By the Paley-Wiener theorem,

$$H^2(\Omega_0) = \{f \in L^2(i\mathbb{R});\ (\mathcal{F}f)(i\xi) = 0, \forall \xi < 0\}.$$

The different integrability conditions in $H^p_i(\Omega_0)$ and $H^p(\Omega_0)$ yield

$$f \in H^2_i(\Omega_0) \quad \Longleftrightarrow \quad \frac{f(z)}{1+z} \in H^2(\Omega_0).$$

When $p = \infty$, we will define $H^\infty(\Omega_0) = H^\infty_i(\Omega_0)$ to be the bounded analytic functions in $\Omega_0$. For more information on $H^p$ spaces of the half-plane see Chapters 10 and 11 of [**Dur70**].

THEOREM 7.15. **(Helson)** *Let $f(s) \sim \sum_{n=1}^\infty a_n n^{-s} \in \mathcal{H}^2$. For a.e. character $\chi \in K$, the function $f_\chi(s) = \sum_{n=1}^\infty a_n \chi(n) n^{-s}$ defined on $\Omega_{1/2}$ extends to an element of $H^2_i(\Omega_0)$, and satisfies*

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f_\chi(it)|^2 \, dt \;=\; \sum_{n=1}^{\infty} |a_n|^2 \;<\; \infty. \tag{7.16}$$

Before we prove the theorem, let us start with a preliminary observation. The set $\{e_q\}_{q\in\mathbb{Q}^+}$ forms an orthonormal basis of $L^2(K)$, where

$$e_q(\chi) = \chi(q), \text{ for all } \chi \in K.$$

Include reference here.

*Proof:* By Tonelli's theorem, we have,

$$\begin{aligned}
&\int_K \int_{-\infty}^{\infty} |f_\chi(it)|^2 d\mu(t) d\rho(\chi) \\
&= \int_{-\infty}^{\infty} \int_K |f_\chi(it)|^2 d\rho(\chi) d\mu(t) \\
&= \int_{-\infty}^{\infty} \int_K \sum_{m,n} a_n \overline{a_m} \chi(n) \overline{\chi(m)} \left(\frac{n}{m}\right)^{-it} d\rho(\chi) d\mu(t) \\
&= \int_{-\infty}^{\infty} \sum_n |a_n|^2 d\mu(t) \\
&= \sum_n |a_n|^2 \\
&= \|f\|^2_{\mathcal{H}^2}.
\end{aligned}$$

We conclude that for a.e. $\chi \in K$, the function $f_\chi$ belongs to $L^2(i\mathbb{R}, d\mu)$.

Fix $k \in \mathbb{N}^+$. By the Cauchy-Schwarz inequality and Tonelli's theorem, we obtain

$$\begin{aligned}
&\int_K \left| \int_{\mathbb{R}} f_\chi(it) \left(\frac{1-it}{1+it}\right)^k d\mu(t) \right|^2 d\rho(\chi) \\
\leq &\quad \int_K \left( \int_{\mathbb{R}} |f_\chi(it)|^2 d\mu(t) \right) \left( \int_{\mathbb{R}} \left| \frac{1-it}{1+it} \right|^{2n} d\mu(t) \right) d\rho(\chi) \\
= &\quad \int_{\mathbb{R}} \int_K |f_\chi(it)|^2 d\rho(\chi) d\mu(t) \\
= &\quad \int_{\mathbb{R}} \int_K \sum_{m,n} a_n \overline{a_m} \chi(n) \overline{\chi(m)} \left(\frac{n}{m}\right)^{-it} d\rho(\chi) d\mu(t) \\
= &\quad \int_{\mathbb{R}} \sum_n |a_n|^2 d\mu(t) \\
= &\quad \sum_n |a_n|^2 \ < \ \infty.
\end{aligned}$$

Thus, the function $G(\chi) := \int_{\mathbb{R}} f_\chi(it)(\frac{1-it}{1+it})^k d\mu(t)$ belongs to $L^2(K) \subset L^1(K)$. To show $G(\chi) = 0$ for a.e. $\chi$, we only need to show that all its Fourier coefficients vanish, i.e,

$$\int_K G(\chi)\overline{\chi(q)}\ d\rho(\chi) = 0, \text{ for all } q \in \mathbb{Q}^+.$$

Let us set $a_q = 0$ for all $q \in \mathbb{Q}^+ \setminus \mathbb{N}^+$. Since $G \in L^1(K)$, we can apply Fubini's theorem:

$$\begin{aligned}
\int_K G(\chi)\overline{\chi(q)}\ d\rho(\chi) \quad &= \quad \int_K \overline{\chi(q)} \int_{\mathbb{R}} \left(\frac{1-it}{1+it}\right)^k f_\chi(it) d\mu(t) d\rho(\chi) \\
&= \quad \int_{\mathbb{R}} \left(\frac{1-it}{1+it}\right)^k \int_K \overline{\chi(q)} \sum_n a_n n^{-it} \chi(n) d\rho(\chi) d\mu(t) \\
&= \quad \int_{\mathbb{R}} \left(\frac{1-it}{1+it}\right)^k a_q q^{-it} d\mu(t).
\end{aligned}$$

If $q \in \mathbb{Q}^+ \setminus \mathbb{N}^+$, then the last term vanishes, since $a_q$ does. If $q \in \mathbb{N}^+$, then $q^{-it} \in H^\infty(\Omega_0) = H_i^\infty(\Omega_0)$ and thus has the form $g \circ \psi$ for some $g \in H^\infty(\mathbb{D}) \subset H^2(\mathbb{D})$. Hence, by (7.11) the last term above also vanishes. Consequently, $G(\chi) = 0$ a.e., and so for a.e. $\chi \in K$, $f_\chi$ belongs to $H_i^2(\Omega_0)$.

To prove (7.16), note that, by Plancherel's theorem, the function $\mathcal{Q}f : K \to \mathbb{C}$ defined by $(\mathcal{Q}f)(\chi) = \sum_{n=1}^\infty a_n \chi(n) = \sum_{n=1}^\infty a_n e_n(\chi)$

belongs to $L^2(K)$. Also note that

$$\begin{aligned} f_\chi(it) &= \sum_n a_n \chi(n) n^{-it} \\ &= \sum_n a_n (T_t\chi)(n) \\ &= (\mathcal{Q}f)(T_t\chi), \end{aligned}$$

where $T_t$ is the Kronecker flow on $K$. We apply the Birkhoff-Khinchin erdodic theorem 7.3 to the ergodic flow $\{T_t\}$ and the function $|\mathcal{Q}f|^2 \in L^1(K)$ to conclude that

$$\begin{aligned} \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f_{\chi_0}(it)|^2 \, dt &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |(\mathcal{Q}f)(T_t\chi_0)|^2 \, dt \\ &= \int_K |(\mathcal{Q}f)(\chi)|^2 \, d\rho(\chi) \\ &= \sum_{n=1}^{\infty} |a_n|^2, \end{aligned}$$

holds for a.e. $\chi_0 \in K$. □

REMARK 7.17. Recall that by Lemma 7.13, $\zeta^k_{1/2+\varepsilon}$ and, consequently, $\tilde{\zeta}^k_{1/2+\varepsilon}$ belong to $\mathcal{H}^2$ for every $k \in \mathbb{N}^+$ and $\varepsilon > 0$. Thus, by Theorem 7.15, for a.e. $\chi \in K$

$$\lim_{T\to\infty} ⨍_{-T}^{T} \left|\zeta_\chi\left(\frac{1}{2}+\varepsilon+it\right)\right|^{2k} dt \;<\; \infty,$$

and

$$\lim_{T\to\infty} ⨍_{-T}^{T} \left|\tilde{\zeta}_\chi\left(\frac{1}{2}+\varepsilon+it\right)\right|^{2k} dt \;<\; \infty.$$

A sequence $\{a_n\}_{n=1}^{\infty}$ is called *totally multiplicative*, if $a_n a_m = a_{nm}$ holds for all $n, m \in \mathbb{N}^+$.

LEMMA 7.18. *Let $\{a_n\}$ be a non-trivial totally multiplicative sequence. If $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, then $1/f(s) \sim \sum_{n=1}^{\infty} a_n \mu(n) n^{-s}$, where $\mu$ denotes the Möbius function.*

*Proof:* Using Corollary 1.18 we obtain

$$\left(\sum_{n=1}^{\infty} a_n n^{-s}\right)\left(\sum_{m=1}^{\infty} a_m \mu(m) m^{-s}\right) \quad = \quad \sum_{k=1}^{\infty} k^{-s}\left(\sum_{n|k} a_n a_{k/n}\, \mu(k/n)\right)$$

$$\begin{aligned} &= \sum_{k=1}^{\infty} a_k k^{-s} \left( \sum_{n|k} \mu(k/n) \right) \\ &= a_1 \\ &= 1. \end{aligned}$$

THEOREM 7.19. *If a sequence $\{a_n\} \in \ell^2$ is totally multiplicative, then for a.e. character $\chi \in K$, $\sum_{n=1}^{\infty} a_n \chi(n) n^{-s}$ extends analytically to a zero-free function in $\Omega_0$.*

*Proof:* Write $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ and, note that $f_\chi = \sum_{n=1}^{\infty} a_n \chi(n) n^{-s}$ also has totally multiplicative coefficients. Thus

$$\frac{1}{f_\chi(s)} = \sum_{n=1}^{\infty} a_n \chi(n) \mu(n) n^{-s} = g_\chi(s),$$

where $g(s) = \sum_{n=1}^{\infty} a_n \mu(n) n^{-s} \in \mathcal{H}^2$, since $\mu(n) \in \{0, \pm 1\}$ for all $n \in \mathbb{N}^+$. By Theorem 7.15, the function $g_\chi$ belongs to $H_i^2(\Omega_0)$ for a.e. $\chi$. Consequently, $f_\chi$ must be zero-free in the right half-plane for the same $\chi$'s. $\square$

We obtain the following "probabilistic version" of the Riemann hypothesis.

