and these lead to the proof of the remaining half of the Franel–Landau theorem.

Consider first the evaluation of I using $G(u) = \sum_{\nu=1}^{A} \bar{B}_1(u + r_\nu)$. This equation shows that G jumps downward by 1 at each Farey fraction and increases like Au between Farey fractions. Moreover, since the r_ν other than $r_A = 1$ are symmetrically distributed around $\frac{1}{2}$, $\sum_{\nu=1}^{A-1} \bar{B}_1(r_\nu) = 0$, so the right-hand limit $G(0^+) = \lim_{u\downarrow 0} G(u) = \lim_{u\downarrow 0} \bar{B}(u+1)$ is $-\frac{1}{2}$. Thus between r_ν and $r_{\nu+1}$ the value of G is given by the formula $G(u) = -\frac{1}{2} + Au - \nu$. Hence

$$I = \sum_{\nu=1}^{A} \int_{r_{\nu-1}}^{r_\nu} \left(-\frac{1}{2} + Au - \nu + 1\right)^2 du$$
$$= \sum_{\nu=1}^{A} \frac{1}{A} \frac{(Au - \nu + \frac{1}{2})^3}{3}\Bigg|_{r_{\nu-1}}^{r_\nu}$$

provided that r_0 is defined to be 0. Since $Ar_\nu = A[r_\nu - (\nu/A) + (\nu/A)] = A\delta_\nu + \nu$, this gives

$$I = \frac{1}{3A} \sum_{\nu=1}^{A} \left[\left(A\delta_\nu + \frac{1}{2}\right)^3 - \left(A\delta_{\nu-1} - \frac{1}{2}\right)^3\right]$$
$$= \frac{1}{3A} \sum_{\nu=1}^{A} \left[\left(A\delta_\nu + \frac{1}{2}\right)^3 - \left(A\delta_\nu - \frac{1}{2}\right)^3\right]$$

(using $A\delta_0 - \frac{1}{2} = -\frac{1}{2} = A\delta_A - \frac{1}{2}$)

$$= \frac{1}{3A} \sum_{\nu=1}^{A} \left[2\cdot 3(A\delta_\nu)^2 \cdot \frac{1}{2} + 2\left(\frac{1}{2}\right)^3\right]$$
$$= A \sum_{\nu=1}^{A} \delta_\nu^2 + \frac{1}{12}$$

as an exact formula for I in terms of the δ_ν.

Now consider the evaluation of I using $G(u) = \sum_{k=1}^{\infty} \bar{B}_1(ku)M(x/k)$. Since the sum is finite, this gives immediately

$$I = \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} M\left(\frac{x}{a}\right)M\left(\frac{x}{b}\right) \int_0^1 \bar{B}_1(au)\bar{B}_1(bu)\, du.$$

The coefficients $I_{ab} = \int_0^1 \bar{B}_1(au)\bar{B}_1(bu)\, du$ of this double series can be evaluated explicitly as follows. If $b = 1$, then it is

$$\int_0^a \bar{B}_1(v)\bar{B}_1\left(\frac{v}{a}\right)a^{-1}\, dv = a^{-1} \sum_{k=0}^{a-1} \int_0^1 \bar{B}_1(k+t)\bar{B}_1\left(\frac{k}{a} + \frac{t}{a}\right) dt$$
$$= a^{-1} \int_0^1 \bar{B}_1(t)\bar{B}_1\left(a \cdot \frac{t}{a}\right) dt$$

[by the periodicity of $\bar{B}_1$ and by (2)]

$$= a^{-1} \int_0^1 \left(t - \frac{1}{2}\right)^2 dt = (12a)^{-1}.$$

If b is relatively prime to a, then the same sequence of steps shows, since (bk/a) mod 1 for $k = 0, 1, \ldots, a-1$ gives each fraction j/a exactly once, that

$$\begin{aligned} I_{ab} &= a^{-1} \int_0^1 \bar{B}_1(t)\bar{B}_1\left(a \cdot \frac{bt}{a}\right) dt \\ &= a^{-1} \int_0^1 B_1(t)\bar{B}_1(bt)\, dt \\ &= a^{-1} I_{b1} = (12ab)^{-1}. \end{aligned}$$

Finally, if $c = (a, b)$ is the greatest common divisor of a and b, then $a = c\alpha$ and $b = c\beta$, where α and β are relatively prime and

$$\begin{aligned} I_{ab} &= \int_0^1 \bar{B}_1(c\alpha u)\bar{B}_1(c\beta u)\, du \\ &= c^{-1} \int_0^c \bar{B}_1(\alpha t)\bar{B}_1(\beta t)\, dt \\ &= I_{\alpha\beta} = (12\alpha\beta)^{-1} = c^2/12ab. \end{aligned}$$

Thus the final formula is

$$I = \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} M\left(\frac{x}{a}\right) M\left(\frac{x}{b}\right) \frac{c^2}{12ab},$$

where $c = (a, b)$ is the greatest common divisor of a and b.

Now if the Riemann hypothesis is true, then for every $\varepsilon > 0$ there is a C such that $M(x) < Cx^{(1/2)+\varepsilon}$ for all x. Hence the Riemann hypothesis implies

$$\begin{aligned} I &< \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} C^2 \left(\frac{x}{a}\right)^{1/2+\varepsilon} \left(\frac{x}{b}\right)^{1/2+\varepsilon} \frac{c^2}{12ab} \\ &= x^{1+2\varepsilon} \frac{C^2}{12} \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \frac{c^2}{(\alpha c \beta c)^{(3/2)+\varepsilon}} \\ &< x^{1+2\varepsilon} K \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\infty} \sum_{c=1}^{\infty} \frac{1}{\alpha^{3/2}\beta^{3/2}c^{1+2\varepsilon}} \end{aligned}$$

(replacing a sum over relatively prime α, β with a sum over all α, β) which shows that I is less than a constant times $x^{1+2\varepsilon}$ for all $\varepsilon > 0$. The other expression for I shows then that for every $\varepsilon > 0$ the Riemann hypothesis implies

$$A \sum_{\nu=1}^{A} \delta_\nu^2 < Kx^{1+2\varepsilon},$$

where K is a constant depending on ε. But then by the Schwarz inequality

$$\begin{aligned} \sum |\delta_\nu| &= |\sum (\pm 1)\delta_\nu| \le [\sum (\pm 1)^2]^{1/2} [\sum (\delta_\nu)^2]^{1/2} \\ &= (A \sum \delta_\nu^2)^{1/2} < K^{1/2} x^{(1/2)+\varepsilon} \end{aligned}$$

as was to be shown.

12.3 DENJOY'S PROBABILISTIC INTERPRETATION OF THE RIEMANN HYPOTHESIS

One of the things which makes the Riemann hypothesis so difficult is the fact that there is no plausibility argument, no hint of a reason, however unrigorous, why it should be true. This fact gives some importance to Denjoy's probabilistic interpretation of the Riemann hypothesis which, though it is quite absurd when considered carefully, gives a fleeting glimmer of plausibility to the Riemann hypothesis.

Suppose an unbiased coin is flipped a large number of times, say N times. By the de Moivre–Laplace limit theorem the probability that the number of heads deviates by less than $KN^{1/2}$ from the expected number of $\frac{1}{2}N$ is nearly equal to $\int_{-(2K^2/\pi)^{1/2}}^{(2K^2/\pi)^{1/2}} \exp(-\pi x^2)\,dx$ in the sense that the limit of these probabilities as $N \longrightarrow \infty$ is equal to this integral. Thus if the total number of heads is subtracted from the total number of tails, the probability that the resulting number is less than $2KN^{1/2}$ in absolute value is nearly equal to $2\int_0^{(2K^2/\pi)^{1/2}} \exp(-\pi x^2)\,dx$, and therefore the probability that it is less than $N^{(1/2)+\varepsilon}$ for some fixed $\varepsilon > 0$ is nearly $2\int_0^{N^\varepsilon(2\pi)^{1/2}} \exp(-\pi x^2)\,dx$. The fact that this approaches 1 as $N \longrightarrow \infty$ can be regarded as saying that *with probability one the number of heads minus the number of tails grows less rapidly than* $N^{(1/2)+\varepsilon}$.

Consider now a very large square-free integer n, that is, a very large integer n with $\mu(n) \neq 0$. Then $\mu(n) = \pm 1$. It is perhaps plausible to say that $\mu(n)$ is plus or minus one "with equal probability" because n will normally have a large number of factors (the density of primes $1/\log x$ approaches zero) and there seems to be no reason why either an even or an odd number of factors would be more likely. Moreover, by the same principle it is perhaps plausible to say that successive evaluations of $\mu(n) = \pm 1$ are "independent" since knowing the value of $\mu(n)$ for one n would not seem to give any† information about its values for other values of n. But then the evaluation of $M(x)$ would be like flipping a coin once for each square-free integer less than x and subtracting the number of heads from the number of tails. It was shown above that for any given $\varepsilon > 0$ the outcome of this experiment for a large‡ number of flips is, with probability nearly one, less than the number of flips raised to the power $\frac{1}{2} + \varepsilon$ and *a fortiori* less than $x^{(1/2)+\varepsilon}$. Thus these probabi-

†An exception to this statement is that for any prime p, $\mu(pn)$ is either $-\mu(n)$ or zero. However, this principle can only be applied once for any p because $\mu(p^2n) = 0$ and this "information" really says little more than that μ is determined by a formula and is not, in fact, a random phenomenon.

‡The number of flips goes to infinity as $x \longrightarrow \infty$ because, among other reasons, there are infinitely many primes, hence *a fortiori* infinitely many square-free integers (products of distinct primes).

listic assumptions about the values of $\mu(n)$ lead to the conclusion, ludicrous as it seems, that $M(x) = O(x^{(1/2)+\varepsilon})$ with probability one and hence that the Riemann hypothesis is true with probability one!

12.4 AN INTERESTING FALSE CONJECTURE

Riemann says in his memoir on $\pi(x)$ that "the known approximation $\pi(x) \sim \mathrm{Li}(x)$ is correct only up to terms of the order $x^{1/2}$ and gives a value which is slightly too large." It appears from the context that he means that the *average* value of $\pi(x)$ is less than $\mathrm{Li}(x)$ because he ignores the "periodic" terms $\mathrm{Li}(x^\rho)$ in the formula for $J(x)$, but in the tables in Sections 1.1 and 1.17 it will be noticed that even the *actual* value of $\pi(x)$ is markedly less than $\mathrm{Li}(x)$ in all cases considered. This makes it natural to ask whether it is indeed true that $\pi(x) < \mathrm{Li}(x)$. This conjecture is supported by all the numerical evidence and by Riemann's observation that the next largest term in the formula for $\pi(x)$ is the negative term $-\frac{1}{2}\,\mathrm{Li}(x^{1/2})$. Nonetheless it has been shown by Littlewood [L13] that this conjecture is *false* and that there exist numbers x for which $\pi(x) > \mathrm{Li}(x)$. In fact Littlewood showed that it is false to such an extent that for very $\varepsilon > 0$ there exist values of x such that $\pi(x) > \mathrm{Li}(x) + x^{(1/2)-\varepsilon}$.

This example shows the danger of basing conjectures on numerical evidence, even such seemingly overwhelming evidence as Lehmer's computations of $\pi(x)$ up to ten million. As a matter of fact (see Lehman [L6]) no actual value of x is known for which $\pi(x) > \mathrm{Li}(x)$, although Littlewood's proof can be used to produce a very large X with the property that some x less than X has this property, which reduces the problem of finding such an x to a finite problem. More importantly, though, this example shows the danger of assuming that relatively small oscillatory terms can be neglected on the assumption that they probably will not reinforce each other enough to overwhelm a larger principal term. In the light of these observations, the evidence for the Riemann hypothesis provided by the computations of Rosser *et al.* and by the empirical verification of Gram's law loses all its force.

12.5 TRANSFORMS WITH ZEROS ON THE LINE

The problem of the Riemann hypothesis motivated a great deal of study of the circumstances under which an invariant operator on R^+ has a transform with zeros on $\mathrm{Re}\, s = \frac{1}{2}$, that is, under which an integral of the form $\int_0^\infty x^{-s} F(x)\, dx$ has all of its zeros on $\mathrm{Re}\, s = \frac{1}{2}$.

A very general theorem on this subject was proved by Polya [P1] in 1918. Stated in the terminology of Chapter 10, Polya's theorem is that *a real self-adjoint operator of the form* $f(x) \mapsto \int_{1/a}^{a} f(ux)F(u)\,du$ [where $F(u)$ is real and satisfies $u^{-1}F(u^{-1}) = F(u)$] *which has the property that*† $u^{1/2}F(u)$ *is nondecreasing on the interval* $[1, a]$ *has the property that the zeros of its transform all lie on the line* $\operatorname{Re} s = \frac{1}{2}$.

Simple and elegant though this theorem is, it gives little promise of leading to a proof of the Riemann hypothesis because the function $H(x)$ which occurs in the formula $2\xi(s) = \int_0^\infty u^{-s}H(u)\,du$ very definitely does not have the property that $u^{1/2}H(u)$ is nondecreasing and, in fact, as obviously must be the case if a positive function $H(u)$ is to have a transform $\int_0^\infty u^{-s}H(u)\,du$ defined for all s, the decrease of $H(u)$ for large u is very strong.

