so this implies that

$$(3)\qquad \frac{1}{2\pi i}\int_{(1/2)-i\infty}^{(1/2)+i\infty} |I_{x,k}(s)|^2\, ds \le \frac{2k^2}{\pi^2}\int_1^{e^{\pi/k}} \left| G(xw) - 1 - \frac{1}{xw}\right|^2 dw$$
$$+ \frac{2}{\pi^2}\int_{e^{\pi/k}}^{\infty} (\log w)^{-2}\left| G(xw) - 1 - \frac{1}{xw}\right|^2 dw.$$

The two integrals on the right can be estimated using the explicit formula for $G(u) - 1$. The first integral will be considered first, after which the second integral is easily estimated by the same techniques.

The parallelogram law $2|A|^2 + 2|B|^2 = |A + B|^2 + |A - B|^2 \ge |A + B|^2$ shows that the first integral on the right side of (3) is at most

$$\frac{4k^2}{\pi^2}\int_1^{e^{\pi/k}} |G(xw) - 1|^2\, dw + \frac{4k^2}{\pi^2}\int_1^{e^{\pi/k}} \frac{dw}{w^2}.$$

The second integral here is simply $4k^2\pi^{-2}[1 - e^{-\pi/k}]$, so it is of the order of magnitude of k^2 and it will suffice to estimate the first integral. Since $G(u) - 1 = 2\sum_{n=1}^{\infty} \exp(-\pi n^2 u^2)$, this integral can be written in the form

$$\frac{16k^2}{\pi^2}\int_1^{e^{\pi/k}} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} e^{-\pi n^2 x^2 w^2} e^{-\pi m^2 x^{-2} w^2}\, dw.$$

Let $x = e^{i\pi/4}e^{-i\delta}$ so that $x^2 = \sin 2\delta + i\cos 2\delta$ and this integral becomes

$$(4)\qquad \frac{16k^2}{\pi^2}\int_1^{e^{\pi/k}} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} e^{-\pi(n^2+m^2)w^2\sin 2\delta} e^{-i\pi(n^2-m^2)w^2\cos 2\delta}\, dw.$$

The double sum converges absolutely and can therefore be rearranged as three sums, one in which $m > n$, one in which $m = n$, and one in which $m < n$. The integral of the terms with $m = n$ is easily estimated by using

$$\sum_{n=1}^{\infty} e^{-\pi(2n^2)w^2 \sin 2\delta} = \tfrac{1}{2}\{G[w(2\sin 2\delta)^{1/2}] - 1\}.$$

Since $u[G(u) - 1]$ is bounded both as $u \to \infty$ and as $u \to 0$, there is a constant K such that $G(u) - 1 < Ku^{-1}$ for all positive u and the terms of (4) with $m = n$ contribute at most

$$\frac{16k^2}{\pi^2}\int_1^{e^{\pi/k}} \frac{1}{2}\,\frac{K}{w(2\sin 2\delta)^{1/2}}\, dw = \frac{4\sqrt{2}\,Kk^2}{\pi^2(\sin 2\delta)^{1/2}}\log(e^{\pi/k})$$

which for small values of δ is less than a constant times $k\delta^{-1/2}$.

It will now be shown that the remaining terms $m \ne n$ of (4) are much smaller than $k\delta^{-1/2}$. The terms with $m > n$ are the complex conjugates of those with $m < n$, so it will suffice to estimate the latter. Termwise integration is easily justified so the quantity to be estimated is equal to the sum over all (m, n) with $m < n$ of

$$(5)\qquad \frac{16k^2}{\pi^2}\int_1^{e^{\pi/k}} e^{-\pi(n^2+m^2)w^2\sin 2\delta} e^{-i\pi(n^2-m^2)w^2\cos 2\delta}\, dw.$$