COROLLARY 7.20. **(Helson)** *For almost every character $\chi \in K$, $\zeta_\chi(s) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$ is zero-free in $\Omega_{1/2}$.*

*Proof:* Since $\{n^{-(1/2+\varepsilon)}\} \in \ell^2$, we conclude that $\sum_n n^{-(1/2+\varepsilon)} \chi(n) n^{-s}$ is zero-free in $\Omega_0$ for a.e. $\chi$. In other words, $\zeta_\chi$ is zero-free in $\Omega_{1/2+\varepsilon}$ for a.e. $\chi$. Taking $\varepsilon = \frac{1}{m}$ and intersecting the sets of corresponding $\chi$'s we conclude that $\zeta_\chi$ is zero-free in $\Omega_{1/2}$ for a.e. $\chi$. $\square$

## 7.3. Dirichlet's theorem on primes in arithmetic progressions

We can write the set of all primes $\mathbb{P}$ as the disjoint union $\mathbb{P} = \mathbb{P}^0 \cup \mathbb{P}^1 \cup \mathbb{P}^2$, where

$$\mathbb{P}^j = \{p \in \mathbb{P};\ p \equiv j \mod 3\}$$

for $j = 0, 1, 2$.

Clearly, $\mathbb{P}^0 = \{3\}$ and the following easy argument shows that $\mathbb{P}^2$ is infinite.

*Proof:* Suppose not and write $\mathbb{P}^2 \setminus \{2\} = \{q_1, \ldots, q_N\}$. Let

$$M = 3q_1 \ldots q_N + 2.$$

Then $M \equiv 2 \mod 3$ and $M$ is not divisible 2 nor by any $q_j$. Thus $M$ factors as $M = \tilde{q}_1 \dots \tilde{q}_k$ with $\tilde{q}_j \in \mathbb{P}^1$ for all $j$. This implies $M \equiv 1 \mod 3$, a contradiction. □

It seems that no similar simple argument exists for $\mathbb{P}^1$. Nevertheless, even more is true: every arithmetic progression without a common factor contains a set of primes whose reciprocals are not summable.

THEOREM 7.21. **(Dirichlet, 1837)** *Let $l, q \in \mathbb{N}^+$ and assume that* $\gcd(l, q) = 1$. *Then*

$$\sum_{p \in \mathbb{P};\ p \equiv l \mod q} \frac{1}{p} \;=\; \infty.$$

Before we prove this theorem, we need some preparation. Let $q$ be a natural number, and let us denote by $\mathbb{Z}_q^*$ the group of units of the ring $\mathbb{Z}_q$, that is, the group of invertible elements of $\mathbb{Z}_q$. It can be checked that $0 \le k \le q-1$ is a unit in $\mathbb{Z}_q$ if and only if $\gcd(k, q) = 1$ (see Exercises **??**), and so $|\mathbb{Z}_q^*| = \phi(q)$.

Let $G$ be a finite abelian group, and let $\ell^2(G)$ be the Hilbert space of functions $f : G \to \mathbb{C}$ normed by

$$\|f\|^2 \;:=\; \frac{1}{|G|} \sum_{g \in G} |f(g)|^2.$$

The dual group of $G$, denoted by $\hat{G}$, is the set of characters, *i.e.* the multiplicative functions from $G$ to $\mathbb{T}$.

PROPOSITION 7.22. *Let $G$ be a finite abelian group. Then $\hat{G}$ forms an orthonormal basis of $\ell^2(G)$.*

Fix $q$, and let $G := \mathbb{Z}_q^*$. Any character $e \in \hat{G} = \widehat{\mathbb{Z}_q^*}$ extends to $\mathbb{Z}$ by

$$e(n) = \begin{cases} e(n \mod q), & \text{if } \gcd(n, q) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $e : \mathbb{Z} \to \mathbb{T} \cup \{0\}$ is totally multiplicative. Any such function is called a *Dirichlet character modulo $q$*. We denote the set of all Dirichlet characters modulo $q$ by $\mathcal{X}_q$. The *trivial* Dirichlet character modulo $q$ is the periodic extension of the trivial character on $\mathbb{Z}_q^*$, that is,

$$\chi_0(n) = \begin{cases} 1, & \text{if } \gcd(n, q) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

We will identify Dirichlet characters modulo $q$ with their restriction to $\mathbb{Z}_q^*$.

Suppose that $\gcd(l,q)=1$ and define $\delta_l:\mathbb{Z}\to\mathbb{T}\cup\{0\}$ by

$$\delta_l(n)=\begin{cases}1, & n\equiv l \mod q,\\ 0, & \text{otherwise.}\end{cases}$$

Then $\delta_l$ is $q$-periodic (but not multiplicative). We can regard it also as an element of $\ell^2(\mathbb{Z}_q^*)$, and by expansion with respect to the orthonormal basis obtain consisting of characters

$$\delta_l(n)=\sum_\chi\langle\delta_l,\chi\rangle\chi(n),$$

if $\gcd(n,q)=1$. If $\gcd(n,q)\neq 1$, the equality also holds, since both sides vanish.

Let $\operatorname{Re} s > 1$, then for any Dirichlet character, the series $\sum_{p\in\mathbb{P}}\chi(p)p^{-s}$ converges absolutely. This justifies exchanging the order of the sums in the following

$$\begin{aligned}\sum_{p\equiv l \mod q;\ p\in\mathbb{P}}\frac{1}{p^s} &= \sum_{p\in\mathbb{P}}\frac{\delta_l(p)}{p^s}\\ &= \frac{1}{\phi(q)}\sum_{p\in\mathbb{P}}\sum_{\chi\in\mathcal{X}_q}\overline{\chi(l)}\chi(p)p^{-s}\\ &= \frac{1}{\phi(q)}\sum_{\chi\in\mathcal{X}_q}\overline{\chi(l)}\sum_{p\in\mathbb{P}}\frac{\chi(p)}{p^s}\\ &= \frac{1}{\phi(q)}\left[\sum_{p\in\mathbb{P}}\frac{\chi_0(p)}{p^s}+\sum_{\chi\neq\chi_0}\overline{\chi(l)}\sum_{p\in\mathbb{P}}\frac{\chi(p)}{p^s}\right] \qquad (7.17)\end{aligned}$$

Except for finitely many primes (the prime factors of $q$), $\chi_0(p)=1$. By Theorem 1.9 we can conclude that $\lim_{s\to1+}\sum_{p\in\mathbb{P}}\chi_0(p)p^{-s}=\infty$. Thus, to prove Theorem 7.21, it is enough to show that the second term in (7.17) is bounded as $s\to 1+$.

Definition 7.23. Let $\chi$ be a Dirichlet character. Define the *Dirichlet L-function* in $\Omega_1$ by

$$L(s,\chi) := \sum_{n=1}^{\infty}\frac{\chi(n)}{n^{-s}}.$$

Since $\chi$ is multiplicative, the same argument that proved the Euler product formula (Theorem 1.5) shows that

$$\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{-s}} = \prod_{p\in\mathbb{P}}\left(\frac{1}{1-\chi(p)p^{-s}}\right).$$

Note that

$$\begin{aligned}
\log L(s,\chi) &= -\sum_{p\in\mathbb{P}} \log(1-\chi(p)p^{-s}) \\
&= -\sum_{p\in\mathbb{P}} \left(-\frac{\chi(p)}{p^{-s}} + O(p^{-2s})\right) \\
&= \sum_{p\in\mathbb{P}} \frac{\chi(p)}{p^{-s}} + O(1).
\end{aligned}$$

Therefore, to prove Theorem 7.21, it is enough to show that $\lim_{s\to 1+} L(s,\chi)$ is finite and non-zero, for every non-trivial Dirichlet character $\chi$. If $q = q_1^{r_1}\dots q_k^{r_k}$, then

$$L(s,\chi_0) \;=\; \prod_{p\in\mathbb{P}} \frac{1}{1-\chi_0(p)p^{-s}} \;=\; (1-q_1^{-s})\dots(1-q_k^{-s})\zeta(s).$$

THEOREM 7.24. *If $\chi$ is a non-trivial Dirichlet character, then $\sigma_c(L(s,\chi)) = 0$.*

*Proof:* Since $\sum_{n=1}^{\infty}\chi(n)$ does not converge, Theorem 3.12 yields $\sigma_c = \limsup_{N\to\infty}\frac{\log|s_N|}{\log N} \geq 0$. We can compute

$$\begin{aligned}
\sum_{n=1}^{q}\chi(n) &= \sum_{n=1}^{q}\chi(n)\chi_0(n) \\
&= \phi(q)\ \langle\chi,\chi_0\rangle_{\ell^2(\mathbb{Z}_q^*)} \\
&= 0.
\end{aligned}$$

Hence, by periodicity of $\chi$, we can conclude that $|s_N| \leq \phi(q)$, and so $\sigma_c = 0$. □

Thus, $\lim_{s\to 1+} L(s,\chi)$ is finite, in fact, $L(1,\chi)$ is defined for every non-trivial Dirichlet character $\chi$. Hence, to prove Theorem 7.21, it remains to show that $L(1,\chi) \neq 0$, for $\chi \neq \chi_0$.

