tells us that

$$E(\mathrm{d}\rho(t),\ \mathrm{d}\rho(t)) \quad = \quad \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{U_\rho(x)-U_\rho(y)}{x-y}\right)^2 \mathrm{d}x\,\mathrm{d}y.$$

What we have, then, to do is to *broaden the scope* of this relation, due to Jesse Douglas, so as to get it to apply to the whole class of measures ρ under consideration in this article. Instead to trying to extend the result directly, or to generalize the argument of problem 23(a) (based on the first lemma of §B.5 in Chapter VIII which was proved there under quite restrictive conditions), we will undertake a new derivation using different ideas.

The machinery employed for this purpose consists of the L_2 theory of Hilbert transforms, sketched in the scholium at the end of §C.1, Chapter VIII. The reader may have already noticed a connection between Hilbert transforms and logarithmic (and Green) potentials, appearing, for instance, in the first lemma of §C.3, Chapter VIII, and in problem 29(b) (Chapter IX, §B.1).

As usual, we write

$$\rho(x) \quad = \quad \int_0^x \mathrm{d}\rho(t) \qquad \text{for } x \geqslant 0$$

when working with real signed measures ρ on $[0, \infty)$. It will also be convenient to extend the definition of such functions ρ to *all of* $\mathbb{R}$ *by making them even* (*sic*!) there.

Lemma. *Let*

$$\int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right| |\mathrm{d}\rho(t)||\mathrm{d}\rho(x)| \quad < \quad \infty$$

for the real signed measure ρ on $[0, \infty)$ without point mass at the origin. Then $\rho(x)$ is $O(\sqrt{x})$ for $x \longrightarrow \infty$.

Remark. This is a weak result.

Proof of lemma. Since

$$|\rho(x)| \quad \leqslant \quad \int_0^x |\mathrm{d}\rho(t)| \qquad \text{for } x > 0,$$

it is just as well to assume to begin with that $\mathrm{d}\rho(t)$ is *positive*, and thus $\rho(x)$ *increasing* on $[0, \infty)$.

Then, by the second lemma of §B.5, Chapter VIII,

$$\int_0^\infty \int_0^\infty \left(\frac{\rho(x)-\rho(y)}{x-y}\right)^2 \frac{x^2+y^2}{(x+y)^2}\,\mathrm{d}x\,\mathrm{d}y \quad = \quad E(\mathrm{d}\rho(t),\ \mathrm{d}\rho(t)) \quad < \quad \infty,$$

so

$$\int_0^\infty \left(\frac{\rho(x)-\rho(y)}{x-y}\right)^2 \mathrm{d}x \quad < \quad \infty$$

for almost all $y > 0$. Fix such a y; we get

$$\int_{2y}^\infty \left(\frac{\rho(x)}{x}\right)^2 \mathrm{d}x \quad \leqslant \quad 2\int_{2y}^\infty \left(\frac{\rho(x)-\rho(y)}{x-y}\right)^2 \mathrm{d}x \; + \; 2\frac{(\rho(y))^2}{y} \quad < \quad \infty,$$

and thence, for $x > 2y$,

$$(\rho(x))^2 \int_x^\infty \frac{\mathrm{d}t}{t^2} \quad \leqslant \quad \int_{2y}^\infty \left(\frac{\rho(t)}{t}\right)^2 \mathrm{d}t \quad < \quad \infty,$$

ρ being increasing. Thus,

$$(\rho(x))^2 \quad \leqslant \quad \text{const.}\, x \qquad \text{for } x > 2y.$$

Done.

Lemma. *Let ρ satisfy the hypothesis of the preceding lemma. Then*

$$U_\rho(x) \quad = \quad -\!\!\!\!\!\!\int_0^\infty \left(\frac{1}{x-t} + \frac{1}{x+t}\right)\rho(t)\,\mathrm{d}t \qquad \text{a.e., } x \in \mathbb{R}.$$

Proof. $U_\rho(x)$ is odd, so it is enough to establish the formula for almost all $x > 0$. Taking any such x for which the integral defining $U_\rho(x)$ converges absolutely, we have

$$U_\rho(x) \quad = \quad \lim_{\varepsilon \to 0}\left(\int_0^{x-\varepsilon} + \int_{x+\varepsilon}^\infty\right) \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t).$$

Fixing for the moment a small $\varepsilon > 0$, we treat the two integrals on the right by partial integration, very much as in the proof of the lemma in §C.3, Chapter VIII (but going in the opposite direction). The integrated term

$$\rho(t)\,\log\left|\frac{x+t}{x-t}\right|$$

which thus arises vanishes at $t = 0$ and also when $t \longrightarrow \infty$, the latter thanks to the preceding lemma. Subtraction of its values for $t = x \pm \varepsilon$ also gives a small result when $\varepsilon > 0$ is small, *as long as $\rho'(x)$ exists and is finite,* and therefore for almost all x. We thus end with the desired formula on making $\varepsilon \longrightarrow 0$, Q.E.D.

Referring to our convention that $\rho(-t) = \rho(t)$, we immediately obtain the

Corollary

$$U_\rho(x) = -\!\!\!\!\!\!\int_{-\infty}^{\infty} \left(\frac{1}{x-t} + \frac{t}{t^2+1} \right) \rho(t)\,dt \qquad \text{a.e.,}\ x \in \mathbb{R}.$$

Remark. The dummy term $t/(t^2+1)$ is introduced in the integrand in order to guarantee absolute convergence of the integral near $\pm\infty$, and does so, $\rho(t)$ being $O(\sqrt{|t|})$ there. We see that, aside from a missing factor of $-1/\pi$, $U_\rho(x)$ is just the harmonic conjugate (Hilbert transform) of $\rho(x)$, which should be very familiar to anyone who has read up to here in the present book.

Lemma. *Let the signed measure ρ satisfy the hypothesis of the first of the preceding two lemmas. Then, for almost every real y, the function of x equal to $(\rho(x)-\rho(y))/(x-y)$ belongs to $L_2(-\infty,\infty)$, and*

$$\frac{U_\rho(x) - U_\rho(y)}{x-y} = -\!\!\!\!\!\!\int_{-\infty}^{\infty} \frac{1}{x-t}\,\frac{\rho(t)-\rho(y)}{t-y}\,dt \qquad \text{a.e.,}\ x \in \mathbb{R}.$$

Proof. As at the beginning of the proof of the first of the above two lemmas,

$$\int_0^\infty \int_0^\infty \left(\frac{\rho(x)-\rho(y)}{x-y} \right)^2 \frac{x^2+y^2}{(x+y)^2}\,dx\,dy < \infty,$$

so, for almost every $y > 0$, $(\rho(x)-\rho(y))/(x-y)$ belongs to $L_2(0,\infty)$ as a function of x. But since ρ is even,

$$\left| \frac{\rho(x) - \rho(y)}{x - y} \right| \leqslant \left| \frac{\rho(|x|) - \rho(|y|)}{|x| - |y|} \right|;$$

we thus see by the statement just made that as a function of x, $(\rho(x)-\rho(y))/(x-y)$ belongs in fact to $L_2(-\infty,\infty)$ for almost all $y \in \mathbb{R}$.

For any real number C, we have (trick!):

$$-\!\!\!\!\!\!\int_{-\infty}^{\infty} \left(\frac{1}{x-t} + \frac{t}{t^2+1} \right) C\,dt = 0.$$

Adding this relation to the formula given by the last corollary, we thus get

$$U_\rho(x) \quad = \quad -\!\!\!\!\!\!\int_{-\infty}^{\infty} \left(\frac{1}{x-t} + \frac{t}{t^2+1}\right)(\rho(t) - C)\,\mathrm{d}t$$

for almost all $x \in \mathbb{R}$, where the exceptional set *does not depend* on the number C. From this relation we subtract the similar one obtained on replacing x by any other value y for which it holds. That yields

$$U_\rho(x) - U_\rho(y) \quad = \quad -\!\!\!\!\!\!\int_{-\infty}^{\infty} \left(\frac{1}{x-t} - \frac{1}{y-t}\right)(\rho(t) - C)\,\mathrm{d}t \quad \text{a.e., } x, y \in \mathbb{R}.$$

In the Cauchy principal value standing on the right, the integrand involves *two* singularities, at $t = x$ and at $t = y$. Consider, however, what happens when y takes one of the values for which $\rho'(y)$ *exists and is finite. Then, we can put* $C = \rho(y)$ *in the preceding relation* (!), and, after dividing by $x - y$, it becomes

$$\frac{U_\rho(x) - U_\rho(y)}{x-y} \quad = \quad -\!\!\!\!\!\!\int_{-\infty}^{\infty} \frac{1}{x-t}\,\frac{\rho(t)-\rho(y)}{t-y}\,\mathrm{d}t,$$

in which the function $(\rho(t) - \rho(y))/(t-y)$ figuring on the right *remains bounded* for $t \longrightarrow y$. What we have on the right is thus just the *ordinary* Cauchy principal value involving an integrand with *one* singularity (at $t = x$), used in the study of Hilbert transforms.

