Expanding the integrand from ($\ast$) in powers of re^{it}, we obtain for that expression the value

$$-\sum_{n=1}^{\infty}\left(\int_0^1 r^{n+2}w(r)\,dr\right)\left(\frac{1}{2\pi}\int_0^{2\pi} e^{i(n-1)t}\Omega(e^{it})\,dt\right)e^{-in\vartheta}.$$

We see that we need to have

$$\Omega(e^{it}) \sim \sum_{-\infty}^{\infty} b_n e^{int}$$

where, for $n > 1$,

$$(\dagger) \qquad b_{1-n} = -\frac{a_{-n}}{\int_0^1 r^{n+2}w(r)\,dr}.$$

We may choose the b_n with *positive* index in any manner compatible with the continuous differentiability of $\Omega(e^{it})$; let us simply *put them all equal to zero.*

By the lemma, the *right side* of ($\dagger$) is in modulus

$$\leqslant \text{const.}|na_{-n}|e^{M(n)}$$

for $n \geqslant 1$. The b_m given by ($\dagger$) therefore satisfy

$$\sum_{-\infty}^{0} |mb_m| < \infty$$

according to the hypothesis of our theorem. This means that there *is* a function $\Omega(e^{it})$ satisfying our requirements whose *differentiated Fourier series is absolutely convergent.* Such a function is surely continuously differentiable; that is what was needed.

The theorem is proved.

Remark 1. We are going to use the extension of F to $\{|z| < 1\}$ furnished by Dynkin's theorem in conjunction with the *corollary at the end of article* 1. That corollary involves the integral

$$\frac{1}{2\pi}\iint_{\mathscr{D}} \frac{F_{\bar\zeta}(\zeta)}{F(\zeta)}\frac{d\xi\,d\eta}{(\zeta - z)}$$

where, on the open set $\mathscr{D}$, $|F(\zeta)| > 0$ and $|F_{\bar\zeta}(\zeta)| \leqslant \text{const.}|F(\zeta)|$. The theorem of article 1 is used to take $\partial/\partial\bar z$ of this integral, and the *hypothesis of the corollary* requires that $F(\zeta)$ be $\mathscr{C}_2$ in $\mathscr{D}$ in order to guarantee the legitimacy of that theorem's application. *Our extension $F(z)$ furnished by Dynkin's theorem is, however, only ensured to be $\mathscr{C}_1$* for $|z| < 1$. Are we not in *trouble*?

Not to worry. All that the corollary really *uses* is continuous differentiability of the *quotient* $F_{\bar\zeta}(\zeta)/F(\zeta)$ in $\mathscr{D}$. Our F *is*, however, $\mathscr{C}_1$ in $\mathscr{D}$, where it is also $\neq 0$. And, *from the proof of Dynkin's theorem*, $F_{\bar{\zeta}}(\zeta) = -|\zeta|^2 w(|\zeta|)\Omega(e^{it})$ for $\zeta = |\zeta|e^{it}$ with $|\zeta| < 1$. *The expression on the right is, however,* $\mathscr{C}_1$ for $|\zeta| < 1$; we indeed checked that it *had* that property during the proof (as we *had* to do in order to justify using the theorem of article 1 to show that $F_{\bar{\zeta}}(\zeta)$ was equal to it!). We are all right.

Remark 2. Under the conditions of Volberg's theorem, there is no essential distinction between the functions $M(v)$ and $2M(v)$, and $e^{M(v)}$ goes to infinity much faster than any power of v as $v \to \infty$. (We will need to require that $M(v) \geqslant \text{const.}v^{\alpha}$ for some $\alpha > \frac{1}{2}$ as has already been remarked in article 2.) In the application of Dynkin's theorem to be made below, we will therefore be able to replace the condition

$$\sum_1^{\infty} |n^2 a_{-n}| e^{M(n)} \;<\; \infty$$

figuring in its hypothesis by

$$|a_{-n}| \;\leqslant\; \text{const.}e^{-2M(n)}, \qquad n \geqslant 1,$$

or even (after a suitable unessential modification in the description of $w(r)$) by

$$|a_{-n}| \;\leqslant\; \text{const.}e^{-M(n)}, \qquad n \geqslant 1.$$

4. Material about weighted planar approximation by polynomials

Lemma. *Let* $w(r) \geqslant 0$ *for* $0 \leqslant r < 1$, *with*

$$\int_0^1 w(r) r \, dr \;<\; \infty.$$

If $F(z)$ *is any function analytic in* $\{|z| < 1\}$ *such that*

$$\iint_{|z|<1} |F(z)|^2 w(|z|) \, dx \, dy \;<\; \infty,$$

there are polynomials $Q(z)$ *making*

$$\iint_{|z|<1} |F(z) - Q(z)|^2 w(|z|) \, dx \, dy$$

arbitrarily small.

Proof. The basic idea is that $w(|z|)$ depends only on the modulus of z.

Given $\varepsilon > 0$, take $\rho < 1$ so close to 1 that

$$\iint_{\rho<|z|<1} |F(z)|^2 w(|z|)\,\mathrm{d}x\,\mathrm{d}y \;<\; \varepsilon.$$

Note that if $0 < \lambda < 1$ and $0 < r < 1$,

$$\int_0^{2\pi} |F(\lambda r e^{i\vartheta})|^2\,\mathrm{d}\vartheta \;\leqslant\; \int_0^{2\pi} |F(re^{i\vartheta})|^2\,\mathrm{d}\vartheta.$$

Therefore,

$$\iint_{\rho<|z|<1} |F(\lambda z)|^2 w(|z|)\,\mathrm{d}x\,\mathrm{d}y \;=\; \int_\rho^1 \int_0^{2\pi} |F(\lambda r e^{i\vartheta})|^2 w(r) r\,\mathrm{d}\vartheta\,\mathrm{d}r$$

$$\leqslant\; \int_\rho^1 \int_0^{2\pi} |F(re^{i\vartheta})|^2 w(r) r\,\mathrm{d}\vartheta\,\mathrm{d}r \;<\; \varepsilon.$$

Once $\rho < 1$ has been fixed, $F(z)$ is *uniformly continuous* for $|z| \leqslant \rho$, so $\iint_{|z|<\rho} |F(z) - F(\lambda z)|^2 w(|z|)\,\mathrm{d}x\,\mathrm{d}y \longrightarrow 0$ as $\lambda \to 1$ (we use the integrability of $rw(r)$ on $[0, 1)$ here). In view of the preceding calculation, we can thus find (and fix) a $\lambda < 1$ such that

$$\iint_{|z|<1} |F(z) - F(\lambda z)|^2 w(|z|)\,\mathrm{d}x\,\mathrm{d}y \;<\; 5\varepsilon.$$

The Taylor series for $F(\lambda z)$ converges *uniformly* for $|z| \leqslant 1$. We may therefore take a suitable *partial sum* $Q(z)$ of that Taylor series so as to make

$$\iint_{|z|<1} |F(\lambda z) - Q(z)|^2 w(|z|)\,\mathrm{d}x\,\mathrm{d}y \;<\; \varepsilon.$$

Then

$$\iint_{|z|<1} |F(z) - Q(z)|^2 w(|z|)\,\mathrm{d}x\,\mathrm{d}y \;<\; 16\varepsilon.$$

That does it.

At this point, we begin to make systematic use of a corollary to the theorem of Levinson given in §A.5. Oddly enough, Beurling's stronger results from §B are never called for in Volberg's work.

Theorem on simultaneous polynominal approximation (Kriete, Volberg). *Let, for* $0 < r < 1$,

$$w(r) \;=\; \exp\left(-H\left(\log\frac{1}{r}\right)\right)$$

where $H(\xi)$ *is decreasing and bounded below on* $(0, \infty)$, *and suppose that*

$$\int_0^a \log H(\xi)\,d\xi \;=\; \infty$$

for all sufficiently small $a > 0$.

Let E *be any proper closed subset of the unit circumference, let* $p(e^{i\vartheta}) \in L_2(E)$, *and suppose that* $f(z)$ *is analytic in* $\{|z| < 1\}$, *and such that*

$$\iint_{|z|<1} |f(z)|^2 w(|z|)\,dx\,dy \;<\; \infty.$$

Then there is a sequence of polynomials $P_n(z)$ *with*

$$\int_E |p(e^{i\vartheta}) - P_n(e^{i\vartheta})|^2\,d\vartheta + \iint_{|z|<1} |f(z) - P_n(z)|^2 w(|z|)\,dx\,dy \;\underset{n}{\longrightarrow}\; 0.$$

Proof. We use the fact that the collection of functions $F(z)$ analytic in $\{|z| < 1\}$ with $\iint_{|z|<1} |F(z)|^2 w(|z|)\,dx\,dy < \infty$ forms a Hilbert space if we bring in the inner product

$$\langle F_1, F_2 \rangle_w \;=\; \iint_{|z|<1} F_1(z)\overline{F_2(z)}\,w(|z|)\,dx\,dy.$$

This is evident, except perhaps for the completeness property. To verify the latter, it is clearly enough to show that, for any L, the functions $F(z)$ analytic in $\{|z| < 1\}$ and satisfying

$$\iint_{|z|<1} |F(z)|^2 w(|z|)\,dx\,dy \;\leqslant\; L$$

form a *normal family* in the open unit disk. However, the weight $w(r)$ we are using is *strictly positive* and *decreasing* on $(0, 1)$. Hence, for any $r < 1$, the previous relation makes

$$\iint_{|z|<r} |F(z)|^2\,dx\,dy \;\leqslant\; \frac{L}{w(r)}.$$

It is well known that such functions $F(z)$ form a normal family in $\{|z| < r\}$. Here, $r < 1$ is arbitrary.

Let us turn to the proof of the theorem, reasoning by duality in the Hilbert space $L_2(E) \oplus \mathscr{H}$ where $\mathscr{H}$ is the Hilbert space just described.* Suppose, then, that there is a $p(e^{i\vartheta}) \in L_2(E)$ and an $f(z) \in \mathscr{H}$ for which the *conclusion of the theorem fails to hold*. There must then be a *non-zero* element (q, F) of $L_2(E) \oplus \mathscr{H}$ *orthogonal* in that space to *all* the elements of the form $(P(e^{i\vartheta}), P(z))$ with *polynomials* P. We are going to obtain a *contradiction* by showing that *in fact* $q = 0$ and $F = 0$.

* We are dealing here with the *direct sum* of $L_2(E)$ and $\mathscr{H}$.

The orthogonality in question is equivalent to the relations

$$\int_E \overline{q(e^{i\vartheta})}e^{in\vartheta}\,d\vartheta + \iint_{|z|<1} \overline{F(z)}z^n w(|z|)\,dx\,dy = 0, \qquad n = 0, 1, 2, 3, \ldots.$$

Define $q(e^{i\vartheta})$ for *all* of $\{|z| = 1\}$ by making it *zero* for $e^{i\vartheta} \notin E$. Then $q(e^{i\vartheta}) \in L_2(-\pi, \pi)$, and if we write

$$(*) \qquad q(e^{i\vartheta}) \sim \sum_{-\infty}^{\infty} \alpha_n e^{in\vartheta}$$

we find from the previous relation that

$$\bar{\alpha}_n = -\frac{1}{2\pi}\iint_{|z|<1} \overline{F(z)}z^n w(|z|)\,dx\,dy, \qquad n \geqslant 0.$$

Since

$$\iint_{|z|<1} |F(z)|^2 w(|z|)\,dx\,dy < \infty,$$

the integral on the right is in modulus

$$\leqslant \text{const.}\sqrt{\int_0^1 r^{2n}w(r)r\,dr}$$

by Schwarz' inequality. However, in terms of $\xi = \log(1/r)$ and the function $H(\xi)$,

$$r^{2n}w(r) = e^{-(H(\xi)+2n\xi)}.$$

Denoting $\inf_{\xi>0}(H(\xi) + \xi\nu)$ by $M(\nu)$ as in the last theorem of article 2, we see that the right side is $\leqslant e^{-M(2n)}$. The preceding expression is therefore $\leqslant \text{const.}e^{-M(2n)/2}$, i.e.

$$(\divideontimes) \qquad |\alpha_n| \leqslant \text{const.}e^{-M(2n)/2}, \qquad n \geqslant 1.$$

Since E is a *proper closed subset of* $\{|z| = 1\}$, *its complement on the unit circumference contains an arc J of positive length.* The function $q(e^{i\vartheta})$ *vanishes* outside E, hence on J, and certainly belongs to $L_1(-\pi, \pi)$. Also, *by the last theorem in article 2,*

$$\int_1^\infty \frac{M(\nu)}{\nu^2}\,d\nu = \infty$$

on account of our hypothesis on $H(\xi)$. Therefore

$$\sum_1^\infty \frac{M(2n)}{2n^2} = \infty,$$

$M(v)$ being increasing, so, *by virtue of the corollary at the end of* §A.5, $(*)$ and $(\ast\!\!\ast)$ imply that $q(e^{i\vartheta}) = 0$ a.e.

We see that $\alpha_n = 0$ for all n, which means that

$$\iint_{|z|<1} \overline{F(z)}\, z^n w(|z|)\, dx\, dy \;=\; 0$$

for $n = 0, 1, 2, \ldots$. Since $H(\xi)$ is bounded below on $(0, \infty)$, $w(r)$ is *bounded above* for $0 < r < 1$, and we can invoke the lemma, concluding that *polynomials are dense in the Hilbert space* $\mathscr{H}$. The previous relation therefore implies that $F(z) \equiv 0$.

We have thus reached a contradiction by showing that $q = 0$ and $F = 0$. The theorem is proved.

Remark. Some applications involve a weight

$$w(r) \;=\; \exp\left(-h\left(\log\frac{1}{r}\right)\right)$$

where, for $\xi > 0$,

$$h(\xi) \;=\; \sup_{v>0}(M(v) - v\xi),$$

the function $M(v)$ being *merely* supposed *increasing*, and such that $M(0) > -\infty$.

In this situation, we can, from the condition

$$\sum_1^\infty \frac{M(n)}{n^2} \;=\; \infty,$$

conclude that *the rest of the above theorem's statement is valid.*

This can be seen without appealing to the last theorems of article 2. We have here, with $\xi = \log(1/r)$,

$$r^{2n} w(r) \;=\; e^{-(h(\xi) + 2n\xi)},$$

and, since, for any $\xi > 0$,

$$h(\xi) \;\geqslant\; M(v) - v\xi$$

for each $v > 0$, $h(\xi) + 2n\xi \geqslant M(2n)$. We now arrive at $(\ast\!\!\ast)$ in the same way as above, so, since $M(v)$ is increasing,

$$\sum_1^\infty \frac{M(n)}{n^2} \;=\; \infty \quad \text{makes} \quad \sum_1^\infty \frac{M(2n)}{2n^2} \;=\; \infty$$

and we can conclude by direct application of the corollary from §A.5. (Here, boundedness of $w(r)$ is ensured by the condition $M(0) > -\infty$.)

Remark on a certain change of cariable

If the weight $w(r) = \exp(-H(\log(1/r)))$ *satisfies the hypothesis of the theorem on simultaneous polynomial approximation, so does the weight* $w(r^L) = \exp(-H(L\log(1/r)))$ *for any positive constant* L. That's simply because

$$\int_0^a H(L\xi)\,d\xi \;=\; \frac{1}{L}\int_0^{aL} H(\xi)\,d\xi \;\;!$$

That theorem therefore remains valid if we replace the weight $w(r)$ *figuring in its statement by* $w(r^L)$, *L being any positive constant.*

We will use this fact several times in what follows.

5. Volberg's theorem on harmonic measures

The result to be proved here plays an important role in the establishment of the main theorem of this §. It is also of interest in its own right.

Definition. Let $\mathcal{O}$ be an open subset of $\{|z| < 1\}$, and J any open *arc* of $\{|z| = 1\}$. We say that $\mathcal{O}$ *abuts* on J if, for each $\zeta \in J$, there is a neighborhood V_ζ of ζ with

$$V_\zeta \cap \{|z| < 1\} \;\subseteq\; \mathcal{O}.$$

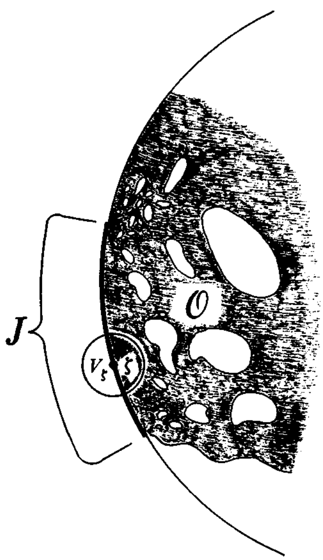

Figure 96

Now we come to the

Theorem on harmonic measures (Volberg). *Let, for* $0 < r < 1$, $w(r) = \exp(-H(\log(1/r)))$, *where* $H(\xi)$ *is* decreasing *and* bounded *below on* $(0, \infty)$, *and tends to* ∞ *sufficiently rapidly as* $\xi \to 0$ *to make* $w(r) = O((1-r)^2)$ *for* $r \to 1$. *(In the situation of Volberg's theorem, we have* $H(\xi) \geqslant \text{const.}\xi^{-c}$ *with* $c > 0$, *so this will certainly be the case.)*

Assume furthermore that

$$\int_0^a \log H(\xi)\,\mathrm{d}\xi \;=\; \infty$$

for all sufficiently small $a > 0$.

Let $\mathcal{O}$ *be any connected open set in* $\{|z| < 1\}$ *whose boundary is regular enough to permit the solution of Dirichlet's problem for* $\mathcal{O}$. *Suppose that there are two open arcs* I *and* J *of positive length on* $\{|z| = 1\}$ *such that:*

(i) $\partial\mathcal{O} \cap J$ *is* empty;

(ii) $\mathcal{O}$ abuts *on* I.

Then, if $\omega_{\mathcal{O}}(\ ,z)$ *denotes* harmonic measure *for* $\mathcal{O}$ *(as seen from* $z \in \mathcal{O}$), *we have*

$$\int_{\{|\zeta|<1\}\cap\partial\mathcal{O}} \log\left(\frac{1}{w(|\zeta|)}\right)\mathrm{d}\omega_{\mathcal{O}}(\zeta, z_0) \;=\; \infty$$

for each $z_0 \in \mathcal{O}$.

Remark 1. The integral is taken over the part of $\partial\mathcal{O}$ lying *inside* $\{|z| < 1\}$.

Remark 2. The assumption that $\mathcal{O}$ *abuts* on an *arc* I can be relaxed. But the proof uses the full strength of the assumption that $\partial\mathcal{O}$ *avoids* J.

Proof of theorem. We work with the weight $w_1(r) = w(r^3)$. By the *theorem on simultaneous polynomial approximation* and *remark on a change of variable* (previous article), there are polynomials $P_n(z)$ with

$$\int_{\{|\zeta|=1\}\sim J} |P_n(e^{i\vartheta})|^2\,\mathrm{d}\vartheta \;\xrightarrow[n]{}\; 0$$

and at the same time

$$\iint_{|z|<1} |P_n(z) - 1|^2 w_1(|z|)\,\mathrm{d}x\,\mathrm{d}y \;\xrightarrow[n]{}\; 0.$$

The second relation certainly implies that

$$\iint_{|z|<1} |P_n(z)|^2 w_1(|z|)\,\mathrm{d}x\,\mathrm{d}y \;\leqslant\; C$$

for some $C < \infty$, and all n.

Take any z_0, $|z_0| < 1$; we use the last inequality to get a uniform upper estimate for the values $|P_n(z_0)|$. Put $\rho = \frac{1}{2}(1 - |z_0|)$.

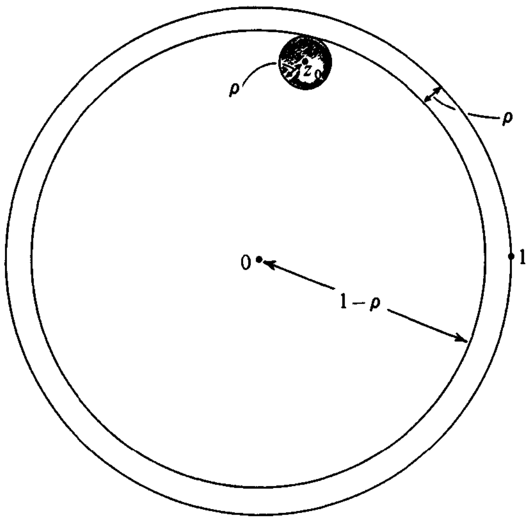

Figure 97

We have

$$|P_n(z_0)|^2 \;\leqslant\; \frac{1}{\pi\rho^2}\iint_{|z-z_0|<\rho} |P_n(z)|^2 \,\mathrm{d}x\,\mathrm{d}y.$$

$w_1(r)$ *decreases*, so the right side is

$$\leqslant\; \frac{1}{\pi\rho^2 w_1(1-\rho)}\iint_{|z-z_0|<\rho} |P_n(z)|^2 w_1(|z|)\,\mathrm{d}x\,\mathrm{d}y$$

which, in turn, is

$$\leqslant\; \frac{C}{\pi\rho^2 w_1(1-\rho)}$$

by the above inequality.

Here, $w_1(1-\rho) = w(1-3\rho+3\rho^2-\rho^3)$ is $\geqslant$ $w(1-2\rho) = w(|z_0|)$ when $3\rho^2 - \rho^3 \leqslant \rho$, i.e., for $\rho \leqslant (3-\sqrt{5})/2$. Also, $w(r)$ is bounded above ($H(\xi)$ being bounded below), so, for $(3-\sqrt{5})/2 < \rho < \frac{1}{2}$,
$w_1(1-\rho) \geqslant w(\sqrt{5}-2) \geqslant \mathrm{const.}w(|z_0|)$. The result just found therefore reduces to

$$|P_n(z_0)|^2 \;\leqslant\; \frac{\mathrm{const.}}{(1-|z_0|)^2 w(|z_0|)},$$

with the right-hand side in turn

$$\leqslant \frac{\text{const.}}{(w(z_0))^2}$$

according to the hypothesis. Thus, since z_0 was arbitrary,

$$(*) \qquad \log|P_n(z)| \leqslant \text{const.} + \log\left(\frac{1}{w(|z|)}\right), \qquad |z|<1.$$

A similar (and simpler) argument, applied to $P_n(z)-1$, shows that

$$(\ast\ast) \qquad P_n(z) \underset{n}{\longrightarrow} 1, \qquad |z|<1.$$

Let us now fix our attention on $\partial\mathcal{O}\cap\{|\zeta|=1\}$, *which we henceforth denote by* S, in order to simplify the notation. The *open unit disk* Δ includes $\mathcal{O}$, therefore, by the *principle of extension of domain* (see §B.1), for $z_0\in\mathcal{O}$,

$$\mathrm{d}\omega_{\mathcal{O}}(\zeta, z_0) \leqslant \mathrm{d}\omega_{\Delta}(\zeta, z_0)$$

for ζ varying on S. In other words,

$$(\dagger) \qquad \mathrm{d}\omega_{\mathcal{O}}(\zeta, z_0) \leqslant K(z_0)|\mathrm{d}\zeta|$$

for ζ on S with a number $K(z_0)$ depending only on z_0.

Since $\mathcal{O}$ *abuts* on the arc I with $|I|>0$, *we surely have*

$$\omega_{\mathcal{O}}(S, z_0) > 0$$

for each $z_0\in\mathcal{O}$ by Harnack's theorem. Since $S \subseteq \{|\zeta|=1\}\sim J$ ((i) of the hypothesis), we have, by (†) and the relation between arithmetic and geometric means,

$$\int_S \log|P_n(e^{i\vartheta})|\,\mathrm{d}\omega_{\mathcal{O}}(e^{i\vartheta}, z_0)$$

$$\leqslant \tfrac{1}{2}\omega_{\mathcal{O}}(S, z_0)\log\left(\frac{1}{\omega_{\mathcal{O}}(S, z_0)}\int_S |P_n(e^{i\vartheta})|^2\,\mathrm{d}\omega_{\mathcal{O}}(e^{i\vartheta}, z_0)\right)$$

$$\leqslant \tfrac{1}{2}\omega_{\mathcal{O}}(S, z_0)\log\left(\frac{K(z_0)}{\omega_{\mathcal{O}}(S, z_0)}\int_{\{|\zeta|=1\}\sim J} |P_n(e^{i\vartheta})|^2\,\mathrm{d}\vartheta\right)$$

for $z_0\in\mathcal{O}$. *This last expression, however, tends to* $-\infty$ *as* $n\to\infty$ because

$$\int_{\{|\zeta|=1\}\sim J} |P_n(e^{i\vartheta})|^2\,\mathrm{d}\vartheta \underset{n}{\longrightarrow} 0$$

At the same time, for any $z_0\in\mathcal{O}$, $\log|P_n(z_0)| \underset{n}{\longrightarrow} 0$ by ($\ast\ast$). Therefore, by the *theorem on harmonic estimation* in §B.1 (whose extension to possibly infinitely connected domains $\mathcal{O}$ of the kind considered here presents no difficulty, at least for *polynomials* $P_n(z)$), we see that

$$\int_S \log|P_n(e^{i\vartheta})|d\omega_{\mathcal{O}}(e^{i\vartheta}, z_0) + \int_{\partial\mathcal{O}\cap\Delta} \log|P_n(\zeta)|d\omega_{\mathcal{O}}(\zeta, z_0)$$
$$= \int_{\partial\mathcal{O}} \log|P_n(\zeta)|d\omega_{\mathcal{O}}(\zeta, z_0) \geqslant \log|P_n(z_0)| \underset{n}{\longrightarrow} 0.$$

As we have just shown, the *first* of the two integrals in the left-hand member *tends to* $-\infty$ as $n \to \infty$. *Hence the second must tend to* ∞ *as* $n \to \infty$ (!). However, by (*),

$$\log|P_n(\zeta)| \leqslant \text{const.} + \log\left(\frac{1}{w(|\zeta|)}\right)$$

for $\zeta \in \partial\mathcal{O} \cap \Delta$. So we must have

$$\int_{\partial\mathcal{O}\cap\Delta} \log\left(\frac{1}{w(|\zeta|)}\right) d\omega_{\mathcal{O}}(\zeta, z_0) = \infty \quad \text{for} \quad z_0 \in \mathcal{O}.$$

The theorem is proved.

Remark. The result just established holds in particular for weights $w(r) = \exp(-h(\log(1/r)))$ with $h(\xi) = \sup_{v>0}(M(v) - v\xi)$ for $\xi > 0$, $M(v)$ being increasing, provided that $M(0) > -\infty$, that $\sum_1^\infty (M(n)/n^2) = \infty$, and that $M(v) \to \infty$ as $v \to \infty$ fast enough to make $w(r) = O((1-r)^2)$ for $r \to 1$. See remark following the theorem on simultaneous polynomial approximation (previous article).

Corollary. *Let the connected open set* $\mathcal{O}$ *and the weight* $w(r)$ *be as in the theorem (or the last remark). Let* $G(z)$ *be* analytic *in* $\mathcal{O}$ *and continuous up to* $\partial\mathcal{O}$, *and suppose that, for some* ρ, $0 < \rho < 1$, *we have*

$$|G(\zeta)| \leqslant w(|\zeta|) \quad \textit{for } \zeta \in \partial\mathcal{O} \textit{ with } 1-\rho < |\zeta| < 1 \ (\textit{sic!}).$$

Then $G(z) \equiv 0$ *in* $\mathcal{O}$.

Proof. Take any $z_0 \in \mathcal{O}$. Since $G(z)$ is continuous on $\bar{\mathcal{O}}$, it is bounded there, so, since $w(r)$ *decreases*, we surely have $G(z) \leqslant \text{const.}w(|z|)$ for $z \in \bar{\mathcal{O}}$ and $|z| \leqslant 1-\rho$. Therefore our hypothesis in fact implies that

$$|G(\zeta)| \leqslant Cw(|\zeta|) \quad \text{for} \quad \zeta \in \partial\mathcal{O} \cap \{|\zeta| < 1\}$$

with a certain constant C.

Write $\partial\mathcal{O} \cap \{|\zeta| < 1\} = \gamma$, and denote the intersection $\partial\mathcal{O} \sim \gamma$ of $\partial\mathcal{O}$ with the unit circumference by S as in the proof of the theorem. By the *theorem on harmonic estimation* (§B.1), we have

$$\log|G(z_0)| \leqslant \int_S \log|G(\zeta)|d\omega_{\mathcal{O}}(\zeta, z_0) + \int_\gamma \log|G(\zeta)|d\omega_{\mathcal{O}}(\zeta, z_0).$$

If M is a bound for $G(z)$ on $\bar{\mathcal{O}}$, the *first* integral on the right is $\leqslant \log M$. The *second* is

$$\leqslant \log C + \int_\gamma \log w(|\zeta|)\, d\omega_{\mathcal{O}}(\zeta, z_0)$$

by the above inequality. The integral just written is, however, equal to $-\infty$ by the theorem on harmonic measures. Hence $G(z_0) = 0$, as required.

