PSEUDODIFFERENTIAL ARITHMETIC AND A REJECTION OF THE RIEMANN HYPOTHESIS

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Abstract. The Weyl symbolic calculus of operators leads to the construction, if one takes for symbol a certain distribution in the plane decomposing over the set of zeros of the Riemann zeta function, of an operator with the following property: the Riemann hypothesis is equivalent to the validity of a collection of estimates involving this operator. Pseudodifferential arithmetic, a novel chapter of pseudodifferential operator theory, makes it possible to make the operator fully explicit. This leads in an unexpected way to a disproof of the conjecture in the following strong sense: zeros of zeta accumulate on the boundary of the critical strip.

1. Introduction

An understanding of the Riemann hypothesis cannot be dependent on complex analysis, in the style of the XIXth century, only. Though the analytic aspects of the Riemann zeta function must indeed contribute, arithmetic, especially prime numbers, must of necessity enter the discussion in a decisive role. The main part of the present paper consists in putting together two aspects of the same object – to wit, a certain hermitian form – one originating from its analytic nature and involving the zeros of the zeta function, and the other described in terms of congruence algebra. A complete understanding of the latter one is reached with the help of pseudodifferential arithmetic, a chapter of pseudodifferential operator theory adapted to the species of symbols needed here.

The central object of the proof to follow is the distribution

$$\mathfrak{T}_\infty(x,\xi) = \sum_{|j|+|k|\neq 0} a((j,k))\,\delta(x-j)\,\delta(\xi-k) \tag{1.1}$$

in the plane, with $(j,k) = \text{g.c.d.}(j,k)$ and $a(r) = \prod_{p|r}(1-p)$ for $r = 1, 2, \dots$. There is a collection $(\mathfrak{E}_\nu)_{\nu\neq\pm 1}$ of so-called Eisenstein distributions,

$\mathfrak{E}_\nu$ homogeneous of degree $-1-\nu$, such that, as an analytic functional,

$$\mathfrak{T}_\infty = 12\,\delta_0 + \sum_{\zeta(\rho)=0} \mathrm{Res}_{\nu=\rho}\left(\frac{\mathfrak{E}_{-\nu}}{\zeta(\nu)}\right), \tag{1.2}$$

where δ_0 is the unit mass at the origin of $\mathbb{R}^2$. This decomposition calls for using $\mathfrak{T}_\infty$ in a possible approach to the zeros of the Riemann zeta function: only, to obtain a full benefit of this distribution, one must appeal to pseudodifferential analysis, more precisely to the Weyl symbolic calculus of operators [14]. This is the rule Ψ that associates to any distribution $\mathfrak{S} \in \mathcal{S}'(\mathbb{R}^2)$ the so-called operator with symbol $\mathfrak{S}$, to wit the linear operator $\Psi(\mathfrak{S})$ from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$ weakly defined by the equation

$$(\Psi(\mathfrak{S})\,u)\,(x) = \frac{1}{2}\int_{\mathbb{R}^2} \mathfrak{S}\left(\frac{x+y}{2},\,\xi\right) e^{i\pi(x-y)\xi}\,u(y)\,dy\,d\xi. \tag{1.3}$$

It defines pseudodifferential analysis, which has been for more than half a century one of the main tools in the study of partial differential equations. However, the methods used in this regard in the present context do not intersect the usual ones (though familiarity with the more formal aspects of the Weyl calculus will certainly help) and may call for the denomination of "pseudodifferential arithmetic" (Sections 6 to 8).

Making use of the Euler operator $2i\pi\mathcal{E} = 1 + x\frac{\partial}{\partial x} + \xi\frac{\partial}{\partial \xi}$ and of the collection of rescaling operators $t^{2i\pi\mathcal{E}}$, to wit

$$\left(t^{2i\pi\mathcal{E}}\mathfrak{S}\right)(x,\xi) = t\,\mathfrak{S}\,(tx,\,t\xi)\,, \qquad t>0. \tag{1.4}$$

we brought attention in [11] to the hermitian form $\left(w \,\middle|\, \Psi\left(Q^{2i\pi\mathcal{E}}\mathfrak{T}_\infty\right)w\right)$. The following criterion for R.H. was obtained: that, for some $\beta > 2$ and any function $w \in C^\infty(\mathbb{R})$ supported in $[0,\beta]$, one should have for every $\varepsilon > 0$

$$\left(w \,\middle|\, \Psi\left(Q^{2i\pi\mathcal{E}}\mathfrak{T}_\infty\right)w\right) = \mathrm{O}\left(Q^{\frac{1}{2}+\varepsilon}\right) \tag{1.5}$$

as $Q \to \infty$ through squarefree integral values. The proof is based on a spectral-theoretic analysis of the operator $\sum_{Q \text{ sqf}} Q^{-s+2i\pi\mathcal{E}}$ as a function of $s \in \mathbb{C}$: ultimately, if one assumes (1.5), a pole with no right to exist will be associated to any zero ρ of zeta such that $\mathrm{Re}\,\rho > \frac{1}{2}$. We shall prove here a slightly different criterion, involving a pair v, u in place of w, w, in Section 5: the case when the supports of v and u do not intersect, nor do those of v and $\overset{\vee}{u}$, is especially important.

Eisenstein distributions will be described in detail in Section 3. They make up just a part of the domain of automorphic distribution theory, which relates to the classical one of modular form theory but is more precise. The operator $\pi^2\mathcal{E}^2$ in the plane transfers under some map (the dual Radon transformation, in an arithmetic context) to the operator $\Delta - \frac{1}{4}$ in the hyperbolic half-plane, where Δ is the automorphic Laplacian. While this is crucial in other applications, it is another feature of automorphic distribution theory that will be essential here: the way it can cooperate with the Weyl symbolic calculus. Automorphic distribution theory has been developed in a series of books, ending with [10]. Its origin got some inspiration from the Lax-Phillips scattering theory of automorphic functions [5]: we shall develop this aspect in Section 12. Let us make it quite clear that the automorphy concepts will not be needed here, though both $\mathfrak{T}_\infty$ and the Eisenstein distributions owe their existence to such considerations.

The following property of the operator $\Psi(\mathfrak{E}_{-\nu})$ with symbol $\mathfrak{E}_{-\nu}$ (shared by all operators with automorphic symbols) will be decisive in algebraic calculations: if v, u is a pair of C^∞ functions on the line such that $0 < x^2 - y^2 < 8$ whenever $\overline{v}(x)u(y) \neq 0$, the hermitian form $\left(v \,\middle|\, \Psi(\mathfrak{E}_{-\nu})\, u\right)$ depends only on the restriction of $\overline{v}(x)u(y)$ to the set $\{(x, y) \colon x^2 - y^2 = 4\}$.

The first step towards a computation of the main hermitian form on the left-hand side of (1.5) (or its polarization) consists in transforming it into a finite-dimensional hermitian form. Given a positive integer N, one sets

$$\mathfrak{T}_N(x, \xi) = \sum_{j,k \in \mathbb{Z}} a((j, k, N))\, \delta(x - j)\, \delta(\xi - k). \tag{1.6}$$

The distribution $\mathfrak{T}_\infty$ is obtained as the limit, as $N \nearrow \infty$ (a notation meant to express that N will be divisible by any given squarefree number, from a certain point on), of the distribution $\mathfrak{T}_N^\times$ obtained from $\mathfrak{T}_N$ by dropping the term corresponding to $j = k = 0$. If Q is squarefree, if the algebraic sum of supports of the functions $v,\, u \in C^\infty(\mathbb{R})$ is contained in $[0, 2\beta]$, finally if $N = RQ$ is a squarefree integer divisible by all primes less than βQ, one has

$$\left(v \,\middle|\, \Psi\left(Q^{2i\pi\mathcal{E}}\mathfrak{T}_\infty\right) u\right) = \left(v \,\middle|\, \Psi\left(Q^{2i\pi\mathcal{E}}\mathfrak{T}_N\right) u\right). \tag{1.7}$$

Now, the hermitian form on the right-hand side is amenable to an algebraic-arithmetic version. Indeed, transferring functions in $\mathcal{S}(\mathbb{R})$ to functions on $\mathbb{Z}/(2N^2)\mathbb{Z}$ under the linear map θ_N defined by the equation

$$(\theta_N u)\,(n) = \sum_{\substack{n_1 \in \mathbb{Z} \\ n_1 \equiv n \bmod 2N^2}} u\left(\frac{n_1}{N}\right), \qquad n \bmod 2N^2, \tag{1.8}$$

one obtains an identity

$$\left(v \,\middle|\, \Psi\left(Q^{2i\pi\mathcal{E}}\mathfrak{T}_N\right) u\right) = \sum_{m,n \bmod 2N^2} c_{R,Q}\left(\mathfrak{T}_N;\, m,n\right)\ \overline{\theta_N v}(m)\ (\theta_N u)\,(n). \tag{1.9}$$

The coefficients $c_{R,Q}\left(\mathfrak{T}_N;\, m,n\right)$ are fully explicit, and the symmetric matrix defining this hermitian form has a Eulerian structure. The important point is to separate, so to speak, the R-factor and Q-factor in this expression. Introduce the reflection $n \mapsto \overset{\vee}{n}$ of $\mathbb{Z}/(2N^2)\mathbb{Z}$ such that $\overset{\vee}{n} \equiv n \bmod R^2$ and $\overset{\vee}{n} \equiv -n \bmod Q^2$. Then, defining $\widetilde{u}$ so that $(\theta_N \widetilde{u})(n) = (\theta_N u)(\overset{\vee}{n})$, one has the identity

$$\left(v \,\middle|\, \Psi\left(Q^{2i\pi\mathcal{E}}\mathfrak{T}_N\right) u\right) = \mu(Q)\ \left(v \,\middle|\, \Psi(\mathfrak{T}_N)\,\widetilde{u}\right), \tag{1.10}$$

which connects the purely arithmetic map $n \mapsto \overset{\vee}{n}$ to the purely analytic rescaling operator $Q^{2i\pi\mathcal{E}}$.

The contributions that precede, to be developed in full in Sections 1 to 6, were for the most already published in [11]: specializing the parity of certain integers has led to a great simplification. It is at this point that, as a research paper, the present paper really starts. Under some strong support conditions relative to v, u, one obtains the identity

$$\begin{aligned}\left(v \,\middle|\, \Psi(Q^{2i\pi\mathcal{E}}\mathfrak{T}_N)\, u\right) = &\sum_{Q_1 Q_2 = Q} \mu(Q_1) \\ &\sum_{R_1 | R} \mu(R_1)\, \overline{v}\left(\frac{R_1}{Q_2} + \frac{Q_2}{R_1}\right)\, u\left(\frac{R_1}{Q_2} - \frac{Q_2}{R_1}\right).\end{aligned} \tag{1.11}$$

In view of the role of this identity, we have given two quite different proofs of it. Nevertheless, just as what one would obtain from the use of the one-dimensional measure $\sum_{k \neq 0} \mu(k)\,\delta(x-k)$, it suffers from the presence of coefficients $\mu(R_1)$, with $C_1 Q < R_1 < C_2 Q$. However, if one constrains the support of u further, replacing for some $\varepsilon > 0$ u by u_Q such that

$u_Q(y) = Q^{\frac{\varepsilon}{2}}\, u(Q^\varepsilon y)$, it follows from this identity that, if one assumes that v is supported in $[2, \sqrt{8}]$ and that u is supported in $[0,1]$, the function F_ε defined for $\mathrm{Re}\, s$ large as the sum of the series

$$F_\varepsilon(s) = \sum_{Q \text{ sqfodd}} Q^{-s} \left(v \,\middle|\, \Psi\left(Q^{2i\pi\mathcal{E}}\mathfrak{T}_\infty\right) u_Q\right) \tag{1.12}$$

is entire. This is a good substitute for our inability to prove the same about the function $F_0(s)$ mentioned after (1.5): the function F_ε is much more tractable than F_0, and $F_\varepsilon(s)$ goes to $F_0(s)$ as $\varepsilon \to 0$ if $\mathrm{Re}\, s > 2$. The whole question then depends on whether $F_\varepsilon(s)$ has a limit as $\varepsilon \to 0$ extends to the case when $\mathrm{Re}\, s > \frac{3}{2}$. Note the importance of having introduced two independent rescaling operators. But, doing so, we have destroyed the possibility to apply the criterion (1.5), and we must reconsider this question.

