and similarly

$$|p(z) - p(-z)| \leqslant \sqrt{2M_\alpha} e^{3k\alpha|z|}$$

Hence

$$|p(z)| \leqslant 2\sqrt{2M_\alpha} e^{3k\alpha|z|}$$

for $|z| \geqslant 1$, and from this, by the principle of maximum,

$$|p(z)| \leqslant 2\sqrt{2M_\alpha} e^{3k\alpha} \qquad \text{for } |z| < 1.$$

The theorem therefore holds with

$$K_\alpha = 2\sqrt{2M_\alpha} e^{3k\alpha}. \qquad \text{Q.E.D.}$$

Corollary. *If* $\alpha > 0$ *is* small enough, *the polynomials* $p(z)$ *satisfying*

$$\sum_{-\infty}^{\infty} \frac{\log^+ |p(n)|}{1+n^2} \leqslant \alpha$$

form a normal family *in the complex plane, and the* limit *of any convergent sequence of such polynomials is an* entire function of exponential type $\leqslant 3k\alpha$, *k being an absolute constant.*

It is thus somewhat as if harmonic measure were available for the domain $\mathbb{C} \sim \mathbb{Z}$, *even though that is not the case.*

11. Weighted polynomial approximation on $\mathbb{Z}$

Given a weight $W(n) \geqslant 1$ defined on $\mathbb{Z}$, we consider the Banach space $\mathscr{C}_W(\mathbb{Z})$ of functions $\varphi(n)$ defined on $\mathbb{Z}$ for which

$$\frac{\varphi(n)}{W(n)} \longrightarrow 0 \quad \text{as} \quad n \to \pm\infty,$$

and write

$$\|\varphi\|_{W,\mathbb{Z}} = \sup_{n\in\mathbb{Z}} \frac{|\varphi(n)|}{W(n)}$$

for such φ. (This is the notation of §A.3.)

Provided that

$$\frac{n^k}{W(n)} \longrightarrow 0 \quad \text{as} \quad n \to \pm\infty$$

for each $k = 0, 1, 2, 3, \ldots$, we can form the $\| \ \|_{W,\mathbb{Z}}$ *closure*, $\mathscr{C}_W(0, \mathbb{Z})$, of the set of *polynomials in* n, in $\mathscr{C}_W(\mathbb{Z})$. The *Bernstein approximation problem for* $\mathbb{Z}$ requires us to find necessary and sufficient conditions on weights $W(n)$ having the property just stated in order that $\mathscr{C}_W(0,\mathbb{Z})$ and $\mathscr{C}_W(\mathbb{Z})$ be the same.

The preceding work enables us to give a complete solution in terms of the Akhiezer function

$$W_*(n) = \sup\{|p(n)|: \ p \text{ a polynomial and } \|p\|_{W,\mathbb{Z}} \leqslant 1\}$$

introduced in §B.1 of Chapter VI.

Theorem. *Let* $W(n)$, *defined and* $\geqslant 1$ *on* $\mathbb{Z}$, *tend to* ∞ faster than any power of n as $n \to \pm\infty$. *Then* $\mathscr{C}_W(0,\mathbb{Z}) = \mathscr{C}_W(\mathbb{Z})$ *if and only if*

$$\sum_{-\infty}^{\infty} \frac{\log W_*(n)}{1+n^2} = \infty.$$

Proof. Let us get the easier *if* part out of the way first – this is not really new, and depends only on the work of Chapter VI, §B.1.

As in §A.3, we take $W(x)$ to be specified on *all* of $\mathbb{R}$ by putting $W(x) = \infty$ for $x \notin \mathbb{Z}$, and define $W_*(z)$ for all $z \in \mathbb{C}$ using the formula

$$W_*(z) = \sup\{|p(z)|: \ p \text{ a polynomial and } \|p\|_{W,\mathbb{Z}} \leqslant 1\}.$$

Then $\mathscr{C}_W(\mathbb{Z})$ can be identified in obvious fashion with the space $\mathscr{C}_W(\mathbb{R})$ constructed from the (discontinuous) weight $W(x)$, and $\mathscr{C}_W(0,\mathbb{Z})$ identified with $\mathscr{C}_W(0)$, the closure of the set of polynomials in $\mathscr{C}_W(\mathbb{R})$. *Proper* inclusion of $\mathscr{C}_W(0,\mathbb{Z})$ in $\mathscr{C}_W(\mathbb{Z})$ is thus *the same* as that of $\mathscr{C}_W(0)$ in $\mathscr{C}_W(\mathbb{R})$, and we can apply the *if* part of Akhiezer's theorem from §B.1 of Chapter III (whose validity *does not* depend on the continuity of $W(x)$!) to conclude that

$$\int_{-\infty}^{\infty} \frac{\log W_*(t)}{t^2+1}\,dt \ < \ \infty$$

when that proper inclusion holds.

If p is any polynomial with $\|p\|_{W,\mathbb{Z}} \leqslant 1$, the hall of mirrors argument at the beginning of the proof of Akhiezer's theorem's *only if* part shows that

$$\log|p(x)| \ \leqslant \ \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{2\log W_*(t)}{(x-t)^2+4}\,dt$$

for $x \in \mathbb{R}$. Taking the supremum over such polynomials p gives us

$$\log W_*(n) \ \leqslant \ \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{2\log W_*(t)}{(n-t)^2+4}\,dt, \qquad n \in \mathbb{Z}.$$

Therefore, since $\log W_*(t) \geqslant 0$ (1 being a polynomial!), we have

$$\sum_{-\infty}^{\infty} \frac{\log W_*(n)}{1+n^2} \leqslant \frac{1}{\pi}\int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{n^2+1}\cdot\frac{2\log W_*(t)}{(n-t)^2+1}\,\mathrm{d}t.$$

The inner sum over n may easily be compared with an integral, and we find in this way that the last expression is

$$\leqslant \text{ const.} \int_{-\infty}^{\infty} \frac{\log W_*(t)}{1+t^2}\,\mathrm{d}t.$$

This, however, is *finite* when $\mathscr{C}_W(0,\mathbb{Z}) \neq \mathscr{C}_W(\mathbb{Z})$, as we have just seen. The *if* part of our theorem is proved.

For the *only if* part, we assume that

$$\sum_{-\infty}^{\infty} \frac{\log W_*(n)}{1+n^2} < \infty,$$

and show that the function

$$\varphi_0(n) = \begin{cases} 1, & n=0 \\ 0, & n \neq 0, \end{cases}$$

cannot belong to $\mathscr{C}_W(0,\mathbb{Z})$. We do this using the *corollary* to the *first* theorem of the preceding article. It is not necessary to resort to the *second* theorem given there.

Suppose, then, that we *have* a sequence of polynomials $p_l(z)$ with

$$\|\varphi_0 - p_l\|_{W,\mathbb{Z}} \xrightarrow[l]{} 0.$$

This implies in particular that

$$p_l(0) \xrightarrow[l]{} \varphi_0(0) = 1,$$

so there is no loss of generality in assuming that $p_l(0) = 1$ for each l, which we *do*. The polynomials

$$Q_l(z) = \tfrac{1}{2}(p_l(z) + p_l(-z))$$

then satisfy the hypothesis of the corollary in question.

We evidently have $\|p_l\|_{W,\mathbb{Z}} \leqslant C$ for some C, so, by definition of W_*, $|p_l(n)| \leqslant CW_*(n)$ for $n\in\mathbb{Z}$ and therefore

$$|Q_l(n)| \leqslant \tfrac{1}{2}C(W_*(n) + W_*(-n)), \qquad n\in\mathbb{Z}.$$

Also, $p_l(n) \xrightarrow[l]{} \varphi_0(n) = 0$ for *each non-zero* $n\in\mathbb{Z}$, so, given any N, we will have

$$|Q_l(n)| < 1 \qquad \text{for } 0 < |n| < N$$

when l is sufficiently large.

Taking any $\alpha > 0$, we choose and fix an N large enough to make

$$\sum_N^\infty \frac{1}{n^2} \log^+(\tfrac{1}{2}C(W_*(n) + W_*(-n))) \;<\; \alpha,$$

this being possible in view of our assumption on W_*. By the preceding two relations we will then have

$$\sum_1^\infty \frac{1}{n^2} \log^+|Q_l(n)| \;=\; \sum_N^\infty \frac{1}{n^2} \log^+|Q_l(n)| \;<\; \alpha$$

for sufficiently large values of l.

If $\alpha > 0$ is sufficiently small, the last condition implies that

$$|Q_l(z)| \;\leqslant\; e^{k\alpha|z|}$$

by the corollary to the first theorem of the preceding article, with k an absolute constant. *This must therefore hold for all sufficiently large values of l.*

A *subsequence* of the polynomials $Q_l(z)$ therefore converges u.c.c. to a certain *entire function $F(z)$ of exponential type* $\leqslant k\alpha$. We evidently have $F(0) = 1$ (so $F \not\equiv 0$!), while $F(n) = 0$ *for each non-zero* $n \in \mathbb{Z}$.

However, by problem 1(a) in Chapter I (!), *such an entire function F cannot exist*, if $\alpha > 0$ is chosen *sufficiently small to begin with.* We have thus reached a contradiction, showing that φ_0 *cannot belong* to $\mathscr{C}_W(0, \mathbb{Z})$. The latter space is thus *properly contained* in $\mathscr{C}_W(\mathbb{Z})$, and the *only if* part of our theorem is proved.

We are done.

