

staying away from $e^{i\vartheta}$. Continuity of $d\omega_\Omega(e^{i\vartheta}, z_0)/d\vartheta$ can then be read off from the formula since γ has no accumulation points inside the I_k .

The function $g(e^{i\vartheta})$ is thus continuous, in addition to enjoying property (i) of our list. Verification of properties (ii) and (v) thereof remains.

Because $d\omega_\Omega(e^{i\vartheta}, z_0)/d\vartheta \leq C$ and $|\Phi(e^{i\vartheta})|$ lies between two constant multiples of $|F(e^{i\vartheta})|$, property (ii) holds on account of the analogous condition satisfied by F and the relation of $g(e^{i\vartheta})$ to $\Phi(e^{i\vartheta})$. Passing to property (v), we note that an earlier relation can be rewritten

$$\left| \int_{-\pi}^{\pi} e^{in\vartheta} g(e^{i\vartheta}) d\vartheta \right| \leq \text{const.} (e^{-n\xi_0} + e^{-M(n)}), \quad n \geq 0.$$

By concavity of $M(v)$, $M(v)/v$ eventually decreases and tends to a limit $l \geq 0$ as $v \rightarrow \infty$. Were $l > 0$, the right side of the inequality just written would be $\leq \text{const.} e^{-nl_0}$ with $l_0 = \min(\xi_0, l) > 0$. Such a bound on the left-hand integral would, with property (ii), force $g(e^{i\vartheta})$ to vanish identically – see the bottom of p. 328. Our $g(e^{i\vartheta})$, however, does not do that, so we must have $l = 0$, making $M(n) < n\xi_0$ for large n . The right side of our inequality can therefore be replaced by $\text{const.} e^{-M(n)}$, and property (v) holds. The construction is now complete.

It is to be noted that the only objects we actually used were the function $h(\xi)$ with its specified properties and $\Phi(z)$, analytic in a certain domain $\mathcal{O} \subseteq \{|z| < 1\}$ and continuous up to $\partial\mathcal{O}$, satisfying $|\Phi(\xi)| \leq \text{const.} \exp\left(-h\left(\log \frac{1}{|\xi|}\right)\right)$ on $\partial\mathcal{O} \cap \{|\xi| < 1\}$ and $|\Phi(\xi)| > 0$ on some arc of $\{|\xi| = 1\}$ included in $\partial\mathcal{O}$. I have a persistent nagging feeling that such functions $h(\xi)$ and $\Phi(z)$, if there really are any, must be lying around somewhere or at least be closely related to others whose constructions are already available. One thinks of various kinds of functions meromorphic in the unit disk but not of bounded characteristic there; especially do the ones described by Beurling at the eighth Scandinavian mathematicians' congress come back continually to mind.

This addendum, however, is already being written at the very last moment. The imminence of press time leaves me no opportunity for pursuing the matter.

3. Extension to functions $F(e^{i\vartheta})$ in $L_1(-\pi, \pi)$.

The theorem of p. 356 holds for L_1 functions $F(e^{i\vartheta})$ not a.e. zero, as does Brennan's refinement of it given in article 1 above. A procedure for handling this more general situation (absence of continuity) is worked out in the beautiful *Mat. Sbornik* paper by Jöricke and Volberg. Here we

adapt their method so as to make it go with the development already familiar from §D.6, Chapter VII, hewing as closely as possible to the latter.

Our aim is to show that

$$\int_{-\pi}^{\pi} \log |F(e^{i\vartheta})| d\vartheta > -\infty$$

for any function $F(e^{i\vartheta}) \in L_1(-\pi, \pi)$ not a.e. zero and satisfying the hypothesis of Brennan's theorem. Let us begin by observing that the treatment of this case can be reduced to that of a *bounded* function F .

Suppose, indeed, that

$$F(e^{i\vartheta}) \sim \sum_{-\infty}^{\infty} a_n e^{in\vartheta}$$

belongs to L_1 , with $|a_n| \leq \text{const. } e^{-M(|n|)}$ for $n < 0$. The series $\sum_{n < 0} a_n e^{in\vartheta}$ is then surely *absolutely convergent*, so

$$\sum_0^{\infty} a_n e^{in\vartheta}$$

is also the Fourier series of an L_1 function, which we denote by $F_+(e^{i\vartheta})$ (this belongs in fact to the space H_1). For $|z| < 1$, put

$$F_+(z) = \sum_0^{\infty} a_n z^n;$$

for this function, analytic in $\{|z| < 1\}$, we have (Chapter II, §B!),

$$F_+(z) \longrightarrow F_+(e^{i\vartheta}) \quad \text{a.e. as } z \nearrow e^{i\vartheta}.$$

Using the integrable function $\log^+ |F_+(e^{i\vartheta})| \geq 0$, we now form

$$b(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\vartheta} + z}{e^{i\vartheta} - z} \log^+ |F_+(e^{i\vartheta})| d\vartheta,$$

analytic and with positive real part for $|z| < 1$. According to the third theorem and scholium of §F.2, Chapter III, $b(z)$ tends for almost every ϑ to a limit $b(e^{i\vartheta})$ as $z \nearrow e^{i\vartheta}$, with

$$\Re b(e^{i\vartheta}) = \log^+ |F_+(e^{i\vartheta})| \quad \text{a.e.}$$

A standard extension of Jensen's inequality to H_1 also tells us that

$$\log |F_+(z)| \leq \Re b(z), \quad |z| < 1$$

(cf. pp. 291–2 where this was proved and used for $z = 0$).

We next perform the Dynkin extension (described on pp. 339–40) on the continuous function

$$F_-(e^{i\vartheta}) = \sum_{-\infty}^{-1} a_n e^{in\vartheta}.$$

This gives us $F_-(z)$, \mathcal{C}_∞ in the unit disk and continuous (hence *bounded!*) up to its boundary, with

$$\left| \frac{\partial F_-(z)}{\partial \bar{z}} \right| \leq \text{const.} \exp \left(-h \left(\log \frac{1}{|z|} \right) \right), \quad |z| < 1,$$

where, in the present circumstances,

$$h(\xi) = \sup_{v>0} (M(v)/2 - v\xi)$$

(see *remark 2*, p. 343). As usual, we write

$$w(r) = \exp \left(-h \left(\log \frac{1}{r} \right) \right);$$

then, putting

$$F(z) = F_-(z) + F_+(z)$$

for $|z| < 1$, we have

$$\left| \frac{\partial F(z)}{\partial \bar{z}} \right| \leq \text{const.} w(|z|)$$

there, and

$$F(z) \longrightarrow F(e^{i\theta}) \quad \text{a.e. for } z \not\rightarrow e^{i\theta}.$$

The bounded function spoken of earlier is simply

$$F_0(z) = e^{-b(z)} F(z).$$

It is bounded in the unit disk by one of the previous relations; another tells us that $F_0(z)$ has a non-tangential boundary value $F_0(e^{i\theta}) = F(e^{i\theta}) \exp(-b(e^{i\theta}))$ equal in modulus to $|F(e^{i\theta})|/\max(|F_+(e^{i\theta})|, 1)$ at almost every point of the unit circumference. Then, since $F(e^{i\theta}) \in L_1$ is not a.e. zero, neither is $F_0(e^{i\theta})$. We note finally that by *analyticity* of $e^{-b(z)}$, $\partial F_0(z)/\partial \bar{z} = e^{-b(z)} \partial F(z)/\partial \bar{z}$, making

$$\left| \frac{\partial F_0(z)}{\partial \bar{z}} \right| \leq \text{const.} w(|z|), \quad |z| < 1.$$

Given that $M(v)$ satisfies the hypothesis of Brennan's theorem, our function $h(\xi)$ enjoys the two properties used in the first part of article 1, namely, that $\xi h(\xi)$ *decreases* and that $\int_0^a \log h(\xi) d\xi = \infty$ for small $a > 0$. If, now, we can *deduce* from these together with the *preceding relation* that

the bounded function $F_0(z)$, not a.e. zero for $|z| = 1$, satisfies

$$\int_{-\pi}^{\pi} \log |F_0(e^{i\vartheta})| d\vartheta > -\infty,$$

we will certainly have *the same conclusion* for

$$\log |F(e^{i\vartheta})| = \log |F_0(e^{i\vartheta})| + \log^+ |F_+(e^{i\vartheta})|.$$

The rest of our work deals exclusively with $F_0(z)$.

