completely independent of the *particular* sum $S(z)$ under consideration. Therefore $W_{C,E}(\mathrm{i})$, the supremum of $|S(\mathrm{i})|$ for such sums S with $\|S\|_{W,E} \leqslant 1$, is *finite*. This is what we needed to show in order to infer the proper inclusion of $\mathscr{C}_W(C,E)$ in $\mathscr{C}_W(E)$. The *first part* of our theorem is thus *proved*.

There remains the *second part*. *That*, however, is not new! Putting, as before, $W(x)$ equal to ∞ on $\mathbb{R} \sim E$, which makes $\mathscr{C}_W(C,E)$ coincide with the subspace $\mathscr{C}_W(C)$ of $\mathscr{C}_W(\mathbb{R})$ considered in Chapter VI, §E, we have, in the notation of that §,

$$\int_{-\infty}^{\infty} \frac{\log W_C(x)}{1+x^2}\,\mathrm{d}x \;<\; \infty$$

if $\mathscr{C}_W(C) \neq \mathscr{C}_W(\mathbb{R})$, according to *Akhiezer's theorem* (Chapter VI, §E.2). Our function $W_{C,E}(x)$ is simply $W_C(x)$. Hence, since $E \subseteq \mathbb{R}$ (!),

$$\int_E \frac{\log W_{C,E}(x)}{1+x^2}\,\mathrm{d}x \;<\; \infty$$

when $\mathscr{C}_W(C,E) \neq \mathscr{C}_W(E)$.

The theorem is completely proved. We are done.

Remark. If we do not assume anything about the continuity of a weight $W(x) \geqslant 1$ defined on E, it is still possible to characterize the equality of $\mathscr{C}_W(C,E)$ with $\mathscr{C}_W(E)$ by an analogue of *Mergelian's second theorem* involving an integral over E. The establishment of such a result proceeds very much along the lines of the proof just finished, and is left to the reader.

Problem 18

Let E be a closed set on $\mathbb{R}$ of the kind specified at the beginning of this §. Show that there are two constants a, b, depending on E, such that, for any entire function $f(z)$ of exponential type $\leqslant C$, bounded on $\mathbb{R}$, we have

$$\int_{-\infty}^{\infty} \frac{\log(1+|f(x)|^2)}{1+x^2}\,\mathrm{d}x \;\leqslant\; aC \;+\; b\int_E \frac{\log(1+|f(x)|^2)}{1+x^2}\,\mathrm{d}x.$$

(Hint: One may apply the *third* and *fourth* theorems from Chapter III, §G.3, and reason as in the above proof. Another procedure is to use the proof of the lemma in §E.1 of Chapter VI so as to first approximate $f(z)$ by finite sums $S(z)$ of the form considered above, having exponents in arithmetic progression.)

If $W(x)$, continuous and $\geqslant 1$ on E, is such that

$$\frac{x^n}{W(x)} \longrightarrow 0 \quad \text{for} \quad x \longrightarrow \pm\infty \quad \text{in} \quad E$$

and $n = 1, 2, 3, \ldots$, we denote by $W_{0,E}(z)$ the *supremum* of $|P(z)|$ for all *polynomials* P with $\|P\|_{W,E} \leqslant 1$.

Theorem. *Let $E \subseteq \mathbb{R}$ be a set of the kind specified at the beginning of this §, and let $W(x)$, continuous and $\geqslant 1$ on E, tend to ∞ faster than any power of x as $x \to \pm\infty$ in E.*

If, for polynomials $P(z)$ *with* $\|P\|_{W,E} \leqslant 1$, *the integrals*

$$\int_E \frac{\log|P(x)|}{1+x^2}\,dx$$

are bounded above, *then* $\mathscr{C}_W(0,E)$ *is* properly contained *in* $\mathscr{C}_W(E)$.

If $\mathscr{C}_W(0,E)$ is *properly contained in* $\mathscr{C}_W(E)$, *then*

$$\int_E \frac{\log W_{0,E}(x)}{1+x^2}\,dx \;<\; \infty.$$

Proof. The *second* part reduces (as at the end of the preceding demonstration) to a known result of Akhiezer (in this case from §B.1, Chapter VI) on putting $W(x) = \infty$ on $\mathbb{R} \sim E$. Hence only the *first* part requires discussion here.

According to *Pollard's theorem* (Chapter VI, §B.3), proper inclusion of $\mathscr{C}_W(0,E)$ in $\mathscr{C}_W(E)$ will certainly follow if the integrals

$$\int_{-\infty}^{\infty} \frac{\log(1+|P(x)|^2)}{1+x^2}\,dx$$

are *bounded above* for P ranging over the *polynomials* with $\|P\|_{W,E} \leqslant 1$. It is therefore enough to *show* this, under the assumption that

$$\int_E \frac{\log|P(x)|}{1+x^2}\,dx \;\leqslant\; M, \text{ say,}$$

for *any* polynomial P with $\|P\|_{W,E} \leqslant 1$.

We may, first of all, argue as in the proof of the above lemma to conclude that our assumption implies a seemingly stronger property: we have

$$\int_E \frac{\log(1+|P(x)|^2)}{1+x^2}\,dx \;\leqslant\; 2M + \pi\log 2$$

for the polynomials P with $\|P\|_{W,E} \leqslant 1$. The proof will therefore be complete if we can verify that

$$\int_{-\infty}^{\infty} \frac{\log(1+|P(x)|^2)}{1+x^2}\,dx \;\leqslant\; b\int_E \frac{\log(1+|P(x)|^2)}{1+x^2}\,dx$$

for polynomials P, b being a certain constant depending on the set E. This we do, using the result of *problem* 18.

Take any polynomial P, of degree N, say. With an arbitrary $\eta > 0$, put

$$f_\eta(z) = \left(\frac{\sin \eta z}{\eta z}\right)^N P(z);$$

$f_\eta(z)$ is then entire, of exponential type $N\eta$, and *bounded* on $\mathbb{R}$. By *problem* 18, we thus have

$$\int_{-\infty}^{\infty} \frac{\log(1 + |f_\eta(x)|^2)}{1+x^2}\,dx \;\leqslant\; aN\eta \;+\; b\int_E \frac{\log(1+|f_\eta(x)|^2)}{1+x^2}\,dx.$$

Here, $|f_\eta(x)| \leqslant |P(x)|$ on $\mathbb{R}$ and $f_\eta(x) \longrightarrow P(x)$ as $\eta \to 0$, so the desired inequality follows on making $\eta \to 0$. We are done.

4. What happens when the set E is sparse

The sets E described at the beginning of this § have the property that

$$|E \cap I|/|I| \;\geqslant\; c \;>\; 0$$

for all intervals I on $\mathbb{R}$ of length exceeding some L. In other words, their *lower uniform density* is *positive*. One suspects that the continual occurrence of the form $dx/(1 + x^2)$ in the integrals over E figuring in the preceding article is somehow connected with this positivity. As a first step towards finding out whether our hunch has any basis in fact, let us try to see what happens to the form $dx/(1 + x^2)$ when E becomes *sparse*. We do this in the special case where

$$E = \bigcup_{n=-\infty}^{\infty} [a_n - \delta,\; a_n + \delta]$$

with $a_n = |n|^p \operatorname{sgn} n$ and $p > 1$. This example was worked out by Benedicks (see his preprint), and all the material in the present article is due to him.

► In order that there may be no doubt, we point out that *the sets E now under consideration are no longer of the sort described at the beginning of the present* §.

Lemma. *Let S be the square*

$$\{(x, y):\; -a < x < a \text{ and } -a < y < a\},$$

and denote by H the union *of its* two horizontal sides, *and by V the* union *of its* two vertical sides. *Then, if* $-a < x < a$,

$$\omega_S(H, x) \;\leqslant\; \omega_S(H, 0)$$

and

$$\omega_S(V,x) \geqslant \omega_S(V,0),$$

where, as usual, $\omega_S(\ ,z)$ *denotes harmonic measure for* S.

Proof (Benedicks). Let, wlog, $0 < x_0 < a$ and consider the harmonic function

$$\Delta(z) = \omega_S(H,z) - \omega_S(H,z+x_0)$$

defined in the rectangle

$$T = \{(x,y):\ -a < x < a - x_0 \text{ and } -a < y < a\}.$$

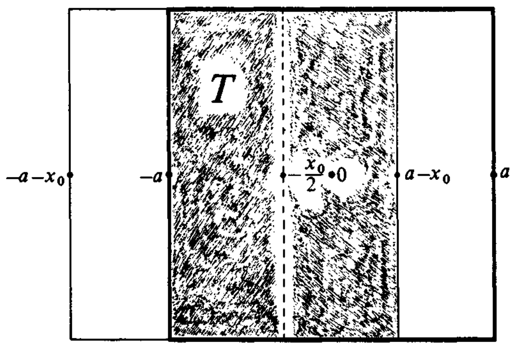

Figure 129

It is clear by symmetry that, for $z \in S$, $\omega_S(H,z) = \omega_S(H,\bar{z})$, and also $\omega_S(H,x+\mathrm{i}y) = \omega_S(H,-x+\mathrm{i}y)$. Therefore $\Delta(-\frac{1}{2}x_0 + \mathrm{i}y) = 0$ on the vertical bisector of T (see figure). Again, on T's *right vertical side*,

$$\begin{aligned}\Delta(a - x_0 + \mathrm{i}y) &= \omega_S(H, a - x_0 + \mathrm{i}y) - \omega_S(H, a + \mathrm{i}y) \\ &= \omega_S(H, a - x_0 + \mathrm{i}y) \geqslant 0\end{aligned}$$

(and similarly, on the opposite side of T,

$$\Delta(-a + \mathrm{i}y) = -\omega_S(H, -a + x_0 + \mathrm{i}y) \leqslant 0).$$

It is clear on the other hand that $\Delta(z) = 0$ on the *top* and *bottom sides* of T $(1 - 1 = 0)$. By the principle of maximum we thus have $\Delta(z) \geqslant 0$ in the *right half* of T; in particular,

$$\Delta(0) = \omega_S(H,0) - \omega_S(H,x_0) \geqslant 0,$$

and $\omega_S(H,x_0) \leqslant \omega_S(H,0)$, proving the *first* inequality asserted by the lemma.

