staying away from $e^{i\vartheta}$. Continuity of $d\omega_\Omega(e^{i\vartheta}, z_0)/d\vartheta$ can then be read off from the formula since γ has no accumulation points inside the I_k.

The function $g(e^{i\vartheta})$ is thus continuous, in addition to enjoying property (i) of our list. Verification of properties (ii) and (v) thereof remains.

Because $d\omega_\Omega(e^{i\vartheta}, z_0)/d\vartheta \leqslant C$ and $|\Phi(e^{i\vartheta})|$ lies between two constant multiples of $|F(e^{i\vartheta})|$, property (ii) holds on account of the analogous condition satisfied by F and the relation of $g(e^{i\vartheta})$ to $\Phi(e^{i\vartheta})$. Passing to property (v), we note that an earlier relation can be rewritten

$$\left|\int_{-\pi}^{\pi} e^{in\vartheta} g(e^{i\vartheta})\,d\vartheta\right| \leqslant \text{const.}\,(e^{-n\xi_0} + e^{-M(n)}), \qquad n \geqslant 0.$$

By *concavity* of $M(v)$, $M(v)/v$ eventually decreases and tends to a limit $l \geqslant 0$ as $v \longrightarrow \infty$. Were $l > 0$, the right side of the inequality just written would be $\leqslant$ const. e^{-nl_0} with $l_0 = \min(\xi_0, l) > 0$. Such a bound on the left-hand integral would, with property (ii), force $g(e^{i\vartheta})$ to vanish identically – see the bottom of p. 328. Our $g(e^{i\vartheta})$, however, does not do that, so we must have $l = 0$, making $M(n) < n\xi_0$ for large n. The right side of our inequality can therefore be replaced by const. $e^{-M(n)}$, and property (v) holds. The construction is now complete.

It is to be noted that the only objects we actually *used* were the function $h(\xi)$ with its specified properties and $\Phi(z)$, analytic in a certain domain $\mathcal{O} \subseteq \{|z| < 1\}$ and continuous up to $\partial\mathcal{O}$, satisfying $|\Phi(\zeta)| \leqslant \text{const.}\exp\left(-h\left(\log\frac{1}{|\zeta|}\right)\right)$ on $\partial\mathcal{O} \cap \{|\zeta| < 1\}$ and $|\Phi(\zeta)| > 0$ on some arc of $\{|\zeta| = 1\}$ included in $\partial\mathcal{O}$. I have a persistent nagging feeling that such functions $h(\xi)$ and $\Phi(z)$, if there really *are* any, must be lying around somewhere or at least be closely related to others whose constructions are already available. One thinks of various kinds of functions meromorphic in the unit disk but not of bounded characteristic there; especially do the ones described by Beurling at the eighth Scandinavian mathematicians' congress come back continually to mind.

This addendum, however, is already being written at the very last moment. The imminence of press time leaves me no opportunity for pursuing the matter.

3. Extension to functions $F(e^{i\vartheta})$ in $L_1(-\pi, \pi)$.

The theorem of p. 356 holds for L_1 functions $F(e^{i\vartheta})$ not a.e. zero, as does Brennan's refinement of it given in article 1 above. A procedure for handling this more general situation (absence of continuity) is worked out in the beautiful *Mat. Sbornik* paper by Jöricke and Volberg. Here we adapt their method so as to make it go with the development already familiar from §D.6, Chapter VII, hewing as closely as possible to the latter.

Our aim is to show that

$$\int_{-\pi}^{\pi} \log|F(e^{i\vartheta})|\,d\vartheta \;>\; -\infty$$

for any function $F(e^{i\vartheta}) \in L_1(-\pi,\pi)$ not a.e. zero and satisfying the hypothesis of Brennan's theorem. Let us begin by observing that the treatment of this case can be reduced to that of a *bounded* function F.

Suppose, indeed, that

$$F(e^{i\vartheta}) \;\sim\; \sum_{-\infty}^{\infty} a_n e^{in\vartheta}$$

belongs to L_1, with $|a_n| \leqslant \text{const.}\,e^{-M(|n|)}$ for $n<0$. The series $\sum_{n<0} a_n e^{in\vartheta}$ is then surely *absolutely convergent*, so

$$\sum_0^{\infty} a_n e^{in\vartheta}$$

is also the Fourier series of an L_1 function, which we denote by $F_+(e^{i\vartheta})$ (this belongs in fact to the space H_1). For $|z|<1$, put

$$F_+(z) \;=\; \sum_0^{\infty} a_n z^n;$$

for this function, analytic in $\{|z|<1\}$, we have (Chapter II, §B!),

$$F_+(z) \longrightarrow F_+(e^{i\vartheta}) \quad \text{a.e. as } z \xrightarrow{\angle} e^{i\vartheta}.$$

