Substitute this into the *integral* in the above formula for $\omega(E, z)$, specialize to $z = \rho$, and use (†) in the first right-hand term of that formula. We find

$$\omega(E,\rho) \;\geqslant\; |E|\left(\frac{C}{1-\rho} \;-\; \frac{1}{\pi}\int_{\gamma_\rho} \frac{d\omega(\zeta,\rho)}{1-|\zeta|}\right).$$

It was, however, seen in *step* 5 that $\rho < 1$ could be chosen in accordance with our requirements so as to make the *integral* in this expression *small.* For a suitable $\rho < 1$ close to 1, we will thus have (and by far !)

$$\omega(E,\rho) \;\geqslant\; \frac{C}{2(1-\rho)}|E|$$

provided that the closed set E lies on the (shorter) arc from $e^{i\log\rho}$ to $e^{-i\log\rho}$ on the unit circle.

This is our local estimate. What it says is that, corresponding to any ζ, $|\zeta| = 1$, we can *get* a $\rho_\zeta < 1$ such that, for closed sets E lying on *the smaller arc* J_ξ *of the unit circle joining* $\zeta e^{i\log\rho_\zeta}$ *to* $\zeta e^{-i\log\rho_\zeta}$,

(††) $\qquad \omega_{\Omega(\rho_\zeta^2)}(E, \rho_\zeta\zeta) \;\geqslant\; C_\zeta|E|,$

with $C_\zeta > 0$ depending (*a priori*) on ζ. Observe that, if $0 < \rho \leqslant \rho_\zeta^2$, $\Omega(\rho)$ (the component of $\mathscr{O} \cap \{\rho < |z| < 1\}$ abutting on the arcs I_k) must by definition contain $\Omega(\rho_\zeta^2)$. Therefore

(§) $\qquad \omega_{\Omega(\rho)}(E, \rho_\zeta\zeta) \;\geqslant\; \omega_{\Omega(\rho_\zeta^2)}(E, \rho_\zeta\zeta)$

by the *principle of extension of domain* when $E \subseteq \{|z| = 1\}$, a subset of *both* boundaries $\partial\Omega(\rho)$, $\partial\Omega(\rho_\zeta^2)$.

A finite number of the arcs J_ζ serve to cover the unit circumference; denote them by $J_1, J_2, \ldots, J_n$, calling the corresponding values of ζ, $\zeta_1, \ldots, \zeta_n$ and the corresponding ρ_ζ's $\rho_1, \rho_2, \ldots, \rho_n$. Let ρ be the *least* of the quantities ρ_j^2, $j = 1, 2, \ldots, n$, and denote the *least* of the C_{ζ_j} by k, which is thus > 0. If E is a closed subset of J_j, (††) and (§) give

$$\omega_{\Omega(\rho)}(E, \rho_j\zeta_j) \;\geqslant\; k|E|.$$

Fix any $z_0 \in \Omega(\rho)$. Using Harnack's inequality in $\Omega(\rho)$ for each of the pairs of points $(z_0,\ \rho_j\zeta_j)$, $j = 1, 2, \ldots, n$, we obtain, from the preceding relation,

(§§) $\qquad \omega_{\Omega(\rho)}(E, z_0) \;\geqslant\; K(z_0)|E|$

for closed subsets E of *any* of the arcs $J_1, J_2, \ldots, J_n$. Here, $K(z_0) > 0$ depends on z_0. Now we see finally that (§§) *in fact holds for any closed subset E of the unit circumference,* large or small. That is an obvious consequence of the additivity of the set function $\omega_{\Omega(\rho)}(\ \ , z_0)$, the arcs J_j forming a covering of $\{|\zeta| = 1\}$.

We are at long last able to conclude our proof of Volberg's theorem on the logarithmic integral. Our chosen z_0 in $\Omega(\rho)$ lies in $\mathcal{O}$, therefore $|\Phi(z_0)| > 0$. By the *theorem on harmonic estimation* applied to the function $\Phi(z)$ analytic in $\Omega(\rho)$ and continuous on $\{|z| \leqslant 1\}$,

$$-\infty < \log|\Phi(z_0)| \leqslant \int_{\partial\Omega(\rho)} \log|\Phi(\zeta)|\,d\omega_{\Omega(\rho)}(\zeta, z_0)$$

$$\leqslant \text{const.} + \int_{|\zeta|=1} \log|\Phi(\zeta)|\,d\omega_{\Omega(\rho)}(\zeta, z_0).$$

According to (§§), this last is in turn

$$\leqslant \text{const.} - K(z_0)\int_{-\pi}^{\pi} \log^{-}|\Phi(e^{i\vartheta})|\,d\vartheta,$$

$\log|\Phi(e^{i\vartheta})|$ being in any case *bounded above*. Thus,

$$\int_{-\pi}^{\pi} \log^{-}|\Phi(e^{i\vartheta})|\,d\vartheta < \infty.$$

However, $|\Phi(e^{i\vartheta})|$ lies, as we know, between *two constant multiples of* $|F(e^{i\vartheta})|$. Therefore

$$\int_{-\pi}^{\pi} \log^{-}|F(e^{i\vartheta})|\,d\vartheta < \infty,$$

i.e.,

$$\int_{-\pi}^{\pi} \log|F(e^{i\vartheta})|\,d\vartheta > -\infty.$$

Volberg's theorem is thus completely proved, and *we are finally done.*

Remark. Of the two regularity conditions required of $M(v)$ for this theorem, viz., that $M(v)/v$ be *decreasing* and that $M(v) \geqslant \text{const.}v^{\alpha}$ for large v with some $\alpha > \frac{1}{2}$, the *first* served to make possible the use of Dynkin's result (article 3) by means of which the analytic function $\Phi(z)$ was brought into the proof.

Decisive use of the *second* was not made until *step* 5, where we estimated

$$\int_{\gamma_\rho} \frac{1}{1-|\zeta|}\,d\omega(\zeta, \rho)$$

in terms of $\int_{\gamma_\rho} h(\log(1/|\zeta|))\,d\omega(\zeta, \rho)$.

Examination of the argument used there shows that *some* relaxation of the condition $M(v) \geqslant \text{const.}v^{\alpha}$ (with $\alpha > \frac{1}{2}$) *is* possible if one is willing to replace it by another with considerably more complicated statement. The *method* of Volberg's proof necessitates, however, that $M(v)$ be *at least* $\geqslant$ const.$v^{\frac{1}{2}}$ for large v. For, by a lemma of article 3, that relation is equivalent to the property that

$$\frac{1}{\xi} = O(h(\xi))$$

for $\xi \to 0$, and we need at least *this* in order to make the abovementioned estimate for $\rho < 1$ near 1.

We needed $\int_{\gamma_\rho}(1/(1-|\zeta|))d\omega(\zeta,\rho)$ in the computation following step 5, where we got a *lower bound* on $\omega(E,\rho)$. The integral came in there on account of the inequality

$$\tilde{\omega}(E,\zeta) \leqslant \frac{|E|}{\pi(1-|\zeta|)}$$

for harmonic measure $\tilde{\omega}(E,\zeta)$ (of sets $E \subseteq \{|\zeta| = 1\}$) in the ring $\{\rho^2 < |\zeta| < 1\}$. And, aside from a constant factor, this inequality is best possible.

7. Scholium. Levinson's log log theorem

Part of the material in articles 2 and 5 is closely related to some older work of Levinson which, because of its usefulness, should certainly be taken up before ending the present chapter.

During the proof of the *first* theorem in §F.4, Chapter VI, we came up with an entire function $L(z)$ satisfying an inequality of the form $|L(z)| \leqslant \text{const.}e^{K|z|}/|\Im z|$, and wished to conclude that $L(z)$ was of exponential type. Here there is an obvious difficulty for the points z lying near the real axis. We dealt with it by using the subharmonicity of $\sqrt{|L(z)|}$ and convergence of

$$\int_{-1}^{1} |y|^{-\frac{1}{2}} dy$$

in order to *integrate out* the denominator $|\Im z|$ from the inequality and thus strengthen the latter to an estimate $|L(z)| \leqslant \text{const.}e^{K|z|}$ for z near $\mathbb{R}$. A more elaborate version of the same procedure was applied in the proof of the *second* theorem of §F.4, Chapter VI, where subharmonicity of $\log|S(z)|$ was used to get rid of a troublesome term tending to ∞ for z approaching the real axis.

It is natural to ask *how far* such tricks can be pushed. Suppose that $f(z)$ is known to be analytic in some rectangle straddling the real axis, and we are assured that

$$|f(z)| \leqslant \text{const.}L(y)$$

in that rectangle, where, unfortunately, the majorant $L(y)$ goes to ∞ as $y \to 0$. What conditions on $L(y)$ will permit us to deduce *finite bounds* on $|f(z)|$, *uniform in the interior* of the given rectangle, from the preceding relation? One's first guess is that a condition of the form $\int_{-a}^{a} \log L(y)\,dy < \infty$ *will do*, but that *nothing much weaker than that can suffice*, because $\log|f(z)|$ *is* subharmonic while functions of $|f(z)|$ which increase *more slowly* than the logarithm are *not*, in general. This conservative appraisal turns out to be *wrong by a whole* (exponential) *order of magnitude*. Levinson found that *it is already enough to have*

$$\int_{-a}^{a} \log\log L(y)\,dy \;<\; \infty,$$

and that this condition *cannot be further weakened*.

