according to whether $\sum |a_n|$ is divergent or convergent. This is a particular case of the previous result.

Example. Determine σ_0 and $\bar{\sigma}$ for the series in which $a_n = 1$, $(-1)^n$, $n^{-\frac{1}{2}}$, $(-1)^n n^{-\frac{1}{2}}$, $a^n \, (0 < a < 1)$, $\log n$, $1/\log n$, respectively; and for the series in which $a_n = 1$ (n a perfect square), $a_n = 0$ otherwise.

9.2. Convergence of the series and regularity of the function. The region of convergence of a power series is determined in a perfectly simple way by the analytic character of the function which it represents—the circle of convergence passes through the singularity nearest to the centre. There is no such simple relation in the case of Dirichlet series. There is not necessarily any singularity on the line of convergence, and in fact $f(s)$ may be an integral function even though the abscissa of convergence of the series is finite. This is shown by the above example of the series for $(1-2^{1-s})\zeta(s)$. This is an integral function, since the pole of $\zeta(s)$ at $s = 1$ is cancelled by the zero of $1-2^{1-s}$. But the corresponding series converges for $\sigma > 0$ only.

On the other hand, the series for $\zeta(s)$, § 9.1 (2), has a singularity on its line of convergence. This is a particular case of the following theorem:

If $a_n \geqslant 0$ for all values of n, then the real point of the line of convergence is a singularity of $f(s)$.

The proof is similar to that of the corresponding theorem for power series (§ 7.21).

We may suppose without loss of generality that $\sigma_0 = 0$. Then if $s = 0$ is a regular point, the Taylor's series for $f(s)$, at the point $s = 1$, has a radius of convergence greater than 1. Hence we can find a negative value of s for which

$$f(s) = \sum_{\nu=0}^{\infty} \frac{(s-1)^\nu}{\nu!} f^{(\nu)}(1) = \sum_{\nu=0}^{\infty} \frac{(1-s)^\nu}{\nu!} \sum_{n=1}^{\infty} \frac{(\log n)^\nu a_n}{n}.$$

But every term in this repeated series is positive. Hence the order of summation may be inverted, and we obtain

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n} \sum_{\nu=0}^{\infty} \frac{(1-s)^\nu (\log n)^\nu}{\nu!}$$

$$= \sum_{n=1}^{\infty} \frac{a_n}{n} e^{(1-s)\log n} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Thus the Dirichlet series is convergent for a negative value of s, contrary to hypothesis.

9.3. Asymptotic behaviour of the function as $t \to \infty$. *The function $f(s)$ is bounded in any half-plane included in the half-plane of absolute convergence.*

For
$$|f(s)| \leqslant \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma}} \leqslant \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\alpha}}$$
for $\sigma \geqslant \alpha > \bar{\sigma}$, and all values of t.

If the series
$$\sum_{n=1}^{\infty} \frac{|a_n|}{n^{\bar{\sigma}}}$$
is convergent, we can take $\alpha = \bar{\sigma}$, and the function is bounded in the half-plane of absolute convergence. This is true, for example, of the function
$$f(s) = \sum_{n=2}^{\infty} \frac{1}{n^s \log^2 n}.$$
But in general the half-plane of absolute convergence is not a region where $f(s)$ is bounded, even if we exclude the neighbourhood of singularities on the line $\sigma = \bar{\sigma}$ (see § 9.32).

Even in the half-plane of absolute convergence, the behaviour of $f(\sigma+it)$ as $t \to \infty$ is, in general, rather complicated. Take, for example, a series with real positive coefficients in which
$$\sum_{n=3}^{\infty} a_n n^{-\sigma} < a_2 2^{-\sigma}$$
for a certain value of σ. Then
$$\mathbf{R} f(s) = \sum_{n=1}^{\infty} \frac{a_n \cos(t \log n)}{n^{\sigma}} > a_1 + \frac{a_2}{2^{\sigma}} - \sum_{n=3}^{\infty} \frac{a_n}{n^{\sigma}}$$
for $t = 2m\pi/\log 2$, $m = 1, 2, \ldots$. Also
$$\mathbf{R} f(s) < a_1 - \frac{a_2}{2^{\sigma}} + \sum_{n=3}^{\infty} \frac{a_n}{n^{\sigma}}$$
for $t = (2m+1)\pi/\log 2$, $m = 1, 2, \ldots$. Hence $\mathbf{R} f(s)$ oscillates as $t \to \infty$.

9.31. The following theorem is due to Dirichlet: *Given N real numbers $c_1, c_2, \ldots, c_N$, a positive integer q, and a positive number τ,* *we can find a number t in the range $\tau \leqslant t \leqslant \tau q^N$, and integers $x_1,\ldots, x_N$, such that*

$$|tc_n - x_n| \leqslant \frac{1}{q} \qquad (n = 1, 2,\ldots, N).$$

The proof is based on the argument that if there are $m+1$ points in m regions, there must be at least one region which contains at least two points.

Consider the N-dimensional unit cube with a vertex at the origin and edges along the coordinate axes. Divide each edge into q equal parts, and thus the cube into q^N equal compartments. Consider the q^N+1 points in the cube

$$(\lambda c_1 - [\lambda c_1], \lambda c_2 - [\lambda c_2],\ldots, \lambda c_N - [\lambda c_N]),$$

where λ takes the values $0, \tau,\ldots, \tau q^N$. At least two of these points must lie in the same compartment. If they are given by $\lambda = \lambda_1$, $\lambda = \lambda_2$ $(\lambda_1 < \lambda_2)$, then there are integers $x_1, x_2,\ldots$ such that

$$(\lambda_2 - \lambda_1)c_n - x_n \leqslant 1/q \qquad (n = 1, 2,\ldots, N),$$

and so $t = \lambda_2 - \lambda_1$ gives the required result.

9.32. We can now deduce the following theorem.

If $f(s) = \sum a_n n^{-s}$, where $a_n \geqslant 0$ for every value of n, and where $\sum a_n n^{-\bar\sigma}$ is divergent, the function $f(s)$ is not bounded in the region $\sigma > \bar\sigma$, $|t| \geqslant t_0 > 0$.

That there may be no singularities on the boundary of this region is shown by the function $\zeta(s) = \sum n^{-s}$, which is regular except at $s = 1$. We make $|t| \geqslant t_0$ to exclude the neighbourhood of the point $s = \bar\sigma$, where there is a singularity.

We have, for every value of N, and $\sigma > \bar\sigma$,

$$f(s) = \sum_{n=1}^{N} \frac{a_n}{n^\sigma} e^{-it\log n} + \sum_{N+1}^{\infty} \frac{a_n}{n^s},$$

and so

$$|f(s)| \geqslant \mathbf{R} \sum_{n=1}^{N} \frac{a_n}{n^\sigma} e^{-it\log n} - \left|\sum_{N+1}^{\infty} \frac{a_n}{n^s}\right|$$

$$\geqslant \sum_{n=1}^{N} \frac{a_n}{n^\sigma} \cos(t\log n) - \sum_{N+1}^{\infty} \frac{a_n}{n^\sigma}.$$

By Dirichlet's theorem there is, for given N and q, a number t $(\tau \leqslant t \leqslant \tau q^N)$ and integers $x_1,\ldots, x_N$, such that

$$\left|\frac{t\log n}{2\pi} - x_n\right| \leqslant \frac{1}{q} \qquad (n = 1, 2,\ldots, N).$$

Hence $\cos(t\log n) \geqslant \cos(2\pi/q)$ for these values of n, and so

$$\sum_{n=1}^{N} \frac{a_n}{n^\sigma}\cos(t\log n) \geqslant \cos\frac{2\pi}{q}\sum_{n=1}^{N}\frac{a_n}{n^\sigma}$$
$$> \cos\frac{2\pi}{q} f(\sigma) - \sum_{N+1}^{\infty}\frac{a_n}{n^\sigma}.$$

Hence, taking $q = 6$, say, so that $\cos 2\pi/q = \frac{1}{2}$,

$$|f(s)| > \tfrac{1}{2}f(\sigma) - 2\sum_{N+1}^{\infty}\frac{a_n}{n^\sigma}.$$

Given any positive number H, we can choose $\sigma - \bar{\sigma}$ so small that $f(\sigma) > 4H$ (since $f(\sigma) \to \infty$ as $\sigma \to \bar{\sigma}$). Having fixed σ, we can choose N so large that

$$\sum_{N+1}^{\infty}\frac{a_n}{n^\sigma} < \tfrac{1}{2}H.$$

Then $|f(s)| > H$, and the result follows.

9.33. In the half-plane of convergence, the function may, for certain values of t, become as large as a power of t. For example, the function $f(s) = (1-2^{1-s})\zeta(s)$, referred to in § 9.13, satisfies the inequality

$$|f(s)| > At^{\frac{1}{2}-\sigma}$$

for some arbitrarily large values of t, and values of σ between 0 and $\frac{1}{2}$.*

On the other hand, the function cannot have values greater than every power of t. This is shown by the following theorem.

