C_{x_0}$ the isotropy group $(G_{x_0})_x$ is compact. The invariant measure on the orbit $G_{x_0} \cdot x \, (x_0 \in X, x \in C_{x_0})$ can then be consistently normalized as follows: Fix a Haar measure dg_0 on G_0. If $x_0 = g \cdot o$ we have $G_{x_0} = gG_0g^{-1}$ and can carry dg_0 over to a measure dg_{x_0} on G_{x_0} by means of the conjugation $z \to gzg^{-1} \quad (z \in G_0)$.

Since dg_0 is bi-invariant, dg_{x_0} is independent of the choice of g satisfying $x_0 = g \cdot o$, and is bi-invariant. Since $(G_{x_0})_x$ is compact it has a unique Haar measure $dg_{x_0,x}$ with total measure 1 and now dg_{x_0} and $dg_{x_0,x}$ determine canonically an invariant measure μ on the orbit $G_{x_0} \cdot x = G_{x_0}/(G_{x_0})_x$. We can therefore state Problem D in a more specific form.

D′. *Express $f(x_0)$ in terms of integrals*

$$\int_{G_{x_0} \cdot x} f(p)\, d\mu(p) \qquad x \in C_{x_0}\,. \tag{21}$$

For the case when X is an *isotropic Lorentz manifold* the assumptions above are satisfied (with C_{x_0} consisting of the "timelike" rays from x_0) and we shall obtain in Ch. IV an explicit solution to Problem D' (Theorem 4.1, Ch. IV).

(c) If in Example (ii) above L is a semisimple Lie group Problem D is a basic step (Gelfand-Graev [1955], Harish-Chandra [1957]) in proving the Plancherel formula for the Fourier transform on L.

§4 Examples of Radon Transforms for Homogeneous Spaces in Duality

In this section we discuss some examples of the abstract formalism and problems set forth in the preceding sections §1–§2.

A. The Funk Transform.

This case goes back to Funk [1916] (preceding Radon's paper [1917]) where he proved that a symmetric function on $\mathbf{S}^2$ is determined by its great circle integrals. This is carried out in more detail and in greater generality in Chapter III, §1. Here we state the solution of Problem B for $X = \mathbf{S}^2$, Ξ the set of all great circles, both as homogeneous spaces of $\mathbf{O}(3)$. Given $p \geq 0$ let $\xi_p \in \Xi$ have distance p from the North Pole o, $H_p \subset \mathbf{O}(3)$ the subgroup leaving ξ_p invariant and $K \subset \mathbf{O}(3)$ the subgroup fixing o. Then in the double fibration

$$\begin{tikzcd}[column sep=small] & \mathbf{O}(3)/(K \cap H_p) \arrow[dl] \arrow[dr] & \\ X = \mathbf{O}(3)/K & & \Xi = \mathbf{O}(3)/H_p \end{tikzcd}$$

$x \in X$ and $\xi \in \Xi$ are incident if and only if $d(x, \xi) = p$. The proof is the same as that of Proposition 1.5. In order to invert the Funk transform $f \to \widehat{f} \quad (= \widehat{f}_0)$ we invoke the transform $\varphi \to \check{\varphi}_p$. Note that $(\widehat{f})^\vee_p(x)$ is the

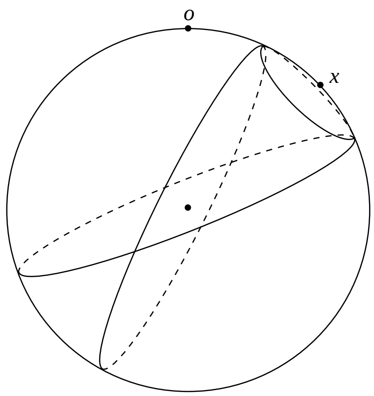

FIGURE II.2.

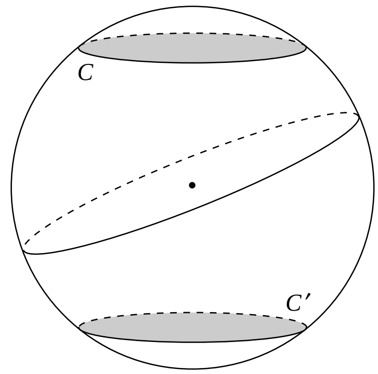

FIGURE II.3.

average of the integrals of f over the great circles ξ at distance p from x (see Figure II.2). As a special case of Theorem 1.11, Chapter III, we have the following inversion.

Theorem 4.1. *The Funk transform $f \to \widehat{f}$ is (for f even) inverted by*

$$f(x) = \frac{1}{2\pi}\left\{\frac{d}{du}\int_0^u (\widehat{f})^\vee_{\cos^{-1}(v)}(x)v(u^2-v^2)^{-\frac{1}{2}}\,dv\right\}_{u=1}. \tag{22}$$

Another inversion formula is

$$f = -\frac{1}{4\pi}LS((\widehat{f})^\vee) \tag{23}$$

(Theorem 1.15, Chapter III), where L is the Laplacian and S the integral operator given by (66)–(68), Chapter III. While (23) is short the operator S is only given in terms of a spherical harmonics expansion. Also Theorem 1.17, Ch. III shows that if f is even and if all its derivatives vanish on the equator then f vanishes outside the "arctic zones" C and C' if and only if $\widehat{f}(\xi) = 0$ for all great circles ξ disjoint from C and C' (Fig. II.3).

The Hyperbolic Plane $\mathbf{H}^2$.

This remarkable object enters into several fields in mathematics. In particular, it offers at least two interesting cases of Radon transforms. We take $\mathbf{H}^2$ as the disk $D : |z| < 1$ with the Riemannian structure

$$\langle u, v\rangle_z = \frac{(u,v)}{(1-|z|^2)^2}\,, \quad ds^2 = \frac{|dz|^2}{(1-|z|^2)^2} \tag{24}$$

if u and v are any tangent vectors at $z \in D$. Here (u, v) denotes the usual inner product on $\mathbf{R}^2$. The Laplace-Beltrami operator for (24) is given by

$$L = (1-x^2-y^2)^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right).$$

The group $G = \mathbf{SU}(1,1)$ of matrices

$$\left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \;:\; |a|^2 - |b|^2 = 1 \right\}$$

acts transitively on the unit disk by

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \cdot z = \frac{az+b}{\bar{b}z+\bar{a}} \tag{25}$$

and leaves the metric (24) invariant. The length of a curve $\gamma(t)$ $(\alpha \leq t \leq \beta)$ is defined by

$$L(\gamma) = \int_\alpha^\beta (\langle \gamma'(t), \gamma'(t) \rangle_{\gamma(t)})^{1/2}\, dt\,. \tag{26}$$

If $\gamma(\alpha) = o$, $\gamma(\beta) = x \in \mathbf{R}$ and $\gamma_o(t) = tx \quad (0 \leq t \leq 1)$ then (26) shows easily that $L(\gamma) \geq L(\gamma_o)$ so γ_o is a geodesic and the distance d satisfies

$$d(o,x) = \int_0^1 \frac{|x|}{1-t^2x^2}\, dt = \tfrac{1}{2} \log \frac{1+|x|}{1-|x|}\,. \tag{27}$$

Since G acts conformally on D the *geodesics* in $\mathbf{H}^2$ are the circular arcs in $|z| < 1$ perpendicular to the boundary $|z| = 1$.