We are, however, assured of the existence and finiteness of $\rho'(y)$ at almost every y. The last relation thus holds a.e. in both x and y, and we are done.

Theorem. *Let the real signed measure ρ on $[0, \infty)$, without point mass at the origin, be such that*

$$\int_0^\infty \log\left|\frac{x+t}{x-t}\right| |\mathrm{d}\rho(t)||\mathrm{d}\rho(x)| \quad < \quad \infty,$$

and put

$$U_\rho(x) \quad = \quad \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t),$$

thus specifying the value of U_ρ almost everywhere on $\mathbb{R}$. Then we have Jesse Douglas' formula:

$$\boxed{E(\mathrm{d}\rho(t), \mathrm{d}\rho(t)) \quad = \quad \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{U_\rho(x) - U_\rho(y)}{x-y}\right)^2 \mathrm{d}x\,\mathrm{d}y.}$$

Proof. The last lemma exhibits the function of x equal to $(U_\rho(x) - U_\rho(y))/(x-y)$ as $-\pi$ times the *Hilbert transform* of the one equal to $(\rho(x)-\rho(y))/(x-y)$ for almost every $y \in \mathbb{R}$, and also tells us that the latter function of x is in $L_2(-\infty, \infty)$ for almost every such y. We may therefore apply to these functions *the L_2 theory of Hilbert transforms* taken up in the scholium at the end of §C.1, Chapter VIII. By that theory,

$$\int_{-\infty}^{\infty} \left(\frac{U_\rho(x) - U_\rho(y)}{\pi(x-y)}\right)^2 \mathrm{d}x \quad = \quad \int_{-\infty}^{\infty} \left(\frac{\rho(x)-\rho(y)}{x-y}\right)^2 \mathrm{d}x$$

for the values of y in question, i.e., almost everywhere in y.

Integrating now with respect to y, this gives

$$\frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{U_\rho(x) - U_\rho(y)}{x-y}\right)^2 \mathrm{d}x\,\mathrm{d}y \quad = \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{\rho(x)-\rho(y)}{x-y}\right)^2 \mathrm{d}x\,\mathrm{d}y.$$

Because ρ is even, the right side is just

$$2\int_0^\infty\int_0^\infty \left\{\left(\frac{\rho(x)-\rho(y)}{x-y}\right)^2 + \left(\frac{\rho(x)-\rho(y)}{x+y}\right)^2\right\}\mathrm{d}x\,\mathrm{d}y$$

$$= \quad 4\int_0^\infty\int_0^\infty \left(\frac{\rho(x)-\rho(y)}{x-y}\right)^2 \frac{x^2+y^2}{(x+y)^2}\,\mathrm{d}x\,\mathrm{d}y.$$

Dividing by 4 and referring to the second lemma of §B.5, Chapter VIII, we immediately obtain the desired result.

Corollary. *For any measure ρ satisfying the hypothesis of the theorem,*

$$E(\mathrm{d}\rho(t),\ \mathrm{d}\rho(t))$$

is determined when the Green potential $U_\rho(x)$ is specified almost everywhere on $\mathbb{R}$, and $\rho = 0$ if $U_\rho(x) = 0$ a.e. in $(0, \infty)$. (Here $U_\rho(x)$ is determined by its values on $(0, \infty)$ because it is odd.)

This corollary finally gives us the right to denote $\sqrt{(E(\mathrm{d}\rho(t),\ \mathrm{d}\rho(t)))}$ by $\|U_\rho\|_E$ for *any* measure ρ fulfilling the conditions of the theorem; indeed, we simply have

$$\|U_\rho\|_E \quad = \quad \frac{1}{2\pi}\sqrt{\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{U_\rho(x)-U_\rho(y)}{x-y}\right)^2\mathrm{d}x\,\mathrm{d}y\right)}.$$

It will be convenient for us to use this formula for *arbitrary* real-valued Lebesgue measurable functions (*odd or not* !) defined on $\mathbb{R}$. Then, of course, it becomes a matter of

Notation. Given v, real-valued and Lebesgue measurable on $\mathbb{R}$, we write

$$\|v\|_E \quad = \quad \frac{1}{2\pi}\sqrt{\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{v(x)-v(y)}{x-y}\right)^2 \mathrm{d}x\,\mathrm{d}y\right)}.$$

This clearly defines $\| \quad \|_E$ as a *norm* on the collection of such functions v (modulo the constants); if $\|v\|_E = 0$ we must have

$$v(x) \quad = \quad \text{const.} \quad \text{a.e.,} \ x \in \mathbb{R}.$$

Near the beginning of this article, we said how the Hilbert space $\mathfrak{H}$ was to be formed: $\mathfrak{H}$ was specified as *the abstract completion in norm* $\| \quad \|_E$ *of the collection of Green potentials* U_ρ *coming from the measures* ρ *satisfying the conditions of the last theorem.* An *element* of $\mathfrak{H}$ is, in other words, defined by a *Cauchy sequence*, $\{U_{\rho_n}\}$, of such potentials. According, however, to the corollary of the *first* theorem in this article, such a Cauchy sequence has in it a *subsequence*, which we may as well *also*, for the moment, denote by $\{U_{\rho_n}\}$, with $U_{\rho_n}(x)$ *pointwise convergent* at almost every $x \in \mathbb{R}$. Writing

$$\lim_{n\to\infty} U_{\rho_n}(x) \quad = \quad U(x)$$

wherever the limit exists, we see that $U(x)$ is *defined* a.e. and *Lebesgue measurable*; it is also *odd*, because the individual Green potentials $U_{\rho_n}(x)$ are odd.

Fixing any index m, we have, making the usual application of Fatou's lemma,

$$\|U - U_{\rho_m}\|_E^2 \quad = \quad \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{U(x) - U_{\rho_m}(x) - U(y) + U_{\rho_m}(y)}{x - y}\right)^2 \mathrm{d}x\,\mathrm{d}y$$

$$\leqslant \quad \liminf_{j\to\infty} \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{U_{\rho_j}(x) - U_{\rho_m}(x) - U_{\rho_j}(y) + U_{\rho_m}(y)}{x - y}\right)^2 \mathrm{d}x\,\mathrm{d}y$$

$$= \quad \liminf_{j\to\infty} \|U_{\rho_j} - U_{\rho_m}\|_E^2.$$

Since we started with a Cauchy sequence, the last quantity is *small* if m is large. This in fact holds for all the U_{ρ_m} from our *original* sequence, for the last chain of inequalities is valid for any of those potentials as long as the U_{ρ_j} appearing therein *run through the subsequence* just described. *Corresponding to the Cauchy sequence* $\{U_{\rho_n}\}$, *we have thus found an odd* *measurable function U with*

$$\|U - U_{\rho_n}\|_E \xrightarrow[n]{} 0.$$

In this fashion we can associate *an odd measurable function U, approximable in the norm $\| \quad \|_E$ by Green potentials U_ρ like the ones appearing in the last theorem, to each element of the space $\mathfrak{H}$.* It is, on the other hand, manifest that *each such function U does indeed correspond to some element of $\mathfrak{H}$* – the Green potentials U_ρ approximating U in norm $\| \quad \|_E$ *furnish us* with a Cauchy sequence of such potentials (in that norm)! There is thus a correspondence between *the collection of such functions U* and *the space $\mathfrak{H}$*.

It is necessary now to show that *this correspondence is one-one.* But that is easy. Suppose, in the first place that *two different odd functions*, say U and V, are associated to *the same* element of the space $\mathfrak{H}$ in the manner described. Then we have *two* Cauchy sequences of Green potentials, say $\{U_{\rho_n}\}$ and $\{U_{\sigma_n}\}$, with

$$\|U_{\rho_n} - U_{\sigma_n}\|_E \xrightarrow[n]{} 0,$$

and such that

$$\|U - U_{\rho_n}\|_E \xrightarrow[n]{} 0$$

while

$$\|V - U_{\sigma_n}\|_E \xrightarrow[n]{} 0.$$

It follows that

$$\|U - V\|_E = 0,$$

but then, as noted above,

$$U(x) - V(x) = \text{const.} \qquad \text{a.e., } x \in \mathbb{R}.$$

Here, $U - V$ is *odd*, so the *constant* must be *zero*, and

$$U(x) = V(x) \qquad \text{a.e., } x \in \mathbb{R}.$$

Given, on the other hand, *two* Cauchy sequences, $\{U_{\rho_n}\}$ and $\{U_{\sigma_n}\}$, of potentials associated to *the same odd function U*, we have

$$\|U - U_{\rho_n}\|_E \xrightarrow[n]{} 0$$

and

$$\|U - U_{\sigma_n}\|_E \xrightarrow[n]{} 0,$$

whence

$$\| U_{\rho_n} - U_{\sigma_n} \|_E \xrightarrow[n]{} 0.$$

Then, however, $\{U_{\rho_n}\}$ and $\{U_{\sigma_n}\}$ define *the same element* of the abstract completion $\mathfrak{H}$.