In 1927 Polya published [P2] a very different sort of theorem on the same general subject. This theorem states that *if* ϕ *is a polynomial which has all its roots on the imaginary axis, or if* ϕ *is an entire function which can be written in a suitable way as a limit of such polynomials, then if* $\int_0^\infty u^{-s} F(u)\,du$ *has all its zeros on* $\operatorname{Re} s = \frac{1}{2}$, *so does* $\int_0^\infty u^{-s}F(u)\phi(\log u)\,du$. Here the conditions on F can be quite weak; it will suffice to consider the case $F(u) = o(\exp(-|\log u|^{2+\delta}))$ in which, in particular, $F(u)$ goes to zero much more rapidly than any power of u as $u \to 0$ or $u \to \infty$. Polya also proved that, conversely, if ϕ is an entire function of genus 0 or 1 which preserves in this way the property of a transform's having zeros on the line $\operatorname{Re} s = \frac{1}{2}$, then ϕ must be a polynomial with purely imaginary roots or a limit of such polynomials.

The idea of the proof of this theorem is roughly as follows. If $P(t)$ is a polynomial with distinct real roots, then so is $rP(t) - P'(t)$ for any real number r; if $r = 0$, this follows from the fact that there is a zero of $P'(t)$ between any two consecutive zeros of $P(t)$ [thus accounting for all the zeros of $P'(t)$], and if $r \neq 0$, it follows from the fact that $rP(t) - P'(t)$ changes sign on each of the intervals (two of them half infinite) into which the real line is divided by the zeros of $P'(t)$ [thus accounting for all the zeros of $rP(t) - P'(t)$]. The change of variable $s = \frac{1}{2} + it$ then shows that if $P(s)$ is a polynomial with distinct roots all of which lie on $\operatorname{Re} s = \frac{1}{2}$, then the same is true of $-irP(s) - P'(s)$ for any real r. In other words, the operator $-d/ds - ir$ preserves the property of a polynomial's having distinct roots all of which lie on $\operatorname{Re} s = \frac{1}{2}$. Thus if $\int_0^\infty uF(u)^{-s}\,du$ has all its roots on $\operatorname{Re} s = \frac{1}{2}$, and if it is a nice entire function which can be written in a suitable way as a limit of polynomials,

†The unnatural-seeming factor $u^{1/2}$ can be eliminated by renormalizing so that $\int_0^\infty x^{-s}F(x)\,dx$ is written $\int_0^\infty x^{(1/2)-s}[x^{1/2}F(x)]\,d\log x = \int_0^\infty x^{-z}\tilde{F}(x)\,d\log x$. Then the self-adjointness condition is simply $\tilde{F}(x) = \tilde{F}(x^{-1})$, Polya's condition is that $\tilde{F}$ be nondecreasing on $[1, a]$, and the conclusion of the theorem is that the zeros lie on $\operatorname{Im} z = 0$.

then it is reasonable to expect that

$$-ir \int_0^\infty u^{-s}F(u)\,du - \frac{d}{ds}\int_0^\infty u^{-s}F(u)\,du$$
$$= \int_0^\infty u^{-s}(\log u - ir)F(u)\,du$$

has the same property. Iterating this statement then gives Polya's theorem for any polynomial ϕ with imaginary roots and hence, on passage to the limit, for any suitable limit of such polynomials. For the actual proof see Polya [P2].

In order to apply Polya's theorem to obtain integrals of the form $\int_0^\infty u^{-s} F(u)\,du$ with zeros on the line $\operatorname{Re} s = \frac{1}{2}$, it is necessary to begin with such an integral. From the theory of Bessel functions it was known that $\int_0^\infty u^{-s}u^{-1/2} \exp(-\pi u^2 - \pi u^{-2})\,du$ is such an integral. Polya proved this directly, without reference to the theory of Bessel functions, as follows. For fixed s let

$$w(a) = \int_0^\infty u^{-s}u^{-1/2}e^{-au^2-au^{-2}}\,du.$$

This can be regarded as a deformation of the given integral $w(\pi)$ to $0 = w(\infty)$. It satisfies a second-order linear differential equation, as can be seen by applying $(a\,d/da)^2$ to find

$$\left(a\frac{d}{da}\right)^2 w(a) = \int_0^\infty u^{-s-(1/2)}\left(a\frac{d}{da}\right)(-au^2 - au^{-2})e^{-au^2-au^{-2}}\,du$$
$$= \int_0^\infty u^{-s-(1/2)}[(-au^2 - au^{-2}) + (au^2 + au^{-2})^2]e^{-au^2-au^{-2}}\,du.$$

The second part of this integrand is similar to $(u\,d/du)^2$ applied to $\exp(-au^2 - au^{-2})$ which is

$$\left(u\frac{d}{du}\right)(-2au^2 + 2au^{-2})e^{-au^2-au^{-2}}$$
$$= [(-4au^2 - 4au^{-2}) + 4(-au^2 + au^{-2})^2]e^{-au^2-au^{-2}}$$
$$= 4[(-au^2 - au^{-2}) + (au^2 + au^{-2})^2 - 4a^2]e^{-au^2-au^{-2}}.$$

Hence

$$\int_0^\infty u^{-s-1/2}\left[\left(u\frac{d}{du}\right)^2 e^{-au^2-au^{-2}}\right]du = 4\left(a\frac{d}{da}\right)^2 w(a) - 16a^2w(a).$$

Integration by parts on the left then gives, since the adjoint of $u\,d/du$ is $-(d/du)u$ which carries $u^{-s-(1/2)}$ to $(s - \frac{1}{2})u^{-s-(1/2)}$,

$$[(s - \tfrac{1}{2})^2 + 16a^2]w(a) = 4\left(a\frac{d}{da}\right)^2 w(a)$$

which is the desired differential equation satisfied by $w(a)$. Let $W(a) =$

$a\,d/da\; w(a)$. Then

$$a\frac{d}{da}[W\bar{w}] = \tfrac{1}{4}[(s-\tfrac{1}{2})^2 + 16a^2]w(a)\bar{w}(a) + W\bar{W}.$$

Divide by a and integrate both sides from π to ∞ to find

$$-W(\pi)\bar{w}(\pi) = \int_\pi^\infty \frac{1}{a}\left\{\left[\frac{1}{4}\left(s-\frac{1}{2}\right)^2 + 4a^2\right]|w|^2 + |W|^2\right\} da$$

$$-\operatorname{Im} W(\pi)\bar{w}(\pi) = \frac{1}{4}2xy\int_\pi^\infty \frac{1}{a}|w|^2\, da,$$

where $s - \frac{1}{2} = x + iy$. If $w(\pi) = 0$, then, because the integral cannot be zero, either x or y must be zero. But if $y = 0$, then u^{-s} is positive real; hence $w(\pi) \neq 0$ directly from its definition. Thus $w(\pi) = 0$ implies $x = 0$, $\operatorname{Re} s = \frac{1}{2}$, as was to be shown.

Thus $\int_0^\infty u^{-s}\phi(\log u)u^{-1/2}\exp(-\pi u^2 - \pi u^{-2})\,du$ has its zeros on the line $\operatorname{Re} s = \frac{1}{2}$ whenever ϕ is as above. Although there seems to be no way to use this fact to prove the Riemann hypothesis, Polya used it to prove that a certain "approximation" to $\xi(s)$ does have its zeros on the line $\operatorname{Re} s = \frac{1}{2}$. In the formula

$$2\xi(s) = \int_0^\infty u^{-s}\frac{d}{du}u^2\frac{d}{du}\sum_1^\infty e^{-\pi n^2u^2}\,du$$

the term with $n = 1$ predominates for u large. This term is

$$\int_0^\infty u^{-s}(4\pi^2u^4 - 6\pi u^2)e^{-\pi u^2}\,du.$$

If this is replaced by

$$2\xi^{**}(s) = \int_0^\infty u^{-s}[4\pi^2(u^4 + u^{-5}) - 6\pi(u^2 + u^{-3})]e^{-\pi u^2 - \pi u^{-2}}\,du,$$

the approximation is still good for large u, and since the integral is now the transform of a self-adjoint operator, the approximation must also be good for u near 0. Thus ξ^{**} is in some sense "like" ξ. However, ξ^{**} *does* have its zeros on the line $\operatorname{Re} s = \frac{1}{2}$, a fact which follows from the above theorems once it is shown that ϕ defined by

$$\phi(\log u) = 4\pi^2(u^{9/2} + u^{-9/2}) - 6\pi(u^{5/2} + u^{-5/2}),$$

that is,

$$\phi(z) = 8\pi^2\cosh(9z/2) - 12\pi\cosh(5z/2)$$

can be written as a suitable limit of polynomials with imaginary zeros. By making appeal to the theory of entire functions, the proof of this statement can be reduced to the statement that ϕ itself has all its zeros on the imaginary axis. This can be done as follows.

Consider the function $P(y) = 8\pi^2y^9 + 8\pi^2y^{-9} - 12\pi y^5 - 12\pi y^{-5}$. This function has precisely 18 nonzero roots in the complex y-plane, and, since $\phi(z) = P(e^{z/2})$, in order to prove that the zeros of ϕ are all pure imaginary it will suffice to prove that these 18 roots all lie on the circle $|y| = 1$. Now on the unit circle $\bar{y} = y^{-1}$, so $P(y) = 2\,\mathrm{Re}\{Q(y)\}$ where $Q(y) = 8\pi^2y^9 - 12\pi y^5$. Since $8\pi^2 > 12\pi$, all 9 roots of Q lie inside the unit circle. Therefore the integral of the logarithmic derivative of Q around the unit circle is $18\pi i$, that is, $Q(y) = re^{i\theta}$, where θ increases by 18π as y goes once around the unit circle. But since $P(y) = \mathrm{Re}\, Q(y) = r\cos\theta$, this implies $P(y)$ has 18 zeros on the circle $|y| = 1$ and accounts for all the zeros of P.

12.6 ALTERNATIVE PROOF OF THE INTEGRAL FORMULA

An interesting alternative proof of the Riemann–Siegel integral formula

$$\frac{2\xi(s)}{s(s-1)} = F(s) + \overline{F(1-\bar{s})}, \tag{1}$$

$$F(s) = \pi^{-s/2}\Pi\left(\frac{s}{2} - 1\right)\int_{0\searrow 1} \frac{e^{-i\pi x^2}x^{-s}\,dx}{e^{i\pi x} - e^{-i\pi x}} \tag{2}$$

(see Section 7.9) was given by Kuzmin [K3] in 1934. Kuzmin's proof is altogether different from the proof given in Section 7.9, and it shows an interesting connection between formula (1)—which can be regarded as Riemann's third proof of the functional equation of ξ—and Riemann's second proof of the functional equation (see Section 1.7). What follows is a simplified proof of (1) based on Kuzmin's. It depends on the functional equation $G(x) = (1/x)$ $G(1/x)$ (see Section 10.4) but not on the definite integral formula (5) of Section 7.4 which is the basis of Riemann's proof of (1).

Let $G(x) = \sum_{n=-\infty}^{\infty} \exp(-\pi n^2x^2)$ as before. Then the formula

$$\frac{2\xi(s)}{s(s-1)} = \int_0^\infty u^{s-1}[G(u) - 1]\,du \qquad (\mathrm{Re}\, s > 1), \tag{3}$$

which is easily proved by using absolute convergence to justify interchange of summation and integration,

$$\begin{aligned}
\int_0^\infty u^{s-1}\left[2\sum_{n=1}^\infty e^{-\pi n^2u^2}\right]du &= 2\sum_{n=1}^\infty \int_0^\infty e^{-\pi n^2u^2}\, u^s\, d\log u \\
&= \sum_{n=1}^\infty \int_0^\infty e^{-v}\left(\frac{v}{\pi n^2}\right)^{s/2} d\log v \\
&= \pi^{-s/2}\sum_{n=1}^\infty \frac{1}{n^s}\int_0^\infty e^{-v}\, v^{(s/2)-1}\,dv \\
&= \pi^{-s/2}\Pi\left(\frac{s}{2} - 1\right)\zeta(s) \\
&= \frac{\pi^{-s/2}\Pi(s/2)(s-1)\zeta(s)}{(s/2)(s-1)} = \frac{2\xi(s)}{s(s-1)},
\end{aligned}$$

is essentially the formula (1) of Section 1.7 on which Riemann bases his second proof of the functional equation. He breaks the integral at $u = 1$ and obtains what amounts to

$$\begin{aligned}
\frac{2\xi(s)}{s(s-1)} &= \int_0^1 u^{s-1}[G(u) - 1]\,du + \int_1^\infty u^{s-1}[G(u) - 1]\,du \\
&= \int_1^\infty v^{-s}\left[\frac{1}{v}G\left(\frac{1}{v}\right) - \frac{1}{v}\right]dv + \int_1^\infty u^{s-1}[G(u) - 1]\,du \\
&= \int_1^\infty (v^{-s} + v^{s-1})[G(v) - 1]\,dv + \int_1^\infty v^{-s}\left[1 - \frac{1}{v}\right]dv \\
&= \int_1^\infty (v^{-s} + v^{s-1})[G(v) - 1]\,dv + \frac{1}{s-1} - \frac{1}{s}
\end{aligned}$$

at first for $\operatorname{Re} s > 1$ but then by analytic continuation for all s. If the integral (3) is broken at $u = b$ instead of $u = 1$, the same sequence of steps gives

$$\begin{aligned}
\frac{2\xi(s)}{s(s-1)} &= \int_{b^{-1}}^\infty v^{-s}\left[\frac{1}{v}G\left(\frac{1}{v}\right) - \frac{1}{v}\right]dv + \int_b^\infty u^{s-1}[G(u) - 1]\,du \\
&= \int_{b^{-1}}^\infty v^{-s}[G(v) - 1]\,dv + \frac{b^{s-1}}{s-1} - \frac{b^s}{s} \\
&\quad + \int_b^\infty u^{s-1}[G(u) - 1]\,du.
\end{aligned}$$

