BOUNDS FOR THE QUARTIC WEYL SUM

D.R. HEATH-BROWN MATHEMATICAL INSTITUTE, OXFORD

ABSTRACT. We improve the standard Weyl estimate for quartic exponential sums in which the argument is a quadratic irrational. Specifically we show that

$$\sum_{n \le N} e(\alpha n^4) \ll_{\varepsilon, \alpha} N^{5/6 + \varepsilon}$$

for any $\varepsilon>0$ and any quadratic irrational $\alpha\in\mathbb{R}-\mathbb{Q}$. Classically one would have had the exponent $7/8+\varepsilon$ for such α . In contrast to the author's earlier work [2] on cubic Weyl sums (which was conditional on the *abc*-conjecture), we show that the van der Corput AB-steps are sufficient for the quartic case, rather than the BAAB-process needed for the cubic sum.

1. Introduction

This paper is concerned with estimates for the Weyl sum

$$S_k(\alpha, N) = \sum_{n < N} e(\alpha n^k),$$

in the particular case k = 4. Here $e(x) = \exp(2\pi i x)$ as usual, and $k \ge 3$ is an integer. In a companion paper [2] we examined the cubic Weyl sum $S_3(\alpha, N)$, but there are significant differences between the two investigations, as well as similarities.

The classical "Weyl bound" takes the form

(1)
$$S_k(\alpha, N) \ll_{\varepsilon} N^{1 - 2^{1 - k} + \varepsilon},$$

where ε is an arbitrary small positive number, whenever α has a rational approximation a/q such that

(2)
$$|\alpha - a/q| \ll \frac{1}{qN^{k-1}}, \quad N \ll q \ll N^{k-1}.$$

This is essentially Lemma 3 in Hardy and Littlewood's first 'Partitio Numerorum' paper [1]. The method however is due to Weyl [5], who did not work out a quantitative estimate.

If $\alpha \in \mathbb{R} - \mathbb{Q}$ is algebraic one may conclude from Roth's theorem that there is always a fraction a/q satisfying (2) when N is large enough. In the case k=4 we deduce that

(3)
$$S_4(\alpha, N) \ll_{\varepsilon, \alpha} N^{7/8+\varepsilon}$$

for such α .

The general exponent 7/8 in (3) has never been improved on, but our goal in the present paper is to show that one can do better for special values of α . We shall prove the following bound.

 $^{2020\} Mathematics\ Subject\ Classification.\ 11L15.$

Key words and phrases. Weyl sum; Quartic; Exponent; Quadratic irrational.

Theorem 1. Let $\alpha \in \mathbb{R} - \mathbb{Q}$ be a quadratic irrational. Then

$$S_4(\alpha, N) \ll_{\varepsilon, \alpha} N^{5/6+\varepsilon}$$

for any $\varepsilon > 0$.

We should compare this with the principal result from the author's previous work [2], in which the Weyl exponent $3/4 + \varepsilon$ for cubic exponential sums was reduced to $5/7 + \varepsilon$ for quadratic irrational α , but only under the assumption of the *abc*-conjecture. For the quartic sum we are able to establish an unconditional result. Moreover, in the quartic case we have reduced the exponent by 7/8 - 5/6 = 1/24, while the corresponding reduction in the cubic case was only 3/4 - 5/7 = 1/28. Thus our new result achieves a better saving, without any unproved hypothesis.

As with the previous paper we use the q-analogue of van der Corput's method. This requires a good approximation a/q to α , in which q factorizes in a suitable way. For the quartic sum we prove the following.

Theorem 2. Suppose that a and q are coprime integers and that q factors as $q = q_1q_2$. Then

$$S_4(\alpha,N) \ll_{\varepsilon} \left(1 + N^4 \left| \alpha - \frac{a}{q} \right| \right) (N^{1/2} q_1^{1/2} + N^{1/2} q_2^{1/4} + N q_2^{-1/6}) q^{\varepsilon},$$

for any $N \geq 2q_1$ and any $\varepsilon > 0$.

For the cubic sum the argument of [2] used the van der Corput BAAB steps. The final B-process produced a complete exponential sum

all *B*-process produced a complete exponential sum
$$\sum_{\substack{w,x,y,z \, (\text{mod } q)\\ q \mid w-x-y+z+t}} e\left(\frac{c(w^3-x^3-y^3+z^3)+uq_2(w-x)+mq_1(w-y)}{q}\right)$$

for certain integers q_1 and q_2 . When q is square-free we could give a suitably good bound for this, but not for general q. In the paper [2] the abc conjecture allowed one to restrict attention to square-free q.