Our Hilbert space $\mathfrak{H}$ is thus in one-to-one correspondence with the collection of odd real measurable functions U approximable, in norm $\| \quad \|_E$, by the potentials U_ρ under consideration here. There is hence nothing to keep us from *identifying the space $\mathfrak{H}$ with that collection of functions U, and we henceforth do so.*

We are now well enough equipped to give a strengthened version, promised earlier, of the corollary to the first theorem in this article.

Lemma. *Let the odd measurable function U be identified with an element of the space $\mathfrak{H}$ in the manner just described, and suppose that ρ is an* absolutely continuous *signed measure on* $[0, \infty)$ *with*

$$\int_0^\infty \int_0^\infty \log \left| \frac{x+t}{x-t} \right| \mathrm{d}\rho(t) \, \mathrm{d}\rho(x)$$

and

$$\int_0^\infty U(x) \, \mathrm{d}\rho(x)$$

both absolutely convergent. Then

$$\int_0^\infty U(x) \mathrm{d}\rho(x) \quad = \quad \langle U, \; U_\rho \rangle_E,$$

and especially

$$\boxed{\left| \int_0^\infty U(x) \, \mathrm{d}\rho(x) \right| \quad \leqslant \quad \| U \|_E \sqrt{(E(\mathrm{d}\rho(t), \; \mathrm{d}\rho(t)))}.}$$

Remark. The second relation is very useful in certain applications.

Proof of lemma. We proceed to establish the first relation, using a somewhat repetitious crank-turning argument.

Starting with an absolutely continuous ρ fulfilling the conditions in the hypothesis, let us put, for $N \geqslant 1$,

$$d\rho_N(t) \quad = \quad \begin{cases} \rho'(t)\,dt & \text{if } |\rho'(t)| \leqslant N, \\ (N \operatorname{sgn} \rho'(t))\,dt & \text{otherwise;} \end{cases}$$

it is claimed that

$$\| U_\rho \; - \; U_{\rho_N} \|_E \;\longrightarrow\; 0$$

for $N \longrightarrow \infty$.

By breaking $\rho'(t)$ up into positive and negative parts, we can reduce the general situation to one in which

$$\rho'(t) \;\geqslant\; 0,$$

so we may as well assume this property. Then, for each $x > 0$, the potentials

$$U_{\rho_N}(x) \quad = \quad \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \min(\rho'(t),\; N)\,dt$$

increase and tend to $U_\rho(x)$ *as* $N \longrightarrow \infty$. Hence, by monotone convergence,

$$\int_0^\infty U_{\rho_N}(x)\,d\rho(x) \;\xrightarrow[N]{}\; \int_0^\infty U_\rho(x)\,d\rho(x).$$

From this, we see that

$$\| U_\rho \; - \; U_{\rho_N} \|_E^2 \quad = \quad E(d\rho(t) - d\rho_N(t),\; d\rho(t) - d\rho_N(t))$$

$$= \quad \int_0^\infty (U_\rho(x) \; - \; U_{\rho_N}(x))(\rho'(x) \; - \; \min(\rho'(x), N))\,dx$$

$$\leqslant \quad \int_0^\infty (U_\rho(x) \; - \; U_{\rho_N}(x))\,d\rho(x) \quad (!)$$

must *tend to zero* as $N \longrightarrow \infty$, verifying our assertion.

From what we have just shown, it follows that

$$\langle U,\; U_{\rho_N} \rangle_E \;\xrightarrow[N]{}\; \langle U,\; U_\rho \rangle_E.$$

But we clearly have

$$\int_0^\infty U(x)\,d\rho_N(x) \;\xrightarrow[N]{}\; \int_0^\infty U(x)\,d\rho(x)$$

by the given absolute convergence of the integral on the right. The desired first relation *will therefore follow* if we can prove that

$$\int_0^\infty U(x)\,d\rho_N(x) \quad = \quad \langle U,\; U_{\rho_N} \rangle_E$$

for each N; that, however, simply amounts to *verifying the relation* in question for *measures* ρ satisfying the hypothesis and *having, in addition, bounded densities* $\rho'(x)$. We have thus brought down by one notch the generality of what is to be proven.

Suppose, then, that ρ satisfies the hypothesis and that $\rho'(x)$ *is also bounded*. For each a, $0 < a < 1$, put

$$\rho'_a(t) \quad = \quad \begin{cases} \rho'(t), \quad a \leqslant t \leqslant \dfrac{1}{a}, \\ \\ 0 \quad \text{otherwise}, \end{cases}$$

and then define a measure ρ_a (*not to be confounded with the* ρ_N *just used!*) by taking $\mathrm{d}\rho_a(t) = \rho'_a(t)\,\mathrm{d}t$. An argument very similar to the one made above now shows that

$$\| U_\rho - U_{\rho_a} \|_E \longrightarrow 0 \quad \text{for } a \longrightarrow 0$$

(it suffices as before to consider the case where $\rho'(t) \geqslant 0$) and hence that

$$\langle U,\, U_{\rho_a} \rangle_E \quad \longrightarrow \quad \langle U,\, U_\rho \rangle_E \quad \text{as } a \longrightarrow 0.$$

At the same time,

$$\int_0^\infty U(x)\,\mathrm{d}\rho_a(x) \quad \longrightarrow \quad \int_0^\infty U(x)\,\mathrm{d}\rho(x),$$

so it is enough to check that

$$\int_0^\infty U(x)\,\mathrm{d}\rho_a(x) \quad = \quad \langle U,\, U_{\rho_a} \rangle_E$$

for each a.

Here, however, $\mathrm{d}\rho_a(x) = \rho'_a(x)\,\mathrm{d}x$ *with* $\rho'_a(x)$ *bounded and of compact support in* $(0, \infty)$, so the *corollary of the first theorem* in this article is applicable. Since U is identified with an element of $\mathfrak{H}$ there is, by the discussion preceding this lemma, a sequence of Green potentials U_{σ_n} of the kind used to form that space such that

$$\| U - U_{\sigma_n} \|_E \xrightarrow[n]{} 0,$$

and also

$$U_{\sigma_n}(x) \xrightarrow[n]{} U(x) \quad \text{a.e.,} \quad x \in \mathbb{R}.$$

Then we must have

$$E(d\sigma_n(t),\ d\rho_a(t)) \quad = \quad \langle U_{\sigma_n},\ U_{\rho_a}\rangle_E \quad \xrightarrow[n]{} \quad \langle U,\ U_{\rho_a}\rangle_E,$$

and, by the corollary referred to,

$$E(d\sigma_n(t),\ d\rho_a(t)) \quad \xrightarrow[n]{} \quad \int_0^\infty U(x)\,d\rho_a(x).$$

Thus,

$$\langle U,\ U_{\rho_a}\rangle_E \quad = \quad \int_0^\infty U(x)\,d\rho_a(x),$$

what we needed to finish showing the first relation in the conclusion of the lemma.

From it, however, the *second* relation follows immediately by Schwarz' inequality and the preceding theorem, since

$$\| U_\rho \|_E \quad = \quad \sqrt{(E(d\rho(t),\ d\rho(t)))}.$$

Our lemma is proved.

What we have done so far still does not amount to a real *description* of the space $\mathfrak{H}$, for we do not yet know *which* odd measurable functions $U(x)$ with $\| U \|_E < \infty$ can be approximated in the norm $\| \quad \|_E$ by potentials U_ρ of the kind appearing in the last theorem. The fact is that *all those functions U can be thus approximated.*

Theorem. *$\mathfrak{H}$ consists precisely of the real odd measurable functions $U(x)$ for which*

$$\| U \|_E^2 \quad = \quad \frac{1}{4\pi^2}\int_{-\infty}^\infty \int_{-\infty}^\infty \left(\frac{U(x) - U(y)}{x - y}\right)^2 dx\,dy$$

is finite.

Proof. That $\mathfrak{H}$ *consists* of such functions U was shown in the course of the previous discussion; what we have to do here is prove the *converse*, to the effect that *any* odd function U with $\| U \|_E < \infty$ *is* in $\mathfrak{H}$. This involves an approximation argument.