Let $F_b(s)$ be the function which is defined by

$$F_b(s) = \int_b^\infty u^{s-1}G(u)\,du$$

for $\operatorname{Re} s < 0$ and therefore by

$$F_b(s) = \int_b^\infty u^{s-1}[G(u) - 1]\,du - \frac{b^s}{s}$$

for all $s \neq 0$. Then the above formula is simply

$$\frac{2\xi(s)}{s(s-1)} = F_b(s) + F_{b^{-1}}(1 - s) \tag{4}$$

and Riemann's second proof of the functional equation is simply the case $b = b^{-1} = 1$ of this formula. But $F_b(s)$ is defined not only for positive real b but for all values of b in the wedge $\{|\operatorname{Im}\log b| < \pi/4\}$ where G is defined; for example, the integral from b to ∞ in the definition of $F_b(s)$ can be taken to be the integral over the half-line $\{b + t\colon t \text{ positive real}\}$ parallel to the real axis. The complex conjugate of $F_b(s)$ is $F_{\bar{b}}(\bar{s})$, so formula (4) has the same form as (1) whenever $b^{-1} = \bar{b}$; that is,

$$\frac{2\xi(s)}{s(s-1)} = F_b(s) + \overline{F_b(1 - \bar{s})} \qquad (|b| = 1) \tag{5}$$

whenever b lies on the unit circle between $(-i)^{1/2}$ and $i^{1/2}$. Kuzmin proves the

Riemann–Siegel formula (1) by proving it is the limiting case of this formula (5) as $b \longrightarrow (-i)^{1/2}$; that is,

$$\lim_{b \to (-i)^{1/2}} F_b(s) = F(s) \qquad (s \neq 0)$$

when b approaches $(-i)^{1/2}$ along the unit circle. This clearly suffices to prove (1).

Kuzmin had already† studied formula (5) in 1930, prior to the publication of the Riemann–Siegel formula, and he had already shown that the limiting case could be written in the form

$$\begin{aligned}(6) \qquad \lim_{b \to (-i)^{1/2}} F_b(s) &= \int_{(-i)^{1/2}}^{\infty} u^{s-1}[G(u) - 1]\, du - \frac{(-i)^{s/2}}{s} \\ &= 2 \int_{(-i)^{1/2}}^{\infty} u^{s-1} \sum_{n=1}^{\infty} e^{-\pi n^2 u^2}\, du - \frac{(-i)^{s/2}}{s} \\ &= 2 \sum_{n=1}^{\infty} \int_{(-i)^{1/2}}^{\infty} u^{s-1} e^{-\pi n^2 u^2}\, du - \frac{(-i)^{s/2}}{s}\end{aligned}$$

for $s \neq 0$. These manipulations will be justified below. If $\operatorname{Re} s < 0$, then the final formula can be written more simply as

$$(7) \qquad \lim_{b \to (-i)^{1/2}} F_b(s) = \sum_{n=-\infty}^{\infty} \int_{(-i)^{1/2}}^{\infty} u^{s-1} e^{-\pi n^2 u^2}\, du.$$

On the other hand, $F(s)$ can also be written as a sum over all integers by using the elementary‡ formula

$$\frac{1}{e^{i\pi x} - e^{-i\pi x}} = \frac{1}{2\pi i} \sum_{-\infty}^{\infty} \frac{(-1)^n x}{x^2 - n^2}$$

†It is interesting to note that this work [K2] of Kuzmin's, which preceded the publication of the Riemann–Siegel formula, was motivated by the wish to be able to compute $\zeta(\frac{1}{2} + it)$ for large t, as was Riemann's. With this and with the Hardy–Littlewood approximate functional equation, one has the feeling that after 70 years other mathematicians were getting up to where Riemann had been. However, it still seems rather doubtful that the Riemann–Siegel asymptotic formula would have been found to this day had it not been found by Riemann.

‡One way to prove this formula is to expand $f(t) = e^{2\pi i x t}$ as a Fourier series on the interval $\{-\frac{1}{2} \leq t \leq \frac{1}{2}\}$ to find $f(t) = \sum_{-\infty}^{\infty} a_n e^{2\pi i n t}$, where

$$a_n = \frac{(-1)^n(e^{i\pi x} - e^{-i\pi x})}{2\pi i(x - n)}$$

from which, with $t = 0$,

$$\frac{1}{e^{i\pi x} - e^{-i\pi x}} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{2\pi i(x - n)} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n x}{2\pi i(x^2 - n^2)}$$

as desired; this holds for real nonintegral x and hence for all nonintegral x by analytic continuation. Another way to prove it is to note that the two sides have the same poles and to then use a method like that of Chapter 2.

in the definition of $F(s)$ to find

$$(8)\qquad F(s) = \sum_{-\infty}^{\infty} \pi^{-s/2}\Pi\left(\frac{s}{2} - 1\right)\frac{(-1)^n}{2\pi i}\int_{0\searrow 1} \frac{e^{-i\pi x^2}x^{1-s}\,dx}{x^2 - n^2}$$

provided termwise integration is valid. Thus in order to deduce the Riemann–Siegel formula from his 1930 formula, Kuzmin had only to justify the termwise integration (8) and to prove

$$(9)\qquad \pi^{-s/2}\Pi\left(\frac{s}{2} - 1\right)\frac{(-1)^n}{2\pi i}\int_{0\searrow 1} \frac{e^{-i\pi x^2}x^{1-s}\,dx}{x^2 - n^2} = \int_{(-i)^{1/2}}^{\infty} u^{s-1}e^{-\pi n^2u^2}\,du$$

for Re $s < 0$, since this proves $\lim F_b(s) = F(s)$ for Re $s < 0$ and hence by analytic continuation for all s.

Consider the function

$$(10)\qquad \int_{i\infty}^{-i\infty} \frac{e^{\alpha x^2}x^{1-s}\,dx}{x^2 - n^2}$$

for Re $s < 0$. This integral converges not only for positive real α but for all α in the halfplane Re $\alpha > 0$. If the line of integration is tilted slightly away from the imaginary axis and toward the line of slope -1 through the origin, then the halfplane of convergence of the integral is rotated slightly and comes to include the negative imaginary α-axis. Thus the function (10) can be continued analytically to have a value at $\alpha = -i\pi$. But then the line of integration can be moved to $0 \searrow 1$ without changing the value of the integral when $\alpha = -i\pi$. In other words, *the integral*

$$\int_{0\searrow 1} \frac{e^{-i\pi x^2}x^{1-s}\,dx}{x^2 - n^2}$$

can be evaluated by finding the analytic continuation of the function (10) *to the negative imaginary α-axis and setting $\alpha = i\pi$* (s being fixed with Re $s < 0$). But a formula for the function (10) which "remains valid for α in the slit plane" can be found simply by the manipulations

$$\begin{aligned}
\int_{i\infty}^{-i\infty} \frac{e^{\alpha x^2}x^{1-s}\,dx}{x^2 - n^2} &= \int_{i\infty}^{-i\infty} \frac{e^{\alpha x^2}x^{-s}\,d\log x}{1 - x^{-2}n^2} \\
&= \int_0^{\infty} \frac{e^{-\alpha v^2}(-iv)^{-s}\,d\log v}{1 + v^{-2}n^2} - \int_0^{\infty} \frac{e^{-\alpha v^2}(iv)^{-s}\,d\log v}{1 + v^{-2}n^2} \\
&= [i^s + (-i)^s]\int_0^{\infty} \frac{e^{-\alpha v^2}v^{-s}\,d\log v}{1 + v^{-2}n^2} \\
&= 2i\sin\frac{s\pi}{2}e^{\alpha n^2}\int_0^{\infty} \frac{e^{-\alpha(v^2+n^2)}v^{1-s}\,dv}{v^2 + n^2} \\
&= 2i\sin\frac{s\pi}{2}e^{\alpha n^2}\int_0^{\infty}\left[\int_{\alpha}^{\infty} e^{-w(v^2+n^2)}\,dw\right]v^{1-s}\,dv \\
&= 2i\sin\frac{s\pi}{2}e^{\alpha n^2}\int_{\alpha}^{\infty} e^{-wn^2}\left[\int_0^{\infty} e^{-wv^2}v^{2-s}\,d\log v\right]dw
\end{aligned}$$

$$
\begin{aligned}
&= 2i \sin \frac{s\pi}{2} e^{\alpha n^2} \int_\alpha^\infty e^{-wn^2} \int_0^\infty e^{-u}\left(\frac{u}{w}\right)^{(2-s)/2} \frac{1}{2}\, d\log u\, dw \\
&= 2i \sin \frac{s\pi}{2} e^{\alpha n^2} \int_\alpha^\infty e^{-wn^2} w^{(s/2)-1} \Pi\left(-\frac{s}{2}\right)\frac{1}{2}\, dw \\
&= 2i\Pi\left(-\frac{s}{2}\right)\left(\sin \frac{s\pi}{2}\right) e^{\alpha n^2} \int_{(\alpha/\pi)^{1/2}}^\infty e^{-\pi z^2 n^2}(\pi z^2)^{s/2}\, d\log z \\
&= 2i\pi^{s/2}\Pi\left(-\frac{s}{2}\right)\left(\sin \frac{s\pi}{2}\right) e^{\alpha n^2} \int_{(\alpha/\pi)^{1/2}}^\infty e^{-\pi z^2 n^2} z^{s-1}\, dz
\end{aligned}
$$

at first for α real and positive (say) but then by analytic continuation for all α in the slit plane. Thus with $\alpha = -i\pi$

$$
\begin{aligned}
&\pi^{-s/2}\Pi\left(\frac{s}{2} - 1\right)\frac{(-1)^n}{2\pi i} \int_{0 \searrow 1} \frac{e^{-i\pi x^2} x^{1-s}\, dx}{x^2 - n^2} \\
&\quad = \Pi\left(\frac{s}{2} - 1\right)\Pi\left(-\frac{s}{2}\right)\left(\sin \frac{s\pi}{2}\right)\frac{(-1)^n e^{-i\pi n^2}}{\pi} \int_{(-i)^{1/2}}^\infty e^{-\pi n^2 u^2} u^{s-1}\, du \\
&\quad = \int_{(-i)^{1/2}}^\infty e^{-\pi n^2 u^2} u^{s-1}\, du
\end{aligned}
$$

for $\operatorname{Re} s < 0$ as was to be shown. (The only complications for $\operatorname{Re} s \geq 0$ occur in the terms with $n = 0$.)

This reduces the proof of the Riemann–Siegel formula (1) to Kuzmin's formula (7) and the termwise integration (8). The termwise integration (8) is easily justified by noting that for any x on the path of integration $0 \searrow 1$ the point x/n lies between $0 \searrow 1$ and the line of slope 1 through the origin ($n \neq 0$). Hence x^2/n^2 is bounded away from 1, say $|(x/n)^2 - 1| \geq K$, and therefore $|x^2 - n^2|^{-1} \leq K^{-1}n^{-2}$, from which it follows that the integrand converges uniformly and can therefore be integrated termwise on finite intervals. Toward the ends of the line of integration, the integrand is dominated by a constant times $\exp(-\pi |x|^2) |x|^{1-s}$ and can therefore be integrated termwise by the Lebesgue dominated convergence theorem. Similarly elementary arguments suffice to prove that the termwise integration

$$
F_b(s) = \sum_{n=-\infty}^{\infty} \int_b^\infty u^{s-1} e^{-\pi n^2 u^2}\, du \qquad (\operatorname{Re} s < 0)
$$

is valid for all b *inside* the wedge $\{\operatorname{Im} \log b < \pi/4\}$ where G is defined, and it suffices to prove that the limit as $b \longrightarrow (-i)^{1/2}$ can be taken termwise. But integration by parts

$$
\int u^{s-1} e^{-\pi n^2 u^2}\, du = -\frac{1}{2\pi n^2} u^{s-2} e^{-\pi n^2 u^2} + \left(\frac{s}{2} - 1\right)\frac{1}{\pi n^2} \int u^{s-3} e^{-\pi n^2 u^2}\, du
$$

shows that if $\operatorname{Re} s < 0$ and if b is inside or on the wedge and outside or on the unit circle, then

$$
\left| \int_b^\infty u^{s-1} e^{-\pi n^2 u^2}\, du \right| \leq \frac{1}{n^2}\left[\frac{1}{2\pi} + \frac{|(s/2) - 1|}{\pi \cdot 2}\right],
$$

and therefore the series for $F_b(s)$ converges uniformly (for fixed s) for b in this region. Therefore the limit as $b \longrightarrow (-i)^{1/2}$ can be taken termwise and the proof is complete.