C. Harmonic estimation in slit regions

We return to domains $\mathscr{D}$ for which the Dirichlet problem *is* solvable, having boundaries formed by removing certain finite open intervals from $\mathbb{R}$. Our interest in the present § is to see whether, from the *existence* of a *Phragmén–Lindelöf function* $Y_{\mathscr{D}}(z)$ for $\mathscr{D}$ (the reader should perhaps look at §A.2 again before continuing), one can *deduce* any *estimates* on the *harmonic measure* for $\mathscr{D}$. We would like in fact to be able to *compare* harmonic measure for $\mathscr{D}$ with $Y_{\mathscr{D}}(z)$. The reason for this desire is the following. Given $A > 0$ and $M(t) \geqslant 0$ on $\partial\mathscr{D}$, suppose that we have a function $v(z)$, *subharmonic* in $\mathscr{D}$ and *continuous* up to $\partial\mathscr{D}$, with

$$v(z) \;\leqslant\; \text{const.} - A|\Im z|, \qquad z \in \mathscr{D},$$

and

$$v(t) \;\leqslant\; M(t), \qquad t \in \partial\mathscr{D}.$$

Then, by harmonic estimation

$$v(z) \leqslant \int_{\partial\mathscr{D}} M(t)\mathrm{d}\omega_{\mathscr{D}}(t,z) - AY_{\mathscr{D}}(z), \qquad z\in\mathscr{D},$$

where (as usual) $\omega_{\mathscr{D}}(\ ,z)$ denotes (*two-sided*) harmonic measure for $\mathscr{D}$ (see §A.1). *It would be very good* if, in this relation, we had some way of *comparing* the *first term* on the right with the *second.*

As we shall see below, such comparison is indeed possible. In order to avoid fastidious justification arguments like the one occurring in the proof of the second theorem from §A.2, *we will assume throughout that $\partial\mathscr{D}$ consists of* $\mathbb{R}$ *minus a finite number of* (bounded) *open intervals.* The results obtained for this situation can usually be extended by means of a simple limiting procedure to cover various more general cases that may arise in practice. The domains $\mathscr{D}$ considered here thus look like this:

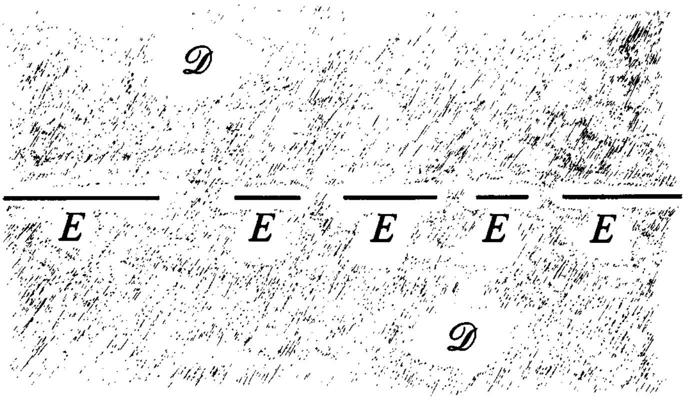

Figure 151

As in §A, we shall frequently denote $\partial\mathscr{D}$ by E. E is a closed subset of $\mathbb{R}$ which, *in this* §, *will contain all real x of sufficiently large absolute value.*

1. Some relations between Green's function and harmonic measure for our domains $\mathscr{D}$

During the present §, we will usually denote the Green's function for one of the domains $\mathscr{D}$ by $G_{\mathscr{D}}(z,w)$, instead of just writing $G(z,w)$ as in §A.2ff. We similarly write $Y_{\mathscr{D}}(z)$ instead of $Y(z)$ for $\mathscr{D}$'s Phragmén–Lindelöf function.

Our domains $\mathscr{D}$ *have* Phragmén–Lindelöf functions. Indeed, for *fixed* $z\in\mathscr{D}$ and *real* t, $G_{\mathscr{D}}(z,t) = G_{\mathscr{D}}(t,z)$ *vanishes* for t outside the *bounded* set $\mathbb{R}\sim E$. (We are using symmetry of the Green's function, established at the end of §A.2.) If we take $z \notin \mathbb{R}$, $G_{\mathscr{D}}(t,z)$ is also a continuous function of $t \in \mathbb{R}$. The integral

$$\int_{-\infty}^{\infty} G_{\mathscr{D}}(z,t)\,\mathrm{d}t$$

is then certainly *finite*, and the *existence* of the *function* $Y_{\mathscr{D}}$ hence assured by the second theorem of §A.2.

According to that same theorem,

$$Y_{\mathscr{D}}(z) \;=\; |\Im z| + \frac{1}{\pi}\int_{-\infty}^{\infty} G_{\mathscr{D}}(z,t)\,\mathrm{d}t.$$

This formula suggests that we first establish some relations between $G_{\mathscr{D}}(z,t)$ and $\omega_{\mathscr{D}}(\ ,z)$ before trying to find out whether the latter is in any way governed by $Y_{\mathscr{D}}(z)$.

We prove *three* such relations here. The first of them is very well known.

Theorem. *For* $w \in \mathscr{D}$,

$$G_{\mathscr{D}}(z,w) \;=\; \log\frac{1}{|z-w|} \;+\; \int_E \log|t-w|\,\mathrm{d}\omega_{\mathscr{D}}(t,z).$$

Proof. The right side of the asserted formula is identical with

$$\log\frac{1}{|z-w|} \;+\; \int_{\partial\mathscr{D}} \log|t-w|\,\mathrm{d}\omega_{\mathscr{D}}(t,z),$$

and, for *bounded* domains $\mathscr{D}$, this expression clearly coincides with $G_{\mathscr{D}}(z,w)$ – just fix $w \in \mathscr{D}$ and check boundary values for z on $\partial\mathscr{D}$! (This argument, and the formula, are due to George Green himself, by the way.)

In our situation, however, $\mathscr{D}$ *is not bounded*, and the *result is not true, in general, for unbounded domains.* (Not even for those with 'nice' boundaries; example:

$$\mathscr{D} \;=\; \{|z|>1\}\cup\{\infty\}.\ )$$

What is needed then in order for it to hold is the presence of 'enough' $\partial\mathscr{D}$ near ∞. That is what we must verify in the present case.

Fixing $w \in \mathscr{D}$, we proceed to find upper and lower bounds on the integral

$$\int_E \log|t-w|\,\mathrm{d}\omega_{\mathscr{D}}(t,z).$$

In order to get an *upper* bound, we take a function $h(z)$, positive and harmonic in $\mathscr{D}$ and continuous up to $\partial\mathscr{D}$, such that

$$h(z) \;=\; \log^{+}|z| + O(1).$$

In the case where E includes the interval $[-1, 1]$ (at which we can always arrive by translation), one may put

$$h(z) = \log|z + \sqrt{(z^2 - 1)}|$$

using, outside $[-1, 1]$, the determination of $\sqrt{}$ that is *positive* for $z = x > 1$. For large $A > 0$, let us write

$$h_A(z) = \min(h(z), A).$$

The function $h_A(t)$ is then *bounded and continuous* on E, so, by the elementary properties of harmonic measure (Chapter VII, §B.1), the function of z equal to

$$\int_E h_A(t) d\omega_{\mathscr{D}}(t, z)$$

is *harmonic* and *bounded above* in $\mathscr{D}$, and takes the boundary value $h_A(z)$ for z on $\partial\mathscr{D}$. The difference $\int_E h_A(t) d\omega_{\mathscr{D}}(t, z) - h(z)$ is thus bounded above in $\mathscr{D}$ and $\leqslant 0$ on $\partial\mathscr{D}$. Therefore, by the *extended principle of maximum* (Chapter III, §C), it is $\leqslant 0$ in $\mathscr{D}$, and we have

$$\int_E h_A(t)\, d\omega_{\mathscr{D}}(t, z) \leqslant h(z), \qquad z \in \mathscr{D}.$$

For $A' \geqslant A$, $h_{A'}(t) \geqslant h_A(t)$. Hence, by the preceding relation and Lebesque's monotone convergence theorem,

$$\int_E h(t)\, d\omega_{\mathscr{D}}(t, z) \leqslant h(z), \qquad z \in \mathscr{D};$$

that is,

$$\int_E \log^+|t| d\omega_{\mathscr{D}}(t, z) \leqslant \log^+|z| + O(1)$$

for $z \in \mathscr{D}$. When $w \in \mathscr{D}$ is fixed, we thus have the upper bound

$$\int_E \log|t - w| d\omega_{\mathscr{D}}(t, z) \leqslant \log^+|z| + O(1)$$

for z ranging over $\mathscr{D}$.

We can get some additional information with the help of the function $h(z)$. Indeed, for *each* A,

$$\int_E h_A(t)\, d\omega_{\mathscr{D}}(t, z) \leqslant \int_E h(t)\, d\omega_{\mathscr{D}}(t, z) \leqslant h(z)$$

when $z \in \mathscr{D}$. As we remarked above, the *left-hand* expression tends to $h_A(x_0)$

whenever $z \longrightarrow x_0 \in \partial\mathscr{D}$; at the same time, the *right-hand* member evidently tends to $h(x_0)$. Taking $A > h(x_0)$, we see that

$$\int_E h(t)\mathrm{d}\omega_{\mathscr{D}}(t, z) \longrightarrow h(x_0)$$

for $z \longrightarrow x_0 \in \partial\mathscr{D}$. On the other hand, for fixed $w \in \mathscr{D}$,

$$\log|t - w| - h(t)$$

is *continuous* and *bounded* on $\partial\mathscr{D}$. Therefore

$$\int_E (\log|t - w| - h(t))\mathrm{d}\omega_{\mathscr{D}}(t, z) \longrightarrow \log|x_0 - w| - h(x_0)$$

when $z \longrightarrow x_0 \in \partial\mathscr{D}$, so, on account of the previous relation, we have

$$\int_E \log|t - w|\mathrm{d}\omega_{\mathscr{D}}(t, z) \longrightarrow \log|x_0 - w|$$

for $z \longrightarrow x_0 \in \partial\mathscr{D}$.

To get a *lower* bound on the left-hand integral, let us, wlog, assume that $\Re z > 0$, and take an $R > 0$ sufficiently large to have $(-\infty, -R] \cup [R, \infty) \subseteq E$. Since $\mathscr{D} \supseteq \{\Im z > 0\}$, we have, for $|t| > R$,

$$\mathrm{d}\omega_{\mathscr{D}}(t, z) \geqslant \frac{1}{\pi}\frac{\Im z}{|z - t|^2}\mathrm{d}t$$

by the principle of *extension of domain* (Chapter VII, §B.1), the right side being just the differential of harmonic measure for the upper half plane. Hence,

$$\begin{aligned}\int_E \log|t + \mathrm{i}|\mathrm{d}\omega_{\mathscr{D}}(t, z) &\geqslant \int_{\{|t| \geqslant R\}} \log|t + \mathrm{i}|\mathrm{d}\omega_{\mathscr{D}}(t, z) \\ &\geqslant \frac{1}{\pi}\int_{|t| \geqslant R} \frac{\Im z \log|t + \mathrm{i}|}{|z - t|^2}\mathrm{d}t \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log|t + \mathrm{i}|}{|z - t|^2}\mathrm{d}t - O(1).\end{aligned}$$

The last integral on the right has, however, the value $\log|z + \mathrm{i}|$, as an elementary computation shows (contour integration). Thus,

$$\int_E \log|t + \mathrm{i}|\mathrm{d}\omega_{\mathscr{D}}(t, z) \geqslant \log|z + \mathrm{i}| - O(1)$$

for $\Im z > 0$, so, for fixed $w \in \mathscr{D}$,

$$\int_E \log|t-w|\,d\omega_{\mathscr{D}}(t,z) \;\geqslant\; \log^+|z| - O(1), \qquad z \in \mathscr{D}.$$

Taking any $w \in \mathscr{D}$, we see by the above that the function of z equal to

$$\log\frac{1}{|z-w|} \;+\; \int_E \log|t-w|\,d\omega_{\mathscr{D}}(t,z)$$

is *harmonic* in $\mathscr{D}$ save at w, differs in $\mathscr{D}$ by $O(1)$ from $\log(1/|z-w|) + \log^+|z|$, and assumes the *boundary value zero* on $\partial\mathscr{D}$. It is in particular *bounded above and below* outside of a neighborhood of w (point where it becomes infinite), and hence $\geqslant 0$ in $\mathscr{D}$ by the extended maximum principle. The expression just written thus has all the properties required of a Green's function for $\mathscr{D}$, and must coincide with $G_{\mathscr{D}}(z,w)$. We are done.

