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The logarithmic integral I

PAUL KOOSIS

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$$\int_{-\infty}^{\infty} \frac{\log M(t)}{1+t^2}\,\mathrm{d}t$$

THE LOGARITHMIC INTEGRAL I

PAUL KOOSIS
McGill University in Montreal
formerly at the University of California, Los Angeles

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Notice

In this paperback edition of volume I a number of small errors – and some actual mathematical mistakes – present in the original hard-cover version have been corrected. Many were pointed out to me by Henrik Pedersen, my former student; it was he who observed in particular that the hint given for Problem 28 (b) was ineffective. I wish to express here my gratitude for the considerable service he has thus rendered.

Let me also call the reader's attention to two annoying oversights in volume II. In the statement of the important theorem on p. 65, the condition that the quantities a_k all be > 0 was inadvertently omitted. On p. 406 it would be better, in the last displayed formula, to replace the difference quotient now standing on the right by $\frac{\mu(x+\Delta x)-\mu(x-\Delta x)}{2\Delta x}$.

March 22, 1997
Outremont, Québec

Contents

Preface

The two volumes that follow make up what is meant primarily as a book for reading. One reason for writing them was to give a connected account of some of the ideas that have dominated my mathematical activity for many years. Another, which was to try to help beginning mathematicians interested in analysis learn how to work by showing how I work, seems now less important because my way is far from being the only one. I do hope, at any rate, to encourage younger analysts by the present book in their efforts to become and remain active.

I have loved $\int_{-\infty}^{\infty}(\log M(t)/(1 + t^2))\,dt$ – the logarithmic integral—ever since I first read Szegő's discussion about the geometric mean of a function and the theorem named after him in his book on orthogonal polynomials, over 30 years ago. Far from being an isolated artifact, this object plays an important role in many diverse and seemingly unrelated investigations about functions of one real or complex variable, and a serious account of its appearances would involve a good deal of the analysis done since 1900. That will be plain to the reader of this book, where some of that subject's developments in which the integral figures are taken up.

No attempt is made here to treat anything like the full range of topics to which the logarithmic integral is relevant. The most serious omission is that of parts of probability theory, especially of what is called prediction theory. For these, an additional volume would have been needed, and we already have the book of Dym and McKean. Considerations involving H_p spaces have also been avoided as much as possible, and the related material from operator theory left untouched. Quite a few books about those matters are now in circulation.

Of this book, begun in 1983, all but Chapter X and part of Chapter IX was written while I was at McGill University; the remainder was done at UCLA. The first 6 chapters are based on a course (and seminar) given

at McGill during the academic year 1982–83, and I am grateful to the mathematics department there for the support provided to me since then out of its rather modest resources. Chapters I–VI and most of the seventh were typed at that department's office.

Chapter VII and parts of Chapter VIII are developed from lectures I gave at the Mittag–Leffler Institute (Sweden) during part of the spring semesters of 1977 and 1983. I am fortunate in having been able to spend almost two years all told working there.

Partial support from the U.S. National Science Foundation was also given me during the first year or two of writing.

I thank first of all John Garnett for having over a long period of time encouraged me to write this book. Lennart Carleson encouraged and helped me with research that led eventually to some of the expositions set out below. I thank him for that and also for my two invitations to the Mittag–Leffler Institute. For the second of those I must also thank Peter Jones who, besides, helped me with at least one item in Chapter VII. The book's very title is from a letter to me by V.P. Havin, and I hope he does not mind my using it. I was unable to think of anything except the mathematical expression it represents!

It was mainly John Taylor who arranged for me to come to McGill in the fall of 1982 and give the course mentioned above. Since then, a good part of my salary at McGill has been paid out of research grants held by him, Jal Choksi, Sam Drury, or Carl Herz. Taylor also came to some of the lectures of my course as did Georg Schmidt. Robert Vermes attended all of them and frequently talked about their material with me. Dr Raymond Couture came part of the time. The students were Janet Henderson, Christian Houdré and Tuan Vu. These people all contributed to the course and helped me to feel that I was doing something of value by giving it. Vermes' constant presence and evident interest in the subject were especially heartening.

Most of the typing for volume I was done by Patricia Ferguson who typed Chapters I through VI and the major part of Chapter VII, and by Babette Dalton who did a very fine job with Chapter VIII. I am beholden to S. Gardiner and P. Jackson of the Press' staff and finally to Dr Tranah, the mathematics editor, for their patience and attention to my desires regarding graphic presentation. The beautiful typesetting was done in India.

August 13, 1987

Laurel, Comté Argenteuil, Québec.

Introduction

The present book has been written so as to necessitate as little consultation by the reader as reasonably possible of other published material. I have hoped to thereby make it accessible to people far from large research centres or any 'good library', and to those who have only their summer vacations to work on mathematics. It is for the same reason that references, where unavoidable, have been made to books rather than periodicals whenever that could be done.

In general, I consider the developments leading up to the various results in the book to be more important than the latter taken by themselves; that is why those developments are set out in more detail than is now customary. My aim has been to enable one to follow them by mostly just reading the text, without having to work on the side to fill in gaps. The reader's active participation is nevertheless solicited, and problems have been given. These are usually accompanied by hints (sometimes copious), so that one may be encouraged to work them out fully rather than feeling stymied by them. It is assumed that the reader's background includes, beyond ordinary undergraduate mathematics, the material which, in North America, is called graduate real and complex variable theory (with a bit of functional analysis). Practically everything needed of this is contained in Rudin's well-known manual. My own preference runs towards a more leisurely approach based on Titchmarsh's *Theory of Functions* and the beautiful *Leçons d'analyse fonctionnelle* of Riesz and Nagy (now available in English). Alongside these books, the use of some supplementary descriptive material on conformal mapping (from Nehari, for instance) is advisable, as is indeed the case with Rudin as well. The Krein–Milman theorem referred to in Chapters VI and X is now included in many books; in Naimark's, for example (on normed algebras or rings), and in Yosida's. In the very few places where more specialized material is called for, additional references will be given. (Exact descriptions of the works just mentioned together with those cited later on can be found in the bibliographies placed at the end of each volume.)

Although the different parts of this book are closely interrelated, they may to a large extent be read independently. Material from Chapter III is, however, called for repeatedly in the succeeding chapters. For finding one's way, the descriptions in the table of contents and the page headings should be helpful; indices to each volume are also provided. Throughout volume I, various arguments commonly looked on as elementary or well-known, but which I nonetheless thought it better to include, have been set in smaller type, and certain readers will miss nothing by passing over them.

The book's units of subdivision are, successively, the chapter, the § (plural §§) and the article. These are indicated respectively by roman numerals, capital letters and arabic numerals. A typical reference would be to '§B.2 of Chapter VI', or to 'Chapter VI, §B.2'. When referring to another article within the same §, that article's number alone is given (e.g., 'see article 3'), and, when it's to another § in the same chapter, just that §'s designation (e.g., 'the discussion in § B') or again, if a particular article in that § is meant, an indication like '§ B.2'. Theorems, definitions and so forth are not numbered, nor are formulas. But certain displayed formulas in a connected development may be labeled by signs like (∗), (†), &c, which are then used to refer to them within that development. The same signs are used over again in different arguments (to designate different formulas), and their order is not fixed. A pause in a discussion is signified by a horizontal space in the text.

About mistakes. There must inevitably be some, although I have tried as hard as I could to eliminate errors in the mathematics as well as misprints. Certain symbols (bars over letters, especially) have an unpleasant tendency to fall off between the typesetters' shop and the camera. I think (and hope) that all the mathematical arguments are clear and correct, at least in their grand lines, and have done my best to make sure of that by rereading everything several times. The reader who, in following a given development, should come upon a misprint or incorrect relation, will thus probably see what should stand in its place and be able to continue unhindered. If something really seems peculiar or devoid of sense, one should try suspending judgement and read ahead for a page or so – what at first appears bizarre may in fact be quite sound and become clear in a moment. Unexpected turnings are encountered as one becomes acquainted with this book's material.

It is beautiful material. May the reader learn to love it as I do.

I

Jensen's formula

On making the substitution $t = \tan(\vartheta/2)$ and then putting $M(t) = P(\vartheta)$, the expression

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\log M(t)}{1+t^2}\,\mathrm{d}t$$

goes over into

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log P(\vartheta)\,\mathrm{d}\vartheta.$$

We begin this book with a discussion of the second integral.

Suppose that $R > 1$ and we are given a function $F(z)$, analytic in $\{|z| < R\}$. If $F(z)$ *has no zeros* for $|z| \leqslant 1$ we can define a *single valued* function $\log F(z)$, analytic for $|z| \leqslant R'$, say, where $1 < R' < R$. By Cauchy's formula we will then have

$$\log F(0) = \frac{1}{2\pi}\int_{0}^{2\pi}\log F(\mathrm{e}^{\mathrm{i}\vartheta})\,\mathrm{d}\vartheta,$$

so, taking the real parts of both sides, we get

$$\log|F(0)| = \frac{1}{2\pi}\int_{-\pi}^{\pi}\log|F(\mathrm{e}^{\mathrm{i}\vartheta})|\,\mathrm{d}\vartheta.$$

What if $F(z)$ *has* zeros in $|z| \leqslant 1$? Assume to begin with that there are none *on* $|z| = 1$, and denote those that $F(z)$ does have inside the unit disk by $a_1, a_2, \ldots, a_n$. According to custom, a zero is *repeated according to its multiplicity* in such an enumeration. Put

$$\Phi(z) = \frac{F(z)}{(z-a_1)(z-a_2)\ldots(z-a_n)}.$$

Then $\Phi(z)$ has no zeros in $\{|z| \leqslant 1\}$, so, by the special case already treated,

$$\log|\Phi(0)| = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|\Phi(e^{i\vartheta})|\,d\vartheta$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|F(e^{i\vartheta})|\,d\vartheta - \sum_{k=1}^{n} \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|e^{i\vartheta} - a_k|\,d\vartheta.$$

Here we make a side calculation. For $|a_k| < 1$ we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|e^{i\vartheta} - a_k|\,d\vartheta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|1 - \bar{a}_k e^{i\vartheta}|\,d\vartheta,$$

and this $= \log 1 = 0$ by the case already discussed ($F(z)$ without zeros in $|z| \leqslant 1$)! Combined with the previous relation this yields

$$\log|\Phi(0)| = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|F(e^{i\vartheta})|\,d\vartheta.$$

Especially, if $F(0) \neq 0$,

$$\log|F(0)| - \sum_{k=1}^{n} \log|a_k| = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|F(e^{i\vartheta})|\,d\vartheta.$$

The sum on the left can be written differently. *Call $n(r)$ the number of zeros of $F(z)$ in $|z| \leqslant r$ (counting multiplicities).* Then, if $F(0) \neq 0$,

$$-\sum_{k=1}^{n} \log|a_k| = \int_0^1 \frac{n(r)}{r}\,dr.$$

Indeed, since $n(r) = 0$ for $r > 0$ *close* to 0,

$$\int_0^1 \frac{n(r)}{r}\,dr = n(1)\log 1 - \int_0^1 \log r\,dn(r) = -\sum_{k=1}^{n} \log|a_k|.$$

We therefore have

$$\log|F(0)| + \int_0^1 \frac{n(r)}{r}\,dr = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|F(e^{i\vartheta})|\,d\vartheta.$$

In case $F(z)$ is regular in a disk including $\{|z| \leqslant R\}$ in its interior and $F(0) \neq 0$ we can (provided that $F(z) \neq 0$ for $|z| = R$) make a change. of variable in the preceding relation and get

$$\boxed{\log|F(0)| + \int_0^R \frac{n(r)}{r}\,dr = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|F(Re^{i\vartheta})|\,d\vartheta.}$$

This is Jensen's formula.

The validity of Jensen's formula *subsists* even when $F(z)$ *has* zeros on the circle $|z| = R$. To see this, observe that then $F(z)$ *will not* have any zeros on the circles $|z| = R'$ with $R' < R$ and sufficiently close to R, for $F(z)$ is analytic in a disk $\{|z| < R + \eta\}$, $\eta > 0$, and not identically zero ($F(0) \neq 0$). So, for such R',

$$\log|F(0)| + \int_0^{R'} \frac{n(r)}{r}\mathrm{d}r = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|F(R'\mathrm{e}^{\mathrm{i}\vartheta})|\mathrm{d}\vartheta.$$

As $R' \to R$, the left side clearly tends to $\log|F(0)| + \int_0^R (n(r)/r)\,\mathrm{d}r$ – the integral on the left is a *continuous* function of its upper limit because $n(r)$ is *bounded*. We need therefore merely verify that

$$\int_{-\pi}^{\pi} \log|F(R'\mathrm{e}^{\mathrm{i}\vartheta})|\mathrm{d}\vartheta \longrightarrow \int_{-\pi}^{\pi} \log|F(R\mathrm{e}^{\mathrm{i}\vartheta})|\mathrm{d}\vartheta$$

as $R' \to R$. The idea here is the same whether $F(z)$ has *several zeros* on $|z| = R$ or *only one*, and in order to simplify the writing we just treat the latter case. Suppose then that $F(\alpha) = 0$ where $|\alpha| = R$, and there are *no other zeros* in a ring of the form $\{R - \eta \leqslant |z| \leqslant R + \eta\}$, $\eta > 0$. On this ring we then have $|F(z)| \geqslant \text{const.}|z - \alpha|^m$, if m is the multiplicity of the zero at α, so, since $|F(z)|$ is also bounded above there,

$$|\log|F(R'\mathrm{e}^{\mathrm{i}\vartheta})|| \leqslant \text{const.} + m\log^+ \frac{1}{|R'\mathrm{e}^{\mathrm{i}\vartheta} - \alpha|}$$

for $R - \eta \leqslant R' \leqslant R$. (*Here, for* $p > 0$, $\log^+ p$ *denotes* $\log p$ *if* $p \geqslant 1$ *and* 0 *if* $p < 1$.) The expression on the right is, however, $\leqslant \text{const.} + m\log^+(1/|R\mathrm{e}^{\mathrm{i}\vartheta} - \alpha|)$, independently of R', when the latter quantity is close to R:

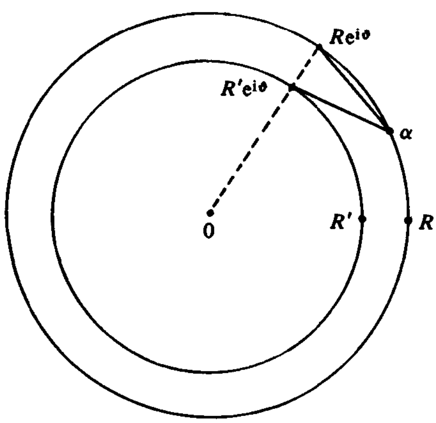

Figure 1

(The constants of course will be *different*; the relation between them need not concern us here.) In other words, for $R' \to R$ the expressions $|\log|F(R'e^{i\vartheta})||$ are bounded above by the *fixed* function const. + $m\log^+(1/|Re^{i\vartheta} - \alpha|)$ of ϑ, *which however, has a finite integral over* $[-\pi, \pi]$, *as we easily check directly.* Since also $\log|F(R'e^{i\vartheta})| \to \log|F(Re^{i\vartheta})|$ pointwise as $R' \to R$, we have

$$\int_{-\pi}^{\pi} \log|F(R'e^{i\vartheta})|\,d\vartheta \longrightarrow \int_{-\pi}^{\pi} \log|F(Re^{i\vartheta})|\,d\vartheta$$

by *Lebesgue's dominated convergence theorem.* This is what we needed to complete our derivation of Jensen's formula. (We see that the *same computation* which shows that

$$\int_{-\pi}^{\pi} \log|F(Re^{i\vartheta})|\,d\vartheta \;>\; -\infty$$

also establishes the *convergence* of $\int_{-\pi}^{\pi}\log|F(R'e^{i\vartheta})|\,d\vartheta$ to that quantity as $R' \to R$!)

Here is a first application of Jensen's formula.

Theorem. *Suppose that $F(z)$ is analytic and $\not\equiv 0$ for $|z| < 1$, and that the integrals*

$$\int_{-\pi}^{\pi} \log^+|F(re^{i\vartheta})|\,d\vartheta$$

are bounded for $0 \leqslant r < 1$. Then for any r_0, $0 < r_0 < 1$, the integrals

$$\int_{-\pi}^{\pi} \log^-|F(re^{i\vartheta})|\,d\vartheta$$

are bounded for $r_0 < r < 1$.

Notation. For $p \geqslant 0$, we write (as remarked above) $\log^+ p = \max(\log p, 0)$. *We also take* $\log^- p = -\min(\log p, 0)$, *so that* $\log^- p \geqslant 0$ *and* $\log p = \log^+ p - \log^- p$. (Everybody means the same thing by $\log^+ p$, but, regarding $\log^- p$, usage is not uniform.)

Proof of theorem. Without loss of generality (*henceforth abbreviated* 'wlog'), let $F(0) \neq 0$. (Otherwise work with $F(z)/z^k$ for a suitable k instead of $F(z)$.) By Jensen's formula,

$$-\infty < \log|F(0)| \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+|F(re^{i\vartheta})|\,d\vartheta$$
$$-\frac{1}{2\pi}\int_{-\pi}^{\pi} \log^-|F(re^{i\vartheta})|\,d\vartheta, \quad 0 < r < 1.$$

By hypothesis, the right-hand side is

$$\leqslant \text{const.} - \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^{-}|F(re^{i\vartheta})|\,d\vartheta.$$

The desired result follows by transposition.

Corollary. *Under the hypothesis of the theorem, suppose that*

$$F(e^{i\vartheta}) = \lim_{r\to 1} F(re^{i\vartheta})$$

exists a.e. Then

$$\int_{-\pi}^{\pi} \log^{-}|F(e^{i\vartheta})|\,d\vartheta \;<\; \infty.$$

Proof. Fatou's lemma.

Remark 1. Actually, the hypothesis of the theorem *forces* a.e. existence of

$$\lim_{r\to 1} F(re^{i\vartheta}).$$

This is a fairly deep result, and depends on Lebesgue's theorem on a.e. existence of derivatives of functions of bounded variation. In the situations we will mostly consider, the existence of this limit can be directly verified ('by inspection'), so the deeper result will not be needed. Therefore we do not prove it now. The interested reader can work up a proof by using the subharmonicity of $\log^{+}|F(z)|$ together with an argument from Chapter III, §F.1, so as to produce a positive measure ν on $[-\pi, \pi]$ for which

$$|F(z)| \;\leqslant\; \left|\exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\vartheta}+z}{e^{i\vartheta}-z}\,d\nu(\vartheta)\right\}\right|, \quad |z|<1.$$

After this, one applies results from §F.2 of Chapter III to the analytic function $\Phi(z)$ within $|\quad|$ on the right, and then to the ratio $F(z)/\Phi(z)$.

Remark 2. The idea of the corollary is that if $|F(z)|$ is *not too big* in $\{|z|<1\}$ (especially if $|F(z)|$ is *bounded* there), then the boundary values $|F(e^{i\vartheta})|$ *cannot be too small* unless $F \equiv 0$.

Problem 1

(a) Let $F(z)$ be entire, $F(0) = 1$, and $|F(z)| \leqslant Ke^{A|z|}$ for all z, where A and K are constants. If $n(R)$ denotes the number of zeros of F having modulus $\leqslant R$, show that, for all R, $n(R) \leqslant eAR + \text{const.}$ (Here, the constant depends on K.)

*(b) Show that in the relation established in (a) the coefficient eA of R cannot in general be diminished. (Hint. Fix $R = m/e$ with m a large integer. Compute the maximum value of $(x/R)^{eR}e^{-x}$ for $x \geqslant 0$. Then look at a function which has m equally spaced zeros on the circle $|z| = R$ and no others.)

II

Szegő's theorem

A. The theorem

Szegő's theorem is a beautiful result in approximation theory, obtained with the help of Jensen's formula. Its proof also uses a limit property of integrals involving the Poisson kernel (for the unit disk) which is now taught in many courses on real variable theory. The reader who does not remember that result will find it in §B, together with its proof.

Theorem (Szegő). *Let $w(\vartheta) \geqslant 0$ belong to $L_1(-\pi,\pi)$. Then the infimum of*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|1-\sum_{n>0} a_n e^{in\vartheta}\right| w(\vartheta)\,d\vartheta,$$

taken with respect to all possible finite sums $\sum_{n>0} a_n e^{in\vartheta}$, is equal to

$$\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\log w(\vartheta)\,d\vartheta\right).$$

Note: $\int_{-\pi}^{\pi}\log^+ w(\vartheta)\,d\vartheta$ *is finite* if $w \in L_1(-\pi,\pi)$. So $\int_{-\pi}^{\pi}\log w(\vartheta)\,d\vartheta$ either *converges,* or else *diverges* to $-\infty$.

Proof of theorem. By the inequality between arithmetic and geometric means,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|1-\sum_{n>0} a_n e^{in\vartheta}\right| w(\vartheta)\,d\vartheta$$
$$\geqslant \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\log\left|1-\sum_{n>0} a_n e^{in\vartheta}\right| + \log w(\vartheta)\right)d\vartheta\right\}.$$

Jensen's formula applied to $F(z) = 1-\sum_{n>0} a_n z^n$ shows that this last expression is always

$$\geqslant \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\log w(\vartheta)\,d\vartheta\right);$$

the desired infimum is thus $\geqslant$ the latter quantity. We must establish the reverse inequality.

Write $w_N(\vartheta) = \max(w(\vartheta), e^{-N})$. By Lebesgue's monotone convergence theorem and the finiteness of $\int_{-\pi}^{\pi} \log^+ w(\vartheta)\,d\vartheta$ we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log w_N(\vartheta)\,d\vartheta \underset{N}{\longrightarrow} \frac{1}{2\pi}\int_{-\pi}^{\pi} \log w(\vartheta)\,d\vartheta.$$

It will therefore be enough to show that for any N and any $\delta > 0$ there exists some finite sum $1 - \sum_{k>0} A_k e^{ik\vartheta}$ such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|1 - \sum_{k>0} A_k e^{ik\vartheta}\right| w(\vartheta)\,d\vartheta \;<\; \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log w_N(\vartheta)\,d\vartheta\right) + \delta.$$

To this end, put first of all

$$(*) \qquad F_N(z) = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\log\left(\frac{1}{w_N(t)}\right)dt\right\}.$$

for $|z| < 1$. We have

$$(\substack{*\\*}) \qquad F_N(0) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left(\frac{1}{w_N(t)}\right)dt\right).$$

Since $w_N(t) \geqslant e^{-N}$, $|F_N(z)| \leqslant e^N$ for $|z| < 1$. Indeed, taking real parts of the logarithms of both sides of $(*)$ gives us

$$\log|F_N(re^{i\vartheta})| = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-r^2-2r\cos(\vartheta - t)}\log\left(\frac{1}{w_N(t)}\right)dt.$$

On the right side we recognize the *Poisson kernel* (that's the *real reason* for using $(e^{it}+z)/(e^{it}-z)$ in $(*)$, aside from the fact that we *want* $F_N(z)$ to be *analytic* in $\{|z|<1\}$). As one knows,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\vartheta - t)}\,dt = 1;$$

the integrand is obviously *positive*. We see that $\log|F_N(re^{i\vartheta})| \leqslant N$ by the previous formula.