The *second* inequality follows from the first one because

$$\omega_S(H,z) + \omega_S(V,z) \equiv 1$$

in S and clearly $\omega_S(V,0) = \omega_S(H,0)$. We are done.

Lemma (Benedicks). *Let $E \subseteq \mathbb{R}$ be any 'reasonable' closed set (for instance, a finite union of closed intervals), let S be the square of the preceding lemma, and put*

$$\Omega = S \cap \sim E.$$

If H denotes the union of the two horizontal sides of S and V that of the vertical ones, we have

$$\omega_\Omega(V,0) \leqslant \omega_\Omega(H,0)$$

for the harmonic measure $\omega_\Omega(\ ,z)$ associated with the domain Ω.

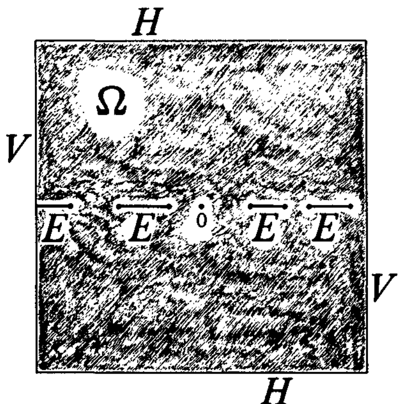

Figure 130

Proof. By a formula derived near the end of §B.1, Chapter VII, for $z \in \Omega$,

$$\omega_\Omega(H,z) = \omega_S(H,z) - \int_E \omega_S(H,\xi)\,d\omega_\Omega(\xi,z)$$

$$\omega_\Omega(V,z) = \omega_S(V,z) - \int_E \omega_S(V,\xi)\,d\omega_\Omega(\xi,z).$$

From the previous lemma,

$$\omega_S(H,\xi) \leqslant \omega_S(H,0) = \omega_S(V,0) \leqslant \omega_S(V,\xi)$$

for real ξ lying in S; in particular, for $\xi \in E$. Substituting this relation into the preceding ones and then making $z = 0$, we get $\omega_\Omega(V,0) \leqslant \omega_\Omega(H,0)$.

Q.E.D.

Corollary. *In the above configuration,*

$$\omega_\Omega(\partial S, 0) \leqslant 2\omega_\Omega(H, 0).$$

Proof. Clear.

Lemma. *Let $p > 1$ and put*

$$E = \bigcup_{n=-\infty}^{\infty} [|n|^p \operatorname{sgn} n - \delta, \ |n|^p \operatorname{sgn} n + \delta],$$

$\delta > 0$ being taken small enough so that the intervals figuring in the union do not intersect. With $x_0 > 0$, let S_{x_0} be the square

$$\left\{\frac{x_0}{2} < \Re z < \frac{3x_0}{2}, \ -\frac{x_0}{2} < \Im z < \frac{x_0}{2}\right\},$$

and Ω_{x_0} the domain

$$S_{x_0} \cap \sim E.$$

For large x_0, the harmonic measure $\omega_{\Omega_{x_0}}(\quad, z)$ associated with Ω_{x_0} satisfies

$$\omega_{\Omega_{x_0}}(\partial S_{x_0}, \ x_0) \leqslant \text{const.} \frac{\log x_0}{x_0^{1/p}}.$$

Proof (Benedicks). By use of a test function and application of the preceding corollary.

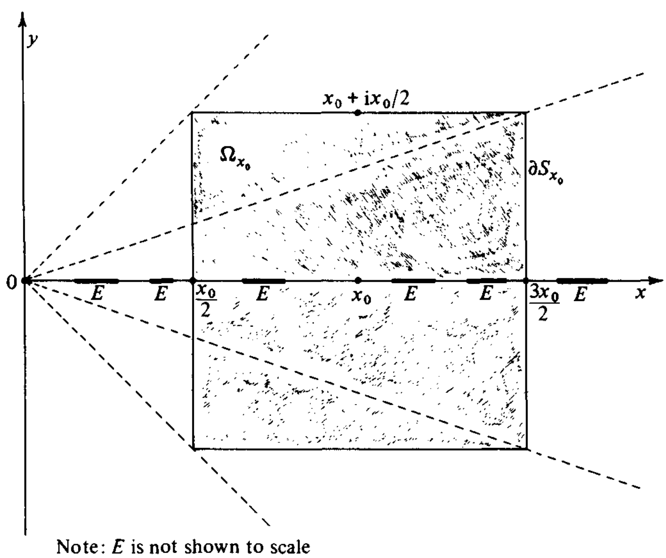

Figure 131

The function $z^{1/p}$ (taken as positive on the positive real axis) is analytic for $\Re z > 0$; so, therefore, is

$$\sin \pi z^{1/p}.$$

In $\Re z > 0$, this function vanishes only at the midpoints n^p of the intervals making up E, and, at $x = n^p$,

$$\frac{\mathrm{d}(\sin \pi x^{1/p})}{\mathrm{d}x} = (-1)^n \frac{\pi}{pn^{p-1}}.$$

This means that, if we take $x_0 > 0$ *large* and put $C_0 = 1/x_0^{(p-1)/p}$, we have $|\sin \pi x^{1/p}| \geqslant kC_0\delta$ for x *outside* E on the interval $(x_0/2,\ 3x_0/2)$, $k > 0$ being a constant depending on p, but independent of x_0 and δ. Recalling the behaviour of the Joukowski transformation

$$w \longrightarrow w + \sqrt{(w^2 - 1)},$$

we see that for a suitable definition of $\sqrt{\ }$, the function

$$v(z) = \log\left|\frac{\sin \pi z^{1/p}}{kC_0\delta} + \sqrt{\left(\frac{\sin^2 \pi z^{1/p}}{(kC_0\delta)^2} - 1\right)}\right|$$

is *positive and harmonic* in Ω_{x_0}.

For this reason, when $x \in \mathbb{R} \cap \Omega_{x_0}$,

$$v(x) \geqslant \inf_{\zeta \in H} v(\zeta) \cdot \omega_{\Omega_{x_0}}(H, x),$$

H denoting the union of the two *horizontal* sides of ∂S_{x_0}. However,

$$v(\zeta) \geqslant \text{const.} x_0^{1/p} \qquad \text{for } \zeta \in H$$

as is easily seen (almost without computation, if one refers to the above diagram). Also,

$$v(x) \leqslant \log \frac{2}{kC_0\delta} = (1 - 1/p)\log x_0 + O(1), \qquad x \in \mathbb{R} \cap \Omega_{x_0}.$$

Therefore

$$\omega_{\Omega_{x_0}}(H, x) \leqslant \text{const.} \frac{\log x_0}{x_0^{1/p}}, \qquad x \in \mathbb{R} \cap \Omega_{x_0}.$$

Since x_0 lies at the centre of the square S_{x_0}, the corollary to the previous lemma gives

$$\omega_{\Omega_{x_0}}(\partial S_{x_0}, x_0) \leqslant 2\omega_{\Omega_{x_0}}(H, x_0).$$

Combining this and the preceding relations, we obtain the desired result.

Theorem (Benedicks). *Let G be the Green's function for the domain*

$$\mathscr{D} = \mathbb{C} \sim E = \mathbb{C} \sim \bigcup_{n=-\infty}^{\infty} [|n|^p \operatorname{sgn} n - \delta,\ |n|^p \operatorname{sgn} n + \delta],$$

where $p > 1$ and $\delta > 0$ is small enough so that the intervals in the union do not intersect.

Then, for real x of large modulus,

$$G(x, \mathrm{i}) \leqslant C\frac{\log|x|}{|x|^{(p+1)/p}},$$

with a constant C depending on p and δ.

Proof. $G(z, \mathrm{i})$ is certainly bounded above – by M say – in the sector $\{0 \leqslant |\Im z| < \Re z\}$. Given $x_0 > 0$, the square S_{x_0} considered in the previous lemma lies in that sector, so $G(\zeta, \mathrm{i}) \leqslant M$ on ∂S_{x_0}. $G(z, \mathrm{i})$ is, moreover, harmonic in $\Omega_{x_0} \subseteq \mathscr{D}$ and *zero on E*, whence $G(x_0, \mathrm{i}) \leqslant M \cdot \omega_{\Omega_{x_0}}(\partial S_{x_0}, x_0)$. By the last lemma we therefore have

$$(*) \qquad G(x_0, \mathrm{i}) \leqslant \text{const.}\frac{\log|x_0|}{|x_0|^{1/p}}$$

for large $x_0 > 0$.