Scholium. L. Carleson observed that the result furnished by the theorem on harmonic measures cannot be essentially improved. By this he meant the following:

If $w(r) = \exp(-h(\log(1/r)))$ *with* $h(\xi)$ *strictly decreasing, convex, and bounded below on* $(0, \infty)$, *and if*

$$\int_0^a \log h(\xi)\, d\xi \;<\; \infty$$

for all sufficiently small $a > 0$, *there is a simply connected open set* $\mathcal{O}$ *in* $\Delta = \{|z| < 1\}$ *fulfilling the conditions of the above theorem for which*

$$\int_{\partial\mathcal{O}\cap\Delta} \log\left(\frac{1}{w(|\zeta|)}\right) d\omega_{\mathcal{O}}(\zeta, z_0) \;<\; \infty, \qquad z_0 \in \mathcal{O}.$$

To see this, observe that the convergence of $\int_0^a \log h(\xi)\, d\xi$ for all sufficiently small $a > 0$ implies that

$$(\S) \qquad \int_0^a \log|h'(\xi)|\, d\xi \;<\; \infty$$

for such a. (See the proof of the *second* theorem in article 2.) We use (§) in order to construct a domain $\mathcal{O}$ like this

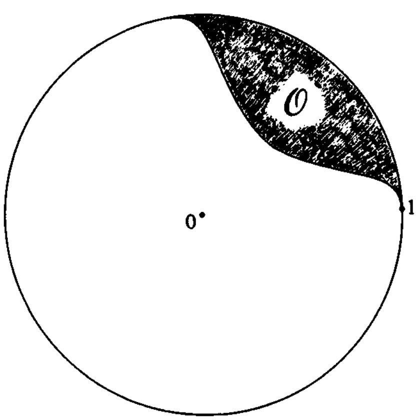

Figure 98

for which

$$\int_{\partial\mathcal{O}\cap\Delta} \log\left(\frac{1}{w(|\zeta|)}\right) d\omega_{\mathcal{O}}(\zeta, z_0) \;<\; \infty, \qquad z_0 \in \mathcal{O},$$

with $w(r) = \exp(-h(\log(1/r)))$. It is convenient to map our (as yet undetermined) region $\mathcal{O}$ *conformally* onto another one, $\mathcal{D}$, by taking $z = re^{i\vartheta}$ to $\varphi = i\log(1/z) = \vartheta + i\log(1/r) = \vartheta + i\xi$. Here ξ has its usual significance.

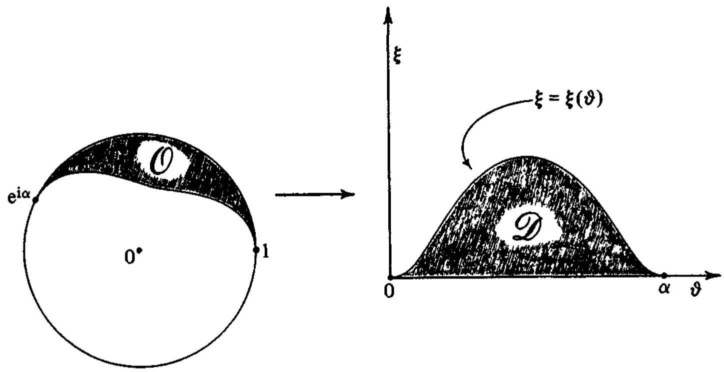

Figure 99

If, in this mapping, the point $z_0 \in \vartheta$ goes over to $p \in \mathcal{D}$, we have, clearly,

$$\int_{\partial\mathcal{O}\cap\Delta} \log\left(\frac{1}{w(|\zeta|)}\right) d\omega_{\mathcal{O}}(\zeta, z_0) \;=\; \int_{\partial\mathcal{D}\cap\{\xi>0\}} h(\xi)\, d\omega_{\mathcal{D}}(\varphi, p).$$

We see that it is enough to determine the equation $\xi = \xi(\vartheta)$ of the *upper bounding curve* of $\mathcal{D}$ (see picture) in such fashion as to have

$$\int_0^{\alpha} h(\xi(\vartheta))\, d\omega_{\mathcal{D}}(\vartheta + i\xi(\vartheta), p) \;<\; \infty$$

when $p \in \mathcal{D}$. The easiest way to proceed is to construct a function $\xi(\vartheta) = \xi(\alpha - \vartheta)$, making the upper bounding curve *symmetric about the vertical line through its midpoint*. Then we need only determine an increasing function $\xi(\vartheta)$ on the range $0 \leqslant \vartheta \leqslant \alpha/2$ in such a way that

$$\int_0^{\theta_0} h(\xi(\vartheta))\, d\omega_{\mathcal{D}}(\vartheta + i\xi(\vartheta), p) \;<\; \infty$$

for some $\theta_0 > 0$ and *some* $p \in \mathcal{D}$. From this, the same inequality will follow for every $p \in \mathcal{D}$ by Harnack's theorem, and we can arrive at the full result by adding two such integrals.

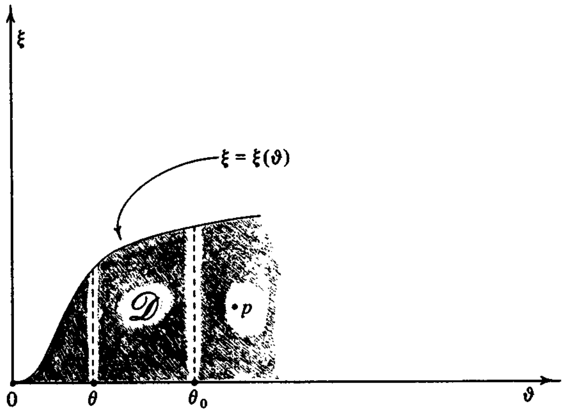

Figure 100

For $0 < \theta < \theta_0$, we have, integrating by parts,

$$\int_\theta^{\theta_0} h(\xi(\vartheta))\mathrm{d}\omega_{\mathscr{D}}(\vartheta + \mathrm{i}\xi(\vartheta), p) \;=\; h(\xi(\theta_0))\omega(\theta_0) - h(\xi(\theta))\omega(\theta)$$

$$- \int_\theta^{\theta_0} \omega(\vartheta)\mathrm{d}h(\xi(\vartheta)),$$

where $\omega(\vartheta)$ denotes the *harmonic measure* (at p) of the segment of the *upper bounding curve* having abscissae between 0 and ϑ, viz.,

$$\omega(\vartheta) \;=\; \int_0^\vartheta \mathrm{d}\omega_{\mathscr{D}}(\tau + \mathrm{i}\xi(\tau), p).$$

Making $\theta \to 0$ and remembering that $h(\xi)$ *decreases*, we see that what we *want* is

$$(\S\S) \qquad \int_0^{\theta_0} \omega(\vartheta)|h'(\xi(\vartheta))|\mathrm{d}\xi(\vartheta) \;<\; \infty.$$

For $\omega(\vartheta)$ we may use the *Carleman–Ahlfors estimate for harmonic measure in curvilinear strips*, to be derived in Chapter IX.* According to that, if p is *fixed* and to the *right* of θ_0,

$$\omega(\vartheta) \;\leqslant\; \text{const.}\exp\left(-\pi \int_\vartheta^{\theta_0} \frac{\mathrm{d}\tau}{\xi(\tau)} \right)$$

for $0 < \vartheta < \theta_0$. The most simple-minded way of ensuring (§§) is then to cook

* See Remark 1 following the third theorem of §E.1 in that chapter. The upper bound arrived at by the method explained there applies in fact to a harmonic measure *larger* than $\omega(\vartheta)$.

the positive increasing function $\xi(\tau)$ so as to have

$$\log|h'(\xi(\vartheta))| \;-\; \pi\int_{\vartheta}^{\theta_0}\frac{d\tau}{\xi(\tau)} \;=\; \text{const.}$$

It is the relation (§) which makes it possible for us to do this.

In order to avoid being fussy, *let us at this point make the additional* (and not really restrictive) *assumption that $h'(\xi)$ is continuously differentiable.* Then we can differentiate the previous equation with respect to ϑ, getting

$$\frac{d\log|h'(\xi)|}{d\xi}\frac{d\xi}{d\vartheta} \;=\; -\frac{\pi}{\xi}$$

for our unknown increasing function $\xi = \xi(\vartheta)$. Calling $\vartheta(\xi)$ the *inverse* to the function $\xi(\vartheta)$, we have

$$\frac{d\vartheta(\xi)}{d\xi} \;=\; -\frac{1}{\pi}\xi\frac{d\log|h'(\xi)|}{d\xi}.$$

In view of (§), this has the solution

$$\vartheta(\xi) \;=\; \frac{1}{\pi}\int_0^{\xi}(\log|h'(t)| - \log|h'(\xi)|)dt$$

with $\vartheta(0) = 0$. Since $h'(t)$ is < 0 and *increasing* (i.e., $\log|h'(t)|$ *decreases*), we see that the function $\vartheta(\xi)$ given by this formula *is* strictly increasing, and therefore *has* an increasing inverse $\xi(\vartheta)$ for which (§§) holds.

This completes our construction, and Carleson's observation is verified.

Let us remark that one can, by the same method, establish a version of the Levinson loglog theorem which we will give at the end of this § (accompanied, however, by a proof based on a different idea). V.P. Gurarii showed me this simple argument (Levinson's original proof of the loglog theorem, found in his book, is quite hard) at the 1966 International Congress in Moscow.

6. Volberg's theorem on the logarithmic integral

We are finally in a position to undertake the proof of the main result of this §. This is what we will establish:*

Theorem. *Let $M(v)$ be increasing for $v \geqslant 0$.*

Suppose that

$$M(v)/v \text{ is decreasing,}$$

that

$$M(v) \;\geqslant\; \text{const.}v^{\alpha}$$

* A refinement of the following result due to Brennan is given in the *Addendum* at the end of the present volume.

with some $\alpha > \frac{1}{2}$ for all large v, and that

$$\sum_1^\infty \frac{M(n)}{n^2} = \infty.$$

Let

$$F(e^{i\vartheta}) \sim \sum_{-\infty}^{\infty} a_n e^{in\vartheta}$$

be continuous and not identically zero.

Then, if

$$|a_{-n}| \leqslant e^{-M(n)} \quad \textit{for} \quad n \geqslant 1,$$

we have

$$\int_{-\pi}^{\pi} \log|F(e^{i\vartheta})|d\vartheta > -\infty.$$

Remark. Volberg states this theorem for functions $F(e^{i\vartheta}) \in L_1(-\pi,\pi)$.* He replaces our second displayed condition on $M(v)$ by a weaker one, requiring only that

$$v^{-\frac{1}{2}}M(v) \longrightarrow \infty$$

for $v \to \infty$, but includes an additional restrictive one, to the effect that

$$v^{1/2}M(v^{1/2}) \leqslant \text{const.} M(v)$$

for large v. This extra requirement serves to ensure that the function

$$h(\xi) = \sup_{v>0}(M(v) - v\xi)$$

satisfies the relation $h(K\xi) \leqslant (h(\xi))^{1-c}$ with some $K > 1$ and $c > 0$ for small $\xi > 0$; here we have entirely dispensed with it.

Proof of theorem (essentially Volberg's). This will be quite long.

We start by making some simple reductions. First of all, we assume that $M(v)/v \longrightarrow 0$ for $v \to \infty$, since, in the contrary situation, the theorem is easily verified directly (see article 2).

According to the first theorem of article 2, our condition that $M(v)/v$ decrease implies that the *smallest concave majorant* $M^*(v)$ of $M(v)$ is $\leqslant 2M(v)$; this means that the hypothesis of the theorem is satisfied if, in it, we replace $M(v)$ by the *concave increasing function* $M^*(v)/2$.

There is thus no loss of generality in supposing to begin with that $M(v)$ is *also concave*. We may also assume that $M(0) \geqslant 3$. To see this, suppose that $M(0) < 3$; in that case we may draw a straight line $\mathscr{L}$ from $(0,3)$ *tangent to the graph* of $M(v)$ vs. v:

* See the addendum for such an extension of Brennan's result.

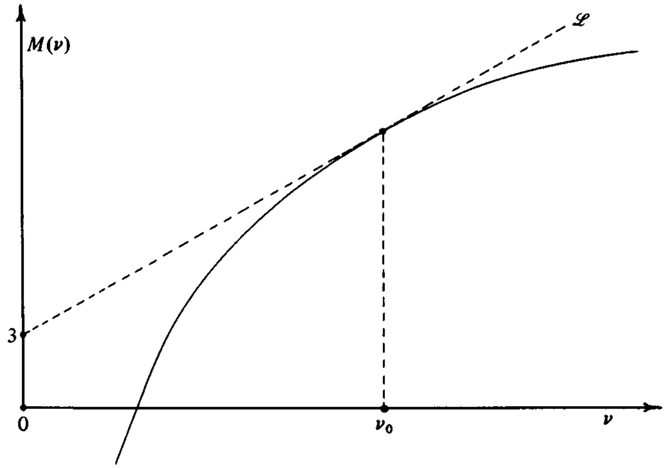

Figure 101

If the point of tangency is at $(v_0, M(v_0))$, we may then take the new increasing concave function $M_0(v)$ equal to $M(v)$ for $v \geqslant v_0$ and to the height of $\mathscr{L}$ at the abscissa v for $0 \leqslant v < v_0$. Our Fourier coefficients a_n will satisfy

$$|a_{-n}| \leqslant \text{const.}e^{-M_0(n)}$$

for $n \geqslant 1$, and the rest of the hypothesis will hold with $M_0(v)$ in place of $M(v)$.

We may now use the simple constructions of $M_1(v)$ and $M_2(v)$ given in article 2 to obtain an infinitely differentiable, increasing and *strictly* concave function $M_2(v)$ (with $M_2''(v) < 0$) which is *uniformly close* (within $\frac{1}{2}$ unit, say) to $M_0(v)$ on $[0, \infty)$. (Here uniformly close on *all* of $[0, \infty)$ because our present function $M_0(v)$ has a *bounded* first derivative on $(0, \infty)$.) We will then still have

$$|a_{-n}| \leqslant \text{const.}e^{-M_2(n)}$$

for $n \geqslant 1$, and the rest of the hypothesis will hold with $M_2(v)$ in place of $M(v)$.

Since $M(v) \geqslant \text{const.}v^{\alpha}$ for large v, where $\alpha > \frac{1}{2}$, we certainly (and by far!) have

$$n^4 \exp(-M_2(n)/2) \longrightarrow 0, \quad n \to \infty.$$

Therefore

$$\sum_{n=1}^{\infty} |n^2 a_{-n}| e^{M_2(n)/2} < \infty.$$

So, putting $\bar{M}(\nu) = M_2(\nu)/2$, we have

$$\sum_1^\infty |n^2 a_{-n}| e^{\bar{M}(n)} < \infty$$

with a function $\bar{M}(\nu)$ which is *increasing, strictly concave, and infinitely differentiable on* $(0, \infty)$, *having* $\bar{M}''(\nu) < 0$ *there*. The hypothesis of the theorem *holds* with $\bar{M}(\nu)$ standing in place of $M(\nu)$. Besides, $\bar{M}(0)$ $(= \lim_{\nu\to 0} \bar{M}(\nu))$ is $\geqslant 1$ since $M_0(0) \geqslant 3$. Later on, this property will be helpful technically.

► *Let us henceforth simply write* $M(\nu)$ *instead of* $\bar{M}(\nu)$. Our *new* function $M(\nu)$ thus *satisfies the hypothesis of Dynkin's theorem* (article 3). We put, as usual,

$$h(\xi) = \sup_{\nu > 0}(M(\nu) - \nu\xi) \quad \text{for} \quad \xi > 0,$$

and then form the weight

$$w(r) = \exp\left(-h\left(\log\frac{1}{r}\right)\right), \qquad 0 < r < 1.$$

Because $M(0) \geqslant 1$, we have $h(\xi) \geqslant 1$ for $\xi > 0$, so $w(r) \leqslant 1/\mathrm{e}$. *Applying* Dynkin's theorem, we obtain a *continuous extension* $F(z)$ of our given function $F(\mathrm{e}^{i\vartheta})$ to $\{|z| \leqslant 1\}$ with $F(z)$ *continuously differentiable in the open unit disk* and

$$\left|\frac{\partial F(z)}{\partial \bar{z}}\right| \leqslant \text{const.} w(|z|), \qquad |z| < 1.$$

We note here that the properties of $M(\nu)$ assumed in the hypothesis certainly make $w(r) \to 0$ (indeed rather rapidly) as $r \to 1$; see article 2.

Let

$$B_0 = \{z\colon\ |z| < 1 \text{ and } |F(z)| \leqslant w(|z|)\}.$$

We cover each of the closed sets

$$B_0 \cap \left\{1 - \frac{1}{n} \leqslant |z| \leqslant 1 - \frac{1}{n+1}\right\}, \qquad n = 1, 2, 3, \ldots,$$

by a finite number of open disks lying in $\{|z| < 1\}$ on which

$$|F(z)| < 2w(|z|).$$

This gives us altogether a countable collection of open disks lying in the unit circle and covering B_0; *the closure of the union of those disks is denoted by* B. We then have

$$|F(z)| \leqslant 2w(|z|), \qquad z \in B,^*$$

* *including* for $z \in B$ of modulus 1, as long as we take $w(1) = 0$! See argument for *Step 1*, p. 361

and

$$|F(z)| \;>\; w(|z|) \quad \text{for } z \notin B \text{ and } |z|<1.$$

Put

$$\mathcal{O} \;=\; \{|z|<1\} \cap (\sim B);$$

$\mathcal{O}$ is an *open* subset of the unit disk and $|F(z)| > w(|z|)$ therein, as we have just seen. We will see presently that $\mathcal{O}$ fills much of the unit disk. Let us at this point simply observe that *$\mathcal{O}$ is certainly not empty.* If, indeed, it *were* empty, B would fill the unit disk and we would have $|F(z)| \leqslant 2w(|z|)$ for $|z|<1$. The fact that $w(r) \to 0$ for $r \to 1$ would then make $F(e^{i\vartheta}) \equiv 0$, *contrary to our hypothesis*, by virtue of the continuity of $F(z)$ on $\{|z| \leqslant 1\}$.

Although the open set $\mathcal{O}$ may have an exceedingly complicated structure, *the Dirichlet problem for it can be solved.* This will follow from a well-known result in elementary potential theory (for a proof of which see, for instance, pp. 35–6 of Gamelin's book, the latter part of the one by Kellog, or any other work on potential theory – I cannot, after all, prove *everything*!), if we show that the *Poincaré cone condition for $\mathcal{O}$ is satisfied at every point ζ of $\partial\mathcal{O}$*, i.e., if, for each such ζ, *there is a small triangle with vertex at ζ lying outside $\mathcal{O}$.* To check this, observe that the small disks used to build up B can accumulate only at points of the unit circumference. Hence, if $|\zeta|<1$ and $\zeta \in \partial\mathcal{O}$, *$\zeta$ is on the boundary of one of those small disks, inside* of which a triangle with vertex at ζ may be drawn. If, however, $|\zeta| = 1$, we may take a triangle lying *outside the unit disk* with vertex at ζ.

Since $|F_{\bar z}(z)| \leqslant Cw(|z|)$ in $\{|z|<1\}$ while $|F(z)| > w(|z|)$ in $\mathcal{O}$, we have

$$\left|\frac{1}{F(z)}\frac{\partial F(z)}{\partial \bar z}\right| \;\leqslant\; C, \qquad z \in \mathcal{O}.$$

Volberg's idea is to take advantage of this relation and use the function

$$\Phi(z) \;=\; F(z)\exp\left\{\frac{1}{2\pi}\int_{\mathcal{O}}\!\!\int \frac{F_{\bar\zeta}(\zeta)}{F(\zeta)}\frac{d\xi\,d\eta}{(\zeta - z)}\right\}$$

on $\{|z| \leqslant 1\}$. (N.B. *Again we are writing $\zeta = \xi + i\eta$, in conflict with the notation $\xi = \log(1/r)$ used in discussing $h(\xi)$.*) According to *Remark* 1 to Dynkin's theorem (end of article 3), $F(z)$ has enough differentiability in $\{|z|<1\}$ for us to be able to use the *corollary from the end of article* 1. By that corollary, *$\Phi(z)$ is analytic in $\mathcal{O}$*, and *$|\Phi(z)|$ lies between two constant multiples of $|F(z)|$ there* (*and, actually, on* $\{|z| \leqslant 1\}$ as the last part of that corollary's proof shows). We thus certainly have $|\Phi(z)| > 0$ in $\mathcal{O}$ since $|F(z)| > w(|z|)$ there.

Volberg now applies the *theorem on harmonic estimation* (§B.1) to

$\Phi(z)$ and the open set $\mathcal{O}$ in order to eventually get at

$$\int_{-\pi}^{\pi} \log|F(e^{i\vartheta})| d\vartheta.$$

His procedure is to show that the whole unit circumference is contained in $\partial\mathcal{O}$, and that the part of $\partial\mathcal{O}$ lying *inside* the unit disk is so unimportant as to make *harmonic measure for* $\mathcal{O}$ act like *ordinary Lebesgue measure* on the unit circumference. We will carry out this program *in 5 steps*. Before going to step 1, we should, however, acknowledge that here the theorem on harmonic estimation will be used under conditions somewhat more general than those allowed for in §B.1. The open set $\mathcal{O}$ may not even be connected, and its components may be infinitely connected! Nevertheless, extension of the theorem in question to the present situation involves no real difficulty – $\Phi(z)$ *is* continuous on $\{|z| \leqslant 1\}$ and the Dirichlet problem for $\mathcal{O}$ *is* solvable.

We proceed.

Step 1. *$B \cap \{|\zeta| = 1\}$ contains no arc of positive length.*

Assume the contrary. By construction of B, each of its points on the unit circumference is a *limit* of a sequence of z_n having modulus < 1 for which $|F(z_n)| < 2w(|z_n|)$. Since $w(r) \to 0$ for $r \to 1$ we thus have $F(e^{i\vartheta}) = 0$ for every point of the form $e^{i\vartheta}$ in B.

If now this happens for each point *belonging to an arc J of positive length* on the unit circumference, we will have $F(e^{i\vartheta}) \equiv 0$ on J. At the same time, the *Fourier coefficients a_n* of $F(e^{i\vartheta})$ satisfy $|a_{-n}| \leqslant \text{const.} e^{-M(n)}$ for $n \geqslant 1$ by hypothesis. *Therefore $F(e^{i\vartheta})$ vanishes identically by the corollary to Levinson's theorem at the end of* §A.5. This, however, is *contrary to our hypothesis.*

Before going on, we note that the properties of $M(v)$ came into play in the preceding argument *only* when we looked at the Fourier coefficients of $F(e^{i\vartheta})$ and *not* when we brought in $w(r) = \exp(-h(\log(1/r))$, even though $h(\xi)$ *is* related to $M(v)$ in the usual way. *Any other* weight $w(r) \geqslant 0$ tending to zero as $r \to 1$ would have worked just as well.

Thanks to what we found in step 1, $\{|\zeta| = 1\} \cap (\sim B)$ is *non-empty* (and even *dense* on the unit circumference). *It is open, hence equal to a countable union of disjoint open arcs I_k on $\{|\zeta| = 1\}$.* (B, remember, is *closed.*) *$\mathcal{O} = \{|z| < 1\} \cap (\sim B)$ clearly abuts on each of the I_k.* (See definition, beginning of article 5.)

Take any ρ_0, $0 < \rho_0 < 1$, and denote by $\Omega_k(\rho_0)$ (or just by Ω_k, if it is not necessary to keep the value of ρ_0 in mind) the *connected component of $\mathcal{O} \cap \{\rho_0 < |z| < 1\}$ abutting on I_k.*

Step 2. *All the $\Omega_k(\rho_0)$ are the same. In other words, $\bigcup_k \Omega_k(\rho_0)$ is connected.*

Assume the contrary. Then we must have two different arcs I_k – call them I_1 and I_2 – for which the corresponding components Ω_1 and Ω_2 are *disjoint*.

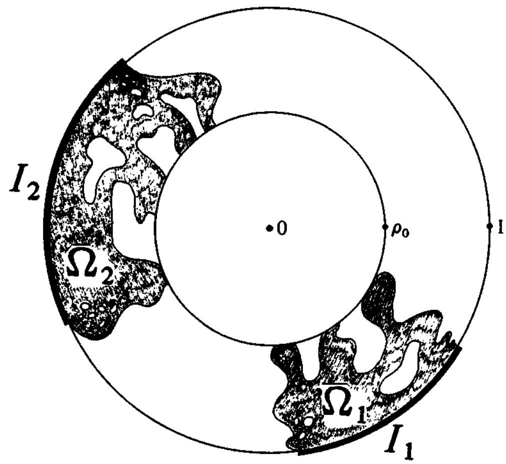

Figure 102

The function $\Phi(z)$ is analytic in Ω_1 and continuous on its closure. As stated above, it's *continuous on* $\{|z| \leqslant 1\}$ – that's because $F(z)$ is, and because the ratio $F_{\bar{\zeta}}(\zeta)/F(\zeta)$ appearing in the formula for $\Phi(z)$ is *bounded* on the region $\mathcal{O}$ over which the double integral figuring therein is taken (see beginning of the proof of the theorem in article 1).

The points ζ on $\partial\Omega_1$ with $\rho_0 < |\zeta| < 1$ (*sic*!) must belong to B, therefore $|F(\zeta)| \leqslant 2w(|\zeta|)$ for them. So, since $A|F(z)| \leqslant |\Phi(z)| \leqslant A'|F(z)|$ for $|z| \leqslant 1$, we have

$$|\Phi(\zeta)| \leqslant \text{const.}w(|\zeta|) \qquad \text{for } \zeta \in \partial\Omega_1 \text{ and } \rho_0 < |\zeta| < 1.$$

We have $w(r) = \exp(-h(\log(1/r)))$ with $h(\zeta)$ *decreasing* and *bounded below* for $\xi > 0$. In the present case, where $h(\xi) = \sup_{v>0}(M(v) - v\xi)$ and $\int_1^\infty (M(v)/v^2)\mathrm{d}v = \infty$, $\int_0^a \log h(\xi)\mathrm{d}\xi = \infty$ for all sufficiently small $a > 0$ (next to last theorem of article 2 – in the present circumstances we could even bypass that theorem as in the remark following the one of article 4). Finally, the condition (given!) that $M(v) \geqslant \text{const.}v^\alpha$ for large v, with $\alpha > \frac{1}{2}$, makes $h(\xi) \geqslant \xi^{-\lambda}$ for small $\xi > 0$, where $\lambda > 1$, by a lemma of article 2. Therefore (and by far!) $w(r) = O((1-r)^2)$ as $r \to 1$.

In our present situation, Ω_1 *abuts* on I_1 and $\partial\Omega_1$ *avoids* I_2. Here, *all the conditions* of *Volberg's theorem on harmonic measures* (previous article) *are fulfilled.* Therefore, by the corollary to that theorem, $\Phi(z) \equiv 0$ in Ω_1. *This, however, is impossible since $\Omega_1 \subseteq \mathcal{O}$ on which $|\Phi(z)| > 0$.*

As we have just seen, the union $\bigcup_k \Omega_k(\rho_0)$ is *connected. We denote that union by* $\Omega(\rho_0)$, or sometimes just by Ω. $\Omega(\rho_0)$ is an open subset of $\mathcal{O}$ lying in the ring $\rho_0 < |z| < 1$ and *abutting on each arc of* $\{|\zeta| = 1\}$ *contiguous to* $\{|\zeta| = 1\} \cap B$.

Step 3. *If* $|\zeta| = 1$, *there are values of* $r < 1$ *arbitrarily close to* 1 *with* $r\zeta \in \Omega(\rho_0)$, *and hence, in particular, with*

$$|F(r\zeta)| \;>\; w(r).$$

Take, wlog, $\zeta = 1$, and assume that for some a, $\rho_0 < a < 1$, *the whole segment* $[a, 1]$ *fails to intersect* Ω. The function $\sqrt{(z-a)/(1-az)}$ can then be defined so as to be analytic and single valued in Ω, and, if we introduce the new variable

$$s \;=\; \sqrt{\frac{z-a}{1-az}},$$

the mapping $z \to s$ takes Ω conformally onto a new domain – call it $\Omega_{\surd}$ – lying in $\{|s| < 1\}$:

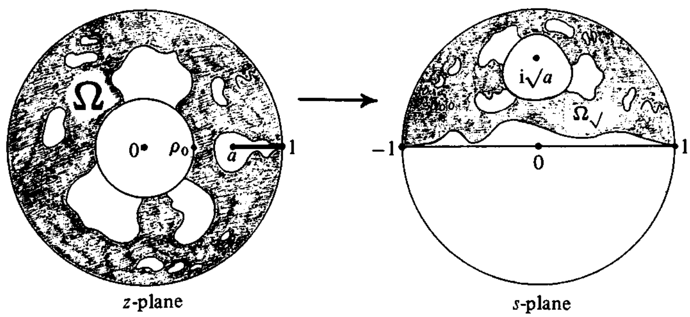

Figure 103

In terms of the variable s, write

$$\Phi(z) = \Psi(s), \qquad z \in \Omega;$$

$\Psi(s)$ is obviously analytic in $\Omega_{\surd}$ and continuous on its closure. If $s \in \Omega_{\surd}$ has $|s| \;>\; \sqrt{a}$, we have, since

$$z \;=\; \frac{s^2 + a}{1 + as^2},$$

that $1 - |z| \;\leqslant\; ((1+a)/(1-a))(1 - |s|^2)$; proof of this inequality is an elementary exercise in the geometry of linear fractional transformations which the reader should do. Hence, for $s \in \Omega_{\surd}$ with $|s| \;>\; \sqrt{a}$,

$$|z| \;\geqslant\; 1 - \frac{1+a}{1-a}(1 - |s|^2),$$

and, if $|s|$ is *close* to 1, this last expression is $\geqslant |s|^L$, where we can take for L a number $> 2(1+a)/(1-a)$ (depending on the closeness of $|s|$ to 1). The same relation between $|s|$ and $|z|$ holds for $s \in \partial\Omega_{\surd}$ with $|s|$ close to 1.