Now we use another, much *finer* property of the Poisson kernel, established in §B below. According to the latter,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\vartheta - t)}\log\left(\frac{1}{w_N(t)}\right)dt \longrightarrow \log\left(\frac{1}{w_N(\vartheta)}\right)$$

for almost all ϑ as $r \to 1$. So $|F_N(re^{i\vartheta})| \to 1/w_N(\vartheta)$ a.e. for $r \to 1$. However, $|F_N(z)|$ is bounded above and $w(\vartheta) \in L_1(-\pi, \pi)$. Therefore, by *dominated* convergence,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|F_N(re^{i\vartheta})|w(\vartheta)\,d\vartheta \longrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{w(\vartheta)}{w_N(\vartheta)}\,d\vartheta$$

as $r \to 1$. The right-hand side is *clearly* $\leqslant 1$. Given $\varepsilon > 0$ we can therefore get an $r < 1$ with

$$(\dagger) \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi}|F_N(re^{i\vartheta})|w(\vartheta)\,d\vartheta < 1 + \varepsilon.$$

Fix such an r.

By the very *form* of the right side of $(*)$, $F_N(z)$ is *analytic* in $\{|z| < 1\}$; it therefore has a Taylor expansion there. And, by $(\substack{*\\*})$, $F_N(0) \neq 0$. Letting $S(z)$ be any *partial sum* of the Taylor series for $F_N(z)$, we see that *for our fixed* r,

$$\frac{S(re^{i\vartheta})}{F_N(0)} \quad \text{is of the form } 1 - \sum_{k>0} A_k e^{ik\vartheta},$$

the sum on the right being finite. Since $F_N(z)$ is regular in $\{|z| < 1\}$ and $r < 1$, we see by $(\dagger)$ that we can choose the partial sum $S(z)$ so that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|S(re^{i\vartheta})|w(\vartheta)\,d\vartheta < 1 + 2\varepsilon.$$

Hence

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|1 - \sum_{k>0} A_k e^{ik\vartheta}\right| w(\vartheta)\,d\vartheta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{S(re^{i\vartheta})}{F_N(0)}\right| w(\vartheta)\,d\vartheta \;\leqslant\; (1 + 2\varepsilon)\cdot\frac{1}{F_N(0)},$$

which equals

$$(1 + 2\varepsilon)\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\log w_N(t)\,dt\right) \quad \text{by } (\substack{*\\*}).$$

This is enough, and we are done.

Remark. This most elegant result was extended by Kolmogorov, and then by Krein, who evaluated the infimum of

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|1 - \sum_{n>0} a_n e^{in\vartheta}\right| d\mu(\vartheta)$$

for all finite sums $\sum_{n>0} a_n e^{in\vartheta}$ when μ is *any finite positive measure*. It turns out that the singular part of μ (with respect to Lebesgue measure) *has no influence here*, that the infimum is simply equal to

$$\exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left(\frac{\mathrm{d}\mu(\vartheta)}{\mathrm{d}\vartheta}\right)\mathrm{d}\vartheta\right\}.$$

I do not give the proof of this result. It depends on the construction of Fatou–Riesz functions which, while not very difficult, is not really part of the material being treated here. The interested reader may find a proof in many books; some of the older ones which have it are Hoffman's and Akhiezer's (on approximation theory). The newer books by Garnett (on bounded analytic functions), and by me (on H_p spaces) both contain proofs.

B. The pointwise approximate identity property of the Poisson kernel

Theorem. Let $P(\vartheta)\in L_1(-\pi,\pi)$, *and, for* $r<1$, *write*

$$U(re^{i\vartheta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-r^2}{1+r^2-2r\cos(\vartheta-t)}P(t)\,\mathrm{d}t.$$

For almost every ϑ, $U(z)$ *tends to* $P(\vartheta)$ *uniformly as* z *tends to* $e^{i\vartheta}$ *within any sector of the form*

$$|\arg(1-e^{-i\vartheta}z)| \leqslant \alpha < \frac{\pi}{2}.^*$$

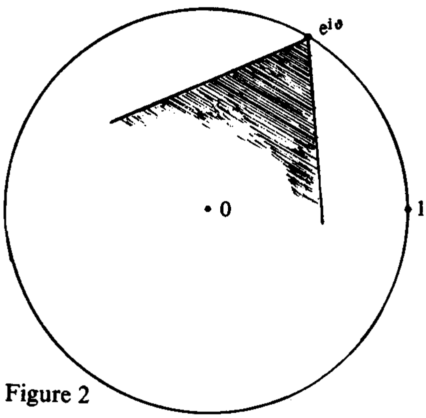

Figure 2

Remark. We write '$U(z)\to P(\vartheta)$ a.e. for $z \xrightarrow{\angle} e^{i\vartheta}$.' Some people say that $U(z)\to P(\vartheta)$ a.e. for z tending *non-tangentially* to $e^{i\vartheta}$, others say that

* It is clear that for z of modulus $>\sin\alpha$ in such a sector we have $|\arg z-\vartheta| \leqslant K(1-|z|)$ with a constant K depending on α.

$U(z) \to P(\vartheta)$ uniformly within any Stoltz domain as $z \to e^{i\vartheta}$ (for almost all ϑ).

Of course, the theorem includes the result that $U(re^{i\vartheta}) \to P(\vartheta)$ a.e. for $r \to 1$, used in proving Szegő's theorem.

Proof of theorem. We will show that $U(re^{i\vartheta_r}) \to P(\vartheta)$ for $r \to 1$ if $|\vartheta_r - \vartheta| \leqslant K(1-r)$, *whenever* $(d/d\vartheta)\int_0^\vartheta P(t)\,dt$ *exists and equals* $P(\vartheta)$, *hence for almost every* ϑ, by Lebesgue's differentiation theorem. The rapidity of the convergence will be seen to depend only on the value of K measuring the opening of the sector with vertex at $e^{i\vartheta}$, and *not* on the particular choice of ϑ_r satisfying the above relation for each value of r.

Without loss of generality, take $\vartheta = 0$, and assume that

$$\int_0^\vartheta P(t)\,dt = \vartheta P(0) + o(|\vartheta|)$$

for $\vartheta \to 0$ (from above *or* below!). Pick any small $\delta > 0$ and write

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\vartheta_r - t)} P(t)\,dt$$

as

$$\left(\frac{1}{2\pi}\int_{\delta \leqslant |t| \leqslant \pi} + \frac{1}{2\pi}\int_{-\delta}^{\delta}\right) \frac{1-r^2}{1+r^2-2r\cos(\vartheta_r - t)} P(t)\,dt.$$

As $r \to 1$, $|\vartheta_r|$ becomes and remains $< \delta/2$, so $(1-r^2)/(1+r^2-2r\cos(\vartheta_r - t)) \to 0$ *uniformly* for $\delta < |t| < \pi$, and the *first integral* tends to *zero.*

The *second* is treated by partial integration. Writing $J(\vartheta) = \int_0^\vartheta P(t)\,dt$, that second integral becomes

$$\frac{1}{2\pi}\left[J(\delta)\frac{1-r^2}{1+r^2-2r\cos(\vartheta_r - \delta)} - J(-\delta)\frac{1-r^2}{1+r^2-2r\cos(\vartheta_r + \delta)}\right]$$

$$-\frac{1}{2\pi}\int_{-\delta}^{\delta} J(t)\frac{\partial}{\partial t}\left(\frac{1-r^2}{1+r^2-2r\cos(\vartheta_r - t)}\right)dt.$$

The two integrated terms in square brackets tend to zero as $r \to 1$. Since $J(t) = P(0)t + o(|t|)$, the *integral* equals

$$-\frac{1}{2\pi}\int_{-\delta}^{\delta} P(0)t\frac{\partial}{\partial t}\left(\frac{1-r^2}{1+r^2-2r\cos(\vartheta_r - t)}\right)dt$$

$$+\frac{1}{2\pi}\cdot\int_{-\delta}^{\delta} o(|t|)\frac{\partial}{\partial t}\left(\frac{1-r^2}{1+r^2-2r\cos(\vartheta_r - t)}\right)dt.$$

Here, the *first* term is readily seen (by *reverse* integration by parts!) to equal

$$o(1) + \frac{P(0)}{2\pi}\int_{-\delta}^{\delta} \frac{1-r^2}{1+r^2-2r\cos(\vartheta_r - t)}\,dt = o(1) + P(0)(1 - o(1)),$$

which tends to $P(0)$ as $r \to 1$.

To estimate the *second* term, we have to use the fact that

$$\frac{1-r^2}{1+r^2-2r\cos(\vartheta_r-t)}$$

is a monotone function of t on each of the intervals $-\delta \leqslant t \leqslant \vartheta_r$ and $\vartheta_r \leqslant t \leqslant \delta$. (We are supposing that r is so close to 1 that $-\delta < \vartheta_r < \delta$.) Given any $\varepsilon > 0$, we can choose $\delta > 0$ so small to begin with that the *second term* is in absolute value

$$\leqslant \frac{\varepsilon}{2\pi}\int_{-\delta}^{\delta}|t|\left|\frac{\partial}{\partial t}\left(\frac{1-r^2}{1+r^2-2r\cos(\vartheta_r-t)}\right)\right|dt.$$

Writing $\tau = \vartheta_r - t$, this becomes

$$\frac{\varepsilon}{2\pi}\int_{\vartheta_r-\delta}^{\vartheta_r+\delta}|\tau-\vartheta_r|\left|\frac{\partial}{\partial \tau}\left(\frac{1-r^2}{1+r^2-2r\cos\tau}\right)\right|d\tau.$$

We break this up as

$$(*)\qquad \left(\frac{\varepsilon}{2\pi}\int_{\vartheta_r-\delta}^{0}+\frac{\varepsilon}{2\pi}\int_{0}^{\vartheta_r+\delta}\right)|\tau-\vartheta_r|\left|\frac{\partial}{\partial \tau}\left(\frac{1-r^2}{1+r^2-2r\cos\tau}\right)\right|d\tau;$$

in the *second integral,*

$$\frac{\partial}{\partial \tau}\left(\frac{1-r^2}{1+r^2-2r\cos\tau}\right) < 0,$$

so that second integral is $\leqslant$

$$-\frac{\varepsilon}{2\pi}\int_{0}^{\vartheta_r+\delta}\tau\frac{\partial}{\partial \tau}\left(\frac{1-r^2}{1+r^2-2r\cos\tau}\right)d\tau$$

$$-\frac{\varepsilon|\vartheta_r|}{2\pi}\int_{0}^{\vartheta_r+\delta}\frac{\partial}{\partial \tau}\left(\frac{1-r^2}{1+r^2-2r\cos\tau}\right)d\tau.$$

Here, the first term is $\varepsilon(\frac{1}{2}+o(1))$ (see above treatment of expression involving $P(0)$!), and the second is

$$\leqslant \frac{\varepsilon|\vartheta_r|}{2\pi}\cdot\frac{1+r}{1-r}.$$

This last, however, is $\leqslant (K/\pi)\varepsilon$ *in view of our condition on* ϑ_r. We see that the second integral in $(*)$ is $\leqslant (K/\pi + \frac{1}{2} + o(1))\varepsilon$ for r close enough to 1.

The first integral in $(*)$ is similarly treated, and seen to also be $\leqq (K/\pi + \frac{1}{2} + o(1))\varepsilon$ for r close to 1. In this way, we have found that the expression

$$\frac{1}{2\pi}\int_{-\delta}^{\delta}o(|t|)\frac{\partial}{\partial t}\left(\frac{1-r^2}{1+r^2-2r\cos(\vartheta_r-t)}\right)dt$$

is in absolute value $\leqslant (1 + 2K/\pi + o(1))\varepsilon$ if $\delta > 0$ is small enough to begin with, and r close enough to 1. However, according to the calculation at the beginning of this proof, the sum of the last expression and $P(0)$ differs by $o(1)$ from $U(re^{i\vartheta_r})$ when $r \to 1$. So, since $\varepsilon > 0$ is arbitrary, we have established the desired result.

Remark. Suppose that

$$U(re^{i\vartheta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\vartheta - t)} \, d\mu(t)$$

with a finite (complex valued) *measure* μ. Form the *primitive*

$$\mu(\vartheta) = \int_0^{\vartheta} d\mu(t).$$

Then it is *still true* that, wherever the *derivative* $\mu'(\vartheta)$ *exists and is finite*, we have $U(z) \to \mu'(\vartheta)$ for $z \xrightarrow{\angle} e^{i\vartheta}$. (Hence $\lim_{r\to 1} U(re^{i\vartheta})$ *exists and is finite* a.e. by Lebesgue's differentiation theorem.) The proof of this slightly more general result is *exactly the same* as that of the above one.

Problem 2

The purpose of this problem is to derive, from Szegő's theorem, the following result. *Let* $w(x) \geqslant 0$ *be in* $L_1(-\infty, \infty)$ *and let* $a > 0$. *There are finite sums* $S(x)$ *of the form* $S(x) = \sum_{\lambda \geqslant a} A_\lambda e^{i\lambda x}$ *with* $\int_{-\infty}^{\infty} |1 - S(x)| w(x) \, dx$ *arbitrarily small iff* $\int_{-\infty}^{\infty} (\log w(x)/(1+x^2)) \, dx = -\infty$. *In case* $\int_{-\infty}^{\infty} (\log w(x)/(1+x^2)) dx = -\infty$, *we can, given any bounded continuous function* $\varphi(x)$, *find finite sums* $S(x)$ *of the above mentioned form with* $\int_{-\infty}^{\infty} |\varphi(x) - S(x)| w(x) \, dx$ *arbitrarily small.* Establishment of this result is in a series of steps.

(a) Let $\alpha > 0$ and let p be a positive integer. There are numbers A_n with

$$\left(\frac{x}{1 - i\alpha x}\right)^p = \sum_0^{\infty} A_n \left(\frac{i - x}{i + x}\right)^n,$$

the series on the right being *uniformly convergent* for $-\infty < x < \infty$. (Hint: Put $w = (i - z)/(i + z)$ and look at where $f(w) = z/(1 - i\alpha z)$ is regular in the w-plane. *Little or no computation is used in doing* (a).)

(b) Let $\lambda > 0$. There are *finite sums* $S_k(x)$, each of the form $\sum_{n \geqslant 0} C_n((i - x)/(i + x))^n$, such that $|S_k(x)| \leqslant 2$ on $\mathbb{R}$ and $S_k(x) \xrightarrow[k]{} e^{i\lambda x}$ u.c.c.* on $\mathbb{R}$. (Hint: $e^{i\lambda x} = \lim_{\alpha \to 0+} e^{i\lambda x/(1 - i\alpha x)}$. For each $\alpha > 0$ the series for $\exp(i\lambda x/(1 - i\alpha x))$ is *uniformly convergent* for $-\infty < x < \infty$. *Little or no computation here.*)

(c) Given any integer $n > 0$ there are finite sums $T_k(x)$ of the form $\sum_{\lambda > 0} A_\lambda e^{i\lambda x}$ with $|T_k(x)| \leqslant C$ independent of k on $\mathbb{R}$ and $T_k(x) \to 1/(i + x)^n$ u.c.c.

* u.c.c means *uniform convergence on compacta.*

on $\mathbb{R}$. (Hint: Start from the integral formula

$$\frac{\mathrm{i}}{\mathrm{i}+x} = \int_0^\infty \mathrm{e}^{-\lambda}\mathrm{e}^{\mathrm{i}\lambda x}\,\mathrm{d}\lambda,$$

$$\frac{\mathrm{i}}{(\mathrm{i}+x)^2} = -\int_0^\infty \mathrm{i}\lambda \mathrm{e}^{-\lambda}\mathrm{e}^{\mathrm{i}\lambda x}\,\mathrm{d}\lambda,$$

&c.)

(d) Given $w \geqslant 0$ in $L_1(\mathbb{R})$, denote by $\mathscr{A}$ the class of bounded continuous φ defined on $\mathbb{R}$ such that $\int_{-\infty}^{\infty} |\varphi(x) - S(x)|\,w(x)\,\mathrm{d}x$ can be made arbitrarily small with suitable finite sums $S(x)$ of the form $\sum_{n\geqslant 0} A_n((\mathrm{i}-x)/(\mathrm{i}+x))^n$. Call $\mathscr{E}$ the set of bounded continuous φ for which finite sums $T(x) = \sum_{\lambda\geqslant 0} A_\lambda \mathrm{e}^{\mathrm{i}\lambda x}$ exist making $\int_{-\infty}^{\infty} |\varphi(x) - T(x)|\,w(x)\,\mathrm{d}x$ arbitrarily small. *Prove that* $\mathscr{A} = \mathscr{E}$.

(e) Let $a > 0$, and denote by $\mathscr{F}$ the set of bounded continuous φ such that there are finite sums $\sigma(x) = \sum_{\lambda\geqslant a} A_\lambda \mathrm{e}^{\mathrm{i}\lambda x}$ with $\int_{-\infty}^{\infty} |\varphi(x) - \sigma(x)|\,w(x)\,\mathrm{d}x$ arbitrarily small. Prove that if $1 \in \mathscr{F}$ then $\mathscr{F}$ contains *all* bounded continuous φ, and this happens iff $\int_{-\infty}^{\infty} (\log w(x)/(1+x^2))\,\mathrm{d}x = -\infty$. (Hint: If $1 \in \mathscr{F}$, then $\mathscr{E}$ (of part (d)) includes all $\mathrm{e}^{\mathrm{i}\lambda x}$ with $\lambda \geqslant -a$, hence, by iteration, all $\mathrm{e}^{\mathrm{i}\lambda x}$ with $\lambda \geqslant -2a$, with $\lambda \geqslant -3a$, &c. So $\mathscr{E}$ includes *all integral* powers $((\mathrm{i}-x)/(\mathrm{i}+x))^n$ with *positive and negative n*. These are enough to approximate $\mathrm{e}^{-\mathrm{i}ax}\varphi(x)$ for any bounded continuous φ.)

III

Entire functions of exponential type

An entire function $f(z)$ is said to be *of exponential type* if there is a constant A such that

$$|f(z)| \leqslant \text{const.}e^{A|z|}$$

everywhere. The *infimum* of the set of A for which such an inequality holds (with the constant in front on the right depending on A) is called the *type* of $f(z)$.

Entire functions of exponential type come up in various branches of analysis, partly on account of the evident fact that integrals of the form

$$\int_K e^{i\lambda z}\, d\mu(\lambda)$$

are equal to such functions whenever K is a compact subset of $\mathbb{C}$. In this chapter we establish some of the most important results concerning them, which find application throughout the rest of the book. We are not of course attempting to give a complete treatment of the subject. Fuller accounts are contained in the books by Boas and by Levin.

A. Hadamard factorization

As in Chapter I, we denote the *number* of *zeros* of $f(z)$ having modulus $\leqslant r$ by $n(r)$ (each zero being counted according to its multiplicity). We sometimes write $n_f(r)$ instead of $n(r)$ when several functions are being dealt with.

Theorem. *If $f(z)$ is entire and of exponential type, $n(r) \leqslant Cr + O(1)$.*

Proof. See Problem 1(a), Chapter I. If $|f(z)| \leqslant \text{const.}e^{A|z|}$ we can take $C = eA$.

Theorem (Hadamard factorization). *Let $f(z)$ be entire, of exponential type, and denote by $\{z_n\}$ the sequence of its zeros $\neq 0$ (multiplicities counted by repetition), so arranged that*

$$0 < |z_1| \leqslant |z_2| \leqslant |z_3| \leqslant \cdots.$$

Then

$$\boxed{f(z) = Cz^k e^{bz} \prod_n \left(1 - \frac{z}{z_n}\right) e^{z/z_n},}$$

the product being uniformly convergent on compact subsets of $\mathbb{C}$.

Terminology. Henceforth we abbreviate the last phrase as 'u.c.c. convergent on $\mathbb{C}$'.

Proof of theorem. By working with $f(z)/z^k$ instead of $f(z)$ (if necessary), we first reduce the situation to one where $f(0) \neq 0$. Then $n(r) \leqslant Kr$ for some K.

If, with a zero z_n of f, we have $|z_n| > 2R$, then, for $|z| \leqslant R$,

$$\begin{aligned}\log\left\{\left(1 - \frac{z}{z_n}\right)e^{z/z_n}\right\} &= -\frac{z}{z_n} - \frac{1}{2}\left(\frac{z}{z_n}\right)^2 - \cdots + \frac{z}{z_n} \\ &= -\frac{1}{2}\left(\frac{z}{z_n}\right)^2 - \frac{1}{3}\left(\frac{z}{z_n}\right)^3 - \cdots \\ &= -\frac{1}{2}\left(\frac{z}{z_n}\right)^2 (1 + O(1))\end{aligned}$$

(We are using the branch of the logarithm which is *zero* at 1). Therefore

$$\left|\log\left\{\left(1 - \frac{z}{z_n}\right)e^{z/z_n}\right\}\right| \leqslant \frac{1}{2}\left|\frac{z}{z_n}\right|^2 (1 + O(1)),$$

whence (assuming always that $|z| \leqslant R$),

$$\begin{aligned}\sum_{|z_n| \geqslant 2R} \left|\log\left\{\left(1 - \frac{z}{z_n}\right)e^{z/z_n}\right\}\right| &\leqslant \frac{1 + O(1)}{2} \sum_{|z_n| \geqslant 2R} \left|\frac{R}{z_n}\right|^2 \\ &= \frac{1 + O(1)}{2} R^2 \int_{2R}^{\infty} \frac{dn(t)}{t^2} \\ &= \frac{1 + O(1)}{2} R^2 \left\{\frac{-n(2R)}{4R^2} + 2\int_{2R}^{\infty} \frac{n(t)}{t^3} dt\right\}\end{aligned}$$

$$\leqslant \frac{1+O(1)}{2}R^2 \cdot \int_{2R}^{\infty} \frac{2K}{t^2}\,dt = \frac{1+O(1)}{2}KR.$$

This inequality establishes absolute and uniform convergence of

$$\sum_{|z_n|\geqslant 2R} \log\left\{\left(1-\frac{z}{z_n}\right)e^{z/z_n}\right\}$$

for $|z|\leqslant R$, and hence the uniform convergence of

$$\prod_{|z_n|\geqslant 2R}\left(1-\frac{z}{z_n}\right)e^{z/z_n}$$

for such values of z.

