are to first order simple poles with residue b_x. Namely for $x \in -Spec$,

$$f(z+x) = \frac{b_x}{z} + O(z^{\epsilon}).$$

Proof. As in Theorem 7 there is the Mellin inversion integral for c sufficiently large,

$$f(z) = \frac{1}{2\pi i}\int_{(c)} L(s)\Gamma(s)z^{-s}ds.$$

Shifting the contour to $1-d$ for large d, noting that the poles of $\Gamma(s)$ contribute a polynomial $p(z)$,

$$f(z) + p(z) = \frac{1}{2\pi i}\int_{(1-d)} L(s)\Gamma(s)z^{-s}ds.$$

Replacing d with $1-d$, we see that

$$f(z) + p(z) = \frac{1}{2\pi i}\int_{(d)} L(1-s)\Gamma(1-s)z^{s-1}ds.$$

For d sufficiently large we may substitute in the general functional equation to get

$$f(z) + p(z) = \frac{1}{2\pi i}\int_{(d)} \sum_{x\in\mathrm{Spec}} b_x x^{-s}\gamma_x(s)\Gamma(1-s)z^{s-1}ds,$$

and by absolute convergence interchange the order of summation and integration to get

$$f(z) + p(z) = \sum_{x\in\mathrm{Spec}} b_x \frac{1}{2\pi i}\int_{(d)} (z/x)^s\gamma_x(s)\Gamma(1-s)z^{-1}ds.$$

Defining the inner integral as the function G_x,

$$f(z) + p(z) = \sum_{x\in\mathrm{Spec}} b_x G_x(z/x)z^{-1},$$

where

$$G_x(z) := \frac{1}{2\pi i}\int_{(d)} \gamma_x(s)\Gamma(1-s)z^{s}ds.$$

As in the proof of Theorem 10, we may apply Lemma 6 to estimate the integrand away from the poles,

$$G_x(z) = \frac{1}{2\pi i}\int_{(d)} \frac{\pi}{\sin(\pi s)}(1+O(1/s))z^s ds.$$

In particular, for $s = \sigma + iT$when $\arg(z) < \pi$ then the integral is absolutely convergent to a holomorphic function. This is because, letting $s = \sigma + iT$ for fixed σ where $\sin(\pi\sigma) \neq 0$, $\sin(\pi s) \ll e^{\pi/2T}$ and $|z^s| \sim e^{\arg(z)T}|z|^\sigma$.

When $|z| < 1$ the integrand converges uniformly to 0 as $\Re s$ increases to ∞, away from the poles. Shifting the contour to the right we pick up on a sum over residues at $-n = 1 - s$,

$$G_x(z) = \sum_{n>d-1} \gamma_x(n-1)z^{n-1}/n!,$$

and this expression is absolutely convergent when $|z| < 1$.

When $|z| > 1$ the integrand converges to 0 as $\Re z$ decreases to $-\infty$, away from the poles. Shifting the contour to the left we pick up on a sum over the residues at the poles of $\gamma_x(y)$,

$$G_x(z) = \sum_y \mathrm{Res}_{z=y}(\gamma_x(y)\Gamma(1-y)z^y),$$

and this expression is absolutely convergent when $|z| > 1$ to a multivalued function.

□

Non-linear twist Dirichlet series

Similar to Kaczorowski and Perelli[52], given a Dirichlet series with a general degree d functional equation, define the non-linear twist Dirichlet series

$$L(s,\alpha) := \sum_{n=1}^{\infty} a_n e^{-\alpha n^{1/d}} n^{-s}.$$

Theorem 13. *As long as $\alpha \notin -Spec$, $L(s,\alpha)$ is entire and of horizontal and vertical degree d. This holds not only for $\Re\alpha \geq 0$ but also to $\Re\alpha \leq 0$ by analytic continuation.*

Kaczorowski and Perelli also derive an analytic continuation, but do not prove that the resulting function has horizontal and vertical degree d, which is the optimal bound.

Proof. First consider $\Re\alpha \geq 0$ and $\alpha \notin -\mathrm{Spec}$. The shifted function $f(z+\alpha) = \sum_{n=1}^{\infty} a_n e^{-n^{1/d}z} e^{-n^{1/d}\alpha}$ is analytic in a neighbourhood of $z = 0$, so therefore the corresponding Dirichlet series $L(s,\alpha)$ is entire and of horizontal and vertical degree d. $f(z+\alpha)$ is a well-defined analytic function even for $\Re(\alpha) < 0$ so the Mellin transform formula $\Gamma(s)L(s,\alpha) = \frac{1}{2\pi i}\int_{(c)} f(z+\alpha) z^{-s} ds$ that recovers $L(s,\alpha)$ when $\Re(\alpha) > 0$ is likewise well-defined for all $\alpha \notin -\mathrm{Spec}$ and defines the analytic continuation. $\square$

A consequence in degree 2

We can also prove the following partial classification in degree 2.

