CHAPTER 3

Convergence of Dirichlet Series

We will now investigate convergence of Dirichlet series. Much of the general theory holds for *generalized Dirichlet series*, that is, series of the form

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}.$$

An ordinary Dirichlet series corresponds, of course, to the case $\lambda_n = \log n$.

When dealing with a generalized Dirichlet series, we shall always assume that λ_n is a strictly increasing sequence tending to infinity, and that $\lambda_1 \geq 0$. Sometimes, an additional assumption is needed, such as the Bohr condition, namely $\lambda_{n+1} - \lambda_n \geq c/n$, for some $c > 0$.

Recall that for a power series $\sum_{n=1}^{\infty} a_n z^n$ there exists a (unique) value $R \in [0, \infty]$, called the *radius of convergence*, such that

(1) if $|z| < R$, then $\sum_{n=1}^{\infty} a_n z^n$ converges,
(2) if $|z| > R$, then $\sum_{n=1}^{\infty} a_n z^n$ diverges,
(3) for any $r < R$, the series $\sum_{n=1}^{\infty} a_n z^n$ converges uniformly and absolutely in $\{|z| \leq r\}$ and the sum is bounded on this set,
(4) on the circle $\{|z| = R\}$, the behavior is more delicate.

As we shall see, the situation for Dirichlet series is more complicated. In particular, compare the third point above with Proposition 3.10.

We start with a basic result.

THEOREM 3.1. *If the series $\sum_{n=1}^{\infty} a_n n^{-s}$ converges at some $s_0 \in \mathbb{C}$, then, for every $\delta > 0$, it converges uniformly in the sector $\{s : -\frac{\pi}{2} + \delta < \arg(s - s_0) < \frac{\pi}{2} - \delta\}$.*

Proof: As usual, we may assume $s_0 = 0$, that is, $\sum_n a_n$ converges. Let $r_n := \sum_{k=n+1}^{\infty} a_k$, and fix $\varepsilon > 0$. Then there exist $n_0 \in \mathbb{N}$ such that $|r_n| < \varepsilon$ for all $n \geq n_0$. Using summation by parts, for s in the sector and $M, N > n_0$

$$\begin{aligned}\sum_{n=M}^{N} a_n n^{-s} &= \sum_{n=M}^{N} (r_{n-1} - r_n) n^{-s} \\ &= \sum_{n=M}^{N-1} r_n \left[\frac{1}{(n+1)^s} - \frac{1}{n^s} \right] \qquad (3.2) \\ &\qquad + \frac{r_{M-1}}{M^s} - \frac{r_N}{N^s}.\end{aligned}$$

The absolute values of the last two terms are bounded by ε, since their numerators are bounded by ε while the denominators have absolute value at least 1. To estimate (3.2), note that

$$\frac{1}{(n+1)^s} - \frac{1}{n^s} = \int_n^{n+1} \frac{-s}{x^{s+1}}\, dx \ ,$$

so that

$$\left| \frac{1}{(n+1)^s} - \frac{1}{n^s} \right| \ \le \ |s| \int_n^{n+1} \frac{dx}{|x^{s+1}|} \ = \ \frac{|s|}{\sigma} \left[\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma} \right]. \qquad (3.3)$$

Thus the absolute value of (3.2) satisfies, for $M, N > n_0$,

$$\begin{aligned}\left| \sum_{n=M}^{N-1} r_n \left[\frac{1}{(n+1)^s} - \frac{1}{n^s} \right] \right| &\le \sum_{n=M}^{N-1} |r_n| \frac{|s|}{\sigma} \left[\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma} \right] \\ &\le \varepsilon \frac{|s|}{\sigma} \sum_{n=M}^{N-1} \left[\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma} \right] \\ &\le \varepsilon \frac{|s|}{\sigma} \left[\frac{1}{M^\sigma} - \frac{1}{N^\sigma} \right] \\ &\le c(\delta)\varepsilon, \qquad (3.4)\end{aligned}$$

since $\frac{|s|}{\sigma} = |1/\cos(\arg s)| \le 1/\cos\left(\frac{\pi}{2} - \delta\right) =: c(\delta)$. This proves that the series is uniformly Cauchy, and hence uniformly convergent. $\square$

COROLLARY 3.5. *If $\sum_{n=1}^{\infty} a_n n^{-s}$ converges at $s_0 \in \mathbb{C}$, then it converges in Ω_{σ_0}.*

Proof: This follows from the inclusion $\Omega_{\sigma_0} \subset \bigcup_{\delta>0} \{s : \arg|s - s_0| < \frac{\pi}{2} - \delta\}$. $\square$

This implies that there exists a unique value $\sigma_c \in [-\infty, \infty]$ such that the Dirichlet series converges to the right of it, and diverges to the left of it.

DEFINITION 3.6. The *abscissa of convergence* of the Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ is the extended real number $\sigma_c \in [-\infty, \infty]$ with the following properties

(1) if Re $s > \sigma_c$, then $\sum_{n=1}^{\infty} a_n n^{-s}$ converges,
(2) if Re $s < \sigma_c$, then $\sum_{n=1}^{\infty} a_n n^{-s}$ diverges.

NOTE 3.7. To determine the abscissa of convergence, it is enough to look at convergence of the series for $s \in \mathbb{R}$.

EXAMPLE 3.8. It may not be true that the series $\sum_{n=1}^{\infty} a_n n^{-s}$ converges absolutely in $\Omega_{\sigma_c+\delta}$ for every $\delta > 0$, in contrast with the behavior of power series. An example of this phenomenon is the *alternating zeta function* defined as

$$\tilde{\zeta}(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}.$$

First note that $\sigma_c = 0$ for this series. Indeed, the alternating series test implies convergence for all $\sigma > 0$, and the series clearly diverges if $\sigma \leq 0$. Absolute convergence of the series is convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^{\sigma}}$, so occurs if and only if $\Re(s) > 1$.

DEFINITION 3.9. Given a Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$, the *abscissa of absolute convergence* is defined as

$$\begin{aligned} \sigma_a &= \inf\left\{\rho : \sum_{n=1}^{\infty} a_n n^{-s} \text{ converges absolutely for some } s \text{ with Re } s = \rho\right\} \\ &= \inf\left\{\rho : \sum_{n=1}^{\infty} a_n n^{-s} \text{ converges absolutely for all } s \text{ with Re } s \geq \rho\right\}. \end{aligned}$$

PROPOSITION 3.10. *For any Dirichlet series, we have*

$$\sigma_c \leq \sigma_a \leq \sigma_c + 1.$$

Proof: The first inequality is obvious. For the second, assume, by the usual trick, that $\sigma_c = 0$. We need to show that for $\sigma > 1$, $\sum_{n=1}^{\infty} |a_n n^{-s}|$ converges. Take $\varepsilon > 0$ such that $\sigma - \varepsilon > 1$. Then,

$$\sum_{n=1}^{\infty} \left|\frac{a_n}{n^s}\right| = \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma}} = \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\varepsilon}} \cdot \frac{1}{n^{\sigma-\varepsilon}}, \leq C \sum_{n=1}^{\infty} \frac{1}{n^{\sigma-\varepsilon}} < \infty,$$

where $C := \sup_n \left|\frac{a_n}{n^{\varepsilon}}\right|$ is finite, since $\sigma_c = 0$. $\square$

REMARK 3.11. If $a_n > 0$ for all $n \in \mathbb{N}^+$, then $\sigma_c = \sigma_a$. This follows immediately by considering $s \in \mathbb{R}$.

Recall that for the radius of convergence of a power series, we have the following formula

$$1/R \;=\; \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}.$$

The following is an analogous formula for the abscissa of convergence of a Dirichlet series.

THEOREM 3.12. *Let* $\sum_{n=1}^{\infty} a_n n^{-s}$ *be a Dirichlet series, and let* σ_c *be its abscissa of convergence. Let* $s_n = a_1 + \cdots + a_n$ *and* $r_n = a_{n+1} + a_{n+2} + \ldots$.

(1) *If* $\sum a_n$ *diverges, then* $0 \le \sigma_c = \limsup_{n\to\infty} \frac{\log |s_n|}{\log n}$.
(2) *If* $\sum a_n$ *converges, then* $0 \ge \sigma_c = \limsup_{n\to\infty} \frac{\log |r_n|}{\log n}$.

Proof: We will show (1); the second part has a similar proof. Hence we assume that $\sum_{n=1}^{\infty} a_n$ diverges and define

$$\alpha := \limsup_{n\to\infty} \frac{\log |s_n|}{\log n}.$$

We will first prove the inequality $\alpha \le \sigma_c$. Assume that $\sum_{n=1}^{\infty} a_n n^{-\sigma}$ converges. Thus $\sigma > 0$ and we need to show that $\sigma \ge \alpha$. Let $b_n = a_n n^{-\sigma}$ and $B_n = \sum_{k=1}^{n} b_k$ (so that $B_0 = 0$). By assumption, the sequence $\{B_n\}$ is bounded, say by M, and we can use summation by parts as follows:

$$\begin{aligned} s_N &= \sum_{n=1}^{N} a_n \\ &= \sum_{n=1}^{N} b_n n^{\sigma} \\ &= \sum_{n=1}^{N-1} B_n[n^{\sigma} - (n+1)^{\sigma}] + B_N N^{\sigma} \end{aligned}$$

so that

$$\begin{aligned} |s_n| &\le M \sum_{n=1}^{N-1} [(n+1)^{\sigma} - n^{\sigma}] + M N^{\sigma} \\ &\le 2MN^{\sigma}. \end{aligned}$$

Applying the natural logarithm to both sides yields

$$\log |s_n| \;\le\; \sigma \log N + \log 2M,$$

so

$$\frac{\log |s_n|}{\log N} \leq \sigma + \frac{\log 2M}{\log N},$$

and this tends to σ as $N \to \infty$, giving the desired upper bound for α.