Benedicks' idea is to now use Poisson's formula for the half plane, so as to improve $(*)$ by iteration. Take any fixed α with $0 < \alpha < 1/p$. Then $(*)$ certainly implies (by symmetry of E) that

$$G(x, \mathrm{i}) \leqslant \frac{\text{const.}}{|x|^\alpha + 1}, \qquad x \in \mathbb{R},$$

$G(x, \mathrm{i})$ being at any rate bounded on the real axis. The function $G(z, \mathrm{i})$ is in fact *bounded and harmonic in* $\Im z < 0$, so

$$G(z, \mathrm{i}) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|}{|z - t|^2} G(t, \mathrm{i})\mathrm{d}t, \qquad \Im z < 0.$$

Plugging in the previous relation, we get

$$G(z, \mathrm{i}) \leqslant \text{const.}\int_{-\infty}^{\infty} \frac{|y|}{(x - t)^2 + y^2}\cdot\frac{\mathrm{d}t}{|t|^\alpha + 1}, \qquad y < 0.$$

Let, wlog, $x > 0$. Then, the integral just written can be broken up as $\int_{-x/2}^{x/2} + \int_{|t| \geqslant x/2}$. Since $\int_{-\infty}^{\infty}(|y|/((x - t)^2 + y^2))\mathrm{d}t = \pi$, the *second* of these terms is obviously $\mathrm{O}(x^{-\alpha})$ for large x. The *first*, on the other hand, is

$$\leqslant \text{const.}\frac{4|y|}{x^2 + 4y^2}\cdot x^{1-\alpha},$$

so, all in all,

$$G(z, \mathrm{i}) \leqslant \text{const.}\,|x|^{-\alpha}$$

for $0 > y > -|x|$, $|x|$ being large.

The inequality just found *remains true*, however, for $0 < y < |x|$, in spite of the logarithmic singularity that $G(z, \mathrm{i})$ has at i. This follows from the fact that $0 < \alpha < 1/p < 1$ and the relation

$$G(z, \mathrm{i}) - G(\bar{z}, \mathrm{i}) = \log\left|\frac{z + \mathrm{i}}{z - \mathrm{i}}\right|.$$

To verify the latter, just subtract the right side from the left. The difference is *harmonic* in $\Im z > 0$ and *bounded* there (the logarithmic poles at i cancel each other out). It is also clearly zero on $\mathbb{R}$, so hence zero for $\Im z > 0$. For large $|z|$, $\log|(z + i)/(z - i)| = O(1/|z|)$, and we see that

$$G(z, \mathrm{i}) \leqslant \text{const.}|x|^{-\alpha} \qquad \text{for } 0 \leqslant |y| < x$$

since this inequality is true for $0 > y > -|x|$.

Suppose that $x_0 > 0$ is large; we can use the previous lemma again. By what has just been shown,

$$G(\zeta, \mathrm{i}) \leqslant \text{const.}|x_0|^{-\alpha}, \qquad \zeta \in \partial S_{x_0}.$$

Arguing as at the beginning of this proof, we get

$$G(x_0, \mathrm{i}) \leqslant \text{const.}|x_0|^{-\alpha}\omega_{\Omega_{x_0}}(\partial S_{x_0}, x_0) \leqslant \text{const.}\frac{\log|x_0|}{|x_0|^{\alpha}|x_0|^{1/p}}.$$

Hence, since $0 < \alpha < 1/p$, we have

$$G(x, \mathrm{i}) \leqslant \frac{\text{const.}}{|x|^{2\alpha} + 1}, \qquad x \in \mathbb{R}.$$

The exponent α in the inequality we started with has been improved to 2α.

If now $2\alpha < 1$, we may start from the inequality just obtained and *repeat* the above argument, ending with the relation

$$G(x, \mathrm{i}) \leqslant \frac{\text{const.}}{|x|^{3\alpha} + 1}, \qquad x \in \mathbb{R}.$$

The process may evidently be continued so as to yield successively the estimates

$$G(x, \mathrm{i}) \leqslant \frac{\text{const.}}{|x|^{n\alpha} + 1}, \qquad x \in \mathbb{R},$$

with $n = 3, 4, \ldots$, *as long as* $(n - 1)\alpha < 1$. Choosing α, $0 < \alpha < 1/p$, *to not* *be of the form* $1/m$, $m = 1, 2, 3, \ldots$, we arrive at an estimate

$$(\ast) \qquad G(x, \mathrm{i}) \leqslant \frac{\text{const.}}{|x|^{n\alpha} + 1}, \qquad x \in \mathbb{R},$$

where $n\alpha$ *is the first integral multiple of* α *strictly* > 1.

Because the exponent $n\alpha$ in $(\ast)$ is > 1, we have

$$\int_{-\infty}^{\infty} G(t, \mathrm{i})\,\mathrm{d}t < \infty.$$

As before, for $y < 0$, we can write

$$G(z, \mathrm{i}) = \frac{1}{\pi}\int_{-|x|/2}^{|x|/2} \frac{|y|}{|z-t|^2} G(t, \mathrm{i})\,\mathrm{d}t + \frac{1}{\pi}\int_{|t| \geqslant |x|/2} \frac{|y|}{|z-t|^2} G(t, \mathrm{i})\,\mathrm{d}t.$$

For $|t| \leqslant |x|/2$, $|y|/|z-t|^2 \leqslant 1/|x|$, so the *first* term on the right is $\leqslant \text{const.}/|x|$ in view of the preceding relation. The *second* is $\leqslant \text{const.}/|x|^{n\alpha} = o(1/|x|)$ by $(\ast)$. Thence, for $|x|$ large,

$$G(z, \mathrm{i}) \leqslant \frac{\text{const.}}{|x|}, \qquad y < 0.$$

Using the relation

$$G(z, \mathrm{i}) - G(\bar{z}, \mathrm{i}) = \log\left|\frac{\mathrm{i} + z}{\mathrm{i} - z}\right|$$

as above, we find that in fact

$$G(z, \mathrm{i}) \leqslant \frac{\text{const.}}{|x|} \qquad \text{for } |z| \text{ large.}$$

Take this relation and apply the preceding lemma *one more time.* For large x_0, we have

$$G(\zeta, \mathrm{i}) \leqslant \text{const.}/x_0 \qquad \text{on } \partial S_{x_0}.$$

Therefore

$$G(x_0, \mathrm{i}) \leqslant \frac{\text{const.}}{x_0}\,\omega_{\Omega_{x_0}}(\partial S_{x_0}, x_0) \leqslant \frac{\text{const.} \log x_0}{x_0^{1+(1/p)}}.$$

This is what we wanted to prove.

We are done.

Corollary. *A Phragmén–Lindelöf function* $Y(z)$ *exists for the domain*

$$\mathbb{C} \sim \bigcup_{n=-\infty}^{\infty} [|n|^p \operatorname{sgn} n - \delta,\ |n|^p \operatorname{sgn} n + \delta].$$

Proof. By the theorem, we certainly have

$$\int_{-\infty}^{\infty} G(x, \mathrm{i})\,\mathrm{d}x \;<\; \infty.$$

The result then follows by the second theorem of article 2.

Remark. Although the theorem tells us that, *on the real axis,*

$$G(x, \mathrm{i}) \;\leqslant\; \text{const.}\frac{\log|x|}{|x|^{1+1/p}}$$

when $|x|$ is large, the inequality

$$G(z, \mathrm{i}) \;\leqslant\; \frac{\text{const.}}{|x|},$$

valid for $|z|$ large, obtained near the end of the theorem's proof, *cannot be improved* in the sector $0 \leqslant |y| \leqslant |x|$.

Indeed, since $G(t, \mathrm{i}) \geqslant 0$ we have, for large $|x|$,

$$G(x - \mathrm{i}|x|,\ \mathrm{i}) \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|x|}{(x-t)^2 + x^2} G(t, \mathrm{i})\,\mathrm{d}t$$

$$\geqslant\; \frac{4}{13\pi|x|}\int_{-|x|/2}^{|x|/2} G(t, \mathrm{i})\,\mathrm{d}t \;\sim\; \frac{4}{13\pi|x|}\int_{-\infty}^{\infty} G(t, \mathrm{i})\,\mathrm{d}t.$$

A better bound on $G(z, \mathrm{i})$ *can* be obtained if $|y|$ is *much smaller* than $|x|$. The following result is used in the next exercise.

Lemma. *For large* $|x|$,

$$G(z, \mathrm{i}) \;\leqslant\; \text{const.}\frac{\log|x|}{|x|^{1+1/p}}, \qquad 0 \leqslant |y| \leqslant |x|^{1-1/p}.$$

Proof. Taking wlog $x > 0$, consider first the case where $y < 0$. By the theorem,

$$G(z, \mathrm{i}) \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|y|}{(x-t)^2 + y^2} G(t, \mathrm{i})\,\mathrm{d}t$$

$$\leqslant\; \text{const.}\int_{-\infty}^{\infty} \frac{|y|}{(x-t)^2 + y^2}\cdot\frac{\log^+|t| + 1}{|t|^{1+1/p} + 1}\,\mathrm{d}t.$$

As usual, we break up the right-hand integral into

$$\int_{-x/2}^{x/2} + \int_{|t| \geqslant x/2}$$

The first term is $\leqslant$ const.$|y|/x^2$ (because $1 + 1/p > 1$!), and this is

$\leqslant$ const.$/x^{1+1/p}$ for $|y| \leqslant x^{1-1/p}$. The second term is clearly

$$\leqslant \frac{\text{const.}\log x}{x^{1+1/p}}.$$

This handles the case of negative y.