In order to stay as close as possible to the notation of §D.6, Chapter VII, we denote the bounded function $F_0(z)$ by $F(z)$ from now on. Using this new $F(z)$, we first form the sets $B \subseteq \{|z| \leq 1\}$ and $\mathcal{O} \subseteq \{|z| < 1\}$ as on pp. 359–60, and then the function $\Phi(z)$ introduced on p. 360. The latter, analytic in \mathcal{O} , is actually defined on the whole unit disk, and has there at least as much continuity as $F(z)$ besides lying in modulus between two constant multiples of $|F(z)|$. It has, in particular, a non-tangential boundary value $\Phi(e^{i\vartheta})$ a.e. on the unit circumference, and this *does not vanish* a.e. The construction of B ensures that

$$|\Phi(\zeta)| \leq \text{const. } w(|\zeta|) \quad \text{on } \partial\mathcal{O} \cap \{|\zeta| < 1\}$$

(indeed, on B), and our task amounts to showing that

$$\int_{-\pi}^{\pi} \log |\Phi(e^{i\vartheta})| d\vartheta > -\infty$$

on account of these properties.

What makes the present situation more complicated than the one studied in §D.6 of Chapter VII is that $\Phi(z)$ need no longer be continuous up to the whole unit circumference. This causes the notion of *abutment* introduced on p. 348 to be less useful here for the examination of our set \mathcal{O} than it was in §D.6, and we have to supplement it with another, that of *fatness*. The latter, based on the famous sawtooth construction of Lusin and Privalov, helps us to take account of $\Phi(z)$'s non-tangential boundary behaviour.

To describe what is meant by fatness, we need to bring in a special kind of domain together with some notation; both will also be used further on. Corresponding to each point $e^{i\alpha}$ on the unit circumference, we have an open set S_α consisting of the z with $1/2 < |z| < 1$ lying in the open 60° sector having vertex at $e^{i\alpha}$ and symmetric about the radius from 0 out to that point. Given any subset E of $\{|\zeta| = 1\}$ we then write

$$S_E = \bigcup_{e^{i\alpha} \in E} S_\alpha.$$

It is evident that if we take any S_E and a ρ , $1/2 < \rho < 1$, the intersection

$$S_E \cap \{\rho < |z| < 1\}$$

breaks up into (at most) a countable number of open *connected* components, each of the form

$$S_{E_k} \cap \{\rho < |z| < 1\},$$

with the E_k making up a (disjoint) partition of the set E .

Definition. A *connected* open set of the form

$$S_E \cap \{\rho < |z| < 1\}$$

(with $1/2 < \rho < 1$) is called a *sawblade of depth* $1 - \rho$. We say that such a sawblade *bites on* the set E .

Now we can state the

Definition. An open subset \mathcal{U} of the unit disk is called *fat* if it contains a sawblade biting on a closed $E \subseteq \{|\zeta| = 1\}$ with $|E| > 0$. In that circumstance we also say that \mathcal{U} is *fat at* E .

Equipped with these tools, we endeavour to investigate the set \mathcal{O} according to the procedure of §D.6, Chapter VII. In this, some modifications are necessary; we have, in the first place, to *skip over step 1* (p. 361). Then, taking ρ , $1/2 < \rho < 1$, we construct a set $\Omega(\rho)$, proceeding differently, however, than as we did on pp. 361–3.

There is, by the properties of $\Phi(z)$, a closed subset E_0 of the unit circumference, $|E_0| > 0$, such that, for the *non-tangential* boundary values $\Phi(\zeta)$, we have, wlog,

$$|\Phi(\zeta)| > 1, \quad \zeta \in E_0.$$

Egorov's theorem enables us to in fact pick E_0 so as to have $|\Phi(z)| > 1$ for $z \in S_{E_0}$ with $\rho' < |z| < 1$ when $\rho' > \rho$ is sufficiently close to 1. But the construction of B and \mathcal{O} makes $|\Phi(z)| \leq \text{const. } w(|z|)$ on B , hence on $\{|z| < 1\} \sim \mathcal{O}$. Therefore, since $w(r) \rightarrow 0$ for $r \rightarrow 1$, we must have

$$S_{E_0} \cap \{\rho' < |z| < 1\} \subseteq \mathcal{O}$$

if ρ' , $\rho < \rho' < 1$, is near enough to 1. One of the components of the intersection on the left is a sawblade of depth $1 - \rho'$ biting on a (Borel) subset E' of E_0 with $|E'| > 0$; a suitable *closed* subset E of E' then has

$|E| > 0$, and there is a sawblade of depth $1 - \rho'$ biting on E and contained in \mathcal{O} . We now take $\Omega(\rho)$ as the component of $\mathcal{O} \cap \{\rho < |z| < 1\}$ including that sawblade; $\Omega(\rho)$ is fat at E .

For the present set $\Omega(\rho)$ there is a substitute for step 2 of p. 362:

Step 2'. $\partial\Omega(\rho)$ includes the whole unit circumference.

This we establish by *reductio ad absurdum*. Let us write Ω for $\Omega(\rho)$, and put

$$\gamma = \partial\Omega \cap \{|z| < 1\},$$

$$\Gamma = \partial\Omega \sim \gamma;$$

Γ is thus the part of $\partial\Omega$ lying on the unit circumference. Assume that there is on the latter a non-empty open arc J with $J \cap \Gamma = \emptyset$; we will then deduce a contradiction.

For that it is quicker to fall back on the device used in the second half of article 2 than to adapt Volberg's theorem on harmonic measures (p. 349) to the present situation. Fixing $z_0 \in \Omega$, we can say that

$$z_0^n \Phi(z_0) = \int_{\partial\Omega} \zeta^n \Phi(\zeta) d\omega_\Omega(\zeta, z_0) \quad \text{for } n \geq 0,$$

whence

$$\begin{aligned} \int_{\Gamma} e^{in\vartheta} \Phi(e^{i\vartheta}) d\omega_\Omega(e^{i\vartheta}, z_0) &= z_0^n \Phi(z_0) \\ &\quad - \int_{\gamma} \zeta^n \Phi(\zeta) d\omega_\Omega(\zeta, z_0), \quad n \geq 0. \end{aligned}$$

Here we are using Poisson's formula for the bounded function $\zeta^n \Phi(\zeta)$ harmonic (even analytic) in Ω and continuous up to γ , but not necessarily up to Γ , where it is only known to have non-tangential boundary values a.e. Such use is legitimate; we postpone verification of that, and of a corresponding version of Jensen's inequality, to the next article, so as not to interrupt the argument now under way.

As in article 2, $d\omega_\Omega(e^{i\vartheta}, z_0)$ is absolutely continuous and $\leq C d\vartheta$ on Γ , and we obtain a bounded measurable function $g(e^{i\vartheta})$ by putting

$$g(e^{i\vartheta}) = \Phi(e^{i\vartheta}) \frac{d\omega_\Omega(e^{i\vartheta}, z_0)}{d\vartheta} \quad \text{for } e^{i\vartheta} \in \Gamma$$

and (here!) taking $g(e^{i\vartheta})$ to be zero outside Γ . From the preceding relation

we then see, as in article 2, that

$$\left| \int_{-\pi}^{\pi} e^{in\vartheta} g(e^{i\vartheta}) d\vartheta \right| \leq \text{const.} (e^{-n\xi_0} + e^{-M_1(n)})$$

for $n \geq 0$, where $\xi_0 > 0$ and

$$M_1(v) = \inf_{\xi > 0} (h(\xi) + \xi v).$$

This function is increasing and concave, so the right side of the last inequality can be replaced by $\text{const.} e^{-M_2(n)}$ for large n , with $M_2(n)$ equal either to $\xi_0 n$ (in case $\lim_{v \rightarrow \infty} (M_1(v)/v) \geq \xi_0$) or else to $M_1(n)$. In either event, $M_2(n)$ increases and $\sum_1^\infty M_2(n)/n^2 = \infty$ on account of the properties of $h(\xi)$. (See the theorem of p. 337 – $M_1(n)$ is actually equal to $M(n)/2$ in the present set-up.) Now we can apply Levinson's theorem, since $g(e^{i\vartheta})$ vanishes on the arc J . The conclusion is that $g(e^{i\vartheta}) \equiv 0$ a.e.