Using the integrable function $\log^+|F_+(e^{i\vartheta})| \geqslant 0$, we now form

$$b(z) \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\vartheta}+z}{e^{i\vartheta}-z}\log^+|F_+(e^{i\vartheta})|\,d\vartheta,$$

analytic and with positive real part for $|z|<1$. According to the third theorem and scholium of §F.2, Chapter III, $b(z)$ tends for almost every ϑ to a limit $b(e^{i\vartheta})$ as $z \xrightarrow{\angle} e^{i\vartheta}$, with

$$\Re b(e^{i\vartheta}) \;=\; \log^+|F_+(e^{i\vartheta})| \quad \text{a.e.}$$

A standard extension of Jensen's inequality to H_1 also tells us that

$$\log|F_+(z)| \;\leqslant\; \Re b(z), \qquad |z|<1$$

(cf. pp. 291–2 where this was proved and used for $z=0$).

We next perform the Dynkin extension (described on pp. 339–40) on the continuous function

$$F_-(e^{i\vartheta}) \;=\; \sum_{-\infty}^{-1} a_n e^{in\vartheta}.$$

This gives us $F_-(z)$, $\mathscr{C}_\infty$ in the unit disk and continuous (hence *bounded*!) up to its boundary, with

$$\left|\frac{\partial F_-(z)}{\partial \bar{z}}\right| \leqslant \text{const.}\exp\left(-h\left(\log\frac{1}{|z|}\right)\right), \qquad |z| < 1,$$

where, in the present circumstances,

$$h(\xi) = \sup_{v>0}(M(v)/2 - v\xi)$$

(see *remark* 2, p. 343). As usual, we write

$$w(r) = \exp\left(-h\left(\log\frac{1}{r}\right)\right);$$

then, putting

$$F(z) = F_-(z) + F_+(z)$$

for $|z| < 1$, we have

$$\left|\frac{\partial F(z)}{\partial \bar{z}}\right| \leqslant \text{const.}\, w(|z|)$$

there, and

$$F(z) \longrightarrow F(e^{i\vartheta}) \quad \text{a.e. for } z \not\longrightarrow e^{i\vartheta}.$$

The bounded function spoken of earlier is simply

$$F_0(z) = e^{-b(z)}F(z).$$

It *is* bounded in the unit disk by one of the previous relations; another tells us that $F_0(z)$ has a non-tangential boundary value $F_0(e^{i\vartheta}) = F(e^{i\vartheta})\exp(-b(e^{i\vartheta}))$ equal in modulus to $|F(e^{i\vartheta})|/\max(|F_+(e^{i\vartheta})|, 1)$ at almost every point of the unit circumference. Then, since $F(e^{i\vartheta}) \in L_1$ is not a.e. zero, neither is $F_0(e^{i\vartheta})$. We note finally that by *analyticity* of $e^{-b(z)}$, $\partial F_0(z)/\partial\bar{z} = e^{-b(z)}\partial F(z)/\partial\bar{z}$, making

$$\left|\frac{\partial F_0(z)}{\partial \bar{z}}\right| \leqslant \text{const.}\, w(|z|), \qquad |z| < 1.$$

Given that $M(v)$ satisfies the hypothesis of Brennan's theorem, our function $h(\xi)$ enjoys the two properties used in the first part of article 1, namely, that $\xi h(\xi)$ *decreases* and that $\int_0^a \log h(\xi)\,d\xi = \infty$ for small $a > 0$. If, now, we can *deduce* from these together with the *preceding relation* that the *bounded* function $F_0(z)$, not a.e. zero for $|z| = 1$, satisfies

$$\int_{-\pi}^{\pi} \log|F_0(e^{i\vartheta})|d\vartheta \;>\; -\infty,$$

we will certainly have *the same conclusion* for

$$\log|F(e^{i\vartheta})| \;=\; \log|F_0(e^{i\vartheta})| \;+\; \log^+|F_+(e^{i\vartheta})|.$$

The rest of our work deals exclusively with $F_0(z)$.

In order to stay as close as possible to the notation of §D.6, Chapter VII, *we denote the bounded function $F_0(z)$ by $F(z)$ from now on.* Using this new $F(z)$, we first form the sets $B \subseteq \{|z| \leqslant 1\}$ and $\mathcal{O} \subseteq \{|z| < 1\}$ as on pp. 359–60, and then the function $\Phi(z)$ introduced on p. 360. The latter, *analytic in $\mathcal{O}$*, is actually defined on the whole unit disk, and has there at least as much continuity as $F(z)$ besides lying in modulus between two constant multiples of $|F(z)|$. It has, in particular, a non-tangential boundary value $\Phi(e^{i\vartheta})$ a.e. on the unit circumference, and this *does not vanish* a.e. The construction of B ensures that

$$|\Phi(\zeta)| \;\leqslant\; \text{const.}\, w(|\zeta|) \qquad \text{on } \partial\mathcal{O} \cap \{|\zeta| < 1\}$$

(indeed, on B), and our task amounts to showing that

$$\int_{-\pi}^{\pi} \log|\Phi(e^{i\vartheta})|d\vartheta \;>\; -\infty$$

on account of these properties.