There are now two decompositions into homogeneous parts (of distributions in the plane or on the line) to be considered. Writing $u = \frac{1}{i}\int_{\mathrm{Re}\,\mu=a} u^\mu d\mu$, where u^μ is homogeneous of degree $-\frac{1}{2}-\mu$, one obtains the formula

$$F_\varepsilon(s) = \frac{1}{2i\pi}\int_{\mathrm{Re}\,\nu=c} \frac{\left(1-2^{-\nu}\right)^{-1}}{\zeta(\nu)}\, H_\varepsilon(s,\,\nu)\, d\nu, \qquad c > 1, \tag{1.13}$$

with

$$H_\varepsilon(s,\,\nu) = \frac{1}{i}\int_{\mathrm{Re}\,\mu=a} f(s-\nu+\varepsilon\mu)\, \left(v \,\middle|\, \Psi\left(\mathfrak{E}_{-\nu}\right) u^\mu\right) d\mu, \qquad a < -\frac{3}{2}, \tag{1.14}$$

and

$$f(s) = \left(1-2^{-s}\right)^{-1} \times \frac{\zeta(s)}{\zeta(2s)}. \tag{1.15}$$

Understanding the Riemann hypothesis then reduces to obtaining the largest half-plane in which $F_\varepsilon(s)$ has a limit as $\varepsilon \to 0$. This depends on Cauchy-type analysis and, borrowing the language of scattering theory, on the decomposition of u^μ into its ingoing and outgoing parts. The present proof diverges from what was expected to be a proof of the Riemann hypothesis only at the very end, leading to the fact that zeros of the Riemann zeta function accumulate on the boundary of the critical strip. Extending the result to the case of Dirichlet L-functions is straightforward.

In a last section, we shall recall how the concept of automorphic distribution relates to the Lax-Phillips automorphic scattering theory.

2. The Weyl symbolic calculus of operators

In space-momentum coordinates, the Weyl calculus, or pseudodifferential calculus, depends on one free parameter with the dimension of action, called Planck's constant. In pure mathematics, even the more so when pseudodifferential analysis is applied to arithmetic, Planck's constant becomes a pure number: there is no question that the good such constant in "pseudodifferential arithmetic" is 2, as especially put into evidence [11, Chapter 6] in the pseudodifferential calculus of operators with automorphic symbols. To avoid encumbering the text with unnecessary subscripts, we shall denote as Ψ the rule denoted as Op_2 in [11, (2.1.1)], to wit the rule that attaches to any distribution $\mathfrak{S} \in \mathcal{S}'(\mathbb{R}^2)$ the linear operator $\Psi(\mathfrak{S})$ from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$ weakly defined as

$$(\Psi(\mathfrak{S})\,u)(x) = \frac{1}{2}\int_{\mathbb{R}^2} \mathfrak{S}\left(\frac{x+y}{2}, \xi\right) e^{i\pi(x-y)\xi}\, u(y)\, d\xi\, dy, \tag{2.1}$$

truly a superposition of integrals (integrate with respect to y first). The operator $\Psi(\mathfrak{S})$ is called the operator with symbol $\mathfrak{S}$. Its integral kernel is the function

$$K(x,y) = \frac{1}{2}\,\left(\mathcal{F}_2^{-1}\mathfrak{S}\right)\left(\frac{x+y}{2}, \frac{x-y}{2}\right), \tag{2.2}$$

where $\mathcal{F}_2^{-1}$ denotes the inverse Fourier transformation with respect to the second variable.

Elementary cases of operators $\Psi(\mathfrak{S})$ are the following. If $\mathfrak{S}(x,\xi) = f(x)$, the operator $\Psi(\mathfrak{S})$ is the operator of multiplication by the function f. If $\mathfrak{S}(x,\xi) = \delta(x)\, g(\xi)$, one has $[\Psi(\mathfrak{S})\, u](x) = (\mathcal{F}^{-1}g)(x)\, u(-x)$.

If one defines the Wigner function $\mathrm{Wig}(v,u)$ of two functions in $\mathcal{S}(\mathbb{R})$ as the function in $\mathcal{S}(\mathbb{R}^2)$ such that

$$\mathrm{Wig}(v,u)(x,\xi) = \int_{-\infty}^{\infty} \overline{v}(x+t)\, u(x-t)\, e^{2i\pi\xi t} dt, \tag{2.3}$$

one has

$$\left(v \,\middle|\, \Psi(\mathfrak{S})\, u\right) = \left\langle \mathfrak{S},\, \mathrm{Wig}(v,u)\right\rangle, \tag{2.4}$$

with $(v|u) = \int_{-\infty}^{\infty} \overline{v}(x)\, u(x)\, dx$, while straight brackets refer to the bilinear operation of testing a distribution on a function. Immediately observe for future reference that if v and u are compactly supported, one can have $\mathrm{Wig}(v,u)(x,\xi) \neq 0$ only if $2x$ lies in the algebraic sum of the supports of v and u.

Given two functions in $\mathcal{S}(\mathbb{R})$, the Wigner function of their Fourier transforms is

$$\mathrm{Wig}(\widehat{v},\,\widehat{u})(x,\xi) = \mathrm{Wig}(v,\,u)(\xi,\,-x). \tag{2.5}$$

An incorrect proof (to make it correct, just use the definition of the Fourier transforms one at a time) goes as follows. The left-hand side of (2.5) is

$$\int_{\mathbb{R}^4} \overline{v}(r)\, u(s)\, \exp[2i\pi\,((x+t)r-(x-t)s)]\, e^{2i\pi t\xi}\, dr\, ds\, dt \\ = \int_{\mathbb{R}^2} \overline{v}(r)\, u(s)\, e^{2i\pi\, x(r-s)}\, \delta(r+s+\xi)\, dr\, ds, \tag{2.6}$$

and setting $r+s=\xi$, $r-s=-\eta$ gives (2.5).

Another useful property of the calculus Ψ is expressed by the following two equivalent identities, obtained with the help of elementary manipulations of the Fourier transformation or with that of (2.2),

$$\Psi\left(\mathcal{F}^{\mathrm{symp}}\mathfrak{S}\right)\, w = \Psi(\mathfrak{S})\, \overset{\vee}{w}, \qquad \mathcal{F}^{\mathrm{symp}}\mathrm{Wig}(v,\,u) = \mathrm{Wig}(v,\,\overset{\vee}{u}), \tag{2.7}$$

where $\overset{\vee}{w}(x) = w(-x)$ and the symplectic Fourier transformation in $\mathbb{R}^2$ is defined in $\mathcal{S}(\mathbb{R}^2)$ or $\mathcal{S}'(\mathbb{R}^2)$ by the equation

$$\left(\mathcal{F}^{\mathrm{symp}}\,\mathfrak{S}\right)(x,\xi) = \int_{\mathbb{R}^2} \mathfrak{S}(y,\eta)\, e^{2i\pi(x\eta-y\xi)}\, dy\, d\eta. \tag{2.8}$$

Introduce the Euler operator

$$2i\pi\mathcal{E} = 1 + x\frac{\partial}{\partial x} + \xi\frac{\partial}{\partial \xi} \tag{2.9}$$

and, for $t>0$, the operator $t^{2i\pi\mathcal{E}}$ such that $\left(t^{2i\pi\mathcal{E}}\mathfrak{S}\right)(x,\xi) = t\,\mathfrak{S}(tx,t\xi)$.

Denoting as $U[2]$ the unitary rescaling operator such that $(U[2]\,u)(x) = 2^{\frac14}u(x\sqrt{2})$, and setting $\mathrm{Resc} = 2^{-\frac12+i\pi\mathcal{E}}$ or $(\mathrm{Resc}\,\mathfrak{S})(x,\xi) = \mathfrak{S}\left(2^{\frac12}x, 2^{\frac12}\xi\right)$, one can connect the rule Ψ to the rule Op_1 used in [11] and denoted as Op there by the equation [11, (2.1.14)]

$$U[2]\,\Psi(\mathfrak{S})\,U[2]^{-1} = \mathrm{Op}_1(\mathrm{Resc}\,\mathfrak{S}). \tag{2.10}$$

This would enable us not to redo, in Section 6, the proof of the main result already written with another normalization in [11]: but, for self-containedness and simplicity (specializing certain parameters), we shall rewrite all proofs dependent on the symbolic calculus. The choice of the rule Ψ makes it possible to avoid splitting into cases, according to the parity of the integers present there, the developments of pseudodifferential arithmetic in Section 6.

The following lemma will be useful later.

Lemma 2.1. *Given $v, u \in \mathcal{S}(\mathbb{R})$, one has*

$$(2i\pi\mathcal{E})\,\mathrm{Wig}(v,\,u) = \mathrm{Wig}(v',\,xu) + \mathrm{Wig}(xv,\,u'). \tag{2.11}$$

Proof. One has $\xi\frac{\partial}{\partial\xi}\,e^{2i\pi t\xi} = t\,\frac{\partial}{\partial t}\,e^{2i\pi t\xi}$. The term $\xi\frac{\partial}{\partial\xi}\,\mathrm{Wig}(v,\,u)$ is obtained from (2.3) with the help of an integration by parts, noting that the transpose of the operator $t\,\frac{\partial}{\partial t}$ is $-1-t\,\frac{\partial}{\partial t}$. Overall, using (2.10), one obtains

$$[(2i\pi\mathcal{E})\,\mathrm{Wig}(v,\,u)]\,(x,\,\xi) = \int_{-\infty}^{\infty} A(x,\,t)\,e^{2i\pi t\xi}\,dt, \tag{2.12}$$

with

$$\begin{aligned} A(x,\,t) &= x\,[\overline{v'}(x+t)\,u(x-t) + \overline{v}(x+t)\,u'(x-t)] \\ &\quad + t\,[-\overline{v'}(x+t)\,u(x-t) + \overline{v}(x+t)\,u'(x-t)] \\ &= (x-t)\,\overline{v'}(x+t)\,u(x-t) + (x+t)\,\overline{v}(x+t)\,u'(x-t). \end{aligned} \tag{2.13}$$

□

3. Eisenstein distributions

The objects in the present section can be found with more details in [11, Section 2.2]. Automorphic distributions are distributions in the Schwartz space $\mathcal{S}'(\mathbb{R}^2)$, invariant under the linear changes of coordinates associated to matrices in $SL(2,\mathbb{Z})$. It is the theory of automorphic and modular distributions, developed over a 20-year span, that led to the definition of the basic distribution $\mathfrak{T}_\infty$ (4.5), and to that of Eisenstein distributions. However, the present disproof of R.H. depends strictly, and only, on pseudodifferential arithmetic and on the definition of Eisenstein distributions.