But $g(e^{i\vartheta})$ does not vanish a.e. Indeed, Ω contains a sawblade \mathcal{E} biting on a closed set E , $|E| > 0$, where $|\Phi(e^{i\vartheta})| \geq 1$. Thence,

$$\int_E |g(e^{i\vartheta})| d\vartheta = \int_E |\Phi(e^{i\vartheta})| d\omega_\Omega(e^{i\vartheta}, z_0) \geq \omega_\Omega(E, z_0).$$

Harnack's theorem assures us that the quantity on the right is > 0 if, for some $z_1 \in \mathcal{E}$, $\omega_\Omega(E, z_1) > 0$. However, by the principle of extension of domain, $\omega_\Omega(E, z_1) \geq \omega_{\mathcal{E}}(E, z_1)$. At the same time, $\partial\mathcal{E}$ is *rectifiable*, so a conformal mapping of \mathcal{E} onto the unit disk must take the subset E of $\partial\mathcal{E}$, having linear measure > 0 , to a set of measure > 0 on the unit circumference. (This follows by the celebrated F. and M. Riesz theorem; a proof can be found in Zygmund or in any of the books about H_p spaces.) We therefore have $\omega_{\mathcal{E}}(E, z_1) > 0$, making $\omega_\Omega(E, z_0) > 0$ and hence, as we have seen, $\int_E |g(e^{i\vartheta})| d\vartheta > 0$.

Our contradiction is thus established. By it we see that the arc J cannot exist, i.e., that Γ is the whole unit circumference, as was to be shown.

With step 2' accomplished, we are ready for step 3. One starts out as on p. 363, using the square root mapping employed there. That gives us a domain $\Omega_\sqrt{}$, certainly *fat* at a closed subset E'' , of $E_\sqrt{}$ (the image of E under our mapping), with $|E''| > 0$ (recall the earlier use of Egorov's theorem). Thereafter, one applies to $\Omega_\sqrt{}$ the argument just made for Ω in doing step 2'.

The weight $w_1(r)$ is next introduced as on p. 365, and the sets B_1 and \mathcal{O}_1 constructed (pp. 365–6). After doing steps 2' and 3 again with these objects, we come to step 4.

Jöricke and Volberg are in fact able to circumvent this step, thanks to a clever rearrangement of *step 5*. Here, however, let us continue according to the plan of §D.6, Chapter VII, for the work done there carries over practically without change to the present situation.

What is important for *step 4* is that a ζ , $|\zeta| = 1$, *not* in B must, even here, lie on an arc of the unit circumference *abutting* on \mathcal{O} . Such a $\zeta \notin B$ must thus, as on p. 367, have a neighborhood V_ζ with

$$V_\zeta \cap \{|z| < 1\} \subseteq \mathcal{O} \cap \{\rho^2 < |z| < 1\}.$$

The left-hand intersection therefore lies in some *connected component* of the one on the right, which, however, *can only be* $\Omega(\rho^2)$, since $\zeta \in \partial\Omega(\rho^2)$ by *step 2'*. The rest of the argument goes as on pp. 367–8.

Now we can do *step 5*, or rather the *substitute* for it carried out at the beginning of article 1. For this it is necessary to have the Jensen inequality

$$\log |\Phi(\rho)| \leq \int_{\partial\Omega(\rho^2)} \log |\Phi(\zeta)| d\omega(\zeta, \rho)$$

(notation of p. 369) available in the present circumstances, where continuity of $\Phi(z)$ up to $\{|\zeta| = 1\}$ may fail. The legitimacy of this will be established in the next article; *granting* it for now, we may proceed exactly as at the beginning of article 1.

From here on, one continues as on pp. 370–2, and reaches the desired conclusion that $\int_{-\pi}^{\pi} \log |\Phi(e^{i\vartheta})| d\vartheta > -\infty$ as on p. 373, after one more application of our extended Jensen inequality.

We thus arrive at the

Theorem. Let $F(e^{i\vartheta}) \in L_1(-\pi, \pi)$ not be zero a.e., and suppose that

$$F(e^{i\vartheta}) \sim \sum_{-\infty}^{\infty} a_n e^{in\vartheta}$$

with

$$|a_n| \leq \text{const. } e^{-M(|n|)}, \quad n \leq 0.$$

Suppose that $M(v)$ is *concave*, that $M(v)/v^{1/2}$ is *increasing* for large v , and that

$$\sum_1^{\infty} M(n)/n^2 = \infty.$$

Then

$$\int_{-\pi}^{\pi} \log |F(e^{i\vartheta})| d\vartheta > -\infty.$$

Remark. In their preprint, Borichev and Volberg consider formal trigonometric series

$$\sum_{-\infty}^{\infty} a_n e^{in\vartheta}$$

in which the a_n with *negative* n satisfy the requirement of the theorem, but the a_n with $n > 0$ are allowed to grow like $e^{M(n)}$ as $n \rightarrow \infty$. Assuming *more* regularity for $M(v)$ ($M(v) \geq \text{const. } v^\alpha$ with an $\alpha < 1$ *close to 1* is enough), they are able to show that under the remaining conditions of the theorem, all the a_n must vanish if

$$\liminf_{r \rightarrow 1} \int_{-\pi}^{\pi} \log \left| \sum_{-\infty}^0 a_n e^{in\vartheta} + \sum_1^{\infty} a_n r^n e^{in\vartheta} \right| d\vartheta = -\infty.$$

Before ending this article let us, as promised in the *last* one, see how the example of Borichev and Volberg shows that the *monotoneity requirement* on $M(v)/v^{1/2}$ *cannot, in the above theorem at least, be relaxed* to $M(v)/v^{1/2} \geq C > 0$, *even though continuity* up to $\{|\zeta| = 1\}$ *should fail* for the function $F(z)$ supplied by their construction.

The reader should refer back to the second part of article 2. Corresponding to the bounded function $F(z)$ used there, *no longer assumed continuous* up to $\{|\zeta| = 1\}$ but having at least non-tangential boundary values a.e. on that circumference, one can, as in the preceding discussion, form the sets B , \mathcal{O} and $\Omega(\rho)$ and do *step 2'*. One may then form the function $g(e^{i\vartheta})$ as in article 2; *here* it is bounded and measurable at least. The work of *step 2'* shows that $g(e^{i\vartheta})$ is *not* a.e. zero, while properties (ii)–(v) of article 2 *hold* for it (for the last one, see again the end of that article).

This is all we need.

4. Lemma about harmonic functions

Suppose we have a domain Ω regular for Dirichlet's problem, lying in the (open) unit disk Δ and having part of $\partial\Omega$ on the unit circumference. As in the last article, we write

$$\Gamma = \partial\Omega \cap \partial\Delta \quad \text{and} \quad \gamma = \partial\Omega \cap \Delta.$$

For the following discussion, let us agree to call ζ , $|\zeta| = 1$, a *radial accumulation point of Ω* if, for a sequence $\{r_n\}$ tending to 1, we have $r_n \zeta \in \Omega$ for each n . We then denote by Γ' the set of such radial accumulation points, noting that $\Gamma' \subseteq \Gamma$ with the inclusion frequently *proper*.

Lemma. (Jöricke and Volberg) Let $V(z)$, harmonic and bounded in Ω , be continuous up to γ , and suppose that

$$\lim_{\substack{r \rightarrow 1 \\ r\zeta \in \Omega}} V(\zeta)$$

exists for almost all $\zeta \in \Gamma'$. Put $v(\zeta)$ equal to that limit for such ζ , and to zero for the remaining $\zeta \in \Gamma$. On γ , take $v(\zeta)$ equal to $V(\zeta)$. Then, for $z \in \Omega$,

$$V(z) = \int_{\partial\Omega} v(\zeta) d\omega_{\Omega}(\zeta, z).$$

Proof. It suffices to establish the result for *real* harmonic functions $V(z)$, and, for those, to show that

$$V(z) \leq \int_{\partial\Omega} v(\zeta) d\omega_{\Omega}(\zeta, z), \quad z \in \Omega,$$

since the reverse inequality then follows on changing the signs of V and v .