Write $P(z) = \prod_n (1 - z/z_n)e^{z/z_n}$; according to what has just been shown, $P(z)$ is an entire function of z. Since $f(0) \neq 0$, *$f(z)/P(z)$ is entire and has no zeros in* $\mathbb{C}$. There is thus an entire function $\varphi(z)$ with

$$\frac{f(z)}{P(z)} = e^{\varphi(z)},$$

and *it is claimed that* $\varphi(z) = a + bz$ with constants a and b.

To show that $\varphi(z)$ has the asserted form, we use the fact that $f(z)$ is of exponential type in conjunction with the inequality $n(r) \leqslant Kr$ in order to get some control on $|\Re\varphi(z)|$ for large $|z|$. For $|z|\leqslant R$,

$$e^{\Re\varphi(z)} = \left|\frac{f(z)}{\prod_{|z_n|<2R}\left(1-\frac{z}{z_n}\right)e^{z/z_n}}\right| \cdot \left|\frac{1}{\prod_{|z_n|\geqslant 2R}\left(1-\frac{z}{z_n}\right)e^{z/z_n}}\right| = \mathrm{I}\cdot\mathrm{II}, \text{ say.}$$

The computation made above shows that $|\log \mathrm{II}| \leqslant CR$ with some constant C (we estimated $|\log\{1 - z/z_n\}e^{z/z_n}|$!), and it suffices to estimate I.

For I we use a *trick*. The ratio $\psi(z) = f(z)/\prod_{|z_n|<2R}(1 - z/z_n)e^{z/z_n}$ is *entire*, so, by the *principle of maximum*, $\sup_{|z|\leqslant R}|\psi(z)| \leqslant \sup_{|z|=4R}|\psi(z)|$. Here, estimation of the quantity on the *right* will furnish an upper bound for I, which is at most equal to the left-hand side. We have, for $|z| = 4R$, $|z/z_n| = 4R/|z_n|$, whence

$$\prod_{|z_n|<2R} |e^{z/z_n}| \geqslant \exp\left\{-4R\sum_{|z_n|\leqslant 2R} 1/|z_n|\right\}$$

$$= \exp\left\{-4R\int_0^{2R}\frac{dn(t)}{t}\right\} = \exp\left\{-2n(2R) - 4R\int_0^{2R}\frac{n(t)}{t^2}\,dt\right\}.$$

Since $n(t) = 0$ for $0 \leqslant t < |z_1|$, a quantity > 0, and $n(t)/t \leqslant K$, the last expression is

$$\geqslant e^{-4R(K\log R + O(1))}.$$

At the same time, for $|z| = 4R$ and $|z_n| \leqslant 2R$,

$$\left|1 - \frac{z}{z_n}\right| \;\geqslant\; 1,$$

so

$$\left|\prod_{|z_n|<2R}\left(1 - \frac{z}{z_n}\right)e^{z/z_n}\right| \;\geqslant\; e^{-4R(K\log R + O(1))}$$

when $|z| = 4R$. Because $f(z)$ is of exponential type we therefore have

$$|\psi(z)| \;\leqslant\; e^{4KR(\log R + O(1))}, \quad |z| = 4R,$$

whence

$$\mathrm{I} \;\leqslant\; e^{4KR(\log R + O(1))},$$

and finally, since $e^{\Re\varphi(z)} = \mathrm{I}\cdot\mathrm{II}$,

$$\Re\varphi(z) \;\leqslant\; 4KR\log R + O(R) \quad \text{for} \quad |z| \leqslant R$$

in view of the fact that $\log \mathrm{II} \leqslant CR$.

At this point, we use a device already applied to the study of $\log|F(re^{i\vartheta})|$ near the end of Chapter I. By analyticity of $\varphi(z)$,

$$\begin{aligned}\Re\varphi(0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\Re\varphi(Re^{i\vartheta})\,d\vartheta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}[\Re\varphi(Re^{i\vartheta})]_+\,d\vartheta - \frac{1}{2\pi}\int_{-\pi}^{\pi}[\Re\varphi(Re^{i\vartheta})]_-\,d\vartheta\end{aligned}$$

(with self-evident notation). Therefore,

$$\begin{aligned}\int_{-\pi}^{\pi}|\Re\varphi(Re^{i\vartheta})|\,d\vartheta &= \int_{-\pi}^{\pi}\{[\Re\varphi(Re^{i\vartheta})]_+ + [\Re\varphi(Re^{i\vartheta})]_-\}\,d\vartheta \\ &= 2\int_{-\pi}^{\pi}[\Re\varphi(Re^{i\vartheta})]_+\,d\vartheta - 2\pi\Re\varphi(0).\end{aligned}$$

By the one-sided inequality just found for $\Re\varphi(z)$, $|z| \leqslant R$, this last expression is $\leqslant 16\pi KR\log R + O(R) + O(1)$.

Now we can conclude the proof. Since $\varphi(z)$ is entire,

$$\varphi(z) = \sum_0^{\infty}\gamma_n z^n,$$

so

$$2\Re\varphi(Re^{i\vartheta}) = \sum_1^{\infty}\bar{\gamma}_n R^n e^{-in\vartheta} + 2\Re\gamma_0 + \sum_1^{\infty}\gamma_n R^n e^{in\vartheta}$$

Using the relations $\int_{-\pi}^{\pi} e^{ik\vartheta}e^{-il\vartheta}d\vartheta = \begin{cases} 0, k \neq l \\ 2\pi, k = l \end{cases}$, we get, for $n \geqslant 2$,

$$\gamma_n = \frac{1}{\pi R^n}\int_{-\pi}^{\pi} \Re\varphi(Re^{i\vartheta})e^{-in\vartheta}d\vartheta.$$

Therefore

$$|\gamma_n| \leqslant \frac{1}{\pi R^n}\int_{-\pi}^{\pi} |\Re\varphi(Re^{i\vartheta})|d\vartheta$$

which, by the above work, is $\leqslant R^{-n}(16KR\log R + O(R) + O(1))$. Making $R \to \infty$, we see that $\gamma_n = 0$ for $n \geqslant 2$. Our power series for $\varphi(z)$ thus reduces to the *linear* expression $\gamma_0 + \gamma_1 z$, and finally

$$f(z) = e^{\varphi(z)}P(z) = e^{\gamma_0}e^{\gamma_1 z}\prod_n\left(1 - \frac{z}{z_n}\right)e^{z/z_n},$$

the required representation. We are done.

B. Characterization of the set of zeros of an entire function of exponential type. Lindelöf's theorems

While establishing the Hadamard factorization in the preceding § we found that

$$\mathrm{II} \leqslant e^{CR}$$

which was to be expected (having *started* with a function of exponential growth), but we could only show that

$$\mathrm{I} \leqslant e^{O(R\log R)}.$$

This, however, forced $\varphi(z)$ to be a first degree polynomial, whence, *in fact*,

$$\mathrm{I} \leqslant e^{O(R)},$$

because the method used to estimate II showed at the same time that

$$\mathrm{II} \geqslant e^{-O(R)}.$$

The refinement on our estimate of I from $e^{O(R\log R)}$ to $e^{O(R)}$ is due to the fact that $|f(z)| \leqslant e^{O(|z|)}$ for large $|z|$. Otherwise, the $R\log R$ growth is *best possible*, and if we *only know* that $n(r) \leqslant Kr$, *we can only conclude that*

$$\left|\prod_n\left(1 - \frac{z}{z_n}\right)e^{z/z_n}\right| \leqslant e^{O(|z|\log|z|)}$$

for $|z|$ large, most of the contribution coming from the factors with $|z_n| < 2|z|$.

The fact that $f(z)$ is of *exponential* type imposes *not only* the growth condition $n(r) \leqslant O(r)$ (for large r), but also *a certain symmetry* in the *distribution of the zeros* z_n. This symmetry is a deeper property of that set than the growth condition.

Theorem (Lindelöf). *Let $f(z)$ be entire, of exponential type, with $f(0) \neq 0$, and denote by $\{z_n\}$ the sequence of zeros of $f(z)$, with, as usual, each zero repeated therein according to its multiplicity. Put*

$$S(r) = \sum_{|z_n| \leqslant r} \frac{1}{z_n}.$$

Then $|S(r)|$ is bounded as $r \to \infty$.

Proof. By double integration. Since $n(r) \leqslant Kr$, f being of exponential type, we clearly have

$$|S(r) - S(R)| \leqslant 2K \quad \text{for} \quad R \leqslant r \leqslant 2R,$$

whence

$$\int_R^{2R} S(r) r \, \mathrm{d}r = \tfrac{3}{2} R^2 S(R) + O(R^2).$$

We proceed to calculate the integral on the left.

Provided that $f(z)$ has no zeros for $|z| = r$, we have, by the calculus of residues,

$$S(r) = \sum_{|z_n| < r} \frac{1}{z_n} = \frac{1}{2\pi \mathrm{i}} \int_0^{2\pi} \frac{f'(re^{\mathrm{i}\vartheta})}{f(re^{\mathrm{i}\vartheta})} \cdot \frac{\mathrm{i}re^{\mathrm{i}\vartheta} \mathrm{d}\vartheta}{re^{\mathrm{i}\vartheta}} - \frac{f'(0)}{f(0)}.$$

By the Cauchy–Riemann equations,

$$\frac{f'(z)}{f(z)} = \left(\frac{\partial}{\partial x} - \mathrm{i} \frac{\partial}{\partial y} \right) \log |f(z)|,$$

whence, putting $z = re^{\mathrm{i}\vartheta}$,

$$S(r) = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{\partial}{\partial x} - \mathrm{i} \frac{\partial}{\partial y} \right) \log |f(re^{\mathrm{i}\vartheta})| \mathrm{d}\vartheta - \frac{f'(0)}{f(0)}.$$

This holds for all save a finite number of values of r on the interval $[R, 2R]$. Multiply by $r\mathrm{d}r$ and integrate from R to $2R$. We find

$$\tfrac{3}{2} R^2 S(R) = \int_R^{2R} S(r) r \mathrm{d}r + O(R^2)$$

$$= \frac{1}{2\pi} \iint_{R \leqslant |z| \leqslant 2R} \left(\frac{\partial}{\partial x} - \mathrm{i} \frac{\partial}{\partial y} \right) \log |f(z)| \mathrm{d}x \, \mathrm{d}y + O(R^2).$$

Since $S(r) = S(r+)$, there is no loss of generality in assuming that $f(z)$ has no zeros on $|z| = R$ or on $|z| = 2R$. We may then apply Green's theorem to the double integral on the right (this is justified by first excising a small disk of radius ρ, say, about each of the z_n in the annulus $R < |z| < 2R$, and then making $\rho \to 0$), obtaining for it the value

$$\frac{1}{2\pi}\int_0^{2\pi}(2R\log|f(2Re^{i\vartheta})| - R\log|f(Re^{i\vartheta})|)e^{-i\vartheta}\,d\vartheta,$$

whence, by the previous relation,

$$\tfrac{3}{2}R^2|S(R)| \leqslant \frac{R}{2\pi}\int_0^{2\pi}(2|\log|f(2Re^{i\vartheta})|| + |\log|f(Re^{i\vartheta})||)\,d\vartheta$$
$$+\,O(R^2).$$

Here, by Jensen's inequality (see Chapter I),

$$\int_0^{2\pi}|\log|f(re^{i\vartheta})||\,d\vartheta \leqslant 2\int_0^{2\pi}\log^+|f(re^{i\vartheta})|\,d\vartheta - 2\pi\log|f(0)|,$$

which is $\leqslant 4\pi Ar + O(1)$ if $|f(z)| \leqslant \text{const.}e^{A|z|}$. Combined with the preceding, this yields

$$\tfrac{3}{2}R^2|S(R)| \leqslant O(R^2) + 8AR^2 + 2AR^2 + O(R),$$

and

$$|S(R)| \leqslant O(1) \quad \text{for} \quad R \to \infty. \qquad \text{Q.E.D.}$$

The result just proven has an easy converse.

Theorem (also due to Lindelöf!). *Let*

$$0 < |z_1| \leqslant |z_2| \leqslant |z_3| \leqslant \cdots,$$

denote by $n(r)$ the number of z_k having modulus $\leqslant r$ (taking account of multiplicities, as usual), and suppose that $n(r) \leqslant Kr$. Suppose also that the sums

$$\sum_{|z_n| \leqslant r}\frac{1}{z_n}$$

remain bounded in absolute value as $r \to \infty$. Then the product

$$C(z) = \prod_n\left(1 - \frac{z}{z_n}\right)e^{z/z_n}$$

is equal to an entire function of exponential type.

Terminology. $C(z)$ is frequently called a *canonical* product.

Proof of theorem. Uniform convergence of the product on compact subsets of $\mathbb{C}$ has already been shown during the establishment of the Hadamard factorization (§A).

Let R be given, and, for $|z| = R$, write

$$|C(z)| = \left|\prod_{|z_n|<2R}\left(1 - \frac{z}{z_n}\right)e^{z/z_n}\right| \cdot \left|\prod_{|z_n|\geqslant 2R}\left(1 - \frac{z}{z_n}\right)e^{z/z_n}\right| = \mathrm{I}\cdot\mathrm{II},$$

say. It has already been shown that $\mathrm{II} \leqslant e^{O(R)}$ while we were deriving the Hadamard factorization, so we need only consider I. Clearly,

$$\mathrm{I} \leqslant \prod_{|z_n|\leqslant 2R}\left(1 + \frac{R}{|z_n|}\right)\cdot\exp\left\{R\left|\sum_{|z_n|\leqslant 2R}\frac{1}{z_n}\right|\right\}.$$

By hypothesis, the exponential factor on the right is $\leqslant e^{O(R)}$, and we need only estimate the *product*.

The logarithm of that product is

$$\sum_{|z_n|\leqslant 2R}\log\left(1 + \frac{R}{|z_n|}\right) = \int_0^{2R}\log\left(1 + \frac{R}{t}\right)dn(t)$$

$$= n(R)\log\tfrac{3}{2} + \int_0^{2R}\frac{R}{R+t}\frac{n(t)}{t}dt,$$

since $n(t)$ is zero for t near 0. Plugging in $n(t) \leqslant Kt$, we see that the last expression is $\leqslant KR\log\frac{3}{2} + 2KR$ so that, finally,

$$\log\mathrm{I} \leqslant KR\log\tfrac{3}{2} + 2KR + O(R) = O(R).$$

Since II has a similar estimate, we see that $|C(z)| \leqslant e^{O(R)}$ for $|z| = R$.

We're done.

Here is an important consequence of the above results.

Theorem. *Let $f(z)$ and $g(z)$ be entire and of exponential type. If the ratio $f(z)/g(z)$ is also entire, it is of exponential type.*

Proof. Combine the Hadamard factorization theorem with the two Lindelöf theorems.

Problem 3

Let p be an integer > 1; suppose that $f(z)$ is entire with $f(0) \neq 0$ and that $|f(z)| \leqslant Ce^{A|z|^p}$.

Prove that $n_f(r) \leqslant Kr^p$ and that the sums

$$T(r) = \sum_{|z_n|\leqslant r}\frac{1}{z_n^p}$$

are bounded. (Hint: In studying $T(r)$, express $f'(re^{i\vartheta})/f(re^{i\vartheta})$ in terms of $(\partial/\partial r)\log|f(re^{i\vartheta})|$ and $(\partial/\partial\vartheta)\log|f(re^{i\vartheta})|$, assuming, of course, that f has no zeros on $|z| = r$.)

C. Phragmén–Lindelöf theorems

The entire functions of exponential type one meets with in the following chapters (and, for that matter, in many parts of analysis where they find application) have their *size on the real axis* subject to some *restriction.* During the remainder of this chapter we will be concerned with such functions, and we start here by seeing what it means to impose *boundedness* on $\mathbb{R}$. Some of the following material is contained in textbooks on elementary complex variable theory; we include it for completeness.

Theorem (extended maximum principle). *Let $\mathscr{D}$ be a domain in $\mathbb{C}$ not equal to all of $\mathbb{C}$, and suppose that $f(z)$ is analytic and bounded in $\mathscr{D}$. Assume that, for each $\zeta \in \partial\mathscr{D}$, $\limsup_{\substack{z\to\zeta \\ z\in\mathscr{D}}} |f(z)| \leqslant m$. Then $|f(z)| \leqslant m$ in $\mathscr{D}$.*

Remark. If $\mathscr{D}$ is a *bounded* domain, this is the ordinary maximum principle, and then the assumption that $f(z)$ is bounded in $\mathscr{D}$ is superfluous. When $\mathscr{D}$ is *unbounded*, however, this assumption is *really necessary*, as the simplest examples show.

Proof of theorem. Wlog, say that $0 \in \partial\mathscr{D}$. Pick any $\eta > 0$ and fix it. According to the hypothesis, we can find a $\rho > 0$ such that $|f(z)| \leqslant m + \eta$ for $z \in \mathscr{D}$ and $|z| \leqslant \rho$; we fix such a ρ and write

$$\mathscr{D}_\rho = \mathscr{D} \cap \{|z| > \rho\}.$$

The open set $\mathscr{D}_\rho$ may not be connected, but that doesn't matter; its boundary consists of part of $\partial\mathscr{D}$ and the arcs of $\{|z| = \rho\}$ lying in $\mathscr{D}$.

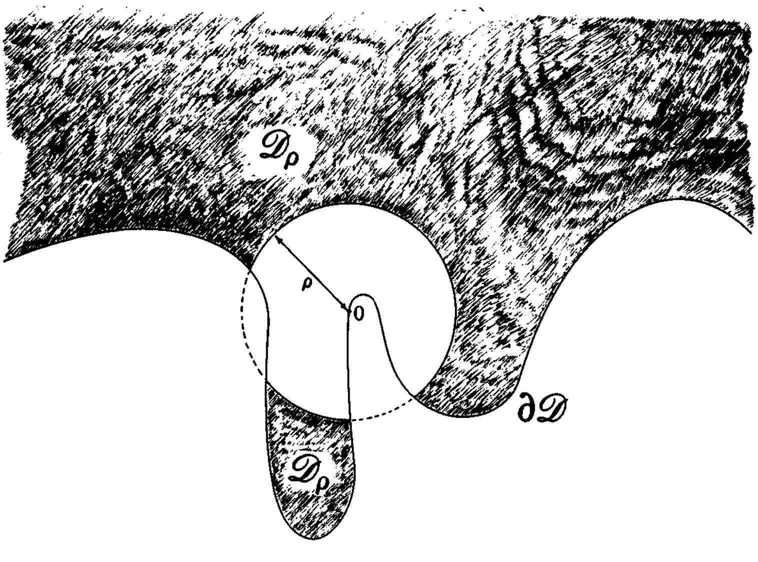

Figure 3

By choice of ρ, $\limsup_{\substack{z\to\zeta\\ z\in\mathscr{D}_\rho}} |f(z)| \leqslant m+\eta$ for $\zeta\in\partial\mathscr{D}_\rho$.

Take now a small $\varepsilon>0$ and consider, in $\mathscr{D}_\rho$, the subharmonic function

$$v_\varepsilon(z) = \log|f(z)| - \varepsilon\log|z|.$$

The right side is $\leqslant \log|f(z)| + \varepsilon\log(1/\rho)$ for $z\in\mathscr{D}_\rho$, and this is in turn $\leqslant \log|f(z)|+\eta$ if ε is chosen sufficiently small, which we assume henceforth. Referring to the previous relation, we see that

$$(*)\qquad \limsup_{\substack{z\to\zeta\\ z\in\mathscr{D}_\rho}} v_\varepsilon(z) \leqslant \log(m+\eta)+\eta$$

for each $\zeta\in\partial\mathscr{D}_\rho$.

Let $z_0\in\mathscr{D}_\rho$. *Since $f(z)$ is bounded in $\mathscr{D}$, say $|f(z)|\leqslant M$ there, we can find an* $R>|z_0|$ (depending of course on ε) *so large that* $v_\varepsilon(z)\leqslant \log M - \varepsilon\log|z|$ is $\leqslant \log m$ *for* $z\in\mathscr{D}_\rho$ and $|z|=R$. (This is the crucial step in the proof.) Denoting by $\mathscr{D}_{\rho,R}$ the *bounded* open set $\mathscr{D}_\rho\cap\{|z|<R\}$, we see that ( ) holds for *every* $\zeta\in\partial\mathscr{D}_{\rho,R}$, because any such ζ which is not on $\partial\mathscr{D}_\rho$ lies in the intersection of $\mathscr{D}_\rho$ with the circle $|\zeta|=R$.

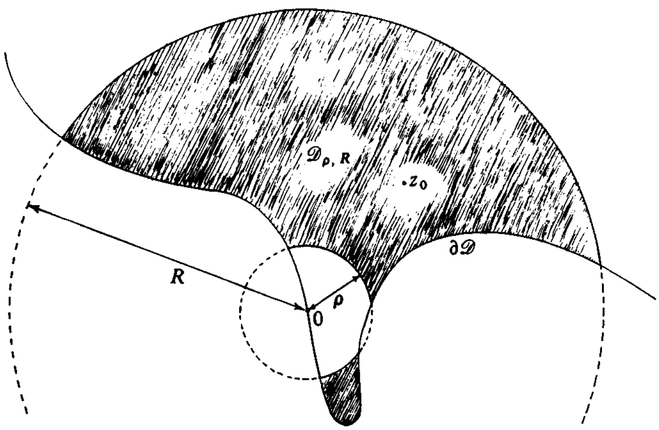

Figure 4

Since $\mathscr{D}_{\rho,R}$ is a bounded open set, we therefore have, by the (ordinary) maximum principle, $v_\varepsilon(z)\leqslant\log(m+\eta)+\eta$, $z\in\mathscr{D}_{\rho,R}$. This holds in particular for $z=z_0$, so

$$\log|f(z_0)| \leqslant \log(m+\eta)+\eta+\varepsilon\log|z_0|.$$

However, $\varepsilon>0$ could be chosen *as small as we pleased.* Therefore,

$$\log|f(z_0)| \leqslant \log(m+\eta)+\eta$$

and, since $\eta>0$ was arbitrary,

$$|f(z_0)|\leqslant m.$$

Q.E.D.