We will now fix a non-trivial character $\eta \in \mathcal{X}_q$. We distinguish two cases.

**Case I:** $\eta$ is not a real-valued character.

LEMMA 7.25. *For $s > 1$, $\prod_{\chi\in\mathcal{X}_q} L(s,\chi) \geq 1$.*

*Proof:* By definition of the Dirichlet $L$-function and the power series expansion of the natural logarithm, we have

$$\prod_{\chi\in\mathcal{X}_q} L(s,\chi) \;=\; \prod_{\chi}\exp\left(\sum_{p\in\mathbb{P}}\log\frac{1}{1-\chi(p)p^{-s}}\right)$$

$$
\begin{aligned}
&= \exp\left(\sum_{\chi}\sum_{p\in\mathbb{P}}\sum_{k=1}^{\infty}\frac{\chi(p^k)}{kp^{ks}}\right)\\
&= \exp\left(\sum_{p\in\mathbb{P}}\sum_{k=1}^{\infty}\frac{1}{kp^{ks}}\sum_{\chi}\chi(p^k)\right), \qquad (7.18)
\end{aligned}
$$

where rearranging the order of summation is justified by absolute convergence. For any character $\chi$, $\chi(1)=1$ and hence

$$
\begin{aligned}
\langle\delta_1,\chi\rangle &= \frac{1}{\phi(q)}\sum_{m\in\mathbb{Z}_q^*}\delta_1(m)\overline{\chi(m)}\\
&= \frac{1}{\phi(q)}.
\end{aligned}
$$

Consequently, $1=\phi(q)\langle\delta_1,\chi\rangle$, so that for any $n\in\mathbb{N}^+$, we obtain

$$
\begin{aligned}
\sum_{\chi}\chi(n) &= \phi(q)\sum_{\chi}\langle\delta_1,\chi\rangle\chi(n)\\
&= \phi(q)\delta_1(n)\\
&\geq 0.
\end{aligned}
$$

We conclude by (7.18) that $\prod_{\chi}L(s,\chi)=\exp(r)$, where $r$ is non-negative. $\square$

Suppose that $L(1,\eta)=0$. Then $L(s,\eta)=O(s-1)$ as $s$ tends to 1. Its conjugate $\overline{\eta}$ is also a character (and different from $\eta$, since we assumed $\eta$ takes on some non-real value somewhere). Moreover, $L(1,\overline{\eta})=\sum_{n=1}^{\infty}\frac{\overline{\eta}(n)}{n}=\overline{L(1,\eta)}=0$. Thus, as $s\to 1+$,

$$
\begin{aligned}
\prod_{\chi}L(s,\chi) &= L(s,\chi_0)\cdot L(s,\eta)\cdot L(s,\overline{\eta})\cdot\prod_{\chi\neq\eta,\overline{\eta},\chi_0}L(s,\chi)\\
&= O((s-1)^{-1})\ O(s-1)\ O(s-1)\ O(1)\\
&= O(s-1),
\end{aligned}
$$

which contradicts Lemma 7.25. This concludes case I.

**Case II:** $\eta$ is real character.

LEMMA 7.26. *Let $m\in\mathbb{N}^+$. Then $\sum_{n|m}\eta(n)\geq 0$. If $m=l^2$ with $l\in\mathbb{N}^+$, then $\sum_{n|m}\eta(n)\geq 1$.*

*Proof:* Write $m=p_1^{r_1}\dots p_k^{r_k}$, then

$$
\sum_{n|m}\eta(n) = \prod_{j=1}^{k}\left[\eta(1)+\eta(p_j)+\cdots+\eta(p_j^{r_j})\right].
$$

Since $\eta$ is real, the only possible values for $\eta(p_j)$ are 1, 0, and $-1$. Corresponding to these cases, we observe that

$$\eta(1)+\eta(p_j)+\cdots+\eta(p_j^{r_j}) = \begin{cases} r_j+1, & \text{if } \eta(p_j)=1, \\ 1, & \text{if } \eta(p_j)=0, \\ 1, & \text{if } \eta(p_j)=-1, \text{ and } r_j \text{ is even}, \\ 0, & \text{if } \eta(p_j)=-1, \text{ and } r_j \text{ is odd}. \end{cases}$$

Thus $\sum_{n|m}\eta(n)$ is a product of non-negative factors. If $m$ is a square, all $r_j$'s are even, so that $\sum_{n|m}\eta(n)$ is a product of numbers larger than 1. ☐

LEMMA 7.27. *For all $M \le N \in \mathbb{N}^+$ and every $\sigma > 0$,*

$$\sum_{n=M}^{N} \frac{\eta(n)}{n^\sigma} = O(M^{-\sigma}).$$

*Proof:* Let $s_n = \sum_{k=1}^{n}\eta(k)$ and use summation by parts as follows

$$\begin{aligned} \left|\sum_{n=M}^{N} \frac{\eta(n)}{n^\sigma}\right| &= \left|\sum_{n=M}^{N-1} s_n\big[n^{-\sigma}-(n+1)^{-\sigma}\big]\right| + O(M^{-\sigma}) \\ &\le \phi(q)\sum_{n=M}^{N-1}\big[n^{-\sigma}-(n+1)^{-\sigma}\big] + O(M^{-\sigma}) \\ &= \phi(q)\big[M^{-\sigma}-N^{-\sigma}\big] + O(M^{-\sigma}) \\ &= O(M^{-\sigma}), \end{aligned}$$

where the estimate $|s_n| \le \phi(q)$ was demonstrated in the proof of Theorem 7.24. ☐

For $N \in \mathbb{N}^+$, set

$$S_N = \sum_{m,n\ge 1;\ mn\le N} \frac{\eta(n)}{\sqrt{mn}}.$$

The following two claims imply that $L(1,\eta) \ne 0$ and thus conclude the proof of Theorem 7.21.

*Claim 1:* $S_N \ge c\log N$, for some $c > 0$.

*Proof:* Write

$$S_N = \sum_{k=1}^{N}\sum_{mn=k}\frac{\eta(n)}{\sqrt{mn}} = \sum_{k=1}^{N} k^{-1/2}\sum_{n|k}\eta(n) \ge \sum_{l=1}^{\lfloor\sqrt{N}\rfloor}\frac{1}{l},$$

since Lemma 7.26 implies that if $k = l^2$, then $\sum_{n|k} \eta(n) \geq 1$ and in general, the sum is non-negative. But we can estimate the last sum from below by comparing to the integral $\int_1^{\sqrt{N}} \frac{dx}{x} \approx \frac{1}{2}\log N$. $\square$

*Claim 2:* $S_N = 2\sqrt{N} L(1,\eta) + O(1)$.

Before we prove this claim, we need the following approximation.

LEMMA 7.28. *For* $K \geq 1$, *we have*

$$\sum_{m=1}^{K} \frac{1}{\sqrt{m}} \;=\; \int_1^{K+1} \frac{dx}{\sqrt{x}} + \tau + O(\frac{1}{\sqrt{K}}),$$

*where* $\tau$ *is some positive constant.*

*Proof:* Let $\tau_m = \frac{1}{\sqrt{m}} - \int_m^{m+1} \frac{dx}{\sqrt{x}}$. Then

$$0 \;<\; \tau_m \;<\; \frac{1}{\sqrt{m}} - \frac{1}{\sqrt{m+1}}, \tag{7.29}$$

which is an alternating series. So $\sum_{m=1}^{\infty} \tau_m$ converges to some number $\tau$ between 0 and 1. We have

$$\begin{aligned} \sum_{m=1}^{K} \frac{1}{\sqrt{m}} &= \int_1^{K+1} \frac{dx}{\sqrt{x}} + \sum_{m=1}^{K} \tau_m \\ &= \int_1^{K+1} \frac{dx}{\sqrt{x}} + \tau - \sum_{m=K+1}^{\infty} \tau_m. \end{aligned}$$

and by (7.29) we know that $\sum_{m=K+1}^{\infty} \tau_m = O(\frac{1}{\sqrt{K}})$. $\square$

We can now prove Claim 2.