Levinson's result is extremely useful. One application could be to *eliminate* the rough and ready but somewhat clumsy *hall of mirrors* argument from many of the places where it occurs in Chapter VI. Let us, for instance, consider again the proof of Akhiezer's first theorem from §B.1 of that chapter. If, in the circumstances of that theorem, we have $\|P\|_W \leqslant 1$ for a polynomial P, the relation

$$\log|P(z)| \;\leqslant\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|}{|z-t|^2}\log|P(t)|\,dt$$

$$\leqslant\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|}{|z-t|^2}\log W_+(t)\,dt$$

and the estimate of $\sup_{t\in\mathbb{R}}|(t-i)/(t-z)|$ from §A.2 (Chapter VI) tell us immediately that

$$\log|P(z)| \;\leqslant\; M\frac{(1+|z|)^2}{|\Im z|},$$

where

$$M \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log W_*(t)}{1+t^2}\,dt.$$

Taking any rectangle

$$\mathscr{D}_R \;=\; \{z\colon\ |\Re z| \leqslant R,\quad |\Im z| \leqslant 2\}$$

and putting

$$L(y) \;=\; \exp\left(M\frac{R^2+5}{|y|}\right),$$

we have $|P(z)| \leqslant L(|\Im z|)$ on $\mathscr{D}_R$. Here,

$$\int_{-1}^{1} \log\log L(y)\,dy \;<\; \infty,$$

so the result of Levinson gives us a control on the size of $|P(z)|$ *in the interior of* $\mathscr{D}_R$ (even *right on* the real axis). In this way we can see that the polynomials P with $\|P\|_W \leqslant 1$ form a *normal family* in any strip straddling the real axis. The relation $\log|P(z)| \;\leqslant\; M(1+|z|)^2/|\Im z|$ already shows that those polynomials form a normal family *outside* such a strip, and the main part of the proof of Akhiezer's first theorem is complete.

One can easily envision the possibility of other applications like the one just shown to situations where the *hall of mirrors* argument would not be available. There is thus no doubt about the worth of the result in question; let us, then, proceed to its precise statement and proof without further ado.

Levinson's* log log *theorem. *Consider any rectangle*

$$\mathscr{D} \;=\; \{z\colon\ a<x<a' \text{ and } -b<y<b\}.$$

Let $L(y)$ be Lebesgue measurable and $\geqslant$ e *for* $-b<y<b$.

Suppose that

$$\int_{-b}^{b} \log\log L(y)\,dy \;<\; \infty.$$

Then there is a decreasing function $m(\delta)$, depending only *on $L(y)$ and finite for $\delta>0$, such that, if $f(z)$ is analytic in $\mathscr{D}$ and if*

$$|f(z)| \;\leqslant\; L(\Im z)$$

there, we also have

$$|f(z)| \;\leqslant\; m(\operatorname{dist.}(z,\partial\mathscr{D})) \quad \textit{for } z\in\mathscr{D}.$$

Remark. In this version (due to Y. Domar), *no regularity properties whatever are required of* $L(y)$. The assumption that $L(y)\geqslant \mathrm{e}$ is of course made *merely to ensure positivity* of $\log\log L(y)$.

Proof of theorem (Y. Domar). Denote by $\mu(\lambda)$ the *distribution function* for $\log L(y)$; i.e.,

$$\mu(\lambda) \;=\; |\{y\colon\ -b<y<b \text{ and } \log L(y)>\lambda\}|.^*$$

Let $z_0\in\mathscr{D}$ and $R<\operatorname{dist}(z_0,\partial\mathscr{D})$, and write $u(z)=\log|f(z)|$. Since $u(z)$ is

* We are continuing to denote by $|E|$ the Lebesgue measure of sets $E\subseteq\mathbb{R}$.

subharmonic in $\mathscr{D}$, we have

$$u(z_0) \leqslant \frac{1}{\pi R^2}\iint_{|z-z_0|<R} u(z)\,dx\,dy.$$

We are going to show that, if $u(z_0)$ is *large* and z_0 *far enough* from $\partial\mathscr{D}$, this inequality leads to a *contradiction* when $u(z) \leqslant \log L(\Im z)$ in $\mathscr{D}$.

Call Δ the disk

$$\{z: |z-z_0| < R\}.$$

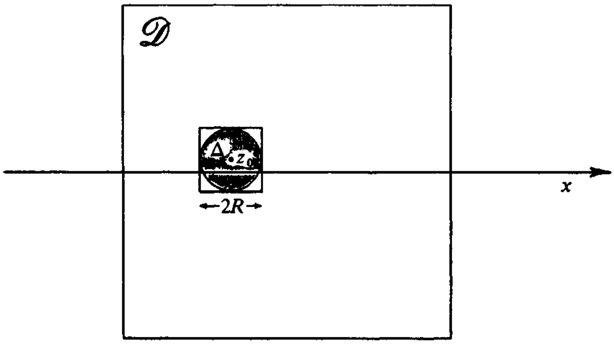

Figure 106

Use $\|E\|$ to denote the two-dimensional Lebesque measure of $E \subseteq \mathbb{C}$ (since $|\ |$ is being used for one-dimensional Lebesque measure of sets on $\mathbb{R}$). Then, by boxing Δ into a square of side $2R$ in the manner shown, we see from the ineqυality $u(z) \leqslant \log L(\Im z)$ that

$$\|\{z \in \Delta: u(z) > M/2\}\| \leqslant 2R\mu(M/2) \quad \text{for } M > 0.$$

Suppose now that $u(z_0) \geqslant M$, but that at the same time we have $u(z) \leqslant 2M$ on Δ. From the previous subharmonicity relation we will then have

$$(*) \qquad u(z_0) \leqslant \frac{M}{2} + \frac{2M}{\|\Delta\|}\left\|\left\{z \in \Delta: u(z) \geqslant \frac{M}{2}\right\}\right\| < \frac{M}{2} + \frac{4M}{\pi R}\mu(M/2).$$

Because $\log\log L(y)$ is integrable on $[-b, b]$ we certainly have $\mu(\lambda) \to 0$ for $\lambda \to \infty$. We can therefore take M *so large* that $\mu(M/2)$ is *much smaller* than $\operatorname{dist}(z_0, \partial\mathscr{D})$. *With such a value* of M, put

$$R = R_0 = \frac{16}{\pi}\mu(M/2).$$

The *right side of* $(*)$ *will then* be $\leqslant \frac{3}{4}M$. This means that *if* $u(z_0) \geqslant M$, the *assumption under which* $(*)$ *was derived is untenable*, i.e., *that* $u(z) > 2M$ *somewhere in* Δ, say for $z = z_1$, $|z_1 - z_0| < R_0$.

Supposing, then, that $u(z_0) \geqslant M$, we have a z_1, $|z_1 - z_0| < R_0$, with $u(z_1) > 2M$. We can then *repeat* the argument just given, making z_1 play the rôle of z_0, $2M$ that of M,

$$R_1 = \frac{16}{\pi}\mu(M)$$

that of R_0, and $\{z: |z - z_1| < R_1\}$ that of Δ. As long as $R_1 > \operatorname{dist}(z_1, \partial\mathscr{D})$, hence, surely, provided that

$$R_0 + R_1 < \operatorname{dist}(z_0, \partial\mathscr{D}),$$

we will get a z_2, $|z_2 - z_1| < R_1$, with $u(z_2) > 4M$. Then we can take $R_2 = (16/\pi)\mu(2M)$, have $4M$ play the rôle held by $2M$ in the previous step, and keep on going.

If, for the numbers

$$R_k = \frac{16}{\pi}\mu(2^{k-1}M),$$

we have

$$\sum_{k=0}^{\infty} R_k < \operatorname{dist}(z_0, \partial\mathscr{D}),$$

the process never stops, and we get a sequence of points $z_k \in \mathscr{D}$, $|z_k - z_{k-1}| < R_{k-1}$, with

$$u(z_k) \geqslant 2^k M.$$

Evidently, $z_k \xrightarrow[k]{} \text{a point } z_\infty \in \mathscr{D}$.

The function $f(z)$ is *analytic* in $\mathscr{D}$, so $u(z) = \log|f(z)|$ is *continuous* (in the extended sense) at z_∞, where $u(z_\infty) < \infty$. This is certainly incompatible with the inequalities $u(z_k) \geqslant 2^k M$ when $z_k \xrightarrow[k]{} z_\infty$. *Therefore we cannot have* $u(z_0) \geqslant M$ *if we can take* M *so large that* $\sum_0^\infty R_k < \operatorname{dist}(z_0, \partial\mathscr{D})$, i.e., *that*

$$(\dagger) \qquad \sum_{k=0}^{\infty} \frac{16}{\pi}\mu(2^{k-1}M) < \operatorname{dist}(z_0, \partial\mathscr{D}).$$

In order to complete the proof, it suffices, then, to show that the left-hand sum in (†) tends to *zero* as $M \to \infty$. By Abel's rearrangement,

$$\begin{aligned}\sum_{k=0}^{n} \mu(2^{k-1}M) = {} & \left\{\mu\left(\frac{M}{2}\right) - \mu(M)\right\} + 2\{\mu(M) - \mu(2M)\} \\ & + 3\{\mu(2M) - \mu(4M)\} + \cdots \\ & + n\{\mu(2^{n-2}M) - \mu(2^{n-1}M)\} + (n+1)\mu(2^{n-1}M).\end{aligned}$$