Remark. The peculiar reasoning followed in the above proof is called a *Phragmén–Lindelöf argument.* Most Phragmén–Lindelöf theorems are proved in the same way. Note the special rôle played by the harmonic function $\varepsilon \log|z|$; a function used in this way is called a *Phragmén–Lindelöf function.*

Theorem (Phragmén–Lindelöf). *Let $f(z)$ be analytic in a sector S of opening 2γ, and suppose that*

$$|f(z)| \leqslant Ce^{A|z|^\alpha}$$

in S, where $\alpha < \pi/2\gamma$. If, for every $\zeta \in \partial S$, $\limsup\limits_{\substack{z \to \zeta \\ z \in S}} |f(z)| \leqslant m$, *then* $|f(z)| \leqslant m$ *in S.*

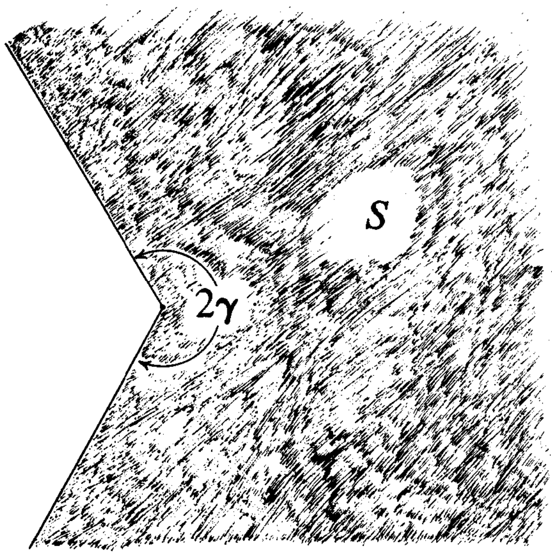

Figure 5

Proof. By making a change of variable, we may reduce our situation to the case where S is the sector

$$\{z: -\gamma < \arg z < \gamma\}$$

with vertex at the origin. Pick any number β, $\alpha < \beta < \pi/2\gamma$, and, with $\varepsilon > 0$ *fixed* but *arbitrary*, consider, in S, the subharmonic function

$$v_\varepsilon(z) = \log|f(z)| - \varepsilon \Re(z^\beta).$$

(Note: z^β is certainly analytic and single valued in S.) For $z = re^{i\vartheta}$ in S, we have

$$\Re(z^\beta) = r^\beta \cos\beta\vartheta > r^\beta \cos\beta\gamma,$$

and $\cos\beta\gamma > 0$ *since* $0 < \beta\gamma < \pi/2$. Therefore, in the first place, $v_\varepsilon(z) \leqslant \log|f(z)|$ in S, so, for $\zeta \in \partial S$,

$$\limsup_{\substack{z\to\zeta \\ z\in S}} v_\varepsilon(z) \leqslant \log m.$$

In the second place, since

$$\log|f(z)| \leqslant O(1) + A|z|^\alpha$$

in S and $\beta > \alpha$, we have

$$v_\varepsilon(z) \leqslant \log m$$

for $z \in S$ whenever $|z|$ is *large enough* (*how* large depends on ε!).

Suppose now that $z_0 \in S$. With our fixed $\varepsilon > 0$, choose an $R > |z_0|$ so large that $v_\varepsilon(z) \leqslant \log m$ for $z \in S$ and $|z| = R$. Then

$$\limsup_{\substack{z\to\zeta \\ z\in S}} v_\varepsilon(z) \quad \text{is} \leqslant \log m$$

for any ζ on the boundary of the *bounded region* $S \cap \{|z| < R\}$,

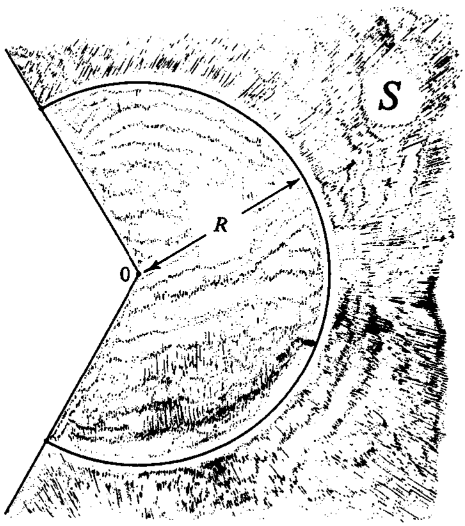

Figure 6

so, by the principle of maximum, $v_\varepsilon(z) \leqslant \log m$ throughout that region. In particular, $v_\varepsilon(z_0) \leqslant \log m$, so $\log|f(z_0)| \leqslant \log m + \varepsilon\Re(z_0^\beta)$. Now, keeping z_0 fixed, squeeze ε. Get $|f(z_0)| \leqslant m$, as required.

Important remark. The preceding two theorems *remain valid* if we merely suppose that $\log|f(z)|$ is *subharmonic* instead of taking $f(z)$ to be *analytic*. The proofs are exactly the same.

In the hypothesis of the *second* of the above two results we required $\alpha < \pi/2\gamma$ (with strict inequality); this in fact *cannot be relaxed* to the condition $\alpha \leqslant \pi/2\gamma$. What happens when $\alpha = \pi/2\gamma$ is seen from the following result, which, for simplicity, is stated for the case where $2\gamma = \pi$ (the only one which will arise on our work). We give its version for subharmonic functions.

Theorem. *Let $u(z)$ be subharmonic for $\Im z > 0$ with $u(z) \leqslant A|z| + o(|z|)$ there when $|z|$ is large. Suppose that, for each real x,*

$$\limsup_{\substack{z \to x \\ \Im z > 0}} u(z) \leqslant 0.$$

Then $u(z) \leqslant A\Im z$ for $\Im z > 0$.

Proof. Take any $\varepsilon > 0$. The function $v_\varepsilon(z) = u(z) - (A + \varepsilon)\Im z$ is subharmonic in the *first quadrant and $v_\varepsilon(z) \leqslant O(|z|)$* there when $|z|$ is large. If ζ lies on the *boundary of the first quadrant*, we clearly have

$$\limsup_{\substack{z \to \zeta \\ \Im z > 0}} v_\varepsilon(z) \leqslant M$$

for some M since $u(iy) \leqslant Ay + o(y)$ for $y > 0$ and large.

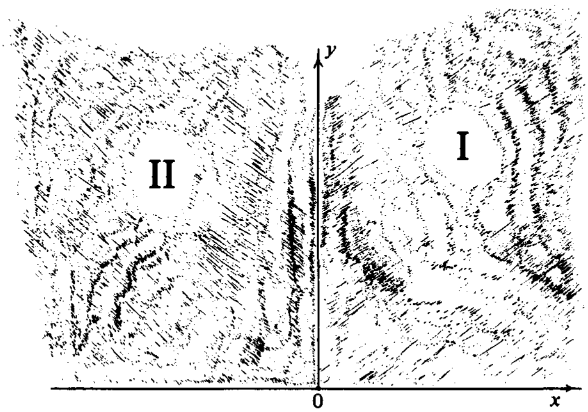

Figure 7

The first quadrant has opening $< \pi$, so by the preceding theorem (or rather by its version for subharmonic functions), $v_\varepsilon(z) \leqslant M$ throughout that region.

We see in like manner that $v_\varepsilon(z)$ is bounded above in the *second* quadrant, so, finally, $v_\varepsilon(z)$ is *bounded above for $\Im z > 0$.*

However, for x real,

$$\limsup_{\substack{z \to x \\ \Im z > 0}} v_\varepsilon(z) \leqslant 0.$$

Therefore, by the version for subharmonic functions of the first theorem in this §,

$$v_\varepsilon(z) \leqslant 0 \quad \text{for} \quad \Im z > 0.$$

That is,

$$u(z) \leqslant (A + \varepsilon)\Im z \quad \text{for} \quad \Im z > 0.$$

Squeeze ε. Get $u(z) \leqslant A\Im z$, $\Im z > 0$. Q.E.D.

Corollary. *Let $f(z)$ be analytic in $\Im z > 0$, continuous up to the real axis, and satisfy*

$$|f(z)| \leqslant Ce^{A|z|}$$

for $\Im z > 0$. If $|f(x)| \leqslant M$ for real x, then

$$|f(z)| \leqslant Me^{A\Im z}$$

when $\Im z > 0$.

Proof. Apply the theorem to $u(z) = \log|f(z)/M|$.

Remark. The example $f(z) = e^{-iAz}$ shows that the inequality furnished by the corollary cannot be improved. (Note also the relation between this particular function – or rather $\log|f(z)|$ – and the Phragmén–Lindelöf function $(A + \varepsilon)\Im z$ used in proving the theorem. That's no accident!)

The preceding theorem has an extension with a more elaborate statement, but the same proof. We give the version for analytic functions.

Theorem. *Let $f(z)$ be analytic in $\Im z > 0$ and continuous in $\Im z \geqslant 0$. Suppose that*

(i) $\log|f(z)| \leqslant O(|z|)$ *for large* $|z|$, $\Im z > 0$,
(ii) $|f(x)| \leqslant M$, $-\infty < x < \infty$,
(iii) $\limsup_{y\to\infty}(\log|f(iy)|)/y = A$.

Then, for $\Im z \geqslant 0$,

$$|f(z)| \leqslant Me^{A\Im z}.$$

Remark. The *growth of f on the imaginary axis* is thus *enough* to control the exponential furnished by the conclusion, as long as $|f(z)|$ has at most *some* finite exponential growth in $\Im z > 0$.

The *proof* of this result is *exactly* like that of the preceding one. It is enough to put $u(z) = \log|f(z)/M|$ and then *copy* the preceding argument word for word.

Any sector of the form $0 < \arg z < \alpha$ or $\alpha < \arg z < \pi$ has opening $< \pi$. Looking at the reasoning used to establish the above two theorems, *we see that we can even replace* (iii) *in the hypothesis of the preceding one* by

(iii)′ $\limsup_{R\to\infty}(\log|f(Re^{i\alpha})|)/R \sin\alpha = A$ for some α, $0 < \alpha < \pi$,

and the same conclusion holds good.

Theorem. *Let $f(z)$ be analytic for $\Im z > 0$ and continuous for $\Im z \geqslant 0$. Suppose that $|f(z)| \leqslant Ce^{A|z|}$ for $\Im z > 0$, that $|f(x)|$ is bounded on the real axis, and that*

$$f(x) \to 0 \quad \text{as} \quad x \to \infty.$$

Then $f(x + iy) \to 0$ uniformly in each strip $0 \leqslant y \leqslant L$ as $x \to \infty$.

Proof. If, say, $|f(x)| \leqslant M$ on $\mathbb{R}$, we have $|f(z)| \leqslant Me^{A\Im z}$ for $\Im z \geqslant 0$ by the corollary preceding the above theorem. Take any $B > A$ and some large K, and look at the function

$$g_K(z) = \frac{z}{z + iK} e^{iBz} f(z)$$

in $\Im z > 0$. Since $B > A$ and $K > 0$, we have $|g_K(z)| \leqslant M$, $\Im z \geqslant 0$. We can, however, do better than this.

Given $\varepsilon > 0$, we can find a Y so large that $e^{-(B-A)Y} < \varepsilon/M$; take such a Y and fix it. Then,

$$|g_K(z)| \;\leqslant\; \left|\frac{z}{z + iK}\right| e^{-(B-A)\Im z} M \;<\; \varepsilon$$

for $\Im z \geqslant Y$ as long as $K > 0$, Choose now $X > 0$ so large that $|f(x)| < \varepsilon$ for $x \geqslant X$; this we can do because $f(x) \to 0$ as $x \to \infty$.

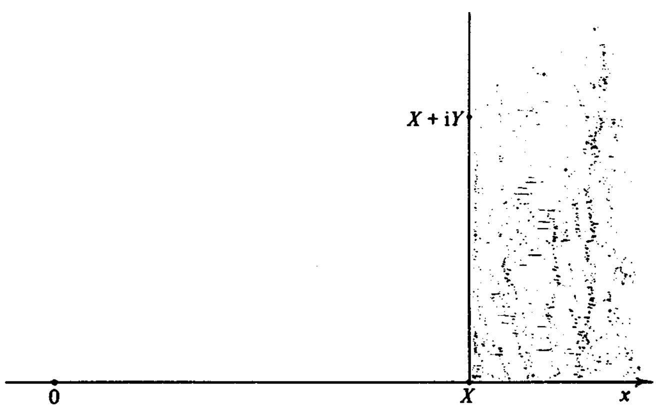

Figure 8

Having fixed X and Y, we now take K so large that $|(X+\mathrm{i}y)/(X+\mathrm{i}y+\mathrm{i}K)|M<\varepsilon$ for $0\leqslant y\leqslant Y$; fixing this K we will then have

$$|g_K(X+\mathrm{i}y)| \leqslant \left|\frac{X+\mathrm{i}y}{X+\mathrm{i}y+\mathrm{i}K}\right| \mathrm{e}^{-(B-A)y}M < \varepsilon$$

for $0\leqslant y\leqslant Y$. By choice of Y the same inequality also holds if $y\geqslant Y$. Finally, $|g_K(x)|\leqslant|f(x)|<\varepsilon$ for $x>X$.

We see that $|g_K(z)|<\varepsilon$ on the boundary of the quadrant $\{\Re z>X, \Im z>0\}$. However, $|g_K(z)|\leqslant M$ *in* that quadrant, so, by the first theorem of this §, $|g_K(z)|<\varepsilon$ *throughout* it. Let then $\Re z>X$ and $\Im z>0$. We have

$$|f(z)| = \left|\frac{z+\mathrm{i}K}{z}\right| \mathrm{e}^{B\Im z}|g_K(z)| < \left|\frac{z+\mathrm{i}K}{z}\right| \mathrm{e}^{B\Im z}\varepsilon.$$

Suppose that $0\leqslant\Im z\leqslant L$. Then, if $\Re z>\max(X,K)$ we have, by the previous relation,

$$|f(z)|<2\mathrm{e}^{BL}\varepsilon.$$

Here, $\varepsilon>0$ is arbitrary. Therefore $f(x+\mathrm{i}y)\to 0$ uniformly for $0\leqslant y\leqslant L$ as $x\to\infty$.

We are done.

D. The Paley–Wiener theorem

Theorem. *Let $f(z)$ be entire and of exponential type A. Suppose that*

$$\int_{-\infty}^{\infty} |f(x)|^2\,\mathrm{d}x < \infty.$$

Then there is a function $\varphi(\lambda)\in L_2(-A,A)$ with

$$f(z)=\frac{1}{2\pi}\int_{-A}^{A} \mathrm{e}^{-\mathrm{i}z\lambda}\varphi(\lambda)\,\mathrm{d}\lambda.$$

Remark. If $f(z)$ is *given* by such an integral, it is obviously of exponential type A and belongs to $L_2(-\infty,\infty)$ on account of Plancherel's theorem. So the *converse of the theorem is evident*. The two results (the *theorem* and its *converse*) taken together constitute the celebrated and much used *Paley–Wiener theorem.*

Proof of theorem. Is essentially based on Plancherel's theorem, combined with contour integration and the third and fifth results of the previous §. An easy but rather fussy preliminary reduction is necessary.

Plancherel's theorem says that

$$\varphi(\lambda) = \operatorname*{l.i.m.}_{M\to\infty} \int_{-M}^{M} e^{i\lambda x} f(x)\,dx$$

exists and belongs to $L_2(-\infty, \infty)$, and that, for $x \in \mathbb{R}$,

$$f(x) = \frac{1}{2\pi} \operatorname*{l.i.m.}_{M\to\infty} \int_{-M}^{M} e^{-ix\lambda} \varphi(\lambda)\,d\lambda.$$

(Here, 'l.i.m.' stands for 'limit in mean (square)', and denotes a limit in $L_2(-\infty, \infty)$.) Our main task is to show that $\varphi(\lambda) \equiv 0$ a.e. for $\lambda > A$ and $\lambda < -A$.

To this end, let us introduce the function

$$f_h(z) = \frac{1}{2h} \int_{-h}^{h} f(z+t)\,dt;$$

$f_h(z)$ is clearly entire. Because $f(z)$ is of exponential type A, we have, for any $A' > A$,

$$|f(z)| \leqslant \text{const.}e^{A'|z|},$$

and from this it is clear that also

$$|f_h(z)| \leqslant \text{const.}e^{A'|z|}$$

(with a different constant).

By Schwarz' inequality we also have

$$|f_h(x)| \leqslant \sqrt{\left(\frac{1}{2h} \int_{x-h}^{x+h} |f(t)|^2\,dt\right)},$$

so, since $f(x) \in L_2(-\infty, \infty)$, $f_h(x)$ is *bounded on* $\mathbb{R}$ *and in fact* $f_h(x) \to 0$ for $x \to \pm\infty$. (This is the *main reason* for doing $(1/2h)\int_{-h}^{h}$ on f!) If we call $\sup_{x\in\mathbb{R}} |f_h(x)| = \tilde{M}_h$, we see by the previous inequality for $|f_h(z)|$ and the *third* theorem (p. 27) of the previous § (applied in *each* half plane $\Im z > 0$ and $\Im z < 0$), that

$$|f_h(z)| \leqslant \tilde{M}_h e^{A'|\Im z|}.$$

Here, A' can be *any* number $> A$, so in fact

$$(*) \qquad |f_h(z)| \leqslant \tilde{M}_h e^{A|\Im z|}.$$

The *fifth* theorem of § C (p. 29) shows moreover that

$$(\ast\ast) \qquad f_h(x+iy) \to 0 \text{ uniformly for } -L \leqslant y \leqslant L \text{ when } x \to \pm\infty.$$

In order to prove that

$$\varphi(\lambda) = \underset{M\to\infty}{\mathrm{l.i.m.}} \int_{-M}^{M} e^{i\lambda x} f(x)\,dx$$

vanishes a.e. for $\lambda > A$ and for $\lambda < -A$ it is more than sufficient to show the same thing with $f(x)$ replaced by $f_h(x)$ in the right-hand integral, $h > 0$ being arbitrary. That's because

$$\underset{M\to\infty}{\mathrm{l.i.m.}} \int_{-M}^{M} e^{i\lambda x} f_h(x)\,dx = \frac{\sin\lambda h}{\lambda h}\varphi(\lambda),$$

which we can check using Fubini's theorem and the fact that

$$\frac{1}{2h}\int_{-h}^{h} e^{i\lambda x}\,dx = \frac{\sin\lambda h}{\lambda h}.*$$

Taking a large M, look at $\int_{-M}^{M} e^{i\lambda x} f_h(x)\,dx$, assuming that $\lambda > A$. Let γ consist of the three *upper* pieces of the rectangular contour shown.

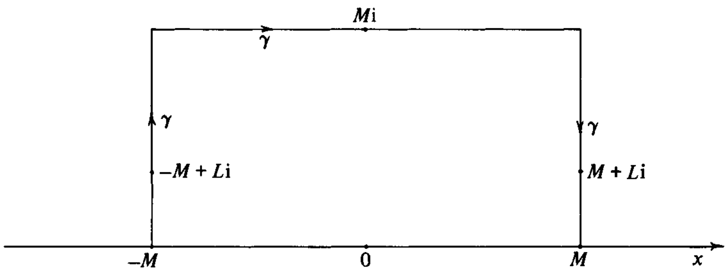

Figure 9

By Cauchy's theorem,

$$\int_{-M}^{M} e^{i\lambda x} f_h(x)\,dx = \int_\gamma e^{i\lambda z} f_h(z)\,dz.$$

The contribution to $\int_\gamma$ from the *top horizontal portion* of γ has absolute value

$$\left| \int_{-M}^{M} e^{i\lambda(x+iM)} f_h(x + iM)\,dx \right|,$$

and this, by (*), is

$$\leqslant 2M\cdot\tilde{M}_h e^{-(\lambda-A)M},$$

a quantity tending to zero as $M\to\infty$, since $\lambda > A$.

Fix any large number L. If $M > L$, we write the contribution to $\int_\gamma$ from

* To do this, one should start from the *second* formula at the top of p. 31 and conclude by applying Plancherel's theorem.

the *right-hand vertical portion* of γ as

$$-\left(\int_0^L + \int_L^M\right) e^{i\lambda(M+iy)} f_h(M+iy)\cdot i\,dy.$$

The *second* integral is in modulus

$$\leqslant \tilde{M}_h \int_L^\infty e^{-(\lambda-A)y}\,dy = \frac{e^{-(\lambda-A)L}}{\lambda - A}\tilde{M}_h,$$

again by (∗), and we can make the quantity on the right as small as we like by taking L *large*. The *first* integral, however, has modulus

$$\leqslant \int_0^L e^{-\lambda y}|f_h(M+iy)|\,dy$$

and this, for any fixed L, tends to 0 as $M \to \infty$ according to (⁑). We see that the contribution from the *right vertical portion* of γ *tends to zero* as $M \to \infty$; that of the *left vertical portion does the same*, as a similar argument shows.

In fine, $\int_\gamma e^{i\lambda z} f_h(z)\,dz \longrightarrow 0$ as $M \to \infty$, i.e.,

$$\int_{-M}^M e^{i\lambda x} f_h(x)\,dx \longrightarrow 0 \quad \text{as} \quad M \to \infty$$

when $\lambda > A$. For $\lambda < -A$ we establish the same result using a similar argument and this contour:

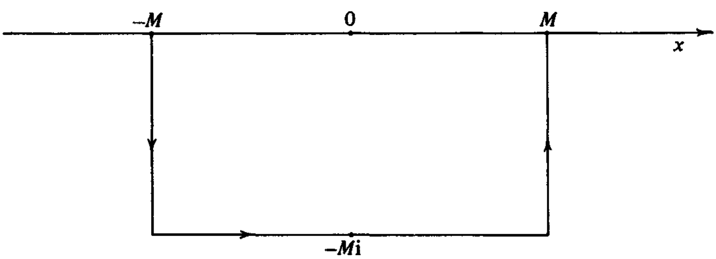

Figure 10

Thus, $\int_{-M}^M e^{i\lambda x} f_h(x)\,dx \longrightarrow 0$ *pointwise* in λ for $|\lambda| > A$ as $M \to \infty$. However, for *some sequence of Ms* tending to ∞, the integrals in question must tend a.e. to

$$\underset{M\to\infty}{\text{l.i.m.}} \int_{-M}^M e^{i\lambda x} f_h(x)\,dx = \frac{\sin \lambda h}{\lambda h}\varphi(\lambda).$$

(L_2 convergence of a *sequence* implies the a.e. *pointwise* convergence of some *subsequence to the same limit.*) This means that $(\sin \lambda h/\lambda h)\varphi(\lambda) = 0$

a.e. for $|\lambda| > A$, whence $\varphi(\lambda) = 0$ a.e. for $|\lambda| > A$. ($\sin \lambda h/\lambda h$ vanishes only on a countable set of points!)

The Fourier–Plancherel inversion formula now gives us, for $x \in \mathbb{R}$,

$$f(x) = \underset{M\to\infty}{\text{l.i.m.}} \frac{1}{2\pi} \int_{-M}^{M} e^{-ix\lambda} \varphi(\lambda)\, d\lambda$$
$$= \frac{1}{2\pi} \int_{-A}^{A} e^{-ix\lambda} \varphi(\lambda)\, d\lambda \quad \text{a.e.}$$

In fact, we have

$$f(z) = \frac{1}{2\pi} \int_{-A}^{A} e^{-iz\lambda} \varphi(\lambda)\, d\lambda$$

for all complex z. That's because *each* of the two sides is an *entire* function of z. Since these two entire functions coincide a.e. on $\mathbb{R}$, they must be everywhere equal by the identity theorem for analytic functions. Our theorem is completely proved.