Suppose $|s| < 1$ is *close* to 1 and $s \in \partial\Omega_{\surd}$. The corresponding z then lies on $\partial\mathcal{O}$ with $|z| < 1$, so $|\Phi(z)| \leqslant \text{const.}w(|z|)$; therefore $|\Psi(s)| = |\Phi(z)| \leqslant \text{const.}w(|s|^L)$ by the relation just found, $w(r)$ being *decreasing*. The open set $\Omega_{\surd}$ *certainly abuts* on some arcs of $\{|s| = 1\}$ having positive length, since Ω abuts on the I_k. And $\partial\Omega_{\surd}$ *does not intersect the arc* $\pi < \arg s < 2\pi$ on $\{|s| = 1\}$ – that's why we *did* the conformal mapping $z \to s$! Here, the weight $w(r^L)$ is just as good (or just as bad) as $w(r)$ – see the *remark on a certain change of variable* at the end of article 4. We can therefore apply the *corollary* of the *theorem on harmonic measures* (end of article 5) to $\Psi(s)$ and the domain $\Omega_{\surd}$, and conclude that $\Psi(s) \equiv 0$ in $\Omega_{\surd}$. This, however, would make $\Phi(z) \equiv 0$ in Ω *which is impossible*, since $\Omega \subseteq \mathcal{O}$ where $|\Phi(z)| > 0$. *Step* 3's assertion must therefore hold.

The result just proved certainly implies that $\partial\Omega(\rho_0)$ *includes the unit circumference*. Since the Dirichlet problem can be solved for $\mathcal{O}$, it can be solved for Ω. It therefore makes sense to speak of the *harmonic measure* $\omega_\Omega(E, z)$ of an arbitrary closed subset E of $\{|\zeta| = 1\}$ ($\subseteq \partial\Omega$) relative to Ω, as seen from $z \in \Omega$. As was said above, our *aim* is to show that $\omega_\Omega(E, z_0) \geqslant k(z_0)|E|$ for such sets E; the analyticity of $\Phi(z)$ in Ω together with the fact that $|\Phi(z)|$ is > 0 and lies between two constant multiples of $|F(z)|$ there will *then* make

$$\int_{-\pi}^{\pi} \log|F(e^{i\vartheta})|\,d\vartheta \;>\; -\infty$$

by the *theorem on harmonic estimation* (§B.1), $|F(z)|$ being in any case *bounded above* in the closed unit disk. According to Harnack's theorem, the desired inequality for $\omega_\Omega(E, z_0)$ will follow from a *local version* of it, which is thus all that we need establish.

At this point the condition, assumed in the hypothesis, that $M(v) \geqslant \text{const.}v^\alpha$ with some $\alpha > \frac{1}{2}$ for large v, begins to play a more important rôle in our construction. We have already made *some* use of that property; it has not yet, however, been used *essentially*.

According to a lemma in article 2, the condition is equivalent to the property that $h(\xi) \geqslant \text{const.}\xi^{-\alpha/(1-\alpha)}$ for small $\xi > 0$. There is, in other words, an η, $0 < \eta < \frac{1}{2}$, with

$$\frac{1}{\xi} \leqslant \text{const.}(h(\xi))^{1-2\eta}$$

for small $\xi > 0$. *We fix such an η and put*

$$\boxed{H(\xi) \;=\; (h(2\xi))^{\eta}}$$

for $\xi > 0$.

Since $h(\xi)$ is *decreasing, so is* $H(\xi)$. Also, the property $h(\xi) \geqslant 1$ (due to the condition $M(0) \geqslant 1$) makes $H(\xi) \leqslant h(2\xi)$, whence, *a fortiori*,

$$H(\xi) \;\leqslant\; h(\xi) \quad \text{for} \quad \xi > 0.$$

We have

$$\int_0^a \log H(\xi)\,d\xi \;=\; \frac{\eta}{2}\int_0^{2a} \log h(x)\,dx \;=\; \infty$$

for all sufficiently small $a > 0$ by a theorem in article 2, since, as we are assuming, $\sum_1^\infty M(n)/n^2 \;=\; \infty$.

Using $H(\xi)$, let us form the new weight

$$w_1(r) \;=\; \exp\left(-H\left(\log\frac{1}{r}\right)\right), \quad 0 \leqslant r < 1.$$

Then

$$w_1(r) \;\geqslant\; w(r), \quad 0 \leqslant r < 1;$$

we still have, however,

$$w_1(r) \;=\; O((1-r)^2) \quad \text{for} \quad r \to 1,$$

although, when r is close to 1, $w_1(r)$ is *much larger than* $w(r)$. Starting with $w_1(r)$, we proceed to construct a new open set $\mathcal{O}_1 \subseteq \mathcal{O}$ on which $|F(z)| > w_1(|z|)$ in much the same fashion as $\mathcal{O}$ was formed by use of $w(r)$.

Take first the set

$$B_0' \;=\; \{z\colon\ |z| < 1 \text{ and } |F(z)| \leqslant w_1(|z|)\}.$$

Since $w_1(r) \geqslant w(r)$, B_0' contains the set B_0 used above in the construction of B. Note that on each of the little open disks used to cover B_0 and form the set B – call those disks Δ_j – we have $|F(z)| < 2w_1(|z|)$ since $|F(z)| < 2w(|z|)$ on them. If the Δ_j also cover B_0', we take $B_1 = B$. Otherwise, we form the difference

$$B_0'' \;=\; B_0' \cap \sim \bigcup_j \Delta_j$$

and cover each closed set

$$B_0'' \cap \left\{1 - \frac{1}{n} \;\leqslant\; |z| \;\leqslant\; 1 - \frac{1}{n+1}\right\}, \qquad n = 1, 2, 3, \ldots,$$

by a *finite number* of open disks $\tilde{\Delta}_k(n)$ lying in $\{|z|<1\}$, on which $|F(z)| < 2w_1(|z|)$. We then take B_1 as the *closure of the union*

$$\left(\bigcup_j \Delta_j\right) \cup \left(\bigcup_{n=1}^{\infty} \bigcup_k \tilde{\Delta}_k(n)\right).$$

For $z \in B_1$, $|F(z)| \leqslant 2w_1(|z|)$. B_1 contains B and has clearly the same general structure as B; B_1 includes all the points z of $\{|z|<1\}$ for which $|F(z)| \leqslant w_1(|z|)$.

We now put

$$\mathcal{O}_1 \;=\; \{|z|<1\} \cap \sim B_1 .$$

The set $\mathcal{O}_1$ is open and contained in $\mathcal{O}$ since $B_1 \supseteq B$. For $z \in \mathcal{O}_1$, $|F(z)| > w_1(|z|)$, and on $\partial\mathcal{O}_1 \cap \{|z|<1\}$ we have $|F(z)| \leqslant 2w_1(|z|)$ since the points of the later set must belong to B_1. The function $\Phi(z)$ introduced above is thus *analytic in* $\mathcal{O}_1$ (and continuous on its closure) and, since $A|F(z)| \;\leqslant\; |\Phi(z)| \;\leqslant\; A'|F(z)|$ for $|z|<1$, satisfies

$$|\Phi(z)| \;>\; \text{const.}w_1(|z|), \qquad z \in \mathcal{O}_1,$$

as well as

$$|\Phi(z)| \;\leqslant\; \text{const.}w_1(|z|) \quad \text{for } z \in \partial\mathcal{O}_1 \text{ and } |z|<1 \ (\textit{sic!}).$$

Our new weight $w_1(r)$ and the function $H(\xi)$ to which it is associated *fulfill the conditions for the theorem on harmonic measures* (article 5). *Hence, in view of the above two inequalities satisfied by* $\Phi(z)$, *there is nothing to prevent our going through steps* 1, 2, *and* 3 *again, with* $\mathcal{O}_1$ *in place of* $\mathcal{O}$ *and* $w_1(r)$ *in place of* $w(r)$. *We henceforth consider this done.*

Once step 3 for $\mathcal{O}_1$ and $w_1(r)$ is carried out, we know that *for each* ζ, $|\zeta| = 1$, *there are* $r<1$ *arbitrarily close to* 1 *for which* $|F(r\zeta)| > w_1(r)$. The open set $\mathcal{O}_1$ was brought into our discussion in order to obtain this result, which will be used to play off $w_1(r)$ against $w(r)$. Having now served its purpose, $\mathcal{O}_1$ will not appear again.

Given ζ_0, $|\zeta_0| = 1$, consider any $\rho < 1$ for which

$$(*) \qquad |F(\rho\zeta_0)| \;>\; w_1(\rho)$$

and form the domain $\Omega(\rho^2)$; this is the *connected* (step 2) component of $\mathcal{O} \cap \{\rho^2 < |z| < 1\}$ ($\mathcal{O}$ and *not* $\mathcal{O}_1$ here!) which *abuts* on *each* of the arcs I_k making up $\{|\zeta| = 1\} \cap (\sim B)$.

Step 4. *If, for given* ζ_0 *of modulus* 1, $(*)$ *holds with* ρ *close enough to* 1, *we have* $\rho\zeta_0 \in \Omega(\rho^2)$.

Assuming the contrary, we shall obtain a contradiction. Wlog, $\zeta_0 = 1$.

When $\rho \to 1$, the ratio

$$w_1(\rho)/w(\rho) \;=\; \exp\left(h\left(\log\frac{1}{\rho}\right) - \left(h\left(2\log\frac{1}{\rho}\right)\right)^{\eta}\right)$$

tends to ∞ since $h(\log(1/\rho))$ tends to ∞ then, h is decreasing, and $0 < \eta < 1$. Hence, if $|F(\rho)| > w_1(\rho)$ and $\rho > 1$ is close enough to 1, we surely have $|F(\rho)| > 2w(\rho)$, i.e., $\rho \notin B$, so $\rho \in \mathcal{O}$. The point ρ must then belong to *some* component of $\mathcal{O} \cap \{\rho^2 < |z| < 1\}$, so, *if it is not* in the *component* $\Omega(\rho^2)$, abutting on the I_k, of that intersection, *it must be in some other one, which* we may call $\mathscr{D}$. $\mathscr{D}$, being *disjoint* from $\Omega(\rho^2)$, *can thus abut on none of the arcs I_k of $\{|\zeta| = 1\}$ contiguous to $\{|\zeta| = 1\} \cap B$.*

It is now claimed that

$$\partial\mathscr{D} \cap \{\rho^2 \leqslant |z| \leqslant 1\} \quad (sic!)$$

is contained in B. Let ζ be in that intersection; if $|\zeta| < 1$, $\zeta \in \partial\mathcal{O} \cap \{|z| < 1\} \subseteq B$, so suppose that $|\zeta| = 1$. If $\zeta \notin B$, then ζ must lie in some contiguous arc I_k, say $\zeta \in I_1$. Then, for some open disk V_ζ with centre at ζ, $V_\zeta \cap \{|z| < 1\} \subseteq \mathcal{O}$. The intersection on the left is, however, *connected*, and, since $\zeta \in \partial\mathscr{D}$, it contains some points from the (*connected*!) open set $\mathscr{D}$. Therefore, $V_\zeta \cap \{|z| < 1\}$ must lie entirely in $\mathscr{D}$, and $\mathscr{D}$ *intersects* with the component $\Omega(\rho^2)$ of $\mathcal{O} \cap \{\rho^2 < |z| < 1\}$ abutting on I_1. But it doesn't! This contradiction shows that we must have $\zeta \in B$, as claimed.

Because $\partial\mathscr{D} \cap \{\rho^2 < |z| \leqslant 1\}$ is contained in B, we have

$$|\Phi(\zeta)| \;\leqslant\; \text{const.}|F(\zeta)| \;\leqslant\; \text{const.}w(|\zeta|)$$

for $\zeta \in \partial\mathscr{D}$ and $\rho^2 < |z| \leqslant 1$ (*sic*!).* The function $\Phi(z)$ is of course *analytic* in $\mathscr{D} \subseteq \mathcal{O}$, and continuous up to $\partial\mathscr{D}$. In order to help the reader follow the argument, let us *try* to draw a picture of $\mathscr{D}$:

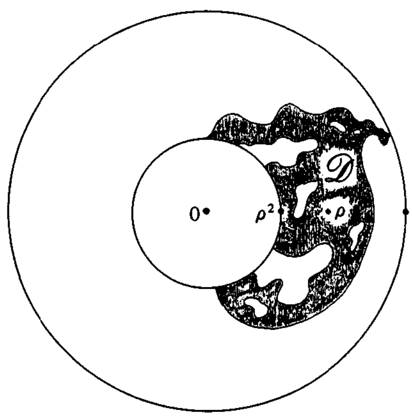

Figure 104

* We are taking $w(1) = 0$. See footnote, p. 359.

(N.B. $\mathscr{D}$ *can't really look like this* as we can see by applying an argument like the one of *step* 3 to $\Omega(\rho^2)$. One of the reasons why the present material is so hard is the *difficulty in drawing correct pictures* of what can happen.)

Call $\Gamma = \partial\mathscr{D} \cap \{\rho^2 < |z| \leqslant 1\}$, and denote harmonic measure for $\mathscr{D}$ by $\omega_{\mathscr{D}}(\ ,z)$. Since $|\Phi(z)|$ is bounded above on the unit disk and $|\Phi(\zeta)| \leqslant \text{const.} w(|\zeta|)$ on Γ, we have, by the theorem on harmonic estimation (§B.1),

$$\log|\Phi(\rho)| \;\leqslant\; \text{const.} \;+\; \omega_{\mathscr{D}}(\Gamma,\rho)\log w(\rho^2),$$

for $|\zeta| > \rho^2$ on Γ and $w(r)$ decreases. An almost trivial application of the *principle of extension of domain* shows that $\omega_{\mathscr{D}}(\Gamma,\rho)$ is *larger* than the *harmonic measure of the circle* $\{|\zeta| = 1\}$ *for the ring* $\{\rho^2 < |z| < 1\}$, *seen from* ρ. However, *that harmonic measure is* $\frac{1}{2}$! We thus see by the previous inequality that

$$\log|\Phi(\rho)| \;\leqslant\; \text{const.} \;+\; \tfrac{1}{2}\log w(\rho^2),$$

i.e., since

$$|\Phi(\rho)| \;\geqslant\; \text{const.}\,|F(\rho)| \;\geqslant\; \text{const.} w_1(\rho)$$

by $(*)$, that

$$\frac{1}{2}h\left(\log\frac{1}{\rho^2}\right) \;\leqslant\; \text{const.} \;+\; \left(h\left(\log\frac{1}{\rho^2}\right)\right)^{\eta}$$

in terms of the function h, with a constant independent of ρ.

Now $0 < \eta < 1$ and $h(\xi) \longrightarrow \infty$ for $\xi \to 0$. *This means that the relation just obtained is impossible for values of* ρ *sufficiently close to* 1. Therefore, if $\rho < 1$ is close enough to 1 and $|F(\rho)| > w_1(\rho)$, we *must have* $\rho \in \Omega(\rho^2)$, the conclusion we desired to make. (Part of the idea for the preceding argument is due to Peter Jones.)

Take now any ζ_0, $|\zeta_0| = 1$, and pick a $\rho < 1$ *very close to* 1 such that

$$|F(\rho\zeta_0)| \;>\; w_1(\rho)$$

(which is possible, as we have already observed), and that therefore $\rho\zeta_0 \in \Omega(\rho^2)$ (by step 4, just completed). We are going to show that if E is a closed set on the arc of the unit circumference going from $\zeta_0 e^{i\log\rho}$ to $\zeta_0 e^{-i\log\rho}$, then

$$\omega_{\Omega(\rho^2)}(E,\rho\zeta_0) \;\geqslant\; C(\zeta_0,\rho)|E|$$

with some constant $C(\zeta_0,\rho)$ depending on ζ_0 and on ρ. Here, of course, $\omega_{\Omega(\rho^2)}(\ ,z)$ denotes harmonic measure for the domain $\Omega(\rho^2)$.

In order to do this, we write

$$\gamma_\rho \;=\; \partial\Omega(\rho^2)\cap\{\rho^2<|z|<1\} \quad (sic!)$$

and carry out

Step 5. *If ρ, chosen according to the above specifications, is close enough to* 1,

$$\int_{\gamma_\rho}\frac{1}{1-|\zeta|}\,d\omega_{\Omega(\rho^2)}(\zeta,\rho\zeta_0)$$

is as small as we please.

In order to simplify the notation, let us write

$$\boxed{\omega(\;\;,z) \quad \text{for} \quad \omega_{\Omega(\rho^2)}(\;\;,z)}$$

during the *remainder* of the *present discussion.* The proof of our statement uses *almost the full strength* of the property that

$$\frac{1}{\xi} \;\leqslant\; \text{const.}\,(h(\xi))^{1-2\eta}$$

for small $\xi>0$ (with $0<\eta<\frac{1}{2}$), equivalent to our condition that

$$M(v) \;\geqslant\; \text{const.}\,v^\alpha$$

(with $\alpha>\frac{1}{2}$) for large v.

Take, wlog, $\zeta_0=1$. Then, if $\rho\in\Omega(\rho^2)$, we have, by the theorem on harmonic estimation (§ B.1),

$$\log|\Phi(\rho)| \;\leqslant\; \int_{\partial\Omega(\rho^2)}\log|\Phi(\zeta)|\,d\omega(\zeta,\rho),$$

$\Phi(z)$ being analytic in $\Omega(\rho^2)$ and continuous up to that set's boundary. The subset γ_ρ of $\partial\Omega(\rho^2)$ is of course contained in B, so, for $\zeta\in\gamma_\rho$,

$$|\Phi(\zeta)| \;<\; \text{const.}\,|F(\zeta)| \;\leqslant\; \text{const.}\,w(|\zeta|);$$

that is,

$$\log|\Phi(\zeta)| \;\leqslant\; \text{const.} \;-\; h\left(\log\frac{1}{|\zeta|}\right), \qquad \zeta\in\gamma_\rho.$$

The function $|\Phi(z)|$ is in any event bounded on $\{|z|\leqslant 1\}$, so this relation, together with the previous one, yields

$$\log|\Phi(\rho)| \;\leqslant\; \text{const.} \;-\; \int_{\gamma_\rho} h\left(\log\frac{1}{|\zeta|}\right)d\omega(\zeta,\rho).$$

If ρ is chosen in such a way that we also have $|F(\rho)| > w_1(\rho)$, this becomes

$$(\ast\!\!\ast)\qquad \int_{\gamma_\rho} h\left(\log\frac{1}{|\zeta|}\right)\mathrm{d}\omega(\zeta,\rho) \;\leqslant\; \text{const.} \;+\; \left(h\left(\log\frac{1}{\rho^2}\right)\right)^{\eta}.$$

Since $1/\xi \leqslant \text{const.}(h(\xi))^{1-2\eta}$ for small $\xi > 0$, we have (when $\rho < 1$ is close to 1),

$$\int_{\gamma_\rho} \frac{1}{1-|\zeta|}\mathrm{d}\omega(\zeta,\rho) \;\leqslant\; \text{const.}\int_{\gamma_\rho}\left(h\left(\log\frac{1}{|\zeta|}\right)\right)^{1-2\eta}\mathrm{d}\omega(\zeta,\rho).$$

Rewrite (!) the right-hand integral as

$$\text{const.}\int_{\gamma_\rho}\frac{\left(h\left(\log\frac{1}{|\zeta|}\right)\right)^{1-\eta}}{\left(h\left(\log\frac{1}{|\zeta|}\right)\right)^{\eta}}\mathrm{d}\omega(\zeta,\rho).$$

Since γ_ρ lies in the ring $\{\rho^2 < |z| < 1\}$ and $h(\xi)$ decreases, the expression just written is

$$\leqslant\; \frac{\text{const.}}{\left(h\left(\log\frac{1}{\rho^2}\right)\right)^{\eta}}\int_{\gamma_\rho}\left(h\left(\log\frac{1}{|\zeta|}\right)\right)^{1-\eta}\mathrm{d}\omega(\zeta,\rho),$$

so, since $h(\xi) \longrightarrow \infty$ for $\xi \to 0$, we see that

$$\int_{\gamma_\rho} \frac{1}{1-|\zeta|}\mathrm{d}\omega(\zeta,\rho) \;\leqslant\; \frac{\delta_\rho}{\left(h\left(\log\frac{1}{\rho^2}\right)\right)^{\eta}}\int_{\gamma_\rho} h\left(\log\frac{1}{|\zeta|}\right)\mathrm{d}\omega(\zeta,\rho)$$

with a quantity δ_ρ going to zero for $\rho \to 1$. Plugging $(\ast\!\!\ast)$ into the right-hand side, we obtain finally

$$\int_{\gamma_\rho} \frac{1}{1-|\zeta|}\mathrm{d}\omega(\zeta,\rho) \;\leqslant\; \delta_\rho\left(1 \;+\; \frac{\text{const.}}{(h(\log(1/\rho^2)))^{\eta}}\right).$$

Here, the right side tends to *zero* as $\rho \to 1$. *Step* 5 is finished.

Now we can make the *local estimate* of $\omega(E, z)$ for closed E on the unit circumference, promised just before step 5. Taking any fixed ζ_0, $|\zeta_0| = 1$, we pick a $\rho < 1$ very close to 1, in such fashion that the conclusions reached in *steps* 4 and 5 apply. Let E be any closed set on the arc of the unit circumference going from $\zeta_0 e^{i\log\rho}$ to $\zeta_0 e^{-i\log\rho}$.

In order to keep the notation simple, consider, as before, the case where $\zeta_0 = 1$.

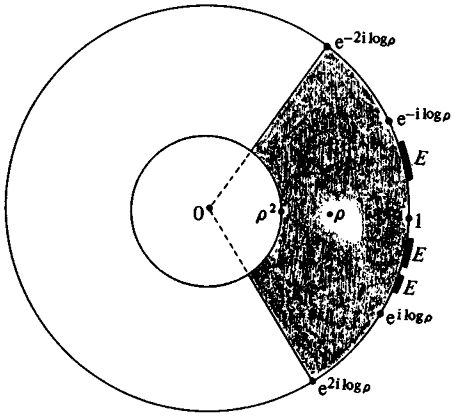

Figure 105

Denote harmonic measure for the *ring* $\{\rho^2 < |z| < 1\}$ by $\bar{\omega}(\ \ , z)$. If we *compare* $\bar{\omega}(E, z)$ with the harmonic measure of E *for the sectorial box shown in the above diagram*, we see immediately that

$$(\dagger) \qquad \bar{\omega}(E,\rho) \;\geqslant\; C\frac{|E|}{1-\rho}$$

with a numerical constant C independent of $|E|$ and of ρ.

Put, as in *step* 5,

$$\gamma_\rho \;=\; \partial\Omega(\rho^2)\cap\{\rho^2 < |z| < 1\},$$

and continue to denote the harmonic measure for $\Omega(\rho^2)$ by $\omega(\ \ , z)$. Since

$$\Omega(\rho^2) \;\subseteq\; \{\rho^2 < |z| < 1\}$$

while

$$\partial\Omega(\rho^2) \;\supseteq\; \{|\zeta| = 1\},$$

we can apply a formula established near the end of § B.1, getting

$$\omega(E,z) \;=\; \bar{\omega}(E,z) - \int_{\gamma_\rho} \bar{\omega}(E,\zeta)\,\mathrm{d}\omega(\zeta,z)$$

for $z\in\Omega(\rho^2)$.

Comparing $\bar{\omega}(E, z)$ with harmonic measure of E for the *whole unit disk*, we get the estimate

$$\bar{\omega}(E,\zeta) \;\leqslant\; \frac{|E|}{\pi(1-|\zeta|)}.$$

Substitute this into the *integral* in the above formula for $\omega(E,z)$, specialize to $z=\rho$, and use (†) in the first right-hand term of that formula. We find

$$\omega(E,\rho) \;\geqslant\; |E|\left(\frac{C}{1-\rho} \;-\; \frac{1}{\pi}\int_{\gamma_\rho}\frac{d\omega(\zeta,\rho)}{1-|\zeta|}\right).$$

It was, however, seen in *step* 5 that $\rho<1$ could be chosen in accordance with our requirements so as to make the *integral* in this expression *small.* For a suitable $\rho<1$ close to 1, we will thus have (and by far !)

$$\omega(E,\rho) \;\geqslant\; \frac{C}{2(1-\rho)}|E|$$

provided that the closed set E lies on the (shorter) arc from $e^{i\log\rho}$ to $e^{-i\log\rho}$ on the unit circle.

This is our local estimate. What it says is that, corresponding to any ζ, $|\zeta|=1$, we can *get* a $\rho_\zeta<1$ such that, for closed sets E lying on *the smaller arc* J_ξ *of the unit circle joining* $\zeta e^{i\log\rho_\zeta}$ *to* $\zeta e^{-i\log\rho_\zeta}$,

(††) $\qquad \omega_{\Omega(\rho_\zeta^2)}(E,\rho_\zeta\zeta) \;\geqslant\; C_\zeta|E|,$

with $C_\zeta>0$ depending (*a priori*) on ζ. Observe that, if $0<\rho\leqslant\rho_\zeta^2$, $\Omega(\rho)$ (the component of $\mathscr{O}\cap\{\rho<|z|<1\}$ abutting on the arcs I_k) must by definition contain $\Omega(\rho_\zeta^2)$. Therefore

(§) $\qquad \omega_{\Omega(\rho)}(E,\rho_\zeta\zeta) \;\geqslant\; \omega_{\Omega(\rho_\zeta^2)}(E,\rho_\zeta\zeta)$

by the *principle of extension of domain* when $E\subseteq\{|z|=1\}$, a subset of *both* boundaries $\partial\Omega(\rho)$, $\partial\Omega(\rho_\zeta^2)$.

A finite number of the arcs J_ζ serve to cover the unit circumference; denote them by J_1, $J_2,\ldots,J_n$, calling the corresponding values of ζ, $\zeta_1,\ldots,\zeta_n$ and the corresponding ρ_ζ's ρ_1, $\rho_2,\ldots,\rho_n$. Let ρ be the *least* of the quantities ρ_j^2, $j=1,2,\ldots,n$, and denote the *least* of the C_{ζ_j} by k, which is thus >0. If E is a closed subset of J_j, (††) and (§) give

$$\omega_{\Omega(\rho)}(E,\rho_j\zeta_j) \;\geqslant\; k|E|.$$

Fix any $z_0\in\Omega(\rho)$. Using Harnack's inequality in $\Omega(\rho)$ for each of the pairs of points $(z_0,\ \rho_j\zeta_j)$, $j=1,2,\ldots,n$, we obtain, from the preceding relation,

(§§) $\qquad \omega_{\Omega(\rho)}(E,z_0) \;\geqslant\; K(z_0)|E|$

for closed subsets E of *any* of the arcs J_1, $J_2,\ldots,J_n$. Here, $K(z_0)>0$ depends on z_0. Now we see finally that (§§) *in fact holds for any closed subset E of the unit circumference,* large or small. That is an obvious consequence of the additivity of the set function $\omega_{\Omega(\rho)}(\quad,z_0)$, the arcs J_j forming a covering of $\{|\zeta|=1\}$.

We are at long last able to conclude our proof of Volberg's theorem on the logarithmic integral. Our chosen z_0 in $\Omega(\rho)$ lies in $\mathcal{O}$, therefore $|\Phi(z_0)| > 0$. By the *theorem on harmonic estimation* applied to the function $\Phi(z)$ analytic in $\Omega(\rho)$ and continuous on $\{|z| \leqslant 1\}$,

$$-\infty < \log|\Phi(z_0)| \leqslant \int_{\partial\Omega(\rho)} \log|\Phi(\zeta)|\,d\omega_{\Omega(\rho)}(\zeta, z_0)$$

$$\leqslant \text{const.} + \int_{|\zeta|=1} \log|\Phi(\zeta)|\,d\omega_{\Omega(\rho)}(\zeta, z_0).$$

According to (§§), this last is in turn

$$\leqslant \text{const.} - K(z_0)\int_{-\pi}^{\pi} \log^-|\Phi(e^{i\vartheta})|\,d\vartheta,$$

$\log|\Phi(e^{i\vartheta})|$ being in any case *bounded above*. Thus,

$$\int_{-\pi}^{\pi} \log^-|\Phi(e^{i\vartheta})|\,d\vartheta < \infty.$$

However, $|\Phi(e^{i\vartheta})|$ lies, as we know, between *two constant multiples of* $|F(e^{i\vartheta})|$. Therefore

$$\int_{-\pi}^{\pi} \log^-|F(e^{i\vartheta})|\,d\vartheta < \infty,$$

i.e.,

$$\int_{-\pi}^{\pi} \log|F(e^{i\vartheta})|\,d\vartheta > -\infty.$$

Volberg's theorem is thus completely proved, and *we are finally done.*

Remark. Of the two regularity conditions required of $M(v)$ for this theorem, viz., that $M(v)/v$ be *decreasing* and that $M(v) \geqslant \text{const.}v^\alpha$ for large v with some $\alpha > \frac{1}{2}$, the *first* served to make possible the use of Dynkin's result (article 3) by means of which the analytic function $\Phi(z)$ was brought into the proof.

Decisive use of the *second* was not made until *step* 5, where we estimated

$$\int_{\gamma_\rho} \frac{1}{1-|\zeta|}\,d\omega(\zeta, \rho)$$

in terms of $\int_{\gamma_\rho} h(\log(1/|\zeta|))\,d\omega(\zeta, \rho)$.

Examination of the argument used there shows that *some* relaxation of the condition $M(v) \geqslant \text{const.}v^\alpha$ (with $\alpha > \frac{1}{2}$) *is* possible if one is willing to replace it by another with considerably more complicated statement. The *method* of Volberg's proof necessitates, however, that $M(v)$ be *at least* $\geqslant$ const.$v^{\frac{1}{2}}$ for large v. For, by a lemma of article 3, that relation is equivalent to the property that

$$\frac{1}{\xi} = O(h(\xi))$$

for $\xi \to 0$, and we need at least *this* in order to make the abovementioned estimate for $\rho < 1$ near 1.