Definition 3.1. If $\nu \in \mathbb{C}$, $\mathrm{Re}\,\nu > 1$, the Eisenstein distribution $\mathfrak{E}_{-\nu}$ is defined by the equation, valid for every $h \in \mathcal{S}(\mathbb{R}^2)$,

$$\langle\, \mathfrak{E}_{-\nu}\,,\, h\,\rangle = \sum_{|j|+|k|\neq 0} \int_0^\infty t^\nu h(jt,\, kt)\, dt. \tag{3.1}$$

It is immediate that the series of integrals converges if $\mathrm{Re}\,\nu > 1$, in which case $\mathfrak{E}_{-\nu}$ is well defined as a tempered distribution. Indeed, writing $|h(x,\xi)| \leq C\,(1+|x|+|\xi|)^{-A}$ with A large, one has

$$\left| \int_0^\infty t^\nu h(jt,kt)\, dt \right| \leq C\,(|j|+|k|)^{-\mathrm{Re}\,\nu-1} \int_0^\infty t^{\mathrm{Re}\,\nu}(1+t)^{-A} dt. \tag{3.2}$$

Obviously, $\mathfrak{E}_{-\nu}$ is $SL(2,\mathbb{Z})$–invariant as a distribution, i.e., an automorphic distribution. It is homogeneous of degree $-1+\nu$, i.e., $(2i\pi\mathcal{E})\,\mathfrak{E}_{-\nu} = \nu\,\mathfrak{E}_{-\nu}$: note that the transpose of $2i\pi\mathcal{E}$ is $-2i\pi\mathcal{E}$. Its name stems from its relation (not needed here) with the classical notion of non-holomorphic Eisenstein series, as made explicit in [11, p.93]: it is, however, a more precise concept.

Proposition 3.2. *As a tempered distribution, $\mathfrak{E}_{-\nu}$ extends as a meromorphic function of $\nu \in \mathbb{C}$, whose only poles are the points $\nu = \pm 1$: these poles are simple, and the residues of $\mathfrak{E}_\nu$ there are*

$$\mathrm{Res}_{\nu=1}\,\mathfrak{E}_{-\nu} = 1 \qquad \text{and} \qquad \mathrm{Res}_{\nu=-1}\,\mathfrak{E}_{-\nu} = -\delta_0, \tag{3.3}$$

the negative of the unit mass at the origin of $\mathbb{R}^2$. Recalling the definition (2.8) *of the symplectic Fourier transformation $\mathcal{F}^{\mathrm{symp}}$, one has, for $\nu \neq \pm 1$, $\mathcal{F}^{\mathrm{symp}}\mathfrak{E}_{-\nu} = \mathfrak{E}_\nu$.*

Proof. Denote as $(\mathfrak{E}_{-\nu})_{\mathrm{princ}}$ (*resp.* $(\mathfrak{E}_{-\nu})_{\mathrm{res}}$) the distribution defined in the same way as $\mathfrak{E}_{-\nu}$, except for the fact that in the integral (3.1), the interval of integration $(0,\infty)$ is replaced by the interval $(0,1)$ (resp. $(1,\infty)$), and observe that the distribution $(\mathfrak{E}_{-\nu})_{\mathrm{res}}$ extends as an entire function of ν: indeed, it suffices to replace (3.2) by the inequality $(1+(|j|+|k|)t)^{-A} \leq C\,(1+(|j|+|k|))^{-A}(1+t)^{-A}$ if $t > 1$. As a consequence of Poisson's formula, one has when $\mathrm{Re}\,\nu > 1$ the identity

$$\int_1^\infty t^{-\nu} \sum_{(j,k)\in\mathbb{Z}^2} (\mathcal{F}^{\mathrm{symp}} h)\,(tk,\,tj)\,dt = \int_1^\infty t^{-\nu} \sum_{(j,k)\in\mathbb{Z}^2} t^{-2}\,h(t^{-1}j,\,t^{-1}k)\,dt$$
$$= \int_0^1 t^{\nu} \sum_{(j,k)\in\mathbb{Z}^2} h(tj,\,tk)\,dt, \qquad (3.4)$$

from which one obtains that

$$< \mathcal{F}^{\mathrm{symp}}\ (\mathfrak{E}_\nu)_{\mathrm{res}}\ ,\, h > -\frac{(\mathcal{F}^{\mathrm{symp}} h)(0,\,0)}{1-\nu} =< (\mathfrak{E}_{-\nu})_{\mathrm{princ}}\ ,\, h > + \frac{h(0,\,0)}{1+\nu}. \qquad (3.5)$$

From this identity, one finds the meromorphic continuation of the function $\nu \mapsto \mathfrak{E}_\nu$, including the residues at the two poles, as well as the fact that $\mathfrak{E}_\nu$ and $\mathfrak{E}_{-\nu}$ are the images of each other under $\mathcal{F}^{\mathrm{symp}}$.

□

So far as the zeta function is concerned, we recall its definition $\zeta(s) = \sum_{n\geq 1} n^{-s} = \prod_p (1-p^{-s})^{-1}$, valid for $\mathrm{Re}\,s > 1$, and the fact that it extends as a meromorphic function in the entire complex plane, with a single simple pole, of residue 1, at $s = 1$; also that, with $\zeta^*(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\,\zeta(s)$, one has the functional equation $\zeta^*(s) = \zeta^*(1-s)$.

Lemma 3.3. *One has if $\nu \neq \pm 1$ the Fourier expansion*

$$\mathfrak{E}_{-\nu}(x,\xi) = \zeta(\nu)\,|\xi|^{\nu-1} + \zeta(1+\nu)\,|x|^{\nu}\delta(\xi) + \sum_{r\neq 0} \sigma_{-\nu}(r)\,|\xi|^{\nu-1} \exp\left(2i\pi \frac{rx}{\xi}\right) \qquad (3.6)$$

where $\sigma_{-\nu}(r) = \sum_{1\leq d|r} d^{-\nu}$: the first two terms must be grouped when $\nu = 0$. Given $b > \varepsilon > 0$, the distribution $(\nu-1)\mathfrak{E}_{-\nu}$ remains in a bounded subset of $\mathcal{S}'(\mathbb{R}^2)$ for $\varepsilon \leq \mathrm{Re}\,\nu \leq b$.

Proof. Isolating the term for which $k = 0$ in (3.1), we write if $\mathrm{Re}\,\nu > 1$, after a change of variable,

$$\big\langle \mathfrak{E}_{-\nu},\, h \big\rangle = \zeta(1+\nu) \int_{-\infty}^{\infty} |t|^{\nu} h(t,0)\,dt + \frac{1}{2} \sum_{j\in\mathbb{Z},\,k\neq 0} \int_{-\infty}^{\infty} |t|^{\nu} h(jt,\,kt)\,dt$$
$$= \zeta(1+\nu) \int_{-\infty}^{\infty} |t|^{\nu} h(t,0)\,dt + \frac{1}{2} \sum_{j\in\mathbb{Z},\,k\neq 0} \int_{-\infty}^{\infty} |t|^{\nu-1} \left(\mathcal{F}_1^{-1} h\right)\left(\frac{j}{t},\,kt\right) dt, \qquad (3.7)$$

where we have used Poisson's formula at the end and denoted as $\mathcal{F}_1^{-1}h$ the inverse Fourier transform of h with respect to the first variable. Isolating now the term such that $j = 0$, we obtain

$$\frac{1}{2}\sum_{j\in\mathbb{Z},\,k\neq 0}\int_{-\infty}^{\infty}|t|^{\nu-1}\left(\mathcal{F}_1^{-1}h\right)\left(\frac{j}{t},\,kt\right)dt = \zeta(\nu)\int_{-\infty}^{\infty}|t|^{\nu-1}\left(\mathcal{F}_1^{-1}h\right)(0,\,t)\,dt$$
$$+\frac{1}{2}\sum_{jk\neq 0}\int_{-\infty}^{\infty}|t|^{\nu-1}\left(\mathcal{F}_1^{-1}h\right)\left(\frac{j}{t},\,kt\right)dt, \quad (3.8)$$

from which the main part of the lemma follows if $\mathrm{Re}\,\nu > 1$ after we have made the change of variable $t \mapsto \frac{t}{k}$ in the main term. The continuation of the identity uses also the fact that, thanks to the trivial zeros $-2, -4, \dots$ of zeta, the product $\zeta(\nu)\,|t|^{\nu-1}$ is regular at $\nu = -2, -4, \dots$ and the product $\zeta(1+\nu)\,|t|^{\nu}$ is regular at $\nu = -3, -5, \dots,$. That the sum $\zeta(\nu)\,|\xi|^{\nu-1} + \zeta(1+\nu)\,|x|^{\nu}\delta(\xi)$ is regular at $\nu = 0$ follows from the facts that $\zeta(0) = -\frac{1}{2}$ and that the residue at $\nu = 0$ of the distribution $|\xi|^{\nu-1} = \frac{1}{\nu}\frac{d}{d\xi}\left(|x|^{\nu}\mathrm{sign}\,\xi\right)$ is $\frac{d}{d\xi}\,\mathrm{sign}\,\xi = 2\delta(\xi)$.

The second assertion is a consequence of the Fourier expansion. The factor $(\nu-1)$ has been inserted so as to kill the pole of $\mathfrak{E}_{-\nu}$, or that of $\zeta(\nu)$, at $\nu = 1$. Bounds for the first two terms of the right-hand side of (3.6) are obtained from bounds for the zeta factors (cf. paragraph that precedes the lemma) and integrations by parts associated to powers of the operator $\xi\frac{\partial}{\partial\xi}$ or $x\frac{\partial}{\partial x}$. For the main terms, we use the integration by parts associated to the equation

$$\left(1 + \xi\frac{\partial}{\partial x}\right)\exp\left(2i\pi\frac{rx}{\xi}\right) = (1 + 2i\pi r)\,\exp\left(2i\pi\frac{rx}{\xi}\right). \quad (3.9)$$

□

Decompositions into homogeneous components of functions or distributions in the plane will be ever-present. Any function $h \in \mathcal{S}(\mathbb{R}^2)$ can be decomposed in $\mathbb{R}^2\backslash\{0\}$ into homogeneous components according to the equations, in which $c > -1$,

$$h = \frac{1}{i}\int_{\mathrm{Re}\,\nu=c} h_\nu\,d\nu, \qquad h_\nu(x,\xi) = \frac{1}{2\pi}\int_0^{\infty} t^{\nu}h(tx,t\xi)\,dt. \quad (3.10)$$

Indeed, the integral defining $h_\nu(x,\xi)$ is convergent for $|x| + |\xi| \neq 0, \mathrm{Re}\,\nu > -1$, and the function h_ν so defined is C^∞ in $\mathbb{R}^2\backslash\{0\}$ and homogeneous of degree $-1-\nu$; it is also analytic with respect to ν. Using twice the integration by parts associated to Euler's equation $-(1+\nu)\, h_\nu = \left(x\frac{\partial}{\partial x} + \xi\frac{\partial}{\partial \xi}\right) h_\nu(x,\xi)$, one sees that the integral $\frac{1}{i}\int_{\mathrm{Re}\,\nu=c} h_\nu(x,\xi)\, d\nu$ is convergent for $c > -1$: its value does not depend on c. Taking $c = 0$ and setting $t = e^{2\pi\tau}$, one has for $|x| + |\xi| \neq 0$

$$h_{i\lambda}(x,\xi) = \int_{-\infty}^{\infty} e^{2i\pi\tau\lambda}\,.\, e^{2\pi\tau} h(e^{2\pi\tau}x, e^{2\pi\tau}\xi)\, d\tau, \tag{3.11}$$

and the Fourier inversion formula shows that $\int_{-\infty}^{\infty} h_{i\lambda}(x,\xi)\, d\lambda = h(x,\xi)$: this proves (3.10).

As a consequence, some automorphic distributions of interest (not all: so-called Hecke distributions are needed too in general) can be decomposed into Eisenstein distributions. A basic one is the "Dirac comb"

$$\mathfrak{D}(x,\xi) = 2\pi \sum_{|j|+|k|\neq 0} \delta(x-j)\,\delta(\xi-k) = 2\pi\,\left[\mathcal{D}ir(x,\xi) - \delta(x)\delta(\xi)\right] \tag{3.12}$$

where, as found convenient in some algebraic calculations, one introduces also the "complete" Dirac comb $\mathcal{D}ir(x,\xi) = \sum_{j,k\in\mathbb{Z}} \delta(x-j)\delta(\xi-k)$.