What makes the present situation more complicated than the one studied in §D.6 of Chapter VII is that $\Phi(z)$ need no longer be continuous up to the whole unit circumference. This causes the notion of *abutment* introduced on p. 348 to be less useful here for the examination of our set $\mathcal{O}$ than it was in §D.6, and we have to supplement it with another, that of *fatness.* The latter, based on the famous sawtooth construction of Lusin and Privalov, helps us to take account of $\Phi(z)$'s non-tangential boundary behaviour.

To describe what is meant by fatness, we need to bring in a special kind of domain together with some notation; both will also be used further on. Corresponding to each point $e^{i\alpha}$ on the unit circumference, we have an open set S_α consisting of the z with $1/2 \;<\; |z| \;<\; 1$ lying in the open 60° sector having vertex at $e^{i\alpha}$ and symmetric about the radius from 0 out to that point. Given any subset E of $\{|\zeta| = 1\}$ we then write

$$S_E \;=\; \bigcup_{e^{i\alpha}\in E} S_\alpha.$$

It is evident that if we take any S_E and a ρ, $1/2 < \rho < 1$, the intersection

$$S_E \cap \{\rho < |z| < 1\}$$

breaks up into (at most) a countable number of open *connected* components, each of the form

$$S_{E_k} \cap \{\rho < |z| < 1\},$$

with the E_k making up a (disjoint) partition of the set E.

Definition. A *connected* open set of the form

$$S_E \cap \{\rho < |z| < 1\}$$

(with $1/2 < \rho < 1$) is called a *sawblade* of *depth* $1 - \rho$. We say that such a sawblade *bites on* the set E.

Now we can state the

Definition. An open subset $\mathscr{U}$ of the unit disk is called *fat* if it contains a sawblade biting on a closed $E \subseteq \{|\zeta| = 1\}$ with $|E| > 0$. In that circumstance we also say that $\mathscr{U}$ is *fat at E*.

Equipped with these tools, we endeavour to investigate the set $\mathscr{O}$ according to the procedure of §D.6, Chapter VII. In this, some modifications are necessary; we have, in the first place, to *skip over step 1* (p. 361). Then, taking ρ, $1/2 < \rho < 1$, we construct a set $\Omega(\rho)$, proceeding differently, however, than as we did on pp. 361–3.

There is, by the properties of $\Phi(z)$, a closed subset E_0 of the unit circumference, $|E_0| > 0$, such that, for the *non-tangential* boundary values $\Phi(\zeta)$, we have, wlog,

$$|\Phi(\zeta)| > 1, \qquad \zeta \in E_0.$$

Egorov's theorem enables us to in fact pick E_0 so as to have $|\Phi(z)| > 1$ for $z \in S_{E_0}$ with $\rho' < |z| < 1$ when $\rho' > \rho$ is sufficiently close to 1. But the construction of B and $\mathscr{O}$ makes $|\Phi(z)| \leqslant \text{const.}\, w(|z|)$ on B, hence on $\{|z| < 1\} \sim \mathscr{O}$. Therefore, since $w(r) \longrightarrow 0$ for $r \longrightarrow 1$, we must have

$$S_{E_0} \cap \{\rho' < |z| < 1\} \subseteq \mathscr{O}$$

if ρ', $\rho < \rho' < 1$, is near enough to 1. One of the components of the intersection on the left is a sawblade of depth $1 - \rho'$ biting on a (Borel) subset E' of E_0 with $|E'| > 0$; a suitable *closed* subset E of E' then has

$|E|>0$, and *there is a sawblade of depth* $1-\rho'$ *biting on* E *and contained in* $\mathcal{O}$. We now take $\Omega(\rho)$ as *the component of* $\mathcal{O}\cap\{\rho<|z|<1\}$ *including that sawblade*; $\Omega(\rho)$ is *fat at* E.

For the present set $\Omega(\rho)$ there is *a substitute* for *step 2* of p. 362:

Step 2'. *$\partial\Omega(\rho)$ includes the whole unit circumference.*

This we establish by *reductio ad absurdum*. Let us write Ω for $\Omega(\rho)$, and put

$$\gamma = \partial\Omega\cap\{|z|<1\},$$

$$\Gamma = \partial\Omega \sim \gamma;$$

Γ is thus the part of $\partial\Omega$ lying on the unit circumference. *Assume* that there is on the latter a non-empty open arc J with $J\cap\Gamma=\emptyset$; we will then deduce a contradiction.