We needed $\int_{\gamma_\rho}(1/(1-|\zeta|))d\omega(\zeta,\rho)$ in the computation following step 5, where we got a *lower bound* on $\omega(E,\rho)$. The integral came in there on account of the inequality

$$\tilde{\omega}(E,\zeta) \leqslant \frac{|E|}{\pi(1-|\zeta|)}$$

for harmonic measure $\tilde{\omega}(E,\zeta)$ (of sets $E \subseteq \{|\zeta| = 1\}$) in the ring $\{\rho^2 < |\zeta| < 1\}$. And, aside from a constant factor, this inequality is best possible.

7. Scholium. Levinson's log log theorem

Part of the material in articles 2 and 5 is closely related to some older work of Levinson which, because of its usefulness, should certainly be taken up before ending the present chapter.

During the proof of the *first* theorem in §F.4, Chapter VI, we came up with an entire function $L(z)$ satisfying an inequality of the form $|L(z)| \leqslant \text{const.}e^{K|z|}/|\Im z|$, and wished to conclude that $L(z)$ was of exponential type. Here there is an obvious difficulty for the points z lying near the real axis. We dealt with it by using the subharmonicity of $\sqrt{|L(z)|}$ and convergence of

$$\int_{-1}^{1} |y|^{-\frac{1}{2}} dy$$

in order to *integrate out* the denominator $|\Im z|$ from the inequality and thus strengthen the latter to an estimate $|L(z)| \leqslant \text{const.}e^{K|z|}$ for z near $\mathbb{R}$. A more elaborate version of the same procedure was applied in the proof of the *second* theorem of §F.4, Chapter VI, where subharmonicity of $\log|S(z)|$ was used to get rid of a troublesome term tending to ∞ for z approaching the real axis.

It is natural to ask *how far* such tricks can be pushed. Suppose that $f(z)$ is known to be analytic in some rectangle straddling the real axis, and we are assured that

$$|f(z)| \leqslant \text{const.}L(y)$$

in that rectangle, where, unfortunately, the majorant $L(y)$ goes to ∞ as $y \to 0$. What conditions on $L(y)$ will permit us to deduce *finite bounds* on $|f(z)|$, *uniform in the interior* of the given rectangle, from the preceding relation? One's first guess is that a condition of the form $\int_{-a}^{a} \log L(y)\,dy < \infty$ *will do*, but that *nothing much weaker than that can suffice*, because $\log|f(z)|$ *is* subharmonic while functions of $|f(z)|$ which increase *more slowly* than the logarithm are *not*, in general. This conservative appraisal turns out to be *wrong by a whole* (exponential) *order of magnitude*. Levinson found that *it is already enough to have*

$$\int_{-a}^{a} \log\log L(y)\,dy \;<\; \infty,$$

and that this condition *cannot be further weakened*.

Levinson's result is extremely useful. One application could be to *eliminate* the rough and ready but somewhat clumsy *hall of mirrors* argument from many of the places where it occurs in Chapter VI. Let us, for instance, consider again the proof of Akhiezer's first theorem from §B.1 of that chapter. If, in the circumstances of that theorem, we have $\|P\|_W \leqslant 1$ for a polynomial P, the relation

$$\begin{aligned}\log|P(z)| \;&\leqslant\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|}{|z-t|^2}\log|P(t)|\,dt \\ &\leqslant\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|}{|z-t|^2}\log W_+(t)\,dt\end{aligned}$$

and the estimate of $\sup_{t\in\mathbb{R}}|(t-i)/(t-z)|$ from §A.2 (Chapter VI) tell us immediately that

$$\log|P(z)| \;\leqslant\; M\frac{(1+|z|)^2}{|\Im z|},$$

where

$$M \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log W_*(t)}{1+t^2}\,dt.$$

Taking any rectangle

$$\mathscr{D}_R \;=\; \{z\colon\; |\Re z| \leqslant R,\quad |\Im z| \leqslant 2\}$$

and putting

$$L(y) \;=\; \exp\left(M\frac{R^2+5}{|y|}\right),$$

we have $|P(z)| \leqslant L(|\Im z|)$ on $\mathscr{D}_R$. Here,

$$\int_{-1}^{1} \log\log L(y)\,dy \;<\; \infty,$$

so the result of Levinson gives us a control on the size of $|P(z)|$ *in the interior of* $\mathscr{D}_R$ (even *right on* the real axis). In this way we can see that the polynomials P with $\|P\|_W \leqslant 1$ form a *normal family* in any strip straddling the real axis. The relation $\log|P(z)| \;\leqslant\; M(1+|z|)^2/|\Im z|$ already shows that those polynomials form a normal family *outside* such a strip, and the main part of the proof of Akhiezer's first theorem is complete.

One can easily envision the possibility of other applications like the one just shown to situations where the *hall of mirrors* argument would not be available. There is thus no doubt about the worth of the result in question; let us, then, proceed to its precise statement and proof without further ado.

Levinson's* log log *theorem. *Consider any rectangle*

$$\mathscr{D} \;=\; \{z\colon\ a < x < a' \text{ and } -b < y < b\}.$$

Let $L(y)$ be Lebesgue measurable and $\geqslant$ e for $-b < y < b$.

Suppose that

$$\int_{-b}^{b} \log\log L(y)\,dy \;<\; \infty.$$

Then there is a decreasing function $m(\delta)$, depending only *on $L(y)$ and finite for $\delta > 0$, such that, if $f(z)$ is analytic in $\mathscr{D}$ and if*

$$|f(z)| \;\leqslant\; L(\Im z)$$

there, we also have

$$|f(z)| \;\leqslant\; m(\text{dist.}(z, \partial\mathscr{D})) \quad \textit{for } z \in \mathscr{D}.$$

Remark. In this version (due to Y. Domar), *no regularity properties whatever are required of* $L(y)$. The assumption that $L(y) \geqslant$ e is of course made *merely to ensure positivity* of $\log\log L(y)$.

Proof of theorem (Y. Domar). Denote by $\mu(\lambda)$ the *distribution function* for $\log L(y)$; i.e.,

$$\mu(\lambda) \;=\; |\{y\colon\ -b < y < b \text{ and } \log L(y) > \lambda\}|.*$$

Let $z_0 \in \mathscr{D}$ and $R < \text{dist}(z_0, \partial\mathscr{D})$, and write $u(z) = \log|f(z)|$. Since $u(z)$ is

* We are continuing to denote by $|E|$ the Lebesgue measure of sets $E \subseteq \mathbb{R}$.

subharmonic in $\mathscr{D}$, we have

$$u(z_0) \leqslant \frac{1}{\pi R^2}\iint_{|z-z_0|<R} u(z)\,dx\,dy.$$

We are going to show that, if $u(z_0)$ is *large* and z_0 *far enough* from $\partial\mathscr{D}$, this inequality leads to a *contradiction* when $u(z) \leqslant \log L(\Im z)$ in $\mathscr{D}$.

Call Δ the disk

$$\{z: |z-z_0| < R\}.$$

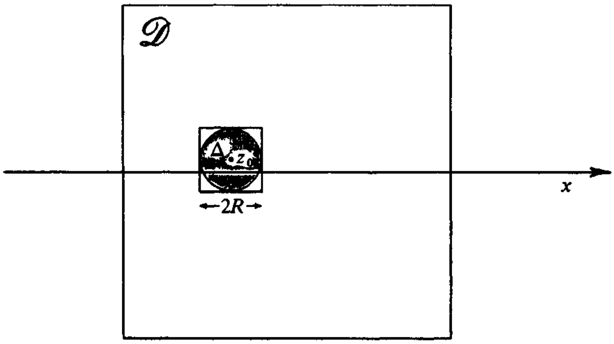

Figure 106

Use $\|E\|$ to denote the two-dimensional Lebesque measure of $E \subseteq \mathbb{C}$ (since $|\ \ |$ is being used for one-dimensional Lebesque measure of sets on $\mathbb{R}$). Then, by boxing Δ into a square of side $2R$ in the manner shown, we see from the ineqдality $u(z) \leqslant \log L(\Im z)$ that

$$\|\{z\in\Delta:\ u(z) > M/2\}\| \ \leqslant\ 2R\mu(M/2) \quad \text{for } M > 0.$$

Suppose now that $u(z_0) \geqslant M$, but that at the same time we have $u(z) \leqslant 2M$ on Δ. From the previous subharmonicity relation we will then have

$$(*) \qquad u(z_0) \ \leqslant\ \frac{M}{2} + \frac{2M}{\|\Delta\|}\left\|\left\{z\in\Delta:\ u(z) \geqslant \frac{M}{2}\right\}\right\| \ <\ \frac{M}{2} + \frac{4M}{\pi R}\mu(M/2).$$

Because $\log\log L(y)$ is integrable on $[-b, b]$ we certainly have $\mu(\lambda) \to 0$ for $\lambda \to \infty$. We can therefore take M *so large* that $\mu(M/2)$ is *much smaller* than $\operatorname{dist}(z_0, \partial\mathscr{D})$. *With such a value* of M, put

$$R \ =\ R_0 \ =\ \frac{16}{\pi}\mu(M/2).$$

The *right side of* $(*)$ *will then* be $\leqslant \frac{3}{4}M$. This means that *if* $u(z_0) \geqslant M$, the *assumption under which* $(*)$ *was derived is untenable*, i.e., *that* $u(z) > 2M$ *somewhere in* Δ, say for $z = z_1$, $|z_1 - z_0| \ <\ R_0$.

Supposing, then, that $u(z_0) \geqslant M$, we have a z_1, $|z_1 - z_0| < R_0$, with $u(z_1) > 2M$. We can then *repeat* the argument just given, making z_1 play the rôle of z_0, $2M$ that of M,

$$R_1 = \frac{16}{\pi}\mu(M)$$

that of R_0, and $\{z\colon |z - z_1| < R_1\}$ that of Δ. As long as $R_1 > \operatorname{dist}(z_1, \partial\mathscr{D})$, hence, surely, provided that

$$R_0 + R_1 < \operatorname{dist}(z_0, \partial\mathscr{D}),$$

we will get a z_2, $|z_2 - z_1| < R_1$, with $u(z_2) > 4M$. Then we can take $R_2 = (16/\pi)\mu(2M)$, have $4M$ play the rôle held by $2M$ in the previous step, and keep on going.

If, for the numbers

$$R_k = \frac{16}{\pi}\mu(2^{k-1}M),$$

we have

$$\sum_{k=0}^{\infty} R_k < \operatorname{dist}(z_0, \partial\mathscr{D}),$$

the process never stops, and we get a sequence of points $z_k \in \mathscr{D}$, $|z_k - z_{k-1}| < R_{k-1}$, with

$$u(z_k) \geqslant 2^k M.$$

Evidently, $z_k \xrightarrow[k]{} $ a point $z_\infty \in \mathscr{D}$.

The function $f(z)$ is *analytic* in $\mathscr{D}$, so $u(z) = \log|f(z)|$ is *continuous* (in the extended sense) at z_∞, where $u(z_\infty) < \infty$. This is certainly incompatible with the inequalities $u(z_k) \geqslant 2^k M$ when $z_k \xrightarrow[k]{} z_\infty$. *Therefore we cannot have $u(z_0) \geqslant M$ if we can take M so large that $\sum_0^\infty R_k < \operatorname{dist}(z_0, \partial\mathscr{D})$*, i.e., *that*

$$(\dagger) \qquad \sum_{k=0}^{\infty} \frac{16}{\pi}\mu(2^{k-1}M) < \operatorname{dist}(z_0, \partial\mathscr{D}).$$

In order to complete the proof, it suffices, then, to show that the left-hand sum in (†) tends to *zero* as $M \to \infty$. By Abel's rearrangement,

$$\begin{aligned}\sum_{k=0}^{n} \mu(2^{k-1}M) = {} & \left\{\mu\left(\frac{M}{2}\right) - \mu(M)\right\} + 2\{\mu(M) - \mu(2M)\} \\ & + 3\{\mu(2M) - \mu(4M)\} + \cdots \\ & + n\{\mu(2^{n-2}M) - \mu(2^{n-1}M)\} + (n+1)\mu(2^{n-1}M).\end{aligned}$$

Remembering that $\mu(\lambda)$ is a *decreasing* function of μ, we see that as long as $M \geqslant 4$, the sum on the right is

$$\leqslant \sum_{k=0}^{n-1} \int_{2^{k-1}M}^{2^k M} \frac{\log\lambda - \log(M/4)}{\log 2}(-\,\mathrm{d}\mu(\lambda))$$

$$+ \int_{2^{n-1}M}^{\infty} \frac{\log\lambda - \log(M/4)}{\log 2}(-\,\mathrm{d}\mu(\lambda))$$

$$\leqslant -\frac{1}{\log 2}\int_{M/2}^{\infty} \log\lambda\,\mathrm{d}\mu(\lambda) = \frac{1}{\log 2}\int_{\substack{\log L(y) \geqslant M/2 \\ -b<y<b}} \log\log L(y)\,\mathrm{d}y.$$

Since $\int_{-b}^{b} \log\log L(y)\,\mathrm{d}y < \infty$, the previous expression, and hence the left-hand side of (†), tends to 0 as $M \to \infty$. Therefore, given $z_0 \in \mathscr{D}$, we can get an M sufficiently large for (†) to *hold, and, with that* M, $u(z_0) < M$, i.e., $|f(z_0)| < \mathrm{e}^M$. Let, then,

$$m(\delta) = \inf\left\{\mathrm{e}^M : \sum_{k=0}^{\infty} \frac{16}{\pi}\mu(2^{k-1}M) < \delta\right\}.$$

As we have just seen, $m(\delta)$ is *finite* for $\delta > 0$; it is obviously decreasing. And, if $f(z)$ is analytic in $\mathscr{D}$ with $|f(z)| \leqslant L(\Im z)$ there, we have $|f(z_0)| \leqslant m(\operatorname{dist}(z_0, \partial\mathscr{D}))$ for $z_0 \in \mathscr{D}$. We are done.

Remark. This beautiful proof is quite recent. The procedure of the scholium at the end of article 5 will yield the same result for sufficiently regular majorants $L(y)$.

We now consider the possibility of relaxing the condition

$$\int_{-b}^{b} \log\log L(y)\,\mathrm{d}y < \infty$$

required in the above theorem. If, for some majorant $L(y)$, *the conclusion of the theorem holds*, any set of *polynomials* P with $|P(z)| \leqslant L(\Im z)$ for $z \in \mathscr{D}$ *must form a normal family* in $\mathscr{D}$. This observation enables us to give a simple proof of the fact that the requirement.

$$\int_{-b}^{b} \log\log L(y)\,\mathrm{d}y < \infty$$

is essential in Levinson's result, at least for majorants $L(y)$ of sufficiently regular behaviour.

Theorem. *Let $L(y)$ be continuous and* $\geqslant$ e *for* $0 < |y| \leqslant b$, *with* $L(y) \to \infty$

for $y \to 0$ *and* $L(y)$ decreasing *on* $(0, b)$. *Suppose also that*

$$\int_0^b \log\log L(y)\,dy \;=\; \infty.$$

Then there is a sequence of polynomials $P_n(z)$ *with, for* $\Im z \neq 0$,

$$|P_n(z)| \;\leqslant\; \text{const.}L(\Im z)$$

on the rectangle

$$\mathscr{D} \;=\; \{z\colon\; -1 < \Re z < 1 \text{ and } -b < \Im z < b\},$$

while at the same time

$$P_n(z) \xrightarrow[n]{} \begin{cases} \;\;1, & z \in \mathscr{D} \text{ and } \Im z > 0, \\ -1, & z \in \mathscr{D} \text{ and } \Im z < 0. \end{cases}$$

Thus the conclusion of Levinson's theorem does not hold *for the majorant* $L(y)$.

Remark. We only require $L(y)$ to be monotone *on one side* of the origin, on the side over which the integral of $\log\log L(y)$ *diverges*. Levinson already had this result under the assumption of more regularity for $L(y)$.

Proof of theorem (Beurling, 1972 – compare with the proof of the theorem on simultaneous polynomial approximation in article 4). The last sentence in the statement follows from the existence of a sequence of polynomials P_n having the asserted properties. For, if the conclusion of Levinson's theorem *held*, the sequence $\{P_n\}$ would form a normal family in $\mathscr{D}$ and there would hence be a function *analytic* in $\mathscr{D}$, equal to $+1$ *above* the real axis, and to -1 *below* it. This is absurd.

Put

$$\varphi(z) \;=\; \begin{cases} \;\;1, & z \in \mathscr{D} \text{ and } \Im z > 0 \\ -1, & z \in \mathscr{D} \text{ and } \Im z < 0, \end{cases}$$

and let us argue by *duality* to obtain a sequence of polynomials $P_n(z)$ for which

$$\sup_{z \in \mathscr{D}} \left(\frac{|\varphi(z) - P_n(z)|}{L(\Im z)} \right) \xrightarrow[n]{} 0.$$

Such P_n will clearly satisfy the conclusion of our theorem.

Note that, since $L(y) \to \infty$ for $y \to 0$, the ratio $\varphi(z)/L(\Im z)$ is *continuous on* $\bar{\mathscr{D}}$ if we *define* $L(0)$ *to be* ∞, which we *do, for the rest of this proof.* Therefore, if a sequence of polynomials P_n fulfilling the above condition *does not exist*, we can, by the *Hahn–Banach theorem*, find a finite complex-valued measure μ on $\bar{\mathscr{D}}$ with

$$(\S) \qquad \iint_{\mathscr{D}} \frac{z^n}{L(\Im z)} \mathrm{d}\mu(z) \;=\; 0 \quad \text{for} \quad n = 0, 1, 2, \ldots,$$

whilst

$$\iint_{\mathscr{D}} \frac{\varphi(z)}{L(\Im z)} \mathrm{d}\mu(z) \;\neq\; 0.$$

The proof will be completed by showing that *in fact* (§) implies

$$\iint_{\mathscr{D}} \frac{\varphi(z)}{L(\Im z)} \mathrm{d}\mu(z) \;=\; 0.$$

Given any measure μ satisfying (§), write

$$\mathrm{d}\nu(z) \;=\; \frac{1}{L(\Im z)} \mathrm{d}\mu(z).$$

The measure ν has *very little mass near the real axis, and none at all on it*. For each complex λ, the power series for $\mathrm{e}^{\mathrm{i}\lambda z}$ converges uniformly for $z \in \bar{\mathscr{D}}$, so from (§) we get

$$\iint_{\mathscr{D}} \mathrm{e}^{\mathrm{i}\lambda z} \mathrm{d}\nu(z) \;=\; 0.$$

Write now

$$\mathscr{D}_+ \;=\; \bar{\mathscr{D}} \cap \{\Im z > 0\} \quad (\textit{sic}!)$$

and

$$\mathscr{D}_- \;=\; \bar{\mathscr{D}} \cap \{\Im z < 0\} \quad (\textit{sic}!).$$

Then put

$$\Phi_+(\lambda) \;=\; \iint_{\mathscr{D}_+} \mathrm{e}^{\mathrm{i}\lambda z} \mathrm{d}\nu(z),$$

$$\Phi_-(\lambda) \;=\; \iint_{\mathscr{D}_-} \mathrm{e}^{\mathrm{i}\lambda z} \mathrm{d}\nu(z);$$

since ν has no mass on $\mathbb{R}$, the previous relation becomes

$$(\dagger\dagger) \qquad \Phi_+(\lambda) \;+\; \Phi_-(\lambda) \;\equiv\; 0.$$

We are going to show that in fact each of the left-hand terms in (††) vanishes identically.

Consider $\Phi_+(\lambda)$. Since $\mathscr{D}_+$ is a *bounded* domain; $\Phi_+(\lambda)$ is *entire and of exponential type*. It is also *bounded* on the *real axis*. Indeed, for $\lambda > 0$,

$$|\Phi_+(\lambda)| \leqslant \iint_{\mathscr{D}_+} e^{-\lambda \Im z}|d\nu(z)| \leqslant \iint_{\mathscr{D}_+} |d\nu(z)|,$$

and for $\lambda < 0$ we can, *on account of* (††), use the relation $\Phi_+(\lambda) = -\Phi_-(\lambda)$ and make a similar estimate involving $\mathscr{D}_-$. These properties of $\Phi_+(\lambda)$ and the theorem of Chapter III, §G.2, imply that $\Phi_+(\lambda) \equiv 0$ provided that

$$(\ast) \qquad \int_{-\infty}^{\infty} \frac{\log|\Phi_+(\lambda)|}{1+\lambda^2}\,d\lambda = -\infty.$$

We proceed to establish this relation.

Write

$$H(y) = \begin{cases} \log L(y), & 0 < y \leqslant b, \\ \log L(b), & y > b. \end{cases}$$

Then $H(y)$ is *decreasing* for $y > 0$ by hypothesis. For $\lambda > 0$,

$$|\Phi_+(\lambda)| = \left|\iint_{\mathscr{D}_+} \frac{e^{i\lambda z}}{L(\Im z)}\,d\mu(z)\right| \leqslant \iint_{\mathscr{D}_+} e^{-H(\Im z)-\lambda\Im z}|d\mu(z)|.$$

If, as in article 5, we put

$$M(\lambda) = \inf_{y>0}\,(H(y) + y\lambda),$$

we see by the previous relation that

$$(\S\S) \qquad |\Phi_+(\lambda)| \leqslant \text{const.}e^{-M(\lambda)}, \qquad \lambda > 0.$$

Since $H(y)$ is decreasing for $y > 0$ and $\geqslant 1$ there, and

$$\int_0^b \log H(y)\,dy = \int_0^b \log\log L(y)\,dy = \infty$$

by hypothesis, we have

$$\int_1^{\infty} \frac{M(\lambda)}{\lambda^2}\,d\lambda = \infty$$

according to the *last* theorem of article 2. This, together with (§§), gives us ($\ast$), and hence $\Phi_+(\lambda) \equiv 0$.

Referring again to (††), we see that also $\Phi_-(\lambda) \equiv 0$. Specializing to $\lambda = 0$ (!), we obtain the two relations

$$\iint_{\mathscr{D}_+} \frac{1}{L(\Im z)}\,d\mu(z) = 0, \qquad \iint_{\mathscr{D}_-} \frac{1}{L(\Im z)}\,d\mu(z) = 0,$$

from which, by subtraction,

$$\iint_{\mathscr{D}} \frac{\varphi(z)}{L(\Im z)}\, d\mu(z) \;=\; 0,$$

what we had set out to show. The proof of our theorem is thus finished, and we are done.

And thus ends this long (aye, too long!) seventh chapter of the present book.

VIII

Persistence of the form $dx/(1+x^2)$

Up to now, integrals like

$$\int_{-\infty}^{\infty} \frac{\log|F(x)|}{1+x^2}\,dx$$

have appeared so frequently in this book mainly on account of the specific form of the Poisson kernel for a half plane. If $\omega(S, z)$ denotes the harmonic measure (at z) of $S \subseteq \mathbb{R}$ for the half plane $\{\Im z > 0\}$, we simply have

$$\omega(S, \mathrm{i}) = \frac{1}{\pi}\int_S \frac{dt}{1+t^2}.$$

Suppose now that we *remove* certain finite open intervals – perhaps infinitely many – from $\mathbb{R}$, leaving a certain residual set E, and that E *looks* something like $\mathbb{R}$ when seen from far enough away. E should, in particular, have infinite extent on both sides of the origin and not be too sparse. Denote by $\mathscr{D}$ the (multiply connected – perhaps even infinitely connected) domain $\mathbb{C} \sim E$, and by $\omega_{\mathscr{D}}(\ \ , z)$ the harmonic measure for $\mathscr{D}$.

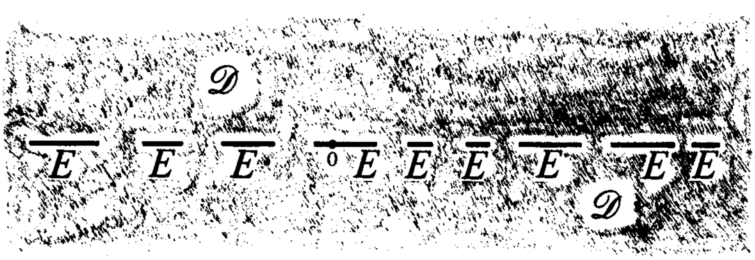

Figure 107

It is a remarkable fact that a formula like the above one for $\omega(S, \mathrm{i})$ *subsists*, to a certain extent, for $\omega_{\mathscr{D}}(\ \ , \mathrm{i})$, provided that the degradation suffered by $\mathbb{R}$

in its reduction to E is not too great. We have, for instance,

$$\omega_{\mathscr{D}}(J \cap E, \mathrm{i}) \;\leqslant\; C_E(\alpha) \int_{J\cap E} \frac{\mathrm{d}t}{1+t^2}$$

for *intervals* J with $|J \cap E| \geqslant \alpha > 0$, where $C_E(\alpha)$ depends on α as well as on the set E. In other words, $\mathrm{d}\omega_{\mathscr{D}}(t, \mathrm{i})$ *still acts* (crudely) *like the restriction of* $\mathrm{d}x/(1+x^2)$ *to* E. It is this *tendency of the form* $\mathrm{d}x/(1+x^2)$ *to persist* when we reduce $\mathbb{R}$ to certain smaller sets E (and enlarge the upper half plane to $\mathscr{D} = \mathbb{C} \sim E$) that constitutes the theme of the present chapter.

The persistence is well illustrated in the situation of *weighted approximation* (whether by polynomials or by functions of exponential type) on the sets E. If a function $W(x) \geqslant 1$ is given on E, with $W(x) \to \infty$ as $x \to \pm\infty$ in E (for weighted *polynomial* approximation on E this must of course take place faster than any power of x), we can look at *approximation on* E (by polynomials or by functions of exponential type bounded on $\mathbb{R}$) *using the weight* W. It turns out that *precise formal analogues* of many of the results established for weighted approximation *on* $\mathbb{R}$ in §§A, B and E of Chapter VI *are valid here*; the only change consists in the *replacement* of the integrals of the form

$$\int_{-\infty}^{\infty} \frac{\log M(t)}{1+t^2}\,\mathrm{d}t$$

occurring in Chapter VI by the corresponding expressions

$$\int_E \frac{\log M(t)}{1+t^2}\,\mathrm{d}t.$$

The integrand, involving $\mathrm{d}t/(1+t^2)$, remains unchanged.

This chapter has three sections. The *first* is mainly devoted to the case where E has *positive lower uniform density* on $\mathbb{R}$ – a typical example is furnished by the set

$$E \;=\; \bigcup_{n=-\infty}^{\infty} [n-\rho,\; n+\rho]$$

where $0 < \rho < \frac{1}{2}$.

In §B, we study the limiting case of the example just mentioned which arises when $\rho = 0$, i.e., when $E = \mathbb{Z}$. There is of course no longer any harmonic measure for $\mathscr{D} = \mathbb{C} \sim \mathbb{Z}$. It is therefore remarkable that *something nevertheless remains true of the results established in* §A. If $P(z)$ is a *polynomial* such that

$$\sum_{n=-\infty}^{\infty} \frac{1}{1+n^2} \log^+ |P(n)| \;\leqslant\; \eta$$

with $\eta > 0$ *sufficiently small* (this restriction turns out to be crucial!), we *still* have, for $z \in \mathbb{C}$,

$$|P(z)| \leqslant K(z, \eta),$$

where $K(z,\eta)$ depends *only* on z and η, and not on P. The proof of this fact is very long, and hard to grasp as a whole. It uses specific properties of polynomials. Since $\mathscr{D}$ has no harmonic measure, a corresponding statement with $\log^+|P(z)|$ replaced by a *general* continuous subharmonic function of at most logarithmic growth is *false.*

We return in §C to the study of harmonic estimation in $\mathscr{D}$ when its boundary, E, *does not* reduce to a discrete set. *Here*, we assume that E contains all $x \in \mathbb{R}$ of sufficiently large absolute value, that situation being general enough for applications. The purpose of §C is to connect up the behavior of a *Phragmén–Lindelöf function* for $\mathscr{D}$ (i.e., one harmonic in $\mathscr{D}$ and acting like $|\Im z|$ there, with boundary value *zero* on E) to that of *harmonic measure* for $\mathscr{D}$. There is a quantitative relation between the former and the latter. Harmonic measure still acts (very crudely!) like the restriction of $dt/(1+t^2)$ to E. This § is independent of §B to a large extent, but does use a fair amount of material from §A. Results obtained in it are needed for Chapter XI.

A. The set E has positive lower uniform density

During most of this §, we consider sets E of the special form

$$\bigcup_{n=-\infty}^{\infty} [a_n - \delta_n,\ a_n + \delta_n],$$

the intervals $[a_n - \delta_n,\ a_n + \delta_n]$ being *disjoint. We will assume that there are four constants, A, B, δ and Δ, with*

$$\boxed{0 < A < a_n - a_{n-1} < B, \qquad 0 < \delta < \delta_n < \Delta,}$$

for all n.