▶ *It will be convenient during the remainder of this § to take $d\omega_{\mathscr{D}}(t,z)$ as defined on all of $\mathbb{R}$, simply putting it equal to zero outside of E. This enables us to simplify our notation by writing $\omega_{\mathscr{D}}(S,z)$ for $\omega_{\mathscr{D}}(S\cap E,z)$ when $S \subseteq \mathbb{R}$.*

Lemma. *Let $0 \in \mathscr{D}$, and write*

$$\omega_{\mathscr{D}}(x) \;=\; \begin{cases} \omega_{\mathscr{D}}([x,\infty),\ 0), & x>0,\\ \omega_{\mathscr{D}}((-\infty,x],\ 0), & x<0 \end{cases}$$

(*note that $\omega_{\mathscr{D}}(x)$* need not be continuous *at* 0). *Then, for* $\Im z \neq 0$,

$$G_{\mathscr{D}}(z,0) \;=\; -\int_{-\infty}^{\infty} \frac{x-t}{(x-t)^2+y^2}\,\omega_{\mathscr{D}}(t)\operatorname{sgn} t\,dt.$$

Proof. By the preceding theorem and symmetry of the Green's function (proved at the end of §A.2), we have

$$G_{\mathscr{D}}(z,0) \;=\; G_{\mathscr{D}}(0,z) \;=\; \log\frac{1}{|z|} + \int_E \log|t-z|\,d\omega_{\mathscr{D}}(t,0).$$

Thanks to our convention, we can rewrite the right-hand integral as

$$\left(\int_{-\infty}^{0} + \int_{0}^{\infty}\right)\log|t-z|\,d\omega_{\mathscr{D}}(t,0).$$

Let us accept for the moment the inequality

$$\omega_{\mathscr{D}}(t) \;\leqslant\; \frac{\text{const.}}{|t|+1},$$

postponing its verification to the end of this proof. Then partial integration yields

$$\int_0^{\infty} \log|t-z|\,d\omega_{\mathscr{D}}(t,0) \;=\; \omega_{\mathscr{D}}(0+)\log|z| \;+\; \int_0^{\infty} \frac{t-x}{|t-z|^2}\,\omega_{\mathscr{D}}(t)\,dt,$$

and

$$\int_{-\infty}^{0} \log|t-z|\,d\omega_{\mathscr{D}}(t,0) \;=\; \omega_{\mathscr{D}}(0-)\log|z| \;-\; \int_{-\infty}^{0} \frac{t-x}{|t-z|^2}\,\omega_{\mathscr{D}}(t)\,dt.$$

Here,

$$\omega_{\mathscr{D}}(0+)+\omega_{\mathscr{D}}(0-) \;=\; \omega_{\mathscr{D}}((-\infty,\infty),\ 0) \;=\; \omega_{\mathscr{D}}(E,0) \;=\; 1,$$

so, adding, we get

$$\begin{aligned} G_{\mathscr{D}}(z,0) \;&=\; \log\frac{1}{|z|} + \int_{-\infty}^{\infty} \log|t-z|\,d\omega_{\mathscr{D}}(t,0) \\ &=\; \log\frac{1}{|z|} + \log|z| + \int_{-\infty}^{\infty} \frac{t-x}{|t-z|^2}\,\omega_{\mathscr{D}}(t)\,\mathrm{sgn}\,t\,dt \\ &=\; -\int_{-\infty}^{\infty} \frac{x-t}{|z-t|^2}\,\omega_{\mathscr{D}}(t)\,\mathrm{sgn}\,t\,dt, \end{aligned}$$

as claimed.

We still have to check the above inequality for $\omega_{\mathscr{D}}(t)$. To do this, pick an $R>0$ large enough to have

$$(-\infty,-R]\cup[R,\infty) \;\subseteq\; E,$$

and take a domain $\mathscr{E}$ equal to the *complement* of

$$(-\infty,-R]\cup[R,\infty)$$

in $\mathbb{C}$. Then $\mathscr{D}\subseteq\mathscr{E}$, so, by the *principle of extension of domain* (Chapter VII, §B.1), $\omega_{\mathscr{D}}(t)+\omega_{\mathscr{D}}(-t) \leqslant \omega_{\mathscr{E}}((-\infty,-t]\cup[t,\infty),\ 0)$ for $t>R$. The quantity on the right can, however, be worked out explicitly by mapping $\mathscr{E}$ conformally onto the unit disk so as to take $-R$ to -1, 0 to 0 and R to 1. In this way, one finds it to be $\leqslant CR/t$ (with a constant C independent of R), verifying the inequality in question. Details are left to the reader – he or she is referred to the proof of the *first* lemma from §A.1, where most of the computation involved here has already been done.

The integral figuring in the lemma just proved, viz.,

$$-\int_{-\infty}^{\infty} \frac{x-t}{|z-t|^2}\,\omega_{\mathscr{D}}(t)\,\mathrm{sgn}\,t\,dt$$

is like one used in the scholium of §H.1, Chapter III, to express a certain *harmonic conjugate.* It differs from the latter by its sign, by the absence of the constant $1/\pi$ in front, and because its integrand involves the factor $(x-t)/|z-t|^2$ instead of the sum

$$\frac{x-t}{|z-t|^2} + \frac{t}{t^2+1}.$$

In §H of Chapter III, the main purpose of the term $t/(t^2+1)$ was really *to ensure convergence*; *here*, since $\omega_{\mathscr{D}}(t)$ is $\mathrm{O}(1/(|t|+1))$, we already have convergence *without* it, and our *omission* of the term $t/(t^2+1)$ amounts merely to the *subtraction of a constant* from the value of the integral. Since harmonic conjugates are *only determined to within additive constants* anyway, *we may just as well take*

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x-t}{|z-t|^2}\,\omega_{\mathscr{D}}(t)\,\mathrm{sgn}\,t\,\mathrm{d}t$$

as the *harmonic conjugate* of

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\,\omega_{\mathscr{D}}(t)\,\mathrm{sgn}\,t\,\mathrm{d}t$$

in $\{\Im z > 0\}$. This brings the investigation of the former integral's boundary behavior on the real axis very close to the study of the *Hilbert transform* already touched on in Chapter III, §§F.2 and H.1.

In our present situation, we already know that, for real $x \neq 0$,

$$\lim_{y\to 0} G_{\mathscr{D}}(x+\mathrm{i}y,\ 0) \;=\; G_{\mathscr{D}}(x,0)$$

exists. The identity furnished by the lemma hence shows, *independently of the general considerations in the articles just mentioned*, that

$$\lim_{y\to 0}\int_{-\infty}^{\infty} \frac{x-t}{|z-t|^2}\,\omega_{\mathscr{D}}(t)\,\mathrm{sgn}\,t\,\mathrm{d}t$$

exists (and equals $-G_{\mathscr{D}}(x,0)$) for real $x \neq 0$. According to an observation in the scholium of §H.1, Chapter III, we can express the preceding limit as an *integral*, namely

$$\int_0^{\infty} \frac{\omega_{\mathscr{D}}(x-\tau)\,\mathrm{sgn}\,(x-\tau) - \omega_{\mathscr{D}}(x+\tau)\,\mathrm{sgn}\,(x+\tau)}{\tau}\,\mathrm{d}\tau.$$

That's because this expression *converges absolutely* for $x \neq 0$, on account of the above inequality for $\omega_{\mathscr{D}}(t)$ and also of the

Lemma. *Let* $0 \in \mathscr{D}$. *Then* $\omega_{\mathscr{D}}(t)$ *is* $\mathrm{Lip}\,\frac{1}{2}$ *for* $t > 0$ *and for* $t < 0$.

Proof. The statement amounts to the claim that

$$\omega_{\mathscr{D}}(I,0) \;\leqslant\; \text{const.}\sqrt{|I|}$$

for any small interval $I \subseteq E$. To show this, take any *interval* $J_0 \subseteq E$ and consider small intervals $I \subseteq J_0$. Letting $\mathscr{E}$ be the region $(\mathbb{C} \cup \{\infty\}) \sim J_0$, the usual application of the *principle of extension of domain* gives us

$$\omega_{\mathscr{D}}(I,0) \;\leqslant\; \omega_{\mathscr{E}}(I,0),$$

with, in turn,

$$\omega_{\mathscr{E}}(I,0) \;\leqslant\; \text{const.}\,\omega_{\mathscr{E}}(I,\infty)$$

by Harnack's theorem.

To simplify the estimate of the right side of the *last* inequality, we may take J_0 to be $[-1,1]$; this just amounts to making a preliminary translation and change of scale – *never mind* here that $0 \in \mathscr{D}$! Then one can map $\mathscr{E}$ onto the unit disk by the Joukowski transformation

$$z \;\longrightarrow\; z - \sqrt{(z^2-1)}$$

which takes ∞ to 0, -1 to -1, and 1 to 1. In this way one easily finds that

$$\omega_{\mathscr{E}}(I,\infty) \;\leqslant\; \text{const.}\sqrt{|I|},$$

proving the lemma.

Remark. The square root is *only* necessary when I is *near one of the endpoints* of J_0. For small intervals I near the *middle* of J_0, $\omega_{\mathscr{E}}(I,\infty)$ acts like a multiple of $|I|$.