For $0 < y < x^{1-1/p}$, use the relation

$$G(z,\mathrm{i}) - G(\bar{z},\mathrm{i}) = \log\left|\frac{z+\mathrm{i}}{z-\mathrm{i}}\right|$$

already applied in the proof of the theorem. Note that the *right hand* side is

$$\Re\log\left(\frac{1+(\mathrm{i}/z)}{1-(\mathrm{i}/z)}\right) = 2\frac{\Im z}{|z|^2} + O\left(\frac{1}{|z|^3}\right)$$

for large $|z|$. For $0 \leqslant \Im z \leqslant |x|^{1-1/p}$, this is

$$\leqslant \text{const.}\frac{1}{|x|^{1+1/p}}.$$

The lemma thus follows because it is true for negative y.

In the following problem the reader is asked to work out the analogue, for our present sets E, of Benedicks' beautiful result about the ones with positive lower uniform density (Problem 16).

Problem 19

If t is on the component $[n^p - \delta,\ n^p + \delta]$ of

$$\mathscr{D} = \mathbb{C} \sim \bigcup_{-\infty}^{\infty} [|k|^p \operatorname{sgn} k - \delta,\quad |k|^p \operatorname{sgn} k + \delta],$$

show that

$$\frac{\mathrm{d}\omega_{\mathscr{D}}(t,\mathrm{i})}{\mathrm{d}t} \leqslant \frac{\text{const.}}{t^{1+1/p}+1}\cdot\frac{1}{\sqrt{(\delta^2 - (t-n^p)^2)}},$$

where $\omega_{\mathscr{D}}(\ ,z)$ denotes harmonic measure for $\mathscr{D}$. Here, the constant depends only on $p > 1$ and $\delta > 0$.

Remark. The result is due to Benedicks. We see that the factor $\log|t|$ in the estimate for $G(t,\mathrm{i})$ furnished by the above theorem *disappears* when we evaluate *harmonic measure.*

Hint for the problem: One proceeds as in the solution of Problem 16, here comparing $G(z,\mathrm{i})$ with

$$U(z) = \log\left|\frac{z-n^p}{\delta} + \sqrt{\left(\left(\frac{z-n^p}{\delta}\right)^2 - 1\right)}\right|$$

on the ellipse Γ_n with foci at $n^p \pm \delta$ and semi-minor-axis equal to $n^{p-1}\delta$:

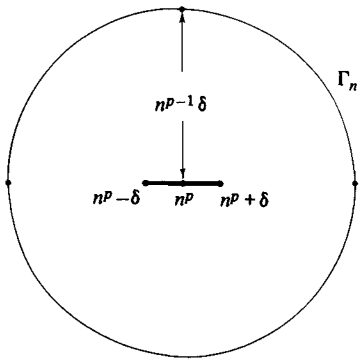

Figure 132

By Problem 19, we have, for the harmonic measure of the component

$$E_n \;=\; [|n|^p \operatorname{sgn} n - \delta,\; |n|^p \operatorname{sgn} n + \delta]$$

of $E = \partial\mathscr{D}$,

$$\boxed{\omega_{\mathscr{D}}(E_n, \mathrm{i}) \;\leqslant\; \frac{\text{const.}}{|n|^{p+1}+1}.}$$

Using this estimate, one can establish a result corresponding to the *first part* of the first theorem in article 3.

Theorem. *Let $W(x) \geqslant 1$ be continuous on*

$$E \;=\; \bigcup_{-\infty}^{\infty} [|n|^p \operatorname{sgn} n - \delta,\; |n|^p \operatorname{sgn} n + \delta],$$

and suppose that $W(x) \longrightarrow \infty$ for $x \to \pm\infty$ in E. If, for some $C > 0$, the supremum of

$$\int_E \frac{\log|S(t)|}{1+|t|^{1+1/p}}\,\mathrm{d}t$$

for S ranging over all finite sums of the form

$$S(t) \;=\; \sum_{-C \leqslant \lambda \leqslant C} a_\lambda \mathrm{e}^{\mathrm{i}\lambda t}$$

with $\|S\|_{W,E} \leqslant 1$ is finite, *then $\mathscr{C}_W(E, C) \neq \mathscr{C}_W(E)$.*

Proof. We have the above boxed estimate for the harmonic measure (in $\mathscr{D} = \mathbb{C} \sim E$) of the components of E, and a previous corollary gives us a Phragmén–Lindelöf function $Y(z)$ for $\mathscr{D}$. Using these facts, one proceeds exactly as in the proof of the first theorem of article 3.

Remark. The sparsity of the set E involved here has caused the form $\mathrm{d}t/(1+t^2)$ occurring in the result of article 3 to be replaced by $\mathrm{d}t/(1+|t|^{1+1/p})$.

Remark. The statement of the above theorem goes in only one direction, unlike that of the corresponding one in article 3. There, since we were dealing with the restriction of the form $\mathrm{d}t/(1+t^2)$ to E, we were able to obtain a *converse* by simply appealing to Akhiezer's theorem from §E.2 of Chapter VI. In the present situation we can't do that, because we are dealing with $\mathrm{d}t/(1+|t|^{1+1/p})$ instead of $\mathrm{d}t/(1+t^2)$, and $1/p<1$. *It would be interesting to see whether* (as seems likely) *the converse is true here.*

In case $W(x)\to\infty$ faster than any power of $|x|$ as $x\to\pm\infty$ in E, we can formulate a result like the above one for *polynomial approximation* on E in the weight W. The statement of it is exactly like that of the *first part* of the *second* theorem in article 3, save that the integrals

$$\int_E \frac{\log|P(x)|}{1+x^2}\,\mathrm{d}x$$

figuring there are here replaced by

$$\int_E \frac{\log|P(x)|}{1+|x|^{1+1/p}}\,\mathrm{d}x.$$

The proof runs much like that of the result in article 3. Details are left to the reader.

B. The set E reduces to the integers

Consider the set

$$E_\rho \;=\; \bigcup_{n=-\infty}^{\infty}\,[n-\rho,\; n+\rho],$$

where $0\leqslant\rho<\frac{1}{2}$. If $\rho>0$, the results of §§A.1–A.3 apply to E_ρ, and there is, in particular, a constant b_ρ such that the inequality

$$\int_{-\infty}^{\infty} \frac{\log(1+|P(x)|^2)}{1+x^2}\,\mathrm{d}x \;\leqslant\; b_\rho \int_{E_\rho} \frac{\log(1+|P(x)|^2)}{1+x^2}\,\mathrm{d}x,$$

used in proving the second theorem of §A.3, holds for polynomials P.

For this reason, given any M, the set of polynomials P such that

$$\int_{E_\rho} \frac{\log^+|P(x)|}{1+x^2}\,dx \;\leqslant\; M$$

forms a normal family in the complex plane.

Suppose now that $\rho = 0$. Then $E_\rho = \mathbb{Z}$, and the proof in §A of the above inequality involving b_ρ, available when $\rho > 0$, cannot be made to work so as to yield a relation of the form

$$\int_{-\infty}^{\infty} \frac{\log(1+|P(x)|^2)}{1+x^2}\,dx \;\leqslant\; C\sum_{-\infty}^{\infty} \frac{\log(1+|P(n)|^2)}{1+n^2}$$

That proof depends on the properties of harmonic measure for $\mathscr{D}_\rho = \mathbb{C} \sim E_\rho$ worked out in §A.1 (for $\rho > 0$); there is, however, *no harmonic measure* for $\mathscr{D} = \mathbb{C} \sim \mathbb{Z}$. This makes it seem very unlikely that the set of polynomials P satisfying

$$\sum_{-\infty}^{\infty} \frac{\log^+|P(n)|}{1+n^2} \;\leqslant\; M$$

for arbitrary given M would form a normal family in the complex plane, and it is in fact easy to construct a counter example to such a claim.

Take simply

$$P_N(x) \;=\; (1-x^2)^{[N/\log N]} \prod_{k=1}^{N}\left(1-\frac{x^2}{k^2}\right)$$

for $N \geqslant 2$, with $[p]$ denoting the greatest integer $\leqslant p$ as usual. Then it is not hard to verify that

$$(*) \qquad \sum_{-\infty}^{\infty} \frac{\log^+|P_N(n)|}{1+n^2} \;\leqslant\; 20 \qquad \text{for } N \geqslant 8.$$

At the same time,

$$P_N(\mathrm{i}) \;\geqslant\; 2^{[N/\log N]} \;\xrightarrow[N]{}\; \infty.$$

Problem 20

Prove $(*)$.

$$\left(\text{Hint:} \sum_1^{\infty} \frac{1}{n^2}\log^+|P_N(n)| \;\leqslant\; \left[\frac{N}{\log N}\right]\sum_{n=N+1}^{\infty} \frac{\log(n^2-1)}{n^2}\right.$$

$$+ \sum_{n=N+1}^{\infty} \frac{1}{n^2}\left[\sum_{k=1}^{N} \log\left|\frac{n^2}{k^2}-1\right|\right]^+.$$

After replacing the sums on the right by suitable integrals and doing some calculation, one obtains the upper bound

$$2 + \frac{2}{\log N} + 2 + 2\int_1^\infty \log\left(\frac{\xi+1}{\xi-1}\right)\frac{d\xi}{\xi}.$$

Here, the integral can be worked out by contour integration.)

