ON THE STRUCTURE AND COMPLEX ANALYSIS OF DIRICHLET SERIES

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I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

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Preface

> “There is a subject in mathematics (it’s a perfectly good and valid subject and it’s perfectly good mathematics) which is misleadingly called Analytic Number Theory. In a sense, it was born with Riemann who was definitely not a number-theorist; it was carried on, among others, by Hadamard, and later by Hardy, who were also not number-theorists (I knew Hadamard well); and to the best of my understanding, analytic number theory is not number-theory.”
> – Andre Weil in “Two Lectures on Number Theory, Past and Present” [99]

This quote, wholly unfair but still remarkable to modern sensibilities, points to the curious interplay between the analytic and algebraic sides of number theory. A number of results that began as curiosities or seemingly random tricks became greatly formalized, becoming a foundation of much of modern number theory.

In the first chapter we provide a general overview of some of the theory and history of Dirichlet series and L-functions, and their relations to other mathematical objects.

The second chapter deals with those results that come from pure complex analysis, neither fundamentally requiring either a Dirichlet series nor a functional equation but using only the regularity of behaviour that these provide. I also discuss the limits of such an approach. The main theorems (1) and (2) restrict the behaviour of an analytic function in two parts of its domain, based on how large the domain itself is. Applied to $\log \zeta(s)$ this provides another proof of Littlewood’s Theorem that the spacing of zeta zeros at height T is unconditionally $O(1/\log_3(T))$. While Littlewood’s proof is restricted to finding rectangular strips as zero-free regions, this approach explicitly gives a bound given any candidate shape for a zero-free region. The approach also gives more refined results provided information on growth rates in the critical strip. Finally a careful analysis of gamma factor asymptotics is done in (4) to give a bound on low-lying zeros of L-functions, uniform in all of the parameters.

The third chapter studies Dirichlet series that analytically continue with some given control on the growth of the analytic continuation (resembling the growth of degree 1 L-functions). Theorem 7 shows these to exactly correspond to power series that analytically continue slightly outside their disc of convergence. As a corollary this shows that growth rate strictly between degree 0 and 1 is impossible, even not assuming a functional equation. Section 3.2.1 then uses this to show that all Dirichlet series with growth rate bounded by that of the gamma function are continuous linear combinations of Hurwitz zeta functions, and discusses the consequence for the Lindelof hypothesis. Attempts to bring in the Euler product relate to a conjecture of Schwarz from 1978. It is also shown in Theorem 10 how the Dirichlet series having a functional equation yields an analytic continuation of the power series to almost the entire plane and as Corollary 2 this provides a new proof of the classification of degree 1 elements of the extended Selberg class.

The fourth chapter expands on theorem 10 from chapter 3 and in Theorem 12 a degree d functional equation is shown to be equivalent to a generalized Fourier series having analytic continuation and some given singularities. Unlike in the degree 1 case this does not yield a complete classification but in Section 4.2 partial results are presented for degree 2. A method for recovering an eigenbasis is presented, and applied to the space of cusp forms on $\Gamma_0(N)$ via the Eichler-Selberg trace formula.

The fifth chapter discusses Selberg zeta functions, which also have analytic continuation and are known to satisfy a Riemann Hypothesis, but are higher order functions so their complex analysis is very different. Theorem uses spectral bounds to bound the Selberg zeta function on the critical line, similarly to the results of chapter two.

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I also would not have travelled so far on this path without the support of family, nor for the excellent instruction and support of Tom Griffiths while I was in middle school and high school. Finally, I was in the past at least a bit confused by the acknowledgements of significant others, but now I truly understand and appreciate how important it is to have such a patient, supportive and loving girlfriend, Phan Ha.

Contents

Chapter 1

Introduction

Dirichlet series $L(s) := \sum_{n=1}^{\infty} a_n n^{-s}$ are important as generating functions in number theory, similarly to how standard power series $\sum_{n=1}^{\infty} a_n z^n$ are important in combinatorics [100]. However, analytic properties of Dirichlet series are still much more mysterious than the case of power series. Power series have a well-defined radius of convergence. For Dirichlet series past the abscissa of absolute convergence, it is difficult to tell how far the function can extend analytically. It is difficult to recognize if a Dirichlet series defines an entire function, although some results will be shown when there is a specific growth rate in the region of analytic continuation. The space of functions defined by power series on a given domain defined by power series consists of all analytic functions on the domain, and the zero set may be an arbitrary discrete subset of the domain. For Dirichlet series it is unknown whether there even can be a half-plane with a finite number of zeros. It is unclear what possible growth-rates on vertical lines are, what restrictions on the locations of the zeros this places, and how a Dirichlet series with its zeros on a line can behave to the right of that line. Many typical Dirichlet series studied have extra structure in the form of a functional equation, or an Euler product, and often come from integral transforms of certain special functions on $GL(n)$. This brings up the further questions of understanding what types of functional equations are possible, understanding how to characterize functions with a certain functional equation, and the question of how much of the original $GL(n)$ structure can be canonically recovered.

1.1 Historical Background and Some Motivating Problems

The prime number theorem states that the counting function of the primes $\pi(x) := \sum_{p\leq x} 1$ is asymptotically $\frac{x}{\log x}$.

If we define the counting function $\Psi(x) := \sum^*_{p^k\leq x} \log p$ of the prime powers with weight $\log p$ the corresponding result is that $\Psi(x)$ is asymptotically x.

Riemann in 1859 outlined how to prove the prime number theorem based on his zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p \text{ prime}} (1-p^{-s})^{-1}.$$

This series is defined for $\Re(s) > 1$ but the function has analytic continuation to all $s \neq 1$, and satisfies the functional equation

$$\Lambda(s) := \zeta(s)\Gamma(s/2)\pi^{-s/2} = \Lambda(1-s).$$

This functional equation is equivalent to the Poisson Summation Formula for arbitrary test function f,

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \hat{f}(m),$$

in the sense that each can prove the other directly.

Furthermore there is the Von Mangoldt explicit formula

$$\Psi(x) = x - \log(2\pi) - \log(1-x^{-2})/2 - \sum_{\rho} \frac{x^{\rho}}{\rho}.$$

The sum is over the non-trivial zeros $\rho = \sigma + iT$ of the Riemann Zeta function, and is really a Fourier decomposition with frequencies T and amplitudes x^{σ}/ρ.

Big open problems, and the limits of the conjectures

There are three particularly long-standing and difficult questions about the Riemann Zeta function.

Definition 1. *(Riemann Hypothesis) All non-trivial zeros of the zeta function have real part* $1/2$*. This is equivalent to having an error bound of* $x^{1/2+\epsilon}$ *in the prime number theorem.*

Definition 2. *(Lindelof Hypothesis)* $\zeta(1/2 + iT) \ll |T|^{\epsilon}$*. This is implied by the Riemann Hypothesis.*

Definition 3. *(GUE Conjecture) The (suitably rescaled) zeros of the Riemann Zeta Function are distributed like the eigenvalues of large random matrices from the Gaussian Unitary Ensemble.*

Based on the GUE Conjecture we have very fine conjectures for the zeta zeros. If, however, there were infinitely many Siegel zeros (particular contradictions to the General Riemann Hypothesis) then we would be in the following very different situation.

Definition 4. *(The Alternative Hypothesis) Instead of there being a nice smooth probability measure for the normalized adjacent zero spacings, the limiting distribution (normalized to have mean 1) is supported on integers or half-integers.*

The Alternative Hypothesis is generally not expected to be true, but it demonstrates the limitations of current techniques.

It is expected that all L-functions in the Selberg Class satisfy a Riemann Hypothesis. It would be interesting to determine if there is any analytic function with similar growth rates satisfying a Riemann Hypothesis

Question 1. *Is there an entire function* $f : \mathcal{C} \to \mathcal{C}$ *such that for some* $\delta > 0$ *the following are satisfied.*

1. $f(z)$ *is bounded when* $\Re z > 1 + \delta$*.*
2. $f(z)$ *is unbounded when* $\Re z = 1$*.*
3. $f(z)$ *grows polynomially in vertical strips: for all* σ *there is* $C_\sigma > 0$ *so that* $|f(\sigma + it)| \ll |t|^{C_\sigma}$*.*

4. *(weak RH)* $f(z)$ *does not vanish when* $\Re z > \frac{1}{2}$.

5. *(strong RH)* $f(z)$ *vanishes only when* $\Re(z) = \frac{1}{2}$ *or for* $\Im(z)$ *in a finite set.*

The road to the Lindelof Hypothesis is often seen to pass through the Riemann Hypothesis it is conceivable that every suitable Dirichlet Series satisfies the Lindelof Hypothesis, which motivates this investigation into the general theory and structure of Dirichlet series.

Question 2. *If* $L(s) = \sum a_n n^{-s}$ *extends to an entire function of order 1 and the coefficients satisfy* $a_n = O(n^\epsilon)$ *for every* $\epsilon > 0$ *is it true that* $L(1/2 + iT) = O(T^\epsilon)$ *for all* $\epsilon > 0$*?*

Dirichlet L-functions

In 1837, predating Riemann's work on the Zeta function, Dirichlet introduced his series

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p (1 - \chi(p)p^{-s})^{-1},$$

where χ is a Dirichlet character modulo k, a character of the multiplicative group $\mathbb{Z}/N\mathbb{Z}$ extended to the natural numbers by periodicity.

These share many properties with the Zeta function, and satisfy a functional equation for either $a = 0$ or $a = 1$, depending on parity,

$$\Lambda(s,\chi) := \left(\frac{\pi}{k}\right)^{-(s+a)/2} \Gamma\left(\frac{s+a}{2}\right) L(s,\chi) = \Lambda(1-s,\overline{\chi})\tau(\chi)k^{-1/2}i^{-a}.$$

Here $\tau(\chi) = \sum_{n=1}^{k} \chi(n)\exp(2\pi i n/k)$ and $|\tau(\chi)| = \sqrt{k}$.

These also conjecturally satisfy a Riemann Hypothesis and Lindelof Hypothesis and were most prominently used by Dirichlet to show that there are infinitely many primes congruent to a mod k, as long as a and k are relatively prime. Dirichlet did not study these as analytic functions, so the analytic continuation and functional equation would come later, and fit very nicely into the modern framework of Tate's thesis [93].

As another example of their application, the Miller-Rabin primality test is known to run in polynomial time only if these $L(s,\chi)$ satisfy the Riemann Hypothesis, although the more recent AKS algorithm [2] has unconditional polynomial running time.

Dedekind Zeta Functions

Let K be an algebraic number of field of degree $n = r_1 + 2r_2$ with r_1 real embeddings and $2r_2$ complex embeddings. The Dedekind Zeta function for $K/\mathbb{Q}$ is defined as

$$\zeta_K(s) = \sum F(m)m^{-s} = \sum_{\mathfrak{a}\neq 0}(N\mathfrak{a})^{-s} = \prod_{\mathfrak{p}}(1-(N\mathfrak{p})^{-s})^{-1},$$

where $F(m)$ denotes the number of nonzero integral ideals of norm m in K.

Defining the completed zeta function to be $\Phi(s) := B^s\Gamma(\frac{1}{2}s)^{r_1}\Gamma(s)^{r_2}\zeta_K(s)$, there is the functional equation $\Phi(s) = \Phi(1-s)$.

The zeta functions for imaginary quadratic fields are the same as for modular forms and for elliptic curves. The zeta functions for real quadratic fields give the same functions as one gets from Maass forms.

These can also be placed in a broader context as follows. The field K, as a vector space over $\mathbb{Q}$, is a n-dimensional representation of $\mathrm{Gal}(K/\mathbb{Q})$ under the standard action. The Artin L-function for this representation is the same as ζ_K. In particular if $\mathrm{Gal}(K/\mathbb{Q})$ is abelian then the irreducible representations are 1 dimensional, and the decomposition of the standard representation into irreducibles gives a decomposition $\zeta_K(s) = \zeta(s)\prod L(s,\chi)$, for some collection of $L(s,\chi)$.

Ramanujan τ function

Consider the power series defined by the infinite product,

$$\sum_{n=1}^{\infty}\tau(n)q^n = q\prod_{n\geq 1}(1-q^n)^{24} = \Delta(z).$$

Ramanujan conjectured in 1916 that the coefficients $\tau(n)$ satisfy the following.

- (Multiplicativity) $\tau(mn) = \tau(m)\tau(n)$ if $\gcd(m,n) = 1$.
- $\tau(p^{r+1}) = \tau(p)\tau(p^r) - p^{11}\tau(p^{r-1})$ for p prime and $r > 0$.
- (Ramanujan Hypothesis) $|\tau(p)| \leq 2p^{11/2}$ for all primes p.

The first two conjectures are a consequence of $\Delta(z)$ being the unique (up to a constant multiple) holomorphic cusp form of weight 12 and level 1 and were proved by Mordell in 1917. The third was proved by Deligne in 1974 as a consequence of his proof of the Weil conjectures.

The corresponding Dirichlet series

$$L(s) := \sum_{n=1}^{\infty} \frac{\tau(n)}{n^{s+11/2}}$$

has a degree 2 functional equation

$$\Lambda(s) := L(s)(2\pi)^{11/2-s}\Gamma(s+11/2) = \Lambda(1-s).$$

Relation to summation formulae

The functional equation for $\zeta(s)$ is equivalent to the Poisson summation formula for the Fourier transform, which can be written as

$$\sum_{n\in\mathbb{Z}} \delta_n \xmapsto{\mathcal{F}} \sum_{n\in\mathbb{Z}} \delta_n.$$

For a primitive Dirichlet character χ mod q, the functional equation for $L(s,\chi)$ is equivalent to the Twisted Poisson summation formula

$$\sum_{n\in\mathbb{Z}} \chi(n)\delta_{n/q} \xmapsto{\mathcal{F}} \tau\chi \sum_{n\in\mathbb{Z}} \overline{\chi}(-n)\delta_n.$$

The functional Equation for $\zeta(s)^2$ is equivalent to the Voronoi summation formula for the Fourier-Bessel transform, written as

$$\sum_{n\in\mathbb{N}} d(n)\delta_{\sqrt{n}} \overset{\mathcal{FB}}{\longmapsto} \log + 2\gamma + \sum_{n\in\mathbb{N}} d(n)\delta_{\sqrt{n}}.$$

Since it is possible to construct infinitely many degree d Dirichlet series $L(s)$ so that $L(s/d)$ has the same functional equation as the Zeta function [20], the relation gives a twist of Poisson summation formula by coefficients of $L(s)$. Namely, for every d there are complex $a_{n^{1/d}}, b_{n^{1/d}}$ and positive Q so that

$$\sum_{n\in\pm\mathbb{N}^{1/d}} a_{|n|}\delta_n \overset{\mathcal{F}}{\longmapsto} \sum_{n\in\pm\mathbb{N}^{1/d}} b_{|n|}\delta_{Qn}.$$

The degree conjecture then claims that d must be a natural number, which potentially says something interesting about resonances at non-integers. Such sums are sometimes called aperiodic Dirac combs and 1-dimensional quasi-crystals, but are typically studied for support having bounded lower density.

1.2 Axiomatization: the Selberg Class

The class $\mathcal{S}_d$ of Dirichlet Series of degree d in the Selberg class consists of those Dirichlet series $L(s) = \sum a_n n^{-s}$ satisfying the following properties.

1. (Analyticity) $(s-1)^m L(s)$ is an entire function of finite order for some non-negative integer m.

2. (Ramanujan Hypothesis) $a_n \ll n^\epsilon$ for any fixed $\epsilon > 0$.

3. (Functional Equation) There must be a function $\gamma(s)$ of the form $\gamma(s) = \epsilon Q^s \prod_{i=1}^r \Gamma(\lambda_i s + \mu_i)$ where $|\epsilon| = 1$, $Q > 0$, $\lambda_i > 0$, $2\sum_{i=1}^r \lambda_i = d$, and $\Re\mu_i \geq 0$ such that
$$\Lambda(s) := \gamma(s)L(s) = \overline{\Lambda}(1-s).$$

4. (Euler product) $a_1 = 1$ and

$$\log F(s) = \sum_{n=p^k}^{\infty} \frac{b_n}{n^s},$$

where the sum ranges over prime powers and $b_n \ll n^\theta$ for some $\theta < 1/2$.

If L and L' are two different elements of the Selberg class that cannot be expressed as products of simpler elements, Selberg made a number of conjectures [87]. There is conjectured to be regularity, $\sum_{p\le x} |a_p|^2/p \sim \log\log x + O(1)$, and orthonormality, $\sum_{p\le x} \frac{a_p a'_p}{p} = O(1)$, of the coefficients at primes. It is conjectured that the Riemann Hypothesis holds for all such L. It is also conjectured that the degree d must be an integer, which is currently known for $d \le 2$ [57]. The first two conjectures imply both the Dedekind conjecture and the Artin conjecture. It is also expected that everything in the Selberg class comes from an automorphic L-function on $GL(n, \mathbb{R})$.

1.3 Recovering greater structure

Hecke's Correspondence Theorem

Let $\{a_n\}$ and $\{b_n\}$ be complex sequences $O(n^c)$ for some $c > 0$. Let $\lambda > 0$, $k \in \mathbb{R}$, $\gamma \in \mathbb{C}$. For $\sigma > c + 1$ define the functions

$$L_a(s) = \sum a_n n^{-s} \quad \text{and} \quad L_b(s) = \sum b_n n^{-s},$$

and the completed functions

$$\Lambda_a(s) = (2\pi/\lambda)^{-s}\Gamma(s)L_a(s) \quad \text{and} \quad \Lambda_b(s) = (2\pi/\lambda)^{-s}\Gamma(s)L_b(s).$$

For $\tau \in \mathcal{H}$, define the corresponding Fourier series,

$$f_a(\tau) = \sum_{n=1}^{\infty} a_n e(n\tau/\lambda) \quad \text{and} \quad f_b(\tau) = \sum_{n=1}^{\infty} b_n e(n\tau/\lambda).$$

Then the following two assertions are equivalent [7].

1. f_a and f_b are involutions of each other: $f_a(\tau) = \gamma(\tau/i)^{-k} f_b(-1/\tau)$.

2. $L_a(s)$ is entire and bounded in vertical strips. Moreover, the completed functions are related by
$$\Lambda_a(s) = \gamma \Lambda_b(k - s).$$

Furthermore, define $M_0(\lambda, k, \gamma)$ to be the set of such self-symmetric functions, $f(\tau) = \sum a_n e(n\tau/\lambda)$ such that

1. $f(-1/\tau) = \gamma(\tau/i)^k f(\tau)$ where $k > 0$ and $\gamma = \pm 1$.

2. $a_n = O(n^c)$ for some c as $n \to \infty$.

This is another way of saying that f is an entire automorphic form of weight k and multiplier γ with respect to $G(\lambda)$, the group of linear transforms generated by $T : \tau \mapsto -1/\tau$ and $S_\lambda : \tau \mapsto \tau + \lambda$.

Hecke [7] is able to fully classify and construct these spaces $M_0(\lambda, k, \gamma)$. First, there is a fundamental domain for the action of $G(\lambda)$ only if $\lambda \geq 2$ or $\lambda = 2\cos(\pi/q)$ with $q \geq 3$ and integer, since otherwise the action is not discrete. If $\lambda > 2$ then $\dim M_0(\lambda, k, \gamma) = \infty$ for every $k > 0$ and $\gamma = \pm 1$. If $\lambda = 2\cos(\pi/q)$ then k must be of the form $\frac{4m}{q-2} + 1 - \gamma$ for $m \in \mathbb{N}$ and

$$\dim M(\lambda, k, \gamma) = 1 + \left[\frac{m + (\gamma - 1)/2}{q}\right]$$

Weil Converse Theorem

In addition to a functional equation for $L(s) := \sum a_n n^{-s}$ suppose that for given N there is an appropriate functional equation for all character twists $L(s, \chi) := \sum a_n \chi(n) n^{-s}$,

$$\Lambda(s, \chi) := (2\pi)^{-s}\Gamma(s)L(s, \chi) = i^k \chi(N) \frac{\tau(\chi)^2}{D} (D^2 N)^{-s+k/2} \overline{\lambda}(k - s, \overline{\chi}),$$

where χ is a character modulo m and m is relatively prime to N. Suppose also that all of this twisted Dirichlet series extend to entire functions for bounded on vertical lines. Then $f(\tau)$ is a modular form of weight k with respect to $\Gamma_0(N)$.