*Proof:* (of Claim 2) Write

$$\begin{aligned} S_N &= \sum_{m<\sqrt{N}, n>\sqrt{N}} \frac{1}{\sqrt{m}} \sum_{nm \leq N} \frac{\eta(n)}{\sqrt{n}} + \sum_{n \leq \sqrt{N}} \frac{\eta(n)}{\sqrt{n}} \sum_{nm \leq N} \frac{1}{\sqrt{m}} \\ &= S_I + S_{II}. \end{aligned}$$

The first term is easy to estimate using lemmata 7.27 and 7.28:

$$S_I \;=\; \sum_{m<\sqrt{N}} \frac{1}{\sqrt{m}} \sum_{\sqrt{N}<n\leq N/m} \frac{\eta(n)}{\sqrt{n}} \;=\; \sum_{m<\sqrt{N}} \frac{1}{\sqrt{m}} O(N^{-1/4}) \;=\; O(1).$$

As for the second term, we have

$$S_{II} \;=\; \sum_{n \leq \sqrt{N}} \frac{\eta(n)}{\sqrt{n}} \sum_{m \leq \sqrt{N}} \frac{1}{\sqrt{m}}$$

$$
\begin{aligned}
&= \sum_{n\le\sqrt{N}} \frac{\eta(n)}{\sqrt{n}} \left[ \int_1^{\frac{N}{n}+1} \frac{dx}{\sqrt{x}} + \tau + O\left(\sqrt{\frac{n}{N}}\right) \right] \\
&= \sum_{n\le\sqrt{N}} \frac{\eta(n)}{\sqrt{n}} \left[ 2\left(\sqrt{\frac{N}{n}+1} - 1\right) + \tau + O\left(\sqrt{\frac{n}{N}}\right) \right] \\
&= \sum_{n\le\sqrt{N}} \frac{\eta(n)}{\sqrt{n}} \left[ 2\sqrt{\frac{N}{n}} + (\tau - 2) + O\left(\sqrt{\frac{n}{N}}\right) \right] \\
&= 2\sqrt{N} \sum_{n\le\sqrt{N}} \frac{\eta(n)}{n} + (\tau-2) \sum_{n\le\sqrt{N}} \frac{\eta(n)}{\sqrt{n}} + \sum_{n\le\sqrt{N}} \eta(n) O\big(N^{-1/2}\big).
\end{aligned}
$$

The second term on the last line is $O(1)$ by Lemma 7.27, and the third term is $O(1)$ since there are only $\sqrt{N}$ terms in the sum. The first term is a truncation of the series for $L(1,\eta)$, so we get

$$S_{II} \;=\; 2\sqrt{N}[L(1,\eta) - \sum_{n=\sqrt{N}+1}^{\infty} \frac{\eta(n)}{\sqrt{n}}] + O(1),$$

which equals

$$2\sqrt{N} L(1,\eta) + O(1)$$

by another application of Lemma 7.27. □

We have thus proved Theorem 7.21.

## 7.4. Exercises

1. Let $q$ be in $\mathbb{N}^+$. Prove that $\gcd(n,q) = 1$ if and only if there exists $m \in \mathbb{N}^+$ such that $mn \equiv 1, \mod q$.
2. Prove Proposition 7.22. (Hint: It is easy if $G$ is cyclic. Then show that $\widehat{G_1 \times G_2} = \hat{G}_1 \times \hat{G}_2$).

## 7.5. Notes

Theorem 7.15 is from [**Hel69**]. Our proof of Dirichlet's theorem is from [**SS03**]. This theorem was where Dirichlet series were first used (and, in honor of this, were named after Dirichlet).

CHAPTER 8

# Zero Sets

There is an interplay between the number of zeroes of a holomorphic function and its size. Roughly speaking, the more zeroes a function has, the larger it must be. The simplest example are polynomials — if a polynomial has $n$ zeroes, it must be of degree at least $n$ thus $|P(z)| \geq C|z|^n$ as $|z| \to \infty$. More generally, assume that $f \in \mathrm{Hol}\,(\mathbb{D})$ is normalized so that $f(0) = 1$. Then, $\log|f(z)|$ is subharmonic in $\mathbb{D}$ and so

$$0 \;=\; \log|f(0)| \;\leq\; \int_{\mathbb{D}} \log|f(z)|\, dA(z).$$

Thus $\log|f(z)|$ has to be "large enough" to offset the negativity of $\log|f(z)|$ around points where $f$ vanishes.

Definition 8.1. Let $\mathcal{F}$ be a family of holomorphic functions defined on a set $U$ and $Z \subset U$. We say that $Z$ is a *zero set* for $\mathcal{F}$, if there exists a function $f \in \mathcal{F}$ that vanishes exactly on $Z$, that is, such that $Z = f^{-1}\{0\}$.

It is well-known that for any connected open set $U \subset \mathbb{C}$, the zero sets for $\mathcal{F} = \mathrm{Hol}\,(U)$ are the sets $Z \subset U$ that have no accumulation points inside $U$.

For the Hardy spaces on the unit disk, the zero sets are well understood:

Theorem 8.2. *Let $\{\lambda_n\}_n \subset \mathbb{D}$ be a sequence, $0 < p \leq \infty$. The following are equivalent*

- *$\{\lambda_n\}$ is zero set for $H^p(\mathbb{D})$,*
- *$\{\lambda_n\}$ is zero set for $H^\infty(\mathbb{D})$,*
- *$\sum_n (1 - |\lambda_n|) < \infty$,*
- *$\prod_n \frac{\lambda_n}{|\lambda_n|} \frac{z-\lambda_n}{1-\overline{\lambda_n} z}$ converges to a non-zero function.*

The fact that the zero sets for $H^p(\mathbb{D})$ are independent of $p$ follows from inner-outer factorization. An analogous factorization theorem does not hold for the polydisk and the zero sets for $H^p(\mathbb{D}^n)$ depend on $p$ when $n > 1$.

Precise descriptions of the zero sets for the Bergman space or the Dirichlet space are not known.

Consider a Dirichlet series of the form $f \sim \sum_{n=1}^{\infty} a_{2^n} 2^{-ns}$. Clearly, if $\lambda \in \sigma_c(f)$ is a zero of $f$, then so is $\lambda + \frac{2\pi i}{\log 2} k$, for any $k \in \mathbb{Z}$. The following theorem shows that the zero sets of Dirichlet series have similar behavior at least in the half-plane $\Omega_{\sigma_u(f)}$.

THEOREM 8.3. *Let $f \sim \sum_{n=1}^{\infty} a_n n^{-s}$, $f(s_0) = 0$ and $s_0 > \sigma_u(f)$. Then, for every $\delta > 0$, the strip $\{|Re\ (s - s_0)| < \delta\}$ contains infinitely many zeroes.*

*Proof:* Since the set of zeroes is discrete, we can find $0 < \tau < \min\ \{\delta, \sigma_0 - \sigma_u\}$ such that $C = \partial B(s_0, \tau)$ does not contain any zero of $f$. By compactness, $m := \inf_{s \in C} |f(s)| > 0$. As the series converges uniformly in $\overline{\Omega_{s_0 - \tau}}$, we can find $N \in \mathbb{N}$ such that

$$\left| f(s) - \sum_{n=1}^{N} a_n n^{-s} \right| \ < \ \frac{m}{4}, \qquad \text{for all } s \in \overline{\Omega_{s_0 - \tau}}$$

By Theorem 6.14, we can find an arbitrarily large $t_0 \in \mathbb{R}$ so that for all primes $p \leq N$, $t_0 \log p \approx 0 \mod 1$. More precisely,

$$\left| n^{-\sigma} e^{it_0 \log n} - n^{-\sigma} \right| \ < \ \frac{m}{4N(|a_n| + 1)}, \qquad \text{for all } 1 \leq n \leq N,\ \sigma \in [\sigma_0 - \tau, \sigma_0 + \tau].$$

Consequently, by triangle inequality,

$$|f(s) - f(s + it_0)| \ \leq \ \frac{m}{2} + \sum_{n=1}^{N} a_n \left| n^{-s} - n^{-s+it_0} \right| \ \leq \ \frac{3m}{4}.$$

By Rouché's theorem, it follows that $f(s + it_0)$ has a zero inside $C$, that is, $f(s)$ has a zero inside of $C + it_0$. Since $t_0$ is arbitrarily large, we can find infinitely many disjoint disks of this form. $\square$

We have an immediate corollary.

COROLLARY 8.4. *If $\varphi \in \text{Mult}\,(\mathcal{H}^2)$ and $\varphi(s_0) = 0$ for some $s_0 \in \Omega_0$, then $\varphi$ vanishes at infinitely many points.*

QUESTION 8.5. Does the above theorem hold for $\sigma_c(f) < \sigma_0 < \sigma_u(f)$?

Note that the function $\frac{1}{\zeta(s)}$ has a zero $s_0 = 1$ and no other zero in the set $\{\text{Re } s > 1/2\}$, if the Riemann hypothesis holds, so the answer to the question should be negative. In [**MV**, Problem 24], M. Balazard poses the similar question of whether a convergent Dirichlet series can have a single zero in a half-plane.

Let us define the *uniformly local $H^p$ space on* $\Omega_{1/2}$ by

$$H^p_\infty(\Omega_{1/2}) := \left\{ g \in \mathrm{Hol}\,(\Omega_{1/2}) : \left[ \sup_{\theta\in\mathbb{R}} \sup_{\sigma>1/2} \int_\theta^{\theta+1} |g(\sigma+it)|^p \, dt \right]^{\frac{1}{p}} < \infty \right\}.$$

Then, clearly,

$$H^p(\Omega_{1/2}) \subset H^p_\infty(\Omega_{1/2}),$$

and

$$f \in H^p_\infty(\Omega_{1/2}) \implies \frac{f(s)}{s} \in H^p(\Omega_{1/2}), \text{ for } p > 1.$$

A deeper result is the following, which is a variant of Hilbert's inequality. See [**Mon94**] or [**HLS97**] for a proof.