In contrast, for our Theorem 1 it suffices to use the much simpler van der Corput AB-process. This will produce a complete exponential sum

$$S(a, h; q) = \sum_{n=1}^{q} e((an^3 + hn)/q),$$

for which we have good bounds for all moduli q. This allows us to give an unconditional treatment of the quartic sum $S_4(\alpha, N)$.

Perhaps the most surprising aspect of this paper is that in handling $S_k(\alpha, N)$ for k = 4 rather than k = 3 one replaces the BAAB process by the simpler AB process.

We remark that one can improve on the factor $1 + N^4 |\alpha - a/q|$ in Theorem 2 when one has $|\alpha - a/q| \gg N^{-4}$. Indeed, even when q is not known to factor one can use a standard version of the van der Corput AB-process to obtain non-trivial results. However this will not be relevant for the application to Theorem 1.

Acknowledgement. The author would like to thank the anonymous referee for their careful reading of the paper, which has resulted in the elimination of a number of misprints.

2. The van der corput A-Process

In this section we use the q-analogue of the van der Corput A-process, which produces the following result, in which we set

$$\Sigma(q; u, v; I) = \sum_{n \in I} e_q(un^3 + vn).$$

Lemma 1. Let $\alpha = a/q + \delta$ with (q, a) = 1 and $q = q_1q_2$, where $1 \leq q_1 \leq N/2$. Then

$$S_4(\alpha, N)^2 \ll (1 + |\delta|N^4)^2 q_1 \left\{ N + \sum_{1 \le h \le N/(2q_1)} \max_{I} |\Sigma(q_2; 4ah, 4ah^3 q_1^2; I)| \right\},$$

where I runs over sub-intervals of (0, N].

Proof. We have

$$S_4(\alpha, N) = S_4(a/q, N)e(\delta N^4) - \int_0^N (2\pi i \delta) 4t^3 e(\delta t^4) S_4(a/q, t) dt$$

$$\ll (1 + N^4 |\delta|) \max_{t \le N} |S_4(a/q, t)|,$$

by partial summation. We now begin the van der corput A-process by setting $H = \lfloor N/2q_1 \rfloor \geq 1$ and writing

$$HS_4(a/q,t) = \sum_{h \le H} \sum_{n:0 \le n+2hq_1 \le t} e_q \left(a(n+2hq_1)^4 \right).$$

If $0 < n + 2hq_1 \le t$ with $0 < h \le H$ and $0 \le t \le N$ we will have -N < n < N, so that

$$HS_4(a/q,t) = \sum_{\substack{|n| < N \ 0 \le n = 2h \ a_1 \le t \le t}} e_q \left(a(n + 2hq_1)^4 \right).$$

Cauchy's inequality then yields

$$H^2|S_4(a/q,t)|^2 \le 2N \sum_{\substack{|n| < N \\ 0 < n + 2hq_1 \le t}} \left| \sum_{\substack{1 \le h \le H \\ 0 < n + 2hq_1 \le t}} e_q \left(a(n+2hq_1)^4 \right) \right|^2.$$

We now expand the square, and change the order of summation to produce

$$H^2|S_4(a/q,t)|^2 \leq 2N \sum_{\substack{0 < h_1,h_2 \leq H \\ 0 < n + 2h_1q_1 \leq t \\ 0 < n + 2h_2q_1 \leq t}} e_q \left(a\{(n+2h_1q_1)^4 - (n+2h_2q_1)^4\} \right).$$

We then write $m = n + (h_1 + h_2)q_1$ and $h = h_1 - h_2$, so that

$$(n+2h_1q_1)^4 - (n+2h_2q_1)^4 = (m+hq_1)^4 - (m-hq_1)^4 = 4hq_1(m^3+h^2q_1^2m).$$

The conditions $0 < n + 2h_1q_1 \le t$ and $0 < n + 2h_2q_1 \le t$ are equivalent to the requirement that $|h|q_1 < m \le t - |h|q_1$, and we deduce that

$$H^2|S_4(a/q,t)|^2 \le 2N \sum_{0 < h_1, h_2 \le H} \sum_{|h|q_1 < m \le t - |h|q_1} e_{q_2} \left(4ah(m^3 + h^2q_1^2m) \right).$$

Each value of h arises at most H times, and $|h| \leq H$ in each case, whence

$$H^{2}|S_{4}(a/q,t)|^{2} \leq 4NH \sum_{0 \leq h \leq H} \left| \sum_{|h|q_{1} < m \leq t-|h|q_{1}} e_{q_{2}} \left(4ah(m^{3} + h^{2}q_{1}^{2}m) \right) \right|.$$

The term h=0 produces an overall contribution at most $4N^2H$, and the lemma follows.