Starting, then, with an odd function U such that $\| U \|_E < \infty$, we must *obtain* signed measures ρ on $[0, \infty)$ meeting the conditions of the last theorem, *for which*

$$\| U \ - \ U_\rho \|_E$$

is as small as we please. That will be done in essentially *three steps.*

The first step makes use of the notion of *contraction*, brought into potential theory by Beurling and Deny. For $M > 0$, put

$$U_M(x) \quad = \quad \begin{cases} U(x) & \text{if } |U(x)| < M, \\ M \operatorname{sgn} U(x) & \text{if } |U(x)| \geqslant M. \end{cases}$$

Then

$$|U_M(x) - U_M(y)| \quad \leqslant \quad |U(x) - U(y)|$$

(this is the contraction property), so

$$\left(\frac{U(x) - U_M(x) - U(y) + U_M(y)}{x - y}\right)^2 \quad \leqslant \quad 4\left(\frac{U(x) - U(y)}{x - y}\right)^2.$$

But $U_M(x) \longrightarrow U(x)$ as $M \longrightarrow \infty$, so the *left side* of this relation *tends to zero* a.e. in x and y (with respect to Lebesgue measure for $\mathbb{R}^2$) as $M \longrightarrow \infty$. (The *odd* function $U(x)$ *cannot be infinite* on a set of *positive measure*, for in that event U would be infinite on such a subset of $(-\infty, 0]$ or of $[0, \infty)$, and this would clearly make $\|U\|_E = \infty$.) Because $\|U\|_E < \infty$, the double integral of the *right side* of the relation over $\mathbb{R}^2$ is *finite*. Hence

$$\|U - U_M\|_E^2 \longrightarrow 0 \quad \text{for } M \longrightarrow \infty$$

by dominated convergence. *Any function U satisfying the hypothesis is thus $\| \;\; \|_E$ - approximable by bounded ones.*

It therefore suffices to show how to do the desired approximation of *bounded* functions U fulfilling the conditions of this theorem. Taking such a one, for which

$$|U(x)| \leqslant M, \quad \text{say,}$$

on the real axis, we look at the products

$$U_H(x) \quad = \quad \frac{H^2}{H^2 + x^2} U(x)$$

where H is *large*. (*These should not be confounded with the contractions U_M used in the previous step.*) Denoting by $v_H(x)$ the (even!) function

$$\frac{H^2}{H^2 + x^2},$$

we have

$$U_H(x) - U_H(y) \quad = \quad v_H(x)(U(x) - U(y)) + U(y)(v_H(x) - v_H(y)),$$

so that

$$\left|\frac{U_H(x)-U_H(y)}{x-y}\right| \quad\leqslant\quad \left|\frac{U(x)-U(y)}{x-y}\right| \quad+\quad M\left|\frac{v_H(x)-v_H(y)}{x-y}\right|.$$

From this last, it is easily seen that

$$\|U_H\|_E^2 \quad\leqslant\quad 2\|U\|_E^2 \;+\; 2M^2\|v_H\|_E^2,$$

and we must compute the norms $\|v_H\|_E$.

To do that, we note that $v_H(x)$ has a harmonic extension to the upper half plane given by

$$v_H(z) \quad=\quad \frac{H(H+y)}{(H+y)^2+x^2} \quad=\quad -\Im\left(\frac{H}{z+\mathrm{i}H}\right),$$

and that this function is sufficiently well behaved for the identities employed in the solution of problem 23(a) (§B.8, Chapter VIII) to apply to it. By the help of those, one finds that

$$\|v_H\|_E^2 \quad=\quad \frac{1}{2\pi}\int_0^\infty\int_{-\infty}^\infty\left\{\left(\frac{\partial v_H(z)}{\partial x}\right)^2 \;+\; \left(\frac{\partial v_H(z)}{\partial y}\right)^2\right\}\mathrm{d}x\,\mathrm{d}y.$$

Using the Cauchy–Riemann equations, we convert the integral on the right to

$$\frac{H^2}{2\pi}\int_0^\infty\int_{-\infty}^\infty\left|\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{z+\mathrm{i}H}\right)\right|^2\mathrm{d}x\,\mathrm{d}y \quad=\quad \frac{H^2}{2\pi}\int_0^\infty\int_{-\infty}^\infty\frac{1}{|z+\mathrm{i}H|^4}\mathrm{d}x\,\mathrm{d}y,$$

which becomes

$$\frac{1}{2\pi}\int_0^\infty\int_{-\infty}^\infty\frac{\mathrm{d}\xi\,\mathrm{d}\eta}{(\xi^2+(\eta+1)^2)^2}$$

after putting $z = H(\xi+\mathrm{i}\eta)$. The further substitution $\xi = (\eta+1)\tau$ converts the last double integral to

$$\frac{1}{2\pi}\int_0^\infty\frac{\mathrm{d}\eta}{(\eta+1)^3}\int_{-\infty}^\infty\frac{\mathrm{d}\tau}{(\tau^2+1)^4} \quad=\quad \frac{1}{8},$$

so we have

$$\|v_H\|_E \quad=\quad \frac{1}{2\sqrt{2}}$$

independently of the number H.

Plugging this into the previous inequality, we get

$$\|U_H\|_E^2 \quad\leqslant\quad 2\|U\|_E^2 \;+\; \tfrac{1}{4}M^2,$$

so the *norms on the left remain bounded as* $H \longrightarrow \infty$. From this it follows that *the functions* U_H *corresponding to some suitable sequence of values of* H *going out to infinity tend weakly* (in the Hilbert space of *all* real *odd* functions with finite $\| \quad \|_E$ norm!) *to some odd function* W *with* $\|W\|_E < \infty$. That in turn implies that *some sequence of* (finite) *convex linear combinations* u_n *of those* U_H (formed by using *ever larger values* of H from the sequence just mentioned) *actually tends in norm* $\| \quad \|_E$ *to* W. It is clear, however, that

$$U_H(x) \longrightarrow U(x)$$

(wherever the value on the right is defined) as $H \longrightarrow \infty$. Hence

$$u_n(x) \xrightarrow[n]{} U(x) \quad \text{a.e.,} \ x \in \mathbb{R},$$

so, since

$$\| W - u_n \|_E \xrightarrow[n]{} 0,$$

we see by the usual application of Fatou's lemma that

$$\| W - U \|_E = 0,$$

making (as in the previous discussion about the construction of $\mathfrak{H}$)

$$W(x) = U(x) \quad \text{a.e.,} \ x \in \mathbb{R},$$

since W and U are *both odd*. Therefore, we in fact have

$$\| U - u_n \|_E \xrightarrow[n]{} 0,$$

and *we can approximate the function* U *in norm* $\| \quad \|_E$ *by finite linear combinations* u_n *of the functions*

$$\frac{H^2}{H^2 + x^2} U(x).$$

Since the function $U(x)$ is *bounded*, we see that each function u_n, *besides being odd* (like U), *satisfies a condition of the form*

$$|u_n(x)| \leqslant \frac{K_n}{x^2 + 1}, \qquad x \in \mathbb{R}$$

(where, of course, the constant K_n may be enormous, but we don't care about that).

For this reason it is enough if, using the potentials U_ρ, one can approximate in norm $\| \quad \|_E$ any odd function u with $\|u\|_E < \infty$ and, in addition,

$$|u(x)| \quad \leqslant \quad \frac{K}{x^2+1} \qquad \text{on } \mathbb{R}.$$

Showing how to do that is the last step in our proof.

Take any such u. For $\Im z > 0$, put

$$u(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\, u(t)\,\mathrm{d}t.$$

The *size* of $|u(t)|$ on $\mathbb{R}$ is here well enough controlled so that the Fourier integral argument used in working problem 23(a) may be applied to $u(z)$. In that way, one readily verifies that

$$\|u\|_E^2 \quad = \quad \frac{1}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty} \{(u_x(z))^2 \;+\; (u_y(z))^2\}\,\mathrm{d}x\,\mathrm{d}y;$$

this, in particular, makes the Dirichlet integral appearing on the right *finite*.

For $h > 0$ (which in a moment will be made to tend to zero) we now put

$$u_h(z) \;=\; u(z+\mathrm{i}h), \qquad \Im z \;\geqslant\; 0.$$

An evident adaptation of the argument just referred to then shows that

$$\|u \;-\; u_h\|_E^2 \quad = \quad \frac{1}{2\pi}\iint_{\Im z>0} \left\{\left(\frac{\partial}{\partial x}(u(z)-u_h(z))\right)^2 \right.$$
$$\left. +\quad \left(\frac{\partial}{\partial y}(u(z)-u_h(z))\right)^2\right\}\mathrm{d}x\,\mathrm{d}y.$$

The right-hand integral is just

$$\frac{1}{2\pi}\iint_{\Im z>0} \{(u_x(z)-u_x(z+\mathrm{i}h))^2 \;+\; (u_y(z)-u_y(z+\mathrm{i}h))^2\}\,\mathrm{d}x\,\mathrm{d}y,$$

and we see that *it must tend to zero when* $h \longrightarrow 0$ (by continuity of translation in $L_2(\mathbb{R}^2)$!), since

$$\iint_{\Im z>0} (u_x(z))^2\,\mathrm{d}x\,\mathrm{d}y \qquad \text{and} \qquad \iint_{\Im z>0} (u_y(z))^2\,\mathrm{d}x\,\mathrm{d}y$$

are *both finite*, according to the observation just made. Thus,

$$\|u \;-\; u_h\|_E \;\longrightarrow\; 0 \quad \text{as } h \longrightarrow 0$$

It is now claimed that *each function* $u_h(x)$ *is equal* (on $\mathbb{R}$) *to a potential* $U_\rho(x)$ of the required kind. Since $u(t)$ is odd, we have

$$u(x+iy) \;=\; u(-x+iy) \qquad \text{for } y>0,$$

so $u_h(x)$ is odd. Our condition on $u(x)$ implies a similar one,

$$|u_h(x)| \;\leqslant\; \frac{\text{const}}{x^2+1}, \qquad x\in\mathbb{R},$$

on u_h, so that function is (and by far!) in $L_2(-\infty,\infty)$, and we can apply to it the L_2 theory of Hilbert transforms from the scholium at the end of §C.1, Chapter VIII. In the present circumstances, $u_h(x) = u(x+ih)$ is $\mathscr{C}_\infty$ in x, so the Hilbert transform

$$\tilde{u}_h(x) \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u_h(t)}{x-t}\,dt \;=\; \frac{1}{\pi}\int_0^{\infty}\frac{u_h(x-\tau)-u_h(x+\tau)}{\tau}\,d\tau$$

is defined and continuous at each real x, the last integral on the right being absolutely convergent. From the Hilbert transform theory referred to (even a watered-down version of it will do here!) we thence get, by the inversion formula,

$$u_h(x) \;=\; -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\tilde{u}_h(t)}{x-t}\,dt \qquad \text{a.e., } x\in\mathbb{R}.$$

This relation, like the one preceding it, holds in fact at *each* $x\in\mathbb{R}$, for $\tilde{u}_h(x)$ is nothing but the value of a *harmonic conjugate to* $u(z)$ at $z = x+ih$ and is hence (like $u_h(x)$) $\mathscr{C}_\infty$ in x. Wishing to integrate the right-hand member by parts, we look at the behaviour of $\tilde{u}_h'(t)$.