By modifying $v(\zeta)$ on a subset of Γ having zero Lebesgue measure, we get a bounded Borel function defined on $\partial\Omega$. But on Γ , we have $d\omega_{\Omega}(\zeta, z) \leq C_z |d\zeta|$ (see articles 2 and 3), so such modification cannot alter the value of $\int_{\partial\Omega} v(\zeta) d\omega_{\Omega}(\zeta, z)$. We may hence just as well take $v(\zeta)$ as a bounded Borel function (on $\partial\Omega$) to begin with.

That granted, we desire to show that the integral just written is $\geq V(z)$. For this it seems necessary to hark back to the very foundations of integration theory. Call the limit of any increasing sequence of functions continuous on $\partial\Omega$ an upper function (on $\partial\Omega$). There is then a decreasing sequence of upper functions $w_n(\zeta) \geq v(\zeta)$ such that

$$\int_{\partial\Omega} w_n(\zeta) d\omega_{\Omega}(\zeta, z) \xrightarrow{n} \int_{\partial\Omega} v(\zeta) d\omega_{\Omega}(\zeta, z), \quad z \in \Omega.$$

Indeed, corresponding to any given $z \in \Omega$, such a sequence is furnished by a basic construction of the Lebesgue–Stieltjes integral, $\omega_{\Omega}(\cdot, z)$ being a Radon measure on $\partial\Omega$. But then that sequence works also for any other $z \in \Omega$, since $d\omega_{\Omega}(\zeta, z') \leq C(z, z') d\omega_{\Omega}(\zeta, z)$ (Harnack).

Our inequality involving v and V will thus be established, provided that we can verify

$$V(z) \leq \int_{\partial\Omega} w_n(\zeta) d\omega_{\Omega}(\zeta, z), \quad z \in \Omega,$$

for each n . *Fixing*, then, any n , we write simply $w(\zeta)$ for $w_n(\zeta)$ and put

$$W(z) = \int_{\partial\Omega} w(\zeta) d\omega_{\Omega}(\zeta, z)$$

for $z \in \Omega$, making $W(z)$ *harmonic* there. Our task is to prove that

$$V(z) \leq W(z), \quad z \in \Omega.$$

It is convenient to define $W(z)$ on *all* of $\bar{\Omega}$ by putting

$$W(\zeta) = w(\zeta), \quad \zeta \in \partial\Omega.$$

At each $\zeta \in \partial\Omega$ we then have

$$\liminf_{\substack{z \rightarrow \zeta \\ z \in \bar{\Omega}}} W(z) \geq W(\zeta)$$

by the elementary approximate identity property of harmonic measure, since $w(\zeta)$, as limit of an *increasing* sequence of continuous functions, satisfies

$$\liminf_{\substack{\zeta \rightarrow \zeta_0 \\ \zeta \in \partial\Omega}} w(\zeta) \geq w(\zeta_0) \quad \text{for } \zeta_0 \in \partial\Omega.$$

The function $W(z)$ enjoys a certain *reproducing property* in $\bar{\Omega}$. Namely, if the domain $\mathcal{D} \subseteq \Omega$ is also regular for Dirichlet's problem, with perhaps (and especially!) part of $\partial\mathcal{D}$ on $\partial\Omega$, we have

$$W(z) = \int_{\partial\mathcal{D}} W(\zeta) d\omega_{\mathcal{D}}(\zeta, z) \quad \text{for } z \in \mathcal{D}.$$

To see this, take an increasing sequence of functions $f_k(\zeta)$ continuous on $\partial\Omega$ and tending to $w(\zeta)$ thereon, and let

$$F_k(z) = \int_{\partial\Omega} f_k(\zeta) d\omega_{\Omega}(\zeta, z), \quad z \in \Omega.$$

Then the $F_k(z)$ tend monotonically to $W(z)$ in Ω by the monotone convergence theorem. That convergence actually holds *on* $\bar{\Omega}$ if we put $F_k(\zeta) = f_k(\zeta)$ on $\partial\Omega$; this, however, makes each function $F_k(z)$ *continuous on* $\bar{\Omega}$ besides being *harmonic* in Ω . In the domain \mathcal{D} , we therefore have

$$F_k(z) = \int_{\partial\mathcal{D}} F_k(\zeta) d\omega_{\mathcal{D}}(\zeta, z)$$

for each k . Another appeal to monotone convergence now establishes the corresponding property for W .

Fix any $z_0 \in \Omega$; we wish to show that $V(z_0) \leq W(z_0)$. For this purpose,

we use the formula just proved with \mathcal{D} equal to the component Ω_r of $\Omega \cap \{|z| < r\}$ containing z_0 , where $|z_0| < r < 1$. Because Ω is regular for Dirichlet's problem, so is each Ω_r ; that follows immediately from the characterization of such regularity in terms of *barriers*, and, in the circumstances of the last article, can also be checked directly (cf. p. 360). We write

$$\Gamma_r = \partial\Omega_r \cap \Omega,$$

making Γ_r the union of some *open arcs* on $\{|\zeta| = r\}$, and then take

$$\gamma_r = \partial\Omega_r \sim \Gamma_r;$$

γ_r is a subset (perhaps proper) of $\gamma \cap \{|\zeta| \leq r\}$.

The function $V(z)$, given as harmonic in Ω and continuous up to γ , is certainly continuous up to $\partial\Omega_r$. Therefore, since $V(\zeta) = v(\zeta)$ on $\gamma \supseteq \gamma_r$, we have, for $z \in \Omega_r$,

$$V(z) = \int_{\gamma_r} v(\zeta) d\omega_{\Omega_r}(\zeta, z) + \int_{\Gamma_r} V(\zeta) d\omega_{\Omega_r}(\zeta, z).$$

At the same time, by the reproducing property of W ,

$$W(z) = \int_{\gamma_r} W(\zeta) d\omega_{\Omega_r}(\zeta, z) + \int_{\Gamma_r} W(\zeta) d\omega_{\Omega_r}(\zeta, z), \quad z \in \Omega_r.$$

We henceforth write $\omega_r(\quad , \quad)$ for $\omega_{\Omega_r}(\quad , \quad)$. Then, since on $\gamma_r \subseteq \partial\Omega$, $W(\zeta) = w(\zeta)$ is $\geq v(\zeta)$, the two last relations yield

$$W(z) - V(z) \geq \int_{\Gamma_r} (W(\zeta) - V(\zeta)) d\omega_r(\zeta, z)$$

for $z \in \Omega_r$. Our idea is to now make $r \rightarrow 1$ in this inequality.

For $|\zeta| = 1$, define

$$\Delta_r(\zeta) = \begin{cases} W(r\zeta) - V(r\zeta) & \text{if } r\zeta \in \Gamma_r, \\ 0 & \text{otherwise.} \end{cases}$$

Since $V(z)$ is given as *bounded*, the functions $\Delta_r(\zeta)$ are *bounded below*. Moreover (and this is the clincher),

$$\liminf_{r \rightarrow 1} \Delta_r(\zeta) \geq 0 \text{ a.e., } |\zeta| = 1.$$

That is indeed *clear* for the ζ on the unit circumference *outside* Γ' (the set of radial accumulation points of Ω); since for such a ζ , $r\zeta$ cannot even belong to Ω (let alone to Γ_r) when r is near 1. Consider therefore a $\zeta \in \Gamma'$, and take any sequence of $r_n < 1$ tending to 1 with, wlog, *all the* $r_n\zeta$ in Ω

and even in their corresponding Γ_{r_n} . Then our hypothesis and the specification of v tell us that

$$V(r_n \zeta) \xrightarrow{n} v(\zeta),$$

except when ζ belongs to a certain set of measure zero, independent of $\{r_n\}$. For such a sequence $\{r_n\}$, however,

$$\liminf_{n \rightarrow \infty} W(r_n \zeta) \geq W(\zeta) = w(\zeta)$$

as seen earlier, yielding, with the preceding,

$$\liminf_{n \rightarrow \infty} \Delta_{r_n}(\zeta) \geq w(\zeta) - v(\zeta) \geq 0.$$

The asserted relation thus holds on Γ' as well, save perhaps in a set of measure zero.