For that it is quicker to fall back on the device used in the second half of article 2 than to adapt Volberg's theorem on harmonic measures (p. 349) to the present situation. *Fixing* $z_0\in\Omega$, we can say that

$$z_0^n\Phi(z_0) = \int_{\partial\Omega}\zeta^n\Phi(\zeta)\,d\omega_\Omega(\zeta,z_0) \qquad \text{for } n\geqslant 0,$$

whence

$$\int_\Gamma e^{in\vartheta}\Phi(e^{i\vartheta})\,d\omega_\Omega(e^{i\vartheta},z_0) = z_0^n\Phi(z_0) - \int_\gamma \zeta^n\Phi(\zeta)\,d\omega_\Omega(\zeta,z_0), \qquad n\geqslant 0.$$

Here we are using Poisson's formula for the bounded function $\zeta^n\Phi(\zeta)$ harmonic (even analytic) in Ω and continuous up to γ, but not necessarily up to Γ, where it is only known to have non-tangential boundary values a.e. Such use is legitimate; we postpone verification of that, and of a corresponding version of Jensen's inequality, to the next article, so as not to interrupt the argument now under way.

As in article 2, $d\omega_\Omega(e^{i\vartheta},z_0)$ is absolutely continuous and $\leqslant C\,d\vartheta$ on Γ, and we obtain a bounded measurable function $g(e^{i\vartheta})$ by putting

$$g(e^{i\vartheta}) = \Phi(e^{i\vartheta})\frac{d\omega_\Omega(e^{i\vartheta},z_0)}{d\vartheta} \qquad \text{for } e^{i\vartheta}\in\Gamma$$

and (here!) taking $g(e^{i\vartheta})$ to be *zero outside* Γ. From the preceding relation we then see, as in article 2, that

$$\left|\int_{-\pi}^{\pi} e^{in\vartheta} g(e^{i\vartheta})\,d\vartheta\right| \leqslant \text{const.}\,(e^{-n\xi_0} + e^{-M_1(n)})$$

for $n \geqslant 0$, where $\xi_0 > 0$ and

$$M_1(\nu) = \inf_{\xi > 0} (h(\xi) + \xi\nu).$$

This function is increasing and concave, so the right side of the last inequality can be replaced by const. $e^{-M_2(n)}$ for large n, with $M_2(n)$ equal either to $\xi_0 n$ (in case $\lim_{\nu\to\infty}(M_1(\nu)/\nu) \geqslant \xi_0$) or else to $M_1(n)$. In either event, $M_2(n)$ increases and $\sum_1^\infty M_2(n)/n^2 = \infty$ on account of the properties of $h(\xi)$. (See the theorem of p. 337 – $M_1(n)$ is actually equal to $M(n)/2$ in the present set-up.) Now we can apply Levinson's theorem, since $g(e^{i\vartheta})$ *vanishes on the arc J*. The conclusion is that $g(e^{i\vartheta}) \equiv 0$ a.e.

But $g(e^{i\vartheta})$ *does not vanish* a.e. Indeed, Ω contains a *sawblade* $\mathscr{E}$ *biting on* a closed set E, $|E| > 0$, where $|\Phi(e^{i\vartheta})| \geqslant 1$. Thence,

$$\int_E |g(e^{i\vartheta})|\,d\vartheta = \int_E |\Phi(e^{i\vartheta})|\,d\omega_\Omega(e^{i\vartheta}, z_0) \geqslant \omega_\Omega(E, z_0).$$

Harnack's theorem assures us that the quantity on the right is >0 if, for some $z_1 \in \mathscr{E}$, $\omega_\Omega(E, z_1) > 0$. However, by the principle of extension of domain, $\omega_\Omega(E, z_1) \geqslant \omega_{\mathscr{E}}(E, z_1)$. At the same time, $\partial\mathscr{E}$ is *rectifiable*, so a conformal mapping of $\mathscr{E}$ onto the unit disk must take the *subset* E of $\partial\mathscr{E}$, *having linear measure* > 0, to a *set of measure* > 0 *on the unit circumference*. (This follows by the celebrated F. and M. Riesz theorem; a proof can be found in Zygmund or in any of the books about H_p spaces.) We therefore have $\omega_{\mathscr{E}}(E, z_1) > 0$, making $\omega_\Omega(E, z_0) > 0$ and hence, as we have seen, $\int_E |g(e^{i\vartheta})|\,d\vartheta > 0$.

Our contradiction is thus established. By it we see that the arc J cannot exist, i.e., that Γ *is the whole unit circumference*, as was to be shown.