By the above two lemmas and related discussion, we have the formula

$$G_{\mathscr{D}}(x,0) \;=\; -\int_0^\infty \frac{\omega_{\mathscr{D}}(x-\tau)\operatorname{sgn}(x-\tau) - \omega_{\mathscr{D}}(x+\tau)\operatorname{sgn}(x+\tau)}{\tau}\,d\tau,$$

valid for $x \neq 0$ if 0 belongs to $\mathscr{D}$. It is customary to write the right-hand member in a different way. That expression is identical with

$$-\lim_{\delta\to 0}\int_{|t-x|\geqslant\delta} \frac{\omega_{\mathscr{D}}(t)\operatorname{sgn} t}{x-t}\,dt.$$

If a function $f(t)$, having a possible singularity at $a \in \mathbb{R}$, is integrable over each set of the form $\{|t-a| \geqslant \delta\}$, $\delta > 0$, and *if*

$$\lim_{\delta\to 0}\int_{|t-a|\geqslant\delta} f(t)\,dt$$

exists, that limit is called a Cauchy principal value, and denoted by

$$⨍_{-\infty}^{\infty} f(t)\,dt \qquad \text{or by} \qquad \text{v.p.}\int_{-\infty}^{\infty} f(t)\,dt.$$

It is important to realize that $⨍_{-\infty}^{\infty} f(t)\,\mathrm{d}t$ *is frequently not an integral in the ordinary sense.*

In terms of this notation, the formula for $G_{\mathscr{D}}(x,0)$ just obtained can be expressed as in the following

Theorem. *Let* $0 \in \mathscr{D}$. *Then, for real* $x \neq 0$,

$$\boxed{G_{\mathscr{D}}(x,0) \;=\; -⨍_{-\infty}^{\infty} \frac{\omega_{\mathscr{D}}(t)\operatorname{sgn} t}{x-t}\,\mathrm{d}t,}$$

where $\omega_{\mathscr{D}}(t)$ *is the function defined in the first of the above two lemmas.*

This result will be used in article 3 below. Now, however, we wish to use it to *solve for* $\omega_{\mathscr{D}}(t)\operatorname{sgn} t$ *in terms of* $G_{\mathscr{D}}(x,0)$, obtaining the relation

$$\omega_{\mathscr{D}}(t)\operatorname{sgn} t \;=\; \frac{1}{\pi^2}⨍_{-\infty}^{\infty} \frac{G_{\mathscr{D}}(x,0)}{t-x}\,\mathrm{d}x.$$

By the inversion theorem for the L_2 Hilbert transform, the latter formula is indeed a consequence of the boxed one above. Here, a direct proof is not very difficult, and we give one for the reader who does not know the inversion theorem.

Lemma. $\int_{-\infty}^{\infty} |G_{\mathscr{D}}(x+\mathrm{i}y,\ 0) - G_{\mathscr{D}}(x,0)|\,\mathrm{d}x \;\longrightarrow\; 0$ *for* $y \to 0$.

Proof. The result follows immediately from the representation

$$G_{\mathscr{D}}(x+\mathrm{i}y,\ 0) \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yG_{\mathscr{D}}(t,0)}{(x-t)^2+y^2}\,\mathrm{d}t, \qquad y>0,$$

by elementary properties of the Poisson kernel, in the usual way.

The representation itself is practically obvious; here is one derivation. From the first theorem of this article,

$$G_{\mathscr{D}}(t,0) \;=\; \log\frac{1}{|t|} + \int_E \log|s-t|\,\mathrm{d}\omega_{\mathscr{D}}(s,0)$$

and

$$G_{\mathscr{D}}(z,0) \;=\; \log\frac{1}{|z|} + \int_E \log|s-z|\,\mathrm{d}\omega_{\mathscr{D}}(s,0).$$

For $\Im z > 0$, we have the elementary formula

$$\log|s-z| \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log|s-t|}{|z-t|^2}\,\mathrm{d}t, \qquad s \in \mathbb{R}.$$

Use this in the right side of the preceding relation (in *both* right-hand terms !), change the order of integration (which is easily justified here), and then refer to the formula for $G_{\mathscr{D}}(t,0)$ just written. One ends with the relation in question.

Lemma. *Let* $0 \in \mathscr{D}$. *Then* $G_{\mathscr{D}}(x,0)$ *is* $\mathrm{Lip}\frac{1}{2}$ *for* $x > 0$ *and for* $x < 0$.

Proof. The open intervals of $\mathbb{R} \sim E$ belong to $\mathscr{D}$, where $G_{\mathscr{D}}(z,0)$ is *harmonic* (save at 0), and hence $\mathscr{C}_{\infty}$. So $G_{\mathscr{D}}(x,0)$ is certainly $\mathscr{C}_1$ (hence Lip 1) in the *interior* of each of those open segments (although *not uniformly* so!) for x outside any neighborhood of 0. Also, $G_{\mathscr{D}}(x,0) = 0$ on each of the *closed segments* making up E; it is thus surely Lip 1 on the interior of each of *those.*

Our claim therefore boils down to the statement that

$$|G_{\mathscr{D}}(x,0) - G_{\mathscr{D}}(a,0)| \;\leqslant\; \text{const.}\sqrt{|x-a|}$$

near any of the endpoints a of any of the segments making up E. Since $G_{\mathscr{D}}(a,0) = 0$, we have to show that

$$G_{\mathscr{D}}(x,0) \;\leqslant\; \text{const.}\sqrt{|x-a|}$$

for $x \in \mathbb{R} \sim E$ *near* such an endpoint a.

Assume, wlog, that a is a *right* endpoint of a component of E and that $x > a$. Pick $b < a$ such that

$$[b,a] \;\subseteq\; E$$

and denote the domain $(\mathbb{C} \cup \{\infty\}) \sim [b,a]$ by $\mathscr{E}$. We have $\mathscr{D} \subseteq \mathscr{E}$, so

$$G_{\mathscr{D}}(x,0) \;\leqslant\; G_{\mathscr{E}}(x,0)$$

by the principle of extension of domain. Here, one may compute $G_{\mathscr{E}}(x,0)$ by mapping $\mathscr{E}$ onto the unit disk conformally with the help of a Joukowski transformation. In this way one finds without much difficulty that

$$G_{\mathscr{E}}(x,0) \;\leqslant\; \text{const.}\sqrt{(x-a)}$$

for $x > a$, proving the lemma. (Cf. proof of the lemma immediately preceding the previous theorem.)

Theorem. *Let* $0 \in \mathscr{D}$. *Then, for* $x \neq 0$,

$$\omega_{\mathscr{D}}(x)\,\mathrm{sgn}\,x \;=\; \frac{1}{\pi^2}\int_{-\infty}^{\infty} \frac{G_{\mathscr{D}}(t,0)}{x-t}\,\mathrm{d}t,$$

where $\omega_{\mathscr{D}}(x)$ *is the function defined in the* first *lemma of this article.*

Proof. By the first of the preceding lemmas, for $t \in \mathbb{R}$ and $h > 0$,

$$G_{\mathscr{D}}(t+\mathrm{i}h,\;0) \;=\; -\int_{-\infty}^{\infty} \frac{t-\xi}{(t-\xi)^2+h^2}\,\omega_{\mathscr{D}}(\xi)\,\mathrm{sgn}\,\xi\,\mathrm{d}\xi.$$

Multiply both sides by

$$\frac{x-t}{(x-t)^2+y^2}$$

and integrate the variable t. We get

$$\int_{-\infty}^{\infty}\frac{x-t}{(x-t)^2+y^2}G_{\mathscr{D}}(t+ih,\ 0)\,dt$$
$$= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{x-t}{(x-t)^2+y^2}\cdot\frac{t-\xi}{(t-\xi)^2+h^2}\,\omega_{\mathscr{D}}(\xi)\operatorname{sgn}\xi\,d\xi\,dt.$$

Suppose for the moment that *absolute convergence* of the double integral has been established. Then we can change the order of integration therein. We have, however, for $y>0$,

$$\int_{-\infty}^{\infty}\frac{(x-t)}{(x-t)^2+y^2}\cdot\frac{t-\xi}{(t-\xi)^2+h^2}\,dt \;=\; -\pi\frac{y+h}{(x-\xi)^2+(y+h)^2},$$

as follows from the identity

$$\int_{-\infty}^{\infty}\frac{1}{x+iy-t}\cdot\frac{1}{\xi+ih-t}\,dt \;=\; 0,$$

verifiable by contour integration (h and y are >0 here), and the semigroup convolution property of the Poisson kernel. The previous relation thus becomes

$$\int_{-\infty}^{\infty}\frac{x-t}{(x-t)^2+y^2}G_{\mathscr{D}}(t+ih,\ 0)\,dt$$
$$= \pi\int_{-\infty}^{\infty}\frac{y+h}{(x-\xi)^2+(y+h)^2}\,\omega_{\mathscr{D}}(\xi)\operatorname{sgn}\xi\,d\xi.$$

Fixing $y>0$ for the moment, make $h\to 0$. According to the *third* of the above lemmas, the last formula then becomes

$$\int_{-\infty}^{\infty}\frac{x-t}{(x-t)^2+y^2}G_{\mathscr{D}}(t,0)\,dt \;=\; \pi\int_{-\infty}^{\infty}\frac{y}{(x-\xi)^2+y^2}\,\omega_{\mathscr{D}}(\xi)\operatorname{sgn}\xi\,d\xi.$$

Now make $y\to 0$, assuming that $x\neq 0$. Since $\omega_{\mathscr{D}}(\xi)$ is continuous at x, the *right side* tends to

$$\pi^2\omega_{\mathscr{D}}(x)\operatorname{sgn}x.$$

Also, by the *fourth* lemma, $G_{\mathscr{D}}(t,0)$ is Lip $\frac{1}{2}$ at x. The left-hand integral therefore tends to the Cauchy principal value

$$\fint_{-\infty}^{\infty} \frac{G_{\mathscr{D}}(t,0)}{x-t}\,\mathrm{d}t$$

(which exists !), according to an observation in §H.1 of Chapter III and the discussion preceding the last theorem above. We thus have

$$\omega_{\mathscr{D}}(x)\operatorname{sgn} x \;=\; \frac{1}{\pi^2}\fint_{-\infty}^{\infty} \frac{G_{\mathscr{D}}(t,0)}{x-t}\,\mathrm{d}t$$

for $x \neq 0$, as asserted.

