has no zeros if $\alpha = 2$, and an infinity of real zeros, but no complex zeros, if $\alpha = 4, 6, 8, \ldots$.

[We have, by § 8.47,

$$F_\alpha(z) = \frac{1}{\alpha} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma\left(\frac{2n+1}{\alpha}\right)}{\Gamma(2n+1)} z^{2n}.$$

If $\alpha = 2$, we have, as in § 8.47,

$$F_2(z) = \tfrac{1}{2}\sqrt{\pi}e^{-\frac{1}{4}z^2},$$

which has no zeros.

If $\alpha = 2k$, where k is a positive integer, let

$$\phi(w) = \Gamma\{(2w+1)/2k\}\Gamma(w+1)/\Gamma(2w+1).$$

Then $\phi(w)$ is an integral function satisfying the conditions of Laguerre's theorems of § 8.6. Hence the zeros of

$$\sum_{n=0}^{\infty} \frac{\phi(n)}{n!} z^n = 2k\, F_{2k}(i\sqrt{z})$$

are all real and negative, so that the zeros of $F_{2k}(z)$ are all real. Also (§ 8.47) $\rho = 2k/2k-1$, so that $1 < \rho < 2$, and there must be an infinity of zeros.

If α is not an even integer, it can be proved that there are an infinity of complex zeros, and a finite number of real zeros.]

8.64. Functions with real negative zeros. If all the zeros of a function are real and negative, the modulus of the function is related to the distribution of its zeros in a specially simple way.

Suppose that $f(z)$ is such a function, and that its order ρ is less than 1. Then

$$f(z) = \prod_{n=1}^{\infty}\left(1+\frac{z}{z_n}\right).$$

Hence, if z is real,

$$\log f(z) = \sum_{n=1}^{\infty} \log\left(1+\frac{z}{z_n}\right) = \sum_{n=1}^{\infty} n\left\{\log\left(1+\frac{z}{z_n}\right) - \log\left(1+\frac{z}{z_{n+1}}\right)\right\}^*$$

$$= \sum_{n=1}^{\infty} n \int_{z_n}^{z_{n+1}} \frac{z\,dt}{t(z+t)} = z\int_0^{\infty} \frac{n(t)\,dt}{t(z+t)},$$

where $n(t)$ has its usual meaning.

Suppose now that as $t \to \infty$, $n(t) \sim \lambda t^\rho$. Then

$$\log f(x) \sim \pi\lambda \operatorname{cosec} \pi\rho\, x^\rho.$$

For we have $\qquad (\lambda-\epsilon)t^\rho < n(t) < (\lambda+\epsilon)t^\rho$

* The reader should justify this step, which is a simple example of partia summation.

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for $t > t_0(\epsilon)$. Hence

$$\log f(x) < x\int_0^{t_0}\frac{n(t)\,dt}{t(x+t)}+x\int_{t_0}^{\infty}\frac{(\lambda+\epsilon)t^\rho}{t(x+t)}\,dt$$

$$= x\int_0^{t_0}\frac{n(t)-(\lambda+\epsilon)t^\rho}{t(x+t)}\,dt+x\int_0^{\infty}\frac{(\lambda+\epsilon)t^\rho}{t(x+t)}\,dt.$$

The first term is plainly $O(1)$, and, putting $t = xu$ in the second integral, we obtain

$$x^\rho(\lambda+\epsilon)\int_0^\infty\frac{u^{\rho-1}\,du}{1+u} = x^\rho(\lambda+\epsilon)\pi\operatorname{cosec}\pi\rho$$

by § 3.123. A similar result holds with $\lambda-\epsilon$, and the theorem follows.

*More generally,**

$$\log f(re^{i\theta}) \sim e^{i\rho\theta}\pi\lambda\operatorname{cosec}\pi\rho\, r^\rho$$

for any fixed θ *in* $(-\pi, \pi)$, $\log f(z)$ *denoting the branch which is real on the positive real axis.*

In fact the above expression for $\log f(z)$ as an integral, obtained for real z, holds by analytic continuation for $-\pi < \arg z < \pi$. Hence we obtain as before

$$\log f(re^{i\theta}) \sim re^{i\theta}\int_0^\infty\frac{\lambda t^\rho\,dt}{t(re^{i\theta}+t)}.$$

Turning the line of integration to $t = ue^{i\theta}$, we obtain

$$\lambda re^{\rho i\theta}\int_0^\infty\frac{u^\rho\,du}{u(r+u)} = \lambda r^\rho e^{\rho i\theta}\pi\operatorname{cosec}\pi\rho$$

as before.

It is also possible to prove theorems of the converse type, viz. to deduce the asymptotic behaviour of $n(r)$ from that of $\log|f(z)|$. The most interesting is that *if, as* $x\to\infty$ *by real values,* $\log f(x)\sim\pi\lambda\operatorname{cosec}\pi\rho\, x^\rho$, *then* $n(r)\sim\lambda r^\rho$. This theorem† is closely connected with the Tauberian results of §§ 7.41–7.44, but the proof is too complicated to give here.

* Pólya and Szegö, *Aufgaben*, IV Abschn., no. 61.
† See Valiron (1), Titchmarsh (5), (6).

8.7. The minimum modulus. Let $m(r)$ denote the minimum of $|f(z)|$ on the circle $|z|=r$.

The function $m(r)$ cannot be expected to behave in as simple a way as $M(r)$, since it vanishes whenever r is the modulus of a zero of $f(z)$. But we shall see that, if we exclude the immediate neighbourhood of these exceptional points, we can set a lower limit to $m(r)$; and, in general, $m(r)$ tends to zero in somewhat the same way as $1/M(r)$.

8.71. Consider first a canonical product $P(z)$ of order ρ, with zeros $z_1, z_2, \ldots, z_n, \ldots$.

If about each zero z_n ($|z_n|>1$) we describe a circle of radius $|z_n|^{-h}$, where $h>\rho$, then in the region excluded from these circles

$$|P(z)|>e^{-r^{\rho+\epsilon}} \qquad (r>r_0(\epsilon)).$$

Following the method of § 8.25, it is clear that

$$\log|P(z)| \geqslant \sum_{r_n\leqslant kr} \log\left|1-\frac{z}{z_n}\right| - \sum_{r_n\leqslant kr} O\left\{\left(\frac{r}{r_n}\right)^p\right\} - \sum_{r_n>kr} O\left\{\left(\frac{r}{r_n}\right)^{p+1}\right\}$$

$$= \sum_{r_n\leqslant kr} \log\left|1-\frac{z}{z_n}\right| - O(r^{\rho+\epsilon}).$$

Since $\sum r_n^{-h}$ is convergent, the sum of the radii of the circles is finite, and so there are circles with centre the origin and arbitrarily large radius which lie entirely in the excluded region. Now if z lies outside every circle $|z-z_n|=r_n^{-h}$, and $r_n\leqslant kr$,

$$\left|1-\frac{z}{z_n}\right| > r_n^{-1-h} > (kr)^{-1-h}.$$

Hence
$$\sum_{1<r_n\leqslant kr} \log\left|1-\frac{z}{z_n}\right| > -(1+h)\log kr\,.\,n(kr)$$
$$> -K\log kr\,.\,r^{\rho+\epsilon} > -r^{\rho+2\epsilon}.$$

Finally
$$\sum_{r_n\leqslant 1} \log\left|1-\frac{z}{z_n}\right| > 0 \qquad (r>2),$$

and the result follows.

8.711. *If $f(z)$ is a function of order ρ, then*

$$m(r)>e^{-r^{\rho+\epsilon}}$$

on circles of arbitrarily large radius.

For
$$f(z)=P(z)e^{Q(z)},$$

where $Q(z)$ is a polynomial of degree $q \leqslant \rho$; hence

$$|e^{Q(z)}| > e^{-Ar^q} > e^{-Ar^\rho},$$

for sufficiently large values of r; and by the previous result

$$|P(z)| > e^{-r^{\rho_1+\epsilon}} > e^{-r^{\rho+\epsilon}}$$

on circles of arbitrarily large radius. Hence the result.

8.72. Another proof of Hadamard's factorization theorem. The theorem of § 8.71 leads to an alternative proof of Hadamard's factorization theorem. Let

$$f(z) = P(z)e^{Q(z)}$$

where $P(z)$ is the canonical product formed with the zeros of $f(z)$. Then $Q(z)$ is an integral function. Let ρ be the order of $f(z)$, ρ_1 the exponent of convergence. Then $P(z)$ is of order ρ_1, and $\rho_1 \leqslant \rho$. Hence

$$|P(z)| > e^{-r^{\rho_1+\epsilon}} > e^{-r^{\rho+\epsilon}}$$

on circles of arbitrarily large radius. Also

$$f(z) = O(e^{r^{\rho+\epsilon}}).$$

Hence $$\mathbf{R}\{Q(z)\} = \log\left|\frac{f(z)}{P(z)}\right| = \log\{O(e^{r^{\rho+\epsilon}})\} < Kr^{\rho+\epsilon}$$

on circles of arbitrarily large radius. Hence, by the theorem of § 2.54, $Q(z)$ is a polynomial of degree $\leqslant \rho$.