In degree 1 we characterized series of the form $\sum a_n e^{-nz}$ by their periodicity. In degree 2 we can characterize series of the form$\sum a_n e^{-n^{1/2}z}$ by a convolutional transform.

For simplicity suppose that we have a general functional equation and the Gamma factors $\gamma_x(s)$ are all simply $\Gamma(s)$. In this case the singularities are simple poles

$$G_x(z) = \frac{1}{2\pi i}\int_{(d)} \frac{\pi}{\sin(\pi s)} z^{-s} ds = \frac{1}{1-z} + p_x(z).$$

Define the following function of two variables as an extension of the non-linear twist,

$$f(z,w) \;=\; \sum_{n=1}^{\infty} a_n e^{-\sqrt{n}z} e^{-inw}.$$

When $w > 0$, take the contour $\mathcal{C}$ following a path from $z-\delta+i\infty$ to $z-\delta$ to $z+\delta$ to $z+\delta-i\infty$. This is chosen so that $\Im((t-z)^2) < 0$ and hence $e^{(t-z)^2/(4wi)}$ decays exponentially. Since the Gaussian is self-dual under the Fourier transform,

$$\sqrt{-4\pi i w}e^{-\sqrt{n}z}e^{-inw} = \int_{\mathcal{C}} e^{-\sqrt{n}t} e^{(t-z)^2/(4wi)} dt.$$

Hence, summing with linear coefficients a_n,

$$\sqrt{-4\pi i w} f(z,w) = \int_{\mathcal{C}} f(t) e^{(t-z)^2/(4wi)} dt.$$

Rotate the contour by a small angle clockwise, so that for every z and as t ranges over the contour, $(t-z)^2$ eventually has steadily decreasing imaginary part, so $e^{(t-z)^2/(4wi)}$ decays rapidly, yielding an entire function of z. When performing this shift of the contour we pick up the contributions from the poles on the positive imaginary axis, denoted by $\mathrm{Spec}_{+i} = \mathrm{Spec} \cap (i\mathbb{R}_{>0})$. Doing this we get the formula

$$\sqrt{-4\pi i w} f(z,w) - (\text{entire}) = \sum_{x \in \mathrm{Spec}_{+i}} b_x e^{(x-z)^2/(4wi)},$$

which simplifies to

$$\sqrt{-4\pi i w} f(z,w) - (\text{entire}) = e^{z^2/(4wi)} \sum_{x \in \mathrm{Spec}_{+i}} b_x e^{ixz/(2w)} e^{x^2/(4wi)}.$$

Suppose $\mathrm{Spec}_{+i} \subset 4\pi\sqrt{\mathbb{N}} i$ and for some Q and some constant C, $b_{4\pi\sqrt{n}Q} = a_n C$, as would happen if $L(s)$ satisfies a functional equation. Then the above comes to

$$\sqrt{-4\pi i w} f(z,w) - (\text{entire}) = e^{z^2/(4wi)} \sum_{n=1}^{\infty} C a_n e^{-2\pi Q\sqrt{n} z/w} e^{-16\pi^2 Q^2 n/(4wi)}.$$

Letting $w = 2\pi$ we get

$$\sqrt{-8\pi i} f(z) - (\text{entire}) = e^{z^2/(8\pi i)} C f(Qz),$$

but the exponential factor blows up singularities not on the real and imaginary axis, so therefore the only singularities may lie on the real and imaginary axis.

4.3 Computing Hecke eigenbases

Given a linear combination of finitely many distinct multiplicative sequences, it is possible to recover each of the original sequences. In particular, this can be an effective way to compute the Hecke eigenbases in $\Gamma_0(\chi)$ from knowledge of the Hecke traces. This approach comes from Andrew Booker.

For ease of exposition, we restrict to the case of weight k cusp forms on $\Gamma_0(N)$ for some fixed character χ, so that we have a known degree 2 Euler product. There is an unknown eigenbasis $f_1, \cdots, f_k$ with corresponding Fourier coefficients $a_i(n)$. We assume we are given $T(n)$ which is known to be a sum of multiplicative functions $T(n) = a_1(n) + \cdots + a_k(n)$.

Define $T(m,n) = a_1(m)a_1(n) + \cdots a_k(m)a_k(n)$. By multiplicativity conditions $a_i(m)a_i(n) = \sum_{d|(m,n)} a_i(mn/d^2)\chi(d)d^{k-1}$, so $T(m,n) = \sum_{d|(m,n)} T(mn/d^2)\chi(d)d^{k-1}$.

