by what we have already seen. The *second* term – call it $p(z)$ – is *finite* (and harmonic!) for $\Im z > 0$, thanks to the above condition on $\log^- \omega(t)$.

Our inequality for $\log|F(z)|$ thus boils down to the relation

$$|F(z)| \leq e^{p(z)} \sqrt{\left(\frac{2}{\pi \Im z} \int_{-\infty}^{\infty} (U(t))^2 w(t)\,dt\right)}, \qquad \Im z > 0,$$

whence, in the upper half-plane,

$$\left|\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} U(t)\,dt\right| = |\Re F(z)| \leq e^{p(z)} \sqrt{\left(\frac{2}{\pi \Im z} \int_{-\infty}^{\infty} |U(t)|^2 w(t)\,dt\right)}$$

for the *real-valued* sums

$$U(t) = \sum_{\lambda \geq a} (A_\lambda \cos \lambda t + B_\lambda \sin \lambda t).$$

This last relation must indeed then hold for such *complex*-valued sums $U(t)$, because any of those can be written as $U_1(t) + iU_2(t)$ with *real-valued* ones U_1, U_2 of the same form.*

Invoke now the second theorem of §D! According to it, the estimate just obtained implies the existence of a non-zero entire function f of exponential type $\leq a$ such that

$$\int_{-\infty}^{\infty} \frac{|f(t)|^2}{(1+t^2)w(t)}\,dt < \infty.$$

Necessity is proved.

We turn to the ***sufficiency***. Suppose we *have* a non-zero entire function $f(z)$ of exponential type $\leq a$ with

$$\int_{-\infty}^{\infty} \frac{|f(t)|^2}{(1+t^2)w(t)}\,dt < \infty.$$

As in the sufficiency proof for the first theorem in §D, f may be taken to be *real* on $\mathbb{R}$. Also, by reasoning as in the sufficiency argument for the second theorem of that §, we see that $f(t)/(t+i)$ is *bounded* on $\mathbb{R}$.

The last statement certainly implies that

$$\int_{-\infty}^{\infty} \frac{\log^+|f(t)|}{1+t^2}\,dt < \infty,$$

* In that circumstance, $|\int_{-\infty}^{\infty}(\Im z\,U(t)/|z-t|^2)dt|^2 = \{\int_{-\infty}^{\infty}(\Im z\,U_1(t)/|z-t|^2)dt\}^2 + \{\int_{-\infty}^{\infty}(\Im z\,U_2(t)/|z-t|^2)dt\}^2$. Use of the inequality on each of the integrals on the right (for which it is already known to hold) yields the upper bound $(2\pi/\Im z)e^{2p(z)}\int_{-\infty}^{\infty}\{(U_1(t))^2 + (U_2(t))^2\}w(t)\,dt$.

so the first theorem of §G.2, Chapter III applies to our function f, and we have

$$\int_{-\infty}^{\infty} \frac{\log^{-}|f(t)|}{1+t^2}\,dt \quad < \quad \infty.$$

At the same time, by the inequality between arithmetic and geometric means,

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+t^2}\log\left(\frac{|f(t)|^2}{w(t)}\right)dt \quad \leqslant \quad \log\left(\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|f(t)|^2}{(1+t^2)w(t)}\,dt\right) \quad < \quad \infty$$

which, with the previous relation, yields

$$\int_{-\infty}^{\infty} \frac{\log^{-} w(t)}{1+t^2}\,dt \quad < \quad \infty.$$

This condition, however, gives us an outer function $\varphi \in H_2$ for which

$$|\varphi(t)| \quad = \quad \sqrt{(w(t))} \quad > \quad 0 \quad \text{a.e.}.$$

According, then, to the second theorem of the preceding article, a function $\omega \geqslant 0$ possessing the desired properties *will exist* provided that we can find a *non-zero* $h \in H_\infty$ such that

$$\left| e^{2iat}\frac{\overline{\varphi(t)}}{\varphi(t)} \; - \; h(t) \right| \quad \leqslant \quad 1 \quad \text{a.e.}.$$

We proceed to exhibit such an h.

For $\Im z > 0$, write, as in article 1,

$$\varphi(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\varphi(t)\,dt;$$

by a theorem from that article, $\varphi(z)$ is *analytic* in the upper half-plane and

$$\varphi(t+iy) \;\longrightarrow\; \varphi(t) \quad \text{a.e.}$$

as $y \longrightarrow 0$. Saying that φ is *outer* means, as we recall, that

$$\log|\varphi(z)| \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log|\varphi(t)|\,dt, \qquad \Im z > 0;$$

$\varphi(z)$ has, in particular, *no zeros in the upper half plane*. The ratio

$$R(z) \;=\; e^{2iaz}\left(\frac{f(z)}{\varphi(z)}\right)^2$$

is thus *analytic* for $\Im z > 0$. Since $f(z)$ is *entire* and $|\varphi(t)| > 0$ a.e., $R(t + \mathrm{i}y)$ approaches for almost every $t \in \mathbb{R}$ a *definite limit*,

$$R(t) = e^{2iat}\left(\frac{f(t)}{\varphi(t)}\right)^2,$$

as $y \longrightarrow 0$. Because our function f is real on $\mathbb{R}$,

$$R(t) \quad \text{and} \quad e^{2iat}\frac{\overline{\varphi(t)}}{\varphi(t)}$$

have *the same argument* there, and we see, referring to our requirement on h, that *if* $R(t)$ *were in* H_∞, we could take for h a suitable constant multiple of R. Usually, however, $R(t)$ is not bounded, so this will not be the case, and we have to do a supplementary construction.

We have

$$|R(t)| = \frac{(f(t))^2}{w(t)}, \qquad t \in \mathbb{R},$$

so by hypothesis,

$$\int_{-\infty}^{\infty} \frac{|R(t)|}{1+t^2}\,dt < \infty.$$

Following an idea from a paper of Adamian, Arov and Krein we now put

$$Q(z) = \frac{1}{\pi i}\int_{-\infty}^{\infty}\left(\frac{1}{t-z} - \frac{t}{t^2+1}\right)|R(t)|\,dt$$

for $\Im z > 0$; the previous relation guarantees absolute convergence of the integral on the right, and $Q(z)$ is *analytic* in the upper half plane, with $\Re Q(z) > 0$ there. The quotient $R(z)/Q(z)$ is thus analytic for $\Im z > 0$.

It is now claimed that

$$\left|\frac{R(z)}{Q(z)}\right| \leqslant 1, \qquad \Im z > 0.$$

The function f is of exponential type $\leqslant a$ and fulfills the above condition involving $\log^+|f(t)|$. Hence, by §G.2, Chapter III,

$$\log|f(z)| \leqslant a\Im z + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2}\log|f(t)|\,dt, \qquad \Im z > 0,$$

which, with the previous formula for $\log|\varphi(z)|$, yields

$$\log|R(z)| \leqslant \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2}\log|R(t)\,dt, \qquad \Im z > 0.$$

Returning to our function $Q(z)$, we get, by the inequality between arithmetic and geometric means,

$$|Q(z)| \geqslant \Re Q(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}|R(t)|\,dt$$

$$\geqslant \exp\left\{\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log|R(t)|\,dt\right\}.$$

The preceding relation says, however, that the right-hand member is $\geqslant |R(z)|$. We thus have $|Q(z)| \geqslant |R(z)|$ for $\Im z > 0$, and the above inequality is verified.

Thanks to that inequality we see, by a result from article 1, that there is an $h \in H_\infty$ with

$$\frac{R(z)}{Q(z)} = h(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}h(t)\,dt$$

for $\Im z > 0$, and that $\|h\|_\infty \leqslant 1$. This function h cannot be a.e. zero on $\mathbb{R}$ because $R(z)$ is not identically zero – the entire function $f(z)$ isn't! A result from article 1 therefore implies that

$$|h(t)| > 0 \qquad \text{a.e., } t \in \mathbb{R}.$$

As $y \longrightarrow 0$,

$$\frac{R(t+iy)}{Q(t+iy)} = h(t+iy) \longrightarrow h(t) \neq 0 \quad \text{a.e..}$$

At the same time,

$$R(t+iy) \longrightarrow R(t) \quad \text{a.e.,}$$

so $Q(t+iy)$ must approach a certain definite limit, $Q(t)$, for almost all $t \in \mathbb{R}$ as $y \longrightarrow 0$. (This also follows directly from §F.2 of Chapter III.) Since

$$\Re Q(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}|R(t)|\,dt,$$

we see, by the usual property of the Poisson kernel, that

$$\Re Q(t) = |R(t)| \quad \text{a.e..}$$

Finally, then,

$$h(t) = \frac{R(t)}{|R(t)| + i\Im Q(t)} \qquad \text{a.e., } t \in \mathbb{R}.$$

Recall that

$$R(t) \quad = \quad e^{2iat}\left(\frac{f(t)}{\varphi(t)}\right)^2 \quad = \quad e^{2iat}\frac{\overline{\varphi(t)}}{\varphi(t)}|R(t)|,$$

$f(t)$ being real. This and the preceding formula thus give

$$\left|e^{2iat}\frac{\overline{\varphi(t)}}{\varphi(t)} - h(t)\right| \quad = \quad \left|1 - \frac{|R(t)|}{|R(t)| + i\mathfrak{Z}Q(t)}\right| \qquad \text{a.e.},$$

and the right side is clearly $\leqslant 1$. Our function $h \in H_\infty$ therefore has the required properties, and our proof of sufficiency is finished.