8.73. In special cases it is possible to prove much more precise results than the theorem of § 8.711.

If $\rho < \frac{1}{2}$, there is a sequence of values of r tending to infinity through which $m(r) \to \infty$.

In the first place, there is no line $\arg z =$ constant on which $f(z)$ is bounded; for the whole plane, bounded by this line, forms an angle 2π, and $2\pi < \pi/\rho$ if $\rho < \frac{1}{2}$. Hence, by § 5.61, if $f(z)$ is bounded on this line it is bounded everywhere, and so reduces to a constant.

Suppose now that

$$f(z) = cz^k \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right),$$

and let $$\phi(z) = cz^k \prod_{n=1}^{\infty} \left(1 + \frac{z}{r_n}\right),$$

where $r_n = |z_n|$. Then

$$\min_{|z|=r} |f(z)| \geqslant |\phi(-r)|,$$

since

$$\left|1-\frac{z}{z_n}\right| \geqslant \left|1-\frac{r}{r_n}\right|$$

for every n. Also $\phi(-r)$ is unbounded, since $\phi(z)$ is an integral function of the same order as $f(z)$. This proves the theorem.

8.74. The following result is still more precise.

If $0<\rho<1$, *there are arbitrarily large values of r for which*

$$m(r) > \{M(r)\}^{\cos\pi\rho-\epsilon}.$$

The following proof is due to Pólya (3). Define $f(z)$ and $\phi(z)$ as before, and we may plainly take $c = 1$, $k = 0$. It is sufficient to prove the theorem for $\phi(z)$. If $0 < \rho < \frac{1}{2}$, i.e. $\cos\pi\rho > 0$, this follows at once from the relations $m(r) \geqslant |\phi(-r)|$, $M(r) \leqslant \phi(r)$. In any case, if z' is a point where $|f(z')| = m(r)$, we have

$$|\phi(r)\phi(-r)| = \left|\prod_{n=1}^{\infty}\left(1-\frac{r^2}{r_n^2}\right)\right| \leqslant |f(z')f(-z')| \leqslant m(r)M(r).$$

Hence, if the theorem is true for $\phi(z)$, then

$$m(r)M(r) \geqslant |\phi(r)|^{1+\cos\pi\rho-\epsilon} \geqslant \{M(r)\}^{1+\cos\pi\rho-\epsilon}$$

for arbitrarily large r, and the result for $f(z)$ follows.

If the theorem is false for $\phi(z)$, there are positive constants ϵ and a such that

$$\log|\phi(-x)| < (\cos\pi\rho-\epsilon)\log\phi(x) \qquad (x > a).$$

By § 8.4, ex. (xii), for $\rho < s < 1$, and so also for $\rho < \mathbf{R}(s) < 1$,

$$\int_0^{\infty} \{\cos\pi s \log\phi(x)-\log|\phi(-x)|\}x^{-s-1}\,dx = 0.$$

Since the integral over $(0,a)$ is regular for $0 < \mathbf{R}(s) < 1$, so is the integral over (a,∞). Hence

$$F(s) = \int_{\alpha}^{\infty} \{\phi_1(e^{\xi})+\psi(s)\phi_2(e^{\xi})\}e^{-s\xi}\,d\xi,$$

where

$$\phi_1(x) = (\cos\pi\rho-\epsilon)\log\phi(x)-\log|\phi(-x)|, \qquad \phi_2(x) = \log\phi(x),$$
$$\psi(s) = \cos\pi s-\cos\pi\rho+\epsilon, \qquad \alpha = \log a,$$

is regular for $0 < \mathbf{R}(s) < 1$, and in particular at $s = \rho$. Here

ϕ_1 and ϕ_2' are positive for $x > a$, and $\psi(s)$ is positive for s real and sufficiently near to ρ.

Let $h > 0$, $D = d/ds$. Then

$$\left|\left(1-\frac{hD}{m}\right)^m F(s)\right| = \left|\sum_{\mu=0}^{m} \frac{(-h)^\mu}{\mu!} \frac{m(m-1)...(m-\mu+1)}{m^\mu} F^\mu(s)\right|$$
$$\leqslant |F(s)|+\frac{h}{1!}|F'(s)|+\frac{h^2}{2!}|F''(s)|+... = M,$$

say, the series being convergent for sufficiently small positive h and $s-\rho$. Also

$$\left(1-\frac{hD}{m}\right)^m \psi(s)e^{-s\xi} = e^{-s\xi}\left(1+\frac{h\xi-hD}{m}\right)^m \psi(s)$$
$$= e^{-s\xi}\left(1+\frac{h\xi}{m}\right)^m \psi(s)+e^{-s\xi}\sum_{\mu=1}^{m}\binom{m}{\mu}\left(1+\frac{h\xi}{m}\right)^{m-\mu}\left(\frac{-h}{m}\right)^\mu \psi^{(\mu)}(s)$$
$$\geqslant e^{-s\xi}\left(1+\frac{h\xi}{m}\right)^m \psi(s)-e^{-s\xi}\left(1+\frac{h\xi}{m}\right)^m \sum_{\mu=1}^{m}\frac{h^\mu}{\mu!}|\psi^\mu(s)|.$$

Since $|\psi^\mu(s)| \leqslant \pi^\mu$ for real s, this plainly exceeds

$$\tfrac{1}{2}e^{-s\xi}\left(1+\frac{h\xi}{m}\right)^m \psi(s)$$

if h is small enough. Hence

$$\left(1-\frac{hD}{m}\right)^m F(s) \geqslant \frac{1}{2}\int_{\alpha}^{\infty} \{\phi_1(e^\xi)+\psi(s)\phi_2(e^\xi)\}\left(1+\frac{h\xi}{m}\right)^m e^{-s\xi}\,d\xi.$$

In particular for any $\omega > \alpha$

$$\int_{\alpha}^{\omega} \phi_2(e^\xi)\left(1+\frac{h\xi}{m}\right)^m e^{-s\xi}\,d\xi \leqslant 2M/\psi(s).$$

Making $m \to \infty$, then $\omega \to \infty$, it follows that

$$\int_{\alpha}^{\infty} \phi_2(e^\xi)e^{(h-s)\xi}\,d\xi = \int_{a}^{\infty} \frac{\log\phi(x)}{x^{s-h+1}}\,dx$$

is convergent for a value of $s-h$ less than ρ. Hence $\sum r_n^{-s+h}$ is convergent, contrary to § 8.26. This proves the theorem.

8.75. A similar result holds for functions of order 1 and exponential type, i.e. such that $f(z) = O(e^{K|z|})$.

If $f(z) = O(e^{k|z|})$, then $m(r) > e^{-(k+\epsilon)r}$ for some arbitrarily large r. If $r_1, r_2,\ldots$ are the moduli of the zeros of $f(z)$, we find as in § 8.21 that $n(r) = O(r)$, $1/r_n \leqslant K/n$. Hence

$$\phi(z) = \prod_{n=1}^{\infty}\left(1+\frac{z}{r_n^2}\right) = O\prod_{n=1}^{\infty}\left(1+\frac{K^2r}{n^2}\right) = \frac{\sinh(\pi K\sqrt{r})}{\pi K\sqrt{r}}.$$

Define $h(\theta)$ (§ 5.7) for $\phi(z)$ with $V(r) = \sqrt{r}$. Then $h(\theta) \leqslant \pi K$ for all θ. Since $|\phi(z)| \geqslant 1$ if $\mathbf{R}(z) \geqslant 0$, $h(\theta)$ is finite for $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$, and so everywhere (§ 5.712). Also $h(-\theta) = h(\theta)$; and § 5.713, with $\theta_1 = -\pi$, $\theta_2 = 0$, $\theta_3 = \pi$, $\rho = \frac{1}{2}$, gives $h(\pi) \geqslant 0$. Hence

$$|f(z)f(-z)| \geqslant |f(0)|^2|\phi(-r^2)| > e^{-\epsilon r}$$

for some arbitrarily large values of r, and the result follows.

8.8. The a-points of an integral function. Our discussion of integral functions has so far centred round the distribution of the zeros of the function. A more general question is that of the distribution of the points where the function takes any given value a—the 'a-points', as we may call them.

There is one case in which we have already obtained fairly precise results, namely, that of functions of finite non-integral order. If $f(z)$ is of order ρ, where ρ is not an integer, then it has an infinity of zeros, and the exponent of convergence of the zeros is ρ. But clearly $f(z)-a$ is also of order ρ, where a is any constant. Hence $f(z)$ has an infinity of a-points, and their exponent of convergence is ρ; i.e. their density is roughly the same for all values of a.

A similar argument may be applied to functions of zero order. Such a function has an infinity of zeros unless it reduces to a polynomial; and $f(z)-a$ is a polynomial for every value of a or for none.

If $f(z)$ is of positive integral order, and $\neq a$, then $f(z)-a = e^{Q(z)}$, where $Q(z)$ is a polynomial. If $b \neq a$, then $Q(z) = \log(b-a)$ for some z, i.e. $f(z) = b$ for some z. Hence $f(z)$ takes every value with one possible exception.

8.81. Picard's theorem. The main theorem of the subject is due to Picard; it is independent of any considerations of order.

An integral function which is not a polynomial takes every value, with one possible exception, an infinity of times.