The following *notation* will be used throughout:

$$\begin{aligned} E_n &= [a_n - \delta_n,\ a_n + \delta_n], \\ \mathcal{O}_n &= (a_n + \delta_n,\ a_{n+1} - \delta_{n+1}), \\ \mathscr{D} &= \mathbb{C} \sim E. \end{aligned}$$

Here is a picture of the setup we are studying:

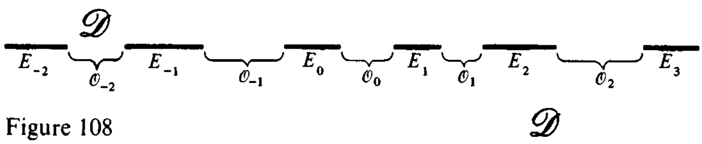

Figure 108

The above boxed conditions on the a_n and δ_n *clearly* imply the existence of two constants C_1 and $C_2 > 0$ (*each* depending on the four numbers A, B, δ and Δ) such that:

(i) if $k \neq l$ and $x \in \mathscr{O}_k$, $x' \in \mathscr{O}_l$, we have $C_1|k-l| \leqslant |x-x'| \leqslant C_2|k-l|$;
(ii) if $k \neq l-1$, l, or $l+1$ and $x \in E_k$, $x' \in E_l$, we have $C_1|k-l| \leqslant |x-x'| \leqslant C_2|k-l|$.

The restriction on the pair (k, l) in (ii) is due to the fact that the *lengths* of the $\mathscr{O}_k$ are *not assumed to be bounded away from zero*; their lengths are only bounded above. It is *the lengths of the* E_k that are bounded above *and* away from zero.

Heavy use will be made of properties (i) and (ii) during the following development. Clearly, if E is any set for which the above boxed condition holds (with given A, B, δ and Δ), so is each of its translates $E + h$ (with the same constants A, B, δ and Δ). The properties (i) and (ii) are thus valid for each of those translates, with the same constants C_1 and C_2 as for E. For this reason there is no real loss of generality in supposing that $0 \in \mathscr{O}_0$, and we will frequently do so when that is convenient.

1. Harmonic measure for $\mathscr{D}$

The Dirichlet problem can be solved for the kind of domains $\mathscr{D}$ we are considering and (*at least*, certainly!) for continuous boundary data on E given by functions tending to 0 as $x \to \pm\infty$ in E. Let us, without going into too much detail, indicate how this fact can be verified.

Take large values of R, and put

$$\mathscr{D}_R = \mathscr{D} \cap \sim \{(-\infty, -R] \cup [R, \infty)\}:$$

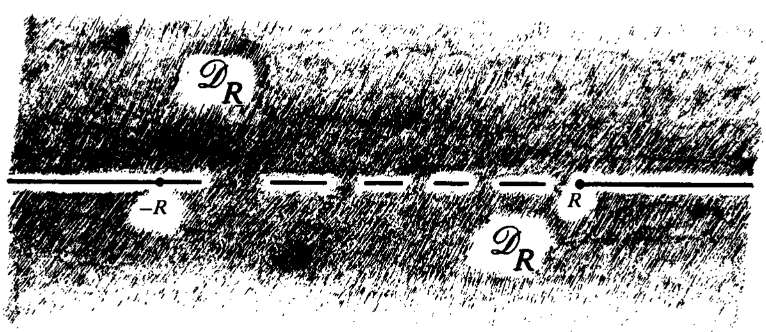

Figure 109

Each of the regions $\mathscr{D}_R$ is *finitely connected*, and the Dirichlet problem can be solved in it. This is *known*; it is true because the straight segment boundary components ('slits') of $\mathscr{D}_R$ are practically as nice as Jordan curve boundary components. One can indeed map $\mathscr{D}_R$ conformally onto a region *bounded* by Jordan curves by using a succession of Joukowski transformations, one for each slit (including the infinite one through ∞):

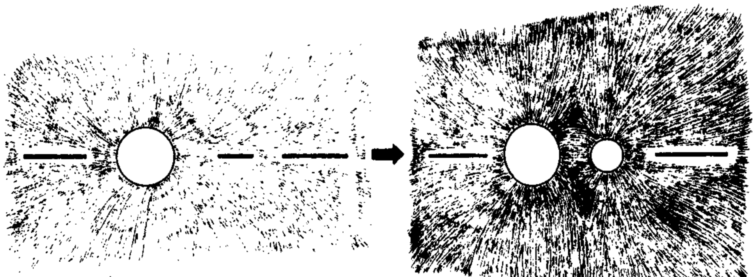

Figure 110

The inverse to the conformal mapping thus obtained *does* take the Jordan curve boundary components back *continuously* onto the original slits, so the Dirichlet problem *can* be solved for $\mathscr{D}_R$ if it can be solved for regions bounded by a finite number of Jordan curves. (This same idea will be used again at the end of article 2, in proving the symmetry of Green's function.)

Once we are sure that the Dirichlet problem can be solved in each $\mathscr{D}_R$ we can, by examining how certain solutions behave for $R \to \infty$, convince ourselves that the Dirichlet problem for $\mathscr{D}$ is also solvable, at least for boundary data of the abovementioned kind. Details of this examination are left to the reader.

Since $\mathscr{D}$ is regular for the Dirichlet problem, harmonic measure is available for it. We know from the rudiments of conformal mapping theory *that a slit should be considered as having two sides, or edges.* Given (say) an interval $J \subseteq$ one of the boundary components E_n of $\mathscr{D}$, we should *distinguish* between *two* intervals coinciding with J: J_+ (lying on the *upper* side of E_n), and J_- (lying on the *lower* side of E_n):

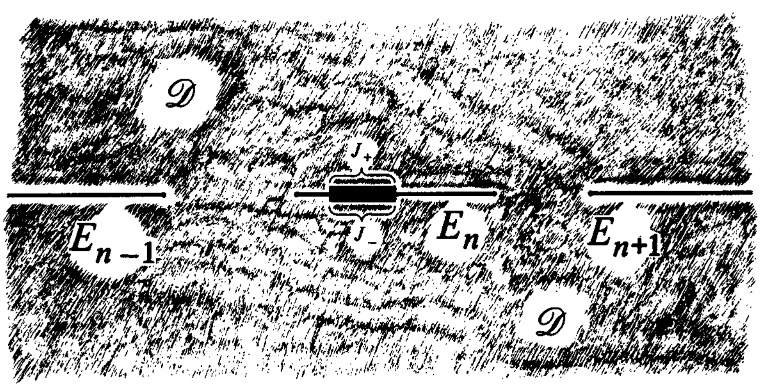

Figure 111

It makes sense, then, to talk about the *two* harmonic measures:

$$\omega_{\mathscr{D}}(J_+, z)$$

(which tends to *zero* when z tends to an interior point of J_-), and

$$\omega_{\mathscr{D}}(J_-, z).$$

In most of our work, however, *separation of J into J_+ and J_- will serve no purpose.* It will in fact be sufficient to work with the *sum*

$$\omega_{\mathscr{D}}(J_+, z) + \omega_{\mathscr{D}}(J_-, z).$$

▶ *This harmonic function tends to* 1 *when z tends from either side of the real axis to an interior point of J, and it is what we take as the harmonic measure*

$$\omega_{\mathscr{D}}(J, z)$$

of J. The harmonic measure $\omega_{\mathscr{D}}(S, z)$ *of any $S \subseteq E$ is defined in the same way.*

Consider now any of the boundary components E_k of $E = \partial\mathscr{D}$, and write

$$\omega_k(z) = \omega_{\mathscr{D}}(E_k, z)$$

for the harmonic measure of E_k, as seen from $z \in \mathscr{D}$. We are going to show that there is a constant C, depending on the four numbers A, B, δ and Δ associated with E, such that

$$\omega_k(x) \leqslant \frac{C}{(l-k)^2+1} \quad \text{for } x \in \mathscr{O}_l.$$

($\mathscr{O}_l$, recall, is the part of $\mathscr{D} \cap \mathbb{R}$ lying between E_l and E_{l+1}.) The proof is due to Carleson; one or two of its ideas go back to earlier work. We need two lemmas, the first of which could almost be given as an exercise.

Lemma. *Denote by $\Omega_k(\quad, z)$ harmonic measure for the domain*

$$\mathscr{D}_k = \{\Im z > 0\} \cup \mathscr{O}_k \cup \{\Im z < 0\}.$$

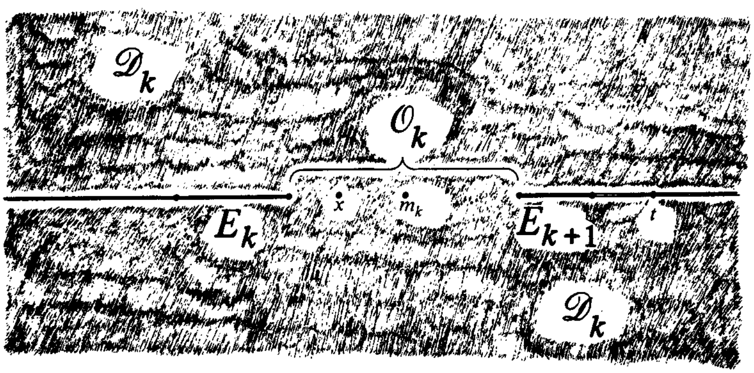

Figure 112

There is a constant K', depending only on the numbers B and δ associated with the set E, such that for $x \in \mathcal{O}_k$ and $t \in \partial\mathscr{D}_k$ lying outside both *of the segments E_k and E_{k+1} we have*

$$d\Omega_k(t,x) \leqslant \frac{K'}{(x-t)^2}\,dt.$$

Proof. By conformal mapping of $\mathscr{D}_k$ onto the unit disk. Calling m_k the midpoint of $\mathcal{O}_k$, we apply to $z \in \mathscr{D}_k$ the chain of mappings

$$z \longrightarrow \zeta = 2\frac{z-m_k}{|\mathcal{O}_k|} \longrightarrow w = \frac{1}{\zeta} - \sqrt{\left(\frac{1}{\zeta^2}-1\right)}.$$

We write $w = \varphi(z)$ and, if t is on $\partial\mathscr{D}_k$, denote $\varphi(t)$ by ω. (In the latter case we must of course distinguish between points t lying on the *upper side* of $\partial\mathscr{D}_k$ and those on its *lower side* – see the preceding remarks. On this distinction depends the choice of the branch of $\sqrt{\ }$ to be used in computing $\omega = \varphi(t)$.)

For t on $\partial\mathscr{D}_k$ outside both E_k and E_{k+1} we have

$$\omega = \varphi(t) = \frac{1}{\tau} - \sqrt{\left(\frac{1}{\tau^2}-1\right)},$$

where $\tau = 2(t-m_k)/|\mathcal{O}_k|$ satisfies the inequality

$$|\tau| - 1 \geqslant 2\frac{\min(|E_k|,|E_{k+1}|)}{|\mathcal{O}_k|} > \frac{4\delta}{B},$$

in view of the relations $|E_l| = 2\delta_l > 2\delta$, $|\mathcal{O}_k| < a_{k+1} - a_k < B$. In terms of τ,

$$d\omega = -\frac{d\tau}{\tau^2}\left(1 \pm \frac{i}{\sqrt{(\tau^2-1)}}\right),$$

i.e.,

$$|d\omega| = \left(\frac{\tau^2}{\tau^2-1}\right)^{\frac{1}{2}}\frac{|d\tau|}{\tau^2}.$$

For t outside both E_k and E_{k+1}, the expression on the right is

$$< \left(\frac{1+4\delta/B}{4\delta/B}\right)^{\frac{1}{2}}\frac{|d\tau|}{\tau^2} = \left(\frac{B}{4\delta}+1\right)^{\frac{1}{2}}\frac{|d\tau|}{\tau^2}$$

by the inequality for $|\tau| - 1$.

Let $x \in \mathcal{O}_k$. Then, remembering that $d\Omega_k(t,x)$ is the harmonic measure of *two infinitesimal intervals* $[t, t+dt]$ lying on $\partial\mathscr{D}_k$ – one on *the upper edge* and one on *the lower*, we see that

$$d\Omega_k(t,x) = \frac{1}{\pi}\frac{1-|\varphi(x)|^2}{|\varphi(x)-\omega|^2}|d\omega|,$$

with $|d\omega|$ being given by the above formula. Since, for $t \notin E_k \cup E_{k+1}$,

$|\tau| - 1 > 4\delta/B$, the image, ω, of t on the unit circumference must lie *outside* two arcs thereof entered at 1 and at -1, and having *lengths* that depend on the ratio $4\delta/B$. We do not need to know the exact form of this dependence.

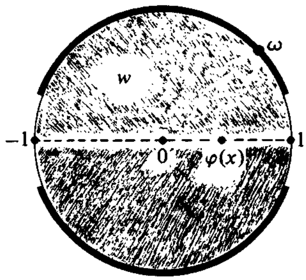

Figure 113

Hence, since $-1 < \varphi(x) < 1$, it is clear from the picture that $|\varphi(x) - \omega|$ is $\geqslant$ a positive quantity depending on the lengths of the excluded arcs about 1 and -1, and thence on the ratio $4\delta/B$. The factor $(1 - |\varphi(x)|^2)/|\varphi(x) - \omega|^2$ in the above formula for $\mathrm{d}\Omega_k(t, x)$ is thus bounded above by a number depending on $4\delta/B$ when $t \notin E_k \cup E_{k+1}$. Substituting into that formula the inequality for $|\mathrm{d}\omega|$ already found, we get

$$\mathrm{d}\Omega_k(t, x) \leqslant C\frac{|\mathrm{d}\tau|}{\tau^2}$$

for $t \notin E_k \cup E_{k+1}$, with a constant C depending on $4\delta/B$. In terms of $t = m_k + \frac{1}{2}|\mathcal{O}_k|\tau$, the right side is

$$\leqslant \frac{C|\mathcal{O}_k|}{2}\frac{\mathrm{d}t}{(t - m_k)^2} \leqslant \frac{CB}{2}\frac{\mathrm{d}t}{(t - m_k)^2}.$$

Since $x \in \mathcal{O}_k$ and $t \notin \mathcal{O}_k$, $\quad |t - x| \leqslant 2|t - m_k|$:

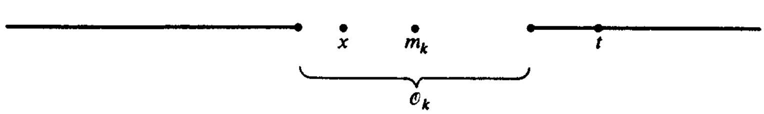

Figure 114

So finally,

$$\mathrm{d}\Omega_k(t, x) \leqslant 2BC\frac{\mathrm{d}t}{(t - x)^2},$$

Q.E.D.

Computational lemma (Carleson, 1982; see also Benedicks' 1980 *Arkiv* paper). *Let $A_{k,l} \geqslant 0$ for $k, l \in \mathbb{Z}$. Suppose there are constants K and λ, with $0 < \lambda < 1$, such that*

$$(*) \qquad A_{k,l} \leqslant \frac{K}{(l-k)^2+1},$$

and

$$(⁑) \qquad \sum_{l=-\infty}^{\infty} A_{k,l} \leqslant \lambda \qquad \textit{for all} \quad k.$$

Then there is a number $\eta > 0$ depending on K and λ such that, for any sequence $\{y_l\}$ with $0 \leqslant y_l \leqslant \eta$ and $0 \leqslant y_l \leqslant 1/(l^2+1)$, we have.

$$\sum_{l=-\infty}^{\infty} A_{k,l}y_l \leqslant \lambda \sup_l y_l \qquad \textit{for all} \quad k,$$

and

$$\sum_{l=-\infty}^{\infty} A_{k,l}y_l \leqslant \frac{1+\lambda}{2}\cdot\frac{1}{k^2+1}.$$

Remark. The *first* of the asserted inequalities is manifest; it is the *second* that is non-trivial. If the constant K is small enough, the second inequality is also clear; it is when K is *not small* that the latter is difficult to verify.

Proof of lemma. As we have just remarked, the *first* inequality is obvious (by (⁑)); let us therefore see to the *second*, endeavoring first of all to prove it for *large* values of $|k|$, say $|k| \geqslant$ some k_0.

Assume, wlog, that $k > 0$, and take some *small* number μ, $0 < \mu < \frac{1}{2}$, about whose precise value we will decide later on. Write the sum

$$\sum_{l=-\infty}^{\infty} A_{k,l}y_l$$

as

$$\sum_{|l|<\mu k} + \sum_{\substack{|l|\geqslant\mu k \\ |l-k|\geqslant\mu k}} + \sum_{|l-k|<\mu k} = \mathrm{I} + \mathrm{II} + \mathrm{III},$$

say. Use of $(*)$ together with the inequality $0 \leqslant y_l \leqslant \eta$ (where η is *as yet* unspecified) gives us first of all

$$\mathrm{I} \leqslant \frac{K}{(1-\mu)^2k^2+1}\cdot 2k\mu\cdot\eta$$

We get III out of the way by combining (⁑) and the inequality

$0 \leqslant y_l \leqslant 1/(l^2+1)$, with the result that

$$\text{III} \leqslant \frac{\lambda}{(1-\mu)^2k^2+1}.$$

It is the middle sum, II, that gives us trouble. We break II up further as

$$\sum_{l \leqslant -\mu k} + \sum_{\mu k \leqslant l < k/2} + \sum_{k/2 \leqslant l \leqslant (1-\mu)k} + \sum_{l \geqslant (1+\mu)k}.$$

The *first* of these sums is

$$\leqslant \sum_{l \leqslant -\mu k} \frac{1}{l^2+1} \cdot \frac{K}{(k-l)^2+1} \leqslant \frac{K}{k^2+1} \sum_{m \geqslant \mu k} \frac{1}{m^2} \leqslant \frac{2K}{\mu k(k^2+1)}.$$

The *second* is similarly

$$\leqslant \sum_{\mu k \leqslant l < k/2} \frac{1}{l^2+1} \cdot \frac{K}{(k-l)^2+1} \leqslant \frac{K}{(k/2)^2+1} \sum_{m \geqslant \mu k} \frac{1}{m^2} \leqslant \frac{8K}{\mu k(k^2+4)}$$

The *third* sum is

$$\leqslant \sum_{l \leqslant (1-\mu)k} \frac{1}{(k/2)^2+1} \cdot \frac{K}{(k-l)^2+1} \leqslant \frac{8K}{\mu k(k^2+4)},$$

and the *fourth*

$$\leqslant \sum_{l \geqslant (1+\mu)k} \frac{1}{l^2+1} \cdot \frac{K}{(l-k)^2+1} \leqslant \frac{2K}{\mu k(k^2+1)}.$$

All told, then,

$$\text{II} \leqslant \frac{20K}{\mu k(k^2+1)}.$$

Adding this last estimate to those already obtained for I and III, we get finally

$$\sum_{l=-\infty}^{\infty} A_{k,l}y_l \leqslant \frac{2k\mu\eta K}{(1-\mu)^2(k^2+1)} + \frac{20K}{\mu k(k^2+1)} + \frac{\lambda}{(1-\mu)^2(k^2+1)}.$$

The idea now is to *first* put η equal to a *very small quantity* η_0, and *then*, assuming k is *large*, put $\mu = 1/\eta_0^{1/2}k$; this will *also be small* for large enough k. For such large k, the previous inequality will reduce to

$$\sum_{l=-\infty}^{\infty} A_{k,l}y_l \leqslant \frac{2\eta_0^{1/2}K + 20\eta_0^{1/2}K + \lambda}{(1-\mu)^2(k^2+1)}.$$

Choosing first $\eta_0^{1/2}$ *small* enough and then taking k_0 so *large* that $1/\eta_0^{1/2}k_0$

is *also* small, we will make the right-hand side of this inequality

$$\leqslant \frac{1+\lambda}{2}\cdot\frac{1}{k^2+1}$$

for $|k| \geqslant k_0$ by putting $\mu = 1/\eta_0^{1/2}|k|$. When $\eta < \eta_0$ and $0 \leqslant y_l \leqslant \eta$ we then have, *a fortiori*,

$$\sum_{l=-\infty}^{\infty} A_{k,l}y_l \;\leqslant\; \frac{1+\lambda}{2}\cdot\frac{1}{k^2+1} \qquad \text{for} \quad |k| \geqslant k_0.$$

With such y_l, however, the sum on the left is also $\leqslant \lambda\eta$. So, taking finally

$$\eta \;=\; \min\left(\eta_0, \frac{1}{k_0^2+1}\right)$$

makes the left side $\leqslant$ $((1+\lambda)/2)(1/(k^2+1))$ for $|k| < k_0$ *as well*, i.e.,

$$\sum_{l=-\infty}^{\infty} A_{k,l}y_l \;\leqslant\; \frac{1+\lambda}{2}\cdot\frac{1}{k^2+1}$$

for all k, provided that $0 \leqslant y_l \leqslant \eta$ and $0 \leqslant y_l \leqslant 1/(l^2+1)$.

The lemma is proved.

Theorem (Carleson, 1982; see also Benedicks' 1980 *Arkiv* paper). *In the domain $\mathscr{D}$, the harmonic measure $\omega_0(z)$ of the component E_0 of $\partial\mathscr{D}$ satisfies*

$$\omega_0(x) \;\leqslant\; \frac{C}{k^2+1} \qquad \textit{for } x \in \mathcal{O}_k,$$

with a constant C depending only on the four numbers A, B, δ and Δ associated with $E = \partial\mathscr{D}$.

Proof (Carleson). Call u_k the *maximum value* of $\omega_0(x)$ on $\mathcal{O}_k$; we are to show that

$$u_k \;\leqslant\; \frac{C}{k^2+1}.$$

For $k = -1$ and $k = 0$ this is certainly true if we put $C = 2$; we may therefore restrict our discussion to the *values of k different from* -1 and 0.

As in the first of the above lemmas, denote by $\mathscr{D}_k$ the domain

$$\{\Im z > 0\} \cup \mathcal{O}_k \cup \{\Im z < 0\}$$

and by $\Omega_k(\;, z)$ the harmonic measure for $\mathscr{D}_k$. $\mathscr{D}_k$ is of course *contained* in $\mathscr{D}$:

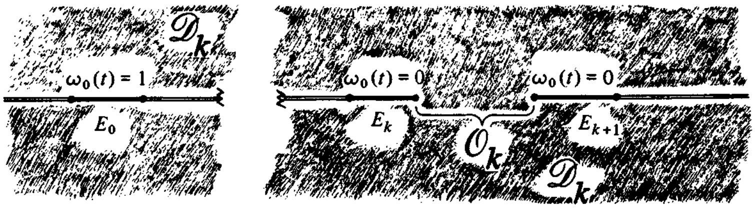

Figure 115

$\omega_0(z)$ is thus harmonic in $\mathscr{D}_k$; since it is clearly bounded there and continuous up to $\partial\mathscr{D}_k$, we may recover it from its values on $\partial\mathscr{D}_k$ by the Poisson formula

$$\omega_0(z) = \int_{\partial\mathscr{D}_k} \omega_0(t) d\Omega_k(t, z), \qquad z \in \mathscr{D}_k.$$

Since we are assuming that $k \neq -1, 0$, we have (by *definition* of harmonic measure!) $\omega_0(t) = 0$ for $t \in E_k \cup E_{k+1}$. In fact, $\omega_0(t)$ is identically zero on *all* the E_l save E_0, where $\omega_0(t) = 1$. The above formula hence becomes

$$\omega_0(z) = \int_{E_0} d\Omega_k(t, z) + \sum_{l \neq k} \int_{\mathscr{O}_l} \omega_0(t) d\Omega_k(t, z),$$

$\mathbb{R}$ being the disjoint union of the intervals $\mathscr{O}_l$ and E_l.

Let x_l be the point in $\mathscr{O}_l$ where $\omega_0(x)$ *assumes its maximum* u_l *therein*, and write

$$A_{k,l} = \int_{\mathscr{O}_l} d\Omega_k(t, x_k).$$

Then, by the previous relation,

$$u_k \leqslant \Omega_k(E_0, x_k) + \sum_{l \neq k} A_{k,l} u_l.$$

Here, the integrals

$$\int_{E_0} d\Omega_k(t, x_k) = \Omega_k(E_0, x_k)$$

and

$$\int_{\mathscr{O}_l} d\Omega_k(t, x_k) = A_{k,l}, \qquad l \neq k,$$

are taken over sets *disjoint from* E_k and E_{k+1}, whereas $x_k \in \mathscr{O}_k$. We may therefore apply the *first* lemma to estimate $d\Omega_k(t, x_k)$ in these integrals, getting

$$\Omega_k(E_0, x_k) \leqslant K' \int_{E_0} \frac{dt}{(t - x_k)^2}$$

and

$$A_{k,l} \leqslant K' \int_{\mathcal{O}_l} \frac{dt}{(t-x_k)^2}, \qquad l \neq k,$$

where K' is a constant depending on the numbers B and δ associated with E. By properties (i) and (ii), given at the beginning of this §, we have

$$(t-x_k)^2 \geqslant C_1^2 k^2 \quad \text{for} \quad t \in E_0$$

and

$$(t-x_k)^2 \geqslant C_1^2 (k-l)^2 \quad \text{for} \quad t \in \mathcal{O}_l.$$

So, since $|E_0| = 2\delta_0 < 2\Delta$ and $|\mathcal{O}_l| < B$, the preceding relations become

$$\Omega_k(E_0, x_k) \leqslant \frac{K}{k^2+1}$$

and

$$(*) \qquad A_{k,l} \leqslant \frac{K}{(k-l)^2+1}, \qquad l \neq k;$$

here, K is a constant depending on the four numbers A, B, δ and Δ.

The numbers $A_{k,l}$ also satisfy the inequality

$$(\ast\ast) \qquad \sum_{l \neq k} A_{k,l} \leqslant \lambda < 1$$

with λ depending *only on the ratio* δ/B. Indeed,

$$\sum_{l \neq k} A_{k,l} = \sum_{l \neq k} \Omega_k(\mathcal{O}_l, x_k)$$

is $\leqslant \Omega_k(\partial\mathcal{D}_k \sim E_k \sim E_{k+1}, x_k)$. Since $|\mathcal{O}_k| < B$ and $|E_k| > 2\delta$, $|E_{k+1}| > 2\delta$, a simple change of variable shows that the latter quantity is *less* than the harmonic measure of the set $1 + 4\delta/B \leqslant |t| < \infty$ on the boundary of the domain $\mathbb{C} \sim (-\infty, -1] \sim [1, \infty)$, seen from some point in $(-1, 1)$:

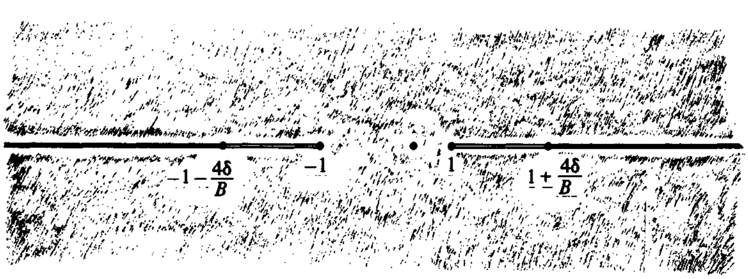

Figure 116

And *that* harmonic measure is clearly *at most* equal to some number $\lambda < 1$ depending on $4\delta/B$.

Let us return to the inequality

$$u_k \leqslant \Omega_k(E_0, x_k) + \sum_{l \neq k} A_{k,l} u_l$$

established above. By plugging into it the estimates just found, we get

$$(\dagger) \qquad u_k - \sum_{l \neq k} A_{k,l} u_l \leqslant \frac{K}{k^2 + 1},$$

where the $A_{k,l}$ are $\geqslant 0$, *and satisfy* $(*)$ *and* $(\ast\!\!\ast)$. This has been proved for $k \neq -1$ and 0, but it *also holds* (*a fortiori*!) for *those* values of k, provided that we take $K \geqslant 2$. *Then our* (unknown) *maxima* $u_k \geqslant 0$ *will satisfy* ($\dagger$) *for all* k; *this we henceforth assume.*

The idea now is to invert the relations ($\dagger$) in order to obtain bounds on the u_k. *It is convenient to define* $A_{k,l}$ *for* $l = k$ *by putting* $A_{k,k} = 0$. Then, calling

$$(\dagger\dagger) \qquad v_k = u_k - \sum_l A_{k,l} u_l,$$

we can recover the u_k *from the* v_k by virtue of $(\ast\!\!\ast)$. Write $A_{k,l}^{(1)} = A_{k,l}$; then put

$$A_{k,l}^{(2)} = \sum_{j=-\infty}^{\infty} A_{k,j} A_{j,l},$$

and in general

$$A_{k,l}^{(n+1)} = \sum_{j=-\infty}^{\infty} A_{k,j} A_{j,l}^{(n)}.$$

The numbers $A_{k,l}^{(n)}$ are $\geqslant 0$ (since the $A_{k,l}$ are), and from $(\ast\!\!\ast)$, we have

$$(\S) \qquad \sum_{l=-\infty}^{\infty} A_{k,l}^{(n)} \leqslant \lambda^n.$$

This makes it possible for us to invert ($\dagger\dagger$), getting

$$u_k = v_k + \sum_l A_{k,l}^{(1)} v_l + \sum_l A_{k,l}^{(2)} v_l + \cdots + \sum_l A_{k,l}^{(n)} v_l + \cdots,$$

the Neumann series on the right being absolutely convergent. Since the $A_{k,l}^{(n)}$ are $\geqslant 0$ and

$$v_l \leqslant \frac{K}{l^2 + 1}$$

by (†) and (††), the previous relation gives

$$u_k \leqslant K\left\{\frac{1}{k^2+1} + \sum_{n=1}^{\infty}\sum_{l=-\infty}^{\infty}\frac{A_{k,l}^{(n)}}{l^2+1}\right\}.$$

We proceed to examine the right-hand side of this inequality.