The legitimacy of the above reasoning required absolute convergence of

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{x-t}{(x-t)^2+y^2}\cdot\frac{t-\xi}{(t-\xi)^2+h^2}\,\omega_{\mathscr{D}}(\xi)\operatorname{sgn}\xi\,\mathrm{d}t\,\mathrm{d}\xi$$

which we must now establish. Fixing y and $h > 0$ and $x \in \mathbb{R}$, we have

$$\left|\frac{x-t}{(x-t)^2+y^2}\cdot\frac{t-\xi}{(t-\xi)^2+h^2}\right| \;\leqslant\; \frac{\text{const.}}{(|t|+1)(|\xi-t|+1)}.$$

Wlog, let $\xi > 0$. Then

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{(|t|+1)(|\xi-t|+1)} \;\leqslant\; 2\int_0^{\infty} \frac{\mathrm{d}t}{(t+1)(|\xi-t|+1)},$$

which we break up in turn as

$$2\int_0^{\xi/2} \;+\; 2\int_{\xi/2}^{3\xi/2} \;+\; 2\int_{3\xi/2}^{\infty}.$$

In the *first* of these integrals we use the inequality

$$|\xi - t| \;\geqslant\; \xi/2,$$

and, in the *second*,

$$t \;\geqslant\; \xi/2,$$

taking in the latter a new variable $s = t - \xi$. Both are thus easily seen to have values

$$\leqslant\; \text{const.}\frac{\log^+\xi + 1}{\xi + 1}$$

In the *third* integral, use the relation

$$t - \xi \;\geqslant\; t/3.$$

This shows that expression to be $\leqslant$ const.$1/(\xi + 1)$.

In fine, then,

$$\int_{-\infty}^{\infty}\left|\frac{x-t}{(x-t)^2+y^2}\cdot\frac{t-\xi}{(t-\xi)^2+h^2}\right|\mathrm{d}t \;\leqslant\; \text{const.}\frac{\log^+|\xi|+1}{|\xi|+1}$$

for *fixed* $x \in \mathbb{R}$ and $y, h > 0$. From the proof of the *first* lemma in this article, we know, however, that

$$|\omega_{\mathscr{D}}(\xi)\operatorname{sgn}\xi| \;=\; \omega_{\mathscr{D}}(\xi) \;\leqslant\; \frac{\text{const.}}{|\xi|+1}.$$

Absolute convergence of our double integral thus depends on the convergence of

$$\int_{-\infty}^{\infty} \frac{1+\log^+|\xi|}{(|\xi|+1)^2}\,\mathrm{d}\xi$$

which evidently *holds*. Our proof is complete.

Notation. *If* $\mathscr{D}$ is one of our domains with $0 \in \mathscr{D}$, we write, for $x > 0$,

$$\boxed{\Omega_{\mathscr{D}}(x) \;=\; \omega_{\mathscr{D}}((-\infty, -x] \cup [x, \infty),\; 0).}$$

Further work in this § will be based on the function $\Omega_{\mathscr{D}}$. For it, the theorem just proved has the

Corollary. *If* $0 \in \mathscr{D}$,

$$\boxed{\Omega_{\mathscr{D}}(x) \;=\; \frac{2}{\pi^2}\int_{-\infty}^{\infty} \frac{xG_{\mathscr{D}}(t,0)}{x^2-t^2}\,\mathrm{d}t \qquad \text{for } x > 0.}$$

Proof. When $x > 0$,

$$\Omega_{\mathscr{D}}(x) \;=\; \omega_{\mathscr{D}}(x) + \omega_{\mathscr{D}}(-x).$$

Plug the formula furnished by the theorem into the right side.

Scholium. The preceding arguments practically suffice to work up a complete treatment of the L_2 theory of Hilbert transforms. The reader who has never studied that theory thus has an opportunity to learn it now.

If $f \in L_2(-\infty, \infty)$, let us write

$$u(z) \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\,f(t)\,\mathrm{d}t$$

and

$$\tilde{u}(z) \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x-t}{|z-t|^2}\,f(t)\,\mathrm{d}t$$

for $\Im z > 0$; $\tilde{u}(z)$ is a *harmonic conjugate* of $u(z)$ in the upper half plane. By taking Fourier transforms and using Plancherel's theorem, one easily checks that

$$\int_{-\infty}^{\infty} |\tilde{u}(x+\mathrm{i}y)|^2\,\mathrm{d}x \;\leqslant\; \|f\|_2^2$$

for each $y > 0$. Following a previous discussion in this article and those of §§F.2 and H.1, Chapter III, we also see that

$$\tilde{f}(x) \;=\; \lim_{y\to 0} \tilde{u}(x+\mathrm{i}y)$$

exists a.e. Fatou's lemma then yields

$$\|\tilde{f}\|_2 \;\leqslant\; \|f\|_2$$

in view of the previous inequality.

It is in fact true that

$$\int_{-\infty}^{\infty} |\tilde{f}(x) - \tilde{u}(x+\mathrm{i}y)|^2\,\mathrm{d}x \;\longrightarrow\; 0$$

for $y \to 0$. This may be seen by noting that

$$\int_{-\infty}^{\infty} |\tilde{u}(x+\mathrm{i}y) - \tilde{u}(x+\mathrm{i}y')|^2\,\mathrm{d}x \;=\; \int_{-\infty}^{\infty} |u(x+\mathrm{i}y) - u(x+\mathrm{i}y')|^2\,\mathrm{d}x$$

for y and $y' > 0$, which may be verified using Fourier transforms and Plancherel's theorem. According to elementary properties of the Poisson kernel, the right-hand integral is *small* when $y > 0$ and $y' > 0$ are, as long as $f \in L_2$. Fixing a small $y > 0$ and then making $y' \to 0$ in the *left-hand* integral, we find that

$$\int_{-\infty}^{\infty} |\tilde{f}(x) - \tilde{u}(x+\mathrm{i}y)|^2\,\mathrm{d}x$$

is small by applying Fatou's lemma.

Once this is known, it is easy to prove that

$$\tilde{\tilde{f}}(x) \;=\; -f(x) \quad \text{a.e.}$$

by following almost exactly the argument used in proving the last theorem above. (Note that $(\log^+|\xi|+1)/(|\xi|+1) \in L_2(-\infty,\infty)$.) This must then imply that

$$\|f\|_2 \;\leqslant\; \|\tilde{f}\|_2,$$

so that finally

$$\|f\|_2 = \|\tilde{f}\|_2.$$

To complete this development, we need the result that the Cauchy principal value

$$\frac{1}{\pi}\fint_{-\infty}^{\infty}\frac{f(t)}{x-t}\,dt$$

exists and equals $\tilde{f}(x)$ a.e. That is the content of

Problem 25

Let $f \in L_p(-\infty,\infty)$, $p \geqslant 1$. Show that

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x-t}{(x-t)^2+y^2}f(t)\,dt \;-\; \frac{1}{\pi}\int_{|t-x|\geqslant y}\frac{f(t)}{x-t}\,dt$$

tends to zero as $y \to 0$ if

$$\frac{1}{y}\int_{x-y}^{x+y}|f(t)-f(x)|\,dt \longrightarrow 0$$

for $y \to 0$, and hence for *almost every* real x. (The set of x for which the last condition holds is called the *Lebesgue set* of f.) (Hint. One may wlog take f to be of compact support, making $\|f\|_1 < \infty$. Choosing a small $\delta > 0$, one considers values of y *between* 0 and δ, for which the difference in question can be written as

$$\frac{1}{\pi}\int_0^y \frac{\tau(f(x-\tau)-f(x+\tau))}{\tau^2+y^2}\,d\tau$$

$$+\;\frac{1}{\pi}\left(\int_y^{\delta}+\int_{\delta}^{\infty}\right)\left(\frac{\tau}{\tau^2+y^2}-\frac{1}{\tau}\right)(f(x-\tau)-f(x+\tau))\,d\tau.$$

If the stipulated condition holds at x, the *first* of these integrals clearly $\to 0$ as $y \to 0$. For *fixed* $\delta > 0$, the integral from δ to ∞ is $\leqslant 2y^2\|f\|_1/\delta^3$ and this $\to 0$ as $y \to 0$. The integral from y to δ is in absolute value

$$\leqslant y^2\int_y^{\delta}\frac{|f(x-\tau)-f(x+\tau)|}{\tau^3}\,d\tau.$$

Integrate this by parts.)

2. An estimate for harmonic measure

Given one of our domains $\mathscr{D}$ with $0 \in \mathscr{D}$, the function $\Omega_{\mathscr{D}}(x) = \omega_{\mathscr{D}}((-\infty,-x]\cup[x,\infty),\,0)$ is equal to

$$\frac{2}{\pi^2}\fint_{-\infty}^{\infty}\frac{xG_{\mathscr{D}}(t,0)}{x^2-t^2}\,dt$$

by the corollary near the end of the preceding article. The Green's function $G_{\mathscr{D}}(t, 0)$ of course vanishes on $\partial\mathscr{D} = \mathbb{R} \cap (\sim \mathscr{D})$, and our attention is restricted to domains $\mathscr{D}$ having *bounded* intersection with $\mathbb{R}$. The above Cauchy principal value thus reduces to an ordinary integral for large x, and we have

$$\Omega_{\mathscr{D}}(x) \sim \frac{2}{\pi^2 x}\int_{-\infty}^{\infty} G_{\mathscr{D}}(t, 0)\,dt \qquad \text{for } x \to \infty,$$

i.e., *in terms of the Phragmén–Lindelöf function* $Y_{\mathscr{D}}(z)$ *for* $\mathscr{D}$, defined in §A.2,

$$\Omega_{\mathscr{D}}(x) \sim \frac{2Y_{\mathscr{D}}(0)}{\pi x}, \qquad x \to \infty.$$

It is remarkable that an *inequality* resembling this asymptotic relation holds for *all* positive x; this means that the kind of comparison spoken of at the beginning of the present § is available.

Theorem. *If* $0 \in \mathscr{D}$,

$$\Omega_{\mathscr{D}}(x) \leqslant \frac{Y_{\mathscr{D}}(0)}{x} \qquad \textit{for } x > 0.$$

Proof. By comparison of harmonic measure for $\mathscr{D}$ with that for another smaller domain that depends on x.