THEOREM 8.6. $\mathcal{H}^2 \hookrightarrow H^2_\infty(\Omega_{1/2})$.

We define $\mathcal{H}^p$ by to be the completion of the set of all finite Dirichlet series with respect to the norm

$$\|f\|_{\mathcal{H}^p} := \left[ \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(it)|^p \, dt \right]^{1/p}.$$

COROLLARY 8.7. $\mathcal{H}^{2n} \hookrightarrow H^{2n}_\infty(\Omega_{1/2})$, *for all* $n \in \mathbb{N}^+$.

QUESTION 8.8. Does $\mathcal{H}^p \hookrightarrow H^p_\infty(\Omega_{1/2})$ hold for all $p > 1$ ($p \geq 1$)?

One might expect that the answer to the question has to be affirmative. As a warning, we recall a conjecture of Hardy and Littlewood in 1935 that for any $q \geq 2$ there exist a constant $c_q > 0$ such that

$$\int_0^{2\pi} \left| \sum a_n e^{int} \right|^q dt \;\leq\; c_q \int_0^{2\pi} \left| \sum |a_n| e^{int} \right|^q dt.$$

The conjecture turns out to be true precisely when $q$ is an even integer (this was shown by Bachelis in 1973 [**Bac73**]).

Suppose that $f \in \mathcal{H}^2$. Then $\frac{f(s)}{s} \in H^2(\Omega_{1/2})$, and hence its zeroes $s_k = \sigma_k + it_k$ satisfy

$$\sum_k \frac{\sigma_k - 1/2}{1 + |s_k|^2} \;<\; \infty.$$

Also, if we define

$$A(\theta) := \sum_{\theta < t_k < \theta+1} \Big( \sigma_k - 1/2 \Big),$$

then, by the above condition, $A(\theta) < \infty$, for all $\theta \in \mathbb{R}$.

THEOREM 8.9. **(Hedelmalm, Lindqvist, Seip)** *If* $f \in \mathcal{H}^2$, *and* $f \not\equiv 0$, *then* $\sup_{\theta\in\mathbb{R}} A(\theta) < \infty$.

*Proof:* Suppose not, then there exist a sequence $\{\theta_j\}_j \subset \mathbb{R}$ such that $A(\theta_j) \to \infty$. Define $f_j(s) := f(s+\theta_j)$; then

$$\|f_j\|_{\mathcal{H}^2} \;=\; \|f\|_{\mathcal{H}^2}.$$

Thus $\{f_j\}_j$ is a bounded sequence in $\mathcal{H}^2$, hence $\{f_j(s)/s\}_j$ i is also bounded in $H^2(\Omega_{1/2})$. Let $\{s_k^j\}_k$ be the zeroes of $f_j(s)/s$. The condition $A(\theta_j) \to \infty$ implies that

$$\sum_k \frac{\sigma_k^j - 1/2}{1+|s_k^j|^2} \to \infty \qquad \text{as } j \to \infty. \tag{8.10}$$

Using inner-outer factorization, this implies that $f_j$'s converge to 0 uniformly on compact sets, since the Blaschke product part does by (8.10), and the outer parts are uniformly bounded on compact sets, by the norm control. But by Proposition 7.11, some subsequence of $\{f_j\}_j$ converges uniformly on compact subsets to a vertical limit function $f_\chi = \sum_n a_n \chi(n) n^{-s}$, where $\chi$ is a character. We conclude that $f_\chi \equiv 0$, a contradicton. $\square$

QUESTION 8.11. How can the zero sets of $\mathcal{H}^2$, $\mathcal{H}^2_w$, etc. be classified?

CHAPTER 9

# Interpolating Sequences

## 9.1. Interpolating Sequences for Multiplier algebras

Definition 9.1. Let $\mathcal{H}$ be a Hilbert space of analytic functions on $X$ with reproducing kernel $k$. We say that $(\lambda_n) \subset X$ is an *interpolating sequence* for $\mathrm{Mult}\,(\mathcal{H})$, if

$$\{(\varphi(\lambda_n)),\ \varphi \in \mathrm{Mult}\,(\mathcal{H})\} \ = \ \ell^\infty.$$

In other words, we require that the map $E : \phi \mapsto (\phi(z_n))$ maps $\mathrm{Mult}\,(\mathcal{H})$ onto $\ell^\infty$ (it always maps into, by Proposition 11.9). By basic functional analysis, whenever one has an interpolating sequence, it comes with an interpolation constant.

Indeed, consider the quotient Banach space $\mathcal{X} = \mathrm{Mult}\,(\mathcal{H})/N$, where

$$N \ := \ \{f :\ f(z_n) = 0,\ \forall\ n \in \mathbb{N}\}$$

is the kernel of $E$. We obtain a bounded operator $\tilde{E} : \mathcal{X} \to \ell^\infty$, which is one-to-one and onto. By the open mapping theorem, it has a bounded inverse. We conclude that if $\{z_n\}_n$ is an interpolating sequence for $\mathrm{Mult}\,(\mathcal{H})$, then there exists a constant $C > 0$ such that for any sequence $(a_n)_n \in \ell^\infty$, there exists a function $f \in \mathrm{Mult}\,(\mathcal{H})$ such that $f(z_n) = a_n$ for all $n \in \mathbb{N}$ and $\|f\|_\infty \leq C\|(a_n)\|_\infty$. The infimum of those $C$ for which this holds is called the *interpolation constant* of the sequence.

The exact description of interpolating sequences for particular spaces is hard. There is a general result due to S. Axler [**Axl92**] showing that sequences that tend to the boundary will, in many spaces, have subsequences that are interpolating, but verifying the condition of the theorem can be difficult.

Theorem 9.2. **(Axler)** *Let $\mathcal{H}$ be a separable reproducing kernel Hilbert space on $X$, and assume that* $\mathrm{Mult}\,(\mathcal{H})$ *separates points of $X$. Suppose that $(x_n)$ is a sequence with the property that for any subsequence $(x_{n_k})$, there exists some $\phi \in$* $\mathrm{Mult}\,(\mathcal{H})$ *such that* $\lim_{k\to\infty} \phi(x_{n_k})$ *does not exist. Then $(x_n)$ has a subsequence that is an interpolating sequence for* $\mathrm{Mult}\,(\mathcal{H})$.

L. Carleson in 1958 characterized interpolating sequences for $H^\infty(\mathbb{D})$. For later convenience, we shall apply a Cayley transform and quote the result for $H^\infty(\Omega_0)$. First we need to introduce a metric.

DEFINITION 9.3. Let $\mathcal{A}$ be a normed algebra of functions on the set $X$. We define the Gleason distance $\rho_{\mathcal{A}}$ between two points $x$ and $y$ by

$$\rho_{\mathcal{A}}(x,y) \;=\; \sup\{|\phi(y)\| \;:\; \phi(x)=0,\ \|\phi\|\leq 1\}.$$

When the algebra is understood, we shall write $\rho$.

For the algebra $H^\infty(\mathbb{D})$, the Gleason distance is called the pseudo-hyperbolic metric, and , and it is given by

$$\rho_{H^\infty(\mathbb{D})}(z,w) \;=\; \left|\frac{z-w}{1-\bar{w}z}\right|.$$

In the right half-plane, this becomes

$$\rho_{H^\infty(\Omega_0)}(s,u) \;=\; \left|\frac{s-u}{s+\bar{u}}\right|.$$

In the polydisk, it is straightforward to show

$$\rho_{H^\infty(\mathbb{D}^m)}(z,w) \;=\; \max_{1\leq j\leq m}\left|\frac{z_j-w_j}{1-\bar{w}_j z_j}\right|. \tag{9.4}$$

THEOREM 9.5. **(Carleson)** *Let $(s_j)\subset\Omega_0$. Then the following are equivalent:*

(1) *$(s_j)$ is an interpolating sequence for $H^\infty(\Omega_0)$.*
(2) $\inf_j \prod_{i\neq j}\left|\frac{s_i-s_j}{s_i+\bar{s}_j}\right| > 0$.
(3) $\inf_{i\neq j}\left|\frac{s_i-s_j}{s_i+\bar{s}_j}\right| > 0$ *and there exists $C>0$ such that for every $f\in H^2(\Omega_0)$,*

$$\sigma_j\sum_j |f(s_j)|^2 \leq C\|f\|_2^2.$$

Carleson's theorem is very important, and the various conditions in it have names.

DEFINITION 9.6. Let $\mathcal{A}$ be a normed algebra of functions on the set $X$, and let $\rho=\rho_{\mathcal{A}}$ be the Gleason distance. We say a sequence $(x_n)$ is *weakly separated* if $\inf_{m\neq n}\rho(x_m,x_n)>0$.