Remembering that $\mu(\lambda)$ is a *decreasing* function of μ, we see that as long as $M \geqslant 4$, the sum on the right is

$$\leqslant \sum_{k=0}^{n-1} \int_{2^{k-1}M}^{2^k M} \frac{\log\lambda - \log(M/4)}{\log 2}(-\,\mathrm{d}\mu(\lambda))$$

$$+ \int_{2^{n-1}M}^{\infty} \frac{\log\lambda - \log(M/4)}{\log 2}(-\,\mathrm{d}\mu(\lambda))$$

$$\leqslant -\frac{1}{\log 2}\int_{M/2}^{\infty} \log\lambda\,\mathrm{d}\mu(\lambda) = \frac{1}{\log 2}\int_{\substack{\log L(y) \geqslant M/2 \\ -b<y<b}} \log\log L(y)\,\mathrm{d}y.$$

Since $\int_{-b}^{b} \log\log L(y)\,\mathrm{d}y < \infty$, the previous expression, and hence the left-hand side of (†), tends to 0 as $M \to \infty$. Therefore, given $z_0 \in \mathscr{D}$, we can get an M sufficiently large for (†) to *hold, and, with that M*, $u(z_0) < M$, i.e., $|f(z_0)| < \mathrm{e}^M$. Let, then,

$$m(\delta) = \inf\left\{\mathrm{e}^M : \sum_{k=0}^{\infty} \frac{16}{\pi}\mu(2^{k-1}M) < \delta\right\}.$$

As we have just seen, $m(\delta)$ is *finite* for $\delta > 0$; it is obviously decreasing. And, if $f(z)$ is analytic in $\mathscr{D}$ with $|f(z)| \leqslant L(\Im z)$ there, we have $|f(z_0)| \leqslant m(\operatorname{dist}(z_0, \partial\mathscr{D}))$ for $z_0 \in \mathscr{D}$. We are done.

Remark. This beautiful proof is quite recent. The procedure of the scholium at the end of article 5 will yield the same result for sufficiently regular majorants $L(y)$.

We now consider the possibility of relaxing the condition

$$\int_{-b}^{b} \log\log L(y)\,\mathrm{d}y < \infty$$

required in the above theorem. If, for some majorant $L(y)$, *the conclusion of the theorem holds*, any set of *polynomials P* with $|P(z)| \leqslant L(\Im z)$ for $z \in \mathscr{D}$ *must form a normal family* in $\mathscr{D}$. This observation enables us to give a simple proof of the fact that the requirement.

$$\int_{-b}^{b} \log\log L(y)\,\mathrm{d}y < \infty$$

is essential in Levinson's result, at least for majorants $L(y)$ of sufficiently regular behaviour.

Theorem. *Let* $L(y)$ *be continuous and* $\geqslant$ e *for* $0 < |y| \leqslant b$, *with* $L(y) \to \infty$

for $y \to 0$ *and* $L(y)$ decreasing *on* $(0, b)$. *Suppose also that*

$$\int_0^b \log\log L(y)\,\mathrm{d}y \;=\; \infty.$$

Then there is a sequence of polynomials $P_n(z)$ *with, for* $\Im z \neq 0$,

$$|P_n(z)| \;\leqslant\; \text{const.}L(\Im z)$$

on the rectangle

$$\mathscr{D} \;=\; \{z\colon\; -1 < \Re z < 1 \text{ and } -b < \Im z < b\},$$

while at the same time

$$P_n(z) \underset{n}{\longrightarrow} \begin{cases} \;\;1, & z \in \mathscr{D} \text{ and } \Im z > 0, \\ -1, & z \in \mathscr{D} \text{ and } \Im z < 0. \end{cases}$$

Thus the conclusion of Levinson's theorem does not hold *for the majorant* $L(y)$.

Remark. We only require $L(y)$ to be monotone *on one side* of the origin, on the side over which the integral of $\log\log L(y)$ *diverges*. Levinson already had this result under the assumption of more regularity for $L(y)$.

Proof of theorem (Beurling, 1972 – compare with the proof of the theorem on simultaneous polynomial approximation in article 4). The last sentence in the statement follows from the existence of a sequence of polynomials P_n having the asserted properties. For, if the conclusion of Levinson's theorem *held*, the sequence $\{P_n\}$ would form a normal family in $\mathscr{D}$ and there would hence be a function *analytic* in $\mathscr{D}$, equal to $+1$ *above* the real axis, and to -1 *below* it. This is absurd.

Put

$$\varphi(z) \;=\; \begin{cases} \;\;1, & z \in \mathscr{D} \text{ and } \Im z > 0 \\ -1, & z \in \mathscr{D} \text{ and } \Im z < 0, \end{cases}$$

and let us argue by *duality* to obtain a sequence of polynomials $P_n(z)$ for which

$$\sup_{z \in \mathscr{D}} \left(\frac{|\varphi(z) - P_n(z)|}{L(\Im z)} \right) \underset{n}{\longrightarrow} 0.$$

Such P_n will clearly satisfy the conclusion of our theorem.

Note that, since $L(y) \to \infty$ for $y \to 0$, the ratio $\varphi(z)/L(\Im z)$ is *continuous on* $\bar{\mathscr{D}}$ if we *define* $L(0)$ *to be* ∞, which we *do, for the rest of this proof.* Therefore, if a sequence of polynomials P_n fulfilling the above condition *does not exist*, we can, by the *Hahn–Banach theorem*, find a finite complex-valued measure μ on $\bar{\mathscr{D}}$ with

$$(\S) \qquad \iint_{\mathscr{D}} \frac{z^n}{L(\Im z)} \, d\mu(z) \;=\; 0 \quad \text{for} \quad n = 0, 1, 2, \ldots,$$

whilst

$$\iint_{\mathscr{D}} \frac{\varphi(z)}{L(\Im z)} \, d\mu(z) \;\neq\; 0.$$

The proof will be completed by showing that *in fact* (§) implies

$$\iint_{\mathscr{D}} \frac{\varphi(z)}{L(\Im z)} \, d\mu(z) \;=\; 0.$$

Given any measure μ satisfying (§), write

$$d\nu(z) \;=\; \frac{1}{L(\Im z)} \, d\mu(z).$$

The measure ν has *very little mass near the real axis, and none at all on it*. For each complex λ, the power series for $e^{i\lambda z}$ converges uniformly for $z \in \bar{\mathscr{D}}$, so from (§) we get

$$\iint_{\mathscr{D}} e^{i\lambda z} \, d\nu(z) \;=\; 0.$$

Write now

$$\mathscr{D}_+ \;=\; \bar{\mathscr{D}} \cap \{\Im z > 0\} \quad (\textit{sic}!)$$

and

$$\mathscr{D}_- \;=\; \bar{\mathscr{D}} \cap \{\Im z < 0\} \quad (\textit{sic}!).$$

Then put

$$\Phi_+(\lambda) \;=\; \iint_{\mathscr{D}_+} e^{i\lambda z} \, d\nu(z),$$

$$\Phi_-(\lambda) \;=\; \iint_{\mathscr{D}_-} e^{i\lambda z} \, d\nu(z);$$

since ν has no mass on $\mathbb{R}$, the previous relation becomes

$$(\dagger\dagger) \qquad \Phi_+(\lambda) \;+\; \Phi_-(\lambda) \;\equiv\; 0.$$

We are going to show that in fact each of the left-hand terms in (††) vanishes identically.

Consider $\Phi_+(\lambda)$. Since $\mathscr{D}_+$ is a *bounded* domain; $\Phi_+(\lambda)$ is *entire and of exponential type*. It is also *bounded* on the *real axis*. Indeed, for $\lambda > 0$,

$$|\Phi_+(\lambda)| \;\leqslant\; \iint_{\mathscr{D}_+} e^{-\lambda\Im z}|d\nu(z)| \;\leqslant\; \iint_{\mathscr{D}_+} |d\nu(z)|,$$

and for $\lambda < 0$ we can, *on account of* (††), use the relation $\Phi_+(\lambda) = -\Phi_-(\lambda)$ and make a similar estimate involving $\mathscr{D}_-$. These properties of $\Phi_+(\lambda)$ and the theorem of Chapter III, §G.2, imply that $\Phi_+(\lambda) \equiv 0$ provided that

$$(\ast) \qquad \int_{-\infty}^{\infty} \frac{\log|\Phi_+(\lambda)|}{1+\lambda^2}\,d\lambda \;=\; -\infty.$$

We proceed to establish this relation.

Write

$$H(y) \;=\; \begin{cases} \log L(y), & 0 < y \leqslant b, \\ \log L(b), & y > b. \end{cases}$$

Then $H(y)$ is *decreasing* for $y > 0$ by hypothesis. For $\lambda > 0$,

$$|\Phi_+(\lambda)| \;=\; \left| \iint_{\mathscr{D}_+} \frac{e^{i\lambda z}}{L(\Im z)}\,d\mu(z) \right| \;\leqslant\; \iint_{\mathscr{D}_+} e^{-H(\Im z) - \lambda\Im z}|d\mu(z)|.$$

If, as in article 5, we put

$$M(\lambda) \;=\; \inf_{y>0}\,(H(y) + y\lambda),$$

we see by the previous relation that

$$(\S\S) \qquad |\Phi_+(\lambda)| \;\leqslant\; \text{const.}e^{-M(\lambda)}, \qquad \lambda > 0.$$

Since $H(y)$ is decreasing for $y > 0$ and $\geqslant 1$ there, and

$$\int_0^b \log H(y)\,dy \;=\; \int_0^b \log\log L(y)\,dy \;=\; \infty$$

by hypothesis, we have

$$\int_1^{\infty} \frac{M(\lambda)}{\lambda^2}\,d\lambda \;=\; \infty$$

according to the *last* theorem of article 2. This, together with (§§), gives us ($\ast$), and hence $\Phi_+(\lambda) \equiv 0$.