3. The van der corput B-Process

In this section we apply the *B*-process to $\Sigma(q_2; 4ah, 4ah^3q_1^2; I)$. We begin by writing $d = (q_2, 4h)$ and setting $q_2 = dr$, 4ah = du, and $4ah^3q_1^2 = dv$, whence

(4)
$$\Sigma(q_2; 4ah, 4ah^3q_1^2; I) = \Sigma(r; u, v; I),$$

with r and u coprime.

For our application, the B-process is given by the following lemma.

Lemma 2. When I is a sub-interval of (0, N] we have

$$\Sigma(r; u, v; I) \ll_{\varepsilon} (r^{1/2} + Nr^{-1/3})r^{\varepsilon}$$

for any fixed $\varepsilon > 0$.

Proof. We start from the identity

$$\Sigma(r; u, v; I) = r^{-1} \sum_{-r/2 < m \le r/2} \sum_{s \pmod{r}} e_r(us^3 + vs) \sum_{n \in I} e_r(m(s - n)).$$

Thus

$$\Sigma(r; u, v; I) = r^{-1} \sum_{-r/2 < m \le r/2} \sum_{n \in I} e_r(-mn) \Sigma(r; u, v + m),$$

where we define

$$\Sigma(r; u, v) = \sum_{n \pmod{r}} e_r(un^3 + vn).$$

However

$$\sum_{n \in I} e_r(-mn) \ll \min(N, r/|m|),$$

whence

(5)
$$\Sigma(r; u, v; I) \ll r^{-1} \sum_{|m| \le r/2} \min(N, r/|m|) |\Sigma(r; u, v + m)|.$$

We now require estimates for $\Sigma(r; u, v)$. Whenever (r, u) = 1 and $\varepsilon > 0$ we have

$$\Sigma(r; u, v) \ll_{\varepsilon} r^{2/3 + \varepsilon}$$

by Hua [3], and

$$\Sigma(r; u, v) \ll_{\varepsilon} r^{1/2 + \varepsilon}(r, v)$$

by Hua [4]. To apply these we cover the range $|m| \le r/2$ in (5) with intervals $|m| \le r/N$ and $M < |m| \le 2M$ for suitable dyadic M with $r/N \ll M \ll r$. We then observe that

$$\sum_{M_0 < t \le M_0 + M} \min\{r^{1/6}, (r, t)\} \le \sum_{d \mid r} \sum_{M_0 < t \le M_0 + M} \min\{r^{1/6}, d\}$$

$$\le \sum_{d \mid r} (M/d + 1) \min\{r^{1/6}, d\}$$

$$\le \sum_{d \mid r} (M/d) \cdot d + \sum_{d \mid r} 1 \cdot r^{1/6}$$

$$\ll_{\varepsilon} r^{\varepsilon} (M + r^{1/6}) .$$

Thus

$$\sum_{|m| \le r/N} \min(N, r/|m|) |\Sigma(r; u, v + m)| \ll_{\varepsilon} r^{1/2 + \varepsilon} (r/N + r^{1/6}) N,$$

and

$$\sum_{M < |m| \le 2M} \min(N, r/|m|) \left| \Sigma(r; u, v + m) \right| \ll_{\varepsilon} r^{1/2 + \varepsilon} (M + r^{1/6}) \frac{r}{M}.$$

Since $r/N \ll M \ll r$ both the above bounds are $O_{\varepsilon}(r^{1/2+\varepsilon}(r+Nr^{1/6}))$. We deduce that the sum (5) is

$$\ll_{\varepsilon} r^{-1} \left\{ 1 + \sum_{\text{dyadic } M} 1 \right\} r^{1/2 + \varepsilon} (r + Nr^{1/6}) \ll_{\varepsilon} (r^{1/2} + Nr^{-1/3}) r^{2\varepsilon},$$

and the lemma follows on replacing ε by $\varepsilon/2$.

We can now complete the proof of Theorem 2.

Proof of Theorem 2. In view of the relation (4) we deduce from Lemmas 1 and 2 that

$$S_4(\alpha, N)^2 \ll_{\varepsilon} (1 + |\delta| N^4)^2 q_1 \left\{ N + \sum_{1 \le h \le N/(2q_1)} q^{\varepsilon} (r^{1/2} + Nr^{-1/3}) \right\},$$

where $r = q_2/(q_2, 4h)$. It follows that

$$S_4(\alpha, N)^2 \ll_{\varepsilon} (1 + |\delta| N^4)^2 q^{\varepsilon} q_1 \left\{ N + q_2^{1/2} N/q_1 + N q_2^{-1/3} \sum_{1 \le h \le N/(2q_1)} (q_2, h)^{1/3} \right\}.$$