We say the sequence is *strongly separated* if

$$\inf_n\,[\sup\{|\phi(x_n)| \;:\; \phi(x_m)=0\ \forall\ m\neq n,\ \|\phi\|\leq 1\}] \;>\; 0. \tag{9.7}$$

In $H^\infty(\Omega_0)$, a sequence is strongly separated if and only if the a priori stronger condition

$$\inf_n \left[ \prod_{m \neq n} \rho(s_m, s_n) \right] \; > \; 0$$

holds; this is an elementary consequence of the fact that dividing out by a Blaschke product does not increase the norm. In the polydisk, as we shall see in Theorem 9.11, these two conditions are different.

DEFINITION 9.8. Let $\mathcal{H}$ be a reproducing kernel Hilbert space on $X$, and let $\mu$ be a measure on $X$. We say $\mu$ is a *Carleson measure* for $\mathcal{H}$ if there exists a constant $C$ such that

$$\int |f|^2 d\mu \; \leq \; C \|f\|^2_{\mathcal{H}} \qquad \forall \, f \; \in \; \mathcal{H}.$$

With these definitions, condition (2) in Carleson's theorem becomes the statement that the sequence is strongly separated, and condition (3) is that the sequence is weakly separated and the measure $\sum \sigma_j \delta_{s_j}$ is a Carleson measure for $H^2(\Omega_0)$.

QUESTION 9.9. What are the interpolating sequences for $\mathrm{Mult}\,(\mathcal{H}^2_w)$?

The answer is not known in general, but K. Seip [**Sei09**] showed that for *bounded* sequences, the interpolating sequences for $\mathrm{Mult}\,(\mathcal{H}^2)$ are the same as for the much larger space $H^\infty(\Omega_0)$. Let us use $\mathcal{H}^\infty$ to denote $\mathrm{Mult}\,(\mathcal{H}^2)$, which by Theorem 6.42 is the bounded functions in $\Omega_0$ that have a Dirichlet series:

$$\mathcal{H}^\infty \; = \; H^\infty(\Omega_0) \cap \mathbb{D}.$$

We shall write $\mathcal{H}^\infty_m$ for those $f$ in $\mathcal{H}^\infty$ whose Dirichlet series is supported on $\mathbb{N}_m$.

THEOREM 9.10. **(Seip)** *Let $(s_j)$ be a bounded sequence in $\Omega_0$. Then the following are equivalent:*

*(i) It is an interpolating sequence for $\mathcal{H}^\infty$.*

*(ii) It is an interpolating sequence for $\mathcal{H}^\infty_2$.*

*(iii) It is an interpolating sequence for $H^\infty(\Omega_0)$.*

*Moreover, if $\{s_j\}$ is contained in a vertical strip of height less than $\frac{2\pi}{\log 2}$, and is bounded horizontally, these three conditions are equivalent to $(s_j)$ being an interpolating sequence for $\mathcal{H}^\infty_1$.*

To prove Seip's theorem, we need a result by B. Berndtsson, S.-Y. Chang and K.-C. Lin [**BCL87**] that gives a sufficient condition for a sequence to be interpolating on the polydisk. We shall explain what condition (3) means in Section 9.2.

THEOREM 9.11. **(Berndtsson, Chang and Lin)** *Consider the three statements*

*(1) There exists $c > 0$ such that*

$$\prod_{j \neq i} \rho_{H^\infty(\mathbb{D}^m)}(\lambda_i, \lambda_j) \geq c$$

*for all $i$.*

*(2) The sequence $\{\lambda_i\}_{i=1}^\infty$ is an interpolating sequence for $H^\infty(\mathbb{D}^m)$.*

*(3) The sequence $\{\lambda_i\}_{i=1}^\infty$ is weakly separated and the associated Grammian with respect to Lebesgue measure $\sigma$ is bounded.*

*Then (1) implies (2) and (2) implies (3). Moreover the converse of both these implications is false.*

To prove Seip's theorem, we need to compare Gleason differences in different algebras. For the remainder of the section, we shall adopt the following notation:

$$\begin{aligned}
d_m(z,w) &= \rho_{H^\infty(\mathbb{D}^m)}(z,w) && = \max_{1 \leq j \leq m} \left| \frac{z_j - w_j}{1 - \bar{w}_j z_j} \right| \\
\rho(s,u) &= \rho_{H^\infty(\Omega_0)}(s,u) && = \left| \frac{s-u}{s+\bar{u}} \right| \\
\rho_m(s,u) &= d_m((2^{-s}, ., p_m^{-s}), (2^{-u}, ., p_m^{-u}))
\end{aligned}$$

For points $s, u$ in $\Omega_0$, we shall write

$$s = \sigma + it, \quad u = \upsilon + iy.$$

LEMMA 9.12. *For each $n \geq 2$,*

$$\begin{aligned}
d_1(n^{-s}, n^{-u}) &\leq \rho(s,u) \\
\rho_2(s,u) &\leq \rho(s,u).
\end{aligned}$$

PROOF: The first inequality is because the map $s \mapsto n^{-s}$ is a holomorphic map from $\Omega_0$ to $\mathbb{D}$, so it is tautologically distance decreasing in the Gleason distances for the corresponding $H^\infty$ spaces.

The second inequality follows from the first. $\square$

LEMMA 9.13. *For every $M > 0$, there exists $\gamma > 0$ such that if $s, u \in \Omega_0$ and $|s|, |u| \leq M$, then*

$$\rho_2(s,u) \geq \rho(s,u)^\gamma.$$

*If in addition $|t - y| \leq H < \frac{2\pi}{\log 2}$, then we can choose $\gamma$ so that*

$$\rho_1(s,u) \geq \rho(s,u)^\gamma.$$

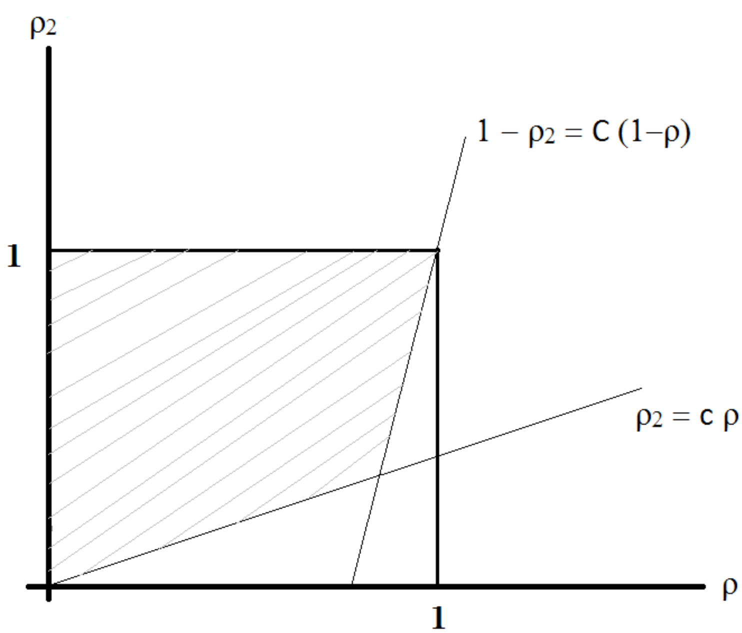


FIGURE 1. Curve $\rho_2 = \rho^\gamma$ fits outside shaded area

PROOF: It is sufficient to prove that there exist constants $c, C > 0$ such that

(1) $\rho_2(s,u) \geq c\rho(s,u)$
(2) $1 - \rho_2(s,u) \leq C(1 - \rho(s,u))$.

(See Figure 9.1).

To prove (1), let $K$ be the closed semi-disk

$$K \ = \ \{s \ : \ \sigma \geq 0, |s| \leq M\}.$$

Define the function $\psi$ on $K \times K$ by

$$\psi \ = \ \begin{cases} \frac{\rho(s,u)}{\rho_2(s,u)} & \text{if } s \neq u, \text{ and } s,u \in \Omega_0 \\ 1 & \text{if } \Re s \text{ or } \Re u = 0 \\ \frac{2^\sigma - 2^{-\sigma}}{2\sigma \log 2} & \text{if } s = u \in \Omega_0. \end{cases}$$

It is straightforward to check that $\psi$ is continuous, so we can set

$$c \ = \ 1/\max_{K \times K} \psi(s,u)$$

and get (1).

For (2), we observe that

$$\rho_2(s,u) \;=\; \max_{p=2,3}\left|\frac{p^{-s}-p^{-u}}{1-p^{-s-\bar u}}\right|.$$

Writing $s=\sigma+it$ and $u=\upsilon+iy$, we get

$$\begin{aligned} 1-\rho(s,u)^2 &= 1-\left|\frac{s-u}{s+\bar u}\right|^2 \\ &= \frac{4\sigma\upsilon}{(\sigma+\upsilon)^2+(t-y)^2}. \end{aligned} \tag{9.14}$$

We also have

$$\begin{aligned} 1-\rho_2(s,u)^2 &= \min_{p=2,3}\left[1-\frac{p^{-2\sigma}+p^{-2\upsilon}-2\Re p^{-s-\bar u}}{1-2\Re p^{-s-\bar u}+p^{-2\sigma-2\upsilon}}\right] \\ &= \min_{p=2,3}\frac{1+p^{-2\sigma-2\upsilon}-p^{-2\sigma}-p^{-2\upsilon}}{1-2\Re p^{-s-\bar u}+p^{-2\sigma-2\upsilon}} \\ &= \min_{p=2,3}\frac{(1-p^{-2\sigma})(1-p^{-2\upsilon})}{(1-p^{-\sigma-\upsilon})^2+2p^{-\sigma-\upsilon}(1-\cos[\log p(t-y)])} \end{aligned} \tag{9.15}$$

We would like to show that for some constant $C_M$ we have that

$$1-\rho_2(s,u)^2 \;\le\; C_M\frac{\sigma\upsilon}{(\sigma+\upsilon)^2+(t-y)^2}, \tag{9.16}$$

as this, together with (9.14), would give (2).