Noting the inequality $\left|\int_0^\infty t^\nu h(tx,t\xi)\, dt\right| \leq C\,(|x| + |\xi|)^{-\mathrm{Re}\,\nu-1}$, one obtains if $h \in \mathcal{S}(\mathbb{R}^2)$ and $c > 1$, pairing (3.12) with (3.10), the identity

$$\left\langle \mathfrak{D},\, h\right\rangle = \frac{1}{i} \sum_{|j|+|k|\neq 0} \int_{\mathrm{Re}\,\nu=c} d\nu \int_0^\infty t^\nu h(tj,tk)\, dt. \tag{3.13}$$

It follows from (3.1) that, for $c > 1$,

$$\mathfrak{D} = \frac{1}{i}\int_{\mathrm{Re}\,\nu=c} \mathfrak{E}_{-\nu}\, d\nu = 2\pi + \frac{1}{i}\int_{\mathrm{Re}\,\nu=0} \mathfrak{E}_{-\nu}\, d\nu, \tag{3.14}$$

the second equation being a consequence of the first in view of (3.3).

Integral superpositions of Eisenstein distributions, such as the one in (3.14), are to be interpreted in the weak sense in $\mathcal{S}'(\mathbb{R}^2)$, i.e., they make sense when tested on arbitrary functions in $\mathcal{S}(\mathbb{R}^2)$. Of course, pole-chasing is essential when changing contours of integration. But, as long as one satisfies oneself with weak decompositions in $\mathcal{S}'(\mathbb{R}^2)$, no difficulty concerning the integrability with respect to $\mathrm{Im}\,\nu$ on the line ever occurs, because of the last assertion of Lemma 3.3 and of the identities

$$(b-\nu)^A < \mathfrak{E}_{-\nu}, W >=< (b-2i\pi\mathcal{E})^A \mathfrak{E}_{-\nu}, W >=< \mathfrak{E}_{-\nu},\, (b+2i\pi\mathcal{E})^A W >, \tag{3.15}$$

in which $A = 0, 1, \dots$ may be chosen arbitrarily large and b is arbitrary.

Theorem 3.4. *For every tempered distribution $\mathfrak{S}$ such that $\mathfrak{S} \circ \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) = \mathfrak{S}$, especially every automorphic distribution, the integral kernel of the operator $\Psi(\mathfrak{S})$ is supported in the set of points (x, y) such that $x^2 - y^2 \in 4\mathbb{Z}$. In particular, given $\nu \in \mathbb{C}$, $\nu \neq \pm 1$, one has for every pair v, u in $\mathcal{S}(\mathbb{R})$,*

$$\left(v \,\middle|\, \Psi\left(\mathfrak{E}_{-\nu}\right) u\right) = \zeta(\nu)\ \left(|x|^{\nu-1} v \,\middle|\, u\right) + \zeta(-\nu)\ \left(|x|^{-\nu-1} v \,\middle|\, \overset{\vee}{u}\right) + \sum_{r \neq 0} \sigma_{-\nu}(r) \int_{-\infty}^{\infty} \overline{v}\left(t + \frac{r}{t}\right)\ |t|^{\nu-1}\, u\left(t - \frac{r}{t}\right) dt. \tag{3.16}$$

Under the support condition that $x > 0$ and $0 < x^2 - y^2 < 8$ if $v(x)u(y) \neq 0$, one has

$$\left(v \,\middle|\, \Psi\left(\mathfrak{E}_{-\nu}\right) u\right) = \int_0^{\infty} \overline{v}\left(t + t^{-1}\right)\ t^{\nu-1}\, u\left(t - t^{-1}\right) dt. \tag{3.17}$$

Proof. The invariance of $\mathfrak{S}$ under the change $(x, \xi) \mapsto (x, \xi + x)$ implies that the distribution $\left(\mathcal{F}_2^{-1} \mathfrak{S}\right)(x, z)$ is invariant under the multiplication by $e^{2i\pi xz}$. Applying (2.2), we obtain the first assertion.

Let us use the expansion (3.6), but only after we have substituted the pair $(\xi, -x)$ for (x, ξ), which does not change $\mathfrak{E}_{-\nu}(x, \xi)$ for any $\nu \neq \pm 1$ in view of (3.1): hence,

$$\mathfrak{E}_{-\nu}(x, \xi) = \zeta(\nu)\, |x|^{\nu-1} + \zeta(1+\nu)\, \delta(x)\, |\xi|^{\nu} + \sum_{r \neq 0} \sigma_{-\nu}(r)\, |x|^{\nu-1} \exp\left(-2i\pi \frac{r\xi}{x}\right). \tag{3.18}$$

The contributions to $\left(v \,\middle|\, \Psi\left(\mathfrak{E}_{-\nu}\right) u\right)$ of the first two terms of this expansion are obtained by what immediately followed (2.2). The sum of terms of the series for $r \neq 0$, to be designated as $\mathfrak{E}_{-\nu}^{\mathrm{trunc}}(x, \xi)$, remains to be examined.

One has

$$\left(\mathcal{F}_2^{-1} \mathfrak{E}_{-\nu}^{\mathrm{trunc}}\right)(x, z) = \sum_{r \neq 0} \sigma_{-\nu}(r)\, |x|^{\nu-1} \delta\left(z - \frac{r}{x}\right). \tag{3.19}$$

Still using (2.2), the integral kernel of the operator $\Psi(\mathfrak{S}_r)$, with

$$\mathfrak{S}_r(x, \xi) \colon = |x|^{\nu-1} \exp\left(-2i\pi \frac{r\xi}{x}\right) = \mathfrak{T}_r\left(x, \frac{\xi}{x}\right), \tag{3.20}$$

is half of

$$2\,K_r(x,y) = \left(\mathcal{F}_2^{-1}\mathfrak{S}_r\right)\left(\frac{x+y}{2}, \frac{x-y}{2}\right) = \left|\frac{x+y}{2}\right| \left(\mathcal{F}_2^{-1}\mathfrak{S}_r\right)\left(\frac{x+y}{2}, \frac{x^2-y^2}{4}\right)$$
$$= \left|\frac{x+y}{2}\right|^{\nu} \delta\left(\frac{x^2-y^2}{4} - r\right) = \left|\frac{x+y}{2}\right|^{\nu-1} \delta\left(\frac{x-y}{2} - \frac{2r}{x+y}\right). \tag{3.21}$$

Making in the integral $\int_{\mathbb{R}^2} K(x,y)\,\overline{v}(x)\,u(y)\,dx\,dy$ the change of variable which amounts to taking $\frac{x+y}{2}$ and $x-y$ as new variables, one obtains (3.16). Under the support assumptions made about v, u in the second part, only the term such that $r = 1$ subsists. □

4. A distribution decomposing over the zeros of zeta

The distributions $\mathfrak{T}_N$ and $\mathfrak{T}_\infty$ to be introduced in this section, regarded as symbols in the Weyl calculus, are the basic ingredients of this approach to the Riemann hypothesis. We are primarily interested in scalar products $\left(v \,\middle|\, \Psi(\mathfrak{T}_N)\,u\right)$ and $\left(v \,\middle|\, \Psi(\mathfrak{T}_\infty)\,u\right)$ under support conditions compatible with those in Theorem 3.4 (to wit, $x > 0$ and $0 < x^2 - y^2 < 8$ when $v(x)\,u(y) \neq 0$). The definition (4.2) of $\mathfrak{T}_N$ as a measure is suitable for the discussion of the arithmetic side of the identity (4.13) crucial in this disproof of R.H., and the definition of $\mathfrak{T}_\infty$ as an integral superposition of Eisenstein distributions is the one one must appeal to when discussing the analytic side of the identity.

Set for $j \neq 0$

$$a(j) = \prod_{p|j}(1-p), \tag{4.1}$$

where p, in the role of defining the range of the subscript in a product, is always tacitly assumed to be prime. The distribution

$$\mathfrak{T}_N(x,\xi) = \sum_{j,k\,\in\mathbb{Z}} a((j,k,N))\,\delta(x-j)\,\delta(\xi-k), \tag{4.2}$$

where the notation (j,k,N) refers to the g.c.d. of the three numbers, depends only on the "squarefree version" of N, defined as $N_\bullet = \prod_{p|N} p$. We also denote as $\mathfrak{T}_N^\times$ the distribution obtained from $\mathfrak{T}_N$ by discarding the term $a(N)\,\delta(x)\delta(\xi)$, in other words by limiting the summation to all pairs of integers j,k such that $|j| + |k| \neq 0$.

Proposition 4.1. [11, Lemma 3.1.1] *For any squarefree integer $N \geq 1$, defining*

$$\zeta_N(s) = \prod_{p|N} (1-p^{-s})^{-1}, \qquad \text{so that } \frac{1}{\zeta_N(s)} = \sum_{1 \leq T|N} \mu(T)\, T^{-s}, \tag{4.3}$$

where μ (also denoted as Möb *when μ has another role) is the Möbius indicator function ($\mu(T) = 0$ unless T is squarefree, in which case it is 1 or -1 according to the parity of its number of prime factors), one has*

$$\mathfrak{T}_N^\times = \frac{1}{2i\pi} \int_{\mathrm{Re}\,\nu = c} \frac{1}{\zeta_N(\nu)}\, \mathfrak{E}_{-\nu}\, d\nu, \qquad c > 1. \tag{4.4}$$

As $N \nearrow \infty$, a notation meant to convey that $N \to \infty$ in such a way that any given finite set of primes constitutes eventually a set of divisors of N, the distribution $\mathfrak{T}_N^\times$ converges weakly in the space $\mathcal{S}'(\mathbb{R}^2)$ towards the distribution

$$\mathfrak{T}_\infty = \frac{1}{2i\pi} \int_{\mathrm{Re}\,\nu = c} \frac{\mathfrak{E}_{-\nu}}{\zeta(\nu)}\, d\nu, \qquad c \geq 1. \tag{4.5}$$

Proof. Using the equation $T^{-x\frac{d}{dx}} \delta(x-j) = \delta\left(\frac{x}{T} - j\right) = T\,\delta(x - Tj)$, one has with $\mathfrak{D}$ as introduced in (3.12)

$$\begin{aligned}
\frac{1}{2\pi} \prod_{p|N} \left(1 - p^{-2i\pi\mathcal{E}}\right) \mathfrak{D}(x,\xi) &= \sum_{T|N} \mu(T)\, T^{-2i\pi\mathcal{E}} \sum_{|j|+|k|\neq 0} \delta(x-j)\,\delta(\xi-k) \\
&= \sum_{T|N} \mu(T)\, T \sum_{|j|+|k|\neq 0} \delta(x-Tj)\,\delta(\xi-Tk) \\
&= \sum_{T|N} \mu(T)\, T \sum_{\substack{|j|+|k|\neq 0 \\ j \equiv k \equiv 0 \bmod T}} \delta(x-j)\,\delta(\xi-k) \\
&= \sum_{\substack{|j|+|k|\neq 0 \\ T|(N,j,k)}} \mu(T)\, T\, \delta(x-j)\,\delta(\xi-k).
\end{aligned} \tag{4.6}$$

Since $\sum_{T|(N,j,k)} \mu(T)\, T = \prod_{p|(N,j,k)} (1-p) = a((N,j,k))$, one obtains

$$\frac{1}{2\pi} \prod_{p|N} \left(1 - p^{-2i\pi\mathcal{E}}\right) \mathfrak{D}(x,\xi) = \mathfrak{T}_N^\times(x,\xi). \tag{4.7}$$

One has

$$\begin{aligned}\left[\mathfrak{T}_N - \mathfrak{T}_N^{\times}\right](x,\xi) &= a(N)\,\delta(x)\delta(\xi)\\ &= \delta(x)\delta(\xi)\prod_{p|N}(1-p) = \prod_{p|N}\left(1-p^{-2i\pi\mathcal{E}}\right)(\delta(x)\delta(\xi)), \qquad (4.8)\end{aligned}$$

so that, adding the last two equations,

$$\mathfrak{T}_N = \prod_{p|N}\left(1-p^{-2i\pi\mathcal{E}}\right)\,\mathcal{D}ir. \qquad (4.9)$$

Combining (4.7) with (3.14) and with $(2i\pi\mathcal{E})\mathfrak{E}_{-\nu} = \nu\,\mathfrak{E}_{-\nu}$, one obtains if $c > 1$

$$\begin{aligned}\mathfrak{T}_N^{\times} &= \prod_{p|N}\left(1-p^{-2i\pi\mathcal{E}}\right)\left[\frac{1}{2i\pi}\int_{\operatorname{Re}\nu=c}\mathfrak{E}_{-\nu}\,d\nu\right]\\ &= \frac{1}{2i\pi}\int_{\operatorname{Re}\nu=c}\mathfrak{E}_{-\nu}\prod_{p|N}\left(1-p^{-\nu}\right)\,d\nu, \qquad (4.10)\end{aligned}$$

which is just (4.4). The $d\nu$-summability is guaranteed by (3.15). Equation (4.5) follows as well, taking the limit as $N \nearrow \infty$. Recall also (3.15).