Returning to our fixed $z_0 \in \Omega$, we note that for $(1 + |z_0|)/2 < r < 1$ (say), we have, on Γ_r ,

$$d\omega_r(\zeta, z_0) \leq K |d\zeta|$$

with K independent of r (just compare $\omega_r(\cdot, \cdot)$ with harmonic measure for $\{|z| < r\}$). There are hence measurable functions $\mu_r(\zeta)$ defined on $\{|\zeta| = 1\}$ for these values of r , with $0 \leq \mu_r(\zeta) \leq K$ (and $\mu_r(\zeta) = 0$ for $r\zeta \notin \Gamma_r$), such that

$$\int_{\Gamma_r} (W(\zeta) - V(\zeta)) d\omega_r(\zeta, z_0) = \int_{|\zeta|=1} \Delta_r(\zeta) r \mu_r(\zeta) |d\zeta|.$$

Here the products $\Delta_r(\zeta) r \mu_r(\zeta)$ are *uniformly bounded below* since the $\Delta_r(\zeta)$ are. And, by what has just been shown,

$$\liminf_{r \rightarrow 1} \Delta_r(\zeta) r \mu_r(\zeta) \geq 0 \quad \text{a.e., } |\zeta| = 1.$$

Thence, by *Fatou's lemma* (!),

$$\liminf_{r \rightarrow 1} \int_{|\zeta|=1} \Delta_r(\zeta) r \mu_r(\zeta) |d\zeta| \geq 0.$$

We have seen, however, that when $r > |z_0|$, $W(z_0) - V(z_0)$ is \geq the left-hand integral in the previous relation. It follows therefore that

$$W(z_0) - V(z_0) \geq 0,$$

as was to be proven.

We are done.

Remark 1. When $V(z)$ is only assumed to be subharmonic in Ω but satisfies otherwise the hypothesis of the lemma, the argument just made shows that

$$V(z) \leq \int_{\partial\Omega} v(\zeta) d\omega_{\Omega}(\zeta, z) \quad \text{for } z \in \Omega.$$

Remark 2. In the applications made in article 3, the function $V(z)$ actually has a continuous extension to the open unit disk Δ with modulus bounded, in $\Delta \sim \Omega$, by a function of $|z|$ tending to zero for $|z| \rightarrow 1$. That extension also has non-tangential boundary values a.e. on $\partial\Delta$. In these circumstances the lemma's *ad hoc* specification of $v(\zeta)$ on $\Gamma \sim \Gamma'$ is *superfluous*, for the non-tangential limit of $V(z)$ must *automatically be zero* at any $\zeta \in \Gamma \sim \Gamma'$ where it exists.

Remark 3. To arrive at the version of Jensen's inequality used in article 3, apply the relation from *remark 1* to the subharmonic functions $V_M(z) = \log^+ |M\Phi(z)|$, referring to *remark 2*. That gives us

$$\max \left(\log |\Phi(z)|, \log \frac{1}{M} \right) \leq \int_{\partial\Omega} \max \left(\log |\Phi(\zeta)|, \log \frac{1}{M} \right) d\omega_{\Omega}(\zeta, z)$$

for $z \in \Omega$. Then, since $|\Phi(z)|$ is bounded above, one may obtain the desired result by making $M \rightarrow \infty$.

Addendum completed June 8, 1987.

Bibliography for volume I

- Akhiezer, N.I. (also spelled Achieser). *Klassicheskaia problema momentov*. Fizmatgiz, Moscow, 1961. *The Classical Moment Problem*. Oliver & Boyd, Edinburgh, 1965.
- Akhiezer, N.I. O vzveshennom priblizhenii nepreryvnykh funktsii na vsei chislovoi osi. *Uspekhi Mat. Nauk* **11** (1956), 3–43. On the weighted approximation of continuous functions by polynomials on the entire real axis. *AMS Translations* **22** Ser. 2 (1962), 95–137.
- Akhiezer, N.I. *Theory of Approximation* (first edition). Ungar, New York, 1956. *Lektsii po teorii approksimatsii* (second edition). Nauka, Moscow, 1965. *Vorlesungen über Approximationstheorie* (second edition). Akademie Verlag, Berlin, 1967.
- Benedicks, M. Positive harmonic functions vanishing on the boundary of certain domains in \mathbb{R}^n . *Arkiv för Mat.* **18** (1980), 53–72.
- Benedicks, M. Weighted polynomial approximation on subsets of the real line. Preprint, Uppsala Univ. Math. Dept., 1981, 12pp.
- Bernstein, S. *Sobranie sochinenii*. Akademia Nauk, USSR. Volume I, 1952; volume II, 1954.
- Bernstein, V. *Leçons sur les progrès récents de la théorie des séries de Dirichlet*. Gauthier-Villars, Paris, 1933.
- Bers, L. An outline of the theory of pseudo-analytic functions. *Bull. AMS* **62** (1956), 291–331.
- Bers, L. *Theory of Pseudo-Analytic Functions*. Mimeographed lecture notes, New York University, 1953.
- Beurling, A. Analyse spectrale des pseudomesures. *C.R. Acad. Sci. Paris* **258** (1964), 406–9.
- Beurling, A. Analytic continuation across a linear boundary. *Acta Math.* **128** (1972), 153–82.
- Beurling, A. *On Quasianalyticity and General Distributions*. Mimeographed lecture notes, Stanford University, summer of 1961.
- Beurling, A. Sur les fonctions limites quasi analytiques des fractions rationnelles. *Huitième Congrès des Mathématiciens Scandinaves, 1934*. Lund, 1935, pp. 199–210.
- Beurling, A. and Malliavin, P. On Fourier transforms of measures with compact support. *Acta Math.* **107** (1962), 291–309.
- Boas, R. *Entire Functions*. Academic Press, New York, 1954.
- Borichev, A. and Volberg, A. Uniqueness theorems for almost analytic functions. Preprint, Leningrad branch of Steklov Math. Institute, 1987, 39pp.