Picard's proof of the theorem depends on the properties of the elliptic modular function. This function, which we shall denote by $\varpi(z)$, has the following properties: *it is regular everywhere except at* $z=0$, 1, *and* ∞; *and its imaginary part is never negative.*

By means of this function we can easily prove that *an integral function which is not a constant takes every value, with one possible exception, at least once.*

Suppose that $f(z)$ is an integral function which does not take either of the values a or b, where $a \neq b$. Then

$$g(z) = \frac{f(z)-a}{b-a}$$

is an integral function, which does not take either of the values 0 or 1. Consider the function $\varpi\{g(z)\}$. It is regular except at infinity, since $g(z)$ does not take either of the finite values for which ϖ is singular. Also its imaginary part is positive. Hence, by § 2.54, it is a constant. But ϖ is not constant, and so $g(z)$ must be constant. This proves the theorem.

As we have not discussed the construction of the modular function, we shall not complete this proof, but shall give a more direct proof, depending on a theorem of Schottky.*

8.82. The characteristic feature of Picard's theorem is that it admits the possibility of there being an exceptional value. This exceptional value may actually exist; for example, the function e^z never takes the value 0. A value with this property is said to be 'exceptional P'.

There is another sense in which a value may be exceptional. A function may take the value a, but only at points which have a convergence-exponent less than ρ. For example, the function $e^z \cos\sqrt{z}$, of order 1, has zeros, but their convergence exponent is $\frac{1}{2}$. A value with this property is said to be 'exceptional B', i.e. exceptional in the sense of Borel. It is clear that a value which is exceptional P is *a fortiori* exceptional B.

8.83. For functions of positive integral order, Picard's theorem is a consequence of the following theorem of Borel, which shows not merely that there is at most one value 'exceptional P', but at most one 'exceptional B'.

* Another direct proof, depending on a theorem of Bloch, is given by **Landau,** *Ergebnisse* . . ., ed. 2 (1929), Ch. VII, and by Dienes, *The Taylor Series*, Ch. VIII.

Borel's theorem. *If the order of $f(z)$ is a positive integer, then the exponent of convergence of the a-points of $f(z)$ is equal to the order, except possibly for one value of a.*

Suppose that there are two exceptional values, a and b. Then

$$f(z)-a = z^{k_1}e^{Q_1(z)}P_1(z) \tag{1}$$

and

$$f(z)-b = z^{k_2}e^{Q_2(z)}P_2(z), \tag{2}$$

where $Q_1(z)$ and $Q_2(z)$ are polynomials of degree ρ, and $P_1(z)$ and $P_2(z)$ are canonical products of order less than ρ.

Subtracting, we have

$$b-a = z^{k_1}e^{Q_1(z)}P_1(z)-z^{k_2}e^{Q_2(z)}P_2(z), \tag{3}$$

or

$$z^{k_1}P_1(z)e^{Q_1(z)-Q_2(z)} = z^{k_2}P_2(z)+(b-a)e^{-Q_2(z)}.$$

Since $Q_2(z)$ is of degree ρ, the right-hand side is of order ρ. Hence so is the left-hand side, and so $Q_1(z)-Q_2(z)$ is of degree ρ, since $P_1(z)$ is of order less than ρ.

Differentiating (3), we have

$$(z^{k_1}P_1Q_1'+k_1z^{k_1-1}P_1+z^{k_1}P_1')e^{Q_1}$$
$$= (z^{k_2}P_2Q_2'+k_2z^{k_2-1}P_2+z^{k_2}P_2')e^{Q_2}. \tag{4}$$

Now the order of P_1' is the same as that of P_1, and so is less than ρ. Hence the coefficient of e^{Q_1} is of order less than ρ, and, similarly, so is that of e^{Q_2}. Hence we may write (4) in the form

$$z^{k_3}P_3e^{Q_1+Q_3} = z^{k_4}P_4e^{Q_2+Q_4},$$

where Q_3 and Q_4 are polynomials of degree $\rho-1$ at most, and P_3 and P_4 are canonical products. The two sides must have the same zeros, so that $k_3 = k_4$, $P_3 = P_4$, and so $Q_1+Q_3 = Q_2+Q_4$, i.e. $Q_1-Q_2 = Q_4-Q_3$, which is of degree less than ρ. This contradicts the previous result, that Q_1-Q_2 is of degree ρ, and so proves the theorem.

8.84. For the proof of Schottky's theorem we require the following lemma:

Let $\phi(r)$ be a real function of r for $0 \leqslant r \leqslant R_1$, and let

$$0 \leqslant \phi(r) \leqslant M \qquad (0 < r \leqslant R_1), \tag{1}$$

and also

$$\phi(r) < \frac{C\sqrt{\phi(R)}}{(R-r)^2} \qquad (0<r<R \leqslant R_1). \tag{2}$$

Then

$$\phi(r) < \frac{AC^2}{(R_1-r)^4} \qquad (0<r<R_1). \tag{3}$$

The actual form of the result (3) is not particularly important. What is important is that it depends only on r, R_1, and C, and not on M.

From (1) and (2) we obtain

$$\phi(r) < \frac{C\sqrt{M}}{(R-r)^2} \qquad (0 < r < R \leqslant R_1), \tag{4}$$

so that the upper bound M given in (1) is reduced at once to a multiple of $\sqrt{M}$. If we repeat the process, using first (4) with r_1, r_2 for r, R, and then (2) with r_1 for R, we obtain

$$\phi(r) < \frac{C}{(r_1-r)^2}\left\{\frac{C}{(r_2-r_1)^2}\right\}^{\frac{1}{2}} M^{\frac{1}{4}} \qquad (0 < r < r_1 < r_2 \leqslant R_1).$$

So generally

$$\phi(r) < \frac{C}{(r_1-r)^2}\left\{\frac{C}{(r_2-r_1)^2}\right\}^{\frac{1}{2}} \cdots \left\{\frac{C}{(r_n-r_{n-1})^2}\right\}^{\frac{1}{2^{n-1}}} M^{\frac{1}{2^n}}.$$

Taking $r_1 = \frac{1}{2}(R_1+r)$, $r_2 = \frac{1}{2}(R_1+r_1)$,..., this gives

$$\phi(r) < 4^{1+1+\frac{3}{4}+\ldots+\frac{n}{2^{n-1}}}\left\{\frac{C}{(R_1-r)^2}\right\}^{1+\frac{1}{2}+\ldots+\frac{1}{2^{n-1}}} M^{\frac{1}{2^n}},$$

and, making $n \to \infty$, the result follows.

8.85. Schottky's theorem. *If $f(z)$ is regular and does not take either of the values 0 or 1 for $|z| \leqslant R_1$, then for $|z| \leqslant R < R_1$*

$$|f(z)| < \exp\left\{\frac{KR_1^4}{(R_1-R)^4}\right\},$$

where K depends on $f(0)$ only. For all functions which satisfy the given conditions and are such that $\delta < |f(0)| < 1/\delta$, $|1-f(0)| > \delta$, we can take K to depend on δ only.

We shall not require the actual form of the upper bound for $f(z)$, which could be considerably improved if necessary; what is important is that it depends only on $f(0)$ in the manner stated, and on R/R_1.

Let $\qquad g_1(z) = \log\{f(z)\}, \qquad g_2(z) = \log\{1-f(z)\},$

where each logarithm has its principal value at $z = 0$. Then $g_1(z)$ and $g_2(z)$ are regular for $|z| \leqslant R_1$. Let $M_1(r)$ and $M_2(r)$ be the maxima of $|g_1(z)|$ and $|g_2(z)|$ respectively on $|z| = r$, and let

$$M(r) = \max\{M_1(r), M_2(r)\}.$$

Let $\qquad B_1(r) = -\min\limits_{|z|=r} \mathbf{R}\{g_1(z)\} = \max\limits_{|z|=r} \log\frac{1}{|f(z)|}.$

Then Carathéodory's theorem (§ 5.5), applied to $g_1(z)$, gives

$$M_1(\rho) \leqslant \frac{2\rho}{r-\rho} B_1(r) + \frac{r+\rho}{r-\rho} |g_1(0)| \quad (0 < \rho < r). \tag{1}$$

There are now two possibilities. Either $B_1(r)$ is not large—say $B_1(r) \leqslant 1$, in which case (1) is a result of the required type; or $B_1(r)$ is large, in which case there is a point z' on $|z| = r$ where $|f(z')|$ is small. But if $|f(z')|$ is small, $g_2(z')$ is (apart from a term $2n\pi i$) approximately equal to $-f(z')$; and then Carathéodory's theorem, applied to $\log g_2$, gives on the left M_1 (not $\log M_1$ as we should in general expect), and on the right $\log M_2 = O(\sqrt{M_2})$. We thus obtain an inequality of the type considered in the elementary lemma. This is the general plan of the proof, and we now proceed to fill in the details.