We need to show the other inequality: $\sigma_c \leq \alpha$. Suppose that $\sigma > \alpha$; we need to show that $\sum_{n=1}^{\infty} a_n n^{-\sigma}$ converges. Choose an $\varepsilon > 0$ such that $\alpha + \varepsilon < \sigma$. By definition, there exist $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\frac{\log |s_n|}{\log n} \leq \alpha + \varepsilon.$$

This implies that

$$\log |s_n| \;\leq\; (\alpha + \varepsilon) \log n \;=\; \log(n^{\alpha+\varepsilon}).$$

Thus, $|s_n| \leq n^{\alpha+\varepsilon}$, for all $n \geq n_0$. Observe that

$$\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma} \;=\; \sigma \int_n^{n+1} \frac{du}{u^{\sigma+1}} \;\leq\; \sigma n^{-(\sigma+1)}.$$

Using summation by parts, we can compute

$$\begin{aligned}
\sum_{n=M+1}^{N} \frac{a_n}{n^\sigma} &= \sum_{n=M}^{N} s_n \left[n^{-\sigma} - (n+1)^{-\sigma} \right] + s_N (N+1)^{-\sigma} - s_M M^{-\sigma} \\
&\leq \sum_{n=M}^{N} n^{\alpha+\varepsilon} \left[\sigma n^{-\sigma-1} \right] + N^{\alpha+\varepsilon} N^{-\sigma} + M^{\alpha+\varepsilon} M^{-\sigma} \\
&\lesssim (M-1)^{\alpha+\varepsilon-\sigma},
\end{aligned}$$

and the last quantity tends to zero as M tends to ∞.

We estimated $\sum_{n=M}^{N} n^{\alpha+\varepsilon-\sigma-1}$ by the integral $\int_{M-1}^{N-1} x^{\alpha+\varepsilon-\sigma-1}\, dx \lesssim (M-1)^{\alpha+\varepsilon-\sigma}$, and the symbol $\lesssim$ means less than or equal to a constant times the right hand-side (where the constant depends on $\alpha + \varepsilon - \sigma$, but, critically, not on M). $\square$

EXERCISE 3.13. Prove (2) of Theorem 3.12.

From the formulae above we can simply deduce formulae for the abscissa of absolute convergence, although these can be derived easily on their own.

COROLLARY 3.14. *For a Dirichlet series* $\sum_{n=1}^{\infty} a_n n^{-s}$, *we have*

1. *if* $\sum |a_n|$ *diverges, then* $\sigma_a = \limsup_{n\to\infty} \frac{\log(|a_1|+\cdots+|a_n|)}{\log n} \geq 0$,
2. *if* $\sum |a_n|$ *converges, then* $\sigma_a = \limsup_{n\to\infty} \frac{\log(|a_{n+1}|+|a_{n+2}|+\ldots)}{\log n} \leq 0$.

Proof: Recall that to determine the abscissae, one only needs to consider $s \in \mathbb{R}$ and then absolute convergence of the series is exactly convergence of the Dirichlet series whose coefficient are the absolute values of the original coefficients. $\square$

EXAMPLE 3.15. *The series*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{p_n^s}$$

has $\sigma_c = 0$ *and* $\sigma_a = 1$.

Proof: The series of coefficients diverges and so we use the first of the pair of formulae for each abscissae:

$$\sigma_c \;=\; \limsup_{n\to\infty} \frac{\log 1}{\log n} \;=\; 0,$$

and, using the prime number theorem,

$$\sigma_a \;=\; \limsup_{n\to\infty} \frac{\log(\pi(n))}{\log n} \;=\; \limsup_{n\to\infty} \frac{\log n - \log(\log n)}{\log n} \;=\; 1.$$

EXERCISE 3.16. Show that Theorem 3.1 holds for the generalized Dirichlet series $\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$, (assuming, as we always do, that λ_n is an increasing sequence tending to infinity).

(Hint: Find a substitute for (3.3), by considering the integral $\int se^{-sx}dx$.)

Therefore generalized Dirichlet series also have an abscissa of convergence.

EXERCISE 3.17. Show that Theorem3.12 implies that if the abscissa of convergence $\sigma_c \geq 0$, then

$$\forall\, \varepsilon > 0, \quad s_n \;=\; O(n^{\sigma_c+\varepsilon}). \tag{3.18}$$

CHAPTER 4

Perron's and Schnee's formulae

Suppose you know the function values $f(s)$ of some function f that, at least in some half-plane, can be represented by the Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$. How do you determine the coefficients a_n? We have seen one way already in Proposition 1.15:

$$\begin{aligned} a_1 &= \lim_{s\to\infty} f(s) \\ a_2 &= \lim_{s\to\infty} 2^s[f(s) - a_1] \\ a_3 &= \lim_{s\to\infty} 3^s[f(s) - a_1 - a_2 2^{-s}] \end{aligned}$$

and so on. The disadvantage is that these formulae are inductive. Schnee's theorem (Theorem 4.11) gives an integral formula for a_n, and Perron's formula (Theorem 4.5) gives a formula for the partial sums.

First, we need to recall the Mellin transform.

Definition 4.1. Suppose that $g(x)x^{\sigma-1} \in L^1(0,\infty)$, then

$$(\mathcal{M}g)(s) := \int_0^\infty g(x)x^{s-1}\, ds$$

is the *Mellin transform* of g at $s = \sigma + it$.

Remark 4.2. The Mellin transform is closely related to the Fourier transform and the Laplace transform. From one point of view, the Fourier transform is the Gelfand transform for the group $(\mathbb{R},+)$, while the Mellin transform is the Gelfand transform for the group $(\mathbb{R}^+,\times)$. The two groups are isomorphic and homeomorphic via the exponential map, and we can use this to derive the formula for the inverse of the Mellin transform.

Here is an inverse transform theorem for the Fourier transform. BV_{loc} means locally of bounded variation, *i.e.* every point has a neighborhood on which the total variation of the function is finite.

Theorem 4.3. *If* $h \in BV_{loc}(-\infty,\infty) \cap L^1(-\infty,\infty)$, *then*

$$\frac{1}{2}\left[h(\lambda^+) + h(\lambda^-)\right] = \frac{1}{2\pi}\lim_{T\to\infty}\int_{-T}^{T} (\mathcal{F}h)(\xi)e^{i\lambda\xi}\, d\xi,$$

for all $\lambda \in \mathbb{R}$.

Proof: See [**Tit37**, Thm. 24]. $\square$

This gives us the following formula for the inverse of the Mellin transform.

THEOREM 4.4. *Suppose that* $g \in BV_{loc}(0,\infty)$. *Let* $\sigma \in \mathbb{R}$, *and assume that* $g(x)x^{\sigma-1} \in L^1(0,\infty)$. *Then*

$$\frac{1}{2}\left[g(x^+) + g(x^-)\right] = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\sigma-iT}^{\sigma+iT} (\mathcal{M}g)(s)x^{-s}\ ds,$$

for all $x > 0$.

Proof: Let $\lambda = \log x$, then $G(\lambda) := g(e^\lambda)$ belongs to $BV_{loc}(-\infty,\infty)$. Let $h(\lambda) := G(\lambda)e^{\lambda\sigma}$. Then

$$\begin{aligned}
\int_{-\infty}^{\infty} |h(\lambda)|\ d\lambda &= \int_{-\infty}^{\infty} |g(e^\lambda)|e^{\lambda(\sigma-1)}e^\lambda\ d\lambda \\
&= \int_0^\infty |g(x)|x^{\sigma-1}\ dx \\
&< \infty,
\end{aligned}$$

so h belongs to $L^1(-\infty,\infty)$. It also belong to $BV_{loc}(-\infty,\infty)$, because, locally, it is the product of a function of bounded variation and a bounded increasing function. We have

$$\begin{aligned}
(\mathcal{M}g)(s) &= \int_0^\infty g(x)x^{s-1}\ dx \\
&= \int_0^\infty G(\lambda)e^{\lambda s}\ d\lambda \\
&= \int_{-\infty}^{\infty} \left(G(\lambda)e^{\lambda\sigma}\right)e^{i\lambda t}\ dt \\
&= \mathcal{F}\left(G(\lambda)e^{\lambda\sigma}\right)(-t) \\
&= (\mathcal{F}h)(-t).
\end{aligned}$$

Now, we apply Theorem 4.3 to h.

$$\begin{aligned}
\frac{1}{2}\left[g(x^+) + g(x^-)\right] &= e^{-\lambda\sigma}\frac{1}{2}\left[h(\lambda^+) + h(\lambda^-)\right] \\
&= e^{-\lambda\sigma}\frac{1}{2\pi}\lim_{T\to\infty}\int_{-T}^{T}(\mathcal{F}h)(\xi)e^{i\lambda\xi}\ d\xi \\
&= e^{-\lambda\sigma}\frac{1}{2\pi}\lim_{T\to\infty}\int_{-T}^{T}(\mathcal{M}g)(\sigma - it)e^{i\lambda t}\ dt \\
&= e^{-\lambda\sigma}\frac{1}{2\pi}\lim_{T\to\infty}\int_{-T}^{T}(\mathcal{M}g)(\sigma + it)e^{-i\lambda t}\ dt
\end{aligned}$$

$$\begin{aligned} &= \frac{1}{2\pi}\lim_{T\to\infty}\int_{-T}^{T}(\mathcal{M}g)(\sigma+it)e^{-\lambda(\sigma+it)}\,dt \\ &= \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\sigma-iT}^{\sigma+iT}(\mathcal{M}g)(s)e^{-\lambda s}\,ds \\ &= \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\sigma-iT}^{\sigma+iT}(\mathcal{M}g)(s)x^{-s}\,ds, \end{aligned}$$

and we are done. □

Given a Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$, let $F(x) = \sum_{n\le x}' a_n$, where $\sum'$ means that for $x = m \in \mathbb{N}^+$, the last term of the sum is replaced by $\frac{a_m}{2}$ so that the function $F(x)$ satisfies

$$F(x) = \frac{1}{2}\left[F(x^+) + F(x^-)\right]$$

for all x. This function $F(x)$ is called the *summatory function* of the Dirichlet series.