Recovering a Maass form (sketch)

Hecke's Correspondence deals with the case of a gamma factor $\Gamma(s)$, which is what you have for Dedekind Zeta functions of imaginary quadratic fields, and for elliptic curve L-functions. Maass developed the theory of non-holomorphic Maass forms to deal with the situation of the Dedekind Zeta function of a real quadratic field, where the gamma factor is instead $\Gamma(s/2)^2$.

The function $\sum a_n e(nx)$ is periodic but, unlike in the Hecke case, it has no functional equation under the transform $x \mapsto \lambda/x$. The function $\sum a_n K_0(ny)$, on the other hand, has a functional equation under the transform $y \mapsto \lambda/y$ but is not periodic.

Combine these together to form the function (appropriately made to be either even or odd)

$$f(x,y) = \sum_{n\in\mathbb{Z}} a_n e(nx) K_0(2\pi ny)$$

This function is an eigenfunction of the hyperbolic Laplacian.

The involution $(0,y) \mapsto (0,\lambda/y)$ is the restriction to $x = 0$ of the hyperbolic involution $(x,y) \mapsto \left(\frac{-\lambda x}{x^2+y^2}, \frac{\lambda y}{x^2+y^2}\right)$. The involution leaves the hyperbolic Laplacian invariant, so sends $f(x,y)$ to a similarly even or odd function that agrees with $f(x,y)$ on $x = 0$ and satisfies the same second order differential equation, so must be the same function.

In effect we have reconstructed a function on the higher dimensional space $\Gamma\backslash SL_2(\mathbb{R})$.

$\mathbf{SL}_3(\mathbb{Z})$ Converse Theorem

(Jacquet, Piatetski-Shapiro, and Shalika) Consider a Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ with $a_1 = 1$. Define $A(n,m)$ so that $A(n,1) = a_n$ and these coefficients satisfy the multiplicativity relations[36] that result in

$$\sum_{m,n=1}^{\infty} A(m,n) m^{-s} n^{-t} = \frac{L(s)\overline{L}(s)}{\zeta(s+t)}.$$

Suppose that all of the Dirichlet twists

$$\sum_{n=1}^{\infty} A(n,1)\chi(n) n^{-s}$$

satisfy a suitable degree 3 functional equation.

Then

$$\begin{aligned} f(z) &= \sum_{\gamma \in U_2(\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{Z})} \sum_{m_1=1}^{\infty} \sum_{m_2 \neq 0} \frac{A(m_1,m_2)}{m_1|m_2|} \\ &\quad \times W_{\text{Jacquet}}\left(\begin{pmatrix} m1|m_2| & & \\ & m_2 & \\ & & 1 \end{pmatrix} \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z, \nu, \Psi_{1,\frac{m_2}{|m_2|}} \right), \end{aligned}$$

is a Maass form for $\mathrm{SL}(3,\mathbb{Z})$. In other words it is a function on $\mathrm{SL}_3(\mathbb{Z})\backslash \mathrm{SL}_3(\mathbb{R})/O_3$ that is an eigenfunction of the center of the universal enveloping algebra.

1.4 Functional Equations

Since a number of examples of functional equation have been mentioned, it will be good here to introduce the general definitions that will be use throughout this thesis.

Definition: symmetric form

Consider two Dirichlet series $L_a(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ and $L_b(s) = \sum_{n=1}^{\infty} b_n n^{-s}$ absolutely convergent in some right half-plane, where $a_n, b_n \in \mathbb{C}$. Suppose $L_a(s)$ and $L_b(s)$ have analytic continuation to entire function with polynomial growth on vertical lines. Define Gamma-factors

$$\begin{aligned}
\gamma_a(s) &:= \prod_{i=1}^{r_a} \pi\Gamma\left(\lambda_{a,i}s + \mu_{a,i}\right) \\
\gamma_b(s) &:= \prod_{i=1}^{r_b} \Gamma\left(\lambda_{b,i}s + \mu_{b,i}\right)
\end{aligned}$$

and suppose that $L_a(s)$ and $L_b(s)$ are related by a functional equation where $\lambda_{a,i}, \lambda_{b,i}, Q_a, Q_b$ all exceed zero.

$$\begin{aligned}
\Lambda(s) &:= L_a(s)\gamma_a(s)Q_a^s \\
\Lambda(1-s) &= L_b(s)\gamma_b(s)Q_b^{s-1}
\end{aligned}$$

Definition: non-symmetric form

Equivalently, the functional equation can also be written in the non-symmetric form

$$L_a(s) = L_b(1-s)\gamma(1-s)\theta(1-s)Q^s,$$

where $Q = Q_b/Q_a$, there is the nonsymmetric gamma factor,

$$\begin{aligned}
\gamma(s) &:= \prod_{i=1}^{r} \Gamma\left(\lambda_i s + \mu_i\right) \\
&= \prod_{i=1}^{r_b} \Gamma\left(\lambda_{b,i}s + \mu_{b,i}\right) \prod_{i=1}^{r_a} \Gamma\left(\lambda_{a,i}s + 1 - \lambda_{a,i} - \mu_{a,i}\right),
\end{aligned}$$

and a product of sines and cosines that we denote

$$\begin{aligned} \theta(s) &:= \prod_{i=1}^{r_a} \sin\left(\pi\left(s\lambda_i - \mu_i - \lambda_i\right)/2\right) \\ &:= \sum \alpha_j e\left(\omega_j s/4\right), \end{aligned}$$

where $\{\omega_j\} = \{\pm\lambda_1 \pm \lambda_2 \pm \cdots\}$. The degree of the functional equation is defines to be $d := \sum \lambda_i = 2\sum \lambda_{a,i} = 2\sum \lambda_{b,i}$.

Let $q^{-1} = Q\prod_{i=1}^{r} \lambda_i^{\lambda_i}$. Let $\mu = \sum_{i=1}^{r_a}(\mu_i - 1/2)$.

Definition Gamma-like functions

Lemma 6 in Appendix A lead to the useful result for $1 = \sum_{i=1}^{r} \lambda_i$ that

$$\prod_{i=1}^{r} \Gamma(\lambda_i s + \mu_i) = \Gamma(s+\mu) q^s s^{\mu}(A + O(1/s)),$$

where $q = \prod_{i=1}^{r} \lambda_i^{\lambda_i}$, $\mu - 1/2 = \sum_{i=1}^{r}(\mu_i - \frac{1}{2})$, and $A = (2\pi)^{(r-1)/2}\prod \lambda_i^{\mu_i - 1/2}$.

To avoid the technical complication of dealing with these types of products directly, call a meromorphic function $g(s)$ "gamma-like" if $g(s) \sim \Gamma(s)(1 + O(1/s))$ away from poles of both sides.

Definition: Functional equation, fully general

Without loss of generality let $\mu = 0$.

Expanding out $\theta(s)$ and L_b we arrive at the expression

$$\sum a_n n^{-s} = \gamma(1-s)\sum_x c_x x^{1-s},$$

where x ranges over the set Spec $:= ne(\omega_j/4)q$, on the Riemann surface of $\log x$, and when $x = ne(\omega j/4)q$ then $c_x = b_n \alpha_j C$ for some constant C. Analogously to Kaczorowski and Perelli [52] we call this set the Spec of L. The exact form of $\gamma(1-s)$ will not matter and it may be replaced by similarly growing functions that are parametrized by x.

The most general context is convenient since non-linear twists do not retain a standard functional equation but they do retain a functional equation in this more general sense.

Definition 5. *(Most general functional equation)* $L(s) = \sum a_n n^{-s}$ *for some set Spec, complex* b_x *bounded by some power of* $|x|$, *and some uniformly Gamma-like* $\gamma_x(s)$ *satisfies the equality*

$$L(s) = \sum_{x \in Spec} b_x x^{s-1} \gamma_x(1 - ds + \mu).$$

By translating s *we may without loss of generality take* $\mu = 0$.

Conjectures

Conjecture 1. *The following is a list of conjectures in strictly increasing order of strength:*

1. *Selberg's degree conjecture that* $d \in \mathbb{N}$.
2. *All* ω_j *are in* $\mathbb{N}$ *and have equal parity. Equivalently Spec always lies over the real or imaginary axis, depending on the parity of* d.
3. *All* $\lambda_{a,i}$ *and* $\lambda_{b,i}$ *may be taken to be 1/2.*
4. *All* λ_i *can be taken to be 1.*

Chapter 2

Unconditional zero-spacing results

2.1 Introduction

The analytic theory of the zeta function, $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, was first developed by Riemann [84] who was motivated by its connection to the distribution of prime numbers, and used it to outline a proof of the prime number theorem. This connection is made fully explicit in Weil's Explicit Formula [48] that relates the summation of a test function over the log of prime powers to the summation of its Fourier transform over the zeros of $\zeta(s)$. Thus in theory, if not in practice, perfect knowledge of the zeros gives perfect knowledge of the primes.

The broad distribution of zeros is well understood. Let $N(T)$ count the number of of zeros with imaginary parts in $[0, T]$. Then

$$N(T) = \frac{T}{2\pi} \log\left(\frac{T}{2\pi e}\right) + \frac{7}{8} + S(T) + O(1/T),$$

where $\pi S(T) = \arg(1/2 + iT)$ is obtained by continuous variation on the straight line from $2 + iT$[42]. $S(T)$ is on average 0, meaning that the average gap between zeros is $2\pi/\log T$, and $S(T)$ measures the cumulative discrepancy. The GUE conjecture in Random matrix theory suggests a particularly nice limiting distribution for the normalized zero spacings, but this is very far from what can be proven. Being able to prove that there are infinitely many zero pairs within half of the average spacing would be number theoretically significant but as far as we know the Alternative Hypothesis could be true: the limiting distribution is supported on half-integers [31]. While typical values of $S(T)$ are on the order of $(\log\log T)^{1/2}$[79][86] and extreme values are as large as $\Omega\left(\sqrt{\frac{\log T}{\log\log T}}\right)$[74], $|S(T) < 2$ for $|T| < 6.8 \cdot 10^6$ and the largest value of $|S(T)|$ found computationally is around 3.2 [81]. Given that $S(T)$ jumps by 1 at each zero, this demonstrates how relatively tame $\zeta(s)$ is in the region in which we have been able to compute, particularly given that a counterexample of the Riemann Hypothesis would make $S(T)$ jump by 2.

Unconditionally $S(T) = O(\log T)$ (so the zero spacings are bounded) and conditional on the Riemann Hypothesis Littlewood showed [65] that $S(T) = O(\log T/\log\log T)$ so that the zero spacings are $O(1/\log\log T)$. This bound has subsequently been improved to $|S(T)| \leq (\frac{1}{2} + o(1))\frac{\log T}{\log\log T}$ by Goldston and Gonek[42], and to $|S(T)| \leq (\frac{1}{4} + o(1))\frac{\log T}{\log\log T}$ by Carneiro, Chandee, and Milinovich.

In his 1924 paper, "Two notes on the Riemann Zeta-function", Littlewood [66] proved that the maximal spacing between consecutive zeros at height T is bounded by $\frac{16}{\log\log\log T}$. Remarkably this is purely a result in complex analysis and not specific to the Zeta function, requiring as its only inputs that $\zeta(s)$ is bounded for $\Re(s) > 2$, of known size for $\Re(s) < -1$, not growing exponentially in the critical strip, and providing as its output that the domain of $\log\zeta(s)$ cannot be too large, and hence that zeros cannot be too spread out. Newer proofs are provided in [60] and [95], with [94] being a good reference. All of these proofs are based on a mix of the Three Circles Theorem, Borel-Caratheodory, and either a study of particular conformal maps or some cleverness with piecing together bounds on various discs.

Littlewood's result is outlined in section 1.1.

In Section 2 I give a new proof, essentially a continuous hybrid of the above proofs. Rather than being restricted to rectangular domains, the result constrains the modulus of an analytic function at two arbitrary parts of an arbitrary domain, based on how constrained the path between these is. Applied to a horizontal rectangular strip this gives the same result as above. Conditional on RH we can take a larger domain and determine a maximal spacing of $\ll \frac{1}{(\log\log T)^A}$. Extra information on the modulus of the function can be used as well, for example assuming that

$|\log\zeta(1/2+iT)| \ll (\log T)^{1-\epsilon}$ the maximal spacing is $\ll \frac{1}{(\log\log T)}$ unconditionally, and $\ll (\log T)^{-C}$ for some C conditional on the Riemann Hypothesis.

In section 3 I apply Littlewood's result to get a uniform bound for low-lying zeros in the Selberg class. While more or less a direct application, previously published bounds have all had implicit constants dependent on the degree or on the shape of the functional equation, so would not have worked. Instead of T the pertinent parameter becomes $q^{1/d}$ where d is the degree and q is essentially the analytic conductor. In particular the lowest lying zero goes to the real axis as long as $q^{1/d} \to \infty$. Without making any further assumptions this is the best possible result, since high powers of an L-function have ever increasing q but have their zeros bounded away from the real axis. It is still plausible that for primitive L-functions the lowest lying zero is $o(1)$ as $q \to \infty$, but this would be far more subtle to prove. For example, if the smallest natural number to not split in a number field is very large then the L-function looks like a power of zeta and the approximate functional equation shows that there are no zeros near the real axis.

2.1.1 Littlewood's Proof for ζ

Following [95] I will first outline the simpler proof that the gaps between zeros are bounded. This result follows directly from the formula for $N(T)$ and so is not interesting in its own right, but it introduces the main techniques and leads directly into the stronger result. Consider a system of four concentric discs C_1, C_2, C_3, C_4 centered at $3+iT$ and with radii 1,4,5, and 6 respectively. Suppose $\zeta(s)$ does not vanish in or on C_4. Then $\log\zeta(s)$ is regular in C_4. Let M_1, M_2, M_3 be its maximum modulus on C_1, C_2, and C_3 respectively. Now, $\Re\log\zeta(s) \ll \log T$ in C_4. The Borel Caratheodory theorem gives $M_3 \ll \log T$. But $M_1 \ll 1$ so Hadamard's Three-Circles Theorem applied to C_1, C_2, C_3 gives $M_2 \ll \log^{0.9} T$ so in particular $\zeta(-1+iT) \ll T^{\epsilon}$, but from the functional equation we know $|\zeta(-1+iT)| \gg T^{3/2}$. This gives a contradiction.

Taking the existence of a conformal map between rectangles and disks almost immediately shows the spacings to be $o(1)$, and a careful investigation of this map allowed Littlewood to prove the following Lemma.

Lemma 1. *Let Γ_1, Γ_2, Γ_4 be three concentric, similar, and similarly oriented rectangles, the ratio of the sides of any one of them being v, and the dimensions of Γ_2 and Γ_4 being respectively 2 and 4 times those of Γ_1. Suppose now that a function f is regular, and satisfies the inequality $\Re f \leqslant P$ in the interior of Γ_4, and satifies $|f| \leqslant M_1$ on the boundary of Γ_1. Then if $v > 100$,*

$$|f| \leqslant (M_1 + P)\, e^{v} \left(\frac{M_1}{M_1 + P}\right)^{e^{-v}}$$

on the boundary of Γ_2.

Take the rectangles to be axis-aligned, having center $5+iT$, and top left-corners $2+i\,(T+h)$, $-1+i\,(T+h)$, $-7+i\,(T+h)$ respectively. If ζ does not vanish on Γ_4 then taking f to be $\log\zeta$ we have $M_1 \asymp 1$ and $P \sim 7.5\log T$. Also, $|\log\zeta\,(-1+iT)\,| > \log T$, so in the lemma (with $v = 3/h$):

$$\log T \ll \log(T)\, e^{v} \log(T)^{-e^{-v}} \ll e^{v} \log(T)^{1-e^{-v}}.$$

Hence $e^{v} \gg e^{\log(\log(T))e^{-v}}$, so $v \gg \log\log\log T$. Making the constants effective gives $h \leq \frac{16}{\log\log\log T}$.

2.2 Growth rates of analytic functions

In the following formulae, c_k will refer to a constant, and $c_k(\cdots)$ will refer to a constant depending only on the specified parameters.

Theorem 1. *Let f be an analytic function on some domain $\mathcal{D}$. Suppose that for $x \in \mathcal{D}$, $\mathcal{D}$ contains a ball around x of radius $r_x > 0$. Let M be the maximum of $\log|f|$ on the domain. Let $m(x,r)$ denote the maximum of $\log|f|$ on a disc of radius r around x. Then for $x, y \in \mathcal{D}$ and $0 < \alpha < 1$:*

$$\left|\log\left(\frac{M - m(y, \alpha r_y)}{M - m(x, \alpha r_x)}\right)\right| \leq d_\alpha(x,y),$$

where $d_\alpha(x,y) = \frac{1+\alpha}{\alpha\log(1/\alpha)} \int_x^y \frac{1}{r_t}|dt|$.

Essentially this says that for all x and y, $f(x)$ and $f(y)$ must be equally close to the maximum modulus or equally far from the maximum modulus unless all paths from x to y go through a narrow section of the domain (a pinch-point). Taking $\alpha = 1/3$ makes the constant $\frac{1+\alpha}{\alpha \log(1/\alpha)}$ on the right hand side just under 4.

Proof. Let m_x and m_r denote $\frac{\partial m}{\partial x}$ and $\frac{\partial m}{\partial r}$ respectively. By the maximum principle applied to concentric circles: $m_r > 0$. By the maximum principle applied to tangent circles: $m_r \pm m_x > 0$, so $|m_x| < m_r$, also $\frac{\partial r_x}{\partial x} \leq 1$ [1]. Hadamards Three Circle Theorem states that $m(x,r)$ is convex as a function of $\log r$, namely $\frac{\partial m}{\partial \log r}(x,r) = m_r(x,r) \cdot r$ is a non-decreasing function in r.

By definition of M,

$$M \geq m(x, r_x).$$

By the fundamental theorem of calculus this is

$$M \geq m(x,r) + \int_r^{r_x} m_r(x,t)dt,$$

and then bringing in the Three Circle bound this is

$$M \geq m(x,r) + \int_r^{r_x} m_r(x,r) \cdot \frac{r}{t} dt$$

which equals

$$M \geq m(x,r) + m_r(x,r) r \log(r_x/r).$$

Rearranging, this tells us that

$$m_r(x,r) \leq \frac{M - m(x,r)}{r \log(r_x/r)}$$

and specifically,

$$m_x(x, \alpha r_x) \leq m_r(x, \alpha r_x) \leq \frac{M - m(x, \alpha r_x)}{\alpha r_x \log(1/\alpha)}.$$

[1] Note that the derivative may not exist, so this should be viewed as a bound on the derivates: $\limsup_{h \to 0} \left| \frac{f(x+h,r) - f(x)}{h} \right| < m_r(x,r)$

Comparing this bound to the following derivative we find

$$-\frac{\partial}{\partial x}\log(M - m(x,\alpha r_x) = \frac{m_x(x,\alpha r_x) + m_r(x,\alpha r_x)\alpha\frac{\partial}{\partial x}r_x}{M - m(x,\alpha r_x)} \leq \frac{1+\alpha}{\alpha\log(1/\alpha)r_x}.$$

And the result follows by integrating both sides:

$$\log(M - m(y,\alpha r_y)) - \log(M - m(x,\alpha r_x)) \geq -\frac{1+\alpha}{\alpha\log(1/\alpha)}\int_x^y \frac{1}{r_t}dt.$$

Similarly

$$\log(M - m(y,\alpha r_y)) - \log(M - m(x,\alpha r_x)) \leq \frac{1+\alpha}{\alpha\log(1/\alpha)}\int_x^y \frac{1}{r_t}dt.$$

□

When applying this to $\log\zeta(s)$ it will be convenient to have a form of this theorem that requires a maximum for $\Re f$ on the domain and not a maximum for $|f|$. Rather than going through the Borel-Caratheodory Theorem, a tighter bound can be derived by composing with the Mobius transform between half planes and discs.