12.7 TAUBERIAN THEOREMS

Perhaps the simplest formulation of the idea of the prime number theorem is the approximation $d\psi(x) \sim dx$. Since $d\psi(x) = (\log x)\, dJ(x)$ (see Section 3.1), this is equivalent to Riemann's approximation $dJ(x) \sim dx/\log x$ (see Section 1.18). The theory of *Tauberian theorems* gives a natural interpretation of the approximate formula $d\psi(x) \sim dx$ and shows a direct heuristic connection between it and the simple pole of $\zeta(s)$ at $s = 1$.

Of course the statement $d\psi(x) \sim dx$ makes no sense at all except as a statement about the *average* density of the point measure $d\psi(x)$. The theory of Tauberian theorems deals precisely with the notion of "average" and its various interpretations. Let the Abel average of a sequence $s_1, s_2, s_3, \ldots$ be defined to be

$$\lim_{r \uparrow 1} \frac{s_1 r + s_2 r^2 + s_3 r^3 + \cdots}{r + r^2 + r^3 + \cdots} = L \tag{1}$$

when this limit exists (and when, in particular, the infinite series in the numerator is convergent for all $r < 1$). In other words, the Abel average is found by taking the weighted average of the sequence $\{s_n\}$, counting the nth term with weight r^n, and then letting $r \uparrow 1$. Since for fixed r the weights r^n approach zero rather rapidly as $n \longrightarrow \infty$, the sum $\sum s_n r^n$ will converge unless the s_n grow rapidly in absolute value; on the other hand, for fixed n the weight r^n of the nth term approaches 1 as $r \uparrow 1$, and so for any fixed N the terms beyond the Nth eventually far outweigh the terms up to the Nth once r is near enough to 1. *Abel's theorem*† states that if the sequence $\{s_n\}$ converges to a limit L, then the Abel average exists and is equal to L. *Tauber's theorem* [T1] states that if $\{s_n\}$ is slowly changing in the sense that $|s_{n+1} - s_n| = o(1/n)$, then the converse is true. More precisely, Tauber's theorem says that if for every $\varepsilon > 0$ there is an N such that $|s_{n+1} - s_n| < \varepsilon/n$ whenever $n \geq N$ and if the limit (1) exists [it is easily shown that the condition on $\{s_n\}$ implies that the

†Abel's theorem is, however, more frequently stated for the series $a_1 = s_1, a_2 = s_2 - s_1, a_3 = s_3 - s_2, \ldots, a_n = s_n - s_{n-1}, \ldots$ of which the s_n are the partial sums (see, for example, [E1]). If the series converges, the s_n are bounded, so $\sum s_n r^n$ converges for $r < 1$ and by multiplication of power series $(\sum s_n r^n)(1 - r) = s_1 r + (s_2 - s_1) r^2 + \cdots = \sum a_n r^n$. Thus the statement (1) is identical to $\lim_{r \uparrow 1} \sum a_n r^n = L$, which is the statement that the series $\sum a_n$ is Abel summable. The usual method of proof of Abel's theorem is to put it in the form (1) by partial summation.

numerator of (1) converges for $r < 1$], then in fact the sequence must be convergent to L.

Both Abel's theorem and Tauber's theorem are "Tauberian theorems" in the modern sense, provided that the statement

$$\lim_{n\to\infty} s_n = L \tag{2}$$

is thought of as one possible interpretation of the statement that the "average" of $\{s_n\}$ is L. Then Abel's theorem states that if $\{s_n\}$ has the average L in the sense of (2), it has the average L in the sense of (1), and Tauber's theorem states that if it has the average L in the sense of (1) *and* if $|s_{n+1} - s_n| = o(1/n)$, then it has the average L in the sense of (2). In general, a "Tauberian theorem" is a theorem like these which permits a conclusion about one kind of average, given information about another kind of average.

An important step forward in the theory of Tauberian theorems, and perhaps the real beginning of the theory as such, was Littlewood's discovery in 1910 that the condition in Tauber's theorem can be very significantly weakened to $|s_{n+1} - s_n| = O(1/n)$. At about the same time Hardy proved the analog of Tauber's theorem—with Littlewood's modification—for the "average" defined by

$$\lim_{N\to\infty} \frac{s_1 + s_2 + s_3 + \cdots + s_N}{1 + 1 + 1 + \cdots + 1} = L. \tag{3}$$

That is, Hardy showed that if $\{s_n\}$ has the average L in the sense of (3) and if there is a K such that $|s_{n+1} - s_n| < K/n$ for all n, then $\{s_n\}$ has the average L in the sense of (2). An average in the sense of (3) is called a *Cesaro average*. In 1914 Hardy and Littlewood [H4] in collaboration proved that for positive sequences an Abel average implies a Cesaro average; that is, if $s_n \geq 0$ and if (1), then (3).

Now let $d\phi(x)$ be the point measure which is s_n at n and zero elsewhere. Then the three types of average (1), (2), and (3) can be restated in terms of $d\phi(x)$ as

$$\lim_{r\uparrow 1} \frac{\int_0^\infty r^x\, d\phi(x)}{\int_0^\infty r^x\, d([x])} = L, \tag{1'}$$

where $d([x])$ is the point measure which is 1 at integers and zero elsewhere

$$\lim_{A\to\infty} \int_A^{A+1} d\phi(x) = L \tag{2'}$$

[if A is an integer $A = n$, this integral is by definition $\frac{1}{2}(s_n + s_{n+1})$] and

$$\lim_{A\to\infty} \frac{\int_0^A d\phi(x)}{\int_0^A d([x])} = L. \tag{3'}$$

The second type of average will not be needed in what follows, and the first and third can be rewritten somewhat more simply as

$$\lim_{r\uparrow 1} \frac{\int_0^\infty r^x\, d\phi(x)}{\int_0^\infty r^x\, dx} = L, \qquad \lim_{A\to\infty} \frac{\int_0^A d\phi(x)}{\int_0^A dx} = L,$$

respectively [because $\int_0^\infty r^x\, d([x]) \sim \int_0^\infty r^x\, dx$ and $\int_0^A d([x]) \sim \int_0^A dx$]. It is natural to take these two statements as the definition of what it means to say that $d\phi(x) \sim L\, dx$ as an Abel average or a Cesaro average, respectively.

With this terminology the prime number theorem $\psi(x) \sim x$ is equivalent to the statement that $d\psi(x) \sim dx$ as a Cesaro average. Now since $d\psi(x) \geq 0$, the Hardy–Littlewood theorem cited above implies that in order to prove the prime number theorem, it would suffice to prove that $d\psi(x) \sim dx$ as an Abel average. Hardy and Littlewood were able to prove $d\psi(x) \sim dx$ as an Abel average more simply than it is possible to prove the prime number theorem directly and thereby were able to give a simpler proof of the prime number theorem or, more exactly, a proof of the prime number theorem in which a significant amount of the work is done by a Tauberian theorem. However, it is natural to hope to give a proof in which *all* of the work is done by a Tauberian theorem because there is a sense of "average" in which it is trivial to prove $d\psi(x) \sim dx$, namely, the sense of

$$\lim_{s\downarrow 1} \frac{\int_1^\infty x^{-s}\, d\psi(x)}{\int_1^\infty x^{-s}\, dx} = 1. \tag{4}$$

Since $\int_1^\infty x^{-s}\, dx = (s-1)^{-1}$, this amounts to saying $\lim_{s\downarrow 1}(s-1)[-\zeta(s)/\zeta'(s)] = 1$, which can be proved by taking the logarithmic derivative of the analytic function $(s-1)\zeta(s)$, multiplying by $(s-1)$, and letting $s \longrightarrow 1$. In short, to say that $d\psi(x) \sim dx$ in the sense of (4) amounts to saying that $-\zeta'(s)/\zeta(s)$ has a pole like $(s-1)^{-1}$ at $s = 1$ or, what is the same, that $\zeta(s)$ has a simple pole at $s = 1$. Thus the study of the prime number theorem suggests that one study conditions on measures $d\phi(x)$ under which one can assert that $d\phi(x) \sim dx$ in the sense of

$$\lim_{s\downarrow 1} \frac{\int_1^\infty x^{-s}\, d\phi(x)}{\int_1^\infty x^{-s}\, dx} = 1 \tag{4'}$$

implies $d\phi(x) \sim dx$ in the sense of

$$\lim_{A\to\infty} \frac{\int_0^A d\phi(x)}{\int_0^A dx} = 1. \tag{3''}$$

The attempt to prove the prime number theorem in this way stimulated a great deal of study of Tauberian theorems in the 1920s and early 1930s, culminating in Wiener's general Tauberian theorem [W3]. Although Wiener's theory was immensely successful in revealing the true nature of Tauberian theorems, its conclusions with respect to the prime number theorem were largely negative in that it showed that to justify the implication (4′) ⇒ (3″), it is essential to study the Fourier transform $\int_0^\infty x^{-s}\,d\phi(x)$ on the line $\operatorname{Re} s = 1$ and hence that this approach to the prime number theorem leads to essentially the same ideas and techniques as those used by Hadamard and de la Vallée Poussin of Fourier inversion of $-\zeta'(s)/\zeta(s)$ and use of $\zeta(1 + it) \neq 0$. However, the general theory did give a concise theorem concerning the implication (4′) ⇒ (3″).

Ikehara's Theorem If the measure $d\phi(x)$ is positive, then the implication (4′) ⇒ (3″) is valid provided the function

$$g(s) = \frac{\int_1^\infty x^{-s}\,d\phi(x)}{\int_1^\infty x^{-s}\,dx}$$

has, in addition to the property that $g(s)$ is defined for $s > 1$ and $\lim_{s\downarrow 1} g(s)$ exists, the property that the function $[g(s) - g(1)]/(s - 1)$ has a continuous extension from the open halfplane $\operatorname{Re} s > 1$ (where it is necessarily defined and analytic) to the closed halfplane $\operatorname{Re} s \geq 1$. [Here $g(1)$ is written for $\lim_{s\downarrow 1} g(s)$.]

Ikehara's original proof [I1] of this theorem was a deduction from Wiener's general Tauberian theorem, but Bochner [B6] and others have given direct proofs independent of the general theory. Since in the case $d\phi(x) = d\psi(x)$ the function $g(s)$ is $(s - 1)[-\zeta'(s)/\zeta(s)]$ which is analytic in the entire plane except for poles at the zeros of $\zeta(s)$, the proof of the prime number theorem amounts to the proof that $\zeta(1 + it) \neq 0$ and to the proof that Ikehara's theorem is true in the particular case $d\phi(x) = d\psi(x)$.

12.8 CHEBYSHEV'S IDENTITY

Chebyshev's work on the distribution of primes consists of just two papers which occupy a total of only about 40 pages in his collected works (available in French [C4] as well as Russian [C5]). These two papers are very clearly written and are well worth reading.

The first of them is a study of the approximation $\pi(x) \sim \int_2^x (dx/\log x)$. It is based on an analysis of the function $\zeta(s) - (s - 1)^{-1}$ for *real* s as $s \downarrow 1$, in the course of which Chebyshev succeeds in proving that if there is a best value for A in the approximation

$$\pi(x) \sim \frac{x}{\log x - A},$$

that value is $A = 1$ and that, more generally, no other approximation to $\pi(x)$

of the same form as

$$\pi(x) \sim \frac{x}{\log x} + \frac{x}{(\log x)^2} + \frac{2x}{(\log x)^3} + \cdots + \frac{n!x}{(\log x)^{n+1}} \qquad [\sim \mathrm{Li}(x)]$$

can be a better approximation than this one (see Section 5.4). Thus the basic idea of the paper involves the relationship between the approximation $\pi(x) \sim \mathrm{Li}(x)$ and the pole of the zeta function, a relationship which was much more thoroughly exploited by Riemann and de Vallée Poussin and which was well incorporated into the mainstream of the study of the prime number theorem.

The second paper, on the other hand, is based on a very different idea, one which was until rather recently relegated to the status of a curiosity, showing what sorts of results can be obtained by "elementary" methods, that is, by methods which do not use the theory of Fourier analysis or functions of a complex variable. In the late 1940s, however, Selberg and Erdös showed that this idea of Chebyshev's can be taken further and that from it one can deduce the prime number theorem itself by entirely "elementary" arguments which do not appeal to Fourier analysis or functions of a complex variable (see Section 12.10). Consequently, there has been a great renewal of interest in it. Briefly, the idea is as follows.

Let T be the step function which for positive nonintegral values of x is $\sum_{n<x} \log n$ and which for integral values of x is, as usual, the middle value $T(n) = \frac{1}{2}[T(n+\varepsilon) + T(n-\varepsilon)]$. The value of $T(x)$ for x not an integer can also be described as the logarithm of $[x]$ factorial where $[x]$ is the integer part of x, that is, $T(x) = \log \Pi([x])$ (x not an integer). The identity on which Chebyshev's proof is based is

$$T(x) = \psi(x) + \psi(x/2) + \psi(x/3) + \psi(x/4) + \cdots. \tag{1}$$

This formula can be proved as follows.