THEOREM 4.5. **(Perron's formula)** *For a Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, the summatory function satisfies*

$$F(x) \;=\; \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\sigma-iT}^{\sigma+iT}\frac{f(w)}{w}x^w dw, \tag{4.6}$$

for all $\sigma > \max(0, \sigma_c)$.

Before we prove Perron's formula, we need the following two propositions.

PROPOSITION 4.7. *Let $F_\sigma(x) = \sum_{n\le x}' a_n n^{-\sigma}$, then*

(1) $F_\sigma(x) = x^{-\sigma}F(x) + \sigma\int_0^x F(y)y^{-\sigma-1}\,dy$,
(2) $F(x) = x^{\sigma}F_\sigma(x) - \sigma\int_0^x F_\sigma(y)y^{\sigma-1}\,dy$.

Proof: First note that if $\sigma = 0$, the formulae hold trivially.

To prove (1), evaluate the integral on the RHS by parts, assuming that $x \notin \mathbb{N}$:

$$\begin{aligned} \text{RHS} &= x^{-\sigma}F(x) + \Big[-F(y)y^{-\sigma}\Big]_0^x + \int_0^x y^{-\sigma}dF(y) \\ &= \sum_{n\le x} a_n n^{-\sigma} = F_\sigma(x). \end{aligned}$$

If $x_0 \in \mathbb{N}^+$, note that the difference between the limit of the LHS as $x \to x_0-$ and the value of the LHS at x_0 is $\frac{1}{2}a_n n^{-\sigma}$ and the same is true for the RHS, since the integral on the RHS depends continuously on x. Since the two sides were equal for all $x \in (x_0 - 1, x_0)$ and they jump by the same amount at x_0, they are equal at x_0 as well.

To prove (2) one can either do an analogous calculation, or set $b_n = a_n n^{-\sigma}$, let $G_\sigma(x) = \sum'_{n\le x} b_n n^{-\sigma}$, and let $G(x) = G_0(x)$. Now we apply (1) with G in place of F and $\tilde\sigma = -\sigma$ instead of σ to get

$$\begin{aligned} F(x) &= G_{\tilde\sigma}(x) \\ &= x^{-\tilde\sigma} G(x) \; + \; \tilde\sigma \int_0^x G(y) y^{-\tilde\sigma - 1} \, dy \\ &= x^{\sigma} F_\sigma(x) \; - \; \sigma \int_0^x F_\sigma(y) y^{\sigma-1} \, dy, \end{aligned}$$

since $F(x) = G_{-\sigma}(x)$ and $F_\sigma(x) = G(x)$. □

The following is a necessary condition for a function to be representable by a Dirichlet series.

Proposition 4.8. *Consider the Dirichlet series $f(s) \sim \sum_{n=1}^\infty a_n n^{-s}$ and take a positive σ satisfying $\sigma > \sigma_1 := max(0, \sigma_c)$. Then*

$$f(\sigma + it) = o(|t|), \text{ as } |t| \to \infty. \tag{4.9}$$

Proof: By Theorem 3.12, we know that $F(x)x^{-\sigma} \to 0$ as $x \to \infty$ (see Exercise 3.17). Since $F(x)$ is 0 if $x < 1$, we have that $F(x)x^{-\sigma-1} \in L^1(0,\infty)$. By Proposition 4.7,

$$f(\sigma) \; = \; \lim_{x\to\infty} F_\sigma(x) \; = \; \lim_{x\to\infty} x^{-\sigma} F(x) \; + \; \sigma \int_0^\infty F(y) y^{-\sigma-1} \, dy.$$

Since the first term tends to 0, we obtain

$$\frac{f(\sigma)}{\sigma} \; = \; (\mathcal{M}F)(-\sigma), \text{ for all } \sigma > \sigma_1. \tag{4.10}$$

In fact, (4.10) holds for all $s \in \Omega_{\sigma_1}$, since both sides are analytic there. As The function $H(\lambda) := F(e^\lambda)e^{-\lambda s}$ is integrable and $\mathcal{F}(H)(t) = (\mathcal{M}F)(-s)$, by a similar change of variables argument to the one we used in the proof of Theorem 4.4. So by the Riemann-Lebesgue lemma we have

$$\begin{aligned} \lim_{t\to\pm\infty} \frac{f(s)}{s} &= \lim_{t\to\pm\infty} (\mathcal{M}F)(-s) \\ &= \lim_{t\to\pm\infty} \mathcal{F}(H)(t) \\ &= 0. \end{aligned}$$

Therefore we get (4.9), since $|s| \approx |t|$ as $t \to \pm\infty$. □

We will now prove Perron's formula (4.6).

Proof: The function F is in $BV_{loc}(0,\infty)$, and $F(x)x^{-\sigma-1} \in L^1(0,\infty)$. So we can apply the Mellin inversion formula to $\mathcal{M}F(-s) = \frac{f(s)}{s}$ and use the substitution $u = -w$ as follows:

$$\begin{aligned} F(x) = \frac{1}{2}[F(x^+) + F(x^-)] &= \frac{1}{2\pi i}\lim_{T\to\infty}\int_{-\sigma-iT}^{-\sigma+iT} (\mathcal{M}F)(u)x^{-u}\,du \\ &= -\frac{1}{2\pi i}\lim_{T\to\infty}\int_{\sigma+iT}^{\sigma-iT} \frac{f(w)}{w}x^w\,dw \\ &= \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\sigma-iT}^{\sigma+iT} \frac{f(w)}{w}x^w\,dw, \end{aligned}$$

and we are done. □

One can use this formula to estimate the growth of a_n from estimates of the growth of $f(w)$. Also, note that the formula might hold for smaller σ's, provided that f extends holomorphically to larger half-planes. This follows from the Cauchy integral formula applied to integrals along long vertical rectangles.

Recall that one can use the Cauchy integral formula to obtain the coefficients of a power series from the values of the function it represents, namely

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}\,d\theta.$$

The following theorem is a Dirichlet series analogue.

THEOREM 4.11. **(Schnee)** *Consider the Dirichlet series $f(s) \sim \sum_{n=1}^{\infty} a_n n^{-s}$. One has, for $\sigma > \sigma_c$,*

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(\sigma+it)e^{i\lambda t}\,dt = \begin{cases} a_n n^{-\sigma}, \text{if } \lambda = \log n, \\ 0, \text{ otherwise.} \end{cases} \tag{4.12}$$

Proof: Formally, exchanging the order of summation and integration, one gets

$$\begin{aligned} \frac{1}{2T}\int_{-T}^{T} f(\sigma+it)e^{i\lambda t}\,dt &= ⨍_{-T}^{T} \sum a_n e^{(-\sigma-it)\log n + i\lambda t}\,dt \\ &= \sum a_n n^{-\sigma} ⨍_{-T}^{T} e^{i(\lambda-\log n)t}\,dt. \end{aligned} \tag{4.13}$$

We write $⨍_{-T}^{T}$ to denote the normalized integral, obtained by dividing by the size of the set over which we are integrating. The integral in

(4.13) is 1 if $\lambda = \log n$, and tends to 0 as $T \to \infty$ otherwise, since for $\alpha \neq 0$, one has

$$\fint_{-T}^{T} e^{i\alpha t}\, dt \;=\; \frac{1}{2Ti\alpha}\left[e^{i\alpha T} - e^{-i\alpha T}\right] \;=\; \frac{\sin(\alpha T)}{\alpha T}.$$

This computation works fine for finite sums, and hence we can change finitely many coefficients of the series. So we may assume that $a_n = 0$, if $\log n \le \lambda + 1$, and then we must show that the LHS of (4.12) is zero.

Case (i): $\sigma > 0$.

Consider the integral inside the limit. Then t lies in the finite interval $[-T, T]$ and on this interval the series converges uniformly (since it is contained in an appropriate sector, and we can apply Theorem 3.1). Thus we may interchange the order of summation and integration and then use integration by parts as follows

$$\begin{aligned}
\frac{1}{2T}\int_{-T}^{T} \sum_{n \ge e^{\lambda+1}} a_n n^{-\sigma} e^{i(\lambda - \log n)t}\, dt &= \sum a_n n^{-\sigma} \fint_{-T}^{T} e^{i(\lambda-\log n)t}\, dt \\
&= \int_0^\infty x^{-\sigma} \fint_{-T}^{T} e^{i(\lambda - \log x)t}\, dt\; dF(x) \\
&= \int_0^\infty x^{-\sigma} \frac{\sin[(\lambda - \log x)T]}{(\lambda - \log x)T}\, dF(x) \\
&= \left[\frac{x^{-\sigma}\sin[(\lambda - \log x)T]}{(\lambda - \log x)T} F(x)\right]_0^\infty \qquad (4.14) \\
&- \int_0^\infty F(x) \frac{d}{dx}\left[\frac{x^{-\sigma}\sin[(\lambda - \log x)T]}{(\lambda - \log x)T}\right] dx \qquad (4.15)
\end{aligned}$$

Since $F(x) = 0$ for $x < 1$, the term in brackets in (4.14) vanishes at 0. At infinity, $F(x) = O(x^{\sigma_1+\varepsilon}) = o(x^\sigma)$, by choosing ε small enough. (Again we let σ_1 denote $\max(0, \sigma_c)$). Hence the expression is $o((\log x)^{-1})$ and so the whole term (4.14) vanishes.