Theorem 2. *Let f be an analytic function on some domain $\mathcal{D}$. Suppose that for $x \in \mathcal{D}$, $\mathcal{D}$ contains a ball around x of radius $r_x > 0$. Let $P > 0$ be the supremum of $\Re f$ on the domain. Let $m(x,r)$ denote the maximum of $\log|f|$ on a disc of radius r around x. Then for $x, y \in \mathcal{D}$:*

$$\Re f(y) \leq P\left(\frac{1}{P}\exp(m(x,\alpha r_x))\right)^{e^{-d}},$$

where $d = d_\alpha(x,y) = \frac{1+\alpha}{\alpha\log(1/\alpha)}\int_x^y \frac{1}{r_t}|dt|$. Thus, as long as $\Re f(y) > 0$, we may extract a bound on d:

$$d \geq \log\left(\frac{\log(P) - m(x,\alpha r_x)}{\log(P/\Re f(y))}\right).$$

Proof. Let $g(z) = f(z)/(2P - f(z))$, chosen so that $|g| \leq 1$ on $\mathcal{D}$. Let $d = \frac{1+\alpha}{\alpha\log(1/\alpha)}\int_x^y \frac{1}{r_t}dt$. Then applying Theorem 1 to g (using m_f and m_g to refer to suprema of f and g respectively) yields

$$\log(-m_g(y, \alpha r_y)) \geq \log(-m_g(x, \alpha r_x)) - d.$$

Exponentiated this gives

$$m_g(y, \alpha r_y) \leq m_g(x, \alpha r_x) e^{-d},$$

so therefore

$$|g(y)| \leq \exp(m_g(y, \alpha r_y)) \leq \exp(m_g(x, \alpha r_x) e^{-d}).$$

Note that $|g(z)| \leq |f(z)|/P$ so $m_g \leq m_f - \log P$ and so

$$|g(y)| \leq \left(\frac{1}{P} \exp(m_f(x, \alpha r_x)) \right)^{e^{-d}}.$$

Now, $f(z) = 2P \cdot \frac{g(z)}{g(z)+1}$ so

$$\Re f(y) \leq |f(y)| \quad \leq \quad P|g(y)| \leq P \left(\frac{1}{P} \exp(m_f(x, \alpha r_x)) \right)^{e^{-d}}.$$

□

In order to apply this to $\zeta(s)$ we will need the following bounds from chapter 5 of [95], which come from the functional equation.

Lemma 2. *When $\sigma > 1.5$, $\zeta(\sigma + iT) \asymp 1$. When $\sigma < -0.5$, $\zeta(\sigma + iT) \asymp T^{1/2+\sigma}$. When $\sigma \geq -0.5$, $\zeta(\sigma + iT) = O(T)$.*

Theorem 3. *There is a constant C such that every horizontal strip $T - h \leq \Im(s) \leq T + h$ contains a zero of $\zeta(s)$ as long as $h > C/\log\log\log T$.*

More generally, any zero-free region $\mathcal{D}$ of $\zeta(s)$ must satisfy $d_{1/4}(-1+iT, 2+iT) \gg \log\log\log T$.

Proof. For given T let $f(s) = \log \zeta(s + iT)$. Suppose f is analytic on the rectangle defined by $|\Im(s)| < h = o(1/\log T)$ and $|\Re(s)| < 3$. Following the notation of Theorem 2, we bound r_x, P, $\Re f$ and m, in order to apply the theorem.

By definition, $r_x = h$ when $-1 \leq x \leq 2$. For some constants c_1, c_2, c_3, c_4 and for h sufficiently small, lemma 2 implies that $c_3 < m(2, h/4) < c_4$, that $\Re f(-1) \sim c_2 \log T$, and that $P \sim \sum c_1 \log T$.

Theorem 2 then concludes that

$$18/h \geq d_{1/4}(2, -1) \geq \log\left(\frac{\log\log T + C}{c_1 - c_2}\right) \sim \log\log\log(T).$$

$\square$

Assuming the Riemann Hypothesis, instead of a rectangle one may take a bowtie-shaped domain such that $r_{x+1/2}\ h + c|x|$ for some $0 < c < 1$ and all $-1 \leq x + 1/2 \leq 2$. Then

$$\int_{-1}^{2} \frac{1}{r_t} dt = 2\int_0^{1.5} \frac{1}{h + ct} dt = 2\log(h + ct)/c|_0^{1.5} \sim 2\log h,$$

so therefore

$$\log\log\log T \ll d_{1/4}(2, -1) \asymp \log h,$$

and we can conclude that for some C

$$h \ll (\log\log T)^{-C}.$$

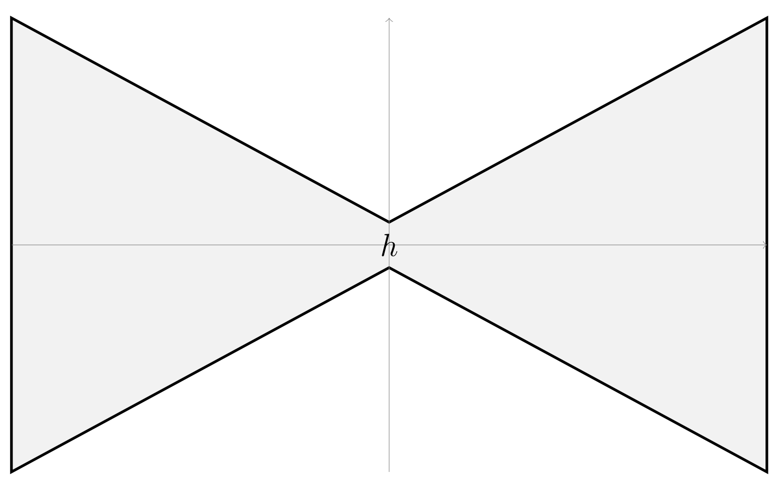


Figure 2.1: A “bowtie” domain

These results can also be strengthened given bounds on $\zeta(s)$ in the critical strip.

Suppose $|\zeta(\frac{1}{2}+iT)| \ll \exp(\log(T)/f(T))$ and $m(1/2, \alpha r_{1/2}) \leq \log(T)/f(T) + O(1)$ for some $f(T)$. Then

$$d_\alpha(1/2, -1) \geq \log\left(\frac{f(T)+C_1}{C_2}\right).$$

For example, if $\zeta(1/2+iT) \ll \exp((\log T)^{1-\epsilon})$ and this holds in a neighbourhood of $1/2+iT$, we may set $f(T) = (\log T)^\epsilon$ and conclude that

$$d_\alpha(1/2, -1) \gg \log\log T.$$

Thus a rectangular strip in the zero-free region would have height $\ll \frac{1}{\log\log T}$, and assuming the Riemann Hypothesis would allow the use of a "bow-tie region" having minimal height at most $\ll (\log T)^{-C}$ for some C.

2.3 The case of general L-functions

To extend the above to the general case, we need a uniform upper bound on $L(s)$ for s close to the real axis with $\Re(s) \geqslant -7$, and to have a good lower bound on some $L(s)$ in this region. In this case the analytic conductor plays the role of T above, but the estimates become more involved as they need to work for small s, uniformly in all parameters, and must allow for a high density of trivial zeros in the region being studied. Because of this we will use the following more restricted definition of the Selberg Class, which is still sufficiently broad to cover all known examples, corresponding to all λ_i in the gamma factor being 1/2.

Consider a Dirichlet series $L(s) = \sum a_n n^{-s}$ that extends to an entire function (wlog a_1=1), with polynomial growth on vertical lines, and functional equation

$$\Lambda(s) \quad := \quad Q^{s/2}\gamma(s)\, L(s) = \omega\overline{\Lambda}(1-s),$$

where $|\omega| = 1$ and γ is the degree d gamma factor

$$\gamma(s) \quad = \quad \prod_{i=1}^{d} \Gamma\left(\frac{s}{2}+\mu_i\right),$$

and $\Re\mu_i \geqslant 0$ for $i = 1, \cdots, d$.

Assuming an Euler product and Ramanujan hypothesis, in the strip $20 > \Re s > 2$,

$$|\log L(s)| = |\sum \log(1 - \alpha_{p,i} p^{-s})| \leq \sum |\log(1 - p^{-2})| = \log \zeta(2).$$

This gives the bound $|\log L(s)| \ll a_2 d$, which is needed, where specifically $a_2 = \log \zeta(2) \approx 0.4977$. It suffices that for s_1, s_2 in this strip there is a uniform bound on $\frac{\log L(s_1)}{\log L(s_2)}$, but the given bound on $|\log L(s)|$ will be used here for simplicity. The dependence on the Ramanujan Hypothesis is unfortunate but while there are results of Molteni [73] and Li [64] that give unconditional bounds on $GL(n)$ L-functions at the edge of the critical strip, these have a dependence on the conductor that would weaken the result here.

Let $q = Q \prod (11 + |\mu_i|)^{1.5} e^{0.1 \Re \mu_i}$

Theorem 4. *With the above definitions and conditions for $L(s)$, the lowest lying zero ρ_1 has $\Im \rho_1 \ll \frac{1}{\log \log \log \left(q^{1/d}\right)}$ where the implied constant is absolute, as long as $q^{1/d} > e^{\frac{1}{2}(a_2 + 0.308)} \sim 1.5$.*

In particular this means that given a sequence of L functions where $q \gg A^d$ for all A, the lowest lying zero converges to the real axis. This is the strongest result possible since $L(s, \chi_q)^d$ has conductor q^d so the family of these has growing conductor but the same zeros. It is still plausible that for all primitive L-functions the lowest lying zero converges to the real axis as $q \to \infty$. This should require a good way of recognizing primitive L-function and a way of seperating these from high powers.

Proof. Define rectangles in the complex plane $\Gamma_1, \Gamma_2, \Gamma_4$ with common center $9 + ih$ and having lower-left corners respectively at 2, -5, -19.

Consider the function $f(s) = \frac{\log L(s)}{d}$ and suppose $L(s)$ does not vanish inside Γ_4, so that f is well defined inside Γ_4. By assumption $|f(s)| \leqslant a_2$ on Γ_1.

First we use the functional equation and Stirling's approximation to find an effective upper bound when $-19 \leqslant \Re s \leqslant 0$ and $|\Im s| \leqslant 2h \ll 1$. Let $m = \sum_{i=1}^{d} \Re \mu_i + |\Im \mu_i + t/2|$. Then from the functional equation

$$|L(s)| = \left|\overline{L}(1-s)\right| Q^{(1-\sigma)/2} \prod_{i=1}^{d} \left| \frac{\Gamma\left(-s/2+1/2+\overline{\mu_i}\right)}{\Gamma\left(s/2+\mu_i\right)} \right|.$$

Note that if $A > B$ and $A+B > 0$ then $\left|\frac{\Gamma(A+iy)}{\Gamma(B-iy)}\right| < \left|\frac{\Gamma(A+iy)}{\Gamma(B-iy)}\right| \left(\frac{A+iy}{B-iy}\right) = \left|\frac{\Gamma(A+iy+1)}{\Gamma(B-iy+1)}\right|$ since $\left(\frac{A+iy}{B-iy}\right) > 1$. When $\Re s \geq -19$ we may repeat this 11 times to get

$$|L(s)| \leqslant \left|\overline{L}(1-s)\right| Q^{11} \prod_{i=1}^{d} \left| \frac{\Gamma\left(-s/2+1/2+\overline{\mu_i}+11\right)}{\Gamma\left(s/2+\mu_i+11\right)} \right|.$$

The gamma functions in the product are evaluated in the domain $\Re \geq 1.5$ and the relative error in Stirling's approximation there is less than 20%, therefore

$$|L(s)| \leqslant e^{a_2 d} Q^{11} \prod_{i=1}^{d} 2 \left| \frac{\left(-s/2+1/2+\overline{\mu_i}+11\right)^{-s/2+1/2+\overline{\mu_i}+11} e^{s-1/2} \sqrt{s/2+\mu_i+11}}{\left(s/2+\mu_i+11\right)^{s/2+\mu_i+11} \sqrt{-s/2+1/2+\mu_i+11}} \right|.$$

Now bounding all of the parameters gives

$$|L(s)| \leqslant e^{a_2 d} Q^{11} \prod_{i=1}^{d} 2 \left| \frac{\left|21+\mu_i\right|^{21+\Re\mu_i} \left|11+\mu_i\right|}{\left|3/2+\mu_i\right|^{11+\Re\mu_i} \left|11.5+\mu_i\right|} e^{\operatorname{Arg}\left(\frac{-s/2+1/2+\overline{\mu_i}+11}{s/2+\mu_i+11}\right)\left(\Im s/2+\Im\mu_i\right)} \right|,$$

which further simplifies to

$$|L(s)| \leqslant e^{a_2 d} Q^{11} 40^{10+\Re\sum\mu_i} \prod_{i=1}^{d} \left|21+\mu_i\right|^{10} e^{\pi|h|+\pi\Im\mu_i}.$$

This gives the following needed upper-bound

$$\begin{aligned} \Re f(s) = \frac{1}{d}\Re\log L(s) &\ll a_2 + \log Q^{1/d} + \frac{1}{d}\sum \log\left|10+\mu_i\right| \\ &\ll 1 + \log q^{1/d}. \end{aligned}$$

It now remains to find a lower-bound on the maximum of f on Γ_2. This is very similar to finding the asymptotics as above, but for the zeros coming from the poles of Γ. These may be dealt with by instead analyzing a certain geometric smoothing of the function. For a function g and for $\delta \in \mathbb{R}$ denote $g_{m\delta} := \exp\left(\frac{1}{\delta}\int_s^{s+\delta} \log g(s)\, ds\right)$. Note $\Gamma_{m1}(s) = \sqrt{2\pi}\left(\frac{s}{e}\right)^s$. As before, the functional equation states that

$$|L(s)| = \left|\overline{L}(1-s)\right| Q^{(1-\sigma)/2} \prod_{i=1}^{d} \left|\frac{\Gamma(-s/2+1/2+\overline{\mu_i})}{\Gamma(s/2+\mu_i)}\right|.$$

For $-19 \leqslant \Re s \leqslant -3$ we bound L as before

$$|L_{m2}(s)| \geqslant e^{-a_2 d} Q^2 \prod_{i=1}^{d} \left|\frac{\Gamma_{m1}(-s/2-1/2+\overline{\mu_i})}{\Gamma_{m1}(s/2+\mu_i)}\right|.$$

Since Γ_{m1} simplifies, this simplifies to

$$e^{-a_2 d} Q^2 \prod_{i=1}^{d} \left|\frac{(-s/2-1/2+\overline{\mu_i})^{-s/2-1/2+\overline{\mu_i}+10}\, e^{s+1/2}}{(s/2+\mu_i)^{s/2+\mu_i}}\right|.$$

In the range $-19 \leq \Re s \leq -3$ this gives the bound

$$\begin{aligned} |L_{m2}(s)| &\geqslant e^{-a_2 d} Q^2 \prod_{i=1}^{d} \left|\frac{1\cdot}{|8.5+m|^{8.5+\Re\mu_i}} \frac{|11+m|^3}{} e^{\mathrm{Arg}\left(\frac{-s/2+1/2+\overline{\mu_i}+10}{s/2+\mu_i+10}\right)(\Im s/2+\Im\mu_i)} e^{-2.5}\right| \\ &\geqslant e^{-a_2 d} Q^2 1.29^{8.5d+\Re\sum\mu_i} \prod_{i=1}^{d} |11+m|^3 e^{-2.5}. \end{aligned}$$

Taking the logarithm of both sides,

$$\begin{aligned} \frac{1}{d}\Re\log L_{m2}(s) &\geqslant -a_2 - 0.308 + \log Q^{2/d} + \frac{1}{d}\log\prod|11+m|^{3/d} + \frac{0.1}{d}\Re\sum\mu_i \\ &\geqslant -a_2 - 0.308 + 2\log q^{1/d}. \end{aligned}$$

Hence there is some $s \in [-3, 1]$ such that $\Re f(s) \geqslant -a_2 - 0.308 + 2\log q^{1/d}$.

Applying lemma 1,

$$-a_2 - 0.308 + 2\log q^{1/d} \leqslant C_1\left(a_2 + \log q^{1/d}\right) e^v \left(\frac{C_2 a_2}{a_2 + \log q^{1/d}}\right)^{e^{-v}},$$

and rearranging the two sides,

$$\left(2\log q^{1/d} - a_2 - 0.308\right)\left(\log q^{1/d} + a_2\right)^{e^{-v}-1} \leqslant C_1 \left(C_2 a_2\right)^{e^{-v}} e^{v}$$

Hence if $2\log q^{1/d} > a_2 + 0.308$ then $\left(\log q^{1/d}\right)^{e^{-v}} \ll e^{v}\left(C_2 a_2\right)^{e^{-v}}$, so $\log q^{1/d} \ll e^{ve^{v}}$ and we conclude that

$$\log\log\log q^{1/d} \ll v.$$

□

2.4 Results from constraints on functions on a disc

Several theorems about the rigidity of entire non-vanishing functions have direct analogues for functions on a disc that are well behaved near the center of the disc but grow very quickly. For example, Schottky's Theorem is the analogue of Little Picard.

Theorem 5. *(Schottky) If f is analytic on the unit disc and misses the values $0, 1$ in its range then*

$$\log|f(z)| \leq \frac{1+|z|}{1-|z|}(7 + \max(0, \log|f(0)|))$$

These theorems apply naturally to $\zeta(z+3/2+iT)$ for sufficiently large T assuming the Riemann Hypothesis and adds significant rigidity. However, since applications of these theorems require $f(s)$ to be very large near $\Re s = 1$ the effects only appear for very large T (around 10^{300}) and so we cannot see these curious functions directly. Some similar phenomena are studied by Ivić[47].

Theorem 6. *(Borel) If $f_1, \cdots, f_n$ are nowhere vanishing entire functions none of which is a scalar multiple of the other, then the sum $f_1 + \cdots + f_n$ cannot be identically zero*

This theorem shows that functions satisfying a Riemann Hypothesis are quite exceptional and must be linearly independent, which is known by more direct means in the Selberg class given an Euler product[51].

2.5 Riemann Hypothesis for "any" entire function

A question to ask at this point is whether there is any entire function that behaves asymptotically like $\zeta(s)$ (without requiring a Dirichlet series nor a functional equation) which satisfies some form of a Riemann Hypothesis. Namely I propose the following as a question of interest.

Question 3. *Is there an entire function $f : \mathbb{C}5 \to \mathbb{C}$ such that for some $\delta > 0$ the following are satisfied.*

1. *$f(z)$ is bounded when $\Re z > 1 + \delta$.*
2. *$f(z)$ is unbounded when $\Re z = 1$.*
3. *$f(z)$ grows polynomially in vertical strips: for all σ there exists $C_\sigma > 0$ so that $|f(\sigma + it)| \ll |t|^{C_\sigma}$.*
4. *(weak RH) $f(z)$ does not vanish when $\Re z > \frac{1}{2}$.*
5. *(strong RH) $f(z)$ vanishes only when $\Re(z) = \frac{1}{2}$ or for $\Im(z)$ in a finite set.*

Conjecturally all L-functions satisfy all 5 conditions, but this has not been proven for any L-function. Many explicit functions satisfy all but condition (2), for example $1 + e^{-z}$ or $\zeta(\frac{1}{2} + z) \cdot (z - \frac{1}{2})$s. If you drop condition (3) then you have as examples the Selberg zeta function, or $\exp(L(s, \chi))$. If you drop condition (1) then you have as an example $\prod_{n=1}^{\infty} \cosh(z/n^2)$.

The remainder of this section consists of an explicit construction of a function satisfying (1)-(4) (namely the weak but not the strong Riemann Hypothesis).

Proposition 1. *There is a function $f(z)$ satisfying properties (1)-(4) above.*

Proof. Consider the entire function $g(z) = \frac{1}{z}(1 - e^{-z}) = \int_0^1 e^{-tz} dt$. From the first expression, in any right half plane $g(z) = O(1/z)$. From the definite integral,

$$|g(z)| \leq \int_0^1 e^{-t\Re(z)} dt = g(\Re(z)).$$

Comparing the integrand termwise, $g(z)$ is dominated in $\Re(z) > 1$ strictly by $a := g(1) > 0$ and tends rapidly to 0 as $\Re(z)$ increases.