We consider now the following subgroups of G:

$$\begin{aligned}
K &= \{k_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : 0 \leq \theta < 2\pi\} \\
M &= \{k_0, k_\pi\}\,, \qquad M' = \{k_0, k_\pi\,, k_{-\frac{\pi}{2}}, k_{\frac{\pi}{2}}\} \\
A &= \{a_t = \begin{pmatrix} \operatorname{ch} t & \operatorname{sh} t \\ \operatorname{sh} t & \operatorname{ch} t \end{pmatrix} \;:\; t \in \mathbf{R}\}, \\
N &= \{n_x = \begin{pmatrix} 1+ix, & -ix \\ ix, & 1-ix \end{pmatrix} \;:\; x \in \mathbf{R}\} \\
\Gamma &= C\mathbf{SL}(2,\mathbb{Z})C^{-1}\,,
\end{aligned}$$

where C is the transformation $w \to (w-i)/(w+i)$ mapping the upper half-plane onto the unit disk.

The orbit $A \cdot o$ is the horizontal diameter and the orbits $N \cdot (a_t \cdot o)$ are the circles tangential to $|z| = 1$ at $z = 1$. Thus $NA \cdot o$ is the entire disk D so we see that $G = NAK$ and also $G = KAN$.

B. The X-ray Transform in $\mathbf{H}^2$.

The (unoriented) geodesics for the metric (24) were mentioned above. Clearly the group G permutes these geodesics transitively (Fig. II.4). Let

Ξ be the set of all these geodesics. Let o denote the origin in $\mathbf{H}^2$ and ξ_o the horizontal geodesic through o. Then

(28)
$$X = G/K\,, \quad \Xi = G/M'A\,.$$

We can also fix a geodesic ξ_p at distance p from o and write

(29)
$$X = G/K\,, \quad \Xi = G/H_p\,,$$

where H_p is the subgroup of G leaving ξ_p stable. Then for the homogeneous spaces (29), x and ξ are incident if and only if $d(x, \xi) = p$. The transform $f \to \widehat{f}$ is inverted by means of the dual transform $\varphi \to \check{\varphi}_p$ for (29). The inversion below is a special case of Theorem 1.10, Chapter III, and is the analog of (22). Note however the absence of v in the integrand. Observe also that the metric ds is renormalized by the factor 2 (so curvature is -1).

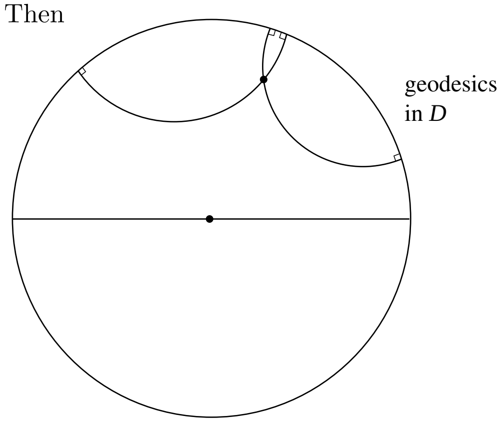

FIGURE II.4.

Theorem 4.2. *The X-ray transform in* $\mathbf{H}^2$ *with the metric*

$$ds^2 = \frac{4|dz|^2}{(1-|z|^2)^2}$$

is inverted by

(30)
$$f(z) = \frac{1}{\pi}\left\{\frac{d}{du}\int_0^u (\widehat{f})^\vee_{\operatorname{lm} v}(z)(u^2 - v^2)^{-\frac{1}{2}}\,dv\right\}_{u=1},$$

where $\operatorname{lm} v = \cosh^{-1}(v^{-1})$.

Another inversion formula is

(31)
$$f = -\frac{1}{4\pi} LS((\widehat{f})^\vee)\,,$$

where S is the operator of convolution on $\mathbf{H}^2$ with the function $x \to \coth(d(x, o)) - 1$, (Theorem 1.14, Chapter III).

C. The Horocycles in $\mathbf{H}^2$.

Consider a family of geodesics with the same limit point on the boundary B. The *horocycles* in $\mathbf{H}^2$ are by definition the orthogonal trajectories of such families of geodesics. Thus the horocycles are the circles tangential to $|z| = 1$ from the inside (Fig. II.5).

One such horocycle is $\xi_0 = N \cdot o$, the orbit of the origin o under the action of N. Since $a_t \cdot \xi$ is the horocycle with diameter $(\tanh t, 1)$ G acts transitively on the set Ξ of horocycles. Now we take $\mathbf{H}^2$ with the metric (24). Since $G = KAN$ it is easy to see that MN is the subgroup leaving ξ_o invariant. Thus we have here

(32)
$$X = G/K\,, \quad \Xi = G/MN\,.$$

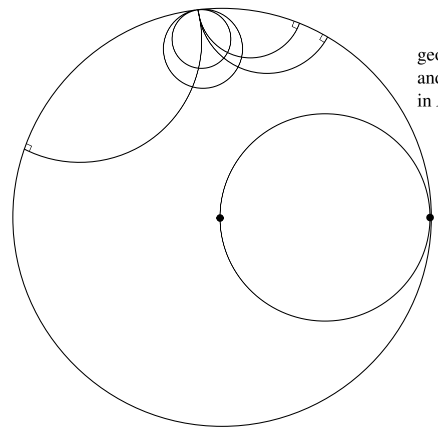

FIGURE II.5.

Furthermore each horocycle has the form $\xi = ka_t \cdot \xi_0$ where $kM \in K/M$ and $t \in \mathbf{R}$ are unique. Thus $\Xi \sim K/M \times A$, which is also evident from the figure.

We observe now that the maps

$$\psi : t \to a_t \cdot o\,, \quad \varphi : x \to n_x \cdot o$$

of $\mathbf{R}$ onto γ_0 and ξ_0, respectively, are isometries. The first statement follows from (27) because

$$d(o, a_t) = d(o, \tanh t) = t\,.$$

For the second we note that

$$\varphi(x) = x(x+i)^{-1}\,, \quad \varphi'(x) = i(x+i)^{-2}$$

so

$$\langle \varphi'(x), \varphi'(x) \rangle_{\varphi(x)} = (x^2+1)^{-4}(1 - |x(x+i)^{-1}|^2)^{-2} = 1\,.$$

Thus we give A and N the Haar measures $d(a_t) = dt$ and $d(n_x) = dx$.