Given $x > 0$, we let $E_x = E \cup (-\infty, -x] \cup [x, \infty)$ and then put $\mathscr{D}_x = \mathbb{C} \sim E_x$:

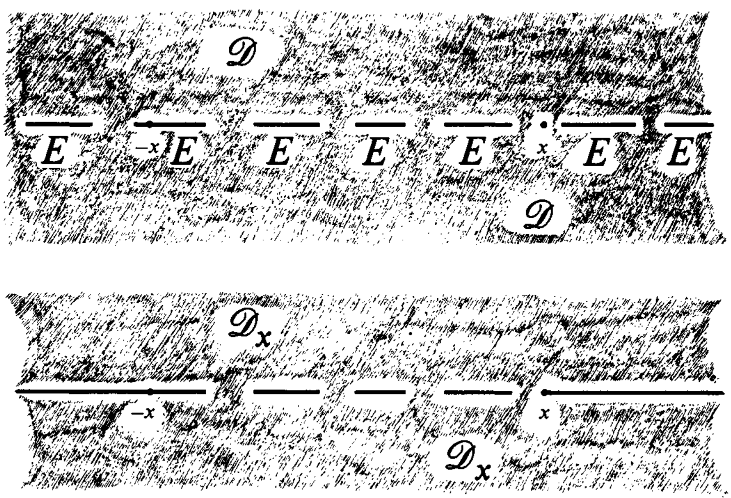

Figure 152

We have $\mathscr{D}_x \subseteq \mathscr{D}$. On comparing $\omega_{\mathscr{D}_x}((-\infty, -x] \cup [x, \infty), \zeta)$ with $\omega_{\mathscr{D}}((-\infty, -x] \cup [x, \infty), \zeta)$ on E_x, we see that the *former* is *larger* than the *latter* for $\zeta\in\mathscr{D}_x$. Hence, putting $\zeta=0$, we get

$$\Omega_{\mathscr{D}}(x) \;\leqslant\; \Omega_{\mathscr{D}_x}(x).$$

Take any number $\rho>1$. Applying the corollary near the end of the previous article and noting that $G_{\mathscr{D}_x}(t,0)$ *vanishes* for $t\in E_x \supseteq (-\infty,-x]\cup[x,\infty)$, we have

$$\Omega_{\mathscr{D}_x}(\rho x) \;=\; \frac{2}{\pi^2}\int_{-x}^{x}\frac{\rho x G_{\mathscr{D}_x}(t,0)}{\rho^2x^2-t^2}\,\mathrm{d}t.$$

Since $\mathscr{D}_x\subseteq\mathscr{D}$, $G_{\mathscr{D}_x}(t,0)\leqslant G_{\mathscr{D}}(t,0)$, so the right-hand integral is

$$\leqslant\; \frac{2\rho}{\pi^2(\rho^2-1)x}\int_{-x}^{x}G_{\mathscr{D}}(t,0)\,\mathrm{d}t \;\leqslant\; \frac{2\rho}{\pi^2(\rho^2-1)x}\int_{-\infty}^{\infty}G_{\mathscr{D}}(t,0)\,\mathrm{d}t.$$

By the formula for $Y_{\mathscr{D}}(z)$ furnished by the *second* theorem of §A2, we thus get

$$\Omega_{\mathscr{D}_x}(\rho x) \;\leqslant\; \frac{2\rho}{\pi(\rho^2-1)}\frac{Y_{\mathscr{D}}(0)}{x}.$$

In order to complete the proof, we show that $\Omega_{\mathscr{D}_x}(\rho x)/\Omega_{\mathscr{D}_x}(x)$ is *bounded below* by a quantity *depending only on* ρ, and then use the inequality just established together with the previous one.

To compare $\Omega_{\mathscr{D}_x}(\rho x)$ with $\Omega_{\mathscr{D}_x}(x)$, take a *third* domain

$$\mathscr{E} \;=\; \mathbb{C}\sim((-\infty,-x]\cup[x,\infty)):$$

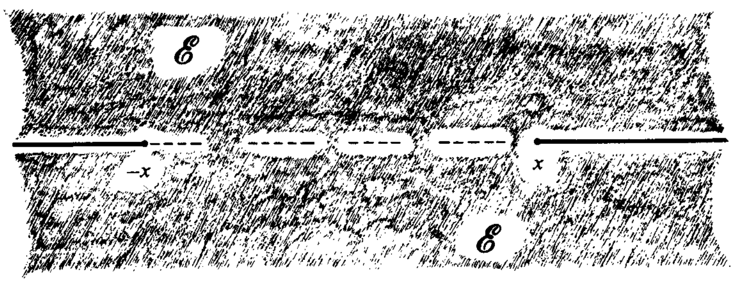

Figure 153

Note that $\mathscr{D}_x\subseteq\mathscr{E}$ and $\partial\mathscr{D}_x=E_x$ consists of $\partial\mathscr{E}$ together with the part of E lying in the segment $[-x,x]$. For $\zeta\in\mathscr{D}_x$ (and $\rho>1$), a formula from §B.1 of Chapter VII tells us that

$$\begin{aligned}
&\omega_{\mathscr{D}_x}((-\infty,-\rho x]\cup[\rho x,\infty),\ \zeta)\\
&\quad=\ \omega_{\mathscr{E}}((-\infty,-\rho x]\cup[\rho x,\infty),\ \zeta)\\
&\qquad\qquad -\int_{E\cap\mathscr{E}}\omega_{\mathscr{E}}((-\infty,-\rho x]\cup[\rho x,\infty),\ t)\,\mathrm{d}\omega_{\mathscr{D}_x}(t,\zeta),
\end{aligned}$$

whence, taking $\zeta = 0$,

$$\Omega_{\mathscr{D}_x}(\rho x) = \omega_{\mathscr{E}}((-\infty, -\rho x] \cup [\rho x, \infty), 0)$$
$$- \int_{E \cap \mathscr{E}} \omega_{\mathscr{E}}((-\infty, -\rho x] \cup [\rho x, \infty), t) d\omega_{\mathscr{D}_x}(t, 0).$$

Also,

$$\Omega_{\mathscr{D}_x}(x) = 1 - \int_{E \cap \mathscr{E}} d\omega_{\mathscr{D}_x}(t, 0).$$

The harmonic measure $\omega_{\mathscr{E}}((-\infty, -\rho x] \cup [\rho x, \infty), t)$ can be computed explicitly by making the Joukowski mapping

$$\zeta \longrightarrow w = \frac{x}{\zeta} - \sqrt{\left(\frac{x^2}{\zeta^2} - 1\right)}$$

of $\mathscr{E}$ onto $\Delta = \{|w| < 1\}$:

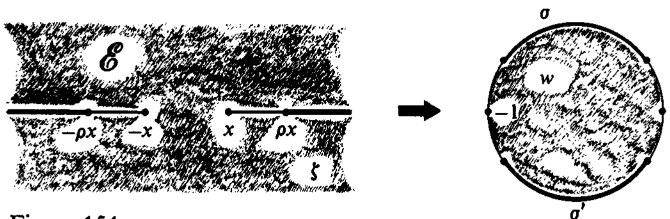

Figure 154

This conformal map takes $[-x, x]$ to the diameter $[-1, 1]$, and 0 to 0. The union of the (two-sided!) intervals $(-\infty, -\rho x]$ and $[\rho x, \infty)$ on $\partial\mathscr{E}$ is taken onto that of two arcs, σ and σ', on $\{|w| = 1\}$, the first symmetric about i and the second symmetric about $-$i. For $\zeta \in \mathscr{E}$, $\omega_{\mathscr{E}}((-\infty, -\rho x] \cup [\rho x, \infty), \zeta)$ is the *sum* of the harmonic measures of these two arcs in Δ, seen from the point w therein corresponding to ζ. When $\zeta = t$ is *real*, this sum is just $2\omega_\Delta(\sigma, u)$, u being the point of $(-1, 1)$ corresponding to t. However, from the rudiments of complex variable theory, the level lines of $\omega_\Delta(\sigma, w)$ are just the *circles* through the endpoints of σ. From a glance at the following diagram, it is hence obvious that $\omega_\Delta(\sigma, u)$ has its *maximum* for $-1 \leqslant u \leqslant 1$ when $u = 0$:

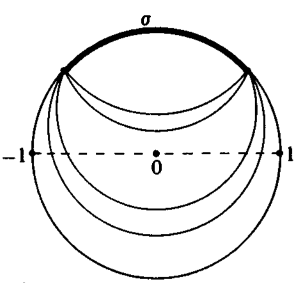

Figure 155

Going back to $\mathscr{E}$, we see that

$$\omega_{\mathscr{E}}((-\infty, -\rho x] \cup [\rho x, \infty), \ t) \ \leqslant \ \omega_{\mathscr{E}}((-\infty, -\rho x] \cup [\rho x, \infty), \ 0)$$

when $-x \leqslant t \leqslant x$. Plugging this into the above formula for $\Omega_{\mathscr{D}_x}(\rho x)$, we find that

$$\Omega_{\mathscr{D}_x}(\rho x) \ \geqslant \ \omega_{\mathscr{E}}((-\infty, -\rho x] \cup [\rho x, \infty), \ 0) \times \left\{ 1 \ - \ \int_{E \cap \mathscr{E}} \mathrm{d}\omega_{\mathscr{D}_x}(t, 0) \right\}.$$

The quantity in curly brackets is just $\Omega_{\mathscr{D}_x}(x)$, so we have

$$\frac{\Omega_{\mathscr{D}_x}(\rho x)}{\Omega_{\mathscr{D}_x}(x)} \ \geqslant \ \omega_{\mathscr{E}}((-\infty, -\rho x] \cup [\rho x, \infty), \ 0).$$

Here, the right side clearly depends only on ρ; this is the relation we set out to obtain.