Suppose that $B_1(r) > 1$, and let z' be a point where

$$B_1(r) = \log 1/|f(z')|.$$

Then
$$|f(z')| = e^{-B_1(r)} < e^{-1} < \tfrac{1}{2}. \tag{2}$$

There is therefore an integer n such that

$$g_2(z') - 2n\pi i = -\sum_{m=1}^{\infty} \frac{\{f(z')\}^m}{m}.$$

Hence
$$|g_2(z') - 2n\pi i| < \sum_{m=1}^{\infty} 2^{-m} = 1,$$

and so
$$2|n|\pi < 1 + |g_2(z')| \leqslant 1 + M_2(r). \tag{3}$$

Let
$$h(z) = \log\{g_2(z) - 2n\pi i\},$$

where the logarithm has its principal value at $z = 0$. Then $h(z)$ is regular for $|z| \leqslant R_1$, since $f(z) \neq 0$ and so $g_2(z) \neq 2n\pi i$; and Carathéodory's theorem gives

$$\max_{|z|=r} |h(z)| \leqslant \frac{2r}{R-r} \max_{|z|=R} \log|g_2(z) - 2n\pi i| + \frac{R+r}{R-r} |h(0)|. \tag{4}$$

The left-hand side is not less than

$$\log\left|\frac{1}{g_2(z') - 2n\pi i}\right| \geqslant \log \frac{1}{|f(z')| + |f(z')|^2 + |f(z')|^3 + \dots}$$
$$\geqslant \log \frac{1}{2|f(z')|} = B_1(r) - \log 2,$$

by (2). On the right we have

$$\max_{|z|=R} \log|g_2(z) - 2n\pi i| \leqslant \log\{M_2(R) + 2|n|\pi\} < \log\{2M_2(R) + 1\},$$

by (3); if $n \neq 0$, $|g_2(0)-2n\pi i| \geqslant \pi > 1$, and so

$$|h(0)| \leqslant \log|g_2(0)-2n\pi i|+\pi \leqslant \log\{|g_2(0)|+1+M_2(r)\}+\pi.$$

If $n = 0$, $|h(0)| \leqslant |\log|g_2(0)||+\pi$. Hence (4) gives

$$B_1(r) \leqslant \frac{2R}{R-r}[2\log\{2M_2(R)+|g_2(0)|+1\}+|\log|g_2(0)||+\pi]+\log 2$$

$$< \frac{4R}{R-r}[\log\{2M_2(R)+|g_2(0)|+1\}+|\log|g_2(0)||+\pi]. \qquad (5)$$

This inequality, proved for $B_1(r) > 1$, is obviously true for $B_1(r) \leqslant 1$. Hence (1) and (5) give in any case

$$M_1(\rho) < \frac{8Rr}{(R-r)(r-\rho)}[\log\{2M_2(R)+|g_2(0)|+1\}+ \\ +|\log|g_2(0)||+|g_1(0)|+\pi].$$

Since we may interchange $g_1(z)$ and $g_2(z)$ in the whole argument, the inequality is still true if the suffixes 1 and 2 are interchanged. Combining the two results, we have

$$M(\rho) < \frac{8Rr}{(R-r)(r-\rho)}\{\log M(R)+K\},$$

where K depends on $|g_1(0)|$ and $|g_2(0)|$ only. Taking $r = \frac{1}{2}(R+\rho)$, we obtain

$$M(\rho) < \frac{32R_1^2}{(R-\rho)^2}\{\log M(R)+K\} < \frac{KR_1^2\sqrt{M(R)}}{(R-\rho)^2},$$

since $\log M(R) = O\{\sqrt{M(R)}\}$. Hence, by the lemma,

$$M(\rho) < \frac{KR_1^4}{(R-\rho)^4},$$

and
$$|f(z)| \leqslant e^{M(r)} < \exp\left\{\frac{KR_1^4}{(R-r)^4}\right\}.$$

Since K depends on $|g_1(0)|$ and $|g_2(0)|$ only, the last part of the theorem also is true.

8.86. Picard's first theorem. *An integral function which is not constant takes every value, with one possible exception, at least once.*

Suppose that $f(z)$ does not take either of the values a or b $(a \neq b)$. Then $g(z) = \{f(z)-a\}/(b-a)$ does not take either of the values 0 or 1. Hence, by Schottky's theorem,

$$|g(z)| < \exp\left\{\frac{KR_1^4}{(R_1-R)^4}\right\} \qquad (|z| \leqslant R < R_1).$$

Taking $R_1 = 2R$, $|g(z)| < K$. Hence $g(z)$ is a constant.

8.87. We can also prove the following generalization of Picard's theorem.

Landau's theorem.* *If α is any number, and β any number other than 0, there is a number $R = R(\alpha, \beta)$ such that every function*

$$f(z) = \alpha + \beta z + a_2 z^2 + a_3 z^3 + \ldots,$$

regular for $|z| \leqslant R$, takes in this circle one of the values 0 or 1.

We may suppose that $\alpha \neq 0$, $\alpha \neq 1$, for otherwise we have the result at once. If $f(z)$ does not take either of the values 0 or 1, then by Schottky's theorem $|f(z)| < K(\alpha)$ for $|z| \leqslant \frac{1}{2}R$. Hence

$$|\beta| = \left| \frac{1}{2\pi i} \int\limits_{|z| = \frac{1}{2}R} \frac{f(z)}{z^2}\, dz \right| \leqslant \frac{K(\alpha)}{\frac{1}{2}R},$$

$$R \leqslant 2K(\alpha)/|\beta|,$$

and the result follows.

8.88. We have so far stated Picard's theorem in terms of integral functions, i.e. functions with an essential singularity at infinity. But a corresponding theorem holds for any function with an isolated essential singularity.

Picard's second theorem. *In the neighbourhood of an isolated essential singularity, a one-valued function takes every value, with one possible exception, an infinity of times.*

In other words, *if $f(z)$ is regular for $0 < |z - z_0| < \rho$, and there are two unequal numbers a, b, such that $f(z) \neq a$, $f(z) \neq b$, for $|z - z_0| < \rho$, then z_0 is not an essential singularity.*

We may suppose that $z_0 = 0$, $\rho = 1$, $a = 0$, and $b = 1$. We prove that there is a sequence of circles $|z| = r_n$, where $r_n \to 0$, on which $f(z)$ is bounded. By § 2.71 this precludes the existence of a singularity at $z = 0$.

We start from Weierstrass's theorem that, in the neighbourhood of an essential singularity, a function approaches arbitrarily near to any given value an infinity of times. Thus there is a sequence of points z_1, z_2,... such that $|z_1| > |z_2| > \ldots$, $|z_n| \to 0$, and

$$|f(z_n) - 2| < \tfrac{1}{2}. \tag{1}$$

It is clear that Schottky's theorem would enable us to construct a sequence of circles, with these points as centres, in which $f(z)$ is bounded. These circles do not, of course, include

* Landau (1), and *Ergebnisse*, § 25.

the origin; but this is, so to speak, an accident arising from the fact that we have proved Schottky's theorem for a class of convex curves (viz. circles). We can remove the difficulty by making a conformal transformation, which has the effect of replacing a circle by an elongated curve which, though it excludes the origin, passes right round it and overlaps itself on the far side.

Let $z = e^w$ $(w = u+iv)$, and consider the half-strip of the w-plane $u < 0$, $-\pi \leqslant v \leqslant \pi$. This corresponds to the interior of the circle $|z| = 1$. Let $w_n = \log z_n$ $(-\pi < \mathbf{I}(w_n) \leqslant \pi)$, so that $\mathbf{R}(w_n) \to -\infty$; and let $f(z) = g(w)$.

We apply Schottky's theorem to the function

$$h(w') = g(w_n + w').$$

This is regular for $|w'| \leqslant 4\pi$ if n is large enough, and it does not take either of the values 0 or 1. Hence

$$|h(w')| < K = K\{h(0)\} \qquad (|w'| \leqslant 2\pi);$$

and, the numbers $h(0) = g(w_n) = f(z_n)$ satisfying (1), we can replace the right-hand side by an absolute constant. Hence $|g(w)| < A$ for $|w - w_n| \leqslant 2\pi$, and in particular for $u = \mathbf{R}(w_n)$, $-\pi \leqslant v \leqslant \pi$. Hence

$$|f(z)| < A \qquad (|z| = |z_n|),$$

and the result follows.

8.89. Asymptotic values. A number a is said to be an asymptotic value of an integral function $f(z)$ if there is a continuous curve from a given point to infinity, i.e. along which $|z| \to \infty$, and along which $f(z) \to a$ as $z \to \infty$. Thus 0 is an asymptotic value of e^z, since $e^z \to 0$ as $z \to \infty$ along the negative real axis. The function

$$\int_0^z e^{-t^q}\, dt,$$

where q is a positive integer, has the q asymptotic values

$$e^{\frac{2\pi i k}{q}} \int_0^\infty e^{-t^q}\, dt \qquad (k = 0, 1, \ldots, q-1),$$

as $z \to \infty$ along the lines $\arg z = e^{2\pi i k/q}$.

We may define the 'asymptotic value ∞' similarly.

We shall now prove the following theorems.

Every function with an isolated essential singularity at infinity, which is not a constant, has the asymptotic value infinity.