Define $T(m,n,p) = a_1(m)a_1(n)a_1(p) + \cdots a_k(m)a_k(n)a_k(p)$, which can similarly be computed in terms of T, as $T(m,n,p) = \sum_{d|(m,n)} T(mn/d^2, p)\chi(d)d^{k-1}$.

Now, $T(n) = \sum a_i(n)$, $T(2,n) = \sum a_i(2)a_i(n)$, $T(3,n) = \sum a_i(3)a_i(n)$ and so forth. These are all linear combinations of the $a_i(n)$, so given sufficiently many we span the whole space and can recover the a_i as linear combinations $\sum_r T(r,n)\alpha_{r,i}$.

Let $\vec{a}_i$ be the infinite vector $(a_i(n))_n$.

Form the matrices $T = (T(m,n))_{m,n}$ and $T_p = (T(m,n,p))_{m,n}$. T_p is symmetric, and T is symmetric and positive definite since it is the sum of outer products $\sum a_i^T a_i$. Let $\vec{v}_i \in \mathbb{C}^k$ be the unknown vector such that $T\vec{v}_i = \vec{a}_i$. Note also that $T_p\vec{v}_i = a_i(p)\vec{v}_i$. Thus for given p we may solve the generalized eigenvalue problem $T_p\vec{v} = \lambda T\vec{v}$ to recover the $\vec{v}_i$ assuming all $a_i(p)$ are distinct.

If the $a_i(p)$ for the selected p are not all distinct then this fails to recover unique eigenvectors. However, this cannot simultaneously happen for very many p, so taking a random linear combination of a few different T_p will result in distinct dimension 1 eigenspaces.

It remains to show that the $T(n)$ may be effectively computed, and so we present a trace formula for these.

4.3.1 Eichler-Selberg Trace Formula

This section follows [102][63] for the case of the Eichler-Selberg Trace Formula on $SL_2(\mathbb{Z})$ and [85][22] for $S_k(\Gamma_0(N), \chi)$. The proofs are given explicitly in Oesterle's thesis [82].

We define a function $H(n)$ on integers as follows. Let $H(0)$ if $n > 0$ and $H(0) = -1/12$. For $n < 0$ let $H(n)$ denote the number of equivalence classes with respect to $SL_2(\mathbb{Z})$ of positive definite binary quadratic forms

$$ax^2 + bxy + cy^2$$

with discriminant

$$b^2 - 4ac = n,$$

counting forms equivalent to a multiple of x^2+y^2 (resp. x^2+xy+y^2) with multiplicity $\frac{1}{2}$ (resp. $\frac{1}{3}$).

This function is closely related to the ordinary class numbers $h(n)$ as follows. Define $h_w(-3) = \frac{1}{3}$, $h_w(-4) = \frac{1}{2}$, $h_w(n) = h(n)$ if $n < -4$ and congruent to 0 or 1 (mod 4), and $h_w(x) = 0$ otherwise. Then

$$H(n) = \sum_{f|n} h_w\left(\frac{n}{f^2}\right)$$

Also define polynomials $Q_k(t, n)$ for $k = 0, 1, \cdots$ so that

$$\frac{u^{k+1} - v^{k+1}}{x - y} = Q_k(u + v, uv).$$

These polynomials have generating series

$$(1 - tx + Nx^2)^{-1} = \sum x^k Q_k(t, n).$$

The polynomials can also be computed recursively, with base conditions $Q_0(t, n) =$

1 and $Q_1(t,n) = t$, and recurrence

$$Q_{k+1}(t,n) = tQ_k(t,n) - nQ_{k-1}(t,n).$$

Theorem 14. *(Trace Formula)*

Let $k \geq 4$ be an even integer and let m be a natural number. Then the trace of the Hecke operator $T(m)$ on the space of cusp forms S_k is given by

$$\operatorname{Tr} T(m) = -\frac{1}{2} \sum_{\substack{t \in \mathbb{Z} \\ t^2-4m \leq 0}} Q_{k-2}(t,m) H(t^2-4m) - \frac{1}{2} \sum_{dd'=m} \min(d,d')^{k-1}.$$

Theorem 15. *(Trace Formula) Let $N \in \mathbb{Z}_{\geq 1}$ and let $\chi : (\mathbb{Z}/N\mathbb{Z})^* \mapsto \mathbb{C}^*$ be a character of conductor N_χ. Furthermore, let $k \geq 2$ be an integer for which $\chi(-1) = (-1)^k$.*

For every integer $n \geq 1$ the trace Tr of the Hecker operator T_n acting on the space of cusp forms $S_k(\Gamma_0(N), \chi)$ is given by

$$\operatorname{Tr}(T_n) = A_1(\chi) + A_2(\chi) + A_3(\chi) + A_4(\chi),$$

where

$$A_1(\chi) = n^{k/2-1}\chi(\sqrt{n})\frac{k-1}{12}\psi(N).$$

(Here $\chi(\sqrt{n}) = 0$ whenever $\sqrt{n} \notin \mathbb{Z}$ and $\psi(N)$ denotes $N\prod_{p|N}(1+1/p)$, where the product runs over the prime divisors p of N).