Geometrically, the Radon transform on X relative to the horocycles is defined by

(33) $$\widehat{f}(\xi) = \int_\xi f(x)\, dm(x)\,,$$

where dm is the measure on ξ induced by (24). Because of our remarks about φ, (33) becomes

(34) $$\widehat{f}(g \cdot \xi_0) = \int_N f(gn \cdot o)\, dn\,,$$

so the geometric definition (33) coincides with the group-theoretic one in (9). The dual transform is given by

$$\check{\varphi}(g\cdot o)=\int_K \varphi(gk\cdot\xi_o)\,dk\,,\qquad (dk=d\theta/2\pi)\,. \tag{35}$$

In order to invert the transform $f\to\widehat{f}$ we introduce the non-Euclidean analog of the operator Λ in Chapter I, §3. Let T be the distribution on $\mathbf{R}$ given by

$$T\varphi=\tfrac{1}{2}\int_{\mathbf{R}}(\operatorname{sh}t)^{-1}\varphi(t)\,dt\,,\quad \varphi\in\mathcal{D}(\mathbf{R})\,, \tag{36}$$

considered as the Cauchy principal value, and put $T'=dT/dt$. Let Λ be the operator on $\mathcal{D}(\Xi)$ given by

$$(\Lambda\varphi)(ka_t\cdot\xi_0)=\int_{\mathbf{R}}\varphi(ka_{t-s}\cdot\xi_0)e^{-s}\,dT'(s)\,. \tag{37}$$

Theorem 4.3. *The Radon transform $f\to\widehat{f}$ for horocycles in $\mathbf{H}^2$ is inverted by*

$$f=\frac{1}{\pi}(\Lambda\widehat{f})^{\vee}\,,\quad f\in\mathcal{D}(\mathbf{H}^2)\,. \tag{38}$$

We begin with a simple lemma.

Lemma 4.4. *Let τ be a distribution on $\mathbf{R}$. Then the operator $\widetilde{\tau}$ on $\mathcal{D}(\Xi)$ given by the convolution*

$$(\widetilde{\tau}\varphi)(ka_t\cdot\xi_0)=\int_{\mathbf{R}}\varphi(ka_{t-s}\cdot\xi_0)\,d\tau(s)$$

is invariant under the action of G.

Proof. To understand the action of $g\in G$ on $\Xi\sim(K/M)\times A$ we write $gk=k'a_{t'}n'$. Since each $a\in A$ normalizes N we have

$$gka_t\cdot\xi_0=gka_tN\cdot o=k'a_{t'}n'a_tN\cdot o=k'a_{t+t'}\cdot\xi_0\,.$$

Thus the action of g on $\Xi\simeq(K/M)\times A$ induces this fixed translation $a_t\to a_{t+t'}$ on A. This translation commutes with the convolution by τ so the lemma follows.

Since the operators Λ, $\widehat{\ }$, $^{\vee}$ in (38) are all G-invariant it suffices to prove the formula at the origin o. We first consider the case when f is K-invariant, i.e., $f(k\cdot z)\equiv f(z)$. Then by (34)

$$\widehat{f}(a_t\cdot\xi_0)=\int_{\mathbf{R}} f(a_tn_x\cdot o)\,dx\,. \tag{39}$$

Because of (27) we have

$$|z| = \tanh d(o,z)\,, \quad \cosh^2 d(o,z) = (1-|z|^2)^{-1}\,. \tag{40}$$

Since

$$a_t n_x \cdot o = (\operatorname{sh} t - ix\, e^t)/(\operatorname{ch} t - ix\, e^t)$$

(40) shows that the distance $s = d(o, a_t n_x \cdot o)$ satisfies

$$\operatorname{ch}^2 s = \operatorname{ch}^2 t + x^2 e^{2t}\,. \tag{41}$$

Thus defining F on $[1,\infty)$ by

$$F(\operatorname{ch}^2 s) = f(\tanh\, s)\,, \tag{42}$$

we have

$$F'(\operatorname{ch}^2 s) = f'(\tanh\, s)(2\operatorname{sh} s\operatorname{ch}^3 s)^{-1}$$

so, since $f'(0) = 0$, $\lim_{u\to 1} F'(u)$ exists. The transform (39) now becomes (with $xe^t = y$)

$$e^t \widehat{f}(a_t \cdot \xi_0) = \int_{\mathbf{R}} F(\operatorname{ch}^2 t + y^2)\, dy\,. \tag{43}$$

We put

$$\varphi(u) = \int_{\mathbf{R}} F(u + y^2)\, dy$$

and invert this as follows:

$$\begin{aligned} \int_{\mathbf{R}} \varphi'(u+z^2)\, dz &= \int_{\mathbf{R}^2} F'(u+y^2+z^2)\, dy\, dz \\ &= 2\pi \int_0^\infty F'(u+r^2) r\, dr = \pi \int_0^\infty F'(u+\rho)\, d\rho\,, \end{aligned}$$

so

$$-\pi F(u) = \int_{\mathbf{R}} \varphi'(u+z^2)\, dz\,.$$

In particular,

$$\begin{aligned} f(o) &= -\frac{1}{\pi}\int_{\mathbf{R}} \varphi'(1+z^2)\, dz = -\frac{1}{\pi}\int_{\mathbf{R}} \varphi'(\operatorname{ch}^2\tau)\operatorname{ch}\tau\, d\tau\,, \\ &= -\frac{1}{\pi}\int_{\mathbf{R}}\int_{\mathbf{R}} F'(\operatorname{ch}^2 t + y^2)\, dy \operatorname{ch} t\, dt \end{aligned}$$

so

$$f(o) = -\frac{1}{2\pi}\int_{\mathbf{R}} \frac{d}{dt}(e^t \widehat{f}(a_t\cdot\xi_0))\frac{dt}{\operatorname{sh} t}\,.$$

Since $(e^t\widehat{f})(a_t\cdot\xi_0)$ is even (cf. (43)) its derivative vanishes at $t=0$ so the integral is well defined. With T as in (36), the last formula can be written

$$f(o) = \frac{1}{\pi} T_t'(e^t \widehat{f}(a_t\cdot\xi_0))\,, \tag{44}$$

the prime indicating derivative. If f is not necessarily K-invariant we use (44) on the average

$$f^{\natural}(z)=\int_K f(k\cdot z)\,dk=\frac{1}{2\pi}\int_0^{2\pi} f(k_\theta\cdot z)\,d\theta\,.$$

Since $f^{\natural}(o)=f(o)$, (44) implies

$$f(o)=\frac{1}{\pi}\int_{\mathbf{R}}[e^t(f^{\natural})\widehat{\ }(a_t\cdot\xi_0)]\,dT'(t)\,. \tag{45}$$

This can be written as the convolution at $t=0$ of $(f^{\natural})\widehat{\ }(a_t\cdot\xi_0)$ with the image of the distribution e^tT_t' under $t\to -t$. Since T' is even the right hand side of (45) is the convolution at $t=0$ of $\widehat{f^{\natural}}$ with $e^{-t}T_t'$. Thus by (37)

$$f(o)=\frac{1}{\pi}(\Lambda\widehat{f^{\natural}})(\xi_0)\,.$$

Since Λ and $\widehat{\ }$ commute with the K action this implies

$$f(o)=\frac{1}{\pi}\int_K(\Lambda\widehat{f})(k\cdot\xi_0)=\frac{1}{\pi}(\Lambda\widehat{f})^{\vee}(o)$$

and this proves the theorem.