The real part of this integral is

$$\frac{16k^2}{\pi^2}\int_1^{e^{\pi/k}} f(w)\cos V(w)\,dw,$$

where $f(w) = \exp[-\pi(n^2 + m^2)w^2 \sin 2\delta]$ and $V(w) = \pi(n^2 - m^2)w^2 \cos 2\delta$. Now $\cos 2\delta$ is positive for small δ, so $V(w)$ is a monotone increasing function of w, and this integral can be written in terms of the variable V as

$$\frac{16k^2}{\pi^2}\int_{V(1)}^{V(e^{\pi/k})} \frac{f}{V'}\cos V\,dV,$$

where f and V' are functions of V by composition with the inverse function $V \longrightarrow w$. Since f is decreasing and V' is increasing, the lemma of Section 9.7 says that this integral is at most

$$\frac{16k^2}{\pi^2}\cdot 2\frac{f(1)}{V'(1)} = \frac{32k^2}{\pi^2}\,\frac{e^{-\pi(n^2+m^2)\sin 2\delta}}{2\pi(n^2 - m^2)\cos 2\delta}.$$

A similar estimate applies to the imaginary part and hence to the modulus of the integral (5). It follows that for small δ the total modulus of the terms to be estimated is at most a constant times

$$\sum_{n=1}^{\infty}\sum_{m<n}\frac{k^2e^{-\pi(n^2+m^2)\sin 2\delta}}{(n^2 - m^2)} \le \sum_{n=1}^{\infty}\sum_{m<n}\frac{k^2e^{-\pi n^2\sin 2\delta}}{(n + m)(n - m)}$$
$$\le \sum_{n=1}^{\infty}\frac{k^2e^{-\pi n^2\sin 2\delta}}{n}\sum_{m<n}\frac{1}{n - m}.$$

Now $\sum_{m<n} 1/(n - m) = \sum_{m<n} 1/m$ is less than a constant times $\log n$, so the quantity to be estimated is less than a constant times

$$k^2\sum_{n=1}^{\infty} e^{-\pi n^2\sin 2\delta}\frac{\log n}{n}.$$

The function $\exp(-\pi u^2 \sin 2\delta)(\log u)/u$ is decreasing for $u > e$, so this sum is at most k^2 times

$$\frac{e^{-\pi 4\sin 2\delta}\log 2}{2} + \frac{e^{-\pi 9\sin 2\delta}\log 3}{3} + \int_3^{\infty} e^{-\pi u^2\sin 2\delta}\log u\, d\log u$$
$$\le \text{const} + \int_3^{(\sin 2\delta)^{-1/2}} e^{-\pi u^2\sin 2\delta}\log u\, d\log u$$
$$+ \int_1^{\infty} e^{-\pi v^2}\log\left(\frac{v}{(\sin 2\delta)^{1/2}}\right) d\log v$$
$$\le \text{const} + \frac{1}{2}(\log u)^2\Big|_{u=3}^{(\sin 2\delta)^{-1/2}}$$
$$+ \frac{1}{2}\log\left(\frac{1}{\sin 2\delta}\right)\int_1^{\infty} \exp^{-\pi v^2} d\log v + \text{const}$$
$$\le \text{const} + \text{const}\left(\log\frac{1}{\sin 2\delta}\right)^2 + \text{const}\left(\log\frac{1}{\sin 2\delta}\right).$$

Given any $\epsilon > 0$ this is much less than $\epsilon\delta^{-1/2}$ for all sufficiently small δ. Putting all these estimates together then shows that *there is a constant K_1 such that for every $\epsilon > 0$ the first integral on the right side of* (3) *has modulus less than $\epsilon k^2\delta^{-1/2} + K_1 k\delta^{-1/2}$ for all sufficiently small positive δ.*

Analogous arguments prove that the same estimate applies to the second integral on the right side of (3). Briefly,

$$\frac{2}{\pi^2}\int_{e^{\pi/k}}^{\infty}(\log w)^{-2}\left|\frac{1}{xw}\right|^2 dw \le \frac{2}{\pi^2}(\log e^{\pi/k})^{-2}\int_{e^{\pi/k}}^{\infty} w^{-2}\,dw$$
$$\le \text{const}\, k^2 \le \epsilon k^2\delta^{-1/2}.$$