- Brennan, J. Functions with rapidly decreasing negative Fourier coefficients. Preprint, University of Kentucky Math. Dept., 1986, 14pp.
- Carleson L. Estimates of harmonic measures. *Annales Acad. Sci. Fennicae*, Series A.I. *Mathematica* **7** (1982), 25–32.
- Cartan, H. *Sur les classes de fonctions définies par des inégalités portant sur leurs dérivées successives*. Hermann, Paris, 1940.
- Cartan, H. and Mandelbrojt, S. Solution du problème d'équivalence des classes de fonctions indéfiniment dérivables. *Acta Math.* **72** (1940), 31–49.
- Cartwright, M. *Integral Functions*. Cambridge Univ. Press, 1956.
- Choquet, G. *Lectures on Analysis*. 3 vols. Benjamin, New York, 1969.
- De Branges, L. *Hilbert Spaces of Entire Functions*. Prentice-Hall, Englewood Cliffs, NJ, 1968.
- Domar, Y. On the existence of a largest subharmonic minorant of a given function. *Arkiv för Mat.* **3** (1958), 429–40.
- Duren, P. *Theory of H^p Spaces*. Academic Press, New York, 1970.
- Dym, H. and McKean, H. *Gaussian Processes, Function Theory and the Inverse Spectral Problem*. Academic Press, New York, 1976.
- Dynkin, E. Funktsii s zadannoï otsenkoï $\partial f/\partial \bar{z}$ i teoremy N. Levinsona. *Mat. Sbornik* **89** (1972), 182–90. Functions with given estimate for $\partial f/\partial \bar{z}$ and N. Levinson's theorem. *Math. USSR Sbornik* **18** (1972), 181–9.
- Gamelin, T. *Uniform Algebras*. Prentice-Hall, Englewood Cliffs, NJ, 1969.
- Garnett, J. *Bounded Analytic Functions*. Academic Press, New York, 1981.
- Garsia, A. *Topics in Almost Everywhere Convergence*. Markham, Chicago, 1970 (copies available from author).
- Gorny, A. Contribution à l'étude des fonctions dérivables d'une variable réelle. *Acta Math.* **71** (1939), 317–58.
- Green, George. *Mathematical Papers of*. Chelsea, New York, 1970.
- Helson, H. *Lectures on Invariant Subspaces*. Academic Press, New York, 1964.
- Helson, H. and Lowdenslager, D. Prediction theory and Fourier Series in several variables. Part I, *Acta Math.* **99** (1958), 165–202; Part II, *Acta Math.* **106** (1961), 175–213.
- Hoffman, K. *Banach Spaces of Analytic Functions*. Prentice-Hall, Englewood Cliffs, NJ, 1962.
- Jörnicke, B. and Volberg, A. Summiruemost' logarifma pochtii analiticheskoi funktsii i obobshchenie teoremy Levinsona–Kartraït. *Mat. Sbornik* **130** (1986), 335–48.
- Kahane, J. Sur quelques problèmes d'unicité et de prolongement, relatifs aux fonctions approchables par des sommes d'exponentielles. *Annales Inst. Fourier* **5** (1953–54), 39–130.
- Kargaev, P. Nelokalnye pochtii differentsialnye operatory i interpoliatsii funktsii s redkim spektrom. *Mat. Sbornik* **128** (1985), 133–42. Nonlocal almost differential operators and interpolation by functions with sparse spectrum. *Math. USSR Sbornik* **56** (1987), 131–40.
- Katznelson, Y. *An Introduction to Harmonic Analysis*. Wiley, New York, 1968 (Dover reprint available).
- Kellog, O. *Foundations of Potential Theory*. Dover, New York, 1953.
- Khachatryan, I.O. O vztvshonnom priblizhenii tselykh funktsii nulevoi stepeni mnogochlenami na deïstvitelnoi osi. *Doklady A.N.* **145** (1962), 744–7. Weighted approximation of entire functions of degree zero by polynomials on the real axis. *Soviet Math (Doklady)* **3** (1962), 1106–10.
- Khachatryan, I.O. O vztvshonnom priblizhenii tselykh funktsii nulevoi stepeni mnogochlenami na deïstvitelnoi osi. *Kharkovskii Universitet, Uchonye Zapiski* **29**, Ser. 4 (1963), 129–42.

- Koosis, P. Harmonic estimation in certain slit regions and a theorem of Beurling and Malliavin. *Acta Math.* **142** (1979), 275–304.
- Koosis, P. *Introduction to H_p Spaces*. Cambridge University Press, 1980.
- Koosis, P. Solution du problème de Bernstein sur les entiers. *C.R. Acad. Sci. Paris* **262** (1966), 1100–2.
- Koosis, P. Sur l'approximation pondérée par des polynômes et par des sommes d'exponentielles imaginaires. *Annales Ecole Norm. Sup.* **81** (1964), 387–408.
- Koosis, P. Weighted polynomial approximation on arithmetic progressions of intervals or points. *Acta Math.* **116** (1966), 223–77.
- Levin, B. *Raspredelenie kornei tselykh funktsii*. Gostekhizdat, Moscow, 1956. *Distribution of Zeros of Entire Functions* (second edition). Amer. Math. Soc., Providence, RI, 1980.
- Levinson, N. *Gap and Density Theorems*. Amer. Math. Soc., New York, 1940, reprinted 1968.
- Levinson, N. and McKean, H. Weighted trigonometrical approximation on the line with application to the germ field of a stationary Gaussian noise. *Acta Math.* **112** (1964), 99–143.
- Lindelöf, E. Sur la représentation conforme d'une aire simplement connexe sur l'aire d'un cercle. *Quatrième Congrès des Mathématiciens Scandinaves, 1916*. Uppsala, 1920, pp. 59–90. [Note: The principal result of this paper is also established in the books by Tsuji and Zygmund (second edition), as well as in my own (on H_p spaces).]
- Mandelbrojt, S. *Analytic Functions and Classes of Infinitely Differentiable Functions*. Rice Institute Pamphlet XXIX, Houston, 1942.
- Mandelbrojt, S. *Séries adhérentes, régularisation des suites, applications*. Gauthier-Villars, Paris, 1952.
- Mandelbrojt, S. *Séries de Fourier et classes quasi-analytiques de fonctions*. Gauthier-Villars, Paris, 1935.
- McGehee, O., Pigno, L. and Smith, B. Hardy's inequality and the L^1 norm of exponential sums. *Annals of Math.* **113** (1981), 613–18.
- Mergelian, S. Vesovye priblizhenie mnogochlenami. *Uspekhi Mat. Nauk* **11** (1956), 107–52. Weighted approximation by polynomials. *AMS Translations* **10** Ser 2 (1958), 59–106.
- Nachbin, L. *Elements of Approximation Theory*. Van Nostrand, Princeton, 1967.
- Naimark, M. *Normirovannyye koltsa*.
First edition: Gostekhizdat, Moscow, 1956.
First edition: *Normed Rings*, Noordhoff, Groningen, 1959.
Second edition: Nauka, Moscow, 1968.
Second edition: *Normed Algebras*. Wolters-Noordhoff, Groningen, 1972.
- Nehari, Z. *Conformal Mapping*. McGraw-Hill, New York, 1952.
- Nevanlinna, R. *Eindeutige analytische Funktionen* (second edition). Springer, Berlin, 1953. *Analytic Functions*. Springer, New York, 1970.
- Paley, R. and Wiener, N. *Fourier Transforms in the Complex Domain*. Amer. Math. Soc., New York, 1934.
- Phelps, R. *Lectures on Choquet's Theorem*. Van Nostrand, Princeton, 1966.
- Pollard, H. Solution of Bernstein's approximation problem. *Proc. AMS* **4** (1953), 869–75.
- Riesz, F. and M. Über die Randwerte einer analytischen Funktion. *Quatrième Congrès des Mathématiciens Scandinaves, 1916*. Uppsala, 1920, pp. 27–44. [Note: The material of this paper can be found in the books by Duren, Garnett, Tsuji, Zygmund (second edition) and myself (on H_p spaces).]
- Riesz, F. and Sz-Nagy, B. *Leçons d'analyse fonctionnelle* (second edition). Akadémiai Kiadó, Budapest, 1953. *Functional Analysis*. Ungar, New York, 1965.

- Riesz, M. Sur le problème des moments.
First note: *Arkiv för Mat., Astr. och Fysik* **16** (12) (1921), 23pp.
Second note: *Arkiv för Mat., Astr. och Fysik* **16** (19) (1922), 21pp.
Third note: *Arkiv för Mat., Astr. och Fysik* **17** (16) (1923), 52pp.
- Rudin, W. *Real and Complex Analysis* (second edition). McGraw Hill, New York, 1974.
- Shohat, J. and Tamarkin, J. *The Problem of Moments*. Math. Surveys No. 1, Amer. Math. Soc., Providence, RI, 1963.
- Szegő, G. *Orthogonal Polynomials*. Amer. Math. Soc., Providence, RI, 1939; revised edition published 1958.
- Titchmarsh, E. *Introduction to the Theory of Fourier Integrals* (second edition). Oxford Univ. Press, 1948.
- Titchmarsh, E. *The Theory of Functions* (second edition). Oxford Univ. Press, 1939; corrected reimpression, 1952.
- Tsuji, M. *Potential Theory in Modern Function Theory*. Maruzen, Tokyo, 1959; reprinted by Chelsea, New York, 1975.
- Vekua, I. *Obobshchennye analiticheskie funktsii*. Fizmatgiz, Moscow, 1959. *Generalized Analytic Functions*. Pergamon, London, 1962.
- Volberg, A. Logarifm pochti-analiticheskoi funktsii summiruem. *Doklady A.N.* **265** (1982), 1297–302. The logarithm of an almost analytic function is summable. *Soviet Math (Doklady)* **26** (1982), 238–43.
- Volberg, A. and Erikke, B., see Jöricke, B. and Volberg, A.
- Widom, H. Norm inequalities for entire functions of exponential type. *Orthogonal Expansions and their Continuous Analogues*. Southern Illinois Univ. Press, Carbondale, 1968, pp. 143–65.
- Yosida, K. *Functional Analysis*. Springer, Berlin, 1965.
- Zygmund, A. *Trigonometric Series* (second edition of following item). 2 vols. Cambridge Univ. Press, 1959; now reprinted in a single volume. *Trigonometrical Series* (first edition of preceding). Monografie matematyczne, Warsaw, 1935; reprinted by Chelsea, New York, in 1952, and by Dover, New York, in 1955.