By Laurent's theorem, such a function is of the form $f(z)+g(z)$, where $f(z)$ is an integral function, and $g(z)$ tends uniformly to a limit as $|z| \to \infty$. Hence it is sufficient to consider integral functions. For an integral function not a constant, the maximum modulus $M(r)$ tends steadily to infinity. Consider an indefinitely increasing sequence of numbers $X_1 = M(r_1)$, $X_2, \ldots$. It follows from Liouville's theorem that there is a point outside the circle $|z| = r_1$ at which $|f(z)| > X_1$. The set of points where $|f(z)| > X_1$ constitutes the interior of one or more regions bounded by curves on which $|f(z)| = X_1$; and these regions must be exterior to the circle $|z| = r_1$. Let one such region be D_1. Now D_1 must extend to infinity; for otherwise we should have a finite region with $|f(z)| = X_1$ on the boundary and $|f(z)| > X_1$ inside, contrary to the maximum-modulus theorem. Further, $f(z)$ is unbounded in D_1. For otherwise the principle of Phragmén and Lindelöf would also show that $|f(z)| \leqslant X_1$ at all points inside D_1. In fact, the argument of § 5.6 applies with P at infinity and $\omega(z) = r_1/z$. Hence there is a point of D_1 at which $|f(z)| > X_2$, and consequently a domain D_2, interior to D_1, such that $|f(z)| > X_2$ at all points of D_2. We can now repeat the argument with $X_3, \ldots$. Hence there is a sequence of infinite regions $D_1, D_2, \ldots$, each interior to the preceding one and such that $|f(z)| > X_m$ in D_m, and $|f(z)| = X_m$ on its boundary. Now, take a point on the boundary of D_1, and join it to a point on the boundary of D_2 by a continuous curve lying in D_1; then this point to a point of D_3 by a continuous curve lying in D_2, and so on. We clearly obtain a continuous curve along which $f(z) \to \infty$.

If an integral function does not take the value a, then a is an asymptotic value.

For $1/\{f(z)-a\}$ is an integral function, and so has the asymptotic value ∞.

The argument of § 5.64 shows that if an integral function has asymptotic values on two curves, and is bounded between the curves, then these asymptotic values must be the same. Asymptotic values not so connected we should consider as distinct, whether they are equal or not.

It was conjectured by Denjoy that *an integral function of finite order ρ can have at most 2ρ asymptotic values.* The theorem with 5ρ instead of 2ρ was proved by Carleman; and Denjoy's conjecture was finally proved by Ahlfors. The general proof is not easy. It is, however, easy to see that there can be at most 2ρ straight lines from 0 to ∞ along which a function of order ρ has distinct asymptotic values. For by § 5.61 the angle between two such lines must be at least equal to π/ρ.

8.9. Meromorphic functions. We shall now give a short introduction to the theory of meromorphic functions, i.e. functions whose only singularities, except at infinity, are poles.

The theory depends largely on the general Jensen formula (§ 3.61 (4)). Let $f(z)$ be a meromorphic function, with zeros $a_1, a_2, \ldots$ and poles $b_1, b_2, \ldots$ (other than 0) arranged with non-decreasing moduli. Suppose that in the neighbourhood of the origin it is of the form $cz^k+\ldots$, where k may be any integer. Then Jensen's formula for $z^{-k}f(z)$ is

$$\log\left|\frac{b_1\ldots b_n}{a_1\ldots a_m}\right| r^{m-n}+\log|c| = \frac{1}{2\pi}\int\limits_0^{2\pi} \log|f(re^{i\theta})|\,d\theta - k\log r.$$

As in § 3.61

$$\log\frac{r^m}{|a_1\ldots a_m|} = \sum_{\nu=1}^{m-1} \nu \int\limits_{|a_\nu|}^{|a_{\nu+1}|} \frac{dx}{x} + m\int\limits_{|a_m|}^{r} \frac{dx}{x}.$$

Let $n(r,0)$ be the number of zeros of $f(z)$ in $|z| \leqslant r$. If $k > 0$, $\nu = n(x,0)-k$ for $|a_\nu| \leqslant x < |a_{\nu+1}|$; hence

$$\log\frac{r^m}{|a_1\ldots a_m|} = \int\limits_0^r \frac{n(x,0)-k}{x}\,dx.$$

If $n(r,\infty)$ is the number of poles of $f(z)$ in $|z| \leqslant r$, we obtain similarly

$$\log\frac{r_n}{|b_1\ldots b_n|} = \int\limits_0^r \frac{n(x,\infty)}{x}\,dx.$$

If k is negative, it appears in the second integral instead of the first. Writing

$$N(r,a) = \int\limits_0^r \frac{n(x,a)-n(0,a)}{x}\,dx + n(0,a)\log r$$

we obtain in any case

$$N(r,0)-N(r,\infty) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta - \log|c|. \tag{1}$$

Let us write $\qquad \log^+\alpha = \max(\log\alpha, 0)$

for any positive α. Thus

$$\log\alpha = \log^+\alpha - \log^+ 1/\alpha.$$

Let $\qquad m(r,a) = m\left(r,\frac{1}{f-a}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|\frac{1}{f(re^{i\theta})-a}\right| d\theta$

and $\qquad m(r,\infty) = m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\theta})|\,d\theta.$

Then (1) may also be written

$$m(r,0)+N(r,0) = m(r,\infty)+N(r,\infty)-\log|c|. \tag{2}$$

Now apply this formula to $f(z)-a$, where a is any number. If $f(z)-a = c_a z^k+\dots$ in the neighbourhood of the origin, we obtain

$$m(r,a)+N(r,a) = m(r,f-a)+N(r,\infty)-\log|c_a|,$$

the term $N(r,\infty)$ being unaltered since the poles of $f(z)-a$ are the same for each a.

Now we have

$$|f|+|a| \leqslant 2|fa|,\quad 2|f|,\quad 2|a| \quad \text{or} \quad 2$$

according as $|f| \geqslant 1$ and $|a| \geqslant 1$, $|f| \geqslant 1$ and $|a| < 1$, $|f| < 1$ and $|a| \geqslant 1$, or $|f| < 1$ and $|a| < 1$. Hence

$$\log(|f|+|a|) \leqslant \log^+|f|+\log^+|a|+\log 2.$$

Hence $\qquad \log^+|f-a| \leqslant \log^+|f|+\log^+|a|+\log 2,$

and similarly

$$\log^+|f| \leqslant \log^+|f-a|+\log^+|a|+\log 2.$$

Hence $\qquad |m(r,f-a)-m(r,f)| \leqslant \log^+|a|+\log 2.$

It follows that

$$m(r,a)+N(r,a) = m(r,\infty)+N(r,\infty)+\phi(r,a),$$

where $\qquad |\phi(r,a)| \leqslant |\log|c_a||+\log^+|a|+\log 2.$

Hence if $f(z)$ is a meromorphic function and not a constant, the

values of the sum
$$m(r,a)+N(r,a)$$
for two given values of a differ by a bounded function of r.

All the sums being to this extent equivalent, we can represent them all, e.g. by the one with $a = \infty$. Thus *if we put*
$$T(r) = m(r,\infty)+N(r,\infty), \tag{3}$$
we have for all values of a
$$m(r,a)+N(r,a) = T(r)+\phi(r,a),$$
where $\phi(r,a)$ *is (for each* a*) bounded as* $r \to \infty$.

$T(r)$ is called the *characteristic function* of $f(z)$.

We shall next show that $T(r)$ *is an increasing convex function of* $\log r$.

Jensen's formula for $f(z)-e^{i\lambda}$ (λ real) is
$$N(r,e^{i\lambda})-N(r,\infty) = \frac{1}{2\pi}\int\limits_0^{2\pi} \log|f(re^{i\theta})-e^{i\lambda}|\,d\theta - \log|f(0)-e^{i\lambda}|, \tag{4}$$
if $f(0) \neq e^{i\lambda}$. Also, for any a,
$$\frac{1}{2\pi}\int\limits_0^{2\pi} \log|e^{i\theta}-a|\,d\theta = \log^+|a|,$$
e.g. by Jensen's theorem with $f(z) = z-a$, $r = 1$. Hence, multiplying (4) by $1/(2\pi)$, and integrating with respect to λ over $(0, 2\pi)$, we obtain
$$\frac{1}{2\pi}\int\limits_0^{2\pi} N(r,e^{i\lambda})\,d\lambda - N(r,\infty) = \frac{1}{2\pi}\int\limits_0^{2\pi} \log^+|f(re^{i\theta})|\,d\theta - \log^+|f(0)|,$$
i.e.
$$T(r) = \frac{1}{2\pi}\int\limits_0^{2\pi} N(r,e^{i\lambda})\,d\lambda + \log^+|f(0)|.$$

Now, for any a, $N(r,a)$ is an increasing convex function of r, since
$$\frac{dN(r,a)}{d\log r} = n(r,a),$$
which is non-negative and non-decreasing. Hence $T(r)$ has the same property.

In the above formulae $N(r,a)$ measures the number of times the function $f(z)$ takes the value a. Since the largest contributions to $m(r, a)$ come from arcs where $f(z)$ is nearly equal to a, $m(r, a)$ measures in a sense the intensity of the approximation of $f(z)$ to a. We could describe $m(r, a)+N(r, a)$ as the total affinity of the function $f(z)$ for the value a.

For a given function, certain values may be exceptional, e.g. in the sense that the function does not take these values. The above theorem shows that there can be no exceptional values in the sense that the total affinity of the function for every value is the same, apart from bounded functions of r.