We will show that the limit of (4.15) as $T \to \infty$ vanishes as well. We need to differentiate the square bracket. We obtain three terms:

$$\begin{aligned}
\frac{d}{dx}\left[\frac{x^{-\sigma}\sin[(\lambda - \log x)T]}{(\lambda - \log x)T}\right] &= \frac{-\sigma x^{-\sigma-1}\sin[(\lambda - \log x)T]}{(\lambda - \log x)T} \\
&+ \frac{-x^{-\sigma-1}T\cos[(\lambda - \log x)T]}{(\lambda - \log x)T} \\
&+ \frac{x^{-\sigma-1}\sin[(\lambda - \log x)T]}{(\lambda - \log x)^2 T}.
\end{aligned}$$

For each of these terms, we estimate the corresponding integral. Recall that $F(x) = O(x^{\sigma_1+\varepsilon})$ for any positive ε. In particular, $x^{-\sigma}F(x) = O(x^{-\delta})$, for any $\delta < \sigma - \sigma_1$. The first and third terms are similar and we get, as $T \to \infty$

$$
\begin{aligned}
I: \quad & \left|\int_{e^{\lambda+1}}^{\infty} F(x)\frac{-\sigma x^{-\sigma-1}\sin[(\lambda-\log x)T]}{(\lambda-\log x)T}\,dx\right| &=& \frac{1}{T}\int_{e^{\lambda+1}}^{\infty} O(x^{-1-\delta}) \to 0,\\
III: \quad & \left|\int_{e^{\lambda+1}}^{\infty} F(x)\frac{x^{-\sigma-1}\sin[(\lambda-\log x)T]}{(\lambda-\log x)^2T}\,dx\right| &=& \frac{1}{T}\int_{e^{\lambda+1}}^{\infty} O(x^{-1-\delta}) \to 0.
\end{aligned}
$$

The remaining term is more delicate. We will use the change of variables $u = (\log x - \lambda)$, so that $dx = e^{u+\lambda}du$. We have

$$
\begin{aligned}
II: \quad \int_{e^{\lambda+1}}^{\infty} F(x)\frac{-x^{-\sigma-1}T\cos[(\lambda-\log x)T]}{(\lambda-\log x)T}\,dx &= -\int_1^{\infty} F(e^{u+\lambda})e^{-(1+\sigma)(u+\lambda)}\frac{\cos Tu}{u}e^{\lambda+u}\,du\\
&= -\int_1^{\infty}\frac{F(e^{u+\lambda})e^{-\sigma(u+\lambda)}}{u}\cos Tu\,du,
\end{aligned}
$$

and the last integral tends to 0 as $T \to \infty$ by the Riemann-Lebesgue lemma. Indeed,

$$g(u) \;:=\; F(e^{u+\lambda})\frac{e^{-\sigma(u+\lambda)}}{u} \;=\; O(e^{-\delta(u+\lambda)}), \text{ as } u \to \infty,$$

and thus belongs to L^1.

Case (ii): $\sigma \leq 0$.

Choose some a such that $\sigma + a > 0$, and define $g(s) = f(s-a)$. Then

$$\frac{1}{2T}\int_{-T}^{T} f(\sigma+it)e^{i\lambda t}\,dt \;=\; \frac{1}{2T}\int_{-T}^{T} g((\sigma+a)+it)e^{i\lambda t}\,dt,$$

and we can reduce to Case (i). □

EXERCISE 4.16. Check that the same proof yields Schnee's theorem for generalized Dirichlet series. Let λ_n be a strictly increasing sequence with $\lim_{n\to\infty}\lambda_n = \infty$. Define the abscissa of convergence for $f(s) = \sum a_n e^{-\lambda_n s}$ just as for an ordinary Dirichlet series. Then, for $\sigma > \sigma_c$,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(\sigma+it)e^{i\mu t}\,dt \;=\; \begin{cases} a_n e^{-\lambda_n\sigma}, \text{if } \mu = \lambda_n,\\ 0, \text{ otherwise.}\end{cases} \tag{4.17}$$

4.1. Notes

Perron's formula, like Schnee's theorem, also holds for generalized Dirichlet series. For further results in this vein, see [**Hel05**, Ch. 1].

CHAPTER 5

Abscissae of uniform and bounded convergence

5.1. Uniform Convergence

We introduced the alternating zeta function $\tilde{\zeta}$ in Example 3.8, and showed its abscissa of convergence was 0, whilst its abscissa of absolute convergence was 1. In the strip $\{0 < \Re s < 1\}$, one can ask whether there is another form of convergence, intermediate between absolute and pointwise conditional convergence. For example, in what half-planes does the series converge uniformly or to a bounded function?

The values of the alternating zeta function are closely related to the values of the Riemann zeta function; more precisely,

$$\tilde{\zeta}(s) = (2^{1-s} - 1)\zeta(s). \tag{5.1}$$

Indeed, for $\sigma > 1$, both series converge absolutely, so we can reorder the terms freely, and hence

$$\begin{aligned} \tilde{\zeta}(s) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} \\ &= \sum_{n=1}^{\infty} \frac{-1}{n^s} + 2\sum_{n=1}^{\infty} \frac{1}{(2n)^s} \\ &= (-1 + 2^{1-s})\zeta(s). \end{aligned}$$

We will see later [**?**] that $\zeta(s)$ can be analytically continued to $\mathbb{C} \setminus \{1\}$, and that this continuation is unbounded on any of the lines $\{s : \text{ Re } s = \alpha\}$ with $\alpha \in (0,1)$. The relationship (5.1) will hold for the continuation as well, since both sides are analytic, and shows that $\tilde{\zeta}(s)$ must also be unbounded on $\{\text{Re } s = \alpha\}$. Hence the convergence cannot be uniform on this line either. We can get uniform convergence, however, provided we divide by s, as the following proposition shows.

Proposition 5.2. *If $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ converges at $s_0 = 0$, then, for any $\delta > 0$,*

$$\frac{1}{s}\sum_{n=1}^{\infty} a_n n^{-s}$$

converges uniformly to $\frac{f(s)}{s}$ in Ω_δ.

Proof: We use the same estimates as when we proved uniform convergence in the sector, but we replace the inequality (3.4) by

$$\varepsilon\frac{|s|}{\sigma}\left[\frac{1}{M^\sigma}-\frac{1}{(N+1)^\sigma}\right] \ \le\ \frac{|s|}{\delta}\varepsilon.$$

This is an estimate for the main term of $\sum_{n=M}^N a_n n^{-s}$. Each of the two other terms was estimated by ε, so with the extra $1/s$, we obtain, using $1/|s|<1/\delta$,

$$\left|\frac{1}{s}\sum_{n=M}^{N} a_n n^{-s}\right| \ \le\ 3\frac{\varepsilon}{\delta},$$

for $M, N \ge n_0$ and $s \in \Omega_\delta$. Thus we are done. □

Definition 5.3. For a Dirichlet series $f(s) \sim \sum_{n=1}^\infty a_n n^{-s}$ we define the *abscissa of uniform convergence* σ_u as

$$\sigma_u := \inf\left\{\rho :\ \sum_{n=1}^{\infty} a_n n^{-s} \text{ converges uniformly in } \Omega_\rho\right\},$$

and the *abscissa of bounded convergence* σ_b as

$$\sigma_b := \inf\left\{\rho :\ \sum_{n=1}^{\infty} a_n n^{-s} \text{ converges to a bounded function in } \Omega_\rho\right\}.$$

If a Dirichlet series converges absolutely at some $s_0 \in \mathbb{C}$, then it converges uniformly in the closed half-plane $\overline{\Omega_{\sigma_0}}$ by the comparison criterion. Also, if a Dirichlet series converges uniformly in some half-plane Ω_{σ_0}, for N large enough, the sum differs by at most 1 from the partial sum $\sum_{n=1}^N a_n n^{-s}$, for all $s \in \Omega_{\sigma_0}$. But (the absolute value of) this partial sum is bounded by $\sum_{n=1}^N |a_n| n^{-\sigma_0} < \infty$, and so the Dirichlet series converges to a bounded function in Ω_{σ_0}. Combining these two observations with the previously known inequalities between σ_c and σ_a and the obvious inequality $\sigma_c \le \sigma_b$ we obtain

$$\sigma_c \ \le\ \sigma_b \ \le\ \sigma_u \ \le\ \sigma_a \ \le\ \sigma_c + 1.$$

In fact, $\sigma_b = \sigma_u$, a result due to Bohr in 1913 [**Boh13b**].

Theorem 5.4. **(H. Bohr)** *Suppose that a Dirichlet series converges somewhere and extends analytically to a bounded function in Ω_ρ. Then for all $\delta > 0$, the Dirichlet series converges uniformly in $\Omega_{\rho+\delta}$.*

Proof: Suppose that $|f| \le K$ in $\overline{\Omega_\rho}$ and fix $0<\delta<1$. If $\rho \ge \sigma_a$, we are done by the chain of inequalities above. Thus, we may assume that $\rho < \sigma_a$. Observe, that it is enough to prove the following estimate for $\sigma \geq \rho + \delta$:

$$\left| f(s) - \sum_{n=1}^{N} a_n n^{-s} \right| \;\leq\; C(K,\delta) N^{-\delta} \log N, \tag{5.2}$$

since the right-hand side is $o(1)$ as $N \to \infty$.