From the inequality just found together with the two others established at the beginning of this proof, we now get

$$\Omega_{\mathscr{D}}(x) \ \leqslant \ \Omega_{\mathscr{D}_x}(x) \ \leqslant \ \frac{\Omega_{\mathscr{D}_x}(\rho x)}{\omega_{\mathscr{E}}((-\infty, -\rho x] \cup [\rho x, \infty), \ 0)}$$

$$\leqslant \ \frac{2\rho}{\pi(\rho^2 - 1)\omega_{\mathscr{E}}((-\infty, -\rho x] \cup [\rho x, \infty), \ 0)} \cdot \frac{Y_{\mathscr{D}}(0)}{x}.$$

The front factor in the right-hand member depends only on the parameter ρ; let us compute its value. The two arcs σ and σ' both subtend angles $2\arcsin(\Gamma/\rho)$ at 0. Therefore

$$\omega_{\mathscr{E}}((-\infty, -\rho x] \cup [\rho x, \infty), \ 0) \ = \ 2\omega_{\Delta}(\sigma, 0) \ = \ \frac{2}{\pi}\arcsin\frac{1}{\rho},$$

and the factor in question equals

$$\frac{\rho}{(\rho^2 - 1)\arcsin\dfrac{1}{\rho}}.$$

It is readily ascertained (put $1/\rho = \sin\alpha$!) that the expression just written *decreases* for $\rho > 1$. Making $\rho \to \infty$, we get the limit 1, whence

$$\Omega_{\mathscr{D}}(x) \ \leqslant \ Y_{\mathscr{D}}(0)/x, \qquad \text{Q.E.D.}$$

Remark. An inequality almost as good as the one just established can be obtained with considerably less effort. By the first theorem of the preceding article, we have, for $y > 0$,

$$G_{\mathscr{D}}(\mathrm{i}y, 0) = \log\frac{1}{y} + \int_{-\infty}^{\infty} \log|\mathrm{i}y - t|\,\mathrm{d}\omega_{\mathscr{D}}(t, 0)$$

$$= \int_{-\infty}^{\infty} \log\sqrt{\left(1 + \frac{t^2}{y^2}\right)}\,\mathrm{d}\omega_{\mathscr{D}}(t, 0),$$

a quantity clearly $\geqslant \Omega_{\mathscr{D}}(y)\log\sqrt{2}$. On the other hand,

$$G_{\mathscr{D}}(\mathrm{i}y, 0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yG_{\mathscr{D}}(t, 0)}{y^2 + t^2}\,\mathrm{d}t$$

as in the proof of the third lemma from that article. Here, the right side is

$$\leqslant \frac{1}{\pi y}\int_{-\infty}^{\infty} G_{\mathscr{D}}(t, 0)\,\mathrm{d}t = \frac{Y_{\mathscr{D}}(0)}{y},$$

so the previous relation yields

$$\Omega_{\mathscr{D}}(y) \leqslant \frac{2}{\log 2}\frac{Y_{\mathscr{D}}(0)}{y}.$$

Problem 26

For $0 < \rho < \frac{1}{2}$, let E_ρ be the union of the segments

$$\left[\frac{2n-1}{2} - \rho,\quad \frac{2n-1}{2} + \rho\right], \qquad n \in \mathbb{Z};$$

these are just the intervals of length 2ρ centered at the *half odd* integers. Denote the component $[(2n-1)/2 - \rho,\ (2n-1)/2 + \rho]$ of E_ρ by J_n (it would be more logical to write $J_n(\rho)$). $\mathscr{D}_\rho = \mathbb{C} \sim E_\rho$ is a domain of the kind considered in §A, and, by *Carleson's theorem* from §A.1,

$$\omega_{\mathscr{D}_\rho}(J_n, 0) \leqslant \frac{K_\rho}{n^2 + 1}.$$

The purpose of this problem is to obtain quantitative information about the asymptotic behaviour of the best value for K_ρ as $\rho \to 0$.

(a) Show that $Y_{\mathscr{D}_\rho}(0) \sim (1/\pi)\log(1/\rho)$ as $\rho \to 0$. (Hint. In $\mathscr{D}_\rho$, consider the harmonic function

$$\log\left|\frac{\cos\pi z}{\sin\pi\rho} + \sqrt{\left(\frac{\cos^2\pi z}{\sin^2\pi\rho} - 1\right)}\right|. \text{)}$$

(b) By making an appropriate limiting argument, adapt the theorem just proved to the domain $\mathscr{D}_\rho$ and hence show that

$$\Omega_{\mathscr{D}_\rho}(x) \leqslant Y_{\mathscr{D}_\rho}(0)/x \qquad \text{for } x > 0.$$

(c) For $n \geqslant 1$, show that

$$\omega_{\mathscr{D}_\rho}(J_{n+1},\ 0) \ \leqslant\ \omega_{\mathscr{D}_\rho}(J_n,\ 0).$$

(Hint:

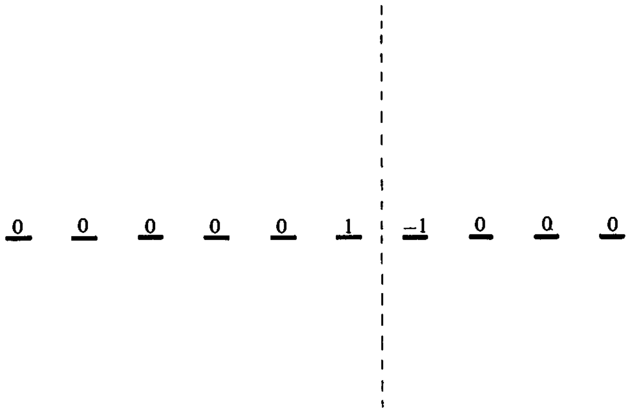

Figure 156 .)

(d) Hence show that, for $n \geqslant 3$,

$$\omega_{\mathscr{D}_\rho}(J_n, 0) \ \leqslant\ \left(C \log\frac{1}{\rho}\right)\Big/ n^2 .)$$

with a numerical constant C independent of ρ.
(Hint: $\Omega_{\mathscr{D}_\rho}(n) \ \geqslant\ 2\sum_{k=n}^{2n} \omega_{\mathscr{D}_\rho}(J_{k+1},\ 0)$.)

(e) Show that the *smallest* constant K_ρ such that $\omega_{\mathscr{D}_\rho}(J_n,\ 0) \ \leqslant\ K_\rho/(n^2+1)$ for all n satisfies

$$K_\rho \ \geqslant\ C' \log\frac{1}{\rho}$$

with a constant C' independent of ρ.
(Hint. This is harder than parts (a)–(d). Fixing any $\rho > 0$, write, for large R, $E_R = E_\rho \cup (-\infty, -R] \cup [R, \infty)$, and then put $\mathscr{D}_R = \mathbb{C} \sim E_R$. As $R \to \infty$, $G_{\mathscr{D}_R}(t, 0)$ increases to $G_{\mathscr{D}_\rho}(t, 0)$, so $Y_{\mathscr{D}_R}(0)$ increases to $Y_{\mathscr{D}_\rho}(0)$. For each R, by the *first* theorem of the previous article,

$$G_{\mathscr{D}_R}(z, w) \ =\ \log\frac{1}{|z-w|} \ +\ \int_{E_R} \log|w-s|\, d\omega_{\mathscr{D}_R}(s, z),$$

whence

$$G_{\mathscr{D}_R}(t, 0) + G_{\mathscr{D}_R}(-t, 0) \ =\ \int_{E_R} \log\left|1 - \frac{s^2}{t^2}\right| d\omega_{\mathscr{D}_R}(s, 0).$$

Fix any integer $A > 0$. Then $\int_{-A}^{A} G_{\mathscr{D}_\rho}(t,0)\mathrm{d}t$ is the limit, as $R \to \infty$, of $\int_{-\infty}^{\infty}\int_0^A \log|1-(s^2/t^2)|\mathrm{d}t\,\mathrm{d}\omega_{\mathscr{D}_R}(s,0)$. Taking an arbitrary large M, which for the moment we *fix*, we break up this double integral as

$$\int_{-M}^{M}\int_0^A + \int_{|s|>M}\int_0^A$$

To study the two terms of this sum, first evaluate

$$\int_0^A \log\left|1-\frac{s^2}{t^2}\right|\mathrm{d}t;$$

for $|s| > A$ this can be done by direct computation, and, for $|s| < A$, by using the identity

$$\int_0^A \log\left|1-\frac{s^2}{t^2}\right|\mathrm{d}t \;=\; -\int_A^{\infty} \log\left|1-\frac{s^2}{t^2}\right|\mathrm{d}t.$$

Regarding $\int_{-M}^{M}\int_0^A \log|1-(s^2/t^2)|\mathrm{d}t\,\mathrm{d}\omega_{\mathscr{D}_R}(s,0)$, we may use the fact that $\omega_{\mathscr{D}_R}(S,0) \longrightarrow \omega_{\mathscr{D}_\rho}(S,0)$ as $R \to \infty$ for *bounded* $S \subseteq \mathbb{R}$, and then plug in the inequality

$$\omega_{\mathscr{D}_\rho}(J_n,0) \;\leqslant\; K_\rho/(n^2+1)$$

together with the result of the computation just indicated. In this way we easily see that $\lim_{R\to\infty}\int_{-M}^{M}\int_0^A \;\leqslant\; CK_\rho$ with a constant C independent of A, M, and ρ.

In order to estimate

$$\int_{|s|>M}\int_0^A \log\left|1-\frac{s^2}{t^2}\right|\mathrm{d}t\,\mathrm{d}\omega_{\mathscr{D}_R}(s,0),$$

use the fact that

$$\Omega_{\mathscr{D}_R}(s) \;\leqslant\; \frac{Y_{\mathscr{D}_R}(0)}{s} \;\leqslant\; \frac{Y_{\mathscr{D}_\rho}(0)}{s}$$

(where $Y_{\mathscr{D}_\rho}(0)$, as we already know, is *finite*) together with the value of the inner integral, already computed, and integrate by parts. In this way one finds an estimate *independent* of R which, *for fixed* A, is *very small* if M is *large enough*. Combining this result with the previous one and then making $M \to \infty$, one sees that

$$\int_{-A}^{A} G_{\mathscr{D}_\rho}(t,0)\,\mathrm{d}t \;\leqslant\; CK_\rho$$

with C independent of A and of ρ.)

Remark. In the circumstances of the preceding problem $G_{\mathscr{D}_\rho}(z,0)$ must, when $\rho \to 0$, tend to ∞ for each z not equal to a half odd integer, and it is interesting to see how fast that happens. Fix any such $z \neq 0$. Then, given $\rho > 0$ we have, working with the domains $\mathscr{D}_R$ used in part (e) of the problem,

$$G_{\mathscr{D}_\rho}(z,0) = \lim_{R\to\infty} G_{\mathscr{D}_R}(z,0).$$

Here,

$$G_{\mathscr{D}_R}(z,0) = \log\frac{1}{|z|} + \int_{-\infty}^{\infty} \log|z-t|\,d\omega_{\mathscr{D}_R}(t,0)$$

$$= O(1) + \int_{-\infty}^{\infty} \log^+|t|\,d\omega_{\mathscr{D}_R}(t,0),$$

where the O(1) term depends on z but is independent of R, and of ρ, when the latter is small enough.