In view of Hadamard's theorem, according to which $\zeta(s)$ has no zero on the line $\operatorname{Re} s = 1$, to be completed by the estimate $\zeta(1+iy) = \mathrm{O}(\log|y|)$, observing also that the pole of $\zeta(\nu)$ at $\nu = 1$ kills that of $\mathfrak{E}_{-\nu}$ there, one can in (4.5) replace the condition $c > 1$ by $c \geq 1$.

$\square$

Using (2.4), one obtains if $v, u \in \mathcal{S}(\mathbb{R})$ the identity

$$\left(v \,\middle|\, \Psi(\mathfrak{T}_N)\,u\right) = \sum_{j,k\in\mathbb{Z}} a((j,\,k,\,N))\,\mathrm{Wig}(v,\,u)(j,\,k). \qquad (4.11)$$

It follows from (2.3) that if v and u are compactly supported and the algebraic sum of the supports of v and u is supported in an interval $[0, 2\beta]$ (this is the situation that will interest us), $\left(v \,\middle|\, \Psi(\mathfrak{T}_N)\,u\right)$ does not depend on N as soon as N is divisible by all primes $< \beta$. Taking the limit of this stationary sequence as $N \nearrow \infty$, one can in this case write also

$$\left(v \,\middle|\, \Psi(\mathfrak{T}_\infty)\,u\right) = \sum_{j,k\in\mathbb{Z}} a((j,\,k))\,\mathrm{Wig}(v,\,u)(j,\,k). \qquad (4.12)$$

More generally, the following reduction, under some support conditions, of $\mathfrak{T}_\infty$ to $\mathfrak{T}_N$, is immediate, and fundamental for our purpose. Assume that v and $u \in C^\infty(\mathbb{R})$ are such that the algebraic sum of the supports of v and u is contained in $[0, 2\beta]$. Then, given a squarefree integer $N = RQ$ (with R, Q integers) divisible by all primes $< \beta Q$, one has

$$\left(v \,\middle|\, \Psi\left(Q^{2i\pi\mathcal{E}}\mathfrak{T}_\infty\right) u\right) = \left(v \,\middle|\, \Psi\left(Q^{2i\pi\mathcal{E}}\mathfrak{T}_N\right) u\right). \tag{4.13}$$

Indeed, (4.12) and the fact that the transpose of $2i\pi\mathcal{E}$ is $-2i\pi\mathcal{E}$ yield

$$\left(v \,\middle|\, \Psi\left(Q^{2i\pi\mathcal{E}}\mathfrak{T}_\infty\right) u\right) = Q^{-1}\sum_{j,k} a((j,k))\,\mathrm{Wig}(v,u)\left(\frac{j}{Q}, \frac{k}{Q}\right). \tag{4.14}$$

Next, from the observation that follows (2.3), one has $0 < \frac{j}{Q} < \beta$, or $0 < j < \beta Q$, for all nonzero terms of this sum, which implies that all prime divisors of j divide N. Note also that using here $\mathfrak{T}_N$ or $\mathfrak{T}_N^\times$ would not make any difference since $\Psi(\delta_0)\, u = \overset{\vee}{u}$ and the interiors of the supports of v and $\overset{\vee}{u}$ do not intersect.

Remark 4.1. In (4.5), introducing a sum of residues over all zeros of zeta with a real part above some large negative number, one can replace the line $\mathrm{Re}\,\nu = c$ with $c > 1$ by a line $\mathrm{Re}\,\nu = c'$ with c' very negative: one cannot go further in the distribution sense. But [11, Theor. 3.2.2, Theor. 3.2.4], one can get rid of the integral if one agrees to interpret the identity in the sense of a certain analytic functional. Then, all zeros of zeta, non-trivial and trivial alike, enter the formula: the "trivial" part

$$\mathfrak{R}_\infty = 2\sum_{n\geq 0} \frac{(-1)^{n+1}}{(n+1)\,!}\,\frac{\pi^{\frac{5}{2}+2n}}{\Gamma(\frac{3}{2}+n)\zeta(3+2n)}\,\mathfrak{E}_{2n+2} \tag{4.15}$$

has a closed expression [11, p.22] as a series of line measures. This will not be needed in the sequel.

We shall also need the distribution

$$\mathfrak{T}_{\frac{\infty}{2}}(x,\xi) = \sum_{|j|+|k|\neq 0} a\big((j,k,\frac{\infty}{2})\big)\,\delta(x-j)\delta(\xi-k), \tag{4.16}$$

where $a\big((j,k,\frac{\infty}{2})\big)$ is the product of all factors $1-p$ with p prime $\neq 2$ dividing (j,k). Since $\prod_{p\neq 2}(1-p^{-\nu}) = \frac{(1-2^{-\nu})^1}{\zeta(\nu)}$, one has

$$\mathfrak{T}_{\frac{\infty}{2}} = \frac{1}{2i\pi}\int_{\mathrm{Re}\,\nu=c}\left(1-2^{-\nu}\right)^{-1}\frac{\mathfrak{E}_{-\nu}}{\zeta(\nu)}\,d\nu, \qquad c > 1. \tag{4.17}$$

The version of (4.13) we shall use, under the same support conditions about v, u, is

$$\left(v \,\middle|\, \Psi\left(Q^{2i\pi\mathcal{E}}\mathfrak{T}_{\frac{\infty}{2}}\right) u\right) = \left(v \,\middle|\, \Psi\left(Q^{2i\pi\mathcal{E}}\mathfrak{T}_N\right) u\right) \tag{4.18}$$

if N and Q are squarefree odd and N is divisible by all odd primes $< \beta Q$.

In [11, Prop. 3.4.2 and 3.4.3], it was proved (with a minor difference due to the present change of Op_1 to Ψ) that, if for some $\beta > 2$ and every function $w \in C^\infty(\mathbb{R})$ supported in $[0, \beta]$, one has

$$\left(w \,\middle|\, \Psi\left(Q^{2i\pi\mathcal{E}}\mathfrak{T}_\infty\right) w\right) = \mathrm{O}\left(Q^{\frac{1}{2}+\varepsilon}\right) \tag{4.19}$$

as $Q \to \infty$ through squarefree integral values, the Riemann hypothesis follows. Except for the change of the pair w, w to a pair v, u with specific support properties, this is the criterion to be discussed in the next section.

5. A criterion for the Riemann hypothesis

Using (4.5) and the homogeneity of $\mathfrak{E}_{-\nu}$, one has

$$Q^{2i\pi\mathcal{E}}\mathfrak{T}_\infty = \frac{1}{2i\pi}\int_{\mathrm{Re}\,\nu=c} Q^\nu \,\frac{\mathfrak{E}_{-\nu}}{\zeta(\nu)}\, d\nu, \qquad c > 1. \tag{5.1}$$

From Lemma 3.3 and (3.15), the product by $Q^{-1-\varepsilon}$ of the distribution $Q^{2i\pi\mathcal{E}}\mathfrak{T}_\infty$ remains for every $\varepsilon > 0$ in a bounded subset of $\mathcal{S}'(\mathbb{R}^2)$ as $Q \to \infty$. If the Riemann hypothesis holds, the same is true after $Q^{-1-\varepsilon}$ has been replaced by $Q^{-\frac{1}{2}-\varepsilon}$.

We shall state and prove some suitably modified version of the converse, involving for some choice of the pair v, u of functions in $\mathcal{S}(\mathbb{R})$ the function

$$F_0(s) = \sum_{Q \in \mathrm{Sq}^{\mathrm{odd}}} Q^{-s}\left(v \,\middle|\, \Psi\left(Q^{2i\pi\mathcal{E}}\mathfrak{T}_{\frac{\infty}{2}}\right) u\right), \tag{5.2}$$

where we denote as $\mathrm{Sq}^{\mathrm{odd}}$ the set of squarefree odd integers. We are discarding the prime 2 so as to make Theorem 7.2 below applicable later. Under the assumption, generalizing (4.19), that $\left(v \,\middle|\, \Psi\left(Q^{2i\pi\mathcal{E}}\mathfrak{T}_\infty\right) u\right) = \mathrm{O}\left(Q^{\frac{1}{2}+\varepsilon}\right)$, the function $F_0(s)$ is analytic in the half-plane $\mathrm{Re}\, s > \frac{3}{2}$ and polynomially bounded in vertical strips in this domain, by which we mean, classically, that given $[a, b] \subset]\frac{3}{2}, \infty[$, there exists $M > 0$ such that, for some $C > 0$, $|F_0(\sigma + it)| \leq C\,(1 + |t|)^M$ when $\sigma \in [a, b]$ and $t \in \mathbb{R}$.

First, we transform the series defining $F_0(s)$ into a line integral of convolution type.

Lemma 5.1. *Given $v, , u \in \mathcal{S}(\mathbb{R})$, the function $F_0(s)$ introduced in* (5.2) *can be written for* $\mathrm{Re}\, s$ *large*

$$F_0(s) = \frac{1}{2i\pi} \int_{\mathrm{Re}\,\nu=1} \left(1+2^{-s+\nu}\right)^{-1} \frac{\zeta(s-\nu)}{\zeta(2(s-\nu))} \\ \times \frac{\left(1-2^{-\nu}\right)^{-1}}{\zeta(\nu)} \left\langle \mathfrak{E}_{-\nu},\, \mathrm{Wig}(v,\, u)\right\rangle d\nu. \quad (5.3)$$

Proof. For $\mathrm{Re}\, s > 1$, one has the identity

$$\sum_{Q \in \mathrm{Sq}^{\mathrm{odd}}} Q^{-s} = \prod_{\substack{q\,\mathrm{prime} \\ q \neq 2}} \left(1+q^{-s}\right) \\ = \left(1+2^{-s}\right)^{-1} \prod_q \frac{1-q^{-2s}}{1-q^{-s}} = \left(1+2^{-s}\right)^{-1} \frac{\zeta(s)}{\zeta(2s)}. \quad (5.4)$$

We apply this with s replaced by $s-\nu$, starting from (5.2). One has $\left(v \,\middle|\, \Psi\left(Q^{2i\pi\mathcal{E}}\mathfrak{E}_{-\nu}\right) u\right) = Q^{\nu} \left\langle \mathfrak{E}_{-\nu},\, \mathrm{Wig}(v,u)\right\rangle$ according to (2.4). Using (5.1) and (5.4), one obtains (5.3) if $\mathrm{Re}\, s > 2$, noting that the denominator $\zeta(\nu)$ takes care of the pole of $\mathfrak{E}_{-\nu}$ at $\nu = 1$, which makes it possible to replace the line $\mathrm{Re}\,\nu = c$, $c > 1$ by the line $\mathrm{Re}\,\nu = 1$. Note that the integrability at infinity is taken care of by (3.15), together with Hadamard's theorem as recalled immediately after (4.10).