Consider $h_{n,N}(z) = (1 + g(z)e^{-n^2 z})^N$ where N is a large integer and $n := n(N)$ is such that $N = e^{n^2+n}/a$. For every half-plane $\Re(z) \geq \sigma$, both $g(z)$ and $e^{-n^2 z}$ are maximized at $z = \sigma$ and are positive there, hence $h_{n,N}(z)$ also has this property. Computing at $z = 1$ and at $z = 1+2/n$ yields $h_{n,N}(1) = (1+ae^{-n^2})^N = (1+e^n/N)^N \sim e^{e^n}$ and $h_{n,N}(1 + 2/n) < (1 + ae^{-n^2-2n})^N \sim (1 + e^{-n}/N)^N \sim 1 + e^{-n}$ respectively.

By construction, on any right half-plane $g(z) = O_{n,N}(1/z)$ (where the constant depends on the half-plane) and so $h_{n,N}(x + iy) = 1 + O_{n,N}(1/y)$. Thus we can pick sufficiently large $t_{n,N}$ such that $|\log |h_{n,N}(z + it_{n,N})|| < 2^{-n} \log(1 + |\Im(z)|)$ for $\Re(z) > -N$.

Now take our candidate function to be

$$f(z) := \prod_{N=1}^{\infty} h_{n,N}(z + it_{n,N}).$$

where the N form a sufficiently fast growing subsequence of the natural numbers so that $\sum e^{-n}$ converges and $e^{n^2/2} > g(1/2)$ for all n. This construction is courtesy of Fedja Nazarov.

At this point all that remains is to verify that $f(z)$ satisfies the 4 properties.

1. As long as $\Re(z) > 1 + 1/n$, the product for $f(z)$ is dominated by $\prod(1 + e^{-n})$. Since $\sum e^{-n}$ is absolutely convergent, the product converges absolutely to a bounded analytic function.

2. $h_{n,N}(1-it_{n,N}) \sim e^{e^n}$ and for sufficiently large $t_{*,*}$ the other terms in the product for $f(1 - it_{n,N})$ can be made arbitrarily close to 1. Thus we can ensure that $f(1 - it_{n,N}) \asymp e^{e^n}$ making f unbounded on $\Re(z) = 1$.

3. For $\Re(z) > -N_0$: $|f(z)| \leq \prod_{N=1}^{N_0} h_{n,N}(-N_0) \prod_{N>N_0}(1 + |\Im(z)|)^{2^{-N}} \ll_{N_0} (1 + \Im(z))^2$, giving absolute converge with polynomial growth on vertical lines, uniformly on half-planes.

4. The $\{n\}$ were chosen so that $|g(z)e^{-n^2 z}|$ is strictly bounded above by 1 when

$\Re(z) > 1/2$. This means that $h_{n,N}(z)$ has no zeros with $\Re(z) > 1/2$ and thus neither does $f(z)$.

This function, however, grows very rapidly to the left, and has a multitude of zeros in every half-plane $\Re(z) < \sigma < 1/2$.

$\square$

Chapter 3

Dirichlet Series with Analytic Continuation and Degree 1 growth rate

3.1 Introduction

The analytic study of Dirichlet series began with Riemann [84] who proved the functional equation of the zeta function. He gave two proofs of the functional equation, the latter has been greatly generalized into the theory of modular forms, whose further generalization into automorphic forms is a central area of study of modern number theory. The first proof is likewise very interesting, extends generally to Dirichlet L-functions, forms the basis for Shintani zeta functions (which can be used to develop the theory of Dedekind zeta functions) and will be central to the current work.

Hamburger[43] was the first to consider the problem of determining a function from such a functional equation, when he proved that a Dirichlet series with moderate growth constraints and the functional equation of $\zeta(s)$ must be $\zeta(s)$. Hecke[45][7] dealt with the case of degree two functional equation $\Gamma(s+k)\lambda^s$. He related precisely Dirichlet series with such a functional equation to analytic functions on the upper half plane modulo a triangle group, and thus was able to determine the dimension of the space of these functions for each set of parameters. Weil's converse theorem additionally states that if sufficiently many Dirichlet twists of the L-function have appropriate functional equations then the Dirichlet series comes from a modular form of some level. Maass[68] proved a Hamburger-type theorem for the functions $\zeta(s)^2$, $2^{-s}\zeta(s)^2$, and $(1+2^{1-2s})\zeta(s)^2$ by showing that any L-function with the same functional equation as these must arise as a specific Maass form. All other cases remain open. Moving to higher degree, the existing converse theorems for degree 3 require infinitely many Dirichlet twists along with the Dirichlet series factoring as an appropriate Euler product[36]. In general $GL(n-2)$ twists are used for degree n converse theorems and it is unclear whether $GL(n-2)$ or $GL(n/2)$ or $GL(1)$ are necessary and sufficient.

The interest in such problems is manifold. Hecke's theorem, while no long as popular to use directly, is the foundation on which Weil's Converse theorem is developed [48], so having an analogue for higher degree may open the way to a better understanding of higher converse theorems. There are many open questions about the behaviour of Dirichlet Series (Lindelof hypothesis, Riemann Hypothesis, Moments Conjecture). Certain results can be proven without any arithmetic content so having a large concrete family of purely analytic examples can help to test conjectures and also to see which results truly do or do not depend on arithmetic.

The general problem is to understand the Selberg class [87][75] of Dirichlet series with functional equations and Euler product (although all of the following work does not actually require an Euler Product). The most coarse measure of complexity for these is the degree d of the functional equation. Conrey and Ghosh [26] determined that there is nothing with degree strictly between 0 and 1 and that degree 0 consists only of Dirichlet polynomials. Kaczorowski and Perelli [54] determined everything of degree 1, and Soundararajan [90] has a particularly short and clean proof of this fact. More recently Kaczorowski and Perelli [57] have shown that there is nothing of degree strictly between 1 and 2.

In section 3.2 I outline a more general framework, which instead of a functional equation requires only a growth bound similar to that of a function in the Selberg class. Expanding on Kuznetsov[61] I determine all such functions for the equivalent of degree 1.

In section 3.3 I reintroduce the functional equation, showing how it relates to singularities of functions, and in corollary 2 construct all Dirichlet series with degree 1 functional equation.

Section 3.5 holds the proofs of the main theorems, and appendix A provides some useful lemmas for dealing with products of gamma functions.

Every result here, aside from corollaries 1 and 2, work also for generalized Dirichlet series $\sum a_n b_n^{-s}$ for increasing positive numbers $b_1, b_2, \cdots$. This will be used in the next chapter to prove corresponding results for higher degree.

3.2 Results not assuming Functional Equation

Before bringing in functional equations we make a definition of degree based only on growth rate.

Definition 6. *Given a Dirichlet series $L(s) = \sum a_n n^{-s}$ absolutely convergent in some right half-plane define the following degrees if $L(s)$ extends to an entire function with polynomial growth bounds on vertical lines.*

Horizontal Degree $\leq d$ (and conductor Q)

$|L(s)| \ll \Gamma(1 + d|s|)Q^{|s|}$ *for some Q.*

Vertical Degree $\leq d$

$|L(\sigma + iT)| = O_\sigma(T^{d\sigma+A})$ *for some A and all $\sigma < 0$*

If one of the degrees satisfies the above for given d but not for any strictly smaller d then we say that the degree equals d.

If $L(s)$ has a functional equation of degree d then its horizontal and vertical degrees are both d, so in this sense the above definition extends the standard notion of degree. They are named as such because horizontal degree gives a bound that is most useful on the negative real axis, and vertical degree bounds the behaviour on vertical lines in the complex plane.

As an example that any horizontal degree $d > 0$ is possible consider

$$L(s) := \sum_{n=1}^{\infty} \exp(-n^{1/d}) n^{-s},$$

which is absolutely convergent everywhere, bounded by its value for real s, and for s real going to $-\infty$ is asymptotic to

$$\int_0^\infty \exp(-x^{1/d})x^{-s-1}dx = d\int_0^\infty e^{-x}x^{-ds-1}dx = d\Gamma(-ds).$$

Thus $L(s)$ has horizontal degree d and vertical degree 0.

We can now state the main results, whose proofs will come in section 3.5.

Proposition 2. *Vertical degree $\leq$ Horizontal degree, in the sense that if the horizontal degree is $\leq d$ then the vertical degree is $\leq d$.*

This proposition will be discussed further in Section 4.2 where we deal more explicitly with degree d.

Conjecture 2. *The vertical degree is always a non-negative integer.*

Theorem 7. *A Dirichlet series $L(s) := \sum_{n=1}^\infty a_n n^{-s}$ has horizontal degree 1 if and only if $p(x) := \sum_{n=1}^\infty a_n x^n$ has radius of convergence 1 and continues analytically in a neighbourhood of 1. Furthermore if this is the case then it also has vertical degree 1.*

More precisely, the following are equivalent, for given Q, and some A not necessarily the same in all cases:

1. *$L(s) := \sum a_n/n^s$ extends to an entire function satisfying $|L(s)| \ll \Gamma(1 + |s|)|s|^A Q^{|s|}$.*

2. *$L(s) := \sum a_n/n^s$ extends to an entire function satisfying $|L(s)|\Gamma(s)Q^s \ll |s|^A e^{-\pi i|Im(s)|/2}$ for s in a left half-plane, away from the poles.*

3. *$f(z) := \sum a_n e^{-nz}$ extends analytically in a $1/Q$ neighbourhood of 0, and $f(z) \ll \frac{1}{1-Q|z|}^A$ in this neighbourhood.*

And in this case f has Taylor expansion: $f(z) = \sum_{n=0}^\infty \frac{(-z)^n}{n!}L(-n)$.

Corollary 1. *If horizontal degree is strictly between 0 and 1, or horizontal degree is 1 and $Q < \frac{1}{\pi}$, then $\sum a_n n^{-s}$ is absolutely convergent everywhere and in particular has vertical degree 0.*

Proof. In either case the bound in theorem 7 is satisfied for some $Q > \frac{1}{\pi}$. The theorem concludes that $f(z)$ is analytic on $1/Q$ neighbourhoods of $2\pi i\mathbb{Z}$, and these cover some half-plane $\Re z > \delta$ where $\delta < 0$. This makes $p(x)$ have radius of convergence > 1 so the a_n decay exponentially. □

The result extends to meromorphic function in the sense that $L(s)$ having a simple pole at $s = 1$ corresponds to $f(z)$ having a simple pole at 1.

Note that this gives a vast family of Dirichlet series. For example consider

$$p(x) := e^{1/(x+1)} = \sum_{n=0}^{\infty} {}_1F_1(n+1; 2; 1)(-x)^n,$$

which leads to a Dirichlet series with horizontal and vertical degree 1

$$L(s) := \sum_{n=1}^{\infty} {}_1F_1(n+1; 2; 1)(-x)^n.$$

It would be interesting to see whether or not this function satisfies the Lindelof Hypothesis. In fact, we could go so far as to conjecture that "everything" satisfies a Lindelof Hypothesis:

Definition 7. *Consider $L(s)$ with finite horizontal degree (or simply entire continuation to a function of order 1). As in [95] define for each σ*

$$\mu(\sigma) := \inf\{\mu : |\zeta(\sigma + iT)| \ll |t|^{\mu}\}.$$

As a function of σ, $\mu(\sigma)$ is non-increasing and is 0 when σ is sufficiently large. By the Phragmen-Lindelof Principle it is convex upwards, and it has slope $\geq -d$.

Conjecture 3 (Analytic Lindelof Hypothesis)**.** *$\mu(\sigma)$ is piecewise linear with all linear parts having integer slope, necessarily increasing from $-d$ to 0.*

3.2.1 Explicit Construction of all with Horizontal degree ≤ 1

Consider the twisted zeta function $\zeta_a(s) := \sum_{n=1}^{\infty} e^{-na} n^{-s}$ where $-\pi \leq \Im a \leq \pi$ and $\Re a \geq 0$. The corresponding $p(x)$ is $p_a(x) = \sum_{n=1}^{\infty} (e^{-a}x)^n = \frac{x}{e^a - x}$. All of these are of horizontal degree 1 with parameter $Q = |a|$. These twisted zeta functions will in fact generate all such Dirichlet series.

Theorem 8. *Consider $L(s) = \sum a_n n^{-s}$ that has horizontal degree ≤ 1 with some given Q and A, and $a_n \ll n^\delta$ for some δ. Consider the union of the right half-plane, and the $1/Q$ disc around 0. Take $\mathcal{C}$ to be the portion of the boundary that lies in the strip $-\pi \leq \Im s \leq \pi$. Then for every $\delta' > \delta + 1/2$ there is some $f \in L^2(\mathcal{C})$ so that*

$$L(s) = \int_{\mathcal{C}} f(t)\zeta_{-t}(s+\delta)dt.$$

where the integral converges uniformly on compact subsets of $\mathcal{C}$.

Proof. Without loss of generality shift s sufficiently so that $A < -1/2$ and $\delta < -1/2$ so that we can take $\delta' = 0$. Then $p(x) = \sum a_n x^n$ is in the Hardy space H^2 of the disc and so converges almost everywhere on the boundary to an L^2 function that we also denote by $p(x)$.

The Cauchy integral formula says

$$p(x) = \frac{1}{2\pi i}\int_{|t|=1} \frac{p(t)}{t-x}dt.$$

Making the substitution $x = e^{-z}$ gives

$$f(z) = \frac{1}{2\pi i}\int_{-\pi i}^{\pi i} \frac{f(t)}{e^{z-t}-1}dt.$$

$f(t)$ on the imaginary axis is taken to be in $L^2(\mathcal{C}/(2\pi i))$, and by assumption is analytic in a $1/Q$ neighbourhood of 0. $f(t)$ is also L^2 on the boundary of this neighbourhood, since its power series expansion at 0 has coefficients growing like $Q^n n^A$. Thus we may shift the contour to $\mathcal{C}$

$$f(z) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f(t)}{e^{z-t}-1}dt.$$

Now take the Mellin transform of both sides

$$L(s)\Gamma(s) = \frac{1}{2\pi i}\int_0^\infty z^{s-1}\int_{\mathcal{C}} \frac{f(t)}{e^{z-t}-1}dt\ dz.$$

By absolute convergence we may interchange the two integrals to get

$$L(s)\Gamma(s) = \frac{1}{2\pi i}\int_{\mathcal{C}} f(t)\int_0^\infty \frac{z^{s-1}}{e^{z-t}-1}dz\ dt.$$

The interior integral gives the twisted zeta function

$$L(s)\Gamma(s) = \frac{1}{2\pi i}\int_{\mathcal{C}} f(t)\zeta_{-t}(s)\Gamma(s)dt,$$

and cancelling out $\Gamma(s)$ from both sides gives

$$L(s) = \frac{1}{2\pi i}\int_{\mathcal{C}} d(t)\zeta(-t)(s).$$

$\square$

3.2.2 The case of vertical degree < 1

While the techniques above work only with strict horizontal control, vertical growth suffices.

Theorem 9. *Let $L(s) = \sum a_n n^{-s}$ be a Dirichlet series absolutely convergent for some s that analytically continues to an entire function, and has vertical degree less than 1, namely for each $\sigma < 0$ $|L(\sigma + iT)| = O_\sigma(T^{C-d\sigma})$ for some $0 \le d < 1$ and some C. Then the series $\sum a_n n^{-s}$ is absolutely convergent everywhere, and in particular the vertical degree is exactly 0.*

Proof. As before take $\Re z > 0$ and consider

$$f(z) = \sum a_n e^{-nz} = \frac{1}{2\pi i}\int_c L(s)\Gamma(s)z^{-s}ds.$$

Recall that for $s = \sigma + iT$, $\Gamma(s)\sin(\pi/2s) = O_\sigma(T^{-1/2+\sigma})$, so that $L(s)\Gamma(s)\sin(\pi/2s) = O_\sigma(T^{C-1/2+(1-d)\sigma})$. Thus we may shift the contour sufficiently far to the left so that on the the line of integration $L(s)\Gamma(s)\sin(\pi/2s) = O(\Im(s)^{-2})$ is absolutely integrable and hence:

$$f(z)=\sum_{k=0}^{K}L(-k)(-z)^k/k!+\int_{-K-1/2}L(s)\Gamma(s)z^{-s}ds=O(|z|^{K+1/2}).$$

This bound is uniform in the right half plane and extends continuously to $\Re(z)=0$. In addition, $f(z)$ is periodic with period $2\pi i$, so is therefore bounded on the imaginary axis. The Fourier coefficients $a_n=\int_0^1 f(2\pi iz)e^{2\pi inz}$ also carry the same bound, so the series $\sum a_n n^s$ is absolutely convergent at $\Re s>2$. Repeating the argument for $L(s+N)$ proves absolute convergence at $\Re s>2-N$ for all N, and the result follows.

□

3.3 Results Assuming a Functional Equation

Definition 8. *Consider two Dirichlet series $L_a(s)=\sum a_n n^{-s}$ and $L_b(s)=\sum b_n n^{-s}$ absolutely convergent in some right half-plane, where $a_n,b_n\in\mathbb{C}$. Define Gamma-factors*

$$\begin{aligned}\gamma_a(s) &:= \prod_{i=1}^{r_a}\pi\Gamma(\lambda_{a,i}s+\mu_{a,i})\\ \gamma_b(s) &:= \prod_{i=1}^{r_b}\Gamma(\lambda_{b,i}s+\mu_{b,i})\end{aligned}$$

and suppose that $L_a(s)$ and $L_b(s)$ have analytic continuation to entire function with polynomial growth on vertical lines, and are related by a functional equation

$$\begin{aligned}\Lambda(s) &:= L_a(s)\gamma_a(s)Q_a^s\\ \Lambda(1-s) &= L_b(s)\gamma_b(s)Q_b^{s-1},\end{aligned}$$

where $\lambda_{a,i},\lambda_{b,i},Q_a,Q_b$ all exceed zero.

The functional equation can also be written in the non-symmetric form

$$\begin{aligned} L_a(s) &= L_b(1-s)\frac{\gamma_b(1-s)}{\gamma_a(s)}Q^s \\ &= L_b(1-s)\,\gamma(1-s)\,\theta(1-s)\,Q^s, \end{aligned}$$

where $Q = Q_b/Q_a$, there is the nonsymmetric gamma factor

$$\begin{aligned} \gamma(s) &:= \prod_{i=1}^{r}\Gamma(\lambda_i s+\mu_i) \\ &= \prod_{i=1}^{r_b}\Gamma(\lambda_{b,i}s+\mu_{b,i})\prod_{i=1}^{r_a}\Gamma(\lambda_{a,i}s+1-\lambda_{a,i}-\mu_{a,i}), \end{aligned}$$

and a product of sines and cosines that we denote

$$\begin{aligned} \theta(s) &:= \prod_{i=1}^{r_a}\sin(\pi(s\lambda_i-\mu_i-\lambda_i)/2) \\ &:= \sum \alpha_j e(\omega_j s/4), \end{aligned}$$

where $\{\omega_j\} = \{\pm\lambda_1 \pm \lambda_2 \pm \cdots\}$. The degree of the functional equation is defined to be $d := \sum \lambda_i = 2\sum \lambda_{a,i} = 2\sum \lambda_{b,i}$.

Let $q^{-1} = Q\prod_{i=1}^{r}\lambda_i^{\lambda_i}$ and let $\mu = \sum_{i=1}^{r_a}(\mu_i - 1/2)$.

Conjecture 4. *The following is a list of conjectures in strictly increasing order of strength:*

1. *Selberg's degree conjecture [87][75] that $d \in \mathbb{N}$.*
2. *All ω_j are in $\mathbb{N}$ and have equal parity.*
3. *All $\lambda_{a,i}$ an $\lambda_{b,i}$ may be taken to be 1/2.*
4. *All λ_i can be taken to be 1.*

The main result in this chapter is that $L(s)$ having a functional equation leads to $f(z)$ being entire aside from isolated singularities on a finite number of rays

Theorem 10.