Since $\psi(x/n)$ is a step function which jumps only when x is a multiple of n, both sides of (1) are step functions which jump only at integer values of x. Since, moreover, both sides are 0 at $x = 0$ and both assume the middle value at jumps, in order to prove they are equal, it suffices to prove that their jumps are equal at each integer. But at $x = n$ the left side jumps by $\log n$ and the right side jumps by $\sum \Lambda(d)$ where d runs over all divisors of n and where $\Lambda(d)$ is defined as in Section 3.2. Now if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the prime factorization of n, then obviously the divisors of n include precisely α_1 powers of p_1 (namely, $p_1, p_1^2, p_1^3, \ldots, p_1^{\alpha_1}$), α_2 powers of p_2, etc., and no other prime powers. Hence $\sum \Lambda(d) = \alpha_1 \log p_1 + \alpha_2 \log p_2 + \cdots + \alpha_k \log p_k = \log n$, which proves (1).

In terms of Fourier analysis Chebyshev's identity† is the inverse transform

†Credit for the discovery of the identity is shared by de Polignac and Chebyshev (see Landau [L3]), but Chebyshev made better use of it.

of the identity

$$-\zeta'(s) = [-\zeta'(s)/\zeta(s)]\zeta(s)$$

because the left side $-\zeta'(s) = \sum (\log n)n^{-s}$ is the transform of the operator $f(x) \mapsto \int_0^\infty f(ux)\,dT(u)$, whereas the right side is the transform of the composition of the operators $f(x) \mapsto \int_0^\infty f(ux)\,d\psi(u)$ and $f(x) \mapsto \sum_1^\infty f(nx)$, which can be written $f(x) \mapsto \sum_n \int_0^\infty f(nux)\,d\psi(u) = \int_0^\infty f(vx)\,d[\sum \psi(v/n)]$. This suggests $dT(u) = d[\sum \psi(u/n)]$ and hence (1). It is not difficult to make this into a proof of (1), but the elementary proof above, which is essentially the one given by Chebyshev, is to be preferred.

Now Möbius inversion applied to Chebyshev's identity (1) gives

$$\psi(x) = \sum_{n=1}^{\infty} \mu(n) T\left(\frac{x}{n}\right). \tag{2}$$

On the other hand, a good approximation to $T(x)$ can be obtained using Stirling's formula (Euler–Maclaurin summation of $\log n$), and hence this formula should give some information about $\psi(x)$, perhaps even the prime number theorem $\psi(x) \sim x$. The difficulty is of course the irregularity of the coefficients $\mu(n)$ which prevents any straightforward analysis of formula (2). Chebyshev circumvents this difficulty by replacing the right side of (2) by

$$T(x) - T\left(\frac{x}{2}\right) - T\left(\frac{x}{3}\right) - T\left(\frac{x}{5}\right) + T\left(\frac{x}{30}\right) \tag{3}$$

and observing that when (1) is substituted in this expression the resulting series in ψ

$$\psi(x) - \psi\left(\frac{x}{6}\right) + \psi\left(\frac{x}{7}\right) - \psi\left(\frac{x}{10}\right) + \psi\left(\frac{x}{11}\right) - \psi\left(\frac{x}{12}\right) + \cdots$$

has the remarkable property that it *alternates*. (More specifically, the series in ψ is $\sum A_n\psi(x/n)$, where A_n depends only on the congruence class of n mod 30 and where, by explicit computation

$$\begin{aligned} A_n = {} & 1, 0, 0, 0, 0, -1, 1, 0, 0, -1, 1, -1, 1, 0, -1, \\ & 0, 1, -1, 1, -1, 0, 0, 1, -1, 0, 0, 0, 0, 1, -1 \end{aligned}$$

for $n = 1, 2, 3, \ldots, 30$, respectively. Chebyshev does not say how he discovered this particular fact.) Thus (3) is less than $\psi(x)$ but greater than $\psi(x) - \psi(x/6)$, and this, together with Stirling's formula for $T(x)$, gives Chebyshev his estimates of $\psi(x)$.

Specifically, the weak form $T(x) = x \log x - x + O(\log x)$ of Stirling's formula gives easily

$$T(x) - T\left(\frac{x}{2}\right) - T\left(\frac{x}{3}\right) - T\left(\frac{x}{5}\right) + T\left(\frac{x}{30}\right) = Ax + O(\log x)$$

where A is the constant $A = \frac{1}{2}\log 2 + \frac{1}{3}\log 3 + \frac{1}{5}\log 5 - \frac{1}{30}\log 30 = 0.921\ldots$. Thus

$$\psi(x) > Ax + O(\log x), \qquad \psi(x) - \psi\left(\frac{x}{6}\right) < Ax + O(\log x).$$

If the second inequality is iterated

$$\psi\left(\frac{x}{6}\right) - \psi\left(\frac{x}{6^2}\right) < A\frac{x}{6} + O(\log x),$$
$$\psi\left(\frac{x}{6^2}\right) - \psi\left(\frac{x}{6^3}\right) < A\frac{x}{6^2} + O(\log x),$$
$$\vdots$$

only $\log x/\log 6$ steps are required to reach $\psi(x/6^n) = 0$. Adding these then gives

$$\begin{aligned}\psi(x) &< Ax\left(1 + \frac{1}{6} + \frac{1}{6^2} + \cdots + \frac{1}{6^n}\right) + O((\log x)^2)\\ &< \frac{6}{5}Ax + O((\log x)^2)\end{aligned}$$

and shows that in the limit as $x \longrightarrow \infty$, the quotient $\psi(x)/x$ lies between $A = 0.921\ldots$ and $6A/5 = 1.105\ldots$. In particular $\psi(x) = O(x)$, a fact which will be needed in the following sections.

12.9 SELBERG'S INEQUALITY

Chebyshev's formula $\sum \psi(x/n) = T(x)$ taken together with Stirling's formula $T(x) = (x + \frac{1}{2})\log x - x + O(1)$ lends credence to the prime number theorem $\psi(x) \sim x$ because if $\psi(x/n)$ is replaced by x/n, then the sum $\sum \psi(x/n)$ is replaced by $x \sum n^{-1} \sim x \log x \sim T(x)$. More specifically, choose as an approximation to $\psi(x)$ a function of the form

$$g(x) = \begin{cases} 0, & x \le a, \\ x - a, & x \ge a, \end{cases}$$

where a is a positive constant. Then for large x

$$\sum_{n=1}^{\infty} g\left(\frac{x}{n}\right) = \sum_{x/n \ge a}\left[\left(\frac{x}{n}\right) - a\right] = x \sum_{n \le x/a} \frac{1}{n} - a\left[\frac{x}{a}\right].$$

Now by Euler–Maclaurin summation [in its simplest version (3) of Section 6.2]

$$\begin{aligned}
\sum_{n\le y}\frac{1}{n} &= \sum_{n\le [y]}\frac{1}{n} = \int_1^{[y]}\frac{dy}{y} + \frac{1}{2}\left(1+\frac{1}{[y]}\right) - \int_1^{[y]}\frac{\bar{B}_1(u)}{u^2}\,du \\
&= \log[y] + \frac{1}{2} - \int_1^{\infty}\frac{\bar{B}_1(u)}{u^2}\,du + \frac{1}{2[y]} + \int_{[y]}^{\infty}\frac{\bar{B}_1(u)}{u^2}\,du \\
&= \log y + \text{const} + \frac{1}{2y}\left(\frac{y-[y]}{[y]}+1\right) + \log\left(1-\frac{y-[y]}{y}\right) \\
&\quad + O\left(\frac{1}{[y]^2}\right) \\
&= \log y + \gamma + O\left(\frac{1}{y}\right),
\end{aligned}$$

where the constant γ is by definition (see Section 3.8) Euler's constant. This gives

$$\begin{aligned}
\sum_{n=1}^{\infty} g\left(\frac{x}{n}\right) &= x\left[\log\frac{x}{a} + \gamma + O\left(\frac{a}{x}\right)\right] - x - O(a) \\
&= x\log x + (\gamma - \log a)x - x + O(1)
\end{aligned}$$

and shows, therefore, that if a is chosen to be $a = e^{\gamma}$, then

$$\sum_{n=1}^{\infty} g\left(\frac{x}{n}\right) = x\log x - x + O(1) = T(x) + O(\log x) = \sum_{n=1}^{\infty}\psi\left(\frac{x}{n}\right) + O(\log x).$$

Thus setting

$$r(x) = \sum_{n=1}^{\infty}\psi\left(\frac{x}{n}\right) - \sum_{n=1}^{\infty} g\left(\frac{x}{n}\right),$$

Chebyshev's identity and Stirling's formula give $r(x) = O(\log x)$. On the other hand, by Möbius inversion $\psi(x) - g(x) = \sum \mu(n)r(x/n)$, so the prime number theorem is the statement that $\sum \mu(n)r(x/n) = o(x)$. This leads to the question of whether estimates of the growth of $\sum \mu(n)r(x/n)$ can be deduced from estimates of the growth of $r(x)$.

It is very difficult to obtain sharp estimates of the growth of $\sum \mu(n)r(x/n)$ because the real reason for its slow growth involves *cancellation* between terms, so that the distribution of the signs $\mu(n) = \pm 1$ and the rate of change of r are crucial. As was shown in the preceding section, Chebyshev dealt with this difficulty by replacing the actual Möbius inverse $\sum \mu(n)r(x/n)$ by an approximate Möbius inverse $r(x) - r(x/2) - r(x/3) - r(x/5) + r(x/30)$. The first step in the elementary proof of the prime number theorem is to replace $\sum \mu(n)r(x/n)$ by the approximate Möbius inverse suggested by Selberg's proof in Section 11.3, namely, to replace it by the expression

$$\sum_{n=1}^{\infty}\mu(n)\left(1-\frac{\log n}{\log x}\right)r\left(\frac{x}{n}\right). \tag{1}$$

(Note that $r(x/n) = 0$ for $n \geq x$ so the weights $[1 - (\log n/\log x)]$ are positive in the nonzero terms.) This leads to an estimate of $\psi(x)$ known as Selberg's inequality which, as will be shown in the next section, is a major step toward the prime number theorem.

The first step in the derivation of Selberg's inequality is to note that the expression (1) grows less rapidly than x as $x \to \infty$ and that in fact it is $O(x/\log x)$. This follows easily from $r(y) < K \log y$ ($y \geq 1$, K a constant independent of y) because this shows that the absolute value of (1) is at most

$$\sum_{n<x} \frac{\log(x/n)}{\log x} K \log\left(\frac{x}{n}\right) = \frac{K}{\log x} \sum_{n<x} \left(\log \frac{x}{n}\right)^2.$$

This sum can be estimated using Euler–Maclaurin summation, but the result is only that it is $O(x/\log x)$, a result which can be obtained much more easily by using $\log y < K'y^\varepsilon$ ($y \geq 1$, K' a constant depending on ε) to find

$$\begin{aligned}\sum_{n<x} \left(\log \frac{x}{n}\right)^2 &< K' \sum_{n<x} \left(\frac{x}{n}\right)^{2\varepsilon} \\ &= K'x^{2\varepsilon} \sum_{n<x} n^{-2\varepsilon} < K'x^{2\varepsilon}\left[1 + \int_1^x u^{-2\varepsilon}\, du\right] \\ &< K'x^{2\varepsilon}\left(1 + \frac{x^{-2\varepsilon+1}}{-2\varepsilon+1}\right) < K''x\end{aligned}$$

which gives the desired result that (1) is $O(x/\log x)$.

The second step in the derivation of Selberg's inequality is to give a precise sense to the idea that the operation (1) is an "approximate Möbius inversion." One can in fact give an explicit expression for

$$\sum_{n=1}^{\infty} \mu(n)\left(1 - \frac{\log n}{\log x}\right) F\left(\frac{x}{n}\right) \tag{2}$$

when F is a function of the form $F(x) = \sum_{n=1}^{\infty} f(x/n)$ with f a function which is identically zero near zero. This explicit expression can be derived as follows. The sum (2) is equal to

$$\begin{aligned}&\frac{1}{\log x} \sum_{n=1}^{\infty} \mu(n) \log\left(\frac{x}{n}\right) \sum_{m=1}^{\infty} f\left(\frac{x}{mn}\right) \\ &\quad = \frac{1}{\log x} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \mu(n)\left[\log\left(\frac{x}{mn}\right) + \log m\right] f\left(\frac{x}{mn}\right) \\ &\quad = \frac{1}{\log x} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \mu(n)\left[\log\left(\frac{x}{mn}\right) f\left(\frac{x}{mn}\right)\right] \\ &\qquad + \frac{1}{\log x} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \mu(n)(\log m) f\left(\frac{x}{mn}\right).\end{aligned}$$

Now by ordinary Möbius inversion the first of these two sums is simply $(\log x)^{-1} (\log x) f(x) = f(x)$, so the second sum gives the amount by which (2) is only an "approximate" Möbius inverse. Note that it is $(\log x)^{-1}$ times a composition of the operators $f(x) \mapsto \sum_{m=1}^{\infty} (\log m) f(x/m)$ and $f(x) \mapsto \sum_{n=1}^{\infty} \mu(n) f(x/n)$, or, what is the same, a composition of $f(x) \mapsto \int_0^\infty f(x/u)\, dT(u)$ and $f(x) \mapsto \int_0^\infty f(x/u)\, dM(u)$. Since these operators have transforms $-\zeta'(-s)$ and $1/\zeta(-s)$, respectively, their composition has the transform $-\zeta'(-s)/\zeta(-s)$ which is the transform of $f(x) \mapsto \int_0^\infty f(x/u)\, d\psi(u)$. This leads to the conjecture that the second sum above is $(\log x)^{-1} \int_0^\infty f(x/u)\, d\psi(u)$, a conjecture which is easily verified (without appeal to Fourier analysis) by using Chebyshev's identity and Möbius inversion to write, for x not an integer,

$$\begin{aligned}\psi(x) &= \sum_{n=1}^{\infty} \mu(n) T\left(\frac{x}{n}\right) = \sum_{n=1}^{\infty} \sum_{m<x/n} \mu(n) \log m = \sum\sum_{mn<x} \mu(n) \log m \\ &= \sum_{k<x} \sum_{mn=k} \mu(n) \log m,\end{aligned}$$

so that

$$\begin{aligned}\int_0^\infty f\left(\frac{x}{u}\right) d\psi(u) &= \sum_{k=1}^{\infty} f\left(\frac{x}{k}\right)\left[\sum_{mn=k} \mu(n) \log m\right] \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \mu(n)(\log m) f\left(\frac{x}{mn}\right).\end{aligned} \tag{3}$$

Thus the final formula is

$$\sum_{n=1}^{\infty} \mu(n)\left(1 - \frac{\log n}{\log x}\right) F\left(\frac{x}{n}\right) = f(x) + \frac{1}{\log x} \int_0^\infty f\left(\frac{x}{u}\right) d\psi(u),$$

where f is a function which is identically zero near zero and where $F(x) = \sum_{m=1}^{\infty} f(x/m)$.