Theorem 4.3 is of course the exact analog to Theorem 3.6 in Chapter I, although we have not specified the decay conditions for f needed in generalizing Theorem 4.3.

D. The Poisson Integral as a Radon Transform.

Here we preserve the notation introduced for the hyperbolic plane $\mathbf{H}^2$. Now we consider the homogeneous spaces

$$X=G/MAN\,,\quad \Xi=G/K\,. \tag{46}$$

Then Ξ is the disk $D:|z|<1$. On the other hand, X is identified with the boundary $B:|z|=1$, because when G acts on B, MAN is the subgroup fixing the point $z=1$. Since $G=KAN$, each coset $gMAN$ intersects eK. Thus each $x\in X$ is incident to each $\xi\in\Xi$. Our abstract Radon transform (9) now takes the form

$$\begin{aligned}\widehat{f}(gK)\ &=\ \int_{K/L} f(gkMAN)\,dk_L=\int_B f(g\cdot b)\,db\,,\\ &=\ \int_B f(b)\frac{d(g^{-1}\cdot b)}{db}\,db\,.\end{aligned} \tag{47}$$

Writing g^{-1} in the form

$$g^{-1}:\zeta\to\frac{\zeta-z}{-\overline{z}\zeta+1}\,,\quad g^{-1}\cdot e^{i\theta}=e^{i\varphi}\,,$$

we have

$$e^{i\varphi} = \frac{e^{i\theta} - z}{-\overline{z}e^{i\theta} + 1}\,, \qquad \frac{d\varphi}{d\theta} = \frac{1 - |z|^2}{|z - e^{i\theta}|}\,,$$

and this last expression is the classical Poisson kernel. Since $gK = z$, (47) becomes the classical Poisson integral

$$\widehat{f}(z) = \int_B f(b) \frac{1 - |z|^2}{|z - b|^2}\, db\,. \tag{48}$$

Theorem 4.5. *The Radon transform $f \to \widehat{f}$ for the homogeneous spaces (46) is the classical Poisson integral (48). The inversion is given by the classical Schwarz theorem*

$$f(b) = \lim_{z \to b} \widehat{f}(z)\,, \qquad f \in C(B)\,, \tag{49}$$

solving the Dirichlet problem for the disk.

We repeat the geometric proof of (49) from our booklet [1981] since it seems little known and is considerably shorter than the customary solution in textbooks of the Dirichlet problem for the disk. In (49) it suffices to consider the case $b = 1$. Because of (47),

$$\begin{aligned} \widehat{f}(\tanh\, t) &= \widehat{f}(a_t \cdot 0) = \frac{1}{2\pi} \int_0^{2\pi} f(a_t \cdot e^{i\theta})\, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f\left(\frac{e^{i\theta} + \tanh\, t}{\tanh\, t\, e^{i\theta} + 1} \right) d\theta\,. \end{aligned}$$

Letting $t \to +\infty$, (49) follows by the dominated convergence theorem.

The range question A for $f \to \widehat{f}$ is also answered by classical results for the Poisson integral; for example, the classical characterization of the Poisson integrals of bounded functions now takes the form

$$L^\infty(B)\widehat{\ } = \{\varphi \in L^\infty(\Xi) \,:\, L\varphi = 0\}\,. \tag{50}$$

The range characterization (50) is of course quite analogous to the range characterization for the X-ray transform described in Theorem 6.9, Chapter I. Both are realizations of the general expectations at the end of §2 that when $\dim X < \dim \Xi$ the range of the transform $f \to \widehat{f}$ should be given as the kernel of some differential operators. The analogy between (50) and Theorem 6.9 is even closer if we recall Gonzalez' theorem [1990b] that if we view the X-ray transform as a Radon transform between two homogeneous spaces of $\mathbf{M}(3)$ (see next example) then the range (83) in Theorem 6.9, Ch. I, can be described as the null space of a differential operator which is invariant under $\mathbf{M}(3)$. Furthermore, the dual transform $\varphi \to \check{\varphi}$ maps $\mathcal{E}(\Xi)$ on $\mathcal{E}(X)$. (See Corollary 4.7 below.)

Furthermore, John's mean value theorem for the X-ray transform (Corollary 6.12, Chapter I) now becomes the exact analog of Gauss' mean value theorem for harmonic functions.

What is the dual transform $\varphi \to \check{\varphi}$ for the pair (46)? The invariant measure on $MAN/M = AN$ is the functional

$$\varphi \to \int_{AN} \varphi(an \cdot o)\, da\, dn\,. \tag{51}$$

The right hand side is just $\check{\varphi}(b_0)$ where $b_0 = eMAN$. If $g = a'n'$ the measure (51) is seen to be invariant under g. Thus it is a constant multiple of the surface element $dz = (1-x^2-y^2)^{-2}\, dx\, dy$ defined by (24). Since the maps $t \to a_t \cdot o$ and $x \to n_x \cdot o$ were seen to be isometries, this constant factor is 1. Thus the measure (51) is invariant under each $g \in G$. Writing $\varphi_g(z) = \varphi(g \cdot z)$ we know $(\varphi_g)^\vee = \check{\varphi}_g$ so

$$\check{\varphi}(g \cdot b_0) = \int_{AN} \varphi_g(an)\, da\, dn = \check{\varphi}(b_0)\,.$$

Thus the dual transform $\varphi \to \check{\varphi}$ assigns to each $\varphi \in \mathcal{D}(\Xi)$ its integral over the disk.

Table II.1 summarizes the various results mentioned above about the Poisson integral and the X-ray transform. The inversion formulas and the ranges show subtle analogies as well as strong differences. The last item in the table comes from Corollary 4.7 below for the case $n = 3$, $d = 1$.

E. The d-plane Transform.

We now review briefly the d-plane transform from a group theoretic standpoint. As in (1) we write

$$X = \mathbf{R}^n = \mathbf{M}(n)/\mathbf{O}(n)\,, \quad \Xi = \mathbf{G}(d,n) = \mathbf{M}(n)/(\mathbf{M}(d)\times\mathbf{O}(n-d))\,, \tag{52}$$

where $\mathbf{M}(d)\times\mathbf{O}(n-d)$ is the subgroup of $\mathbf{M}(n)$ preserving a certain d-plane ξ_0 through the origin. Since the homogeneous spaces

$$\mathbf{O}(n)/\mathbf{O}(n) \cap (\mathbf{M}(d) \times \mathbf{O}(n-d)) = \mathbf{O}(n)/(\mathbf{O}(d) \times \mathbf{O}(n-d))$$

and

$$(\mathbf{M}(d) \times \mathbf{O}(n-d))/\mathbf{O}(n) \cap (\mathbf{M}(d) \times \mathbf{O}(n-d)) = \mathbf{M}(d)/\mathbf{O}(d)$$

have unique invariant measures the group-theoretic transforms (9) reduce to the transforms (52), (53) in Chapter I. The range of the d-plane transform is described by Theorem 6.3 and the equivalent Theorem 6.5 in Chapter I. It was shown by Richter [1986a] that the differential operators in