If we refer to the *fourth* theorem of §C (p. 28), we see that we can give the result just proved a more general formulation. The statement thus obtained, which we give as a corollary, also goes under the names of Paley and Wiener.

Corollary. *Let $f(z)$ be entire and of (some) exponential type, with $f(x) \in L_2(\mathbb{R})$, and let*

$$\limsup_{y\to\infty} \frac{\log|f(iy)|}{y} = b,$$
$$\limsup_{y\to-\infty} \frac{\log|f(iy)|}{|y|} = -a.$$

Then

$$f(z) = \frac{1}{2\pi} \int_{a}^{b} e^{-iz\lambda} \varphi(\lambda)\, d\lambda,$$

where $\varphi \in L_2(a, b)$.

Proof. If f is of exponential type A, say, we certainly have

$$f(z) = \frac{1}{2\pi} \int_{-A}^{A} e^{-iz\lambda} \varphi(\lambda)\, d\lambda$$

by the theorem, so, if $x \in \mathbb{R}$,

$$|f(x)| \leq \frac{1}{2\pi} \sqrt{\left(2A \int_{-A}^{A} |\varphi(\lambda)|^2\, d\lambda \right)},$$

a *finite quantity*, i.e., f is *bounded on* $\mathbb{R}$ if we only assume $f \in L_2(\mathbb{R})$, *provided* that it is *entire* and of *exponential type*. Applying the fourth theorem of §C in each of the half planes $\Im z > 0$, $\Im z < 0$, we now see that

$$|f(z)| \leqslant \text{const.} e^{b\Im z}, \qquad \Im z > 0;$$
$$|f(z)| \leqslant \text{const.} e^{-a|\Im z|}, \qquad \Im z < 0.$$

Symmetrize by taking $g(z) = e^{i\gamma z} f(z)$ with $\gamma = (b + a)/2$. Then $g(z)$ is also entire, $g(x) \in L_2(\mathbb{R})$, and, by the previous relations,

$$|g(z)| \leqslant \text{const.} e^{(b-a)|\Im z|/2}$$

in *both* upper and lower half planes, i.e., $g(z)$ is of *exponential type* $(b - a)/2$. (We see at this point that $(b-a)/2$ *cannot be* $\leqslant 0$ *unless* $f(z) \equiv 0$ – the reader is urged to think out why this is so.) Use the above theorem once more, this time for $g(z)$. We find

$$g(z) = \frac{1}{2\pi} \int_{-(1/2)(b-a)}^{(1/2)(b-a)} e^{-iz\lambda} \psi(\lambda) \, d\lambda$$

with a certain $\psi \in L_2$. Going back to $f(z) = e^{-i\gamma z} g(z)$, we have

$$f(z) = \frac{1}{2\pi} \int_a^b e^{-iz\lambda} \psi(\lambda - \gamma) \, d\lambda,$$

establishing the result with $\varphi(\lambda) = \psi(\lambda - \gamma)$.

Scholium. The Paley–Wiener theorem has more content than meets the eye. Suppose that $\varphi \in L_2(a, b)$; then

$$f(z) = \frac{1}{2\pi} \int_a^b e^{-iz\lambda} \varphi(\lambda) \, d\lambda$$

is entire, of exponential type, and belongs to L_2 on $\mathbb{R}$. We can also easily verify directly that

$$\limsup_{y \to \infty} \frac{\log |f(x + iy)|}{y} \leqslant b$$

and

$$\limsup_{y \to -\infty} \frac{\log |f(x + iy)|}{|y|} \leqslant -a$$

for each real x.

These inequalities remain true as long as φ vanishes a.e. outside $[a, b]$. If, however, we take for $[a, b]$ the *smallest closed interval containing* φ's *support – the so-called supporting interval for* φ *– the inequalities become equalities*!

Without loss of generality, take $x = 0$, and suppose, for instance, that

$$\limsup_{y\to\infty} \frac{\log|f(\mathrm{i}y)|}{y} = b' < b.$$

The above corollary then shows that

$$\varphi(\lambda) = \operatorname*{l.i.m.}_{M\to\infty} \int_{-M}^{M} \mathrm{e}^{\mathrm{i}\lambda x} f(x)\,\mathrm{d}x$$

in fact vanishes a.e. for $\lambda > b'$. The support of φ would thus be contained in $[a, b']$, so $[a, b]$ would not be φ's supporting interval, and we have a contradiction.

If $[a, b]$ *is* the supporting interval of φ, we must therefore have

$$\limsup_{y\to\infty} \frac{\log|f(\mathrm{i}y)|}{y} = b.$$

It is clear that this b can only come from the portions

$$\frac{1}{2\pi}\int_{b-\varepsilon}^{b} \mathrm{e}^{-\mathrm{i}z\lambda}\varphi(\lambda)\,\mathrm{d}\lambda$$

of the integral giving $f(z)$, $\varepsilon > 0$. We *know* that $\varphi(\lambda)$ cannot vanish identically a.e. on any interval of the form $[b-\varepsilon, b]$, but it is *still quite conceivable that*

$$\left|\int_{b-\varepsilon}^{b} \mathrm{e}^{y\lambda}\varphi(\lambda)\,\mathrm{d}\lambda\right|$$

could come out much smaller than e^{by} *for large* y *on account of cancellation.* The Paley–Wiener theorem teaches us that *such cancellation cannot take place.* This is a remarkable and deep property of (square) integrable functions $\varphi(\lambda)$.

There are versions of the Paley–Wiener theorem for other spaces besides $L_2(\mathbb{R})$. The following is frequently used.

Theorem. *Let* $\varphi(\lambda) \in L_1(\mathbb{R})$ *have compact support, put*

$$f(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}z\lambda}\varphi(\lambda)\,\mathrm{d}\lambda,$$

and suppose that

$$\limsup_{y\to\infty} \frac{\log|f(\mathrm{i}y)|}{y} = b, \quad \limsup_{y\to-\infty} \frac{\log|f(\mathrm{i}y)|}{|y|} = -a.$$

Then $\varphi(\lambda)$ *vanishes a.e. outside* $[a, b]$.

Proof. This would be part of the corollary to the Paley–Wiener theorem, save that $f(x)$ is not necessarily in $L_2(\mathbb{R})$. For $h > 0$, put

$$\varphi_h(\lambda) = \frac{1}{2h}\int_{-h}^{h} \varphi(\lambda+\tau)\,\mathrm{d}\tau;$$

then $\|\varphi_h - \varphi\|_1 \to 0$ as $h \to 0$ and we need only show that $\varphi_h(\lambda) \equiv 0$ for $\lambda \notin [a-h, b+h]$.

Write

$$f_h(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}z\lambda}\varphi_h(\lambda)\,\mathrm{d}\lambda;$$

then

$$f_h(z) = \frac{\sin hz}{hz} f(z),$$

so, since $f(x)$ is clearly *bounded* on $\mathbb{R}$ (φ being in $L_1(\mathbb{R})$), $f_h(x) \in L_2(\mathbb{R})$. By the hypothesis we now have

$$\limsup_{y\to\infty} \frac{\log|f_h(\mathrm{i}y)|}{y} = b + h,$$

$$\limsup_{y\to-\infty} \frac{\log|f_h(\mathrm{i}y)|}{|y|} = -a + h.$$

Therefore $\varphi_h(\lambda) \equiv 0$ outside $[a-h, b+h]$, by Paley–Wiener. We're done.

Remark. The same result holds (with almost the same proof) if we replace $\varphi(\lambda)\,\mathrm{d}\lambda$ (with $\varphi \in L_1$ and of compact support) by $\mathrm{d}\mu(\lambda)$, μ being any *finite signed measure* of compact support.

E. Introduction to the condition $\int_{-\infty}^{\infty} (\log^+ |f(x)|/(1+x^2))\,\mathrm{d}x < \infty$

The entire functions of exponential type considered in the previous § certainly satisfy this condition, as do those arising in the study of many questions in analysis. We will meet repeatedly with such functions in the following chapters of this book, and the rest of the present chapter is mainly concerned with them. It turns out that the boundedness condition

$$\int_{-\infty}^{\infty} \frac{\log^+|f(x)|}{1+x^2}\,\mathrm{d}x \;<\; \infty$$

implies many results for entire functions f of exponential type.

The following simple result is very useful, and all that one needs for many investigations.

Theorem. *Let $f(z)$ be regular in $\Im z > 0$ and continuous up to the real axis. Suppose that $\log|f(z)| \leqslant O(|z|)$ for $|z|$ large when $\Im z > 0$, that*

$$\limsup_{y\to\infty} \frac{\log|f(\mathrm{i}y)|}{y} = A,$$

and that

$$\int_{-\infty}^{\infty} \frac{\log^+|f(x)|}{1+x^2}\mathrm{d}x < \infty.$$

Then, for $\Im z > 0$,

$$\boxed{\log|f(z)| \leqslant A\Im z + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log^+|f(t)|}{|z-t|^2}\mathrm{d}t.}$$

Proof. $\Im z/|z-t|^2 = \Re(\mathrm{i}/(z-t))$ is, for each $t\in\mathbb{R}$, a *positive and harmonic function* in $\Im z > 0$. For fixed z with positive real part we have, by calculus,

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\mathrm{d}t = 1,$$

and, if $z\to x_0\in\mathbb{R}$,

$$\sup_{|t-x_0|\geqslant\delta}\left\{\frac{\Im z}{|z-t|^2}\Big/\frac{1}{t^2+1}\right\}\longrightarrow 0$$

for each $\delta > 0$. Therefore, if $P(t)$ is any positive continuous function with

$$\int_{-\infty}^{\infty} \frac{P(t)}{1+t^2}\mathrm{d}t < \infty,$$

we have by the usual elementary approximate identity argument (no need to refer to Chapter II, §B, here!),

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}P(t)\mathrm{d}t \longrightarrow P(x_0)$$

for $z\to x_0\in\mathbb{R}$.

In our present situation $P(t) = \log^+|f(t)|$ is continuous on $\mathbb{R}$, so if we put

$$U(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log^+|f(t)|}{|z-t|^2}\mathrm{d}t$$

for $\Im z > 0$, $U(z)$ is *positive and harmonic* in the upper half plane and

$U(z) \to \log^+|f(x_0)|$ for $z \to x_0 \in \mathbb{R}$. We see that in $\Im z > 0$, $\log|f(z)| - U(z)$ is *subharmonic*, is $\leqslant O(|z|)$ for large $|z|$, and *has boundary values* $\leqslant 0$ *everywhere on* $\mathbb{R}$. Moreover,

$$\log|f(iy)| - U(iy) \leqslant Ay + o(y)$$

for $y \to \infty$. The *fourth* theorem of §C (p. 28) (or rather its version for subharmonic functions) now yields without further ado

$$\log|f(z)| - U(z) \leqslant A\Im z, \quad \Im z > 0,$$

that is,

$$\log|f(z)| \leqslant A\Im z + U(z), \quad \Im z > 0.$$

We are done.

Later on we will give some refined versions of this result. Their derivation requires more effort.

F. Representation of positive harmonic functions as Poisson integrals

In order to proceed further with the discussion begun in §E, it is simplest to apply the Riesz–Evans–Herglotz representation for positive harmonic functions, although its use can in fact be avoided. We explain that representation here, together with some of its function-theoretic consequences.

1. The representation

Theorem. *Let $V(w)$ be positive and harmonic for $|w| < 1$. There is a finite positive measure ν on $[-\pi, \pi]$ with*

$$V(w) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - |w|^2}{|w - e^{it}|^2}\, d\nu(t), \quad |w| < 1.$$

Sketch of Proof. By the ordinary Poisson formula, if $R < 1$, we have, for $|w| < R$,

$$V(w) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{R^2 - |w|^2}{|w - Re^{it}|^2} V(Re^{it})\, dt,$$

that is, for $|w| < 1$,

$$(*) \qquad V(Rw) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - |w|^2}{|w - e^{it}|^2} V(Re^{it})\, dt.$$

In particular,

$$(\ast\!\ast)\qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} V(Re^{it})dt = V(0) < \infty,$$

no matter how close $R < 1$ is to 1.

We must now use some version of Tychonoff's theorem in order to obtain the measure ν.

Take any sequence $\{R_n\}$ tending monotonically to 1, for example $R_n = 1 - 1/n$. The functions $\varphi_n(t) = V(R_n e^{it})$ are all $\geqslant 0$ and have bounded integrals over $[-\pi, \pi]$ by ($\ast\!\ast$); we can therefore (by using Cantor's diagonal process) extract a *subsequence of these functions*, which we also denote by $\{\varphi_n\}$ (so as not to write subscripts of subscripts!), having the property that

$$L(G) = \lim_{n\to\infty} \int_{-\pi}^{\pi} G(t)\varphi_n(t)dt$$

exists and is finite for G ranging over a *countable dense subset* of $\mathscr{C}(-\pi, \pi)$.

If, however, G and $G' \in \mathscr{C}(-\pi, \pi)$ and $\| G - G' \| < \varepsilon$, we have, for *every* n,

$$\left| \int_{-\pi}^{\pi} G(t)\varphi_n(t)dt - \int_{-\pi}^{\pi} G'(t)\varphi_n(t)dt \right| \leqslant \int_{-\pi}^{\pi} \varepsilon\varphi_n(t)dt = 2\pi\varepsilon V(0),$$

so *in fact* $L(G) = \lim_{n\to\infty}\int_{-\pi}^{\pi} G(t)\varphi_n(t)dt$ exists for *every* $G \in \mathscr{C}(-\pi, \pi)$, and $|L(G)| \leqslant 2\pi V(0)\| G \|$.

L is thus a bounded linear functional on $\mathscr{C}(-\pi, \pi)$; it is moreover positive because, if $G \in \mathscr{C}(-\pi, \pi)$ and $G \geqslant 0$, $L(G) \geqslant 0$ since $\varphi_n(t) \geqslant 0$ for each n. By the Riesz representation theorem there is thus a positive finite measure ν on $[-\pi, \pi]$ with

$$L(G) = \int_{-\pi}^{\pi} G(t)d\nu(t), \quad G \in \mathscr{C}(-\pi, \pi).$$

Taking in particular $G(t) = (1/2\pi)(1 - |w|^2)/|w - e^{it}|^2$ with a fixed w, $|w| < 1$, we obtain

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - |w|^2}{|w - e^{it}|^2} d\nu(t) &= L(G) \\
&= \lim_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - |w|^2}{|w - e^{it}|^2}\varphi_n(t)\,dt \\
&= \lim_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - |w|^2}{|w - e^{it}|^2} V(R_n e^{it})\,dt.
\end{aligned}$$

Referring to ($\ast$), we see that the last expression equals $\lim_{n\to\infty} V(R_n w) = V(w)$, V being certainly *continuous* for $|w| < 1$.

This completes the proof.

Scholium. Once we *know* that the measure ν giving rise to the desired representation *exists*, we see that the *passage to a subsequence* of the $\varphi_n(t)$ in the above proof was not really *necessary* (although we are only able to see this once the proof has been carried out!).

Suppose $G(t)$ is any continuous function on $[-\pi,\pi]$ with $G(-\pi)=G(\pi)$; then, by the elementary approximate identity property of the Poisson kernel

$$\frac{1}{2\pi}\frac{1-|w|^2}{|w-e^{it}|^2},$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-R^2}{|e^{it}-Re^{i\vartheta}|^2}G(\vartheta)d\vartheta\longrightarrow G(t)$$

uniformly for $-\pi\leqslant t\leqslant\pi$ as $R\to 1$, so, by Fubini's theorem,

$$\begin{aligned}\int_{-\pi}^{\pi}G(\vartheta)V(Re^{i\vartheta})d\vartheta &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{1-R^2}{|e^{it}-Re^{i\vartheta}|^2}G(\vartheta)d\vartheta d\nu(t)\\ &\longrightarrow \int_{-\pi}^{\pi}G(t)\,d\nu(t)\end{aligned}$$

as $R\to 1$. This simple fact can frequently be used to get information about the measure ν.

The reader should think through what happens with the argument just given when $G\in\mathscr{C}(-\pi,\pi)$ but $G(-\pi)\neq G(\pi)$. Here is a hint: we at least have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-R^2}{|e^{it}-Re^{i\vartheta}|^2}G(\vartheta)d\vartheta \longrightarrow \begin{cases} G(t), & t\neq-\pi,\pi,\\ \dfrac{G(\pi)+G(-\pi)}{2}, & t=-\pi,\pi,\end{cases}$$

as $R\to 1$, although the convergence is no longer uniform. The integrals on the left are, however, bounded.

Terminology. The situation of the scholium is frequently described by saying that

$$V(Re^{i\vartheta})d\vartheta \longrightarrow d\nu(\vartheta)\quad w^*$$

for $R\to 1$, or by writing

$$\text{`}V(Re^{i\vartheta})\rightharpoonup d\nu(\vartheta)\quad\text{as}\quad R\to 1\text{'}$$

(with a *half arrow*).

Theorem. *Let $v(z)$ be positive and harmonic in $\Im z>0$. There is a positive number α and a positive measure μ on $\mathbb{R}$ with*

$$\int_{-\infty}^{\infty}\frac{d\mu(t)}{1+t^2}<\infty$$

such that

$$v(z) = \alpha\Im z + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\mathrm{d}\mu(t) \quad \textit{for} \quad \Im z > 0.$$

Proof. From the previous theorem by making the change of variable

$$z \to w = \frac{\mathrm{i}-z}{\mathrm{i}+z}$$

which takes $\Im z > 0$ conformally onto the open unit disk.

Everybody should do this calculation at least once in his or her life, so let us give the good example. The conformal mapping just described takes $v(z)$ to a positive harmonic function $V(w) = v(z)$ defined for $|w| < 1$, so we have

$$V(w) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-|w|^2}{|\mathrm{e}^{\mathrm{i}\tau}-w|^2}\mathrm{d}\nu(\tau)$$

with a positive measure ν according to the result just proved. We write τ here because t will denote a variable *running along the real axis.*

We have $w = (\mathrm{i}-z)/(\mathrm{i}+z)$, and the real t corresponds in a similar way to

$$\mathrm{e}^{\mathrm{i}\tau} = \frac{\mathrm{i}-t}{\mathrm{i}+t}.$$

Therefore

$$\frac{1-|w|^2}{|\mathrm{e}^{\mathrm{i}\tau}-w|^2} = \frac{1-\left|\dfrac{\mathrm{i}-z}{\mathrm{i}+z}\right|^2}{\left|\dfrac{\mathrm{i}-t}{\mathrm{i}+t}-\dfrac{\mathrm{i}-z}{\mathrm{i}+z}\right|^2}$$

$$= \frac{(|\mathrm{i}+z|^2-|\mathrm{i}-z|^2)|\mathrm{i}+t|^2}{|(\mathrm{i}-t)(\mathrm{i}+z)-(\mathrm{i}+t)(\mathrm{i}-z)|^2} = \frac{4\Im z(t^2+1)}{|2\mathrm{i}(z-t)|^2}$$

$$= \frac{\Im z}{|z-t|^2}(1+t^2).$$

Since $\mathrm{e}^{\pm\mathrm{i}\pi} = -1$ corresponds to $t = \infty$, we see that

$$\frac{1}{2\pi}\int_{(-\pi,\pi)} \frac{1-|w|^2}{|\mathrm{e}^{\mathrm{i}\tau}-w|^2}\mathrm{d}\nu(\tau) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2}\mathrm{d}\mu(t),$$

where $\mathrm{d}\mu(t) = \frac{1}{2}(1+t^2)\mathrm{d}\nu(\tau)$.

We are finally left with the (possible) point masses coming from ν at $-\pi$ and π; their contribution gives us the term $\alpha\Im z$ with $\alpha \geqslant 0$. Recalling that $v(z) = V(w)$, we see that the proof is complete.

Remark. If $\Phi(t)$ is a continuous function of compact support,

$$\int_{-\infty}^{\infty} \Phi(t)\mathrm{d}\mu(t) = \lim_{y\to 0+} \int_{-\infty}^{\infty} \Phi(t)v(t+\mathrm{i}y)\mathrm{d}t.$$

To see this, just use the approximate identity property of $(1/\pi)(\Im z/|z-t|^2)$ (§E) – compare with the above scholium.

2. Digression on the a.e. existence of boundary values

The representation derived in the preceding article can be combined with the result in Chapter II, §B, to obtain some theorems about the a.e. existence of (non-tangential) finite boundary values for certain classes of harmonic and analytic functions defined in $\{|w|<1\}$ or in $\{\Im z>0\}$. Although this is *not* a book about boundary behaviour or H_p spaces, it is perhaps a good idea to show here how such results are deduced, especially since that can be done with so little additional effort.

Theorem. *Let $V(w)$ be positive and harmonic in $\{|w|<1\}$. Then, for almost every ϑ, the non-tangential boundary value*

$$\lim_{w \not\angle\to e^{i\vartheta}} V(w)$$

exists and is finite.

Proof. By the previous article,

$$V(w) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-|w|^2}{|w-\mathrm{e}^{\mathrm{i}\tau}|^2}\mathrm{d}\nu(\tau)$$

where ν is a finite positive measure. A theorem of Lebesgue says that

$$\nu'(\vartheta) = \frac{\mathrm{d}}{\mathrm{d}\vartheta}\left(\int_0^{\vartheta}\mathrm{d}\nu(\tau)\right)$$

exists and is finite a.e. And by the remark at the end of Chapter 2, §B, $V(w)\to\nu'(\vartheta)$ as $w \not\angle\to \mathrm{e}^{\mathrm{i}\vartheta}$ wherever $\nu'(\vartheta)$ exists and is finite. We're done.

Corollary (Fatou). *Let $F(w)$ be analytic and bounded for $|w|<1$. Then*

$$\lim_{w \not\angle\to e^{i\vartheta}} F(w)$$

exists for almost all ϑ.

Proof. If $|F(w)|<M$ in $\{|w|<1\}$, $M+\Re F(w)$ and $M+\Im F(w)$ are both positive and harmonic there.

Notation. Let $F(w)$ be analytic and bounded for $\{|w|<1\}$. The *non-tangential limit*

$$\lim_{w \not\angle\!\to e^{i\vartheta}} F(w)$$

(which, by the corollary, exists a.e.) *is denoted by* $F(e^{i\vartheta})$. The function $F(e^{i\vartheta})$, thus defined a.e., is Lebesgue measurable (and, of course, bounded).

Theorem. *Let $F(w)$ be analytic and bounded for $|w|<1$. Then*

$$F(w) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{F(e^{i\vartheta})e^{i\vartheta}d\vartheta}{e^{i\vartheta}-w}, \quad |w|<1.$$

Remark. Thus, the boundary values $F(e^{i\vartheta})$ (which are defined a.e.) serve to recover $F(w)$.