$$A_2(\chi) = -\frac{1}{2} \sum_{t \in \mathbb{Z}, t^2 < 4n} Q_{k-2}(t,n) \sum_f h_w\left(\frac{t^2-4n}{f^2}\right) \mu(t,f,n,\chi).$$

(Here the sum runs over the positive divisors f of t^2-4n for which $(t^2-4n)/f^2 \in \mathbb{Z}$ is congruent to 0 or 1 (mod 4). The numbers $\mu(t,f,n,\chi)$ are given by

$$\mu(t,f,n,\chi) = \frac{\psi(N)}{\Psi(N/N_f)} \sum_{\substack{x \ (mod\ N) \\ x^2-tx+n \equiv 0 \ (mod\ N_f N))}} \chi(x),$$

where N_f *denotes* $\gcd(N, f)$.

$$A_3(\chi) = -\sideset{}{'}\sum_{\substack{d|n \\ 0<d\leq\sqrt{n}}} d^{k-1} \sum_{\substack{c|N \\ \gcd(c,N/c)|\gcd(N/N_\chi, n/d-d)}} \phi\left(\gcd\left(c, \frac{N}{c}\right)\right)\chi(y).$$

(Here ϕ *denotes Euler's phi-function; the prime in the first summation indicates that the contribution of the term* $d = \sqrt{n}$*, if it occurs, should be multiplied by* $\frac{1}{2}$*. The number* y *is defined modulo* $N/\gcd(c, N/c)$ *by* $y \equiv d \pmod{c}$*,* $y \equiv (n/d) \pmod{N/c}$*.)*

$$A_4(\chi) = \begin{cases} \sum_{\substack{0<t|n \\ \gcd(N,n/t)=1}} t & \text{if } k = 2 \text{ and } \chi = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(The unit character is denoted by 1*. We recall that every character* χ *is extended* $\mathbb{Z}/N\mathbb{Z}$ *by* $\chi(m) = 0$ *whenever* $\gcd(m, N) > 1$*).*

To allow for computations to be done over integers it will be convenient for fixed N to take the Fourier transform of the A_i, over the characters χ.

Theorem 16.

$$\hat{A}_1(a) = \begin{cases} n^{k/2-1}\frac{k-1}{12}\psi(N) & \text{if } n \equiv a \pmod{N} \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{\mu}(t, f, n, a) = \begin{cases} \frac{\psi(N)}{\Psi(N/N_f)} & \text{if } N * N_f | a^2 - ta + n \\ 0 & \text{otherwise} \end{cases}$$

where N_f *denotes* $\gcd(N, f)$.

$$\hat{A}_2(a) == -\frac{1}{2}\sum_{t\in\mathbb{Z}, t^2<4n} Q_{k-2}(t, n)\sum_f h_w\left(\frac{t^2-4n}{f^2}\right)\hat{\mu}(t, f, n, a).$$

$$\hat{A}_3(a) = -\sideset{}{'}\sum_{\substack{d|n \\ 0<d\leq\sqrt{n}}} d^{k-1} \sum_{\substack{c|N \\ a\equiv d \pmod{c} \\ a\equiv n/d \pmod{N/c}}} \phi\left(\gcd\left(c, \frac{N}{c}\right)\right)/\gcd(c, \frac{N}{c}).$$

$$\hat{A}_4(a) = \frac{1}{\phi(N)} \begin{cases} \sum_{\substack{0<t|n \\ \gcd(N,n/t)=1}} t & \textit{if } k=2, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The formulae for $\hat{A}_1$, $\hat{\mu}$, A_2 only rely on the Fourier transform of χ being the indicator of the congruence class a. The formula for $\hat{A}_4$ has a factor of $\frac{1}{\phi(N)}$ coming in from only the trivial character contributing.

In $\hat{A}_3$ care needs to be taken when dealing with non-primitive characters. □

4.3.2 Algorithm

To compute first M coefficients of all Hecke eigenforms of weight k on $\Gamma_0(N)$ for all characters χ mod N.

Compute sufficiently many primes, and sufficiently many Dirichlet coefficients of $1/\zeta(s) = \sum_{n=1}^{\infty} \mu(n) n^{-s}$ and $1/\zeta(s)^2 = \sum_{n=1}^{\infty} \mu_2(n) n^{-s}$.

Compute sufficiently many $\hat{A}_i(\chi)$, in exact integer arithmetic since these are all in $\mathbb{Z}/12$.