	Poisson Integral	*X-ray Transform*
Coset spaces	$X = \mathbf{SU}(1,1)/MAN$ $\Xi = \mathbf{SU}(1,1)/K$	$X = \mathbf{M}(3)/\mathbf{O}(3)$ $\Xi = \mathbf{M}(3)/(\mathbf{M}(1) \times \mathbf{O}(2))$
$f \to \widehat{f}$	$\widehat{f}(z) = \int_B f(b) \frac{1-\|z\|^2}{\|z-b\|^2} db$	$\widehat{f}(\ell) = \int_\ell f(p)\, dm(p)$
$\varphi \to \check{\varphi}$	$\check{\varphi}(x) = \int_\Xi \varphi(\xi)\, d\xi$	$\check{\varphi}(x) =$ average of φ over set of ℓ through x
Inversion	$f(b) = \lim_{z \to b} \widehat{f}(z)$	$f = \frac{1}{\pi}(-L)^{1/2}((\widehat{f})^\vee)$
Range of $f \to \widehat{f}$	$L^\infty(X)\widehat{} =$ $\{\varphi \in L^\infty(\Xi) : L\varphi = 0\}$	$\mathcal{D}(X)\widehat{} =$ $\{\varphi \in \mathcal{D}(\Xi) : \Lambda(\|\xi - \eta\|^{-1}\varphi) = 0\}$
Range characterization	Gauss' mean value theorem	Mean value property for hyperboloids of revolution
Range of $\varphi \to \check{\varphi}$	$\mathcal{E}(\Xi)^\vee = \mathbf{C}$	$\mathcal{E}(\Xi)^\vee = \mathcal{E}(X)$

TABLE II.1. Analogies between the Poisson Integral and the X-ray Transform.

Theorem 6.5 could be replaced by $\mathbf{M}(n)$-induced second order differential operators and then Gonzalez [1990b] showed that the whole system could be replaced by a single fourth order $\mathbf{M}(n)$-invariant differential operator on Ξ.

Writing (52) for simplicity in the form

$$X = G/K\,, \quad \Xi = G/H \tag{53}$$

we shall now discuss the range question for the dual transform $\varphi \to \check{\varphi}$ by invoking the d-plane transform on $\mathcal{E}'(X)$.

Theorem 4.6. *Let $\mathcal{N}$ denote the kernel of the dual transform on $\mathcal{E}(\Xi)$. Then the range of $S \to \widehat{S}$ on $\mathcal{E}'(X)$ is given by*

$$\mathcal{E}'(X)\widehat{} = \{\Sigma \in \mathcal{E}'(\Xi) \,:\, \Sigma(\mathcal{N}) = 0\}\,.$$

The inclusion $\subset$ is clear from the definitions (14),(15) and Proposition 2.5. The converse is proved by the author in [1983a] and [1994b], Ch. I, §2 for $d = n-1$; the proof is also valid for general d.

For Fréchet spaces E and F one has the following classical result. A continuous mapping $\alpha : E \to F$ is surjective if the transpose ${}^t\alpha : F' \to E'$ is injective and has a closed image. Taking $E = \mathcal{E}(\Xi)$, $F = \mathcal{E}(X)$, α as the dual transform $\varphi \to \check{\varphi}$, the transpose ${}^t\alpha$ is the Radon transform on $\mathcal{E}'(X)$. By Theorem 4.6, ${}^t\alpha$ does have a closed image and by Theorem 5.5, Ch. I (extended to any d) ${}^t\alpha$ is injective. Thus we have the following result (Hertle [1984] for $d = n-1$) expressing the surjectivity of α.

Corollary 4.7. *Every $f \in \mathcal{E}(\mathbf{R}^n)$ is the dual transform $f = \check{\varphi}$ of a smooth d-plane function φ.*

F. Grassmann Manifolds.

We consider now the (affine) Grassmann manifolds $\mathbf{G}(p,n)$ and $\mathbf{G}(q,n)$ where $p + q = n - 1$. If $p = 0$ we have the original case of points and hyperplanes. Both are homogeneous spaces of the group $\mathbf{M}(n)$ and we represent them accordingly as coset spaces

$$X = \mathbf{M}(n)/H_p\,, \quad \Xi = \mathbf{M}(n)/H_q\,. \tag{54}$$

Here we take H_p as the isotropy group of a p-plane x_0 through the origin $0 \in \mathbf{R}^n$, H_q as the isotropy group of a q-plane ξ_0 through 0, *perpendicular to* x_0. Then

$$H_p \sim \mathbf{M}(p) \times \mathbf{O}(n-p)\,, \quad H_q = \mathbf{M}(q) \times \mathbf{O}(n-q)\,.$$

Also

$$H_q \cdot x_0 = \{x \in X \,:\, x \perp \xi_0, x \cap \xi_0 \neq \emptyset\}\,,$$

the set of p-planes intersecting ξ_0 orthogonally. It is then easy to see that

$$x \text{ is incident to } \xi \Leftrightarrow x \perp \xi\,, \quad x \cap \xi \neq \emptyset\,.$$

Consider as in Chapter I, §6 the mapping

$$\pi \,:\, \mathbf{G}(p,n) \to \mathbf{G}_{p,n}$$

given by parallel translating a p-plane to one such through the origin. If $\sigma \in \mathbf{G}_{p,n}$, the fiber $F = \pi^{-1}(\sigma)$ is naturally identified with the Euclidean space $\sigma^\perp$. Consider the linear operator $\square_p$ on $\mathcal{E}(\mathbf{G}(p,n))$ given by

$$(\square_p f)|F = L_F(f|F)\,. \tag{55}$$

Here L_F is the Laplacian on F and bar denotes restriction. Then one can prove that $\square_p$ is a differential operator on $\mathbf{G}(p,n)$ which is invariant under the action of $\mathbf{M}(n)$. Let $f \to \widehat{f}$, $\varphi \to \check{\varphi}$ be the Radon transform and its dual corresponding to the pair (54). Then $\widehat{f}(\xi)$ represents the integral of f over all p-planes x intersecting ξ under a right angle. For n odd this is inverted as follows (Gonzalez [1984, 1987]).

Theorem 4.8. *Let $p, q \in \mathbb{Z}^+$ such that $p+q+1=n$ is odd. Then the transform $f \to \widehat{f}$ from $\mathbf{G}(p,n)$ to $\mathbf{G}(q,n)$ is inverted by the formula*

$$C_{p,q} f = ((\Box_q)^{(n-1)/2} \widehat{f})^\vee\,, \quad f \in \mathcal{D}(\mathbf{G}(p,n))$$

where $C_{p,q}$ is a constant.

If $p=0$ this reduces to Theorem 3.6, Ch. I.