By the Cauchy–Riemann equations,

$$\tilde{u}_h'(x) \;=\; \tilde{u}_x(x+ih) \;=\; -u_y(x+ih).$$

After differentiating the (Poisson) formula for $u(z)$ and then plugging in the given estimate on $|u(t)|$, we get (for small $h>0$)

$$|u_y(x+ih)| \;\leqslant\; \frac{\text{const.}}{h(x^2+1)},$$

so

$$|\tilde{u}_h'(t)| \;\leqslant\; \frac{\text{const.}}{t^2+1} \qquad \text{for } t\in\mathbb{R}.$$

As we have noted, $u_h(x)$ is *odd.* Its Hilbert transform $\tilde{u}_h(t)$ is therefore *even*, and the preceding formula for u_h can be written

$$u_h(x) \quad = \quad -\frac{1}{\pi}\int_0^\infty \left(\frac{1}{x-t} \;+\; \frac{1}{x+t}\right)\tilde{u}_h(t)\,\mathrm{d}t.$$

Here, we integrate by parts as in proving the lemma of §C.3, Chapter VIII and the third lemma of the present article. By the last inequality we actually have

$$\int_0^\infty |\tilde{u}_h'(t)|\,\mathrm{d}t \quad < \quad \infty,$$

so, $\tilde{u}_h(t)$ being $\mathscr{C}_\infty$, the partial integration readily yields the formula

$$u_h(x) \quad = \quad \frac{1}{\pi}\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \tilde{u}_h'(t)\,\mathrm{d}t,$$

valid for all real x.

This already exhibits $u_h(x)$ as a Green potential $U_\rho(x)$ with

$$\mathrm{d}\rho(t) \quad = \quad \frac{1}{\pi}\,\tilde{u}_h'(t)\,\mathrm{d}t,$$

and in order to complete this last step of the proof, it is only necessary to check that

$$\int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right| |\tilde{u}_h'(t)|\,|\tilde{u}_h'(x)|\,\mathrm{d}t\,\mathrm{d}x \quad < \quad \infty.$$

That, however, follows in straightforward fashion from the above estimate on $|\tilde{u}_h'(t)|$. Breaking up (for $x > 0$)

$$\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \frac{\mathrm{d}t}{1+t^2}$$

as $\int_0^{2x} + \int_{2x}^\infty$, we have

$$\int_0^{2x} \quad = \quad \int_0^2 \log\left|\frac{1+\tau}{1-\tau}\right| \frac{x\,\mathrm{d}\tau}{1+x^2\tau^2} \quad \leqslant \quad \frac{1}{2}\int_0^2 \log\left|\frac{1+\tau}{1-\tau}\right| \frac{\mathrm{d}\tau}{\tau},$$

a finite constant, whilst

$$\int_{2x}^\infty \quad \leqslant \quad \int_{2x}^\infty O\left(\frac{x}{t}\right)\frac{\mathrm{d}t}{1+t^2} \quad = \quad O\left(x\log\frac{1+x^2}{x^2}\right) \quad = \quad O\left(\frac{1}{1+x}\right).$$

The double integral in question is thus

$$\leqslant \quad \text{const.}\int_0^\infty \left(1 \;+\; \frac{1}{1+x}\right)\frac{\mathrm{d}x}{1+x^2} \quad < \quad \infty,$$

showing that the measure ρ given by the above formula has the required property.

The three steps of our approximation have thus been carried out, and the theorem completely proved.

Remark. The Green potentials U_ρ furnished by this proof and approximating, in norm $\| \quad \|_E$, a given odd function $U(x)$ with $\|U\|_E < \infty$, are formed from signed measures ρ on $[0, \infty)$ having, in addition to the properties enumerated at the beginning of this article, the following special one:

each ρ is absolutely continuous, with $\mathscr{C}_\infty$ density satisfying a relation of the form

$$|\rho'(t)| \leqslant \frac{\text{const.}}{t^2+1}, \qquad t \geqslant 0.$$

The corresponding potentials $U_\rho(x)$ are also $\mathscr{C}_\infty$, and especially,

$$|U_\rho(x)| \leqslant \frac{\text{const.}}{x^2+1}, \qquad x \in \mathbb{R}.$$

This property will be used to advantage in the next article.

Scholium. Does the space $\mathfrak{H}$ actually *consist entirely* of Green potentials

$$U_\mu(x) = \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\mu(t)$$

formed from certain signed Borel measures μ on $(0, \infty)$, perhaps more general than the measures ρ considered in this article?

One easily proves that if the *positive* measures $\mathrm{d}\rho_n$ (without point mass at 0) form a Cauchy sequence in the norm $\sqrt{(E(\ , \ ))}$, *then* at least a *subsequence* of them (and in fact the original sequence) *does converge w* on every compact subinterval* $[a, 1/a]$ of $(0, \infty)$, thus yielding a *positive Borel measure* μ on $(0, \infty)$ (perhaps with $\mu((0,1)) = \infty$ as well as $\mu([1, \infty)) = \infty$). It is thence not hard to show that

$$U_\mu(x) = \lim_{n\to\infty} U_{\rho_n}(x) \qquad \text{a.e., } x \in \mathbb{R},$$

and U_μ may hence be identified with the limit of the U_{ρ_n} in the space $\mathfrak{H}$.

Verification of the w^* convergence statement goes as follows: by the first lemma of this article, the integrals

$$\int_0^1 U_{\rho_n}(x)\,\mathrm{d}x$$

are surely bounded. However,

$$\int_0^1 U_{\rho_n}(x)\,\mathrm{d}x \quad = \quad \int_0^\infty \int_0^1 \log\left|\frac{x+t}{x-t}\right| \mathrm{d}x\,\mathrm{d}\rho_n(t)$$

with

$$\int_0^1 \log\left|\frac{x+t}{x-t}\right| \mathrm{d}x \quad \geqslant \quad 0 \qquad \text{for } t \geqslant 0$$

and clearly *bounded below by a number* > 0 *on any segment of the form* $\{a \leqslant t \leqslant 1/a\}$ with $a > 0$. Therefore, since $\mathrm{d}\rho_n(t) \geqslant 0$ for each n, the quantities

$$\rho_n([a,\ 1/a])$$

must stay bounded as $n \longrightarrow \infty$ for each $a > 0$. The existence of a subsequence of the ρ_n having the stipulated property now follows by the usual application of Helly's selection principle and the Cantor diagonal process.

As soon, however, as the measures $\mathrm{d}\rho_n$ *are allowed to be of variable sign*, the argument just made, and its conclusion as well, *cease to be valid*. There are thus plently of functions U in $\mathfrak{H}$ which are *not* of the form U_μ (unless one accepts to bring in certain *Schwartz distributions* μ). This fact is even familiar from physics: *lots* of functions $U(z)$, harmonic in the upper half plane and continuous up to the real axis, with $U(x)$ odd thereon, *cannot be obtained as logarithmic potentials of charge distributions on* $\mathbb{R}$, *even though they have finite Dirichlet integrals*

$$\iint_{\Im z > 0} \{(U_x(z))^2 \ + \ (U_y(z))^2\}\,\mathrm{d}x\,\mathrm{d}y.$$

Instead, physicists are obliged to resort to what they call a *double-layer distribution* on $\mathbb{R}$ (formed from 'dipoles'); mathematically, this simply amounts to using the Poisson representation

$$U(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^\infty \frac{\Im z}{|z-t|^2}\,U(t)\,\mathrm{d}t$$

in place of the formula

$$U(z) \quad = \quad -\frac{1}{\pi}\int_0^\infty \log\left|\frac{z+t}{z-t}\right| U_y(t+\mathrm{i}0)\,\mathrm{d}t,$$

which is not available unless $\partial U(z)/\partial y$ is sufficiently well behaved for $\Im z \longrightarrow 0$. ($U(x)$ may be *continuous* and the above Dirichlet integral *finite*, and yet the boundary value $U_y(x+\mathrm{i}0)$ *exist almost nowhere* on $\mathbb{R}$. This is most easily seen by first mapping the upper half plane conformally onto the unit disk and then working with lacunary Fourier series.)