We are done.

Remark. From this theorem and the second one of §D, we see that *only* by virtue of the existence of a non-zero $\omega \geqslant 0$ making

$$\int_{-\infty}^{\infty} |\tilde{U}(t)|^2 \omega(t)\,dt \quad \leqslant \quad \int_{-\infty}^{\infty} |U(t)|^2 w(t)\,dt$$

can the harmonic extension of a general sum

$$U(t) \quad = \quad \sum_{\lambda \geqslant a} (A_\lambda \cos \lambda t + B_\lambda \sin \lambda t)$$

to the upper half plane be controlled there by the integral on the right.

In the next problem, we consider bounded functions $u(\vartheta)$ defined on $[-\pi, \pi]$, using for them a Hilbert transform given by the formula

$$\tilde{u}(\vartheta) \quad = \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{u(\tau)}{\tan((\vartheta-\tau)/2)}\,d\tau,$$

as is customary in the study of Fourier series. (The expression on the right is a Cauchy principal value.) If one puts $\tan(\vartheta/2) = x$, $\tan(\tau/2) = t$, and then writes $u(\tau) = U(t)$, the function $\tilde{u}(\vartheta)$ goes over into the *first kind* of Hilbert transform $\tilde{U}(x)$ for functions U defined on $\mathbb{R}$, described at the beginning of article 2.

If a function $w(\vartheta) \geqslant 0$ belonging to $L_1(-\pi, \pi)$ is given, one may ask whether there exists an $\omega(\vartheta) \geqslant 0$, not a.e. zero on $[-\pi, \pi]$, such that

$$\int_{-\pi}^{\pi} |\tilde{u}(\vartheta)|^2 \omega(\vartheta)\,d\vartheta \quad \leqslant \quad \int_{-\pi}^{\pi} |u(\vartheta)|^2 w(\vartheta)\,d\vartheta$$

for all bounded functions u. It is clear that any given ω has this property iff, with it, the relation just written holds for all u of the *special form*

$$u(\vartheta) \quad = \quad \sum_{-N}^{N} a_n e^{in\theta}.$$

(Here N is *finite*, but *arbitrary*.) Such a function u is called a *trigonometric polynomial*; for it we have

$$\tilde{u}(\vartheta) \quad = \quad -i\sum_{-N}^{N} a_n \operatorname{sgn} n \, e^{in\theta}.$$

Problem 43

Given $w \geqslant 0$ in $L_1(-\pi, \pi)$, one is to prove that there exists an $\omega \geqslant 0$, not a.e. zero on $[-\pi, \pi]$, such that

$$\int_{-\pi}^{\pi} |\tilde{u}(\vartheta)|^2 \omega(\vartheta) \, d\vartheta \quad \leqslant \quad \int_{-\pi}^{\pi} |u(\vartheta)|^2 w(\vartheta) \, d\vartheta$$

for all *trigonometric polynomials* u iff

$$\int_{-\pi}^{\pi} \frac{d\vartheta}{w(\vartheta)} \quad < \quad \infty.$$

(a) First prove Kolmogorov's theorem, which says that there is a sequence of *trigonometric polynomials* $u_k(\vartheta)$ *without constant term* (i.e., in which $a_0 = 0$) such that

$$\int_{-\pi}^{\pi} |1 \; - \; u_k(\vartheta)|^2 w(\vartheta) \, d\vartheta \;\xrightarrow[k]{}\; 0$$

iff

$$\int_{-\pi}^{\pi} \frac{d\vartheta}{w(\vartheta)} \quad = \quad \infty.$$

(Hint: Work with the inner product

$$\langle u, v \rangle_w \quad = \quad \int_{-\pi}^{\pi} u(\vartheta)\overline{v(\vartheta)} w(\vartheta) \, d\vartheta$$

and use orthogonality.)

(b) Show that the condition $\int_{-\pi}^{\pi} (d\vartheta / w(\vartheta)) \; < \; \infty$ is *necessary* for the existence of an ω enjoying the properties in question. (Hint: Let $u_0(\vartheta) \; = \; \sum_{n \neq 0} a_n e^{in\vartheta}$ be any trigonometric polynomial *without constant term*, and put

$$u_1(\vartheta) \; = \; 1 - u_0(\vartheta); \qquad u_2(\vartheta) \; = \; e^{-i\vartheta}(1 - u_0(\vartheta)).$$

Then observe that

$$e^{i\vartheta}\tilde{u}_2(\vartheta) - \tilde{u}_1(\vartheta) = i(1 - a_1 e^{i\vartheta}),$$

so, for an ω having the above properties, we would have

$$\int_{-\pi}^{\pi} |1 - a_1 e^{i\vartheta}|^2 \omega(\vartheta)\, d\vartheta \leqslant 4\int_{-\pi}^{\pi} |1 - u_0(\vartheta)|^2 w(\vartheta)\, d\vartheta.$$

We have

$$|1 - a_1 e^{i\vartheta}|^2 = (1 - |a_1|)^2 + 4|a_1|\sin^2 \frac{\vartheta - \alpha}{2}$$

with $-\pi \leqslant \alpha \leqslant \pi$, so the integral on the left has a *strictly positive minimum* for $a_1 \in \mathbb{C}$ *unless* $\omega(\vartheta)$ *vanishes* a.e. on $[-\pi, \pi]$. Apply the result from (a).)

(c) Suppose now that $w(\vartheta)$ and $1/w(\vartheta)$ both belong to $L_1(-\pi, \pi)$. For $|z| < 1$, put

$$\Omega(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\tau} + z}{e^{i\tau} - z} \frac{d\tau}{w(\tau)}.$$

$\Omega(z)$ is analytic for $\{|z| < 1\}$ and, by Chapter III, §F.2,

$$\lim_{r \to 1} \Omega(re^{i\vartheta}) = \Omega(e^{i\vartheta})$$

exists for almost all ϑ, and

$$\Re\Omega(e^{i\vartheta}) = 1/w(\vartheta) \quad \text{a.e.}.$$

This makes $|1/\Omega(e^{i\vartheta})| \leqslant w(\vartheta)$ a.e., so that $1/\Omega(e^{i\vartheta}) \in L_1(-\pi, \pi)$. *Show that*

$$\int_{-\pi}^{\pi} \frac{e^{in\vartheta}}{\Omega(e^{i\vartheta})}\, d\vartheta = 0 \quad \text{for } n = 1, 2, 3, \ldots.$$

(Hint: The reader familiar with the theory of H_p spaces for the unit disk may use Smirnov's theorem. Otherwise, one may start from scratch, arguing as follows. For $|z| < 1$, by the inequality between arithmetic and harmonic means,

$$\left|\frac{1}{\Omega(z)}\right| \leqslant \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - |z|^2}{|e^{i\tau} - z|^2} \frac{d\tau}{w(\tau)}\right)^{-1} \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - |z|^2}{|e^{i\tau} - z|^2} w(\tau)\, d\tau.$$

Use this relation to show, *firstly*, that there is a *complex measure* ν on $[-\pi, \pi]$ for which

$$\frac{1}{\Omega(z)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - |z|^2}{|e^{i\tau} - z|^2}\, d\nu(\tau), \qquad |z| < 1,$$

(see proof of *first* theorem in §F.1, Chapter III), and, *secondly*, that ν must be *absolutely continuous* (without appealing to the F. and M. Riesz theorem). These facts imply that

$$\frac{1}{\Omega(z)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-|z|^2}{|e^{i\tau}-z|^2}\,\frac{d\tau}{\Omega(e^{i\tau})}, \qquad |z|<1,$$

and from this the desired relation follows on taking an $r<1$, observing that

$$\int_{-\pi}^{\pi} (e^{in\vartheta}/\Omega(re^{i\vartheta}))\,d\vartheta = 0 \quad \text{for } n = 1, 2, 3, \ldots,$$

and then using Fubini's theorem.)

(d) Let $\Omega(e^{i\vartheta})$ be the function from (c). In analogy with what was done in article 3, put

$$\sigma(\vartheta) = 1 - \left|1 - \frac{1}{w(\vartheta)\Omega(e^{i\vartheta})}\right|.$$

Show that $0 \leqslant \sigma(\vartheta) \leqslant 1$ a.e. and that $\sigma(\vartheta)$ is *not* a.e. zero on $[-\pi, \pi]$.