G. Half-lines in a Half-plane.

In this example X denotes the half-plane $\{(a,b) \in \mathbf{R}^2 \,:\, a>0\}$ viewed as a subset of the plane $\{(a,b,1) \in \mathbf{R}^3\}$. The group G of matrices

$$(\alpha, \beta, \gamma) = \begin{pmatrix} \alpha & 0 & 0 \\ \beta & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix} \in \mathbf{GL}(3, \mathbf{R})\,, \quad \alpha > 0$$

acts transitively on X with the action

$$(\alpha, \beta, \gamma) \odot (a,b) = (\alpha a, \beta a + b + \gamma)\,.$$

This is the restriction of the action of $\mathbf{GL}(3,\mathbf{R})$ on $\mathbf{R}^3$. The isotropy group of the point $x_0 = (1,0)$ is the group

$$K = \{(1, \beta - \beta) \,:\, \beta \in \mathbf{R}\}\,.$$

Let Ξ denote the set of half-lines in X which end on the boundary $\partial X = 0 \times \mathbf{R}$. These lines are given by

$$\xi_{v,w} = \{(t, v+tw) \,:\, t>0\}$$

for arbitrary $v, w \in \mathbf{R}$. Thus Ξ can be identified with $\mathbf{R} \times \mathbf{R}$. The action of G on X induces a transitive action of G on Ξ which is given by

$$(\alpha, \beta, \gamma) \Diamond (v,w) = (v + \gamma, \frac{w+\beta}{\alpha})\,.$$

(Here we have for simplicity written (v,w) instead of $\xi_{v,w}$.) The isotropy group of the point $\xi_0 = (0,0)$ (the x-axis) is

$$H = \{(\alpha, 0, 0) \,:\, \alpha > 0\} = \mathbf{R}^\times_+\,,$$

the multiplicative group of the positive real numbers. Thus we have the identifications

$$X = G/K\,, \quad \Xi = G/H\,. \tag{56}$$

The group $K \cap H$ is now trivial so the Radon transform and its dual for the double fibration in (56) are defined by

$$\widehat{f}(gH) = \int_H f(ghK)\, dh\,, \tag{57}$$

$$\check{\varphi}(gK) = \chi(g) \int_K \varphi(gkH)\, dk\,, \tag{58}$$

where χ is the homomorphism $(\alpha, \beta\,, \gamma) \to \alpha^{-1}$ of G onto $\mathbf{R}_+^\times$. The reason for the presence of χ is that we wish Proposition 2.2 to remain valid even if G is not unimodular. In (57) and (58) we have the Haar measures

$$dk_{(1,\beta-\beta)} = d\beta\,, \quad dh_{(\alpha,0,0)} = d\alpha/\alpha\,. \tag{59}$$

Also, if $g = (\alpha, \beta, \gamma)$, $h = (a, 0, 0)$, $k = (1, b, -b)$ then

$$\begin{aligned} gH = (\gamma, \beta/\alpha)\,, &\qquad ghK = (\alpha a, \beta a + \gamma) \\ gK = (\alpha, \beta + \gamma)\,, &\qquad gkH = (-b + \gamma, \tfrac{b+\beta}{\alpha}) \end{aligned}$$

so (57)–(58) become

$$\begin{aligned} \widehat{f}(\gamma, \beta/\alpha) &= \int_{\mathbf{R}^+} f(\alpha a, \beta a + \gamma) \frac{da}{a} \\ \check{\varphi}(\alpha, \beta + \gamma) &= \alpha^{-1} \int_{\mathbf{R}} \varphi(-b + \gamma, \tfrac{b+\beta}{\alpha})\, db\,. \end{aligned}$$

Changing variables these can be written

$$\widehat{f}(v, w) = \int_{\mathbf{R}^+} f(a, v + aw) \frac{da}{a}\,, \tag{60}$$

$$\check{\varphi}(a, b) = \int_{\mathbf{R}} \varphi(b - as, s)\, ds \qquad a > 0\,. \tag{61}$$

Note that in (60) the integration takes place over all points on the line $\xi_{v,w}$ and in (61) the integration takes place over the set of lines $\xi_{b-as,s}$ all of which pass through the point (a, b). This is an *a posteriori* verification of the fact that our incidence for the pair (56) amounts to $x \in \xi$.

From (60)–(61) we see that $f \to \widehat{f}, \varphi \to \check{\varphi}$ are adjoint relative to the measures $\frac{da}{a}\, db$ and $dv\, dw$:

$$\int_{\mathbf{R}} \int_{\mathbf{R}_+^\times} f(a, b) \check{\varphi}(a, b) \frac{da}{a}\, db = \int_{\mathbf{R}} \int_{\mathbf{R}} \widehat{f}(v, w) \varphi(v, w)\, dv\, dw\,. \tag{62}$$

The proof is a routine computation.

We recall (Chapter V) that $(-L)^{1/2}$ is defined on the space of rapidly decreasing functions on $\mathbf{R}$ by

$$((-L)^{1/2}\psi)^{\sim}\ (\tau) = |\tau| \widetilde{\psi}(\tau) \tag{63}$$

and we define Λ on $\mathcal{S}(\Xi)(=\mathcal{S}(\mathbf{R}^2))$ by having $(-L)^{1/2}$ only act on the second variable:

$$(\Lambda\varphi)(v,w) = ((-L)^{1/2}\varphi(v,\cdot))(w)\,. \tag{64}$$

Viewing $(-L)^{1/2}$ as the Riesz potential I^{-1} on $\mathbf{R}$ (Chapter V, §5) it is easy to see that if $\varphi_c(v,w) = \varphi(v,\frac{w}{c})$ then

$$\Lambda\varphi_c = |c|^{-1}(\Lambda\varphi)_c\,. \tag{65}$$

The Radon transform (57) is now inverted by the following theorem.

Theorem 4.9. *Let $f \in \mathcal{D}(X)$. Then*

$$f = \frac{1}{2\pi}(\Lambda\widehat{f})^\vee\,.$$

Proof. In order to use the Fourier transform $F \to \widetilde{F}$ on $\mathbf{R}^2$ and on $\mathbf{R}$ we need functions defined on all of $\mathbf{R}^2$. Thus we define

$$f^*(a,b) = \begin{cases} \frac{1}{a}f(\frac{1}{a}\,,\,\frac{-b}{a}) & a > 0\,, \\ 0 & a \leq 0\,. \end{cases}$$

Then

$$\begin{aligned} f(a,b) &= \frac{1}{a}f^*\left(\frac{1}{a}\,,\,-\frac{b}{a}\right) \\ &= a^{-1}(2\pi)^{-2}\iint \widetilde{f^*}(\xi,\eta)e^{i(\frac{\xi}{a}-\frac{b\eta}{a})}\,d\xi\,d\eta \\ &= (2\pi)^{-2}\iint \widetilde{f^*}(a\xi+b\eta,\eta)e^{i\xi}\,d\xi\,d\eta \\ &= a(2\pi)^{-2}\iint |\xi|\widetilde{f^*}((a+ab\eta)\xi,a\eta\xi)e^{i\xi}\,d\xi\,d\eta\,. \end{aligned}$$

Next we express the Fourier transform in terms of the Radon transform. We have

$$\begin{aligned} \widetilde{f^*}((a+ab\eta)\xi,a\eta\xi) &= \iint f^*(x,y)e^{-ix(a+ab\eta)\xi}e^{-iya\eta\xi}\,dx\,dy \\ &= \int_{\mathbf{R}}\int_{x\geq 0}\frac{1}{x}f\left(\frac{1}{x}\,,\,-\frac{y}{x}\right)e^{-ix(a+ab\eta)\xi}e^{-iya\eta\xi}\,dx\,dy \\ &= \int_{\mathbf{R}}\int_{x\geq 0} f\left(\frac{1}{x}\,,\,b+\frac{1}{\eta}+\frac{z}{x}\right)e^{iza\eta\xi}\,\frac{dx}{x}\,dz\,. \end{aligned}$$