Referring again to (††), we see that also $\Phi_-(\lambda) \equiv 0$. Specializing to $\lambda = 0$ (!), we obtain the two relations

$$\iint_{\mathscr{D}_+} \frac{1}{L(\Im z)}\,d\mu(z) = 0, \qquad \iint_{\mathscr{D}_-} \frac{1}{L(\Im z)}\,d\mu(z) \;=\; 0,$$

from which, by subtraction,

$$\iint_{\mathscr{D}} \frac{\varphi(z)}{L(\Im z)} \, d\mu(z) \;=\; 0,$$

what we had set out to show. The proof of our theorem is thus finished, and we are done.

And thus ends this long (aye, too long!) seventh chapter of the present book.

VIII

Persistence of the form $\mathrm{d}x/(1+x^2)$

Up to now, integrals like

$$\int_{-\infty}^{\infty} \frac{\log|F(x)|}{1+x^2}\,\mathrm{d}x$$

have appeared so frequently in this book mainly on account of the specific form of the Poisson kernel for a half plane. If $\omega(S,z)$ denotes the harmonic measure (at z) of $S \subseteq \mathbb{R}$ for the half plane $\{\Im z > 0\}$, we simply have

$$\omega(S,\mathrm{i}) \;=\; \frac{1}{\pi}\int_S \frac{\mathrm{d}t}{1+t^2}.$$

Suppose now that we *remove* certain finite open intervals – perhaps infinitely many – from $\mathbb{R}$, leaving a certain residual set E, and that E *looks* something like $\mathbb{R}$ when seen from far enough away. E should, in particular, have infinite extent on both sides of the origin and not be too sparse. Denote by $\mathscr{D}$ the (multiply connected – perhaps even infinitely connected) domain $\mathbb{C} \sim E$, and by $\omega_{\mathscr{D}}(\ ,z)$ the harmonic measure for $\mathscr{D}$.

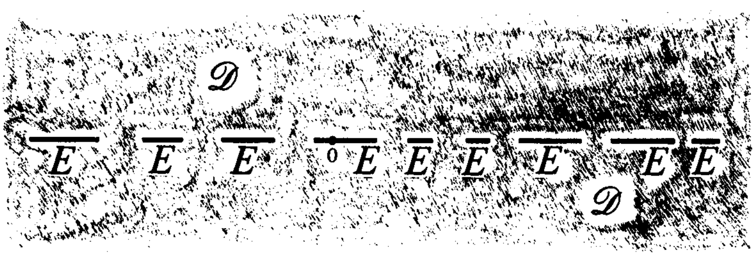

Figure 107

It is a remarkable fact that a formula like the above one for $\omega(S,\mathrm{i})$ *subsists*, to a certain extent, for $\omega_{\mathscr{D}}(\ ,\mathrm{i})$, provided that the degradation suffered by $\mathbb{R}$

in its reduction to E is not too great. We have, for instance,

$$\omega_{\mathscr{D}}(J \cap E, \mathrm{i}) \;\leqslant\; C_E(\alpha) \int_{J \cap E} \frac{\mathrm{d}t}{1+t^2}$$

for *intervals* J with $|J \cap E| \geqslant \alpha > 0$, where $C_E(\alpha)$ depends on α as well as on the set E. In other words, $\mathrm{d}\omega_{\mathscr{D}}(t, \mathrm{i})$ *still acts* (crudely) *like the restriction of* $\mathrm{d}x/(1+x^2)$ *to* E. It is this *tendency of the form* $\mathrm{d}x/(1+x^2)$ *to persist* when we reduce $\mathbb{R}$ to certain smaller sets E (and enlarge the upper half plane to $\mathscr{D} = \mathbb{C} \sim E$) that constitutes the theme of the present chapter.

The persistence is well illustrated in the situation of *weighted approximation* (whether by polynomials or by functions of exponential type) on the sets E. If a function $W(x) \geqslant 1$ is given on E, with $W(x) \to \infty$ as $x \to \pm\infty$ in E (for weighted *polynomial* approximation on E this must of course take place faster than any power of x), we can look at *approximation on* E (by polynomials or by functions of exponential type bounded on $\mathbb{R}$) *using the weight* W. It turns out that *precise formal analogues* of many of the results established for weighted approximation *on* $\mathbb{R}$ in §§A, B and E of Chapter VI *are valid here*; the only change consists in the *replacement* of the integrals of the form

$$\int_{-\infty}^{\infty} \frac{\log M(t)}{1+t^2}\,\mathrm{d}t$$

occurring in Chapter VI by the corresponding expressions

$$\int_E \frac{\log M(t)}{1+t^2}\,\mathrm{d}t.$$

The integrand, involving $\mathrm{d}t/(1+t^2)$, remains unchanged.

This chapter has three sections. The *first* is mainly devoted to the case where E has *positive lower uniform density* on $\mathbb{R}$ – a typical example is furnished by the set

$$E \;=\; \bigcup_{n=-\infty}^{\infty} [n-\rho,\; n+\rho]$$

where $0 < \rho < \frac{1}{2}$.

In §B, we study the limiting case of the example just mentioned which arises when $\rho = 0$, i.e., when $E = \mathbb{Z}$. There is of course no longer any harmonic measure for $\mathscr{D} = \mathbb{C} \sim \mathbb{Z}$. It is therefore remarkable that *something nevertheless remains true of the results established in* §A. If $P(z)$ is a *polynomial* such that

$$\sum_{n=-\infty}^{\infty} \frac{1}{1+n^2} \log^{+}|P(n)| \;\leqslant\; \eta$$

with $\eta > 0$ *sufficiently small* (this restriction turns out to be crucial!), we *still* have, for $z \in \mathbb{C}$,

$$|P(z)| \leqslant K(z, \eta),$$

where $K(z, \eta)$ depends *only* on z and η, and not on P. The proof of this fact is very long, and hard to grasp as a whole. It uses specific properties of polynomials. Since $\mathscr{D}$ has no harmonic measure, a corresponding statement with $\log^+|P(z)|$ replaced by a *general* continuous subharmonic function of at most logarithmic growth is *false.*

We return in §C to the study of harmonic estimation in $\mathscr{D}$ when its boundary, E, *does not* reduce to a discrete set. *Here*, we assume that E contains all $x \in \mathbb{R}$ of sufficiently large absolute value, that situation being general enough for applications. The purpose of §C is to connect up the behavior of a *Phragmén–Lindelöf function* for $\mathscr{D}$ (i.e., one harmonic in $\mathscr{D}$ and acting like $|\Im z|$ there, with boundary value *zero* on E) to that of *harmonic measure* for $\mathscr{D}$. There is a quantitative relation between the former and the latter. Harmonic measure still acts (very crudely!) like the restriction of $\mathrm{d}t/(1 + t^2)$ to E. This § is independent of §B to a large extent, but does use a fair amount of material from §A. Results obtained in it are needed for Chapter XI.

A. The set E has positive lower uniform density

During most of this §, we consider sets E of the special form

$$\bigcup_{n=-\infty}^{\infty} [a_n - \delta_n,\ a_n + \delta_n],$$

the intervals $[a_n - \delta_n,\ a_n + \delta_n]$ being *disjoint*. *We will assume that there are four constants, A, B, δ and Δ, with*

$$\boxed{0 < A < a_n - a_{n-1} < B, \qquad 0 < \delta < \delta_n < \Delta,}$$

for all n.

The following *notation* will be used throughout:

$$E_n = [a_n - \delta_n,\ a_n + \delta_n],$$
$$\mathcal{O}_n = (a_n + \delta_n,\ a_{n+1} - \delta_{n+1}),$$
$$\mathscr{D} = \mathbb{C} \sim E.$$

Here is a picture of the setup we are studying:

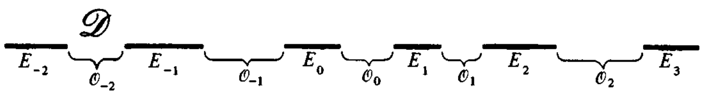

Figure 108

The above boxed conditions on the a_n and δ_n *clearly* imply the existence of two constants C_1 and $C_2 > 0$ (*each* depending on the four numbers A, B, δ and Δ) such that:

(i) if $k \neq l$ and $x \in \mathcal{O}_k$, $x' \in \mathcal{O}_l$, we have $C_1|k-l| \leqslant |x-x'| \leqslant C_2|k-l|$;
(ii) if $k \neq l-1$, l, or $l+1$ and $x \in E_k$, $x' \in E_l$, we have $C_1|k-l| \leqslant |x-x'| \leqslant C_2|k-l|$.

The restriction on the pair (k, l) in (ii) is due to the fact that the *lengths* of the $\mathcal{O}_k$ are *not assumed to be bounded away from zero*; their lengths are only bounded above. It is *the lengths of the E_k* that are bounded above *and* away from zero.

Heavy use will be made of properties (i) and (ii) during the following development. Clearly, if E is any set for which the above boxed condition holds (with given A, B, δ and Δ), so is each of its translates $E + h$ (with the same constants A, B, δ and Δ). The properties (i) and (ii) are thus valid for each of those translates, with the same constants C_1 and C_2 as for E. For this reason there is no real loss of generality in supposing that $0 \in \mathcal{O}_0$, and we will frequently do so when that is convenient.