To prove 5.2, we fix s and N and define

$$g(z) \;:=\; \frac{f(z)}{z-s} \left(N + \frac{1}{2} \right)^{z-s}.$$

Let d denote $\sigma_a - \rho + 2$, and integrate g around the rectangle with vertices $s - \delta \pm iN^d$ and $s + (\sigma_a - \rho) \pm iN^d$.

It would be nice to put a picture in here

By the residue theorem, we obtain

$$\int_{\square} g(z)\; dz \;=\; 2\pi i f(s).$$

Consider the left-hand edge of the rectangle (LHE), on it we can estimate

$$|g(z)| \;\leq\; \frac{K}{\sqrt{\delta^2 + \mathrm{Im}^2(z-s)}} \left(N + \frac{1}{2} \right)^{-\delta}$$

so that

$$\begin{aligned}
\left| \int_{LHE} g(z)\; dz \right| \;&\lesssim\; KN^{-\delta} \int_{-N^d}^{N^d} \frac{1}{\sqrt{\delta^2 + y^2}}\; dy \\
&=\; KN^{-\delta} \Big[\log \left(y + \sqrt{\delta^2 + y^2} \right) \Big]_{-N^d}^{N^d} \\
&\leq\; CKN^{-\delta} [\log N + \log \delta] \\
&=\; C(K,\delta) N^{-\delta} \log N.
\end{aligned}$$

As for the integration over both of the horizontal edges (HE), we can use the same estimate

$$\begin{aligned}
\left| \int_{HE} g(z)\; dz \right| \;&\leq\; KN^{-d} \int_{\sigma-\delta}^{\sigma+d-2} \left(N + \frac{1}{2} \right)^{x-\sigma} dx \\
&\lesssim\; KN^{-d} \left[\frac{1}{\log N} N^{x-\sigma} \right]_{x=\sigma-\delta}^{x=\sigma+d-2} \\
&\lesssim\; \frac{KN^{-2}}{\log N}.
\end{aligned}$$

Hence, we can conclude that

$$2\pi i f(s) \;=\; \int_{RHE} g(z)\,dz + O(N^{-\delta}\log N).$$

Since the series converges absolutely on RHE, we can interchange the order of integration and summation

$$\begin{aligned}\int_{RHE} g(z)\,dz &= \int_{RHE}\sum_{n=1}^{\infty} a_n n^{-z}\left(N+\frac{1}{2}\right)^{z-s}\frac{1}{z-s}\,dz\\ &= \sum_{n=1}^{\infty} a_n \int_{RHE} n^{-z}\left(N+\frac{1}{2}\right)^{z-s}\frac{1}{z-s}\,dz\\ &= \sum_{n=1}^{\infty} a_n n^{-s}\int_{RHE}\left(\frac{N+\frac{1}{2}}{n}\right)^{z-s}\frac{1}{z-s}\,dz\end{aligned}$$

We will show that the contribution of the tail of the series above — the sum for $n > N$ — is small, while the sum over $n \le N$ is approximately the partial sum of the Dirichlet series.

First, assume that $n > N$, i.e., $n \geq N+1$. Apply Cauchy's theorem to the rectangular path whose left-hand edge is RHE and whose horizontal sides have length L, and let L tend to infinity. Since the integrand has no poles in the region encompassed by this rectangle, the integral over the closed path vanishes. On the new right-hand edge, the integrand decays exponentially with L and so the limit of the integral over this edge tends to 0. On the top edge (and similarly, on the bottom one), we estimate as follows,

$$\begin{aligned}\left|\int_{s+(\sigma_a-\rho)\pm iN^d}^{\infty+it\pm iN^d}\left(\frac{N+\frac{1}{2}}{n}\right)^{z-s}\frac{dz}{z-s}\,dz\right| &\leq \frac{1}{N^d}\int_{\sigma+(\sigma_a-\rho)}^{\infty}\left(\frac{N+\frac{1}{2}}{n}\right)^{x-\sigma}dx\\ &= \frac{1}{N^d}\int_{\sigma+(\sigma_a-\rho)}^{\infty} e^{(x-\sigma)\log\left(\frac{N+\frac{1}{2}}{n}\right)}dx\\ &= \frac{1}{N^d}\frac{1}{-\log\left(\frac{N+\frac{1}{2}}{n}\right)}e^{(\sigma_a-\rho)\log\left(\frac{N+\frac{1}{2}}{n}\right)}.\end{aligned}$$

The expression $\log\left(\frac{N+\frac{1}{2}}{n}\right)$ is minimized when $n = N+1$. So

$$\begin{aligned}\left|\log\left(\frac{N+\frac{1}{2}}{n}\right)\right| &\geq -\log\left(1-\frac{1}{2N+2}\right)\\ &> \frac{1}{2(N+1)}.\end{aligned}$$

Hence, we can estimate the tail of the series by

$$\begin{aligned}
\left|\sum_{n=N+1}^{\infty} a_n n^{-s} \int_{RHE} \left(\frac{N+\frac{1}{2}}{n}\right)^{z-s} \frac{1}{z-s}\, dz\right| &\lesssim \sum_{n=N+1}^{\infty} \frac{|a_n|}{n^{\sigma}} N^{1-d} \left(\frac{N+\frac{1}{2}}{n}\right)^{\sigma_a-\rho} \\
&= N^{1-d}\left(N+\frac{1}{2}\right)^{\sigma_a-\rho} \sum_{n=N+1}^{\infty} \frac{|a_n|}{n^{\sigma+\sigma_a-\rho}} \\
&\lesssim N^{-1},
\end{aligned}$$

since $\sum \frac{|a_n|}{n^{\sigma+\sigma_a-\rho}}$ converges.

If $n \le N$, we use Cauchy's theorem again, but now with a rectangular path whose right-hand edge is RHE and whose width L tends to infinity. The residue theorem now implies that

$$\int_{\square} \left(\frac{N+\frac{1}{2}}{n}\right)^{z-s} \frac{1}{z-s}\, dz \;=\; 2\pi i.$$

The integrand decays exponentially on the left-hand edge, and so the integral over that edge tends to zero. As for the top edge (and also the bottom one)

$$\begin{aligned}
\left|\int_{-\infty+it\pm iN^d}^{s+(\sigma_a-\rho)\pm iN^d} \left(\frac{N+\frac{1}{2}}{n}\right)^{z-s} \frac{dz}{z-s}\, dz\right| &\le \frac{1}{N^d}\int_{-\infty}^{\sigma+(\sigma_a-\rho)} \left(\frac{N+\frac{1}{2}}{n}\right)^{x-\sigma} dx \\
&= \frac{1}{N^d}\int_{-\infty}^{\sigma+(\sigma_a-\rho)} e^{(x-\sigma)\log\left(\frac{N+\frac{1}{2}}{n}\right)}\, dx \\
&= \frac{1}{N^d}\frac{1}{\log\left(\frac{N+\frac{1}{2}}{n}\right)} e^{(\sigma_a-\rho)\log\left(\frac{N+\frac{1}{2}}{n}\right)} \\
&\le N^{-d}\frac{1}{\log\left(\frac{N+\frac{1}{2}}{N}\right)}\left(\frac{N+\frac{1}{2}}{n}\right)^{\sigma_a-\rho} \\
&\lesssim N^{1-d}\left(\frac{N+\frac{1}{2}}{n}\right)^{\sigma_a-\rho} \\
&\lesssim N^{-1}n^{-\sigma_a+\rho}
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{n=1}^{N} a_n n^{-s} \int_{RHE} \left(\frac{N+\frac{1}{2}}{n}\right)^{z-s} \frac{1}{z-s}\, dz &= 2\pi i \sum_{n=1}^{N} \frac{a_n}{n^s} + O(N^{-1}n^{-\sigma_a+\rho}) \sum_{n=1}^{N} \frac{|a_n|}{n^{\sigma}} \\
&= 2\pi i \sum_{n=1}^{N} \frac{a_n}{n^s} + O(N^{-1}),
\end{aligned}$$

where we used boundedness of the partial sums of the convergent series $\sum_n \frac{|a_n|}{n^{\sigma+\sigma_a-\rho}}$. We have shown that $\frac{1}{2\pi i}\int_{RHE} g(z)\, dz$ is close to both the partial sum of the Dirichlet series and $f(s)$ (and the error is as in (5.2), and does not depend on s). □

The promised equality of the two new abscissae is now an immediate corollary.

COROLLARY 5.5. *The equality $\sigma_b = \sigma_u$ holds for any Dirichlet series.*

Note, however, that the above corollary does not imply that if a Dirichlet series converges to a bounded function in some half-plane, it will converge uniformly in that half-plane. We only know that it will converge uniformly in every strictly smaller half-plane.

REMARK 5.6. The function $g(z)$ used in the proof of the theorem above comes from Perron's formula which can be restated as (in the special case of $x = N + \frac{1}{2}$)

$$\sum_{n\le N} a_n n^{-s} \;=\; \frac{1}{2\pi i}\int_{\sigma+it-iT}^{\sigma+it+iT} f(z)\frac{(N+\frac{1}{2})^{z-s}}{z-s}\, dz \;+\; e_{N,T},$$

where $e_{N,T}$ is an error term that comes from not taking the limit in T. One can also prove this formula using the estimates above.