This last expression is

$$\int_{\mathbf{R}}\widehat{f}(b+\eta^{-1},z)e^{iza\eta\xi}\,dz = (\widehat{f})^\sim(b+\eta^{-1},-a\eta\xi)\,,$$

where $\sim$ denotes the 1-dimensional Fourier transform (in the second variable). Thus

$$f(a,b) = a(2\pi)^{-2} \iint |\xi| (\widehat{f})^{\sim}(b+\eta^{-1}, -a\eta\xi) e^{i\xi}\, d\xi\, d\eta\,.$$

However $\widetilde{F}(c\xi) = |c|^{-1}(F_c)^{\sim}(\xi)$ so by (65)

$$\begin{aligned} f(a,b) &= a(2\pi)^{-2} \iint |\xi|((\widehat{f})_{a\eta})^{\sim}(b+\eta^{-1}, -\xi) e^{i\xi}\, d\xi |a\eta|^{-1}\, d\eta \\ &= (2\pi)^{-1} \int \Lambda((\widehat{f})_{a\eta})(b+\eta^{-1}, -1)|\eta|^{-1}\, d\eta \\ &= (2\pi)^{-1} \int |a\eta|^{-1}(\Lambda\widehat{f})_{a\eta}(b+\eta^{-1}, -1)|\eta|^{-1}\, d\eta \\ &= a^{-1}(2\pi)^{-1} \int (\Lambda\widehat{f})(b+\eta^{-1}, -(a\eta)^{-1})\eta^{-2}\, d\eta \end{aligned}$$

so

$$\begin{aligned} f(a,b) &= (2\pi)^{-1} \int_{\mathbf{R}} (\Lambda\widehat{f})(b-av, v)\, dv \\ &= (2\pi)^{-1}(\Lambda\widehat{f})^{\vee}(a,b)\,. \end{aligned}$$

proving the theorem.

Remark 4.10. It is of interest to compare this theorem with Theorem 3.6, Ch. I. If $f \in \mathcal{D}(X)$ is extended to all of $\mathbf{R}^2$ by defining it 0 in the left half plane then Theorem 3.6 does give a formula expressing f in terms of its integrals over half-lines in a strikingly similar fashion. Note however that while the operators $f \to \widehat{f}, \varphi \to \check{\varphi}$ are in the two cases defined by integration over the same sets (points on a half-line, half-lines through a point) the measures in the two cases are different. Thus it is remarkable that the inversion formulas look exactly the same.

H. Theta Series and Cusp Forms.

Let G denote the group $\mathbf{SL}(2,\mathbf{R})$ of 2×2 matrices of determinant one and Γ the *modular group* $\mathbf{SL}(2,\mathbb{Z})$. Let N denote the unipotent group $\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ where $n \in \mathbf{R}$ and consider the homogeneous spaces

$$X = G/N\,, \qquad \Xi = G/\Gamma\,. \tag{66}$$

Under the usual action of G on $\mathbf{R}^2$, N is the isotropy subgroup of $(1,0)$ so X can be identified with $\mathbf{R}^2 - (0)$, whereas Ξ is of course 3-dimensional.

In number theory one is interested in decomposing the space $L^2(G/\Gamma)$ into G-invariant irreducible subspaces. We now give a rough description of this by means of the transforms $f \to \widehat{f}$ and $\varphi \to \check{\varphi}$.

As customary we put $\Gamma_\infty = \Gamma \cap N$; our transforms (9) then take the form

$$\widehat{f}(g\Gamma) = \sum_{\Gamma/\Gamma_\infty} f(g\gamma N)\,, \quad \check{\varphi}(gN) = \int_{N/\Gamma_\infty} \varphi(gn\Gamma)\, dn_{\Gamma_\infty}\,.$$

Since N/Γ_∞ is the circle group, $\check{\varphi}(gN)$ is just the constant term in the Fourier expansion of the function $n\Gamma_\infty \to \varphi(gn\Gamma)$. The null space $L^2_d(G/\Gamma)$ in $L^2(G/\Gamma)$ of the operator $\varphi \to \check{\varphi}$ is called the space of *cusp forms* and the series for $\widehat{f}$ is called *theta series*. According to Prop. 2.2 they constitute the orthogonal complement of the image $C_c(X)\widehat{}$.

We have now the G-invariant decomposition

$$L^2(G/\Gamma) = L^2_c(G/\Gamma) \oplus L^2_d(G/\Gamma)\,, \tag{67}$$

where ($-$ denoting closure)

$$L^2_c(G/\Gamma) = (C_c(X)\widehat{}\,)^-\tag{68}$$

and as mentioned above,

$$L^2_d(G/\Gamma) = (C_c(X)\widehat{}\,)^\perp\,. \tag{69}$$

It is known (cf. Selberg [1962], Godement [1966]) that the representation of G on $L^2_c(G/\Gamma)$ is the *continuous* direct sum of the irreducible representations of G from the principal series whereas the representation of G on $L^2_d(G/\Gamma)$ is the *discrete* direct sum of irreducible representations each occurring with finite multiplicity.

In conclusion we note that the determination of a function in $\mathbf{R}^n$ in terms of its integrals over unit spheres (John [1955]) can be regarded as a solution to the first half of Problem B in §2 for the double fibration (4).

Bibliographical Notes

The Radon transform and its dual for a double fibration

$$\begin{tikzcd}[column sep=small] & Z = G/(K\cap H) \arrow[dl] \arrow[dr] & \\ X = G/K & & \Xi = G/H \end{tikzcd} \tag{70}$$

was introduced in the author's paper [1966a]. The results of §1–§2 are from there and from [1994b]. The definition uses the concept of *incidence* for $X = G/K$ and $\Xi = G/H$ which goes back to Chern [1942]. Even when the elements of Ξ can be viewed as subsets of X and vice versa (Lemma 1.3) it can be essential for the inversion of $f \to \widehat{f}$ not to restrict the incidence to the naive one $x \in \xi$. (See for example the classical case $X = \mathbf{S}^2, \Xi =$ set of great circles where in Theorem 4.1 a more general incidence is essential.) The double fibration in (1) was generalized in Gelfand, Graev and Shapiro [1969], by relaxing the homogeneity assumption.

For the case of geodesics in constant curvature spaces (Examples A, B in §4) see notes to Ch. III.

The proof of Theorem 4.3 (a special case of the author's inversion formula in [1964], [1965b]) makes use of a method by Godement [1957] in another context. Another version of the inversion (38) for $\mathbf{H}^2$ (and $\mathbf{H}^n$) is given in Gelfand-Graev-Vilenkin [1966]. A further inversion of the horocycle transform in $\mathbf{H}^2$ (and $\mathbf{H}^n$), somewhat analogous to (30) for the X-ray transform, is given by Berenstein and Tarabusi [1994].