□

Lemma 5.2. *Let $\rho \in \mathbb{C}$ and consider a product $h(\nu)\, f(s-\nu)$, where the function $f = f(z)$, defined and meromorphic near the point $z = 1$, has a simple pole at this point, and the function h, defined and meromorphic near ρ, has at that point a pole of order $\ell \geq 1$. Then, the function $s \mapsto \mathrm{Res}_{\nu=\rho}\, [h(\nu)\, f(s-\nu)]$ has at $s = 1+\rho$ a pole of order ℓ.*

Proof. If $h(\nu) = \sum_{j=1}^{\ell} a_j (\nu-\rho)^{-j} + \mathrm{O}(1)$ as $\nu \to \rho$, one has for s close to $1+\rho$ but distinct from this point

$$\mathrm{Res}_{\nu=\rho}\, [h(\nu)\, f(s-\nu)] = \sum_{j=1}^{\ell} (-1)^{j-1} a_j \frac{f^{(j-1)}(s-\rho)}{(j-1)\,!}, \quad (5.5)$$

and the function $s \mapsto f^{(j-1)}(s-\rho)$ has at $s = 1+\rho$ a pole of order j.

$\square$

Remark 5.1. An integral $\int_\gamma h(\nu)\, f(s-\nu)\, d\nu$ over a finite part of a line cannot have a pole at $s = s_0$ unless both h and the function $\nu \mapsto g(s_0 - \nu)$ have poles at the same point $\nu \in \gamma$. This follows from the residue theorem together with the case $\ell = 0$ of Lemma 5.2.

Theorem 5.3. *Assume that, for every given pair v, u of functions in $\mathcal{S}(\mathbb{R})$, the function $F_0(s)$ defined in the integral version* (5.3)*, initially defined and analytic for* $\mathrm{Re}\, s > 2$*, extends as an analytic function in the half-plane* $\{s\colon \mathrm{Re}\, s > \frac{3}{2}\}$*. Then, all zeros of zeta have real parts $\leq \frac{1}{2}$: in other words, the Riemann hypothesis does hold.*

Proof. Assume that a zero ρ of zeta with a real part $> \frac{1}{2}$ exists: one may assume that the real part of ρ is the largest one among those of all zeros of zeta (if any other should exist !) with the same imaginary part and a real part $> \frac{1}{2}$. Choose β such that $0 < \beta < \mathrm{Re}\,(\rho - \frac{1}{2})$. Assuming that $\mathrm{Re}\, s > 2$, change the line $\mathrm{Re}\,\nu = 1$ to a simple contour γ on the left of the initial line, enclosing the point ρ but no other point ρ with $\zeta(\rho) = 0$, coinciding with the line $\mathrm{Re}\,\nu = 1$ for $|\mathrm{Im}\,\nu|$ large, and such that $\mathrm{Re}\,\nu > \mathrm{Re}\,\rho - \beta$ for $\nu \in \gamma$.

Let Ω be the relatively open part of the half-plane $\mathrm{Re}\,\nu \leq 1$ enclosed by γ and the line $\mathrm{Re}\,\nu = 1$. Let $\mathcal{D}$ be the domain consisting of the numbers s such that $s - 1 \in \Omega$ or $\mathrm{Re}\, s > 2$. When $s \in \mathcal{D}$, one has $\mathrm{Re}\, s > 1 + \mathrm{Re}\,\rho - \beta > \frac{3}{2}$.

Still assuming that $\mathrm{Re}\, s > 2$, one obtains the equation

$$F_0(s) = \frac{1}{2i\pi} \int_\gamma f(s-\nu)\, h_0(\nu)\, d\nu + \mathrm{Res}_{\nu=\rho}\left[h_0(\nu) f(s-\nu) \right], \tag{5.6}$$

with

$$\begin{aligned} h_0(\nu) &= \left(1 - 2^{-\nu}\right)^{-1} \times \frac{\left\langle \mathfrak{E}_{-\nu},\, \mathrm{Wig}(v,\, u) \right\rangle}{\zeta(\nu)}, \\ f(s-\nu) &= \left(1 + 2^{-s+\nu}\right)^{-1} \times \frac{\zeta(s-\nu)}{\zeta(2(s-\nu))}. \end{aligned} \tag{5.7}$$

We show now that the integral term in (5.6) is holomorphic in the domain $\mathcal{D}$. The first point is that the numerator $\zeta(s-\nu)$ of $f(s-\nu)$ will not contribute singularities. Indeed, one can have $s - \nu = 1$ with $s \in \mathcal{D}$ and

$\nu \in \gamma$ only if $s \in 1+\Omega$, since $\mathrm{Re}\,(s-\nu) > 1$ if $\mathrm{Re}\,s > 2$ and $\nu \in \gamma$. Then, the conditions $s-1 \in \Omega$ and $s-1 \in \gamma$ are incompatible because on one hand, the imaginary part of $s-1$ does not agree with that of any point of the two infinite branches of γ, while the rest of γ is a part of the boundary of Ω. Finally, when $s-1 \in \Omega$ and $\nu \in \gamma$, that $\zeta(2(s-\nu)) \neq 0$ follows from the inequalities $\mathrm{Re}\,(s-\nu) \geq \mathrm{Re}\,\rho - \beta > \frac{1}{2}$, since $\mathrm{Re}\,s > 1+(\mathrm{Re}\,\rho-\beta)$ and $\mathrm{Re}\,\nu \leq 1$.

Since $F_0(s)$ is analytic for $\mathrm{Re}\,s > \frac{3}{2}$, it follows that the residue present in (5.6) extends as an analytic function of s in $\mathcal{D}$. But an application of Lemma 5.2, together with the first equation (5.7) and (3.17), shows that this residue is singular at $s = 1+\rho$ for some choice of the pair v, u. We have reached a contradiction.

Remark 5.2. This remark should prevent a possible misunderstanding. Though we are ultimately interested in a residue at $\nu = \rho$ and in the continuation of $F_0(s)$ near $s = 1+\rho$, we have established (5.6) under the assumption that $\mathrm{Re}\,s > 2$, in which case $s-1$ does not lie in the domain covered by ν between the line $\mathrm{Re}\,\nu = 1$ and the line γ. We must therefore not add to the right-hand side of (5.6) the residue of the integrand at $\nu = s-1$ (the two residues would have killed each other). Separating the poles of the two factors has been essential. The conclusion resulted from analytic continuation and the assumption that $F_0(s)$ extends as an analytic function for $\mathrm{Re}\,s > \frac{3}{2}$.

$\square$

Corollary 5.4. *Let $\rho \in \mathbb{C}$ with $\mathrm{Re}\,\rho > 0$ be given. Let $v,\, u \in \mathcal{S}(\mathbb{R})$ be such that $\left(v \,\middle|\, \Psi(\mathfrak{E}_{-\rho})\, u\right) = \left\langle \mathfrak{E}_{-\rho},\, \mathrm{Wig}(v,u) \right\rangle \neq 0$. If the function $F_0(s)$, defined for $\mathrm{Re}\,s > 2$ by (5.2), can be continued analytically along a path connecting the point $1+\rho$ to a point with a real part > 2, the point ρ cannot be a zero of zeta. For any ρ, a pair v, u with v supported in $[2, \sqrt{8}]$ and u supported in $[0,1]$, such that $\left(v \,\middle|\, \Psi(\mathfrak{E}_{-\rho})\, u\right) \neq 0$, can be found.*

Proof. It follows the proof of Theorem 5.3. It suffices there to take Ω containing the path in the assumption, and to observe that $h_0(\nu)$, as defined in (5.7), would under the nonvanishing condition $\left\langle \mathfrak{E}_{-\rho},\, \mathrm{Wig}(v,u) \right\rangle \neq 0$ have a pole at ρ if one had $\zeta(\rho) = 0$.

$\square$

The following proposition shows a way to reinforce the $d\nu$-summability in the integral (5.3) defining $F_0(s)$ without losing the conclusion of Theorem 5.3.

Proposition 5.5. *Given $M = 1, 2, \dots,$ define*

$$F_{0,M}(s) = \frac{1}{2i\pi} \int_{\mathrm{Re}\,\nu=1} \left(1+2^{-s+\nu}\right)^{-1} \frac{\zeta(s-\nu)}{\zeta(2(s-\nu))} \\ \times\, \nu^{-M}\, \frac{\left(1-2^{-\nu}\right)^{-1}}{\zeta(\nu)} \left\langle \mathfrak{E}_{-\nu},\, \mathrm{Wig}(v,\, u) \right\rangle d\nu. \quad (5.8)$$

Theorem 5.3 *remains valid if one substitutes $F_{0,M}$ for F_0.*

Proof. It suffices to see how F_0 can be rebuilt in terms of $F_{0,M}$, for which one observes the effect of multiplying by ν^M an expression such as $\left\langle \mathfrak{E}_{-\nu},\, \mathrm{Wig}(v,u) \right\rangle$. This is immediate since

$$\nu^M \left\langle \mathfrak{E}_{-\nu},\, \mathrm{Wig}(v,u) \right\rangle = \left\langle \mathfrak{E}_{-\nu},\, (-2i\pi\mathcal{E})^M \mathrm{Wig}(v,u) \right\rangle, \quad (5.9)$$

while the effect of applying $(2i\pi\mathcal{E})^M$ on a Wigner function is provided by Lemma 2.1.

□

6. Pseudodifferential arithmetic

As noted in (4.13), one can substitute for the analysis of the hermitian form $\left(v \,\middle|\, \Psi\left(Q^{2i\pi\mathcal{E}} \mathfrak{T}_{\frac{\infty}{2}}\right) u\right)$ that of $\left(v \,\middle|\, \Psi\left(Q^{2i\pi\mathcal{E}} \mathfrak{T}_N\right) u\right)$, under the assumption that the algebraic sum of the supports of v and u is contained in $[0, 2\beta]$, provided that N is a squarefree odd integer divisible by all odd primes $< \beta Q$. The new hermitian form should be amenable to an algebraic treatment. In this section, we make no support assumptions on v, u, just taking them in $\mathcal{S}(\mathbb{R})$.

We consider operators of the kind $\Psi(Q^{2i\pi\mathcal{E}}\mathfrak{S})$ with

$$\mathfrak{S}(x,\xi) = \sum_{j,k\in\mathbb{Z}} b(j,k)\, \delta(x-j)\, \delta(\xi-k), \quad (6.1)$$

under the following assumptions: that N is a squarefree integer decomposing as the product $N = RQ$ of two positive integers, and that b satisfies the periodicity conditions

$$b(j,k) = b(j+N,k) = b(j,k+N). \quad (6.2)$$

A special case consists of course of the symbol $\mathfrak{S} = \mathfrak{T}_N$. The aim is to transform the hermitian form associated to the operator $\Psi\left(Q^{2i\pi\mathcal{E}}\mathfrak{T}_N\right)$ to an arithmetic version.

The following proposition reproduces [11, Prop. 4.1.2, Prop. 4.1.3], the parameter denoted as ω there being set to the value 2.