5.2. The Bohr correspondence

Bohr's idea was to use the following correspondence between Dirichlet series and power series in infinitely many variables. For a positive integer with prime factorization $n = p_1^{k_1}\dots p_l^{k_l}$, we define

$$z^{r(n)} \;:=\; z_1^{k_1}\dots z_l^{k_l}.$$

We have an isomorphism between formal power series in infinitely many variables $z_1, z_2, \dots$ and Dirichlet series, given by

$$\mathcal{B} : \sum_n a_n z^{r(n)} \;\mapsto\; \sum_n a_n n^{-s}. \tag{5.7}$$

We shall write $\mathcal{Q}$ for the inverse of $\mathcal{B}$:

$$\mathcal{Q} : \sum_n a_n n^{-s} \;\mapsto\; \sum_n a_n z^{r(n)}. \tag{5.8}$$

The map $\mathcal{B}$ is an evaluation homomorphism — indeed, we evaluate the power series on the one-dimensional set $\{(z_i) : \; z_i = p_i^{-s}\}$. It is clearly onto, and it has a trivial kernel because the right-hand side is 0 iff all the coefficients vanish.

For finite series, we can norm both spaces so that $\mathcal{B}$ will be isometric. Indeed, we have

$$\begin{aligned} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left| \sum_{n=1}^{N} a_n n^{-it} \right|^2 dt &= \sum_{n=1}^{N} |a_n|^2 \\ &= \int_{\mathbb{T}^\infty} \left| \sum_{n=1}^{N} a_n e^{2\pi i t \cdot r(n)} \right|^2 dt. \quad (5.9) \end{aligned}$$

By $\mathbb{T}^\infty$ we mean the infinite torus

$$\mathbb{T}^\infty = \{(e^{2\pi i t_1}, e^{2\pi i t_2}, \dots) : 0 \leq t_j < 1 \ \forall j \in \mathbb{N}^+\}$$

which we identify with the infinite product

$$[0,1) \times [0,1) \times \cdots$$

on which we put the product probability measure of Lebesgue measure on each interval.

We shall investigate when (5.9) holds for infinite sums in Theorem 6.39.

Flesh this section out.

5.3. Bohnenblust-Hille Theorem

We will now proceed to show that $\sigma_a - \sigma_b \leq \frac{1}{2}$, and that this bound is sharp. Originally, Hille and Bohnenblust exhibited an example of a Dirichlet series for which equality holds in the above inequality. Their construction was extremely complicated.

Instead of going through their construction, we shall show that such an example exists using a probabilistic method. This is a non-constructive method, used in other fields, in particular in combinatorics/graph theory.

Before describing the probabilistic method we mention two analogous methods: the "cardinality method" and the "Baire category method". Recall that one can prove the existence of transcendental numbers by showing that there are only countably many algebraic numbers (and uncountably many real numbers). This is much easier than proving that a concrete number is transcendental. Similarly, the existence of a nowhere differentiable continuous function on an interval I can be proved by showing that the set of all continuous functions with a derivative at at least one point is of the first category (and thus cannot equal the complete metric space of all continuous function on I). The construction of a particular example is again fairly technical.

The probabilistic method is similar in spirit. Instead of exhibiting a concrete example of an object with some given property, we consider some set S of objects and equip it with a convenient probability measure. We strive to show that a randomly chosen object will have the desired property with a non-zero probability. Although it might seem that this will rarely work, the probability method has been very successful, especially when examples with the given property have a complicated structure or description.

In our case, we will need to consider random series of functions of the form

$$f_\varepsilon(s) \;=\; \sum_{n=1}^{\infty} \varepsilon_n a_n n^{-s},$$

where $\{\varepsilon_n\}$ is a *Rademacher sequence*, that is, a sequence of independent random variables, such that each $\varepsilon_n \in \{\pm 1\}$ and $\mathrm{Prob}\,(\varepsilon_n = 1) = \mathrm{Prob}\,(\varepsilon_n = -1) = 1/2$. One can also consider the random series

$$f_\omega(s) \;=\; \sum_{n=1}^{\infty} a_n e^{in\omega_n} n^{-s},$$

where $\omega = \{\omega_n\}_{n=1}^{\infty}$ is a sequence of random variables that are independent and such that each ω_n is uniformly distributed on $[0, 2\pi]$. Both $\{\varepsilon_n\}$ and $\{\omega_n\}$ are i.i.d.'s, that is, independent and identically distributed.

When we have a Rademacher sequence, we use $\mathbb{E}$ to denote the expectation, that is the average over all choices of sign, of some function that depends on the sequence:

$$\mathbb{E}[\sum \varepsilon_n g_n].$$

If the sequence is finite of length K, this just means adding up all 2^K choices and dividing by 2^K. If the sequence is infinite, one must replace this by integrating over the space $\{-1, 1\}^\infty$ with the product probability measure.

Note that a sequence of i.i.d.'s has a canonical probability distribution associated to it, namely the product probability. Heuristically, to choose a random sequence, we can choose it element by element, and since these elements should be independent, we arrive at the product probability.

As an example of a theorem about random Dirichlet series we prove the following proposition.

PROPOSITION 5.10. *Let $\{a_n\}$ be a sequence of complex numbers and let $\{\omega_n\}_{n=1}^{\infty}$ be sequence of i.i.d.'s which are uniformly distributed on*

$[0;2\pi]$. *Denote* $f_\omega(s) := \sum_{n=1}^{\infty} a_n e^{in\omega_n} n^{-s}$, *as above. Then there exists some* $\tilde{\sigma} = \tilde{\sigma}(\{a_n\})$ *such that* $\sigma_c(f_\omega) = \tilde{\sigma}$ *almost surely.*

Proof: Given a sequence of random variables, a *tail event* is an event whose incidence is not changed by changing the values assumed by any finitely many elements of the sequence. The zero-one law of probability asserts that any tail event associated to a sequence of i.i.d.'s happens with probability either 0 or 1 [**Kah85**, p.7]. Consider the events $B_a = \{f_\omega : \sigma_c(f_\omega) \le a\}$ for $a \in \mathbb{R}$. These are clearly tail events. Let

$$\tilde{\sigma} \ := \ \inf \{a : \ \mathrm{Prob}\,(B_a) = 1\},$$

where we agree that $\inf \emptyset = \infty$. Since the events B_a are nested, we have

$$\begin{aligned} \mathrm{Prob}\,(B_a) = 0, &\ \text{ for all } a < \tilde{\sigma}, \\ \mathrm{Prob}\,(B_a) = 1, &\ \text{ for all } a > \tilde{\sigma}, \end{aligned}$$

$$\text{and} \quad \{f_\omega : \ \sigma_c(f_\omega) = \tilde{\sigma}\} \ = \ \left(\bigcap\nolimits_n B_{\tilde{\sigma}+\frac{1}{n}}\right) \setminus \left(\bigcup\nolimits_n B_{\tilde{\sigma}-\frac{1}{n}}\right),$$

which easily implies that $\tilde{\sigma}$ has the desired property. $\square$

Let f be a holomorphic function on a domain $\Omega \subset \mathbb{C}$. We say that $\partial\Omega$ is a *natural boundary* for f, if no point $z_0 \in \partial\Omega$ has a neighborhood to which it can be holomorphically continued. Proposition 5.10 can be strengthened in the following way [**Kah85**, p. 44].

THEOREM 5.11. *Let* ω_n *and* $\overline{\sigma}$ *be as above. Then, with probability* 1, *the line* $\{Re\ s = \overline{\sigma}\}$ *is the natural boundary for the Dirichlet series* $\sum a_n e^{i\omega_n} n^{-s}$.

We will need the following theorem, which we shall prove as Corollary 5.23 below. We shall use multi-index notation, where $\alpha \in \mathbb{Z}^r$ — see Appendix 11.1. We shall use $\mathbb{T}^r$ to denote the r-torus, which by an abuse of notation we shall identify with both $\{(e^{2\pi i t_1}, \cdots, e^{2\pi i t_r}) : 0 \le t_j \le 1\ \forall\ j\}$ and $\{(t_1, \cdots, t_r) \ : \ 0 \le t_j \le 1\ \forall\ j\}$.

THEOREM 5.12. *There exists a universal constant* $C > 0$ *such that for every* $r \in \mathbb{N}^+$, *every* $N \ge 2$, *and every choice of coefficients* $c_\alpha \in \mathbb{C}$, *with* $|\alpha| = |\alpha_1| + \cdots + |\alpha_r| \le N$, *there exists some choice of signs such that*

$$\sup_{t \in \mathbb{T}^r} \left| \sum_{|\alpha| \le N} \pm c_\alpha e^{2\pi i(\alpha_1 t_1 + \cdots + \alpha_r t_r)} \right| \ \le \ C \left[r \log N \sum |c_\alpha|^2 \right]^{\frac{1}{2}}. \tag{5.13}$$

By Fubini's theorem and the orthogonality of $\{e^{2\pi i\alpha\cdot t}\}$, for any choice of signs

$$\int_{\mathbb{T}^r}\left|\sum_{\alpha}\pm c_n e^{2\pi i\alpha\cdot t}\right|^2 dt \;=\; \sum_{\alpha}|c_\alpha|^2,$$

so the left-hand side of (5.13) is at least $\sum_\alpha |c_\alpha|^2$. The theorem says that for some choice of signs, this estimate is only off by a factor of $\sqrt{r\log N}$.