First, assume that

$$|t-y| \;\le\; H \;<\; \frac{2\pi}{\log p}. \tag{9.17}$$

Then, as $1-p^{-x}$ is comparable to $x$ on $[0,2M]$, we see that the numerator in (9.15) is comparable to $\sigma\upsilon$, and the first term in the denominator is comparable to $(\sigma+\upsilon)^2$. As for the second term, a Taylor series argument shows that for $t-y$ close to 0,

$$1-\cos[\log p(t-y)] \;\approx\; (t-y)^2. \tag{9.18}$$

Continuity and compactness show that (9.18) remains true (with some constants) if (9.17) holds, as the left-hand side can then vanish only at $t-y=0$. This gives us the second part of the lemma, where we only need to use the prime $p=2$.

If (9.17) fails with $p=2$, there will be points where $t\neq y$ but

$$1-\cos[\log 2(t-y)] \;=\; 0.$$

However, one cannot simultaneously have

$$1-\cos[\log 3(t-y)] \;=\; 0,$$

since $\log 2$ and $\log 3$ are rationally linearly independent. So by compactness and continuity again, we get (9.16). $\square$

PROOF OF THM. 9.10: Suppose $(s_j)$ is bounded and interpolating for $H^\infty(\Omega_0)$. Then by Theorem 9.5 and Lemma 9.13, we have

$$\inf_j \prod_{i \neq j} \rho_2(s_i, s_j) \ \geq \ 0.$$

Therefore by Theorem 9.11, the sequence $((2^{-s_j}, 3^{-s_j}))$ is interpolating for $H^\infty(\mathbb{D}^2)$. So if $(a_j)$ is any target in $\ell^\infty$, there exists some $\psi \in H^\infty(\mathbb{D}^2)$ satisfying

$$\psi(2^{-s_j}, 3^{-s_j}) \ = \ a_j.$$

Then

$$\phi(s) \ = \ \psi(2^{-s}, 3^{-s})$$

solves the interpolation problem in $H^\infty(\Omega_0) \cap \mathcal{D}$.

Finally, if the vertical height of a rectangle containing all the points is less than $2\pi / \log 2$, the second part of Lemma 9.13 shows that one can interpolate with a function of the form $\psi(2^{-s})$, where $\psi \in H^\infty(\mathbb{D})$. $\square$

## 9.2. Interpolating sequences in Hilbert spaces

Let $\mathcal{H}_k$ be a reproducing kernel Hilbert space on a set $X$. Given a sequence $(\lambda_i)$ in $X$, let $g_i$ denote the normalized kernel function at $\lambda_i$:

$$g_i \ := \ \frac{1}{\|k_{\lambda_i}\|} k_{\lambda_i}.$$

Define a linear operator $\mathcal{E}$ by

$$\mathcal{E} \ : \ f \mapsto \langle f, g_i \rangle. \tag{9.19}$$

We say the sequence $(\lambda_i)$ is an *interpolating sequence for* $\mathcal{H}_k$ if the map $\mathcal{E}$ is into and onto $\ell^2$. (Note that because $g_i$ is normalized, $\mathcal{E}$ necessarily maps into $\ell^\infty$; but it does not have to map into $\ell^2$).

We say that a set of vectors $\{v_i\}$ in a Banach space is *topologically free* if no one is contained in the closed linear span of the others. This is equivalent to the existence of a dual system, vectors $\{h_i\}$ in the dual satisfying

$$\langle h_j, v_i \rangle \ = \ \delta_{ij}.$$

The dual system is called *minimal* if each $h_j$ is in $\vee\{v_i\}$.

THEOREM 9.20. *The sequence $(\lambda_i)$ is an interpolating sequence for $\mathcal{H}_k$ if and only if the Gram matrix $G = \langle g_j, g_i \rangle$ is bounded and bounded below.*

PROOF: ($\Rightarrow$) Suppose $\mathcal{E}$ is bounded and onto $\ell^2$. As $\mathcal{E}^* e_j = g_j$, we have

$$\langle g_j, g_i \rangle \;=\; \langle \mathcal{E}\mathcal{E}^* e_j, e_i \rangle$$

is bounded. By the open mapping theorem, $\mathcal{E}$ has an inverse

$$\mathcal{E}^{-1} : \ell^2 \;\to\; \vee\{g_i\} \subseteq \mathcal{H}_k.$$

Let $h_j = \mathcal{E}^{-1} e_j$. Then

$$G^{-1} \;=\; \langle h_j, h_i \rangle$$

is bounded.

($\Leftarrow$) Suppose $G$ is bounded and bounded below. Define

$$\begin{array}{rcl} L : \ell^2 & \to & \mathcal{H}_k \\ e_j & \mapsto & g_j. \end{array}$$

Since $G$ is bounded, $L$ is bounded, and $\mathcal{E} = L^*$ is therefore a bounded map into $\ell^2$. Since $G$ is bounded below, the minimal dual system $\{h_j\}$ to $\{g_j\}$ has a bounded Gram matrix (see Exercise 9.37), and if $(a_j)$ is any sequence in $\ell^2$, we have

$$\mathcal{E}\left(\sum a_j h_j\right) \;=\; (a_j),$$

so $\mathcal{E}$ is onto. $\square$

THEOREM 9.21. *Any interpolating sequence for* $\mathrm{Mult}\,(\mathcal{H}_k)$ *is an interpolating sequence for* $\mathcal{H}_k$.

PROOF: Suppose $(\lambda_i)$ is an interpolating sequence for $\mathrm{Mult}\,(\mathcal{H}_k)$. Then there is a constant $M$ such that for every sequence $(w_i)$ in the unit ball of $\ell^\infty$, the map

$$R : g_i \;\mapsto\; \bar{w}_i g_i$$

extends to a linear operator on $\mathcal{H}_k$ of norm at most $M$ (since it is the adjoint of a multiplication operator that solves the interpolation problem). Therefore, for all finite sequences of scalars $(c_j)$, we have

$$\|\sum c_j \bar{w}_j g_j\|^2 \;\leq\; M^2 \|\sum c_j g_j\|^2.$$

Write this as

$$\sum_{i,j} c_j \bar{c}_i \bar{w}_j w_i \langle g_j, g_i \rangle \;\leq\; M^2 \sum_{i,j} c_j \bar{c}_i \langle g_j, g_i \rangle,$$

let $w_j = e^{2\pi i t_j}$ and integrate with respect to each $t_j$ to get

$$\sum |c_j|^2 \leq\; M^2 \sum_{i,j} c_j \bar{c}_i \langle g_j, g_i \rangle.$$

This proves $G$ is bounded below. A similar argument, with $w_j = e^{2\pi i t_j}$ and $c_j = a_j e^{2\pi i t_j}$ gives

$$\sum_{i,j} a_j \bar{a}_i \langle g_j, g_i \rangle \ \leq \ M^2 \sum |a_j|^2,$$

so $G$ is also bounded. By Theorem 9.20, we are done. □

Interpolating sequences for $\mathcal{H}^2$ and $\mathcal{H}^2_w$ where the weights $w_n$ are $(\log n)^\alpha$, as in (6.34), are studied in [**OS08**]. In particular, they show that for bounded sequences, the interpolating sequences are the same as in the corresponding space of analytic functions that do not have to have Dirichlet series representations.

## 9.3. The Pick property

A particularly useful feature of the Hardy space $H^2$ is that it has the Pick property.