Theorem 6.1. *With $N = RQ$ and $b(j,k)$ satisfying the condition* (6.2), *define the function*

$$f_N(j,\, s) = \frac{1}{N} \sum_{k \bmod N} b(j,k)\, \exp\left(\frac{2i\pi ks}{N}\right), \qquad j, s \in \mathbb{Z}/N\mathbb{Z}. \tag{6.3}$$

Set, noting that the condition $m - n \equiv 0 \bmod 2Q$ implies that $m+n$ too is even,

$$c_{R,Q}\,(\mathfrak{S};\, m\,, n) = \mathrm{char}(m{+}n \equiv 0 \bmod R,\ m{-}n \equiv 0 \bmod 2Q)\, f_N\left(\frac{m+n}{2R},\, \frac{m-n}{2Q}\right). \tag{6.4}$$

On the other hand, set, for $u \in \mathcal{S}(\mathbb{R})$,

$$(\theta_N u)(n) = \sum_{\ell\in\mathbb{Z}} u\left(\frac{n}{N} + 2\ell N\right), \qquad n \bmod 2N^2. \tag{6.5}$$

Then, one has

$$\left(v \,\middle|\, \Psi\left(Q^{2i\pi\mathcal{E}}\mathfrak{S}\right) u\right) = \sum_{m,n\in\mathbb{Z}/(2N^2)\mathbb{Z}} c_{R,Q}\,(\mathfrak{S};\, m\,, n)\, \overline{\theta_N v(m)}\, (\theta_N u)(n). \tag{6.6}$$

Proof. There is no restriction here on the supports of v, u, and one can replace these two functions by $v[Q]\,, u[Q]$ defined as $v[Q]\,(x) = v(Qx)$ and $u[Q]\,(x) = u(Qx)$. One has

$$(\theta_N\, u[Q]\,)(n) = (\kappa u)(n) \colon = \sum_{\ell\in\mathbb{Z}} u\left(\frac{n}{R} + 2QN\ell\right), \qquad n \bmod 2N^2. \tag{6.7}$$

It just requires the definition (1.3) of Ψ and changes of variables amounting to rescaling x, y, ξ by the factor Q^{-1} to obtain $\left(v[Q] \,\middle|\, \Psi\left(Q^{2i\pi\mathcal{E}}\mathfrak{S}\right) u[Q]\,\right) = (v \,|\, \mathcal{B}u)$, with

$$(\mathcal{B}\, u)\,(x) = \frac{1}{2\,Q^2} \int_{\mathbb{R}^2} \mathfrak{S}\left(\frac{x+y}{2},\, \xi\right) u(y)\, \exp\left(\frac{i\pi}{Q^2}(x-y)\xi\right) dy\, d\xi. \tag{6.8}$$

The identity (6.6) to be proved is equivalent, with κ as defined in (6.7), to

$$(v\,|\,\mathcal{B}u) = \sum_{m,n\in\mathbb{Z}/(2N^2)\mathbb{Z}} c_{R,Q}\,(\mathfrak{S};\,m\,,n)\;\overline{\kappa v(m)}\,(\kappa u)(n). \tag{6.9}$$

From (6.4), one has

$$\begin{aligned}(v\,|\,\mathcal{B}u) &= \frac{1}{2Q^2}\int_{-\infty}^{\infty}\overline{v}(x)\,dx\int_{\mathbb{R}^2}\mathfrak{S}\left(\frac{x+y}{2},\,\xi\right)u(y)\,\exp\left(\frac{i\pi}{Q^2}(x-y)\xi\right)dy\,d\xi\\ &= \frac{1}{Q^2}\int_{-\infty}^{\infty}\overline{v}(x)\,dx\int_{\mathbb{R}^2}\mathfrak{S}(y,\xi)\,u(2y-x)\,\exp\left(\frac{2i\pi}{Q^2}(x-y)\xi\right)dy\,d\xi\\ &= \frac{1}{Q^2}\int_{-\infty}^{\infty}\overline{v}(x)\,dx\sum_{j,k\in\mathbb{Z}}b(j,k)\,u(2j-x)\,\exp\left(\frac{2i\pi}{Q^2}(x-j)k\right).\end{aligned} \tag{6.10}$$

Since $b(j,k)=b(j,k+N)$, one replaces k by $k+N\ell$, the new k lying in the interval $[0,N-1]$ of integers. One has (Poisson's formula)

$$\sum_{\ell\in\mathbb{Z}}\exp\left(\frac{2i\pi}{Q^2}(x-j)\ell N\right) = \sum_{\ell\in\mathbb{Z}}\exp\left(\frac{2i\pi}{Q}(x-j)\ell R\right) = \frac{Q}{R}\sum_{\ell\in\mathbb{Z}}\delta\left(x-j-\frac{\ell Q}{R}\right), \tag{6.11}$$

and, from (6.10),

$$\begin{aligned}&(\mathcal{B}u)(x)\\ &= \frac{1}{N}\sum_{\substack{j\in\mathbb{Z}\\ 0\le k<N}} b(j,k)\sum_{\ell\in\mathbb{Z}}u\left(j-\frac{\ell Q}{R}\right)\,\exp\left(\frac{2i\pi(x-j)k}{Q^2}\right)\delta\left(x-j-\frac{\ell Q}{R}\right)\\ &= \sum_{m\in\mathbb{Z}}t_m\,\delta\left(x-\frac{m}{R}\right),\end{aligned} \tag{6.12}$$

with $m=Rj+\ell Q$ and t_m to be made explicit: we shall drop the summation with respect to ℓ for the benefit of a summation with respect to m. Since, when $x=j+\frac{\ell Q}{R}=\frac{m}{R}$, one has $\frac{x-j}{Q^2}=\frac{\ell}{N}=\frac{m-Rj}{NQ}$ and $j-\frac{\ell Q}{R}=2j-x=$

$2j - \frac{m}{R}$, one has

$$\begin{aligned} t_m & \\ = \frac{1}{N} \sum_{\substack{j \in \mathbb{Z} \\ 0 \leq k < N}} & b(j,k)\, \mathrm{char}(m \equiv Rj \bmod Q) u\left(2j - \frac{m}{R}\right) \exp\left(\frac{2i\pi k(m-Rj)}{NQ}\right) \\ &= \frac{1}{N} \sum_{\substack{0 \leq j < QN \\ 0 \leq k < N}} b(j,k)\, \mathrm{char}(m \equiv Rj \bmod Q) \\ & \sum_{\ell_1 \in \mathbb{Z}} u\left(2(j + \ell_1 QN) - \frac{m}{R}\right) \exp\left(\frac{2i\pi k(m-Rj)}{QN}\right). \qquad (6.13) \end{aligned}$$

Recalling the definition (6.7) of κu, one obtains

$$t_m = \frac{1}{N} \sum_{\substack{0 \leq j < QN \\ 0 \leq k < N}} b(j,k)\, \mathrm{char}(m \equiv Rj \bmod Q) \\ (\kappa u)(2Rj - m) \exp\left(\frac{2i\pi k(m-Rj)}{QN}\right). \qquad (6.14)$$

The function κu is $(2N^2)$-periodic, so one can replace the subscript $0 \leq j < RQ^2$ by $j \bmod RQ^2$. Using (6.12), we obtain

$$(v \,|\, \mathcal{B}u) = \frac{1}{N} \sum_{j \bmod RQ^2} \sum_{0 \leq k < N} b(j,k) \sum_{\substack{m_1 \in \mathbb{Z} \\ m_1 \equiv Rj \bmod Q}} \\ \overline{v}\left(\frac{m_1}{R}\right) (\kappa u)(2Rj - m_1) \exp\left(\frac{2i\pi k(m_1 - Rj)}{QN}\right). \qquad (6.15)$$

The change of m to m_1 is just a change of notation.

Fixing k, we trade the set of pairs m_1, j with $m_1 \in \mathbb{Z}$, $j \bmod RQ^2$, $m_1 \equiv Rj \bmod Q$ for the set of pairs $m, n \in \left(\mathbb{Z}/(2N^2)\mathbb{Z}\right) \times \left(\mathbb{Z}/(2N^2)\mathbb{Z}\right)$, where m is the class mod $2N^2$ of m_1 and n is the class mod $2N^2$ of $2Rj - m_1$. Of necessity, $m + n \equiv 0 \bmod 2R$ and $m - n \equiv 2(m - Rj) \equiv 0 \bmod 2Q$. Conversely, given a pair of classes m, n mod $2N^2$ satisfying these conditions, the equation $2Rj - m = n$ uniquely determines j mod $\frac{2N^2}{2R} = RQ^2$, as it should. The sum $\sum_{m_1\equiv m \bmod 2N^2} v\left(\frac{m_1}{R}\right)$ is just $(\kappa v)(m)$, and we have obtained the identity

$$(v\,|\,\mathcal{B}u) = \sum_{m,n \bmod 2N^2} c_{R,Q}\left(\mathfrak{S};\, m\,,n\right)\,\overline{(\kappa v)(m)}\,(\kappa u)(n), \tag{6.16}$$

provided we define

$$c_{R,Q}\left(\mathfrak{S};\, m\,,n\right) = \frac{1}{N}\,\mathrm{char}(m+n\equiv 0 \bmod R,\ m-n\equiv 0 \bmod 2Q)$$
$$\sum_{k \bmod N} b\left(\frac{m+n}{2R},\, k\right)\exp\left(\frac{2i\pi k}{N}\,\frac{m-n}{2Q}\right), \tag{6.17}$$

which is just the way indicated in (6.3), (6.4).

□

7. Computation of the arithmetic side of the main identity

Lemma 7.1. *With the notation of Theorem* 6.1*, one has if* $N = RQ$ *is squarefree odd*

$$c_{R,Q}(\mathfrak{T}_N;\, m,\, n) = \mathrm{char}(m+n\equiv 0 \bmod 2R)\,\mathrm{char}(m-n\equiv 0 \bmod 2Q)$$
$$\sum_{\substack{R_1R_2=R\\ Q_1Q_2=Q}} \mu(R_1Q_1)\,\mathrm{char}\left(\frac{m+n}{R}\equiv 0 \bmod R_1Q_1\right)\,\mathrm{char}\left(\frac{m-n}{2Q}\equiv 0 \bmod R_2Q_2\right). \tag{7.1}$$

Proof. We compute the function $f_N(j,\, s)$ defined in (6.3). If $N = N_1N_2$ and $cN_1+dN_2 = 1$, so that $\frac{k}{N} = \frac{dk}{N_1}+\frac{ck}{N_2}$, one identifies $k\in \mathbb{Z}/N\mathbb{Z}$ with the pair $(k_1,k_2)\in \mathbb{Z}/N_1\mathbb{Z}\times\mathbb{Z}/N_2\mathbb{Z}$ such that $k_1\equiv dk \bmod N_1$, $k_2\equiv ck \bmod N_2$. On one hand,

$$\exp\left(\frac{2i\pi ks}{N}\right) = \exp\left(\frac{2i\pi k_1 s}{N_1}\right)\times\exp\left(\frac{2i\pi k_2 s}{N_2}\right). \tag{7.2}$$

On the other hand, as $(d,N_1) = (c,N_2) = 1$,

$$a((j,k.N)) = a((j,k,N_1))\,a((j,k,N_2)) = a((j,k_1,N_1))\,a((j,k_2,N_2)). \tag{7.3}$$

It follows from (6.3) that $f_N(j,\,s) = f_{N_1}(j,\,s)\, f_{N_2}(j,\,s)$, and one has the Eulerian formula $f_N = \otimes_{p|N} f_p$, with

$$\begin{aligned} f_p(j,\,s) &= \frac{1}{p} \sum_{k \bmod p} (1 - p \ \mathrm{char}(j \equiv k \equiv 0 \bmod p)) \ \exp\left(\frac{2i\pi ks}{p}\right) \\ &= \frac{1}{p} \sum_{k \bmod p} \exp\left(\frac{2i\pi ks}{p}\right) - \mathrm{char}(j \equiv 0 \bmod p) \\ &= \mathrm{char}(s \equiv 0 \bmod p) - \mathrm{char}(j \equiv 0 \bmod p). \end{aligned} \tag{7.4}$$

Expanding the product,

$$f_N(j,\,s) = \sum_{N_1 N_2 = N} \mu(N_1)\,\mathrm{char}(s \equiv 0 \bmod N_2)\,\mathrm{char}(j \equiv 0 \bmod N_1). \tag{7.5}$$

The equation (7.1) follows from (6.4).