For each character χ, Fourier invert to compute $A_i(\chi), \cdots, A_4(\chi)$ and hence sufficiently many $\operatorname{Tr} T(m)$. Define $T(m,n) = \sum_{d|(m,n)} \operatorname{Tr} T(mn/d^2)\chi(d)d^{k-1}$. Define $T(m,n,p) = \sum_{d|(m,n)} T(mn/d^2, p)\chi(d)d^{k-1}$.

For each m and given χ, $(\operatorname{Tr} T(m))^2 \overline{\chi}(m)$ is real. Define $\sqrt{\chi(m)}$, picking the sign of $\sqrt{\chi(p)}$ arbitrarily, and picking the sign of $\sqrt{\chi(m)}$ to retain multiplicativity. Computations may be done instead in terms of $T_m = \operatorname{Tr} T(m) \overline{\sqrt{\chi(m)}}$ so that all T_m are real.

Note that $d := \operatorname{Tr} T(1)$ is the dimension of the space of cuspforms, and if this is 0 then continue.

Iteratively form a sequence of primes $p_1, p_2, \cdots, p_d$, taking p_i to be the least prime so that the i by i matrix $(T(p_j, p_k))_{j,k}$ is invertible. Let T be the resulting d by d matrix.

For a small number of primes p, Let C be a random linear combination of the matrices $T_p = (T(p_j, p_k, p))$.

Solve the generalized self-adjoint eigenvalue problem $C\vec{v} = \lambda T\vec{v}$. Each eigenvector v represents one cusp eigenform, whose coefficients may be recovered as $a_i = \sum_{j=1}^{d} T(i, p_j) v_j$.

Appendix A

Properties of $\Gamma(s)$

In this chapter we collect some results about $\Gamma(s)$ that are needed throughout the thesis. First recall some standard results from [1].

$\Gamma(s)$ is defined by the Mellin transform

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx.$$

This transform may be inverted to get the following integral on $\Re(s) = c$

$$e^{-x} = \frac{1}{2\pi i} \int_{(c)} \Gamma(s) z^{-s} ds.$$

This is the continuous analogue of the factorial function since for $n \in \mathbb{N}$

$$\Gamma(1+n) = n!.$$

There is the reflection formula

$$\Gamma(s)\Gamma(1-s) = \pi / \sin(\pi s).$$

There is the duplication formula

$$\Gamma(2s) = \frac{1}{\sqrt{4\pi}} 4^s \Gamma(s)\Gamma(s+1/2).$$

Stirling's Formula gives the asymptotic away from the negative real axis

$$\Gamma(s) \sim \sqrt{\frac{2\pi}{s}} \left(\frac{s}{e}\right)^s .$$

Euler-Maclaurin summation gives an symptotic series away from the negative real axis

$$\log \Gamma(s) \sim (s + a - 1/2) \log s - s + \frac{1}{2} \log(2\pi) + \sum_{n=1}^{\infty} \frac{B_{n+1}(a)}{n(n+1)s^n}.$$

Following [9] and [44] gives the following useful lemma.

Lemma 5. *Let $\mu_i \in \mathbb{C}$ and $\lambda_i > 0$, for $i = 1, 2, \cdots, r$, where d=$\sum_{i=1}^{r} \lambda_i$. Then away from a sector containing the poles of the left-hand side:*

$$\prod_{i=1}^{r} \Gamma\left(\lambda_i s + \mu_i\right) \quad = \quad s^{ds} q^s s^{\mu} \left(A + O_{\lambda,\mu}\left(1/s\right)\right),$$

where $q = e^{-d} \prod_{i=1}^{r} \lambda_i^{\lambda_i}$, $\mu = \sum_{i=1}^{r} (\mu_i - 1/2)$, and $A = \sqrt{2\pi}^r \prod \lambda_i^{\mu_i - 1/2}$.

Proof. Euler-Maclaurin summation gives Stirling's formula away from the poles, as the asymptotic formula[1],

$$\log \Gamma\left(s + a\right) \quad = \quad \left(s + a - 1/2\right) \log s - s + \frac{\log 2\pi}{2} + \sum_{n=1}^{\infty} \frac{B_{n+1}\left(a\right)}{n\left(n+1\right) s^n}.$$

Hence

$$\log \Gamma\left(\lambda s + \mu\right) \quad = \quad \left(\lambda s + \mu - 1/2\right) \log\left(\lambda s\right) - \lambda s + \frac{\log 2\pi}{2} + O_{\lambda,\mu}\left(1/s\right).$$