Examples. (i) Let $f(z)$ be a rational function, say $= P(z)/Q(z)$, where $P(z)$ is of degree μ, $Q(z)$ of degree ν, P and Q having no common factor. If $\mu > \nu$, then

$$m(r, a) = O(1), \qquad N(r, a) = \mu \log r + O(1)$$

for every finite a, while

$$m(r, \infty) = (\mu - \nu)\log r + O(1), \qquad N(r, \infty) = \nu \log r + O(1).$$

If $\mu < \nu$, then

$$m(r, a) = O(1), \qquad N(r, a) = \nu \log r + O(1)$$

for $a \neq 0$, while

$$m(r, 0) = (\nu - \mu)\log r + O(1), \qquad N(r, 0) = \mu \log r + O(1).$$

If $\mu = \nu$, let a_0 and b_0 be the coefficients of x^μ in P and Q. Then if $a \neq a_0/b_0$,

$$m(r, a) = O(1), \qquad N(r, a) = \mu \log r + O(1),$$

while, if $a_0 Q - b_0 P$ is of degree α,

$$m\left(r, \frac{a_0}{b_0}\right) = (\mu - \alpha)\log r + O(1), \qquad N\left(r, \frac{a_0}{b_0}\right) = \alpha \log r + O(1).$$

In any case $T(r) = O(\log r)$.

(ii) The function e^z does not take the values 0 or ∞; on the other hand these are limiting values of the function as $z \to \infty$. Here

$$N(r, 0) = N(r, \infty) = 0, \qquad m(r, 0) = m(r, \infty) = \frac{r}{\pi},$$

while for $a \neq 0, \infty$,

$$m(r, a) = O(1), \qquad N(r, a) = \frac{r}{\pi} + O(1).$$

Here $T(r) = r/\pi$.

(iii) Consider similarly $\tan z$ ($\pm i$ are exceptional values).

8.91. Order of a meromorphic function. The meromorphic function $f(z)$ is said to be of order ρ if

$$\overline{\lim_{r \to \infty}} \frac{\log T(r)}{\log r} = \rho,$$

so that

$$T(r) = O(r^{\rho + \epsilon})$$

for every positive ϵ, but not for $\epsilon < 0$.

To show that this agrees with the definition of order in the case of an integral function, we shall prove:

If $f(z)$ is an integral function,

$$T(r) \leqslant \log^+ M(r) \leqslant \frac{R+r}{R-r} T(R)$$

for $0 < r < R$.

For an integral function, $N(r, \infty) = 0$, and $T(r) = m(r, \infty)$. The left-hand inequality is thus

$$\frac{1}{2\pi} \int\limits_0^{2\pi} \log^+|f(re^{i\theta})|\, d\theta \leqslant \log^+ \max|f(re^{i\theta})|,$$

which is plainly true. Also by the Poisson-Jensen formula

$$\log|f(re^{i\theta})| = \frac{1}{2\pi} \int\limits_0^{2\pi} \frac{R^2-r^2}{R^2-2Rr\cos(\theta-\phi)+r^2} \log|f(Re^{i\phi})|\, d\phi - \\ - \sum_{\mu=1}^{m} \log\left|\frac{R^2-\bar{a}_\mu re^{i\theta}}{R(Re^{i\theta}-a_\mu)}\right|.$$

Each term in $\sum$ is negative, and

$$R^2-2Rr\cos(\theta-\phi)+r^2 \geqslant (R-r)^2.$$

Hence, taking θ so that the left-hand side is a maximum,

$$\log|M(r)| \leqslant \frac{R+r}{R-r} \frac{1}{2\pi} \int\limits_0^{2\pi} \log|f(Re^{i\phi})|\, d\phi$$

$$\leqslant \frac{R+r}{R-r} T(R)$$

and the second inequality follows.

Taking $R = 2r$, the identity of the two definitions of the order of an integral function is clear.

Now let $r_n(a)$ be the moduli of the zeros of $f(z)-a$, $r_n(\infty)$ the moduli of the poles of $f(z)$. Then we have the following results.

If $f(z)$ is of order ρ, then for every a

$$m(r, a) = O(r^{\rho+\epsilon}), \qquad N(r, a) = O(r^{\rho+\epsilon}), \qquad n(r, a) = O(r^{\rho+\epsilon})$$

and

$$\sum \left(\frac{1}{r_n(a)}\right)^{\rho+\epsilon}$$

is convergent.

The first two results are immediate since

$$m(r,a) \leqslant T(r)+O(1), \qquad N(r,a) \leqslant T(r)+O(1).$$

The remaining results then follow from that on $N(r,a)$ as in the case of an integral function.

More precise results of the same kind are given by Nevanlinna, *Fonctions Méromorphes*, Ch. II.

8.92. Factorization of meromorphic functions. Let $f(z)$ be a meromorphic function of order ρ, with zeros a_n and poles b_n ($f(0) \neq 0$). Then it follows from the above results that there are integers p_1 and p_2 not exceeding ρ such that

$$P_1(z) = \prod_{n=1}^{\infty} E\left(\frac{z}{a_n}, p_1\right), \qquad P_2(z) = \prod_{n=1}^{\infty} E\left(\frac{z}{b_n}, p_2\right)$$

are convergent for all values of z. Hence $P_1(z)$ and $P_2(z)$ are integral functions of order not exceeding ρ. Also

$$f_1(z) = f(z)P_2(z)$$

is an integral function. Now

$$\begin{aligned} T(r,f_1) &= m(r,\infty,f_1) \leqslant m(r,\infty,f)+m(r,\infty,P_2) \\ &\leqslant T(r,f)+T(r,P_2) = O(r^{\rho+\epsilon})+O(r^{\rho+\epsilon}). \end{aligned}$$

Hence $f_1(z)$ is of order ρ at most, and hence

$$f_1(z) = e^{Q(z)}P_1(z),$$

where $Q(z)$ is a polynomial of degree not exceeding ρ.

We have thus proved that

$$f(z) = e^{Q(z)}P_1(z)/P_2(z),$$

an extension of Hadamard's factorization theorem to meromorphic functions.

A slightly deeper theorem, in which the numerator and denominator do not necessarily converge separately, is proved by Nevanlinna, *Fonctions Méromorphes*, Ch. III.

Further developments of the theory of meromorphic functions are largely concerned with extensions of the theorems of Picard and Borel. For these we must refer to the books of Nevanlinna.

MISCELLANEOUS EXAMPLES

1. Prove that, if a is not a multiple of π,

$$\sin(a-z) = \sin a \, e^{-z\cot a} \prod_{n=-\infty}^{\infty} \left(1-\frac{z}{a+n\pi}\right) e^{\frac{z}{a+n\pi}}.$$

2. Show that the equations

$$\sin z = z^2, \qquad \log z = z^3, \qquad \tan z = az+b,$$

where a and b are any complex numbers, each have an infinity of roots.

3. Find all the zeros of the function

$$e^{e^z}-1,$$

and show that they have no finite exponent of convergence.

4. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a function of non-integral order, show that the coefficients in the polynomial $Q(z) = b_1 z+...+b_q z^q$ of Hadamard's theorem can be expressed in terms of $a_1, a_2,..., a_q$.

[If ρ is not an integer, $q < p+1$, and $P(z) = 1+O(z^{p+1})$ as $z \to 0$. Hence on equating coefficients $P(z)$ is not involved.]

5. If $f(z) = \sum a_n z^n$ is of order ρ, what is the order of $F(z) = \sum |a_n|^p z^n$?

6. The generalized hypergeometric function is defined by the formula

$${}_pF_q(\alpha_1,..., \alpha_p ; \beta_1,..., \beta_q ; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!},$$

where $(\alpha)_n = \alpha(\alpha+1)...(\alpha+n)$, $(\alpha)_0 = 1$. Show that it is an integral function if $q \geqslant p$, and find its order.

7. Show that

$$f(z) = \sum_{n=-\infty}^{\infty} q^{n^k} e^{inz}$$

is an integral function if $|q| < 1$, $k > 1$, and find its order.

8. If $\sigma > 1$, the function

$$P_\sigma(z) = \prod_{n=1}^{\infty} \left(1+\frac{z}{n^\sigma}\right)$$

is an integral function of order $1/\sigma$. [For further properties of the function see Hardy (4).]

9. The function

$$\prod_{n=1}^{\infty} \left(1+\frac{z}{e^n}\right)$$

is an integral function of zero order.

10. Show that, if $\alpha > 0$,

$$f_\alpha(z) = \prod_{n=1}^{\infty} \left(1+\frac{e^z}{e^{n^\alpha}}\right)$$

is an integral function of order $1+1/\alpha$; and express it in the standard factor form in the cases $\alpha = 1$, $\alpha = 2$.

11. If $a > 0$, the function

$$E_a(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+an)}$$

is an integral function of order $1/a$. [Several memoirs on this function are to be found in *Acta Math.* 29.]

12. If a is real, all the roots of the equation $\Gamma'(z) = a\Gamma(z)$ are real.

13. Show that

$$\sum_{n=0}^{\infty} \frac{\cosh \sqrt{n}}{n!} z^n$$

is an integral function of order 1, and that it has an infinity of zeros, all of them being real and negative.