Proof. For each $R<1$, we have, by Cauchy's theorem,

$$F(Rw) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{F(Re^{i\vartheta})e^{i\vartheta}d\vartheta}{e^{i\vartheta}-w}, \quad |w|<1.$$

Fix w, and take $R=R_n$ with $R_n \xrightarrow[n]{} 1$. We have $|F(R_n e^{i\vartheta})| \leqslant M$, say, and $F(R_n e^{i\vartheta}) \xrightarrow[n]{} F(e^{i\vartheta})$ a.e. by the corollary. The result follows by Lebesgue's dominated convergence theorem.

Lemma. *Let $F(w)$ be analytic for $|w|<1$ and suppose that $|F(w)|\leqslant 1$ there. Let $|\alpha|=1$, and take*

$$E = \{\vartheta : F(e^{i\vartheta}) = \alpha, \ -\pi < \vartheta \leqslant \pi\}.$$

*Then, unless $F(w)\equiv\alpha$, $|E|=0$.**

Remark. The result is also true when $|\alpha|<1$. But then the proof is more difficult.

Proof of lemma. Take, wlog, $\alpha=1$. We must then prove that $F(w)\equiv 1$ if $|E|>0$.

The function $((F(w)+1)/2)^n$ is analytic in $\{|w|<1\}$ and in modulus $\leqslant 1$ there, so, by the above theorem applied to it,

$$\left(\frac{F(0)+1}{2}\right)^n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{F(e^{i\vartheta})+1}{2}\right)^n d\vartheta.$$

Here, $(F(e^{i\vartheta})+1)/2 = 1$ if $\vartheta\in E$, and, if ϑ, $-\pi<\vartheta\leqslant\pi$, is *not* in E, $|(F(e^{i\vartheta})+1)/2|<1$, so $((F(e^{i\vartheta})+1)/2)^n \xrightarrow[n]{} 0$. We see by bounded conver-

* We follow the customary practice of denoting the Lebesgue measure of $E\subseteq\mathbb{R}$ by $|E|$.

gence that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{F(e^{i\vartheta})+1}{2}\right)^n d\vartheta = \frac{|E|}{2\pi} + o(1)$$

for $n \to \infty$. Suppose $|E| > 0$. Then the last relation combines with the previous to yield

$$\frac{F(0)+1}{2} = \sqrt[n]{\frac{|E|}{2\pi}} + o(1), \quad n \to \infty,$$

after extracting an nth root. Since the right side tends to 1 for $n \to \infty$ we have finally $F(0) = 1$.

However, $|F(w)| \leqslant 1$, $|w| < 1$. Therefore $F(w) \equiv 1$ there by the strong maximum principle, Q.E.D.

Theorem. *Let $f(t) \in L_1(-\pi, \pi)$, and put, for $|w| < 1$,*

$$G(w) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+w}{e^{it}-w}f(t)\,dt.$$

Then $\lim_{w \not\angle\to e^{i\vartheta}} G(w)$ exists and is finite a.e.

Proof. Wlog, $f(t) \geqslant 0$. Notice that $G(w)$ is certainly analytic in $\{|w| < 1\}$, and that

$$\Re G(w) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-|w|^2}{|w-e^{it}|^2}f(t)\,dt$$

is $\geqslant 0$ there. (Compare Chapter II, §A!)

The function

$$F(w) = \frac{G(w)-1}{G(w)+1}$$

is therefore analytic and in modulus $\leqslant 1$ for $|w| < 1$. So, by a previous corollary,

$$F(e^{i\vartheta}) = \lim_{w \not\angle\to e^{i\vartheta}} F(w)$$

exists a.e. It follows that, whenever this limit exists,

$$\lim_{w \not\angle\to e^{i\vartheta}} G(w)$$

must *also* exist and equal the finite quantity

$$\frac{1+F(e^{i\vartheta})}{1-F(e^{i\vartheta})}$$

unless $F(e^{i\vartheta}) = 1$.

However, $F(e^{i\vartheta})$ *can equal* 1 *only on a set of measure zero by the lemma–otherwise* $G(w)$ would equal ∞ everywhere in $\{|w|<1\}$, which is absurd. So $\lim_{w \not\to e^{i\vartheta}} G(w)$ exists and is finite a.e., as required.

Scholium. Write $w = re^{i\vartheta}$ and *suppose that* $f \in L_1(-\pi, \pi)$ *is real-valued.* Then we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+w}{e^{it}-w} f(t)dt$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\vartheta - t)} f(t)dt$$
$$+ \frac{i}{2\pi}\int_{-\pi}^{\pi} \frac{2r\sin(\vartheta - t)}{1+r^2-2r\cos(\vartheta - t)} f(t)dt.$$

We see that both

$$U(re^{i\vartheta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\vartheta - t)} f(t)dt$$

and

$$\tilde{U}(re^{i\vartheta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{2r\sin(\vartheta - t)}{1+r^2-2r\cos(\vartheta - t)} f(t)dt$$

are *harmonic* in $\{|w|<1\}$, $U(w)$ being equal to $\Re G(w)$ there, and $\tilde{U}(w)$ equal to $\Im G(w)$, with $G(w)$ the analytic function considered in the above theorem.

$\tilde{U}(w)$ is frequently called a *harmonic conjugate* to $U(w)$; it has the property that $U(w) + i\tilde{U}(w)$ is *analytic* in $\{|w|<1\}$. It is an easy exercise to see that any two harmonic conjugates to $U(w)$ must differ by a constant; the particular one we are considering has the property that

$$\tilde{U}(0) = 0.$$

By Chapter II, §B, we *already know* that

$$\lim_{w \not\to e^{i\vartheta}} U(w)$$

exists and is finite a.e.; it is *in fact equal to* $f(\vartheta)$ almost everywhere. *The above theorem now tells us that* $\lim_{w \not\to e^{i\vartheta}} \tilde{U}(w)$ *also exists and is finite a.e., indeed,* *under the present circumstances,* $\tilde{U}(w) = \Im G(w)$. This conclusion is so important that it should be stated as a separate

Theorem. *Let $f \in L_1(-\pi, \pi)$. Then, for almost every φ, the limit of*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{2r\sin(\vartheta - t)}{1 + r^2 - 2r\cos(\vartheta - t)} f(t)\,dt$$

exists and is finite for $re^{i\vartheta} \xrightarrow{\angle} e^{i\varphi}$

Notation. The non-tangential limit in question is frequently denoted by $\tilde{f}(\varphi)$; for obvious reasons we often call $\tilde{f}$ the *harmonic conjugate* of f. It is also called the *Hilbert transform* of f. We will come back to the consideration of $\tilde{f}$ later on in this chapter.

G. Return to the subject of §E

1. Functions without zeros in $\Im z > 0$

Theorem. *Let $f(z)$ be analytic in $\Im z > 0$ and at the points of the real axis. Suppose that*

$$\log|f(z)| \leqslant O(|z|)$$

for $\Im z \geqslant 0$ and $|z|$ large, and that

$$\int_{-\infty}^{\infty} \frac{\log^+|f(x)|}{1 + x^2}\,dx \; < \; \infty.$$

Then, if $f(z)$ has no zeros in $\Im z > 0$,

$$\log|f(z)| = A\Im z + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log|f(t)|}{|z - t|^2}\,dt$$

there, where

$$A = \limsup_{y \to \infty} \frac{\log|f(iy)|}{y}.$$

Remark. $f(z)$ is *allowed* to have zeros *on* $\mathbb{R}$.

Proof of theorem. With

$$U(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log^+|f(t)|}{|z - t|^2}\,dt$$

we have by §E

$$\log|f(z)| - A\Im z - U(z) \; \leqslant \; 0$$

for $\Im z > 0$. Since $f(z)$ has no zeros in $\Im z > 0$, $v(z) = \log|f(z)| - A\Im z - U(z)$ is *harmonic* there, and we have, by §F.1,

$$v(z) = -\alpha\Im z - \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2}\,\mathrm{d}\mu(t)$$

for $\Im z > 0$, where $\alpha \geqslant 0$ and μ is a positive measure on $\mathbb{R}$.

We use the remark at the end of §F.1 to obtain the description of μ. According to that remark, if $\Phi(t)$ is *continuous and of compact support,*

$$-\int_{-\infty}^{\infty}\Phi(t)\,\mathrm{d}\mu(t) = \lim_{y\to 0+}\int_{-\infty}^{\infty}\Phi(t)v(t+\mathrm{i}y)\,\mathrm{d}t.$$

In view of the formula for $U(z)$, we also have

$$\int_{-\infty}^{\infty}\Phi(t)\log^+|f(t)|\,\mathrm{d}t = \lim_{y\to 0+}\int_{-\infty}^{\infty}\Phi(t)U(t+\mathrm{i}y)\,\mathrm{d}t.$$

Therefore

$$\int_{-\infty}^{\infty}\Phi(t)(\log^+|f(t)|\,\mathrm{d}t - \mathrm{d}\mu(t))$$

$$= \lim_{y\to 0+}\int_{-\infty}^{\infty}\Phi(t)(U(t+\mathrm{i}y) + v(t+\mathrm{i}y))\,\mathrm{d}t$$

$$= \lim_{y\to 0+}\int_{-\infty}^{\infty}\Phi(t)(\log|f(t+\mathrm{i}y)| - Ay)\,\mathrm{d}t.$$

Under the hypothesis of the present theorem (analyticity *up to* and *on* $\mathbb{R}$), the last limit is just

$$\int_{-\infty}^{\infty}\Phi(t)\log|f(t)|\,\mathrm{d}t.$$

Indeed, we easily verify directly (using dominated convergence) that

$$\int_J \big|\log|f(t+\mathrm{i}y)| - \log|f(t)|\big|\,\mathrm{d}t \longrightarrow 0$$

as $y \to 0$ for any finite interval J on $\mathbb{R}$. (The argument is essentially the same as that used in the proof of Jensen's formula, Chapter I.) We thus have

$$\int_{-\infty}^{\infty}\Phi(t)(\log^+|f(t)|\,\mathrm{d}t - \mathrm{d}\mu(t)) = \int_{-\infty}^{\infty}\Phi(t)\log|f(t)|\,\mathrm{d}t$$

for each continuous function Φ of compact support, and hence

$$\log^+|f(t)|\,\mathrm{d}t - \mathrm{d}\mu(t) = \log|f(t)|\,\mathrm{d}t.$$

Therefore, for $\Im z > 0$,

$$\begin{aligned}\log|f(z)| &= A\Im z + U(z) + v(z) \\ &= (A-\alpha)\Im z + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log|f(t)|}{|z-t|^2}\,\mathrm{d}t,\end{aligned}$$

by the formulas for $U(z)$ and $v(z)$.

In order to complete the proof, we must show that $\alpha = 0$. To see this, recall that by §F.1 the positive measure μ introduced above satisfies

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}\mu(t)}{1+t^2} < \infty.$$

(We are already tacitly using this property – without it the formulas for $v(z)$ and especially for $\log|f(z)|$ make no sense!) Therefore, by the evaluation of $\mathrm{d}\mu(t)$ just made,

$$(*) \qquad \int_{-\infty}^{\infty} \frac{|\log|f(t)||}{1+t^2}\,\mathrm{d}t \;<\; \infty.$$

The formula for $\log|f(z)|$ just obtained now yields

$$\frac{\log|f(\mathrm{i}y)|}{y} = A - \alpha + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log|f(t)|}{t^2+y^2}\,\mathrm{d}t.$$

Making $y \to \infty$, we see from $(*)$ that $\log|f(\mathrm{i}y)|/y \to A - \alpha$. Since we called

$$A = \limsup_{y\to\infty} \frac{\log|f(\mathrm{i}y)|}{y},$$

we have $\alpha = 0$, and the theorem is proved.

Remark. Under the conditions of the present theorem, we see that $\limsup_{y\to\infty}(\log|f(\mathrm{i}y)|/y)$ is actually a *limit*.

2. Convergence of $\int_{-\infty}^{\infty}(\log^{-}|f(x)|/(1+x^2))\,\mathrm{d}x$

We are going to extend the work of the previous article to functions $f(z)$ *having* zeros in $\Im z > 0$. For this purpose, we need some preparatory material.

Lemma. *Let $S(w)$ be positive and superharmonic in $\{|w| < 1\}$ and suppose that $\lim_{r\to 1} S(re^{i\tau}) = S(e^{i\tau})$ exists a.e. Then, if $|w| < 1$,*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-|w|^2}{|w-e^{i\tau}|^2}\, S(e^{i\tau})\,\mathrm{d}\tau \leqslant S(w).$$

Remark. The assumption on the a.e. existence of the radial limit $S(e^{i\tau})$ is superfluous. This is a consequence of a difficult theorem of Littlewood, which can be found in the books of Tsuji and Garnett. In our applications, this existence will, however, be manifest, so we may as well require it in the hypothesis of the lemma.

Proof of lemma. Let $|w_0| < 1$. By superharmonicity (mean value property) $\liminf_{w \to w_0} S(w) \leqslant S(w_0)$. If r_n increases towards 1 we can therefore find a sequence $\{w_n\}$ with $|w_n| < 1$ and $w_n \xrightarrow[n]{} w_0$ such that $\liminf_{n \to \infty} S(r_n w_n) \leqslant S(w_0)$.

By superharmonicity of $S(w)$ for $|w| < 1$ we have, for each $r_n < 1$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-|w_n|^2}{|w_n - e^{i\tau}|^2} S(r_n e^{i\tau})\, d\tau \leqslant S(r_n w_n).$$

Also, $((1-|w_n|^2)/|w_n - e^{i\tau}|^2) S(r_n e^{i\tau})$ is $\geqslant 0$ and tends to $((1-|w_0|^2)/|w_0 - e^{i\tau}|^2) S(e^{i\tau})$ for almost all τ as $n \to \infty$ (separate convergence of the two factors!). Therefore, by Fatou's lemma,

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-|w_0|^2}{|w_0 - e^{i\tau}|^2} S(e^{i\tau})\, d\tau \\ &\leqslant \liminf_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-|w_n|^2}{|w_n - e^{i\tau}|^2} S(r_n e^{i\tau})\, d\tau \\ &\leqslant \liminf_{n\to\infty} S(r_n w_n) \leqslant S(w_0), \end{aligned}$$

and we are done.

Lemma. *If $v(z)$ is subharmonic and $\leqslant 0$ in $\Im z > 0$ and $v(t) = \lim_{z \angle t} v(z)$ exists a.e. on $\mathbb{R}$, then, for $\Im z > 0$,*

$$v(z) \;\leqslant\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} v(t)\, dt.$$

Proof. Apply the previous lemma to $S(w)$, given by the formula

$$S\left(\frac{i-z}{i+z}\right) = -v(z),$$

and use the calculation made to establish the second theorem of §F.1.

From this lemma we have first of all the very important and much used

Theorem. *Let $f(z)$ be analytic for $\Im z > 0$ and continuous up to $\mathbb{R}$. Suppose also that* $\log|f(z)| \leqslant O(|z|)$ *for $|z|$ large, $\Im z \geqslant 0$, and that*

$$\int_{-\infty}^{\infty} \frac{\log^+|f(x)|}{1+x^2} dx \;<\; \infty.$$

Then, unless $f(z) \equiv 0$,

$$\boxed{\int_{-\infty}^{\infty} \frac{\log^-|f(x)|}{1+x^2}\mathrm{d}x \;<\; \infty.}$$

Proof. Without loss of generality, $f(\mathrm{i}) \neq 0$; otherwise work with

$$\left(\frac{z+\mathrm{i}}{z-\mathrm{i}}\right)^k f(z)$$

instead of $f(z)$ if f should have a zero of multiplicity k at i. By the theorem of §E, if we write $A = \limsup_{y\to\infty}(\log|f(\mathrm{i}y)|/y)$, the function

$$v(z) = \log|f(z)| - A\Im z - \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log^+|f(t)|}{|z-t|^2}\mathrm{d}t$$

is $\leqslant 0$ for $\Im z > 0$. It is not, however, *harmonic* there as in the previous subsection, but merely *subharmonic.*

For $z \to t \in \mathbb{R}$, the right-hand integral in the previous formula tends to $\log^+|f(t)|$, since $\log^+|f(x)|$ is continuous on $\mathbb{R}$. Therefore, when $z \to t$,

$$v(z) \to \log|f(t)| - \log^+|f(t)| = -\log^-|f(t)|.$$

We may now apply the preceding lemma with $v(t) = -\log^-|f(t)|$. Since $f(\mathrm{i}) \neq 0$, we find that

$$-\infty \;<\; v(\mathrm{i}) \;<\; -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log^-|f(t)|}{1+t^2}\mathrm{d}t.$$

We are done.

Theorem. *Under the hypothesis of the preceding result,*

$$\boxed{\log|f(z)| \leqslant A\Im z + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log|f(t)|}{|z-t|^2}\mathrm{d}t}$$

for $\Im z > 0$, *where*

$$A = \limsup_{y\to\infty} \frac{\log|f(\mathrm{i}y)|}{y},$$

the integral on the right being absolutely convergent.

Remark. This is an improvement of the result in §E, where we have $\log^+|f(t)| \geqslant 0$ instead of the (signed) quantity $\log|f(t)|$ in the right-hand integral.

Proof. Taking $v(z)$ as in the proof of the preceding theorem we have, by the discussion there,

$$v(z) \leqslant -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log^-|f(t)|}{|z-t|^2}\,dt,$$

$\Im z > 0$, according to the above lemma. Adding

$$A\Im z + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log^+|f(t)|}{|z-t|^2}\,dt$$

to both sides of this inequality gives the desired result.

3. Taking the zeros in $\Im z > 0$ into account. Use of Blaschke products

Theorem. *Let $f(z)$ be analytic in $\{\Im z > 0\}$ and continuous up to $\mathbb{R}$, and suppose that $\log|f(z)| \leqslant O(|z|)$ for $|z|$ large, $\Im z > 0$, and that*

$$\int_{-\infty}^{\infty} \frac{\log^+|f(x)|}{1+x^2}\,dx < \infty.$$

Assume also that $f(0) \neq 0$.

Denote by $\{z_n\}$ the sequence of zeros of $f(z)$ in $\Im z > 0$ (with repetitions according to multiplicities). Then

$$\sum_n \frac{\Im z_n}{|z_n|^2} < \infty.$$

Remark. The requirement that $f(0) \neq 0$ is *essential.*

Proof of theorem. Since $f(0) \neq 0$ and $f(z)$ is continuous up to $\mathbb{R}$, the z_n cannot accumulate at 0, i.e., $|z_n| > c$ for some $c > 0$. We may, wlog, take $c = 3$, for, if c is smaller than 3, we can work with $f((3/c)z)$ instead of $f(z)$.

The integral

$$\frac{i}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{z-t}+\frac{t}{t^2+1}\right)\log^+|f(t)|\,dt$$

is absolutely convergent for $\Im z > 0$ because of our condition

$$\int_{-\infty}^{\infty}\frac{\log^+|f(t)|}{1+t^2}\,dt \;<\; \infty;$$

it therefore represents a function *analytic* in that half plane whose *real part* is none other than

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2}\log^+|f(t)|\,dt.$$

From this observation and the result of §E we see that

$$g(z) \;=\; \frac{e^{iAz}f(z)}{\exp\left\{\dfrac{i}{\pi}\displaystyle\int_{-\infty}^{\infty}\left(\dfrac{1}{z-t}+\dfrac{t}{t^2+1}\right)\log^+|f(t)|\,dt\right\}}$$

is *analytic and in modulus* $\leqslant 1$ for $\Im z > 0$, where the constant A is defined in the usual fashion. We have $g(i) \neq 0$, since (here) all the $|z_n|$ are > 3. For each N, apply the principle of maximum to

$$g(z)\bigg/\prod_1^N\left(\frac{z-z_n}{z-\bar{z}_n}\right)$$

in $\Im z > 0$. Since $|g(z)| \leqslant 1$ there, we find

$$0 \;<\; |g(i)| \;\leqslant\; \prod_1^N\left|\frac{i-z_n}{i-\bar{z}_n}\right|.$$

For each n, we have

$$\left|\frac{i-z_n}{i-\bar{z}_n}\right| \;=\; \left|\frac{1-i/z_n}{1-i/\bar{z}_n}\right| \;=\; \left|1-\frac{2}{1-i/\bar{z}_n}\cdot\frac{\Im z_n}{|z_n|^2}\right|.$$

Here, however, $|z_n| > 3$, so the last expression is certainly

$$\leqslant\; 1-\frac{2}{1+1/|z_n|}\cdot\frac{\Im z_n}{|z_n|^2},$$

as is evident if we look at the image of the circle $|\omega| = 1/|z_n|$ $(<\frac{1}{3})$ under the linear fractional transformation

$$\omega \longrightarrow \frac{2}{1+\omega}\frac{\Im z_n}{|z_n|^2}:$$

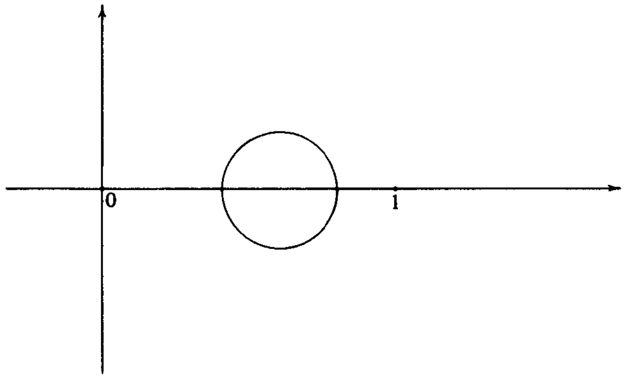

Figure 11

Substituting into the previous inequality and taking logarithms, we get

$$-\infty \ < \ \log|g(\mathrm{i})| \ \leqslant \ \sum_1^N \log\left(1 - \frac{2|z_n|}{|z_n|+1}\cdot\frac{\Im z_n}{|z_n|^2}\right)$$

and this is in turn

$$\leqslant \ -\sum_1^N \frac{2|z_n|}{|z_n|+1}\,\frac{\Im z_n}{|z_n|^2} \ \leqslant \ -\frac{3}{2}\sum_1^N \frac{\Im z_n}{|z_n|^2}$$

since $|z_n| > 3$.

We thus have

$$\sum_1^N \frac{\Im z_n}{|z_n|^2} \ \leqslant \ \frac{2}{3}\log\frac{1}{|g(\mathrm{i})|} \ < \ \infty$$

for all N, and our theorem follows on making $N \to \infty$.

Theorem. *Let $f(z)$ be an entire function of exponential type with*

$$\int_{-\infty}^{\infty} \frac{\log^+|f(x)|}{1+x^2}\,\mathrm{d}x \ < \ \infty.$$

There is an entire function $g(z)$ of exponential type with no zeros in $\Im z > 0$ and $|g(x)| = |f(x)|$ for $-\infty < x < \infty$.