Applying this formula in the case $F = r$ and using the fact that (1) is $O(x/\log x)$ gives

$$\begin{aligned}&[\psi(x) - g(x)] + \frac{1}{\log x} \int_0^\infty \left[\psi\left(\frac{x}{u}\right) - g\left(\frac{x}{u}\right)\right] d\psi(u) \\ &\qquad = O(x/\log x)\end{aligned} \tag{4}$$

which, in essence, is Selberg's inequality. To obtain the inequality in the form stated by Selberg [S3], it is necessary first to estimate the integral

$$\begin{aligned}\int_0^\infty g\left(\frac{x}{u}\right) d\psi(u) &= -\int_0^\infty \psi(u)\, dg\left(\frac{x}{u}\right) = \int_0^\infty \psi\left(\frac{x}{v}\right) dg(v) = \int_a^\infty \psi\left(\frac{x}{v}\right) dv \\ &= \sum_1^\infty \psi\left(\frac{x}{n}\right) - \sum_1^\infty \psi\left(\frac{x}{n}\right) + \int_1^\infty \psi\left(\frac{x}{v}\right) dv - \int_1^a \psi\left(\frac{x}{v}\right) dv \\ &= T(x) - \frac{1}{2}\psi(x) - \int_1^\infty \bar{B}_1(v)\, d\psi\left(\frac{x}{v}\right) + O(\psi(x)) \\ &= x \log x + O(x) + O\left[-\int_1^\infty d\psi\left(\frac{x}{v}\right)\right] = x \log x + O(x)\end{aligned}$$

using Chebyshev's theorem $\psi(x) = O(x)$. (This calculation assumes that x is not an integer—so that the discontinuities of $\bar{B}_1(v)$ never coincide with those of $\psi(x/v)$—and it assumes that x is not an integral multiple of a—so that $v = a$ is not a discontinuity of $\psi(x/v)$. This excludes only a discrete set of values of x, and since $\int_0^\infty g(x/u)\, d\psi(u)$ is an increasing function of x, the final estimate is obviously valid for these values of x as well.) This shows that Selberg's inequality (4) can also be written in the form

$$\begin{aligned}\psi(x) \log x + \int_0^\infty \psi\left(\frac{x}{u}\right) d\psi(u) \\ &= g(x) \log x + \int_0^\infty g\left(\frac{x}{u}\right) d\psi(u) + O(x) \\ &= x \log x + O(\log x) + x \log x + O(x)\end{aligned}$$

and hence finally

$$\psi(x) \log x + \int_0^\infty \psi\left(\frac{x}{u}\right) d\psi(u) = 2x \log x + O(x). \tag{5}$$

This is almost Selberg's statement of it except that Selberg [S3] deals with θ rather than ψ (see Section 4.4 for the definition of θ) and his inequality is

$$\theta(x) \log x + \int_0^\infty \theta\left(\frac{x}{u}\right) d\theta(u) = 2x \log x + O(x).$$

[Note that the integral is simply the finite sum $\sum_{p<x} \theta(x/p) \log p$.] The proof of the inequality in this form is somewhat longer than the proof of (5) and will not be needed in what follows.

12.10 ELEMENTARY PROOF OF THE PRIME NUMBER THEOREM

The deduction of the prime number theorem from Selberg's inequality, although it is "elementary" in the technical sense that it does not use Fourier analysis or complex variables, is by no means simple or straightforward. Selberg's original proof depended on a weakened version of the prime number theorem which Erdös had previously proved by elementary methods, but Selberg never published this proof in full, preferring to give a complete proof *ab initio* and also preferring to eliminate the appeal to the notion of "lim sup" which the original proof contained. Since 1949 many variations, extensions, and refinements of the elementary proof have been given, but none of them seems very straightforward or natural, nor does any of them give much insight into the theorem.

The proof which follows is a combination of Wirsing's proof [W5] and the proof given by Levinson in his expository paper [L10]. Following Wirsing, it is based on the consideration of approximations not to $\psi(x)$ but to the function $\int_0^x u^{-1}\, d\psi(u)$. This function has the advantage that its discontinuities are small for large x $[\Lambda(n)/n \leq (\log n)/n < \varepsilon]$ whereas the discontinuities of $\psi(x)$ are large. As in Section 5.6, this function $\int_0^x u^{-1}\,d\psi(u)$ will be denoted $P(x)$. It was shown in Section 5.6—but not by elementary methods—that $P(x) = \log x - \gamma + \eta(x)$, where γ is Euler's constant and where the remainder $\eta(x)$ goes to zero faster than $(\log x)^{-n}$ for any n. This and the form of the approximation g to ψ in the preceding section suggest as an approximation to $P(x)$

$$G(x) = \begin{cases} \log x - \gamma, & x \geq e^\gamma, \\ 0, & x \leq e^\gamma. \end{cases}$$

As in Section 5.6, let $\eta(x) = P(x) - G(x)$ be the error in this approximation. In order to prove the prime number theorem it will suffice to prove that $\eta(x) \to 0$ as $x \to \infty$ because then

$$\begin{aligned} \psi(x) &= \int_0^x d\psi(u) = \int_0^x u\,dP(u) \\ &= \int_0^x u\,dG(u) + \int_0^x u\,d\eta(u) \\ &= \int_{e^\gamma}^x u\,d\log u + \int_0^x d[u\eta(u)] - \int_0^x \eta(u)\,du \\ &= x - e^\gamma + x\eta(x) - \int_0^x \eta(u)\,du \\ &= x + x\left\{-\frac{e^\gamma}{x} + \eta(x) - \text{average of } \eta \text{ on } [0, x]\right\} \\ &= x + o(x). \end{aligned}$$

Thus the goal is to prove by elementary methods that $\eta(x) \to 0$.

Note first that η is bounded. This follows easily from the estimate $\int_0^\infty g(x/u)\,d\psi(u) = x \log x + O(x)$ at the end of the preceding section which gives

$$\begin{aligned} P(x) &= \int_0^x \frac{1}{u}\,d\psi(u) = \frac{1}{x}\int_0^x \frac{x}{u}\,d\psi(u) \\ &= \frac{1}{x}\int_0^x g\left(\frac{x}{u}\right) d\psi(u) + \frac{1}{x}\int_0^x \left[\frac{x}{u} - g\left(\frac{x}{u}\right)\right] d\psi(u) \\ &= \frac{1}{x}\int_0^\infty g\left(\frac{x}{u}\right) d\psi(u) + \int_0^{x/a} a\,d\psi(u) + \frac{1}{x}\int_{x/a}^x \frac{x}{u}\,d\psi(u) \\ &= \log x + O(1) + \frac{a\psi(x/a)}{x} + O\left(\frac{a[\psi(x) - \psi(x/a)]}{x}\right) \\ &= \log x + O(1) = G(x) + O(1), \end{aligned}$$

where, as before, $a = e^\gamma$. Thus $\eta(x) = O(1)$ as was to be shown.

Chebyshev's identity $\sum \psi(x/n) = T(x) = \sum_{n<x} \log n$ ($x \neq$ integer) implies an analogous identity for P which can be derived as follows:

$$xP(x) = \int_0^x \frac{x}{u}\, d\psi(u) = \int_\infty^1 v\, d\psi\left(\frac{x}{v}\right),$$

$$\sum_{n=1}^\infty \frac{x}{n} P\left(\frac{x}{n}\right) = \sum_{n=1}^\infty \int_\infty^1 v\, d\psi\left(\frac{x}{nv}\right) = \int_\infty^1 v\, dT\left(\frac{x}{v}\right)$$

$$= \int_0^x \frac{x}{u}\, dT(u) = x \sum_{n<x} \frac{\log n}{n},$$

$$\sum_{n=1}^\infty \frac{1}{n} P\left(\frac{x}{n}\right) = \sum_{n<x} \frac{\log n}{n}.$$

The form of this identity suggests that one consider $\sum(1/n)\eta(x/n)$. This gives the following estimate analogous to the estimate $r(x) = O(\log x)$ of the last section:

$$\sum_{n=1}^\infty \frac{1}{n}\eta\left(\frac{x}{n}\right) = \sum_{n=1}^\infty \frac{1}{n} G\left(\frac{x}{n}\right) - \sum_{n=1}^\infty \frac{1}{n} P\left(\frac{x}{n}\right)$$

$$= \sum_{x/n>a} \frac{1}{n}\left(\log \frac{x}{n} - \gamma\right) - \sum_{n<x} \frac{\log n}{n}$$

$$= (\log x - \gamma) \sum_{n<x/a} \frac{1}{n} - \sum_{n<x/a} \frac{\log n}{n} - \sum_{n<x} \frac{\log n}{n}.$$

Now by Euler–Maclaurin summation

$$\sum_{n=1}^N \frac{\log n}{n} = \int_1^N \frac{\log u}{u}\, du + \frac{1}{2}\left[\frac{\log N}{N} + 0\right] + \int_1^N \bar{B}_1(u) \frac{1 - \log u}{u^2}\, du$$

$$= \frac{1}{2}(\log u)^2 \Big|_1^N + \int_1^\infty \frac{\bar{B}_1(u)(1 - \log u)}{u^2}\, du$$

$$- \int_N^\infty \frac{\bar{B}_1(u)(1 - \log u)}{u^2}\, du + \frac{\log N}{2N}$$

$$= \frac{1}{2}(\log N)^2 + \text{const} + O\left(\frac{\log N}{N}\right)$$

and this together with $\sum_{n<x} 1/n = \log x + \gamma + O(1/x)$ gives

$$\sum \frac{1}{n}\eta\left(\frac{x}{n}\right) = \log(x - \gamma)\left[\log \frac{x}{a} + \gamma + O\left(\frac{a}{x}\right)\right] - \frac{1}{2}\left(\log \frac{x}{a}\right)^2$$

$$- \text{const} - O\left(\frac{\log(x/a)}{x/a}\right) - \frac{1}{2}(\log x)^2 - \text{const} - O\left(\frac{\log x}{x}\right)$$

$$= (\log x - \gamma)\left[\log x + O\left(\frac{1}{x}\right)\right] - \frac{1}{2}(\log x - \gamma)^2$$

$$- \frac{1}{2}(\log x)^2 + \text{const} + O\left(\frac{\log x}{x}\right)$$

$$= \text{const} + O\left(\frac{\log x}{x}\right).$$

Let $s(x)$ denote this function $\sum (1/n)\eta(x/n)$. Then Möbius inversion gives $\eta(x) = \sum [\mu(n)/n]s(x/n)$, and the sort of approximate Möbius inversion of the preceding section [see in particular formula (3)] gives

$$\begin{aligned}
&\sum \frac{\mu(n)}{n}\left(1 - \frac{\log n}{\log x}\right)s\left(\frac{x}{n}\right) \\
&= \frac{1}{\log x}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \mu(n)\left(\log\frac{x}{n}\right)\frac{1}{mn}\eta\left(\frac{x}{mn}\right) \\
&= \frac{1}{x\log x}\sum\sum \mu(n)\left[\log\frac{x}{mn} + \log m\right]\frac{x}{mn}\eta\left(\frac{x}{mn}\right) \\
&= \frac{1}{x\log x}(\log x)x\eta(x) + \frac{1}{x\log x}\int_0^\infty \frac{x}{u}\eta\left(\frac{x}{u}\right)d\psi(u) \\
&= \eta(x) + \frac{1}{\log x}\int_0^\infty \eta\left(\frac{x}{u}\right)dP(u).
\end{aligned}$$

Using the estimate $s(x) = \text{const} + O[(\log x)/x]$, it is possible to show that this function of x is $O(1/\log x)$. In fact, since

$$\begin{aligned}
\frac{1}{\log x}\sum_{n<x}\frac{1}{n}\left(\log\frac{x}{n}\right)\frac{\log(x/n)}{x/n} &= \frac{1}{x\log x}\sum_{n<x}\left(\log\frac{x}{n}\right)^2 \\
&= O\left(\frac{1}{\log x}\right)
\end{aligned}$$

(see the estimate of $\sum [\log(x/n)]^2$ in the preceding section), the proof of this reduces immediately to the proof that

$$\sum \frac{\mu(n)}{n}\log\frac{x}{n} = O(1). \tag{1}$$

This can be accomplished as follows.