Applying this to each summand of the left-hand side gives

$$\begin{aligned} \sum_{i=1}^{r} \log \Gamma(\lambda_i s + \mu_i) \quad = \quad & \Big(ds + \sum \left(\mu_i - 1/2\right)\Big) \log s + \sum_{i=1}^{r} \lambda_i s \log \lambda_i \\ & + \sum_{i=1}^{r} \left(\mu_i - 1/2\right) \log \lambda_i - ds + \frac{r \log 2\pi}{2} + O_{\lambda,\mu}\left(1/s\right), \end{aligned}$$

which simplifies to

$$\begin{aligned}\sum_{i=1}^{r} \log \Gamma(\lambda_i s + \mu_i) &= ds \log s + \sum \lambda_i \log \lambda_i s - ds + \sum (\mu_i - 1/2) \log s \\ &\quad + \sum_{i=1}^{r} (\mu_i - 1/2) \log \lambda_i + \frac{r \log 2\pi}{2} + O(1/s) \\ &= ds \log s + s \log q + \mu \log s + \log A + O(1/s),\end{aligned}$$

where $\log q = -d + \sum_{i=1}^{r} \lambda_i \log \lambda_i$, $\mu = \sum_{i=1}^{r} (\mu_i - 1/2)$, and $\log A = \frac{r \log 2\pi}{2} + \sum_{i=1}^{r} (\mu_i - 1/2) \log \lambda_i$. □

Furthermore, carrying through the correction terms yields the asymptotic series

$$\prod_{i=1}^{r} \Gamma(\lambda_i s + \mu_i) = s^{ds} q^s s^{\mu} \left(A_0 + A_1 s^{-1} + A_2 s^{-2} + \cdots\right),$$

where the A_n are functions of the λ_i and μ_i.

Lemma 6. *For positive λ_i summing to d*

$$\prod_{i=1}^{r} \Gamma(\lambda_i s + \mu_i) = \Gamma(ds + \mu) q^s (A + O(1/s))$$

where $q = d^{-d} \prod_{i=1}^{r} \lambda_i^{\lambda_i}$, $\mu - 1/2 = \sum_{i=1}^{r} (\mu_i - \frac{1}{2})$, *and* $A = (2\pi)^{(r-1)/2} \prod \lambda_i^{\mu_i - 1/2} d^{1/2-\mu}$.

Proof. Applying Lemma 5 to the left-hand side we get

$$\prod_{i=1}^{r} \Gamma(\lambda_i s + \mu_i) = s^{ds} \left(e^{-d} \prod_{i=1}^{r} \lambda_i^{\lambda_i}\right)^s s^{\sum_{i=1}^{r} (\mu_i - 1/2)} ((2\pi)^{r/2} \prod_{i=1}^{r} \lambda_i^{\mu_i - 1/2} + O(1/s)).$$

Applying the lemma to the right side we get the equivalent expression

$$\Gamma(ds + \mu) q^s (A + O(1/s)) = (s/e)^{ds} e^{-ds} s^{\mu - 1/2} (\sqrt{2\pi}^r + O(1/s)) \prod_{i=1}^{r} \lambda_i^{\lambda_i s + \mu_i - 1/2}.$$

□

Lemma 7. *For i in $1, 2, \cdots, r_a$ and j in $1, 2, \cdots, r_b$, given $\mu_{a,i} \in \mathbb{C}$, $\mu_{b,j} \in \mathbb{C}$, $\lambda_{a,i}, \lambda_{b,j} > 0$, where $\sum_{i=1}^{r_a} \lambda_{a,i} = \sum_{j=1}^{r_b} \lambda_{b,j}$, when $\mathrm{Re}\,(s)$ is sufficiently large:*

$$\prod_{i=1}^{r_a} \Gamma\left(\lambda_{a,i} s + \mu_{a,i}\right) \prod_{j=1}^{r_b} \Gamma\left(-\lambda_{b,j} s + \mu_{b,j}\right) = q^s s^\mu \left(A + O\left(1/s\right)\right) \prod_{j=1}^{r_b} \frac{\pi}{\sin\left(-\pi \lambda_{b,j} s + \pi \mu_{b,j}\right)},$$

where $q = \prod_{i=1}^{r_a} \lambda_{a,i}^{\lambda_{a,i}} / \prod_{j=1}^{r_b} \lambda_{b,j}^{\lambda_{b,j}}$, $\mu = \sum_{i=1}^{r_a} \left(\mu_{a,i} - 1/2\right) + \sum_{j=1}^{r_b} \left(\mu_{b,i} - 1/2\right)$, and $A = \sqrt{2\pi}^{r_a - r_b} \prod_{i=1}^{r_{a,i}} \lambda_{a,i}^{\mu_{a,i} - 1/2} / \prod_{j=1}^{r_b} \lambda_{b,j}^{\mu_{b,j} - 1/2}$.