Problem 61

Let $V(x)$ be *even and* > 0, with $\|V\|_E < \infty$. Given $U \in \mathfrak{H}$, define a function $U_V(x)$ by putting

$$U_V(x) \quad = \quad \begin{cases} U(x) & \text{if } |U(x)| < V(x), \\ V(x)\operatorname{sgn} U(x) & \text{if } |U(x)| \geqslant V(x); \end{cases}$$

the formation of U_V is illustrated in the following figure:

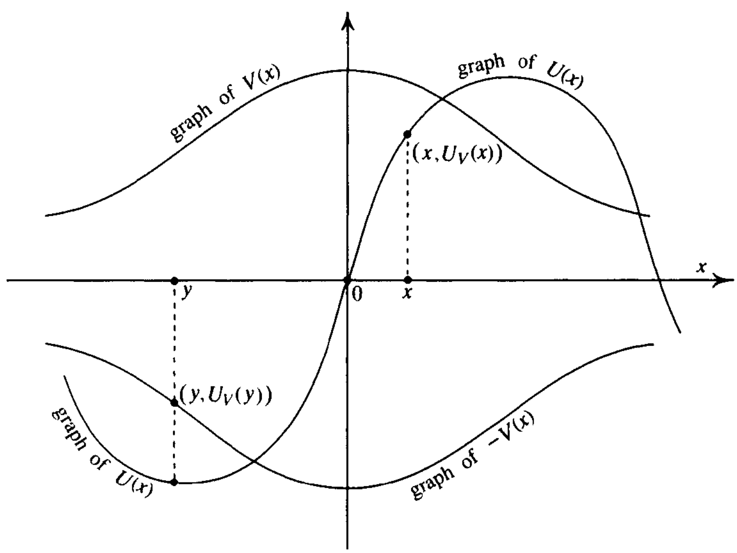

Figure 247

Show that $U_V \in \mathfrak{H}$.
(Hint: Show that

$$|U_V(x) \; - \; U_V(y)| \quad \leqslant \quad \max(|U(x) - U(y)|, \; |V(x) - V(y)|).$$

To check this, it is enough to look at six cases:

(i) $|U(x)| < V(x)$ and $|U(y)| < V(y)$

(ii) $U(x) \geqslant V(x)$ and $U(y) \geqslant V(y)$

(iii) $U(x) \geqslant V(x)$ and $U(y) \leqslant -V(y)$

(iv) $0 \leqslant U(x) < V(x)$ and $U(y) \leqslant -V(y)$

(v) $0 \leqslant U(x) < V(x)$, $V(y) \leqslant U(x)$ and $U(y) \geqslant V(y)$

(vi) $0 \leqslant U(x) < V(x)$, $V(y) > U(x)$ and $U(y) \geqslant V(y)$.)

Problem 62

(Beurling and Malliavin) Let $\omega(x)$ be *even*, $\geqslant 0$, and *uniformly* Lip 1, with.

$$\int_0^\infty \frac{\omega(x)}{x^2}\,dx \quad < \quad \infty.$$

Show that then $\omega(x)/x$ belongs to $\mathfrak{H}$.
(Hint: The function $\omega(x)$ is certainly *continuous*, so, if the integral condition on it is to hold, we must have $\omega(0) = 0$. Thence, by the Lip 1 property, $\omega(x) \leqslant C|x|$, i.e.,

$$\left|\frac{\omega(x)}{x}\right| \leqslant C, \qquad x \in \mathbb{R}.$$

In the circumstances of this problem,

$$2\pi^2 \left\|\frac{\omega(x)}{x}\right\|_E^2 = \int_0^\infty \int_0^\infty \left(\frac{\dfrac{\omega(x)}{x} - \dfrac{\omega(y)}{y}}{x - y} \right)^2 dx\,dy$$

$$+ \int_0^\infty \int_0^\infty \left(\frac{\dfrac{\omega(x)}{x} + \dfrac{\omega(y)}{y}}{x + y} \right)^2 dx\,dy.$$

Using the inequality $(A+B)^2 \leqslant 2(A^2+B^2)$, the second double integral is immediately seen by symmetry to be

$$\leqslant 4\int_0^\infty \int_0^\infty \frac{1}{(x+y)^2}\left(\frac{\omega(x)}{x}\right)^2 dy\,dx \leqslant 4C\int_0^\infty \frac{\omega(x)}{x^2}\,dx.$$

The first double integral, by symmetry, is

$$2\int_0^\infty\int_y^\infty\left(\frac{\dfrac{\omega(x)}{x}-\dfrac{\omega(y)}{y}}{x-y}\right)^2 \mathrm{d}x\,\mathrm{d}y \;\leqslant\; 4\int_0^\infty\int_y^\infty \frac{1}{x^2}\left(\frac{\omega(x)-\omega(y)}{x-y}\right)^2 \mathrm{d}x\,\mathrm{d}y$$

$$+\quad 4\int_0^\infty\int_y^\infty\left(\frac{\dfrac{1}{x}-\dfrac{1}{y}}{x-y}\right)^2(\omega(y))^2\,\mathrm{d}x\,\mathrm{d}y.$$

The *second* of the expressions on the right boils down to

$$4\int_0^\infty\int_y^\infty\left(\frac{\omega(y)}{y}\right)^2\frac{\mathrm{d}x}{x^2}\mathrm{d}y$$

which is handled by reasoning already used; we are thus left with the *first* expression on the right.

That one we break up further as

$$4\int_0^\infty\int_y^{y+\omega(y)} \quad + \quad 4\int_0^\infty\int_{y+\omega(y)}^\infty \quad ;$$

again, the first of these terms is readily estimated, and the second is

$$\leqslant\quad 8\int_0^\infty\int_{y+\omega(y)}^\infty\left\{\frac{(\omega(y))^2}{y^2}\,\frac{1}{(x-y)^2} \;+\; \frac{(\omega(x))^2}{x^2}\,\frac{1}{(x-y)^2}\right\}\mathrm{d}x\,\mathrm{d}y.$$

Integration of the first term in $\{\quad\}$ still does not give any problem, and we only need to deal with

$$8\int_0^\infty\int_{y+\omega(y)}^\infty \frac{(\omega(x))^2}{x^2}\,\frac{1}{(x-y)^2}\,\mathrm{d}x\,\mathrm{d}y.$$

By reversing the order of integration, show that this is

$$\leqslant\quad 8\int_0^\infty \frac{(\omega(x))^2}{x^2}\cdot\frac{1}{\omega(Y(x))}\,\mathrm{d}x,$$

where $Y(x)$ denotes the *largest value* of y for which $y\,+\,\omega(y)\;\leqslant\; x$. Then use the Lip 1 property of ω to get

$$|\omega(x)\;-\;\omega(Y(x))|\quad\leqslant\quad C\omega(Y(x)),$$

whence $1/\omega(Y(x))\;\leqslant\;(C+1)/\omega(x)$. This relation is then substituted into the last integral.)

5. Even weights W with $\|\log W(x)/x\|_E < \infty$

Theorem (Beurling and Malliavin). *Let* $W(x) \geqslant 1$ *be continuous and even, with*

$$\int_0^\infty \frac{\log W(x)}{x^2}\,dx \quad < \quad \infty,$$

and suppose that the odd function $\log W(x)/x$ *belongs to the Hilbert space* $\mathfrak{H}$ *discussed in the preceding article. Then, given any* $A > 0$, *there is an increasing function* $v(t)$, *zero on a neighborhood of the origin, such that*

$$\frac{v(t)}{t} \longrightarrow \frac{A}{\pi} \quad \text{as } t \longrightarrow \infty$$

and

$$\log W(x) \; + \; \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| dv(t) \quad \leqslant \quad \text{const.} \qquad \text{on } \mathbb{R}.$$

Proof. The argument, again based on the procedure explained in article 1, is much like those made in articles 2 and 3. For that reason, certain of its details may be omitted.

In order to have a weight going to ∞ for $x \longrightarrow \pm\infty$, we first take

$$W_\eta(x) \quad = \quad (1 + x^2)^\eta\, W(x)$$

with a small number $\eta > 0$. Then, given a value of $A > 0$, we form the function*

$$F(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^\infty \frac{|\Im z|}{|z-t|^2} \log W_\eta(t)\,dt \;\; - \;\; A|\Im z|.$$

As before, the proof of our theorem boils down to showing that

$$(\mathfrak{M}F)(0) \; < \; \infty.$$

This, in turn, reduces to the *determination of an upper bound, independent of* $\mathscr{D}$, on

$$\int_{\partial\mathscr{D}} \log W_\eta(t)\,d\omega_{\mathscr{D}}(t,\, 0) \quad - \quad A\,Y_{\mathscr{D}}(0),$$

where $\mathscr{D}$ is any domain of the kind studied in §C of Chapter VIII. We

* as usual, we put $F(x)$ equal to $\log W_\eta(x)$ on the real axis

see that what is needed is a comparison of

$$\int_{\partial\mathscr{D}} \log W_\eta(t)\, d\omega_{\mathscr{D}}(t,\ 0)$$

with the quantity $Y_{\mathscr{D}}(0)$.