14. A function $f(z)$ of order $\frac{1}{2}$ has all its zeros real and negative, and such that $n(r) \sim k\sqrt{r} \log r$. Determine the asymptotic behaviour of $M(r)$.

[Use the method of § 8.64.]

15. Show that, if $f(z)$ is a canonical product with zeros z_n such that $\sum 1/|z_n|$ is convergent, then $f(z) = O(e^{\epsilon|z|})$, and $|f(z)| > e^{-\epsilon|z|}$ on circles of arbitrarily large radius.

[We have

$$|f(z)| \leqslant \prod_{n=1}^{N} \left(1+\left|\frac{z}{z_n}\right|\right) \exp\left(\sum_{N+1}^{\infty} \left|\frac{z}{z_n}\right|\right),$$

whence the first result easily follows. The second part then follows from § 8.75.]

16. In Laguerre's theorem of § 8.52, show that $f'(z)$ has the same genus as $f(z)$.

[The only case in which there is anything to prove is the case $\rho = 1$, when the genus may be 0 or 1. Then we use the fact that the series $\sum 1/|z_n|$ and $\sum 1/|z_n'|$ converge or diverge together, compare $M(r)$ and $M'(r)$ by § 8.51, and apply the previous example.]

17. Show that the genus of a function of exponential type (§ 8.75) is 1.

18. Show that, if $f(z) = \sum_{0}^{\infty} a_n z^n$ is of exponential type, $f^{(n)}(0) = O(e^{An})$, and hence that $\phi(z) = \sum n! a_n z^n$ has a finite radius of convergence.

19. In order that $f(z)$ should be of exponential type, it is necessary and sufficient that it should be expressible in the form

$$f(z) = \frac{1}{2\pi i} \int_C e^{zw} \chi(w)\, dw,$$

where $\chi(w)$ is regular for sufficiently large values of w (including infinity), and C is a circle with centre the origin and sufficiently large radius.

[We have $\chi(w) = 1/w\phi(1/w)$, where ϕ is the function defined in the previous example.]

20. Let $f(z)$ be of exponential type, and let $h(\theta)$, supposed $\geqslant 0$, be the Phragmén-Lindelöf function associated with $f(z)$, with $V(r) = r$. Consider the radii vectores of length $h(\theta)$ making angles $-\theta$ with the real axis, and the perpendiculars to these radii vectores at their ends (cf. § 5.72). Then $\chi(w)$ is regular if w lies on the side of one of these perpendiculars opposite to the origin.

[We have

$$\phi(z) = \int_0^\infty e^{-t}f(zt)\,dt$$

by term-by-term integration, if $|z|$ is small enough; turning the contour through an angle λ,

$$\phi(z) = \int_0^\infty e^{-te^{i\lambda}}f(zte^{i\lambda})e^{i\lambda}\,dt.$$

Here the integrand is $\quad O(e^{-t\cos\lambda+rt\{h(\theta+\lambda)+\epsilon\}}),$

and the integral is convergent if

$$rh(\theta+\lambda) < \cos\lambda. \tag{1}$$

Hence $\phi(z)$ is regular at $z = re^{i\theta}$ if (1) holds for some value of λ. If $w = r'e^{i\theta'}$, then $\chi(w)$ is regular if $r' > h(\lambda-\theta')\sec\lambda$ for some λ. This is equivalent to the above statement.

For a detailed discussion see Pólya (4).]

21. The function

$$\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s n!}$$

is of exponential type. [For further properties of the function see Hardy (2).]

22. Show that the function

$$f(z) = \int_a^b e^{izt}g(t)\,dt,$$

where $g(t)$ is continuous, is of exponential type, and that the corresponding function $\chi(w)$ is regular except in the interval (ia, ib) of the imaginary axis.

23. Show that the function $f(z)$ of the above example tends to zero as $z \to \infty$ in either direction along the real axis, and deduce that $f(z)$ has an infinity of zeros.

24. A function $f(z)$ is said to be of zero type if $f(z) = O(e^{\epsilon r})$.

In order that $f(z)$ should be of zero type, it is necessary and sufficient that

$$f(z) = \frac{1}{2\pi i}\int_C e^{zw}\chi(w)\,dw,$$

where $\chi(w)$ is an integral function of $\dfrac{1}{w}$.

[The situation is similar to that of examples 18–19, except that here $f^{(n)}(0) = O(e^{\epsilon n})$.]

25. A necessary and sufficient condition that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ should be an integral function of $1/(1-z)$ is that there should be an integral function $g(z)$ of zero type such that $a_n = g(n)$ for $n = 1, 2, \ldots$.

[Carlson (1), Wigert (1), Hardy (14). If there is such a function $g(z)$, let

$$g(z) = \frac{1}{2\pi i}\int_C e^{zw}\chi(w)\,dw.$$

Then
$$f(z)-a_0 = \sum_{n=1}^{\infty} a_n z^n = \frac{1}{2\pi i}\int_C \frac{ze^w}{1-ze^w}\chi(w)\,dw,$$

if C is a contour enclosing the origin, and on which $\mathbf{R}(w) < \log|1/z|$. This is an integral of the type used in § 4.6, and by deforming it we can show that any branch of $f(z)$ is regular except at $z = 1$ (where the contour is nipped between 0 and $\log 1/z$). Also

$$f(z)-a_0 = \chi\left(\log\frac{1}{z}\right)+\frac{1}{2\pi i}\int_{C'} \frac{ze^w}{1-ze^w}\chi(w)\,dw,$$

where C' is a contour enclosing $w = \log 1/z$. This shows that $f(z)$ is one-valued near $z = 1$, and so an integral function of $1/(1-z)$.

Conversely, if $f(z)$ is of the type prescribed, we have

$$a_n = \frac{1}{2\pi i}\int \frac{f(z)}{z^{n+1}}\,dz,$$

and we can put $z = e^{-w}$, and deform the resulting contour into any simple closed contour which encloses the origin but lies entirely inside the circle $|w| = 2\pi$. Finally $f(e^{-w})$ is regular except for $w = 0$ and $w = \pm 2k\pi i$ $(k = 1, 2,\ldots)$, and so $f(e^{-w}) = g(w)+\psi(w)$, where $g(w)$ is an integral function of $1/w$, and $\psi(w)$ is regular for $|w| < 2\pi$. Hence the result.]

CHAPTER IX

DIRICHLET SERIES

9.1. Introduction. By a Dirichlet series we mean, in this chapter, a series of the form

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}, \tag{1}$$

where the coefficients a_n are any given numbers, and s is a complex variable. The more general type of series

$$\sum a_n e^{-\lambda_n s}$$

is also known as a Dirichlet series. The special type is obtained by putting $\lambda_n = \log n$. For the theory of the general type we must refer to Hardy and Riesz's *General Theory of Dirichlet's Series*.

Throughout the chapter we shall write $s = \sigma + it$, where σ and t are real. If the Dirichlet series is convergent, we shall denote its sum by $f(s)$. We have already had one important example of a Dirichlet series, the zeta-function of Riemann,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{2}$$

Dirichlet series are not of such importance in general analysis as power series because they only represent a very special class of analytic functions. They are, however, of great importance in applications of analysis to theory of numbers. In several ways their theory is more complicated than that of power series. For example, the circle of convergence, circle of absolute convergence, and circle of regularity of sum-function are all the same for a power series. In the theory of Dirichlet series, in which 'circle' must be replaced by 'half-plane', the three corresponding half-planes may be all different.

9.11. The association of half-planes with a Dirichlet series depends on the following theorem:

If the Dirichlet series is convergent for $s = s_0$, then it is uniformly convergent throughout the angular region in the s-plane defined by the inequality

$$|\arg(s - s_0)| \leqslant \tfrac{1}{2}\pi - \delta,$$

where δ is any positive number less than $\tfrac{1}{2}\pi$.

It is sufficient to consider the case where $s_0 = 0$; for

$$\sum \frac{a_n}{n^s} = \sum \frac{a'_n}{n^{s'}},$$

where $$a'_n = a_n n^{-s_0}, \qquad s' = s - s_0,$$

and the latter series is convergent for $s' = 0$.