This example, however, does not invalidate the analogue (with obvious *statement*) of Akhiezer's theorem for weighted polynomial approximation on $\mathbb{Z}$. In order to disprove such a conjecture, one would (at least) need similar examples with the number 20 standing on the right side of $(*)$ replaced by *arbitrarily small quantities* > 0. No matter how one tries to construct such examples, something always seems to go wrong. It seems impossible to diminish the number in $(*)$ to less than a certain strictly positive quantity without forcing boundedness of the $|P_N(\mathrm{i})|$. One comes in such fashion to believe in the existence of a number $C > 0$ such that the set of polynomials P with

$$\sum_{-\infty}^{\infty} \frac{\log^+|P(n)|}{1+n^2} \leqslant C$$

does form a normal family in the complex plane.

This partial extension of the result from §A.3 to the limiting case $E_\rho = \mathbb{Z}$ turns out to be *valid*. With its help one can establish the *complete analogue* of Akhiezer's theorem for weighted polynomial approximation on $\mathbb{Z}$; its interest is not, however, limited to that application. The extension is easily reduced to a special version of it for polynomials P of the particular form

$$P(x) = \prod_k \left(1 - \frac{x^2}{x_k^2}\right)$$

with *real* roots $x_k > 0$, and most of the *real work* is involved in the treatment of this case, taking up all but the last two of the following articles. The investigation is straightforward but very laborious; although I have tried hard to simplify it, I have not succeeded too well.

The difficulties are what they are, and there is no point in stewing over them. It is better to just take hold of the traces and forge ahead.

1. Using certain sums as upper bounds for integrals corresponding to them

Our situation from now up to almost the end of the present § is as follows: we have a polynomial $P(z)$ of the special form

$$P(z) = \prod_k \left(1 - \frac{z^2}{x_k^2}\right),$$

where the x_k are >0 (in other words, $P(z)$ is *even*, with all of its zeros *real*, and $P(0) = 1$), and we are given an *upper bound* for the sum

$$\sum_{-\infty}^{\infty} \frac{\log^+|P(m)|}{1+m^2},$$

or, what amounts to the same thing *here*, for

$$\sum_1^{\infty} \frac{1}{m^2} \log^+|P(m)|.$$

From this information we desire to *obtain* a bound on $|P(z)|$ for each complex z.

The first idea that comes to mind is to try to use our knowledge about the preceding *sum* in order to control *the integral*

$$\int_{-\infty}^{\infty} \frac{\log^+|P(x)|}{1+x^2} \mathrm{d}x;$$

we have indeed seen in Chapter VI, §B.1, how to deduce an upper bound on $|P(z)|$ from one for this integral. This plan, although probably too simple to be carried out as it stands, does suggest a start on the study of our problem. For *certain intervals* $I \subset (0, \infty)$,

$$\int_I \frac{\log|P(x)|}{x^2} \mathrm{d}x$$

is comparable with

$$\sum_{m \in I \cap \mathbb{Z}} \frac{\log^+|P(m)|}{m^2}.$$

We have

$$\frac{\mathrm{d}^2 \log|P(x)|}{\mathrm{d}x^2} = -2\sum_k \frac{x^2 + x_k^2}{(x^2 - x_k^2)^2} < 0,$$

so $\log|P(x)|$ is *concave* (downward) on any real interval *free of the zeros* $\pm x_k$ *of* P. This means that, if $a < b$ and P has no *zeros* on $[a, b]$,

$$\int_a^b \log|P(x)|\mathrm{d}x \leqslant (b-a)\log|P(m)|$$

for the *midpoint* m of $[a, b]$:

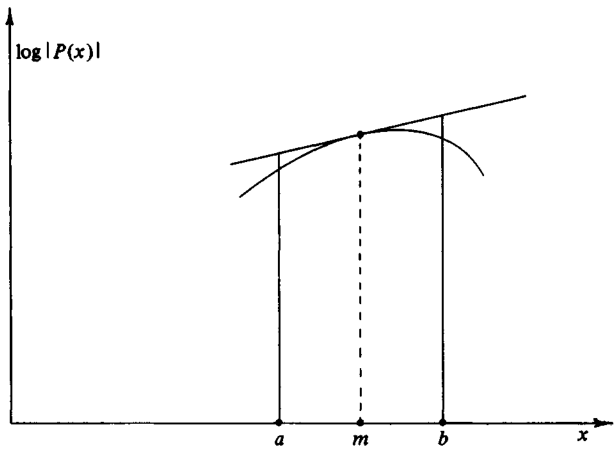

Figure 133

Of course $\int_a^b \log|P(x)|\mathrm{d}x$ is not the integral we are dealing with here. If, however, $a > 0$ is large and $b - a$ not too big, the presence of the factor $1/x^2$ in front of $\log|P(x)|$ does not make much difference. A similar formula still holds, except that m is no longer *exactly the midpoint* of $[a, b]$.

Lemma. *Let* $0 < a < m < b$, *and suppose that* P *has no zeros on* $[a, b]$. *Then*

$$\int_a^b \frac{\log|P(x)|}{x^2}\mathrm{d}x \;\leqslant\; (\log|P(m)|)\int_a^b \frac{\mathrm{d}x}{x^2},$$

provided that

$$\boxed{\log\frac{b}{a} \;=\; \frac{m}{a} - \frac{m}{b}.}$$

Proof. Let M denote the *slope* of the graph of $\log|P(x)|$ vs. x at $x = m$. Then, since $\log|P(x)|$ is *concave* on $[a, b]$, we have there

$$\log|P(x)| \;\leqslant\; \log|P(m)| + M(x - m)$$

(see the previous figure). Hence

$$\int_a^b \frac{\log|P(x)|}{x^2}\mathrm{d}x \;\leqslant\; (\log|P(m)|)\int_a^b \frac{\mathrm{d}x}{x^2} \;+\; M\int_a^b \left(\frac{1}{x} - \frac{m}{x^2}\right)\mathrm{d}x.$$

The second term on the right is

$$M\log\frac{b}{a} - M\left(\frac{m}{a}-\frac{m}{b}\right),$$

and this is zero if the boxed condition on m is satisfied. Done.

We will be interested in situations where the number m figuring in the above boxed relation is a *positive integer*, and where *one* of the two numbers a, b ($a \leqslant m \leqslant b$) is to be *found*, the other being *given*. Regarding these, we have two estimates.

Lemma. *If $m \geqslant 7$ and $m-1 \leqslant a \leqslant m$, the number $b \geqslant m$ such that*

$$\log\frac{b}{a} = \frac{m}{a} - \frac{m}{b}$$

is $\leqslant m+2$.

Proof. Write $\rho = a/b$; then $0 < \rho \leqslant 1$, and the relation to be satisfied becomes $\log(1/\rho) = (m/a)(1-\rho)$. If $a = m$, this is obviously satisfied for $\rho = 1$, i.e., $m = b$; otherwise $0 < \rho < 1$, and we have

$$\frac{m}{a} = \frac{\log\frac{1}{\rho}}{1-\rho}.$$

Now

$$\log\frac{1}{\rho} = 1-\rho+\tfrac{1}{2}(1-\rho)^2+\tfrac{1}{3}(1-\rho)^3+\cdots,$$

so the preceding relation implies that

$$\frac{m}{a} \geqslant 1+\tfrac{1}{2}(1-\rho),$$

i.e.,

$$1-\rho \leqslant 2\frac{m-a}{a},$$

and

$$\rho \geqslant \frac{3a-2m}{a}.$$

Therefore

$$b = \frac{a}{\rho} \leqslant \frac{a^2}{3a-2m},$$

and

$$b-m \leqslant \frac{(2m-a)(m-a)}{3a-2m} \leqslant \frac{m+1}{m-3}.$$

Here the right-hand side is $\leqslant 2$ for $m \geqslant 7$. We are done.

Lemma. *If $m \geqslant 2$ and $m \leqslant b \leqslant m+1$, the number $a \leqslant m$ such that*

$$\log\frac{b}{a} = \frac{m}{a} - \frac{m}{b}$$

is $> m-2$.

Proof. Put $\rho = a/b$ as in proving the preceding lemma; here, it is also convenient to write

$$y = \frac{m}{b}.$$

Then $0 < \rho \leqslant 1$ and $0 < y \leqslant 1$. In terms of y and ρ, our equation becomes

$$\log\frac{1}{\rho} = \frac{y}{\rho} - y.$$

When $y < 1$, we must also have $\rho < 1$, and then

$$y = \frac{\rho\log(1/\rho)}{1-\rho}.$$

This yields, for $0 < \rho < 1$,

$$\frac{dy}{d\rho} = \frac{\log(1/\rho)-(1-\rho)}{(1-\rho)^2} = \tfrac{1}{2}+\tfrac{1}{3}(1-\rho)+\tfrac{1}{4}(1-\rho)^2+\cdots \geqslant \tfrac{1}{2}.$$

Hence, since the value $y=1$ corresponds to $\rho = 1$, we have, for $0 < y < 1$,

$$\tfrac{1}{2}(1-\rho) \leqslant 1-y,$$

i.e.,

$$\rho \geqslant 1-2(1-y).$$

It was given that $m \leqslant b \leqslant m+1$, so

$$1-y = \frac{b-m}{b} \leqslant \frac{1}{m+1}$$

(the middle term here is a monotone function of b). Therefore, by the previous relation,

$$\rho \geqslant 1 - \frac{2}{m+1},$$

and finally,

$$a = \rho b \geqslant \rho m \geqslant m - \frac{2m}{m+1} > m - 2.$$

We are done.