The analogy suggested above between the X-ray transform and the horocycle transform in $\mathbf{H}^2$ goes even further in $\mathbf{H}^3$. There the 2-dimensional transform for totally geodesic submanifolds has *the same* inversion formula as the horocycle transform (Helgason [1994b], p. 209).

For a treatment of the horocycle transform on a Riemannian symmetric space see the author's monograph [1994b], Chapter II, where Problems A, B, C in §2 are discussed in detail along with some applications to differential equations and group representations. See also Quinto [1993a] and Gonzalez and Quinto [1994] for new proofs of the support theorem.

Example G is from Hilgert's paper [1994], where a related Fourier transform theory is also established. It has a formal analogy to the Fourier analysis on $\mathbf{H}^2$ developed by the author in [1965b] and [1972].

CHAPTER III

THE RADON TRANSFORM ON TWO-POINT HOMOGENEOUS SPACES

Let X be a complete Riemannian manifold, x a point in X and X_x the tangent space to X at x. Let Exp_x denote the mapping of X_x into X given by $\mathrm{Exp}_x(u) = \gamma_u(1)$ where $t \to \gamma_u(t)$ is the geodesic in X through x with tangent vector u at $x = \gamma_u(0)$.

A connected submanifold S of a Riemannian manifold X is said to be *totally geodesic* if each geodesic in X which is tangential to S at a point lies entirely in S.

The totally geodesic submanifolds of $\mathbf{R}^n$ are the planes in $\mathbf{R}^n$. Therefore, in generalizing the Radon transform to Riemannian manifolds, it is natural to consider integration over totally geodesic submanifolds. In order to have enough totally geodesic submanifolds at our disposal we consider in this section Riemannian manifolds X which are *two-point homogeneous* in the sense that for any two-point pairs $p, q \in X$ $p', q' \in X$, satisfying $d(p, q) = d(p', q')$, (where $d =$ distance), there exists an isometry g of X such that $g \cdot p = p',\ g \cdot q = q'$. We start with the subclass of Riemannian manifolds with the richest supply of totally geodesic submanifolds, namely the spaces of constant curvature.

While §1, which constitutes most of this chapter, is elementary, §2–§5 will involve a bit of Lie group theory.

§1 Spaces of Constant Curvature. Inversion and Support Theorems

Let X be a simply connected complete Riemannian manifold of dimension $n \geq 2$ and constant sectional curvature.

Lemma 1.1. *Let $x \in X$, V a subspace of the tangent space X_x. Then* $\mathrm{Exp}_x(V)$ *is a totally geodesic submanifold of X.*

Proof. For this we choose a specific embedding of X into $\mathbf{R}^{n+1}$, and assume for simplicity the curvature is $\epsilon(= \pm 1)$. Consider the quadratic form

$$B_\epsilon(x) = x_1^2 + \cdots + x_n^2 + \epsilon x_{n+1}^2$$

and the quadric Q_ϵ given by $B_\epsilon(x) = \epsilon$. The orthogonal group $\mathbf{O}(B_\epsilon)$ acts transitively on Q_ϵ. The form B_ϵ is positive definite on the tangent space $\mathbf{R}^n \times (0)$ to Q_ϵ at $x^0 = (0, \ldots, 0, 1)$; by the transitivity B_ϵ induces a positive definite quadratic form at each point of Q_ϵ, turning Q_ϵ into a

Riemannian manifold, on which $\mathbf{O}(B_\epsilon)$ acts as a transitive group of isometries. The isotropy subgroup at the point x^0 is isomorphic to $\mathbf{O}(n)$ and its acts transitively on the set of 2-dimensional subspaces of the tangent space $(Q_\epsilon)_{x^0}$. It follows that all sectional curvatures at x^0 are the same, namely ϵ, so by homogeneity, Q_ϵ has constant curvature ϵ. In order to work with connected manifolds, we replace Q_{-1} by its intersection Q_{-1}^+ with the half-space $x_{n+1} > 0$. Then Q_{+1} and Q_{-1}^+ are simply connected complete Riemannian manifolds of constant curvature. Since such manifolds are uniquely determined by the dimension and the curvature it follows that we can identify X with Q_{+1} or Q_{-1}^+.

The geodesic in X through x^0 with tangent vector $(1, 0, \ldots, 0)$ will be left point-wise fixed by the isometry

$$(x_1, x_2, \ldots, x_n, x_{n+1}) \to (x_1, -x_2, \ldots, -x_n, x_{n+1}) .$$

This geodesic is therefore the intersection of X with the two-plane $x_2 = \cdots = x_n = 0$ in $\mathbf{R}^{n+1}$. By the transitivity of $\mathbf{O}(n)$ all geodesics in X through x^0 are intersections of X with two-planes through 0. By the transitivity of $\mathbf{O}(Q_\epsilon)$ it then follows that the geodesics in X are precisely the nonempty intersections of X with two-planes through the origin.

Now if $V \subset X_{x^0}$ is a subspace, $\mathrm{Exp}_{x^0}(V)$ is by the above the intersection of X with the subspace of $\mathbf{R}^{n+1}$ spanned by V and x^0. Thus $\mathrm{Exp}_{x^0}(V)$ is a quadric in $V + \mathbf{R}x^0$ and its Riemannian structure induced by X is the same as induced by the restriction $B_\epsilon|(V + \mathbf{R}x^0)$. Thus, by the above, the geodesics in $\mathrm{Exp}_{x^0}(V)$ are obtained by intersecting it with two-planes in $V + \mathbf{R}x^0$ through 0. Consequently, the geodesics in $\mathrm{Exp}_{x^0}(V)$ are geodesics in X so $\mathrm{Exp}_{x^0}(V)$ is a totally geodesic submanifold of X. By the homogeneity of X this holds with x^0 replaced by an arbitrary point $x \in X$. The lemma is proved.

In accordance with the viewpoint of Ch. II we consider X as a homogeneous space of the identity component G of the group $\mathbf{O}(Q_\epsilon)$. Let K denote the isotropy subgroup of G at the point $x^0 = (0, \ldots, 0, 1)$. Then K can be identified with the special orthogonal group $\mathbf{SO}(n)$. Let k be a fixed integer, $1 \leq k \leq n-1$; let $\xi_0 \subset X$ be a fixed totally geodesic submanifold of dimension k passing through x^0 and let H be the subgroup of G leaving ξ_0 invariant. We have then

$$X = G/K\,, \quad \Xi = G/H\,, \tag{1}$$

Ξ denoting the set of totally geodesic k-dimensional submanifolds of X. Since $x^0 \in \xi_0$ it is clear that the abstract incidence notion boils down to the naive one, in other words: The cosets $x = gK$ $\xi = \gamma H$ have a point in common if and only if $x \in \xi$. In fact

$$x \in \xi \Leftrightarrow x^0 \in g^{-1}\gamma \cdot \xi_0 \Leftrightarrow g^{-1}\gamma \cdot \xi_0 = k \cdot \xi_0 \quad \text{for some } k \in K\,.$$