1. Harmonic measure for $\mathscr{D}$

The Dirichlet problem can be solved for the kind of domains $\mathscr{D}$ we are considering and (*at least*, certainly!) for continuous boundary data on E given by functions tending to 0 as $x \to \pm\infty$ in E. Let us, without going into too much detail, indicate how this fact can be verified.

Take large values of R, and put

$$\mathscr{D}_R = \mathscr{D} \cap \sim\{(-\infty, -R] \cup [R, \infty)\}:$$

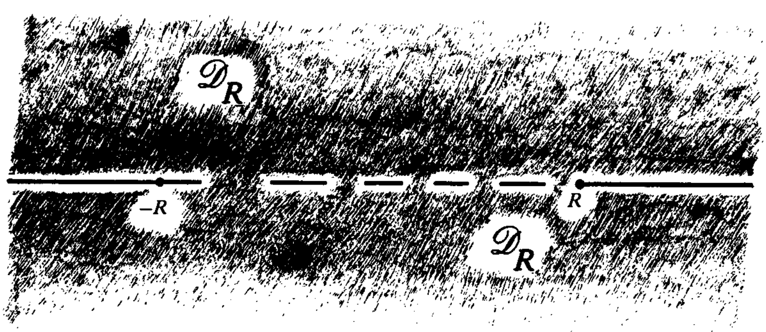

Figure 109

Each of the regions $\mathscr{D}_R$ is *finitely connected*, and the Dirichlet problem can be solved in it. This is *known*; it is true because the straight segment boundary components ('slits') of $\mathscr{D}_R$ are practically as nice as Jordan curve boundary components. One can indeed map $\mathscr{D}_R$ conformally onto a region *bounded* by Jordan curves by using a succession of Joukowski transformations, one for each slit (including the infinite one through ∞):

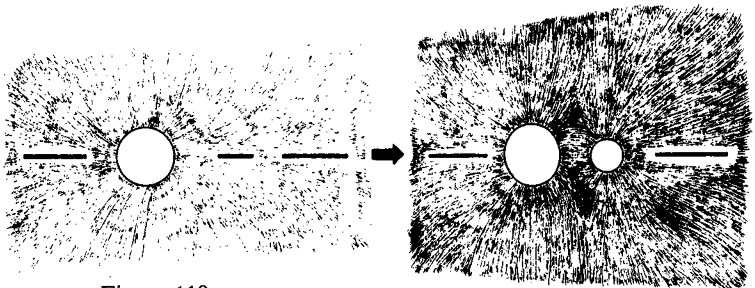

Figure 110

The inverse to the conformal mapping thus obtained *does* take the Jordan curve boundary components back *continuously* onto the original slits, so the Dirichlet problem *can* be solved for $\mathscr{D}_R$ if it can be solved for regions bounded by a finite number of Jordan curves. (This same idea will be used again at the end of article 2, in proving the symmetry of Green's function.)

Once we are sure that the Dirichlet problem can be solved in each $\mathscr{D}_R$ we can, by examining how certain solutions behave for $R \to \infty$, convince ourselves that the Dirichlet problem for $\mathscr{D}$ is also solvable, at least for boundary data of the abovementioned kind. Details of this examination are left to the reader.

Since $\mathscr{D}$ is regular for the Dirichlet problem, harmonic measure is available for it. We know from the rudiments of conformal mapping theory *that a slit should be considered as having two sides, or edges.* Given (say) an interval $J \subseteq$ one of the boundary components E_n of $\mathscr{D}$, we should *distinguish* between *two* intervals coinciding with J: J_+ (lying on the *upper* side of E_n), and J_- (lying on the *lower* side of E_n):

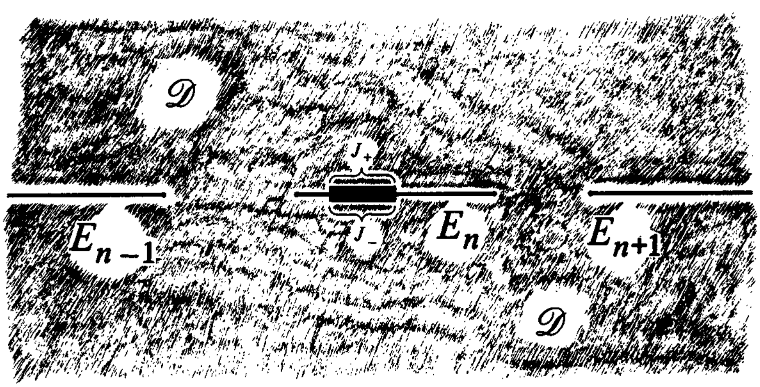

Figure 111

It makes sense, then, to talk about the *two* harmonic measures:

$$\omega_{\mathscr{D}}(J_+,\ z)$$

(which tends to *zero* when z tends to an interior point of J_-), and

$$\omega_{\mathscr{D}}(J_-,\ z).$$

In most of our work, however, *separation of J into J_+ and J_- will serve no purpose.* It will in fact be sufficient to work with the *sum*

$$\omega_{\mathscr{D}}(J_+,\ z) + \omega_{\mathscr{D}}(J_-,\ z).$$

▶ *This harmonic function tends to* 1 *when z tends from either side of the real axis to an interior point of J, and it is what we take as the harmonic measure*

$$\omega_{\mathscr{D}}(J, z)$$

of J. The harmonic measure $\omega_{\mathscr{D}}(S, z)$ *of any $S \subseteq E$ is defined in the same way.*

Consider now any of the boundary components E_k of $E = \partial\mathscr{D}$, and write

$$\omega_k(z) = \omega_{\mathscr{D}}(E_k, z)$$

for the harmonic measure of E_k, as seen from $z \in \mathscr{D}$. We are going to show that there is a constant C, depending on the four numbers A, B, δ and Δ associated with E, such that

$$\omega_k(x) \leqslant \frac{C}{(l-k)^2+1} \quad \text{for } x \in \mathcal{O}_l.$$

($\mathcal{O}_l$, recall, is the part of $\mathscr{D} \cap \mathbb{R}$ lying between E_l and E_{l+1}.) The proof is due to Carleson; one or two of its ideas go back to earlier work. We need two lemmas, the first of which could almost be given as an exercise.

Lemma. *Denote by $\Omega_k(\quad, z)$ harmonic measure for the domain*

$$\mathscr{D}_k = \{\Im z > 0\} \cup \mathcal{O}_k \cup \{\Im z < 0\}.$$

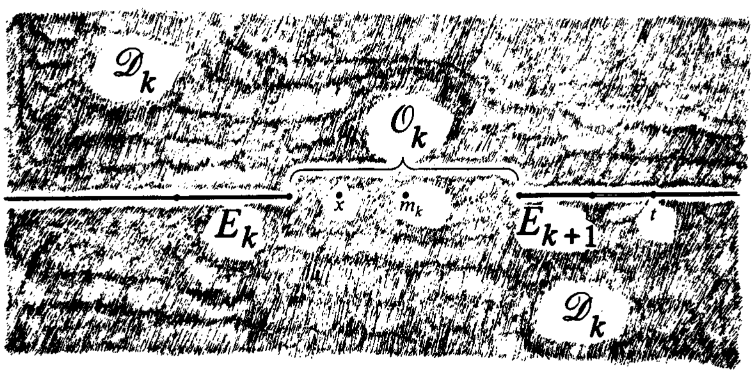

Figure 112

There is a constant K', depending only on the numbers B and δ associated with the set E, such that for $x \in \mathcal{O}_k$ and $t \in \partial\mathscr{D}_k$ lying outside both *of the segments E_k and E_{k+1} we have*

$$d\Omega_k(t,x) \leqslant \frac{K'}{(x-t)^2}\,dt.$$

Proof. By conformal mapping of $\mathscr{D}_k$ onto the unit disk. Calling m_k the midpoint of $\mathcal{O}_k$, we apply to $z \in \mathscr{D}_k$ the chain of mappings

$$z \longrightarrow \zeta = 2\frac{z-m_k}{|\mathcal{O}_k|} \longrightarrow w = \frac{1}{\zeta} - \sqrt{\left(\frac{1}{\zeta^2}-1\right)}.$$

We write $w = \varphi(z)$ and, if t is on $\partial\mathscr{D}_k$, denote $\varphi(t)$ by ω. (In the latter case we must of course distinguish between points t lying on the *upper side* of $\partial\mathscr{D}_k$ and those on its *lower side* – see the preceding remarks. On this distinction depends the choice of the branch of $\sqrt{\ }$ to be used in computing $\omega = \varphi(t)$.)

For t on $\partial\mathscr{D}_k$ outside both E_k and E_{k+1} we have

$$\omega = \varphi(t) = \frac{1}{\tau} - \sqrt{\left(\frac{1}{\tau^2}-1\right)},$$

where $\tau = 2(t-m_k)/|\mathcal{O}_k|$ satisfies the inequality

$$|\tau| - 1 \geqslant 2\frac{\min(|E_k|,|E_{k+1}|)}{|\mathcal{O}_k|} > \frac{4\delta}{B},$$

in view of the relations $|E_l| = 2\delta_l > 2\delta$, $|\mathcal{O}_k| < a_{k+1} - a_k < B$. In terms of τ,

$$d\omega = -\frac{d\tau}{\tau^2}\left(1 \pm \frac{i}{\sqrt{(\tau^2-1)}}\right),$$

i.e.,

$$|d\omega| = \left(\frac{\tau^2}{\tau^2-1}\right)^{\frac{1}{2}}\frac{|d\tau|}{\tau^2}.$$

For t outside both E_k and E_{k+1}, the expression on the right is

$$< \left(\frac{1+4\delta/B}{4\delta/B}\right)^{\frac{1}{2}}\frac{|d\tau|}{\tau^2} = \left(\frac{B}{4\delta}+1\right)^{\frac{1}{2}}\frac{|d\tau|}{\tau^2}$$

by the inequality for $|\tau| - 1$.