Theorem. *Let $6 \leqslant a < b$. There is a number b^*, $b \leqslant b^* < b+3$, such that*

$$\int_a^{b^*} \frac{\log|P(x)|}{x^2} \mathrm{d}x \leqslant 5 \sum_{a<m<b^*} \frac{\log^+|P(m)|}{m^2},$$

provided that P has no zeros on $[a, b^*]$. The sum on the right is taken over the integers m *with* $a < m < b^*$.

▶ ***Definition.*** During the rest of this §, we will say that b^* is *well disposed* with respect to a.

Proof. By repeated application of the first two of the above lemmas.

Let the integer m_1 be such that $m_1 - 1 \leqslant a < m_1$; then $m_1 \geqslant 7$, so, by the *second* lemma, we can find a number a_1, $m_1 < a_1 \leqslant m_1 + 2$, with

$$\log \frac{a_1}{a} = \frac{m_1}{a} - \frac{m_1}{a_1}.$$

We have $a_1 \leqslant a + 3$, so, since $b > a$, $a_1 < b + 3$.

By the *first* lemma, if $P(x)$ is free of zeros on $[a, a_1]$,

$$\int_a^{a_1} \frac{\log|P(x)}{x^2} \mathrm{d}x \leqslant \log|P(m_1)| \int_a^{a_1} \frac{\mathrm{d}x}{x^2}.$$

Here, $a_1 - a \leqslant 3$ and $m_1/a \leqslant \frac{7}{6}$, so

$$\int_a^{a_1} \frac{\mathrm{d}x}{x^2} \leqslant \frac{5}{m_1^2}.$$

Therefore,

$$\int_a^{a_1} \frac{\log|P(x)|}{x^2} \mathrm{d}x \leqslant \frac{5\log^+|P(m_1)|}{m_1^2}.$$

If now $a_1 \geqslant b$, we simply put $b^* = a_1$ and the theorem is proved. Otherwise, $a_1 < b$ and we take the integer m_2 such that $m_2 - 1 \leqslant a_1 < m_2$. Since $a_1 > m_1$, $m_2 > m_1$, and we can find an a_2, $m_2 < a_2 \leqslant m_2 + 2$, with

$$\log\frac{a_2}{a_1} = \frac{m_2}{a_1} - \frac{m_2}{a_2}.$$

We have $a_2 \leqslant a_1 + 3 < b + 3$, and, by the first lemma,

$$\int_{a_1}^{a_2} \frac{\log|P(x)|}{x^2}\mathrm{d}x \leqslant \log|P(m_2)| \int_{a_1}^{a_2} \frac{\mathrm{d}x}{x^2} \leqslant \frac{5\log^+|P(m_2)|}{m_2^2}$$

just as in the preceding step, provided that P has no zeros on $[a_1, a_2]$.

If $a_2 \geqslant b$, we put $b^* = a_2$. If not, we continue as above, getting numbers $a_3 > a_2$, $a_4 > a_3$, and so forth, $a_{k+1} \leqslant a_k + 3$, until we first reach an a_l with $a_l \geqslant b$. We will then have $a_l < b + 3$, and we put $b^* = a_l$. There are integers m_k, $m_2 < m_3 < \cdots < m_l$, with $a_{k-1} < m_k < a_k$, $k = 3, \ldots, l$, and, as in the previous steps,

$$\int_{a_{k-1}}^{a_k} \frac{\log|P(x)|}{x^2}\mathrm{d}x \leqslant \frac{5\log^+|P(m_k)|}{m_k^2}$$

for $k = 3, \ldots, l$, as long as P has no zeros on $[a_{k-1}, a_k]$.

Write $a_0 = a$. Then, if P has no zeros on $[a, b^*] = [a_0, a_l]$,

$$\int_a^{b^*} \frac{\log|P(x)|}{x^2}\mathrm{d}x = \sum_{k=1}^{l} \int_{a_{k-1}}^{a_k} \frac{\log|P(x)|}{x^2}\mathrm{d}x$$

$$\leqslant \sum_{k=1}^{l} \frac{5\log^+|P(m_k)|}{m_k^2} \leqslant \sum_{\substack{a<m<b^* \\ m\in\mathbb{Z}}} \frac{5\log^+|P(m)|}{m^2}.$$

We are done.

In the result just proved, a is kept fixed and we move from b to a point b^* well disposed with respect to a, lying between b and $b+3$. One can obtain the same effect keeping b fixed and moving downward from a.

Theorem. *Let $10 \leqslant a < b$. There is an a^*, $a - 3 < a^* \leqslant a$, such that b is well disposed with respect to a^*, i.e.,*

$$\int_{a^*}^{b} \frac{\log|P(x)|}{x^2}\mathrm{d}x \leqslant 5 \sum_{a^*<m<b} \frac{\log^+|P(m)|}{m^2},$$

provided that $P(x)$ has no zeros on $[a^*, b]$.

The proof uses the *first* and *third* of the above lemmas, and is otherwise very much like the one of the previous theorem. Its details are left to the reader.

2. Construction of certain intervals containing the zeros of $P(x)$

We have seen in the preceding article how certain intervals $I \subseteq (0, \infty)$ can be obtained for which

$$\int_I \frac{\log|P(x)|}{x^2}\,dx \;\leqslant\; 5\sum_{m\in I}\frac{\log^+|P(m)|}{m^2}$$

as long as they are free of zeros of P. Our next step is to split up $(0, \infty)$ into two kinds of intervals: zero-free ones of the sort just mentioned and then some residual ones which, together, contain all the positive zeros of $P(x)$. The latter are closely related to some intervals used earlier by Vladimir Bernstein (*not* the S. Bernstein after whom the weighted polynomial approximation problem is named) in his study of Dirichlet series, and it is to their construction we now turn.

▶ As is customary, we denote by $n(t)$ *the number of zeros* x_k *of* $P(x)$ *in the interval* $[0, t]$ for $t \geqslant 0$ (*counting multiplicities as in Chapter* III). When $t < 0$, we take $n(t) = 0$. The function $n(t)$ is thus *integer-valued and increasing*. It is *zero* for all $t > 0$ *sufficiently close to* 0 (because the $x_k > 0$), and *constant* for sufficiently large t (P being a polynomial).

The graph of $n(t)$ vs. t consists of some *horizontal portions* separated by *jumps*. At each jump, $n(t)$ increases by an integral multiple of unity. In this quantization must lie the essential difference between the behaviour of subharmonic functions of the *special form* $\log|P(x)|$ with P a polynomial, and that of *general* ones having at most logarithmic growth at ∞, for which there holds no valid analogue of the theorem to be established in this §. (Just look at the subharmonic functions $\eta\log|P_N(z)|$, where $\eta > 0$ is arbitrarily small and the P_N are the polynomials considered in Problem 20.) During the present article we will see precisely how the quantization affects matters.

For the following work we *fill in the vertical portions* of the *graph* of $n(t)$ vs. t. In other words, if $n(t)$ has a jump discontinuity at t_0, we consider the *vertical segment joining* $(t_0, n(t_0-))$ *to* $(t_0, n(t_0+))$ *as forming part of that graph.*

Our constructions are arranged in three stages.

First stage. Construction of the Bernstein intervals

We begin by taking an arbitrary *small* number $p > 0$ (requiring, say, that $p < 1/20$). Once chosen, p is kept *fixed* during most of the discussion of this and the following articles.

Denote by $\mathcal{O}$ the set of points $t_0 \in \mathbb{R}$ with the property that *a straight line of slope* p through $(t_0, n(t_0))$ *cuts or touches the graph of* $n(t)$ *vs.* t *only once.* $\mathcal{O}$ is open and its *complement* in $\mathbb{R}$ consists of a *finite number* of closed intervals B_0, B_1, B_2,... called the Bernstein intervals for slope p associated with the polynomial $P(x)$. (Together, the B_k make up what V. Bernstein called a *neighborhood set* for the positive zeros of P – see page 259 of his book on Dirichlet series. His construction of the B_k is different from the one given here.) It is best to show the formation of the B_k by a diagram:

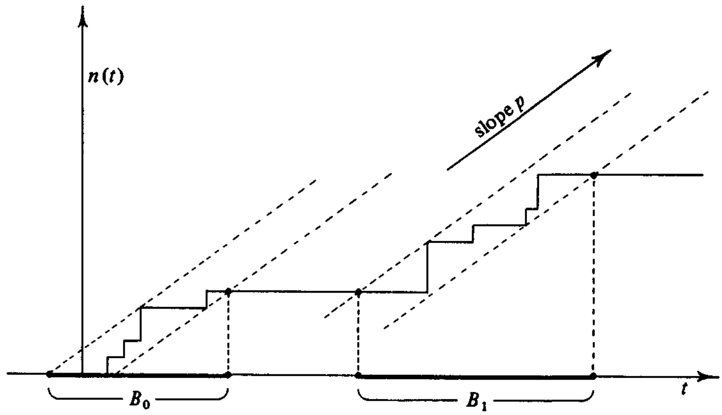

Figure 134

We see that all the positive zeros of P (points of discontinuity of $n(t)$) are contained in the union of the B_k. Also, *taking any B_k and denoting it by $[a, b]$*:

The part of the graph of $n(t)$ vs. t corresponding to the values of t in B_k lies between the two parallel lines of slope p through the points $(a,\ n(a))$ and $(b,\ n(b))$.