Proof. Let $\{\lambda_n\}$ denote the set of zeros of $f(z)$ in $\Im z > 0$, and $\{\mu_n\}$ *all the other zeros* of $f(z)$ (repetitions according to multiplicities, as usual). Wlog, $f(0) \neq 0$, otherwise work with $f(z)/z^k$ instead of $f(z)$. The Hadamard factorization of $f(z)$ can then be written

$$f(z) = Ce^{bz}\lim_{R\to\infty}\left\{\prod_{|\lambda_n|\leqslant R}\left(1-\frac{z}{\lambda_n}\right)e^{z/\lambda_n}\cdot\prod_{|\mu_n|\leqslant R}\left(1-\frac{z}{\mu_n}\right)e^{z/\mu_n}\right\}.$$

By the previous theorem, the sums $\sum_{|\lambda_n| \leqslant R} (\Im \lambda_n / |\lambda_n|^2)$, and by Lindelöf's theorem (§B), the sums

$$\sum_{|\lambda_n| \leqslant R} \frac{1}{\lambda_n} + \sum_{|\mu_n| \leqslant R} \frac{1}{\mu_n}$$

are bounded in absolute value.

The sums

$$\sum_{|\lambda_n| \leqslant R} \frac{1}{\bar{\lambda}_n} + \sum_{|\mu_n| \leqslant R} \frac{1}{\mu_n}$$

are therefore also bounded in absolute value, so, by the (easy) converse of Lindelöf's theorem, the products

$$C e^{bz} \prod_{|\lambda_n| \leqslant R} \left(1 - \frac{z}{\bar{\lambda}_n}\right) e^{z/\bar{\lambda}_n} \cdot \prod_{|\mu_n| \leqslant R} \left(1 - \frac{z}{\mu_n}\right) e^{z/\mu_n}$$

converge u.c.c. in the complex plane to an *entire function $g(z)$ of exponential type* as $R \to \infty$.

For real x,

$$\frac{g(x)}{f(x)} = \lim_{R \to \infty} \prod_{|\lambda_n| \leqslant R} \left(\frac{1 - x/\bar{\lambda}_n}{1 - x/\lambda_n}\right) \exp\left(2i \frac{\Im \lambda_n}{|\lambda_n|^2} x\right),$$

the product being u.c.c. convergent on $\mathbb{R}$ in view of the condition

$$\sum_n \frac{\Im \lambda_n}{|\lambda_n|^2} < \infty.$$

The right side of the above expression is clearly of modulus 1 on $\mathbb{R}$, so $|f(x)| = |g(x)|$ there. And $g(z)$ has no zeros in $\Im z > 0$.

Theorem (Riesz–Fejér). *Let $F(z)$ be entire and of exponential type, let $F(x) \geqslant 0$ on $\mathbb{R}$, and suppose that*

$$\int_{-\infty}^{\infty} \frac{\log^+ F(x)}{1 + x^2} \mathrm{d}x < \infty.$$

Then there is an entire function $f(z)$ of exponential type without zeros in $\Im z > 0$ such that $F(z) = f(z) \cdot \overline{f(\bar{z})}$. In particular, $F(x) = |f(x)|^2$, $x \in \mathbb{R}$.

Proof. Since $F(x)$ is *real* on $\mathbb{R}$, $F(\bar{z}) = \overline{F(z)}$ by the Schwarz reflection principle, so, if $\Im \lambda > 0$, λ is a *zero* of $F(z)$ iff $\bar{\lambda}$ is also a zero thereof, with the same multiplicity. Because $F(x) \geqslant 0$ on $\mathbb{R}$, any *real* zero of F must have *even* multiplicity.

Denote by $\{\lambda_n\}$ the set of zeros of $F(z)$ in $\Im z > 0$ (repetitions according to multiplicities); then, by the observations just made, the Hadamard factorization of F must take the form

$$F(z) = Ce^{bz} \lim_{R\to\infty} \left\{ \prod_{|\lambda_n|\leqslant R} \left(1 - \frac{z}{\lambda_n}\right) e^{z/\lambda_n} \cdot \prod_{|\lambda_n|\leqslant R} \left(1 - \frac{z}{\bar{\lambda}_n}\right) e^{z/\bar{\lambda}_n} \right.$$
$$\left. \cdot \prod_{|\alpha_n|\leqslant R} \left(1 - \frac{z}{\alpha_n}\right)^2 e^{2z/\alpha_n} \right\}.*$$

Here, the α_n are certain *real* numbers corresponding to the possible real zeros of $F(z)$ – of course, there may not really *be* any α_n. Using the fact that $F(x) \geqslant 0$ we easily check that $C \geqslant 0$ and that b is *real*.

As in the proof of the preceding theorem, we have $\sum_n \Im\lambda_n/|\lambda_n|^2 < \infty$, and this, together with the Lindelöf theorems of §B, implies that the products

$$\sqrt{C}e^{bz/2} \prod_{|\lambda_n|\leqslant R} \left(1 - \frac{z}{\bar{\lambda}_n}\right) e^{z/\bar{\lambda}_n} \prod_{|\alpha_n|\leqslant R} \left(1 - \frac{z}{\alpha_n}\right) e^{z/\alpha_n}$$

converge u.c.c. in $\mathbb{C}$ to a certain entire function $f(z)$ of *exponential type* as $R \to \infty$. We clearly have

$$F(z) = f(z)\overline{f(\bar{z})},$$

and $f(z)$ has no zeros in $\Im z > 0$. As required.

Theorem. *Let $f(z)$ be entire and of exponential type and suppose that*

$$\int_{-\infty}^{\infty} \frac{\log^+|f(x)|}{1+x^2} dx < \infty.$$

Denote by $\{\lambda_n\}$ the set of zeros of $f(z)$ in $\Im z > 0$ (repetitions according to multiplicities), and put

$$A = \limsup_{y\to\infty} \frac{\log|f(iy)|}{y}.$$

Then, for $\Im z > 0$,

$$\boxed{\log|f(z)| = A\Im z + \sum_n \log\left|\frac{1 - z/\lambda_n}{1 - z/\bar{\lambda}_n}\right| + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} \log|f(t)|\,dt.}$$

Proof. With the entire function $g(z)$ used in proving the theorem before the last one, put

$$G(z) = g(z)\cdot\exp\left\{-2i\sum_n \frac{\Im\lambda_n}{|\lambda_n|^2} z\right\}.$$

* As long as $F(0) \neq 0$. Otherwise, an additional factor z^{2k} appears on the right, and the description of $f(z)$ following in text must be modified accordingly.

We have, of course,

$$(*)\qquad \sum_n \frac{\Im \lambda_n}{|\lambda_n|^2} < \infty,$$

so $G(z)$ is an entire function of exponential type because $g(z)$ is, and, for $x \in \mathbb{R}$, $|G(x)| = |g(x)| = |f(x)|$. $G(z)$ is without zeros for $\Im z > 0$, and, in view of the description of $g(z)$ given where it was introduced,

$$G(z) = f(z) \cdot \lim_{R \to \infty} \prod_{|\lambda_n| \leqslant R} \left(\frac{1 - z/\bar{\lambda}_n}{1 - z/\lambda_n} \right).$$

Using $(*)$ and the fact that $\lambda_n \xrightarrow[n]{} \infty$, we readily verify directly that the infinite product

$$\prod_n \left(\frac{1 - z/\lambda_n}{1 - z/\bar{\lambda}_n} \right)$$

is u.c.c. convergent for $\Im z \geqslant 0$; in the upper half plane, $G(z)$ evidently equals $f(z)$ *divided* by this infinite product.

For $\Im z > 0$, $|(1 - z/\bar{\lambda}_n)/(1 - z/\lambda_n)| > 1$, therefore $|G(z)| > |f(z)|$. Hence, if we call

$$A' = \limsup_{y \to \infty} \frac{\log|G(iy)|}{y},$$

we have

$$A' \geqslant \limsup_{y \to \infty} \frac{\log|f(iy)|}{y} = A.$$

Apply now the theorem of article 1 to $G(z)$, which has no zeros in $\Im z > 0$. Since $|G(t)| = |f(t)|$ for real t, we find that

$$\log|G(z)| = A'\Im z + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\Im z}{|z - t|^2} \log|f(t)| \mathrm{d}t$$

for $\Im z > 0$. In view of the relation between $G(z)$ and $f(z)$, this yields

$$\log|f(z)| = A'\Im z + \sum_n \log \left| \frac{1 - z/\lambda_n}{1 - z/\bar{\lambda}_n} \right|$$
$$+ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\Im z}{|z - t|^2} \log|f(t)| \mathrm{d}t, \quad \Im z > 0.$$

We will be done when we show that $A' = A$. Indeed, we have already seen that $A' \geqslant A$ so it is only necessary to prove that $A' \leqslant A$.

To this end, consider the functions

$$G_N(z) = f(z) \Big/ \prod_1^N \left(\frac{1 - z/\lambda_n}{1 - z/\bar{\lambda}_n} \right).$$

For each fixed N, $|G_N(x)| = |f(x)|$ on $\mathbb{R}$, and $G_N(iy)/f(iy) \to 1$ for $y \to \infty$, since the product on the right has only a finite number of factors. Because of this,

$$\limsup_{y\to\infty} \frac{\log|G_N(iy)|}{y} = \limsup_{y\to\infty} \frac{\log|f(iy)|}{y} = A,$$

whence, *by the second theorem of article* 2,

$$\log|G_N(z)| \leqslant A\Im z + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log|f(t)|\mathrm{d}t$$

for $\Im z > 0$. Make now $N \to \infty$; then $G_N(z) \xrightarrow[N]{} G(z)$ u.c.c. in $\Im z > 0$, so finally

$$\log|G(z)| \leqslant A\Im z + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log|f(t)|\mathrm{d}t$$

there. However, the *left* side *equals*

$$A'\Im z + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log|f(t)|\mathrm{d}t.$$

Therefore $A' \leqslant A$, whence finally $A' = A$, and the theorem is proved.

Remark. The expression

$$\prod_n \left(\frac{1 - z/\lambda_n}{1 - z/\bar{\lambda}_n}\right)$$

is called a *Blaschke product.*

Problem 4

Let φ and $\psi \in L_1(\mathbb{R})$ be functions of compact support. Let $[a,b]$ be the *supporting interval* for φ (that's the *smallest* closed interval *outside* of which $\varphi \equiv 0$ a.e.), and denote by $[a',b']$ the supporting interval for ψ. *Prove* that the supporting interval for $\varphi * \psi$ is precisely $[a+a', b+b']$. (Note: By $\varphi * \psi$ we mean the *convolution*

$$(\varphi * \psi)(\lambda) = \int_{-\infty}^{\infty} \varphi(\lambda - \tau)\psi(\tau)\mathrm{d}\tau.)$$

H. Levinson's theorem on the density of the zeros

We are going to close this chapter by proving a version of Levinson's theorem on the distribution of the zeros of entire functions of exponential type for such functions f which also satisfy the condition

$$\int_{-\infty}^{\infty} \frac{\log^+|f(x)|}{1+x^2}\mathrm{d}x \;<\; \infty.$$

This version is in fact due to Miss Cartwright and, although sufficient for most applications in analysis, is not the most general form of Levinson's theorem. For the latter one should consult the books by Boas or Levin, or, for that matter, the one by Levinson himself.

The proof to be given here depends on Kolmogorov's theorem on the *harmonic conjugates* of integrable functions, so we turn first to the establishment of that result.

1. Kolmogorov's theorem on the harmonic conjugate

Let $f(z)$ be analytic and *bounded* for $\Im z > 0$. An obvious application of a result of Fatou (the corollary in §F.2) shows that the boundary value

$$f(t) = \lim_{y \to 0+} f(t + iy)$$

exists for almost every real t. *In the application of the following lemma to be given below, this fact can also be verified directly*; the reader interested in economy of thought may therefore include it in the hypothesis if he or she wants to.

Lemma. *Let $f(z)$ be bounded and analytic in $\Im z > 0$. Then*

$$f(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} f(t)\,dt$$

there.

Proof. If z lies inside the contour Γ shown below, we have, by Cauchy's theorem,

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)\,d\zeta}{\zeta - z}.$$

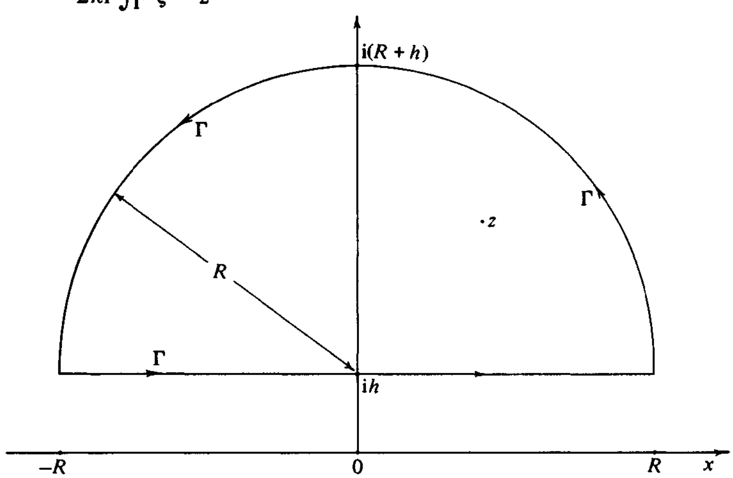

Figure 12

We fix R and make $h \to 0$ through some sequence of values; since $|f(t+\mathrm{i}h)| \leqslant C$ say and $f(t+\mathrm{i}h) \to f(t)$ a.e. for $h \to 0$, we have, by dominated convergence,

$$\int_{-R}^{R} \frac{f(t+\mathrm{i}h)}{t+\mathrm{i}h-z}\,\mathrm{d}t \longrightarrow \int_{-R}^{R} \frac{f(t)}{t-z}\,\mathrm{d}t.$$

Similarly,

$$\int_0^{\pi} \frac{f(\mathrm{i}h+R\mathrm{e}^{\mathrm{i}\vartheta})}{\mathrm{i}h+R\mathrm{e}^{\mathrm{i}\vartheta}-z}\,\mathrm{i}R\mathrm{e}^{\mathrm{i}\vartheta}\,\mathrm{d}\vartheta \longrightarrow \int_0^{\pi} \frac{f(R\mathrm{e}^{\mathrm{i}\vartheta})}{R\mathrm{e}^{\mathrm{i}\vartheta}-z}\,\mathrm{i}R\mathrm{e}^{\mathrm{i}\vartheta}\,\mathrm{d}\vartheta$$

as $h \to 0$. We thus see that

$$(*) \qquad f(z) = \frac{1}{2\pi\mathrm{i}} \int_{-R}^{R} \frac{f(t)}{t-z}\,\mathrm{d}t + \frac{1}{2\pi} \int_0^{\pi} \frac{f(R\mathrm{e}^{\mathrm{i}\vartheta})}{R\mathrm{e}^{\mathrm{i}\vartheta}-z}\,R\mathrm{e}^{\mathrm{i}\vartheta}\,\mathrm{d}\vartheta.$$

Taking the relation

$$0 = \frac{1}{2\pi\mathrm{i}} \int_{\Gamma} \frac{f(\zeta)}{\zeta-\bar{z}}\,\mathrm{d}\zeta$$

and making $h \to 0$, we see in the same way that

$$0 = \frac{1}{2\pi\mathrm{i}} \int_{-R}^{R} \frac{f(t)\mathrm{d}t}{t-\bar{z}} + \frac{1}{2\pi} \int_0^{\pi} \frac{f(R\mathrm{e}^{\mathrm{i}\vartheta})}{R\mathrm{e}^{\mathrm{i}\vartheta}-\bar{z}}\,R\mathrm{e}^{\mathrm{i}\vartheta}\,\mathrm{d}\vartheta.$$

Subtract this equation from $(*)$. We get

$$f(z) = \frac{1}{\pi} \int_{-R}^{R} \frac{\Im z}{|t-z|^2} f(t)\,\mathrm{d}t + \frac{\mathrm{i}}{\pi} \int_0^{\pi} \frac{\Im z\, f(R\mathrm{e}^{\mathrm{i}\vartheta})R\mathrm{e}^{\mathrm{i}\vartheta}}{(R\mathrm{e}^{\mathrm{i}\vartheta}-z)(R\mathrm{e}^{\mathrm{i}\vartheta}-\bar{z})}\,\mathrm{d}\vartheta.$$

Since f is bounded, the second term on the right is $O(1/R)$ for large R, so, making $R \to \infty$, we end with

$$f(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} f(t)\,\mathrm{d}t. \qquad \text{Q.E.D.}$$

Scholium. The reader is invited to obtain the lemma from the second theorem of §F.1 and the remark thereto (on representation of positive harmonic functions in $\Im z > 0$).

Suppose now that $u(t)$ is real valued and that

$$\int_{-\infty}^{\infty} \frac{|u(t)|}{1+t^2}\,\mathrm{d}t \; < \; \infty.$$

Then the integral

$$\frac{\mathrm{i}}{\pi} \int_{-\infty}^{\infty} \left(\frac{1}{z-t} + \frac{t}{t^2+1} \right) u(t)\,\mathrm{d}t$$

converges absolutely for $\Im z > 0$ and equals an analytic function of z – call it $F(z)$ – there. $(\Re F)(z)$ is simply the by now familiar harmonic function

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2}u(t)\mathrm{d}t;$$

$(\Im F)(z)$ is equal to

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\Re z - t}{|z-t|^2}+\frac{t}{t^2+1}\right)u(t)\,\mathrm{d}t.$$

Let us call the former expression $U(z)$ and the latter $\tilde{U}(z)$. Both are real valued and harmonic in $\{\Im z > 0\}$, and the latter is a *harmonic conjugate* of the former for that region since $U(z) + \mathrm{i}\tilde{U}(z)$ is equal to the function $F(z)$, *analytic* therein.

In order to examine the *boundary behaviour* of $U(z)$ and $\tilde{U}(z)$ we may first map $\{\Im z > 0\}$ onto $\{|w| < 1\}$ by taking $w = (\mathrm{i} - z)/(\mathrm{i} + z)$ and then appeal to the results in Chapter II §B and in §F.2 of this chapter. From the first of those §§ we see that one simply has

$$U(t+\mathrm{i}y) \to u(t)$$

at almost every $t \in \mathbb{R}$ when $y \to 0$. The behaviour of $\tilde{U}(t+\mathrm{i}y)$ for $y \to 0$ is less transparent. According to the last theorem and scholium following it in §F.2, $\tilde{U}(t+\mathrm{i}y)$ *must, however, tend to a definite finite limit for almost every* $t \in \mathbb{R}$ as $y \to 0$. It is not very easy to see how that limit is related to our original function u; we get around this difficulty by denoting the limit by $\tilde{u}(t)$ and calling $\tilde{u}$ the *Hilbert transform* (or 'harmonic conjugate') of u.

Under certain circumstances one *can* in fact write a formula for $\tilde{u}(t)$ and verify almost *by inspection* that $\tilde{U}(t+\mathrm{i}y)$ tends to $\tilde{u}(t)$ (as given by the formula) when $y \to 0$. When this happens, *we do not need to use the general result of* §F.2 *to establish existence of* $\lim_{y\to 0}\tilde{U}(t+\mathrm{i}y)$. *That will indeed be the case in the application we make here*; the reader who is merely interested in arriving at Levinson's result may therefore include *existence* of the appropriate Hilbert transforms in the *hypothesis* of Kolmogorov's theorem, to be given below. It *is*, however, *true* that the Hilbert transforms in question *do* always exist a.e.

Here is a situation in which the *existence* of $\lim_{y\to 0}\tilde{U}(t+\mathrm{i}y)$ is *elementary*. Suppose that the integral

$$\int_0^1 \frac{u(x_0-\tau)-u(x_0+\tau)}{\tau}\mathrm{d}\tau$$

is absolutely convergent; this will *certainly* be the case, for instance, if $u(t)$ is

Lip 1 (or even Lip α, $\alpha > 0$) at x_0. Then, if we write

$$\tilde{u}(x_0) = \frac{1}{\pi}\int_0^1 \frac{u(x_0 - \tau) - u(x_0 + \tau)}{\tau}\,\mathrm{d}\tau$$
$$+ \frac{1}{\pi}\int_{x_0 - 1}^{x_0 + 1} \frac{tu(t)}{t^2 + 1}\,\mathrm{d}t + \frac{1}{\pi}\int_{|t - x_0| > 1} \left(\frac{1}{x_0 - t} + \frac{t}{t^2 + 1}\right) u(t)\,\mathrm{d}t,$$

we easily verify that

$$\tilde{U}(x_0 + \mathrm{i}y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \left(\frac{x_0 - t}{(x_0 - t)^2 + y^2} + \frac{t}{t^2 + 1}\right) u(t)\,\mathrm{d}t$$

tends to $\tilde{u}(x_0)$ as $y \to 0$. Just break up the integral on the right into two pieces, one with $x_0 - 1 \leqslant t \leqslant x_0 + 1$ and the other with $|t - x_0| > 1$. The *first* piece is readily seen to tend to the sum of the *first two* right-hand terms in the formula for $\tilde{u}(x_0)$ when $y \to 0$, and the *second* piece tends to the *third* term. In proving Levinson's theorem, the functions $u(t)$ which concern us are of the form $u(t) = \log^+|f(t)|$ or $u(t) = \log^-|f(t)|$ with $f(z)$ *entire* and such that

$$\int_{-\infty}^{\infty} \frac{|\log|f(t)||}{1 + t^2}\,\mathrm{d}t \;<\; \infty.$$

The function $\log^+|f(t)|$ is *certainly* Lip 1 at *every point* of $\mathbb{R}$ – $f(z)$ is *analytic*! And $\log^-|f(t)|$ is Lip 1 at all the points of $\mathbb{R}$ save those *isolated* ones where f has a *zero*. In *either* case, then, $\tilde{u}(x_0)$ is defined by the elementary procedure just described for all $x_0 \in \mathbb{R}$ except those belonging to some *countable* set of isolated points.

Our purpose in dwelling on the above matter at such length has been to explain that the proof of Levinson's theorem to be given below does *not really depend* on deep theorems about the *existence* of the Hilbert transform. The question of that existence is, however, *close enough* to the subject at hand to require our giving it *some* attention. The reader who wants to learn *more* about this question should consult the books of Zygmund or Garnett (*Bounded Analytic Functions*) or my own (on H_p spaces). There is also a beautiful real-variable treatment in Garsia's book on almost everywhere convergence.

Without further ado, let us now give

Kolmogorov's theorem. *Let $u(t)$ be real valued, let*

$$\int_{-\infty}^{\infty} \frac{|u(t)|}{1 + t^2}\,\mathrm{d}t \;<\; \infty,$$

and put

$$\tilde{u}(x) = \lim_{y\to 0+} \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{x-t}{(x-t)^2+y^2}+\frac{t}{t^2+1}\right)u(t)\mathrm{d}t,$$

the limit on the right existing a.e. Then, if $\lambda > 0$,

$$\boxed{\int_{\{|\tilde{u}(t)|>\lambda\}}\frac{\mathrm{d}t}{t^2+1} \leqslant \frac{4}{\lambda}\int_{-\infty}^{\infty}\frac{|u(t)|}{t^2+1}\mathrm{d}t.}$$

Proof. Consider first the *special case where* $u(t) \geqslant 0$; this is actually where most of the work has to be done. The following argument was first published by Katznelson, and is due to Carleson.