With *step 2′* accomplished, we are ready for *step 3*. One starts out as on p. 363, using the square root mapping employed there. That gives us a domain $\Omega_{\surd}$, certainly *fat* at a closed subset E'', of $E_{\surd}$ (the image of E under our mapping), with $|E''| > 0$ (recall the earlier use of Egorov's theorem). Thereafter, one applies to $\Omega_{\surd}$ the argument just made for Ω in doing *step 2′*.

The weight $w_1(r)$ is next introduced as on p. 365, and the sets B_1 and $\mathscr{O}_1$ constructed (pp. 365–6). After doing *steps 2′* and *3* again with these objects, we come to *step 4*.

Jöricke and Volberg are in fact able to circumvent this step, thanks to a clever rearrangement of *step 5*. Here, however, let us continue according to the plan of §D.6, Chapter VII, for the work done there carries over practically without change to the present situation.

What is important for *step 4* is that a ζ, $|\zeta| = 1$, *not* in B must, even here, lie on an arc of the unit circumference *abutting* on $\mathcal{O}$. Such a $\zeta \notin B$ must thus, as on p. 367, have a neighborhood V_ζ with

$$V_\zeta \cap \{|z| < 1\} \subseteq \mathcal{O} \cap \{\rho^2 < |z| < 1\}.$$

The left-hand intersection therefore lies in some *connected component* of the one on the right, which, however, *can only be* $\Omega(\rho^2)$, since $\zeta \in \partial\Omega(\rho^2)$ by *step 2'*. The rest of the argument goes as on pp. 367–8.

Now we can do *step 5*, or rather the *substitute* for it carried out at the beginning of article 1. For this it is necessary to have the Jensen inequality

$$\log|\Phi(\rho)| \leq \int_{\partial\Omega(\rho^2)} \log|\Phi(\zeta)| \mathrm{d}\omega(\zeta, \rho)$$

(notation of p. 369) available in the present circumstances, where continuity of $\Phi(z)$ up to $\{|\zeta| = 1\}$ may fail. The legitimacy of this will be established in the next article; *granting* it for now, we may proceed exactly as at the beginning of article 1.

From here on, one continues as on pp. 370–2, and reaches the desired conclusion that $\int_{-\pi}^{\pi} \log|\Phi(e^{i\vartheta})| \mathrm{d}\vartheta > -\infty$ as on p. 373, after one more application of our extended Jensen inequality.

We thus arrive at the

Theorem. Let $F(e^{i\vartheta}) \in L_1(-\pi,\pi)$ not be zero a.e., and suppose that

$$F(e^{i\vartheta}) \sim \sum_{-\infty}^{\infty} a_n e^{in\vartheta}$$

with

$$|a_n| \leq \text{const.}\, e^{-M(|n|)}, \qquad n \leq 0.$$

Suppose that $M(v)$ is *concave*, that $M(v)/v^{1/2}$ is *increasing* for large v, and that

$$\sum_1^{\infty} M(n)/n^2 = \infty.$$

Then

$$\int_{-\pi}^{\pi} \log|F(e^{i\vartheta})| \mathrm{d}\vartheta > -\infty.$$

Remark. In their preprint, Borichev and Volberg consider formal trigonometric series

$$\sum_{-\infty}^{\infty} a_n \mathrm{e}^{in\vartheta}$$

in which the a_n with *negative n* satisfy the requirement of the theorem, but the a_n with $n > 0$ are allowed to *grow* like $\mathrm{e}^{M(n)}$ as $n \longrightarrow \infty$. Assuming *more* regularity for $M(v)$ $(M(v) \geqslant \text{const.}\, v^\alpha$ with an $\alpha < 1$ *close to* 1 is enough), they are able to show that under the remaining conditions of the theorem, all the a_n must vanish if

$$\liminf_{r\to 1} \int_{-\pi}^{\pi} \log\left| \sum_{-\infty}^{0} a_n \mathrm{e}^{in\vartheta} + \sum_{1}^{\infty} a_n r^n \mathrm{e}^{in\vartheta} \right| \mathrm{d}\vartheta = -\infty.$$

Before ending this article let us, as promised in the *last* one, see how the example of Borichev and Volberg shows that the *monotoneity requirement* on $M(v)/v^{1/2}$ *cannot, in the above theorem at least*, be *relaxed* to $M(v)/v^{1/2} \geqslant C > 0$, *even though continuity* up to $\{|\zeta| = 1\}$ *should fail* for the function $F(z)$ supplied by their construction.