We have

$$f(s) = O(|t|^{1-(c-\sigma_0)+\epsilon})$$

as $|t| \to \infty$, *for any value of* σ *between* σ_0 *and* σ_0+1; *and also uniformly in the half-plane to the right of any such line.*

Suppose first that $\sum a_n$ is convergent. Then a_n and s_n are bounded. Now (§ 9.14 (1))

$$\sum_{1}^{N}\frac{a_n}{n^s} = \sum_{1}^{M}\frac{a_n}{n^s} + \sum_{M+1}^{N} s_n\left\{\frac{1}{n^s} - \frac{1}{(n+1)^s}\right\} - \frac{s_M}{(M+1)^s} + \frac{s_N}{(N+1)^s}.$$

If $\sigma > 0$, the last term tends to zero as $N \to \infty$, and we obtain

$$f(s) = \sum_{1}^{M}\frac{a_n}{n^s} + \sum_{M+1}^{\infty} s_n\left\{\frac{1}{n^s} - \frac{1}{(n+1)^s}\right\} - \frac{s_M}{(M+1)^s}.$$

* See Miscellaneous Examples, no. 18.

Hence, by § 9.11 (2), if $0<\sigma<1$,

$$|f(s)|<A\sum_{1}^{M}\frac{1}{n^{\sigma}}+A\frac{|s|}{\sigma}\sum_{M+1}^{\infty}\left\{\frac{1}{n^{\sigma}}-\frac{1}{(n+1)^{\sigma}}\right\}+\frac{A}{(M+1)^{\sigma}}$$

$$<AM^{1-\sigma}+AtM^{-\sigma}+A.$$

Taking $M=[t]$, we obtain

$$f(s)=O(t^{1-\sigma}) \qquad (0<\sigma<1),$$

and similarly $f(s)=O(t^{1-\alpha})$ $(0<\alpha<1,\ \sigma\geqslant\alpha)$. In the general case, the series $\sum a_n n^{-s}$ is convergent for $s=\sigma_0+\epsilon$, and we obtain the above case by changing the origin to this point. Hence the general result follows.

9.4. Functions of finite order. At this point we adopt a slightly different point of view. We have so far considered $f(s)$ as being defined by the series $\sum a_n n^{-s}$, and we have confined our attention to the half-plane of convergence of the series. It may, however, be possible to continue the function outside this half-plane. The function, so defined, may be regular in a wider half-plane; or it may be regular in a wider half-plane except for a certain finite region. We shall now consider the relations between a function defined in this way, and the Dirichlet series from which it originated.

The theorem of § 9.33 suggests that it will be particularly interesting to consider functions which satisfy the condition

$$f(s)=O(|t|^{A})$$

for some positive value of A. A function which satisfies this condition for a particular value of σ is said to be of finite order for that value; if the condition is satisfied uniformly for $\sigma_1\leqslant\sigma\leqslant\sigma_2$, we say that the function is of finite order in this strip. Similarly we can define a function of finite order in a half-plane $\sigma\geqslant\sigma_1$.

We have seen that any function defined by a Dirichlet series is of finite order in a half-plane included in the half-plane of convergence. It may be of finite order outside this half-plane; for $\zeta(s)$, for example, $\sigma_0=1$; but (§ 9.13)

$$\zeta(s)=(1-2^{1-s})^{-1}\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n^{s}} \qquad (\sigma>0),$$

and hence, by § 9.33,

$$\zeta(s) = O(|t|^{1-\sigma+\epsilon}) \qquad (0 < \sigma \leqslant 1).$$

9.41. The function $\mu(\sigma)$. The lower bound μ of numbers ξ such that $f(s) = O(|t|^{\xi})$ is called the *order* of $f(s)$ for that particular value of σ. Thus μ is a function of σ.

The main properties of the function $\mu(\sigma)$ follow from the Phragmén-Lindelöf theorem proved in § 5.65. Suppose that $f(s)$ is regular and of finite order for $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, $t \geqslant t_0$, and let $\mu(\sigma_1) = \mu_1$, $\mu(\sigma_2) = \mu_2$. Then, if ϵ is any positive number,

$$f(\sigma_1+it) = O(t^{\mu_1+\epsilon}), \qquad f(\sigma_2+it) = O(t^{\mu_2+\epsilon}).$$

Hence, by the theorem referred to,

$$f(s) = O(t^{k(\sigma)}) \qquad (\sigma_1 \leqslant \sigma \leqslant \sigma_2),$$

where $$k(\sigma) = \frac{(\sigma_2-\sigma)(\mu_1+\epsilon)+(\sigma-\sigma_1)(\mu_2+\epsilon)}{\sigma_2-\sigma_1}.$$

Making $\epsilon \to 0$, it follows that

$$\mu(\sigma) \leqslant \frac{(\sigma_2-\sigma)\mu_1+(\sigma-\sigma_1)\mu_2}{\sigma_2-\sigma_1} \qquad (\sigma_1 \leqslant \sigma \leqslant \sigma_2). \tag{1}$$

Hence *the function* $\mu(\sigma)$ *is convex downwards.*

It follows also that $\mu(\sigma)$ *is continuous* (§ 5.31).

Secondly, $\mu(\sigma) = 0$ for sufficiently large values of σ; for since $f(s)$ is bounded for $\sigma > \bar{\sigma}$, $\mu(\sigma) \leqslant 0$ for $\sigma > \bar{\sigma}$; on the other hand, if a_m is the first coefficient in the Dirichlet series which does not vanish, and $\bar{\sigma} < \alpha < \sigma$,

$$|f(s)| \geqslant \frac{|a_m|}{m^\sigma} - \sum_{n=m+1}^{\infty} \frac{|a_n|}{n^\sigma} \geqslant \frac{|a_m|}{m^\sigma} - (m+1)^{\alpha-\sigma} \sum_{n=m+1}^{\infty} \frac{|a_n|}{n^\alpha},$$

which can be made positive by taking σ large enough. Thus $|f(s)|$, considered as a function of t, has a positive lower bound if σ is large enough. Hence $\mu(\sigma) \geqslant 0$, so that in fact $\mu(\sigma) = 0$.

If now $\mu(\sigma)$ were negative for any value σ_1 in the region where $f(s)$ is of finite order, it would follow from (1), with σ_2 so large that $\mu_2 = 0$, that $\mu(\sigma) < 0$ for $\sigma_1 < \sigma < \sigma_2$; and we have shown that this is impossible if σ_2 is large enough. *Hence* $\mu(\sigma)$ *is never negative.*

In particular, $\mu(\sigma) = 0$ for $\sigma > \bar{\sigma}$; for we have already shown that $\mu(\sigma) \leqslant 0$ for $\sigma > \bar{\sigma}$.

Again, take $\sigma_2 > \bar{\sigma}$ in (1), so that $\mu_2 = 0$. Then if $\mu_1 > 0$,

$$\mu(\sigma) \leqslant \frac{\sigma_2 - \sigma}{\sigma_2 - \sigma_1} \mu_1 < \mu_1 \qquad (\sigma > \sigma_1).$$

Hence $\mu(\sigma)$ is a steadily decreasing function of σ.

9.42. Perron's formula. We next require an expression for the sum s_n as an integral. This is a particular case of the following theorem.

If x is not an integer, c is any positive number, and $\sigma > \sigma_0 - c$, then

$$\sum_{n<x} \frac{a_n}{n^s} = \frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} f(s+w) \frac{x^w}{w} \, dw. \tag{1}$$

Suppose first that $\sigma > \bar{\sigma} - c$. Then the series for $f(s+w)$ is absolutely and uniformly convergent, and we have

$$\frac{1}{2\pi i} \int\limits_{c-iU}^{c+iT} f(s+w) \frac{x^w}{w} \, dw = \frac{1}{2\pi i} \int\limits_{c-iU}^{c+iT} \sum_{1}^{\infty} \frac{a_n}{n^{s+w}} \frac{x^w}{w} \, dw$$

$$= \frac{1}{2\pi i} \sum_{1}^{\infty} \frac{a_n}{n^s} \int\limits_{c-iU}^{c+iT} \left(\frac{x}{n}\right)^w \frac{dw}{w}. \tag{2}$$

Now by § 3.126

$$\frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} \left(\frac{x}{n}\right)^w \frac{dw}{w} = \begin{matrix} 1 \\ 0 \end{matrix} \qquad \begin{matrix} (n < x) \\ (n > x). \end{matrix}$$

It is therefore sufficient to prove that we can replace U and T in (2) by ∞; that is, we want

$$\lim_{T\to\infty} \sum_{1}^{\infty} \frac{a_n}{n^s} \int\limits_{c+iT}^{\infty+iT} \left(\frac{x}{n}\right)^w \frac{dw}{w} = 0,$$

with a similar result for U.

Now for a fixed x,

$$\int\limits_{c+iT}^{c+i\infty} \left(\frac{x}{n}\right)^w \frac{dw}{w} = -\left(\frac{x}{n}\right)^{c+iT} \frac{1}{(\log x/n)(c+iT)} + \frac{1}{\log x/n} \int\limits_{c+iT}^{c+i\infty} \left(\frac{x}{n}\right)^w \frac{dw}{w^2}$$

$$= O\left(\frac{1}{n^c T}\right) + O\left(\frac{1}{n^c} \int\limits_{T}^{\infty} \frac{dv}{c^2+v^2}\right) = O\left(\frac{1}{n^c T}\right).$$

Hence $$\sum_{n=1}^{\infty} \frac{a_n}{n^s} \int_{c+iT}^{c+i\infty} \left(\frac{x}{n}\right)^w \frac{dw}{w} = O\left(\frac{1}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma+c}}\right),$$

and the result (for $\sigma > \bar{\sigma}-c$) follows.