(e) If $f(\vartheta)$ is a finite sum of the form $\sum_{n\geqslant 1} c_n e^{in\vartheta}$, show that

$$\Re \int_{-\pi}^{\pi} (f(\vartheta))^2 w(\vartheta)\,d\vartheta \leqslant \int_{-\pi}^{\pi} (1 - \sigma(\vartheta)) w(\vartheta) |f(\vartheta)|^2\,d\vartheta.$$

(Hint: By (c), the integral figuring on the left equals

$$\int_{-\pi}^{\pi} w(\vartheta)\left(1 - \frac{1}{w(\vartheta)\Omega(e^{i\vartheta})}\right)(f(\vartheta))^2\,d\vartheta.$$

Refer to (d).)

(f) Hence show that

$$\int_{-\pi}^{\pi} \sigma(\vartheta) w(\vartheta) |\tilde{u}_0(\vartheta)|^2\,d\vartheta \leqslant 2\int_{-\pi}^{\pi} w(\vartheta)|u_0(\vartheta)|^2\,d\vartheta$$

for any trigonometric polynomial $u_0(\vartheta)$ *without constant term.*

(Hint: It is enough to do this for *real-valued* $u_0(\vartheta)$. Given such a one, use

$$f(\vartheta) = \tilde{u}_0(\vartheta) - iu_0(\vartheta)$$

in result from (e).)

(g) Show that

$$\int_{-\pi}^{\pi} \sigma(\vartheta) w(\vartheta) |\tilde{u}(\vartheta)|^2\,d\vartheta \leqslant C\int_{-\pi}^{\pi} w(\vartheta)|u(\vartheta)|^2\,d\vartheta$$

for *general* trigonometric polynomials $u(\vartheta)$, where C is a suitable constant. (Hint: If $u_0(\vartheta)$ denotes $u(\vartheta)$ minus its constant term, $\tilde{u}(\vartheta) = \tilde{u}_0(\vartheta)$. Use result from (a) to show that

$$\int_{-\pi}^{\pi} |u_0(\vartheta)|^2 w(\vartheta)\,d\vartheta \quad \leqslant \quad \text{const.} \int_{-\pi}^{\pi} |u(\vartheta)|^2 w(\vartheta)\,d\vartheta. \;)$$

(*h) Show that

$$\int_{-\pi}^{\pi} \log\big(\sigma(\vartheta)w(\vartheta)\big)\,d\vartheta \quad > \quad -\infty.$$

(Hint: Look at the proof of the last theorem in article 3; here $1/w(\vartheta)\Omega(e^{i\vartheta})$ already lies on the circle with diameter $[0, 1]$ for almost all $\vartheta \in [-\pi, \pi]$. Argument uses some H_p space theory for the unit disk.)

The result established in this problem was generalized to the case of weighted L_p norms ($1 < p < \infty$) by Carleson and Jones and, using a method different from theirs, by Rubio de Francia*. Related investigations have been made by Arocena, Cotlar, Sadoski, and their co-workers. A general result for operators in Hilbert space is due to Treil.

F. Relation of material in preceding § to the geometry of unit sphere in L_∞/H_∞

Combination of the second theorem in §E.3 with the one from §E.4 shows immediately that if we take any outer function $\varphi \in H_2$, the *existence* of a non-zero entire function f of exponential type $\leqslant a$ making

$$\int_{-\infty}^{\infty} \frac{|f(t)|^2}{(1+t^2)|\varphi(t)|^2}\,dt \quad < \quad \infty$$

is *equivalent* to that of a *non-zero* $h \in H_\infty$ such that

$$\left| e^{2iat}\frac{\overline{\varphi(t)}}{\varphi(t)} - h(t) \right| \quad \leqslant \quad 1 \qquad \text{a.e..}$$

* Chapter VI of his recent book with Garcia-Cuerva has in it a rather general treatment of the corresponding question about singular integrals on $\mathbb{R}^n$.

The second of these two conditions has an interpretation in terms of the quotient space L_∞/H_∞ that deserves mention; we look at it briefly in the present §.

The space L_∞/H_∞ has already appeared in §E.3; as explained there, its elements are the *cosets*

$$\psi + H_\infty,$$

where ψ ranges over L_∞. Instead of the *ad hoc* norm $\| \;\|_\infty^\sigma$ employed for such cosets in §E.3, we will here use the standard one to which $\| \;\|_\infty^\sigma$ reduces when $\sigma = 0$, viz.

$$\|\psi + H_\infty\|_\infty = \inf\{\|\psi + h\|_\infty : \; h \in H_\infty\}.$$

Equipped with $\| \;\|_\infty$, L_∞/H_∞ becomes a Banach space, and we denote by Σ the *unit sphere* (unit ball) of that space; that is simply the collection of cosets $\psi + H_\infty$ for which

$$\|\psi + H_\infty\|_\infty \leqslant 1.$$

In §E.3, essential use was made of the fact that L_∞/H_∞ is the *dual* of H_1; now we observe that this makes Σ w^* *compact* by what boils down to Tychonoff's theorem.

A member P of Σ is called an *extreme point* of Σ if, whenever

$$P = \lambda Q + (1-\lambda)R$$

with Q and R in Σ and $0 < \lambda < 1$, we *must have* $Q = R = P$. Geometrically, this means that *there cannot be any straight segment lying in* Σ *and passing through* P (i.e., with P *strictly between* its endpoints).

If $0 < \|P\|_\infty < 1$ it is clear that P *cannot* be an extreme point of Σ; the *zero coset* cannot be one either, for, since $L_\infty \neq H_\infty$, Σ contains cosets P and $-P$ with $P \neq 0 + H_\infty$. Any extreme points that Σ can have must thus be included among the set of P with $\|P\|_\infty = 1$ which we may refer to as the *surface* of Σ. Knowledge about Σ's extreme points can be used to gain insight into the geometrical structure of that surface. Such an approach is familiar to functional analysts, and one may get an idea of some of its possibilities by consulting the *Proceedings* of the A.M.S. symposium on convexity. Phelps' beautiful little book is also recommended.

The convexity and w^* compactness of Σ ensure that it has lots of extreme points according to the celebrated Krein–Milman theorem. So many, in fact, that Σ is their w^* closed convex hull. From a theorem of Bishop and Phelps (about which more later) we can furthermore deduce a much stronger result in the present circumstances: the extreme points of Σ are actually $\| \ \|_\infty$ dense on its surface. We may thus think of that surface as being 'filled out, for all practical purposes' by Σ's extreme points.

For this very reason, it seems of interest to have a procedure for exhibiting points on Σ's surface which are *not* extreme points of Σ. An outer function $\varphi \in H_2$ satisfying *either* (and hence *both*) of the two conditions set down above will frequently *give* us such a point, thanks to the following simple

Lemma. *Let* $|u(t)| \equiv 1$ *a.e.. Then* $u + H_\infty$ *is an extreme point of* Σ, *the unit sphere of* L_∞/H_∞, *iff there is* no non-zero $h \in H_\infty$ *for which*

$$|u(t) + h(t)| \leqslant 1 \quad \text{a.e.}.$$

Proof. Since $\|u\|_\infty = 1$, $u + H_\infty$ is certainly in Σ.

Suppose in the first place that $u + H_\infty$ is *not* an extreme point of Σ, then there are two *different* cosets $v_1 + H_\infty$, $v_2 + H_\infty$, both of norm $\leqslant 1$, and a λ, $0 < \lambda < 1$, with

$$u + H_\infty = \lambda(v_1 + H_\infty) + (1-\lambda)(v_2 + H_\infty).$$

According to a result proved in §E.3 (recall that $\| \ \|_\infty^\sigma$ is just $\| \ \|_\infty$ when $\sigma = 0$!),

$$\inf\{\|v_1 + h\|_\infty : \ h \in H_\infty\} = \|v_1 + H_\infty\|_\infty$$

is actually realized for some $h \in H_\infty$. Therefore, since $v_1 + h + H_\infty = v_1 + H_\infty$, there is no loss of generality in assuming that $\|v_1\|_\infty \leqslant 1$. Similarly, we may suppose that $\|v_2\|_\infty \leqslant 1$.

The previous relation means that there is some $h_0 \in H_\infty$ for which

$$u + h_0 = \lambda v_1 + (1-\lambda)v_2.$$

Here, the right side has norm $\leqslant 1$. Therefore

$$\|u + h_0\|_\infty \leqslant 1.$$

In this relation, however, h_0 cannot be zero. Indeed, assuming it *were*, we would have

$$u(t) = \lambda v_1(t) + (1-\lambda)v_2(t) \quad \text{a.e.},$$

with $0 < \lambda < 1$, $|u(t)| = 1$, and $|v_1(t)| \leqslant 1$, $|v_2(t)| \leqslant 1$. *Strict convexity* of the unit circle would then make $v_1(t) = v_2(t)$ a.e., so the cosets $v_1 + H_\infty$ and $v_2 + H_\infty$ would be *equal*, contrary to our initial assumption.