I *If $L_a(s) := \sum a_n n^{-s}$ has analytic continuation and a functional equation as above relating it to $L_b(s)$ then $f(z) := \sum_{n=1}^{\infty} a_n \exp(-nz)$ has analytic continuation to $\mathbb{C}$ aside from singularities at the points $-nqe\left(\frac{\omega_j}{4}\right)$ for $n \in \mathbb{N}$.*

II *More precisely, the singularities are of the same type and have "residues" b_n. Up to a polynomial $p(z)$,*

$$f(z) = p(z) + \sum_{n=1}^{\infty} \sum_{j=1}^{r} db_n \alpha_j G(ze(\omega_j s/4)q/n) z^{-1},$$

where $G(z)z^{-1}$ is holomorphic aside from a branch cut singularity at -1.

III *By shifting and possibly replacing $L_a(s)$ with some $L_a(s+A)$, we may wolog assume $\mu = 0$. Then around $z = 1$ when $|\operatorname{Arg}(z)| < \pi$,*

$$G(z/q)z^{-1} = \frac{B_0}{z+1} + B_1 \log(z+1) + O(1)$$

for some B_0, B_1.

We also assume that $L_a(s)$ cannot be expressed as $L_{\alpha(s)}/n_a^s$ for $n_a = 2, 3, \cdots$ and L_α also a Dirichlet series. Similarly we assume that $L_b(s)$ cannot be expressed as $L_{\beta(s)}/n_b^s$. Otherwise we could replace $L_a(s)$ with $L_a(s)n_a^s$, replace $L_b(s)$ with $L_b(s)n_b^s$, and replace Q with $\frac{n_a}{n_b}Q$.

Corollary 2. *If $L_b(s) = \sum a_n n^{-s}$ has degree 1, then $q \in 2\pi/\mathbb{N}$ and for some A the sequence $a_n n^A$ is periodic mod $2\pi/q$. Hence it is a linear combination of $L(s+A,\chi)$ for various χ (not necessarily primitive) of modulus $2\pi/q$.*

Proof. It has been shown that the singularities of $f(z)$ are evenly spread out along rays with angles $\omega_j/4$. But $f(z)$ is periodic which leaves only the possibility $(\omega_j)_{j=1}^r = (-1, 1)$ so that $(e(\omega j s_4)) = (-i, i)$. The location of the singularities is periodic mod 2π, and the singularities are supported on a lattice with spacing $2\pi/q$, therefore $2\pi/q \in \mathbb{N}$.

As in the final remark of the above theorem, wlog shift so that $\mu = 0$. Then to first order the singularity at nqi behaves as a simple pole with residue $B_0 d b_n \alpha$ and we conclude that the b_n are periodic mod $2\pi/q$ as claimed. □

3.4 Results assuming an Euler product

Schwarz conjectured in 1978 that a power series with multiplicative coefficients has a natural boundary on its circle of convergence or is a rational function, but progress has only been made in the case where the mean value of the coefficients exists and is non-zero [67]. In our context the question becomes: aside from $L(s,\chi)$ is there a Dirichlet series of horizontal degree 1 with an Euler product.

Subject to extra conditions this can be answered in the negative. Suppose that $g(x) = \sum a_n x^n$ has completely multiplicative coefficients, radius of convergence 1, and only finitely many singularities on the unit circle.[1] Let m be the number of singularities.

Since the coefficients are bounded, $|a_p| \leqslant 1$ for all p. Also, by gap theorems, only finitely many of the a_p can be 0. In fact, it is necessary that $|a_p| = 1$ for all but finitely many p, since the coefficients need to satisfy an almost linear-recurrence. Without loss of generality, when $|a_p| \neq 1$ we can make $a_p = 0$ without changing the hypotheses.

For every prime p with $a_p \neq 0$: $\sum_{n=0}^{p-1} f(ze(n/p)) = pa_p f(z^p)$. Hence if z is a singularity of f on the unit circle then at least one of the $z^{1/p}e(n/p)$ is also a singularity. Repeating this argument gives a sequence of related singularities. Since there are only finitely many singularities this sequence has to repeat at some point, after at most m terms, which implies that $z = e(r)$ for some $r \in \mathbb{Q}$.

Take some p not dividing any of the denominators of these r. Then for $z = e(r)$, exactly one of the $z^{1/p}e(n/p) = e((r+n)/p)$ has denominator relatively prime to p, hence this must be the singularity. Thus for this n, $pa_p f(z) = f(z^{1/p}e(n/p))$ up to

[1] This is the case that often comes up when you have extra structure to the L-function such as a functional equation. In addition, the Hadamard Multiplication Theorem gives you Rankin-Selberg convolution for this class of functions.

a quantity that is bounded in a neighbourhood of z. Repeating this as before, after $m!$ steps we get back to $z = e(r)$, so $p^{m!}a_p^{m!}f(e(r+x)) = f(e(r+x^{1/p^{m!}}))$ up to a bounded quantity around $x = 0$. Similarly if one takes another prime q satisfying the hypotheses. Since the group generated by p and q is dense in $\mathbb{R}^*$ (eg: one may find natural numbers a, b such that $p^a q^{-b}$ is arbitrarily close to 1) one gets that $a_p^{m!} = p^z$ and $a_q^{m!} = q^z$ for some z that works for all such primes. By the Hadamard Multiplication Theorem $\sum a_n x^n$ and $\sum a_n n^{-z/m!} x^n$ have the same singularities, so without loss of generality we may assume that $z = 0$.

But this implies that all a_n are either $m!$-th roots of unity or 0. Szego's theorem then implies that $g(z)$ is a rational function.

The same argument works for other Euler products by applying the appropriate Hecke operators in place of $\sum_{n=0}^{p-1} f(ze(n/p)) - pa_p f(z^p)$.

3.5 Proofs of Theorems and Propositions

3.5.1 Theorem 7, the forward direction

This is essentially Lemma 3 in [61], which comes from a Mittag-Leffler expansion of $L(s)\Gamma(s)$ and some careful estimates of sums and integrals.

First we want to turn the weaker growth bound into the stronger growth bound.

Consider the following two functions, for $N > A$ sufficiently large so that $L(s)$ is bounded on $\Re(s) = N - 1$:

$$M(s) := L(s)\Gamma(s)\sin(\pi s)(s+i)^{-N}Q^s,$$

$$M_+(s) := M(s)e^{i\pi s/2}.$$

By the multiplication formula we know that $\Gamma(s)\Gamma(1-s)\sin(\pi s) = \pi$ and so when $s \in \mathbb{R}_{<0}$,

$$M_+(s) \ll \Gamma(1-s)\Gamma(s)(s+i)^{-N}\sin(\pi s)Q^s Q^{-s} \ll 1.$$

For $s = N - 1 + iT$ and $T > 0$, $L(s)$ is bounded, so the function $M_+(s)$ is also bounded, since both $\Gamma(s)e^{-\pi i s/2}s^{-N}$ and $\sin(\pi s)e^{i\pi s}$ are bounded.

$M_+(s)$ is of order 1 since it is a product of order 1 functions, so by Phragmen-Lindelof it follows that $M_+(s)$ is bounded on the quadrant $\Im s \geq 0$ and $\Re s \leq \sigma$.

Similarly $M(s)e^{-i\pi s/2}$ is bounded for $\Im(s) < 0$ and $\Re(s) < \sigma$, so we conclude that $M(s) \ll e^{i\pi|\Im(s)|/2}$. This is also the exact bound needed to show that the vertical degree is ≤ 1.

From the integral formula for $\Gamma(s)$,

$$\Gamma(s)/n^s = \int_0^\infty z^{s-1}e^{-nz}dz,$$

and so summing over n with linear coefficients a_n gives

$$\Gamma(s)L(s) = \int_0^\infty z^{s-1}f(z)dz.$$

Hence by Mellin inversion, when $\mathrm{Re}(z) > 0$:

$$f(z) = \frac{1}{2\pi i}\int_{(c)} L(s)\Gamma(s)z^{-s}ds$$

When $Q|z| < 1$ shift the contour to $-\infty$ to get the sum over residues:

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} z^n \operatorname{Res}_{z=-n}(L(s)\Gamma(s))) \\ &= \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} L(-n). \end{aligned}$$

By assumption $L(-n) \ll n!n^AQ^n$ so this sum converges to an analytic function in this region, bounded by $\sum_{n=0}^\infty \frac{|z|^n}{n!}Q^n n^A \ll \frac{1}{1-Q|z|}^{1+A}$.

3.5.2 Theorem 7, the backward direction

Take $f(z) = \sum_{n=1}^\infty a_n e^{-nz}$, which extends analytically in a small neighbourhood of 0. We wish to define and bound the analytic continuation of $L(s) = \sum_{n=1}^\infty a_n n^{-s}$.

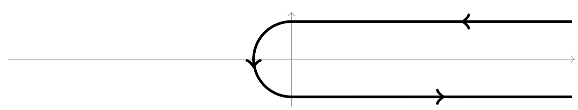
As above we may use the integral for the Gamma function

$$\Gamma(s)n^{-s} = \int_0^\infty x^{s-1}e^{-x}dx$$

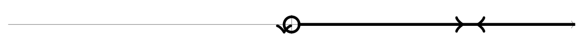
to get the expression as a Mellin transform when $\Re(s) > 1$:

$$\Gamma(s)L(s) = \int_0^\infty x^{s-1}f(x)dx.$$

(a) Circling the positive real axis at distance δ.



(b) Following the positive real axis.



(c) Following the positive real axis and circling the origin at distance q.

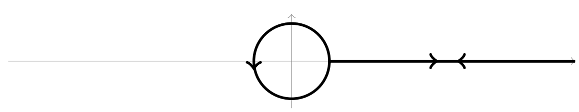


Figure 3.1: The contours used

Following Riemann consider an integral along the Hankel contour (a), which wraps around the positive real axis in the positive direction, while staying distance δ away from the real axis,

$$I(s) := \int_{+\infty+i\delta}^{+\infty-i\delta} (-z)^s f(z)dz/z.$$

This defines an entire function of s.

When $s > 0$ the integrand is continuous to the real axis, aside from a branch cut discontinuity, so we may shift to contour (b) to coincide with the real axis traversed twice wrapping around the origin, and pick up on factors of $(-1)^s$ from the branch-cut of $(-z)^s$,

$$\begin{aligned} I(s) &= \int_{+\infty+0i}^{+\infty-0i} (-z)^s f(z) dz/z \\ &= \int_{\infty}^{0} z^s e^{-\pi i s} f(z) dz/z + \int_{0}^{\infty} z^s e^{\pi i s} f(z) dz/z. \end{aligned}$$

Simplifying the expression, pulling out the factors of $e^{\pm \pi i s}$,

$$I(s) = (e^{\pi i s} - e^{-\pi i s}) \int_0^\infty z^{s-1} f(z) dz = 2i \sin(\pi s) \Gamma(s) L(s),$$

which is further simplified by the multiplication formula, to get

$$I(s) = 2\pi i L(s)/\Gamma(1-s),$$

and we conclude that

$$L(s) = \frac{\Gamma(1-s)}{2\pi i} I(s).$$

The function $I(s)$ is entire and, by Cauchy's formula, $I(s)$ is zero when $s = 1, 2, 3, \cdots$. This cancels out the poles of $\Gamma(1-s)$, thus proving that $L(s)$ is entire.

Move the contour to (c) to circle around the origin at distance $q = \frac{1-\epsilon}{Q}$ and otherwise follow the positive real axis. The integral defining $I(s)$ is then bounded as

$$\begin{aligned} I(s) &= (e^{\pi i s} - e^{-\pi i s}) \int_q^\infty z^s f(z) dz/z + \int_{|z|=q} (-z)^s f(z) dz/z \\ &\ll e^{\pi |\Im t} \int_q^\infty z^{\Re s} e^{-z} dz/z + q^s \epsilon^A. \end{aligned}$$

When $\Re s > 0$ the integral is bounded by $\Gamma(\Re s)$, when $\Re s < 1$ it is bounded by $q^{\Re(s)-1}$. Take $\epsilon \sim 1/|s|$ so that $q^s = \frac{(1-\epsilon)^s}{Q^s} \asymp Q^{-s}$.

Thus if $\Re(s) \geq 0.5$ then

$$L(s) \ll \Gamma(1-s)\Gamma(\Re s) Q^{|s|} + \Gamma(1-s) Q^{-s} |s|^A \ll \Gamma(1+|s|) Q^{|s|} |s|^A,$$

and if $\Re(s) \leq 0.5$ then

$$L(s) \ll \Gamma(1-s)Q^{-s}|s|^A,$$

which combine to give the required (weaker) form of the bound.

3.5.3 Theorem 10

By substituting the functional equation into the expression

$$f(z) = \frac{1}{2\pi i}\int_{(c)} L(s)\Gamma(s)z^{-s}ds,$$

we end up with a sum of scaled Meijer G functions. Rather than work explicitly with these we follow Hardy and Titchmarsh [44] in showing that $\gamma(s)$ for all intents and purposes is $\Gamma(s)$ in our formulae (lemma 5), which will reduce all computations to the case of $\zeta(s)$. This approach is also expanded on in Braaksma [9].

Take c sufficiently large so that $L_a(c)$ and $L_b(c)$ are absolutely convergent series, and as before define

$$f(z) \quad := \quad \sum_{n=1}^{\infty} a_n e^{-nz} = \frac{1}{2\pi i}\int_{(c)} L_a(s)\,\Gamma(s)\,z^{-s}ds.$$

Shift the contour to $1-c$, noting that the poles of $\Gamma(s)$ contribute a polynomial $p(z)$, showing

$$f(z) + p(z) = \frac{1}{2\pi i}\int_{(1-c)} L_a(s)\,\Gamma(s)\,z^{-s}ds.$$

Now substitute $1-s$ for s to get

$$f(z) + p(z) = \frac{1}{2\pi i}\int_{(c)} L_a(1-s)\,\Gamma(1-s)\,z^{s-1}ds,$$

and apply the functional equation to see that

$$f(z) + p(z) = \frac{1}{2\pi i}\int_{(c)} L_b(s)\,\gamma(s)\,\theta(s)\,Q^s\Gamma(1-s)\,z^{s-1}ds.$$

Assuming that the interchange of summation and integration is justified, as is about to proven, expand out L_b to get

$$f(z)+p(z)=\frac{1}{2\pi i}\sum_{n=1}^{\infty}\sum_{j}b_n\alpha_j\int_{(c)}\gamma\left(s\right)n^{-s}e\left(\omega_j s/4\right)Q^s\Gamma\left(1-s\right)z^{s-1}ds.$$

We denote the inner integral as the function G,

$$G\left(z\right):=\frac{1}{2\pi i}\int_{(c)}\gamma\left(s\right)\Gamma\left(1-s\right)z^s ds,$$

so that

$$f(z)+p(z)=\sum_{n=1}^{\infty}\sum_{j}b_n\alpha_j G\left(ze\left(\omega_j s/4\right)\left(Q/n\right)\right)z^{-1}.$$

The result now follows from the following analysis of $G\left(z\right)$.

Lemma 3. *$G\left(z\right)z^{-1}$ is holomorphic aside from a branch cut singularity at $-q_1^{-1}$ where $q_1=\prod\lambda_i^{\lambda_i}$.*

Proof. This proof is done in three stages, dealing with the cases of $\mathrm{Arg}\left(z\right)<\pi$, $|z|<q_1$, and $|z|>q_1$.

Case 1. Letting $s=\sigma+iT$, $\gamma(s)$ is bounded like $e^{-|T|(\pi/2-\epsilon)}$, $\Gamma\left(1-s\right)$ is bounded like $e^{-|T|(\pi/2-\epsilon)}$, z^{s-1} is bounded as $e^{(\mathrm{Arg}(z)-\epsilon)|T|}$, so the integral is absolutely convergent as long as $|\mathrm{Arg}\left(z\right)|<\pi$.

Case 2. For $\mathrm{Re}\left(s\right)$ sufficiently large, lemma 5 in appendix A says that

$$\gamma\left(s\right)\Gamma\left(1-s\right)z^s \quad = \quad \left(zq_1\right)^s s^{\mu}\left(C_1+O\left(1/s\right)\right)\frac{\pi}{\sin\left(-\pi s+\pi d\right)}.$$

Hence when $|zq_1|<1$ you may shift the contour to the right as both the integrand (away from poles) and the residues at $\mathbb{Z}$ are exponentially decaying, to get z times an absolutely convergent power series in z.

Case 3. For $\mathrm{Re}\left(s\right)$ sufficiently negative, lemma 5 upon substituting $-s$ says that

$$\gamma\left(s\right)\Gamma\left(1-s\right)z^s \quad = \quad \left(zq_1\right)^s s^{\mu}\left(C_2+O\left(1/s\right)\right)\prod_{i=1}^{r}\frac{\pi}{\sin\left(-\pi\lambda_i s+\pi\mu_i\right)}$$

Hence when $|zq_1| > 1$ you may shift the contour to the left as both the integrand (away from poles) and the residues at $\bigcup_{i=1}^{r}\left(\mu_i + \frac{\mathbb{Z}}{\lambda_i}\right)$ are exponentially decaying, to get an absolutely convergent generalized power series, on the Riemann surface for $\log z$. $\square$

It can also be useful to understand the behaviour of $G(z)$ around $-q_1^{-1}$.

Lemma 4. *Without loss of generality, translate L_a so that $\mu = 0$. Then around $z = -1$, when $|\mathrm{Arg}(z)| < \pi$,*

$$G(z/q_1)\, z^{-1} = \frac{B_0}{z+1} + B_1 \log(z+1) + O(1)$$

for some B_0, B_1.

Proof. By definition of G, the left hand side is

$$G(z/q_1)\, z^{-1} = \frac{1}{2\pi i}\int_{(c)} \gamma(s)\,\Gamma(1-s)\,(z/q_1)^s\, z^{-1} ds.$$

Lemma 5 shows that $\gamma(s)$, up to correction terms $A_0 + A_1/s$, is essentially $\Gamma(s)$, ergo

$$G(z/q_1)\, z^{-1} = \frac{1}{2\pi i}\int_{(c)} z^{s-1}\left(A_0 + A_1/s + O\left(1/s^2\right)\right)\frac{\pi}{\sin(-\pi s + \pi)} ds.$$

Replacing s with $s+1$, for some new constants B_0 and B_1,

$$G(z/q_1)\, z^{-1} = \frac{1}{2\pi i}\int_{(c)} z^{s}\left(B_0 + B_1/s + O\left(1/s^2\right)\right)\frac{\pi}{\sin(\pi s)}.$$

Bringing the error term outside the integral then gives us

$$G(z/q_1)\, z^{-1} = \frac{1}{2\pi i}\int_{(c)} z^{s}\left(B_0 + B_1/s\right)\frac{\pi}{\sin(\pi s)} + O(1),$$

and computing the integral we conclude that

$$G(z/q_1)\, z^{-1} = \frac{B_0}{1+z} + B_1 \log(1+z) + O(1)$$

□

Chapter 4

Results on degree 2 and higher

4.1 Introduction

The theory of degree 1 Dirichlet series in the previous chapter is relatively straightforward and satisfying, particularly since we can understand these in terms of periodic functions, there's a natural linear space of these, and we have a natural inner product that allows us to decompose in terms of known building blocks. In higher degree each of these things break down in profound ways, and we have to pick our tradeoffs.

First of all, the standard $GL(d)$ L-function is formed by taking the Fourier-Whittaker coefficients $A(n_1, \cdots, n_d)$ of a Maass form and then using a subset of these coefficients as Dirichlet coefficients to form $\sum A(n_1, 1, 1, \cdots, 1)n^{-s}$. The process of expressing these as a zeta integral generally involves restricting to some subspace and then integrating against a known lower-dimensional objects, expressing the L-function as a type of matrix coefficient. Multiple Dirichlet series can be formed using the full set of Fourier coefficients, but past $n = 3$ there seem to be fundamental difficulties to getting the meromorphic continuation that one would ideally have[16][18][10].