Note that choosing all c_α positive and using the Cauchy-Schwarz inequality yields the following much cruder estimate:

$$\begin{aligned}\sup_{t\in\mathbb{T}^r}\left|\sum_{|\alpha|\le N} c_\alpha e^{i(\alpha_1 t_1+\cdots+\alpha_r t_r)}\right| \;&=\; \sum_\alpha c_\alpha \\ &\le\; \sqrt{C_N}\left(\sum_\alpha |c_\alpha|^2\right)^{\frac{1}{2}},\end{aligned}$$

where C_N is the number of terms, roughly N^r, if $N \gg r$.

We will need the following lemma.

Lemma 5.14. *Let*

$$P(t) \;=\; \sum_{|\alpha|\le N} c_\alpha e^{2\pi i(\alpha_1 t_1+\cdots+\alpha_r t_r)}$$

be a trigonometric polynomial on $\mathbb{T}^r$. *If* P *is real, then there exists an* r*-dimensional cube* $I\subset\mathbb{T}^r$ *of volume* $(N+1)^{-2r}$ *on which* $|P(t_1,\ldots,t_r)|\ge\frac{1}{2}\|P\|_\infty$.

Proof: By multiplying P by (-1), if necessary, we may assume that there exists $\theta=(\theta_1,\ldots,\theta_r)\in\mathbb{T}^r$ such that

$$P(\theta)=\|P\|_\infty.$$

By the mean value theorem, we conclude that for any $t=(t_1,\ldots,t_r)\in\mathbb{T}^r$, there exists $\tilde\theta$ belonging to the segment connecting t and θ such that

$$P(t)-P(\theta) \;=\; \sum_{j=1}^{r}(t_j-\theta_j)\frac{\partial P}{\partial t_j}(\tilde\theta),$$

Thus,

$$|P(t)-P(\theta)| \;\le\; \max_j |t_j-\theta_j| \sum_{j=1}^{r}\left|\frac{\partial P}{\partial t_j}(\tilde\theta)\right| \tag{5.15}$$

There exists a choice of signs $s_j \in \{\pm 1\}$ so that

$$\frac{d}{dx}_{|x=0} P(\tilde{\theta}_1 + s_1 x, \ldots, \tilde{\theta}_r + s_r x) \ = \ \sum_j \left| \frac{\partial P}{\partial t_j}(\tilde{\theta}) \right| .$$

We fix this choice, and define a trigonometric polynomial of degree at most N

$$Q(x) = P(\tilde{\theta}_1 + s_1 x, \ldots, \tilde{\theta}_r + s_r x).$$

Then $Q(x) = \sum_k b_k e^{ikx}$ and $Q'(x) = \sum_k ikb_k e^{ikx}$. Note that by integrating against e^{-ikx} we obtain $|b_k| \leq \|Q\|_\infty$, and hence

$$\begin{aligned} |Q'(0)| \quad &\leq \quad \sum_k |kb_k| \\ &\leq \quad \max_k |b_k| \sum_{k=-N}^{N} k \\ &\leq \quad \|Q\|_\infty N(N+1) \\ &\leq \quad \|P\|_\infty N(N+1). \end{aligned}$$

Thus, we can continue our estimate from (5.15)

$$|P(t) - P(\theta)| \ \leq \ \|P\|_\infty N(N+1) \sup_j |t_j - \theta_j|. \tag{5.16}$$

Since $|P(\theta)| = \|P\|_\infty$, whenever the right-hand side of (5.16) is bounded by $\frac{1}{2}\|P\|_\infty$, we have $P(t) \geq \frac{\|P\|_\infty}{2}$. This will occur if

$$\sup_j |t_j - \theta_j| \leq \frac{1}{2N(N+1)}.$$

The set of such t's is a cube of volume $[N(N+1)]^{-r} \geq (N+1)^{-2r}$. $\square$

THEOREM 5.17. *Let $\{P_n\}_{n=1}^K$ be a finite set of complex trigonometric polynomials in r variables of degree less than or equal to N, with $N \geq 1$. Let $Q(t_1, \ldots, t_r) = \sum_n \varepsilon_n P_n(t_1, \ldots, t_r)$, where ε_n is a Rademacher sequence. Then*

$$\text{Prob}\left(\|Q\|_\infty \geq \left[32r \log \gamma N \sum_n \|P_n\|_\infty^2 \right]^{\frac{1}{2}} \right) \leq \frac{2}{\gamma},$$

for all real $\gamma \geq 8$.

Proof: First suppose that all P_n's are real, let $\tau = \sum_n \|P_n\|_\infty^2$ and $M = \|Q\|_\infty$ (here $M = M(\varepsilon)$ is a random variable). Let λ be an arbitrary real number. Then, using the inequality $\frac{1}{2}(e^x + e^{-x}) \leq e^{\frac{x^2}{2}}$ yields

$$\begin{aligned}
\mathbb{E}\left(e^{\lambda Q(t)}\right) &= \mathbb{E}\left(e^{\lambda \sum_n \varepsilon_n P_n(t)}\right) \\
&= \mathbb{E}\left(\prod_n e^{\lambda \varepsilon_n P_n(t)}\right) \\
&= \prod_n \mathbb{E}\left(e^{\lambda \varepsilon_n P_n(t)}\right) \\
&= \prod_n \left(\frac{1}{2}[e^{\lambda P_n(t)} + e^{-\lambda P_n(t)}]\right) \\
&\leq \prod_n e^{\lambda^2 \frac{P_n^2(t)}{2}} \\
&\leq \prod_n e^{\frac{\lambda^2}{2}\|P_n\|_\infty^2} \\
&= e^{\frac{\lambda^2}{2}\sum_n \|P_n\|_\infty^2} \\
&= e^{\frac{\tau\lambda^2}{2}}. \qquad (5.18)
\end{aligned}$$

By Lemma 5.14, there exists an interval $I = I(\varepsilon) \subset \mathbb{T}^r$ of volume at least $(N+1)^{-2r}$ such that $|Q| \geq \frac{1}{2}\|Q\|_\infty$ of I. For fixed $\varepsilon = \{\varepsilon_n\}$ we thus have

$$\begin{aligned}
e^{\frac{\lambda M(\varepsilon)}{2}} &\leq \frac{1}{vol(I(\varepsilon))}\int_{I(\varepsilon)} e^{\lambda Q(t)} + e^{-\lambda Q(t)}\ dt \\
&\leq (N+1)^{2r}\int_{\mathbb{T}^r} e^{\lambda Q(t)} + e^{-\lambda Q(t)}\ dt
\end{aligned}$$

Taking the expected value and using estimate (5.18) yields

$$\begin{aligned}
\mathbb{E}\left(e^{\frac{\lambda M}{2}}\right) &\leq (N+1)^{2r}\mathbb{E}\left(\int_{\mathbb{T}^r} e^{\lambda Q(t)} + e^{-\lambda Q(t)}\ dt\right) \\
&= (N+1)^{2r}\int_{\mathbb{T}^r} \mathbb{E}\left(e^{\lambda Q(t)} + e^{-\lambda Q(t)}\right)\ dt \\
&\leq (N+1)^{2r}\int_{\mathbb{T}^r} 2e^{\frac{\tau\lambda^2}{2}}\ dt \\
&= 2(N+1)^{2r}e^{\frac{\tau\lambda^2}{2}} \\
&= e^{\frac{\tau\lambda^2}{2}+\log 2+2r\log(N+1)}
\end{aligned}$$

Thus,

$$\mathbb{E}\left(e^{\frac{\lambda M}{2}-\frac{\lambda^2\tau}{2}-\log 2-2r\log(N+1)}\right) \leq 1,$$

and hence, by Chebyshev's inequality,

$$\text{Prob}\ \left(e^{\frac{\lambda M}{2}-\frac{\lambda^2\tau}{2}-\log 2-2r\log(N+1)} \geq \gamma\right) \ \leq \ \frac{1}{\gamma}. \tag{5.19}$$

The event on the left-hand side of (5.19) is equivalent to

$$\frac{\lambda M - \lambda^2\tau}{2} - \log 2 - 2r\log(N+1) \ \geq \ \log\gamma. \tag{5.20}$$

Choose $\lambda = \sqrt{\frac{2}{\tau}\log[2\gamma(N+1)^{2r}]}$, then, after algebraic manipulations, (5.20) becomes

$$M\sqrt{\frac{2}{\tau}\log[2\gamma(N+1)^{2r}]} \ \geq \ 4\log[2\gamma(N+1)^{2r}],$$

which is the same as

$$M \ \geq \ 2\sqrt{2\tau}\sqrt{\log[2\gamma(N+1)^{2r}]}. \tag{5.21}$$

For $\gamma \geq 8$ we have

$$2\gamma(N+1)^{2r} \leq (\gamma N)^{2r},$$

so (5.21) will hold if

$$\begin{aligned} M &\geq 2\sqrt{2\tau}\sqrt{\log[\gamma N]^{2r}} \\ &= 4\sqrt{r\tau\log[\gamma N]}. \end{aligned}$$

Recalling that $M = \|Q\|_\infty$ and $\tau = \sum_n \|P_n\|^2$ we obtain

$$\text{Prob}\ \left(\|Q\|_\infty \ \geq \ 4\left[r\log[\gamma N]\sum_n \|P_n\|^2\right]^{\frac{1}{2}}\right) \ \leq \ \frac{1}{\gamma},$$

when Q is real.