However a standard argument shows that

$$\sum_{1 \le h \le N/(2q_1)} (q_2, h)^{1/3} \le \sum_{\substack{d \mid q_2}} d^{1/3} \sum_{\substack{h \le N/(2q_1) \\ d \mid h}} 1$$

$$\le \sum_{\substack{d \mid q_2}} d^{1/3} \frac{N}{2q_1 d}$$

$$\ll_{\varepsilon} Nq_1^{-1} q^{\varepsilon}.$$

We therefore have

$$S_4(\alpha, N)^2 \ll_{\varepsilon} (1 + |\delta| N^4)^2 q^{2\varepsilon} q_1 \left\{ N + q_2^{1/2} N q_1^{-1} + N^2 q_2^{-1/3} q_1^{-1} \right\},$$

and the theorem follows.

4. Deduction of Theorem 1

The key result which utilizes the special approximation properties of real quadratic irrationals is the following.

Theorem 3. Let $\alpha \in \mathbb{R}$ be a quadratic irrational, and let $\varepsilon > 0$ be given. Then there is a constant $C(\alpha, \varepsilon)$ such that, for any $Q \in \mathbb{N}$, one can solve

$$\left|\alpha - \frac{a}{q}\right| \le \frac{C(\alpha, \varepsilon)}{qQ}, \quad (a \in \mathbb{Z}, \ q \in \mathbb{N}, \ Q \ll_{\varepsilon, \alpha} q \le Q)$$

with q having no prime factors $p > q^{\varepsilon}$.

This is essentially Theorem 3 of [2]. However the assertion that $q \gg_{\varepsilon,\alpha} Q$ was accidentally omitted from the statement there, although it is explicitly mentioned in the proof.

To deduce Theorem 1 we approximate α as above, with Q taken to be N^2 . We proceed to build a divisor q_1 of q, one prime factor at a time, to produce a product in the range

$$q^{1/3} \le q_1 \le q^{1/3 + \varepsilon}.$$

We will therefore have $q = q_1q_2$ with

$$q^{2/3-\varepsilon} \le q_2 \le q^{2/3}.$$

Moreover $q_1 \ll_{\varepsilon,\alpha} q^{1/3+\varepsilon} \leq N^{2/3+2\varepsilon}$ so that $2q_1 \leq N$ for large enough N. Since $N^4|\alpha-a/q| \ll_{\varepsilon,\alpha} 1$ it now follows from Theorem 2 that

$$S_4(\alpha, N) \ll_{\varepsilon, \alpha} \left(N^{1/2} q^{1/6 + \varepsilon/2} + N q^{-1/9 + \varepsilon/6} \right) q^{\varepsilon} \ll_{\varepsilon, \alpha} N^{5/6 + 3\varepsilon}.$$

This suffices for Theorem 1, on re-defining ε .

5. Addendum

Since preparing the original version of this paper it has been pointed out to the author by Professor Ping Xi that one can handle $S_k(\alpha, N)$ to good effect by the q-analogue of van der Corput's method for all $k \geq 5$, but subject to the abc-conjecture following the author's work [2]. The general form of the q-analogue of van der Corput's method has been developed by Wu and Xi [6]. Subject to the abc-conjecture one finds that the A^sB -process produces a result for the sum $S_k(\alpha, N)$, having the same shape as Theorem 1 but with exponent

$$1 - \frac{2s - k}{2(2^s - 2)} + \varepsilon.$$

The choice s = [(k+3)/2] is optimal, and produces an improvement on the Weyl bound for all k.

References

- [1] G.H. Hardy and J.E. Littlewood, Some problems of 'Partitio Numerorum'; I; A new solution of Waring's problem, *Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl.*, 1920 (1920), 33–54.
- [2] D.R. Heath-Brown, Bounds for the cubic Weyl sum, J. Math. Sciences, 171 (2010), 813–823.
- [3] L.-K. Hua, On an exponential sum, J. Chinese Math. Soc., 2 (1940), 301–312.
- [4] L.-K. Hua, On exponential sums, Sci. Record (Peking) (N.S.), 1 (1957), 1–4.
- [5] H. Weyl, Über die Gleichverteilung der Zahlen mod. Eins, Math. Ann., 77 (1916), 313–352.
- [6] J. Wu and P. Xi, Arithmetic exponent pairs for algebraic trace functions and applications, Algebra Number Theory, 15 (2021), no. 9, 2123–2172.

MATHEMATICAL INSTITUTE, RADCLIFFE OBSERVATORY QUARTER, WOODSTOCK ROAD, OXFORD, OX2 6GG

Email address: rhb@maths.ox.ac.uk