We thus have a *non-zero* $h_0 \in H_\infty$ such that

$$|u(t) + h_0(t)| \leqslant 1 \quad \text{a.e.},$$

and our lemma is proved in one direction.

Going the other way, assume that there *is* a non-zero $h \in H_\infty$ such that $\|u + h\|_\infty \leqslant 1$. Put then

$$\sigma(t) = 1 - |u(t) + \tfrac{1}{2}h(t)|;$$

we have

$$0 \leqslant \sigma(t) \leqslant 1 \quad \text{a.e.},$$

and see, as in proving *sufficiency* for the second theorem of §E.3, that $\sigma(t) > 0$ *on a set of positive measure.*

We now have

$$|u(t) + \sigma(t) + \tfrac{1}{2}h(t)| \leqslant 1 \quad \text{a.e.}$$

and

$$|u(t) - \sigma(t) + \tfrac{1}{2}h(t)| \leqslant 1 \quad \text{a.e.},$$

so, since

$$u + H_\infty = \tfrac{1}{2}(u + \sigma + H_\infty) + \tfrac{1}{2}(u - \sigma + H_\infty),$$

it will follow that $u + H_\infty$ is *not* an extreme point of Σ as long as the two cosets on the right are *different*, i.e., as long as $\sigma \notin H_\infty$.

If, however, $\sigma \in H_\infty$,

$$\sigma(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z - t|^2}\sigma(t)\,\mathrm{d}t$$

is *analytic* in $\Im z > 0$ by §E.1; it is, at the same time, *real* there, since $\sigma(t)$ is real. This makes $\sigma(z)$ *constant* and hence $\sigma(t)$, equal a.e. to $\lim_{y\to 0}\sigma(t + \mathrm{i}y)$, *also constant.*

It thus follows from our present assumption that $u + H_\infty$ is *not* an extreme point of Σ, save perhaps in the case where $\sigma(t)$ is constant. *But then $u + H_\infty$ cannot be an extreme point either*, for the constant must be > 0, $\sigma(t)$ being > 0 on a set of positive measure. We have, in other words,

$$|u(t) + \tfrac{1}{2}h(t)| = 1 - c \quad \text{a.e.}$$

where $c > 0$, so $\|u + H_\infty\|_\infty < 1$. As already noted, such a coset $u + H_\infty$ is not an extreme point of Σ.

The lemma is proved.

This result and the equivalence noted at the beginning of the present § yield without further ado the

Theorem. *Let φ be an outer function in H_2. Given $a > 0$, the coset*

$$e^{2iat}\frac{\overline{\varphi(t)}}{\varphi(t)} \quad + \quad H_\infty$$

fails *to be an extreme point of Σ, the unit sphere in L_∞/H_∞, iff*

$$\int_{-\infty}^{\infty} \frac{|f(t)|^2}{(1+t^2)|\varphi(t)|^2}\,dt \quad < \quad \infty$$

for some non-zero *entire function f of exponential type $\leqslant a$.*

From the theorem we have the following recipe for obtaining points on Σ's surface that are *not* extreme points of Σ: take any outer $\varphi \in H_2$ such that, for some $a > 0$, an entire $f \not\equiv 0$ of exponential type $\leqslant a$ satisfying the relation in the statement exists. Then the point

$$P \quad = \quad e^{2iat}\frac{\overline{\varphi(t)}}{\varphi(t)} \quad + \quad H_\infty$$

will have the property in question as long as $\|P\|_\infty = 1$.

It will indeed *frequently happen* that $\|P\|_\infty = 1$. Should that *fail* to come about, in which case

$$\|(e^{2iat}\overline{\varphi(t)}/\varphi(t)) \; + \; H_\infty\|_\infty \;<\; 1,$$

the relation

$$\int_{-\infty}^{\infty} |\tilde{U}(t)|^2|\varphi(t)|^2\,dt \quad \leqslant \quad \text{const.}\int_{-\infty}^{\infty} |U(t)|^2|\varphi(t)|^2\,dt$$

will in fact hold for the sums

$$U(t) \quad = \quad \sum_{\lambda \geqslant a} (A_\lambda \cos \lambda t + B_\lambda \sin \lambda t)$$

according to the second theorem of §E.3. This can only occur for rather special φ in H_2, closely related to the entire functions of exponential type $\leqslant a$ of a particular kind, *integrable* on the real axis but at the same time *not too small* there. The possibility may be fully investigated by the method used in proving the Helson–Szegő theorem. In spite of the matter's relevance to the study of various questions, we cannot go further into it here; very similar material is taken up in the paper of Hruščev, Nikolskii and Pavlov.

Except in the circumstance just mentioned, an outer function $\varphi \in H_2$ will satisfy

$$\|(e^{2iat}\overline{\varphi(t)}/\varphi(t)) + H_\infty\|_\infty = 1.$$

Use of the above procedure with such φ can lead to interesting examples even when only the simple Paley–Wiener multiplier theorem from §A.1 is called on. The reader is encouraged to carry out one or two constructions in such fashion.

For any coset $u + H_\infty$ lying on Σ's surface but not an extreme point of Σ one has a beautiful parametric representation, due to Adamian, Arov and Krein, of the functions $h \in H_\infty$ with $\|u + h\|_\infty = 1$. This was first obtained by means of operator theory, but Garnett has since found an easier function-theoretic derivation, given in his book.*

The work in §E was originally done at the end of the 1960s, in hopes that the connection established by the last theorem would make possible a *proof* of the Beurling–Malliavin multiplier theorem (stated in §A.2) *based on* Banach space and Banach algebra techniques. That approach did not work, and I think now that it is probably not feasible. Whatever value the result may have seems rather to lie in the possibility of its helping us understand the structure of L_∞/H_∞ *when used with multiplier theorems* to construct various examples, according to the above scheme.

The quotient space L_∞/H_∞ is the foundation for the theory of Hankel and Toeplitz forms. Study of these is not really part of this book's subject matter, and the present § is included merely to show some ways of applying multiplier theorems therein. The reader interested in that study should first of all consult Sarason's Blacksburg notes. Wishing to go further, he or she should next take up the papers of that author and his co-workers, perusing, at the same time, a book on H_p spaces so as to get a good grounding in their theory. It is then essential that one become familiar with the remarkable papers of Adamian, Arov and Krein. Those make heavy use of Hilbert space operator theory.

There is, in general, much mutual interplay between operator theory and the investigation of L_∞/H_∞; so vast, indeed, is the region common to these two fields that it seems hopeless to try to furnish even sketchy references here. Let us at least mention the so-called *Nagy–Foiaş model*; the book about it by those two authors is well known. For more recent

* the one on bounded analytic functions

treatments, see Nikolskii's book and (especially) his survey article with Hruščev.

Before closing this § and the present chapter, let us see how the extreme points of Σ are related to its *support points*, to be defined in a moment. As we saw in §E.3, each coset $u + H_\infty$ in L_∞/H_∞ corresponds to a *linear functional* Λ on H_1 given by the formula

$$\Lambda(f) = \int_{-\infty}^{\infty} u(t)f(t)\,dt, \qquad f \in H_1,$$

and the *supremum* of $|\Lambda(f)|$ for the $f \in H_1$ with $\|f\|_1 \leqslant 1$ is equal to $\|u + H_\infty\|_\infty$. (The reader is again reminded that the norm $\| \ \|_\infty^\sigma$ used in §E.3 reduces to $\| \ \|_\infty$ when $\sigma(t) \equiv 0$.) We know that there *is* some $h \in H_\infty$ for which $\|u + h\|_\infty = \|u + H_\infty\|_\infty$; *that, however, does not mean that there need be an* $f \in H_1$ *of norm* 1 *with* $\Lambda(f) = \|u + H_\infty\|_\infty$. Since the space H_1 is not reflexive there is no reason why this should be the case; it is in fact *true* for *some* cosets $u + H_\infty$ and *false* for *others.*

Definition. A coset $u + H_\infty$ with $\|u + H_\infty\|_\infty = 1$ is said to be a *support point* for Σ if there is an $f \in H_1$ with

$$\int_{-\infty}^{\infty} u(t)f(t)\,dt = \|f\|_1 = 1.$$

There is then the

Theorem. *A support point of* Σ *is an extreme point of* Σ.

Proof. Let $u + H_\infty$ be a support point of Σ. There is, as we know, a $v \in u + H_\infty$ with $\|v\|_\infty = \|u + H_\infty\|_\infty = 1$; it is enough to show that such a v is of modulus 1 a.e. and *uniquely determined*, for then there can be *no* non-zero $h \in H_\infty$ with $\|v + h\|_\infty \leqslant 1$, and $u + H_\infty = v + H_\infty$ must hence be an extreme point of Σ by the previous lemma.