Definition 9.22. The reproducing kernel Hilbert space $\mathcal{H}_k$ on $X$ has the Pick property if, for every subset $F \subseteq X$, and every function $\psi : F \to \mathbb{C}$, if the linear operator defined by

$$T : k_\lambda \mapsto \overline{\psi(\lambda)} k_\lambda$$

is bounded by $C$ on $\vee\{k_\lambda : \lambda \in F\}$, then there is a multiplier $\phi$ of $\mathcal{H}_k$, with multiplier norm bounded by $C$, and satisfying

$$\phi(\lambda) \ = \ \psi(\lambda) \quad \forall \, \lambda \in F.$$

Theorem 9.23. *If $\mathcal{H}_k$ has the Pick property, then the interpolating sequences for* Mult $(\mathcal{H}_k)$ *and $\mathcal{H}_k$ coincide.*

Proof: Suppose $(\lambda_i)$ is an interpolating sequence for $\mathcal{H}_k$, so there are constants $c_1$ and $c_2$ so that

$$c_1 \sum |a_i|^2 \ \leq \ \| \sum a_i g_i \|^2 \ \leq \ c_2 \sum |a_i|^2.$$

Let $(w_i)$ be a sequence in the unit ball of $\ell^\infty$. Define $R$ by

$$R : g_i \ \mapsto \ \bar{w}_i g_i.$$

Then

$$\begin{aligned}
\langle \left[ \frac{c_2}{c_1} - R^* R \right] g_j, g_i \rangle \quad &= \quad \frac{c_2}{c_1} \langle g_j, g_i \rangle - w_i \bar{w}_j \langle g_j, g_i \rangle \\
&\geq \quad \frac{c_2}{c_1} (c_1 \delta_{ij}) - w_i \bar{w}_j (c_2 \delta_{ij}) \\
&= \quad c_2 \delta_{ij} (1 - |w_i|^2) \\
&\geq \quad 0.
\end{aligned}$$

Therefore $R$ is bounded by $\sqrt{c_2/c_1}$, so by the Pick property, there is a multiplier $\phi$ of $\mathcal{H}_k$ with norm bounded by $\sqrt{c_2/c_1}$ such that $\phi(\lambda_i) = w_i$. $\square$

The idea of using the Pick property to reduce the characterization of interpolating sequences for a multiplier algebra to the more tractable problem of characterizing them for a Hilbert space was originally due to H.S. Shapiro and A. Shields, in the case of $H^\infty(\mathbb{D})$ [**SS61**]. It was developed more systematically by D. Marshall and C. Sundberg in [**MS94**].

The space $\mathcal{H}^2$ does not have the Pick property — one way to see this is that the bounded interpolating sequences for $\mathcal{H}^2$ are interpolating sequences for $H^2(\Omega_{1/2})$ [**OS08**], whereas bounded interpolating sequences for the multiplier algebra can only accumulate on the boundary of $\Omega_0$ by Theorem 9.10. However, there are several Hilbert spaces of Dirichlet series that have the Pick property (and a stronger, matrix-valued version, called the complete Pick property).

THEOREM 9.24. *If $k(s,u) = \eta(s + \bar{u})$, then this has the complete Pick property for each of the following $\eta$'s:*

$$\eta(s) = \frac{1}{2 - \zeta(s)} \tag{9.25}$$

$$\eta(s) = \frac{\zeta(s)}{\zeta(s) + \zeta'(s)}$$

$$\eta(s) = \frac{\zeta(2s)}{2\zeta(2s) - \zeta(s)}$$

$$\eta(s) = \frac{P(2)}{P(2) - P(2+s)}. \tag{9.26}$$

In (9.26), the function $P(s)$ is the *prime zeta function* , defined by

$$P(s) = \sum_{p \in \mathbb{P}} p^{-s}.$$

DEFINITION 9.27. A sequence $(\lambda_i)$ satisfies Carleson's condition in the reproducing kernel Hilbert space $\mathcal{H}_k$ if there exists a constant $C$ so that

$$\sum_i \frac{|f(\lambda_i)|^2}{\|k_{\lambda_i}\|^2} \leq C\|f\|^2 \quad \forall\ f \in \mathcal{H}_k.$$

DEFINITION 9.28. The sequence $(\lambda_i)$ is weakly separated in the reproducing kernel Hilbert space $\mathcal{H}_k$ if there exists a constant $c > 0$ so that, for all $i \neq j$, the normalized reproducing kernels satisfy

$$|\langle g_i, g_j \rangle| \leq 1 - c.$$

THEOREM 9.29. [**AHMR17**] *Let $\mathcal{H}_k$ have the complete Pick property. Then a sequence $(\lambda_i)$ is an interpolating sequence if and only if it is weakly separated and satisfies Carleson's condition.*

## 9.4. Sampling sequences

DEFINITION 9.30. Let $\mathcal{K}$ be a Hilbert space of functions on a set $X$ with bounded point evaluations and denote the reproducing kernel at $\zeta \in X$ by $k_\zeta$. We say that a sequence $\{z_n\}_n \subset X$ is a *sampling sequence*, if for all $f \in \mathcal{K}$, we have

$$\sum_n \frac{|f(z_n)|^2}{\|k_{z_n}\|^2} \approx \|f\|_{\mathcal{K}}^2.$$

Equivalently, one can say that the operator $E : \mathcal{K} \to \ell^2$ given by $E : f \mapsto \left(\frac{f(z_n)}{\|k_{z_n}\|}\right)_n$ is bounded and bounded below. Another way to rephrase this is to require that the sequence of normalized reproducing kernels $\left\{\frac{k_{z_n}}{\|k_{z_n}\|}\right\}_n$ forms a *frame*.

For weighted Bergman spaces on the disk there is a complete description of sampling sequences in terms of lower density of the sequence.

PROPOSITION 9.31. *There are no sampling sequences for the Hardy space of the disk.*

*Proof:* Suppose that $\{z_n\}_n \subset \mathbb{D}$ is a sampling sequence for $H^2(\mathbb{D})$. Then $\{z_n\}_n$ cannot be a Blachke sequence, since the corresponding Blaschke product $f$ would satisfy $0 < \|f\|_2 < \infty$ and $\sum_n |f(z_n)|^2 \cdot \|k_{z_n}\|^{-2} = 0$. If $\{z_n\}_n$ is not a Blaschke sequence, we consider the function $f(z) = 1$. Then

$$\sum_n \frac{|f(z_n)|^2}{\|k_{z_n}\|^2} \;=\; \sum_n (1 - |z_n|^2) \;=\; \infty,$$

while $\|f\|_2 < \infty$, a contradiction. $\square$

However, there exists "generalized sampling sequences" for the Hardy space, that is, sequences satisfying

$$|f(0)|^2 + \sum_n \frac{|f'(z_n)|^2}{\|\tilde{k}_{z_n}\|^2} \;\approx\; \|f\|_2^2,$$

where $\tilde{k}_{z_n}$ is the reproducing kernel for the derivative. But this is because differentiation maps the Hardy space (modulo constants) isometrically onto a weighted Bergman space.

There is another proof of the fact that $H^2(\mathbb{D})$ does not admit any sampling sequence, using the fact that multiplication by $z$ is isometric.

The same idea works for $\mathcal{H}^2$. The reader is invited to recast that proof for the Hardy space.

PROPOSITION 9.32. *There are no sampling sequences on $\mathcal{H}^2$.*

*Proof:* The multiplication operator $M_{N^{-s}}$ is an isometry on $\mathcal{H}^2$. On the other hand, the sequence $f_N := M_{N^{-s}} f$ tends to 0 uniformly on compact sets in $\Omega_{1/2}$. Thus $\sum_n \frac{|f_N(s_n)|^2}{\|k_{s_n}\|^2} \to 0$ as $N \to \infty$ and thus cannot be comparable to $\|f_N\|^2 = \|f\|^2$. □

A similar argument shows that "generalized sampling sequences" involving $f'(s_n)$ do not exists.

QUESTION 9.33. Is there a sensible interpretation of "$\{s_n\}_n$ is a generalized sampling sequence for $\mathcal{H}^2$"? If so, how are these characterized?

## 9.5. Exercises

EXERCISE 9.34. Prove Equation (9.4). (Hint: use an automorphism of $\mathbb{D}^m$ to move one point to the origin).

EXERCISE 9.35. Prove that any sequence that tends sufficiently quickly to $\partial\mathbb{D}$ is an interpolating sequence for $H^\infty(\mathbb{D})$.

EXERCISE 9.36. Fill in the details of the proof of (9.16).

EXERCISE 9.37. Prove that if $(h_i)$ is the minimal dual system of $(g_i)$, then the inverse of the Gram matrix $G = \langle g_j, g_i \rangle$ is the matrix $\langle h_j, h_i \rangle$.

## 9.6. Notes

For a much more comprehensive treatment of interpolating sequences, we recommend the excellent monograph [**Sei04**] by K. Seip. For a concise treatment for $H^\infty(\mathbb{D})$ and $H^2(\mathbb{D})$, including Theorem 9.20, see [**Nik85**].

Axler's theorem 9.2 was proved for multipliers of the Dirichlet space [**Axl92**], but the argument readily adapts to the stated version. Carleson's theorem is in [**Car58**]. Seip's paper [**Sei09**] contains much more information on interpolating sequences for $\mathcal{H}^\infty$ than Theorem 9.10 alone.

Necessary and sufficient conditions for a sequence to be interpolating for $H^\infty(\mathbb{D}^2)$ are given in [**AM01**], but they do not completely resolve the issue. For example, the following is still open:

QUESTION 9.38. If $\lambda_n$ is strongly separated in $H^\infty(\mathbb{D}^2)$, is it an interpolating sequence?

Interpolating sequences for $\mathcal{H}^2$ and $\mathcal{H}^2_w$ were first considered in [**OS08**]. See also [**Ols11**].

The fact that (9.25) gives rise to a Pick kernel was observed in [**McCa04**]. Necessary and sufficient conditions for a general kernel to have the complete Pick property, a matrix valued version of the Pick property, are given by P. Quiggin [**Qui93, Qui94**] and S. McCullough [**McCu92, McCu94**]; see also [**AM00**]. The application to kernels of the form discussed in Theorem 9.24 is discussed in [**?**]. The kernel coming from (9.26) is particularly interesting, as it is in some sense universal amongst all kernels with the complete Pick property. See [**?**] for details.