Taking an $M > 1$, we rewrite the last integral on the right as

$$\int_{|t|<M} + \int_{|t|\geqslant M},$$

and thus find it to be

$$\leqslant \log M - \int_M^{\infty} \log t\,d\Omega_{\mathscr{D}_R}(t)$$

$$= \log M + \Omega_{\mathscr{D}_R}(M)\log M + \int_M^{\infty} \frac{\Omega_{\mathscr{D}_R}(t)}{t}\,dt.$$

Plug the inequalities $\Omega_{\mathscr{D}_R}(t) \leqslant Y_{\mathscr{D}_R}(0)/t$ and $Y_{\mathscr{D}_R}(0) \leqslant Y_{\mathscr{D}_\rho}(0)$ into the expression on the right. Then, referring to the previous relation and making $R\to\infty$, we see that

$$G_{\mathscr{D}_\rho}(z,0) \leqslant O(1) + \log M + Y_{\mathscr{D}_\rho}(0)\frac{\log M + 1}{M}.$$

By part (a) of the problem, $Y_{\mathscr{D}_\rho}(0) = O(1) + (1/\pi)\log(1/\rho)$. Hence, choosing $M = (1/\pi)\log(1/\rho)\log\log(1/\rho)$ in the last relation, we get

$$G_{\mathscr{D}_\rho}(z,0) \leqslant O(1) + \log\log\frac{1}{\rho} + \log\log\log\frac{1}{\rho}.$$

This order of growth seems rather slow. One would have expected $G_{\mathscr{D}_\rho}(z,0)$ to behave like $\log(1/\rho)$ for small ρ when z is fixed.

3. The energy integral again

The result of the preceding article already has some applications to the project described at the beginning of this §. Suppose that the majorant $M(t) \geqslant 0$ is defined and *even* on $\mathbb{R}$. Taking $M(t)$ to be identically zero in a neighborhood of 0 involves no real loss of generality. If $M(t)$ is also *increasing* on $[0, \infty)$, the Poisson integral

$$\int_{\partial\mathscr{D}} v(t)\mathrm{d}\omega_{\mathscr{D}}(t, 0)$$

for a function $v(z)$ subharmonic in one of our domains $\mathscr{D}$ with $0 \in \mathscr{D}$ and satisfying

$$v(t) \leqslant M(t), \qquad t \in \partial\mathscr{D},$$

has the simple majorant

$$Y_{\mathscr{D}}(0) \cdot \int_0^\infty \frac{M(t)}{t^2}\mathrm{d}t.$$

The entire dependence of the Poisson integral on the domain $\mathscr{D}$ is thus expressed by means of the single factor $Y_{\mathscr{D}}(0)$ occurring in this second expression.

To see this, recall that $\omega_{\mathscr{D}}((-\infty, -t] \cup [t, \infty), 0) = \Omega_{\mathscr{D}}(t)$ for $t > 0$; the given majoration on $v(t)$ therefore makes the Poisson integral $\leqslant -\int_0^\infty M(t)\mathrm{d}\Omega_{\mathscr{D}}(t)$, which here is equal to

$$\int_0^\infty \Omega_{\mathscr{D}}(t)\mathrm{d}M(t).$$

Since $M(t)$ is increasing on $[0, \infty)$, we may substitute the relation $\Omega_{\mathscr{D}}(t) \leqslant Y_{\mathscr{D}}(0)/t$ proved in the preceding article into the last expression, showing it to be

$$\leqslant Y_{\mathscr{D}}(0)\int_0^\infty \frac{\mathrm{d}M(t)}{t} = Y_{\mathscr{D}}(0)\int_0^\infty \frac{M(t)}{t^2}\mathrm{d}t.$$

This argument cannot be applied to *general* even majorants $M(t) \geqslant 0$, because the relation $\Omega_{\mathscr{D}}(t) \leqslant Y_{\mathscr{D}}(0)/t$ cannot be differentiated to yield $\mathrm{d}\omega_{\mathscr{D}}(t, 0) \leqslant (Y_{\mathscr{D}}(0)/t^2)\mathrm{d}t$. Indeed, when $x \in \partial\mathscr{D} = E$ gets near any of the *endpoints a* of the intervals making up that set, $\mathrm{d}\omega_{\mathscr{D}}(x, 0)/\mathrm{d}x$ gets *large* like a multiple of $|x - a|^{-1/2}$ (see the second lemma of article 1 and the remark following it). We are not supposing *anything* about the *disposition* of these intervals except that they be *finite in number*; there may *otherwise* be *arbitrarily many* of them. It is therefore not possible to bound $\int_{-\infty}^\infty M(t)\mathrm{d}\omega_{\mathscr{D}}(t, 0)$ by an expression involving *only* $\int_0^\infty (M(t)/t^2)\mathrm{d}t$ for *general* even majorants $M(t) \geqslant 0$; some *additional regularity properties* of $M(t)$ are required and must be taken into account. A very useful instrument for this purpose turns out to be the *energy* introduced in §B.5 which has already played such an important rôle in §B. Application of that notion to matters like the one now under discussion goes back to the 1962 paper of Beurling and Malliavin. The material of that paper will be taken up in Chapter XI, where we will use the results established in the present §.

Appearance of the energy here is due to the following

Lemma. *Let* $0 \in \mathscr{D}$. For $x \neq 0$,

$$G_{\mathscr{D}}(x,0) + G_{\mathscr{D}}(-x,0) \;=\; \frac{1}{x}\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}(t\Omega_{\mathscr{D}}(t)).$$

Proof. By the *second* theorem of article 1,

$$G_{\mathscr{D}}(x,0) \;=\; -\!\!\!\!\!\!\int_{-\infty}^{\infty} \frac{\omega_{\mathscr{D}}(t)\,\mathrm{sgn}\,t}{x-t}\,\mathrm{d}t \qquad \text{for } x \neq 0,$$

where

$$\omega_{\mathscr{D}}(t) \;=\; \begin{cases} \omega_{\mathscr{D}}((-\infty,t],\ 0), & t<0, \\ \omega_{\mathscr{D}}([t,\infty),\ 0), & t>0. \end{cases}$$

Thence,

$$G_{\mathscr{D}}(x,0) + G_{\mathscr{D}}(-x,0) \;=\; -\!\!\!\!\!\!\int_{-\infty}^{\infty} \frac{2t\,\mathrm{sgn}\,t\,\omega_{\mathscr{D}}(t)}{t^2-x^2}\,\mathrm{d}t$$

$$=\; -\!\!\!\!\!\!\int_0^\infty \frac{2t}{t^2-x^2}\,\Omega_{\mathscr{D}}(t)\,\mathrm{d}t,$$

since $\omega_{\mathscr{D}}(t) + \omega_{\mathscr{D}}(-t) = \Omega_{\mathscr{D}}(t)$ for $t>0$.

Assuming wlog that $x>0$, we take a small $\varepsilon>0$ and apply partial integration to the two integrals in

$$\left(\int_0^{x-\varepsilon} + \int_{x+\varepsilon}^\infty\right)\frac{2}{t^2-x^2}\,t\Omega_{\mathscr{D}}(t)\,\mathrm{d}t,$$

getting

$$\left(\frac{t\Omega_{\mathscr{D}}(t)}{x}\log\left|\frac{t-x}{t+x}\right|\right)\left(\Big]_0^{x-\varepsilon} + \Big]_{x+\varepsilon}^{\infty}\right)$$

$$+\;\left(\int_0^{x-\varepsilon} + \int_{x+\varepsilon}^\infty\right)\frac{1}{x}\log\left|\frac{x+t}{x-t}\right| \mathrm{d}(t\Omega_{\mathscr{D}}(t)).$$

The function $\Omega_{\mathscr{D}}(t)$ is 1 for $t>0$ near 0 and $O(1/t)$ for large t; it is moreover Lip $\frac{1}{2}$ at each $x>0$ by the second lemma of article 1. The sum of the integrated terms therefore tends to 0 as $\varepsilon \to 0$, and we see that

$$\int_0^\infty \frac{2t}{t^2 - x^2}\Omega_{\mathscr{D}}(t)\mathrm{d}t \;=\; \frac{1}{x}\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}(t\Omega_{\mathscr{D}}(t)).$$

Since the left side equals $G_{\mathscr{D}}(x,0) + G_{\mathscr{D}}(-x,0)$, the lemma is proved.

In the language of §B.5, $x(G_{\mathscr{D}}(x,0) + G_{\mathscr{D}}(-x,0))$ is the *Green potential* of $\mathrm{d}(t\Omega_{\mathscr{D}}(t))$. Here, since we are assuming $\mathscr{D} \cap \mathbb{R} = \mathbb{R} \sim E$ to be bounded,

$$\Omega_{\mathscr{D}}(x) \;=\; \frac{1}{\pi^2}\int_0^\infty \frac{2x}{x^2 - t^2} G_{\mathscr{D}}(t,0)\mathrm{d}t$$

has, for large x, a convergent expansion of the form

$$\frac{a_1}{x} + \frac{a_3}{x^3} + \frac{a_5}{x^5} + \cdots,$$

so that

$$\mathrm{d}(t\Omega_{\mathscr{D}}(t)) \;=\; -\left(\frac{2a_3}{t^3} + \frac{4a_5}{t^5} + \cdots\right)\mathrm{d}t$$

for large t. Using this fact it is easy to verify that

$$\int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}(t\Omega_{\mathscr{D}}(t))\,\mathrm{d}(x\Omega_{\mathscr{D}}(x))$$

is *absolutely convergent; this double integral thus coincides with the energy*

$$E(\mathrm{d}(t\Omega_{\mathscr{D}}(t)),\ \mathrm{d}(t\Omega_{\mathscr{D}}(t)))$$

defined in §B.5.

Theorem. *If* $0 \in \mathscr{D}$,

$$\boxed{E(\mathrm{d}(t\Omega_{\mathscr{D}}(t)),\ \mathrm{d}(t\Omega_{\mathscr{D}}(t))) \;\leqslant\; \pi(Y_{\mathscr{D}}(0))^2.}$$

Proof. By the lemma, the left side, equal to the above double integral, can be rewritten as

$$\int_0^\infty x[G_{\mathscr{D}}(x,0) + G_{\mathscr{D}}(-x,0)]\mathrm{d}(x\Omega_{\mathscr{D}}(x)).$$

Here, $G_{\mathscr{D}}(x,0) + G_{\mathscr{D}}(-x,0) \geqslant 0$ and $\Omega_{\mathscr{D}}(x)$ is *decreasing*, so the last expression is

$$\leqslant \int_0^\infty [G_{\mathscr{D}}(x,0) + G_{\mathscr{D}}(-x,0)]x\Omega_{\mathscr{D}}(x)\mathrm{d}x.$$