We have

$$\sum_{l=-\infty}^{\infty}\frac{1}{(k-l)^2+1}\cdot\frac{1}{l^2+1} \leqslant \frac{\text{const.}}{k^2+1}$$

(look at the reproduction property of the Poisson kernel $y/((x-t)^2+y^2)$ on which the *hall of mirrors* argument used in Chapter 6 is based!). Hence, by (*), there is a constant L with

$$0 \leqslant A_{k,l}^{(n)} \leqslant \frac{L^n}{(k-l)^2+1},$$

and the summand

$$\sum_l A_{k,l}^{(n)}\cdot\frac{1}{l^2+1}$$

on the right side of the above estimate for u_k is

$$\leqslant \frac{\text{const.}\,L^n}{k^2+1}.$$

We have, however, to add up infinitely many of these summands. *It is here that we must resort to the computational lemma.*

Call

$$v_k^{(n)} = \sum_{l=-\infty}^{\infty}\frac{A_{k,l}^{(n)}}{l^2+1};$$

we certainly have $v_k^{(n)} \geqslant 0$. According to the computational lemma, there is an $\eta > 0$ depending on λ and the K in (*) such that

$$\sum_{l=-\infty}^{\infty} A_{k,l}y_l \leqslant \frac{1+\lambda}{2}\cdot\frac{1}{k^2+1}$$

if $0 \leqslant y_l \leqslant \eta$ and $y_l \leqslant 1/(l^2+1)$. *Fix* such an η. By (§) we can certainly find an m such that $0 \leqslant v_k^{(m)} \leqslant \eta$ for all k. *Fix* such an m. As we have just seen, there is an M depending on m such that

$$v_k^{(m)} = \sum_l \frac{A_{k,l}^{(m)}}{l^2+1} \leqslant \frac{M}{k^2+1},$$

and we may of course suppose that $M \geqslant 1$. Apply now the computational lemma with

$$y_l = v_l^{(m)}/M;$$

we get (after multiplying by M again – this trick works because $\eta/M \leqslant \eta$!),

$$v_k^{(m+1)} = \sum_l A_{k,l} v_l^{(m)} \leqslant \frac{1+\lambda}{2} \cdot \frac{M}{l^2+1}.$$

We also have, of course,

$$0 \leqslant v_k^{(m+1)} \leqslant \lambda\eta$$

by (⁑).

We may now use the computational lemma again with

$$y_l = \frac{2}{(1+\lambda)M} v_l^{(m+1)};$$

note that *here*

$$0 \leqslant y_l \leqslant \frac{2\eta\lambda}{(1+\lambda)M} < \eta.$$

After multiplying back by $(1+\lambda)M/2$, we find

$$v_k^{(m+2)} = \sum_l A_{k,l} v_l^{(m+1)} \leqslant \left(\frac{1+\lambda}{2}\right)^2 \frac{M}{k^2+1}.$$

In this fashion, we can continue indefinitely and prove that

$$v_k^{(m+p)} \leqslant \left(\frac{1+\lambda}{2}\right)^p \cdot \frac{M}{k^2+1}$$

for $p = 1, 2, 3, \ldots$. Therefore, since $\lambda < 1$,

$$\sum_{n=m}^{\infty} v_k^{(n)} \leqslant \frac{2M}{(1-\lambda)(k^2+1)}.$$

This, however, implies that

$$\begin{aligned} u_k &\leqslant K\left\{\frac{1}{k^2+1} + \sum_{n=1}^{\infty}\sum_l \frac{A_{k,l}^{(n)}}{l^2+1}\right\} \\ &= K\left\{\frac{1}{k^2+1} + v_k^{(1)} + \cdots + v_k^{(m-1)} + \sum_{n=m}^{\infty} v_k^{(n)}\right\} \leqslant \frac{C}{k^2+1} \end{aligned}$$

with a certain constant C, since

$$v_k^{(n)} \leqslant \frac{\text{const. } L^n}{k^2+1}$$

for each n. We have proved that $\omega_0(x)$ (which is *at most* u_k on $\mathcal{O}_k$) is

$$\leqslant \frac{C}{k^2+1} \qquad \text{for } x \in \mathcal{O}_k. \qquad \text{Q.E.D}$$

Problem 15

In this problem, the set E is as described at the beginning of the present §, with the boxed condition given there.

(a) Let $U_R(z)$ be the harmonic measure (for $\mathscr{D}$) of the subset $E \cap [-R/2, R/2]$ of $\partial\mathscr{D}$, seen from $z \in \mathscr{D}$. Show that there is a number $\alpha > 0$ depending *only* on the four quantities A, B, δ and Δ associated with E, such that $U_R(z) \leqslant \frac{1}{2}$ for $|z| = R$ and $|\Im z| \leqslant \alpha R$. (Hint: First look at $U_R(z)$ for $|z| = R$ and $|\Im z| \leqslant 1$; then use Harnack.)

(b) Let $V_R(z)$ be the harmonic measure (for $\mathscr{D}$) of

$$E \cap \{(-\infty, -R/2] \cup [R/2, \infty)\},$$

seen from $z \in \mathscr{D}$. Show that there is a number $\beta > 0$ depending only on A, B, δ and Δ such that $V_R(z) \geqslant \beta$ for $|z| = R$. (Hint: Use (a) and Harnack.)

(c) For $R > 0$, call $\mathscr{D}_R = \mathscr{D} \cap \{|z| < R\}$ and let $\omega_R(z)$ be the harmonic measure of $\{|z| = R\}$ *for* $\mathscr{D}_R$, as seen from $z \in \mathscr{D}_R$. Prove *Benedicks' lemma*, which says that

$$\omega_R(0) \leqslant \frac{C}{R}$$

with a constant C depending *only* on the four quantities A, B, δ and Δ. (Hint: Compare the $V_R(z)$ of (b) with $\omega_R(z)$ in $\mathscr{D}_R$.)

2. Green's function and a Phragmén–Lindelöf function for $\mathscr{D}$

A Green's function is available for domains $\mathscr{D}$ of the kind considered here. Let us remind the reader who may not remember that, for given $w \in \mathscr{D}$, the Green's function $G(z, w)$ is a positive function of z, *harmonic in* $\mathscr{D}$ *save at* w *where it acts like*

$$\log \frac{1}{|z-w|},$$

which is *bounded* in $\mathscr{D}$ *outside of disks of positive radius centered at* w, and *tends to zero when* z *tends to any point on the boundary* E of $\mathscr{D}$. *Existence* of $G(z, w)$ for our domains $\mathscr{D}$ follows by standard general arguments, found in many books on complex variable theory. For the sake of completeness, we will show that $G(z, w)$ is *symmetric* in z and w at the end of this article. The *last part* of the argument given there may easily be adapted so as to furnish an existence proof for $G(z, w)$.

Theorem. *Let $\mathscr{D} = \mathbb{C} \sim E$, where $E \subseteq \mathbb{R}$ has the properties given at the beginning of this §. Assume that $0 \in \mathcal{O}_0$. Then there is a constant C, depending only on the four numbers A, B, δ and Δ associated with E, such that $G(x, 0) \leqslant C/(x^2 + 1)$ for $x \in \mathbb{R} \sim \mathcal{O}_0$, $G(z, w)$ being the Green's function for $\mathscr{D}$.*

Proof. Draw a circle Γ with diameter running from the *left endpoint* of E_0 to the *right endpoint* of E_1:

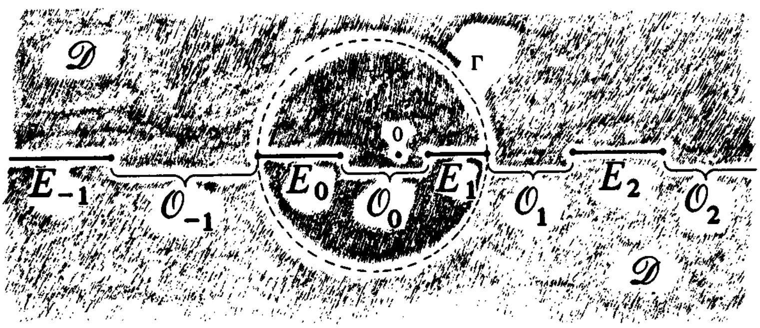

Figure 117

Let us show first of all that there is a number α, depending only on δ, Δ and B, such that

$$G(z,0) \leqslant \alpha, \quad z \in \Gamma.$$

To see this, observe first of all that $G(z,0) \leqslant g(z,0)$, the Green's function for $(\mathbb{C} \sim E_1) \cup \{\infty\}$. This follows by looking at the *difference* $g(z,0) - G(z,0)$ on E. The latter is *harmonic* and *bounded* in $\mathscr{D}$, since the logarithmic poles of $g(z,0)$ and $G(z,0)$ at 0 cancel each other out. It is thus enough to get an upper bound for $g(z,0)$ on Γ, and that bound will also serve for $G(z,0)$ there.

Translation along $\mathbb{R}$ to the midpoint of E_1 followed by scaling down, using the factor $2/|E_1|$, takes $(\mathbb{C} \sim E_1) \cup \{\infty\}$ conformally onto the standard domain $\mathscr{E} = (\mathbb{C} \sim [-1, 1]) \cup \{\infty\}$:

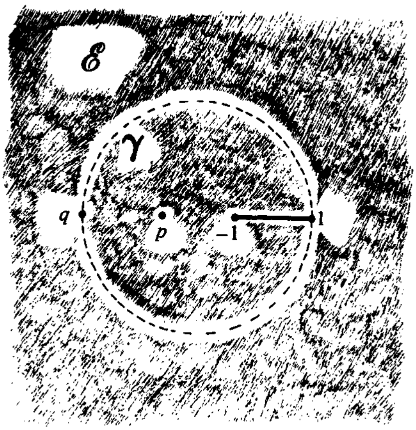

Figure 118

In this reduction, 0 goes to a point p on the real axis, $p < -1$, and the circle Γ goes to another, γ, having $[q, 1]$ as its diameter, where $q < p$. $g(z, 0)$ is of course *equal* to the Green's function for $\mathscr{E}$ with pole at p.

We have

$$|p| - 1 \;\leqslant\; \frac{|\mathscr{O}_0|}{|E_1|/2} \;\leqslant\; \frac{B}{\delta}$$

and

$$|q| - |p| \;\geqslant\; \frac{2|E_0|}{|E_1|} \;\geqslant\; \frac{2\delta}{\Delta}.$$

Therefore the Green's function for $\mathscr{E}$ with pole at p is bounded above on γ by some number α depending on B/δ and $2\delta/\Delta$. (The nature of this dependence could be worked out by mapping $\mathscr{E}$ conformally onto $\{1 < |w| \leqslant \infty\}$; we, however, do not need to know it.) This means that $g(z, 0) \leqslant \alpha$ on Γ and finally $G(z, 0) \leqslant \alpha$, $z \in \Gamma$.

This being verified, we take the centre m of Γ and, with R equal to that circle's radius, map the exterior of Γ conformally onto the domain $\mathscr{E}$ just considered by taking z to $w = \frac{1}{2}\{(z-m)/R + R/(z-m)\}$. That mapping takes Γ to the slit $E'_1 = [-1, 1]$ *and each of the components* E_n of $\partial\mathscr{D}$, $n \neq 0$, 1, *onto segments* E'_n *on the real axis.* The function $\varphi(z) = \frac{1}{2}\{(z-m)/R + R/(z-m)\}$ thus takes

$$\mathscr{D}_\Gamma \;=\; \mathscr{D} \cap \{|z - m| > R\}$$

conformally onto a domain

$$\mathscr{D}' \;=\; \mathbb{C} \sim \bigcup_{-\infty}^{\infty}{}' E'_n:$$

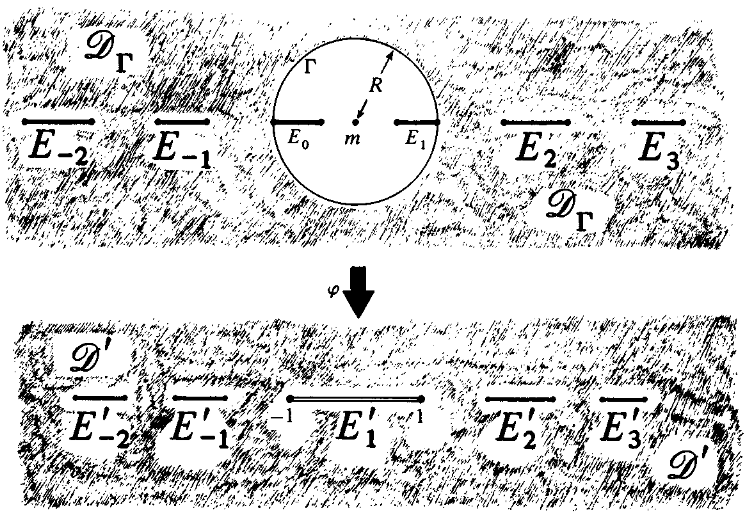

Figure 119

Define a harmonic function $\Omega(w)$ for $w \in \mathscr{D}'$ by putting

$$\Omega(\varphi(z)) \;=\; G(z,0)$$

for $z \in \mathscr{D}_\Gamma$. $\Omega(w)$ is evidently bounded in $\mathscr{D}'$, and has boundary value *zero* on *each* of the components $E'_n (= \varphi(E_n))$ of $\partial\mathscr{D}'$, *save on* E'_1. $\Omega(w)$ *is*, however, continuous up to the latter one, and *on* it

$$\Omega(w) \;\leqslant\; \alpha,$$

since $E'_1 = \varphi(\Gamma)$ and $G(z,0) \leqslant \alpha$ on Γ. We therefore have $\Omega(w) \leqslant \alpha\omega_{\mathscr{D}'}(E'_1, w)$ in $\mathscr{D}'$, where $\omega_{\mathscr{D}'}(\;, w)$ denotes *harmonic measure* for $\mathscr{D}'$.

The set

$$E' \;=\; \bigcup_{-\infty}^{\infty}{}' E'_n$$

has, however, the properties specified for our sets E at the beginning of this §. Indeed, for real x,

$$\varphi(x) \;=\; \frac{1}{2R}x - \frac{m}{2R} + O\left(\frac{1}{x}\right)$$

when $|x|$ is large, with R lying between the two numbers 2δ and $B/2 + 2\Delta$. Hence, each of the intervals $E'_n (n \neq 0)$ is of the form $[a'_n - \delta'_n,\; a'_n + \delta'_n]$, where, for certain numbers A', B', γ' and Δ' depending on the original A, B, δ and Δ,

$$0 < A' < a'_{n+1} - a'_n < B'$$

and

$$0 < \delta' < \delta'_n < \Delta'.$$

(Again, the exact *form* of the dependence does not concern us here.) *We can therefore apply Carleson's theorem from the previous article to the domain $\mathscr{D}'$*, and find that

$$\omega_{\mathscr{D}'}(E'_1, u) \leqslant \frac{C'}{1+u^2}, \qquad u \in \mathbb{R},$$

with a constant C' depending on A', B', δ' and Δ' and hence, finally, on A, B, δ and Δ. Thence, in view of the previous relation,

$$\Omega(u) \leqslant \frac{\alpha C'}{1+u^2} \quad \text{for} \quad u \in \mathbb{R},$$

i.e.

$$G(x, 0) \leqslant \frac{\alpha C'}{1 + (\varphi(x))^2}$$

for real x lying *outside* the circle Γ. Using the fact that $0 \in \mathcal{O}_0$ (whence $|m| \leqslant B + 2\Delta$), the bounds on R given above, and the asymptotic formula for $\varphi(x)$, we see that the right side of the preceding inequality is in turn

$$\leqslant \frac{C}{1+x^2}$$

with a constant C depending only on A, B, δ and Δ. Thus

$$G(x, 0) \leqslant \frac{C}{1+x^2}$$

for real x outside of E_0, $\mathcal{O}_0$ and E_1. But this also holds on E_0 and E_1 since $G(x, 0) = 0$ on those sets! So it holds for real x outside of $\mathcal{O}_0$, which is what we had to prove. We are done.

Problem 16

Let $E \subseteq \mathbb{R}$ fulfill the conditions set forth at the beginning of this §, and assume that $0 \in \mathcal{O}_0$. Let $\omega_{\mathscr{D}}(\ , z)$ be *harmonic measure* for $\mathscr{D} = \mathbb{C} \sim E$. *Prove Benedicks' theorem*, which says that there is a constant C depending only on the four numbers A, B, δ and Δ associated with E, such that, for t in any component

$$E_n = [a_n - \delta_n,\ a_n + \delta_n]$$

of E, and $n \neq 0, 1$,

$$d\omega_{\mathscr{D}}(t,0) \;\leqslant\; \frac{C}{1+t^2}\cdot\frac{dt}{\sqrt{(\delta_n^2-(t-a_n)^2)}}.$$

(This is a most beautiful result, by the way!) (Hint: Let G be the Green's function for $\mathscr{D}$. According to a classical elementary formula, if, for instance, we consider points t_+ lying on the *upper edge* of E_n, we have

$$\frac{d\omega_{\mathscr{D}}(t_+,0)}{dt} \;=\; \frac{1}{2\pi}G_y(t_+,0) \;=\; \lim_{\Delta y\to 0+}\frac{G(t+i\Delta y,0)}{2\pi\Delta y},$$

since $G(t,0)=0$. (Green *introduced* the functions bearing his name for this very reason!) Take the ellipse Γ given by the equation

$$\frac{(x-a_n)^2}{2\delta_n^2}+\frac{y^2}{\delta_n^2} \;=\; 1$$

and compare $G(z,0)$ with

$$U(z) \;=\; \log\left|\frac{z-a_n}{\delta_n}+\sqrt{\left(\left(\frac{z-a_n}{\delta_n}\right)^2-1\right)}\right|$$

on Γ. Note that $G(x,0)$ and $U(x)$ *both vanish* on E_n, that $U(z)$ is *harmonic* in the region $\mathscr{E}$ between E_n and Γ, and that $G(z,0)$ is at least *subharmonic* there (not *necessarily* harmonic because some of the E_k with $k\neq n$ may intrude into $\mathscr{E}$).)

The work in Chapter VI frequently involved entire functions of exponential type bounded on the real axis. If $f(z)$ is such a function, of exponential type A say, and we know that

$$|f(x)| \;\leqslant\; 1, \quad x\in\mathbb{R},$$

we can deduce that

$$|f(z)| \;\leqslant\; e^{A|\Im z|}$$

for all z. This follows by the *third* Phragmén–Lindelöf theorem of Chapter III, §C, whose *proof* depends on the availability of the function $|\Im z|$, *harmonic and* $\geqslant 0$ *in each of the half planes* $\{\Im z>0\}$, $\{\Im z<0\}$, *and zero on the real axis.*

Suppose now that we are presented with such a function $f(z)$, known to be *bounded* (with, however, an *unknown bound*) on $\mathbb{R}$, such that, for some closed $E\subseteq\mathbb{R}$,

$$|f(x)|\leqslant 1, \quad x\in E.$$

If there is a function $Y(z)$, *harmonic in* $\mathscr{D}=\mathbb{C}\sim E$, *having boundary value zero on E and such that* $Y(z)\geqslant|\Im z|$, $z\in\mathscr{D}$, we can argue as in the proof of the

Phragmén–Lindelöf theorem just mentioned, and conclude that

$$|f(z)| \leqslant e^{AY(z)}$$

for $z \in \mathscr{D}$ if $f(z)$ is of exponential type A.* We are therefore interested in the *existence* of such functions $Y(z)$ for given closed sets $E \subseteq \mathbb{R}$.

In order to avoid situations involving irregular boundary points for the Dirichlet problem, whose investigation has nothing to do with the material of the present book, we *limit* the following discussion to closed sets E which can be expressed as *disjoint unions of segments on* $\mathbb{R}$ *not accumulating at any finite point*. We do *not*, however, assume in that discussion that the sets E have the form specified at the beginning of this §.

Definition. A Phragmén–Lindelöf function $Y(z)$ for $\mathscr{D} = \mathbb{C} \sim E$ is one *harmonic* in $\mathscr{D}$ and *continuous* up to E, such that

(i) $Y(x) = 0, \quad x \in E$

(ii) $Y(z) \geqslant |\Im z|, \quad z \in \mathscr{D}$,

(iii) $Y(iy) = |y| + o(|y|)$ for $y \to \pm\infty$.

It turns out that for given closed $E \subseteq \mathbb{R}$ of the form just described, the existence of $Y(z)$ is governed by the *behaviour of the Green's function* $G(z, w)$ for $\mathscr{D} = \mathbb{C} \sim E$. Before going into this matter, let us mention a simple example (not without its own usefulness) which the reader should keep in mind.

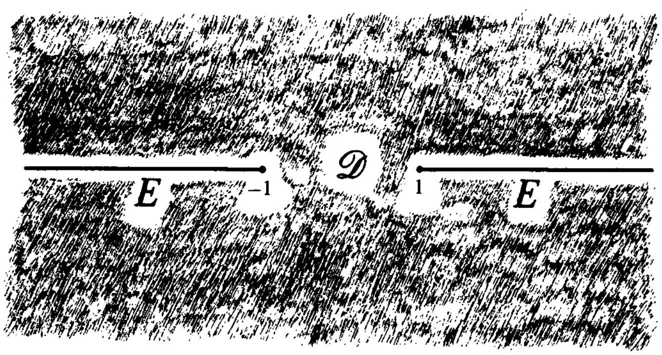

Figure 120

* In fact, *boundedness of* f on $\mathbb{R}$ is *not necessary* here. If $|f(x)| \leqslant 1$, $x \in E$, and $|f(z)| \leqslant C\exp(A|z|)$, the function $v(z) = \log|f(z)| - (A \sec\delta)\, Y(z)$ is subharmonic in $\mathscr{D}$ and bounded above on each of the lines $x = \pm y \tan\delta$ – here, $0 < \delta < \pi/2$. Since $v(z) \leqslant \text{const.}|z|$ in $\mathscr{D}$, the *second* Phragmén–Lindelöf theorem of Chapter III §C shows that v is bounded above in the vertical sectors $|x| < \pm y \tan\delta$. Because $v(x) \leqslant 0$ on E, the *proof* of that same theorem can be adapted without change to show that v is *also* bounded above in $\mathscr{D} \cap \{x > |y| \tan\delta\}$ and $\mathscr{D} \cap \{x < -|y| \tan\delta\}$, even though the latter domains are not full sectors. Therefore v is bounded above in $\mathscr{D}$, so by the *first* theorem of §C, Chapter III, $v(z) \leqslant 0$ in $\mathscr{D}$. Hence $|f(z)| \leqslant \exp(A \sec\delta \cdot Y(z))$, $z \in \mathscr{D}$, and, making $\delta \to 0$, we get $|f(z)| \leqslant \exp(A\,Y(z))$.

Here, $E = (-\infty, -1] \cup [1, \infty)$. In $\mathscr{D} = \mathbb{C} \sim E$, we can put

$$Y(z) = \Im(\sqrt{(z^2 - 1)}),$$

using the branch of the square root which is *positive imaginary* for $z \in (-1, 1)$. It is easy to check that this $Y(z)$ is a Phragmén–Lindelöf function for $\mathscr{D}$.

The Green's function $G(z, w)$ for one of our domains $\mathscr{D}$ enjoys a *symmetry property*:

$$G(z, w) = G(w, z), \qquad z, w \in \mathscr{D}.$$

The reader who does not remember how this is proved may find a proof, general enough to cover our situation, at the end of this article. It is convenient to define $G(z, w)$ for all z and w in $\bar{\mathscr{D}}$ (which *here* is just $\mathbb{C}$!) by taking $G(z, w) = 0$ if *either* z or w belongs to $\partial\mathscr{D}$. Then we have

$$G(z, w) = G(w, z) \qquad \text{for} \quad z, w \in \bar{\mathscr{D}}.$$

(N.B. $G(z, w)$ as thus defined is not *quite* continuous from $\bar{\mathscr{D}} \times \bar{\mathscr{D}}$ to $[0, \infty]$ (*sic*!). We can take sequences $\{z_n\}$ and $\{w_n\}$ of points in $\mathscr{D}$, both tending to limits on $E = \partial\mathscr{D}$, but with $|z_n - w_n| \underset{n}{\longrightarrow} 0$ sufficiently rapidly to make $G(z_n, w_n) \underset{n}{\longrightarrow} \infty$.)

The connection between $G(z, w)$ and $Y(z)$ (when the latter exists) can be made to depend on the elementary formula

$$\lim_{R \to \infty} \int_{-R}^{R} \log\left|1 - \frac{z}{t}\right| dt = \pi |\Im z|,$$

which may be derived by contour integration. The reader is invited (nay, urged!) to do the computation. Here is the result.

Theorem. *A Phragmén–Lindelöf function $Y(z)$ exists for $\mathscr{D}$, a domain of the kind considered here, iff*

$$\int_{-\infty}^{\infty} G(z, t)\, dt < \infty$$

for some *$z \in \mathscr{D}$, G being the Green's function for that domain. If the integral just written converges for* any *such z, it converges for* all *$z \in \mathscr{D}$, and then*

$$Y(z) = |\Im z| + \frac{1}{\pi} \int_{-\infty}^{\infty} G(z, t)\, dt.$$

Remark. In his 1980 *Arkiv* paper, Benedicks has versions of this result for $\mathbb{R}^{n+1}$

Proof of theorem. The idea is very simple, and is expressed by the identity

$$
\begin{aligned}
|\Im z| &+ \frac{1}{\pi}\int_{-\infty}^{\infty} G(z,t)\,\mathrm{d}t \\
&= \frac{1}{\pi}\int_{-A}^{A}\left(\log\frac{1}{|t|} + \log|z-t| + G(z,t)\right)\mathrm{d}t \\
&\quad + \frac{1}{\pi}\int_{A}^{\infty}\log\left|1-\frac{z^2}{t^2}\right|\mathrm{d}t \;+\; \frac{1}{\pi}\int_{|t|\geqslant A} G(z,t)\,\mathrm{d}t,
\end{aligned}
$$

an obvious consequence of the formula just mentioned. Here, $A > 0$ is arbitrary.

Suppose, indeed, that the left-hand integral is convergent. That integral then equals a positive harmonic function in each of the half planes $\{\Im z > 0\}$, $\{\Im z < 0\}$, and we can use Harnack to show that it is $o(|y|)$ for $z = iy$ and $y \to \pm\infty$. Denoting the left side of our identity by $Y(z)$, we thus see that $Y(z)$ is harmonic in the upper and lower half planes and has property (iii). It is clear that $Y(z)$ has the properties (i) and (ii). Only the harmonicity of $Y(z)$ at points of $\mathscr{D}\cap\mathbb{R}$ remains to be verified. This, however, can be checked in the neighborhood of any such point by taking $A > 0$ sufficiently large and looking at the *right side* of our identity. The *first* right-hand term will be harmonic in $\mathscr{D}\cap(-A,A)$, because, for each t therein, the logarithmic pole of $G(z,t)$ at t is cancelled by the term $\log|z-t|$. The *second* term on the right is *clearly* harmonic for $|z| < A$, and the *third* harmonic in $\mathscr{D}\cap(-A,A)$.

This explanation will probably satisfy the experienced analyst. The general mathematical reader may, however, well desire more justification, based if possible on general principles, so that he or she may avoid having to search through specialized books on potential theory. We proceed to furnish this justification. Its details make the following development somewhat long.

Let us begin with a preliminary remark. Suppose we have any open subset $\mathcal{O}$ of $\mathscr{D}$, and a compact $F \subseteq \bar{\mathscr{D}}$ *disjoint* from $\mathcal{O}$. By the symmetry of G, $G(z,w) = G(w,z)$ is, for each *fixed* $z\in\mathcal{O}$, *continuous* (as a function of w) on F, so, if μ is any finite positive measure on F, the integral

$$\int_F G(z,w)\,\mathrm{d}\mu(w)$$

is obtainable as a limit of Riemann sums in the usual way. As a *function* of z, *any one* of those sums is *positive and harmonic in* $\mathcal{O}$. So the *integral*, being a *pointwise limit* of such functions (of z), *is itself* a *positive and harmonic function of* z in $\mathcal{O}$. We will make repeated use of this observation.

Suppose, now, that $\int_{-\infty}^{\infty} G(z,t)\mathrm{d}t < \infty$ for some *non-real* z, say wlog that

$$\int_{-\infty}^{\infty} G(\mathrm{i},t)\,\mathrm{d}t \;<\; \infty.$$

For each N, the function

$$H_N(z) \;=\; \int_{-N}^{N} G(z,t)\mathrm{d}t$$

is positive and harmonic in both $\{\Im z > 0\}$ and $\{\Im z < 0\}$ according to the remark just made. Therefore, since $G(z,t) \geqslant 0$, $H_{N+1}(z) \geqslant H_N(z)$, and

$$H(z) \;=\; \lim_{N\to\infty} H_N(z)$$

is either *harmonic* (and finite!) in $\{\Im z > 0\}$ or else *everywhere infinite* there. Because $H(\mathrm{i}) < \infty$, the first alternative holds, and $H(z)$ is then *also* finite (and harmonic) in $\Im z < 0$, since obviously

$$G(z,t) \;=\; G(\bar{z},t)$$

for real t, $E \;=\; \partial\mathscr{D}$ being on $\mathbb{R}$.