Proof. The reflection formula for Γ applied to each term gives

$$\prod_{j=1}^{r_b} \Gamma\left(-\lambda_{b,j} s + \mu_{b,j}\right) = \prod_{j=1}^{r_b} \frac{\pi}{\sin\left(-\pi \lambda_{b,j} s + \pi \mu_{b,j}\right) \Gamma\left(\lambda_{b,j} s - \mu_{b,j} + 1\right)}.$$

Applying lemma 5 to both numerator and denominator of the quotient

$$\frac{\prod_{i=1}^{r_a} \Gamma\left(\lambda_{a,i} s + \mu_{a,i}\right)}{\prod_{j=1}^{r_b} \Gamma\left(\lambda_{b,j} s - \mu_{b,j} + 1\right)}$$

gives the asymptotic

$$q^s s^\mu \left(A + O\left(1/s\right)\right),$$

and the result follows. $\square$

Appendix B

On a formula for $\zeta(3)$

Apery's [3][4] proof of irrationality of $\zeta(3)$ begins with the identity, proven by combinatorial manipulation of hypergeometric series:

$$\frac{2}{5}\zeta(3) = \sum \frac{(-1)^{n-1}}{n^3\binom{2n}{n}}.$$

A new proof is presented here via a Mellin Transform, expressing $\zeta(3)$ in terms of truncating the classical integral $\Gamma(s)\zeta(s) = \int_0^\infty \frac{x^{s-1}}{e^x-1}dx$.

Proposition 3. *Apery's formula*

$$\frac{2}{5}\zeta(3) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3\binom{2n}{n}} \tag{B.1}$$

is equivalent to

$$\frac{8}{5}\zeta(3) + \frac{1}{6}\psi^3 = \int_\psi^\infty \frac{x^2}{e^x-1}dx, \tag{B.2}$$

where $\psi = \log\frac{3+\sqrt{5}}{2}$.

Proposition 4. *Formula (B.2) is true. Moreover we have*

$$\int_{\log(\alpha)}^\infty \frac{x^2}{e^x-1}dx = r\zeta(3) + s\log(\alpha)^3,$$

for the following sets of parameters (α, r, s)*:*

α	r	s
1	2	*
$\frac{3+\sqrt{5}}{2}$	$\frac{8}{5}$	$\frac{1}{6}$
2	$\frac{7}{4}$	$\frac{1}{3}$

Proof of Proposition 3. The starting point is the familiar formula [1]

$$\frac{\Gamma(s)}{n^s} = \int_0^\infty e^{-nx} x^{s-1} dx.$$

Summing both sides over n with linear coefficients a_n relates the Dirichlet series $L(s) = \sum a_n n^{-s}$ and the power series $f(z) = \sum a_n z^n$ as

$$\Gamma(s)L(s) = \int_0^\infty f(e^{-x}) x^{s-1} dx.$$

The interchange of summation and integration is valid as long as $\Re(s) > 1$ and $f(z)$ has radius of convergence greater than 1.

Consider then the power series with radius of convergence 2 [1]

$$\begin{aligned} f(z) &:= 2 \operatorname{arcsinh}(x/2)^2 \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{n^2 \binom{2n}{n}}. \end{aligned}$$

The corresponding Dirichlet series is

$$L(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2 \binom{2n}{n} (2n)^s}.$$

Setting $s = 1$ and recalling that $\Gamma(1) = 1$,

$$\begin{aligned} L(1)\Gamma(1) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n^3 \binom{2n}{n}} \\ &= 2 \int_0^\infty \operatorname{arcsinh}(e^{-x}/2)^2 dx. \end{aligned}$$

Making the subsitution $y = 2\operatorname{arcsinh}(e^{-x}/2)$ so that $x = -\log(2\sinh(y/2))$ and $dx = \frac{\cosh(y/2)}{2\sinh(y/2)}dy$,

$$\begin{aligned} 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3\binom{2n}{n}} &= 8\int_0^{\infty}\operatorname{arcsinh}(e^{-x}/2)^2 dx \\ &= \int_0^{\psi}\frac{\cosh(y/2)}{\sinh(y/2)}y^2 dy. \end{aligned}$$

Now, rewriting cosh and sinh in terms of exponentials, this is

$$\int_0^{\psi}\frac{e^y+1}{e^y-1}\cdot y^2 dy = \int_0^{\psi} y^2 dy + 2\int_0^{\psi}\frac{y^2}{e^y-1}dy.$$