Let $x \in \mathcal{O}_k$. Then, remembering that $d\Omega_k(t,x)$ is the harmonic measure of *two infinitesimal intervals* $[t,\ t+dt]$ lying on $\partial\mathscr{D}_k$ – one on *the upper edge* and one on *the lower*, we see that

$$d\Omega_k(t,x) = \frac{1}{\pi}\frac{1-|\varphi(x)|^2}{|\varphi(x)-\omega|^2}|d\omega|,$$

with $|d\omega|$ being given by the above formula. Since, for $t \notin E_k \cup E_{k+1}$,

$|\tau| - 1 > 4\delta/B$, the image, ω, of t on the unit circumference must lie *outside* two arcs thereof entered at 1 and at -1, and having *lengths* that depend on the ratio $4\delta/B$. We do not need to know the exact form of this dependence.

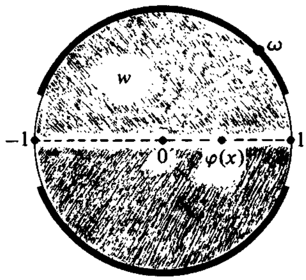

Figure 113

Hence, since $-1 < \varphi(x) < 1$, it is clear from the picture that $|\varphi(x) - \omega|$ is $\geqslant$ a positive quantity depending on the lengths of the excluded arcs about 1 and -1, and thence on the ratio $4\delta/B$. The factor $(1 - |\varphi(x)|^2)/|\varphi(x) - \omega|^2$ in the above formula for $d\Omega_k(t, x)$ is thus bounded above by a number depending on $4\delta/B$ when $t \notin E_k \cup E_{k+1}$. Substituting into that formula the inequality for $|d\omega|$ already found, we get

$$d\Omega_k(t, x) \leqslant C\frac{|d\tau|}{\tau^2}$$

for $t \notin E_k \cup E_{k+1}$, with a constant C depending on $4\delta/B$. In terms of $t = m_k + \frac{1}{2}|\mathscr{O}_k|\tau$, the right side is

$$\leqslant \frac{C|\mathscr{O}_k|}{2}\frac{dt}{(t - m_k)^2} \leqslant \frac{CB}{2}\frac{dt}{(t - m_k)^2}.$$

Since $x \in \mathscr{O}_k$ and $t \notin \mathscr{O}_k$, $\quad |t - x| \leqslant 2|t - m_k|$:

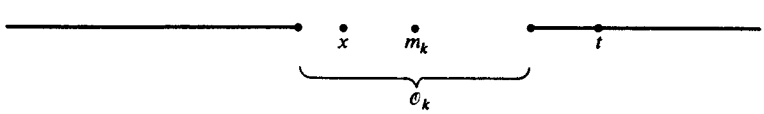

Figure 114

So finally,

$$d\Omega_k(t, x) \leqslant 2BC\frac{dt}{(t - x)^2},$$

Q.E.D.

Computational lemma (Carleson, 1982; see also Benedicks' 1980 *Arkiv* paper). *Let $A_{k,l} \geqslant 0$ for $k, l \in \mathbb{Z}$. Suppose there are constants K and λ, with $0 < \lambda < 1$, such that*

$$(*) \qquad A_{k,l} \leqslant \frac{K}{(l-k)^2+1},$$

and

$$(\ast\ast) \qquad \sum_{l=-\infty}^{\infty} A_{k,l} \leqslant \lambda \qquad \textit{for all} \quad k.$$

Then there is a number $\eta > 0$ depending on K and λ such that, for any sequence $\{y_l\}$ with $0 \leqslant y_l \leqslant \eta$ and $0 \leqslant y_l \leqslant 1/(l^2+1)$, we have

$$\sum_{l=-\infty}^{\infty} A_{k,l} y_l \leqslant \lambda \sup_l y_l \qquad \textit{for all} \quad k,$$

and

$$\sum_{l=-\infty}^{\infty} A_{k,l} y_l \leqslant \frac{1+\lambda}{2} \cdot \frac{1}{k^2+1}.$$

Remark. The *first* of the asserted inequalities is manifest; it is the *second* that is non-trivial. If the constant K is small enough, the second inequality is also clear; it is when K is *not small* that the latter is difficult to verify.

Proof of lemma. As we have just remarked, the *first* inequality is obvious (by ($\ast\ast$)); let us therefore see to the *second*, endeavoring first of all to prove it for *large* values of $|k|$, say $|k| \geqslant$ some k_0.

Assume, wlog, that $k > 0$, and take some *small* number μ, $0 < \mu < \frac{1}{2}$, about whose precise value we will decide later on. Write the sum

$$\sum_{l=-\infty}^{\infty} A_{k,l} y_l$$

as

$$\sum_{|l| < \mu k} + \sum_{\substack{|l| \geqslant \mu k \\ |l-k| \geqslant \mu k}} + \sum_{|l-k| < \mu k} = \mathrm{I} + \mathrm{II} + \mathrm{III},$$

say. Use of $(*)$ together with the inequality $0 \leqslant y_l \leqslant \eta$ (where η is *as yet* unspecified) gives us first of all

$$\mathrm{I} \leqslant \frac{K}{(1-\mu)^2 k^2 + 1} \cdot 2k\mu \cdot \eta$$

We get III out of the way by combining ($\ast\ast$) and the inequality

$0 \leqslant y_l \leqslant 1/(l^2+1)$, with the result that

$$\mathrm{III} \;\leqslant\; \frac{\lambda}{(1-\mu)^2k^2+1}.$$

It is the middle sum, II, that gives us trouble. We break II up further as

$$\sum_{l \leqslant -\mu k} + \sum_{\mu k \leqslant l < k/2} + \sum_{k/2 \leqslant l \leqslant (1-\mu)k} + \sum_{l \geqslant (1+\mu)k}.$$

The *first* of these sums is

$$\leqslant \sum_{l \leqslant -\mu k} \frac{1}{l^2+1} \cdot \frac{K}{(k-l)^2+1} \;\leqslant\; \frac{K}{k^2+1} \sum_{m \geqslant \mu k} \frac{1}{m^2} \;\leqslant\; \frac{2K}{\mu k(k^2+1)}.$$

The *second* is similarly

$$\leqslant \sum_{\mu k \leqslant l < k/2} \frac{1}{l^2+1} \cdot \frac{K}{(k-l)^2+1} \;\leqslant\; \frac{K}{(k/2)^2+1} \sum_{m \geqslant \mu k} \frac{1}{m^2} \;\leqslant\; \frac{8K}{\mu k(k^2+4)}$$

The *third* sum is

$$\leqslant \sum_{l \leqslant (1-\mu)k} \frac{1}{(k/2)^2+1} \cdot \frac{K}{(k-l)^2+1} \;\leqslant\; \frac{8K}{\mu k(k^2+4)},$$

and the *fourth*

$$\leqslant \sum_{l \geqslant (1+\mu)k} \frac{1}{l^2+1} \cdot \frac{K}{(l-k)^2+1} \;\leqslant\; \frac{2K}{\mu k(k^2+1)}.$$

All told, then,

$$\mathrm{II} \;\leqslant\; \frac{20K}{\mu k(k^2+1)}.$$

Adding this last estimate to those already obtained for I and III, we get finally

$$\sum_{l=-\infty}^{\infty} A_{k,l}y_l \;\leqslant\; \frac{2k\mu\eta K}{(1-\mu)^2(k^2+1)} + \frac{20K}{\mu k(k^2+1)} + \frac{\lambda}{(1-\mu)^2(k^2+1)}.$$

The idea now is to *first* put η equal to a *very small quantity* η_0, and *then*, assuming k is *large*, put $\mu = 1/\eta_0^{1/2}k$; this will *also be small* for large enough k. For such large k, the previous inequality will reduce to

$$\sum_{l=-\infty}^{\infty} A_{k,l}y_l \;\leqslant\; \frac{2\eta_0^{1/2}K + 20\eta_0^{1/2}K + \lambda}{(1-\mu)^2(k^2+1)}.$$

Choosing first $\eta_0^{1/2}$ *small* enough and then taking k_0 so *large* that $1/\eta_0^{1/2}k_0$

is *also* small, we will make the right-hand side of this inequality

$$\leqslant \frac{1+\lambda}{2}\cdot\frac{1}{k^2+1}$$

for $|k| \geqslant k_0$ by putting $\mu = 1/\eta_0^{1/2}|k|$. When $\eta < \eta_0$ and $0 \leqslant y_l \leqslant \eta$ we then have, *a fortiori*,

$$\sum_{l=-\infty}^{\infty} A_{k,l} y_l \leqslant \frac{1+\lambda}{2}\cdot\frac{1}{k^2+1} \qquad \text{for} \quad |k| \geqslant k_0.$$

With such y_l, however, the sum on the left is also $\leqslant \lambda\eta$. So, taking finally

$$\eta = \min\left(\eta_0, \frac{1}{k_0^2+1}\right)$$

makes the left side $\leqslant ((1+\lambda)/2)(1/(k^2+1))$ for $|k| < k_0$ *as well*, i.e.,

$$\sum_{l=-\infty}^{\infty} A_{k,l} y_l \leqslant \frac{1+\lambda}{2}\cdot\frac{1}{k^2+1}$$

for all k, provided that $0 \leqslant y_l \leqslant \eta$ and $0 \leqslant y_l \leqslant 1/(l^2+1)$.