Let $D(x)$ again represent the function which is 1 for $x > 1$, $\frac{1}{2}$ for $x = 1$, and 0 for $x < 1$. Then $\sum D(x/n)$ is simply the greatest integer function (x not an integer) and Möbius inversion gives

$$D(x) = \sum_{n=1}^{\infty}\mu(n)\left[\frac{x}{n}\right]$$

which for large x is

$$1 = \sum_{n<x}\mu(n)\frac{x}{n} - \sum_{n<x}\mu(n)\left(\frac{x}{n} - \left[\frac{x}{n}\right]\right) = x\sum_{n<x}\frac{\mu(n)}{n} + O(x)$$

so that division by x gives $\sum_{n<x}\mu(n)/n = O(1)$. Then Möbius inversion of the estimate

$$\sum_{n=1}^{\infty}\frac{1}{n}D\left(\frac{x}{n}\right) = \sum_{n<x}\frac{1}{n} = \begin{cases}\log x + \gamma + O\left(\frac{1}{x}\right) & \text{for } x \ge 1, \\ 0 & \text{for } x < 1,\end{cases}$$

gives

$$D(x) = \sum_{n<x} \frac{\mu(n)}{n}\left[\log\left(\frac{x}{n}\right) + \gamma + O\left(\frac{n}{x}\right)\right],$$

$$1 = \sum_{n<x} \frac{\mu(n)}{n} \log \frac{x}{n} + \gamma\cdot O(1) + O(1),$$

from which (1) follows.

In summary, then, it has been shown that

$$\eta(x) + \frac{1}{\log x}\int_0^\infty \eta\left(\frac{x}{u}\right) dP(u) = O\left(\frac{1}{\log x}\right). \tag{2}$$

This is the analog of Selberg's inequality [in the form (4) of Section 12.9] for the error in the approximation $P \sim G$ instead of the error in the approximation $\psi \sim g$. The objective is to use it to prove that $\eta(x) \to 0$ as $x \to \infty$.

The first step in the proof is to *iterate* Selberg's inequality, which in the present case can be carried out as follows:

$$\int_0^\infty \eta\left(\frac{x}{u}\right) dP(u) = -\eta(x)\log x + O(1),$$

$$\begin{aligned}\int_0^\infty\int_0^\infty \eta\left(\frac{x}{uv}\right) dP(u)\,dP(v) &= \int_0^{x+\varepsilon}\int_0^\infty \eta\left(\frac{x}{uv}\right) dP(u)\,dP(v)\\
&= -\int_0^{x+\varepsilon} \eta\left(\frac{x}{v}\right)\log\left(\frac{x}{v}\right) dP(v)\\
&\quad + \int_0^{x+\varepsilon} O(1)\,dP(v)\\
&= -\log x\int_0^\infty \eta\left(\frac{x}{v}\right) dP(v)\\
&\quad + \int_0^\infty \eta\left(\frac{x}{v}\right)\log v\,dP(v) + O(P(x))\\
&= -\log x[-\eta(x)\log x + O(1)]\\
&\quad + \int_0^\infty \eta\left(\frac{x}{v}\right)\log v\,dP(v) + O(G(x))\\
&= \eta(x)(\log x)^2 + \int_0^\infty \eta\left(\frac{x}{v}\right)\log v\,dP(v)\\
&\quad + O(\log x).\end{aligned}$$

The double integral on the left is in fact a finite sum, so it can be rearranged and written as $\int_0^\infty \eta(x/w)\,dN(w)$ where $N(w) = \int_0^\infty P(w/u)\,dP(u)$. Thus

$$\int_0^\infty \eta\left(\frac{x}{w}\right) dN(w) - \int_0^\infty \eta\left(\frac{x}{v}\right)\log v\,dP(v) = \eta(x)(\log x)^2 + O(\log x).$$

Since N and P are both increasing functions, taking absolute values gives

$$\begin{aligned}(\log x)^2\,|\eta(x)| &\le \int_0^\infty \left|\eta\left(\frac{x}{w}\right)\right| dN(w)\\
&\quad + \int_0^\infty \left|\eta\left(\frac{x}{w}\right)\right| \log w\,dP(w) + O(\log x).\end{aligned} \tag{3}$$

Now $dP(w) = w^{-1}\, d\psi(w)$ is roughly like $w^{-1}\, dw = d \log w$, from which it follows that $dN(w)$ and $\log w\, dP(w)$ are both roughly $\log w\, d \log w$; so the right side is roughly $2 \int_0^\infty |\eta(x/w)| \log w\, d \log w$. More precisely,

$$\begin{aligned}
&\int_0^x [dN(w) + \log w\, dP(w)] \\
&\quad = N(x) + \int_0^x \log w\, dP(w) \\
&\quad = \int_0^\infty \left\{G\left(\frac{x}{w}\right) + \eta\left(\frac{x}{w}\right)\right\} dP(w) + \int_0^x \log w\, dP(w) \\
&\quad = \int_0^\infty \eta\left(\frac{x}{w}\right) dP(w) + \int_0^{x/a} \log \frac{x}{aw}\, dP(w) + \int_0^x \log w\, dP(w) \\
&\quad = -\eta(x) \log x + O(1) + \int_0^x \log \frac{x}{a}\, dP(w) - \int_{x/a}^x \log \frac{x}{aw}\, dP(w) \\
&\quad = -\eta(x) \log \frac{x}{a} + O(1) + \left(\log \frac{x}{a}\right) P(x) + O\left(\int_{x/a}^x \log a\, dP(w)\right) \\
&\quad = G(x) \log \frac{x}{a} + O(1) + O\left(P(x) - P\left(\frac{x}{a}\right)\right) \\
&\quad = \{G(x)\}^2 + O(1) + O\left(\log x - \gamma + \eta(x) - \log \frac{x}{a} + \gamma - \eta\left(\frac{x}{a}\right)\right) \\
&\quad = \{G(x)\}^2 + O(1)
\end{aligned}$$

for all $x > a$. Moreover, the measures $dN(w)$, $\log w\, dP(w)$, and $d\{G(w)^2\}$ are all identically zero for $x \leq 1$, so integration by parts shows that the right side of (3) differs from $2\int_0^\infty |\eta(x/w)|\, G(w)\, dG(w)$ by at most $O(\log x)$ plus

$$\begin{aligned}
&\int_1^\infty \left|\eta\left(\frac{x}{w}\right)\right| [dN(w) + \log w\, dP(w) - 2G(w)\, dG(w)] \\
&\qquad = -\int_1^\infty \{O(1)\}\, d\left[\left|\eta\left(\frac{x}{w}\right)\right|\right] = O\left(-\int_x^0 d[|\eta(v)|]\right) \\
&\qquad \leq O\left(\int_0^x dG(v) + \int_0^x dP(v)\right) = O(2G(x) + \eta(x)) = O(\log x).
\end{aligned}$$

On the other hand $2\int_0^\infty |\eta(x/w)|\, G(w)\, dG(w)$ can be rewritten in the form

$$\begin{aligned}
&2\int_0^\infty \left|\eta\left(\frac{x}{w}\right)\right| G(w)\, dG(w) \\
&\quad = 2\int_a^\infty \left|\eta\left(\frac{x}{w}\right)\right| \log \frac{w}{a}\, d \log w = 2\int_{x/a}^0 |\eta(v)| \log \frac{x}{va}\, d \log\left(\frac{1}{v}\right) \\
&\quad = 2\int_0^{x/a} |\eta(v)| \left\{\int_v^{x/a} d \log u\right\} d \log v = 2\int_1^{x/a} \int_v^{x/a} |\eta(v)|\, d \log u\, d \log v \\
&\quad = 2\int_1^{x/a} \int_1^u |\eta(v)|\, d \log v\, d \log u;
\end{aligned}$$

so (3) can be rewritten as

$$\text{(4)} \qquad (\log x)^2 |\eta(x)| \le 2\int_1^{x/a} \int_1^u |\eta(v)|\, d\log v\, d\log u + O(\log x).$$

Now the integral

$$\text{(5)} \qquad \frac{2}{[\log(x/a)]^2} \int_1^{x/a} \int_1^u |\eta(v)|\, d\log v\, d\log u$$
$$= \frac{2}{[\log(x/a)]^2} \int_1^{x/a} \log u \left(\frac{1}{\log u} \int_1^u |\eta(v)|\, d\log v\right) d\log u$$

can be regarded as the result of applying to $|\eta(v)|$ two averaging processes one after the other, the first being the average $f(x) \mapsto (\log x)^{-1} \int_1^x f(v)\, d\log v$ and the second being the average $f(x) \mapsto [\log(x/a)]^{-2} \int_1^{x/a} f(u)\, d\{(\log u)^2\}$. In particular, since $|\eta|$ is bounded the integral (5) is bounded; so multiplication of (4) by $(\log x)^{-2} = [\log(x/a)]^{-2}[1 + O(1/\log x)]$ gives finally

$$|\eta(x)| \le \frac{2}{\left(\log \frac{x}{a}\right)^2} \int_1^{x/a} \log u \left(\frac{1}{\log u} \int_1^u |\eta(v)|\, d\log v\right) d\log u + O\left(\frac{1}{\log x}\right).$$

(6)

This inequality is the keystone of the proof. It states that $|\eta|$ is dominated, in the limit as $x \to \infty$, by an average of an average of $|\eta|$. Since averaging normally reduces functions unless they are constant, this indicates that $|\eta(x)|$ must be nearly constant in the limit as $x \to \infty$. Since $\eta(x)$ changes rather gradually, this indicates that $\eta(x)$ must also be nearly constant in the limit as $x \to \infty$, that is, $\lim_{x\to\infty} \eta(x)$ exists. However, because of the analogy with Chebyshev's elementary proof that $[\pi(x) - \mathrm{Li}(x)]/\mathrm{Li}(x)$ can have no limit other than zero, one would expect to be able to prove by elementary means that $\lim_{x\to\infty} \eta(x)$, given that it exists, must be zero and thus to complete the proof.

The actual proof will require two more estimates, an estimate of the rate of change of η and an estimate which proves that η can approach no limit other than zero.

Wirsing gives the estimate

$$\text{(7)} \qquad |\eta(x) - \eta(y)| \le \log(x/y) + O(1/\log x)$$

for $x > y > 0$. Since P increases, $\eta(x) - \eta(y) = [G(x) - G(y)] - [P(x) - P(y)] \le G(x) - G(y) = \int_y^x dG(u) \le \int_y^x d\log u = \log(x/y)$, and the upper estimate $\eta(x) - \eta(y) \le \log(x/y)$ is trivial. Since η is bounded, the lower estimate $\eta(x) - \eta(y) \ge -\log(x/y)$ holds trivially whenever $\log(x/y)$ is sufficiently large, say whenever $\log(x/y) \ge K$. Thus it suffices to find a lower estimate $\eta(x) - \eta(y) \ge -\log(x/y) - O(1/\log x)$ under the additional assumption $\log(x/y) \le K$. This can be done by using Selberg's inequality to find

$$\begin{aligned}
\eta(x) - \eta(y) &= \frac{1}{\log x}\left[\eta(x)\log x - \eta(y)\log y - \eta(y)\log\frac{x}{y}\right] \\
&\ge \frac{1}{\log x}\left[-\int_0^\infty \eta\left(\frac{x}{u}\right)dP(u) + O(1)\right. \\
&\qquad \left.+ \int_0^\infty \eta\left(\frac{y}{u}\right)dP(u) + O(1)\right] - O(\eta(y))\frac{K}{\log x} \\
&= \frac{1}{\log x}\left[-\int_0^\infty \left\{\eta\left(\frac{x}{u}\right) - \eta\left(\frac{y}{u}\right)\right\}dP(u)\right] - O\left(\frac{1}{\log x}\right) \\
&\ge -\frac{1}{\log x}\int_0^x \log\frac{x}{y}\,dP(u) - O\left(\frac{1}{\log x}\right) \\
&= -\frac{[\log(x/y)][G(x)+\eta(x)]}{\log x} - O\left(\frac{1}{\log x}\right) \\
&= -\log\frac{x}{y} - \left(\log\frac{x}{y}\right)\frac{-\gamma+\eta(x)}{\log x} - O\left(\frac{1}{\log x}\right) \\
&\ge -\log\frac{x}{y} - K\frac{O(1)}{\log x} - O\left(\frac{1}{\log x}\right) \\
&= -\log\frac{x}{y} - O\left(\frac{1}{\log x}\right).
\end{aligned}$$

This completes the proof of (7).