The first term may be explicitly computed making this

$$\frac{1}{3}\psi^3 + 2\int_0^{\psi}\frac{y^2}{e^y-1}dy,$$

and making note that $2\zeta(3) = \int_0^{\infty}\frac{y^2}{e^y-1}dy$, we may rewrite this to give the desired expression

$$\frac{1}{3}\psi^3 + 4\zeta(3) - 2\int_{\psi}^{\infty}\frac{y^2}{e^y-1}dy.$$

$\square$

Proof of Proposition 4. We are interested in evaluating $\int_{\log(\alpha)}^{\infty}\frac{x^2}{e^x-1}dx$. Substituting $x = \log t$ and then replacing t with $1/t$ gives the equalities

$$\int_{\log(\alpha)}^{\infty}\frac{x^2}{e^x-1}dx = \int_{\alpha}^{\infty}\frac{\log^2(t)}{t(t-1)}dt = \int_0^{1/\alpha}\frac{\log^2(t)}{1-t}dt.$$

For $0 \leqslant t < \infty$ define the function $A(t)$ as the antiderivative

$$A(t) := \int\frac{\log^2(t)}{1-t}dt,$$

where the specific antiderivative is chosen so that $A(0) = 0$, and hence $A(1) = \int_0^{\infty}\frac{x^2}{e^x-1}dx = 2\zeta(3)$.

Substituting $1/t$ for t gives

$$A(1/t) = \int \frac{\log^2(t)}{t(1-t)} dt,$$

which can be expanded as partial fractions to get

$$A(1/t) = \int \frac{\log^2(t)}{1-t} dt + \int \frac{\log^2(t)}{t} dt,$$

which then evaluates to

$$A(1/t) = A(t) + \frac{1}{3}\log^3(t). \tag{B.3}$$

Note that $\lim_{t\to 0} A(t) = 0$, so $A(1/t) = o(1)$ for large t, and so for large t: $A(t) = -\frac{1}{3}\log^3(t) + o(1)$.

Substituting t^2 for t gives

$$A(t^2) = \int \frac{8t\log^2(t)}{(1-t^2)} dt,$$

which can be expanded as partial fractions to get

$$A(t^2) = \int \frac{4\log^2(t)}{1-t} dt - \int \frac{4\log^2(t)}{1+t} dt,$$

which then evaluates to

$$A(t^2) = 4A(t) - \int \frac{4\log^2(t)}{t+1}. \tag{B.4}$$

Substituting $1 + 1/t$ for t gives

$$A(1+1/t) = \int \frac{\log^2(1+1/t)}{t} dt,$$

and $\log(1 + 1/t) = \log(t + 1) - \log(t)$ so

$$A(1+1/t) = \int \frac{\log^2(t+1) + \log^2(t) - 2\log(t)\log(t+1)}{t} dt.$$

Expanding the sum inside the integral, the first two terms may be evaluated and the third term may be integrated by parts to get

$$A(1+1/t) = -A(t+1) + \frac{1}{3}\log^3(t) - \log^2(t)\log(t+1) + \int \frac{\log^2(t)}{t+1}dt.$$

Therefore

$$A(1+1/t) + A(t+1) = \frac{1}{3}\log^3(t) - \log^2(t)\log(t+1) + A(t) - \frac{1}{4}A(t^2) + C. \quad \text{(B.5)}$$

Taking t to infinity gives

$$2\zeta(3) - \frac{1}{3}\log^3(t) + o(1) = \frac{1}{3}\log^3(t) - \log^3(t) + o(1) - \frac{1}{3}\log^3(t) + \frac{2}{3}\log^3(t) + C,$$

and we conclude that $C = 2\zeta(3)$.

Now, formula (B.5) relates $A(t), A(t+1), A(t^2), A(1+1/t)$. To evaluate a single value, we need all but one distinct value to cancel out, which will happen if $t=1$ or if $t+1=t^2$ and $1+1/t=t$.

When $t=1$, formula (B.5) gives

$$2A(2) = \frac{3}{4}A(1) + C = \frac{7}{2}\zeta(3).$$

Therefore $A(2) = \frac{7}{4}\zeta(3)$ and, from $(B.3)$, $A(1/2) = \frac{7}{4}\zeta(3) + \frac{1}{3}\log^3(2)$.

For $t+1$ to equal t^2, t must equal $\varphi := \frac{1+\sqrt{5}}{2}$, and this gives

$$A(\varphi) + A(\varphi^2) = \frac{1}{3}\log^3(\varphi) - \log^2(\varphi)\log(\varphi^2) + A(\varphi) - \frac{1}{4}A(\varphi^2) + 2\zeta(3),$$

which simplifies to

$$A(\varphi^2) = \frac{8}{5}\zeta(3) - \frac{1}{6}\log^3(\varphi^2).$$

Combining this with $(B.3)$ gives

$$A(\varphi^{-2}) = \frac{8}{5}\zeta(3) + \frac{1}{6}\log^3(\varphi^2),$$

as claimed. $\square$

This method does not yield any other triples (α, r, s) and a cursory brute-force search found no such linear relationships for other quadratic α where the coefficients of the quadratic are bounded by 100.

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