Suppose next that $\sigma_0-c < \sigma \leqslant \bar{\sigma}-c$. Let $\alpha > \bar{\sigma}-\sigma$, and consider the integral

$$\int f(s+w) \frac{x^w}{w} \, dw$$

taken round the rectangle formed by the lines $\mathbf{R}(w) = c$, $\mathbf{R}(w) = \alpha$, $\mathbf{I}(w) = -U$, $\mathbf{I}(w) = T$. By the theorem of § 9.33, the integrand is

$$O(t^{-(\sigma+c-\sigma_0)+\epsilon}),$$

and so the integrals along the horizontal sides tend to zero as $U \to \infty$, $T \to \infty$. The integrand is regular inside the rectangle, and so, by Cauchy's theorem,

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s+w) \frac{x^w}{w} \, dw = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} f(s+w) \frac{x^w}{w} \, dw.$$

Since $\sigma > \bar{\sigma}-\alpha$, the right-hand side is equal to $\sum_{n<x} a_n n^{-s}$, by the first part. This completes the proof.

The particular case $s = 0$ is

$$\sum_{n<x} a_n = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(w) \frac{x^w}{w} \, dw \qquad (c > \sigma_0). \tag{3}$$

This is Perron's formula.

9.43. There are several other formulae of the same type as Perron's. One which we shall use later is

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} e^{-(n\delta)^\lambda} = \frac{1}{2\pi i\lambda} \int_{c-i\infty}^{c+i\infty} \Gamma\left(\frac{w}{\lambda}\right) f(s+w) \delta^{-w} \, dw,$$

where $\delta > 0$, $\lambda > 0$, and $c > 0$, $c > \bar{\sigma}-\sigma$. To prove this, write the right-hand side as

$$\frac{1}{2\pi i\lambda} \int_{c-i\infty}^{c+i\infty} \Gamma\left(\frac{w}{\lambda}\right) \sum_{n=1}^{\infty} \frac{a_n}{n^{s+w}} \delta^{-w} \, dw,$$

and observe that we can invert the order of summation and integration by 'absolute convergence'. We therefore obtain

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} \frac{1}{2\pi i\lambda} \int\limits_{c-i\infty}^{c+i\infty} \Gamma\left(\frac{w}{\lambda}\right)(n\delta)^{-w}\,dw,$$

and the result follows by the calculus of residues.

9.44. The theorem of § 9.42 enables us to obtain a result of the opposite type to the previous ones—we can pass from the order of the function to the convergence of the series.

The Dirichlet series is convergent in the half-plane where $f(s)$ *is regular and* $\mu(\sigma) = 0$.

Let s be a point in the interior of this half-plane, and let δ be a positive number so small that $\sigma-\delta$ is still in the same half-plane. Let $c > \bar{\sigma}-\sigma+1$ (so that the simpler case of the theorem of § 9.42 can be used). We deform the contour of § 9.42 (1) into the form

$$c-i\infty, \quad c-iT, \quad -\delta-iT, \quad -\delta+iT, \quad c+iT, \quad c+i\infty,$$

where $T > |t|$. In doing so we pass over a pole at $w = 0$, with residue $f(s)$. Hence

$$\sum_{n<x} \frac{a_n}{n^s} - f(s)$$
$$= \frac{1}{2\pi i}\left\{\int\limits_{c-i\infty}^{c-iT} + \int\limits_{c-iT}^{-\delta-iT} + \int\limits_{-\delta-iT}^{-\delta+iT} + \int\limits_{-\delta+iT}^{c+iT} + \int\limits_{c+iT}^{c+i\infty}\right\} f(s+w)\frac{x^w}{w}\,dw.$$

Since we are in the half-plane where $\mu(\sigma) = 0$, we have $f(s) = O(|t|^\epsilon)$ for every positive ϵ. Hence

$$\int\limits_{-\delta-iT}^{-\delta+iT} f(s+w)\frac{x^w}{w}\,dw = \int\limits_{-T}^{T} O\{(|t|+|v|)^\epsilon\}\frac{x^{-\delta}\,dv}{\sqrt{(\delta^2+v^2)}} = O(x^{-\delta}T^\epsilon),$$

and $$\int\limits_{-\delta+iT}^{c+iT} f(s+w)\frac{x^w}{w}\,dw = \int\limits_{-\delta}^{c} O(T^\epsilon)\frac{x^c}{T}\,du = O(x^cT^{\epsilon-1}).$$

A similar result holds for the integral over $(c-iT, -\delta-iT)$. Finally, as in § 9.42,

$$\int\limits_{c+iT}^{c+i\infty} f(s+w)\frac{x^w}{w}\,dw = \sum_{n=1}^{\infty}\frac{a_n}{n^s}\int\limits_{c+iT}^{c+i\infty}\left(\frac{x}{n}\right)^w\frac{dw}{w}$$
$$= O\left(\frac{x^c}{T}\sum_{n=1}^{\infty}\frac{|a_n|}{n^{\sigma+c}|\log x/n|}\right).$$

We may suppose without loss of generality that x is half an odd integer. Then

$$|\log x/n| > \log\{(n+\tfrac{1}{2})/n\} > A/n.$$

Hence the above expression is

$$O\left(\frac{x^c}{T}\sum_{n=1}^{\infty}\frac{|a_n|}{n^{\sigma+c-1}}\right) = O\left(\frac{x^c}{T}\right),$$

since $\sigma+c-1 > \bar{\sigma}$.

A similar result holds for the integral over $(c-i\infty, c-iT)$; and adding, we obtain

$$\sum_{n<x} a_n n^{-s} - f(s) = O(x^{-\delta}T^{\epsilon}) + O(x^c T^{\epsilon-1}).$$

Taking $T = x^{2c}$, this is

$$O(x^{-\delta+2c\epsilon}) + O(x^{-c+2c\epsilon}),$$

which tends to zero as $x \to \infty$ if $\epsilon < \delta/2c$ and $\epsilon < \frac{1}{2}$. This proves the theorem.

A more general theorem of the same type is given in Landau's *Handbuch*, § 238, Satz 57.

9.45. Let σ_ϵ be the abscissa limiting the half-plane where $f(s)$ is regular and of the form $O(t^\epsilon)$. Then we have proved that

$$\sigma_0 \leqslant \sigma_\epsilon \leqslant \bar{\sigma}.$$

It is not easy to give an example where these numbers are all different. There is some reason to suppose that, for the function $(1-2^{1-s})\zeta(s)$, we have $\sigma_\epsilon = \frac{1}{2}$, so that the three numbers are 0, $\frac{1}{2}$, and 1 respectively. But this has not been proved.

9.5. The mean-value formula. *If* $\sigma > \bar{\sigma}$,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(s)|^2\,dt = \sum_{n=1}^{\infty}\frac{|a_n|^2}{n^{2\sigma}}.$$

For
$$|f(s)|^2 = \sum_{m=1}^{\infty}\frac{a_m}{m^{\sigma+it}}\sum_{n=1}^{\infty}\frac{\bar{a}_n}{n^{\sigma-it}}$$
$$= \sum_{n=1}^{\infty}\frac{|a_n|^2}{n^{2\sigma}} + \sum_{m\neq n}\sum\frac{a_m\bar{a}_n}{m^\sigma n^\sigma}\left(\frac{n}{m}\right)^{it},$$

the series being absolutely convergent, and uniformly convergent in any finite t-range. Hence we may integrate term by term, and obtain

$$\frac{1}{2T}\int_{-T}^{T}|f(s)|^2\,dt=\sum_{n=1}^{\infty}\frac{|a_n|^2}{n^{2\sigma}}+\sum_{m\neq n}\sum\frac{a_m\bar{a}_n}{m^\sigma n^\sigma}\frac{2\sin(T\log n/m)}{2T\log n/m}.$$

The factor involving T is bounded for all T, m, and n, so that the double series converges uniformly with respect to T; and each term tends to zero as $T\to\infty$. Hence the sum tends to zero as $T\to\infty$, and the result follows.

9.51. The mean-value half-plane. Let σ_m be the least number such that $f(s)$ is regular and of finite order, and the mean-value formula holds, for every σ greater than σ_m. We shall call the half-plane $\sigma>\sigma_m$ the mean-value half-plane. This expression is justified by the following theorem:*

If $f(s)$ is regular and of finite order for $\sigma\geqslant\alpha$, and

$$\frac{1}{2T}\int_{-T}^{T}|f(\alpha+it)|^2\,dt \tag{1}$$

is bounded as $T\to\infty$, then

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(\sigma+it)|^2\,dt=\sum_{n=1}^{\infty}\frac{|a_n|^2}{n^{2\sigma}} \tag{2}$$

for $\sigma>\alpha$, and uniformly in any strip $\alpha<\sigma_1\leqslant\sigma\leqslant\sigma_2$.

Starting with the formula of § 9.43, and moving the contour to $\mathbf{R}(w)=\alpha-\sigma$, where $\sigma>\alpha$, we pass a pole at $w=0$, with residue $\lambda f(s)$; and if $\lambda>\sigma-\alpha$, no other pole is passed. Hence

$$\sum_{n=1}^{\infty}\frac{a_n}{n^s}e^{-(n\delta)^\lambda}-f(s)=\frac{1}{2\pi i\lambda}\int_{\alpha-\sigma-i\infty}^{\alpha-\sigma+i\infty}\Gamma\left(\frac{w}{\lambda}\right)f(s+w)\delta^{-w}\,dw$$

$$=O\left\{\delta^{\sigma-\alpha}\int_{-\infty}^{\infty}e^{-A|v|}|f\{\alpha+i(t+v)\}|\,dv\right\}$$

by the asymptotic formula for the Γ-function (§ 4.42, ex. (i)). Now if $|t|<T$,

$$\int_{2T}^{\infty}e^{-Av}|f\{\alpha+i(t+v)\}|\,dv=O\left(\int_{2T}^{\infty}e^{-Av}v^A\,dv\right)=O(e^{-AT}),$$

* Carlson (1). The theorem is analogous to Parseval's theorem for Fourier series (§ 13.54).

and a similar result holds for the integral over $(-\infty, -2T)$. Also, by Schwarz's inequality,*

$$\left\{\int_{-2T}^{2T} e^{-A|v|}|f\{\alpha+i(t+v)\}|\, dv\right\}^2$$
$$\leqslant \int_{-2T}^{2T} e^{-A|v|}|f\{\alpha+i(t+v)\}|^2\, dv \int_{-2T}^{2T} e^{-A|v|}\, dv$$
$$< A \int_{-2T}^{2T} e^{-A|v|}|f\{\alpha+i(t+v)\}|^2\, dv.$$

Hence

$$\left|\sum \frac{a_n}{n^s} e^{-(n\delta)^\lambda} - f(s)\right|^2$$
$$< A\delta^{2\sigma-2\alpha} \int_{-2T}^{2T} e^{-A|v|}|f\{\alpha+i(t+v)\}|^2\, dv + A\delta^{2\sigma-2\alpha}e^{-AT},$$

and, integrating with respect to t over $(-T, T)$,

$$\int_{-T}^{T} \left|\sum \frac{a_n}{n^s} e^{-(n\delta)^\lambda} - f(s)\right|^2 dt$$
$$< A\delta^{2\sigma-2\alpha} \int_{-2T}^{2T} e^{-A|v|}\, dv \int_{-T}^{T} |f\{\alpha+i(t+v)\}|^2\, dt + O(\delta^{2\sigma-2\alpha}).$$

Now

$$\int_{-T}^{T} |f\{\alpha+i(t+v)\}|^2\, dt = \int_{-T+v}^{T+v} |f(\alpha+it)|^2\, dt = O(T)$$

uniformly for $|v| < 2T$. Hence

$$\frac{1}{2T}\int_{-T}^{T} \left|\sum \frac{a_n}{n^s} e^{-(n\delta)^\lambda} - f(s)\right|^2 dt = O(\delta^{2\sigma-2\alpha})$$

uniformly with respect to T. Hence†

$$\left\{\frac{1}{2T}\int_{-T}^{T} \left|\sum \frac{a_n}{n^s} e^{-(n\delta)^\lambda}\right|^2 dt\right\}^{\frac{1}{2}} - \left\{\frac{1}{2T}\int_{-T}^{T} |f(s)|^2\, dt\right\}^{\frac{1}{2}} = O(\delta^{\sigma-\alpha}) \quad (3)$$

uniformly with respect to T.

* See *P.M.* Ch. VII, ex. 42; or § 12.41 below.

† By Minkowski's inequality (§ 12.43), but only in the case where $p = 2$ and the functions are continuous.

If $\delta > 0$, the series $\sum a_n n^{-s} e^{-(n\delta)^\lambda}$ is absolutely convergent, and so, by § 9.5,

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left| \sum \frac{a_n}{n^s} e^{-(n\delta)^\lambda} \right|^2 dt = \sum \frac{|a_n|^2}{n^{2\sigma}} e^{-2(n\delta)^\lambda} \tag{4}$$

Taking, say, $\delta = 1$, it follows from (3) and (4) that

$$\frac{1}{2T} \int_{-T}^{T} |f(s)|^2 \, dt < A.$$

Hence by (3), with any positive δ,

$$\sum \frac{|a_n|^2}{n^{2\sigma}} e^{-2(n\delta)^\lambda} < A,$$

and so, since δ may be as small as we please,

$$\sum \frac{|a_n|^2}{n^{2\sigma}}$$

is convergent, and

$$\lim_{\delta\to 0} \sum \frac{|a_n|^2}{n^{2\sigma}} e^{-2(n\delta)^\lambda} = \sum \frac{|a_n|^2}{n^{2\sigma}}.$$

Given ϵ, we can choose δ so that the absolute value of the left-hand side of (3) is less than ϵ, for all values of T, and so that

$$\left| \left\{ \sum \frac{|a_n|^2}{n^{2\sigma}} e^{-2(n\delta)^\lambda} \right\}^{\frac{1}{2}} - \left\{ \sum \frac{|a_n|^2}{n^{2\sigma}} \right\}^{\frac{1}{2}} \right| < \epsilon.$$

Having fixed δ, we can, by (4), choose T_0 so large that

$$\left| \left\{ \frac{1}{2T} \int_{-T}^{T} \left| \sum \frac{a_n}{n^s} e^{-(n\delta)^\lambda} \right|^2 dt \right\}^{\frac{1}{2}} - \left\{ \sum \frac{|a_n|^2}{n^{2\sigma}} e^{-2(n\delta)^\lambda} \right\}^{\frac{1}{2}} \right| < \epsilon.$$

for $T > T_0$. Then

$$\left| \left\{ \frac{1}{2T} \int_{-T}^{T} |f(s)|^2 \, dt \right\}^{\frac{1}{2}} - \left\{ \sum \frac{|a_n|^2}{n^{2\sigma}} \right\}^{\frac{1}{2}} \right| < 3\epsilon$$

for $T > T_0$, and the theorem is proved.

9.52. *If $|f(s)|^2$ has a mean-value for $\sigma = \alpha$, then the Dirichlet series is absolutely convergent for $\sigma > \alpha + \frac{1}{2}$. In symbols,*

$$\bar{\sigma} \leqslant \sigma_m + \tfrac{1}{2}.$$

It follows from the above theorem that

$$\sum \frac{|a_n|^2}{n^{2\alpha+2\epsilon}}$$

is convergent for every positive ϵ. Now

$$\left\{\sum_{n=1}^{N} \left|\frac{a_n}{n^s}\right|\right\}^2 \leqslant \sum_{n=1}^{N} \frac{|a_n|^2}{n^{2\alpha+2\epsilon}} \sum_{n=1}^{N} \frac{1}{n^{2\sigma-2\alpha-2\epsilon}}$$

and this is bounded if ϵ is small enough and $\sigma-\alpha > \frac{1}{2}$. This result was obtained in another way by Hardy (10).

9.53. *If $f(s)$ is bounded for $\sigma > \alpha$, then $\sum |a_n|^2 n^{-2\alpha}$ is convergent; if $|f(s)| \leqslant M$, then*

$$\sum \frac{|a_n|^2}{n^{2\alpha}} \leqslant M^2.$$

This also follows from the theorem of § 9.51. For

$$\sum \frac{|a_n|^2}{n^{2\sigma}} = \lim \frac{1}{2T} \int_{-T}^{T} |f(s)|^2\, dt \leqslant M^2$$

for every $\sigma > \alpha$; and making $\sigma \to \alpha$ the result follows.

If we assume that $f(s)$ is bounded, the analysis of § 9.51 can, of course, be very much simplified.

9.54. Another consequence of these theorems is that a strip in which $f(s)$ is bounded, but in which the Dirichlet series is not absolutely convergent, can be at most of breadth $\frac{1}{2}$.

9.55. *The Dirichlet series converges in the half-plane in which $f(s)$ is regular and of finite order, and*

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(s)|^2\, dt$$

exists. That is, $\sigma_0 \leqslant \sigma_m$.

We have first to deduce an 'order' result for $f(s)$ from the given mean-value result.

We have $f(s) = O(|t|^{\frac{1}{2}})$ uniformly in any strip $\alpha \leqslant \sigma \leqslant \beta$, where $\alpha > \sigma_m$.

Let s be a point of the strip (α, β), R a number less than 1 and less than $\alpha-\sigma_m$, independent of t. Then, if $0 < \rho < R$,

$$f(s) = \frac{1}{2\pi i} \int_{|z-s|=\rho} \frac{f(z)}{z-s}\, dz = \frac{1}{2\pi} \int_{0}^{2\pi} f(s+\rho e^{i\phi})\, d\phi.$$

Hence, by Schwarz's inequality,

$$|f(s)|^2 \leqslant \frac{1}{4\pi^2} \int_0^{2\pi} d\phi \int_0^{2\pi} |f(s+\rho e^{i\phi})|^2 \, d\phi = \frac{1}{2\pi} \int_0^{2\pi} |f(s+\rho e^{i\phi})|^2 \, d\phi.$$

Multiplying by ρ, and integrating with respect to ρ from 0 to R, we have

$$\tfrac{1}{2}R^2|f(s)|^2 \leqslant \frac{1}{2\pi} \int_0^R \int_0^{2\pi} |f(s+\rho e^{i\phi})|^2 \rho \, d\rho d\phi$$

$$< \frac{1}{2\pi} \int_{\sigma-R}^{\sigma+R} \int_{t-R}^{t+R} |f(x+iy)|^2 \, dxdy < \frac{1}{2\pi} \int_{\sigma-R}^{\sigma+R} dx \int_{-|t|-1}^{|t|+1} |f(x+iy)|^2 \, dy.$$

Now
$$\int_{-|t|-1}^{|t|+1} |f(x+iy)|^2 \, dy = O(t)$$

uniformly in x; hence
$$\tfrac{1}{2}R^2|f(s)|^2 = O(t),$$

which gives the required result.

We use the same contour integral as in § 9.44, s and $\sigma-\delta$ now being in the half-plane $\sigma > \sigma_m$. Then

$$\left| \int_{-\delta-iT}^{-\delta+iT} f(s+w) \frac{x^w}{w} \, dw \right| \leqslant x^{-\delta} \left\{ \int_{-T}^{T} \frac{|f(s+w)|^2}{\sqrt{(\delta^2+v^2)}} \, dv \int_{-T}^{T} \frac{dv}{\sqrt{(\delta^2+v^2)}} \right\}^{\frac{1}{2}}.$$

Let
$$\phi(v) = \int_0^v |f(\sigma+u+iy)|^2 \, dy = O(v).$$

Then
$$\int_0^T \frac{|f|^2}{\sqrt{(\delta^2+v^2)}} \, dv = \frac{\phi(T)}{\sqrt{(\delta^2+T^2)}} + \int_0^T \frac{v\,\phi(v)}{(\delta^2+v^2)^{\frac{3}{2}}} \, dv$$

$$= O(1) + \int_{-T}^{T} \frac{O(v^2)}{(\delta^2+v^2)^{\frac{3}{2}}} \, dv = O(\log T).$$

Similarly the integral over $(-T, 0)$ is $O(\log T)$. Hence

$$\int_{-\delta-iT}^{-\delta+iT} f(s+w) \frac{x^w}{w} \, dw = O(x^{-\delta} \log T).$$

Also, by the lemma,

$$\int_{-\delta+iT}^{c+iT} f(s+w)\frac{x^w}{w}\,dw = \int_{-\delta}^{c} O(\sqrt{T})\frac{x^c}{T}\,du = O(x^c T^{-\frac{1}{2}}),$$

with a similar result for the integral over $(c-iT,\, -\delta-iT)$. As in § 9.44, the remaining integrals are $O(x^c T^{-1})$. Hence

$$\sum_{n<x}\frac{a_n}{n^s}-f(s) = O(x^{-\delta}\log T)+O(x^c T^{-\frac{1}{2}}),$$

and, taking $T=x^{2c+1}$, this tends to 0 as $x\to\infty$, so that the result follows.

9.6. The uniqueness theorem. A function $f(s)$ can have at most one representation as a Dirichlet series. More precisely, *if*

$$\sum_1^\infty \frac{a_n}{n^s} = \sum_1^\infty \frac{b_n}{n^s}$$

in any region of values of s, then $a_n = b_n$ for all values of n.

For the series $\sum(a_n-b_n)n^{-s}$ is uniformly convergent in a region including part of the given region and extending arbitrarily far to the right; and so its sum is the same analytic function, viz. 0, in the whole region. But, if m is the first value of n for which $a_n \neq b_n$,

$$\left|\sum (a_n-b_n)n^{-s}\right| \geqslant |a_m-b_m|m^{-\sigma}-\sum_{m+1}^{\infty}|a_n-b_n|n^{-\sigma},$$

and, as in § 9.41, this is positive if σ is large enough. This leads to a contradiction, and so proves the theorem.

9.61. The zeros of $f(s)$. The above argument shows that *$f(s)$ always has a half-plane free from zeros.*

The problem of the distribution of the zeros of any given $f(s)$ is usually a very difficult one, and the results for different functions may be very different. For example, it is supposed that all the complex zeros of the function

$$\frac{1}{1^s}-\frac{1}{2^s}+\frac{1}{3^s}-\dots = (1-2^{1-s})\zeta(s)$$

lie on the lines $\sigma=1$ (zeros of $1-2^{1-s}$) and $\sigma=\frac{1}{2}$ (zeros of $\zeta(s)$); the zeros on $\sigma=1$ are easily identified, but the remaining statement has never been proved.

On the other hand, it is known that the function $\zeta'(s)/\zeta(s)$, which is represented by an absolutely convergent Dirichlet series for $\sigma > 1$, has no zeros in a certain half-plane $\sigma > E$ $(E > 1)$, and zeros on lines $\sigma = \sigma'$ which are dense everywhere in the interval $1 < \sigma < E$.

It is interesting to compare the general problem with the particular case where $a_n = 0$ except when n is a power of 2. Then the function is of the form

$$f(s) = \sum_{n=0}^{\infty} \frac{b_n}{2^{ns}} = \sum_{n=0}^{\infty} b_n z^n,$$

where $z = 2^{-s}$. The series is a power series as well as a Dirichlet series. To each zero z_ν of the power series corresponds a sequence of zeros

$$s_{\mu,\nu} = -\frac{\log z_\nu + 2\mu\pi i}{\log 2} \qquad (\mu = 0, \pm 1, \pm 2, \ldots)$$

of $f(s)$. If z_0 is the zero of smallest modulus (other than 0), $f(s)$ has no zero to the right of the line

$$\sigma = \frac{\log 1/|z_0|}{\log 2},$$

there being an infinity of zeros on this line.

9.62. The function $N(\sigma, T)$. Let t_0 be a positive number such that $f(s)$ is regular for $t \geqslant t_0$ and σ sufficiently large, and let $N(\sigma, T)$ be the number of zeros $\sigma' + it'$ of $f(s)$ such that $\sigma' > \sigma$, $t_0 < t' < T$. Then we have the following theorems:

9.621. *If $f(s)$ is of finite order for $\sigma \geqslant \alpha$, then*

$$N(\sigma, T) = O(T \log T) \qquad (\sigma > \alpha).$$

We can find a number β so large that $|f(s)|$ has positive lower and upper bounds on the line $\sigma = \beta$. Let $0 < \delta < \frac{1}{2}(\beta - \alpha)$. We apply Jensen's theorem to the circle with centre $\beta + in\delta$ and radius $\beta - \alpha$. If $n(r)$ is the number of zeros of $f(s)$ in the circle $|s - (\beta + in\delta)| \leqslant r$, Jensen's theorem gives

$$\int_0^{\beta-\alpha} \frac{n(r)}{r}\, dr = \frac{1}{2\pi} \int_0^{2\pi} \log|f\{\beta + in\delta + (\beta-\alpha)e^{i\theta}\}|\, d\theta - \log|f(\beta + in\delta)|.$$

Now $f(s) = O(t^A)$ for $\sigma \geqslant \alpha$, and so

$$\log|f\{\beta + in\delta + (\beta-\alpha)e^{i\theta}\}| = \log|O\{(n\delta + \beta - \alpha)^A\}| < K \log n,$$

where K depends on α, β, δ,... only. Also $\log|f(\beta+in\delta)| = O(1)$; hence

$$\int\limits_0^{\beta-\alpha} \frac{n(r)}{r}\,dr < K\log n.$$

But

$$\int\limits_0^{\beta-\alpha} \frac{n(r)}{r}\,dr \geqslant n(\beta-\alpha-\delta) \int\limits_{\beta-\alpha-\delta}^{\beta-\alpha} \frac{dr}{r} > Kn(\beta-\alpha-\delta),$$

and $n(\beta-\alpha-\delta)$ is, if δ is small enough, greater than the number of zeros in the strip

$$\sigma \geqslant \alpha+2\delta, \qquad (n-\tfrac{1}{2})\delta \leqslant t < (n+\tfrac{1}{2})\delta.$$

Denoting this number by ν_n, we have therefore

$$\nu_n < K\log n.$$

Hence $$N(\alpha+2\delta, T) \leqslant \sum_{t_0/\delta<n<T/\delta} \nu_n < KT\log T,$$

and the theorem follows.

9.622. *If $f(s)$ is bounded for $\sigma \geqslant \alpha$,*

$$N(\sigma, T) = O(T) \qquad (\sigma > \alpha).$$

The proof is similar to the previous one, but here the factor $\log T$ obviously does not occur. The example at the end of § 9.61 shows that we may have $N(\sigma, T) > AT$.

9.623. *If $f(s)$ has a mean value for $\sigma = \alpha$, and is of finite order for $\sigma \geqslant \alpha$, then* $$N(\sigma, T) = O(T) \qquad (\sigma > \alpha).$$

We use the following lemma:

If $\phi(t)$ is a positive continuous function in (a, b),

$$\frac{1}{b-a}\int\limits_a^b \log\phi(t)\,dt \leqslant \log\left\{\frac{1}{b-a}\int\limits_a^b \phi(t)\,dt\right\}.$$

Divide the interval (a, b) into n equal parts by the points $a = x_0, x_1, \dots, x_n = b$. We have

$$\{\phi(x_1)\phi(x_2)\dots\phi(x_n)\}^{1/n} \leqslant \{\phi(x_1)+\phi(x_2)+\dots+\phi(x_n)\}/n.$$

Hence $$\frac{1}{n}\sum \log\phi(x_\nu) \leqslant \log\left\{\frac{1}{n}\sum \phi(x_\nu)\right\},$$

i.e.

$$\frac{1}{b-a}\sum (x_\nu - x_{\nu-1})\log\phi(x_\nu) \leqslant \log\left\{\frac{1}{b-a}\sum (x_\nu - x_{\nu-1})\phi(x_\nu)\right\}.$$

Making $n \to \infty$, the result follows.

The theorem may be deduced from Jensen's theorem by an elaboration of the argument of § 9.621, but it is more convenient to use the theorem of § 3.8. Applying it to the function $f(s)$ and the rectangle $(\alpha, \beta; t_0, T)$, we have, on taking real parts,

$$2\pi\int\limits_{\alpha}^{\beta} N(\sigma, T)\,d\sigma = \int\limits_{t_0}^{T} \log|f(\alpha+it)|\,dt -$$

$$-\int\limits_{t_0}^{T} \log|f(\beta+it)|\,dt + \int\limits_{\alpha}^{\beta} \arg f(\sigma+iT)\,d\sigma - \int\limits_{\alpha}^{\beta} \arg f(\sigma+it_0)\,d\sigma. \quad \textbf{(1)}$$

Applying the lemma to the first term on the right of (1), we have

$$\frac{1}{T-t_0}\int\limits_{t_0}^{T} \log|f(\alpha+it)|\,dt = \frac{1}{2}\frac{1}{T-t_0}\int\limits_{t_0}^{T} \log|f(\alpha+it)|^2\,dt$$

$$\leqslant \tfrac{1}{2}\log\left\{\frac{1}{T-t_0}\int\limits_{T_0}^{T} |f(\alpha+it)|^2\,dt\right\} < A$$

by hypothesis. Thus the term in question is less than AT.

Secondly, as in § 9.621, $\log|f(\beta+it)|$ is bounded if β is large enough. Hence, if β is suitably chosen, the second term on the right of (1) is $O(T)$.

To deal with the third term, suppose first that $f(s)$ is real for real s. We can take β so large that $\mathbf{R}\{f(s)\}$ does not vanish on $\sigma = \beta$. Then, as in § 3.56, $\arg f(s)$ is bounded on $\sigma = \beta$, and, on $t = T$, $\arg f(s) = O(q)$, where q is the number of times $\mathbf{R}\{f(s)\}$ vanishes on $t = T$, $\alpha \leqslant \sigma < \beta$. Now on $t = T$

$$\mathbf{R}\{f(s)\} = \tfrac{1}{2}\{f(\sigma+iT)+f(\sigma-iT)\} = g(\sigma),$$

say, and q is the number of zeros of $g(z)$ on the real z-axis such that $\alpha \leqslant z \leqslant \beta$. Since $g(z) = O(T^A)$, it follows from Jensen's theorem as in § 9.621 that $q = O(\log T)$. Hence the third term on the right of (1) is $O(\log T)$.

If $f(s)$ is not real on the real axis, we can consider instead the function

$$f_1(s) = \sum \frac{a_n}{n^s} \sum \frac{\bar{a}_n}{n^s} = f(s)\bar{f}(s)$$

and apply the same proof to this.

Finally, the last term on the right of (1) is a constant. Hence

$$\int_{\alpha}^{\beta} N(\sigma, T)\, d\sigma = O(T).$$

But
$$\int_{\alpha}^{\beta} N(\sigma, T)\, d\sigma \geqslant \int_{\alpha}^{\alpha+\delta} N(\sigma, T)\, d\sigma \geqslant \delta N(\alpha+\delta, T),$$

and the result now follows.

9.7. Representation of functions by Dirichlet series. What sort of function can be represented by a Dirichlet series?

It would take us much too far to give anything like an adequate answer to this question, but we can give some indications. It is not difficult to see that a Dirichlet series can only represent functions of a very special kind.

If $f(s)$ is representable by a Dirichlet series, it must, in the first place, be regular and bounded in a certain half-plane (viz. $\sigma \geqslant \bar{\sigma}+\epsilon$). Further, it must have a mean-value

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma+it)|^2\, dt$$

for all sufficiently large values of σ, and the value of the limit must decrease steadily as σ increases.

Again, if $f(s) = \sum a_n n^{-s}$, and x is real, then for $\sigma > \bar{\sigma}$

$$\frac{1}{2T} \int_{-T}^{T} f(s)x^s\, dt = \frac{1}{2T} \int_{-T}^{T} \left(\sum_{n=1}^{\infty} \frac{a_n}{n^s}\right) x^s\, dt = \frac{x^\sigma}{2T} \sum_{n=1}^{\infty} \frac{a_n}{n^\sigma} \int_{-T}^{T} \left(\frac{x}{n}\right)^{it} dt$$

$$= a_x + x^\sigma \sum_{n \neq x} \frac{a_n}{n^\sigma} \frac{2\sin(T\log x/n)}{2T\log x/n},$$

the term a_x occurring if x is a positive integer only. The last series, being uniformly convergent in T, tends to zero as $T \to \infty$. Hence

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(s)x^s\, dt = \begin{cases} a_x & (x \text{ a positive integer}), \\ 0 & (\text{otherwise}). \end{cases}$$

This, therefore, is a necessary condition for $f(s)$ to have the form $\sum a_n n^{-s}$ (the formulae are due to Hadamard). It is, however, not sufficient. But it shows what special properties a function representable by a Dirichlet series must have.

If the Dirichlet series reduces to a single term, say $f(s) = ak^{-s}$, then $f(s)$ is periodic, with period $2\pi i/\log k$. The general Dirichlet series with period $2\pi i/\log k$ is $\sum_{n=0}^{\infty} b_n k^{-ns}$. If we insert other terms, the property of periodicity disappears; but $f(s)$ always retains a certain more general property, which resembles that of periodicity, and any such function is said to be 'almost periodic'. It is in the study of almost periodic functions that answers to the question which we have raised are to be found. We have no space to go into this question further. But we may say roughly that, if an almost periodic function takes a certain value, it repeats this value, not *exactly*, but *approximately*, an infinity of times; and the points where it does this are distributed in much the same way as the periods $(a, 2a, 3a, ...)$ of a periodic function.

The theory of almost periodic functions is due to H. Bohr (1), (2), (3).

MISCELLANEOUS EXAMPLES

1. Prove that, if $\phi(x) = \sum_{n=1}^{\infty} a_n e^{-nx}$, then

$$f(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1}\phi(x)\,dx$$

(i) for $\sigma > 0$, $\sigma > \bar{\sigma}$, (ii) for $\sigma > 0$, $\sigma > \sigma_0$.

2. If $0 < \theta < 2\pi$, the function $f(s)$, defined for $\sigma > 0$ by the equation

$$f(s) = \sum_{n=1}^{\infty} \frac{e^{in\theta}}{n^s},$$

is an integral function.

[Use ex. 1 and proceed as in the case of $\zeta(s)$.]

3. The functions defined for $\sigma > 1$ by the series

$$\sum_{n=1}^{\infty} \frac{e^{ain^b}}{n^s} \quad (a > 0,\ 0 < b < 1), \qquad \sum_{n=1}^{\infty} \frac{e^{ai(\log n)^2}}{n^s} \quad (a > 0),$$

are both integral functions. [Hardy (7), (10).]

4. A function represented by a Dirichlet series cannot tend to a limit (in the half-plane of absolute convergence) as $t \to \infty$, unless it is a constant.

[If $f(s) = \sum_1^\infty a_n n^{-s}$, then for $\sigma > \bar{\sigma}$

$$\lim \frac{1}{2T} \int_{-T}^{T} f(s)\, dt = a_1, \qquad \lim \frac{1}{2T} \int_{-T}^{T} |f(s)|^2\, dt = \sum a_n^2.$$

Hence, if $f(s) \to a$, $\quad a_1 = a, \quad \sum |a_n|^2 = |a|^2$.

Hence $\quad |a_2|^2 + |a_3|^2 + \ldots = 0,$

i.e. $a_2 = 0$, $a_3 = 0, \ldots$. Hence $f(s) = a_1$.]

5. Show that

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \qquad (\sigma > 1),$$

where $\mu(1) = 1$, $\mu(n) = (-1)^r$ if n is the product of r different primes, and otherwise $\mu(n) = 0$. Show also that

$$\frac{\zeta(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s}.$$

[The infinite product for $\zeta(s)$ is given in § 1.44, ex. 1.]

6. Verify the formulae*

$$\{\zeta(s)\}^2 = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}, \qquad \frac{\{\zeta(s)\}^3}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{d(n^2)}{n^s}, \qquad \frac{\{\zeta(s)\}^4}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{\{d(n)\}^2}{n^s},$$

where $d(n)$ denotes the number of divisors of n, and $\sigma > 1$.

[If the expression of n in prime factors is

$$n = p_1^{m_1} p_2^{m_2} \ldots p_r^{m_r},$$

then $\quad d(n) = (m_1+1)(m_2+1)\ldots(m_r+1).$

Hence
$$\sum \frac{d(n^2)}{n^s} = \prod_p \sum_{m=0}^{\infty} \frac{(2m+1)}{p^{ms}}$$

and
$$\sum \frac{\{d(n)\}^2}{n^s} = \prod_p \sum_{m=0}^{\infty} \frac{(m+1)^2}{p^{ms}}.]$$

7. Verify the formulae

$$\zeta(s)\zeta(s-a) = \sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^s} \qquad (\sigma > 1,\ \sigma > a+1),$$

* A number of other formulae of this kind are given by Pólya and Szegö, *Aufgaben*, VIII. Abschn., nos. 49–64.

and*

$$\frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)} = \sum_{n=1}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^s}$$

$$(\sigma > 1,\ \sigma > a+1,\ \sigma > b+1,\ \sigma > a+b+1),$$

where $\sigma_a(n)$ denotes the sum of the ath powers of the divisors of n.

[The second formula follows from the identity

$$\frac{1-p^{a+b-2s}}{(1-p^{-s})(1-p^{a-s})(1-p^{b-s})(1-p^{a+b-s})}$$
$$= \frac{1}{(1-p^a)(1-p^b)} \sum_{m=0}^{\infty} \frac{(1-p^{(m+1)a})(1-p^{(m+1)b})}{p^{ms}}.\Big]$$

8. Let $d_k(n)$, where $k = 2, 3, \ldots$, denote the number of ways of expressing n as a product of k factors, the order of the factors being taken into account. Then

$$\sum_{n=1}^{\infty} \frac{d_k(n)}{n^s} = \{\zeta(s)\}^k \qquad (\sigma > 1);$$

and

$$\sum_{n=1}^{\infty} \frac{\{d_k(n)\}^2}{n^s} = \{\zeta(s)\}^k \prod_p \left\{P_{k-1}\left(\frac{1+p^{-s}}{1-p^{-s}}\right)\right\} \qquad (\sigma > 1),$$

where p runs through all prime numbers, and $P_n(z)$ is the Legendre polynomial of degree n. [Titchmarsh (8).]

9. Show that, if $f(s)$ has the period $2\pi i/\log k$, Hadamard's formulae for the coefficients a_n (§ 9.7) are equivalent to Laurent's formulae for the coefficients in a power series.

10. A necessary and sufficient condition that a function $f(s)$ should be of the form

$$\sum_{n=0}^{\infty} \frac{b_n}{k^{ns}}$$

is that $f(s)$ should be regular and bounded for sufficiently large values of σ. and have the period $2\pi i/\log k$.

11. If $a_n = 0$ unless n is a power of k, then

$$\sigma_0 = \bar{\sigma} = \sigma_c = \sigma_m.$$

12. The function $f(s) = \sum_{m=0}^{\infty} 2^{-m!s}$ has the line $\sigma = 0$ as a natural boundary. [See § 4.71.]

13. The function $f(s) = \sum p^{-s}$, where p runs through all prime numbers, has the line $\sigma = 0$ as a natural boundary.

[This is a more recondite example than the previous one; see Landau and Walfisz (1).]

* Ramanujan (1). B. M. Wilson (1).

14. The function

$$f(s) = \sum_{n=1}^{\infty} \frac{\{d_k(n)\}^r}{n^s}$$

is meromorphic if $r = 1$, or if $r = 2$, $k = 2$; for other values of r and k it has the line $\sigma = 0$ as a natural boundary. [Estermann (1).]

15. Show that, for the function $\zeta(s)$,

$$\mu(\sigma) = 0 \ (\sigma \geqslant 1), \qquad = 1-2\sigma \ (\sigma \leqslant 0),$$

and that $\mu(\sigma) \leqslant 1-\sigma$ for $0 < \sigma < 1$.

[The result for $\sigma < 0$ follows from the functional equation for $\zeta(s)$. The actual value of $\mu(\sigma)$ for $0 < \sigma < 1$ is not known.]

16. Calculate the mean value

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(s)|^2 \, dt \qquad (\sigma > 1)$$

for the functions $f(s) = \zeta(s)$, $1/\zeta(s)$, $\{\zeta(s)\}^2$.

17. Show that, if $f(s)$ is unbounded on any line $\sigma = \alpha$ in the half-plane where it is of finite order, it is also unbounded on every line $\sigma = \beta < \alpha$ in the same half-plane.

18. Show that the function $f(s) = (1-2^{1-s})\zeta(s)$ is unbounded on every line $\sigma = \alpha \leqslant 1$; and that $t^{\sigma-\frac{1}{2}}f(s)$ is unbounded on every line $\sigma = \alpha$, where $0 < \alpha < \frac{1}{2}$.

[The theorem of § 9.32 shows that $\zeta(s)$, and so also $(1-2^{1-s})\zeta(s)$, is unbounded for $\sigma > 1$, $|t| > 1$. The theorem of § 9.41 then gives the first result, and the second result then follows from the functional equation for $\zeta(s)$, § 4.44, and the asymptotic formula for the Γ-function, § 4.42.]

19. If $s_n = \sum_{\nu=1}^{n} a_\nu$ is bounded, then $f(s) = \sum a_n n^{-s}$ is regular for $\sigma > 0$; and, if $f(s)$ has a pole on $\sigma = 0$, it is at most of the first order.

[If $\phi(u) = \sum_{\nu \leqslant u} a_\nu$, we have

$$f(s) = s\int_{1}^{\infty} \frac{\phi(u)}{u^{s+1}}\,du = O\left(s\int_{1}^{\infty} \frac{du}{u^{\sigma+1}}\right) = O\left(\frac{s}{\sigma}\right).\Bigg]$$

20. If $s_n \sim n$, then $f(s) \sim 1/(s-1)$ as $s \to 1$ by real values greater than 1.

If $s_n \sim n \log^k n$, where k is a positive integer, then

$$f(s) \sim \frac{k!}{(s-1)^{k+1}}.$$

CHAPTER X

THE THEORY OF MEASURE AND THE LEBESGUE INTEGRAL

10.1. Riemann integration. In the theory of analytic functions we have used the familiar definition of an integral due to Riemann. In the theory of functions of a real variable, however, Riemann's definition has been almost entirely superseded by a more general one, due to Lebesgue.

Lebesgue's definition enables us to integrate functions for which Riemann's method fails; but this is only one of its advantages. The new theory gives us a command over the whole subject which was previously lacking. It deals, so to speak, automatically with many of the limiting processes which present difficulties in the Riemann theory. At this early stage it is difficult to say anything more precise.

Let us begin by recalling the definition of the Riemann integral of a bounded function. Suppose that $f(x)$ is bounded in the interval (a, b); we subdivide this interval by means of the points $x_0, x_1, \dots, x_n$, so that

$$a = x_0 < x_1 < \dots < x_{n-1} < x_n = b.$$

Let m_ν, M_ν be the lower and upper bounds of $f(x)$ in the interval $x_\nu < x \leqslant x_{\nu+1}$, and let

$$s = \sum_{\nu=0}^{n-1} m_\nu(x_{\nu+1} - x_\nu), \qquad S = \sum_{\nu=0}^{n-1} M_\nu(x_{\nu+1} - x_\nu).$$

When the number of division-points is increased indefinitely so that the greatest interval $x_{\nu+1} - x_\nu$ tends to zero, each of the sums s and S tends to a limit. If the limits are the same, their common value is the Riemann integral

$$\int_a^b f(x)\, dx.$$

In certain cases, e.g. if $f(x)$ is continuous, we can say definitely that this integral exists.

Suppose in particular that $f(x)$ takes the values 0 and 1 only, say $f(x) = 1$ in a set E, and $f(x) = 0$ elsewhere. Then it is easily seen that s is equal to the sum of the lengths of those intervals throughout which $f(x) = 1$, i.e. intervals consisting entirely of points of E; while S is the sum of lengths of intervals which include any point of E. If the set E consists of a finite number of intervals, there is no difficulty in proving that s and S tend to the same limit, viz. the sum of the lengths of the intervals of E.

The Riemann integral of such a function ($f(x)=1$ in E, 0 elsewhere) may be called the *extent* of the set E. *Extent* is thus a generalization of the *length* of an interval. The extent of E, if it exists, is written $e(E)$, so that

$$e(E)=\int_a^b f(x)\,dx.$$

Whether the extent exists or not, the limits of s and S exist. These limits are called the interior and exterior extents* of E, and are written $e_i(E)$, $e_e(E)$.

The function $f(x)$ is called the *characteristic function* of the set E.

It is easy to define a set which has no extent. Let E be the set of all rational values of x in (a,b). Since every interval contains both rational and irrational numbers, we have $m_\nu=0$, $M_\nu=1$, for all modes of division and all values of ν. Hence $s=0$, $S=b-a$, and consequently

$$e_i(E)=0, \qquad e_e(E)=b-a.$$

The extent of this set is therefore undefined, and the characteristic function $f(x)$ has no Riemann integral.

In the general case we may say that the definition of the extent of E depends on the consideration of certain sets of intervals related to E, the number of such intervals being always finite.

Lebesgue's generalization is in the first place a generalization of extent; and it consists fundamentally in removing the restriction that our sets of intervals must be finite. Before we can introduce it formally we must make some further remarks about sets of points.

10.2. Sets of points. For the fundamental ideas concerning sets of points we refer to Hardy's *Pure Mathematics*, Chapter I.

We usually denote sets of points by E, E_1,... and suppose

* The exterior extent is sometimes called the content.

them all to lie within a finite interval (a, b). We denote by CE the complement of E, i.e. the set of all points of the interval (a, b) which do not belong to E.

If E_1 and E_2 are two sets, we denote by E_1+E_2 the set of all points belonging to E_1 or E_2, and by E_1E_2 the set of all points belonging to both E_1 and E_2. The notation is suggested by the fact that, if $f_1(x)$, $f_2(x)$ are the characteristic functions of E_1 and E_2, then $f_1(x)f_2(x)$ is the characteristic function of E_1E_2; while, if E_1 and E_2 have no common points, $f_1(x)+f_2(x)$ is the characteristic function of E_1+E_2.

Note that $$C(E_1+E_2) = CE_1 . CE_2.$$

The notation extends in an obvious way to any finite number of sets; also, if there are an infinity of given sets E_1, E_2,..., then $E_1+E_2+...$ denotes the set of points belonging to any of the given sets, and $E_1E_2...$ denotes the set of points belonging to each of the given sets.

By $E_1 < E_2$ we mean that every point of E_1 is a point of E_2. Two sets are said to 'overlap' if they have common points.

An infinite set of points is said to be *enumerable* if it is possible to define a one-to-one correspondence between the points of the set and the integers 1, 2, 3,...; that is, we must be able to arrange the points in a sequence x_1, x_2, x_3,... such that every point occupies a definite place in the sequence. For example, the set of numbers 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$,... is enumerable; so is the set $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$,....

The set of all proper rational fractions is enumerable; for we can arrange them as follows:

$$\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{4}, \tfrac{3}{4}, \ldots$$

taking the denominators in order of magnitude, then the numerators.

The 'sum' of two enumerable sets is enumerable; for if E_1 consists of the points x_1, x_2,..., and E_2 of ξ_1, ξ_2,..., then all points of E_1+E_2 are given by the sequence

$$x_1, \xi_1, x_2, \xi_2, \ldots.$$

A similar argument applies to any finite number of enumerable sets. Further, *the sum of an enumerable infinity of enumerable sets is enumerable*; for let the sets be E_1, E_2,..., and let E_n consist of the points

$$x_{1,n}, x_{2,n}, \ldots, x_{m,n}, \ldots.$$

We can arrange the double infinity of points $x_{m.n}$ as a single infinity in various ways, e.g. by taking together points for which $m+n = k$ $(k = 2, 3,...)$, and in each such group taking m increasing; thus

$$x_{1,1},\ x_{1,2},\ x_{2,1},\ x_{1,3},\ x_{2,2},\ x_{3,1},\ x_{1,4},\ldots .$$

This proves the theorem.

Finally *a sub-set of an enumerable set is enumerable.* For any sub-set of x_1, x_2, x_3,... clearly has a first member, a second member, a third member, and so on, and this gives the required enumeration.

10.201. The reader might begin to suspect that all sets were enumerable; but this is not the case. *The set of all numbers between* 0 *and* 1 *is not enumerable.*

To prove this, suppose on the contrary it were possible to arrange all such numbers in a sequence x_1, x_2,... . Suppose each such number expressed as an infinite decimal ('terminating' decimals end with an infinity of 0's; we exclude a recurring 9). We then form a new decimal ξ, such that, for every value of n, the nth term in the decimal for ξ exceeds by 1 the nth term in the decimal for x_n, if it is 0, 1,..., 7, and is 0 if it is 8 or 9. This rule defines ξ completely, and ξ does not end with a recurring 9. But ξ is a number between 0 and 1, and is different from any of the numbers x_n. This contradicts the assumption that the sequence x_n contains *all* the numbers between 0 and 1.

A similar argument applies to any interval. We call all the points of an interval a *continuum*. Our result is that *a continuum is not enumerable.*

10.202. A point ξ is called a 'limit-point' of a set E if, however small δ may be, there are points of E, other than ξ, in the interval $(\xi-\delta, \xi+\delta)$. (See *P.M.*, p. 30, where a limit-point is called a 'point of accumulation'.)

A set which contains all its limit-points is called a *closed* set. Thus an interval together with its end-points is a closed set. Such an interval is called a closed interval.

An *open* interval is an interval without its end-points. An *open* set is the complement of a closed set with respect to an open interval.

An open set consists of an enumerable set of non-overlapping *open intervals*. For let E be an open set, and let x be a point of E. Then, for sufficiently small values of δ, the interval $(x, x+\delta)$ consists entirely of points of E; for otherwise x would be a limit-point of CE, so that CE would not be closed. Let δ_1 be the upper bound of values of δ with this property. Then ξ belongs to E for $x \leqslant \xi < x+\delta_1$; but $x+\delta_1$ is not a point of E, since, if it were, the interval of points of E would extend beyond it, by the above argument.

Similarly there is a number δ_2 such that ξ is in E for $x-\delta_2 < \xi \leqslant x$, while $x-\delta_2$ is not in E.

Thus x is a point of an open interval $(x-\delta_2, x+\delta_1)$ of points of E.

Similarly all points of E fall into open intervals. To arrange these intervals as an enumerable sequence, take first the interval, if there is one, greater than $\frac{1}{2}(b-a)$; next, in the order in which they occur on the line, those whose length is $\leqslant \frac{1}{2}(b-a)$ and $> \frac{1}{3}(b-a)$; and so on. Every interval of E has a definite place in this enumeration.

The 'sum' of two open sets is an open set. For if $E = E_1+E_2$, and E_1 and E_2 are open, every point of E is an interior point of an interval of points of E.

The same argument shows that *the sum of any finite number, or of an enumerable infinity, of open sets is open.* In particular (the converse of the above theorem), *the sum of an infinity of open intervals is an open set.*

Also *if E_1 and E_2 are open sets, then $E_1 E_2$ is open.* For a point of $E_1 E_2$ is an interior point of intervals both of E_1 and of E_2; and so it is not a limit-point of $C(E_1 E_2)$, which consists of points of either CE_1 or CE_2.

This argument cannot be extended to an infinity of sets; e.g. if E_n is the open interval $-1/n < x < 1/n$, then $E_1 E_2 \ldots$ is the single point $x = 0$.

10.21. The measure of a set of points. We are now in a position to define a new generalization of 'length'. Instead of starting from a finite number of intervals, we start from an open set, which may contain an infinity of intervals.

The measure of an open set is defined to be the sum of the lengths of its intervals. This sum is, in general, the sum of an infinite series. It is always convergent, since the sum of any finite number of terms is the sum of the lengths of a finite number of non-overlapping intervals, all contained in an interval (a, b), and so is not greater than $b-a$. Hence the measure of any open set contained in (a, b) does not exceed $b-a$.

The exterior measure of a set E is the lower bound of the measures of all open sets which contain E. It is denoted by $m_e(E)$. It is clear that

$$0 \leqslant m_e(E) \leqslant b-a,$$

and that, if $E_1 \prec E_2$, then $m_e(E_1) \leqslant m_e(E_2)$.

The interior measure, $m_i(E)$, is defined by the formula

$$m_i(E) = b-a-m_e(CE).$$

If $m_i(E) = m_e(E)$, then the set E is said to be measurable, and the common value of $m_i(E)$ and $m_e(E)$ is called its measure, and is denoted by $m(E)$.

We have also

$$m_i(CE) = b-a-m_e(E).$$

If E is measurable, so that $m_i(E) = m_e(E)$, it follows that $m_i(CE) = m_e(CE)$. Hence CE is measurable, and

$$m(E)+m(CE) = b-a.$$

Notice that we have given two definitions of the measure of an open set, one direct and one indirect. It will appear before long that they are equivalent. Meanwhile, in arguments involving open sets, we use the direct definition.

10.22. *For any set E we have*

$$m_i(E) \leqslant m_e(E).$$

For, by the definition of exterior measure, there are open sets O and O', including E and CE respectively, and such that

$$m(O) < m_e(E)+\epsilon,$$
$$m(O') < m_e(CE)+\epsilon.$$

If $\epsilon > 0$, every point of the interval $(a+\epsilon, b-\epsilon)$ is an interior point of an interval of O or of O'; and so, by the Heine-Borel theorem,* we can select from these intervals a finite set, say Q, which together include $(a+\epsilon, b-\epsilon)$. Then plainly

$$m(Q) \geqslant b-a-2\epsilon$$

and

$$m(Q) \leqslant m(O)+m(O').$$

* *P.M.* § 105. In the proof there given we start with an interval ending at a, whereas here there is an interval including a. This does not affect the proof.