There is by definition an $f \in H_1$ with

$$\int_{-\infty}^{\infty} v(t)f(t)\,dt = \int_{-\infty}^{\infty} u(t)f(t)\,dt = \|f\|_1 = 1.$$

Here, $|v(t)| \leqslant 1$ a.e., so we must have

$$v(t)f(t) = |f(t)| \quad \text{a.e.}.$$

However, $|f(t)| > 0$ a.e. by §E.1, so we get

$$v(t) = \frac{|f(t)|}{f(t)} \quad \text{a.e.,}$$

which determines v makes it of modulus 1 a.e.. Done.

The *converse* of this theorem is *not true*; the example provided by problems 44 and 46 given below shows that. It is, however, true that the *surface* of Σ is *full of support points*; those are, indeed, $\| \quad \|_\infty$ dense on that surface according to a remarkable theorem, due to Bishop and Phelps, whose proof may be found in the above mentioned A.M.S. volume on convexity. Since all these support points are extreme points by the result just obtained, it follows that *the extreme points of* Σ are $\| \quad \|_\infty$ *dense on its surface.* This is a much higher concentration of extreme points than could be surmised from the Krein–Milman theorem. Still, there are lots of points on Σ's surface that are *not* extreme, and we have seen how to find many of them.

When a particular $u \in L_\infty$ with $\|u + H_\infty\|_\infty = 1$ is given, it is hard to tell by just looking at the qualitative behaviour of the function $u(t)$ whether $u + H_\infty$ is a support point of Σ or not. About all that is known *generally* is that $u + H_\infty$ must then be a support point if $u(t)$ is *continuous* on $\mathbb{R}$ *and tends to equal limits for* $t \longrightarrow \infty$ *and* $t \longrightarrow -\infty$. Proof of this fact, which depends on the F. and M. Riesz theorem, may be found in the more recent books about H_p spaces. If we merely require *uniform continuity* of $u(t)$ on $\mathbb{R}$, the conclusion may cease to hold. An example of this will be furnished by problems 44 and 46.

To work the following problems, a generalization of the Schwarz reflection principle due to Carleman will be needed. Suppose that we have a rectangle $\mathscr{D}_0$ in the upper half plane whose *base* is a segment of the real axis:

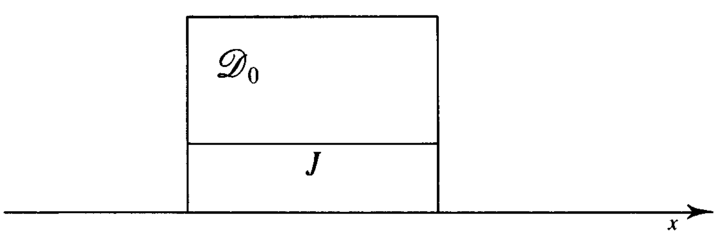


Figure 228

Carleman's result deals with functions $F(z)$ analytic in $\mathscr{D}_0$ for which

$$\int_J |F(z)|\,|\mathrm{d}z| \quad \leqslant \quad \text{const.}$$

for all the *horizontal* line segments J running across the interior of $\mathscr{D}_0$. These functions are just those of the class $\mathscr{S}_1(\mathscr{D}_0)$ studied in §B.4, Chapter VII. The reader should refer again to that §. According to the first theorem proved there, when $F \in \mathscr{S}_1(\mathscr{D}_0)$, $\lim_{y\to x} F(x+iy)$ exists for almost every x on the base of $\mathscr{D}_0$. Following standard practice, that limit is denoted by $F(x)$.

Lemma (Carleman). *If* $F \in \mathscr{S}_1(\mathscr{D}_0)$ *and* $F(x)$ *is* real *a.e. along the base of* $\mathscr{D}_0$, $F(z)$ *can be analytically continued across that base into* $\mathscr{D}_0^*$, *the reflection of* $\mathscr{D}_0$ *in the real axis, by putting* $F(z) = \overline{F(\bar{z})}$ *for* $z \in \mathscr{D}_0^*$.

Proof. Let I be any segment properly included in the base of $\mathscr{D}_0$ in the manner shown in the following figure, and take any rectangle $\mathscr{D}$, entirely contained in $\mathscr{D}_0$, having I as its base.

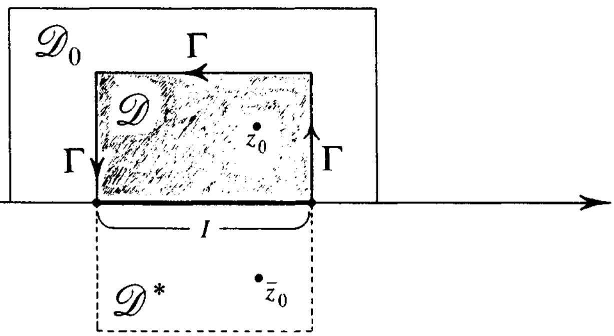


Figure 229

If $z_0 \in \mathscr{D}$, $1/(\zeta - \bar{z}_0)$ is analytic in ζ for ζ in $\mathscr{D}$, so, by the corollary to the *second* theorem of §B.4, Chapter VII,

$$\int_{\partial\mathscr{D}} \frac{F(\zeta)}{\zeta - \bar{z}_0}\,\mathrm{d}\zeta \quad = \quad 0;$$

here the integral is absolutely convergent according to the third lemma and first theorem* of that §.

The function $(F(z) - F(z_0))/(z - z_0)$ clearly belongs to $\mathscr{S}_1(\mathscr{D}_0)$ if F does.

* In Fig. 69, accompanying the proof of that theorem (p. 287 of vol I), B_1 and B_2 should have designated the horizontal sides of $\mathscr{D}_0$ and not of $\mathscr{D}$.

Hence, by the corollary just used, we also have

$$\int_{\partial\mathscr{D}} \frac{F(\zeta) - F(z_0)}{\zeta - z_0} \, d\zeta \quad = \quad 0,$$

from which the Cauchy formula

$$F(z_0) \quad = \quad \frac{1}{2\pi i} \int_{\partial\mathscr{D}} \frac{F(\zeta)}{\zeta - z_0} \, d\zeta$$

immediately follows on using the relation

$$\int_{\partial\mathscr{D}} \frac{d\zeta}{\zeta - z_0} \quad = \quad 2\pi i.$$

Combining our formula for $F(z_0)$ with the one preceding it, and then dropping the subscript on z, we get

$$F(z) \quad = \quad \frac{1}{2\pi i} \int_{\Gamma} \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - \bar{z}} \right) F(\zeta) \, d\zeta \; + \; \frac{1}{\pi} \int_I \frac{\Im z}{|\xi - z|^2} F(\xi) \, d\xi, \quad z \in \mathscr{D},$$

where Γ is the path consisting of the *top* of $\mathscr{D}$ together with its *two vertical sides* (see figure).

Of the two integrals on the right, the *first* certainly represents a complex-valued *harmonic* (*not* analytic!) function of z in any region disjoint from both Γ and its reflection in the real axis. That expression is, in particular, harmonic in the rectangle $\mathscr{D} \cup I \cup \mathscr{D}^*$, where $\mathscr{D}^*$ denotes the reflection of $\mathscr{D}$ in $\mathbb{R}$; its *imaginary part*, $V(z)$, is thus also harmonic in $\mathscr{D} \cup I \cup \mathscr{D}^*$.

The function $\Im F(z)$, harmonic in $\mathscr{D}$, is *equal* there to

$$V(z) \quad + \quad \frac{1}{\pi} \int_I \frac{\Im z}{|\xi - z|^2} \Im F(\xi) \, d\xi;$$

this, however, is just $V(z)$, since $\Im F(\xi) \; = \; 0$ a.e. on I by hypothesis. The function $\Im F(z)$ therefore has a *harmonic continuation* from $\mathscr{D}$ to the larger rectangle $\mathscr{D} \cup I \cup \mathscr{D}^*$. Its *harmonic conjugate*, $-\Re F(z)$, can thus also be continued harmonically into all of $\mathscr{D} \cup I \cup \mathscr{D}^*$, and then we obtain an *analytic continuation* of $F(z)$ into that larger rectangle by putting $F(z) \; = \; \Re F(z) \; + \; i\Im F(z)$.

This means, in particular, that $F(z)$ is *continuous* at the points of I (save perhaps at the endpoints). By hypothesis, however, $F(x)$ is *real* a.e. on I. Therefore it is *real everywhere* on I, besides being *continuous there*. Now we can apply the *classical* Schwarz reflection principle to conclude that $F(\bar{z}) \; = \; \overline{F(z)}$ for $z \in \mathscr{D}$.

Our choices of I, properly contained in $\mathscr{D}_0$'s base, and of $\mathscr{D}$, entirely included in $\mathscr{D}_0$, were arbitrary. The formula $F(\bar{z}) = \overline{F(z)}$ thus gives us an analytic continuation of F across the *whole* base of $\mathscr{D}_0$ into all of $\mathscr{D}_0^*$. Done.

Problem 44

Let Λ be any measurable sequence of distinct integers > 0, having (ordinary) density $D_\Lambda < \frac{1}{2}$ (refer to §E.3, Chapter VI).