Consider now some real $x_0 \notin E$. Take $A > \max(|x_0|, 1)$. The integrals $\int_{-A}^{A} G(x_0,t)\mathrm{d}t$ and $\int_{-A}^{A} G(\mathrm{i},t)\mathrm{d}t$ are *both finite*, so we can show that $\int_{-\infty}^{\infty} G(x_0,t)\mathrm{d}t$ and $\int_{-\infty}^{\infty} G(\mathrm{i},t)\mathrm{d}t$ are either both finite or else both infinite by comparing

$$\int_{|t|\geqslant A} G(x_0,t)\,\mathrm{d}t$$

and

$$\int_{|t|\geqslant A} G(\mathrm{i},t)\,\mathrm{d}t.$$

In $\mathscr{D}_A = \mathscr{D} \cap \{|z| < A\}$, the function $\int_{|t|\geqslant A} G(z,t)\mathrm{d}t$ is the limit of the increasing sequence of functions

$$\int_{A\leqslant|t|\leqslant N} G(z,t)\mathrm{d}t,$$

each of which is *positive and harmonic in* $\mathscr{D}_A$. So $\int_{|t|\geqslant A} G(z,t)\mathrm{d}t$ is either *harmonic* (and finite) in $\mathscr{D}_A$, or else everywhere infinite there. It is thus *finite* for $z = \mathrm{i}$ *if and only if* it is finite for $z = x_0$. We see that $\int_{-\infty}^{\infty} G(x_0,t)\mathrm{d}t < \infty$ iff $\int_{-\infty}^{\infty} G(\mathrm{i},t)\mathrm{d}t < \infty$, and, if this inequality holds,

$$H(z) \;=\; \int_{-\infty}^{\infty} G(x,t)\,\mathrm{d}t$$

is finite for every $z \in \mathscr{D}$.

If $H(z)$ *is* finite, let us show that

$$H(\mathrm{i}y) = \mathrm{o}(|y|) \quad \text{for} \quad y \to \pm\infty.$$

Pick any large N, and write

$$(*) \qquad H(\mathrm{i}y) \;=\; \int_{|t|\leqslant N} G(\mathrm{i}y,t)\,\mathrm{d}t \;+\; \int_{|t|>N} G(\mathrm{i}y,t)\,\mathrm{d}t.$$

Since $H(\mathrm{i})<\infty$ we can, given any $\varepsilon>0$, take N so large that $\int_{|t|\geqslant N} G(\mathrm{i},t)\,\mathrm{d}t<\varepsilon$. The function $\int_{|t|\geqslant N} G(z,t)\mathrm{d}t$ is, by the previous discussion, *positive and harmonic* in $\{\Im z>0\}$. Therefore, by Harnack's theorem,*

$$\int_{|t|\geqslant N} G(\mathrm{i}y,t)\mathrm{d}t \;\leqslant\; y\int_{|t|\geqslant N} G(\mathrm{i},t)\mathrm{d}t \;<\; \varepsilon y$$

for $y>1$. This takes care of the *second* term on the right in $(*)$.

The *first* term from the right side of $(*)$ remains; our claim is that it is *bounded*. This (and more) follows from a simple estimate which will be used several times in the proof.

Take any component E_0 of $E=\partial\mathscr{D}$, and put $\mathscr{D}_0=(\mathbb{C}\sim E_0)\cup\{\infty\}$. If E_0 is of *infinite length*, replace it *by any segment of length* 2 *thereon* in this last expression. We have $\mathscr{D}\subseteq\mathscr{D}_0$, so, if $g(z,w)$ is the Green's function for $\mathscr{D}_0$,

$$G(z,w) \;\leqslant\; g(z,w), \qquad z,w\in\mathscr{D}$$

(cf. beginning of the proof of *first* theorem in this article). We compute $g(z,w)$ by first mapping $\mathscr{D}_0$ conformally onto the unit disk $\{|\zeta|<1\}$, thinking of ζ as a new *coordinate variable* for $\mathscr{D}_0$:

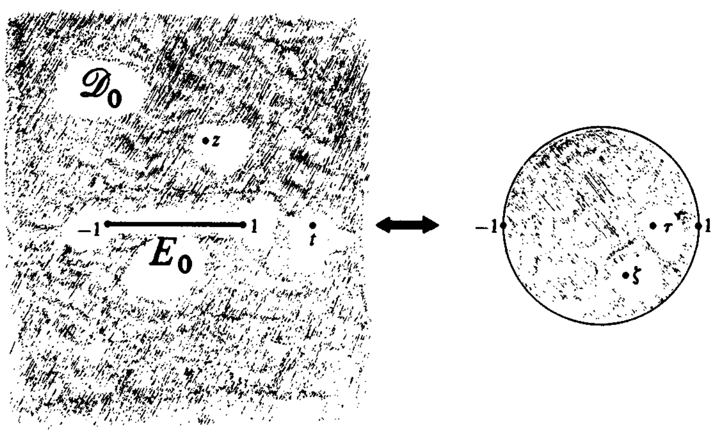

Figure 121

Say, for instance, that E_0 is $[-1,1]$ so that we can use $z=\frac{1}{2}(\zeta+1/\zeta)$. Then, if $t\in\mathscr{D}_0$ is

* Actually, by the Poisson representation for $\{\Im z>0\}$ of functions positive and harmonic there. Using the ordinary form of Harnack's inequality gives us a factor of $2y$ instead of y on the right. That, of course, makes no difference in this discussion.

real, we can put $t = \frac{1}{2}(\tau + 1/\tau)$, where $-1 < \tau < 1$, and, in terms of ζ and τ,

$$g(z, t) = \log\left|\frac{1 - \tau\zeta}{\zeta - \tau}\right|,$$

the expression on the right being simply the Green's function for the unit disk.

If $N > 1$, we have

$$\int_{|t| \leqslant N} G(z, t)\mathrm{d}t \leqslant \int_{1 \leqslant |t| \leqslant N} g(z, t)\mathrm{d}t,$$

since $G(z, t)$ and $g(z, t)$ *vanish* for $t \in E_0 = [-1, 1]$. For $1 \leqslant |t| \leqslant N$, the parameter τ satisfies $C_N \leqslant |\tau| \leqslant 1$, $C_N > 0$ being a number depending on N which we need not calculate. Also, for such t,

$$\mathrm{d}t = -\frac{1}{2}\left(\frac{1 - \tau^2}{\tau^2}\right)\mathrm{d}\tau.$$

Therefore,

$$(\dagger) \qquad \int_{|t| \leqslant N} G(z, t)\mathrm{d}t \leqslant \frac{1}{2}\int_{C_N \leqslant |\tau| \leqslant 1} \log\left|\frac{1 - \tau\zeta}{\zeta - \tau}\right|\left(\frac{1 - \tau^2}{\tau^2}\right)\mathrm{d}\tau.$$

Since $C_N > 0$, the right side is clearly bounded for $|\zeta| < 1$; we see already that the *first right-hand term of* $(*)$ *is bounded*, verifying our claim.

As we have already shown, the *second* term on the right in $(*)$ will be $\leqslant \varepsilon y$ for $y \geqslant 1$ if N is large enough. Combining this result with the preceding, we have, from $(*)$,

$$H(\mathrm{i}y) \leqslant \mathrm{O}(1) + \varepsilon y, \quad y > 1,$$

so, since $\varepsilon > 0$ is arbitrary, $H(\mathrm{i}y) = \mathrm{o}(|y|)$, $y \to \infty$. Because $H(\bar{z}) = H(z)$, the same holds good for $y \to -\infty$.

Having established this fact, let us return for a moment to $(\dagger)$. For each τ, $C_N \leqslant |\tau| < 1$,

$$\log\left|\frac{1 - \tau\zeta}{\zeta - \tau}\right| \longrightarrow 0 \quad \text{as} \quad |\zeta| \longrightarrow 1.$$

Starting from this relation, one can, by a straightforward argument, check that

$$\int_{C_N \leqslant |\tau| \leqslant 1} \log\left|\frac{1 - \tau\zeta}{\zeta - \tau}\right|\left(\frac{1 - \tau^2}{\tau^2}\right)\mathrm{d}\tau \longrightarrow 0$$

as $|\zeta| \to 1$. (One may, for instance, break up the integral into two pieces.)

Problem 17 (a)

Carry out this verification

This means, by $(\dagger)$, that

$$\int_{|t| \leqslant N} G(z, t)\,\mathrm{d}t \longrightarrow 0$$

when $z \in \mathscr{D}$ *tends to any point of* E_0. We could, however, have taken E_0 to be *any of the components of* E with finite length, or any segment of length 2 on one of the unbounded ones (if there are any); that would not have essentially changed the above argument.* *Hence*

$$\int_{|t| \leqslant N} G(z,t)\,\mathrm{d}t$$

tends to zero whenever z *tends to any point on* $E = \partial\mathscr{D}$ (besides being *bounded* in $\mathscr{D}$).

We can now prove that

$$H(z) \;=\; \int_{-\infty}^{\infty} G(z,t)\,\mathrm{d}t \;\longrightarrow\; 0$$

whenever z tends to any point x_0 of E. *Given* such an x_0, take a circle γ about x_0 so small that *precisely one* of the components of E (the one containing x_0) cuts γ, passing into its inside:

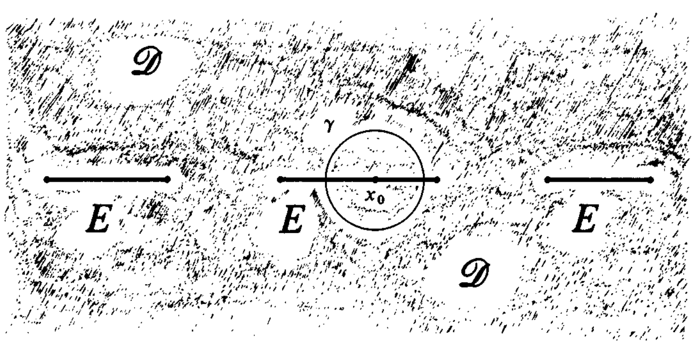

Figure 122

If x_0 is an endpoint of one of the components of E, our picture looks like this:

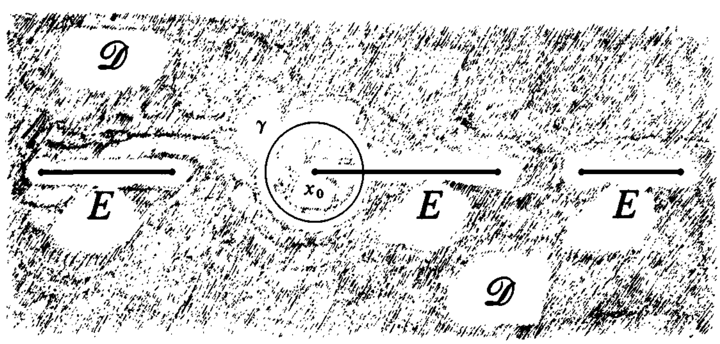

Figure 123

Call $\mathscr{D}_\gamma$ the part of $\mathscr{D}$ lying inside γ, and E_γ the part of E therein.

* As long as $E_0 \subseteq \{|t| < N\}$. If this is not so, we can *increase* N until the argument in the text applies. Since that only makes the integral in question *larger*, the one corresponding to the original value of N must (*a fortiori*!) have the asserted

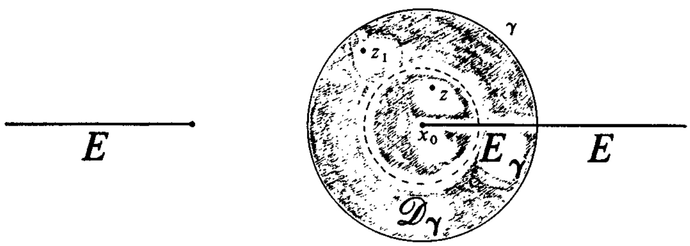

Figure 124

Fix any $z_1 \in \mathscr{D}_\gamma$. Then, there is a constant K, depending on z_1, such that, for any function $V(z)$, *positive* and *harmonic* in $\mathscr{D}_\gamma$ and continuous on its closure, *with* $V(x) = 0$ *on* E_γ, we have $V(z) \leqslant KV(z_1)$ for $|z - x_0| < \frac{1}{2}$ radius of γ.

Problem 17 (b)
Prove the statement just made.

This being granted, choose N so large that

$$\int_{|t| \geqslant N} G(z_1, t)\,\mathrm{d}t \;<\; \varepsilon/K,$$

ε being any number > 0. For each $M > N$, the function

$$V_M(z) \;=\; \int_{N \leqslant |t| \leqslant M} G(z, t)\,\mathrm{d}t$$

is positive and harmonic in $\mathscr{D}_\gamma$, and certainly continuous up to $\gamma \cap \mathscr{D}$. Also, $V_M(z) \;\leqslant\; \int_{|t| \leqslant M} G(z, t)$ which, *by the previous discussion*, tends to *zero* whenever z tends to any point of E. $V_M(z)$ is therefore continuous up to E_γ, *where it equals zero*. By the above statement, we thus have

$$V_M(z) \;\leqslant\; KV_M(z_1) \;\leqslant\; K\int_{|t| \geqslant N} G(z_1, t)\mathrm{d}t \;<\; \varepsilon$$

for $|z - x_0| < \frac{1}{2}$ radius of γ. This holds for all $M > N$, so making $M \to \infty$, we get $\int_{|t| \geqslant N} G(z, t)\mathrm{d}t \leqslant \varepsilon$ for $|z - x_0| < \frac{1}{2}$ radius of γ. Hence, since

$$H(z) \;=\; \int_{|t| \leqslant N} G(z, t)\mathrm{d}t \;+\; \int_{|t| \geqslant N} G(z, t)\mathrm{d}t,$$

and, as we already know, the first integral on the right tends to zero when $z \to x_0$, we must have $H(z) < 2\varepsilon$ for z close enough to x_0. This shows that $H(z) \to 0$ whenever z tends to any point of E.

We now see by the preceding arguments *that*

$$Y(z) \;=\; |\Im z| \;+\; \frac{1}{\pi}H(z)$$

enjoys the properties (i), (ii) *and* (iii) *required of Phragmén–Lindelöf functions, and is also harmonic in both the lower and upper half planes, and continuous everywhere.* Therefore, to complete the proof of the fact that $Y(z)$ *is* a Phragmén–Lindelöf function for $\mathscr{D}$, *we need only verify that it is harmonic at the points of* $\mathscr{D} \cap \mathbb{R}$.

For this purpose, we bring in the formula

$$|\Im z| \;=\; \lim_{A\to\infty} \frac{1}{\pi}\int_{-A}^{A} \log\left|1 \;-\; \frac{z}{t}\right|dt$$

mentioned earlier. From it, and the definition of $H(z)$, we get

$$(*) \qquad Y(z) \;=\; |\Im z| \;+\; \frac{1}{\pi}\int_{-\infty}^{\infty} G(z,t)\,dt$$

$$= \frac{1}{\pi}\int_{-A}^{A}\left(\log\frac{1}{|t|} + \log|z-t| + G(z,t)\right)dt$$

$$+\; \frac{1}{\pi}\int_{A}^{\infty}\log\left|1-\frac{z^2}{t^2}\right|dt \;+\; \frac{1}{\pi}\int_{|t|\geqslant A} G(z,t)\,dt.$$

The number $A > 0$ may be chosen at pleasure.

Let $x_0 \in \mathscr{D} \cap \mathbb{R}$; pick A larger than $|x_0|$. The function $\int_A^{\infty}\log|1-z^2/t^2|\,dt$ is certainly harmonic near x_0; we have also seen previously that $\int_{|t|\geqslant A} G(z,t)\,dt$ is harmonic in $\mathscr{D} \cap \{|z| < A\}$, so, in particular, at x_0. Again, $\int_{-A}^{A}\log(1/|t|)\,dt$ is *finite*. Our task thus boils down to showing harmonicity of

$$\int_{-A}^{A}(\log|z-t| + G(z,t))\,dt$$

at x_0.

Take a $\delta > 0$ such that $(x_0 - 5\delta,\; x_0 + 5\delta) \subseteq \mathscr{D}$. According to observations already made,

$$\int_{\substack{|t-x_0|>\delta \\ |t|<A}} G(z,t)\,dt$$

is harmonic for $|z - x_0| < \delta$; so is (clearly)

$$\int_{\substack{|t-x_0|>\delta \\ |t|<A}} \log|z-t|\,dt.$$

We therefore need only check the harmonicity of

$$\int_{x_0-\delta}^{x_0+\delta}(\log|z-t| + G(z,t))\,dt.$$

Here, we must use the symmetry of $G(z, w)$. In order not to get bogged down in notation, *let us assume that* $x_0 = \alpha > 0$ *and that the segment* $[-2\alpha,\ 6\alpha]$ *lies entirely in* $\mathscr{D}$. The general situation can always be *reduced* to this one by *suitable translation*. It will be enough to show that

$$\int_0^{2\alpha} (\log|z - t| + G(z, t))\,dt$$

is harmonic for $|z - \alpha| < \alpha$.

For each *fixed* z,

$$\log|z - w| + G(z, w) \;=\; \log|w - z| + G(w, z)$$

is a certain *harmonic function*, $h_z(w)$, of $w \in \mathscr{D}$; this is where the symmetry of G comes in. ($h_z(w)$ is harmonic in w *even at the point* z, for addition of the term $\log|w - z|$ *removes* the logarithmic singularity of $G(w, z)$ there.) Hence, if $\rho < \operatorname{dist}(w, E)$,

$$h_z(w) \;=\; \frac{1}{2\pi}\int_0^{2\pi} h_z(w + \rho e^{i\vartheta})\,d\vartheta.$$

This relation makes a *trick* available. In it, put $w = t$ where $0 < t < 2\alpha$, and use $\rho = t + 2\alpha$.

We get

$$h_z(t) \;=\; \frac{1}{2\pi}\int_0^{2\pi} h_z(t + (t + 2\alpha)e^{i\vartheta})\,d\vartheta,$$

whence,

$$\int_0^{2\alpha} (\log|z - t| + G(z, t))\,dt \;=\; \frac{1}{2\pi}\int_0^{2\alpha}\int_0^{2\pi} h_z(t + (t + 2\alpha)e^{i\vartheta})\,d\vartheta\,dt.$$

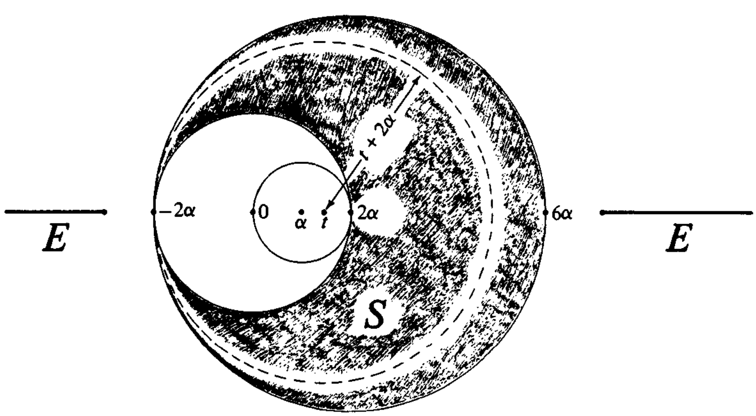

Figure 125

The double integral on the right can be expressed as one over the region

$$S = \{|\zeta - 2\alpha| < 4\alpha\} \cap \{|\zeta| > 2\alpha\}$$

shown in the above picture. Indeed, the mapping

$$(t, \vartheta) \longrightarrow (\xi, \eta)$$

given by $\xi + i\eta = \zeta = t + (t + 2\alpha)e^{i\vartheta}$ takes $\{0 < t < 2\alpha\} \times \{0 < \vartheta < 2\pi\}$ in one–one fashion onto S, and the Jacobian

$$\frac{\partial(\xi, \eta)}{\partial(t, \vartheta)}$$

works out to $(t + 2\alpha)(1 + \cos\vartheta) = \xi + 2\alpha$.
Hence,

$$\frac{1}{2\pi}\int_0^{2\alpha}\int_0^{2\pi} h_z(t + (t + 2\alpha)e^{i\vartheta})\,d\vartheta\,dt \;=\; \frac{1}{2\pi}\iint_S h_z(\zeta)\frac{d\xi\,d\eta}{\xi + 2\alpha},$$

so, by the previous relation,

$$\int_0^{2\alpha}(\log|z - t| + G(z, t))\,dt$$

$$= \frac{1}{2\pi}\iint_S \frac{\log|z - \zeta|}{\xi + 2\alpha}\,d\xi\,d\eta \;+\; \frac{1}{2\pi}\iint_S \frac{G(z, \zeta)}{\xi + 2\alpha}\,d\xi\,d\eta.$$

Here, we have

$$\iint_S \frac{d\xi\,d\eta}{\xi + 2\alpha} \;=\; \int_0^{2\alpha}\int_0^{2\pi} d\vartheta\,dt \;<\; \infty,$$

so both of the above double integrals must equal *harmonic functions of* z in the *disk* $\{|z - \alpha| < \alpha\}$, *disjoint from* $\bar{S}$. (This follows for the *second* of those integrals by the remark at the very beginning of this proof.) We see that the left-hand expression is *harmonic in* z *for* z *near* $x_0 = \alpha$. According, then, to (⁑) and the observations immediately following, *the same is true for* $Y(z)$.

We have finished proving that $Y(z)$ *is a Phragmén–Lindelöf function for* $\mathscr{D}$ *if the integral* $\int_{-\infty}^{\infty} G(z, t)\,dt$ *is finite for any* z *therein.* The *second half* of our theorem thus remains to be established. That is easier.

In the second half, *we assume that* $\mathscr{D}$ *has a Phragmén–Lindelöf function* $Y(z)$, and set out to show that

$$\int_{-\infty}^{\infty} G(z, t)\,dt \;<\; \infty$$

for *each* $z \in \mathscr{D}$, G being that domain's Green's function.

Given any $A > 0$, consider the expression

$$Y_A(z) = |\Im z| + \frac{1}{\pi}\int_{-A}^{A} G(z,t)\,\mathrm{d}t$$

$$= \frac{1}{\pi}\int_{-A}^{A}\left(\log\frac{1}{|t|} + \log|z-t| + G(z,t)\right)\mathrm{d}t$$

$$+ \frac{1}{\pi}\int_{A}^{\infty}\log\left|1-\frac{z^2}{t^2}\right|\mathrm{d}t.$$

From the preceding arguments, we know that the *first integral on the right* is *harmonic* for $z\in\mathscr{D}$ – proof of this fact *did not depend on the convergence of*

$$\int_{-\infty}^{\infty} G(z,t)\,\mathrm{d}t.$$

What we have already done also tells us that $Y_A(z)$ *tends to zero when* z *tends to any point of* E (again, *whether* $\int_{-\infty}^{\infty} G(z,t)\mathrm{d}t$ *converges or not*) and that, for any fixed A,

$$\int_{-A}^{A} G(z,t)\,\mathrm{d}t$$

is bounded in the complex plane. The expression

$$\int_{A}^{\infty}\log\left|1-\frac{z^2}{t^2}\right|\mathrm{d}t$$

is evidently *subharmonic* in the complex plane.

The function $Y_A(z)$ given by the above formula is thus *subharmonic*, and *zero on* E, and moreover,

$$Y_A(z) = |\Im z| + O(1), \quad z\in\mathscr{D}.$$

Our Phragmén–Lindelöf function $Y(z)$ (presumed to exist!) is, *however, harmonic and* $\geqslant |\Im z|$ *in* $\mathscr{D}$, *and zero on* E. The difference $Y_A(z) - Y(z)$ is therefore *subharmonic and bounded above in* $\mathscr{D}$, *and zero on* E. We can conclude by the *extended maximum principle* (subharmonic version of *first* theorem in § C, Chapter III) that $Y_A(z) - Y(z) \leqslant 0$ for $z\in\mathscr{D}$. In other words,

$$|\Im z| + \frac{1}{\pi}\int_{-A}^{A} G(z,t)\,\mathrm{d}t \leqslant Y(z).$$

Fixing $z\in\mathscr{D}$ and then making $A\to\infty$, we see that

$$\frac{1}{\pi}\int_{-\infty}^{\infty} G(z,t)\,\mathrm{d}t \leqslant Y(z) - |\Im z| < \infty.$$

This is what we wanted. The *second half of the theorem is proved.*

We are done.

We apply the result just proved to domains $\mathscr{D}$ of the special form described at the beginning of the present §, using the *first* theorem of this article. In that way we obtain the important

Theorem (Benedicks). *If E is a union of segments on $\mathbb{R}$ fulfilling the conditions given at the beginning of this § (involving the four constants A, B, δ and Δ),* there is *a Phragmén–Lindelöf function for the domain $\mathscr{D} = \mathbb{C} \sim E$.*

Proof. Assume wlog that $0 \in \mathscr{D}$, and call $\mathcal{O}_0$ the component of $\mathbb{R} \sim E$ containing 0. By the first theorem of the present article,

$$G(t,0) \;\leqslant\; \frac{C}{1+t^2}$$

for $t \in \mathcal{O}_0$, and clearly

$$G(t,0) \;\leqslant\; \log^+ \frac{1}{|t|} + \mathrm{O}(1), \qquad t \in \mathcal{O}_0.$$

Therefore (symmetry again!)

$$\int_{-\infty}^{\infty} G(0,t)\,\mathrm{d}t \;=\; \int_{-\infty}^{\infty} G(t,0)\,\mathrm{d}t \;<\; \infty.$$

Now refer to the preceding theorem.

We are done.

This result will be applied to the study of weighted approximation on sets E in the next article. We cannot, however, end *this* one without keeping our promise about proving symmetry of the Green's function. So, here we go:

Theorem. *In $\mathscr{D} = \mathbb{C} \sim E$,*

$$G(z,w) \;=\; G(w,z).$$

Proof. Let us first treat the case where *E consists of a finite number of intervals, of finite or infinite length.* (If E contains *two* semi-infinite intervals at opposite ends of $\mathbb{R}$, we consider them as forming *one* interval passing through ∞.)

We first proceed as at the beginning of article 1, *and map $\mathscr{D}$* (or $\mathscr{D} \cup \{\infty\}$, if $\infty \notin E$) *conformally onto a bounded domain, bounded by a finite number of analytic Jordan curves.* This useful trick simplifies a lot of work; let us describe (in somewhat more detail than at the beginning of article 1) how it is done.

Suppose that $E_1, E_2, \ldots, E_N$ are the components of E. First map $(\mathbb{C} \cup \{\infty\}) \sim E_1$ conformally onto the disk $\{|z| < 1\}$; in this mapping, E_1 (which gets split down its middle, with its two edges spread apart) goes onto $\{|z| = 1\}$, and $E_2, \ldots, E_N$ are taken onto *analytic* Jordan arcs, $A_2, \ldots, A_N$ respectively, lying inside the unit disk. (Actually, in our situation, where the E_k lie on $\mathbb{R}$, we can choose the mapping of $(\mathbb{C} \cup \{\infty\}) \sim E_1$ onto $\{|z| < 1\}$ so that $\mathbb{R} \sim E_1$ is taken onto $(-1, 1)$. Then $A_2, \ldots, A_N$ will be *segments* on $(-1, 1)$.) In this fashion, $\mathscr{D}$ is mapped conformally onto

$$\{|z| < 1\} \sim A_2 \sim A_3 \sim \cdots \sim A_N.$$

Now map $(\mathbb{C} \cup \{\infty\}) \sim A_2$ conformally onto $\{|w| < 1\}$. In this transformation, $\{|z| = 1\}$ goes onto a certain analytic Jordan curve $\mathscr{C}_1$ lying inside the unit disk, A_2 (after having its two sides spread apart) goes onto $\{|w| = 1\}$, and, if $N > 2$, the arcs $A_3, \ldots, A_N$ go onto other analytic arcs $A'_3, \ldots, A'_N$, lying inside $\{|w| < 1\}$. ($A'_3, \ldots, A'_N$ are indeed *segments* on $(-1, 1)$ in our present situation, if this second conformal mapping is properly chosen.) So far, composition of our two mappings yields a conformal transformation of $\mathscr{D}$ onto the region lying in $\{|w| < 1\}$, bounded by the unit circumference, the analytic Jordan curve $\mathscr{C}_1$, and the analytic Jordan arcs $A'_3, \ldots, A'_N$ (in the case where $N > 2$).

It is evident how one may continue this process when $N > 2$. Do the same thing with A'_3 that was done with A_2, and so forth, until all the boundary components are used up. The final result is a conformal mapping of $\mathscr{D}$ onto a region bounded by the *unit circumference* and $N - 1$ *analytic Jordan curves situated within it.*

Under conformal mapping, Green's functions correspond to Green's functions. Therefore, in order to prove that $G(z, w) = G(w, z)$, we *may as well assume that G is the Green's function for a bounded domain* Ω like the one arrived at by the process just described, i.e., with $\partial\Omega$ *consisting of a finite number of analytic Jordan curves.* For such domains Ω we can establish symmetry using methods going back to Green himself. (Green's original proof – the result *is* due to him, by the way – is a little different from the one we are about to give. Adapted to two dimensions, it amounts to the observation that

$$\begin{aligned} G(z,w) &= \log\frac{1}{|z-w|} + \int_{\partial\Omega} \log|\zeta - w|\, d\omega_\Omega(\zeta, z) \\ &= \log\frac{1}{|z-w|} + \int_{\partial\Omega}\int_{\partial\Omega} \log|\zeta - \sigma|\, d\omega_\Omega(\sigma, w)\, d\omega_\Omega(\zeta, z), \end{aligned}$$

where $\omega_\Omega(\ \ , z)$ is the harmonic measure for Ω. This argument can easily be made rigorous for our domains Ω. The interested reader may want to consult Green's collected papers, reprinted by Chelsea in 1970.)