The lemma is proved.

Theorem (Carleson, 1982; see also Benedicks' 1980 *Arkiv* paper). *In the domain $\mathscr{D}$, the harmonic measure $\omega_0(z)$ of the component E_0 of $\partial\mathscr{D}$ satisfies*

$$\omega_0(x) \leqslant \frac{C}{k^2+1} \qquad \textit{for } x \in \mathscr{O}_k,$$

with a constant C depending only on the four numbers A, B, δ and Δ associated with $E = \partial\mathscr{D}$.

Proof (Carleson). Call u_k the *maximum value* of $\omega_0(x)$ on $\mathscr{O}_k$; we are to show that

$$u_k \leqslant \frac{C}{k^2+1}.$$

For $k = -1$ and $k = 0$ this is certainly true if we put $C = 2$; we may therefore restrict our discussion to the *values of k different from* -1 and 0.

As in the first of the above lemmas, denote by $\mathscr{D}_k$ the domain

$$\{\Im z > 0\} \cup \mathscr{O}_k \cup \{\Im z < 0\}$$

and by $\Omega_k(\ ,z)$ the harmonic measure for $\mathscr{D}_k$. $\mathscr{D}_k$ is of course *contained* in $\mathscr{D}$:

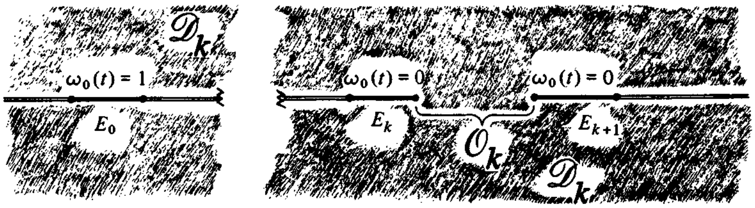

Figure 115

$\omega_0(z)$ is thus harmonic in $\mathscr{D}_k$; since it is clearly bounded there and continuous up to $\partial\mathscr{D}_k$, we may recover it from its values on $\partial\mathscr{D}_k$ by the Poisson formula

$$\omega_0(z) = \int_{\partial\mathscr{D}_k} \omega_0(t) \mathrm{d}\Omega_k(t, z), \qquad z \in \mathscr{D}_k.$$

Since we are assuming that $k \neq -1, 0$, we have (by *definition* of harmonic measure!) $\omega_0(t) = 0$ for $t \in E_k \cup E_{k+1}$. In fact, $\omega_0(t)$ is identically zero on *all* the E_l save E_0, where $\omega_0(t) = 1$. The above formula hence becomes

$$\omega_0(z) = \int_{E_0} \mathrm{d}\Omega_k(t, z) + \sum_{l \neq k} \int_{\mathcal{O}_l} \omega_0(t) \mathrm{d}\Omega_k(t, z),$$

$\mathbb{R}$ being the disjoint union of the intervals $\mathcal{O}_l$ and E_l.

Let x_l be the point in $\mathcal{O}_l$ where $\omega_0(x)$ *assumes its maximum* u_l *therein*, and write

$$A_{k,l} = \int_{\mathcal{O}_l} \mathrm{d}\Omega_k(t, x_k).$$

Then, by the previous relation,

$$u_k \leqslant \Omega_k(E_0, x_k) + \sum_{l \neq k} A_{k,l} u_l.$$

Here, the integrals

$$\int_{E_0} \mathrm{d}\Omega_k(t, x_k) = \Omega_k(E_0, x_k)$$

and

$$\int_{\mathcal{O}_l} \mathrm{d}\Omega_k(t, x_k) = A_{k,l}, \qquad l \neq k,$$

are taken over sets *disjoint from* E_k and E_{k+1}, whereas $x_k \in \mathcal{O}_k$. We may therefore apply the *first* lemma to estimate $\mathrm{d}\Omega_k(t, x_k)$ in these integrals, getting

$$\Omega_k(E_0, x_k) \leqslant K' \int_{E_0} \frac{\mathrm{d}t}{(t - x_k)^2}$$

and

$$A_{k,l} \leqslant K' \int_{\mathcal{O}_l} \frac{dt}{(t-x_k)^2}, \qquad l \neq k,$$

where K' is a constant depending on the numbers B and δ associated with E. By properties (i) and (ii), given at the beginning of this §, we have

$$(t - x_k)^2 \geqslant C_1^2 k^2 \quad \text{for} \quad t \in E_0$$

and

$$(t - x_k)^2 \geqslant C_1^2 (k-l)^2 \quad \text{for} \quad t \in \mathcal{O}_l.$$

So, since $|E_0| = 2\delta_0 < 2\Delta$ and $|\mathcal{O}_l| < B$, the preceding relations become

$$\Omega_k(E_0, x_k) \leqslant \frac{K}{k^2+1}$$

and

$$(*) \qquad A_{k,l} \leqslant \frac{K}{(k-l)^2+1}, \qquad l \neq k;$$

here, K is a constant depending on the four numbers A, B, δ and Δ.

The numbers $A_{k,l}$ also satisfy the inequality

$$(\ast\ast) \qquad \sum_{l \neq k} A_{k,l} \leqslant \lambda < 1$$

with λ depending *only on the ratio* δ/B. Indeed,

$$\sum_{l \neq k} A_{k,l} = \sum_{l \neq k} \Omega_k(\mathcal{O}_l, x_k)$$

is $\leqslant \Omega_k(\partial\mathcal{D}_k \sim E_k \sim E_{k+1}, x_k)$. Since $|\mathcal{O}_k| < B$ and $|E_k| > 2\delta$, $|E_{k+1}| > 2\delta$, a simple change of variable shows that the latter quantity is *less* than the harmonic measure of the set $1 + 4\delta/B \leqslant |t| < \infty$ on the boundary of the domain $\mathbb{C} \sim (-\infty, -1] \sim [1, \infty)$, seen from some point in $(-1, 1)$:

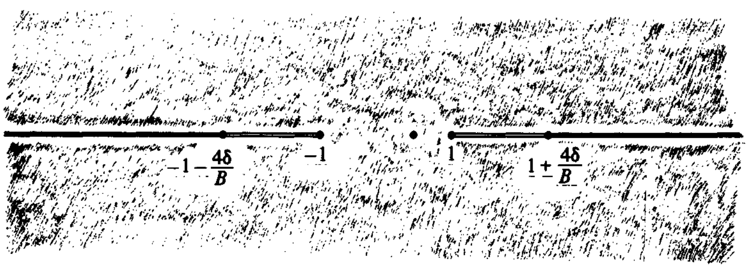

Figure 116

And *that* harmonic measure is clearly *at most* equal to some number $\lambda < 1$ depending on $4\delta/B$.

Let us return to the inequality

$$u_k \;\leqslant\; \Omega_k(E_0, x_k) \;+\; \sum_{l \neq k} A_{k,l} u_l$$

established above. By plugging into it the estimates just found, we get

$$(\dagger) \qquad u_k \;-\; \sum_{l \neq k} A_{k,l} u_l \;\leqslant\; \frac{K}{k^2 + 1},$$

where the $A_{k,l}$ are $\geqslant 0$, *and satisfy* $(*)$ *and* $(\ast\!\!\ast)$. This has been proved for $k \neq -1$ and 0, but it *also holds* (*a fortiori!*) for *those* values of k, provided that we take $K \geqslant 2$. *Then our* (unknown) *maxima* $u_k \geqslant 0$ *will satisfy* $(\dagger)$ *for all* k; *this we henceforth assume.*

The idea now is to invert the relations $(\dagger)$ in order to obtain bounds on the u_k. *It is convenient to define* $A_{k,l}$ *for* $l = k$ *by putting* $A_{k,k} = 0$. Then, calling

$$(\dagger\dagger) \qquad v_k \;=\; u_k \;-\; \sum_l A_{k,l} u_l,$$

we can recover the u_k *from the* v_k by virtue of $(\ast\!\!\ast)$. Write $A^{(1)}_{k,l} = A_{k,l}$; then put

$$A^{(2)}_{k,l} = \sum_{j=-\infty}^{\infty} A_{k,j} A_{j,l},$$

and in general

$$A^{(n+1)}_{k,l} = \sum_{j=-\infty}^{\infty} A_{k,j} A^{(n)}_{j,l}.$$

The numbers $A^{(n)}_{k,l}$ are $\geqslant 0$ (since the $A_{k,l}$ are), and from $(\ast\!\!\ast)$, we have

$$(\S) \qquad \sum_{l=-\infty}^{\infty} A^{(n)}_{k,l} \;\leqslant\; \lambda^n.$$

This makes it possible for us to invert $(\dagger\dagger)$, getting

$$u_k \;=\; v_k \;+\; \sum_l A^{(1)}_{k,l} v_l \;+\; \sum_l A^{(2)}_{k,l} v_l \;+\; \cdots \;+\; \sum_l A^{(n)}_{k,l} v_l \;+\; \cdots,$$

the Neumann series on the right being absolutely convergent. Since the $A^{(n)}_{k,l}$ are $\geqslant 0$ and

$$v_l \;\leqslant\; \frac{K}{l^2 + 1}$$

by (†) and (††), the previous relation gives

$$u_k \leqslant K\left\{\frac{1}{k^2+1} + \sum_{n=1}^{\infty}\sum_{l=-\infty}^{\infty}\frac{A_{k,l}^{(n)}}{l^2+1}\right\}.$$

We proceed to examine the right-hand side of this inequality.