□

Theorem 7.2. *Let $N = RQ$ be squarefree odd. Let $v, u \in C^\infty(\mathbb{R})$, compactly supported, satisfying the conditions that $x > 0$ and $0 < x^2 - y^2 < 8$ when $v(x)u(y) \neq 0$. Then, if N is large enough,*

$$\begin{aligned} \big(v \,\big|\, \Psi(Q^{2i\pi\mathcal{E}}\mathfrak{T}_N)\, u\big) = \sum_{Q_1 Q_2 = Q} \mu(Q_1) \\ \sum_{R_1 | R} \mu(R_1)\, \overline{v}\left(\frac{R_1}{Q_2} + \frac{Q_2}{R_1}\right) \, u\left(\frac{R_1}{Q_2} - \frac{Q_2}{R_1}\right). \end{aligned} \tag{7.6}$$

Proof. Characterize $n \bmod 2N^2$ by $n \in \mathbb{Z}$ such that $-N^2 \leq n < N^2$. The equation $(\theta_N u)\,(n) = \sum_{\ell\in\mathbb{Z}} u\left(\frac{n}{N} + 2\ell N\right)$ imposes $\ell = 0$, so that $(\theta_N u)\,(n) = u\left(\frac{n}{N}\right)$. Similarly, $(\theta_N v)\,(m) = v\left(\frac{m}{N}\right)$ if $-N^2 \leq m < N^2$.

The identity (6.6) and Lemma 7.1 yield

$$\begin{aligned} \big(v \,\big|\, \Psi(Q^{2i\pi\mathcal{E}}\mathfrak{T}_N)\, u\big) = \sum_{\substack{R_1 R_2 = R \\ Q_1 Q_2 = Q}} \mu(R_1 Q_1)\,\mathrm{char}(m{+}n \equiv 0 \bmod 2RR_1Q_1) \\ \mathrm{char}(m - n \equiv 0 \bmod 2QR_2Q_2)\, \overline{v}\left(\frac{m}{N}\right)\, u\left(\frac{n}{N}\right). \end{aligned} \tag{7.7}$$

Set $m + n = (2RR_1Q_1)\, a$, $m - n = (2QR_2Q_2)\, b$, with $a,\, b \in \mathbb{Z}$. Then,

$$\frac{m^2}{N^2} - \frac{n^2}{N^2} = \frac{4QR(R_1R_2)(Q_1Q_2)\, ab}{N^2} = 4ab. \tag{7.8}$$

For all nonzero terms of the last equation, one has $0 < \frac{m^2}{N^2} - \frac{n^2}{N^2} < 8$. On the other hand, the symbol $\left(Q^{2i\pi\mathcal{E}}\mathfrak{T}_N\right)(x,\,\xi) = Q\,\mathfrak{T}_N(Qx,\,Q\xi)$, just as the symbol $\mathfrak{T}_N$, is invariant under the change of x, ξ to $x, x+\xi$. It follows from Theorem 3.4 that its integral kernel $K(x,\,y)$ is supported in the set of points $(x,\,y)$ such that $x^2 - y^2 \in 4\mathbb{Z}$. With $x = \frac{m}{N}$ and $y = \frac{n}{N}$, the only possibility is to take $x^2 - y^2 = 4$, hence $ab = 1$, finally $a = b = 1$ since $a > 0$.

The equations $m + n = (2RR_1Q_1)$, $m - n = (2QR_2Q_2)$ give $\frac{m}{N} = \frac{R_1}{Q_2} + \frac{Q_2}{R_1}$ and $\frac{n}{N} = \frac{R_1}{Q_2} - \frac{Q_2}{R_1}$. Be sure to note that, given $N = RQ$ and R_1, there is no summation with respect to $R_2 = \frac{R}{R_1}$.

□

8. Arithmetic significance of last theorem

In view of the central role of Theorem 7.2 in the proof to follow of the Riemann hypothesis, we give now an independent proof of the major part of it. Reading this section is thus in principle unnecessary: but we find the equation (8.2) below illuminating.

Theorem 8.1. *Let $N = RQ$ be a squarefree odd integer. Introduce the reflection $n \mapsto \overset{\vee}{n}$ of $\mathbb{Z}/(2N^2)\mathbb{Z}$ such that $\overset{\vee}{n} \equiv n \mod R^2$ and $\overset{\vee}{n} \equiv -n \mod 2Q^2$. Then, with the notation in Theorem 6.1, one has*

$$c_{R,Q}\,(\mathfrak{T}_N;\, m,\, n) = \mu(Q)\, c_{N,1}(\mathfrak{T}_N;\, m,\, \overset{\vee}{n}). \tag{8.1}$$

If two functions u and $\widetilde{u}$ in $\mathcal{S}(\mathbb{R})$ are such that $(\theta_N\,\widetilde{u})\,(n) = (\theta_N u)(\overset{\vee}{n})$ for every $n \in \mathbb{Z}$, one has

$$\left(v \,\middle|\, \Psi\left(Q^{2i\pi\mathcal{E}}\mathfrak{T}_N\right) u\right) = \mu(Q)\, \left(v \,\middle|\, \Psi(\mathfrak{T}_N)\,\widetilde{u}\right), \tag{8.2}$$

Proof. With the notation of Theorem 6.1, and making use of (7.4), one has the Eulerian decomposition $f_N = \otimes f_p$, in which, for every p, $f_p(j,\,s) = -f_p(s,\,j)$: it follows that $f_N(j,\,s) = \mu(N)\, f_N(s,\,j)$. To prove (8.1), one may assume that $R = 1$, $N = Q$ since the R-factor is left unaffected by the map $n \mapsto \overset{\vee}{n}$. Then,

$$c_{1,Q}\left(\mathfrak{S};\, m,\, n\right) = \mathrm{char}(m-n \equiv 0 \bmod 2Q)\, f\left(\frac{m+n}{2},\, \frac{m-n}{2Q}\right),$$

$$c_{Q,1}\left(\mathfrak{S};\, m,\, \overset{\vee}{n}\right) = \mathrm{char}(m+\overset{\vee}{n} \equiv 0 \bmod 2Q)\, f\left(\frac{m+\overset{\vee}{n}}{2Q},\, \frac{m-\overset{\vee}{n}}{2}\right)$$

$$= \mathrm{char}(m-n \equiv 0 \bmod 2Q)\, f\left(\frac{m-n}{2Q},\, \frac{m+n}{2}\right). \tag{8.3}$$

That the first and third line are the same, up to the factor $(\mu(Q))$, follows from the set of equations (7.4) $f_p(j,\, s) = -f_p(s,\, j)$.

The equation (8.2) follows from (8.1) in view of (6.6).

In [11, p.62-64], we gave a direct proof of a more general result (involving a more general symbol $\mathfrak{S}$ of type (6.1), not depending on Theorem 6.1.

□

It is not immediately obvious that, given $u \in \mathcal{S}(\mathbb{R})$, there exists $\widetilde{u} \in \mathcal{S}(\mathbb{R})$ such that $(\theta_N\, \widetilde{u})\,(n) = (\theta_N u)(\overset{\vee}{n})$ for every $n \in \mathbb{Z}$. Note that such a function $\widetilde{u}$ could not be unique, since any translate by a multiple of $2N^2$ would do just as well. That such a function does exist [11, p.61] is obtained from an explicit formula, to wit

$$\widetilde{u}(y) = \frac{1}{Q^2} \sum_{0 \leq \sigma, \tau < Q^2} u\left(y + \frac{2R\tau}{Q}\right) \exp\left(2i\pi\, \frac{\sigma(Nx + R^2\tau)}{Q^2}\right). \tag{8.4}$$

Indeed, with such a definition, one has

$$(\theta_N \widetilde{u})(n)$$
$$= \frac{1}{Q^2} \sum_{\ell \in \mathbb{Z}} \sum_{0 \leq \sigma, \tau < Q^2} u\left(\frac{n}{N} + \frac{2R\tau}{Q} + \ell N\right) \exp\left(\frac{2i\pi\sigma}{Q^2}\,(n + \ell N^2 + R^2\tau)\right). \tag{8.5}$$

Summing with respect to σ, this is the sme as

$$\sum_{\ell \in \mathbb{Z}} u\left(\frac{n + 2R^2\tau + \ell N^2}{N}\right), \tag{8.6}$$

where the integer $\tau \in [0, Q^2[$ is characterized by the condition $n + \ell N^2 + 2R^2\tau \equiv 0 \bmod Q^2$, or $n + 2R^2\tau \equiv 0 \bmod Q^2$. Finally, as $\ell \in \mathbb{Z}$, the number $n + 2R^2\tau + \ell N^2$ runs through the set of integers n_2 such that $n_2 \equiv -n \mod Q^2$ and $n_2 \equiv n \mod 2R^2$, in other words the set of numbers n_2 such that $n_2 \equiv \overset{\vee}{n} \mod 2N^2$. But applying this formula leads to rather unpleasant (not published, though leading to the correct result) calculations, as we experienced.

Instead, we shall give arithmetic the priority, starting from a manageable expression of the map $n \mapsto \overset{\vee}{n}$. Recall (4.18) that if v, u are compactly supported and the algebraic sum of their supports is contained in some interval $[0, 2\beta]$, and if one interests oneself in $\left(v \,\middle|\, \Psi\left(Q^{2i\pi\mathcal{E}}\mathfrak{T}_{\frac{\infty}{2}}\right) u\right)$, one can replace $\mathfrak{T}_{\frac{\infty}{2}}$ by $\mathfrak{T}_N$ provided that $N = RQ$ is divisible by all odd primes $< \beta Q$. With this in mind, the following lemma will make it possible to assume without loss of generality that $R \equiv 1 \mod 2Q^2$.

Lemma 8.2. *Let Q be a squarefree odd positive integer and let $\beta > 0$ be given. There exists $R > 0$, with $N = RQ$ squarefree odd divisible by all odd primes $< \beta Q$, such that $R \equiv 1 \mod 2Q^2$.*

Proof. Choose R_1 positive, odd and squarefree, relatively prime to Q, divisible by all odd primes $< \beta Q$ relatively prime to Q, and $\overline{R}_1$ such that $\overline{R}_1 R_1 \equiv 1 \mod 2Q^2$ and $\overline{R}_1 \equiv 1 \mod R_1$. Since $[R_1, 2Q] = 1$, there exists $x \in \mathbb{Z}$ such that $x \equiv 1 \mod R_1$ and $x \equiv \overline{R}_1 \mod 2Q^2$. Choosing (Dirichlet's theorem) a prime r such that $r \equiv x \mod 2R_1Q^2$, the number $R = R_1 r$ satisfies the desired condition.

□

With such a choice of R, we can make the map $n \mapsto \overset{\vee}{n}$ from $\mathbb{Z}/(2N^2)\mathbb{Z}$ to $\mathbb{Z}/(2N^2)\mathbb{Z}$ explicit. Indeed, the solution of a pair of congruences $x \equiv \lambda \mod R^2$, $x \equiv \mu \mod 2Q^2$ is given as $x \equiv (1-R^2)\lambda + \mu R^2 \mod 2N^2$. In particular, taking $\lambda = n$, $\mu = -n$, one obtains $\overset{\vee}{n} \equiv n(1-2R^2) \mod 2N^2$. Then, defining for instance $\widetilde{u}(y) = u(y(1-2R^2))$, one has indeed $(\theta_N \widetilde{u})(n) = (\theta_N u)(\overset{\vee}{n})$. Of course, the support of $\widetilde{u}$ is not the same as that of u, but no support conditions are necessary for (8.2) to hold.

We can now make a quick partial verification of the identity (7.6), starting with the case for which $Q = 1$. The point of Theorem 8.1 is that the general case can be reduced to this special one.