Write

$$C(z) = \prod_{n\in\Lambda}\left(1 - \frac{z^2}{n^2}\right),$$

and then put

$$B(z) = \frac{C(z-\mathrm{i})}{C(z+\mathrm{i})};$$

$B(z)$ is just a *Blaschke product* for the upper half plane (see §G.3, Chapter III), having *zeros* at the points $\pm n + \mathrm{i}$, $n\in\Lambda$, and *poles* at $\pm n - \mathrm{i}$, $n\in\Lambda$. Consider the function

$$u(t) = \frac{\mathrm{e}^{\pi i t}}{B(t)},$$

of modulus 1 on $\mathbb{R}$.

(a) Show that $u(t)$ is uniformly continuous on $\mathbb{R}$. (Hint: $B(t)$ is of the form $\exp(\mathrm{i}\varphi(t))$ where $\varphi(t)$ is real. Express $\varphi'(t)$ in terms of the $n\in\Lambda$.)

(b) Assume that $u + H_\infty$ *is* a support point of Σ; this means that there *is* an $f\in H_1$ with $\int_{-\infty}^{\infty} u(t)f(t)\,\mathrm{d}t = \|f\|_1 = 1$. For $\Im z > 0$, write, as usual,

$$f(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} f(t)\,\mathrm{d}t,$$

and then put

$$F(z) = \mathrm{e}^{\pi i z}(C(z+\mathrm{i}))^2 f(z)$$

in the upper half plane. *Show that* $F(z)$ *can be continued analytically across* $\mathbb{R}$ by putting $F(\bar{z}) = \overline{F(z)}$. (Hint: Use Carleman's lemma.)

(c) Show that the entire function $F(z)$ obtained in (b) is of *exponential type*. (Hint: See proof of the first theorem in §F.4, Chapter VI.)

(d) Hence obtain a *contradiction with the assumption made in* (b) by showing that $F(z)$ must be identically zero. (Hint: Look at the behaviour of $F(z)$ on the imaginary axis, referring to problem 29(a) from §B.1 of Chapter IX.)

By making the right choice of the sequence Λ, various interesting examples can be obtained. We need another lemma, best given as

Problem 45

(a) Let $g(w)$ be analytic in $\{|w| < 1\}$, with $\Re g(w) \geqslant 0$ there. Show that for any $p < 1$, the integrals

$$\int_{-\pi}^{\pi} |g(re^{i\vartheta})|^p \, d\vartheta$$

are bounded for $r < 1$. (Hint: By the principle of conservation of domain, $g(w)$ can never be zero for $|w| < 1$, so we can define an analytic and single valued branch of $(g(w))^p$ there. Apply Cauchy's formula to the latter to get $(g(0))^p$, then take real parts and note that $\cos(p \arg g(w))$ is bounded away from 0.)

(b) If $g(w)$ is as in (a), show that $\lim_{r\to 1} g(re^{i\tau}) = g(e^{i\tau})$ exists a.e., and that for any $p < 1$,

$$(g(w))^p = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-|w|^2}{|w-e^{i\tau}|^2}(g(e^{i\tau}))^p \, d\tau, \qquad |w| < 1,$$

the integral on the right being absolutely convergent. (Hint: Fix a p', $p < p' < 1$, and apply the result from (a) to $(g(w))^{p'}$. Then argue as in the proof of the first theorem from §F.1, Chapter III, using the duality between the spaces $L_r(-\pi, \pi)$ and $L_s(-\pi, \pi)$, where $r = p'/p$ and $(1/r) + (1/s) = 1$. This will yield a function $G(\tau)$ in $L_r(-\pi, \pi)$ such that

$$(g(w))^p = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-|w|^2}{|w-e^{i\tau}|^2} G(\tau) \, d\tau, \qquad |w| < 1.$$

Appeal to standard results about the Poisson integral to describe the boundary behaviour of $(g(w))^p$ and relate $G(\tau)$ thereto.)

(c) Given $v(t)$ defined on $\mathbb{R}$ with $|v(t)| \leqslant \pi/2$ there, consider the function

$$\psi(z) = \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{t-z} - \frac{t}{t^2+1}\right)v(t)\,dt,$$

analytic in $\Im z > 0$. As we know,

$$\lim_{y\to 0} \psi(t+iy) = \psi(t)$$

exists a.e. on $\mathbb{R}$, with $\psi(t) = -\tilde{v}(t) + iv(t)$ a.e. there, $\tilde{v}(t)$ being the *first* kind of Hilbert transform described at the beginning of §E.2. *Show that, when* $p < 1$,

$$\frac{e^{p\psi(t)}}{(t+i)^2}$$

belongs to H_1. (Hint: By mapping the upper half plane conformally onto the unit disk and using the result from (b), show first of all that when $p < 1$,

$$e^{p\psi(z)} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} e^{p\psi(t)}\,dt$$

for $\Im z > 0$, the integral on the right being absolutely convergent. This representation gives us fairly good control on the *size* of $\exp(p\psi(z))$ in $\Im z > 0$ – cf. Chapter VI, §A.2 – A.3. Knowing this, show by integrating around suitable contours that if λ and $\delta > 0$,

$$\int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{(x+i)^{2+\delta}} \exp(p\psi(x+ih)\,dx = 0$$

for any $h > 0$ – cf. proof of theorem that the product of two H_2 functions is in H_1, §E.1. Now one may make $\delta \longrightarrow 0$ and use dominated convergence (*guaranteed* by our representation for $\exp(p\psi(z))$!) to get

$$\int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{(x+i)^{2}} \exp(p\psi(x+ih))\,dx = 0.$$

Plug the representation for $\exp(p\psi(z))$ into this result and use Fubini's theorem, noting that $e^{i\lambda t}/(t+i)^2$ is in H_∞. Finally, make $h \longrightarrow 0$.)

Let us now take a measurable sequence Λ of integers > 0 *having* (ordinary) *density zero*, whose Beurling–Malliavin *effective density* $\tilde{D}_\Lambda$ *is equal* to 1 (see §D.2 of Chapter IX). It is easy to construct such sequences. We need merely pick intervals $[a_k, b_k]$ with integral endpoints,

$$0 < a_1 < b_1 < a_2 < b_2 < a_3 < \cdot\cdot\cdot,$$

such that b_{k-1}/b_k and $(b_k - a_k)/b_k$ both tend to zero as $k \longrightarrow \infty$, while

$$\sum_1^\infty \left(\frac{b_k - a_k}{b_k}\right)^2 = \infty,$$

and then have Λ consist of the integers in the $[a_k, b_k]$.

Using such a sequence Λ, let us form the functions $C(z)$, $B(z)$ and $u(t)$ considered in problem 44. Then,

$u + H_\infty$ is an extreme point of Σ even though it is not a support point thereof.

This will follow from problem 44, the first lemma of the present §, and

Problem 46

To show that there can be *no* non-zero $h \in H_\infty$ with

$|u(t) - h(t)| \leqslant 1$ a.e..

(a) Assuming that there *is* such an h, show how to construct a function $v(t)$ defined a.e. on $\mathbb{R}$, with $|v(t)| \leqslant \pi/2$ and

$e^{-\pi i t}B(t)h(t)^{iv(t)} \geqslant 0$ a.e..

(b) Using the v found in (a), form $\psi(t) = -\tilde{v}(t) + iv(t)$ as in problem 45(c). Show that in the present circumstances,

$$\int_{-\infty}^{\infty} \frac{|h(t)\exp\psi(t)|}{1+t^2}\,dt < \infty.$$

(Hint: From the solution of problem 45(c),

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\exp(p\psi(t))}{1+t^2}\,dt = \exp(p\psi(i))$$

whenever $p < 1$. Take real parts and make $p \longrightarrow 1$; then what we have on the right tends to the *finite* value $\Re\exp\psi(i)$, so that

$$\int_{-\infty}^{\infty} \frac{e^{-\tilde{v}(t)}\cos v(t)}{1+t^2}\,dt \leqslant \pi\Re e^{\psi(i)} < \infty$$

by Fatou's lemma. If, however, we write

$g(t) = e^{-\pi i t}B(t)h(t)$,

we have the following diagram:

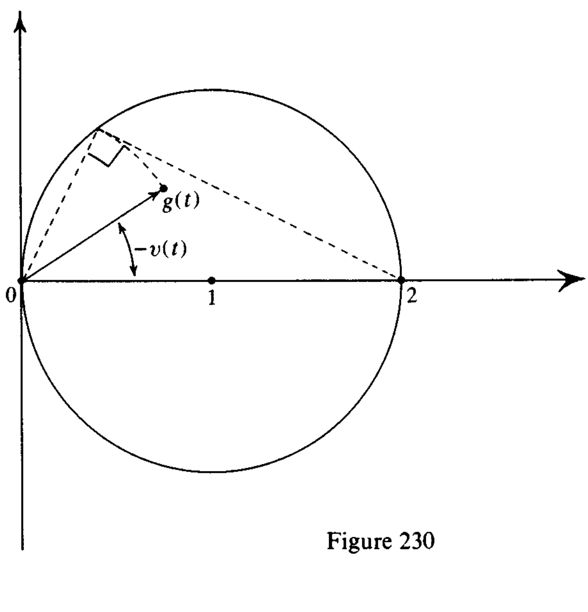


Figure 230

From this it is clear that $|g(t)| \leqslant 2\cos v(t)$.)