The reader should refer back to the second part of article 2. Corresponding to the bounded function $F(z)$ used there, *no longer assumed continuous* up to $\{|\zeta| = 1\}$ but having at least non-tangential boundary values a.e. on that circumference, one can, as in the preceding discussion, form the sets B, $\mathcal{O}$ and $\Omega(\rho)$ and do *step 2'*. One may then form the function $g(\mathrm{e}^{i\vartheta})$ as in article 2; *here* it is bounded and measurable at least. The work of *step 2'* shows that $g(\mathrm{e}^{i\vartheta})$ is *not* a.e. zero, while properties (ii)–(v) of article 2 *hold* for it (for the last one, see again the end of that article).

This is all we need.

4. Lemma about harmonic functions

Suppose we have a domain Ω regular for Dirichlet's problem, lying in the (open) unit disk Δ and having part of $\partial\Omega$ on the unit circumference. As in the last article, we write

$$\Gamma = \partial\Omega \cap \partial\Delta \quad \text{and} \quad \gamma = \partial\Omega \cap \Delta.$$

For the following discussion, let us agree to call ζ, $|\zeta| = 1$, a *radial accumulation point of* Ω if, for a *sequence* $\{r_n\}$ tending to 1, we have $r_n\zeta \in \Omega$ for each n. We then denote by Γ' the set of such radial accumulation points, noting that $\Gamma' \subseteq \Gamma$ with the inclusion frequently *proper*.

Lemma. (Jöricke and Volberg) Let $V(z)$, harmonic and bounded in Ω, be continuous up to γ, and suppose that

$$\lim_{\substack{r\to 1\\ r\zeta\in\Omega}} V(\zeta)$$

exists for almost all $\zeta\in\Gamma'$. Put $v(\zeta)$ *equal* to that limit for such ζ, and to *zero* for the *remaining* $\zeta\in\Gamma$. On γ, take $v(\zeta)$ equal to $V(\zeta)$. Then, for $z\in\Omega$,

$$V(z) \;=\; \int_{\partial\Omega} v(\zeta)\,\mathrm{d}\omega_\Omega(\zeta, z).$$

Proof. It suffices to establish the result for *real* harmonic functions $V(z)$, and, for those, to show that

$$V(z) \;\leqslant\; \int_{\partial\Omega} v(\zeta)\,\mathrm{d}\omega_\Omega(\zeta, z), \qquad z\in\Omega,$$

since the reverse inequality then follows on changing the signs of V and v.

By modifying $v(\zeta)$ on a subset of Γ having zero Lebesgue measure, we get a bounded Borel function defined on $\partial\Omega$. But on Γ, we have $\mathrm{d}\omega_\Omega(\zeta, z) \leqslant C_z|d\zeta|$ (see articles 2 and 3), so such modification cannot alter the value of $\int_{\partial\Omega} v(\zeta)\,\mathrm{d}\omega_\Omega(\zeta, z)$. *We may hence just as well take $v(\zeta)$ as a bounded Borel function* (on $\partial\Omega$) *to begin with.*

That granted, we desire to show that the integral just written is $\geqslant V(z)$. For this it seems necessary to hark back to the very foundations of integration theory. Call the *limit* of any *increasing sequence of functions continuous on* $\partial\Omega$ an *upper function* (on $\partial\Omega$). There is then a *decreasing sequence* of *upper functions* $w_n(\zeta) \geqslant v(\zeta)$ such that

$$\int_{\partial\Omega} w_n(\zeta)\,\mathrm{d}\omega_\Omega(\zeta, z) \;\xrightarrow[n]{}\; \int_{\partial\Omega} v(\zeta)\,\mathrm{d}\omega_\Omega(\zeta, z), \qquad z\in\Omega.$$

Indeed, corresponding to any *given* $z\in\Omega$, such a sequence is furnished by a basic construction of the Lebesgue–Stieltjes integral, $\omega_\Omega(\ \ ,z)$ being a Radon measure on $\partial\Omega$. But then that sequence works also for *any other* $z\in\Omega$, since $\mathrm{d}\omega_\Omega(\zeta, z') \leqslant C(z, z')\mathrm{d}\omega_\Omega(\zeta, z)$ (Harnack).

Our inequality involving v and V will *thus be established*, provided that we can verify

$$V(z) \;\leqslant\; \int_{\partial\Omega} w_n(\zeta)\,\mathrm{d}\omega_\Omega(\zeta, z), \qquad z\in\Omega,$$

for each n. *Fixing*, then, any n, we write simply $w(\zeta)$ for $w_n(\zeta)$ and put

$$W(z) \;=\; \int_{\partial\Omega} w(\zeta)\,d\omega_\Omega(\zeta, z)$$

for $z \in \Omega$, making $W(z)$ *harmonic* there. Our task is to prove that

$$V(z) \;\leqslant\; W(z), \qquad z \in \Omega.$$

It is convenient to define $W(z)$ on *all* of $\bar{\Omega}$ by putting

$$W(\zeta) \;=\; w(\zeta), \qquad \zeta \in \partial\Omega.$$

At each $\zeta \in \partial\Omega$ we then have

$$\liminf_{\substack{z \to \zeta \\ z \in \bar{\Omega}}} W(z) \;\geqslant\; W(\zeta)$$

by the elementary approximate identity property of harmonic measure, since $w(\zeta)$, as limit of an *increasing* sequence of continuous functions, satisfies

$$\liminf_{\substack{\zeta \to \zeta_0 \\ \zeta \in \partial\Omega}} w(\zeta) \;\geqslant\; w(\zeta_0) \qquad \text{for } \zeta_0 \in \partial\Omega.$$