For a closed interval $I = [\alpha, \beta]$, say, let us write $n(I)$ for $n(\beta +) - n(\alpha -)$. The statement just made then implies that

$$n(B_k)/p|B_k| \leqslant 1$$

for each Bernstein interval. An inequality in the opposite sense is less apparent.

Lemma (Bernstein). *For each of the B_k,*

$$n(B_k)/p|B_k| \geqslant 1/2.$$

Proof. It is geometrically evident that a line of slope p which cuts (or touches) the graph of $n(t)$ vs. t *more than once* must come into contact with some *vertical portion* of it – let the reader make a diagram.

Take any interval B_k, denote it by $[a, b]$, and denote the portion of the graph of $n(t)$ vs. t corresponding to the values $a \leqslant t \leqslant b$ by G. We indicate by L and M the lines of slope p through the points $(a, n(a))$ and $(b, n(b))$ respectively. According to our definition, any line N of slope p *between* L and M must cut (or touch) the graph of $n(t)$ vs. t *at least twice*, and hence come into contact with some *vertical portion* of that graph. Otherwise such a line N, which surely cuts G, would intersect the graph *only once*, at some point with abscissa $t_0 \in (a, b)$; t_0 would then belong to $\mathcal{O}$ and thus *not* to B_k. The line N must in fact come into contact with a vertical portion of G, for, as a glance at the preceding figure shows, it can never touch any part of the graph that does not lie over $[a, b]$.

In order to prove the lemma, it is therefore enough to show that *if*

$$n(B_k)/p|B_k| \ < \ 1/2.$$

there must be some line N of slope p, lying between L and M, that does not come into contact with any vertical portion of G.

Let V be the *union* of the *vertical portions* of G, and for $X \in V$, denote by $\Pi(X)$ *the downward projection, along a line with slope p*, of the point X onto the horizontal line through $(a, n(a))$.

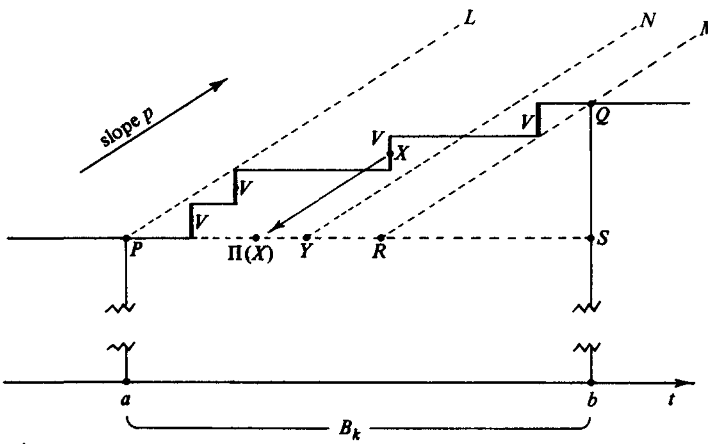

Figure 135

In this figure, $|B_k| = \overline{PS}$ and $n(B_k) = \overline{QS}$. The result, $\Pi(V)$, of applying Π to all the points of V is a certain closed subset of the segment PR, and, if we use $|\ \ |$ to denote linear Lebesgue measure, it is clear that

$$|\Pi(V)| \ \leqslant \ |V|/p.$$

We have $p \cdot \overline{RS} = \overline{QS}$, so, if $n(B_k) = \overline{QS} < \frac{1}{2}p|B_k| = \frac{1}{2}p \cdot \overline{PS}$, $p \cdot \overline{RS} < \frac{1}{2}p \cdot \overline{PS}$, and therefore $\overline{PR} > \frac{1}{2}\overline{PS} > \overline{QS}/p = |V|/p$. With the preceding relation, this yields

$$|\Pi(V)| \;<\; \overline{PR}.$$

There is thus a point Y on PR *not belonging* to the projection $\Pi(V)$. If, then, N is the line of slope p through Y, N cannot come into contact with V. This line N lies between L and M, so we are done.

Second stage. Modification of the Bernstein intervals

The Bernstein intervals B_k just constructed include all the positive zeros of $P(x)$, and

$$\frac{1}{2} \;\leqslant\; \frac{n(B_k)}{p|B_k|} \;\leqslant\; 1.$$

We are going to modify them so as to obtain new closed intervals $I_k \subseteq (0,\infty)$ containing all the positive zeros of $P(x)$, positioned so as to make

$$\int_I \frac{\log|P(x)|}{x^2}\,\mathrm{d}x \;\leqslant\; 5\sum_{m\in I}\frac{\log^+|P(m)|}{m^2}$$

for each of the interval components I of

$$(0,\infty) \sim \bigcup_k I_k.$$

(Note that B_0 need not even be contained in $[0,\infty)$.) For the calculations which come later on, it is also very useful to have *all the ratios* $n(I_k)/|I_k|$ *the same*, and we carry out the construction so as to ensure this.

Specifically, the intervals I_k, which we will write as $[\alpha_k, \beta_k]$ with $k = 0, 1, 2, \ldots$ and $0 < \alpha_0 < \beta_0 < \alpha_1 < \beta_1 < \cdots$, are to have the following properties:

(i) *All the positive zeros of* $P(x)$ *are contained in the union of the* I_k,

(ii) $n(I_k)/p|I_k| = \frac{1}{2}, \quad k = 0, 1, 2, \ldots,$

(iii) *For* $\alpha_0 \leqslant t \leqslant \beta_0$,

$$n(\beta_0) - n(t) \;\leqslant\; \frac{p}{1-3p}(\beta_0 - t),$$

and, for $\alpha_k \leqslant t \leqslant \beta_k$ *with* $k \geqslant 1$,

$$n(t) - n(\alpha_k) \;\leqslant\; \frac{p}{1-3p}(t - \alpha_k),$$

$$n(\beta_k) - n(t) \;\leqslant\; \frac{p}{1-3p}(\beta_k - t),$$

(recall that we are assuming $0 < p < \frac{1}{20}$),

(iv) *For $k \geqslant 1$, α_k is well disposed with respect to β_{k-1}* (see the preceding article).

Denote the Bernstein intervals B_k, $k = 0, 1, 2, \ldots$, by $[a_k, b_k]$, arranging the indices so as to have $b_{k-1} < a_k$. We begin by constructing I_0. Take α_0 as the *smallest* positive zero of $P(x)$; α_0 is the first point of discontinuity of $n(t)$ and $a_0 < \alpha_0 < b_0$. We have

$$\frac{n([\alpha_0, b_0])}{p(b_0 - \alpha_0)} = \frac{n(B_0)}{p(b_0 - \alpha_0)} > \frac{n(B_0)}{p|B_0|} \geqslant \frac{1}{2}$$

by the lemma from the preceding (first) stage. For $\tau \geqslant b_0$, let J_τ be the interval $[\alpha_0, \tau]$. As we have just seen,

$$n(J_\tau)/p|J_\tau| > 1/2$$

for $\tau = b_0$. When τ increases from b_0 to a_1 (assuming that there *is* a Bernstein interval B_1; there need not be!) the numerator of the left-hand ratio remains equal to $n(B_0)$, while the denominator increases. The ratio itself therefore decreases when τ goes from b_0 to a_1, and either gets down to $\frac{1}{2}$ in (b_0, a_1), or else remains $> \frac{1}{2}$ there. (In case there is no Bernstein interval B_1 we may take $a_1 = \infty$, and then the first possibility is realized.)

Suppose that we *do* have $n(J_\tau)/p|J_\tau| = \frac{1}{2}$ for some τ, $b_0 < \tau < a_1$. Then we put β_0 *equal to that value of* τ, and property (ii) certainly holds for $I_0 = [\alpha_0, \beta_0]$. Property (iii) does also. Indeed, by construction of the B_k, the line of slope p through $(\beta_0, n(\beta_0))$ cuts the graph of $n(t)$ vs. t *only once*, so the portion of the graph corresponding to values of $t < \beta_0$ *lies entirely to the left* of that line (look at the *first* of the diagrams in this article). That is,

$$n(\beta_0) - n(t) \leqslant p(\beta_0 - t), \qquad t \leqslant \beta_0,$$

whence, *a fortiori*,

$$n(\beta_0) - n(t) \leqslant \frac{p}{1 - 3p}(\beta_0 - t), \qquad t \leqslant \beta_0$$

(since $0 < p < 1/20$, $0 < 1 - 3p < 1$).