We suppose, then, that $\sum a_n$ is convergent. Let

$$r_n = a_{n+1} + a_{n+2} + \ldots$$

so that $r_n \to 0$. Then

$$\sum_{n=M}^{N} \frac{a_n}{n^s} = \sum_{M}^{N} \frac{r_{n-1} - r_n}{n^s} = \sum_{M}^{N} r_n \left\{\frac{1}{(n+1)^s} - \frac{1}{n^s}\right\} + \frac{r_{M-1}}{M^s} - \frac{r_N}{(N+1)^s}. \quad (1)$$

Now

$$\left|\frac{1}{(n+1)^s} - \frac{1}{n^s}\right| = \left|s \int_n^{n+1} \frac{du}{u^{s+1}}\right| \leqslant |s| \int_n^{n+1} \frac{du}{u^{\sigma+1}} = \frac{|s|}{\sigma}\left\{\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma}\right\}, \quad (2)$$

and $|r_n| < \epsilon$ for $n \geqslant n_0 = n_0(\epsilon)$, n_0 being independent of s. Hence for $M > n_0$

$$\left|\sum_{n=M}^{N} \frac{a_n}{n^s}\right| < \frac{\epsilon|s|}{\sigma} \sum_{n=M}^{N} \left\{\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma}\right\} + \frac{\epsilon}{M^\sigma} + \frac{\epsilon}{(N+1)^\sigma}$$

$$= \frac{\epsilon|s|}{\sigma}\left\{\frac{1}{M^\sigma} - \frac{1}{(N+1)^\sigma}\right\} + \frac{\epsilon}{M^\sigma} + \frac{\epsilon}{(N+1)^\sigma}.$$

$$< 2\epsilon|s|/\sigma + 2\epsilon.$$

If $|\arg s| \leqslant \frac{1}{2}\pi - \delta$, i.e. $t/\sigma \leqslant \tan(\frac{1}{2}\pi - \delta) = \cot\delta$, we have

$$|s|/\sigma = \sqrt{(1 + t^2/\sigma^2)} \leqslant \operatorname{cosec}\delta.$$

Hence $$\sum_{n=M}^{N} \frac{a_n}{n^s} < 2\epsilon(\operatorname{cosec}\delta + 1).$$

The right-hand side is independent of s, and tends to 0 with ϵ; and this establishes uniform convergence.

In particular, *if the series is convergent for $s_0 = \sigma_0 + it_0$, it is convergent for $s = \sigma + it$, provided that $\sigma > \sigma_0$.* For we can choose the δ of the above proof so small that $|\arg(s - s_0)| < \frac{1}{2}\pi - \delta$.

9.12. *The region of convergence of the series is a half-plane.* For we can divide values of σ' into two classes, those for which the series is convergent for $\sigma > \sigma'$, and other values of σ'. By the above theorem, every member of the first class lies to the right of every member of the second class. Let σ_0 be the real number defined by this section. Then the series is convergent for $\sigma > \sigma_0$, divergent for $\sigma < \sigma_0$.

The number σ_0 is called *the abscissa of convergence* of the series.

The series may converge for all values of s (e.g. $a_n = 1/n!$), or for no values of s (e.g. $a_n = n!$).

The sum $f(s)$ of the series is an analytic function of s, regular for $\sigma > \sigma_0$. For every term of the series is analytic, and any point s with $\sigma > \sigma_0$ is included in a domain of uniform convergence.

The questions of the convergence of the series, and the regularity of the function, *on* the line $\sigma = \sigma_0$, remain open; and (as in the case of power series) various different cases are possible.

We have, however, the following analogue of Abel's theorem for power series:

If the series is convergent for $s = s_0$, and has the sum $f(s_0)$, then $f(s) \to f(s_0)$ when $s \to s_0$ along any path lying entirely inside the region $|\arg(s-s_0)| \leqslant \frac{1}{2}\pi - \delta$.

This follows at once from the theorem of uniform convergence.

9.13. Absolute convergence. *The region of absolute convergence of the Dirichlet series is a half-plane.*

For the series is absolutely convergent if the series

$$\sum_1^\infty \frac{|a_n|}{n^\sigma}$$

is convergent. If this is convergent for a particular value of σ, it is clearly convergent for any greater value. Hence, as in the case of convergence, there is a number $\bar{\sigma}$ such that it is convergent for $\sigma > \bar{\sigma}$, and divergent for $\sigma < \bar{\sigma}$.

Hence the original series is absolutely convergent for $\sigma > \bar{\sigma}$, and not absolutely convergent for $\sigma < \bar{\sigma}$.

The number $\bar{\sigma}$ is called the *abscissa of absolute convergence.*

The numbers σ_0 and $\bar{\sigma}$ are not necessarily equal, i.e. there may be a strip of the plane in which the series is convergent, but not absolutely convergent.

This is shown by the following example. If $\sigma > 1$, we have

$$\left(1 - \frac{1}{2^{s-1}}\right)\zeta(s) = \left(\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \dots\right) - 2\left(\frac{1}{2^s} + \frac{1}{4^s} + \dots\right)$$
$$= \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \dots.$$

The last series, as it is arranged here, is convergent for $\sigma > 0$ (and uniformly convergent in any finite region to the right of $\sigma = 0$). For, by a well-known theorem (*P.M.* § 188), it is convergent if s is real and positive. Hence, by the theory of analytic continuation, the formula holds for $\sigma > 0$.

In this case $\sigma_0 = 0$, $\bar{\sigma} = 1$.

In any case $\bar{\sigma} - \sigma_0 \leqslant 1$.

For if $\sum a_n n^{-s}$ is convergent, $|a_n| n^{-\sigma}$ is bounded as $n \to \infty$, and hence

$$\sum \frac{a_n}{n^{s+1+\delta}}$$

is absolutely convergent if $\delta > 0$, which gives the result.

In the above example, $\bar{\sigma} - \sigma_0 = 1$, so that the strip of non-absolute convergence may be as wide as 1, though it can be no wider.

9.14. The abscissa of convergence. The formula for σ_0, analogous to the formula (§ 7.1) for the radius of convergence of a power series, takes slightly different forms according to whether $\sum a_n$ is convergent or not. Let

$$s_n = a_1 + a_2 + \ldots + a_n,$$

and, if $\sum a_n$ is convergent, let $r_n = a_{n+1} + a_{n+2} + \ldots$. Let

$$\alpha = \varlimsup_{n \to \infty} \frac{\log|s_n|}{\log n}, \qquad \beta = \varlimsup_{n \to \infty} \frac{\log|r_n|}{\log n},$$

β being defined only if $\sum a_n$ is convergent.

Then $\sigma_0 = \alpha$ *if* $\sum a_n$ *is divergent, and otherwise* $\sigma_0 = \beta$.

In the former case $\sigma_0 \geqslant 0$, and in the latter case $\sigma_0 \leqslant 0$; for $\sum a_n$ is convergent if $\sigma_0 < 0$.

(i) Let $\sum a_n$ be divergent, and let s have a real positive value for which the Dirichlet series is convergent. Let

$$b_n = a_n n^{-s}, \qquad B_n = b_1 + b_2 + \ldots + b_n, \qquad B_0 = 0,$$

so that B_n is bounded, say $|B_n| \leqslant B$. Then

$$s_N = \sum_{n=1}^{N} b_n n^s = \sum_{n=1}^{N} (B_n - B_{n-1}) n^s$$

$$= \sum_{n=1}^{N-1} B_n \{n^s - (n+1)^s\} + B_N N^s.$$

Hence $$|s_N| \leqslant B\sum_{n=1}^{N-1}\{(n+1)^s-n^s\}+BN^s<2BN^s,$$
$$\log|s_N|<s\log N+\log 2B,$$
and so $\alpha \leqslant s$. Hence $\alpha \leqslant \sigma_0$.

A similar argument holds if $\sum a_n$ is convergent. If s has a real negative value for which the Dirichlet series is convergent,
$$\begin{aligned} r_N &= \sum_{n=N+1}^{\infty} b_n n^s = \sum_{n=N+1}^{\infty} (B_n-B_{n-1})n^s \\ &= \sum_{n=N}^{\infty} B_n\{n^s-(n+1)^s\}-B_N N^s, \end{aligned}$$
so that $$|r_N| \leqslant B\sum_{n=N}^{\infty}\{n^s-(n+1)^s\}+BN^s=2BN^s.$$
Hence, as in the other case, $\beta \leqslant \sigma_0$.

(ii) Since $s_n=O(n^{\alpha+\epsilon})$, and, if s is real,
$$\frac{1}{n^s}-\frac{1}{(n+1)^s}=s\int\limits_{n}^{n+1}\frac{du}{u^{s+1}}=O(n^{-s-1}),$$
we have
$$\begin{aligned} \sum_{n=M+1}^{N}\frac{a_n}{n^s} &= \sum_{n=M+1}^{N}\frac{s_n-s_{n-1}}{n^s} \\ &= \sum_{n=M+1}^{N} s_n\left\{\frac{1}{n^s}-\frac{1}{(n+1)^s}\right\}+\frac{s_N}{(N+1)^s}-\frac{s_M}{(M+1)^s} \qquad (1) \\ &= \sum_{M+1}^{N} O(n^{\alpha+\epsilon-s-1})+O(N^{\alpha+\epsilon-s})+O(M^{\alpha+\epsilon-s}) \\ &= O(M^{\alpha+\epsilon-s})=o(1) \end{aligned}$$
if $s>\alpha$ and ϵ is small enough. Hence the Dirichlet series is convergent if $s>\alpha$; hence $\sigma_0 \leqslant \alpha$. Since $a_n=r_{n-1}-r_n$ if $\sum a_n$ is convergent, we find similarly that $\sigma_0 \leqslant \beta$. This proves the theorem.

If $\alpha=\infty$, the series is nowhere convergent, and if $\beta=-\infty$ it is everywhere convergent. This easily follows from the above argument.

9.15. The abscissa of absolute convergence. We have
$$\bar{\sigma}=\overline{\lim_{n\to\infty}}\frac{\log(|a_1|+|a_2|+\ldots+|a_n|)}{\log n},$$
or $$\bar{\sigma}=\overline{\lim_{n\to\infty}}\frac{\log(|a_{n+1}|+|a_{n+2}|+\ldots)}{\log n},$$