Index

- Akhiezer's description of entire functions
arising in weighted
approximation 160, 174
- Akhiezer's theorems about weighted
polynomial approximation 158ff,
424, 523
- Akhiezer's theorems on weighted
approximation by sums of imaginary
exponentials 174, 424, 432, 445
- approximation index $M(A)$,
Beurling's 275
- approximation index $M_p(A)$,
Beurling's 293
- approximation, weighted 145ff, 385, 424
see also under weighted approximation
- Benedicks, M. 434ff
- Benedicks' lemma on harmonic measure
for slit regions bounded by a
circle 400
- Benedicks' theorem on existence of a
Phragmén–Lindelöf function 418,
431
- Benedicks' theorem on harmonic measure
for slit regions 404
- Bernstein approximation problem 146ff
- Bernstein intervals associated with a set of
points on $(0, \infty)$ 454ff
- Bernstein's lemma 102
- Bernstein's theorem on weighted
polynomial approximation 169
- Beurling, A. and Malliavin, P. 550, 568
- Beurling quasianalyticity 275ff
- Beurling quasianalyticity for L_p
functions 292ff
- Beurling–Dynkin theorem on the Legendre
transform 333
- Beurling's approximation indices *see*
under approximation index
- Beurling's gap theorem 237, 305
- Beurling's identity for certain bilinear
forms 484
- Beurling's theorem about Fourier–Stieltjes
transforms vanishing on a set of
positive measure 268
- Beurling's theorem about his quasi-
analyticity 276
- Beurling's theorem on his L_p quasi-
analyticity 293
- boundary values, non-tangential 10, 43ff,
265, 269, 286ff
- canonical product 21
- Carleman's criterion for
quasianalyticity 80
its necessity 89
- Carleman's inequality 96
- Carleson's lemma on linear forms 392,
398
- Carleson's theorem on harmonic measure
for slit regions 394, 404, 430
- Cartan–Gorny theorem 104
- Cauchy principal value, definition of 533
- Cauchy transform, planar 320ff
- class $\mathcal{C}_f(\{M_n\})$ of infinitely differentiable
functions 79
its quasianalyticity 80
- convex logarithmic regularization $\{M_n\}$ of
a sequence $\{M_n\}$ 83ff, 92ff, 104ff, 130,
226
- de Branges' lemma 187
- de Branges' theorem 192
discussion about 198ff
- density, of a measurable sequence 178
- Dirichlet integral 479, 500, 510ff
- Dirichlet problem 251, 360, 387, 388
- Dynkin's extension theorem 339, 359, 373
- energy of a measure on $(0, \infty)$ 479ff, 549ff,
562, 568

- bilinear form associated thereto 482, 487, 494ff, 508, 512ff, 551, 552, 553, 563, 566
 - formulas for 479, 485, 497, 512
 - positivity of 482, 493
- entire functions of exponential type 15ff
 - arising in weighted approximation 160, 174, 218, 219, 525
 - as majorants on subsets of \mathbb{R} 555ff, 562, 564, 568
 - coming from certain partial fraction expansions 203ff, 205
 - see also under* Hadamard factorization
- exponential type, entire functions of *see preceding*
- extension of domain, principle of 259, 289, 301, 368, 372, 529, 531
- extension of positive linear functionals 111ff, 116
- extreme point of a convex w^* compact set of measures 186ff
- Fejér and Riesz, lemma of 281
- function of exponential type, entire 15ff
 - see also under* entire functions
- function $T(r)$ used in study of quasianalyticity 80ff
- gap theorem, Beurling's 237
- Gauss quadrature formula 134, 137ff
- Green, George, homage to 419–22
- Green's function 400ff, 406, 407, 410, 418ff, 439, 479, 526ff, 547ff, 550
 - estimates for in slit regions 401, 439, 442, 548
 - symmetry of 401, 415, 418ff, 530
- Green potential 479, 551, 552, 553, 560, 562, 563, 566
- Hadamard factorization for entire functions of exponential type 16, 19, 22, 54, 56, 70, 201, 556, 561
- hall of mirrors argument 157, 158, 184, 208, 375, 523
- Hall, T., his theorem on weighted polynomial approximation 169
- Hankel matrix 117
- harmonic conjugate 46, 59, 61
 - existence a.e. of 47, 532, 537
 - see also under* Hilbert transform
- harmonic estimation, statement of theorem on it 256
- harmonic functions, positive, representations for
 - in half plane 41
 - in unit disk 39
- harmonic measure 251ff
 - approximate identity property of 253, 261
- boundary behaviour of 261ff, 265
 - definition of 255
 - in curvilinear strips, use of estimate for 355
 - in slit regions 385, 389ff, 394, 403, 404, 430, 437, 443, 444, 446, 522, 525ff, 530, 541, 545ff, 554, 562, 565
 - Volberg's theorem on 349, 353, 362, 364, 366
- Harnack's inequality 254, 372, 410, 430
- Hilbert transform 47, 61, 62, 63, 65, 532, 534, 538ff
- Jensen's formula 2, 4, 7, 21, 76, 163, 291, 559
- Kargaev's example on Beurling's gap theorem 305ff, 315
- Kolmogorov's theorem on the harmonic conjugate 62ff
- Krein's theorem on certain entire functions 205
- Krein–Milman theorem, its use 186, 199
- Kronecker's lemma 119
- Legendre transform $h(\xi)$ of an increasing function $M(v)$ 323ff
- Levinson (and Cartwright), theorem on distribution of zeros
 - for functions with real zeros only 66
 - general form of 69
 - use of 175, 178
- Levinson's log log theorem 374ff, 376, 379ff
- Levinson's theorem about Fourier–Stieltjes transforms vanishing on an interval 248, 347, 361
- Levinson's theorem on weighted approximation by sums of imaginary exponentials 243
- Lindelöf's theorems about the zeros of entire functions of exponential type, statements 20, 21
- Lindelöf's theorem on conformal mapping 264
- log log theorem *see under* Levinson
- Lower polynomial regularization $W_*(x)$ of a weight $W(x)$, its definition 158
- Lower regularization $W_A(x)$ of a weight $W(x)$ by entire functions of exponential type $\leq A$ 175, 428
 - for Lip 1 weights 236
 - for weights increasing on $[0, \infty)$ 242
- Lower regularizations $W_{A,E}(x)$ of a weight $W(x)$ corresponding to closed unbounded sets $E \subseteq \mathbb{R}$ 428
- Markov–Riesz–Pollard trick 139, 155, 171, 182, 190