The function $W(z)$ enjoys a certain *reproducing property* in $\bar{\Omega}$. Namely, if the domain $\mathscr{D} \subseteq \Omega$ is also regular for Dirichlet's problem, with perhaps (and especially!) part of $\partial\mathscr{D}$ on $\partial\Omega$, we have

$$W(z) \;=\; \int_{\partial\mathscr{D}} W(\zeta)\,d\omega_{\mathscr{D}}(\zeta, z) \qquad \text{for } z \in \mathscr{D}.$$

To see this, take an increasing sequence of functions $f_k(\zeta)$ continuous on $\partial\Omega$ and tending to $w(\zeta)$ thereon, and let

$$F_k(z) \;=\; \int_{\partial\Omega} f_k(\zeta)\,d\omega_\Omega(\zeta, z), \qquad z \in \Omega.$$

Then the $F_k(z)$ tend monotonically to $W(z)$ in Ω by the monotone convergence theorem. That convergence actually holds *on* $\bar{\Omega}$ if we put $F_k(\zeta) \;=\; f_k(\zeta)$ on $\partial\Omega$; this, however, makes each function $F_k(z)$ *continuous on* $\bar{\Omega}$ besides being *harmonic* in Ω. *In the domain* $\mathscr{D}$, we therefore have

$$F_k(z) \;=\; \int_{\partial\mathscr{D}} F_k(\zeta)\,d\omega_{\mathscr{D}}(\zeta, z)$$

for each k. Another appeal to monotone convergence now establishes the corresponding property for W.

Fix any $z_0 \in \Omega$; we wish to show that $V(z_0) \;\leqslant\; W(z_0)$. For this purpose, we use the formula just proved with $\mathscr{D}$ equal to *the component* Ω_r *of* $\Omega \cap \{|z| < r\}$ *containing* z_0, where $|z_0| < r < 1$. Because Ω is regular for Dirichlet's problem, so is each Ω_r; that follows immediately from the characterization of such regularity in terms of *barriers*, and, in the circumstances of the last article, can also be checked directly (cf. p. 360). We write

$$\Gamma_r = \partial\Omega_r \cap \Omega,$$

making Γ_r the union of some *open arcs* on $\{|\zeta| = r\}$, and then take

$$\gamma_r = \partial\Omega_r \sim \Gamma_r;$$

γ_r is a subset (perhaps proper) of $\gamma \cap \{|\zeta| \leqslant r\}$.

The function $V(z)$, given as harmonic in Ω and continuous up to γ, is certainly continuous up to $\partial\Omega_r$. Therefore, since $V(\zeta) = v(\zeta)$ on $\gamma \supseteq \gamma_r$, we have, for $z \in \Omega_r$,

$$V(z) = \int_{\gamma_r} v(\zeta)\, d\omega_{\Omega_r}(\zeta, z) + \int_{\Gamma_r} V(\zeta)\, d\omega_{\Omega_r}(\zeta, z).$$

At the same time, by the reproducing property of W,

$$W(z) = \int_{\gamma_r} W(\zeta)\, d\omega_{\Omega_r}(\zeta, z) + \int_{\Gamma_r} W(\zeta)\, d\omega_{\Omega_r}(\zeta, z), \qquad z \in \Omega_r.$$

We henceforth write $\omega_r(\ ,\ )$ *for* $\omega_{\Omega_r}(\ ,\ )$. Then, since on $\gamma_r \subseteq \partial\Omega$, $W(\zeta) = w(\zeta)$ is $\geqslant v(\zeta)$, the two last relations yield

$$W(z) - V(z) \geqslant \int_{\Gamma_r} (W(\zeta) - V(\zeta))\, d\omega_r(\zeta, z)$$

for $z \in \Omega_r$. Our idea is to now make $r \longrightarrow 1$ in this inequality.