We have, in the first place, to take care of the factor $(1+x^2)^\eta$ used in forming $W_\eta(x)$. That is easy. Writing, for $t \geqslant 0$,

$$\Omega_{\mathscr{D}}(t) \quad = \quad \omega_{\mathscr{D}}(\partial\mathscr{D} \cap ((-\infty, -t] \cup [t, \infty)),\ 0)$$

as in §C of Chapter VIII, we have

$$\int_{\partial\mathscr{D}} \eta \log(1+t^2)\, d\omega_{\mathscr{D}}(t,\ 0) \quad = \quad -\eta \int_0^\infty \log(1+t^2)\, d\Omega_{\mathscr{D}}(t)$$

$$= \quad \eta \int_0^\infty \frac{2t}{1+t^2}\Omega_{\mathscr{D}}(t)\, dt.$$

By the fundamental result of §C.2, Chapter VIII,

$$\Omega_{\mathscr{D}}(t) \quad \leqslant \quad \frac{Y_{\mathscr{D}}(0)}{t},$$

so the last expression is

$$\leqslant \quad 2\eta Y_{\mathscr{D}}(0) \int_0^\infty \frac{dt}{1+t^2} \quad = \quad \pi\eta Y_{\mathscr{D}}(0).$$

Thence,

$$\int_{\partial\mathscr{D}} \log W_\eta(t)\, d\omega_{\mathscr{D}}(t,\ 0)$$

$$= \quad \int_{\partial\mathscr{D}} \eta \log(1+t^2)\, d\omega_{\mathscr{D}}(t,\ 0) \quad + \quad \int_{\partial\mathscr{D}} \log W(t)\, d\omega_{\mathscr{D}}(t,\ 0)$$

$$\leqslant \quad \pi\eta Y_{\mathscr{D}}(0) \quad + \quad \int_{\partial\mathscr{D}} \log W(t)\, d\omega_{\mathscr{D}}(t,\ 0),$$

and our main work is the estimation of the integral in the last member.

For that purpose, we may as well make full use of the third theorem in the preceding article, having done the work to get it. The reader wishing to avoid use of that theorem will find a similar alternative procedure sketched in problems 63 and 64 below. According to our hypothesis, $\log W(x)/x \in \mathfrak{H}$ so, by the theorem referred to, there is, for any $\eta > 0$, a potential $U_\rho(x)$

of the sort considered in the last article, with

$$\left\| \frac{\log W(x)}{x} - U_\rho(x) \right\|_E < \eta$$

and also (by the remark to that theorem)

$$|U_\rho(x)| \leqslant \frac{K_\eta}{1+x^2} \quad \text{for } x \in \mathbb{R}.$$

Let us now proceed as in proving the theorem of §C.4, Chapter VIII, trying, however, to make use of the difference $(\log W(x)/x) - U_\rho(x)$.

We have

$$\int_{\partial\mathscr{D}} \log W(t)\,\mathrm{d}\omega_{\mathscr{D}}(t,\ 0) = \int_{\partial\mathscr{D}} tU_\rho(t)\,\mathrm{d}\omega_{\mathscr{D}}(t,\ 0)$$
$$+ \int_{\partial\mathscr{D}} (\log W(t) - tU_\rho(t))\,\mathrm{d}\omega_{\mathscr{D}}(t,\ 0).$$

Because $|tU_\rho(t)| \leqslant K_\eta/2$ by the last inequality, and $\omega_{\mathscr{D}}(\ \ ,\ 0)$ is a positive measure of total mass 1, the *first* integral on the right is

$$\leqslant K_\eta/2.$$

In terms of $\Omega_{\mathscr{D}}(t)$, the *second* right-hand integral is

$$-\int_0^\infty (\log W(t) - tU_\rho(t))\,\mathrm{d}\Omega_{\mathscr{D}}(t);$$

to this we now apply the trick used in the proof just mentioned, rewriting the last expression as

$$\int_0^\infty \left(\frac{\log W(t)}{t} - U_\rho(t)\right)\Omega_{\mathscr{D}}(t)\,\mathrm{d}t - \int_0^\infty \left(\frac{\log W(t)}{t} - U_\rho(t)\right)\mathrm{d}(t\Omega_{\mathscr{D}}(t)).$$

The *first* of these terms can be disposed of immediately. Taking a large number L, we break it up as

$$\int_0^L \frac{\log W(t)}{t}\Omega_{\mathscr{D}}(t)\,\mathrm{d}t - \int_0^\infty U_\rho(t)\,\Omega_{\mathscr{D}}(t)\,\mathrm{d}t + \int_L^\infty \frac{\log W(t)}{t}\Omega_{\mathscr{D}}(t)\,\mathrm{d}t.$$

We use the inequality $\Omega_{\mathscr{D}}(t) \leqslant 1$ in the first two integrals, plugging the above estimate on $U_\rho(t)$ into the second one. In the third integral, the relation $\Omega_{\mathscr{D}}(t) \leqslant Y_{\mathscr{D}}(0)/t$ is once again employed. In that way, the sum of these integrals is seen to be

$$\leqslant L\int_0^\infty \frac{\log W(t)}{t^2}\mathrm{d}t + \frac{\pi}{2}K_\eta + Y_{\mathscr{D}}(0)\int_L^\infty \frac{\log W(t)}{t^2}\mathrm{d}t.$$

We come to

$$\int_0^\infty \left(\frac{\log W(t)}{t} - U_\rho(t)\right) \mathrm{d}(t\Omega_{\mathscr{D}}(t)),$$

the *second* of the above terms. According to §C.3 of Chapter VIII, the double integral

$$\int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}(t\Omega_{\mathscr{D}}(t))\ \mathrm{d}(x\Omega_{\mathscr{D}}(x))$$

is absolutely convergent, and its value,

$$E(\mathrm{d}(t\Omega_{\mathscr{D}}(t)),\ \mathrm{d}(t\Omega_{\mathscr{D}}(t))),$$

is

$$\leqslant \quad \pi(Y_{\mathscr{D}}(0))^2.$$

The measure $\mathrm{d}(t\Omega_{\mathscr{D}}(t))$ is *absolutely continuous* (albeit with *unbounded* density!), and acts like const.$(\mathrm{d}t/t^3)$ for large t. Near the origin, $\mathrm{d}(t\Omega_{\mathscr{D}}(t)) = \mathrm{d}t$, for $\Omega_{\mathscr{D}}(t) \equiv 1$ in a neighborhood of that point. These properties together with our given conditions on $W(t)$ and the above estimate for $U_\rho(t)$ ensure absolute convergence of

$$\int_0^\infty ((\log W(t)/t) - U_\rho(t))\, \mathrm{d}(t\Omega_{\mathscr{D}}(t)),$$

which may hence be estimated by the *fifth* lemma of the last article. In that way this integral is found to be in absolute value

$$\leqslant \quad \left\| \frac{\log W(t)}{t} - U_\rho(t) \right\|_E \sqrt{(E(\mathrm{d}(t\Omega_{\mathscr{D}}(t)),\ \mathrm{d}(t\Omega_{\mathscr{D}}(t))))}.$$

Referring to the previous relation, we see that for our choice of $U_\rho(t)$, the quantity just found is

$$\leqslant \quad \sqrt{\pi}\ \eta Y_{\mathscr{D}}(0),$$

and we have our upper bound for the second term in question.

Combining this with the estimate already obtained for the *first* term, we get

$$-\int_0^\infty (\log W(t) - tU_\rho(t))\, \mathrm{d}\Omega_{\mathscr{D}}(t)$$

$$\leqslant \quad L\int_0^\infty \frac{\log W(t)}{t^2}\mathrm{d}t + \frac{\pi}{2}K_\eta + Y_{\mathscr{D}}(0)\int_L^\infty \frac{\log W(t)}{t^2}\mathrm{d}t + \sqrt{\pi}\ \eta Y_{\mathscr{D}}(0),$$

whence, by an earlier computation,

$$\int_{\partial\mathscr{D}} \log W(t)\,\mathrm{d}\omega_{\mathscr{D}}(t,\ 0)$$

$$\leqslant \quad L\int_0^\infty \frac{\log W(t)}{t^2}\mathrm{d}t \;+\; \frac{\pi+1}{2}K_\eta \;+\; \left(\sqrt{\pi}\,\eta \;+\; \int_L^\infty \frac{\log W(t)}{t^2}\mathrm{d}t\right)Y_{\mathscr{D}}(0),$$

and thus finally

$$\int_{\partial\mathscr{D}} \log W_\eta(t)\,\mathrm{d}\omega_{\mathscr{D}}(t,\ 0)$$

$$\leqslant \quad \left((\pi+\sqrt{\pi})\eta \;+\; \int_L^\infty \frac{\log W(t)}{t^2}\mathrm{d}t\right)Y_{\mathscr{D}}(0)$$

$$+ \; L\int_0^\infty \frac{\log W(t)}{t^2}\mathrm{d}t \;+\; \frac{\pi+1}{2}K_\eta .$$