(c) Continuing with the notation used in (b), show that

$$\frac{B(t)h(t)\exp\psi(t)}{(t+\mathrm{i})^2}$$

belongs to H_1. (Hint: $Bh \in H_\infty$ and, for each $p < 1$, $\exp(p\psi(t))/(t+\mathrm{i})^2$ is in H_1 by problem 45(c). Thus, if $\lambda \geqslant 0$, we have

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\lambda t}\frac{B(t)h(t)\exp(p\psi(t))}{(t+\mathrm{i})^2}\mathrm{d}t \quad = \quad 0$$

by §E.1, whenever $p < 1$. In this relation, we may let $p \longrightarrow 1$ and use dominated convergence, referring to the result from (b).)

(d) Show that the function

$$F(z) \quad = \quad \mathrm{e}^{-\pi \mathrm{i}z}B(z)h(z)\mathrm{e}^{\psi(z)},$$

analytic in the upper half-plane, can be continued across the real axis, *yielding an entire function of exponential type* $\leqslant \pi$, by putting

$$F(\bar{z}) \quad = \quad \overline{F(z)}.$$

Here, as usual,

$$h(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2}h(t)\,\mathrm{d}t, \qquad \Im z \; > \; 0.$$

(Hint: See problem 44 again.)

(e) Hence show that the function $F(z)$ from (d) *is identically zero*, so that *in fact* $h(z) \equiv 0$, proving that the assumption made in (a) is *untenable*. (Hint: First apply the Riesz–Fejér theorem from §G.3 of Chapter III to get an entire function $f(z)$ of exponential type $\leqslant \pi/2$, *vanishing* at each of the points $\pm n + \mathrm{i}$, $n \in \Lambda$, such that

$$F(z) \quad = \quad f(z)\overline{f(\bar{z})}.$$

$f(z+\mathrm{i})$ then certainly vanishes at each point of Λ. Use the fact that $\tilde{D}_\Lambda = 1$, applying a suitable variant of the theorem quoted at the very beginning of §E, Chapter IX.)

Problem 47

Let $u(t)$ be as in problem 46. Show that $\bar{u} + H_\infty$ (*sic*!) is an extreme point, but not a support point, of Σ.

Remark. When Carleman's lemma is used in the above problems, one is actually dealing with functions $F \in \mathscr{S}_1(\mathscr{D}_0)$ whose boundary values are *positive* (and not merely real) along the base of $\mathscr{D}_0$. In this circumstance, analytic continuation across the base of $\mathscr{D}_0$ is possible under a *weaker* condition on F than that of membership in $\mathscr{S}_1(\mathscr{D}_0)$. It is enough that F belong to the space $H_{1/2}$ associated with each of the smaller rectangles $\mathscr{D}$ used in the proof of the lemma. That fact follows easily from an argument due to Neuwirth and Newman, and, independently, to Helson and Sarason. The reader is referred to the discussion accompanying problem 13 at the end of Chapter II in Garnett's book.*

* *Bounded Analytic Functions*

XI

Multiplier Theorems

It is time to prove the multiplier theorem stated in §A.2 of the preceding chapter and then applied there, in §§B and C. We desire also to establish another result of the same kind and finally to start working towards a *description* of the weights $W(x) \geqslant 1$ that *admit multipliers* (in the sense explained at the beginning of Chapter X). All this will require the use of some elementary material from potential theory.

There is a dearth of modern expositions of that theory accessible to readers having only a general background in analysis. Moreover, the books on it that do exist* are not so readily available. It therefore seems advisable to first explain the basic results we will use from the subject without, however, getting involved in any attempt at a systematic treatment of it. That is the purpose of the first § in this chapter. Other more special potential-theoretic results called for later on will be formulated and proved as they are needed.

A Some rudimentary potential theory

1. Superharmonic functions; their basic properties

A function $U(z)$ harmonic in a domain $\mathscr{D}$ enjoys the *mean value property* there: for $z \in \mathscr{D}$,

$$U(z) \quad = \quad \frac{1}{2\pi}\int_0^{2\pi} U(z+\rho e^{i\vartheta})\,d\vartheta, \qquad 0 \;<\; \rho \;<\; \operatorname{dist}(z,\,\partial\mathscr{D}).$$

* The books by Carleson, Tsuji, Kellogg, Helms and Landkof are in my possession, together with a copy of Frostman's thesis; most of the time I have been able to make do with just the *first three* of these.

To Gauss is due the important *converse* of this statement: among the functions $U(z)$ continuous in $\mathscr{D}$, the mean value property *characterizes* the ones harmonic there. The proof of this contains a key to the understanding of much of the work with superharmonic functions (defined presently) to concern us here; let us therefore recall how that proof goes.

An (apparently) more general result can in fact be established by the same reasoning. Suppose that a function $U(z)$, continuous in a domain $\mathscr{D}$, enjoys a *local mean value property there*; in other words, that to each $z \in \mathscr{D}$ corresponds an r_z, $0 < r_z \leqslant \operatorname{dist}(z, \partial\mathscr{D})$ (with, *a priori*, $r_z < \operatorname{dist}(z, \partial\mathscr{D})$) such that

$$U(z) = \frac{1}{2\pi}\int_0^{2\pi} U(z+\rho e^{i\vartheta})\,d\vartheta \qquad \text{for } 0 < \rho < r_z.$$

It is claimed that $U(z)$ is then *harmonic in* $\mathscr{D}$.

The main part of the argument consists in showing that the local mean value property implies the *strong maximum principle* for U on (connected) domains with compact closures lying in $\mathscr{D}$. Letting Ω be any such domain, we have to verify that for $z \in \Omega$,

$$U(z) < \sup_{\zeta\in\partial\Omega} U(\zeta)$$

unless $U(z) \equiv$ const on $\bar{\Omega}$. Here, $U(z)$ has on $\bar{\Omega}$ a *maximum* – call it M – and the statement in question amounts to the assertion that $U(z) \equiv M$ on $\bar{\Omega}$ if, for any $z_0 \in \Omega$, $U(z_0) = M$.

Suppose there is such a z_0. Then, for each sufficiently small $\rho > 0$,

$$\frac{1}{2\pi}\int_0^{2\pi} U(z_0+\rho e^{i\vartheta})\,d\vartheta = U(z_0) = M$$

with $U(z_0 + \rho e^{i\vartheta})$ continuous in ϑ and $\leqslant M$. This makes $U(z_0+\rho e^{i\vartheta}) \equiv M$ for such ρ, so that $U(z) \equiv M$ in a *small disk* centered at z_0. The set

$$E = \{z_0 \in \Omega\colon\ U(z_0) = M\}$$

is thus *open*. That set is, however, *closed* in Ω's relative topology on account of the continuity of U. Hence $E = \Omega$ since Ω is connected, and $U(z) \equiv M$ in Ω – thus finally on $\bar{\Omega}$, thanks again to the continuity of U.

To complete the proof of Gauss' result, let us take any $z_0 \in \mathscr{D}$ and an $R < \operatorname{dist}(z_0, \partial\mathscr{D})$; it is enough to establish that

$$U(z_0+\rho e^{i\vartheta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-\rho^2}{R^2+\rho^2-2R\rho\cos(\vartheta-\tau)}U(z_0+Re^{i\tau})\,d\tau$$

for $0 \leqslant \rho < R$. Calling the expression on the right $V(z_0 + \rho e^{i\vartheta})$, we proceed to show first that

$$U(z) \leqslant V(z)$$

for $|z - z_0| < R$.