- maximum principle, extended, its
 - statement 23
- measurable sequence 178
- Mergelian's theorems about weighted
 - polynomial approximation 147ff
- Mergelian's theorems on weighted
 - approximation by sums of imaginary
 - exponentials 173, 174, 432
- moment problem *see under* Riesz
- moment sequences
 - definition of 109
 - determinacy of 109, 126, 128, 129, 131,
 - 141, 143
 - indeterminacy of 109, 128, 133, 143
 - Riesz' characterization of 110
 - same in terms of determinants 121
- Newton polygon 83ff
- non-tangential limit 11
- Paley and Wiener, their construction of
 - certain entire functions 100
- Paley and Wiener, theorem of 31
 - L_1 version of same 36
- Phragmén–Lindelöf argument 25, 405,
 - 406, 553
- Phragmén–Lindelöf function 25, 386, 406,
 - 407, 418, 431, 441, 525ff, 541, 555
- Phragmén–Lindelöf theorems
 - first 23
 - second 25
 - third 27
 - fourth 28
 - fifth 29
- Poisson kernel
 - for half plane 38, 42, 384, 534, 536, 539
 - for rectangle 299
 - for unit disk 7, 8, 10ff
 - pointwise approximate identity property
 - of latter 10
- Pollard's theorem 164, 433
 - for weighted approximation by sums of
 - imaginary exponentials 181, 428
- Pólya maximum density for a positive
 - increasing sequence 176ff
- Pólya's theorem 178
- quasianalytic classes $\mathcal{C}_f(\{M_n\})$, their
 - characterization 91
- quasianalyticity, Beurling's 275ff
- quasianalyticity of a class $\mathcal{C}_f(\{M_n\})$ 80
 - Carleman's criterion for it 80
 - necessity of same 89
- representations for positive harmonic
 - functions *see under* harmonic functions
- Riesz, F. and M. 259, 276, 286
- Riesz' criterion for existence of a solution
 - to moment problem 110, 121
- Riesz' criterion for indeterminacy of the
 - moment problem 133
- Riesz–Fejér theorem 55, 556
- simultaneous polynomial approximation,
 - Volberg's theorem on 344, 349
- slit regions (whose boundary consists of
 - slits along real axis) 384, 386ff,
 - 401, 402, 418, 430, 439, 441, 525ff,
 - 540ff, 545, 553, 564, 568
 - see also under* harmonic measure
- spaces $\mathcal{C}_W(0)$ and $\mathcal{C}_W(0+)$ 212
 - conditions on W for their equality 223,
 - 226
 - weights W for which they differ 229ff,
 - 244ff
- spaces of functions used in studying
 - weighted approximation, their
 - definitions
 - $\mathcal{C}_W(\mathbb{R})$ 145
 - $\mathcal{C}_W(0)$, $\mathcal{C}_W(A)$, $\mathcal{C}_W(A+)$ 211
 - $\mathcal{C}_W(E)$, $\mathcal{C}_W(A, E)$, $\mathcal{C}_W(0, E)$ 424
 - $\mathcal{C}_W(\mathbb{Z})$, $\mathcal{C}_W(0, \mathbb{Z})$ 522
- spaces $\mathcal{S}_p(\mathcal{D}_0)$ 281ff
- Szegő's theorem 7, 291, 292
 - extension of same by Krein 9
- two constants, theorem on 257
- Volberg's theorems
 - on harmonic measures 349, 353, 362,
 - 364, 366
 - on simultaneous polynomial
 - approximation 344, 349
 - on the logarithmic integral 317ff, 357
- w^* convergence 41
- weight 145ff
- weighted approximation 145ff, 385, 424
- weighted approximation by
 - polynomials 147ff, 169, 247, 433, 445
 - on \mathbb{Z} 447ff, 523
 - see also under* Akhiezer, Mergelian
- weighted approximation by sums of
 - imaginary exponentials 171ff
 - on closed unbounded subsets of \mathbb{R} 428,
 - 444
 - with a Lip 1 weight 236
 - with a weight increasing on $[0, \infty)$ 243,
 - 247
 - see also under* Akhiezer, Mergelian
- well disposed, definition of term 452

Contents of volume II

IX Jensen's Formula Again

A Pólya's gap theorem

B Scholium. A converse to Pólya's gap theorem

1 Special case. Σ measurable and of density $D > 0$

Problem 29

2 General case; Σ not necessarily measurable. Beginning of Fuchs' construction

3 Bringing in the gamma function

Problem 30

4 Formation of the group products $R_j(z)$

5 Behaviour of $\frac{1}{x} \log \left| \frac{x - \lambda}{x + \lambda} \right|$

6 Behaviour of $\frac{1}{x} \log |R_j(x)|$ outside the interval $[X_j, Y_j]$

7 Behaviour of $\frac{1}{x} \log |R_j(x)|$ inside $[X_j, Y_j]$

8 Formation of Fuchs' function $\Phi(z)$. Discussion

9 Converse of Pólya's gap theorem in general case

C A Jensen formula involving confocal ellipses instead of circles

D A condition for completeness of a collection of imaginary exponentials on a finite interval

Problem 31

1 Application of the formula from §C

2 Beurling and Malliavin's effective density \tilde{D}_Λ .

E Extension of the results in §D to the zero distribution of entire functions $f(z)$ of exponential type with

$$\int_{-\infty}^{\infty} (\log^+ |f(x)| / (1 + x^2)) dx \quad \text{convergent}$$

1 Introduction to extremal length and to its use in estimating harmonic measure

Problem 32

Problem 33

Problem 34

2 Real zeros of functions $f(z)$ of exponential type with

$$\int_{-\infty}^{\infty} (\log^+ |f(x)|/(1+x^2))dx < \infty$$

F Scholium. Extension of results in §E.1. Pfluger's theorem and Tsuji's inequality

1 Logarithmic capacity and the conductor potential

Problem 35

2 A conformal mapping. Pfluger's theorem

3 Application to the estimation of harmonic measure. Tsuji's inequality

Problem 36

Problem 37

X Why we want to have multiplier theorems

A Meaning of term 'multiplier theorem' in this book

Problem 38

1 The weight is even and increasing on the positive real axis

2 Statement of the Beurling–Malliavin multiplier theorem

B Completeness of sets of exponentials on finite intervals

1 The Hadamard product over Σ

2 The little multiplier theorem

3 Determination of the completeness radius for real and complex sequences Λ

Problem 39

C The multiplier theorem for weights with uniformly continuous logarithms

1 The multiplier theorem

2 A theorem of Beurling

Problem 40

D Poisson integrals of certain functions having given weighted quadratic norms

E Hilbert transforms of certain functions having given weighted quadratic norms

1 H_p spaces for people who don't want to really learn about them

Problem 41

Problem 42

2 Statement of the problem, and simple reductions of it

3 Application of H_p space theory; use of duality

4 Solution of our problem in terms of multipliers

Problem 43

- F Relation of material in preceding § to the geometry of unit sphere in L_∞/H_∞
 - Problem 44
 - Problem 45
 - Problem 46
 - Problem 47

XI Multiplier theorems

- A Some rudimentary potential theory
 - 1 Superharmonic functions; their basic properties
 - 2 The Riesz representation of superharmonic functions
 - Problem 48
 - Problem 49
 - 3 A maximum principle for pure logarithmic potentials. Continuity of such a potential when its restriction to generating measure's support has that property
 - Problem 50
 - Problem 51
- B Relation of the existence of multipliers to the finiteness of a superharmonic majorant
 - 1 Discussion of a certain regularity condition on weights
 - Problem 52
 - Problem 53
 - 2 The smallest superharmonic majorant
 - Problem 54
 - Problem 55
 - Problem 56
 - 3 How \mathfrak{MF} gives us a multiplier if it is finite
 - Problem 57
- C Theorems of Beurling and Malliavin
 - 1 Use of the domains from §C of Chapter VIII
 - 2 Weight is the modulus of an entire function of exponential type
 - Problem 58
 - 3 A quantitative version of the preceding result
 - Problem 59
 - Problem 60
 - 4 Still more about the energy. Description of the Hilbert space \mathfrak{H} used in Chapter VIII, §C.5
 - Problem 61
 - Problem 62
 - 5 Even weights W with $\|\log W(x)/x\|_E < \infty$
 - Problem 63
 - Problem 64
- D Search for the presumed essential condition
 - 1 Example. Uniform Lip 1 condition on $\log \log W(x)$ not sufficient

- 2 Discussion
- Problem 65
- 3 Comparison of energies
- Problem 66
- Problem 67
- Problem 68
- 4 Example. The finite energy condition not necessary
- 5 Further discussion and a conjecture
- E A necessary and sufficient condition for weights meeting the local regularity requirement
 - 1 Five lemmas
 - 2 Proof of the conjecture from §D.5
 - Problem 69
 - Problem 70
 - Problem 71