When $|G(xw) - 1|^2$ is written as a double sum over n and m, the total of the terms with $m = n$ is at most a constant times

$$\begin{aligned}
&\int_{e^{\pi/k}}^{\infty}(\log w)^{-2}\sum_{n=1}^{\infty} e^{-\pi n^2 2w^2 \sin 2\delta}\,dw \\
&= \int_{e^{\pi/k}}^{\infty}(\log w)^{-2}[G(w(2\sin 2\delta)^{1/2}) - 1]\,dw \\
&\le K\int_{e^{\pi/k}}^{\infty}(\log w)^{-2}\frac{1}{w(2\sin 2\delta)^{1/2}}\,dw \\
&\le \frac{K}{(2\sin 2\delta)^{1/2}}\int_{e^{\pi/k}}^{\infty}(\log w)^{-2}\,d\log w \\
&= \frac{K}{(2\sin 2\delta)^{1/2}}(\log e^{\pi/k})^{-1} \le \text{const}\, k\delta^{-1/2}
\end{aligned}$$

for all sufficiently small δ. Finally, the terms with $m \ne n$ are at most a constant times

$$\sum_{n=1}^{\infty}\sum_{m<n}\int_{e^{\pi/k}}^{\infty}(\log w)^{-2}e^{-\pi(n^2+m^2)w^2\sin 2\delta}e^{-i\pi(n^2-m^2)w^2\cos 2\delta}\,dw$$
$$\le \text{const}\sum_{n=1}^{\infty}\sum_{m<n}\frac{e^{-\pi(n^2+m^2)\sin 2\delta}}{(\log e^{\pi/k})^2(n^2-m^2)}$$

which by the same sequence of estimates as before is less than $\epsilon k^2\delta^{-1/2}$ for all sufficiently small δ.

In what follows it will be convenient to consider $x = e^{-i\pi/4}e^{i\delta}$ rather than $x = e^{i\pi/4}e^{-i\delta}$ (because then the significant values of the integral occur for positive values of t—see below). Since this replaces $I_{x,k}(s)$ by its complex conjugate, the same estimates apply and what has been proved is that *if $I_{x,k}(s)$ is defined as in* (1) *with $x = e^{-i\pi/4}e^{i\delta}$, then there is a constant K' such that given $\epsilon > 0$ the inequality*

$$\frac{1}{2\pi i}\int_{(1/2)-i\infty}^{(1/2)+i\infty}|I_{x,k}(s)|^2\,ds < \frac{K'k + \epsilon k^2}{\delta^{1/2}}$$

holds for all sufficiently small positive values of δ ($k > 0$ being arbitrary).

Later in the proof this estimate will be used in combination with the Schwarz inequality to obtain an estimate of the integral of $|I|$. First, however, some estimates will be made of the value of $|I|$ implied by the assumption that $\xi(\frac{1}{2}+it)$ does not change sign in its domain of integration. Since

$$I_{x,k}\left(\frac{1}{2}+it\right) = \frac{1}{2\pi}\int_{t-k}^{t+k}\frac{2\xi(\frac{1}{2}+iu)}{-u^2-\frac{1}{4}}x^{-1/2}x^{iu}\,du$$
$$= -x^{-1/2}\frac{1}{\pi}\int_{t-k}^{t+k}\frac{\xi(\frac{1}{2}+iu)}{u^2+\frac{1}{4}}e^{\pi u/4}e^{-u\delta}\,du$$

(where $x = e^{-i\pi/4}e^{i\delta}$), the integral $J(t) = J_{x,k}(t)$ defined by

$$J_{x,k}(t) = \frac{1}{\pi}\int_{t-k}^{t+k}\frac{|\xi(\frac{1}{2}+iu)|}{u^2+\frac{1}{4}}e^{\pi u/4}e^{-u\delta}\,du$$

has the property that $|J(t)| \geq |I(\frac{1}{2}+it)|$ for all t and $|J(t)| = |I(\frac{1}{2}+it)|$ whenever the interval of integration of $I(\frac{1}{2}+it)$ contains no roots ρ. The basic idea of the proof is to show that in a suitable sense $J(t)$ is much larger than $I(\frac{1}{2}+it)$ on the average. Thus estimates of $J(t)$ from below are required.

It was shown in the preceding section that $|\Pi(s/2)|$ for $s = \frac{1}{2}+it$ is e^B where $B = \frac{3}{4}\log|t| - |t|\pi/4 + \cdots$, the omitted terms remaining bounded as $|t| \longrightarrow \infty$. Combining this with the formula $\xi(s) = \Pi(s/2)\pi^{-s/2}(s-1)\zeta(s)$ and obvious estimates of $|\pi^{-s/2}|$ and $|s-1|$ gives