We have

$$\sum_{l=-\infty}^{\infty}\frac{1}{(k-l)^2+1}\cdot\frac{1}{l^2+1} \leqslant \frac{\text{const.}}{k^2+1}$$

(look at the reproduction property of the Poisson kernel $y/((x-t)^2+y^2)$ on which the *hall of mirrors* argument used in Chapter 6 is based!). Hence, by (∗), there is a constant L with

$$0 \leqslant A_{k,l}^{(n)} \leqslant \frac{L^n}{(k-l)^2+1},$$

and the summand

$$\sum_l A_{k,l}^{(n)}\cdot\frac{1}{l^2+1}$$

on the right side of the above estimate for u_k is

$$\leqslant \frac{\text{const.}\,L^n}{k^2+1}.$$

We have, however, to add up infinitely many of these summands. *It is here that we must resort to the computational lemma.*

Call

$$v_k^{(n)} = \sum_{l=-\infty}^{\infty}\frac{A_{k,l}^{(n)}}{l^2+1};$$

we certainly have $v_k^{(n)} \geqslant 0$. According to the computational lemma, there is an $\eta > 0$ depending on λ and the K in (∗) such that

$$\sum_{l=-\infty}^{\infty} A_{k,l}y_l \leqslant \frac{1+\lambda}{2}\cdot\frac{1}{k^2+1}$$

if $0 \leqslant y_l \leqslant \eta$ and $y_l \leqslant 1/(l^2+1)$. *Fix* such an η. By (§) we can certainly find an m such that $0 \leqslant v_k^{(m)} \leqslant \eta$ for all k. *Fix* such an m. As we have just seen, there is an M depending on m such that

$$v_k^{(m)} = \sum_l \frac{A_{k,l}^{(m)}}{l^2+1} \leqslant \frac{M}{k^2+1},$$

and we may of course suppose that $M \geqslant 1$. Apply now the computational lemma with

$$y_l \;=\; v_l^{(m)}/M;$$

we get (after multiplying by M again – this trick works because $\eta/M \leqslant \eta$!),

$$v_k^{(m+1)} \;=\; \sum_l A_{k,l} v_l^{(m)} \;\leqslant\; \frac{1+\lambda}{2} \cdot \frac{M}{l^2+1}.$$

We also have, of course,

$$0 \;\leqslant\; v_k^{(m+1)} \;\leqslant\; \lambda\eta$$

by (⁑).

We may now use the computational lemma again with

$$y_l \;=\; \frac{2}{(1+\lambda)M} v_l^{(m+1)};$$

note that *here*

$$0 \;\leqslant\; y_l \;\leqslant\; \frac{2\eta\lambda}{(1+\lambda)M} \;<\; \eta.$$

After multiplying back by $(1+\lambda)M/2$, we find

$$v_k^{(m+2)} \;=\; \sum_l A_{k,l} v_l^{(m+1)} \;\leqslant\; \left(\frac{1+\lambda}{2}\right)^2 \frac{M}{k^2+1}.$$

In this fashion, we can continue indefinitely and prove that

$$v_k^{(m+p)} \;\leqslant\; \left(\frac{1+\lambda}{2}\right)^p \cdot \frac{M}{k^2+1}$$

for $p = 1, 2, 3, \ldots$. Therefore, since $\lambda < 1$,

$$\sum_{n=m}^{\infty} v_k^{(n)} \;\leqslant\; \frac{2M}{(1-\lambda)(k^2+1)}.$$

This, however, implies that

$$\begin{aligned} u_k \;&\leqslant\; K\left\{\frac{1}{k^2+1} + \sum_{n=1}^{\infty} \sum_l \frac{A_{k,l}^{(n)}}{l^2+1}\right\} \\ &=\; K\left\{\frac{1}{k^2+1} + v_k^{(1)} + \cdots + v_k^{(m-1)} + \sum_{n=m}^{\infty} v_k^{(n)}\right\} \;\leqslant\; \frac{C}{k^2+1} \end{aligned}$$

with a certain constant C, since

$$v_k^{(n)} \leqslant \frac{\text{const. } L^n}{k^2+1}$$

for each n. We have proved that $\omega_0(x)$ (which is *at most* u_k on $\mathcal{O}_k$) is

$$\leqslant \frac{C}{k^2+1} \qquad \text{for } x \in \mathcal{O}_k. \qquad\qquad \text{Q.E.D}$$

Problem 15

In this problem, the set E is as described at the beginning of the present §, with the boxed condition given there.

(a) Let $U_R(z)$ be the harmonic measure (for $\mathscr{D}$) of the subset $E \cap [-R/2, R/2]$ of $\partial\mathscr{D}$, seen from $z \in \mathscr{D}$. Show that there is a number $\alpha > 0$ depending *only* on the four quantities A, B, δ and Δ associated with E, such that $U_R(z) \leqslant \frac{1}{2}$ for $|z| = R$ and $|\Im z| \leqslant \alpha R$. (Hint: First look at $U_R(z)$ for $|z| = R$ and $|\Im z| \leqslant 1$; then use Harnack.)

(b) Let $V_R(z)$ be the harmonic measure (for $\mathscr{D}$) of

$$E \cap \{(-\infty, -R/2] \cup [R/2, \infty)\},$$

seen from $z \in \mathscr{D}$. Show that there is a number $\beta > 0$ depending only on A, B, δ and Δ such that $V_R(z) \geqslant \beta$ for $|z| = R$. (Hint: Use (a) and Harnack.)

(c) For $R > 0$, call $\mathscr{D}_R = \mathscr{D} \cap \{|z| < R\}$ and let $\omega_R(z)$ be the harmonic measure of $\{|z| = R\}$ *for* $\mathscr{D}_R$, as seen from $z \in \mathscr{D}_R$. Prove *Benedicks' lemma*, which says that

$$\omega_R(0) \leqslant \frac{C}{R}$$

with a constant C depending *only* on the four quantities A, B, δ and Δ. (Hint: Compare the $V_R(z)$ of (b) with $\omega_R(z)$ in $\mathscr{D}_R$.)

2. Green's function and a Phragmén–Lindelöf function for $\mathscr{D}$

A Green's function is available for domains $\mathscr{D}$ of the kind considered here. Let us remind the reader who may not remember that, for given $w \in \mathscr{D}$, the Green's function $G(z, w)$ is a positive function of z, *harmonic in* $\mathscr{D}$ *save at* w *where it acts like*

$$\log \frac{1}{|z-w|},$$

which is *bounded* in $\mathscr{D}$ *outside of disks of positive radius centered at* w, and *tends to zero when* z *tends to any point on the boundary* E of $\mathscr{D}$. *Existence* of $G(z, w)$ for our domains $\mathscr{D}$ follows by standard general arguments, found in many books on complex variable theory. For the sake of completeness, we will show that $G(z, w)$ is *symmetric* in z and w at the end of this article. The *last part* of the argument given there may easily be adapted so as to furnish an existence proof for $G(z, w)$.

Theorem. *Let $\mathscr{D} = \mathbb{C} \sim E$, where $E \subseteq \mathbb{R}$ has the properties given at the beginning of this §. Assume that $0 \in \mathcal{O}_0$. Then there is a constant C, depending only on the four numbers A, B, δ and Δ associated with E, such that $G(x, 0) \leqslant C/(x^2 + 1)$ for $x \in \mathbb{R} \sim \mathcal{O}_0$, $G(z, w)$ being the Green's function for $\mathscr{D}$.*

Proof. Draw a circle Γ with diameter running from the *left endpoint* of E_0 to the *right endpoint* of E_1:

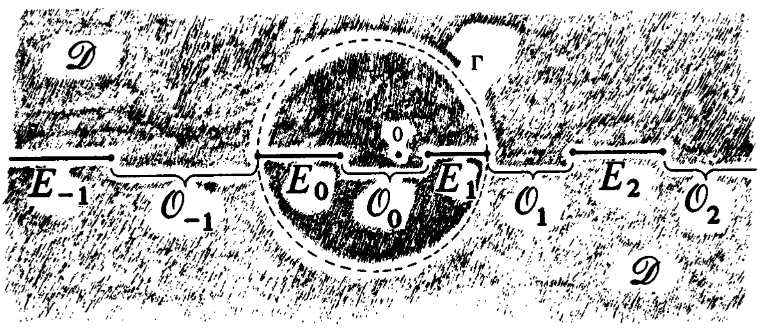

Figure 117

Let us show first of all that there is a number α, depending only on δ, Δ and B, such that

$$G(z, 0) \leqslant \alpha, \quad z \in \Gamma.$$

To see this, observe first of all that $G(z, 0) \leqslant g(z, 0)$, the Green's function for $(\mathbb{C} \sim E_1) \cup \{\infty\}$. This follows by looking at the *difference* $g(z, 0) - G(z, 0)$ on E. The latter is *harmonic* and *bounded* in $\mathscr{D}$, since the logarithmic poles of $g(z, 0)$ and $G(z, 0)$ at 0 cancel each other out. It is thus enough to get an upper bound for $g(z, 0)$ on Γ, and that bound will also serve for $G(z, 0)$ there.

Translation along $\mathbb{R}$ to the midpoint of E_1 followed by scaling down, using the factor $2/|E_1|$, takes $(\mathbb{C} \sim E_1) \cup \{\infty\}$ conformally onto the standard domain $\mathscr{E} = (\mathbb{C} \sim [-1, 1]) \cup \{\infty\}$: