and similarly

$$|p(z) - p(-z)| \leqslant \sqrt{2M_\alpha}\,e^{3k\alpha|z|}$$

Hence

$$|p(z)| \leqslant 2\sqrt{2M_\alpha}\,e^{3k\alpha|z|}$$

for $|z| \geqslant 1$, and from this, by the principle of maximum,

$$|p(z)| \leqslant 2\sqrt{2M_\alpha}\,e^{3k\alpha} \qquad \text{for } |z| < 1.$$

The theorem therefore holds with

$$K_\alpha = 2\sqrt{2M_\alpha}\,e^{3k\alpha}.$$

Q.E.D.

Corollary. *If* $\alpha > 0$ *is* small enough, *the polynomials* $p(z)$ *satisfying*

$$\sum_{-\infty}^{\infty} \frac{\log^+ |p(n)|}{1+n^2} \leqslant \alpha$$

form a normal family *in the complex plane, and the* limit *of any convergent sequence of such polynomials is an* entire function of exponential type $\leqslant 3k\alpha$, *k being an absolute constant.*

It is thus somewhat as if harmonic measure were available for the domain $\mathbb{C} \sim \mathbb{Z}$, *even though that is not the case.*

11. Weighted polynomial approximation on $\mathbb{Z}$

Given a weight $W(n) \geqslant 1$ defined on $\mathbb{Z}$, we consider the Banach space $\mathscr{C}_W(\mathbb{Z})$ of functions $\varphi(n)$ defined on $\mathbb{Z}$ for which

$$\frac{\varphi(n)}{W(n)} \longrightarrow 0 \quad \text{as} \quad n \to \pm\infty,$$

and write

$$\|\varphi\|_{W,\mathbb{Z}} = \sup_{n\in\mathbb{Z}} \frac{|\varphi(n)|}{W(n)}$$

for such φ. (This is the notation of §A.3.)

Provided that

$$\frac{n^k}{W(n)} \longrightarrow 0 \quad \text{as} \quad n \to \pm\infty$$

for each $k = 0, 1, 2, 3, \ldots$, we can form the $\|\ \|_{W,\mathbb{Z}}$ *closure*, $\mathscr{C}_W(0, \mathbb{Z})$, of the set of *polynomials in n*, in $\mathscr{C}_W(\mathbb{Z})$. The *Bernstein approximation problem for* $\mathbb{Z}$ requires us to find necessary and sufficient conditions on weights $W(n)$ having the property just stated in order that $\mathscr{C}_W(0,\mathbb{Z})$ and $\mathscr{C}_W(\mathbb{Z})$ be the same.

The preceding work enables us to give a complete solution in terms of the Akhiezer function

$$W_*(n) = \sup\{|p(n)|: p \text{ a polynomial and } \|p\|_{W,\mathbb{Z}} \leqslant 1\}$$

introduced in §B.1 of Chapter VI.

Theorem. *Let* $W(n)$, *defined and* $\geqslant 1$ *on* $\mathbb{Z}$, *tend to* ∞ faster than any power of n as $n \to \pm\infty$. *Then* $\mathscr{C}_W(0,\mathbb{Z}) = \mathscr{C}_W(\mathbb{Z})$ *if and only if*

$$\sum_{-\infty}^{\infty} \frac{\log W_*(n)}{1+n^2} \;=\; \infty.$$

Proof. Let us get the easier *if* part out of the way first – this is not really new, and depends only on the work of Chapter VI, §B.1.

As in §A.3, we take $W(x)$ to be specified on *all* of $\mathbb{R}$ by putting $W(x) = \infty$ for $x \notin \mathbb{Z}$, and define $W_*(z)$ for all $z \in \mathbb{C}$ using the formula

$$W_*(z) = \sup\{|p(z)|: p \text{ a polynomial and } \|p\|_{W,\mathbb{Z}} \leqslant 1\}.$$

Then $\mathscr{C}_W(\mathbb{Z})$ can be identified in obvious fashion with the space $\mathscr{C}_W(\mathbb{R})$ constructed from the (discontinuous) weight $W(x)$, and $\mathscr{C}_W(0,\mathbb{Z})$ identified with $\mathscr{C}_W(0)$, the closure of the set of polynomials in $\mathscr{C}_W(\mathbb{R})$. *Proper* inclusion of $\mathscr{C}_W(0,\mathbb{Z})$ in $\mathscr{C}_W(\mathbb{Z})$ is thus *the same* as that of $\mathscr{C}_W(0)$ in $\mathscr{C}_W(\mathbb{R})$, and we can apply the *if* part of Akhiezer's theorem from §B.1 of Chapter III (whose validity *does not* depend on the continuity of $W(x)$!) to conclude that

$$\int_{-\infty}^{\infty} \frac{\log W_*(t)}{t^2+1}\,dt \;<\; \infty$$

when that proper inclusion holds.

If p is any polynomial with $\|p\|_{W,\mathbb{Z}} \leqslant 1$, the hall of mirrors argument at the beginning of the proof of Akhiezer's theorem's *only if* part shows that

$$\log|p(x)| \;\leqslant\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{2\log W_*(t)}{(x-t)^2+4}\,dt$$

for $x \in \mathbb{R}$. Taking the supremum over such polynomials p gives us

$$\log W_*(n) \;\leqslant\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{2\log W_*(t)}{(n-t)^2+4}\,dt, \qquad n \in \mathbb{Z}.$$

Therefore, since $\log W_*(t) \geqslant 0$ (1 being a polynomial!), we have

$$\sum_{-\infty}^{\infty} \frac{\log W_*(n)}{1+n^2} \leqslant \frac{1}{\pi}\int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{n^2+1}\cdot\frac{2\log W_*(t)}{(n-t)^2+1}\,dt.$$

The inner sum over n may easily be compared with an integral, and we find in this way that the last expression is

$$\leqslant \text{ const.}\int_{-\infty}^{\infty} \frac{\log W_*(t)}{1+t^2}\,dt.$$

This, however, is *finite* when $\mathscr{C}_W(0,\mathbb{Z}) \neq \mathscr{C}_W(\mathbb{Z})$, as we have just seen. The *if* part of our theorem is proved.

For the *only if* part, we assume that

$$\sum_{-\infty}^{\infty} \frac{\log W_*(n)}{1+n^2} < \infty,$$

and show that the function

$$\varphi_0(n) = \begin{cases} 1, & n=0 \\ 0, & n\neq 0, \end{cases}$$

cannot belong to $\mathscr{C}_W(0,\mathbb{Z})$. We do this using the *corollary* to the *first* theorem of the preceding article. It is not necessary to resort to the *second* theorem given there.

Suppose, then, that we *have* a sequence of polynomials $p_l(z)$ with

$$\|\varphi_0 - p_l\|_{W,\mathbb{Z}} \underset{l}{\longrightarrow} 0.$$

This implies in particular that

$$p_l(0) \underset{l}{\longrightarrow} \varphi_0(0) = 1,$$

so there is no loss of generality in assuming that $p_l(0) = 1$ for each l, which we *do*. The polynomials

$$Q_l(z) = \tfrac{1}{2}(p_l(z) + p_l(-z))$$

then satisfy the hypothesis of the corollary in question.

We evidently have $\|p_l\|_{W,\mathbb{Z}} \leqslant C$ for some C, so, by definition of W_*, $|p_l(n)| \leqslant CW_*(n)$ for $n\in\mathbb{Z}$ and therefore

$$|Q_l(n)| \leqslant \tfrac{1}{2}C(W_*(n) + W_*(-n)), \qquad n\in\mathbb{Z}.$$

Also, $p_l(n) \underset{l}{\longrightarrow} \varphi_0(n) = 0$ for *each non-zero* $n\in\mathbb{Z}$, so, given any N, we will have

$$|Q_l(n)| < 1 \qquad \text{for } 0 < |n| < N$$

when l is sufficiently large.

Taking any $\alpha > 0$, we choose and fix an N large enough to make

$$\sum_N^\infty \frac{1}{n^2} \log^+(\tfrac{1}{2}C(W_*(n) + W_*(-n))) \;<\; \alpha,$$

this being possible in view of our assumption on W_*. By the preceding two relations we will then have

$$\sum_1^\infty \frac{1}{n^2} \log^+ |Q_l(n)| \;=\; \sum_N^\infty \frac{1}{n^2} \log^+ |Q_l(n)| \;<\; \alpha$$

for sufficiently large values of l.

If $\alpha > 0$ is sufficiently small, the last condition implies that

$$|Q_l(z)| \;\leqslant\; e^{k\alpha|z|}$$

by the corollary to the first theorem of the preceding article, with k an absolute constant. *This must therefore hold for all sufficiently large values of l.*

A *subsequence* of the polynomials $Q_l(z)$ therefore converges u.c.c. to a certain *entire function $F(z)$ of exponential type $\leqslant k\alpha$.* We evidently have $F(0) = 1$ (so $F \not\equiv 0$!), while $F(n) = 0$ *for each non-zero $n \in \mathbb{Z}$.*

However, by problem 1(a) in Chapter I (!), *such an entire function F cannot exist*, if $\alpha > 0$ is chosen *sufficiently small to begin with.* We have thus reached a contradiction, showing that φ_0 *cannot belong* to $\mathscr{C}_W(0, \mathbb{Z})$. The latter space is thus *properly contained* in $\mathscr{C}_W(\mathbb{Z})$, and the *only if* part of our theorem is proved.

We are done.

C. Harmonic estimation in slit regions

We return to domains $\mathscr{D}$ for which the Dirichlet problem *is* solvable, having boundaries formed by removing certain finite open intervals from $\mathbb{R}$. Our interest in the present § is to see whether, from the *existence* of a *Phragmén–Lindelöf function* $Y_{\mathscr{D}}(z)$ for $\mathscr{D}$ (the reader should perhaps look at §A.2 again before continuing), one can *deduce* any *estimates* on the *harmonic measure* for $\mathscr{D}$. We would like in fact to be able to *compare* harmonic measure for $\mathscr{D}$ with $Y_{\mathscr{D}}(z)$. The reason for this desire is the following. Given $A > 0$ and $M(t) \geqslant 0$ on $\partial\mathscr{D}$, suppose that we have a function $v(z)$, *subharmonic* in $\mathscr{D}$ and *continuous* up to $\partial\mathscr{D}$, with

$$v(z) \;\leqslant\; \text{const.} - A|\Im z|, \qquad z \in \mathscr{D},$$

and

$$v(t) \;\leqslant\; M(t), \qquad t \in \partial\mathscr{D}.$$

Then, by harmonic estimation

$$v(z) \leqslant \int_{\partial\mathscr{D}} M(t)\mathrm{d}\omega_{\mathscr{D}}(t,z) - AY_{\mathscr{D}}(z), \qquad z\in\mathscr{D},$$

where (as usual) $\omega_{\mathscr{D}}(\ ,z)$ denotes (*two-sided*) harmonic measure for $\mathscr{D}$ (see §A.1). *It would be very good* if, in this relation, we had some way of *comparing* the *first term* on the right with the *second*.

As we shall see below, such comparison is indeed possible. In order to avoid fastidious justification arguments like the one occurring in the proof of the second theorem from §A.2, *we will assume throughout that* $\partial\mathscr{D}$ *consists of* $\mathbb{R}$ *minus a finite number of* (bounded) *open intervals*. The results obtained for this situation can usually be extended by means of a simple limiting procedure to cover various more general cases that may arise in practice. The domains $\mathscr{D}$ considered here thus look like this:

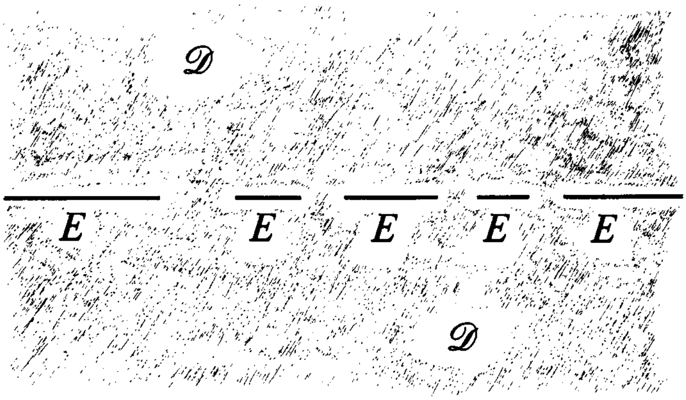

Figure 151

As in §A, we shall frequently denote $\partial\mathscr{D}$ by E. E is a closed subset of $\mathbb{R}$ which, *in this* §, *will contain all real x of sufficiently large absolute value.*

1. Some relations between Green's function and harmonic measure for our domains $\mathscr{D}$

During the present §, we will usually denote the Green's function for one of the domains $\mathscr{D}$ by $G_{\mathscr{D}}(z,w)$, instead of just writing $G(z,w)$ as in §A.2ff. We similarly write $Y_{\mathscr{D}}(z)$ instead of $Y(z)$ for $\mathscr{D}$'s Phragmén–Lindelöf function.

Our domains $\mathscr{D}$ *have* Phragmén–Lindelöf functions. Indeed, for *fixed* $z\in\mathscr{D}$ and *real t*, $G_{\mathscr{D}}(z,t) = G_{\mathscr{D}}(t,z)$ *vanishes* for t outside the *bounded* set $\mathbb{R} \sim E$. (We are using symmetry of the Green's function, established at the end of §A.2.) If we take $z \notin \mathbb{R}$, $G_{\mathscr{D}}(t, z)$ is also a continuous function of $t \in \mathbb{R}$. The integral

$$\int_{-\infty}^{\infty} G_{\mathscr{D}}(z, t)\,\mathrm{d}t$$

is then certainly *finite*, and the *existence* of the *function* $Y_{\mathscr{D}}$ hence assured by the second theorem of §A.2.

According to that same theorem,

$$Y_{\mathscr{D}}(z) \;=\; |\Im z| + \frac{1}{\pi}\int_{-\infty}^{\infty} G_{\mathscr{D}}(z, t)\,\mathrm{d}t.$$

This formula suggests that we first establish some relations between $G_{\mathscr{D}}(z, t)$ and $\omega_{\mathscr{D}}(\ , z)$ before trying to find out whether the latter is in any way governed by $Y_{\mathscr{D}}(z)$.

We prove *three* such relations here. The first of them is very well known.

Theorem. *For* $w \in \mathscr{D}$,

$$G_{\mathscr{D}}(z, w) \;=\; \log\frac{1}{|z - w|} \;+\; \int_E \log|t - w|\,\mathrm{d}\omega_{\mathscr{D}}(t, z).$$

Proof. The right side of the asserted formula is identical with

$$\log\frac{1}{|z - w|} \;+\; \int_{\partial\mathscr{D}} \log|t - w|\,\mathrm{d}\omega_{\mathscr{D}}(t, z),$$

and, for *bounded* domains $\mathscr{D}$, this expression clearly coincides with $G_{\mathscr{D}}(z, w)$ – just fix $w \in \mathscr{D}$ and check boundary values for z on $\partial\mathscr{D}$! (This argument, and the formula, are due to George Green himself, by the way.)

In our situation, however, $\mathscr{D}$ *is not bounded*, and the *result is not true, in general, for unbounded domains.* (Not even for those with 'nice' boundaries; example:

$$\mathscr{D} \;=\; \{|z| > 1\} \cup \{\infty\}.\ )$$

What is needed then in order for it to hold is the presence of 'enough' $\partial\mathscr{D}$ near ∞. That is what we must verify in the present case.

Fixing $w \in \mathscr{D}$, we proceed to find upper and lower bounds on the integral

$$\int_E \log|t - w|\,\mathrm{d}\omega_{\mathscr{D}}(t, z).$$

In order to get an *upper* bound, we take a function $h(z)$, positive and harmonic in $\mathscr{D}$ and continuous up to $\partial\mathscr{D}$, such that

$$h(z) \;=\; \log^{+}|z| + O(1).$$

In the case where E includes the interval $[-1, 1]$ (at which we can always arrive by translation), one may put

$$h(z) = \log|z + \sqrt{(z^2 - 1)}|$$

using, outside $[-1, 1]$, the determination of $\sqrt{\ }$ that is *positive* for $z = x > 1$. For large $A > 0$, let us write

$$h_A(z) = \min(h(z), A).$$

The function $h_A(t)$ is then *bounded and continuous* on E, so, by the elementary properties of harmonic measure (Chapter VII, §B.1), the function of z equal to

$$\int_E h_A(t) d\omega_{\mathscr{D}}(t, z)$$

is *harmonic* and *bounded above* in $\mathscr{D}$, and takes the boundary value $h_A(z)$ for z on $\partial\mathscr{D}$. The difference $\int_E h_A(t) d\omega_{\mathscr{D}}(t, z) - h(z)$ is thus bounded above in $\mathscr{D}$ and $\leqslant 0$ on $\partial\mathscr{D}$. Therefore, by the *extended principle of maximum* (Chapter III, §C), it is $\leqslant 0$ in $\mathscr{D}$, and we have

$$\int_E h_A(t)\, d\omega_{\mathscr{D}}(t, z) \leqslant h(z), \qquad z \in \mathscr{D}.$$

For $A' \geqslant A$, $h_{A'}(t) \geqslant h_A(t)$. Hence, by the preceding relation and Lebesque's monotone convergence theorem,

$$\int_E h(t)\, d\omega_{\mathscr{D}}(t, z) \leqslant h(z), \qquad z \in \mathscr{D};$$

that is,

$$\int_E \log^+|t| d\omega_{\mathscr{D}}(t, z) \leqslant \log^+|z| + O(1)$$

for $z \in \mathscr{D}$. When $w \in \mathscr{D}$ is fixed, we thus have the upper bound

$$\int_E \log|t - w| d\omega_{\mathscr{D}}(t, z) \leqslant \log^+|z| + O(1)$$

for z ranging over $\mathscr{D}$.

We can get some additional information with the help of the function $h(z)$. Indeed, for *each* A,

$$\int_E h_A(t)\, d\omega_{\mathscr{D}}(t, z) \leqslant \int_E h(t)\, d\omega_{\mathscr{D}}(t, z) \leqslant h(z)$$

when $z \in \mathscr{D}$. As we remarked above, the *left-hand* expression tends to $h_A(x_0)$

whenever $z \longrightarrow x_0 \in \partial\mathscr{D}$; at the same time, the *right-hand* member evidently tends to $h(x_0)$. Taking $A > h(x_0)$, we see that

$$\int_E h(t)\,\mathrm{d}\omega_{\mathscr{D}}(t,z) \;\longrightarrow\; h(x_0)$$

for $z \longrightarrow x_0 \in \partial\mathscr{D}$. On the other hand, for fixed $w \in \mathscr{D}$,

$$\log|t-w| \;-\; h(t)$$

is *continuous* and *bounded* on $\partial\mathscr{D}$. Therefore

$$\int_E (\log|t-w| - h(t))\,\mathrm{d}\omega_{\mathscr{D}}(t,z) \longrightarrow \log|x_0-w| \;-\; h(x_0)$$

when $z \longrightarrow x_0 \in \partial\mathscr{D}$, so, on account of the previous relation, we have

$$\int_E \log|t-w|\,\mathrm{d}\omega_{\mathscr{D}}(t,z) \longrightarrow \log|x_0-w|$$

for $z \longrightarrow x_0 \in \partial\mathscr{D}$.

To get a *lower* bound on the left-hand integral, let us, wlog, assume that $\Re z > 0$, and take an $R > 0$ sufficiently large to have $(-\infty, -R] \cup [R, \infty) \subseteq E$. Since $\mathscr{D} \supseteq \{\Im z > 0\}$, we have, for $|t| > R$,

$$\mathrm{d}\omega_{\mathscr{D}}(t,z) \;\geqslant\; \frac{1}{\pi}\frac{\Im z}{|z-t|^2}\mathrm{d}t$$

by the principle of *extension of domain* (Chapter VII, §B.1), the right side being just the differential of harmonic measure for the upper half plane. Hence,

$$\begin{aligned}\int_E \log|t+\mathrm{i}|\,\mathrm{d}\omega_{\mathscr{D}}(t,z) \;&\geqslant\; \int_{\{|t|\geqslant R\}} \log|t+\mathrm{i}|\,\mathrm{d}\omega_{\mathscr{D}}(t,z)\\ &\geqslant\; \frac{1}{\pi}\int_{|t|\geqslant R} \frac{\Im z \log|t+\mathrm{i}|}{|z-t|^2}\mathrm{d}t\\ &=\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log|t+\mathrm{i}|}{|z-t|^2}\mathrm{d}t \;-\; O(1).\end{aligned}$$

The last integral on the right has, however, the value $\log|z+\mathrm{i}|$, as an elementary computation shows (contour integration). Thus,

$$\int_E \log|t+\mathrm{i}|\,\mathrm{d}\omega_{\mathscr{D}}(t,z) \;\geqslant\; \log|z+\mathrm{i}| - O(1)$$

for $\Im z > 0$, so, for fixed $w \in \mathscr{D}$,

$$\int_E \log|t-w|\,\mathrm{d}\omega_{\mathscr{D}}(t,z) \;\geqslant\; \log^+|z| - \mathrm{O}(1), \qquad z\in\mathscr{D}.$$

Taking any $w\in\mathscr{D}$, we see by the above that the function of z equal to

$$\log\frac{1}{|z-w|} \;+\; \int_E \log|t-w|\,\mathrm{d}\omega_{\mathscr{D}}(t,z)$$

is *harmonic* in $\mathscr{D}$ save at w, differs in $\mathscr{D}$ by O(1) from $\log(1/|z-w|) + \log^+|z|$, and assumes the *boundary value zero* on $\partial\mathscr{D}$. It is in particular *bounded above and below* outside of a neighborhood of w (point where it becomes infinite), and hence $\geqslant 0$ in $\mathscr{D}$ by the extended maximum principle. The expression just written thus has all the properties required of a Green's function for $\mathscr{D}$, and must coincide with $G_{\mathscr{D}}(z,w)$. We are done.

► *It will be convenient during the remainder of this § to take* $\mathrm{d}\omega_{\mathscr{D}}(t,z)$ *as defined on all of* $\mathbb{R}$, *simply putting it equal to zero outside of* E. *This enables us to simplify our notation by writing* $\omega_{\mathscr{D}}(S,z)$ *for* $\omega_{\mathscr{D}}(S\cap E,z)$ *when* $S\subseteq\mathbb{R}$.

Lemma. *Let* $0\in\mathscr{D}$, *and write*

$$\omega_{\mathscr{D}}(x) \;=\; \begin{cases}\omega_{\mathscr{D}}([x,\infty),\ 0), & x>0,\\ \omega_{\mathscr{D}}((-\infty,x],\ 0), & x<0\end{cases}$$

(*note that* $\omega_{\mathscr{D}}(x)$ need not be continuous *at* 0). *Then, for* $\Im z\neq 0$,

$$G_{\mathscr{D}}(z,0) \;=\; -\int_{-\infty}^{\infty}\frac{x-t}{(x-t)^2+y^2}\,\omega_{\mathscr{D}}(t)\,\mathrm{sgn}\,t\,\mathrm{d}t.$$

Proof. By the preceding theorem and symmetry of the Green's function (proved at the end of §A.2), we have

$$G_{\mathscr{D}}(z,0) \;=\; G_{\mathscr{D}}(0,z) \;=\; \log\frac{1}{|z|} + \int_E \log|t-z|\,\mathrm{d}\omega_{\mathscr{D}}(t,0).$$

Thanks to our convention, we can rewrite the right-hand integral as

$$\left(\int_{-\infty}^{0} + \int_{0}^{\infty}\right)\log|t-z|\,\mathrm{d}\omega_{\mathscr{D}}(t,0).$$

Let us accept for the moment the inequality

$$\omega_{\mathscr{D}}(t) \;\leqslant\; \frac{\text{const.}}{|t|+1},$$

postponing its verification to the end of this proof. Then partial integration yields

$$\int_0^\infty \log|t-z|\,\mathrm{d}\omega_{\mathscr{D}}(t,0) \;=\; \omega_{\mathscr{D}}(0+)\log|z| \;+\; \int_0^\infty \frac{t-x}{|t-z|^2}\,\omega_{\mathscr{D}}(t)\,\mathrm{d}t,$$

and

$$\int_{-\infty}^0 \log|t-z|\,\mathrm{d}\omega_{\mathscr{D}}(t,0) \;=\; \omega_{\mathscr{D}}(0-)\log|z| \;-\; \int_{-\infty}^0 \frac{t-x}{|t-z|^2}\,\omega_{\mathscr{D}}(t)\,\mathrm{d}t.$$

Here,

$$\omega_{\mathscr{D}}(0+)+\omega_{\mathscr{D}}(0-) \;=\; \omega_{\mathscr{D}}((-\infty,\infty),\;0) \;=\; \omega_{\mathscr{D}}(E,0) \;=\; 1,$$

so, adding, we get

$$\begin{aligned} G_{\mathscr{D}}(z,0) \;&=\; \log\frac{1}{|z|} + \int_{-\infty}^\infty \log|t-z|\,\mathrm{d}\omega_{\mathscr{D}}(t,0) \\ &= \log\frac{1}{|z|} + \log|z| + \int_{-\infty}^\infty \frac{t-x}{|t-z|^2}\,\omega_{\mathscr{D}}(t)\,\mathrm{sgn}\,t\,\mathrm{d}t \\ &= -\int_{-\infty}^\infty \frac{x-t}{|z-t|^2}\,\omega_{\mathscr{D}}(t)\,\mathrm{sgn}\,t\,\mathrm{d}t, \end{aligned}$$

as claimed.

We still have to check the above inequality for $\omega_{\mathscr{D}}(t)$. To do this, pick an $R>0$ large enough to have

$$(-\infty,-R]\cup[R,\infty) \;\subseteq\; E,$$

and take a domain $\mathscr{E}$ equal to the *complement* of

$$(-\infty,-R]\cup[R,\infty)$$

in $\mathbb{C}$. Then $\mathscr{D}\subseteq\mathscr{E}$, so, by the *principle of extension of domain* (Chapter VII, §B.1), $\omega_{\mathscr{D}}(t)+\omega_{\mathscr{D}}(-t) \leqslant \omega_{\mathscr{E}}((-\infty,-t]\cup[t,\infty),\;0)$ for $t>R$. The quantity on the right can, however, be worked out explicitly by mapping $\mathscr{E}$ conformally onto the unit disk so as to take $-R$ to -1, 0 to 0 and R to 1. In this way, one finds it to be $\leqslant CR/t$ (with a constant C independent of R), verifying the inequality in question. Details are left to the reader – he or she is referred to the proof of the *first* lemma from §A.1, where most of the computation involved here has already been done.

The integral figuring in the lemma just proved, viz.,

$$-\int_{-\infty}^\infty \frac{x-t}{|z-t|^2}\,\omega_{\mathscr{D}}(t)\,\mathrm{sgn}\,t\,\mathrm{d}t$$

is like one used in the scholium of §H.1, Chapter III, to express a certain *harmonic conjugate.* It differs from the latter by its sign, by the absence of the constant $1/\pi$ in front, and because its integrand involves the factor $(x-t)/|z-t|^2$ instead of the sum

$$\frac{x-t}{|z-t|^2} + \frac{t}{t^2+1}.$$

In §H of Chapter III, the main purpose of the term $t/(t^2+1)$ was really *to ensure convergence*; *here*, since $\omega_{\mathscr{D}}(t)$ is $O(1/(|t|+1))$, we already have convergence *without* it, and our *omission* of the term $t/(t^2+1)$ amounts merely to the *subtraction of a constant* from the value of the integral. Since harmonic conjugates are *only determined to within additive constants* anyway, *we may just as well take*

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x-t}{|z-t|^2}\,\omega_{\mathscr{D}}(t)\,\mathrm{sgn}\,t\,\mathrm{d}t$$

as the *harmonic conjugate* of

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\,\omega_{\mathscr{D}}(t)\,\mathrm{sgn}\,t\,\mathrm{d}t$$

in $\{\Im z > 0\}$. This brings the investigation of the former integral's boundary behavior on the real axis very close to the study of the *Hilbert transform* already touched on in Chapter III, §§F.2 and H.1.

In our present situation, we already know that, for real $x \neq 0$,

$$\lim_{y\to 0} G_{\mathscr{D}}(x+\mathrm{i}y,\ 0) \;=\; G_{\mathscr{D}}(x,0)$$

exists. The identity furnished by the lemma hence shows, *independently of the general considerations in the articles just mentioned*, that

$$\lim_{y\to 0}\int_{-\infty}^{\infty} \frac{x-t}{|z-t|^2}\,\omega_{\mathscr{D}}(t)\,\mathrm{sgn}\,t\,\mathrm{d}t$$

exists (and equals $-G_{\mathscr{D}}(x,0)$) for real $x \neq 0$. According to an observation in the scholium of §H.1, Chapter III, we can express the preceding limit as an *integral*, namely

$$\int_0^{\infty} \frac{\omega_{\mathscr{D}}(x-\tau)\,\mathrm{sgn}\,(x-\tau) - \omega_{\mathscr{D}}(x+\tau)\,\mathrm{sgn}\,(x+\tau)}{\tau}\,\mathrm{d}\tau.$$

That's because this expression *converges absolutely* for $x \neq 0$, on account of the above inequality for $\omega_{\mathscr{D}}(t)$ and also of the

Lemma. *Let* $0 \in \mathscr{D}$. *Then* $\omega_{\mathscr{D}}(t)$ *is* $\mathrm{Lip}\,\frac{1}{2}$ *for* $t>0$ *and for* $t<0$.

Proof. The statement amounts to the claim that

$$\omega_{\mathscr{D}}(I,0) \;\leqslant\; \text{const.}\sqrt{|I|}$$

for any small interval $I \subseteq E$. To show this, take any *interval* $J_0 \subseteq E$ and consider small intervals $I \subseteq J_0$. Letting $\mathscr{E}$ be the region $(\mathbb{C} \cup \{\infty\}) \sim J_0$, the usual application of the *principle of extension of domain* gives us

$$\omega_{\mathscr{D}}(I,0) \;\leqslant\; \omega_{\mathscr{E}}(I,0),$$

with, in turn,

$$\omega_{\mathscr{E}}(I,0) \;\leqslant\; \text{const.}\,\omega_{\mathscr{E}}(I,\infty)$$

by Harnack's theorem.

To simplify the estimate of the right side of the *last* inequality, we may take J_0 to be $[-1,1]$; this just amounts to making a preliminary translation and change of scale – *never mind* here that $0 \in \mathscr{D}$! Then one can map $\mathscr{E}$ onto the unit disk by the Joukowski transformation

$$z \;\longrightarrow\; z - \sqrt{(z^2-1)}$$

which takes ∞ to 0, -1 to -1, and 1 to 1. In this way one easily finds that

$$\omega_{\mathscr{E}}(I,\infty) \;\leqslant\; \text{const.}\sqrt{|I|},$$

proving the lemma.

Remark. The square root is *only* necessary when I is *near one of the endpoints* of J_0. For small intervals I near the *middle* of J_0, $\omega_{\mathscr{E}}(I,\infty)$ acts like a multiple of $|I|$.

By the above two lemmas and related discussion, we have the formula

$$G_{\mathscr{D}}(x,0) \;=\; -\int_0^\infty \frac{\omega_{\mathscr{D}}(x-\tau)\,\mathrm{sgn}\,(x-\tau) - \omega_{\mathscr{D}}(x+\tau)\,\mathrm{sgn}\,(x+\tau)}{\tau}\,\mathrm{d}\tau,$$

valid for $x \neq 0$ if 0 belongs to $\mathscr{D}$. It is customary to write the right-hand member in a different way. That expression is identical with

$$-\lim_{\delta\to 0}\int_{|t-x|\geqslant\delta} \frac{\omega_{\mathscr{D}}(t)\,\mathrm{sgn}\,t}{x-t}\,\mathrm{d}t.$$

If a function $f(t)$, having a possible singularity at $a \in \mathbb{R}$, is integrable over each set of the form $\{|t-a| \geqslant \delta\}$, $\delta > 0$, and *if*

$$\lim_{\delta\to 0}\int_{|t-a|\geqslant\delta} f(t)\,\mathrm{d}t$$

exists, that limit is called a Cauchy principal value, and denoted by

$$\fint_{-\infty}^{\infty} f(t)\,\mathrm{d}t \qquad \textit{or by} \qquad \text{v.p.}\int_{-\infty}^{\infty} f(t)\,\mathrm{d}t.$$

It is important to realize that $⨍_{-\infty}^{\infty} f(t)\,dt$ is frequently not an integral in the ordinary sense.

In terms of this notation, the formula for $G_{\mathscr{D}}(x, 0)$ just obtained can be expressed as in the following

Theorem. *Let $0 \in \mathscr{D}$. Then, for real $x \neq 0$,*

$$\boxed{G_{\mathscr{D}}(x,0) \;=\; -⨍_{-\infty}^{\infty} \frac{\omega_{\mathscr{D}}(t)\,\mathrm{sgn}\,t}{x-t}\,dt,}$$

where $\omega_{\mathscr{D}}(t)$ is the function defined in the first of the above two lemmas.

This result will be used in article 3 below. Now, however, we wish to use it to *solve for $\omega_{\mathscr{D}}(t)\,\mathrm{sgn}\,t$ in terms of $G_{\mathscr{D}}(x, 0)$*, obtaining the relation

$$\omega_{\mathscr{D}}(t)\,\mathrm{sgn}\,t \;=\; \frac{1}{\pi^2}⨍_{-\infty}^{\infty} \frac{G_{\mathscr{D}}(x,0)}{t-x}\,dx.$$

By the inversion theorem for the L_2 Hilbert transform, the latter formula is indeed a consequence of the boxed one above. Here, a direct proof is not very difficult, and we give one for the reader who does not know the inversion theorem.

Lemma. $\int_{-\infty}^{\infty} |G_{\mathscr{D}}(x + iy,\ 0) - G_{\mathscr{D}}(x, 0)|\,dx \;\longrightarrow\; 0$ *for* $y \to 0$.

Proof. The result follows immediately from the representation

$$G_{\mathscr{D}}(x+iy,\ 0) \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yG_{\mathscr{D}}(t,0)}{(x-t)^2+y^2}\,dt, \qquad y > 0,$$

by elementary properties of the Poisson kernel, in the usual way.

The representation itself is practically obvious; here is one derivation. From the first theorem of this article,

$$G_{\mathscr{D}}(t,0) \;=\; \log\frac{1}{|t|} + \int_E \log|s-t|\,d\omega_{\mathscr{D}}(s,0)$$

and

$$G_{\mathscr{D}}(z,0) \;=\; \log\frac{1}{|z|} + \int_E \log|s-z|\,d\omega_{\mathscr{D}}(s,0).$$

For $\Im z > 0$, we have the elementary formula

$$\log|s-z| \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log|s-t|}{|z-t|^2}\,dt, \qquad s \in \mathbb{R}.$$

Use this in the right side of the preceding relation (in *both* right-hand terms !), change the order of integration (which is easily justified here), and then refer to the formula for $G_{\mathscr{D}}(t,0)$ just written. One ends with the relation in question.

Lemma. *Let $0 \in \mathscr{D}$. Then $G_{\mathscr{D}}(x,0)$ is* $\mathrm{Lip}\,\frac{1}{2}$ *for $x > 0$ and for $x < 0$.*

Proof. The open intervals of $\mathbb{R} \sim E$ belong to $\mathscr{D}$, where $G_{\mathscr{D}}(z,0)$ is *harmonic* (save at 0), and hence $\mathscr{C}_{\infty}$. So $G_{\mathscr{D}}(x,0)$ is certainly $\mathscr{C}_1$ (hence Lip 1) in the *interior* of each of those open segments (although *not uniformly* so!) for x outside any neighborhood of 0. Also, $G_{\mathscr{D}}(x,0) = 0$ on each of the *closed segments* making up E; it is thus surely Lip 1 on the interior of each of *those.*

Our claim therefore boils down to the statement that

$$|G_{\mathscr{D}}(x,0) - G_{\mathscr{D}}(a,0)| \;\leqslant\; \text{const.}\sqrt{|x-a|}$$

near any of the endpoints a of any of the segments making up E. Since $G_{\mathscr{D}}(a,0) = 0$, we have to show that

$$G_{\mathscr{D}}(x,0) \;\leqslant\; \text{const.}\sqrt{|x-a|}$$

for $x \in \mathbb{R} \sim E$ *near* such an endpoint a.

Assume, wlog, that a is a *right* endpoint of a component of E and that $x > a$. Pick $b < a$ such that

$$[b,a] \;\subseteq\; E$$

and denote the domain $(\mathbb{C} \cup \{\infty\}) \sim [b,a]$ by $\mathscr{E}$. We have $\mathscr{D} \subseteq \mathscr{E}$, so

$$G_{\mathscr{D}}(x,0) \;\leqslant\; G_{\mathscr{E}}(x,0)$$

by the principle of extension of domain. Here, one may compute $G_{\mathscr{E}}(x,0)$ by mapping $\mathscr{E}$ onto the unit disk conformally with the help of a Joukowski transformation. In this way one finds without much difficulty that

$$G_{\mathscr{E}}(x,0) \;\leqslant\; \text{const.}\sqrt{(x-a)}$$

for $x > a$, proving the lemma. (Cf. proof of the lemma immediately preceding the previous theorem.)

Theorem. *Let $0 \in \mathscr{D}$. Then, for $x \neq 0$,*

$$\omega_{\mathscr{D}}(x)\,\mathrm{sgn}\,x \;=\; \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{G_{\mathscr{D}}(t,0)}{x-t}\,\mathrm{d}t,$$

where $\omega_{\mathscr{D}}(x)$ is the function defined in the first *lemma of this article.*

Proof. By the first of the preceding lemmas, for $t \in \mathbb{R}$ and $h > 0$,

$$G_{\mathscr{D}}(t + \mathrm{i}h,\; 0) \;=\; -\int_{-\infty}^{\infty} \frac{t-\xi}{(t-\xi)^2 + h^2}\,\omega_{\mathscr{D}}(\xi)\,\mathrm{sgn}\,\xi\,\mathrm{d}\xi.$$

Multiply both sides by

$$\frac{x-t}{(x-t)^2+y^2}$$

and integrate the variable t. We get

$$\int_{-\infty}^{\infty}\frac{x-t}{(x-t)^2+y^2}G_{\mathscr{D}}(t+\mathrm{i}h,\ 0)\,\mathrm{d}t$$
$$= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{x-t}{(x-t)^2+y^2}\cdot\frac{t-\xi}{(t-\xi)^2+h^2}\,\omega_{\mathscr{D}}(\xi)\operatorname{sgn}\xi\,\mathrm{d}\xi\,\mathrm{d}t.$$

Suppose for the moment that *absolute convergence* of the double integral has been established. Then we can change the order of integration therein. We have, however, for $y>0$,

$$\int_{-\infty}^{\infty}\frac{(x-t)}{(x-t)^2+y^2}\cdot\frac{t-\xi}{(t-\xi)^2+h^2}\,\mathrm{d}t = -\pi\frac{y+h}{(x-\xi)^2+(y+h)^2},$$

as follows from the identity

$$\int_{-\infty}^{\infty}\frac{1}{x+\mathrm{i}y-t}\cdot\frac{1}{\xi+\mathrm{i}h-t}\,\mathrm{d}t = 0,$$

verifiable by contour integration (h and y are >0 here), and the semigroup convolution property of the Poisson kernel. The previous relation thus becomes

$$\int_{-\infty}^{\infty}\frac{x-t}{(x-t)^2+y^2}G_{\mathscr{D}}(t+\mathrm{i}h,\ 0)\,\mathrm{d}t$$
$$= \pi\int_{-\infty}^{\infty}\frac{y+h}{(x-\xi)^2+(y+h)^2}\,\omega_{\mathscr{D}}(\xi)\operatorname{sgn}\xi\,\mathrm{d}\xi.$$

Fixing $y>0$ for the moment, make $h\to 0$. According to the *third* of the above lemmas, the last formula then becomes

$$\int_{-\infty}^{\infty}\frac{x-t}{(x-t)^2+y^2}G_{\mathscr{D}}(t,0)\,\mathrm{d}t = \pi\int_{-\infty}^{\infty}\frac{y}{(x-\xi)^2+y^2}\,\omega_{\mathscr{D}}(\xi)\operatorname{sgn}\xi\,\mathrm{d}\xi.$$

Now make $y\to 0$, assuming that $x\neq 0$. Since $\omega_{\mathscr{D}}(\xi)$ is continuous at x, the *right side* tends to

$$\pi^2\omega_{\mathscr{D}}(x)\operatorname{sgn} x.$$

Also, by the *fourth* lemma, $G_{\mathscr{D}}(t,0)$ is $\mathrm{Lip}\,\frac{1}{2}$ at x. The left-hand integral therefore tends to the Cauchy principal value

$$\fint_{-\infty}^{\infty} \frac{G_{\mathscr{D}}(t,0)}{x-t}\,dt$$

(which exists!), according to an observation in §H.1 of Chapter III and the discussion preceding the last theorem above. We thus have

$$\omega_{\mathscr{D}}(x)\operatorname{sgn} x \;=\; \frac{1}{\pi^2}\fint_{-\infty}^{\infty} \frac{G_{\mathscr{D}}(t,0)}{x-t}\,dt$$

for $x \neq 0$, as asserted.

The legitimacy of the above reasoning required absolute convergence of

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{x-t}{(x-t)^2+y^2}\cdot\frac{t-\xi}{(t-\xi)^2+h^2}\,\omega_{\mathscr{D}}(\xi)\operatorname{sgn}\xi\,dt\,d\xi$$

which we must now establish. Fixing y and $h>0$ and $x\in\mathbb{R}$, we have

$$\left|\frac{x-t}{(x-t)^2+y^2}\cdot\frac{t-\xi}{(t-\xi)^2+h^2}\right| \;\leqslant\; \frac{\text{const.}}{(|t|+1)(|\xi-t|+1)}.$$

Wlog, let $\xi>0$. Then

$$\int_{-\infty}^{\infty} \frac{dt}{(|t|+1)(|\xi-t|+1)} \;\leqslant\; 2\int_0^{\infty} \frac{dt}{(t+1)(|\xi-t|+1)},$$

which we break up in turn as

$$2\int_0^{\xi/2} \;+\; 2\int_{\xi/2}^{3\xi/2} \;+\; 2\int_{3\xi/2}^{\infty}.$$

In the *first* of these integrals we use the inequality

$$|\xi-t| \;\geqslant\; \xi/2,$$

and, in the *second*,

$$t \;\geqslant\; \xi/2,$$

taking in the latter a new variable $s=t-\xi$. Both are thus easily seen to have values

$$\leqslant\; \text{const.}\frac{\log^+\xi+1}{\xi+1}$$

In the *third* integral, use the relation

$$t-\xi \;\geqslant\; t/3.$$

This shows that expression to be $\leqslant$ const.$1/(\xi+1)$.

In fine, then,

$$\int_{-\infty}^{\infty}\left|\frac{x-t}{(x-t)^2+y^2}\cdot\frac{t-\xi}{(t-\xi)^2+h^2}\right|dt \;\leqslant\; \text{const.}\frac{\log^+|\xi|+1}{|\xi|+1}$$

for *fixed* $x \in \mathbb{R}$ and $y, h > 0$. From the proof of the *first* lemma in this article, we know, however, that

$$|\omega_{\mathscr{D}}(\xi) \operatorname{sgn} \xi| \;=\; \omega_{\mathscr{D}}(\xi) \;\leqslant\; \frac{\text{const.}}{|\xi|+1}.$$

Absolute convergence of our double integral thus depends on the convergence of

$$\int_{-\infty}^{\infty} \frac{1+\log^+|\xi|}{(|\xi|+1)^2} \mathrm{d}\xi$$

which evidently *holds*. Our proof is complete.

Notation. *If* $\mathscr{D}$ is one of our domains with $0 \in \mathscr{D}$, we write, for $x > 0$,

$$\boxed{\Omega_{\mathscr{D}}(x) \;=\; \omega_{\mathscr{D}}((-\infty, -x] \cup [x, \infty),\; 0).}$$

Further work in this § will be based on the function $\Omega_{\mathscr{D}}$. For it, the theorem just proved has the

Corollary. *If* $0 \in \mathscr{D}$,

$$\boxed{\Omega_{\mathscr{D}}(x) \;=\; \frac{2}{\pi^2} \fint_{-\infty}^{\infty} \frac{x G_{\mathscr{D}}(t, 0)}{x^2 - t^2} \mathrm{d}t \qquad \text{for } x > 0.}$$

Proof. When $x > 0$,

$$\Omega_{\mathscr{D}}(x) \;=\; \omega_{\mathscr{D}}(x) + \omega_{\mathscr{D}}(-x).$$

Plug the formula furnished by the theorem into the right side.

Scholium. The preceding arguments practically suffice to work up a complete treatment of the L_2 theory of Hilbert transforms. The reader who has never studied that theory thus has an opportunity to learn it now.

If $f \in L_2(-\infty, \infty)$, let us write

$$u(z) \;=\; \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} f(t)\, \mathrm{d}t$$

and

$$\tilde{u}(z) \;=\; \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x-t}{|z-t|^2} f(t)\, \mathrm{d}t$$

for $\Im z > 0$; $\tilde{u}(z)$ is a *harmonic conjugate* of $u(z)$ in the upper half plane. By taking Fourier transforms and using Plancherel's theorem, one easily checks that

$$\int_{-\infty}^{\infty} |\tilde{u}(x+\mathrm{i}y)|^2\,\mathrm{d}x \;\leqslant\; \|f\|_2^2$$

for each $y > 0$. Following a previous discussion in this article and those of §§F.2 and H.1, Chapter III, we also see that

$$\tilde{f}(x) \;=\; \lim_{y\to 0} \tilde{u}(x+\mathrm{i}y)$$

exists a.e. Fatou's lemma then yields

$$\|\tilde{f}\|_2 \;\leqslant\; \|f\|_2$$

in view of the previous inequality.

It is in fact true that

$$\int_{-\infty}^{\infty} |\tilde{f}(x) - \tilde{u}(x+\mathrm{i}y)|^2\,\mathrm{d}x \;\longrightarrow\; 0$$

for $y \to 0$. This may be seen by noting that

$$\int_{-\infty}^{\infty} |\tilde{u}(x+\mathrm{i}y) - \tilde{u}(x+\mathrm{i}y')|^2\,\mathrm{d}x \;=\; \int_{-\infty}^{\infty} |u(x+\mathrm{i}y) - u(x+\mathrm{i}y')|^2\,\mathrm{d}x$$

for y and $y' > 0$, which may be verified using Fourier transforms and Plancherel's theorem. According to elementary properties of the Poisson kernel, the right-hand integral is *small* when $y > 0$ and $y' > 0$ are, as long as $f \in L_2$. Fixing a small $y > 0$ and then making $y' \to 0$ in the *left-hand* integral, we find that

$$\int_{-\infty}^{\infty} |\tilde{f}(x) - \tilde{u}(x+\mathrm{i}y)|^2\,\mathrm{d}x$$

is small by applying Fatou's lemma.

Once this is known, it is easy to prove that

$$\tilde{\tilde{f}}(x) \;=\; -f(x) \quad \text{a.e.}$$

by following almost exactly the argument used in proving the last theorem above. (Note that $(\log^+|\xi| + 1)/(|\xi| + 1) \in L_2(-\infty, \infty)$.) This must then imply that

$$\|f\|_2 \;\leqslant\; \|\tilde{f}\|_2,$$

so that finally

$$\|f\|_2 = \|\tilde{f}\|_2.$$

To complete this development, we need the result that the Cauchy principal value

$$\frac{1}{\pi}⨍_{-\infty}^{\infty} \frac{f(t)}{x-t}\,dt$$

exists and equals $\tilde{f}(x)$ a.e. That is the content of

Problem 25

Let $f \in L_p(-\infty, \infty)$, $p \geqslant 1$. Show that

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x-t}{(x-t)^2+y^2} f(t)\,dt - \frac{1}{\pi}\int_{|t-x|\geqslant y} \frac{f(t)}{x-t}\,dt$$

tends to zero as $y \to 0$ if

$$\frac{1}{y}\int_{x-y}^{x+y} |f(t)-f(x)|\,dt \longrightarrow 0$$

for $y \to 0$, and hence for *almost every* real x. (The set of x for which the last condition holds is called the *Lebesgue set* of f.) (Hint. One may wlog take f to be of compact support, making $\|f\|_1 < \infty$. Choosing a small $\delta > 0$, one considers values of y *between* 0 and δ, for which the difference in question can be written as

$$\frac{1}{\pi}\int_0^y \frac{\tau(f(x-\tau)-f(x+\tau))}{\tau^2+y^2}\,d\tau$$
$$+ \frac{1}{\pi}\left(\int_y^\delta + \int_\delta^\infty\right)\left(\frac{\tau}{\tau^2+y^2} - \frac{1}{\tau}\right)(f(x-\tau)-f(x+\tau))\,d\tau.$$

If the stipulated condition holds at x, the *first* of these integrals clearly $\to 0$ as $y \to 0$. For *fixed* $\delta > 0$, the integral from δ to ∞ is $\leqslant 2y^2\|f\|_1/\delta^3$ and this $\to 0$ as $y \to 0$. The integral from y to δ is in absolute value

$$\leqslant y^2 \int_y^\delta \frac{|f(x-\tau)-f(x+\tau)|}{\tau^3}\,d\tau.$$

Integrate this by parts.)

2. An estimate for harmonic measure

Given one of our domains $\mathscr{D}$ with $0 \in \mathscr{D}$, the function $\Omega_{\mathscr{D}}(x) = \omega_{\mathscr{D}}((-\infty, -x] \cup [x, \infty), 0)$ is equal to

$$\frac{2}{\pi^2}⨍_{-\infty}^{\infty} \frac{xG_{\mathscr{D}}(t,0)}{x^2-t^2}\,dt$$

by the corollary near the end of the preceding article. The Green's function $G_{\mathscr{D}}(t,0)$ of course vanishes on $\partial\mathscr{D} = \mathbb{R} \cap (\sim \mathscr{D})$, and our attention is restricted to domains $\mathscr{D}$ having *bounded* intersection with $\mathbb{R}$. The above Cauchy principal value thus reduces to an ordinary integral for large x, and we have

$$\Omega_{\mathscr{D}}(x) \sim \frac{2}{\pi^2 x}\int_{-\infty}^{\infty} G_{\mathscr{D}}(t,0)\,\mathrm{d}t \qquad \text{for } x \to \infty,$$

i.e., *in terms of the Phragmén–Lindelöf function* $Y_{\mathscr{D}}(z)$ *for* $\mathscr{D}$, defined in §A.2,

$$\Omega_{\mathscr{D}}(x) \sim \frac{2Y_{\mathscr{D}}(0)}{\pi x}, \qquad x \to \infty.$$

It is remarkable that an *inequality* resembling this asymptotic relation holds for *all* positive x; this means that the kind of comparison spoken of at the beginning of the present § is available.

Theorem. *If* $0 \in \mathscr{D}$,

$$\Omega_{\mathscr{D}}(x) \leqslant \frac{Y_{\mathscr{D}}(0)}{x} \qquad \textit{for } x > 0.$$

Proof. By comparison of harmonic measure for $\mathscr{D}$ with that for another smaller domain that depends on x.

Given $x > 0$, we let $E_x = E \cup (-\infty, -x] \cup [x, \infty)$ and then put $\mathscr{D}_x = \mathbb{C} \sim E_x$:

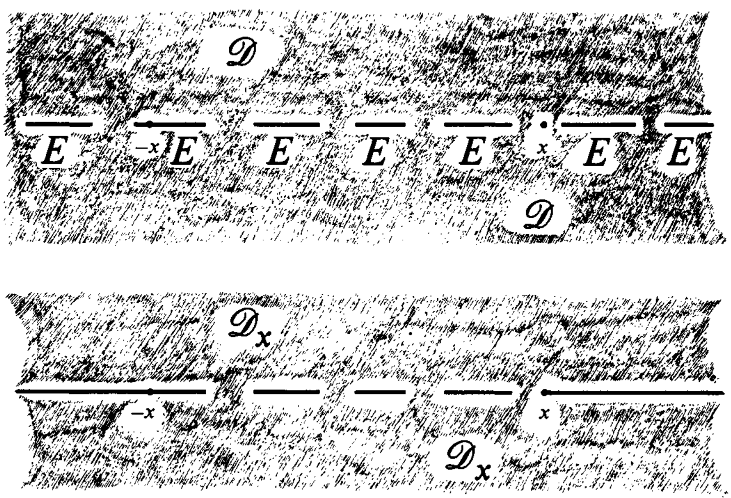

Figure 152

We have $\mathscr{D}_x \subseteq \mathscr{D}$. On comparing $\omega_{\mathscr{D}_x}((-\infty, -x] \cup [x, \infty), \zeta)$ with $\omega_{\mathscr{D}}((-\infty, -x] \cup [x, \infty), \zeta)$ on E_x, we see that the *former* is *larger* than the *latter* for $\zeta \in \mathscr{D}_x$. Hence, putting $\zeta = 0$, we get

$$\Omega_{\mathscr{D}}(x) \leqslant \Omega_{\mathscr{D}_x}(x).$$

Take any number $\rho > 1$. Applying the corollary near the end of the previous article and noting that $G_{\mathscr{D}_x}(t, 0)$ *vanishes* for $t \in E_x \supseteq (-\infty, -x] \cup [x, \infty)$, we have

$$\Omega_{\mathscr{D}_x}(\rho x) = \frac{2}{\pi^2}\int_{-x}^{x} \frac{\rho x G_{\mathscr{D}_x}(t, 0)}{\rho^2 x^2 - t^2}\,\mathrm{d}t.$$

Since $\mathscr{D}_x \subseteq \mathscr{D}$, $G_{\mathscr{D}_x}(t, 0) \leqslant G_{\mathscr{D}}(t, 0)$, so the right-hand integral is

$$\leqslant \frac{2\rho}{\pi^2(\rho^2 - 1)x}\int_{-x}^{x} G_{\mathscr{D}}(t, 0)\,\mathrm{d}t \leqslant \frac{2\rho}{\pi^2(\rho^2 - 1)x}\int_{-\infty}^{\infty} G_{\mathscr{D}}(t, 0)\,\mathrm{d}t.$$

By the formula for $Y_{\mathscr{D}}(z)$ furnished by the *second* theorem of §A2, we thus get

$$\Omega_{\mathscr{D}_x}(\rho x) \leqslant \frac{2\rho}{\pi(\rho^2 - 1)}\frac{Y_{\mathscr{D}}(0)}{x}.$$

In order to complete the proof, we show that $\Omega_{\mathscr{D}_x}(\rho x)/\Omega_{\mathscr{D}_x}(x)$ is *bounded below* by a quantity *depending only on* ρ, and then use the inequality just established together with the previous one.

To compare $\Omega_{\mathscr{D}_x}(\rho x)$ with $\Omega_{\mathscr{D}_x}(x)$, take a *third* domain

$$\mathscr{E} = \mathbb{C} \sim ((-\infty, -x] \cup [x, \infty)):$$

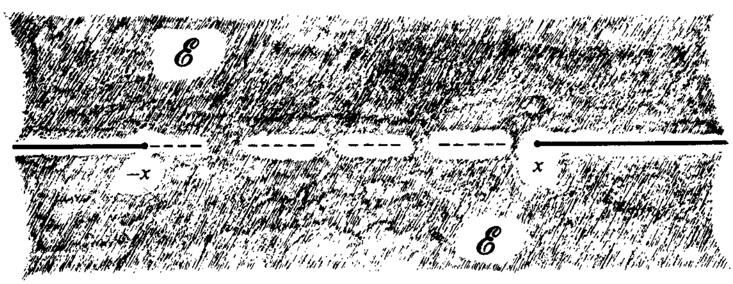

Figure 153

Note that $\mathscr{D}_x \subseteq \mathscr{E}$ and $\partial\mathscr{D}_x = E_x$ consists of $\partial\mathscr{E}$ together with the part of E lying in the segment $[-x, x]$. For $\zeta \in \mathscr{D}_x$ (and $\rho > 1$), a formula from §B.1 of Chapter VII tells us that

$$\begin{aligned} &\omega_{\mathscr{D}_x}((-\infty, -\rho x] \cup [\rho x, \infty),\ \zeta) \\ &= \omega_{\mathscr{E}}((-\infty, -\rho x] \cup [\rho x, \infty),\ \zeta) \\ &\qquad - \int_{E \cap \mathscr{E}} \omega_{\mathscr{E}}((-\infty, -\rho x] \cup [\rho x, \infty),\ t)\,\mathrm{d}\omega_{\mathscr{D}_x}(t, \zeta), \end{aligned}$$

whence, taking $\zeta = 0$,

$$\Omega_{\mathscr{D}_x}(\rho x) = \omega_{\mathscr{E}}((-\infty, -\rho x] \cup [\rho x, \infty), 0) - \int_{E\cap\mathscr{E}} \omega_{\mathscr{E}}((-\infty, -\rho x] \cup [\rho x, \infty), t) d\omega_{\mathscr{D}_x}(t, 0).$$

Also,

$$\Omega_{\mathscr{D}_x}(x) = 1 - \int_{E\cap\mathscr{E}} d\omega_{\mathscr{D}_x}(t, 0).$$

The harmonic measure $\omega_{\mathscr{E}}((-\infty, -\rho x] \cup [\rho x, \infty), t)$ can be computed explicitly by making the Joukowski mapping

$$\zeta \longrightarrow w = \frac{x}{\zeta} - \sqrt{\left(\frac{x^2}{\zeta^2} - 1\right)}$$

of $\mathscr{E}$ onto $\Delta = \{|w| < 1\}$:

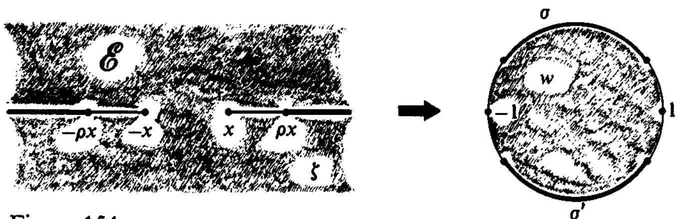

Figure 154

This conformal map takes $[-x, x]$ to the diameter $[-1, 1]$, and 0 to 0. The union of the (two-sided!) intervals $(-\infty, -\rho x]$ and $[\rho x, \infty)$ on $\partial\mathscr{E}$ is taken onto that of two arcs, σ and σ', on $\{|w| = 1\}$, the first symmetric about i and the second symmetric about $-$i. For $\zeta \in \mathscr{E}$, $\omega_{\mathscr{E}}((-\infty, -\rho x] \cup [\rho x, \infty), \zeta)$ is the *sum* of the harmonic measures of these two arcs in Δ, seen from the point w therein corresponding to ζ. When $\zeta = t$ is *real*, this sum is just $2\omega_\Delta(\sigma, u)$, u being the point of $(-1, 1)$ corresponding to t. However, from the rudiments of complex variable theory, the level lines of $\omega_\Delta(\sigma, w)$ are just the *circles* through the endpoints of σ. From a glance at the following diagram, it is hence obvious that $\omega_\Delta(\sigma, u)$ has its *maximum* for $-1 \leqslant u \leqslant 1$ when $u = 0$:

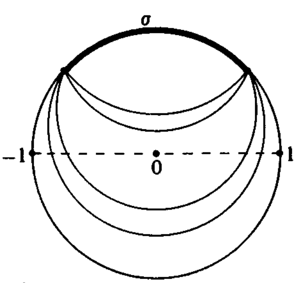

Figure 155

Going back to $\mathscr{E}$, we see that

$$\omega_{\mathscr{E}}((-\infty, -\rho x]\cup[\rho x, \infty),\ t) \ \leqslant\ \omega_{\mathscr{E}}((-\infty, -\rho x]\cup[\rho x, \infty),\ 0)$$

when $-x \leqslant t \leqslant x$. Plugging this into the above formula for $\Omega_{\mathscr{D}_x}(\rho x)$, we find that

$$\Omega_{\mathscr{D}_x}(\rho x) \ \geqslant\ \omega_{\mathscr{E}}((-\infty, -\rho x]\cup[\rho x, \infty),\ 0)$$
$$\times\left\{1 - \int_{E\cap\mathscr{E}} d\omega_{\mathscr{D}_x}(t, 0)\right\}.$$

The quantity in curly brackets is just $\Omega_{\mathscr{D}_x}(x)$, so we have

$$\frac{\Omega_{\mathscr{D}_x}(\rho x)}{\Omega_{\mathscr{D}_x}(x)} \ \geqslant\ \omega_{\mathscr{E}}((-\infty, -\rho x]\cup[\rho x, \infty),\ 0).$$

Here, the right side clearly depends only on ρ; this is the relation we set out to obtain.

From the inequality just found together with the two others established at the beginning of this proof, we now get

$$\Omega_{\mathscr{D}}(x) \ \leqslant\ \Omega_{\mathscr{D}_x}(x) \ \leqslant\ \frac{\Omega_{\mathscr{D}_x}(\rho x)}{\omega_{\mathscr{E}}((-\infty, -\rho x]\cup[\rho x, \infty),\ 0)}$$
$$\leqslant\ \frac{2\rho}{\pi(\rho^2-1)\omega_{\mathscr{E}}((-\infty, -\rho x]\cup[\rho x, \infty),\ 0)}\cdot\frac{Y_{\mathscr{D}}(0)}{x}.$$

The front factor in the right-hand member depends only on the parameter ρ; let us compute its value. The two arcs σ and σ' both subtend angles $2\arcsin(\Gamma/\rho)$ at 0. Therefore

$$\omega_{\mathscr{E}}((-\infty, -\rho x]\cup[\rho x, \infty),\ 0) \ =\ 2\omega_{\Delta}(\sigma, 0) \ =\ \frac{2}{\pi}\arcsin\frac{1}{\rho},$$

and the factor in question equals

$$\frac{\rho}{(\rho^2-1)\arcsin\dfrac{1}{\rho}}.$$

It is readily ascertained (put $1/\rho = \sin\alpha$!) that the expression just written *decreases* for $\rho > 1$. Making $\rho \to \infty$, we get the limit 1, whence

$$\Omega_{\mathscr{D}}(x) \ \leqslant\ Y_{\mathscr{D}}(0)/x, \qquad\qquad \text{Q.E.D.}$$

Remark. An inequality almost as good as the one just established can be obtained with considerably less effort. By the first theorem of the preceding article, we have, for $y > 0$,

$$G_{\mathscr{D}}(\mathrm{i}y, 0) = \log\frac{1}{y} + \int_{-\infty}^{\infty} \log|\mathrm{i}y - t|\,\mathrm{d}\omega_{\mathscr{D}}(t, 0)$$

$$= \int_{-\infty}^{\infty} \log\sqrt{\left(1 + \frac{t^2}{y^2}\right)}\mathrm{d}\omega_{\mathscr{D}}(t, 0),$$

a quantity clearly $\geqslant \Omega_{\mathscr{D}}(y)\log\sqrt{2}$. On the other hand,

$$G_{\mathscr{D}}(\mathrm{i}y, 0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yG_{\mathscr{D}}(t, 0)}{y^2 + t^2}\mathrm{d}t$$

as in the proof of the third lemma from that article. Here, the right side is

$$\leqslant \frac{1}{\pi y}\int_{-\infty}^{\infty} G_{\mathscr{D}}(t, 0)\mathrm{d}t = \frac{Y_{\mathscr{D}}(0)}{y},$$

so the previous relation yields

$$\Omega_{\mathscr{D}}(y) \leqslant \frac{2}{\log 2}\frac{Y_{\mathscr{D}}(0)}{y}.$$

Problem 26

For $0 < \rho < \frac{1}{2}$, let E_ρ be the union of the segments

$$\left[\frac{2n-1}{2} - \rho, \quad \frac{2n-1}{2} + \rho\right], \qquad n \in \mathbb{Z};$$

these are just the intervals of length 2ρ centered at the *half odd* integers. Denote the component $[(2n-1)/2 - \rho,\ (2n-1)/2 + \rho]$ of E_ρ by J_n (it would be more logical to write $J_n(\rho)$). $\mathscr{D}_\rho = \mathbb{C} \sim E_\rho$ is a domain of the kind considered in §A, and, by *Carleson's theorem* from §A.1,

$$\omega_{\mathscr{D}_\rho}(J_n, 0) \leqslant \frac{K_\rho}{n^2 + 1}.$$

The purpose of this problem is to obtain quantitative information about the asymptotic behaviour of the best value for K_ρ as $\rho \to 0$.

(a) Show that $Y_{\mathscr{D}_\rho}(0) \sim (1/\pi)\log(1/\rho)$ as $\rho \to 0$. (Hint. In $\mathscr{D}_\rho$, consider the harmonic function

$$\log\left|\frac{\cos \pi z}{\sin \pi\rho} + \sqrt{\left(\frac{\cos^2 \pi z}{\sin^2 \pi\rho} - 1\right)}\right|. \text{)}$$

(b) By making an appropriate limiting argument, adapt the theorem just proved to the domain $\mathscr{D}_\rho$ and hence show that

$$\Omega_{\mathscr{D}_\rho}(x) \leqslant Y_{\mathscr{D}_\rho}(0)/x \qquad \text{for } x > 0.$$

(c) For $n \geqslant 1$, show that

$$\omega_{\mathscr{D}_\rho}(J_{n+1},\ 0) \ \leqslant\ \omega_{\mathscr{D}_\rho}(J_n,\ 0).$$

(Hint:

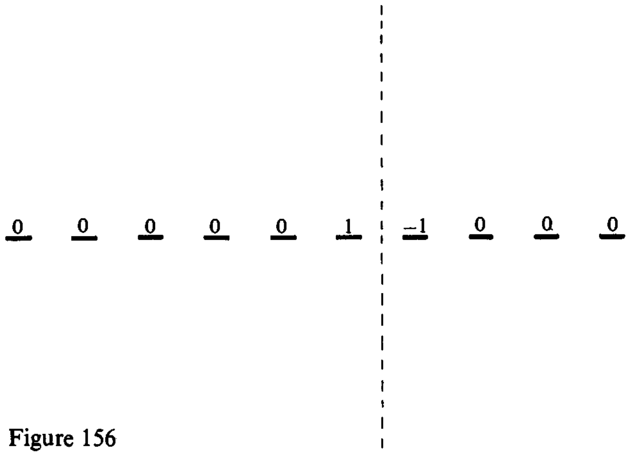

Figure 156

.)

(d) Hence show that, for $n \geqslant 3$,

$$\omega_{\mathscr{D}_\rho}(J_n, 0) \ \leqslant\ \left(C\log\frac{1}{\rho}\right)\Big/ n^2.)$$

with a numerical constant C independent of ρ.
(Hint: $\Omega_{\mathscr{D}_\rho}(n) \ \geqslant\ 2\sum_{k=n}^{2n}\omega_{\mathscr{D}_\rho}(J_{k+1},\ 0)$.)

(e) Show that the *smallest* constant K_ρ such that $\omega_{\mathscr{D}_\rho}(J_n,\ 0) \ \leqslant\ K_\rho/(n^2+1)$ for all n satisfies

$$K_\rho \ \geqslant\ C'\log\frac{1}{\rho}$$

with a constant C' independent of ρ.
(Hint. This is harder than parts (a)–(d). Fixing any $\rho > 0$, write, for large R, $E_R = E_\rho \cup (-\infty, -R] \cup [R, \infty)$, and then put $\mathscr{D}_R = \mathbb{C} \sim E_R$. As $R \to \infty$, $G_{\mathscr{D}_R}(t, 0)$ increases to $G_{\mathscr{D}_\rho}(t, 0)$, so $Y_{\mathscr{D}_R}(0)$ increases to $Y_{\mathscr{D}_\rho}(0)$. For each R, by the *first* theorem of the previous article,

$$G_{\mathscr{D}_R}(z, w) \ =\ \log\frac{1}{|z-w|} \ +\ \int_{E_R} \log|w-s|\,d\omega_{\mathscr{D}_R}(s, z),$$

whence

$$G_{\mathscr{D}_R}(t, 0) + G_{\mathscr{D}_R}(-t, 0) \ =\ \int_{E_R} \log\left|1 - \frac{s^2}{t^2}\right| d\omega_{\mathscr{D}_R}(s, 0).$$

Fix any integer $A > 0$. Then $\int_{-A}^{A} G_{\mathscr{D}_\rho}(t,0)\mathrm{d}t$ is the limit, as $R \to \infty$, of $\int_{-\infty}^{\infty}\int_0^A \log|1-(s^2/t^2)|\mathrm{d}t\,\mathrm{d}\omega_{\mathscr{D}_R}(s,0)$. Taking an arbitrary large M, which for the moment we *fix*, we break up this double integral as

$$\int_{-M}^{M}\int_0^A + \int_{|s|>M}\int_0^A.$$

To study the two terms of this sum, first evaluate

$$\int_0^A \log\left|1-\frac{s^2}{t^2}\right|\mathrm{d}t;$$

for $|s| > A$ this can be done by direct computation, and, for $|s| < A$, by using the identity

$$\int_0^A \log\left|1-\frac{s^2}{t^2}\right|\mathrm{d}t \;=\; -\int_A^{\infty} \log\left|1-\frac{s^2}{t^2}\right|\mathrm{d}t.$$

Regarding $\int_{-M}^{M}\int_0^A \log|1-(s^2/t^2)|\mathrm{d}t\,\mathrm{d}\omega_{\mathscr{D}_R}(s,0)$, we may use the fact that $\omega_{\mathscr{D}_R}(S,0) \longrightarrow \omega_{\mathscr{D}_\rho}(S,0)$ as $R \to \infty$ for *bounded* $S \subseteq \mathbb{R}$, and then plug in the inequality

$$\omega_{\mathscr{D}_\rho}(J_n, 0) \;\leqslant\; K_\rho/(n^2+1)$$

together with the result of the computation just indicated. In this way we easily see that $\lim_{R\to\infty}\int_{-M}^{M}\int_0^A \;\leqslant\; CK_\rho$ with a constant C independent of A, M, and ρ.

In order to estimate

$$\int_{|s|>M}\int_0^A \log\left|1-\frac{s^2}{t^2}\right|\mathrm{d}t\,\mathrm{d}\omega_{\mathscr{D}_R}(s,0),$$

use the fact that

$$\Omega_{\mathscr{D}_R}(s) \;\leqslant\; \frac{Y_{\mathscr{D}_R}(0)}{s} \;\leqslant\; \frac{Y_{\mathscr{D}_\rho}(0)}{s}$$

(where $Y_{\mathscr{D}_\rho}(0)$, as we already know, is *finite*) together with the value of the inner integral, already computed, and integrate by parts. In this way one finds an estimate *independent* of R which, *for fixed* A, is *very small* if M is *large enough*. Combining this result with the previous one and then making $M \to \infty$, one sees that

$$\int_{-A}^{A} G_{\mathscr{D}_\rho}(t,0)\,\mathrm{d}t \;\leqslant\; CK_\rho$$

with C independent of A and of ρ.)

Remark. In the circumstances of the preceding problem $G_{\mathscr{D}_\rho}(z,0)$ must, when $\rho \to 0$, tend to ∞ for each z not equal to a half odd integer, and it is interesting to see how fast that happens. Fix any such $z \neq 0$. Then, given $\rho > 0$ we have, working with the domains $\mathscr{D}_R$ used in part (e) of the problem,

$$G_{\mathscr{D}_\rho}(z,0) = \lim_{R\to\infty} G_{\mathscr{D}_R}(z,0).$$

Here,

$$G_{\mathscr{D}_R}(z,0) = \log\frac{1}{|z|} + \int_{-\infty}^{\infty} \log|z-t|\,d\omega_{\mathscr{D}_R}(t,0)$$

$$= O(1) + \int_{-\infty}^{\infty} \log^+|t|\,d\omega_{\mathscr{D}_R}(t,0),$$

where the $O(1)$ term depends on z but is independent of R, and of ρ, when the latter is small enough.

Taking an $M > 1$, we rewrite the last integral on the right as

$$\int_{|t|<M} + \int_{|t|\geq M},$$

and thus find it to be

$$\leqslant \log M - \int_M^\infty \log t\,d\Omega_{\mathscr{D}_R}(t)$$

$$= \log M + \Omega_{\mathscr{D}_R}(M)\log M + \int_M^\infty \frac{\Omega_{\mathscr{D}_R}(t)}{t}\,dt.$$

Plug the inequalities $\Omega_{\mathscr{D}_R}(t) \leqslant Y_{\mathscr{D}_R}(0)/t$ and $Y_{\mathscr{D}_R}(0) \leqslant Y_{\mathscr{D}_\rho}(0)$ into the expression on the right. Then, referring to the previous relation and making $R \to \infty$, we see that

$$G_{\mathscr{D}_\rho}(z,0) \leqslant O(1) + \log M + Y_{\mathscr{D}_\rho}(0)\frac{\log M + 1}{M}.$$

By part (a) of the problem, $Y_{\mathscr{D}_\rho}(0) = O(1) + (1/\pi)\log(1/\rho)$. Hence, choosing $M = (1/\pi)\log(1/\rho)\log\log(1/\rho)$ in the last relation, we get

$$G_{\mathscr{D}_\rho}(z,0) \leqslant O(1) + \log\log\frac{1}{\rho} + \log\log\log\frac{1}{\rho}.$$

This order of growth seems rather slow. One would have expected $G_{\mathscr{D}_\rho}(z,0)$ to behave like $\log(1/\rho)$ for small ρ when z is fixed.

3. The energy integral again

The result of the preceding article already has some applications to the project described at the beginning of this §. Suppose that the majorant $M(t) \geqslant 0$ is defined and *even* on $\mathbb{R}$. Taking $M(t)$ to be identically zero in a neighborhood of 0 involves no real loss of generality. If $M(t)$ is also *increasing* on $[0, \infty)$, the Poisson integral

$$\int_{\partial\mathscr{D}} v(t)\,\mathrm{d}\omega_{\mathscr{D}}(t, 0)$$

for a function $v(z)$ subharmonic in one of our domains $\mathscr{D}$ with $0 \in \mathscr{D}$ and satisfying

$$v(t) \;\leqslant\; M(t), \qquad t \in \partial\mathscr{D},$$

has the simple majorant

$$Y_{\mathscr{D}}(0) \cdot \int_0^\infty \frac{M(t)}{t^2}\,\mathrm{d}t.$$

The entire dependence of the Poisson integral on the domain $\mathscr{D}$ is thus expressed by means of the single factor $Y_{\mathscr{D}}(0)$ occurring in this second expression.

To see this, recall that $\omega_{\mathscr{D}}((-\infty, -t] \cup [t, \infty),\ 0) = \Omega_{\mathscr{D}}(t)$ for $t > 0$; the given majoration on $v(t)$ therefore makes the Poisson integral $\leqslant\; -\int_0^\infty M(t)\,\mathrm{d}\Omega_{\mathscr{D}}(t)$, which here is equal to

$$\int_0^\infty \Omega_{\mathscr{D}}(t)\,\mathrm{d}M(t).$$

Since $M(t)$ is increasing on $[0, \infty)$, we may substitute the relation $\Omega_{\mathscr{D}}(t) \leqslant Y_{\mathscr{D}}(0)/t$ proved in the preceding article into the last expression, showing it to be

$$\leqslant\; Y_{\mathscr{D}}(0) \int_0^\infty \frac{\mathrm{d}M(t)}{t} \;=\; Y_{\mathscr{D}}(0) \int_0^\infty \frac{M(t)}{t^2}\,\mathrm{d}t.$$

This argument cannot be applied to *general* even majorants $M(t) \geqslant 0$, because the relation $\Omega_{\mathscr{D}}(t) \leqslant Y_{\mathscr{D}}(0)/t$ cannot be differentiated to yield $\mathrm{d}\omega_{\mathscr{D}}(t, 0) \;\leqslant\; (Y_{\mathscr{D}}(0)/t^2)\,\mathrm{d}t$. Indeed, when $x \in \partial\mathscr{D} = E$ gets near any of the *endpoints a* of the intervals making up that set, $\mathrm{d}\omega_{\mathscr{D}}(x, 0)/\mathrm{d}x$ gets *large* like a multiple of $|x - a|^{-1/2}$ (see the second lemma of article 1 and the remark following it). We are not supposing *anything* about the *disposition* of these intervals except that they be *finite in number*; there may *otherwise* be *arbitrarily many* of them. It is therefore not possible to bound $\int_{-\infty}^\infty M(t)\,\mathrm{d}\omega_{\mathscr{D}}(t, 0)$ by an expression involving *only* $\int_0^\infty (M(t)/t^2)\,\mathrm{d}t$ for *general* even majorants $M(t) \geqslant 0$; some *additional regularity properties* of $M(t)$ are required and must be taken into account. A very useful instrument for this purpose turns out to be the *energy* introduced in §B.5 which has already played such an important rôle in §B. Application of that notion to matters like the one now under discussion goes back to the 1962 paper of Beurling and Malliavin. The material of that paper will be taken up in Chapter XI, where we will use the results established in the present §.

Appearance of the energy here is due to the following

Lemma. *Let* $0\in\mathscr{D}$. For $x\neq 0$,

$$G_{\mathscr{D}}(x,0)+G_{\mathscr{D}}(-x,0) \;=\; \frac{1}{x}\int_0^{\infty}\log\left|\frac{x+t}{x-t}\right|\mathrm{d}(t\Omega_{\mathscr{D}}(t)).$$

Proof. By the *second* theorem of article 1,

$$G_{\mathscr{D}}(x,0) \;=\; -\!\!\!\!\!\!\;\;\int_{-\infty}^{\infty}\frac{\omega_{\mathscr{D}}(t)\operatorname{sgn} t}{x-t}\,\mathrm{d}t \qquad \text{for } x\neq 0,$$

where

$$\omega_{\mathscr{D}}(t) \;=\; \begin{cases}\omega_{\mathscr{D}}((-\infty,t],\ 0), & t<0,\\ \omega_{\mathscr{D}}([t,\infty),\ 0), & t>0.\end{cases}$$

Thence,

$$G_{\mathscr{D}}(x,0)+G_{\mathscr{D}}(-x,0) \;=\; -\!\!\!\!\!\!\;\;\int_{-\infty}^{\infty}\frac{2t\operatorname{sgn} t\,\omega_{\mathscr{D}}(t)}{t^2-x^2}\,\mathrm{d}t$$

$$=\; -\!\!\!\!\!\!\;\;\int_{0}^{\infty}\frac{2t}{t^2-x^2}\,\Omega_{\mathscr{D}}(t)\,\mathrm{d}t,$$

since $\omega_{\mathscr{D}}(t)+\omega_{\mathscr{D}}(-t)=\Omega_{\mathscr{D}}(t)$ for $t>0$.

Assuming wlog that $x>0$, we take a small $\varepsilon>0$ and apply partial integration to the two integrals in

$$\left(\int_0^{x-\varepsilon}+\int_{x+\varepsilon}^{\infty}\right)\frac{2}{t^2-x^2}\,t\Omega_{\mathscr{D}}(t)\,\mathrm{d}t,$$

getting

$$\left(\frac{t\Omega_{\mathscr{D}}(t)}{x}\log\left|\frac{t-x}{t+x}\right|\right)\left(\Big]_0^{x-\varepsilon}+\Big]_{x+\varepsilon}^{\infty}\right)$$

$$+\;\left(\int_0^{x-\varepsilon}+\int_{x+\varepsilon}^{\infty}\right)\frac{1}{x}\log\left|\frac{x+t}{x-t}\right|\mathrm{d}(t\Omega_{\mathscr{D}}(t)).$$

The function $\Omega_{\mathscr{D}}(t)$ is 1 for $t>0$ near 0 and $O(1/t)$ for large t; it is moreover Lip $\frac{1}{2}$ at each $x>0$ by the second lemma of article 1. The sum of the integrated terms therefore tends to 0 as $\varepsilon \to 0$, and we see that

$$\fint_0^\infty \frac{2t}{t^2 - x^2}\Omega_{\mathscr{D}}(t)\mathrm{d}t \;=\; \frac{1}{x}\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}(t\Omega_{\mathscr{D}}(t)).$$

Since the left side equals $G_{\mathscr{D}}(x,0) + G_{\mathscr{D}}(-x,0)$, the lemma is proved.

In the language of §B.5, $x(G_{\mathscr{D}}(x,0) + G_{\mathscr{D}}(-x,0))$ is the *Green potential* of $\mathrm{d}(t\Omega_{\mathscr{D}}(t))$. Here, since we are assuming $\mathscr{D} \cap \mathbb{R} = \mathbb{R} \sim E$ to be bounded,

$$\Omega_{\mathscr{D}}(x) \;=\; \frac{1}{\pi^2}\fint_0^\infty \frac{2x}{x^2 - t^2} G_{\mathscr{D}}(t,0)\mathrm{d}t$$

has, for large x, a convergent expansion of the form

$$\frac{a_1}{x} + \frac{a_3}{x^3} + \frac{a_5}{x^5} + \cdots,$$

so that

$$\mathrm{d}(t\Omega_{\mathscr{D}}(t)) \;=\; -\left(\frac{2a_3}{t^3} + \frac{4a_5}{t^5} + \cdots\right)\mathrm{d}t$$

for large t. Using this fact it is easy to verify that

$$\int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}(t\Omega_{\mathscr{D}}(t))\,\mathrm{d}(x\Omega_{\mathscr{D}}(x))$$

is *absolutely convergent; this double integral thus coincides with the energy*

$$E(\mathrm{d}(t\Omega_{\mathscr{D}}(t)),\ \mathrm{d}(t\Omega_{\mathscr{D}}(t)))$$

defined in §B.5.

Theorem. *If* $0 \in \mathscr{D}$,

$$\boxed{E(\mathrm{d}(t\Omega_{\mathscr{D}}(t)),\ \mathrm{d}(t\Omega_{\mathscr{D}}(t))) \;\leqslant\; \pi(Y_{\mathscr{D}}(0))^2.}$$

Proof. By the lemma, the left side, equal to the above double integral, can be rewritten as

$$\int_0^\infty x[G_{\mathscr{D}}(x,0) + G_{\mathscr{D}}(-x,0)]\mathrm{d}(x\Omega_{\mathscr{D}}(x)).$$

Here, $G_{\mathscr{D}}(x,0) + G_{\mathscr{D}}(-x,0) \geqslant 0$ and $\Omega_{\mathscr{D}}(x)$ is *decreasing*, so the last expression is

$$\leqslant\; \int_0^\infty [G_{\mathscr{D}}(x,0) + G_{\mathscr{D}}(-x,0)]x\Omega_{\mathscr{D}}(x)\mathrm{d}x.$$

From the theorem of the preceding article we have $x\Omega_{\mathscr{D}}(x) \leqslant Y_{\mathscr{D}}(0)$, so this is in turn

$$\leqslant Y_{\mathscr{D}}(0)\int_{-\infty}^{\infty} G_{\mathscr{D}}(x,0)\,\mathrm{d}x,$$

which, however equals $\pi(Y_{\mathscr{D}}(0))^2$ by the second theorem of §A.2.

We are done.

This theorem will be used in establishing the remaining results of the present §. For that work it will be convenient to have at hand an *alternative notation for the energy*

$$E(\mathrm{d}\rho(t),\ \mathrm{d}\rho(t)).$$

Suppose that we have a real Green potential

$$u(x) = \int_0^{\infty} \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t).$$

If the double integral

$$\int_0^{\infty}\int_0^{\infty} \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t)\,\mathrm{d}\rho(x)$$

used to define $E(\mathrm{d}\rho(t),\ \mathrm{d}\rho(t))$ *is absolutely convergent, we write*

$$\boxed{\|u\|_E^2 = E(\mathrm{d}\rho(t),\ \mathrm{d}\rho(t)).}$$

If we have another such Green potential

$$v(x) = \int_0^{\infty} \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\sigma(t),$$

we similarly write

$$\boxed{\langle u, v\rangle_E = E(\mathrm{d}\rho(t),\ \mathrm{d}\sigma(t)).}$$

$\langle\ ,\ \rangle_E$ is a bilinear form on the collection of real Green potentials of this kind; according to the remark at the end of §B.5 it is *positive definite*. The reader may wonder whether our use of the symbol $\|u\|_E$ to denote $\sqrt{(E(\mathrm{d}\rho(t), \mathrm{d}\rho(t)))}$ is *legitimate*; could not *the same* function $u(x)$ be the Green potential of *two different measures*? That this cannot occur is easily seen, and boils down to showing that if $\rho(x)$ is *not constant*, the Green potential

$$u(x) = \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t)$$

cannot be $\equiv 0$ on $[0, \infty)$ (provided, of course, that the double integral used to define $E(\mathrm{d}\rho(t), \mathrm{d}\rho(t))$ is absolutely convergent). Here, we have $E(\mathrm{d}\rho(t), \mathrm{d}\rho(t)) = \int_0^\infty u(x)\mathrm{d}\rho(x)$. Hence, if $u(x) \equiv 0$, the left-hand side is also zero. Then, however, $\rho(x)$ is constant by the second lemma of §B.5.

4. Harmonic estimation in $\mathscr{D}$

We are now able to give a fairly general result of the kind envisioned at the beginning of this §. Suppose we have an *even* majorant $M(t) \geqslant 0$ with $M(0) = 0$. In the case where $M(x)/x$ is a *Green potential*

$$\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t)$$

with the double integral defining $E(\mathrm{d}\rho(t), \mathrm{d}\rho(t)) = \|M(x)/x\|_E^2$ *absolutely convergent*, the following is true:

Theorem. *Let $M(t)$ be a majorant of the kind just described. Given one of our domains $\mathscr{D}$ containing* 0, *suppose we have a function $v(z)$, subharmonic in $\mathscr{D}$ and continuous up to $\partial\mathscr{D}$, with*

$$v(z) \leqslant A|\Im z| + \mathrm{O}(1)$$

for some real (sic!) *A, and*

$$v(t) \leqslant M(t) \qquad \textit{for } t \in \partial\mathscr{D}.$$

Then

$$\boxed{v(0) \leqslant Y_{\mathscr{D}}(0)\left\{A + \int_0^\infty \frac{M(t)}{t^2}\mathrm{d}t + \sqrt{\pi}\left\|\frac{M(x)}{x}\right\|_E\right\}.}$$

Remark. The assumptions on $v(z)$'s behaviour can be lightened by means of standard Phragmén–Lindelöf arguments (see footnote near beginning of §A.2, after problem 16). Such extensions are left to the reader; what we have here is general enough for the applications in this book.

Proof of theorem. The difference

$$v(z) - AY_{\mathscr{D}}(z)$$

is (by the definition of $Y_{\mathscr{D}}(z)$ in §A.2) *subharmonic* and *bounded above* in $\mathscr{D}$, and continuous up to $\partial\mathscr{D}$, where it coincides with $v(z)$. Hence, by harmonic majoration (Chapter VII, §B.1),

$$v(z) - AY_{\mathscr{D}}(z) \leqslant \int_{\partial\mathscr{D}} v(t)\mathrm{d}\omega_{\mathscr{D}}(t,z) \leqslant \int_{\partial\mathscr{D}} M(t)\mathrm{d}\omega_{\mathscr{D}}(t,z) \qquad \text{for} \quad z\in\mathscr{D}.$$

Taking $z = 0$, we see that we have to estimate $\int_{\partial\mathscr{D}} M(t)\mathrm{d}\omega_{\mathscr{D}}(t,0)$, which, in view of the definition of $\Omega_{\mathscr{D}}(t)$, equals $-\int_0^\infty M(t)\mathrm{d}\Omega_{\mathscr{D}}(t)$, $M(t)$ being *even.*

The *trick* here is to write

$$-\int_0^\infty M(t)\,\mathrm{d}\Omega_{\mathscr{D}}(t) = \int_0^\infty \frac{M(t)}{t}\Omega_{\mathscr{D}}(t)\,\mathrm{d}t - \int_0^\infty \frac{M(t)}{t}\mathrm{d}(t\Omega_{\mathscr{D}}(t)).$$

Since $M(t) \geqslant 0$, the *first* integral on the right is

$$\leqslant Y_{\mathscr{D}}(0)\int_0^\infty \frac{M(t)}{t^2}\mathrm{d}t$$

by the theorem of article 2. In view of our assumption on $M(t)$, the *second* right-hand integral can be rewritten

$$-\int_0^\infty\int_0^\infty \log\left|\frac{t+x}{t-x}\right|\mathrm{d}\rho(x)\,\mathrm{d}(t\Omega_{\mathscr{D}}(t)) = -E(\mathrm{d}\rho(t),\ \mathrm{d}(t\Omega_{\mathscr{D}}(t))).$$

Using Schwarz' inequality on the *positive definite* bilinear form $E(\ ,\ )$ (see remark, end of §B.5), we see that the last expression is in modulus

$$\leqslant \sqrt{(E(\mathrm{d}\rho(t),\ \mathrm{d}\rho(t)))}\cdot\sqrt{(E(\mathrm{d}(t\Omega_{\mathscr{D}}(t)),\ \mathrm{d}(t\Omega_{\mathscr{D}}(t))))}$$

which, by the result of the preceding article, is $\leqslant \|M(x)/x\|_E\sqrt{\pi}\,Y_{\mathscr{D}}(0)$. Putting our two estimates together, we get

$$\boxed{\int_{\partial\mathscr{D}} M(t)\mathrm{d}\omega_{\mathscr{D}}(t,0) \leqslant Y_{\mathscr{D}}(0)\left\{\int_0^\infty \frac{M(t)}{t^2}\mathrm{d}t + \sqrt{\pi}\left\|\frac{M(x)}{x}\right\|_E\right\}.}$$

As we have seen $v(0) - AY_{\mathscr{D}}(0)$ is $\leqslant$ the left-hand integral. The theorem is thus proved.

Remark. This result shows that for special majorants $M(t)$ of the kind described, the *entire dependence* of our bound for $v(0)$ *on the domain* $\mathscr{D}$ is *expressed through the quantity* $Y_{\mathscr{D}}(0)$, $Y_{\mathscr{D}}$ being the Phragmén–Lindelöf function for $\mathscr{D}$.

5. When majorant is the logarithm of an entire function of exponential type

The result in the preceding article can be extended so as to apply to certain even majorants $M(x)$ of the form

$$x\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t)$$

for which the iterated integral

$$\int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t)\,\mathrm{d}\rho(x)$$

is *not* absolutely convergent. This can, in particular, be done in the important special case where

$$M(x) \;=\; \log|G(x)|$$

with an entire function G of exponential type, 1 at 0, having *even* modulus $\geqslant 1$ on $\mathbb{R}$, and such that

$$\int_{-\infty}^\infty \frac{\log|G(x)|}{1+x^2}\mathrm{d}x \;<\; \infty.$$

Then the right side of the boxed formula at the end of the previous article can be simplified so as to involve only $Y_{\mathscr{D}}(0)$, $\int_0^\infty (M(t)/t^2)\mathrm{d}t$, and the type of G.

The treatment of *any* majorant $M(x)$, *even or not*, of the form $\log^+|F(x)|$ with F entire, of exponential type, and such that

$$\int_{-\infty}^\infty \frac{\log^+|F(x)|}{1+x^2}\mathrm{d}x \;<\; \infty,$$

can be reduced to that of one of the kind just described. Indeed, to any such $M(x)$ corresponds another, $M_1(x) = \log|G(x)|$ with G entire and of exponential type, such that

$$M_1(x) \;\geqslant\; M(x) \qquad \text{for } |x|\geqslant 1,$$
$$M_1(x) \;=\; M_1(-x) \;\geqslant\; 0,$$
$$M_1(0) \;=\; 0,$$

and

$$\int_0^\infty \frac{M_1(x)}{x^2}\mathrm{d}x \;<\; \infty.$$

To see this, put first of all

$$\Phi(z) \;=\; 1 + z^2[F(z)\overline{F(\bar{z})} + F(-z)\overline{F(-\bar{z})}];$$

$\Phi(z)$ is then entire and of exponential type, even, and $\geqslant 1$ on $\mathbb{R}$ with $\Phi(0) = 1$. Clearly

$$\int_{-\infty}^{\infty} \frac{\log \Phi(x)}{1+x^2}\,\mathrm{d}x \;<\; \infty$$

in view of the similar property of F, and

$$\Phi(x) \;\geqslant\; |F(x)|^2 \qquad \text{for } |x| \geqslant 1.$$

By the Riesz–Fejer theorem (the *third* one in §G.3 of Chapter III), there is an entire function $G(z)$ of exponential type, *having all its zeros in* $\Im z < 0$ (since here $\Phi(x) \geqslant 1$), such that

$$\Phi(z) \;=\; G(z)\overline{G(\bar{z})}.$$

The majorant $M_1(x) = \log|G(x)|$ then has the required properties.

The result to be obtained in this article regarding even majorants $\log|G(x)|$ of the abovementioned kind can thus be used in studying problems involving the more general ones of the form $\log^+|F(x)|$.

For entire functions $G(z)$ of exponential type with $G(0) = 1$, $|G(x)| = |G(-x)| \geqslant 1$, and

$$\int_{-\infty}^{\infty} \frac{\log|G(x)|}{1+x^2}\,\mathrm{d}x \;<\; \infty,$$

$\log|G(x)|$ has a simple representation as a Stieltjes integral. When dealing only with the *modulus* of G on $\mathbb{R}$, we may, by the *second* theorem of §G.3, Chapter III, *assume that $G(z)$ has all its zeros in the lower half plane.* Forming, for the moment, the entire function $\Phi(z) = G(z)\overline{G(\bar{z})}$, we see that $\Phi(x) = \Phi(-x)$ on $\mathbb{R}$ so that $\Phi(z) = \Phi(-z)$, and every zero of $\Phi(z)$ is also one of $\Phi(-z)$. The zeros of $\Phi(z)$ are just those of $G(z)$ together with their *complex conjugates*, so, since all the former lie in $\Im z < 0$, we have $G(-\bar{\lambda}) = 0$ whenever $G(\lambda) = 0$. The zeros of $G(z)$ thus fall into three groups: those on the *negative imaginary axis*, those in the *open fourth quadrant*, and the *reflections* of *these latter ones* in the *imaginary axis*. The Hadamard factorization (Chapter III, §A) of $G(z)$ can therefore be written

$$G(z) = \mathrm{e}^{\alpha z}\prod_k \left(1 + \frac{z}{\mathrm{i}\mu_k}\right)\mathrm{e}^{\mathrm{i}z/\mu_k} \cdot \prod_n \left(1 - \frac{z}{\bar{\lambda}_n}\right)\mathrm{e}^{z/\bar{\lambda}_n}\left(1 + \frac{z}{\lambda_n}\right)\mathrm{e}^{-z/\lambda_n},$$

where the $\mu_k > 0$, $\Re\lambda_n > 0$ and $\Im\lambda_n > 0$. One (or even both!) of the two products occurring on the right may of course be empty.

Since $|G(x)| = |G(-x)|$, α is pure imaginary. We also know, by the *first* theorem of §G.3, Chapter III, that

$$\sum_k \frac{1}{\mu_k} \quad \text{and} \quad \sum_n \frac{\Im\lambda_n}{|\lambda_n|^2}$$

both converge. The exponential factors figuring in the above product may therefore be grouped together and multiplied out separately, after which the expression takes the form

$$e^{ibz}\prod_k\left(1+\frac{z}{i\mu_k}\right)\cdot\prod_n\left(1-\frac{z}{\bar{\lambda}_n}\right)\left(1+\frac{z}{\lambda_n}\right),$$

with b real. Here, we are only concerned with the *modulus* $|G(x)|$, $x\in\mathbb{R}$; *we may hence take $b = 0$. This we do throughout the remainder of this article, working exclusively with entire functions of exponential type of the form*

$$\boxed{G(z) = \prod_k\left(1+\frac{z}{i\mu_k}\right)\cdot\prod_n\left(1-\frac{z}{\bar{\lambda}_n}\right)\left(1+\frac{z}{\lambda_n}\right),}$$

where the $\mu_k > 0$, $\Re\lambda_n > 0$ and $\Im\lambda_n > 0$. The products on the right are of course assumed to be *convergent*. Our Stieltjes integral representation for such functions is provided by the

Lemma. *Let $G(z)$, of exponential type, be of the form just described. Then, for $\Im z > 0$,*

$$\log|G(z)| = \int_0^\infty \log\left|1-\frac{z^2}{t^2}\right| d\nu(t)$$

with an increasing function $\nu(t)$ given by

$$\frac{d\nu(t)}{dt} = \frac{1}{\pi}\left(\sum_k \frac{\mu_k}{\mu_k^2+t^2} + \sum_n\left(\frac{\Im\lambda_n}{|\lambda_n - t|^2}+\frac{\Im\lambda_n}{|\lambda_n+t|^2}\right)\right).$$

Proof. *Fix z, $\Im z > 0$.* Then $\log|1+z/\lambda|$ is a *harmonic function of λ* in $\{\Im\lambda > 0\}$, *bounded* therein for λ away from 0, and *continuous* up to $\mathbb{R}$ save at $\lambda = 0$ where it has a logarithmic singularity. We can therefore apply Poisson's formula, getting

$$\log\left|1+\frac{z}{\lambda}\right| = \frac{1}{\pi}\int_{-\infty}^{\infty}\log\left|1-\frac{z}{t}\right|\cdot\frac{\Im\lambda}{|\lambda+t|^2}dt$$

for $\Im\lambda > 0$, from which

$$\log\left|1+\frac{z}{\lambda}\right| + \log\left|1-\frac{z}{\bar{\lambda}}\right| = \frac{1}{\pi}\int_0^\infty \log\left|1-\frac{z^2}{t^2}\right|\left(\frac{\Im\lambda}{|\lambda+t|^2}+\frac{\Im\lambda}{|\lambda-t|^2}\right)\mathrm{d}t.$$

Similarly, for $\mu > 0$,

$$\log\left|1+\frac{z}{\mathrm{i}\mu}\right| = \frac{1}{\pi}\int_0^\infty \log\left|1-\frac{z^2}{t^2}\right| \frac{\mu}{\mu^2+t^2}\mathrm{d}t.$$

We have

$$\log|G(z)| = \sum_k \log\left|1+\frac{z}{\mathrm{i}\mu_k}\right| + \sum_n\left(\log\left|1+\frac{z}{\lambda_n}\right|+\log\left|1-\frac{z}{\bar{\lambda}_n}\right|\right).$$

When $\Im z > 0$, we can rewrite each of the terms on the right using the formulas just given, obtaining a certain *sum of integrals.* If $|\Re z| < \Im z$, the *order* of summation and integration in that sum can be *reversed*, for then

$$\log\left|1-\frac{z^2}{t^2}\right| \geqslant 0, \qquad t\in\mathbb{R}.$$

This gives

$$\log|G(z)| = \int_0^\infty \log\left|1-\frac{z^2}{t^2}\right|\mathrm{d}\nu(t),$$

at least for $|\Re z| < \Im z$, with $\nu'(t)$ as in the statement of the lemma.

Both sides of the relation just found are, however, *harmonic in z* for $\Im z > 0$; the *left* one by our assumption on $G(z)$ and the *right* one because $\int_0^\infty \log|1+y^2/t^2|\mathrm{d}\nu(t)$, being just equal to $\log|G(\mathrm{i}y)|$ for $y>0$, is *convergent* for every such y. (To show that this implies u.c.c. convergence, and hence harmonicity, of the integral involving z for $\Im z > 0$, one may argue as at the beginning of the proof of the second theorem in §A, Chapter III.) The two sides of our relation, equal for $|\Re z| < \Im z$, must therefore coincide for $\Im z > 0$ and finally for $\Im z \geqslant 0$ by a continuity argument.

Remark. Since $G(z)$ has no zeros for $\Im z \geqslant 0$, a branch of $\log G(z)$, and hence of $\arg G(z)$, is defined there. By logarithmic differentiation of the above boxed product formula for $G(z)$, it is easy to check that

$$\frac{\mathrm{d}\arg G(t)}{\mathrm{d}t} = -\pi\nu'(t)$$

with the ν of the lemma. From this it is clear that $\nu'(t)$ is certainly *continuous* (and even $\mathscr{C}_\infty$) on $\mathbb{R}$.

In what follows, we will take $\nu(0) = 0$, $\nu(t)$ being the increasing function in the lemma. Since $\nu'(t)$ is *clearly even*, $\nu(t)$ *is then odd*. With $\nu(t)$ thus specified, we have the easy

Lemma. *If* $G(z)$, *given by the above boxed formula, is of exponential type, the function* $\nu(t)$ *corresponding to it is* $\leqslant$ const.t *for* $t \geqslant 0$.

Proof. By the preceding lemma,

$$\int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}\nu(t) \;=\; \log|G(z)|$$

for $\Im z \geqslant 0$, the right side being $\leqslant K|z|$ by hypothesis, since $G(0) = 1$. Calling the left-hand integral $U(z)$, we have, however, $U(z) = U(\bar{z})$, so

$$U(z) \;\leqslant\; K|z|$$

for all z.

Reasoning as in the proof of Jensen's formula, Chapter I (what we are dealing with here is indeed nothing but a version of that formula for the subharmonic function $U(z)$), we see, for $t \neq 0$, that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|1 - \frac{re^{i\vartheta}}{t}\right| \mathrm{d}\vartheta \;=\; \begin{cases} \log\dfrac{r}{|t|}, & |t| < r, \\ 0, & |t| \geqslant r. \end{cases}$$

Thence, by Fubini's theorem,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} U(re^{i\vartheta})\,\mathrm{d}\vartheta \;=\; \int_{-r}^{r} \log\frac{r}{|t|}\,\mathrm{d}\nu(t).$$

Integrating the right side by parts, we get the value $2\int_0^r (\nu(t)/t)\,\mathrm{d}t$, $\nu(t)$ being odd and $\nu'(0)$ finite. In view of the above inequality on $U(z)$, we thus have

$$\int_0^r \frac{\nu(t)}{t}\,\mathrm{d}t \;\leqslant\; \tfrac{1}{2}Kr.$$

From this relation we easily deduce that $\nu(r) \;\leqslant\; \frac{1}{2}eKr$ as in problem 1, Chapter I. Done.

Using the two results just proved in conjunction with the *first* lemma of §B.4, we now obtain, without further ado, the

Theorem. *Let the entire function* $G(z)$ *of exponential type be given by the above boxed formula, and let* $\nu(t)$ *be the increasing function associated to* G *in the way described above. Then, for* $x > 0$,

$$\log|G(x)| \;=\; -x\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\left(\frac{\nu(t)}{t}\right).$$

For our functions $G(z)$, $(\log|G(x)|)/x$ is thus a *Green potential* on $(0, \infty)$. *This makes it possible for us to apply the result of the preceding article to majorants*

$$M(t) \;=\; \log|G(t)|.$$

With that in mind, let us give a more quantitative version of the second of the above lemmas.

Lemma. *If* $G(z)$, *given by the above boxed formula, is* $\geqslant 1$ *in modulus on* $\mathbb{R}$ *and of exponential type* a, *the increasing function* $\nu(t)$ *associated to it satisfies*

$$\frac{\nu(t)}{t} \;\leqslant\; \frac{\mathrm{e}}{2}a \;+\; \frac{\mathrm{e}}{\pi}\int_{-\infty}^{\infty}\frac{\log|G(x)|}{x^2}\mathrm{d}x. \qquad t \geqslant 0.$$

Remark. We are not striving for a best possible inequality here.

Proof of lemma. The function $U(z)$ used in proving the previous lemma is *subharmonic* and $\leqslant K|z|$. Assuming that

$$\int_{-\infty}^{\infty}\frac{\log|G(t)|}{t^2}\mathrm{d}t \;<\; \infty$$

(the only situation we need consider), let us find an explicit estimate for K.

Under our assumption, we have, for $\Im z > 0$,

$$\log|G(z)| \;\leqslant\; a\Im z + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2}\log|G(t)|\mathrm{d}t$$

by §E of Chapter III. When $-y \leqslant x \leqslant y$, we have, however, for $z = x + \mathrm{i}y$,

$$|z-t|^2 \;=\; t^2 - 2xt + x^2 + y^2 \;\geqslant\; \frac{t^2}{2} + \frac{t^2}{2} - 2xt + 2x^2$$

$$\geqslant\; \frac{t^2}{2}, \qquad t \in \mathbb{R},$$

whence, $\log|G(t)|$ being $\geqslant 0$,

$$\log|G(z)| \;\leqslant\; ay + \frac{2y}{\pi}\int_{-\infty}^{\infty}\frac{\log|G(t)|}{t^2}\mathrm{d}t.$$

Thus, since $U(z) = U(\bar{z}) = \log|G(z)|$ for $\Im z \geqslant 0$,

$$U(z) \;\leqslant\; \left(a + \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\log|G(t)|}{t^2}\mathrm{d}t\right)|\Im z|$$

in both of the sectors $|\Re z| \leqslant |\Im z|$.

Because $U(z) \leqslant \text{const.}|z|$ we can apply the second Phragmén–Lindelöf theorem of §C, Chapter III, to the difference

$$U(z) \;-\; \left(a + \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\log|G(t)|}{t^2}\,\mathrm{d}t\right)\Re z$$

in the 90° sector $|\Im z| \leqslant \Re z$, and find that it is $\leqslant 0$ in that sector. One proceeds similarly in $\Re z \leqslant -|\Im z|$, and we have

$$U(z) \;\leqslant\; \left(a + \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\log|G(t)|}{t^2}\,\mathrm{d}t\right)|\Re z|$$

for $|\Im z| \leqslant |\Re z|$.

Combining the two estimates for $U(z)$ just found, we get

$$U(z) \;\leqslant\; K|z|$$

with

$$K \;=\; a + \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\log|G(t)|}{t^2}\,\mathrm{d}t.$$

This value of K may now be plugged into the *proof* of the previous lemma. That yields the desired result.

Problem 27

Let $\Phi(z)$ be entire and of exponential type, with $\Phi(0) = 1$. *Suppose that* $\Phi(z)$ *has all its zeros in* $\Im z < 0$ *and that* $|\Phi(x)| \geqslant 1$ *on* $\mathbb{R}$. Show that *then*

$$\int_{-\infty}^{\infty} \frac{\log|\Phi(x)|}{x^2}\,\mathrm{d}x \;<\; \infty.$$

(Hint: First use Lindelöf's theorem from Chapter III, §B, to show that the Hadamard factorization for $\Phi(z)$ can be cast in the form

$$\Phi(z) \;=\; \mathrm{e}^{cz}\prod_n\left(1 - \frac{z}{\lambda_n}\right)\mathrm{e}^{z\Re(1/\lambda_n)},$$

where the $\Im\lambda_n < 0$. Taking $\Psi(z) = \Phi(z)\exp(-\mathrm{i}z\Im c)$, show that $\partial\log|\Psi(z)|/\partial y \geqslant 0$ for $y \geqslant 0$, and then look at $1/\Psi(z)$.)

Suppose now that we have an entire function $G(z)$ given by the above boxed representation, of exponential type a and $\geqslant 1$ in modulus on $\mathbb{R}$. *If the double integral*

$$\int_0^{\infty}\int_0^{\infty} \log\left|\frac{x+t}{x-t}\right|\mathrm{d}\left(\frac{\nu(t)}{t}\right)\mathrm{d}\left(\frac{\nu(x)}{x}\right)$$

is absolutely convergent, we may, as in the previous two articles, speak of the *energy*

$$E\left(\mathrm{d}\left(\frac{\nu(t)}{t}\right),\ \mathrm{d}\left(\frac{\nu(t)}{t}\right)\right);$$

in terms of the Green potential

$$\frac{\log|G(x)|}{x} = -\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\left(\frac{\nu(t)}{t}\right),$$

this is just $\|(\log|G(x)|)/x\|_E^2$ according to the notation introduced at the end of article 3.

To Beurling and Malliavin is due the important observation that $\|(\log|G(x)|)/x\|_E$ can be expressed in terms of a and $\int_0^\infty(\log|G(x)|/x^2)\mathrm{d}x$ under the present circumstances. Since $\log|G(t)| \geqslant 0$ and $\nu(t)$ increases, we have indeed

$$\begin{aligned}
\|(\log|G(x)|)/x\|_E^2 &= -\int_0^\infty \frac{\log|G(x)|}{x}\mathrm{d}\left(\frac{\nu(x)}{x}\right) \\
&= \int_0^\infty \frac{\log|G(x)|}{x^2}\left(\frac{\nu(x)}{x}\mathrm{d}x - \mathrm{d}\nu(x)\right) \\
&\leqslant \left(\sup_{x>0}\frac{\nu(x)}{x}\right)\cdot\int_0^\infty \frac{\log|G(x)|}{x^2}\mathrm{d}x.
\end{aligned}$$

Using the preceding lemma and remembering that $|G(x)|$ is even, we find that

$$\boxed{\left\|\frac{\log|G(x)|}{x}\right\|_E^2 \leqslant \left(\frac{ea}{2} + \frac{2e}{\pi}\int_0^\infty \frac{\log|G(x)|}{x^2}\mathrm{d}x\right)\cdot\int_0^\infty \frac{\log|G(x)|}{x^2}\mathrm{d}x.}$$

Take now an even majorant $M(t) \geqslant 0$ equal to $\log|G(t)|$, and consider one of our domains $\mathscr{D}$ with $0\in\mathscr{D}$. From the result just obtained and the boxed formula near the end of the previous article, we get

$$\int_{\partial\mathscr{D}} M(t)\,\mathrm{d}\omega_{\mathscr{D}}(t,0) \leqslant Y_{\mathscr{D}}(0)\left\{J + \sqrt{\left(2eJ\left(J+\frac{\pi a}{4}\right)\right)}\right\},$$

with

$$J = \int_0^\infty \frac{\log|G(t)|}{t^2}\mathrm{d}t = \int_0^\infty \frac{M(t)}{t^2}\mathrm{d}t,$$

at least in the case where

$$\int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu(t)}{t}\right) d\left(\frac{\nu(x)}{x}\right)$$

is absolutely convergent. On the right side of this relation, the coefficient $Y_{\mathscr{D}}(0)$ is multiplied by a factor *involving only a, the type of G, and the integral* $\int_0^\infty (M(t)/t^2)\,dt$ (essentially, the one this book is about!).

It is very important that the requirement of absolute convergence on the above double integral can be lifted, and the preceding relation still remains true. This will be shown by bringing in the *completion*, for the norm $\| \ \|_E$, of the collection of real Green potentials associated with absolutely convergent energy integrals – that completion is a real Hilbert space, since $\| \ \|_E$ comes from a positive definite bilinear form. The details of the argument take up the remainder of this article.

Starting with our entire function $G(z)$ of exponential type and the increasing function $\nu(t)$ associated to it, put

$$Q(x) = \frac{\log|G(x)|}{x} = -\int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu(t)}{t}\right),$$

and, for $n = 1, 2, 3, \ldots,$

$$Q_n(x) = \frac{1}{x}\int_0^n \log\left|1-\frac{x^2}{t^2}\right| d\nu(t).$$

In terms of

$$\nu_n(t) = \begin{cases} \nu(t), & 0 \leqslant t \leqslant n, \\ \nu(n), & t > n, \end{cases}$$

we have

$$Q_n(x) = -\int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu_n(t)}{t}\right)$$

by the first lemma of §B.4; evidently, $Q_n(x) \longrightarrow Q(x)$ u.c.c. in $[0, \infty)$ as $n \to \infty$.

Each of the integrals

$$\int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu_n(t)}{t}\right) d\left(\frac{\nu_n(x)}{x}\right)$$

is absolutely convergent. This is easily verified using the facts that

$$d\left(\frac{\nu_n(t)}{t}\right) \sim \tfrac{1}{2}\nu''(0)\,dt$$

near 0 ($\nu(t)$ being $\mathscr{C}_\infty$ by a previous remark), and that

$$d\left(\frac{\nu_n(t)}{t}\right) = -\frac{\nu(n)}{t^2}dt \qquad \text{for } t > n.$$

Lemma. *If* $|G(x)| \geqslant 1$ *on* $\mathbb{R}$, *the functions* $Q_n(x)$ *are* $\geqslant 0$ *for* $x > 0$, *and*

$$\|Q_n\|_E \leqslant \frac{\pi}{2}\nu'(0).$$

Proof. For $t > 0$, $\log|1 - x^2/t^2| \geqslant 0$ when $x \geqslant \sqrt{2}t$, so

$$xQ_n(x) = \int_0^n \log\left|1 - \frac{x^2}{t^2}\right| d\nu(t)$$

is $\geqslant 0$ for $x \geqslant \sqrt{2}n$. Again, for $0 \leqslant x \leqslant \sqrt{2}t$, $\log|1 - x^2/t^2| \leqslant 0$, so, for $0 \leqslant x \leqslant \sqrt{2}n$,

$$\int_n^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\nu(t) \leqslant 0,$$

and finally $xQ_n(x)$, equal to $\log|G(x)|$ *minus* this integral, is $\geqslant 0$ since $|G(x)| \geqslant 1$.

The second lemma of §B.4 is applicable to the functions $\nu_n(t)$. Using it and the positivity of $Q_n(x)$, already established, we get

$$\begin{aligned}\|Q_n\|_E^2 &= \int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu_n(t)}{t}\right) d\left(\frac{\nu_n(x)}{x}\right) \\ &= \int_0^\infty Q_n(x)\left\{\frac{\nu_n(x)}{x^2}dx - \frac{d\nu_n(x)}{x}\right\} \\ &\leqslant \int_0^\infty Q_n(x)\frac{\nu_n(x)}{x^2}dx = \frac{\pi^2}{4}(\nu_n'(0))^2 = \frac{\pi^2}{4}(\nu'(0))^2.\end{aligned}$$

We are done.

Theorem. *Let* $G(z)$ *be an entire function of exponential type* a, 1 *at* 0, *with* $|G(x)|$ even *and* $\geqslant 1$ *on* $\mathbb{R}$, *and such that*

$$\int_{-\infty}^\infty \frac{\log|G(x)|}{1+x^2}dx < \infty.$$

If $\mathscr{D}$ *is one of our domains containing* 0, *we have*

$$\int_{\partial\mathscr{D}} \log|G(t)|\,\mathrm{d}\omega_{\mathscr{D}}(t,0) \;\leqslant\; Y_{\mathscr{D}}(0)\left\{J + \sqrt{\left(2eJ\left(J+\frac{\pi a}{4}\right)\right)}\right\}$$

where

$$J \;=\; \int_0^\infty \frac{\log|G(t)|}{t^2}\,\mathrm{d}t.$$

Proof. According to the discussion at the beginning of this article we may, without loss of generality,* assume that $G(z)$ has the above boxed product representation.

Beginning as in the proof of the theorem from the preceding article, we have

$$\int_{\partial\mathscr{D}} \log|G(x)|\,\mathrm{d}\omega_{\mathscr{D}}(x,0) \;=\; \int_0^\infty \frac{\log|G(x)|}{x}\Omega_{\mathscr{D}}(x)\,\mathrm{d}x - \int_0^\infty \frac{\log|G(x)|}{x}\,\mathrm{d}(x\Omega_{\mathscr{D}}(x)).$$

The *first* term on the right is of course

$$\leqslant\; Y_{\mathscr{D}}(0)\int_0^\infty \frac{\log|G(x)|}{x^2}\,\mathrm{d}x$$

by the theorem of article 2, $\log|G(x)|$ being positive. The *second*, equal to

$$-\int_0^\infty Q(x)\mathrm{d}(x\Omega_{\mathscr{D}}(x)),$$

can be looked at in two different ways.

In the first place, for $x > 0$,

$$Q(x) \;=\; \lim_{n\to\infty} Q_n(x)$$

with the functions $Q_n(x)$ introduced above. Also, for each n,

* Dropping the factor $\exp(ibz)$ from the second displayed expression on p. 557 can only *diminish* the overall exponential type, for, if $G(z)$ is given by the *boxed formula* on that page, the limsups of $\log|G(iy)|/|y|$ for y tending to ∞ and to $-\infty$ are *equal.* To see that, observe that the limsup for $y\to\infty$ is actually a *limit* (see remark, p. 49), and that $\overline{G(\bar{z})}/G(z) = B(z)$ is a Blaschke product like the one figuring in the remark on p. 58. The argument of pp. 57–8 shows, however, that then the limsup of $\log|B(iy)|/y$ for $y\to\infty$ is *zero.*

$$Q_n(x) \leqslant \frac{1}{x}\int_0^n \log\left|1+\frac{x^2}{t^2}\right| \mathrm{d}\nu(t)$$

$$\leqslant \frac{1}{x}\int_0^\infty \log\left|1+\frac{x^2}{t^2}\right| \mathrm{d}\nu(t), \qquad x>0.$$

Since $\nu(t) \leqslant Kt$, the right-hand member comes out $\leqslant \pi K$ on integrating by parts. This, together with the preceding lemma, shows that

$$0 \leqslant Q_n(x) \leqslant \pi K \qquad \text{for } x>0.$$

However, for *large* x,

$$\mathrm{d}(x\Omega_{\mathscr{D}}(x)) = \left(\frac{\text{const.}}{x^3} + O\left(\frac{1}{x^5}\right)\right)\mathrm{d}x$$

(see just before the theorem of article 3). Therefore

$$\int_0^\infty Q(x)\mathrm{d}(x\Omega_{\mathscr{D}}(x)) = \lim_{n\to\infty}\int_0^\infty Q_n(x)\mathrm{d}(x\Omega_{\mathscr{D}}(x))$$

by dominated convergence.

The right-hand limit can also be expressed as an inner product in a certain real Hilbert space. The latter – call it $\mathfrak{H}$ – is the *completion with respect to the norm* $\| \ \|_E$ of the collection of real Green potentials

$$u(x) = \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t)$$

such that

$$\int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right| |\mathrm{d}\rho(t)||\mathrm{d}\rho(x)| < \infty;$$

the positive definite bilinear form $\langle \ , \ \rangle_E$ extends by continuity to $\mathfrak{H}$ for which it serves as inner product. For each n, we have

$$\int_0^\infty Q_n(x)\mathrm{d}(x\Omega_{\mathscr{D}}(x)) = E\left(\mathrm{d}\left(\frac{\nu_n(t)}{t}\right), \mathrm{d}(t\Omega_{\mathscr{D}}(t))\right) = \langle Q_n, P\rangle_E,$$

where

$$P(x) = x(G_{\mathscr{D}}(x,0)+G_{\mathscr{D}}(-x,0)) ;$$

here only Green potentials associated with *absolutely convergent* energy integrals are involved. By the lemma, however,

$$\|Q_n\|_E \leqslant \frac{\pi}{2}\nu'(0),$$

so a *subsequence* of $\{Q_n\}$, which we may as well *also* denote by $\{Q_n\}$, *converges weakly in* $\mathfrak{H}$ *to some element* q *of that space.* (Here, we do *not need* to 'identify' q with the function $Q(x)$, although that can easily be done.) In view of the previous limit relation, we see that

$$\int_0^\infty Q(x)\mathrm{d}(x\Omega_{\mathscr{D}}(x)) = \lim_{n\to\infty}\langle Q_n, P\rangle_E = \langle q, P\rangle_E.$$

Thence, by Schwarz' inequality and the result of article 3,

$$\left|\int_0^\infty Q(x)\mathrm{d}(x\Omega_{\mathscr{D}}(x))\right| \leqslant \|q\|_E\|P\|_E$$
$$= \|q\|_E\sqrt{(E(\mathrm{d}(t\Omega_{\mathscr{D}}(t)),\ \mathrm{d}(t\Omega_{\mathscr{D}}(t))))} \leqslant \sqrt{\pi Y_{\mathscr{D}}(0)}\|q\|_E.$$

Returning to the beginning of this proof, we see that

$$\int_{\partial\mathscr{D}}\log|G(x)|\mathrm{d}\omega_{\mathscr{D}}(x,0) \leqslant Y_{\mathscr{D}}(0)\int_0^\infty\frac{\log|G(x)|}{x^2}\mathrm{d}x$$
$$+ \sqrt{\pi Y_{\mathscr{D}}(0)}\|q\|_E,$$

and thus need an estimate for $\|q\|_E$. The obvious one, $\|q\|_E \leqslant \liminf_{n\to\infty}\|Q_n\|_E \leqslant \pi\nu'(0)/2$, is not good enough to give us what we want here, so we argue as follows.

The weak convergence of Q_n to q in $\mathfrak{H}$ implies first of all that

$$\|q\|_E^2 = \lim_{n\to\infty}\langle q, Q_n\rangle_E.$$

Fix any n; then, by weak convergence again,

$$\langle q, Q_n\rangle_E = \lim_{k\to\infty}\langle Q_k, Q_n\rangle_E = -\lim_{k\to\infty}\int_0^\infty Q_k(x)\mathrm{d}\left(\frac{\nu_n(x)}{x}\right).$$

Here, $\mathrm{d}(\nu_n(x)/x)$ is just $-(\nu(n)/x^2)\mathrm{d}x$ for $x > n$, so, since $0 \leqslant Q_k(x) \leqslant \pi K$, we have, by dominated convergence,

$$-\lim_{k\to\infty}\int_0^\infty Q_k(x)\mathrm{d}\left(\frac{\nu_n(x)}{x}\right) = -\int_0^\infty Q(x)\mathrm{d}\left(\frac{\nu_n(x)}{x}\right)$$

which, $Q(x)$ being positive, is

$$\leqslant \int_0^\infty Q(x)\frac{\nu_n(x)}{x^2}\mathrm{d}x.$$

Again, $\nu_n(x) \leqslant \nu(x)$ for $x \geqslant 0$, so finally

$$\langle q, Q_n\rangle_E \leqslant \int_0^\infty Q(x)\frac{\nu(x)}{x^2}\mathrm{d}x = \int_0^\infty\frac{\log|G(x)|}{x^2}\frac{\nu(x)}{x}\mathrm{d}x$$

for each fixed n. The right-hand integral was already estimated above, before the preceding lemma, and found to be

$$\leqslant \frac{2e}{\pi}\left(\frac{\pi a}{4} + \int_0^\infty \frac{\log|G(x)|}{x^2}\,dx\right)\int_0^\infty \frac{\log|G(x)|}{x^2}\,dx.$$

This quantity is thus $\geqslant \lim_{n\to\infty}\langle q, Q_n\rangle_E = \|q\|_E^2$, giving us an upper bound on $\|q\|_E$.

Substituting the estimate just obtained into the above inequality for $\int_{\partial\mathscr{D}}\log|G(x)|\,d\omega_{\mathscr{D}}(x,0)$, we have the theorem. The proof is complete.

Corollary. *Let $G(z)$ and the domain $\mathscr{D}$ be as in the hypothesis of the theorem. If $v(z)$, subharmonic in $\mathscr{D}$ and continuous up to $\partial\mathscr{D}$, satisfies*

$$v(t) \leqslant \log|G(t)|, \qquad t\in\partial\mathscr{D},$$

and

$$v(z) \leqslant A|\Im z| + O(1)$$

with some real A, we have

$$v(0) \leqslant Y_{\mathscr{D}}(0)\left\{A + J + \sqrt{\left(2eJ\left(J + \frac{\pi a}{4}\right)\right)}\right\},$$

where

$$J = \int_0^\infty \frac{\log|G(x)|}{x^2}\,dx$$

and a is the type of G.

This result will be used in proving the Beurling–Malliavin multiplier theorem in Chapter XI.

Problem 28

Let $G(z)$, entire and of exponential type, be given by the above boxed product formula and satisfy the hypothesis of the preceding theorem. Suppose also that

$$\frac{\log|G(iy)|}{|y|} \longrightarrow a \quad \text{for} \quad y \longrightarrow \pm\infty.$$

The purpose of this problem is to improve the estimate of $\|(\log|G(x)|)/x\|_E$ obtained above.

(a) Show that $v'(0) = a/\pi + 2J/\pi^2$ and that $v(t)/t \longrightarrow a/\pi$ as $t\to\infty$. Here, J has the same meaning as in the statement of the theorem. (Hint. For the second relation, one may just indicate how to adapt the argument from §H.2 of Chapter III.)

(b) Show that

$$\int_0^\infty \frac{\log|G(x)|}{x}\frac{\nu(x)}{x^2}\,dx = \frac{\pi^2}{4}\left((\nu'(0))^2 - \left(\lim_{t\to\infty}\frac{\nu(t)}{t}\right)^2\right).$$

(Hint. Integral on left is the *negative* of

$$\int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu(t)}{t}\right)\frac{\nu(x)}{x^2}\,dx.$$

Here, direct application of the method used to prove the second lemma of §B.4 is hampered by $(d/dt)(\nu(t)/t)$'s lack of regularity for large t; however, the following procedure works and is quite general.

For small $\delta > 0$ and large L one can get ε, $0 < \varepsilon < \delta$, and $R > L$ making

$$\int_\delta^L\int_\varepsilon^R \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu(t)}{t}\right)\frac{\nu(x)}{x^2}\,dx$$

nearly equal to the above iterated integral. The order of integration can now be reversed and then the second mean value theorem applied to show that $\int_\varepsilon^\delta\int_\delta^L$ and $\int_L^R\int_\delta^L$ are *both small in magnitude* when $\delta > 0$ is small and L large. Our initial expression is thus closely approximated by

$$\int_\delta^L\int_\delta^L \log\left|\frac{x+t}{x-t}\right|\frac{\nu(x)}{x^2}\,dx\,d\left(\frac{\nu(t)}{t}\right).$$

Apply to this a suitable modification of the reasoning in the proof of the aforementioned lemma, and then make $\delta \to 0$, $L \to \infty$.)

(c) Hence show that

$$\int_0^\infty \frac{\log|G(x)|}{x^2}\frac{\nu(x)}{x}\,dx = \frac{1}{\pi^2}J(J+\pi a)$$

so that

$$\int_0^\infty\left(\int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu(t)}{t}\right)\right) d\left(\frac{\nu(x)}{x}\right) \leqslant \frac{1}{\pi^2}J(J+\pi a).$$

Addendum

Improvement of Volberg's Theorem on the Logarithmic Integral. Work of Brennan, Borichev, Jöricke and Volberg.

Writing of §D in Chapter VII was completed early in 1984, and some copies of the MS were circulated that spring. At the beginning of 1987 I learned, first from V.P. Havin and then from N.K. Nikolskii, that the persons named in the title had extended the theorem of §D.6. Expositions of their work did not come into my hands until April and May of 1987, when I had finished going through the second proof sheets for this volume.

In these circumstances, time and space cannot allow for inclusion of a thorough presentation of the recent work here. It nevertheless seems important to describe *some* of it because the strengthened version of Volberg's theorem first obtained by Brennan is very likely close to being best possible. I am thankful to Nikolskii, Volberg and Borichev for having made sure that the material got to me in time for me to be able to include the following account.

The development given below is based on the methods worked out in §D of Chapter VII, and familiarity with that § on the part of the reader is assumed. In order to save space and avoid repetition, we will refer to §D frequently and use the symbols employed there whenever possible.

1. Brennan's improvement, for $M(\nu)/\nu^{1/2}$ monotone increasing

Let us return to the proof of the theorem in §D.6 of Chapter VII, starting from the place on p. 359 where $h(\xi)$ and the weight $w(r) = \exp(-h(\log(1/r)))$ were brought into play. We take over the notation used in that discussion without explaining it anew.

What is shown by the reasoning of pp. 359–73 is that *unless* $F(e^{i\vartheta})$

vanishes identically,

$$\int_{-\pi}^{\pi} \log|F(e^{i\vartheta})|\,d\vartheta \;>\; -\infty$$

provided that

$$h(\xi) \;\geqslant\; \text{const.}\ \xi^{-(1+\delta)}$$

with some $\delta > 0$ as $\xi \longrightarrow 0$, and that

$$\int_0^a \log h(\xi)\,d\xi \;=\; \infty$$

for small $a > 0$. *Brennan's result is that the first condition on h can be replaced by the requirement that $\xi h(\xi)$ be decreasing for small $\xi > 0$.* (The second condition then obviously implies that $\xi h(\xi) \longrightarrow \infty$ as $\xi \longrightarrow 0$.)

Borichev and Volberg made the important observation that Brennan's result is yielded by Volberg's original argument. To see how this comes about, we begin by noting that in §D.6 of Chapter VII, *no real use of the property* $h(\xi) \;\geqslant\;$ const. $\xi^{-(1+\delta)}$ *is made until one comes to step 5* on p. 369. Up to then, it is more than enough to have $h(\xi) \;\geqslant\;$ const. ξ^{-c} with *some* $c > 0$ together with the integral condition on $\log h(\xi)$. *Step 5* itself, however, is carried out in rather clumsy fashion (see p. 370). The reader was probably aware of this, and especially of the wasteful manner of using that step's conclusion in the subsequent local estimate of $\omega(E, z)$ (pp. 370–2). At the top of p. 372, the smallness of $\int_{\gamma_\rho}(1/(1-|\zeta|))\,d\omega(\zeta,\rho)$ was used where its *smallness in relation to* $1/(1-\rho)$ would have sufficed!

Instead of verifying the conclusion of *step* 5, let us show that *the quantity*

$$(1-\rho)\int_{\gamma_\rho} \frac{d\omega(\zeta,\rho\zeta_0)}{1-|\zeta|}$$

can be made as small as we please for ρ sufficiently close to 1 chosen according to the specifications at the bottom of p. 368, *under the assumption* that $\xi h(\xi)$ *decreases*, with the integral of $\log h(\xi)$ *divergent*.

The original argument for *step 5* is *unchanged* up to the point where the relation

$$(\ast) \qquad \int_{\gamma_\rho} h\left(\log\frac{1}{|\zeta|}\right) d\omega(\zeta,\rho) \;\leqslant\; \text{const.} \;+\; (h(\log(1/\rho^2)))^\eta$$

is obtained at the top of p. 370; here η can be chosen *at pleasure* in the interval $(0, 1)$, the construction following *step 3* (pp. 365–6) and subsequent carrying out of *step 4* being in no way hindered. Write now

$$P(\xi) \;=\; \xi h(\xi);$$

under the present circumstances $P(\xi)$ is *decreasing* for small $\xi > 0$. Since γ_ρ, recall, lies in the ring $\{\rho^2 \leqslant |\zeta| < 1\}$, we then have, for ρ near 1,

$$\int_{\gamma_\rho} \frac{\mathrm{d}\omega(\zeta,\rho)}{1-|\zeta|} \;\leqslant\; 2\int_{\gamma_\rho} \frac{\mathrm{d}\omega(\zeta,\rho)}{\log(1/|\zeta|)} \;=\; 2\int_{\gamma_\rho} \frac{h(\log(1/|\zeta|))}{P(\log(1/|\zeta|))}\mathrm{d}\omega(\zeta,\rho)$$

$$\leqslant\; \frac{2}{P(2\log(1/\rho))}\int_{\gamma_\rho} h(\log(1/|\zeta|))\mathrm{d}\omega(\zeta,\rho).$$

Referring to (⁑), we see that the last expression is

$$\leqslant\; \frac{2}{P(2\log(1/\rho))}\{\text{const.} \;+\; (h(2\log(1/\rho)))^\eta\}.$$

Here, the monotoneity of $P(\xi)$ makes it tend to ∞ for $\xi \longrightarrow 0$; otherwise $\int_0^a \log h(\xi)\,\mathrm{d}\xi$ would be *finite* for small $a > 0$ as already remarked. The function $h(\xi)$ also tends to ∞ for $\xi \longrightarrow 0$, so, for ρ close to 1 the preceding quantity is

$$\leqslant\; 3\left\{\frac{h(2\log(1/\rho))}{P(2\log(1/\rho))}\right\}^\eta \;=\; \frac{3}{(\log(1/\rho^2))^\eta} \;\leqslant\; \frac{3}{(1-\rho)^\eta}.$$

We thus have

$$\int_{\gamma_\rho} \frac{\mathrm{d}\omega(\zeta,\rho)}{1-|\zeta|} \;\leqslant\; 3(1-\rho)^{-\eta} \;=\; \mathrm{o}(1/(1-\rho))$$

for values of ρ tending to 1 chosen in the way mentioned above, and our substitute for *step 5* is established.

This, as already noted, is all we need for the reasoning at the top of p. 372. The local estimate for $\omega(E,\rho)$ obtained on pp. 370–2 is therefore valid, and proof of the relation

$$\int_{-\pi}^{\pi} \log|F(\mathrm{e}^{\mathrm{i}\vartheta})|\,\mathrm{d}\vartheta \;>\; -\infty$$

is completed as on pp. 372–3.

It may well appear that the argument just made did not make full use of the monotoneity of $\xi h(\xi)$. However that may be, this requirement does not seem capable of further significant relaxation, as we shall see in the next two articles. At present, let us translate our conclusion into a result involving the majorant $M(v)$ figuring in Volberg's theorem (p. 356).

In the statement of that theorem, two regularity properties are required of the increasing function $M(\nu)$ in addition to the divergence of $\sum_1^\infty M(n)/n^2$, namely, *that $M(\nu)/\nu$ be decreasing* and that

$$M(\nu) \geqslant \text{const.}\, \nu^\alpha$$

for large ν, where $\alpha > 1/2$. The *first* of these properties is (for us) practically equivalent to *concavity* of $M(\nu)$ by the theorem on p. 326. The concavity is needed for Dynkin's theorem (p. 339) and is not at issue here. Our interest is in replacing the *second* property by a *weaker* one. That being the object, there is no point in trying to gild the lily, and we may as well phrase our result for *concave majorants* $M(\nu)$. Indeed, nothing is really lost by sticking to *infinitely differentiable ones* with $M''(\nu) < 0$ and $M'(\nu) \longrightarrow 0$ for $\nu \longrightarrow \infty$, as long as that simplifies matters. See the theorem, p. 326 and the subsequent discussion on pp. 328–30; see also the beginning of the proof of the theorem in the next article.

With this simplification granted, passage from the result just arrived at to one stated in terms of $M(\nu)$ is provided by the easy

Lemma. Let $M(\nu)$ be infinitely differentiable for $\nu > 0$ with $M''(\nu) < 0$ and $M'(\nu) \longrightarrow 0$ for $\nu \longrightarrow \infty$, and put (as usual)

$$h(\xi) = \sup_{\nu > 0} (M(\nu) - \nu\xi).$$

Then $\xi h(\xi)$ is decreasing for small $\xi > 0$ if and only if $M(\nu)/\nu^{1/2}$ is increasing for large ν.

Proof. Under the given conditions, when $\xi > 0$ is sufficiently small, $h(\xi) = M(\nu) - \nu\xi$ for the *unique* ν with $M'(\nu) = \xi$ by the lemmas on pp. 330 and 332. Thus,

$$M'(\nu)h(M'(\nu)) = M(\nu)M'(\nu) - \nu(M'(\nu))^2,$$

so, since $M'(\nu)$ tends monotonically to zero as $\nu \longrightarrow \infty$, $\xi h(\xi)$ is *decreasing* for small $\xi > 0$ if and only if the right side of the last relation is *increasing* for large ν. But

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\nu}(M(\nu)M'(\nu) - \nu(M'(\nu))^2) &= M''(\nu)M(\nu) - 2\nu M''(\nu)M'(\nu) \\ &= -2\nu^{3/2}M''(\nu)\frac{\mathrm{d}}{\mathrm{d}\nu}\left(\frac{M(\nu)}{\nu^{1/2}}\right).\end{aligned}$$

Since $M''(\nu) < 0$, the lemma is clear.

Referring now to the above result, we get, almost without further ado, the

Theorem (Brennan). Let $M(\nu)$ be infinitely differentiable for $\nu > 0$, with $M''(\nu) < 0$,

$$\frac{M(\nu)}{\nu^{1/2}} \text{ increasing for large } \nu,$$

and

$$\sum_1^\infty M(n)/n^2 = \infty.$$

Suppose that

$$F(e^{i\vartheta}) \sim \sum_{-\infty}^{\infty} a_n e^{in\vartheta}$$

is continuous, with

$$|a_n| \leqslant \text{const.}\, e^{-M(|n|)} \qquad \text{for } n < 0.$$

Then, unless $F(e^{i\vartheta})$ vanishes identically,

$$\int_{-\pi}^{\pi} \log|F(e^{i\vartheta})|\, d\vartheta > -\infty.$$

Indeed, this follows directly by the lemma unless $\lim_{\nu\to\infty} M'(\nu) > 0$. Then, however, the theorem is true anyway – see p. 328.

2. Discussion

Brennan's result *really is* more general than the theorem on p. 356. That's because the hypothesis of the former one is fulfilled for any function $F(e^{i\vartheta})$ satisfying the hypothesis of the latter, thanks to the following

Theorem. Let $M(\nu)$, increasing and with $M(\nu)/\nu$ decreasing, satisfy the condition $\sum_1^\infty M(n)/n^2 = \infty$ and have $M(\nu) \geqslant \text{const.}\, \nu^{\frac{1}{2}+\delta}$ for large ν, where $\delta > 0$. Then there is an infinitely differentiable function $M_0(\nu)$, with $M_0''(\nu) < 0$,

$$M_0(\nu) \leqslant M(\nu) \text{ for large } \nu,$$

$M_0(\nu)/\nu^{1/2}$ increasing, and $\sum_1^\infty M_0(n)/n^2 = \infty$.

Proof. By the theorem on p. 326 we can, wlog, take $M(\nu)$ to be *actually concave*. It is then sufficient to obtain *any concave minorant* $M_*(\nu)$ of $M(\nu)$

with $M_*(\nu)/\nu^{1/2}$ *increasing* and $\int_1^\infty (M_*(\nu)/\nu^2)\mathrm{d}\nu$ *divergent*, for from such a minorant one easily obtains an $M_0(\nu)$ with the additional regularity affirmed by the theorem.

The procedure for doing this is like the one of pp. 229–30. Starting with an $M_*(\nu)$, one first puts $M_1(\nu) = M_*(\nu) + \nu^{1/2}$ and then, using a $\mathscr{C}_\infty$ function $\varphi(\tau)$ having the graph shown on p. 329, takes

$$M_0(\nu) = c\int_0^1 M_1(\nu-\tau)\varphi(\tau)\,\mathrm{d}\tau$$

for $\nu > 1$ with a suitable small constant c. This function $M_0(\nu)$ (defined in any convenient fashion for $0 < \nu \leqslant 1$) is readily seen to do the job.

Our main task is thus the construction of an $M_*(\nu)$. For that it is helpful to make a further reduction, arranging for $M(\nu)$ to have a *piecewise linear graph starting out from the origin*. That poses no problem; we simply replace our *given* concave function $M(\nu)$ by *another*, with graph consisting of a straight segment going from the origin to a point on the graph of the original function followed by suitably chosen *successive chords* of that graph. This having been attended to, we let $R(\nu)$ be the *largest increasing minorant* of $M(\nu)/\nu^{1/2}$ and then put

$$M_*(\nu) = \nu^{1/2}R(\nu);$$

this of course makes $M_*(\nu)/\nu^{1/2}$ automatically increasing and $M_*(\nu) \leqslant M(\nu)$.

Thanks to our initial adjustment to the graph of $M(\nu)$, we have $M(\nu)/\nu^{1/2} \longrightarrow 0$ for $\nu \longrightarrow 0$. Hence, since $M(\nu) \geqslant$ const. $\nu^{\frac{1}{2}+\delta}$ for large ν, $R(\nu)$ must tend to ∞ for $\nu \longrightarrow \infty$, and *coincides* with $M(\nu)/\nu^{1/2}$ *save on certain disjoint intervals* $(\alpha_k, \beta_k) \subset (0, \infty)$ for which

$$\frac{M(\alpha_k)}{\alpha_k^{1/2}} = R(\nu) = \frac{M(\beta_k)}{\beta_k^{1/2}}, \qquad \alpha_k \leqslant \nu \leqslant \beta_k.$$

Concavity of $M_*(\nu)$ follows from that of $M(\nu)$. The graph of $M_*(\nu)$ coincides with that of $M(\nu)$, save over the intervals (α_k, β_k), where it has *concave* arcs (along which $M_*(\nu)$ is proportional to $\nu^{1/2}$), lying *below* the corresponding arcs for $M(\nu)$ and *meeting those* at their *endpoints*. The *former* graph is thus clearly concave if the other one is.

Proving that $\sum_1^\infty M_*(n)/n^2 = \infty$ is trickier. There would be no trouble at all here if we could be sure that the ratios β_k/α_k were *bounded*, but we cannot assume that and our argument makes strong use of the fact that $\delta > 0$ in the condition $M(\nu) >$ const. $\nu^{\frac{1}{2}+\delta}$.

We again appeal to the special structure of $M(\nu)$'s graph to argue that the *local maxima* of $M(\nu)/\nu^{1/2}$, and hence the *intervals* (α_k, β_k), *cannot accumulate* at any finite point. Those intervals can therefore be indexed from left to right, and in the event that two adjacent ones should touch at their endpoints, we can consolidate them to form a single larger interval and then relabel. In this fashion, we arrive at a set-up where

$$0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots,$$

with $M_*(v) = M(v)$ *outside* the union of the (perhaps new) (α_k, β_k), and

$$M_*(v) = \left(\frac{v}{\alpha_k}\right)^{1/2} M(\alpha_k) = \left(\frac{\beta_k}{v}\right)^{1/2} M(\beta_k) \qquad \text{for } \alpha_k \leqslant v \leqslant \beta_k.$$

It is convenient to fix a β_0 with $0 < \beta_0 < \alpha_1$. Then, since $M(v)/v$ *decreases*, $M(\alpha_1) \leqslant (\alpha_1/\beta_0)M(\beta_0)$, so, by the preceding relation,

$$M(\beta_1) = \left(\frac{\beta_1}{\alpha_1}\right)^{1/2} M(\alpha_1) \leqslant \left(\frac{\beta_1}{\alpha_1}\right)^{1/2} \frac{\alpha_1}{\beta_0} M(\beta_0).$$

In like manner we find first that $M(\alpha_2) \leqslant (\alpha_2/\beta_1)M(\beta_1)$ and thence that $M(\beta_2) \leqslant (\beta_2/\alpha_2)^{1/2}(\alpha_2/\beta_1)M(\beta_1)$ which, substituted into the previous, yields

$$M(\beta_2) \leqslant \left(\frac{\beta_2}{\alpha_2}\right)^{1/2} \frac{\alpha_2}{\beta_1} \left(\frac{\beta_1}{\alpha_1}\right)^{1/2} \frac{\alpha_1}{\beta_0} M(\beta_0).$$

Continuing in this fashion, we see that

$$M(\beta_n) \leqslant \left(\frac{\beta_n}{\alpha_n}\right)^{1/2} \frac{\alpha_n}{\beta_{n-1}} \left(\frac{\beta_{n-1}}{\alpha_{n-1}}\right)^{1/2} \frac{\alpha_{n-1}}{\beta_{n-2}} \cdots \left(\frac{\beta_1}{\alpha_1}\right)^{1/2} \frac{\alpha_1}{\beta_0} M(\beta_0).$$

Now by hypothesis, $M(\beta_n) \geqslant C\beta_n^{\frac{1}{2}+\delta}$ where, wlog, $C = 1$. Use this with the relation just found and then divide the resulting inequality by $\alpha_n^{\frac{1}{2}+\delta}$, noting that

$$\alpha_n = \frac{\alpha_n}{\beta_{n-1}} \frac{\beta_{n-1}}{\alpha_{n-1}} \frac{\alpha_{n-1}}{\beta_{n-2}} \cdots \frac{\beta_1}{\alpha_1} \frac{\alpha_1}{\beta_0} \beta_0.$$

One gets

$$\left(\frac{\beta_n}{\alpha_n}\right)^{\frac{1}{2}+\delta} \leqslant \left(\frac{\beta_n}{\alpha_n}\right)^{\frac{1}{2}} \frac{\dfrac{\alpha_n}{\beta_{n-1}}\left(\dfrac{\beta_{n-1}}{\alpha_{n-1}}\right)^{1/2} \cdots \left(\dfrac{\beta_1}{\alpha_1}\right)^{1/2} \dfrac{\alpha_1}{\beta_0} M(\beta_0)}{\left\{\dfrac{\alpha_n}{\beta_{n-1}} \dfrac{\beta_{n-1}}{\alpha_{n-1}} \cdots \dfrac{\beta_1}{\alpha_1} \dfrac{\alpha_1}{\beta_0} \beta_0\right\}^{1/2+\delta}}.$$

After cancelling $(\beta_n/\alpha_n)^{1/2}$ from both sides and rearranging, this becomes

$$\left(\frac{\beta_n}{\alpha_n} \frac{\beta_{n-1}}{\alpha_{n-1}} \cdots \frac{\beta_1}{\alpha_1}\right)^{\delta} \leqslant \left(\frac{\alpha_n}{\beta_{n-1}} \frac{\alpha_{n-1}}{\beta_{n-2}} \cdots \frac{\alpha_1}{\beta_0}\right)^{\frac{1}{2}-\delta} \frac{M(\beta_0)}{\beta_0^{\frac{1}{2}+\delta}}.$$

There is of course no loss of generality here in assuming $\delta < 1/2$. The last formula can be rewritten

$$\sum_{k=1}^{n} \log\left(\frac{\beta_k}{\alpha_k}\right) \leqslant c + \frac{1-2\delta}{2\delta}\sum_{k=1}^{n} \log\left(\frac{\alpha_k}{\beta_{k-1}}\right)$$

where $c = (1/\delta)\log(M(\beta_0)/\beta_0^{1/2+\delta})$ is independent of n, and this estimate makes it possible for us to compare some integrals of $M(\nu)/\nu^2$ over complementary sets.

Since $M(\nu)/\nu$ is decreasing, we have

$$\int_{\beta_{n-1}}^{\alpha_n} \frac{M(\nu)}{\nu^2}\mathrm{d}\nu \geqslant \frac{M(\alpha_n)}{\alpha_n}\int_{\beta_{n-1}}^{\alpha_n} \frac{\mathrm{d}\nu}{\nu} = \frac{M(\alpha_n)}{\alpha_n}\log\frac{\alpha_n}{\beta_{n-1}},$$

and at the same time,

$$\int_{\alpha_n}^{\beta_n} \frac{M(\nu)}{\nu^2}\mathrm{d}\nu \leqslant \frac{M(\alpha_n)}{\alpha_n}\int_{\alpha_n}^{\beta_n} \frac{\mathrm{d}\nu}{\nu} = \frac{M(\alpha_n)}{\alpha_n}\log\frac{\beta_n}{\alpha_n}.$$

From the *second* inequality,

$$\sum_{n=1}^{N}\int_{\alpha_n}^{\beta_n} \frac{M(\nu)}{\nu^2}\mathrm{d}\nu \leqslant \sum_{n=1}^{N} \frac{M(\alpha_n)}{\alpha_n}\log\frac{\beta_n}{\alpha_n},$$

and partial summation converts the right side to

$$\sum_{n=1}^{N-1}\left\{\frac{M(\alpha_n)}{\alpha_n} - \frac{M(\alpha_{n+1})}{\alpha_{n+1}}\right\}\sum_{k=1}^{n}\log\frac{\beta_k}{\alpha_k} + \frac{M(\alpha_N)}{\alpha_N}\sum_{k=1}^{N}\log\frac{\beta_k}{\alpha_k}.$$

The ratios $M(\alpha_n)/\alpha_n$ are, however, decreasing, so we may apply the estimate obtained above to see that the last expression is

$$\leqslant \sum_{n=1}^{N-1}\left\{\frac{M(\alpha_n)}{\alpha_n} - \frac{M(\alpha_{n+1})}{\alpha_{n+1}}\right\}\left\{\frac{1-2\delta}{2\delta}\sum_{k=1}^{n}\log\frac{\alpha_k}{\beta_{k-1}} + c\right\}$$
$$+ \frac{M(\alpha_N)}{\alpha_N}\left\{\frac{1-2\delta}{2\delta}\sum_{k=1}^{N}\log\frac{\alpha_k}{\beta_{k-1}} + c\right\},$$

which, by reverse summation by parts, boils down to

$$\frac{1-2\delta}{2\delta}\sum_{n=1}^{N}\frac{M(\alpha_n)}{\alpha_n}\log\frac{\alpha_n}{\beta_{n-1}} + c\frac{M(\alpha_1)}{\alpha_1}.$$

This in turn is

$$\leqslant \frac{1-2\delta}{2\delta}\sum_{n=1}^{N}\int_{\beta_{n-1}}^{\alpha_n}\frac{M(\nu)}{\nu^2}\mathrm{d}\nu + c\frac{M(\alpha_1)}{\alpha_1}$$

by the *first* of the above inequalities, so, since $M(\nu) = M_*(\nu)$ on each of the intervals $[\beta_{n-1}, \alpha_n]$, we have finally

$$\sum_{n=1}^{N} \int_{\alpha_n}^{\beta_n} \frac{M(v)}{v^2} \mathrm{d}v \;\leqslant\; \frac{1-2\delta}{2\delta} \sum_{n=1}^{N} \int_{\beta_{n-1}}^{\alpha_n} \frac{M_*(v)}{v^2} \mathrm{d}v \;+\; c\frac{M(\alpha_1)}{\alpha_1}.$$

Adding $\sum_{n=1}^{N} \int_{\beta_{n-1}}^{\alpha_n} (M(v)/v^2)\mathrm{d}v = \sum_{n=1}^{N} \int_{\beta_{n-1}}^{\alpha_n} (M_*(v)/v^2)\mathrm{d}v$ to both sides of this relation one gets (*a fortiori*!)

$$\int_{\beta_0}^{\beta_N} \frac{M(v)}{v^2} \mathrm{d}v \;<\; c\frac{M(\alpha_1)}{\alpha_1} \;+\; \frac{1}{2\delta} \int_{\beta_0}^{\alpha_N} \frac{M_*(v)}{v^2} \mathrm{d}v,$$

and thence

$$\int_{\beta_0}^{\infty} \frac{M(v)}{v^2} \mathrm{d}v \;\leqslant\; c\frac{M(\alpha_1)}{\alpha_1} \;+\; \frac{1}{2\delta} \int_{\beta_0}^{\infty} \frac{M_*(v)}{v^2} \mathrm{d}v.$$

In the present circumstances, however, divergence of $\sum_1^\infty M(n)/n^2$ is equivalent to that of the left-hand integral and divergence of $\sum_1^\infty M_*(n)/n^2$ equivalent to that of the integral on the right. Our assumptions on $M(v)$ thus make $\sum_1^\infty M_*(n)/n^2 = \infty$, and the proof of the theorem is complete.

The second observation to be made about Brennan's theorem is that its *monotoneity requirement* on $M(v)/v^{1/2}$ is probably *incapable of much further relaxation*. That depends on an example mentioned at the end of Borichev and Volberg's preprint. Unfortunately, they do not describe the construction of the example, so I cannot give it here. *Let us, in the present addendum, assume that their construction is right and show how to deduce from this supposition that Brennan's result is close to being best possible in a sense to be soon made precise.*

The example of Borichev and Volberg, if correct, furnishes a decreasing function $h(\xi)$ with $\xi h(\xi) \geqslant 1$ and $\int_0^1 \log h(\xi)\,\mathrm{d}\xi = \infty$ together with $F(z)$, bounded and $\mathscr{C}_\infty$ in $\{|z|<1\}$ and having the non-tangential boundary value $F(\mathrm{e}^{\mathrm{i}\vartheta})$ a.e. on $\{|z|=1\}$, such that

$$\left|\frac{\partial F(z)}{\partial \bar{z}}\right| \;\leqslant\; \exp\left(-h\left(\log\frac{1}{|z|}\right)\right) \qquad \text{for } |z|<1,$$

while

$$\int_{-\pi}^{\pi} \log|F(\mathrm{e}^{\mathrm{i}\vartheta})|\,\mathrm{d}\vartheta \;=\; -\infty$$

although $F(\mathrm{e}^{\mathrm{i}\vartheta})$ *is not* a.e. *zero.*

The procedure we are about to follow comes from the paper of Jöricke and Volberg, and will be used again to investigate the more complicated situation taken up in the next article. In order that the reader may first see its main idea unencumbered by detail, let us *for now* make *an additional assumption that the function $F(z)$ supplied by the Borichev–Volberg construction is continuous up to $|z| = 1$.* At the end of the next article we will see that a counter-example to further extension of the L_1 version of Brennan's result given there can be obtained *without this continuity*. Assuming it here enables us to just *take over* the constructions of §D.6, Chapter VII.

The present function $F(z)$ is to be subjected to the treatment applied to the one thus denoted in §D.6, beginning on p. 359. We also employ the symbols

$$w(r) = \exp\left(-h\left(\log\frac{1}{r}\right)\right),$$

$\mathcal{O}$, B, Φ, Ω, &c with the meanings adopted there.

Starting with $F(z)$, we construct a *continuous* function $g(e^{i\vartheta})$ on $\{|z| = 1\}$ and a *concave* increasing majorant $M(v)$ having the following properties:

(i) $\quad g(e^{i\vartheta}) \not\equiv 0,$

(ii) $\quad \displaystyle\int_{-\pi}^{\pi} \log|g(e^{i\vartheta})|\,d\vartheta = -\infty,$

(iii) $\quad \displaystyle\sum_1^\infty M(n)/n^2 = \infty,$

(iv) $\quad M(v)/v^{1/2} \geqslant 2,$

(v) $\quad g(e^{i\vartheta}) \sim \displaystyle\sum_{-\infty}^{\infty} a_n e^{in\vartheta}$ with $|a_n| \leqslant \text{const.}\, e^{-M(|n|)}$ for $n < 0$.

It is clear from this *how close* Brennan's result comes to being *best possible* provided that the above assumptions are granted.

The weight $w(r)$ we are now using is decreasing, and, since $\xi h(\xi) \geqslant 1$, goes to zero rapidly enough for the reasoning followed in *steps 2* and *3* of §D.6, Chapter VII to carry over without change.

But the argument made for *step 1* on p. 361 requires modification. Here, since $F(z)$ is continuous on the closed unit disk and $\not\equiv 0$ on its circumference, there is a non-empty open arc I of that circumference on which $|F(e^{i\vartheta})|$ is *bounded away from zero.* Then, because $w(r) \longrightarrow 0$ for $r \longrightarrow 1$, the open set $\mathcal{O}$ must have a *component* – call it $\mathcal{O}'$ – *abutting* on I. If, at the same time, B contained a non-void open arc J of the unit circumference, we would have $\partial\mathcal{O}' \cap J = \emptyset$. In that event one could reason with the analytic function $\Phi(z)$ as at the bottom of p. 362, because $|\Phi(\zeta)| \leqslant \text{const.}\, w(|\zeta|)$ on $\partial\mathcal{O}' \cap \{|\zeta| < 1\}$. In that way, one would find that $\Phi(z) \equiv 0$ in $\mathcal{O}'$, making $F(e^{i\vartheta}) \equiv 0$ on I, a contradiction. Hence no such arc as J can exist.

Once *steps 1, 2* and *3* are carried out, we fix any ρ, $0 < \rho < 1$ and take the connected set $\Omega = \Omega(\rho) \subseteq \{\rho < |z| < 1\}$ described at the top of p. 363. As pointed out on p. 364, $\partial\Omega$ *includes the whole unit circumference.*

Fix now a $z_0 \in \Omega$. Given an integer $n \geqslant 0$, let us apply *Poisson's formula* to the function $z^n\Phi(z)$, *harmonic* (since analytic!) in Ω and *continuous* up to $\partial\Omega$. We get

$$z_0^n\Phi(z_0) \;=\; \int_{\partial\Omega} \zeta^n\Phi(\zeta)\,d\omega_\Omega(\zeta, z_0)$$

where, as usual, $\omega_\Omega(\;\;,\;\;)$ is harmonic measure for Ω. The boundary $\partial\Omega$ consists of the unit circumference together with

$$\gamma \;=\; \partial\Omega \cap \{|z| < 1\},$$

so the last relation can be rewritten

$$\int_{-\pi}^{\pi} e^{in\vartheta}\Phi(e^{i\vartheta})\,d\omega_\Omega(e^{i\vartheta}, z_0) \;=\; z_0^n\Phi(z_0) \;-\; \int_\gamma \zeta^n\Phi(\zeta)\,d\omega_\Omega(\zeta, z_0).$$

Let us first examine the *right side* of this formula.

With $\log\dfrac{1}{|z_0|} = \xi_0 > 0$, the *first* term on the right has modulus $|\Phi(z_0)|e^{-n\xi_0}$. Concerning the *second* term, we recall that by the construction of $\mathcal{O}$, $|\Phi(\zeta)| \leqslant \text{const.}\, w(|\zeta|)$ on γ, including on any arcs thereof lying on $\{|\zeta| = \rho\}$ and in $\mathcal{O}$, as long as the constant is chosen large enough. Therefore, writing

$$M(v) \;=\; \inf_{\xi > 0}\,(h(\xi) + \xi v)$$

we have, since $w(|\zeta|) \;=\; \exp(-h(\xi))$ with $\xi = \log(1/|\zeta|)$,

$$|\zeta^n\Phi(\zeta)| \;\leqslant\; \text{const.}\, e^{-M(n)}, \qquad \zeta \in \gamma.$$

Harmonic measure of course has total mass 1. Our second term is hence $\leqslant$ const. $e^{-M(n)}$ in magnitude, and we find that altogether, for $n \geqslant 0$,

$$\left|\int_{-\pi}^{\pi} e^{in\vartheta}\Phi(e^{i\vartheta})\,d\omega_\Omega(e^{i\vartheta}, z_0)\right| \;\leqslant\; \text{const.}\,(e^{-n\xi_0} + e^{-M(n)}).$$

It will be seen presently that $e^{-M(n)}$ dominates $e^{-n\xi_0}$ for large n, so that the latter term can be dropped from this last relation. On account of that, we next turn our attention to $M(v)$. This function is *concave* by its definition, and, since $h(\xi) \geqslant 1/\xi$, easily seen to be $\geqslant 2v^{1/2}$ and thus enjoy property (iv) of the above list. Because $h(\xi)$ is decreasing and $\int_0^1 \log h(\xi)\,d\xi = \infty$, we have $\int_1^\infty (M(v)/v^2)\,dv = \infty$ by the theorem on p. 337. That, however, implies that $\sum_1^\infty M(n)/n^2 = \infty$, which is property (iii).

We look now at the measure $\Phi(e^{i\vartheta})\,d\omega_\Omega(e^{i\vartheta}, z_0)$ appearing *on the left* in the preceding relation. In the first place, $d\omega_\Omega(e^{i\vartheta}, z_0)$ is *absolutely continuous* with respect to $d\vartheta$ on $\{|\zeta| = 1\}$, and indeed $\leqslant C\,d\vartheta$ there, the constant C depending on z_0. This follows immediately by comparison of $d\omega_\Omega(e^{i\vartheta}, z_0)$ with harmonic measure for the whole unit disk. We can therefore write

$$\Phi(e^{i\vartheta})\,d\omega_\Omega(e^{i\vartheta}, z_0) = g(e^{i\vartheta})\,d\vartheta$$

with a *bounded* function g, and have just the *moduli of* $2\pi g(e^{i\vartheta})$'s *Fourier coefficients* (of negative index) standing on the left in the above relation.

In fact, $d\omega_\Omega(e^{i\vartheta}, z_0)$ has *more regularity* than we have just noted. The *derivative* $d\omega_\Omega(e^{i\vartheta}, z_0)/d\vartheta$ is, for instance, *strictly positive* in the interior of each arc I_k of the unit circumference contiguous to B's intersection therewith. To see this one may, given I_k, construct a very shallow sectorial box $\mathscr{S}$ in the unit disk with base on I_k and *slightly shorter* than the latter. A shallow enough $\mathscr{S}$ will have none of $\partial\Omega$ in its interior since Ω *abuts* on I_k. One may therefore compare $d\omega_\Omega(e^{i\vartheta}, z)$ with harmonic measure for $\mathscr{S}$ when $z \in \mathscr{S}$ and $e^{i\vartheta}$ is on that box's base, and an application of Harnack then leads to the desired conclusion.

From this we can already see that $|g(e^{i\vartheta})|$ is *bounded away from zero* inside some of the arcs I_k, for instance, on the arc I used at the beginning of this discussion. But there is more – $g(e^{i\vartheta})$ is *continuous* on the unit circumference. That follows immediately from *four* properties: the *continuity* of $\Phi(e^{i\vartheta})$, its *vanishing* for $e^{i\vartheta} \in B$, the *boundedness* of $d\omega_\Omega(e^{i\vartheta}, z_0)/d\vartheta$, and, finally, the *continuity* of this derivative in the interior of each arc I_k contiguous to $B \cap \{|\zeta| = 1\}$. The first three of these we are sure of, so it suffices to verify the fourth.

For that purpose, it is easiest to use the formula

$$\frac{d\omega_\Omega(e^{i\vartheta}, z_0)}{d\vartheta} = \frac{d\omega_\Delta(e^{i\vartheta}, z_0)}{d\vartheta} - \int_\gamma \frac{d\omega_\Delta(e^{i\vartheta}, \zeta)}{d\vartheta}\,d\omega_\Omega(\zeta, z_0),$$

where $\omega_\Delta(\ \ , z_0)$ is ordinary harmonic measure for the unit disk Δ (cf. p. 371). For $e^{i\vartheta}$ moving along an arc I_k,

$$d\omega_\Delta(e^{i\vartheta}, \zeta)/d\vartheta = (1 - |\zeta|^2)/2\pi|\zeta - e^{i\vartheta}|^2$$

varies *continuously*, and *uniformly so*, for ζ ranging over any subset of Δ

staying away from $e^{i\vartheta}$. Continuity of $d\omega_\Omega(e^{i\vartheta}, z_0)/d\vartheta$ can then be read off from the formula since γ has no accumulation points inside the I_k.

The function $g(e^{i\vartheta})$ is thus continuous, in addition to enjoying property (i) of our list. Verification of properties (ii) and (v) thereof remains.

Because $d\omega_\Omega(e^{i\vartheta}, z_0)/d\vartheta \leqslant C$ and $|\Phi(e^{i\vartheta})|$ lies between two constant multiples of $|F(e^{i\vartheta})|$, property (ii) holds on account of the analogous condition satisfied by F and the relation of $g(e^{i\vartheta})$ to $\Phi(e^{i\vartheta})$. Passing to property (v), we note that an earlier relation can be rewritten

$$\left|\int_{-\pi}^{\pi} e^{in\vartheta} g(e^{i\vartheta})\, d\vartheta\right| \leqslant \text{const.}\,(e^{-n\xi_0} + e^{-M(n)}), \qquad n \geqslant 0.$$

By *concavity* of $M(v)$, $M(v)/v$ eventually decreases and tends to a limit $l \geqslant 0$ as $v \longrightarrow \infty$. Were $l > 0$, the right side of the inequality just written would be $\leqslant$ const. e^{-nl_0} with $l_0 = \min(\xi_0, l) > 0$. Such a bound on the left-hand integral would, with property (ii), force $g(e^{i\vartheta})$ to vanish identically – see the bottom of p. 328. Our $g(e^{i\vartheta})$, however, does not do that, so we must have $l = 0$, making $M(n) < n\xi_0$ for large n. The right side of our inequality can therefore be replaced by const. $e^{-M(n)}$, and property (v) holds. The construction is now complete.

It is to be noted that the only objects we actually *used* were the function $h(\xi)$ with its specified properties and $\Phi(z)$, analytic in a certain domain $\mathcal{O} \subseteq \{|z| < 1\}$ and continuous up to $\partial\mathcal{O}$, satisfying $|\Phi(\zeta)| \leqslant \text{const.}\exp\left(-h\left(\log\frac{1}{|\zeta|}\right)\right)$ on $\partial\mathcal{O} \cap \{|\zeta| < 1\}$ and $|\Phi(\zeta)| > 0$ on some arc of $\{|\zeta| = 1\}$ included in $\partial\mathcal{O}$. I have a persistent nagging feeling that such functions $h(\xi)$ and $\Phi(z)$, if there really *are* any, must be lying around somewhere or at least be closely related to others whose constructions are already available. One thinks of various kinds of functions meromorphic in the unit disk but not of bounded characteristic there; especially do the ones described by Beurling at the eighth Scandinavian mathematicians' congress come back continually to mind.

This addendum, however, is already being written at the very last moment. The imminence of press time leaves me no opportunity for pursuing the matter.

3. Extension to functions $F(e^{i\vartheta})$ in $L_1(-\pi, \pi)$.

The theorem of p. 356 holds for L_1 functions $F(e^{i\vartheta})$ not a.e. zero, as does Brennan's refinement of it given in article 1 above. A procedure for handling this more general situation (absence of continuity) is worked out in the beautiful *Mat. Sbornik* paper by Jöricke and Volberg. Here we adapt their method so as to make it go with the development already familiar from §D.6, Chapter VII, hewing as closely as possible to the latter.

Our aim is to show that

$$\int_{-\pi}^{\pi} \log|F(e^{i\vartheta})|\,d\vartheta \;>\; -\infty$$

for any function $F(e^{i\vartheta}) \in L_1(-\pi,\pi)$ not a.e. zero and satisfying the hypothesis of Brennan's theorem. Let us begin by observing that the treatment of this case can be reduced to that of a *bounded* function F.

Suppose, indeed, that

$$F(e^{i\vartheta}) \;\sim\; \sum_{-\infty}^{\infty} a_n e^{in\vartheta}$$

belongs to L_1, with $|a_n| \leqslant \text{const.}\,e^{-M(|n|)}$ for $n<0$. The series $\sum_{n<0} a_n e^{in\vartheta}$ is then surely *absolutely convergent*, so

$$\sum_{0}^{\infty} a_n e^{in\vartheta}$$

is also the Fourier series of an L_1 function, which we denote by $F_+(e^{i\vartheta})$ (this belongs in fact to the space H_1). For $|z|<1$, put

$$F_+(z) \;=\; \sum_{0}^{\infty} a_n z^n;$$

for this function, analytic in $\{|z|<1\}$, we have (Chapter II, §B!),

$$F_+(z) \longrightarrow F_+(e^{i\vartheta}) \quad \text{a.e. as } z \xrightarrow{\angle} e^{i\vartheta}.$$

Using the integrable function $\log^+|F_+(e^{i\vartheta})| \geqslant 0$, we now form

$$b(z) \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\vartheta}+z}{e^{i\vartheta}-z}\log^+|F_+(e^{i\vartheta})|\,d\vartheta,$$

analytic and with positive real part for $|z|<1$. According to the third theorem and scholium of §F.2, Chapter III, $b(z)$ tends for almost every ϑ to a limit $b(e^{i\vartheta})$ as $z \xrightarrow{\angle} e^{i\vartheta}$, with

$$\Re b(e^{i\vartheta}) \;=\; \log^+|F_+(e^{i\vartheta})| \quad \text{a.e.}$$

A standard extension of Jensen's inequality to H_1 also tells us that

$$\log|F_+(z)| \;\leqslant\; \Re b(z), \qquad |z|<1$$

(cf. pp. 291–2 where this was proved and used for $z=0$).

We next perform the Dynkin extension (described on pp. 339–40) on the continuous function

$$F_-(e^{i\vartheta}) \;=\; \sum_{-\infty}^{-1} a_n e^{in\vartheta}.$$

This gives us $F_-(z)$, $\mathscr{C}_\infty$ in the unit disk and continuous (hence *bounded*!) up to its boundary, with

$$\left|\frac{\partial F_-(z)}{\partial \bar{z}}\right| \leqslant \text{const.}\exp\left(-h\left(\log\frac{1}{|z|}\right)\right), \qquad |z|<1,$$

where, in the present circumstances,

$$h(\xi) = \sup_{v>0}(M(v)/2 - v\xi)$$

(see *remark* 2, p. 343). As usual, we write

$$w(r) = \exp\left(-h\left(\log\frac{1}{r}\right)\right);$$

then, putting

$$F(z) = F_-(z) + F_+(z)$$

for $|z|<1$, we have

$$\left|\frac{\partial F(z)}{\partial \bar{z}}\right| \leqslant \text{const.}\, w(|z|)$$

there, and

$$F(z) \longrightarrow F(e^{i\vartheta}) \quad \text{a.e. for } z \xrightarrow{\not\angle} e^{i\vartheta}.$$

The bounded function spoken of earlier is simply

$$F_0(z) = e^{-b(z)}F(z).$$

It *is* bounded in the unit disk by one of the previous relations; another tells us that $F_0(z)$ has a non-tangential boundary value $F_0(e^{i\vartheta}) = F(e^{i\vartheta})\exp(-b(e^{i\vartheta}))$ equal in modulus to $|F(e^{i\vartheta})|/\max(|F_+(e^{i\vartheta})|, 1)$ at almost every point of the unit circumference. Then, since $F(e^{i\vartheta}) \in L_1$ is not a.e. zero, neither is $F_0(e^{i\vartheta})$. We note finally that by *analyticity* of $e^{-b(z)}$, $\partial F_0(z)/\partial\bar{z} = e^{-b(z)}\partial F(z)/\partial\bar{z}$, making

$$\left|\frac{\partial F_0(z)}{\partial \bar{z}}\right| \leqslant \text{const.}\, w(|z|), \qquad |z|<1.$$

Given that $M(v)$ satisfies the hypothesis of Brennan's theorem, our function $h(\xi)$ enjoys the two properties used in the first part of article 1, namely, that $\xi h(\xi)$ *decreases* and that $\int_0^a \log h(\xi)\,d\xi = \infty$ for small $a>0$. If, now, we can *deduce* from these together with the *preceding relation* that the *bounded* function $F_0(z)$, not a.e. zero for $|z| = 1$, satisfies

$$\int_{-\pi}^{\pi} \log|F_0(e^{i\vartheta})| d\vartheta \;>\; -\infty,$$

we will certainly have *the same conclusion* for

$$\log|F(e^{i\vartheta})| \;=\; \log|F_0(e^{i\vartheta})| \;+\; \log^+|F_+(e^{i\vartheta})|.$$

The rest of our work deals exclusively with $F_0(z)$.

In order to stay as close as possible to the notation of §D.6, Chapter VII, *we denote the bounded function $F_0(z)$ by $F(z)$ from now on.* Using this new $F(z)$, we first form the sets $B \subseteq \{|z| \leqslant 1\}$ and $\mathcal{O} \subseteq \{|z| < 1\}$ as on pp. 359–60, and then the function $\Phi(z)$ introduced on p. 360. The latter, *analytic in $\mathcal{O}$*, is actually defined on the whole unit disk, and has there at least as much continuity as $F(z)$ besides lying in modulus between two constant multiples of $|F(z)|$. It has, in particular, a non-tangential boundary value $\Phi(e^{i\vartheta})$ a.e. on the unit circumference, and this *does not vanish* a.e. The construction of B ensures that

$$|\Phi(\zeta)| \;\leqslant\; \text{const.}\, w(|\zeta|) \qquad \text{on } \partial\mathcal{O} \cap \{|\zeta| < 1\}$$

(indeed, on B), and our task amounts to showing that

$$\int_{-\pi}^{\pi} \log|\Phi(e^{i\vartheta})| d\vartheta \;>\; -\infty$$

on account of these properties.

What makes the present situation more complicated than the one studied in §D.6 of Chapter VII is that $\Phi(z)$ need no longer be continuous up to the whole unit circumference. This causes the notion of *abutment* introduced on p. 348 to be less useful here for the examination of our set $\mathcal{O}$ than it was in §D.6, and we have to supplement it with another, that of *fatness.* The latter, based on the famous sawtooth construction of Lusin and Privalov, helps us to take account of $\Phi(z)$'s non-tangential boundary behaviour.

To describe what is meant by fatness, we need to bring in a special kind of domain together with some notation; both will also be used further on. Corresponding to each point $e^{i\alpha}$ on the unit circumference, we have an open set S_α consisting of the z with $1/2 \;<\; |z| \;<\; 1$ lying in the open 60° sector having vertex at $e^{i\alpha}$ and symmetric about the radius from 0 out to that point. Given any subset E of $\{|\zeta| = 1\}$ we then write

$$S_E \;=\; \bigcup_{e^{i\alpha} \in E} S_\alpha.$$

It is evident that if we take any S_E and a ρ, $1/2 < \rho < 1$, the intersection

$$S_E \cap \{\rho < |z| < 1\}$$

breaks up into (at most) a countable number of open *connected* components, each of the form

$$S_{E_k} \cap \{\rho < |z| < 1\},$$

with the E_k making up a (disjoint) partition of the set E.

Definition. A *connected* open set of the form

$$S_E \cap \{\rho < |z| < 1\}$$

(with $1/2 < \rho < 1$) is called a *sawblade* of *depth* $1 - \rho$. We say that such a sawblade *bites on* the set E.

Now we can state the

Definition. An open subset $\mathscr{U}$ of the unit disk is called *fat* if it contains a sawblade biting on a closed $E \subseteq \{|\zeta| = 1\}$ with $|E| > 0$. In that circumstance we also say that $\mathscr{U}$ is *fat at E*.

Equipped with these tools, we endeavour to investigate the set $\mathscr{O}$ according to the procedure of §D.6, Chapter VII. In this, some modifications are necessary; we have, in the first place, to *skip over step 1* (p. 361). Then, taking ρ, $1/2 < \rho < 1$, we construct a set $\Omega(\rho)$, proceeding differently, however, than as we did on pp. 361–3.

There is, by the properties of $\Phi(z)$, a closed subset E_0 of the unit circumference, $|E_0| > 0$, such that, for the *non-tangential* boundary values $\Phi(\zeta)$, we have, wlog,

$$|\Phi(\zeta)| > 1, \qquad \zeta \in E_0.$$

Egorov's theorem enables us to in fact pick E_0 so as to have $|\Phi(z)| > 1$ for $z \in S_{E_0}$ with $\rho' < |z| < 1$ when $\rho' > \rho$ is sufficiently close to 1. But the construction of B and $\mathscr{O}$ makes $|\Phi(z)| \leqslant \text{const.}\, w(|z|)$ on B, hence on $\{|z| < 1\} \sim \mathscr{O}$. Therefore, since $w(r) \longrightarrow 0$ for $r \longrightarrow 1$, we must have

$$S_{E_0} \cap \{\rho' < |z| < 1\} \subseteq \mathscr{O}$$

if ρ', $\rho < \rho' < 1$, is near enough to 1. One of the components of the intersection on the left is a sawblade of depth $1 - \rho'$ biting on a (Borel) subset E' of E_0 with $|E'| > 0$; a suitable *closed* subset E of E' then has

$|E|>0$, and *there is a sawblade of depth* $1-\rho'$ *biting on* E *and contained in* $\mathcal{O}$. We now take $\Omega(\rho)$ as *the component of* $\mathcal{O}\cap\{\rho<|z|<1\}$ *including that sawblade*; $\Omega(\rho)$ is *fat at* E.

For the present set $\Omega(\rho)$ there is *a substitute* for *step 2* of p. 362:

Step 2'. *$\partial\Omega(\rho)$ includes the whole unit circumference.*

This we establish by *reductio ad absurdum.* Let us write Ω for $\Omega(\rho)$, and put

$$\gamma = \partial\Omega\cap\{|z|<1\},$$

$$\Gamma = \partial\Omega \sim \gamma;$$

Γ is thus the part of $\partial\Omega$ lying on the unit circumference. *Assume* that there is on the latter a non-empty open arc J with $J\cap\Gamma=\emptyset$; we will then deduce a contradiction.

For that it is quicker to fall back on the device used in the second half of article 2 than to adapt Volberg's theorem on harmonic measures (p. 349) to the present situation. *Fixing* $z_0\in\Omega$, we can say that

$$z_0^n\Phi(z_0) = \int_{\partial\Omega}\zeta^n\Phi(\zeta)\,d\omega_\Omega(\zeta,z_0) \qquad \text{for } n\geqslant 0,$$

whence

$$\int_\Gamma e^{in\vartheta}\Phi(e^{i\vartheta})\,d\omega_\Omega(e^{i\vartheta},z_0) = z_0^n\Phi(z_0) - \int_\gamma \zeta^n\Phi(\zeta)\,d\omega_\Omega(\zeta,z_0), \qquad n\geqslant 0.$$

Here we are using Poisson's formula for the bounded function $\zeta^n\Phi(\zeta)$ harmonic (even analytic) in Ω and continuous up to γ, but not necessarily up to Γ, where it is only known to have non-tangential boundary values a.e. Such use is legitimate; we postpone verification of that, and of a corresponding version of Jensen's inequality, to the next article, so as not to interrupt the argument now under way.

As in article 2, $d\omega_\Omega(e^{i\vartheta},z_0)$ is absolutely continuous and $\leqslant C\,d\vartheta$ on Γ, and we obtain a bounded measurable function $g(e^{i\vartheta})$ by putting

$$g(e^{i\vartheta}) = \Phi(e^{i\vartheta})\frac{d\omega_\Omega(e^{i\vartheta},z_0)}{d\vartheta} \qquad \text{for } e^{i\vartheta}\in\Gamma$$

and (here!) taking $g(e^{i\vartheta})$ to be *zero outside* Γ. From the preceding relation we then see, as in article 2, that

$$\left|\int_{-\pi}^{\pi} e^{in\vartheta} g(e^{i\vartheta})\,d\vartheta\right| \leqslant \text{const.}\left(e^{-n\xi_0} + e^{-M_1(n)}\right)$$

for $n \geqslant 0$, where $\xi_0 > 0$ and

$$M_1(v) = \inf_{\xi>0}\,(h(\xi) + \xi v).$$

This function is increasing and concave, so the right side of the last inequality can be replaced by const. $e^{-M_2(n)}$ for large n, with $M_2(n)$ equal either to $\xi_0 n$ (in case $\lim_{v\to\infty}(M_1(v)/v) \geqslant \xi_0$) or else to $M_1(n)$. In either event, $M_2(n)$ increases and $\sum_1^\infty M_2(n)/n^2 = \infty$ on account of the properties of $h(\xi)$. (See the theorem of p. 337 – $M_1(n)$ is actually equal to $M(n)/2$ in the present set-up.) Now we can apply Levinson's theorem, since $g(e^{i\vartheta})$ *vanishes on the arc* J. The conclusion is that $g(e^{i\vartheta}) \equiv 0$ a.e.

But $g(e^{i\vartheta})$ *does not vanish* a.e. Indeed, Ω contains a *sawblade* $\mathscr{E}$ *biting on* a closed set E, $|E| > 0$, where $|\Phi(e^{i\vartheta})| \geqslant 1$. Thence,

$$\int_E |g(e^{i\vartheta})|\,d\vartheta = \int_E |\Phi(e^{i\vartheta})|\,d\omega_\Omega(e^{i\vartheta}, z_0) \geqslant \omega_\Omega(E, z_0).$$

Harnack's theorem assures us that the quantity on the right is >0 if, for some $z_1 \in \mathscr{E}$, $\omega_\Omega(E, z_1) > 0$. However, by the principle of extension of domain, $\omega_\Omega(E, z_1) \geqslant \omega_\mathscr{E}(E, z_1)$. At the same time, $\partial\mathscr{E}$ is *rectifiable*, so a conformal mapping of $\mathscr{E}$ onto the unit disk must take the *subset* E of $\partial\mathscr{E}$, *having linear measure* > 0, to a *set of measure* > 0 *on the unit circumference*. (This follows by the celebrated F. and M. Riesz theorem; a proof can be found in Zygmund or in any of the books about H_p spaces.) We therefore have $\omega_\mathscr{E}(E, z_1) > 0$, making $\omega_\Omega(E, z_0) > 0$ and hence, as we have seen, $\int_E |g(e^{i\vartheta})|\,d\vartheta > 0$.

Our contradiction is thus established. By it we see that the arc J cannot exist, i.e., that Γ *is the whole unit circumference*, as was to be shown.

With *step 2′* accomplished, we are ready for *step 3*. One starts out as on p. 363, using the square root mapping employed there. That gives us a domain $\Omega_{\surd}$, certainly *fat* at a closed subset E'', of $E_{\surd}$ (the image of E under our mapping), with $|E''| > 0$ (recall the earlier use of Egorov's theorem). Thereafter, one applies to $\Omega_{\surd}$ the argument just made for Ω in doing *step 2′*.

The weight $w_1(r)$ is next introduced as on p. 365, and the sets B_1 and $\mathcal{O}_1$ constructed (pp. 365–6). After doing *steps 2′* and *3* again with these objects, we come to *step 4*.

Jöricke and Volberg are in fact able to circumvent this step, thanks to a clever rearrangement of *step 5*. Here, however, let us continue according to the plan of §D.6, Chapter VII, for the work done there carries over practically without change to the present situation.

What is important for *step 4* is that a ζ, $|\zeta| = 1$, *not* in B must, even here, lie on an arc of the unit circumference *abutting* on $\mathcal{O}$. Such a $\zeta \notin B$ must thus, as on p. 367, have a neighborhood V_ζ with

$$V_\zeta \cap \{|z| < 1\} \subseteq \mathcal{O} \cap \{\rho^2 < |z| < 1\}.$$

The left-hand intersection therefore lies in some *connected component* of the one on the right, which, however, *can only be* $\Omega(\rho^2)$, since $\zeta \in \partial\Omega(\rho^2)$ by *step 2'*. The rest of the argument goes as on pp. 367–8.

Now we can do *step 5*, or rather the *substitute* for it carried out at the beginning of article 1. For this it is necessary to have the Jensen inequality

$$\log|\Phi(\rho)| \leqslant \int_{\partial\Omega(\rho^2)} \log|\Phi(\zeta)| \mathrm{d}\omega(\zeta, \rho)$$

(notation of p. 369) available in the present circumstances, where continuity of $\Phi(z)$ up to $\{|\zeta| = 1\}$ may fail. The legitimacy of this will be established in the next article; *granting* it for now, we may proceed exactly as at the beginning of article 1.

From here on, one continues as on pp. 370–2, and reaches the desired conclusion that $\int_{-\pi}^{\pi} \log|\Phi(e^{i\vartheta})| \mathrm{d}\vartheta > -\infty$ as on p. 373, after one more application of our extended Jensen inequality.

We thus arrive at the

Theorem. Let $F(e^{i\vartheta}) \in L_1(-\pi,\pi)$ not be zero a.e., and suppose that

$$F(e^{i\vartheta}) \sim \sum_{-\infty}^{\infty} a_n e^{in\vartheta}$$

with

$$|a_n| \leqslant \text{const.}\, e^{-M(|n|)}, \qquad n \leqslant 0.$$

Suppose that $M(v)$ is *concave*, that $M(v)/v^{1/2}$ is *increasing* for large v, and that

$$\sum_1^{\infty} M(n)/n^2 = \infty.$$

Then

$$\int_{-\pi}^{\pi} \log|F(e^{i\vartheta})| \mathrm{d}\vartheta > -\infty.$$

Remark. In their preprint, Borichev and Volberg consider formal trigonometric series

$$\sum_{-\infty}^{\infty} a_n e^{in\vartheta}$$

in which the a_n with *negative n* satisfy the requirement of the theorem, but the a_n with $n > 0$ are allowed to *grow* like $e^{M(n)}$ as $n \longrightarrow \infty$. Assuming *more* regularity for $M(v)$ ($M(v) \geqslant$ const. v^{α} with an $\alpha < 1$ *close to* 1 is enough), they are able to show that under the remaining conditions of the theorem, all the a_n must vanish if

$$\liminf_{r \to 1} \int_{-\pi}^{\pi} \log \left| \sum_{-\infty}^{0} a_n e^{in\vartheta} + \sum_{1}^{\infty} a_n r^n e^{in\vartheta} \right| d\vartheta = -\infty.$$

Before ending this article let us, as promised in the *last* one, see how the example of Borichev and Volberg shows that the *monotoneity requirement* on $M(v)/v^{1/2}$ *cannot, in the above theorem at least*, be *relaxed* to $M(v)/v^{1/2} \geqslant C > 0$, *even though continuity* up to $\{|\zeta| = 1\}$ *should fail* for the function $F(z)$ supplied by their construction.

The reader should refer back to the second part of article 2. Corresponding to the bounded function $F(z)$ used there, *no longer assumed continuous* up to $\{|\zeta| = 1\}$ but having at least non-tangential boundary values a.e. on that circumference, one can, as in the preceding discussion, form the sets B, $\mathcal{O}$ and $\Omega(\rho)$ and do *step 2'*. One may then form the function $g(e^{i\vartheta})$ as in article 2; *here* it is bounded and measurable at least. The work of *step 2'* shows that $g(e^{i\vartheta})$ is *not* a.e. zero, while properties (ii)–(v) of article 2 *hold* for it (for the last one, see again the end of that article).

This is all we need.

4. Lemma about harmonic functions

Suppose we have a domain Ω regular for Dirichlet's problem, lying in the (open) unit disk Δ and having part of $\partial\Omega$ on the unit circumference. As in the last article, we write

$$\Gamma = \partial\Omega \cap \partial\Delta \quad \text{and} \quad \gamma = \partial\Omega \cap \Delta.$$

For the following discussion, let us agree to call ζ, $|\zeta| = 1$, a *radial accumulation point of* Ω if, for a *sequence* $\{r_n\}$ tending to 1, we have $r_n\zeta \in \Omega$ for each n. We then denote by Γ' the set of such radial accumulation points, noting that $\Gamma' \subseteq \Gamma$ with the inclusion frequently *proper*.

Lemma. (Jöricke and Volberg) Let $V(z)$, harmonic and bounded in Ω, be continuous up to γ, and suppose that

$$\lim_{\substack{r\to 1\\ r\zeta\in\Omega}} V(\zeta)$$

exists for almost all $\zeta\in\Gamma'$. Put $v(\zeta)$ *equal* to that limit for such ζ, and to *zero* for the *remaining* $\zeta\in\Gamma$. On γ, take $v(\zeta)$ equal to $V(\zeta)$. Then, for $z\in\Omega$,

$$V(z) \;=\; \int_{\partial\Omega} v(\zeta)\,\mathrm{d}\omega_\Omega(\zeta, z).$$

Proof. It suffices to establish the result for *real* harmonic functions $V(z)$, and, for those, to show that

$$V(z) \;\leqslant\; \int_{\partial\Omega} v(\zeta)\,\mathrm{d}\omega_\Omega(\zeta, z), \qquad z\in\Omega,$$

since the reverse inequality then follows on changing the signs of V and v.

By modifying $v(\zeta)$ on a subset of Γ having zero Lebesgue measure, we get a bounded Borel function defined on $\partial\Omega$. But on Γ, we have $\mathrm{d}\omega_\Omega(\zeta, z) \leqslant C_z|d\zeta|$ (see articles 2 and 3), so such modification cannot alter the value of $\int_{\partial\Omega} v(\zeta)\,\mathrm{d}\omega_\Omega(\zeta, z)$. *We may hence just as well take $v(\zeta)$ as a bounded Borel function* (on $\partial\Omega$) *to begin with.*

That granted, we desire to show that the integral just written is $\geqslant V(z)$. For this it seems necessary to hark back to the very foundations of integration theory. Call the *limit* of any *increasing sequence of functions continuous on $\partial\Omega$* an *upper function* (on $\partial\Omega$). There is then a *decreasing sequence* of *upper functions* $w_n(\zeta) \geqslant v(\zeta)$ such that

$$\int_{\partial\Omega} w_n(\zeta)\,\mathrm{d}\omega_\Omega(\zeta, z) \;\xrightarrow[n]{}\; \int_{\partial\Omega} v(\zeta)\,\mathrm{d}\omega_\Omega(\zeta, z), \qquad z\in\Omega.$$

Indeed, corresponding to any *given* $z\in\Omega$, such a sequence is furnished by a basic construction of the Lebesgue–Stieltjes integral, $\omega_\Omega(\;\;, z)$ being a Radon measure on $\partial\Omega$. But then that sequence works also for *any other* $z\in\Omega$, since $\mathrm{d}\omega_\Omega(\zeta, z') \leqslant C(z, z')\mathrm{d}\omega_\Omega(\zeta, z)$ (Harnack).

Our inequality involving v and V will *thus be established*, provided that we can verify

$$V(z) \;\leqslant\; \int_{\partial\Omega} w_n(\zeta)\,\mathrm{d}\omega_\Omega(\zeta, z), \qquad z\in\Omega,$$

for each n. *Fixing*, then, any n, we write simply $w(\zeta)$ for $w_n(\zeta)$ and put

$$W(z) = \int_{\partial\Omega} w(\zeta)\,d\omega_\Omega(\zeta, z)$$

for $z\in\Omega$, making $W(z)$ *harmonic* there. Our task is to prove that

$$V(z) \leqslant W(z), \qquad z\in\Omega.$$

It is convenient to define $W(z)$ on *all* of $\bar{\Omega}$ by putting

$$W(\zeta) = w(\zeta), \qquad \zeta\in\partial\Omega.$$

At each $\zeta\in\partial\Omega$ we then have

$$\liminf_{\substack{z\to\zeta \\ z\in\bar{\Omega}}} W(z) \geqslant W(\zeta)$$

by the elementary approximate identity property of harmonic measure, since $w(\zeta)$, as limit of an *increasing* sequence of continuous functions, satisfies

$$\liminf_{\substack{\zeta\to\zeta_0 \\ \zeta\in\partial\Omega}} w(\zeta) \geqslant w(\zeta_0) \qquad \text{for } \zeta_0\in\partial\Omega.$$

The function $W(z)$ enjoys a certain *reproducing property* in $\bar{\Omega}$. Namely, if the domain $\mathscr{D} \subseteq \Omega$ is also regular for Dirichlet's problem, with perhaps (and especially!) part of $\partial\mathscr{D}$ on $\partial\Omega$, we have

$$W(z) = \int_{\partial\mathscr{D}} W(\zeta)\,d\omega_{\mathscr{D}}(\zeta, z) \qquad \text{for } z\in\mathscr{D}.$$

To see this, take an increasing sequence of functions $f_k(\zeta)$ continuous on $\partial\Omega$ and tending to $w(\zeta)$ thereon, and let

$$F_k(z) = \int_{\partial\Omega} f_k(\zeta)\,d\omega_\Omega(\zeta, z), \qquad z\in\Omega.$$

Then the $F_k(z)$ tend monotonically to $W(z)$ in Ω by the monotone convergence theorem. That convergence actually holds *on* $\bar{\Omega}$ if we put $F_k(\zeta) = f_k(\zeta)$ on $\partial\Omega$; this, however, makes each function $F_k(z)$ *continuous on* $\bar{\Omega}$ besides being *harmonic* in Ω. *In the domain* $\mathscr{D}$, we therefore have

$$F_k(z) = \int_{\partial\mathscr{D}} F_k(\zeta)\,d\omega_{\mathscr{D}}(\zeta, z)$$

for each k. Another appeal to monotone convergence now establishes the corresponding property for W.

Fix any $z_0\in\Omega$; we wish to show that $V(z_0) \leqslant W(z_0)$. For this purpose, we use the formula just proved with $\mathscr{D}$ equal to *the component* Ω_r *of* $\Omega \cap \{|z| < r\}$ *containing* z_0, where $|z_0| < r < 1$. Because Ω is regular for Dirichlet's problem, so is each Ω_r; that follows immediately from the characterization of such regularity in terms of *barriers*, and, in the circumstances of the last article, can also be checked directly (cf. p. 360). We write

$$\Gamma_r = \partial\Omega_r \cap \Omega,$$

making Γ_r the union of some *open arcs* on $\{|\zeta| = r\}$, and then take

$$\gamma_r = \partial\Omega_r \sim \Gamma_r;$$

γ_r is a subset (perhaps proper) of $\gamma \cap \{|\zeta| \leqslant r\}$.

The function $V(z)$, given as harmonic in Ω and continuous up to γ, is certainly continuous up to $\partial\Omega_r$. Therefore, since $V(\zeta) = v(\zeta)$ on $\gamma \supseteq \gamma_r$, we have, for $z \in \Omega_r$,

$$V(z) = \int_{\gamma_r} v(\zeta)\,d\omega_{\Omega_r}(\zeta, z) + \int_{\Gamma_r} V(\zeta)\,d\omega_{\Omega_r}(\zeta, z).$$

At the same time, by the reproducing property of W,

$$W(z) = \int_{\gamma_r} W(\zeta)\,d\omega_{\Omega_r}(\zeta, z) + \int_{\Gamma_r} W(\zeta)\,d\omega_{\Omega_r}(\zeta, z), \qquad z \in \Omega_r.$$

We henceforth write $\omega_r(\ ,\ )$ *for* $\omega_{\Omega_r}(\ ,\ )$. Then, since on $\gamma_r \subseteq \partial\Omega$, $W(\zeta) = w(\zeta)$ is $\geqslant v(\zeta)$, the two last relations yield

$$W(z) - V(z) \geqslant \int_{\Gamma_r} (W(\zeta) - V(\zeta))\,d\omega_r(\zeta, z)$$

for $z \in \Omega_r$. Our idea is to now make $r \longrightarrow 1$ in this inequality.

For $|\zeta| = 1$, define

$$\Delta_r(\zeta) = \begin{cases} W(r\zeta) - V(r\zeta) & \text{if } r\zeta \in \Gamma_r, \\ 0 \quad \text{otherwise.} \end{cases}$$

Since $V(z)$ is given as *bounded*, the functions $\Delta_r(\zeta)$ are *bounded below*. *Moreover* (and this is the clincher),

$$\liminf_{r \to 1} \Delta_r(\zeta) \geqslant 0 \text{ a.e.}, |\zeta| = 1.$$

That is indeed *clear* for the ζ on the unit circumference *outside* Γ' (the set of radial accumulation points of Ω); since for such a ζ, $r\zeta$ cannot even belong to Ω (let alone to Γ_r) when r is near 1. Consider therefore a $\zeta \in \Gamma'$, and take any sequence of $r_n < 1$ tending to 1 with, wlog, *all the* $r_n\zeta$ *in* Ω

and even in their corresponding Γ_{r_n}. Then our hypothesis and the specification of v tell us that

$$V(r_n\zeta) \underset{n}{\longrightarrow} v(\zeta),$$

except when ζ belongs to a certain set of measure zero, independent of $\{r_n\}$. For such a sequence $\{r_n\}$, however,

$$\liminf_{n\to\infty} W(r_n\zeta) \;\geqslant\; W(\zeta) \;=\; w(\zeta)$$

as seen earlier, yielding, with the preceding,

$$\liminf_{n\to\infty} \Delta_{r_n}(\zeta) \;\geqslant\; w(\zeta) - v(\zeta) \;\geqslant\; 0.$$

The asserted relation thus holds *on* Γ' *as well*, save perhaps in a set of measure zero.

Returning to our fixed $z_0 \in \Omega$, we note that for $(1+|z_0|)/2 \;<\; r \;<\; 1$ (say), we have, on Γ_r,

$$d\omega_r(\zeta, z_0) \;\leqslant\; K|d\zeta|$$

with K independent of r (just compare $\omega_r(\;\;,\;\;)$ with harmonic measure for $\{|z|<r\}$). There are hence measurable functions $\mu_r(\zeta)$ defined on $\{|\zeta|=1\}$ for these values of r, with $0 \;\leqslant\; \mu_r(\zeta) \;\leqslant\; K$ (and $\mu_r(\zeta)=0$ for $r\zeta\notin\Gamma_r$), such that

$$\int_{\Gamma_r} (W(\zeta) - V(\zeta))\,d\omega_r(\zeta, z_0) \;=\; \int_{|\zeta|=1} \Delta_r(\zeta) r\mu_r(\zeta)|d\zeta|.$$

Here the products $\Delta_r(\zeta)r\mu_r(\zeta)$ are *uniformly bounded below* since the $\Delta_r(\zeta)$ are. And, by what has just been shown,

$$\liminf_{r\to 1} \Delta_r(\zeta) r\mu_r(\zeta) \;\geqslant\; 0 \quad \text{a.e., } |\zeta|=1.$$

Thence, by *Fatou's lemma* (!),

$$\liminf_{r\to 1} \int_{|\zeta|=1} \Delta_r(\zeta) r\mu_r(\zeta)|d\zeta| \;\geqslant\; 0.$$

We have seen, however, that when $r>|z_0|$, $W(z_0)-V(z_0)$ is $\geqslant$ the left-hand integral in the previous relation. It follows therefore that

$$W(z_0) - V(z_0) \;\geqslant\; 0,$$

as was to be proven.

We are done.

Remark 1. When $V(z)$ is only assumed to be *subharmonic* in Ω but satisfies otherwise the hypothesis of the lemma, the argument just made shows that

$$V(z) \leqslant \int_{\partial\Omega} v(\zeta)\,\mathrm{d}\omega_\Omega(\zeta, z) \qquad \text{for } z\in\Omega.$$

Remark 2. In the applications made in article 3, the function $V(z)$ actually has a continuous extension to the open unit disk Δ with modulus *bounded, in* $\Delta \sim \Omega$, by a function of $|z|$ *tending to zero for* $|z| \longrightarrow 1$. That extension also has non-tangential boundary values a.e. on $\partial\Delta$. In these circumstances the lemma's *ad hoc* specification of $v(\zeta)$ on $\Gamma \sim \Gamma'$ is *superfluous*, for the non-tangential limit of $V(z)$ must *automatically be zero* at any $\zeta \in \Gamma \sim \Gamma'$ where it exists.

Remark 3. To arrive at the version of Jensen's inequality used in article 3, apply the relation from *remark 1* to the subharmonic functions $V_M(z) = \log^+|M\Phi(z)|$, referring to *remark 2*. That gives us

$$\max\left(\log|\Phi(z)|,\ \log\frac{1}{M}\right) \leqslant \int_{\partial\Omega} \max\left(\log|\Phi(\zeta)|,\ \log\frac{1}{M}\right)\mathrm{d}\omega_\Omega(\zeta, z)$$

for $z\in\Omega$. Then, since $|\Phi(z)|$ is bounded above, one may obtain the desired result by making $M \longrightarrow \infty$.

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