The estimate $\sum (1/n)\eta(x/n) = \text{const} + O[(\log x)/x]$ is a good indication that η cannot approach any limit other than zero. A formulation of this estimate which is more convenient for present purposes is $\int_1^\infty (1/u)\eta(x/u)\,du = O(1)$, or what is the same,

$$\int_0^x \eta(v)\,d\log v = O(1). \tag{8}$$

This shows that the average value of η on $[1, x]$ relative to the invariant measure $d\log v$ approaches zero; hence, because η changes slowly, η must be arbitrarily near zero infinitely often. The estimate (8) can be proved as follows:

$$\begin{aligned}
\int_0^x \eta(v)\,d\log v &= \int_1^\infty \frac{1}{u}\eta\left(\frac{x}{u}\right)du \\
&= \sum_1^\infty \frac{1}{n}\eta\left(\frac{x}{n}\right) - \sum_1^\infty \frac{1}{n}\eta\left(\frac{x}{n}\right) + \int_1^\infty \frac{1}{u}\eta\left(\frac{x}{u}\right)du \\
&= O(1) - \frac{1}{2}\eta(x) - \int_1^\infty \bar{B}_1(u)\,d\left[\frac{1}{u}\eta\left(\frac{x}{u}\right)\right] \\
&= O(1) + \int_1^\infty \bar{B}_1(u)\eta\left(\frac{x}{u}\right)\frac{du}{u^2} - \int_1^\infty \frac{\bar{B}_1(u)}{u}\,d\eta\left(\frac{x}{u}\right)
\end{aligned}$$

$$= O(1) + O(1) + \int_1^\infty \frac{\bar{B}_1(u)}{u}\, dG\left(\frac{x}{u}\right) - \int_1^\infty \frac{\bar{B}_1(u)}{u}\, dP\left(\frac{x}{u}\right)$$
$$= O(1) + \int_1^{x/a} \frac{\bar{B}_1(u)}{u}\, d\log\left(\frac{1}{u}\right) - \int_1^\infty \frac{\bar{B}_1(u)}{u} \cdot \frac{u}{x}\, d\psi\left(\frac{x}{u}\right)$$
$$= O(1) - \frac{1}{x}\int_1^\infty \bar{B}_1(u)\, d\psi\left(\frac{x}{u}\right)$$
$$= O(1) + O\left(\frac{\psi(x)}{x}\right) = O(1).$$

The prime number theorem can now be deduced from the following theorem.

Theorem Let B be an upper bound for $|\eta(x)|$ and let M be an upper bound for $|\int_1^x \eta(u)\, d\log u|$. Without loss of generality assume $M > B^2$. Then if $\beta > 0$ is any number less than or equal to B with the property that $|\eta(x)| \leq \beta$ for all sufficiently large x, the number

$$\beta' = \beta - (\beta^3/400M)$$

has the same property.

Proof Let β be given and let x_0 be such that $|\eta(x)| \leq \beta$ for $x \geq x_0$. Divide the interval $[x_0, \infty)$ into subintervals on which η changes by less than $\frac{1}{3}\beta$. Specifically, define λ by $\log \lambda = \frac{1}{4}\beta$ and set $x_j = \lambda^j x_0$. Then $x_j \leq y < x \leq x_{j+1}$ implies by (7) that $|\eta(x) - \eta(y)| \leq \log x - \log y + O(1/\log x) \leq \log x_{j+1} - \log x_j + O(1/\log x)_j \leq \log \lambda + O(1/\log x_0)$. By increasing x_0 if necessary this gives $|\eta(x) - \eta(y)| < \frac{1}{3}\beta$ as desired.

Call $|\eta|$ "small" on the interval $[x_j, x_{j+1}]$ if its value at either or both ends is less than $\frac{1}{2}\beta$ and otherwise call it "large" on the interval. Since η cannot change sign on an interval where $|\eta|$ is large, the average of $|\eta|$ can be estimated using (8). In fact, if $|\eta|$ is large on all intervals between x_j and x_{j+k}, then the average of $|\eta|$ is at most

$$\frac{1}{\log(x_{j+k}/x_j)} \int_{x_j}^{x_{j+k}} |\eta(v)|\, d\log v$$
$$= \frac{1}{k \log \lambda} \left| \int_{x_j}^{x_{j+k}} \eta(v)\, d\log v \right|$$
$$= \frac{4}{k\beta} \left| \int_1^{x_{j+k}} \eta(v)\, d\log v - \int_1^{x_j} \eta(v)\, d\log v \right| \leq \frac{8M}{k\beta}.$$

This can be made strictly less than β by making k large; for example, $(8M/k\beta) \leq \frac{5}{6}\beta$ when $k \geq 48M/5\beta^2$. On the other hand, if $|\eta|$ is small on at least one of the intervals between x_j and x_{j+k}, then the average of $|\eta|$ is at most $k^{-1}[(k-1)\beta + \frac{5}{6}\beta] = \beta - (\beta/6k)$ because $|\eta| \leq \beta$ throughout, and because among the k intervals of equal weight $\log(x_{n+1}/x_n) = \log \lambda$ there is at least one on which $|\eta| \leq \frac{5}{6}\beta$ throughout. Thus the average is strictly less than β in either case. Fix k as the smallest integer satisfying $k \geq 48M/5\beta^2$.

Then $k < (48M + 5\beta^2)/5\beta^2 \leq (48M + 5B^2)/5\beta^2 \leq 53M/5\beta^2$, $6k < 100M/\beta^2$, $\beta/6k > \beta^3/100M$, and finally

$$\beta - (\beta/6k) < \beta - (\beta^3/100M);$$

so the average of $|\eta|$ on $[x_j, x_{j+k}]$ (always with respect to the measure $d \log x$) is either less than this amount or less than $\frac{5}{6}\beta$. But since $\beta^2 \leq B^2 < M$ gives $\beta^2/100M < 1/100 < \frac{1}{6}$, this shows that

$$\frac{1}{\log(x_{j+k}/x_j)} \int_{x_j}^{x_{j+k}} |\eta(v)|\, d \log v \leq \beta - \frac{\beta^3}{100M}$$

for all $j = 0, 1, 2, \ldots$ when k is defined as above.

Consider now the average $(\log u)^{-1} \int_1^u |\eta(v)|\, d \log v$ for large u. Let the interval $[1, u]$ be divided into three parts, the interval $[1, x_0]$, the interval $[x_0, x_{nk}]$, and the interval $[x_{nk}, u]$, where n is the largest integer such that $x_{nk} \leq u$, with k as before. In the whole interval $[1, u]$ these three intervals count with weights $\log x_0/\log u$, $nk \log \lambda/\log u$, and $\log (u/x_{nk})/\log u < \log (x_{nk+k}/x_{nk})/\log u < k \log \lambda/\log u$, respectively. Thus the first and last intervals count with weights which approach zero as $u \to \infty$, and the averages on these intervals are constant on the first one and at most β on the last one. On the middle interval the average is at most $\beta - (\beta^3/100M)$. Therefore as $u \to \infty$, the average on the entire interval $[1, u]$ can be only slightly greater than $\beta - (\beta^3/100M)$, say

$$\frac{1}{\log u} \int_1^u |\eta(v)|\, d \log v \leq \beta - \frac{\beta^3}{200M}$$

for all sufficiently large u. But then the average of this amount over $1 \leq u \leq (x/a)$ relative to the measure $2 \log u\, d \log u$ for large x can be only slightly greater than $\beta - (\beta^2/200M)$. Thus the inequality (6) implies the desired conclusion.

Corollary $\eta(x) \to 0$ as $x \to \infty$.

Proof Start with $\beta = B$ and apply the theorem repeatedly. This gives a decreasing sequence $\beta_0 > \beta_1 > \beta_2 > \cdots$ of positive numbers such that for each β_n the inequality $|\eta(x)| \leq \beta_n$ holds for all sufficiently large x. Since $\beta_n \to 0$ as $n \to \infty$, this proves the corollary.

This completes the elementary proof of the prime number theorem. It is natural to ask whether the stronger theorem $\eta(x) = O(\log^{-n} x)$ can also be proved by elementary methods. This was accomplished in the 1960s by both Wirsing [W5] and Bombieri [B9]. Thus the prime number theorem with the error estimate

$$\psi(x) = x + O(x/\log^n x)$$

can be proved by "elementary" methods.

12.11 OTHER ZETA FUNCTIONS. WEIL'S THEOREM

In concentrating exclusively on the study of the zeta function and its relation to the prime number theorem, this book ignores one of the most fruitful areas of development of Riemann's work, namely, number theory. The use of functions like the zeta function in number theory was a major feature of the work of Dirichlet—both in his L-series and in his formula for the class number of a quadratic number field—many years before Riemann's paper appeared, and the use of such functions has been a prominent theme in number theory ever since. Riemann's contributions in this area were primarily function-theoretic, not number-theoretic, and consisted of focusing attention on the functions as functions of a *complex* variable, on the possibility of their satisfying a functional equation under $s \leftrightarrow 1 - s$, and on the importance of the location of their complex zeros. A few of the most important names in the subsequent study of these number-theoretic functions are those of Dedekind, Hilbert, Hecke, Artin, Weil, and Tate.

Ignorance prevents me from entering into a discussion of these functions and what is known about them. However, it seems that they provide some of the best reasons for believing that the Riemann hypothesis is true—for believing, in other words, that there is a profound and as yet uncomprehended number-theoretic phenomenon, one facet of which is that the roots ρ all lie on $\operatorname{Re} s = \frac{1}{2}$. In particular, there is a "zeta function" associated in a natural number-theoretic way to any function field over a finite field, and Weil [W2] has shown that *the analog of the Riemann hypothesis is true for such "zeta functions."*

APPENDIX

On the Number of Primes Less Than a Given Magnitude

by BERNHARD RIEMANN†

I believe I can best express my gratitude for the honor which the Academy has bestowed on me in naming me as one of its correspondents by immediately availing myself of the privilege this entails to communicate an investigation of the frequency of prime numbers, a subject which because of the interest shown in it by Gauss and Dirichlet over many years seems not wholly unworthy of such a communication.

In this investigation I take as my starting point the observation of Euler that the product

$$\prod \frac{1}{1 - \dfrac{1}{p^s}} = \sum \frac{1}{n^s},$$

where p ranges over all prime numbers and n over all whole numbers. The function of a complex variable s which these two expressions define when they converge I denote by $\zeta(s)$. They converge only when the real part of s is greater than 1; however, it is easy to find an expression of the function which always is valid. By applying the equation

$$\int_0^\infty e^{-nx} x^{s-1}\, dx = \frac{\Pi(s-1)}{n^s},$$

one finds first

$$\Pi(s-1)\zeta(s) = \int_0^\infty \frac{x^{s-1}\, dx}{e^x - 1}.$$

If one considers the integral

$$\int \frac{(-x)^{s-1}\, dx}{e^x - 1}$$

†Translated from *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse* [R1, p. 145] by H. M. Edwards.

from $+\infty$ to $+\infty$ in the positive sense around the boundary of a domain which contains the value 0 but no other singularity of the integrand in its interior, then it is easily seen to be equal to

$$(e^{-\pi si} - e^{\pi si}) \int_0^\infty \frac{x^{s-1}\,dx}{e^x - 1},$$

provided that in the many-valued function $(-x)^{s-1} = e^{(s-1)\log(-x)}$ the logarithm of $-x$ is determined in such a way that it is real for negative values of x. Thus

$$2 \sin \pi s\, \Pi(s-1)\,\zeta(s) = i \int_\infty^\infty \frac{(-x)^{s-1}\,dx}{e^x - 1}$$

when the integral is defined as above.

This equation gives the value of the function $\zeta(s)$ for all complex s and shows that it is single-valued and finite for all values of s other than 1, and also that it vanishes when s is a negative even integer.

When the real part of s is negative, the integral can be taken, instead of in the positive sense around the boundary of the given domain, in the negative sense around the complement of this domain because in that case (when $\operatorname{Re} s < 0$) the integral over values with infinitely large modulus is infinitely small. But inside this complementary domain the only singularities of the integrand are at the integer multiples of $2\pi i$, and the integral is therefore equal to the sum of the integrals taken around these singularities in the negative sense. Since the integral around the value $n2\pi i$ is $(-n2\pi i)^{s-1}(-2\pi i)$, this gives

$$2 \sin \pi s\, \Pi(s-1)\,\zeta(s) = (2\pi)^s \sum n^{s-1}[(-i)^{s-1} + i^{s-1}],$$

and therefore a relation between $\zeta(s)$ and $\zeta(1-s)$ which, by making use of known properties of the function Π, can also be formulated as the statement that

$$\Pi\left(\frac{s}{2} - 1\right)\pi^{-s/2}\zeta(s)$$

remains unchanged when s is replaced by $1-s$.

This property of the function motivated me to consider the integral $\Pi((s/2)-1)$ instead of the integral $\Pi(s-1)$ in the general term of $\sum n^{-s}$, which leads to a very convenient expression of the function $\zeta(s)$. In fact

$$\frac{1}{n^s}\Pi\left(\frac{s}{2} - 1\right)\pi^{-s/2} = \int_0^\infty e^{-nn\pi x} x^{(s/2)-1}\,dx;$$

so when one sets

$$\sum_1^\infty e^{-nn\pi x} = \psi(x),$$

it follows that

$$\Pi\left(\frac{s}{2} - 1\right)\pi^{-s/2}\zeta(s) = \int_0^\infty \psi(x) x^{(s/2)-1}\,dx$$