It may happen, however, that $n(J_\tau)/p|J_\tau|$ remains $> \frac{1}{2}$ for $b_0 < \tau < a_1$. Then that ratio is *still* $\geqslant \frac{1}{2}$ for $\tau = b_1$. This is true because $n(B_1)/p|B_1| \geqslant \frac{1}{2}$ (lemma from the preceding stage), and

$$n([\alpha_0, b_1]) = n(a_1 -) - n(\alpha_0 -) + n(B_1),$$

while

$$b_1 - \alpha_0 = a_1 - \alpha_0 + |B_1|.$$

Thus, in our present case, $n(J_\tau)/p|J_\tau|$ is $\geqslant \frac{1}{2}$ for $\tau = b_1$ and again decreases as τ moves from b_1 towards $a_2 > b_1$. (If there *is* no interval B_2 we may take $a_2 = \infty$.) If, for some $\tau \in [b_1, a_2)$, we have $n(J_\tau)/p|J_\tau| = \frac{1}{2}$, we take β_0 equal to that value of τ, and property (ii) holds for $I_0 = [\alpha_0, \beta_0]$. Also, for $\beta_0 \in [b_1, a_2)$, the part of the graph of $n(t)$ vs. t corresponding to the values $t \leqslant \beta_0$ lies *on or entirely to the left* of the line of slope p through $(\beta_0, n(\beta_0))$, as in the situation already discussed. Therefore, $n(\beta_0) - n(t) \leqslant (p/(1-3p))(\beta_0 - t)$ for $t \leqslant \beta_0$ as before, and property (iii) holds for I_0.

In case $n(J_\tau)/p|J_\tau|$ still remains $> \frac{1}{2}$ for $b_1 \leqslant \tau < a_2$, we will have $n(J_\tau)/p|J_\tau| \geqslant \frac{1}{2}$ for $\tau = b_2$ by an argument like the one used above, and we look for β_0 in the interval $[b_2, a_3)$. The process continues in this way, and we either get a β_0 lying between two successive intervals B_k, B_{k+1} (perhaps coinciding with the right endpoint of B_k), or else pass through the half open interval separating the *last two* of the B_k without ever bringing the ratio $n(J_\tau)/p|J_\tau|$ down to $\frac{1}{2}$. If this second eventuality occurs, suppose that $B_l = [a_l, b_l]$ is the *last* B_k; then $n(J_\tau)/p|J_\tau| \geqslant \frac{1}{2}$ for $\tau = b_l$ by the reasoning already used. Here, $n(J_\tau)$ remains equal to $n([0, b_l])$ for $\tau \geqslant b_l$ while $|J_\tau|$ increases without limit, so a value β_0 of $\tau \geqslant b_l$ will make $n(J_\tau)/p|J_\tau| = \frac{1}{2}$. There is then only one interval I_k, namely, $I_0 = [\alpha_0, \beta_0]$, and our construction is finished, because properties (i) and (ii) obviously hold, while (iii) does by the above reasoning and (iv) is vacuously true.

In the event that the process gives us a β_0 *lying between two successive Bernstein intervals*, we have to construct $I_1 = [\alpha_1, \beta_1]$. In these circumstances we must first choose α_1 so as to have it *well disposed* with respect to β_0, ensuring property (iv) for $k = 1$.

▶ ***It is here that we make crucial use of the property that each jump in $n(t)$ has height $\geqslant$ 1.***

Assume that $b_k \leqslant \beta_0 < a_{k+1}$. We have $p(\beta_0 - \alpha_0) = 2n(I_0) \geqslant 2$ with $0 < p < \frac{1}{20}$; therefore $\beta_0 > 40$ and there is by the *first* theorem of the preceding article a number α_1, $a_{k+1} \leqslant \alpha_1 < a_{k+1} + 3$, which is *well disposed with respect to* β_0.

Now α_1 may well lie to the right of a_{k+1}. It is nevertheless true that $n(\alpha_1) = n(a_{k+1} -)$, and moreover

$$n(t) - n(\alpha_1) \leqslant \frac{p}{1-3p}(t - \alpha_1) \qquad \text{for } t \geqslant \alpha_1.$$

The following diagram shows how these properties follow from two facts:

that $n(t)$ increases by at least 1 *at each jump*, and that $1/p > 3$:

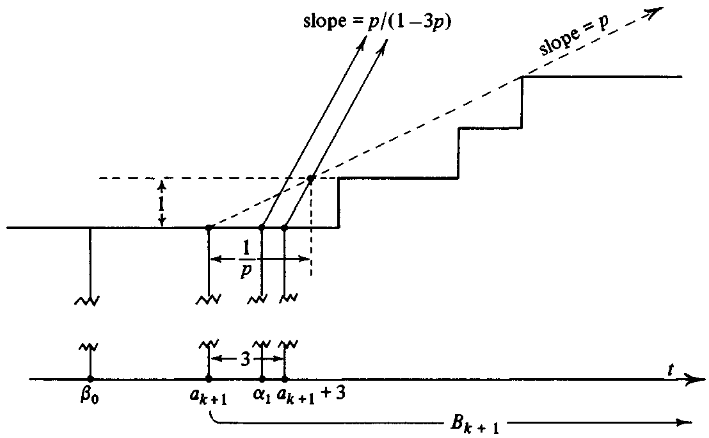

Figure 136

For this choice of α_1, properties (i)–(iv) will hold, provided that β_1, α_2 and so forth are correctly determined.

We go on to specify β_1. This is very much like the determination of β_0. Since

$$n(b_{k+1}) - n(\alpha_1) \;=\; n(b_{k+1}) - n(a_{k+1}-) \;=\; n(B_{k+1}),$$

we certainly have

$$\frac{n([\alpha_1, b_{k+1}])}{p(b_{k+1}-\alpha_1)} = \frac{n(B_{k+1})}{p(b_{k+1}-\alpha_1)} \geqslant \frac{n(B_{k+1})}{p|B_{k+1}|} \geqslant \frac{1}{2}$$

by the lemma from the preceding stage. For $\tau \geqslant b_{k+1}$, denote by J'_τ the interval $[\alpha_1, \tau]$; then $n(J'_\tau)/p|J'_\tau|$ is $\geqslant \frac{1}{2}$ for $\tau = b_{k+1}$ and *diminishes* as τ increases along $[b_{k+1}, a_{k+2})$. (If there *is* no B_{k+2} we take $a_{k+2} = \infty$.) We may evidently proceed just as above to get a $\tau \geqslant b_{k+1}$, lying either in a half open interval separating two successive Bernstein intervals or else beyond all of the latter, such that $n(J'_\tau)/p|J'_\tau| = \frac{1}{2}$. *That value* of τ is taken as β_1. The part of the graph of $n(t)$ vs. t corresponding to values of $t \leqslant \beta_1$ lies, as before, *on or to the left* of the line through $(\beta_1, n(\beta_1))$ with slope p. Hence, *a fortiori*,

$$n(\beta_1) - n(t) \;\leqslant\; \frac{p}{1-3p}(\beta_1 - t) \qquad \text{for } t \leqslant \beta_1.$$

We see that properties (ii) and (iii) hold for I_0 and $I_1 = [\alpha_1, \beta_1]$.

If $I_0 \cup I_1$ does not already include all of the B_k, β_1 must lie between two of them, and we may proceed to find an α_2 in the way that α_1 was found above. Then we can construct an I_2. Since there are only a finite number of B_k, the process will eventually stop, and we will end with a finite number of intervals $I_k = [\alpha_k, \beta_k]$ having properties (ii)–(iv). Property (i) will then also hold, since, when we finish, the union of the I_k includes that of the B_k.

Here is a picture showing the relation of the intervals I_k to the graph of $n(t)$ vs. t:

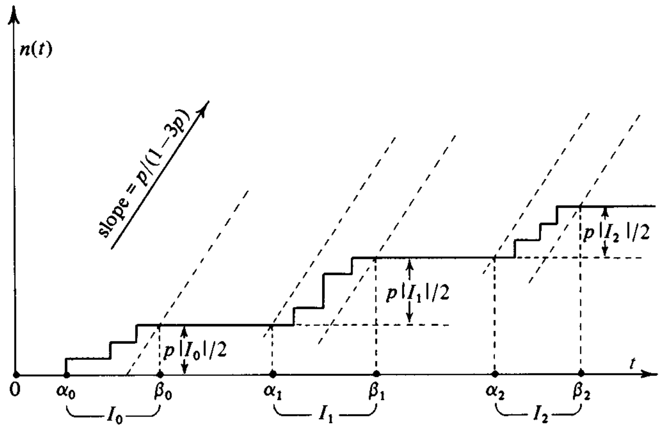

Figure 137

Let us check the statement made before starting the construction of the I_k, to the effect that

$$\int_I \frac{\log|P(x)|}{x^2}\,dx \;\leqslant\; 5\sum_{m\in I}\frac{\log^+|P(m)|}{m^2}$$

for each of the interval components I of the *complement*

$$(0,\infty) \sim \bigcup_k I_k.$$

Since, for $k>0$, α_k is well disposed with respect to β_{k-1}, this is certainly true for the components I of the form (β_{k-1}, α_k), $k \geqslant 1$ (if there *are* any!), by the first theorem of the preceding article. This is also true, and trivially so, for $I = (0, \alpha_0)$, because

$$|P(x)| \;=\; \prod_k \left|1 - \frac{x^2}{x_k^2}\right| \;<\; 1$$

for $0 < x < \alpha_0$, all the positive zeros x_k of $P(x)$ being $\geqslant \alpha_0$. Finally, if I_l is the *last* of the I_k, our relation is true for $I = (\beta_l, \infty)$. This follows because