A very different route is to interpret the Dirichlet coefficients as Hecke eigenvalues for simultaneous eigenfunctions of the Hecke operators $T(n)$ on an appropriate space. As with the Fourier coefficients this is just a narrow subset of the full amount of information in the original object, namely the behaviour under the full Hecke algebra. This however can be recovered from the Euler product using the $T(n)$ to generate the algebra, and utilizing a factorisation of $\sum T(n)n^{-s}$ as something of the form $\prod_p(1-T(p,1,1,\cdots)p^{-s}+T(p,p,1,1,\cdots)p^{-2s}-\cdots\pm T(p,p,\cdots,p)p^{-ds})$. This relies heavily on the Euler product, and one may not take linear combinations of eigenvalues in the same way that one takes linear combinations of Dirichlet series or Fourier series.

Like in the degree 1 case, linear twists $\sum a_n e(n\alpha)n^{-s}$ generally make sense but only for rational α. The naive Rankin-Selberg convolution $\sum a_n b_n n^{-s}$ stops having analytic continuation past $d=2$ and while there is a convolution on degree $d>2$ it requires first recovering a higher dimensional structure by, for example, factoring the Euler product into the Langlands parameters.

The Hurwitz zeta functions that figured prominently in the previous chapter have natural analogue in the Shintani zeta functions[46], which have analytic continuation, and can be used as a basis for Dedekind zeta functions and of Eisenstein series. A particular special case of these are the Witten zeta functions, which are expressed as sums over root systems of a semi-simple Lie group [101][103]. These do not however provide a good analogue to the continuous family of linear twists in degree 1.

The presence of the natural numbers in the Dirichlet series crucial since, as Kaczorowski and Perelli point out, if you extend to generalized Dirichlet series $\sum a_n b_n^{-s}$ every gamma factor and conductor is possible.

Currently what is known of the general theory is that degree strictly between 1 and 2 is impossible[57], degree 2 Hecke-type functional equations are fully understood[7], certain small cases of Maass-type functional equations with conductor 1 or 2 are understood[68], and sporadic cases in degree 2 where it is known that a certain functional equation, such as $\Gamma(s/3)\Gamma(2s/3)$ and conductor 1, is impossible.

4.2 Degree d Functional Equations

We revisit the results from chapter 3, both to apply it to higher degree, and because here we will need more precise information on the dependence on s in order to apply more general functional equations.

Theorem 11. *A Dirichlet series $L(s) := \sum_{n=1}^{\infty} a_n n^{-s}$ has horizontal degree d if and only if $f(z) := \sum_{n=1}^{\infty} a_n e^{-n^{1/d}z}$, well defined for $\Re z > 0$, continues analytically in a* *neighbourhood of 0. Furthermore if this is the case then it also has vertical degree d.*

More precisely, the following are equivalent, for given Q, and some A not necessarily the same in all cases:

1. *$L(s/d)$ extends to an entire function satisfying $|L(s/d)| \ll \Gamma(1+|s|)|s|^A Q^{|s|}$.*
2. *$L(s/d)$ extends to an entire function satisfying $|L(s/d)|\Gamma(s)Q^s \ll |s|^A e^{-\pi i|\Im(s)|/2}$ for s in a left half-plane, away from the poles.*
3. *$f(z)$ extends analytically in a $1/Q$ neighbourhood of 0, and $f(z) \ll \left(\frac{1}{1-Q|z|}\right)^A$ in this neighbourhood.*

And in this case f has Taylor expansion: $f(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} L(-n/d)$.

Proof. The proof is identical to that of Theorem 11, applying it to the generalized Dirichlet series $L(s/d)$. Note that this also shows that vertical degree is no greater than the horizontal degree, as claimed in Proposition 2. □

For applying to functional equations recall the most general definition from the introduction.

Definition 9. *(Most general functional equation) $L(s) = \sum a_n n^{-s}$ satisfies for some set Spec on the Riemann surface for* log*, some complex b_x bounded by some power of $|x|$, and some uniformly Gamma-like $\gamma_x(s)$,*

$$L(s) = \sum_{x \in Spec} b_x x^{s-1} \gamma_x(1 - ds + \mu).$$

By translating s we may without loss of generality take $\mu = 0$.

Theorem 12. *Suppose $L(s) = \sum a_n n^{-s}$ satisfies for some set Spec and Gamma-like functions γ_x,*

$$L(s/d) = \sum_{x \in Spec} b_x x^{1-s} \gamma_x(1 - s).$$

Then $f(z) := \sum a_n e^{-n^{1/d} z}$ has analytic continuation to all of $\mathbb{C}$ aside from some singularities (potentially branch cut discontinuities) for $z \in -Spec$. The singularities *are to first order simple poles with residue b_x. Namely for $x \in -Spec$,*

$$f(z+x) = \frac{b_x}{z} + O(z^\epsilon).$$

Proof. As in Theorem 7 there is the Mellin inversion integral for c sufficiently large,

$$f(z) = \frac{1}{2\pi i}\int_{(c)} L(s)\Gamma(s)z^{-s}ds.$$

Shifting the contour to $1-d$ for large d, noting that the poles of $\Gamma(s)$ contribute a polynomial $p(z)$,

$$f(z) + p(z) = \frac{1}{2\pi i}\int_{(1-d)} L(s)\Gamma(s)z^{-s}ds.$$

Replacing d with $1-d$, we see that

$$f(z) + p(z) = \frac{1}{2\pi i}\int_{(d)} L(1-s)\Gamma(1-s)z^{s-1}ds.$$

For d sufficiently large we may substitute in the general functional equation to get

$$f(z) + p(z) = \frac{1}{2\pi i}\int_{(d)} \sum_{x\in\mathrm{Spec}} b_x x^{-s}\gamma_x(s)\Gamma(1-s)z^{s-1}ds,$$

and by absolute convergence interchange the order of summation and integration to get

$$f(z) + p(z) = \sum_{x\in\mathrm{Spec}} b_x \frac{1}{2\pi i}\int_{(d)} (z/x)^s\gamma_x(s)\Gamma(1-s)z^{-1}ds.$$

Defining the inner integral as the function G_x,

$$f(z) + p(z) = \sum_{x\in\mathrm{Spec}} b_x G_x(z/x)z^{-1},$$

where

$$G_x(z) := \frac{1}{2\pi i}\int_{(d)} \gamma_x(s)\Gamma(1-s)z^s ds.$$

As in the proof of Theorem 10, we may apply Lemma 6 to estimate the integrand away from the poles,

$$G_x(z) = \frac{1}{2\pi i}\int_{(d)} \frac{\pi}{\sin(\pi s)}(1 + O(1/s))z^s ds.$$

In particular, for $s = \sigma + iT$when $\arg(z) < \pi$ then the integral is absolutely convergent to a holomorphic function. This is because, letting $s = \sigma + iT$ for fixed σ where $\sin(\pi\sigma) \neq 0$, $\sin(\pi s) \ll e^{\pi/2T}$ and $|z^s| \sim e^{\arg(z)T}|z|^\sigma$.

When $|z| < 1$ the integrand converges uniformly to 0 as $\Re s$ increases to ∞, away from the poles. Shifting the contour to the right we pick up on a sum over residues at $-n = 1 - s$,

$$G_x(z) = \sum_{n>d-1} \gamma_x(n-1)z^{n-1}/n!,$$

and this expression is absolutely convergent when $|z| < 1$.

When $|z| > 1$ the integrand converges to 0 as $\Re z$ decreases to $-\infty$, away from the poles. Shifting the contour to the left we pick up on a sum over the residues at the poles of $\gamma_x(y)$,

$$G_x(z) = \sum_y \mathrm{Res}_{z=y}(\gamma_x(y)\Gamma(1-y)z^y),$$

and this expression is absolutely convergent when $|z| > 1$ to a multivalued function.

□

Non-linear twist Dirichlet series

Similar to Kaczorowski and Perelli[52], given a Dirichlet series with a general degree d functional equation, define the non-linear twist Dirichlet series

$$L(s, \alpha) := \sum_{n=1}^{\infty} a_n e^{-\alpha n^{1/d}} n^{-s}.$$

Theorem 13. *As long as $\alpha \notin -Spec$, $L(s, \alpha)$ is entire and of horizontal and vertical degree d. This holds not only for $\Re\alpha \geq 0$ but also to $\Re\alpha \leq 0$ by analytic continuation.*

Kaczorowski and Perelli also derive an analytic continuation, but do not prove that the resulting function has horizontal and vertical degree d, which is the optimal bound.

Proof. First consider $\Re\alpha \geq 0$ and $\alpha \notin -\text{Spec}$. The shifted function $f(z+\alpha) = \sum_{n=1}^{\infty} a_n e^{-n^{1/d}z} e^{-n^{1/d}\alpha}$ is analytic in a neighbourhood of $z=0$, so therefore the corresponding Dirichlet series $L(s,\alpha)$ is entire and of horizontal and vertical degree d. $f(z+\alpha)$ is a well-defined analytic function even for $\Re(\alpha) < 0$ so the Mellin transform formula $\Gamma(s)L(s,\alpha) = \frac{1}{2\pi i}\int_{(c)} f(z+\alpha)z^{-s}ds$ that recovers $L(s,\alpha)$ when $\Re(\alpha) > 0$ is likewise well-defined for all $\alpha \notin -\text{Spec}$ and defines the analytic continuation. □

A consequence in degree 2

We can also prove the following partial classification in degree 2.

In degree 1 we characterized series of the form $\sum a_n e^{-nz}$ by their periodicity. In degree 2 we can characterize series of the form$\sum a_n e^{-n^{1/2}z}$ by a convolutional transform.

For simplicity suppose that we have a general functional equation and the Gamma factors $\gamma_x(s)$ are all simply $\Gamma(s)$. In this case the singularities are simple poles

$$G_x(z) = \frac{1}{2\pi i}\int_{(d)} \frac{\pi}{\sin(\pi s)} z^{-s} ds = \frac{1}{1-z} + p_x(z).$$

Define the following function of two variables as an extension of the non-linear twist,

$$f(z,w) \;\; = \;\; \sum_{n=1}^{\infty} a_n e^{-\sqrt{n}z} e^{-inw}.$$

When $w > 0$, take the contour $\mathcal{C}$ following a path from $z-\delta+i\infty$ to $z-\delta$ to $z+\delta$ to $z+\delta-i\infty$. This is chosen so that $\Im((t-z)^2) < 0$ and hence $e^{(t-z)^2/(4wi)}$ decays exponentially. Since the Gaussian is self-dual under the Fourier transform,

$$\sqrt{-4\pi i w} e^{-\sqrt{n}z} e^{-inw} = \int_{\mathcal{C}} e^{-\sqrt{n}t} e^{(t-z)^2/(4wi)} dt.$$

Hence, summing with linear coefficients a_n,

$$\sqrt{-4\pi i w} f(z,w) = \int_{\mathcal{C}} f(t) e^{(t-z)^2/(4wi)} dt.$$

Rotate the contour by a small angle clockwise, so that for every z and as t ranges over the contour, $(t-z)^2$ eventually has steadily decreasing imaginary part, so $e^{(t-z)^2/(4wi)}$ decays rapidly, yielding an entire function of z. When performing this shift of the contour we pick up the contributions from the poles on the positive imaginary axis, denoted by $\mathrm{Spec}_{+i} = \mathrm{Spec} \cap (i\mathbb{R}_{>0})$. Doing this we get the formula

$$\sqrt{-4\pi i w} f(z,w) - (\text{entire}) = \sum_{x \in \mathrm{Spec}_{+i}} b_x e^{(x-z)^2/(4wi)},$$

which simplifies to

$$\sqrt{-4\pi i w} f(z,w) - (\text{entire}) = e^{z^2/(4wi)} \sum_{x \in \mathrm{Spec}_{+i}} b_x e^{ixz/(2w)} e^{x^2/(4wi)}.$$

Suppose $\mathrm{Spec}_{+i} \subset 4\pi\sqrt{\mathbb{N}} i$ and for some Q and some constant C, $b_{4\pi\sqrt{n}Q} = a_n C$, as would happen if $L(s)$ satisfies a functional equation. Then the above comes to

$$\sqrt{-4\pi i w} f(z,w) - (\text{entire}) = e^{z^2/(4wi)} \sum_{n=1}^{\infty} C a_n e^{-2\pi Q\sqrt{n} z/w} e^{-16\pi^2 Q^2 n/(4wi)}.$$

Letting $w = 2\pi$ we get

$$\sqrt{-8\pi i} f(z) - (\text{entire}) = e^{z^2/(8\pi i)} C f(Qz),$$

but the exponential factor blows up singularities not on the real and imaginary axis, so therefore the only singularities may lie on the real and imaginary axis.

4.3 Computing Hecke eigenbases

Given a linear combination of finitely many distinct multiplicative sequences, it is possible to recover each of the original sequences. In particular, this can be an effective way to compute the Hecke eigenbases in $\Gamma_0(\chi)$ from knowledge of the Hecke traces. This approach comes from Andrew Booker.

For ease of exposition, we restrict to the case of weight k cusp forms on $\Gamma_0(N)$ for some fixed character χ, so that we have a known degree 2 Euler product. There is an unknown eigenbasis $f_1, \cdots, f_k$ with corresponding Fourier coefficients $a_i(n)$. We assume we are given $T(n)$ which is known to be a sum of multiplicative functions $T(n) = a_1(n) + \cdots + a_k(n)$.

Define $T(m,n) = a_1(m)a_1(n) + \cdots a_k(m)a_k(n)$. By multiplicativity conditions $a_i(m)a_i(n) = \sum_{d|(m,n)} a_i(mn/d^2)\chi(d)d^{k-1}$, so $T(m,n) = \sum_{d|(m,n)} T(mn/d^2)\chi(d)d^{k-1}$.

Define $T(m,n,p) = a_1(m)a_1(n)a_1(p) + \cdots a_k(m)a_k(n)a_k(p)$, which can similarly be computed in terms of T, as $T(m,n,p) = \sum_{d|(m,n)} T(mn/d^2, p)\chi(d)d^{k-1}$.

Now, $T(n) = \sum a_i(n)$, $T(2,n) = \sum a_i(2)a_i(n)$, $T(3,n) = \sum a_i(3)a_i(n)$ and so forth. These are all linear combinations of the $a_i(n)$, so given sufficiently many we span the whole space and can recover the a_i as linear combinations $\sum_r T(r,n)\alpha_{r,i}$.

Let $\vec{a}_i$ be the infinite vector $(a_i(n))_n$.

Form the matrices $T = (T(m,n))_{m,n}$ and $T_p = (T(m,n,p))_{m,n}$. T_p is symmetric, and T is symmetric and positive definite since it is the sum of outer products $\sum a_i^T a_i$. Let $\vec{v}_i \in \mathbb{C}^k$ be the unknown vector such that $T\vec{v}_i = \vec{a}_i$. Note also that $T_p\vec{v}_i = a_i(p)\vec{v}_i$. Thus for given p we may solve the generalized eigenvalue problem $T_p\vec{v} = \lambda T\vec{v}$ to recover the $\vec{v}_i$ assuming all $a_i(p)$ are distinct.

If the $a_i(p)$ for the selected p are not all distinct then this fails to recover unique eigenvectors. However, this cannot simultaneously happen for very many p, so taking a random linear combination of a few different T_p will result in distinct dimension 1 eigenspaces.

It remains to show that the $T(n)$ may be effectively computed, and so we present a trace formula for these.

4.3.1 Eichler-Selberg Trace Formula

This section follows [102][63] for the case of the Eichler-Selberg Trace Formula on $SL_2(\mathbb{Z})$ and [85][22] for $S_k(\Gamma_0(N), \chi)$. The proofs are given explicitly in Oesterle's thesis [82].

We define a function $H(n)$ on integers as follows. Let $H(0)$ if $n > 0$ and $H(0) = -1/12$. For $n < 0$ let $H(n)$ denote the number of equivalence classes with respect to $SL_2(\mathbb{Z})$ of positive definite binary quadratic forms

$$ax^2 + bxy + cy^2$$

with discriminant

$$b^2 - 4ac = n,$$

counting forms equivalent to a multiple of x^2+y^2 (resp. x^2+xy+y^2) with multiplicity $\frac{1}{2}$ (resp. $\frac{1}{3}$).

This function is closely related to the ordinary class numbers $h(n)$ as follows. Define $h_w(-3) = \frac{1}{3}$, $h_w(-4) = \frac{1}{2}$, $h_w(n) = h(n)$ if $n < -4$ and congruent to 0 or 1 (mod 4), and $h_w(x) = 0$ otherwise. Then

$$H(n) = \sum_{f|n} h_w\left(\frac{n}{f^2}\right)$$

Also define polynomials $Q_k(t, n)$ for $k = 0, 1, \cdots$ so that

$$\frac{u^{k+1} - v^{k+1}}{x - y} = Q_k(u + v, uv).$$

These polynomials have generating series

$$(1 - tx + Nx^2)^{-1} = \sum x^k Q_k(t, n).$$

The polynomials can also be computed recursively, with base conditions $Q_0(t, n) =$

1 and $Q_1(t,n) = t$, and recurrence

$$Q_{k+1}(t,n) = tQ_k(t,n) - nQ_{k-1}(t,n).$$

Theorem 14. *(Trace Formula)*

Let $k \geq 4$ be an even integer and let m be a natural number. Then the trace of the Hecke operator $T(m)$ on the space of cusp forms S_k is given by

$$\operatorname{Tr} T(m) = -\frac{1}{2} \sum_{\substack{t \in \mathbb{Z} \\ t^2 - 4m \leq 0}} Q_{k-2}(t,m) H(t^2 - 4m) - \frac{1}{2} \sum_{dd'=m} \min(d,d')^{k-1}.$$

Theorem 15. *(Trace Formula) Let $N \in \mathbb{Z}_{\geq 1}$ and let $\chi : (\mathbb{Z}/N\mathbb{Z})^* \mapsto \mathbb{C}^*$ be a character of conductor N_χ. Furthermore, let $k \geq 2$ be an integer for which $\chi(-1) = (-1)^k$.*

For every integer $n \geq 1$ the trace Tr of the Hecker operator T_n acting on the space of cusp forms $S_k(\Gamma_0(N), \chi)$ is given by

$$\operatorname{Tr}(T_n) = A_1(\chi) + A_2(\chi) + A_3(\chi) + A_4(\chi),$$

where

$$A_1(\chi) = n^{k/2-1} \chi(\sqrt{n}) \frac{k-1}{12} \psi(N).$$

(Here $\chi(\sqrt{n}) = 0$ whenever $\sqrt{n} \notin \mathbb{Z}$ and $\psi(N)$ denotes $N \prod_{p|N}(1+1/p)$, where the product runs over the prime divisors p of N).

$$A_2(\chi) = -\frac{1}{2} \sum_{t \in \mathbb{Z}, t^2 < 4n} Q_{k-2}(t,n) \sum_f h_w\left(\frac{t^2-4n}{f^2}\right) \mu(t,f,n,\chi).$$

(Here the sum runs over the positive divisors f of $t^2 - 4n$ for which $(t^2-4n)/f^2 \in \mathbb{Z}$ is congruent to 0 or 1 (mod 4). The numbers $\mu(t,f,n,\chi)$ are given by

$$\mu(t,f,n,\chi) = \frac{\psi(N)}{\Psi(N/N_f)} \sum_{\substack{x \ (mod\ N) \\ x^2 - tx + n \equiv 0 \ (mod\ N_f N))}} \chi(x),$$

where N_f *denotes* $\gcd(N, f)$.

$$A_3(\chi) = -\sum_{\substack{d|n\\ 0<d\le\sqrt{n}}}{}' d^{k-1} \sum_{\substack{c|N\\ \gcd(c,N/c)|\gcd(N/N_\chi,n/d-d)}} \phi\left(\gcd\left(c, \frac{N}{c}\right)\right)\chi(y).$$

(Here ϕ *denotes Euler's phi-function; the prime in the first summation indicates that the contribution of the term* $d = \sqrt{n}$*, if it occurs, should be multiplied by* $\frac{1}{2}$*. The number* y *is defined modulo* $N/\gcd(c, N/c)$ *by* $y \equiv d \pmod{c}$*,* $y \equiv (n/d) \pmod{N/c}$*.)*

$$A_4(\chi) \;=\; \begin{cases} \sum_{\substack{0<t|n\\ \gcd(N,n/t)=1}} t & \text{if } k = 2 \text{ and } \chi = 1,\\ 0 & \text{otherwise.}\end{cases}$$

(The unit character is denoted by 1. We recall that every character χ *is extended* $\mathbb{Z}/N\mathbb{Z}$ *by* $\chi(m) = 0$ *whenever* $\gcd(m, N) > 1$*).*

To allow for computations to be done over integers it will be convenient for fixed N to take the Fourier transform of the A_i, over the characters χ.