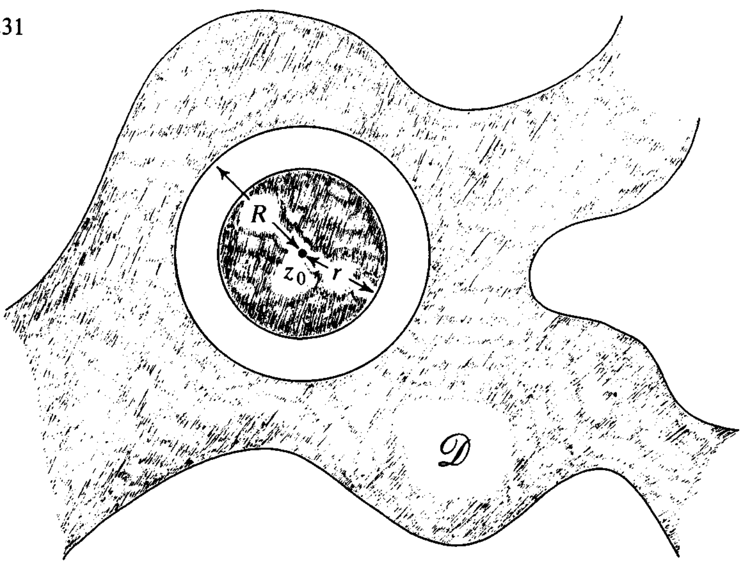
Fix any $\varepsilon > 0$. By continuity of U and the elementary properties of the Poisson kernel we know that

$$V(z_0 + re^{i\vartheta}) \longrightarrow U(z_0 + Re^{i\vartheta})$$

uniformly in ϑ for $r < R$ tending to R; the same is of course true if we replace V by U on the left. On the circles $|z - z_0| = r$ with radii $r < R$ sufficiently close to R we therefore have

$$U(z) - V(z) \leqslant \varepsilon.$$

Figure 231



Here, both $U(z)$ and the *harmonic* function $V(z)$ enjoy the local mean value property in the open disk $\{|z - z_0| < R\}$. Hence, by what has just been shown, we have the strong maximum principle for the *difference* $U(z) - V(z)$ on the smaller disks $\{|z - z_0| < r\}$. The preceding inequality thus implies that $U(z) - V(z) \leqslant \varepsilon$ on each of those disks, and finally that $U(z) - V(z) \leqslant \varepsilon$ for $|z - z_0| < R$. Squeezing ε, we see that

$$U(z) - V(z) \leqslant 0 \quad \text{for } |z - z_0| < R.$$

By working with the difference $V(z) - U(z)$ we can, however, prove the *reverse* inequality in the same fashion. This means that one must have $U(z) = V(z)$ for $|z - z_0| < R$, and our proof is finished. It is this *argument* that the reader will find helpful to keep in mind during the following development.

Next in importance to the harmonic functions as objects of interest in potential theory come those that are *subharmonic* or *superharmonic*. One can actually work exclusively with harmonic functions and the ones belonging to *either* of the last two categories; which of the latter is singled out makes very little difference. Logarithms of the moduli of analytic functions are subharmonic, but most writers on potential theory prefer (probably on account of the customary formulation of Riesz' theorem, to be given in article 2) to deal with *superharmonic functions*, and we follow their example here. The difference between the two kinds of functions is purely one of *sign*: a given $F(z)$ is *subharmonic* if and only if $-F(z)$ is *superharmonic*.

Definition. A function $U(z)$ defined in a domain $\mathscr{D}$ with $-\infty < U(z) \leqslant \infty$ there is said to be superharmonic in $\mathscr{D}$ provided that

(i) $\liminf_{z \to z_0} U(z) \geqslant U(z_0)$ for $z_0 \in \mathscr{D}$;

(ii) to each $z \in \mathscr{D}$ corresponds an r_z, $0 < r_z \leqslant \operatorname{dist}(z, \partial\mathscr{D})$, such that

$$\frac{1}{2\pi}\int_0^{2\pi} U(z + \rho e^{i\vartheta})\, d\vartheta \quad \leqslant \quad U(z) \qquad \text{for } 0 < \rho < r_z.$$

Superharmonic functions are thus permitted to assume the value $+\infty$ at certain points. Although authors on potential theory do not generally agree to call the function *identically equal* to $+\infty$ superharmonic, we will sometimes find it convenient to do so.

Assumption of the value $-\infty$, on the other hand, is not allowed. This restriction plays a serious rôle in the subject. By it, functions like

$$U(z) \quad = \quad \begin{cases} \Im z, & \Im z > 0, \\ -\infty, & \Im z \leqslant 0, \end{cases}$$

are excluded from consideration.

It may seem at first sight that an extensive theory could hardly be based on the definition just given. On thinking back, however, to the proof of

Gauss' result, one begins to suspect that the simple conditions figuring in the definition involve more structure than is immediately apparent. One notices, to begin with, that (i) and (ii) signify *opposite kinds* of local behaviour. The *first* guarantees that $U(z)$ *stays almost as large* as $U(z_0)$ on small neighborhoods of z_0, and the *second* gives us lots of points z in such neighborhoods at which $U(z) \leqslant U(z_0)$. Considerable use of the interplay between these two contrary effects will be made presently; for the moment, let us simply remark that together, they entail *equality* of $\liminf_{z \to z_0} U(z)$ and $U(z_0)$ at the $z_0 \in \mathscr{D}$.

It is probably best to start our work with superharmonic functions by seeing what can be deduced from the requirement that $U(z) > -\infty$ and condition (i), *taken by themselves*. The latter is nothing other than a prescription for *lower semicontinuity* in $\mathscr{D}$; as is well known, and easily verified by the reader, it implies that $U(z)$ has an *assumed minimum* on each compact subset of $\mathscr{D}$. Together with the requirement, that means that $U(z)$ *has a finite lower bound on every compact subset of* $\mathscr{D}$. This property will be used repeatedly. (I can never remember *which* of the two kinds of semicontinuity is *upper*, and which is *lower*, and suspect that some readers of this book may have the same trouble. That is why I systematically avoid using the *terms* here, and prefer instead to specify explicitly each time which behaviour is meant.)

A *monotonically increasing* sequence of functions continuous on a domain $\mathscr{D}$ tends to a limit $U(z) > -\infty$ satisfying (i) there. This is immediate; what is less apparent is a kind of *converse*:

Lemma. *If* $U(z) > -\infty$ *has property* (i) *in* $\mathscr{D}$ *there is, for any compact subset* K *of* $\mathscr{D}$, *a monotonically increasing sequence of functions* $\varphi_n(z)$ *continuous on* K *and tending to* $U(z)$ *there.*

Proof. For each $n \geqslant 1$ put, for $z \in K$,

$$\varphi_n(z) = \inf_{\zeta \in K}(U(\zeta) + n|z-\zeta|).$$

Since $U(\zeta)$ is bounded below on K by the above observation, the functions $\varphi_n(z)$ are all $> -\infty$. It is evident that $\varphi_n(z) \leqslant \varphi_{n+1}(z) \leqslant U(z)$ for $z \in K$ and each n.

To show continuity of φ_n at $z_0 \in K$, we remark that the function of ζ equal to $U(\zeta) + n|\zeta - z_0|$ enjoys, like $U(\zeta)$, property (i) and thus *assumes its minimum* on K. There is hence a $\zeta_0 \in K$ such that

$$\varphi_n(z_0) = n|\zeta_0 - z_0| + U(\zeta_0),$$

so, if $z \in K$,

$$\varphi_n(z) \leqslant n|z-\zeta_0| + U(\zeta_0) \leqslant n|z-z_0| + \varphi_n(z_0).$$

In the same way, we see that

$$\varphi_n(z_0) \leqslant n|z_0-z| + \varphi_n(z)$$

which, combined with the previous, yields

$$|\varphi_n(z)-\varphi_n(z_0)| \leqslant n|z-z_0| \quad \text{for } z_0 \text{ and } z \in K.$$

We proceed to verify that $\varphi_n(z_0) \xrightarrow[n]{} U(z_0)$ at each $z_0 \in K$. Given such a z_0, take any number $V < U(z_0)$. By property (i) there is an $\eta > 0$ such that $U(\zeta) > V$ for $|\zeta - z_0| < \eta$. $U(\zeta)$ has, as just recalled, a finite lower bound, say $-M$, on K. Then, for $n > (V+M)/\eta$, we have $n|\zeta - z_0| + U(\zeta) > V$ for $\zeta \in K$ with $|\zeta - z_0| \geqslant \eta$. But when $|\zeta - z_0| < \eta$ we also have $n|\zeta - z_0| + U(\zeta) > V$. Therefore

$$\varphi_n(z_0) \geqslant V \quad \text{for } n > (M+V)/\eta$$

Since, on the other hand, $\varphi_n(z_0) \leqslant U(z_0)$, we see that the convergence in question holds, $V < U(z_0)$ being arbitrary.

The lemma is proved.

Remark. This result figures in some introductory treatments of the Lebesque integral.

Let us give some examples of superharmonic functions. The class of these includes, to begin with, all the *harmonic* functions. Gauss' result implies indeed that a function $U(z)$ defined on a domain $\mathscr{D}$ is *harmonic* there *if and only if* both $U(z)$ and $-U(z)$ are *superharmonic* in $\mathscr{D}$. The simplest kind of functions $U(z)$ superharmonic, but not harmonic, in $\mathscr{D}$ are those of the form

$$U(z) = \log\frac{1}{|z-z_0|} \quad \text{with } z_0 \in \mathscr{D}.$$

Positive linear combinations of these are also superharmonic, and so, finally, are the expressions

$$U(z) = \int_K \log\frac{1}{|z-\zeta|}\,\mathrm{d}\mu(\zeta)$$

formed from positive measures μ supported on compact sets K. *The reader should not proceed further without verifying the last statement.* This involves