If $\zeta \in \partial\Omega$ and the function F is $\mathscr{C}_1$ in a *neighborhood* of ζ, we denote by

$$\frac{\partial F(\zeta)}{\partial n_\zeta}$$

the *directional derivative of F in the direction of the unit outward normal* $\mathbf{n}_\zeta$ to $\partial\Omega$ at ζ:

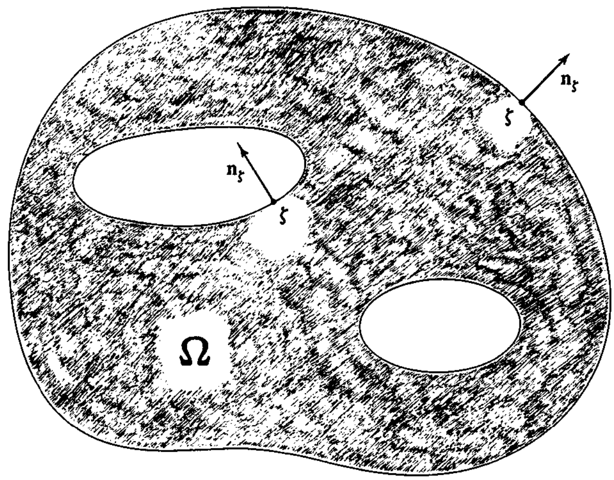

Figure 126

If $w \in \Omega$ is *fixed*, $G(z, w)$ is harmonic as a function of $z \in \Omega$ (for z away from w) and continuous up to $\partial\Omega$, where it equals *zero*. *Analyticity* of the components of $\partial\Omega$ means that, given any $\zeta_0 \in \partial\Omega$, we can find a *conformal mapping* of a small disk centered at ζ_0 which takes the *part of* $\partial\Omega$ *lying in that disk* to a *segment* σ *on the real axis*. If we compose $G(z, w)$ with this conformal mapping for $z \in \Omega$ near ζ_0, we see, by *Schwarz' reflection principle*, that the *composed* function is actually *harmonic in a neighborhood of* σ, and thence that $G(z, w)$ is *harmonic* (in z) *in a neighborhood of* ζ_0. $G(z, w)$ is, in particular, *a* $\mathscr{C}_\infty$ *function of* z *in the neighborhood of every point on* $\partial\Omega$.

This regularity, together with the smoothness of the components of $\partial\Omega$, makes it possible for us to apply *Green's theorem*. Given z and $w \in \Omega$ with $z \neq w$, take two small non-intersecting circles γ_z and γ_w lying in Ω, about z and w respectively. Call Ω' the domain obtained from Ω by *removing* therefrom the small disks bounded by γ_z and γ_w:

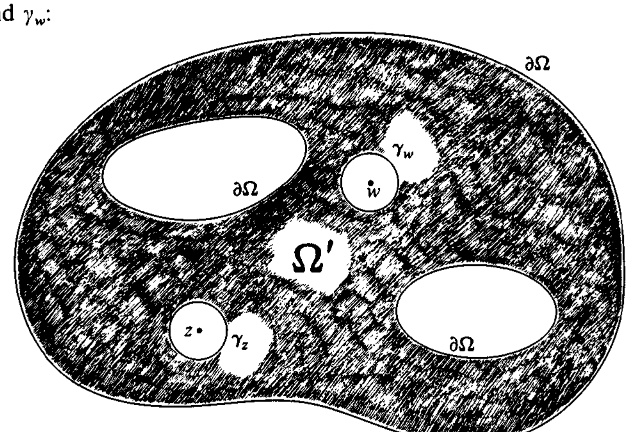

Figure 127

Denote by **grad** the *vector gradient* with respect to (ξ, η), where $\zeta = \xi + i\eta$, and by '$\cdot$' the *dot product* in $\mathbb{R}^2$. We have

$$\int_{\partial\Omega'} \left(G(\zeta, w) \frac{\partial G(\zeta, z)}{\partial n_\zeta} - G(\zeta, z) \frac{\partial G(\zeta, w)}{\partial n_\zeta} \right) |d\zeta|$$

$$= \int_{\partial\Omega'} (G(\zeta, w)\,\mathbf{grad}\,G(\zeta, z) - G(\zeta, z)\,\mathbf{grad}\,G(\zeta, w))\cdot\mathbf{n}_\zeta |d\zeta|.$$

Since the vector-valued function

$$G(\zeta, w)\,\mathbf{grad}\,G(\zeta, z) - G(\zeta, z)\,\mathbf{grad}\,G(\zeta, w)$$

of ζ is $\mathscr{C}_\infty$ in and on $\bar{\Omega}'$, ($\mathscr{C}_\infty$ on $\partial\Omega$ by what was said above), we can apply *Green's theorem* to the second of these integrals, and find that it equals

$$\iint_{\Omega'} \operatorname{div}(G(\zeta, w)\,\mathbf{grad}\,G(\zeta, z) - G(\zeta, z)\,\mathbf{grad}\,G(\zeta, w))\,d\xi\,d\eta,$$

where div denotes *divergence* with respect to (ξ, η). However, by *Green's identity,*

$$\begin{aligned}\operatorname{div}(G(\zeta, w)\,&\mathbf{grad}\,G(\zeta, z) - G(\zeta, z)\,\mathbf{grad}\,G(\zeta, w))\\ &= G(\zeta, w)\nabla^2 G(\zeta, z) - G(\zeta, z)\nabla^2 G(\zeta, w),\end{aligned}$$

where $\nabla^2 = \partial^2/\partial\xi^2 + \partial^2/\partial\eta^2$. Because $z \notin \Omega'$ and $w \notin \Omega'$, $G(\zeta, z)$ and $G(\zeta, w)$ are *harmonic* in ζ, $\zeta \in \Omega'$. Hence

$$\nabla^2 G(\zeta, z) = \nabla^2 G(\zeta, w) = 0, \qquad \zeta \in \Omega',$$

and the above double integral vanishes identically. Therefore the *first* of the above *line integrals* around $\partial\Omega'$ must be *zero.*

Now $\partial\Omega' = \partial\Omega \cup \gamma_z \cup \gamma_w$, and $G(\zeta, w) = G(\zeta, z) = 0$ for $\zeta \in \partial\Omega$. That line integral therefore reduces to

$$\left\{ \int_{\gamma_z} + \int_{\gamma_w} \right\} \left(G(\zeta, w) \frac{\partial G(\zeta, z)}{\partial n_\zeta} - G(\zeta, z) \frac{\partial G(\zeta, w)}{\partial n_\zeta} \right) |d\zeta|,$$

which must thus *vanish.* Near z, $G(\zeta, z)$ equals $\log(1/|\zeta - z|)$ plus a harmonic function of ζ; with this in mind we see that the integral around γ_z is *very nearly* $2\pi G(z, w)$ if the radius of γ_z is small. The integral around γ_w is seen in the same way to be very nearly equal to $-2\pi G(w, z)$ when that circle has small radius, so, making the radii of both γ_z and γ_w tend to *zero,* we find in the limit that

$$2\pi G(z, w) - 2\pi G(w, z) = 0,$$

i.e., $G(z, w) = G(w, z)$ for $z, w \in \Omega$. This same symmetry must then hold for the Green's functions belonging to *finitely connected domains* $\mathscr{D}$ of the kind we are considering. **How much must we admire George Green, self taught, who did such beautiful work isolated in provincial England at the beginning of the nineteenth century. One wonders what he might have done had he lived longer than he did.**

AN ESSAY

ON THE

APPLICATION

OF

MATHEMATICAL ANALYSIS TO THE THEORIES OF ELECTRICITY AND MAGNETISM.

BY

GEORGE GREEN.

Nottingham:

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1828.

Once the symmetry of Green's function for *finitely connected domains* $\mathscr{D}$ is *known*, we can establish that property *in the general case* by a limiting argument. By a slight modification of the following procedure, one can actually *prove existence* of the Green's function for infinitely connected domains $\mathscr{D} = \mathbb{C} \sim E$ of the kind being considered here, and the reader is invited to see how such a proof would go. Let us, however, content ourselves with what we set out to do.

Put $E_R = E \cap [-R, R]$ and take $\mathscr{D}_R = (\mathbb{C} \cup \{\infty\}) \sim E_R$. With our sets E, E_R consists of a finite number of intervals, so $\mathscr{D}_R$ is finitely connected, and, by what we have just shown, $G_R(z, w) = G_R(w, z)$ for the Green's function G_R belonging to $\mathscr{D}_R$. (Provided, of course, that R is large enough to make $|E_R| > 0$, so that $\mathscr{D}_R$ *has* a Green's function! *This we henceforth assume.*) We have $\mathscr{D}_R \supseteq \mathscr{D}$, whence, for z, $w \in \mathscr{D}$,

$$G(z, w) \;\leqslant\; G_R(z, w).$$

If we can show that

$$G_R(z, w) \longrightarrow G(z, w)$$

for z, $w \in \mathscr{D}$ as $R \to \infty$, we will obviously have $G(z, w) = G(w, z)$.

To verify this convergence, observe that

$$G_{R'}(z, w) \leqslant G_R(z, w)$$

for z and w in $\mathscr{D}_{R'}$ (hence *certainly* for $z, w \in \mathscr{D}$!) when $R' \geqslant R$, because then $\mathscr{D}_{R'} \subseteq \mathscr{D}_R$. The limit

$$\tilde{G}(z, w) = \lim_{R \to \infty} G_R(z, w)$$

thus *certainly exists* for $z, w \in \mathscr{D}$, and is $\geqslant 0$. If we can prove that $\tilde{G}(z, w) = G(z, w)$, we will be done.

Fix any $w \in \mathscr{D}$. Outside any small circle about w lying in $\mathscr{D}$, $\tilde{G}(z, w)$ is the limit of a decreasing sequence of positive harmonic functions of z, and is therefore itself harmonic in that variable. Let $x_0 \in E$. Take $R > |x_0|$; then, since $0 \leqslant \tilde{G}(z, w) \leqslant G_R(z, w)$ for $z \in \mathscr{D}$ and $G_R(z, w) \longrightarrow 0$ as $z \to x_0$, we have

$$\tilde{G}(z, w) \longrightarrow 0 \qquad \text{for } z \longrightarrow x_0.$$

If we fix any large R, we have, for $z \in \mathscr{D}$,

$$0 \leqslant \tilde{G}(z, w) \leqslant G_R(z, w) = \log \frac{1}{|z - w|} + O(1).$$

Therefore, since

$$G(z, w) = \log \frac{1}{|z - w|} + O(1),$$

we have

$$\tilde{G}(z, w) \leqslant G(z, w) + O(1), \qquad z \in \mathscr{D}.$$

However, this last inequality can be *turned around*. Indeed, for $z \in \mathscr{D}$ and *every* sufficiently large R,

$$G_R(z, w) \geqslant G(z, w),$$

from which we get

$$\tilde{G}(z, w) \geqslant G(z, w), \qquad z \in \mathscr{D}$$

on making $R \to \infty$.

We see finally that $0 \leqslant \tilde{G}(z, w) - G(z, w) \leqslant O(1)$ for $z \in \mathscr{D}$ (at least when $z \neq w$); the difference in question is, moreover, *harmonic in* z (for $z \in \mathscr{D}$, $z \neq w$) and *tends*, according to what we have shown above, to *zero when* z *tends to any point of* $E = \partial\mathscr{D}$. Hence

$$\tilde{G}(z, w) - G(z, w) = 0, \qquad z \in \mathscr{D},$$

i.e.,

$$G_R(z, w) \longrightarrow G(z, w)$$

for $z \in \mathscr{D}$ when $R \to \infty$, which is what we needed to establish the symmetry of $G(z, w)$.

We are done.

3. Weighted approximation on the sets E

Let E be a closed set on $\mathbb{R}$, having infinite extent in both directions and consisting of (at most) countably many closed intervals not accumulating at any finite point. Suppose that we are given a function $W(x) \geqslant 1$, defined and continuous on E, such that $W(x) \to \infty$ for $x \to \pm\infty$ in E. Then, in analogy with Chapter VI, we make the

Definition. $\mathscr{C}_W(E)$ is the set of functions φ defined and continuous on E, such that

$$\frac{\varphi(x)}{W(x)} \longrightarrow 0 \quad \text{for} \quad x \longrightarrow \pm\infty \textit{ in } E.$$

And we put

$$\|\varphi\|_{W,E} \;=\; \sup_{x\in E}\left|\frac{\varphi(x)}{W(x)}\right|$$

for $\varphi \in \mathscr{C}_W(E)$.

For $A > 0$, we denote by $\mathscr{C}_W(A, E)$ the $\|\ \|_{W,E}$-closure in $\mathscr{C}_W(E)$ of the collection of finite sums of the form

$$\sum_{-A \leqslant \lambda \leqslant A} C_\lambda e^{i\lambda x}.$$

Also, if, for every $n > 0$,

$$\frac{x^n}{W(x)} \longrightarrow 0 \quad \text{as} \quad x \longrightarrow \pm\infty \text{ in } E,$$

we denote by $\mathscr{C}_W(0, E)$ the $\|\ \|_{W,E}$-closure in $\mathscr{C}_W(E)$ of the set of *polynomials*.

We are interested in obtaining criteria for equality of the $\mathscr{C}_W(A, E)$, $A > 0$, (and of $\mathscr{C}_W(0, E)$) with $\mathscr{C}_W(E)$. One *can*, of course, reduce our present situation to the one considered in Chapter VI by putting $W(x) \equiv \infty$ on $\mathbb{R} \sim E$ and working with the space $\mathscr{C}_W(\mathbb{R})$. The equality in question is then governed by Akhiezer's theorems found in §§B and E of Chapter VI, according to the remark in §B.1 of that chapter (see also the corollary at the end of §E.2 therein). In this way, one arrives at results in which the set E does not figure explicitly. Our aim, however, already mentioned at the beginning of the present chapter, is to show how the *form*

$$\int_{-\infty}^{\infty} \frac{\log W_*(x)}{1+x^2}\,dx,$$

occurring in Akhiezer's first theorem, can, in the present situation, be replaced by

$$\int_E \frac{\log W_*(x)}{1+x^2}\,dx$$

when dealing with certain kinds of sets E. That is the subject of the following discussion. Our results will depend strongly on those of the preceding two articles.

Lemma. *Let $A > 0$, and suppose that there is a finite M such that*

$$(*)\qquad \int_E \frac{\log|S(x)|}{1+x^2}\,dx \;\leqslant\; M$$

for all finite sums $S(x)$ of the form

$$\sum_{-A\leqslant\lambda\leqslant A} a_\lambda e^{i\lambda x}$$

with $\|S\|_{W,E} \leqslant 1$. Then there is a finite M' such that

$$(\S)\qquad \int_E \frac{\log^+|S(x)|}{1+x^2}\,dx \;\leqslant\; M'$$

for such S with $\|S\|_{W,E} \leqslant 1$.

Proof. Given a sum $S(x)$ of the specified form with $\|S\|_{W,E} \leqslant 1$, we wish to show that (§) holds for some M' independent of S. Let us assume, to begin with, that the *exponents λ figuring in the sum $S(x)$ are in arithmetic progression*, more precisely, that

$$S(x) \;=\; \sum_{n=-N}^{N} C_n e^{inhx}$$

where $h = A/N$, N being some large integer. There is then another sum

$$T(x) \;=\; \sum_{n=-N}^{N} a_n e^{inhx}$$

(which is thus *also* of the form $\sum_{-A\leqslant\lambda\leqslant A} C_\lambda e^{i\lambda x}$) such that

$$1 + S(x)\overline{S(x)} \;=\; T(x)\overline{T(x)} \qquad \text{for } x \in \mathbb{R}.$$

This we can see by an elementary argument, going back to Fejér and

Riesz. For $x \in \mathbb{R}$, we have

$$1 + S(x)\overline{S(x)} \;=\; \sum_{n=-2N}^{2N} \gamma_n e^{inhx}$$

with certain coefficients γ_n. Write, for the moment,

$$e^{ihx} = \zeta;$$

then

$$1 + S(x)\overline{S(x)} \;=\; R(\zeta),$$

where

$$R(\zeta) \;=\; \sum_{n=-2N}^{2N} \gamma_n \zeta^n$$

is a certain *rational* function of ζ. We have $R(\zeta) \geqslant 1$ for $|\zeta| = 1$, so, by the Schwarz reflection principle,

$$R(\overline{1/\zeta}) \;=\; \overline{R(\zeta)}.$$

Therefore, if α, $0 < |\alpha| < 1$, is a zero of $R(\zeta)$, so is $1/\bar{\alpha}$, and the latter has the same multiplicity as α. Also, if $-m$ denotes the *least* integer n for which $\gamma_n \neq 0$, we must have $\gamma_n = 0$ for $n > m$ (*sic*!), as follows on comparing the orders of magnitude of $R(\zeta)$ for $\zeta \to 0$ and for $\zeta \to \infty$.

The *polynomial* $\zeta^m R(\zeta)$ is thus of degree $2m$, and of the form

$$\text{const.} \prod_{k=1}^{m} (\zeta - \alpha_k)\left(\zeta - \frac{1}{\bar{\alpha}_k}\right).$$

Thence,

$$R(\zeta) \;=\; C \prod_{k=1}^{m} (\zeta - \alpha_k)\left(\frac{1}{\zeta} - \bar{\alpha}_k\right),$$

and $C > 0$ since $R(\zeta) \geqslant 1$ for $|\zeta| = 1$. Going back to the real variable x, we see that

$$1 + S(x)\overline{S(x)} \;=\; C \prod_{k=1}^{m} (e^{ihx} - \alpha_k)(e^{-ihx} - \bar{\alpha}_k) \;=\; T(x)\overline{T(x)},$$

where

$$T(x) \;=\; C^{\frac{1}{2}} e^{-iNhx} \prod_{k=1}^{m} (e^{ihx} - \alpha_k)$$

is of the form

$$\sum_{n=-N}^{N} a_n e^{inhx},$$

since $m \leqslant 2N$.

Once this is known, it is easy to deduce (§) for sums $S(x)$ of the special form just considered with $\|S\|_{W,E} \leqslant 1$. Take any such S; we have another sum $T(x)$ of the same kind with $1 + |S(x)|^2 = |T(x)|^2$ on $\mathbb{R}$. Since $W(x) \geqslant 1$ on E, the condition $\|S\|_{W,E} \leqslant 1$ implies that $\|T\|_{W,E} \leqslant \sqrt{2}$, i.e.,

$$\|T/\sqrt{2}\|_{W,E} \;\leqslant\; 1.$$

For this reason, $T(x)/\sqrt{2}$ satisfies $(*)$, by hypothesis. Hence

$$\int_E \frac{\log|T(x)|}{1+x^2}\,dx \;\leqslant\; M + \int_E \frac{\log\sqrt{2}}{1+x^2}\,dx.$$

But $\log|T(x)| = \log\sqrt{(1+|S(x)|^2)} \geqslant \log^+|S(x)|$. Therefore

$$\int_E \frac{\log^+|S(x)|}{1+x^2}\,dx \;\leqslant\; M + \pi\log\sqrt{2},$$

and we have obtained (§).

We must still consider the case where the exponents λ in the *finite* sum

$$\sum_{-A\leqslant\lambda\leqslant A} a_\lambda e^{i\lambda x} \;=\; S(x)$$

are *not* in arithmetic progression, the condition $\|S\|_{W,E} \leqslant 1$ being, however, satisfied. Here, we may associate to each λ figuring in the expression just written a *rational multiple λ' of A*, with $|\lambda' - \lambda|$ exceedingly small. Since $W(x) \to \infty$ for $x \to \pm\infty$ in E, the sum

$$S'(x) \;=\; \sum_{-A\leqslant\lambda\leqslant A} a_\lambda e^{i\lambda' x}$$

will then be *as close as we like* in $\|\quad\|_{W,E}$-norm to $S(x)$ (depending on the closeness of the individual λ' to their corresponding λ). In this way, we can get a sequence of sums $S_n(x)$ of the form in question, *each one having its exponents in arithmetic progression*, such that $\|S_n\|_{W,E} \leqslant 1$ and $S_n(x) \xrightarrow[n]{} S(x)$ u.c.c. on $\mathbb{R}$. By what we have already shown, $\int_E(\log^+|S_n(x)|/(1+x^2))\,dx \leqslant M + \pi\log\sqrt{2}$ for each n. Therefore

$$\int_E \frac{\log^+|S(x)|}{1+x^2}\,dx \;\leqslant\; M + \pi\log\sqrt{2}$$

by Fatou's lemma. We are done.

For the sets E described at the beginning of the present § we can establish analogues, involving integrals over E, of the Akhiezer and Pollard theorems given in Chapter VI, §§E.2 and E.4. These are included in the following theorem which, in one direction, assumes as little as possible and concludes that $\mathscr{C}_W(A, E) \subset \mathscr{C}_W(E)$ properly. In the other, it assumes the proper inclusion and asserts as much as possible.

For $z \in \mathbb{C}$, *denote by* $W_{A,E}(z)$ *the supremum of* $|S(z)|$ *for the finite sums*

$$S(z) \;=\; \sum_{-A \leqslant \lambda \leqslant A} a_\lambda e^{i\lambda z}$$

with $\|S\|_{W,E} \leqslant 1$. (If we agree that $W(x) \equiv \infty$ on $\mathbb{R} \sim E$, $W_{A,E}(z)$ and $\mathscr{C}_W(A, E)$ reduce respectively to the function $W_A(z)$ and the space $\mathscr{C}_W(A)$ already considered in Chapter VI, §§E.2ff.)

Theorem. *Let E be one of the sets described at the beginning of this §, the conditions involving the four numbers* A, B, δ *and* Δ *being fulfilled. If, for some* $C > 0$, *the supremum of*

$$\int_E \frac{\log|S(x)|}{1+x^2}\,dx$$

for all finite sums

$$S(x) \;=\; \sum_{-C \leqslant \lambda \leqslant C} a_\lambda e^{i\lambda x}$$

with $\|S\|_{W,E} \leqslant 1$ *is* finite, *then* $\mathscr{C}_W(C, E) \subset \mathscr{C}_W(E)$ properly.

If, conversely, that proper inclusion holds, then

$$\int_E \frac{\log W_{C,E}(x)}{1+x^2}\,dx \;<\; \infty.$$

Proof. All the work here is in the establishment of the *first* part of the statement.

Define $W(x)$ on all of $\mathbb{R}$ by putting it equal to ∞ on $\mathbb{R} \sim E$. This makes it possible for us to apply results about $\mathscr{C}_W(C)$ from Chapter VI, §E, in the present situation. According to the Pollard theorem of Chapter VI, §E.4, and the remark thereto, we will have $\mathscr{C}_W(C) \neq \mathscr{C}_W(\mathbb{R})$ as soon as $W_C(i) < \infty$; in other words, $\mathscr{C}_W(C, E) \neq \mathscr{C}_W(E)$ provided that

$$W_{C,E}(i) \;<\; \infty.$$

We proceed to show this inequality, using the results of articles 1 and 2.

According to the lemma, our hypothesis for the first part of the theorem implies that

$$(*) \qquad \int_E \frac{\log^+|S(x)|}{1+x^2}\,dx \;\leqslant\; M' \;<\; \infty$$

for all sums $S(x)$ of the stipulated form with $\|S\|_{W,E} \leqslant 1$.

We have

$$E = \bigcup_{n=-\infty}^{\infty} [a_n - \delta_n,\ a_n + \delta_n],$$

where

$$0 < A < a_{n+1} - a_n < B,$$
$$0 < \delta < \delta_n < \Delta.$$

Given any finite sum

$$S(z) = \sum_{-C \leqslant \lambda \leqslant C} a_\lambda e^{i\lambda z}$$

with $\|S\|_{W,E} \leqslant 1$, let us put

$$v_S(z) = \frac{1}{\delta}\int_{-\delta/2}^{\delta/2} \log^+ |S(z+t)|\, dt$$

(using the δ associated to E by the above inequalities). The function $v_S(z)$ is then *continuous* and *subharmonic* in the complex plane. Obviously,

$$|S(z)| \leqslant \text{const.}e^{C|\Im z|}$$

(the constant may be *enormous*, but we don't care!), so

(†) $\quad v_S(z) \leqslant O(1) + C|\Im z|, \quad z \in \mathbb{C}.$

Put now

$$E' = \bigcup_{n=-\infty}^{\infty} \left[a_n - \frac{\delta_n}{2},\ a_n + \frac{\delta_n}{2}\right].$$

On the component

$$E'_n = \left[a_n - \frac{\delta_n}{2},\ a_n + \frac{\delta_n}{2}\right]$$

of E' we have

$$v_S(x) \leqslant \frac{1}{\delta}\int_{E_n} \log^+ |S(t)|\, dt,$$

where (as usual)

$$E_n = [a_n - \delta_n,\ a_n + \delta_n].$$

Denoting the right side of the previous relation by v_n, we have

(§§) $\quad v_S(x) \leqslant v_n, \quad x \in E'_n.$

The set E' (like E) is one of the kind specified at the beginning of the present §; the numbers $A, B, \delta/2$ and $\Delta/2$ are associated to it. *The* *results of the previous two articles are therefore valid for E' and the domain $\mathscr{D}' = \mathbb{C} \sim E'$.* We can, in particular, apply Carleson's theorem from article 1. Assume, wlog, that

$$0 \in \mathscr{O}_0' = \left(a_0 + \frac{\delta_0}{2},\ a_1 - \frac{\delta_1}{2}\right).$$

Then, if we denote by $\omega_n'(z)$ the *harmonic measure of* E_n' in $\mathscr{D}'$ (as seen from z), that theorem tells us that

$$\omega_n'(0) \leqslant \frac{K}{1+n^2}$$

with a constant K depending on A, B, $\delta/2$ and $\Delta/2$. By Harnack's theorem, there is thus a function $K(z)$, *continuous in* $\mathscr{D}'$, such that

$$\omega_n'(z) \leqslant \frac{K(z)}{1+n^2}, \qquad z \in \mathscr{D}'$$

(see discussion near the beginning of §B.1, Chapter VII).

Using properties (i) and (ii) from the beginning of this § we see that the quantities v_n introduced above satisfy

$$\frac{v_n}{1+n^2} \leqslant \alpha \int_{E_n} \frac{\log^+ |S(t)|}{1+t^2} \mathrm{d}t,$$

α being a certain constant depending only on the set E. Combined with the previous estimate, this yields

$$v_n \omega_n'(z) \leqslant \alpha K(z) \int_{E_n} \frac{\log^+ |S(t)|}{1+t^2} \mathrm{d}t, \qquad z \in \mathscr{D}',$$

so, for the sum

$$P(z) = \sum_{n=-\infty}^{\infty} v_n \omega_n'(z),$$

we have

$$(\dagger\dagger) \qquad P(z) \leqslant \alpha M' K(z), \qquad z \in \mathscr{D}',$$

by virtue of ($\ast$).

Because $|S(x)|$ is bounded on $\mathbb{R}$, the v_n (which are $\geqslant 0$, by the way) are bounded. The series used to define $P(z)$ is therefore u.c.c. convergent, so that function is *continuous up to* $E' = \partial\mathscr{D}'$ as well as being *positive* and *harmonic* in $\mathscr{D}'$. On the component E_n' of E', $P(x)$ takes the constant value v_n. Hence the function

$$v_S(z) - P(z)$$

is *subharmonic in* $\mathscr{D}'$ and continuous up to E', where it is $\leqslant 0$ by (§§). It is, moreover, $\leqslant O(1) + C|\Im z|$ by (†).

Now according to *Benedicks' theorem* (article 2), a *Phragmén–Lindelöf function* $Y(z)$ is available for $\mathscr{D}'$. The function

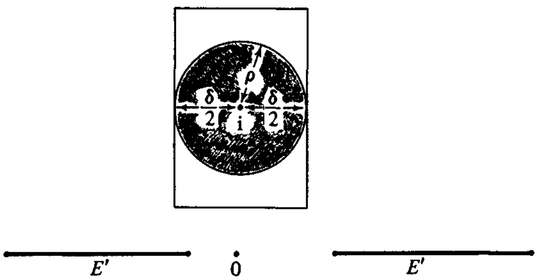
$$v_S(z) - P(z) - CY(z)$$

is *subharmonic and bounded above* in $\mathscr{D}'$ and continuous up to $E' = \partial\mathscr{D}'$ where it is $\leqslant 0$. It is thence $\leqslant 0$ *throughout* $\mathscr{D}'$ by the extended maximum principle (Chapter III, §C). Referring to (††), we see that

$$(\ddagger) \qquad v_S(z) \;\leqslant\; \alpha M'K(z) + CY(z), \qquad z \in \mathscr{D}'.$$

Let $\rho = \min(\tfrac{1}{2}, \delta/2)$. Since $\log^+|S(z)|$ is *subharmonic*, we have

$$\log^+|S(\mathrm{i})| \;\leqslant\; \frac{1}{\pi\rho^2}\iint_{|z-\mathrm{i}|<\rho} \log^+|S(z)|\,\mathrm{d}x\,\mathrm{d}y.$$



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The integral on the right is

$$\leqslant\; \frac{1}{\pi\rho^2}\int_{1-\rho}^{1+\rho}\int_{-\delta/2}^{\delta/2} \log^+|S(x+\mathrm{i}y)|\,\mathrm{d}x\,\mathrm{d}y \;=\; \frac{\delta}{\pi\rho^2}\int_{1-\rho}^{1+\rho} v_S(\mathrm{i}y)\,\mathrm{d}y.$$

Plugging (‡) into the last expression, we obtain

$$\log^+|S(\mathrm{i})| \;\leqslant\; \frac{\delta}{\pi\rho^2}\int_{1-\rho}^{1+\rho} (\alpha M'K(\mathrm{i}y) + CY(\mathrm{i}y))\,\mathrm{d}y.$$

This, then, is valid for *any* finite sum $S(z)$ of the form

$$\sum_{-C\leqslant\lambda\leqslant C} a_\lambda \mathrm{e}^{\mathrm{i}\lambda z}$$

with $\|S\|_{W,E} \leqslant 1$.

The *right side* of the inequality just found is a *finite quantity*, dependent on M' and C, and on the set E (through α, $K(\mathrm{i}y)$ and $Y(\mathrm{i}y)$); it is, however,