Wishing now to have the initial term on the right outweighed by $-AY_{\mathscr{D}}(0)$ we first, for our given value of $A > 0$, pick

$$\eta \quad \leqslant \quad \frac{A}{2(\pi+\sqrt{\pi})} \quad \text{(say)},$$

and then choose (and fix!) L large enough so as to have

$$\int_L^\infty \frac{\log W(t)}{t^2}\mathrm{d}t \quad \leqslant \quad \frac{A}{2}.$$

For these particular values of η and L, it will follow that

$$\int_{\partial\mathscr{D}} \log W_\eta(t)\,\mathrm{d}\omega_{\mathscr{D}}(t,\ 0) \;-\; AY_{\mathscr{D}}(0) \quad \leqslant \quad L\int_0^\infty \frac{\log W(t)}{t^2}\mathrm{d}t \;+\; \frac{\pi+1}{2}K_\eta$$

for any of the domains $\mathscr{D}$, and hence that

$$(\mathfrak{M}F)(0) \quad \leqslant \quad L\int_0^\infty \frac{\log W(t)}{t^2}\mathrm{d}t \quad + \quad \frac{\pi+1}{2}K_\eta,$$

by the method used in articles 2 and 3. This, however, proves the theorem.

We are done.

Referring now to the corollary of the next-to-the last theorem in §B.1, we immediately obtain the

Corollary. *Let the continuous weight* $W(x) \geqslant 1$ *satisfy the hypothesis of the theorem, and also fulfill the regularity requirement formulated in* §B.1. *Then* W *admits multipliers.*

This result and the one obtained in problem 62 (last article) give us once again a proposition due to Beurling and Malliavin, already deduced from their Theorem on the Multiplier (of article 2) in §C.1, Chapter X. That proposition may be stated in the following form:

Theorem. *Let* $W(x) \geqslant 1$ *be even, with* $\log W(x)$ *uniformly* Lip 1 *on* $\mathbb{R}$, *and*

$$\int_0^\infty \frac{\log W(x)}{x^2}\,\mathrm{d}x \quad < \quad \infty.$$

Then W *admits multipliers.*

It suffices to observe that the regularity requirement of §B.1 is certainly met by weights W with $\log W(x)$ uniformly Lip 1.

Originally, this theorem was essentially derived in such fashion from the preceding one by Beurling and Malliavin.

Problem 63

(a) Let ρ be a positive measure on $[0,\infty)$ without point mass at the origin, such that $E(\mathrm{d}\rho(t), \mathrm{d}\rho(t)) < \infty$. Show that there is a sequence of positive measures σ_n of *compact support* in $(0,\infty)$ with $\mathrm{d}\rho(t) - \mathrm{d}\sigma_n(t) \geqslant 0$, $U_{\sigma_n}(x)$ *bounded* on $\mathbb{R}$ for each n, and $\|U_\rho - U_{\sigma_n}\|_E \xrightarrow[n]{} 0$. (Hint: First argue as in the proof of the *fifth* lemma of the last article to verify that if ρ_n denotes the restriction of ρ to $[1/n, n]$, then $\|U_\rho - U_{\rho_n}\|_E \xrightarrow[n]{} 0$. Then, for each n, take σ_n as the restriction of ρ_n to the closed subset of $[1/n, n]$ on which $U_{\rho_n}(x) \leqslant$ some sufficiently large number M_n.)

(b) Let σ be a positive measure of *compact support* $\subseteq (0,\infty)$ with $\|U_\sigma\|_E < \infty$ and $U_\sigma(x)$ *bounded on* $\mathbb{R}$. Show that, corresponding to any $\varepsilon > 0$, there is a *signed* measure τ on $[0,\infty)$, without point mass at the origin, such that $U_\tau(x)$ is also bounded on $\mathbb{R}$, that $\|U_\sigma - U_\tau\|_E < \varepsilon$, and that $U_\tau(x) = 0$ for all sufficiently large x. (Hint: We have $U_\sigma(x) \longrightarrow 0$ for $x \longrightarrow \infty$. Taking a very large $R > 0$, far beyond the support K of σ, consider the domain $\mathscr{D}_R = \{\Re z > 0\} \sim [R,\infty)$, and the harmonic measure $\omega_R(\ \ , z)$ for $\mathscr{D}_R$. Define an absolutely continuous measure σ_R on $[R,\infty)$ by putting, for $t > R$,

$$\frac{\mathrm{d}\sigma_R(t)}{\mathrm{d}t} \quad = \quad \int_K \frac{\mathrm{d}\omega_R(t,\ \xi)}{\mathrm{d}t}\,\mathrm{d}\sigma(\xi).$$

Show that $U_{\sigma_R}(x) = U_\sigma(x)$ for $x > R$, that U_{σ_R} is bounded on $\mathbb{R}$, and that $\| U_{\sigma_R} \|_E < \varepsilon$ if R is taken large enough. Then put $\tau = \sigma - \sigma_R$. *Note*: Potential theorists say that σ_R has been obtained from σ by *balayage* (sweeping) onto the set $[R, \infty)$.)

(c) Hence show that if ρ is any *signed* measure on $[0, \infty)$ without point mass at 0 making $E(|d\rho(t)|, |d\rho(t)|) < \infty$, there is another such signed measure μ on $[0, \infty)$ with $\| U_\rho - U_\mu \|_E < \varepsilon$, $U_\mu(x)$ *bounded* on $\mathbb{R}$, and $U_\mu(x) = 0$ for all $x > R$, a number depending on ε. (Here, parts (a) and (b) are applied in turn to the *positive* part of ρ and to its *negative* part.)

Problem 64

Prove the first theorem of this article using the result of problem 63. (Hint: Given that $\log W(x)/x \in \mathfrak{H}$, take first a signed measure ρ on $[0, \infty)$ like the one in problem 63(c) such that

$$\|(\log W(x)/x) - U_\rho(x)\|_E < \eta/2,$$

and then a μ, furnished by that problem, with $\| U_\rho - U_\mu \|_E < \eta/2$. Argue as in the proof given above, working with the difference

$$\frac{\log W(x)}{x} - U_\mu(x),$$

and taking the number L figuring there to be *larger* than the R obtained in problem 63 (c).)

D. Search for the presumed essential condition

At the beginning of §B.1, it was proposed to limit a good part of the considerations of this chapter to weights $W(x) \geqslant 1$ satisfying a mild local regularity requirement:

There are three constants C, α and $L > 0$ (depending on W) such that, for each real x, one has an interval J_x of length L containing x with

$$W(t) \geqslant C(W(x))^\alpha \qquad \text{for } t \in J_x.$$

That restriction was accepted because, while leaving us with room enough to accommodate many of the weights arising in different circumstances, it serves, we believe, to rule out accidental and, so to say, *trivial* irregularities in a weight's behaviour that could spoil the existence of multipliers which might otherwise be forthcoming. Admittance of multipliers by a weight W was thought to be *really* governed *by some other condition* on its behaviour – an 'essential' one, probably not of strictly local character –

acting in conjunction with the growth requirement

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2} dx \quad < \quad \infty.$$

In adopting this belief, we of course made a tacit asumption that another condition regarding the weight (besides convergence of its logarithmic integral) *is in fact involved.* Up to now, however, we have not seen any reasons why that should be the case. *It is still quite conceivable that the integral condition and the local regularity requirement are, by themselves, sufficient to guarantee admittance of multipliers.*

Such a conclusion would be most satisfying, and indeed make a fitting end to this book. If its truth seemed likely, we would have to abandon our present viewpoint and think instead of looking for a proof. We have arrived at the place where one must decide which path to take.

It is for that purpose that the example given in the first article was constructed. This shows that an additional condition on our weights – what we are thinking of as the 'essential' one – *is really needed.* Our aim during the succeeding articles of this § will then be to find out *what that condition is* or at least arrive at some partial knowledge of it.

In working towards that goal, we will be led to the construction of a *second* example, actually quite similar to the one of the first article, but yielding a weight that *admits* multipliers although the weight furnished by the latter *does not.* Comparison of the two examples will enable us to form an idea of what the 'essential' condition on weights must look like, and, eventually, lead us to the *necessary and sufficient conditions for admittance of multipliers* (on weights meeting the local regularity requirement) formulated in the theorem of §E.

Before proceeding to the first example, it is worthwhile to see what the *absence* of an additional condition on our weights *would have* entailed. The local regularity requirement quoted at the beginning of this discussion is certainly satisfied by weights $W(x) \geqslant 1$ with

$$|\log^+ \log W(x) \; - \; \log^+ \log W(x')| \quad \leqslant \quad \text{const.}\, |x - x'|$$

on $\mathbb{R}$. Absence of an additional condition would therefore make

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2} dx \quad < \quad \infty$$

necessary and sufficient for the admittance of multipliers by such W. This