For $|\zeta| = 1$, define

$$\Delta_r(\zeta) = \begin{cases} W(r\zeta) - V(r\zeta) & \text{if } r\zeta \in \Gamma_r, \\ 0 \quad \text{otherwise.} \end{cases}$$

Since $V(z)$ is given as *bounded*, the functions $\Delta_r(\zeta)$ are *bounded below. Moreover* (and this is the clincher),

$$\liminf_{r \to 1} \Delta_r(\zeta) \geqslant 0 \text{ a.e., } |\zeta| = 1.$$

That is indeed *clear* for the ζ on the unit circumference *outside* Γ' (the set of radial accumulation points of Ω); since for such a ζ, $r\zeta$ cannot even belong to Ω (let alone to Γ_r) when r is near 1. Consider therefore a $\zeta \in \Gamma'$, and take any sequence of $r_n < 1$ tending to 1 with, wlog, *all the* $r_n\zeta$ *in* Ω

and even in their corresponding Γ_{r_n}. Then our hypothesis and the specification of v tell us that

$$V(r_n\zeta) \underset{n}{\longrightarrow} v(\zeta),$$

except when ζ belongs to a certain set of measure zero, independent of $\{r_n\}$. For such a sequence $\{r_n\}$, however,

$$\liminf_{n\to\infty} W(r_n\zeta) \geqslant W(\zeta) = w(\zeta)$$

as seen earlier, yielding, with the preceding,

$$\liminf_{n\to\infty} \Delta_{r_n}(\zeta) \geqslant w(\zeta) - v(\zeta) \geqslant 0.$$

The asserted relation thus holds *on* Γ' *as well*, save perhaps in a set of measure zero.

Returning to our fixed $z_0 \in \Omega$, we note that for $(1+|z_0|)/2 < r < 1$ (say), we have, on Γ_r,

$$d\omega_r(\zeta, z_0) \leqslant K|d\zeta|$$

with K independent of r (just compare $\omega_r(\quad,\quad)$ with harmonic measure for $\{|z|<r\}$). There are hence measurable functions $\mu_r(\zeta)$ defined on $\{|\zeta|=1\}$ for these values of r, with $0 \leqslant \mu_r(\zeta) \leqslant K$ (and $\mu_r(\zeta)=0$ for $r\zeta\notin\Gamma_r$), such that

$$\int_{\Gamma_r} (W(\zeta) - V(\zeta))\, d\omega_r(\zeta, z_0) = \int_{|\zeta|=1} \Delta_r(\zeta) r\mu_r(\zeta)|d\zeta|.$$

Here the products $\Delta_r(\zeta)r\mu_r(\zeta)$ are *uniformly bounded below* since the $\Delta_r(\zeta)$ are. And, by what has just been shown,

$$\liminf_{r\to 1} \Delta_r(\zeta)r\mu_r(\zeta) \geqslant 0 \quad \text{a.e., } |\zeta|=1.$$

Thence, by *Fatou's lemma* (!),

$$\liminf_{r\to 1} \int_{|\zeta|=1} \Delta_r(\zeta)r\mu_r(\zeta)|d\zeta| \geqslant 0.$$

We have seen, however, that when $r>|z_0|$, $W(z_0)-V(z_0)$ is $\geqslant$ the left-hand integral in the previous relation. It follows therefore that

$$W(z_0) - V(z_0) \geqslant 0,$$

as was to be proven.

We are done.

Remark 1. When $V(z)$ is only assumed to be *subharmonic* in Ω but satisfies otherwise the hypothesis of the lemma, the argument just made shows that

$$V(z) \leqslant \int_{\partial\Omega} v(\zeta)\,d\omega_\Omega(\zeta, z) \qquad \text{for } z \in \Omega.$$

Remark 2. In the applications made in article 3, the function $V(z)$ actually has a continuous extension to the open unit disk Δ with modulus *bounded, in* $\Delta \sim \Omega$, by a function of $|z|$ *tending to zero for* $|z| \longrightarrow 1$. That extension also has non-tangential boundary values a.e. on $\partial\Delta$. In these circumstances the lemma's *ad hoc* specification of $v(\zeta)$ on $\Gamma \sim \Gamma'$ is *superfluous*, for the non-tangential limit of $V(z)$ must *automatically be zero* at any $\zeta \in \Gamma \sim \Gamma'$ where it exists.

Remark 3. To arrive at the version of Jensen's inequality used in article 3, apply the relation from *remark 1* to the subharmonic functions $V_M(z) = \log^+|M\Phi(z)|$, referring to *remark 2*. That gives us

$$\max\left(\log|\Phi(z)|, \log\frac{1}{M}\right) \leqslant \int_{\partial\Omega} \max\left(\log|\Phi(\zeta)|, \log\frac{1}{M}\right) d\omega_\Omega(\zeta, z)$$

for $z \in \Omega$. Then, since $|\Phi(z)|$ is bounded above, one may obtain the desired result by making $M \longrightarrow \infty$.

Addendum completed June 8, 1987.

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