$$\frac{|\xi(\frac{1}{2}+iu)|}{u^2+\frac{1}{4}}e^{u\pi/4} \geq \text{const}\,\frac{u^{3/4}e^{-u\pi/4}\pi^{-1/4}u\,|\zeta(\frac{1}{2}+iu)|\,e^{u\pi/4}}{u^2+\frac{1}{4}}$$
$$\geq \text{const}\,u^{-1/4}|\zeta(\tfrac{1}{2}+iu)|$$

for $u \geq 1$. Therefore, for large t,

$$J(t) \geq \text{const}\,(t+k)^{-1/4}e^{-(t+k)\delta}\int_{t-k}^{t+k}|\zeta(\tfrac{1}{2}+iu)|\,du$$

and to estimate $J(t)$ from below it will suffice to estimate $\int|\zeta|$ from below. This can be done using a technique very similar to the technique of Section 9.7, which is also due to Hardy–Littlewood.

It was shown in Section 9.7 that

$$\zeta(\tfrac{1}{2}+iv) = \sum_{n<v} n^{-(1/2)-iv} + R(v),$$

where $R(v)$ is less than a constant times $v^{-1/2}$ as $v \longrightarrow \infty$. Thus for $t-k \leq v \leq t+k$

$$\zeta(\tfrac{1}{2}+iv) = \sum_{n<t} n^{-(1/2)-iv} + E(v) + R(v),$$

where $E(v)$ is plus or minus the sum of $n^{-1/2-iv}$ over all integers n between v and t. Since $E(v)$ consists of at most $k+1$ terms each of modulus at most $n^{-1/2} \leq (t-k)^{-1/2}$, this shows that for $t-k \leq v \leq t+k$

$$\Big|-\zeta(\tfrac{1}{2}+iv) + \sum_{n<t} n^{-(1/2)-iv}\Big| \leq (k+1)(t-k)^{-1/2} + \text{const}\,v^{-1/2}$$
$$\leq (k+\text{const})(t-k)^{-1/2}.$$

Assume $k \geq 1$, so the right side can be written as a constant times $k(t-k)^{-1/2}$. Then

$$\operatorname{Re}\{-\zeta(\tfrac{1}{2}+iv) + \sum_{n<t} n^{-(1/2)-iv}\} \leq \text{const}\, k(t-k)^{-1/2},$$

$$\begin{aligned}\int_{t-k}^{t+k} |\zeta(\tfrac{1}{2}+iv)|\, dv &\geq \int_{t-k}^{t+k} \operatorname{Re}\zeta(\tfrac{1}{2}+iv)\, dv \\ &\geq \int_{t-k}^{t+k} [\operatorname{Re}\sum_{n<t} n^{-(1/2)-iv} - \text{const}\, k(t-k)^{-1/2}]\, dv \\ &= 2k + \sum_{2\leq n<t} \operatorname{Re}\int_{t-k}^{t+k} n^{-(1/2)-iv}\, dv \\ &\quad - 2k(\text{const}\, k)(t-k)^{-1/2} \\ &= 2k + \operatorname{Re}\sum_{2\leq n<t} \frac{2\sin(k\log n)}{n^{(1/2)+it}\log n} \\ &\quad - \text{const}\, k^2(t-k)^{-1/2} \\ &\geq 2k - 2\left|\sum_{2\leq n<t} \frac{1}{n^{(1/2)+it}\log n}\right| \\ &\quad - \text{const}\, k^2(t-k)^{-1/2}.\end{aligned}$$

If t is much larger than k, the last term is insignificant compared to the first. Now although the middle term is not necessarily small, it is *on the average* smaller than the first term, so that on the average the first term $2k$ is a lower bound. Specifically, over any interval $A \leq t \leq B$ with $B > A \geq 1$, the integral

$$\int_A^B \sum_{2\leq n<t} \left|\frac{1}{n^{(1/2)+it}\log n}\right|^2 dt = \int_A^B \sum_{2\leq m<t,\, 2\leq n<t} \frac{1}{n^{1/2}m^{1/2}\log n \log m}\left(\frac{m}{n}\right)^{it} dt$$

can be estimated as follows. The terms with $m = n$ contribute just $(B-A)$ times a partial sum of the series $\sum n^{-1}(\log n)^{-2}$. Since this series is convergent, its partial sums are bounded and this is at most a constant times $(B-A)$. Each of the terms† with $m \neq n$ is of the form

$$\frac{1}{n^{1/2}m^{1/2}\log n \log m}\int_b^B \left(\frac{m}{n}\right)^{it} dt,$$

where $b = \max(A, m, n)$; so regardless of the value of b its modulus is at most

$$\frac{2}{n^{1/2}m^{1/2}\log n \log m\,|\log(m/n)|}$$

and the total of the remaining terms has modulus at most

$$4\sum_{2\leq m<n<B}\frac{1}{n^{1/2}m^{1/2}\log n \log m \log(n/m)}.$$

As in Section 9.7 divide this sum into two parts according to whether $m <$

†The sum is finite, so termwise integration is valid.