Wlog,

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(t)}{1+t^2}\mathrm{d}t = 1.$$

Put

$$F(z) = \frac{\mathrm{i}}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{z-t}+\frac{t}{t^2+1}\right)u(t)\,\mathrm{d}t;$$

this function is analytic in $\Im z > 0$ *and has positive real part there.* Also, $F(\mathrm{i}) = 1$. For almost all $t\in\mathbb{R}$,

$$\Re F(t+\mathrm{i}y) \longrightarrow u(t)$$

and

$$\Im F(t+\mathrm{i}y) \longrightarrow \tilde{u}(t)$$

as $y \to 0+$.

Fix now any $\lambda > 0$, and take

$$f(z) = 1 + \frac{F(z)-\lambda}{F(z)+\lambda}$$

for $\Im z > 0$; $f(z)$ is analytic there and has modulus at most 2. For almost every $t\in\mathbb{R}$, $f(t) = \lim_{y\to 0+} f(t+\mathrm{i}y)$ exists, and can be expressed in terms of $u(t) + \mathrm{i}\tilde{u}(t)$.

By the lemma, for $\Im z > 0$,

$$f(z) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2}f(t)\mathrm{d}t,$$

therefore

$$(*)\qquad \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Re f(t)}{1+t^2}\mathrm{d}t \;=\; \Re f(\mathrm{i}) \;=\; \Re\left(1+\frac{1-\lambda}{1+\lambda}\right) \;=\; \frac{2}{1+\lambda}.$$

The transformation $F \to f = 1 + (F-\lambda)/(F+\lambda)$ makes the half-plane $\Re F > 0$ correspond to the circle $|f-1| < 1$ *and takes the two portions of the imaginary axis where* $|F| > \lambda$ *onto the right half of the circle* $|f-1| = 1$.

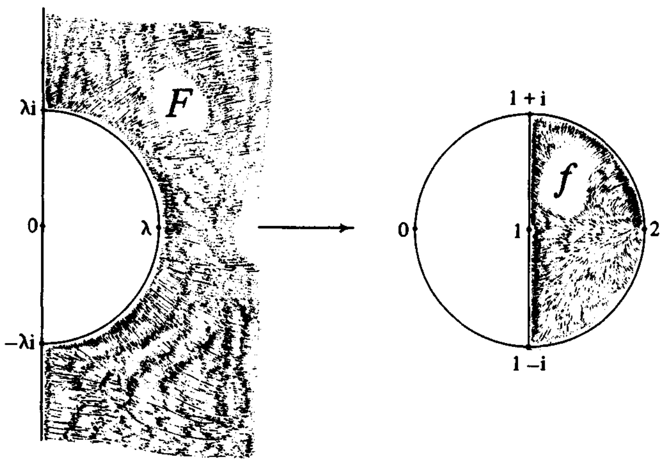

Figure 13

From the picture we therefore see that $\Re f(t) \geqslant 1$ whenever $|F(t)| = |u(t) + \mathrm{i}\tilde{u}(t)| \geqslant \lambda$, hence, *surely whenever* $|\tilde{u}(t)| \geqslant \lambda$. Since we always have $\Re f(t) \geqslant 0$, we get from $(*)$,

$$\frac{1}{\pi}\int_{\{|\tilde{u}(t)|\geqslant\lambda\}}\frac{\mathrm{d}t}{1+t^2} \;\leqslant\; \frac{2}{1+\lambda} \;<\; \frac{2}{\lambda}\cdot 1 \;=\; \frac{2}{\pi\lambda}\int_{-\infty}^{\infty}\frac{u(t)\mathrm{d}t}{1+t^2},$$

since we assumed that the *integral alone* is equal to π. By homogeneity, we therefore get

$$\int_{\{|\tilde{u}(t)|\geqslant\lambda\}}\frac{\mathrm{d}t}{1+t^2} \;\leqslant\; \frac{2}{\lambda}\int_{-\infty}^{\infty}\frac{u(t)\mathrm{d}t}{1+t^2}$$

for the case where $u(t) \geqslant 0$.

In the general case of real u, write $u(t) = u_+(t) - u_-(t)$ and observe that $\tilde{u}(t) = \tilde{u}_+(t) - \tilde{u}_-(t)$, whence

$$\{t: |\tilde{u}(t)| \geqslant \lambda\} \subseteq \{t: |\tilde{u}_+(t)| \geqslant \lambda/2\} \cup \{t: |\tilde{u}_-(t)| \geqslant \lambda/2\}.$$

The inequality just established may be applied to each of the functions u_+, u_- and we obtain the desired result by adding, since

$$\int_{-\infty}^{\infty} \frac{u_+(t)}{1+t^2}\mathrm{d}t + \int_{-\infty}^{\infty} \frac{u_-(t)}{1+t^2}\mathrm{d}t = \int_{-\infty}^{\infty} \frac{|u(t)|\mathrm{d}t}{1+t^2}.$$

Kolmogorov's theorem is proved.

When $\lambda \to \infty$, the *right-hand side* of the inequality furnished by Kolmogorov's theorem may be replaced by $\mathrm{o}(1/\lambda)$.

Corollary. *Let $u(t)$ be real-valued and $\int_{-\infty}^{\infty}(|u(t)|/(1+t^2))\mathrm{d}t < \infty$. Then*

$$\int_{\{|\tilde{u}(t)|>\lambda\}} \frac{\mathrm{d}t}{1+t^2} = \mathrm{o}\left(\frac{1}{\lambda}\right) \quad \textit{for } \lambda \to \infty.$$

Proof. Take any $\varepsilon > 0$. We can find a *continuously differentiable function φ of compact support with*

$$\int_{-\infty}^{\infty} \frac{|u(t)-\varphi(t)|}{1+t^2}\mathrm{d}t < \varepsilon.$$

Referring to the discussion preceding the statement of the above theorem, we see that $\tilde{\varphi}$ can be readily computed; in fact

$$\tilde{\varphi}(x) = \frac{1}{\pi}\int_0^1 \frac{\varphi(x-\tau)-\varphi(x+\tau)}{\tau}\mathrm{d}\tau$$

$$+\frac{1}{\pi}\int_{x-1}^{x+1}\frac{t\varphi(t)\mathrm{d}t}{t^2+1} + \frac{1}{\pi}\int_{|t-x|>1}\left(\frac{1}{x-t}+\frac{t}{t^2+1}\right)\varphi(t)\mathrm{d}t.$$

Because $\varphi'(t)$ is bounded and of compact support, the expression on the right is *bounded*; it is even $\mathrm{O}(1/|x|)$ for large x. So $|\tilde{\varphi}(x)| \leqslant M$, say ($M$, of course, *depends* on φ, hence on ε!), and, if $\lambda > 2M$, the set $\{t\colon |\tilde{u}(t)| > \lambda\}$ is included in $\{t\colon |\tilde{u}(t) - \tilde{\varphi}(t)| > \lambda/2\}$. Applying the theorem to the function $u - \varphi$, we therefore find that

$$\int_{\{|\tilde{u}(t)|>\lambda\}} \frac{\mathrm{d}t}{1+t^2} < \frac{8\varepsilon}{\lambda}$$

for $\lambda > 2M$. ε, however, was arbitrary. We're done.

2. Functions with only real zeros

If we want to study the *distribution of the zeros* of an entire function $f(z)$ of exponential type with

$$\int_{-\infty}^{\infty} \frac{\log^+|f(x)|}{1+x^2}\mathrm{d}x < \infty,$$

and we put

$$\limsup_{y\to\infty}\frac{\log|f(iy)|}{y} = A,$$

$$\limsup_{y\to-\infty}\frac{\log|f(iy)|}{|y|} = A',$$

there is *no loss of generality in assuming that $A = A'$*. The latter situation may always be arrived at by working with

$$e^{i(A-A')z/2}f(z)$$

instead of $f(z)$; here, the new function has the *same* zeros as $f(z)$ and *equals $f(z)$ in modulus on the real axis.*

We begin by looking at such functions f which have only real zeros.

Theorem. *Let $f(z)$ be entire and of exponential type, have only real zeros, and satisfy the condition*

$$\int_{-\infty}^{\infty}\frac{\log^+|f(x)|}{1+x^2}\,dx \;<\; \infty.$$

Suppose that

$$\limsup_{y\to\infty}\frac{\log|f(iy)|}{y} \;=\; \limsup_{y\to-\infty}\frac{\log|f(iy)|}{|y|} \;=\; A.$$

For $t \geqslant 0$, let $\nu(t)$ be the number of zeros of *f on $[0, t]$, and, if $t < 0$, take $\nu(t)$ as* minus *the* number of zeros *of f in $[t, 0)$. (In both cases, multiplicities are counted.) Then $\nu(t)/t \to A/\pi$ for $t\to\infty$ and for $t\to-\infty$.*

Proof. By the theorem of §G.1,

$$(*)\qquad \log|f(z)| = A\Im z + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2}\log|f(t)|\,dt$$

for $\Im z > 0$, and by the same token, for $\Im z < 0$,

$$\log|f(z)| = A|\Im z| + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|\Im z|}{|z-t|^2}\log|f(t)|\,dt.$$

From these two relations, we see that the function $f(z)/\overline{f(\bar{z})}$, *analytic* in $\{\Im z > 0\}$, has constant modulus equal to 1 there. Therefore *in fact* $f(z)/\overline{f(\bar{z})} \equiv \beta$, a constant of modulus 1, for $\Im z > 0$. Making $z\to$ any point x of the real axis where $f(x) \neq 0$, we see that $f(x)/\overline{f(x)} = \beta$. This means that any continuous determination of $\arg f(x)$ on a *zero-free interval* (for f) is *constant* on *that interval.*

Since $f(z) \neq 0$ in $\{\Im z > 0\}$, we can define a (single valued) analytic branch of $\log f(z)$ in that half plane, and then take $\arg f(z)$ as the harmonic function $\Im \log f(z)$ there. For $x \in \mathbb{R}$ such that $f(x) \neq 0$, *define* $\arg f(x)$ as $\lim_{y\to 0+} \arg f(x+iy)$; as we have just seen, this function $\arg f(x)$ is *constant* on each interval of $\mathbb{R}$ where $f(x) \neq 0$. If x *increases* and passes through a zero x_0 of f, $\arg f(x)$ clearly *jumps down* by $\pi \times$ *the multiplicity of the zero* x_0. Therefore

$$\arg f(x) = -\pi\nu(x) + \text{const.}$$

for real x with $f(x) \neq 0$.

From $(*)$ we see that, in $\{\Im z > 0\}$, the harmonic function $\log|f(z)|$ is the *real part* of the *analytic* one

$$-iAz + \frac{i}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{z-t} + \frac{t}{t^2+1}\right)\log|f(t)|\,dt.$$

It is, at the same time, the real part of $\log f(z)$ there. The *imaginary part* of the *latter analytic function* must therefore differ by a *constant* from that of the *former* one in $\{\Im z > 0\}$, and we have

$$\arg f(z) = -A\Re z + \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\Re z - t}{|z-t|^2} + \frac{t}{t^2+1}\right)\log|f(t)|\,dt + \text{const.}$$

there. Taking $z = x + iy$ with x not a zero of f and making $y \to 0$, we obtain

$$\nu(x) = \frac{A}{\pi}x - \frac{1}{\pi^2}\lim_{y\to 0+}\int_{-\infty}^{\infty}\left(\frac{x-t}{(x-t)^2+y^2} + \frac{t}{t^2+1}\right)\log|f(t)|\,dt$$
$$+ \text{const.},$$

in view of the relation between $\arg f(x)$ and $\nu(x)$.

Write

$$\Delta(x) = \lim_{y\to 0+}\frac{1}{\pi^2}\int_{-\infty}^{\infty}\left(\frac{x-t}{(x-t)^2+y^2} + \frac{t}{t^2+1}\right)\log|f(t)|\,dt,$$

so that $\nu(x) = (A/\pi)x - \Delta(x) + \text{const.}$, save perhaps when $f(x) = 0$. (The limit in question certainly exists if $f(x) \neq 0$; see the discussion just preceding Kolmogorov's theorem in the previous article.)

In the course of proving the theorem of §G.1 we showed that the condition

$$\int_{-\infty}^{\infty}\frac{\log^+|f(x)|}{1+x^2}\,dx \;<\; \infty$$

(which is part of our hypothesis) actually *implies* that

$$\int_{-\infty}^{\infty}\frac{|\log|f(t)||}{1+t^2}\,dt \;<\; \infty;$$

we have of course been tacitly using the latter relation all along, since without it, $(*)$ and the formulas following therefrom would not make much sense. We can therefore apply Kolmogorov's theorem, and especially its *corollary*, to $u(t) = \log|f(t)|$. We have $\Delta(x) = (1/\pi)\tilde{u}(x)$ with this u, and therefore, by the corollary,

$$(\ast\ast) \qquad \int_{\{|\Delta(x)| > \lambda\}} \frac{dx}{1+x^2} = o\left(\frac{1}{\lambda}\right)$$

for large λ.

In order to prove that

$$\frac{v(x)}{x} \longrightarrow \frac{A}{\pi} \quad \text{for } x \to \pm\infty,$$

it is enough to show that $\Delta(x)/x \to 0$, $x \to \pm\infty$, and we restrict ourselves to the situation where $x \to \infty$, since the other one is treated in the same manner.

Pick any $\gamma > 1$, as close to 1 as we please, and any $\varepsilon > 0$. For large n, we have

$$\int_{\gamma^n}^{\gamma^{n+1}} \frac{dx}{1+x^2} \sim \frac{\gamma - 1}{\gamma^{n+1}},$$

so, taking $\lambda = \varepsilon\gamma^n$ in $(\ast\ast)$ and making $n \to \infty$, we see that there must be *some* $x_n \in [\gamma^n, \gamma^{n+1}]$ with $|\Delta(x_n)| \leqslant \varepsilon\gamma^n$ if n is *large enough*. Since $v(x) = (A/\pi)x - \Delta(x) + \text{const.}$ is *increasing* (by its definition!), we have, for $\gamma^n \leqslant x \leqslant \gamma^{n+1}$ with n large,

$$-\Delta(x_{n-1}) - \frac{A}{\pi}(\gamma^{n+1} - \gamma^{n-1}) \leqslant -\Delta(x) \leqslant -\Delta(x_{n+1}) + \frac{A}{\pi}(\gamma^{n+2} - \gamma^n).$$

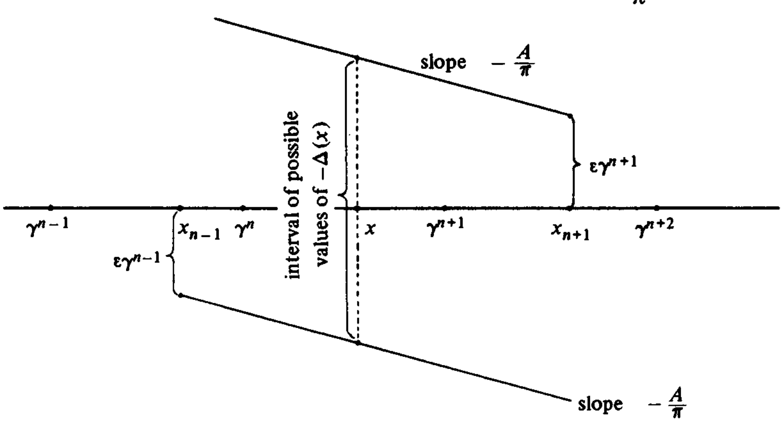

Figure 14

Thus, if n is large enough and $\gamma^n \leqslant x \leqslant \gamma^{n+1}$,

$$|\Delta(x)| \leqslant \varepsilon\gamma^{n+1} + \frac{A}{\pi}(\gamma^2 - 1)\gamma^n,$$

and

$$\left|\frac{\Delta(x)}{x}\right| \leqslant (\gamma^2 - 1)\frac{A}{\gamma} + \varepsilon\gamma.$$

Since $\gamma > 1$ and $\varepsilon > 0$ are arbitrary, we see that $\Delta(x)/x \to 0$ when $x \to \infty$, as required.

Our proof is now complete.

3. The zeros not necessarily real

Given an entire function $f(z)$, let us denote by $n_+(r)$ the number of its zeros with real part $\geqslant 0$ having modulus $\leqslant r$, and by $n_-(r)$ the number of its zeros with real part < 0 having modulus $\leqslant r$. As usual,

$$n(r) = n_+(r) + n_-(r)$$

is the *total number* of zeros of f with modulus $\leqslant r$, and multiple zeros of f are counted according to their multiplicities in reckoning the quantities $n_+(r)$, $n_-(r)$ and $n(r)$.

Theorem (Levinson). *Let the entire function $f(z)$ of exponential type be such that*

$$\int_{-\infty}^{\infty} \frac{\log^+|f(x)|}{1+x^2}\mathrm{d}x \;<\; \infty,$$

and suppose that

$$\limsup_{y\to\infty}\frac{\log|f(\mathrm{i}y)|}{y} \;=\; \limsup_{y\to-\infty}\frac{\log|f(\mathrm{i}y)|}{|y|} \;=\; A.$$

Then

$$\frac{n_+(r)}{r} \longrightarrow \frac{A}{\pi}$$

and

$$\frac{n_-(r)}{r} \longrightarrow \frac{A}{\pi}$$

as $r \to \infty$. Given any $\delta > 0$, the number of zeros of f with modulus $\leqslant r$ lying outside both of the two sectors $|\arg z| < \delta$, $|\arg z - \pi| < \delta$ is $\mathrm{o}(r)$ for large r.

Proof. Without loss of generality, $f(0) \neq 0$, for if $f(0) = 0$ we can work with a suitable quotient $f(z)/z^k$ instead of $f(z)$. *We may thus just as well take $f(0) = 1$ in what follows.*

Denote by $\{\lambda_n\}$ the sequence of zeros of $f(z)$, each zero being repeated in that sequence according to its multiplicity. By the first theorem of §G.3,

$$\sum_{\Im\lambda_n > 0} \frac{\Im\lambda_n}{|\lambda_n|^2} < \infty,$$

and similarly (referring to the *lower* half plane),

$$\sum_{\Im\lambda_n < 0} \frac{\Im\lambda_n}{|\lambda_n|^2} > -\infty.$$

Hence,

$$\sum_n \frac{|\Im\lambda_n|}{|\lambda_n|^2} < \infty.$$

Take any $\delta > 0$. From the previous relation, we have

$$\sum_{\delta \leqslant |\arg\lambda_n| \leqslant \pi - \delta} \frac{1}{|\lambda_n|} < \infty.$$

This certainly implies that the number of λ_n with $\delta \leqslant |\arg\lambda_n| \leqslant \pi - \delta$ and $|\lambda_n| \leqslant r$ is $o(r)$ for $r \to \infty$; the *last* affirmation of the theorem is thus established.

To get the rest (and main part) of the theorem, we follow an idea due to Levinson himself and compare $f(z)$ with another entire function *having only real zeros* to which the conclusion of the theorem in the previous subsection can be applied.

The Hadamard factorization of our function f has the form

$$f(z) = e^{cz} \prod_n \left(1 - \frac{z}{\lambda_n}\right) e^{z/\lambda_n}.$$

Corresponding to each λ_n we now compute a real number λ'_n according to the formula

$$\frac{1}{\lambda'_n} = \Re\left(\frac{1}{\lambda_n}\right);$$

if perchance λ_n is pure imaginary we put $\lambda'_n = \infty$. Let us now write

$$\varphi(z) = e^{cz} \prod_n \left(1 - \frac{z}{\lambda'_n}\right) e^{z/\lambda_n}.$$

We must, first of all, show that the product just written converges u.c.c. in $\mathbb{C}$. But

$$\left(1-\frac{z}{\lambda_n'}\right)e^{z/\lambda_n} = \left(1-\frac{z}{\lambda_n'}\right)e^{z/\lambda_n'}\cdot e^{-iz\Im\lambda_n/|\lambda_n|^2}$$

Here, $\sum_n(|\Im\lambda_n|/|\lambda_n|^2)<\infty$. Also $|\lambda_n'|\geqslant|\lambda_n|$ with $n(r)$, the number of λ_n having modulus $\leqslant r$, at most $O(r)$ (see §A), so $\prod_n(1-z/\lambda_n')e^{z/\lambda_n'}$ does converge u.c.c. in $\mathbb{C}$, by §A. The product defining $\varphi(z)$ therefore converges u.c.c. in $\mathbb{C}$, and $\varphi(z)$ *is an entire function* whose zeros are the real numbers λ_n'.

We want to show that $\varphi(z)$ is of exponential type. This can be done most easily by appealing to the Lindelöf theorems of §B, and the reader is invited to see how that goes. One can also make a direct verification without resorting to the Lindelöf theorems by proceeding as follows.

In the first place, we clearly have

$$|\varphi(x)|\leqslant|f(x)|$$

for $x\in\mathbb{R}$, so, since $f(z)$ *is* of exponential type, $|\varphi(x)|$ is at most $e^{O(|x|)}$ on the real axis. Consider now $z=iy$; here,

$$|\varphi(iy)| = e^{-y\Im c}\prod_n\left(1+\left(\frac{y}{\lambda_n'}\right)^2\right)^{1/2}e^{y\Im\lambda_n/|\lambda_n|^2}.$$

The right side is easily seen to be $\leqslant e^{O(|y|)}$; the only place where calculation is required is in the evaluation of $\sum_n\log(1+(y/\lambda_n')^2)$. To compute this sum, write it as a Stieltjes integral and integrate by parts, using the fact that the number of λ_n' with absolute value $\leqslant r$ is $O(r)$; we find without trouble that the sum is $O(|y|)$. (A very similar calculation was made in proving the second (easy) Lindelöf theorem of §B.)

Having seen that $|\varphi(x)|\leqslant e^{O(|x|)}$ and $|\varphi(iy)|\leqslant e^{O(|y|)}$, we must examine $|\varphi(z)|$ for general complex z. According to the discussion at the beginning of §B, from the fact that the number of λ_n' with modulus $\leqslant r$ is $O(r)$ we can only deduce an inequality of the form

$$\log|\varphi(z)| \leqslant O(|z|\log|z|),$$

valid for large $|z|$. At this point, however, we can apply the *second* Phragmén–Lindelöf theorem of §C. Look at $\varphi(z)$ in each of the quadrants I, II, III and IV. Take, say, the *first* one. For proper choice of the *complex* (!) constant γ,

$$e^{\gamma z}\varphi(z)$$

will be *bounded* on both the *positive real* and *positive imaginary* axes. For