If Q is complex and

$$\|Q\|_\infty \ \geq \ 4\big[2r\log[\gamma N]\sum_n \|P_n\|_\infty^2\big]^{\frac{1}{2}},$$

then one of the two following inequalities must hold:

$$\begin{aligned} \|\text{Re } Q\|_\infty &\geq 4\left[r\log[\gamma N]\sum_n \|\text{Re } P_n\|_\infty^2\right]^{\frac{1}{2}}, \\ \|\text{Im } Q\|_\infty &\geq 4\left[r\log[\gamma N]\sum_n \|\text{Im } P_n\|_\infty^2\right]^{\frac{1}{2}}. \end{aligned}$$

But since these inequalities involve real polynomials, either of them happens with probability at most $\frac{1}{\gamma}$, by the real case. The probability that at least one of them happens is thus at most $\frac{2}{\gamma}$. $\square$

COROLLARY 5.22. *Let $N \geq 2$, and let $c_\alpha \in \mathbb{C}$ be given for every $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}^r$ with $|\alpha| \leq N$. Then, for any $\gamma \geq 8$, there exists $C > 0$ such that*

$$\mathrm{Prob}\left(\left\| \sum_{|\alpha| \leq N} \varepsilon_\alpha c_\alpha e^{i\alpha\cdot t} \right\|_\infty \geq C(r \log N)^{\frac{1}{2}} \left[\sum_{|\alpha|\leq N} |c_\alpha|^2 \right]^{\frac{1}{2}} \right) \;\leq\; \frac{2}{\gamma}.$$

Proof: Fix $\gamma \geq 8$, and choose $C > 0$ such that $C^2 \geq 32\left(1 + \frac{\log\gamma}{\log N}\right)$. Let $P_\alpha(t) = c_\alpha e^{i\alpha\cdot t}$ and use Theorem 5.17. $\square$

COROLLARY 5.23. *There exist a choice of signs $\{\varepsilon_\alpha\}$ such that*

$$\left\| \sum_{|\alpha| \leq N} \varepsilon_\alpha c_\alpha e^{i\alpha\cdot t} \right\|_\infty \;\leq\; C(r \log N)^{\frac{1}{2}} \left[\sum_{|\alpha|\leq N} |c_\alpha|^2 \right]^{\frac{1}{2}}.$$

Proof: For any $\gamma \geq 8$, the probability that a random series will not have the property is at most $\frac{2}{\gamma} < 1$. $\square$

THEOREM 5.24. **(H. Bohr)** *For any Dirichlet series $\sigma_a - \sigma_u \leq \frac{1}{2}$.*

Proof: Let $\rho > \sigma_u$, then $\sum_{n=1}^\infty a_n n^{-s}$ converges uniformly in $\overline{\Omega_\rho}$. Fix $s \in \mathbb{C}$ with $\operatorname{Re} s = \rho + \frac{1}{2} + \varepsilon$. By the Cauchy-Schwarz inequality,

$$\begin{aligned} \sum_n |a_n n^{-s}| &= \sum_n |a_n| n^{-(\rho+\frac{1}{2}+\varepsilon)} \\ &\leq \left(\sum_n |a_n|^{-2\rho}\right)^{\frac{1}{2}} \left(\sum_n n^{-(1+2\varepsilon)}\right)^{\frac{1}{2}}, \qquad (5.25) \end{aligned}$$

where the second sum converges. By uniform convergence, there exists $K > 0$ such that for every $t \in \mathbb{R}$ and $N \in \mathbb{N}^+$

$$\left| \sum_{n=1}^N a_n n^{-(\rho+it)} \right| \;\leq\; K.$$

Consequently,

$$\begin{aligned} K^2 &\geq \left| \sum_{n=1}^N a_n n^{-(\rho+it)} \right|^2 \\ &= \sum_{n=1}^N |a_n|^2 n^{-2\rho} + 2\operatorname{Re} \sum_{1\leq n<m\leq N} a_n \overline{a}_m (nm)^{-\rho} e^{it\log\frac{m}{n}}. \end{aligned}$$

Taking the normalized integral yields

$$K^2 \;\geq\; \sum_{n=1}^{N} |a_n|^2 n^{-2\rho} + 2\,\mathrm{Re} \sum_{1\leq n<m\leq N} a_n \overline{a}_m (nm)^{-\rho} ⨍_{-T}^{T} e^{it\log\frac{m}{n}}\; dt.$$

Taking the limit as T tends to ∞, the mixed terms tend to 0 and so we conclude that

$$\sum_{n=1}^{N} |a_n|^2 n^{-2\rho} \;\leq K^2,$$

for all $N \in \mathbb{N}^+$. Thus the first sum on the right-hand side of (5.25) is bounded, and so $\sum_n |a_n n^{-s}|$ converges. Thus, $\sigma_a \leq \frac{1}{2} + \rho + \varepsilon$. Since this is true for every $\rho > \sigma_u$ and $\varepsilon > 0$, we get $\sigma_a \leq \frac{1}{2} + \sigma_u$. □

Theorem 5.26. **(Bohnenblust-Hille, 1931)** *There exist a Dirichlet series* $\sum_{n=1}^{\infty} a_n n^{-s}$ *for which* $\sigma_u = \frac{1}{2}$ *and* $\sigma_a = 1$.

We shall present a probabilistic proof, due to H. Boas [**Boa97**].

Proof: Each a_n will be an element of $\{\pm 1, 0\}$ and the coefficients will be constructed in groups, starting with $k = 2$. To construct the k^{th} group, choose a homogeneous polynomial Q_k of degree k in 2^k variables with coefficients $\varepsilon_j \in \{\pm 1\}$, with $j = (j_1, \ldots, j_{2^k})$,

$$Q_k(z_1, z_2, \ldots, z_{2^k}) \;=\; \sum_{|j|=k} \varepsilon_j z_1^{j_1} \ldots z_{2^k}^{j_{2^k}}$$

so that

$$\|Q_k\|_\infty \;\leq\; C\left[2^k \log k \sum_{|j|=k} |\varepsilon_j|^2\right]^{\frac{1}{2}}.$$

This is possible, by Corollary 5.23. By Lemma 5.29, the number of (monic) monomials of degree k in 2^k variables is $\binom{2^k+k-1}{k}$. We conclude that

$$\|Q_k\|_\infty \;\leq\; C\left[2^k \log k \binom{2^k+k-1}{k}\right]^{\frac{1}{2}}.$$

We convert the Q_k's into Dirichlet series as in (5.7)

$$f_k(s) \;:=\; (\mathcal{B}Q_k)(s) \;=\; \sum_{|j|=k} \varepsilon_j \left(p_{2^k}^{j_1} \ldots p_{2^k+2^k-1}^{j_{2^k}}\right)^{-s},$$

and let $f = \sum_{k=2}^{\infty} f_k$, thought of as a Dirichlet series. Then the coefficients of f lie in $\{\pm 1, 0\}$, since each n can appear in at most one f_k.

Claim 1: $\sigma_a(f) = 1$.

Proof: In f_k, the number of non-zero coefficients is

$$\binom{2^k+k-1}{k} \geq \frac{(2^k)^k}{k!} \geq \frac{2^{k^2}}{k^k}.$$

By the prime number theorem, $p_k \approx k\log k$, so that $p_{2^{k+1}} \leq M2^k k$, for some $M > 1$. Hence any n that has a non-zero coefficient in f_k must satisfy

$$n \leq \left(M2^k k\right)^k.$$

Thus, we can estimate for $\sigma < 1$,

$$\begin{aligned} \sum_n |a_n| n^{-\sigma} &\geq \sum_k \frac{2^{k^2}}{k^k}\left(M2^k k\right)^{-k\sigma} \qquad (5.27) \\ &= \sum_k \frac{2^{k^2(1-\sigma)}}{k^{k(1+\sigma)} M^{k\sigma}} \end{aligned}$$

By the root test (or ratio test), (5.27) diverges for $\sigma < 1$.

Since for $\sigma > 1$ the series converges absolutely (by comparison to $\sum_n n^{-\sigma}$), we conclude that $\sigma_a = 1$.

Claim 2: $\sigma_u(f) = \frac{1}{2}$.

Proof: Fix $\varepsilon > 0$, let $\sigma = \frac{1}{2} + \varepsilon$, and note that

$$\begin{aligned} |f_k(\sigma+it)| &= |Q_k(p_{2^k}^{-s}, \ldots, p_{2^{k+1}-1}^{-s})| \\ &= \left| \sum_{|j|=k} \varepsilon_j \left(p_{2^k}^{j_1} \ldots p_{2^{k+1}-1}^{j_{2^k}}\right)^\sigma \left(p_{2^k}^{j_1} \ldots p_{2^{k+1}-1}^{j_{2^k}}\right)^{it} \right|. \end{aligned}$$

Thus,

$$\begin{aligned} \sup_t |f_k(\sigma+it)| &\leq \sup_{|z_i|=p_{2^k-1+i}^{-\sigma},\ i=1,\ldots,2^k} |Q_k(z_1,\ldots,z_{2^k})| \\ &\leq p_{2^k}^{-k\sigma} \sup_{|z_i|=1} |Q_k| \\ &\leq C p_{2^k}^{-k\sigma}\left[2^k \log k \binom{2^k+k-1}{k}\right]^{\frac{1}{2}} \qquad (5.28) \\ &\lesssim (2^k k\log 2)^{-k\sigma}\left[2^k \log k 2^{k^2}\right]^{\frac{1}{2}} \\ &= (k\log 2)^{-k\sigma} 2^{k^2(-\sigma+\frac{1}{2}+\frac{1}{2k})}\sqrt{\log k} \\ &= (k\log 2)^{-k\sigma} 2^{k^2(-\varepsilon+\frac{1}{2k})}\sqrt{\log k}. \end{aligned}$$

The series $\sum_k |f_k|$ is thus estimated by a summable series. Hence, $\sum_k f_k$ converges to a holomorphic function which is bounded in $\Omega_{1/2+\varepsilon}$