Theorem 16.

$$\hat{A}_1(a) = \begin{cases} n^{k/2-1}\frac{k-1}{12}\psi(N) & \text{if } n \equiv a \pmod{N}\\ 0 & \text{otherwise}\end{cases}$$

$$\hat{\mu}(t, f, n, a) = \begin{cases} \frac{\psi(N)}{\Psi(N/N_f)} & \text{if } N * N_f | a^2 - ta + n\\ 0 & \text{otherwise}\end{cases}$$

where N_f *denotes* $\gcd(N, f)$.

$$\hat{A}_2(a) == -\frac{1}{2}\sum_{t\in\mathbb{Z},t^2<4n} Q_{k-2}(t,n)\sum_f h_w\left(\frac{t^2-4n}{f^2}\right)\hat{\mu}(t,f,n,a).$$

$$\hat{A}_3(a) = -\sum_{\substack{d|n\\ 0<d\le\sqrt{n}}}{}' d^{k-1} \sum_{\substack{c|N\\ a\equiv d \pmod{c}\\ a\equiv n/d \pmod{N/c}}} \phi\left(\gcd\left(c, \frac{N}{c}\right)\right)/\gcd(c, \frac{N}{c}).$$

$$\hat{A}_4(a) = \frac{1}{\phi(N)} \begin{cases} \sum_{\substack{0<t|n \\ \gcd(N,n/t)=1}} t & \text{if } k = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The formulae for $\hat{A}_1$, $\hat{\mu}$, A_2 only rely on the Fourier transform of χ being the indicator of the congruence class a. The formula for $\hat{A}_4$ has a factor of $\frac{1}{\phi(N)}$ coming in from only the trivial character contributing.

In $\hat{A}_3$ care needs to be taken when dealing with non-primitive characters. □

4.3.2 Algorithm

To compute first M coefficients of all Hecke eigenforms of weight k on $\Gamma_0(N)$ for all characters χ mod N.

Compute sufficiently many primes, and sufficiently many Dirichlet coefficients of $1/\zeta(s) = \sum_{n=1}^{\infty} \mu(n) n^{-s}$ and $1/\zeta(s)^2 = \sum_{n=1}^{\infty} \mu_2(n) n^{-s}$.

Compute sufficiently many $\hat{A}_i(\chi)$, in exact integer arithmetic since these are all in $\mathbb{Z}/12$.

For each character χ, Fourier invert to compute $A_i(\chi), \cdots, A_4(\chi)$ and hence sufficiently many $\operatorname{Tr} T(m)$. Define $T(m,n) = \sum_{d|(m,n)} \operatorname{Tr} T(mn/d^2)\chi(d)d^{k-1}$. Define $T(m,n,p) = \sum_{d|(m,n)} T(mn/d^2, p)\chi(d)d^{k-1}$.

For each m and given χ, $(\operatorname{Tr} T(m))^2 \overline{\chi}(m)$ is real. Define $\sqrt{\chi(m)}$, picking the sign of $\sqrt{\chi(p)}$ arbitrarily, and picking the sign of $\sqrt{\chi(m)}$ to retain multiplicativity. Computations may be done instead in terms of $T_m = \operatorname{Tr} T(m)\overline{\sqrt{\chi(m)}}$ so that all T_m are real.

Note that $d := \operatorname{Tr} T(1)$ is the dimension of the space of cuspforms, and if this is 0 then continue.

Iteratively form a sequence of primes $p_1, p_2, \cdots, p_d$, taking p_i to be the least prime so that the i by i matrix $(T(p_j, p_k))_{j,k}$ is invertible. Let T be the resulting d by d matrix.

For a small number of primes p, Let C be a random linear combination of the matrices $T_p = (T(p_j, p_k, p))$.

Solve the generalized self-adjoint eigenvalue problem $C\vec{v} = \lambda T\vec{v}$. Each eigenvector v represents one cusp eigenform, whose coefficients may be recovered as $a_i = \sum_{j=1}^{d} T(i, p_j)v_j$.

Appendix A

Properties of $\Gamma(s)$

In this chapter we collect some results about $\Gamma(s)$ that are needed throughout the thesis. First recall some standard results from [1].

$\Gamma(s)$ is defined by the Mellin transform

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx.$$

This transform may be inverted to get the following integral on $\Re(s) = c$

$$e^{-x} = \frac{1}{2\pi i} \int_{(c)} \Gamma(s) z^{-s} ds.$$

This is the continuous analogue of the factorial function since for $n \in \mathbb{N}$

$$\Gamma(1+n) = n!.$$

There is the reflection formula

$$\Gamma(s)\Gamma(1-s) = \pi / \sin(\pi s).$$

There is the duplication formula

$$\Gamma(2s) = \frac{1}{\sqrt{4\pi}} 4^s \Gamma(s)\Gamma(s+1/2).$$

Stirling's Formula gives the asymptotic away from the negative real axis

$$\Gamma(s) \sim \sqrt{\frac{2\pi}{s}}\left(\frac{s}{e}\right)^s.$$

Euler-Maclaurin summation gives an symptotic series away from the negative real axis

$$\log\Gamma(s) \sim (s+a-1/2)\log s - s + \frac{1}{2}\log(2\pi) + \sum_{n=1}^{\infty}\frac{B_{n+1}(a)}{n(n+1)s^n}.$$

Following [9] and [44] gives the following useful lemma.

Lemma 5. *Let* $\mu_i \in \mathbb{C}$ *and* $\lambda_i > 0$, *for* $i = 1, 2, \cdots, r$, *where* $d{=}\sum_{i=1}^r \lambda_i$. *Then away from a sector containing the poles of the left-hand side:*

$$\prod_{i=1}^{r}\Gamma\left(\lambda_i s + \mu_i\right) \quad = \quad s^{ds}q^s s^{\mu}\left(A + O_{\lambda,\mu}\left(1/s\right)\right),$$

where $q = e^{-d}\prod_{i=1}^r \lambda_i^{\lambda_i}$, $\mu = \sum_{i=1}^r (\mu_i - 1/2)$, *and* $A = \sqrt{2\pi}^r \prod \lambda_i^{\mu_i - 1/2}$.

Proof. Euler-Maclaurin summation gives Stirling's formula away from the poles, as the asymptotic formula[1],

$$\log\Gamma\left(s+a\right) \quad = \quad \left(s+a-1/2\right)\log s - s + \frac{\log 2\pi}{2} + \sum_{n=1}^{\infty}\frac{B_{n+1}\left(a\right)}{n\left(n+1\right)s^n}.$$

Hence

$$\log\Gamma\left(\lambda s+\mu\right) \quad = \quad \left(\lambda s+\mu-1/2\right)\log\left(\lambda s\right) - \lambda s + \frac{\log 2\pi}{2} + O_{\lambda,\mu}\left(1/s\right).$$

Applying this to each summand of the left-hand side gives

$$\begin{aligned}\sum_{i=1}^{r}\log\Gamma(\lambda_i s+\mu_i) \quad = \quad & \Big(ds + \sum\left(\mu_i - 1/2\right)\Big)\log s + \sum_{i=1}^{r}\lambda_i s\log\lambda_i \\ & + \sum_{i=1}^{r}\left(\mu_i - 1/2\right)\log\lambda_i - ds + \frac{r\log 2\pi}{2} + O_{\lambda,\mu}\left(1/s\right),\end{aligned}$$

which simplifies to

$$\begin{aligned}\sum_{i=1}^{r} \log\Gamma(\lambda_i s+\mu_i) &= ds\log s+\sum \lambda_i \log\lambda_i s - ds + \sum(\mu_i-1/2)\log s \\ &\quad + \sum_{i=1}^{r}(\mu_i-1/2)\log\lambda_i + \frac{r\log 2\pi}{2} + O\left(1/s\right) \\ &= ds\log s + s\log q + \mu\log s + \log A + O\left(1/s\right),\end{aligned}$$

where $\log q = -d + \sum_{i=1}^{r}\lambda_i\log\lambda_i$, $\mu = \sum_{i=1}^{r}(\mu_i-1/2)$, and $\log A = \frac{r\log 2\pi}{2} + \sum_{i=1}^{r}(\mu_i-1/2)\log\lambda_i$. □

Furthermore, carrying through the correction terms yields the asymptotic series

$$\prod_{i=1}^{r}\Gamma\left(\lambda_i s+\mu_i\right) = s^{ds}q^s s^{\mu}\left(A_0 + A_1 s^{-1} + A_2 s^{-2} + \cdots\right),$$

where the A_n are functions of the λ_i and μ_i.

Lemma 6. *For positive λ_i summing to d*

$$\prod_{i=1}^{r}\Gamma(\lambda_i s+\mu_i) = \Gamma(ds+\mu)q^s(A+O(1/s))$$

where $q = d^{-d}\prod_{i=1}^{r}\lambda_i^{\lambda_i}$, $\mu - 1/2 = \sum_{i=1}^{r}(\mu_i-\frac{1}{2})$, *and* $A = (2\pi)^{(r-1)/2}\prod\lambda_i^{\mu_i-1/2}d^{1/2-\mu}$.

Proof. Applying Lemma 5 to the left-hand side we get

$$\prod_{i=1}^{r}\Gamma(\lambda_i s+\mu_i) = s^{ds}\left(e^{-d}\prod_{i=1}^{r}\lambda_i^{\lambda_i}\right)^{s} s^{\sum_{i=1}^{r}(\mu_i-1/2)}((2\pi)^{r/2}\prod_{i=1}^{r}\lambda_i^{\mu_i-1/2} + O(1/s)).$$

Applying the lemma to the right side we get the equivalent expression

$$\Gamma(ds+\mu)q^s(A+O(1/s)) = (s/e)^{ds}e^{-ds}s^{\mu-1/2}(\sqrt{2\pi}^{r} + O(1/s))\prod_{i=1}^{r}\lambda_i^{\lambda_i s+\mu_i-1/2}.$$

□

Lemma 7. *For i in $1, 2, \cdots, r_a$ and j in $1, 2, \cdots, r_b$, given $\mu_{a,i} \in \mathbb{C}$, $\mu_{b,j} \in \mathbb{C}$, $\lambda_{a,i}, \lambda_{b,j} > 0$, where $\sum_{i=1}^{r_a} \lambda_{a,i} = \sum_{j=1}^{r_b} \lambda_{b,j}$, when $\mathrm{Re}\,(s)$ is sufficiently large:*

$$\prod_{i=1}^{r_a} \Gamma\left(\lambda_{a,i}s + \mu_{a,i}\right) \prod_{j=1}^{r_b} \Gamma\left(-\lambda_{b,j}s + \mu_{b,j}\right) \quad = \quad q^s s^\mu \left(A + O\left(1/s\right)\right) \prod_{j=1}^{r_b} \frac{\pi}{\sin\left(-\pi\lambda_{b,j}s + \pi\mu_{b,j}\right)},$$

where $q = \prod_{i=1}^{r_a} \lambda_{a,i}^{\lambda_{a,i}} / \prod_{j=1}^{r_b} \lambda_{b,j}^{\lambda_{b,j}}$, $\mu = \sum_{i=1}^{r_a} \left(\mu_{a,i} - 1/2\right) + \sum_{j=1}^{r_b} \left(\mu_{b,i} - 1/2\right)$, and $A = \sqrt{2\pi}^{r_a - r_b} \prod_{i=1}^{r_{a,i}} \lambda_{a,i}^{\mu_{a,i}-1/2} / \prod_{j=1}^{r_b} \lambda_{b,j}^{\mu_{b,j}-1/2}$.

Proof. The reflection formula for Γ applied to each term gives

$$\prod_{j=1}^{r_b} \Gamma\left(-\lambda_{b,j}s + \mu_{b,j}\right) \quad = \quad \prod_{j=1}^{r_b} \frac{\pi}{\sin\left(-\pi\lambda_{b,j}s + \pi\mu_{b,j}\right)\Gamma\left(\lambda_{b,j}s - \mu_{b,j} + 1\right)}.$$

Applying lemma 5 to both numerator and denominator of the quotient

$$\frac{\prod_{i=1}^{r_a} \Gamma\left(\lambda_{a,i}s + \mu_{a,i}\right)}{\prod_{j=1}^{r_b} \Gamma\left(\lambda_{b,j}s - \mu_{b,j} + 1\right)}$$

gives the asymptotic

$$q^s s^\mu \left(A + O\left(1/s\right)\right),$$

and the result follows. $\square$

Appendix B

On a formula for $\zeta(3)$

Apery's [3][4] proof of irrationality of $\zeta(3)$ begins with the identity, proven by combinatorial manipulation of hypergeometric series:

$$\frac{2}{5}\zeta(3) = \sum \frac{(-1)^{n-1}}{n^3\binom{2n}{n}}.$$

A new proof is presented here via a Mellin Transform, expressing $\zeta(3)$ in terms of truncating the classical integral $\Gamma(s)\zeta(s) = \int_0^\infty \frac{x^{s-1}}{e^x-1}dx$.

Proposition 3. *Apery's formula*

$$\frac{2}{5}\zeta(3) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3\binom{2n}{n}} \tag{B.1}$$

is equivalent to

$$\frac{8}{5}\zeta(3) + \frac{1}{6}\psi^3 = \int_\psi^\infty \frac{x^2}{e^x-1}dx, \tag{B.2}$$

where $\psi = \log\frac{3+\sqrt{5}}{2}$.

Proposition 4. *Formula (B.2) is true. Moreover we have*

$$\int_{\log(\alpha)}^\infty \frac{x^2}{e^x-1}dx = r\zeta(3) + s\log(\alpha)^3,$$

for the following sets of parameters (α, r, s)*:*

α	r	s
1	2	$*$
$\frac{3+\sqrt{5}}{2}$	$\frac{8}{5}$	$\frac{1}{6}$
2	$\frac{7}{4}$	$\frac{1}{3}$

Proof of Proposition 3. The starting point is the familiar formula [1]

$$\frac{\Gamma(s)}{n^s} = \int_0^\infty e^{-nx} x^{s-1} dx.$$

Summing both sides over n with linear coefficients a_n relates the Dirichlet series $L(s) = \sum a_n n^{-s}$ and the power series $f(z) = \sum a_n z^n$ as

$$\Gamma(s)L(s) = \int_0^\infty f(e^{-x}) x^{s-1} dx.$$

The interchange of summation and integration is valid as long as $\Re(s) > 1$ and $f(z)$ has radius of convergence greater than 1.

Consider then the power series with radius of convergence 2 [1]

$$\begin{aligned} f(z) &:= 2\operatorname{arcsinh}(x/2)^2 \\ &= \sum_{n=1}^\infty \frac{(-1)^{n-1} x^{2n}}{n^2 \binom{2n}{n}}. \end{aligned}$$

The corresponding Dirichlet series is

$$L(s) = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^2 \binom{2n}{n} (2n)^s}.$$

Setting $s = 1$ and recalling that $\Gamma(1) = 1$,

$$\begin{aligned} L(1)\Gamma(1) &= \sum_{n=1}^\infty \frac{(-1)^{n-1}}{2n^3 \binom{2n}{n}} \\ &= 2\int_0^\infty \operatorname{arcsinh}(e^{-x}/2)^2 dx. \end{aligned}$$

Making the subsitution $y = 2\,\mathrm{arcsinh}(e^{-x}/2)$ so that $x = -\log(2\sinh(y/2))$ and $dx = \frac{\cosh(y/2)}{2\sinh(y/2)}dy$,

$$\begin{aligned} 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3\binom{2n}{n}} &= 8\int_0^{\infty}\mathrm{arcsinh}(e^{-x}/2)^2dx \\ &= \int_0^{\psi}\frac{\cosh(y/2)}{\sinh(y/2)}y^2dy. \end{aligned}$$

Now, rewriting cosh and sinh in terms of exponentials, this is

$$\int_0^{\psi}\frac{e^y+1}{e^y-1}\cdot y^2dy = \int_0^{\psi}y^2dy + 2\int_0^{\psi}\frac{y^2}{e^y-1}dy.$$

The first term may be explicitly computed making this

$$\frac{1}{3}\psi^3 + 2\int_0^{\psi}\frac{y^2}{e^y-1}dy,$$

and making note that $2\zeta(3) = \int_0^{\infty}\frac{y^2}{e^y-1}dy$, we may rewrite this to give the desired expression

$$\frac{1}{3}\psi^3 + 4\zeta(3) - 2\int_{\psi}^{\infty}\frac{y^2}{e^y-1}dy.$$

□

Proof of Proposition 4. We are interested in evaluating $\int_{\log(\alpha)}^{\infty}\frac{x^2}{e^x-1}dx$. Substituting $x = \log t$ and then replacing t with $1/t$ gives the equalities

$$\int_{\log(\alpha)}^{\infty}\frac{x^2}{e^x-1}dx = \int_{\alpha}^{\infty}\frac{\log^2(t)}{t(t-1)}dt = \int_0^{1/\alpha}\frac{\log^2(t)}{1-t}dt.$$

For $0 \leqslant t < \infty$ define the function $A(t)$ as the antiderivative

$$A(t) := \int\frac{\log^2(t)}{1-t}dt,$$

where the specific antiderivative is chosen so that $A(0) = 0$, and hence $A(1) = \int_0^{\infty}\frac{x^2}{e^x-1}dx = 2\zeta(3)$.

Substituting $1/t$ for t gives

$$A(1/t) = \int \frac{\log^2(t)}{t(1-t)} dt,$$

which can be expanded as partial fractions to get

$$A(1/t) = \int \frac{\log^2(t)}{1-t} dt + \int \frac{\log^2(t)}{t} dt,$$

which then evaluates to

$$A(1/t) = A(t) + \frac{1}{3} \log^3(t). \tag{B.3}$$

Note that $\lim_{t\to 0} A(t) = 0$, so $A(1/t) = o(1)$ for large t, and so for large t: $A(t) = -\frac{1}{3}\log^3(t) + o(1)$.

Substituting t^2 for t gives

$$A(t^2) = \int \frac{8t\log^2(t)}{(1-t^2)} dt,$$

which can be expanded as partial fractions to get

$$A(t^2) = \int \frac{4\log^2(t)}{1-t} dt - \int \frac{4\log^2(t)}{1+t} dt,$$

which then evaluates to

$$A(t^2) = 4A(t) - \int \frac{4\log^2(t)}{t+1}. \tag{B.4}$$

Substituting $1 + 1/t$ for t gives

$$A(1+1/t) = \int \frac{\log^2(1+1/t)}{t} dt,$$

and $\log(1 + 1/t) = \log(t + 1) - \log(t)$ so

$$A(1+1/t) = \int \frac{\log^2(t+1) + \log^2(t) - 2\log(t)\log(t+1)}{t} dt.$$

Expanding the sum inside the integral, the first two terms may be evaluated and the third term may be integrated by parts to get

$$A(1+1/t) = -A(t+1) + \frac{1}{3}\log^3(t) - \log^2(t)\log(t+1) + \int \frac{\log^2(t)}{t+1}dt.$$

Therefore

$$A(1+1/t) + A(t+1) = \frac{1}{3}\log^3(t) - \log^2(t)\log(t+1) + A(t) - \frac{1}{4}A(t^2) + C. \quad \text{(B.5)}$$

Taking t to infinity gives

$$2\zeta(3) - \frac{1}{3}\log^3(t) + o(1) = \frac{1}{3}\log^3(t) - \log^3(t) + o(1) - \frac{1}{3}\log^3(t) + \frac{2}{3}\log^3(t) + C,$$

and we conclude that $C = 2\zeta(3)$.

Now, formula (B.5) relates $A(t), A(t+1), A(t^2), A(1+1/t)$. To evaluate a single value, we need all but one distinct value to cancel out, which will happen if $t = 1$ or if $t+1 = t^2$ and $1 + 1/t = t$.

When $t = 1$, formula (B.5) gives

$$2A(2) = \frac{3}{4}A(1) + C = \frac{7}{2}\zeta(3).$$

Therefore $A(2) = \frac{7}{4}\zeta(3)$ and, from $(B.3)$, $A(1/2) = \frac{7}{4}\zeta(3) + \frac{1}{3}\log^3(2)$.

For $t+1$ to equal t^2, t must equal $\varphi := \frac{1+\sqrt{5}}{2}$, and this gives

$$A(\varphi) + A(\varphi^2) = \frac{1}{3}\log^3(\varphi) - \log^2(\varphi)\log(\varphi^2) + A(\varphi) - \frac{1}{4}A(\varphi^2) + 2\zeta(3),$$

which simplifies to

$$A(\varphi^2) = \frac{8}{5}\zeta(3) - \frac{1}{6}\log^3(\varphi^2).$$

Combining this with $(B.3)$ gives

$$A(\varphi^{-2}) = \frac{8}{5}\zeta(3) + \frac{1}{6}\log^3(\varphi^2),$$

as claimed. $\square$

This method does not yield any other triples (α, r, s) and a cursory brute-force search found no such linear relationships for other quadratic α where the coefficients of the quadratic are bounded by 100.

Bibliography

[1] Milton Abramowitz and Irene A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964.

[2] Manindra Agrawal, Neeraj Kayal, and Nitin Saxena. Primes is in p. *Annals of Mathematics*, 160(2):pp. 781–793, 2004.

[3] Roger Apéry. Irrationalité de $\zeta(2)$ et $\zeta(3)$. *Astérisque*, 61:11–13, 1979.

[4] Tom M. Apostol. A proof that Euler missed: evaluating $\zeta(2)$ the easy way. *Math. Intelligencer*, 5(3):59–60, 1983.

[5] Antal Balog, Andrew Granville, and Kannan Soundararajan. Multiplicative functions in arithmetic progressions. *Ann. Math. Qué.*, 37(1):3–30, 2013.

[6] Jennifer Beineke and Daniel Bump. A summation formula for divisor functions associated to lattices. *Forum Math.*, 18(6):869–906, 2006.

[7] Bruce C. Berndt and Marvin I. Knopp. *Hecke's theory of modular forms and Dirichlet series*, volume 5 of *Monographs in Number Theory*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008.

[8] Jonathan Bober, J. Brian Conrey, David W. Farmer, Akio Fujii, Sally Koutsoliotas, Stefan Lemurell, Michael Rubinstein, and Hiroyuki Yoshida. The highest lowest zero of general L-functions. *J. Number Theory*, 147:364–373, 2015.

[9] B. L. J. Braaksma. Asymptotic expansions and analytic continuations for a class of Barnes-integrals. *Compositio Math.*, 15:239–341 (1964), 1964.

[10] Ben Brubaker, Daniel Bump, and Solomon Friedberg. Weyl group multiple Dirichlet series, Eisenstein series and crystal bases. *Ann. of Math. (2)*, 173(2):1081–1120, 2011.

[11] Benjamin Brubaker, Daniel Bump, Solomon Friedberg, and Jeffrey Hoffstein. Coefficients of the n-fold theta function and Weyl group multiple Dirichlet series. In *Contributions in analytic and algebraic number theory*, volume 9 of *Springer Proc. Math.*, pages 83–95. Springer, New York, 2012.

[12] H. M. Bui, Brian Conrey, and Matthew P. Young. More than 41% of the zeros of the zeta function are on the critical line. *Acta Arith.*, 150(1):35–64, 2011.

[13] D. Bump, J. W. Cogdell, E. de Shalit, D. Gaitsgory, E. Kowalski, and S. S. Kudla. *An introduction to the Langlands program*. Birkhäuser Boston, Inc., Boston, MA, 2003. Lectures presented at the Hebrew University of Jerusalem, Jerusalem, March 12–16, 2001, Edited by Joseph Bernstein and Stephen Gelbart.

[14] Daniel Bump. *Automorphic forms on* $\mathrm{GL}(3, \mathbf{R})$, volume 1083 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1984.

[15] Daniel Bump. *Automorphic forms and representations*, volume 55 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997.

[16] Daniel Bump. Introduction: multiple Dirichlet series. In *Multiple Dirichlet series, L-functions and automorphic forms*, volume 300 of *Progr. Math.*, pages 1–36. Birkhäuser/Springer, New York, 2012.

[17] Daniel Bump, Solomon Friedberg, and David Ginzburg. Rankin-Selberg integrals in two complex variables. *Math. Ann.*, 313(4):731–761, 1999.

[18] Daniel Bump, Solomon Friedberg, and Jeffrey Hoffstein. On some applications of automorphic forms to number theory. *Bull. Amer. Math. Soc. (N.S.)*, 33(2):157–175, 1996.

[19] Daniel Bump, Masaaki Furusawa, and David Ginzburg. Non-unique models in the Rankin-Selberg method. *J. Reine Angew. Math.*, 468:77–111, 1995.

[20] Ettore Carletti, Giacomo Monti Bragadin, and Alberto Perelli. A note on Hecke's functional equation and the Selberg class. *Funct. Approx. Comment. Math.*, 41(part 2):211–220, 2009.

[21] Vorrapan Chandee and K. Soundararajan. Bounding $|\zeta(\frac{1}{2}+it)|$ on the Riemann hypothesis. *Bull. Lond. Math. Soc.*, 43(2):243–250, 2011.

[22] Henri Cohen. Trace des opérateurs de Hecke sur $\Gamma_0(N)$. In *Séminaire de Théorie des Nombres (1976–1977)*, pages Exp. No. 4, 9. Lab. Théorie des Nombres, CNRS, Talence, 1977.

[23] B. Conrey, A. Granville, B. Poonen, and K. Soundararajan. Zeros of Fekete polynomials. *Ann. Inst. Fourier (Grenoble)*, 50(3):865–889, 2000.

[24] B. Conrey and H. Iwaniec. Spacing of zeros of Hecke L-functions and the class number problem. *Acta Arith.*, 103(3):259–312, 2002.

[25] J. B. Conrey and D. W. Farmer. An extension of Hecke's converse theorem. *Internat. Math. Res. Notices*, (9):445–463, 1995.

[26] J. B. Conrey and A. Ghosh. On the Selberg class of Dirichlet series: small degrees. *Duke Math. J.*, 72(3):673–693, 1993.

[27] J. B. Conrey, A. Ghosh, D. Goldston, S. M. Gonek, and D. R. Heath-Brown. On the distribution of gaps between zeros of the zeta-function. *Quart. J. Math. Oxford Ser. (2)*, 36(141):43–51, 1985.

[28] J. B. Conrey, A. Ghosh, and S. M. Gonek. Large gaps between zeros of the zeta-function. *Mathematika*, 33(2):212–238 (1987), 1986.

[29] J. B. Conrey, H. Iwaniec, and K. Soundararajan. Small gaps between zeros of twisted L-functions. *Acta Arith.*, 155(4):353–371, 2012.

[30] J. B. Conrey and N. C. Snaith. Applications of the L-functions ratios conjectures. *Proc. Lond. Math. Soc. (3)*, 94(3):594–646, 2007.

[31] J. Brian Conrey. The Riemann hypothesis. *Notices Amer. Math. Soc.*, 50(3):341–353, 2003.

[32] J. Brian Conrey. Notes on L-functions and random matrix theory. In *Frontiers in number theory, physics, and geometry. I*, pages 107–162. Springer, Berlin, 2006.

[33] J. Brian Conrey and Amit Ghosh. Remarks on the generalized Lindelöf hypothesis. *Funct. Approx. Comment. Math.*, 36:71–78, 2006.

[34] J. Brian Conrey and Henryk Iwaniec. Small populations of zeros of L-functions. In *Number theory, analysis, and combinatorics*, De Gruyter Proc. Math., pages 11–23. De Gruyter, Berlin, 2014.

[35] Nikolaos Diamantis and Dorian Goldfeld. A converse theorem for double Dirichlet series and Shintani zeta functions. *J. Math. Soc. Japan*, 66(2):449–477, 2014.

[36] Dorian Goldfeld. *Automorphic forms and L-functions for the group* $\mathrm{GL}(n, \mathbf{R})$, volume 99 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2006. With an appendix by Kevin A. Broughan.

[37] Dorian Goldfeld. Rank lowering operators on $\mathrm{GL}(n, \mathbb{R})$. *Int. J. Number Theory*, 3(3):365–375, 2007.

[38] Dorian Goldfeld and Joseph Hundley. *Automorphic representations and L-functions for the general linear group. Volume I*, volume 129 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2011. With exercises and a preface by Xander Faber.

[39] Dorian Goldfeld and Joseph Hundley. *Automorphic representations and L-functions for the general linear group. Volume II*, volume 130 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2011. With exercises and a preface by Xander Faber.

[40] Dorian Goldfeld and Xiaoqing Li. The Voronoi formula for $\mathrm{GL}(n,\mathbb{R})$. *Int. Math. Res. Not. IMRN*, (2):Art. ID rnm144, 39, 2008.

[41] Dorian Goldfeld and Meera Thillainatesan. Rank lowering linear maps and multiple Dirichlet series associated to $\mathrm{GL}(n,\mathbb{R})$. *Pure Appl. Math. Q.*, 2(2, Special Issue: In honor of John H. Coates. Part 2):601–615, 2006.

[42] D. A. Goldston and S. M. Gonek. A note on $S(t)$ and the zeros of the Riemann zeta-function. *Bull. Lond. Math. Soc.*, 39(3):482–486, 2007.

[43] Hans Hamburger. Über die Riemannsche Funktionalgleichung der ξ-Funktion. *Math. Z.*, 13(1):283–311, 1922.

[44] G. H. Hardy and E. C. Titchmarsh. A Class of Fourier Kernels. *Proc. London Math. Soc.*, S2-35(1):116.

[45] Erich Hecke. *Lectures on Dirichlet series, modular functions and quadratic forms*. Vandenhoeck & Ruprecht, Göttingen, 1983. Edited by Bruno Schoeneberg, With the collaboration of Wilhelm Maak.

[46] Haruzo Hida. *Elementary theory of L-functions and Eisenstein series*, volume 26 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1993.

[47] Aleksandar Ivić. On some results concerning the Riemann hypothesis. In *Analytic number theory (Kyoto, 1996)*, volume 247 of *London Math. Soc. Lecture Note Ser.*, pages 139–167. Cambridge Univ. Press, Cambridge, 1997.

[48] Henryk Iwaniec and Emmanuel Kowalski. *Analytic number theory*, volume 53 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.

[49] Hervé Jacquet. Dirichlet series for the group GL(n). In *Automorphic forms, representation theory and arithmetic (Bombay, 1979)*, volume 10 of *Tata Inst. Fund. Res. Studies in Math.*, pages 155–163. Tata Inst. Fundamental Res., Bombay, 1981.

[50] Ĭ. Kacharovski and A. Perelli. Linear twists of L-functions of degree 2 from the Selberg class. *Mat. Zametki*, 88(3):399–404, 2010.

[51] J. Kaczorowski, G. Molteni, and A. Perelli. Linear independence in the Selberg class. *C. R. Math. Acad. Sci. Soc. R. Can.*, 21(1):28–32, 1999.

[52] J. Kaczorowski and A. Perelli. On the structure of the Selberg class. VI. Non-linear twists. *Acta Arith.*, 116(4):315–341, 2005.

[53] J. Kaczorowski and A. Perelli. Lower bounds for the conductor of L-functions. *Acta Arith.*, 155(2):185–199, 2012.

[54] Jerzy Kaczorowski and Alberto Perelli. On the structure of the Selberg class. I. $0 \leq d \leq 1$. *Acta Math.*, 182(2):207–241, 1999.

[55] Jerzy Kaczorowski and Alberto Perelli. A note on the degree conjecture for the Selberg class. *Rend. Circ. Mat. Palermo (2)*, 57(3):443–448, 2008.

[56] Jerzy Kaczorowski and Alberto Perelli. A correspondence theorem for L-functions and partial differential operators. *Publ. Math. Debrecen*, 79(3-4):497–505, 2011.

[57] Jerzy Kaczorowski and Alberto Perelli. On the structure of the Selberg class, VII: $1 < d < 2$. *Ann. of Math. (2)*, 173(3):1397–1441, 2011.

[58] Haseo Ki. On the zeros of Weng's zeta functions. *Int. Math. Res. Not. IMRN*, (13):2367–2393, 2010.

[59] Marvin I. Knopp and Mark Sheingorn. On Dirichlet series and Hecke triangle groups of infinite volume. *Acta Arith.*, 76(3):227–244, 1996.

[60] L. Kramaschke. Nullstellen der Zetafunktion. *Deutsche Math*, 2:107–110, 1937.

[61] V. N. Kuznetsov. An analogue of Szegő's theorem for a class of Dirichlet series. *Mat. Zametki*, 36(6):805–813, 941, 1984.

[62] Jeffrey C. Lagarias and Masatoshi Suzuki. The Riemann hypothesis for certain integrals of Eisenstein series. *J. Number Theory*, 118(1):98–122, 2006.

[63] Serge Lang. *Introduction to modular forms*, volume 222 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1995. With appendixes by D. Zagier and Walter Feit, Corrected reprint of the 1976 original.

[64] Xiannan Li. Upper bounds on L-functions at the edge of the critical strip. *Int. Math. Res. Not. IMRN*, (4):727–755, 2010.

[65] J. E. Littlewood. On the Zeros of the Riemann Zeta-function. *Mathematical Proceedings of the Cambridge Philosophical Society*, 22:295–318, 1924.

[66] J. E. Littlewood. Two notes on the Riemann Zeta-function. *Mathematical Proceedings of the Cambridge Philosophical Society*, 22:234–242, 9 1924.

[67] Lutz Lucht. Power series with multiplicative coefficients. *Math. Z.*, 177(3):359–374, 1981.

[68] Hans Maass. Über eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen. *Math. Ann.*, 121:141–183, 1949.

[69] Stephen D. Miller and Wilfried Schmid. Summation formulas, from Poisson and Voronoi to the present. In *Noncommutative harmonic analysis*, volume 220 of *Progr. Math.*, pages 419–440. Birkhäuser Boston, Boston, MA, 2004.

[70] Stephen D. Miller and Wilfried Schmid. The Rankin-Selberg method for automorphic distributions. In *Representation theory and automorphic forms*, volume 255 of *Progr. Math.*, pages 111–150. Birkhäuser Boston, Boston, MA, 2008.

[71] Stephen D. Miller and Wilfried Schmid. Adelization of automorphic distributions and mirabolic Eisenstein series. In *Representation theory and mathematical physics*, volume 557 of *Contemp. Math.*, pages 289–334. Amer. Math. Soc., Providence, RI, 2011.

[72] Stephen D. Miller and Wilfried Schmid. A general Voronoi summation formula for $GL(n,\mathbb{Z})$. In *Geometry and analysis. No. 2*, volume 18 of *Adv. Lect. Math. (ALM)*, pages 173–224. Int. Press, Somerville, MA, 2011.

[73] Giuseppe Molteni. Upper and lower bounds at $s = 1$ for certain Dirichlet series with Euler product. *Duke Math. J.*, 111(1):133–158, 2002.

[74] Hugh L. Montgomery. Extreme values of the Riemann zeta function. *Comment. Math. Helv.*, 52(4):511–518, 1977.

[75] M. Ram Murty. Selberg's conjectures and Artin L-functions. *Bull. Amer. Math. Soc. (N.S.)*, 31(1):1–14, 1994.

[76] Roop Narain. The G-functions as unsymmetrical Fourier kernels. I. *Proc. Amer. Math. Soc.*, 13:950–959, 1962.

[77] A. M. Odlyzko. Lower bounds for discriminants of number fields. *Acta Arith.*, 29(3):275–297, 1976.

[78] A. M. Odlyzko. Lower bounds for discriminants of number fields. II. *Tôhoku Math. J.*, 29(2):209–216, 1977.

[79] A. M. Odlyzko. On the distribution of spacings between zeros of the zeta function. *Math. Comp.*, 48(177):273–308, 1987.

[80] A. M. Odlyzko. Bounds for discriminants and related estimates for class numbers, regulators and zeros of zeta functions: a survey of recent results. *Sém. Théor. Nombres Bordeaux (2)*, 2(1):119–141, 1990.

[81] A.M. Odlyzko. Zeros of the Riemann zeta function: Conjectures and computations. 2002.

[82] Joseph Oesterlé. *Sur la trace des opérateurs de Hecke*. PhD thesis, 1977.

[83] A. Perelli. Non-linear twists of L-functions: a survey. *Milan J. Math.*, 78(1):117–134, 2010.

[84] B. Riemann. Ueber die Anzahl der Primzahlen unter einer gegebenen Grosse. 1859.

[85] René Schoof and Marcel van der Vlugt. Hecke operators and the weight distributions of certain codes. *J. Combin. Theory Ser. A*, 57(2):163–186, 1991.

[86] Atle Selberg. Contributions to the theory of the Riemann zeta-function. *Arch. Math. Naturvid.*, 48(5):89–155, 1946.

[87] Atle Selberg. Old and new conjectures and results about a class of Dirichlet series. In *Proceedings of the Amalfi Conference on Analytic Number Theory (Maiori, 1989)*, pages 367–385. Univ. Salerno, Salerno, 1992.

[88] K. Soundararajan. On the distribution of gaps between zeros of the Riemann zeta-function. *Quart. J. Math. Oxford Ser. (2)*, 47(187):383–387, 1996.

[89] K. Soundararajan. Strong multiplicity one for the Selberg class. *Canad. Math. Bull.*, 47(3):468–474, 2004.

[90] K. Soundararajan. Degree 1 elements of the Selberg class. *Expo. Math.*, 23(1):65–70, 2005.

[91] K. Soundararajan. Extreme values of zeta and L-functions. *Math. Ann.*, 342(2):467–486, 2008.

[92] Kannan Soundararajan. Moments of the Riemann zeta function. *Ann. of Math. (2)*, 170(2):981–993, 2009.

[93] J. T. Tate. Fourier analysis in number fields, and Hecke's zeta-functions. In *Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965)*, pages 305–347. Thompson, Washington, D.C., 1967.

[94] E. C. Titchmarsh. On the Riemann zeta-function. *Mathematical Proceedings of the Cambridge Philosophical Society*, 28:273–274, 1932.

[95] E. C. Titchmarsh. *The theory of the Riemann zeta-function*. The Clarendon Press, Oxford University Press, New York, second edition, 1986. Edited and with a preface by D. R. Heath-Brown.

[96] N. Ja. Vilenkin and A. U. Klimyk. *Representation of Lie groups and special functions. Vol. 1*, volume 72 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1991. Simplest Lie groups, special functions and integral transforms, Translated from the Russian by V. A. Groza and A. A. Groza.

[97] N. Ja. Vilenkin and A. U. Klimyk. *Representation of Lie groups and special functions. Vol. 3*, volume 75 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1992. Classical and quantum groups and special functions, Translated from the Russian by V. A. Groza and A. A. Groza.

[98] N. Ja. Vilenkin and A. U. Klimyk. *Representation of Lie groups and special functions. Vol. 2*, volume 74 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1993. Class I representations, special functions, and integral transforms, Translated from the Russian by V. A. Groza and A. A. Groza.

[99] André Weil. Two lectures on number theory, past and present. *Enseignement Math. (2)*, 20:87–110, 1974.

[100] Herbert S. Wilf. *Generatingfunctionology*. A. K. Peters, Ltd., Natick, MA, USA, 2006.

[101] Edward Witten. On quantum gauge theories in two dimensions. *Comm. Math. Phys.*, 141(1):153–209, 1991.

[102] D. Zagier. Correction to: “The Eichler-Selberg trace formula on $\mathrm{SL}_2(\mathbf{Z})$” (*introduction to modular forms*, Appendix, pp. 44–54, Springer, Berlin, 1976) by S. Lang. In *Modular functions of one variable, VI (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976)*, pages 171–173. Lecture Notes in Math., Vol. 627. Springer, Berlin, 1977.

[103] Don Zagier. Values of zeta functions and their applications. In *First European Congress of Mathematics, Vol. II (Paris, 1992)*, volume 120 of *Progr. Math.*, pages 497–512. Birkhäuser, Basel, 1994.