$\frac{1}{2}n$ or $m \geq \frac{1}{2}n$. The total of the terms with $m < \frac{1}{2}n$ is at most

$$4\sum\sum \frac{1}{n^{1/2}m^{1/2}\log n \log m \log 2} = \frac{4}{\log 2}\Big(\sum_{2\leq n<B}\frac{1}{n^{1/2}\log n}\Big)^2$$
$$\leq \frac{4}{\log 2}\Big(\int_1^B u^{-1/2}\,du\Big)^2 \leq \text{const}\, B.$$

To estimate the total of the terms with $m \geq \frac{1}{2}n$, set $r = n - m$ so that $\log(n/m) = -\log[1-(r/n)] > r/n$ and the total is less than

$$4\sum_{3\leq n<B}\sum_{n/2\leq m<n}\frac{1}{n^{1/2}(n-r)^{1/2}\log n \log(n/2)(r/n)}$$
$$= 4\sum_{3\leq n<B}\frac{1}{\log n\log(n/2)}\sum_{1\leq r\leq n/2}\frac{1}{r[1-(r/n)]^{1/2}}$$
$$\leq 4\sqrt{2}\sum_{3\leq n<B}\frac{1}{\log n \log(n/2)}\sum_{1\leq r\leq n/2}\frac{1}{r}$$
$$\leq 4\sqrt{2}\sum_{3\leq n\leq B}\frac{\log(n/2)+1}{\log n\log(n/2)},$$

which, since the terms of the sum are bounded, is less than a constant times B too. This proves that

$$\int_A^B\Big|\sum_{2\leq n<t}\frac{1}{n^{(1/2)+it}\log n}\Big|^2 dt < K_2B$$

for some positive constant K_2. Thus by the Schwarz inequality

$$\int_A^B\Big|\sum\frac{1}{n^{(1/2)+it}\log n}\Big|dt \leq \Big(\int_A^B 1^2\,dt\Big)^{1/2}\Big(\int_A^B\Big|\sum\frac{1}{n^{(1/2)+it}\log n}\Big|^2 dt\Big)^{1/2}$$
$$\leq (B-A)^{1/2}(K_2B)^{1/2} \leq K_2^{1/2}B,$$

so the average order of magnitude of the middle term is bounded and, therefore, when k is sufficiently large, the first term $2k$ is on the average dominant.

Now let ν be the number of zeros of $\xi(\frac{1}{2}+it)$ in the interval $\{0 \leq t \leq B+k\}$. Let the entire real axis be divided into intervals of length k and for each of the ν zeros strike out the interval which contains it and the two intervals which adjoin this one. Let S be the subset of $\{A \leq t \leq B\}$ consisting of points which do not lie in the stricken intervals. Then the total length of the intervals of S is at least $B - A - 3\nu k$ since a length of at most $3k$ was stricken for each zero. On the other hand $|I(\frac{1}{2}+it)| = J(t)$ for all t in S (there is no zero between $t-k$ and $t+k$) so

$$\int_S |I(\tfrac{1}{2}+it)|\,dt = \int_S J(t)\,dt \geq \int_S \text{const}\,(B+k)^{-1/4}$$
$$\times\, e^{-(B+k)\delta}\int_{t-k}^{t+k}|\zeta(\tfrac{1}{2}+iu)|\,du$$
$$\geq \text{const}\,(B+k)^{-1/4}e^{-(B+k)\delta}$$
$$\times\int_S\Big[2k - 2\Big|\sum\frac{1}{n^{(1/2)+it}\log n}\Big| - \text{const}\,k^2(t-k)^{-1/2}\Big]dt$$
$$\geq \text{const}\,(B+k)^{-1/4}e^{-(B+k)\delta}$$
$$\times\,[2k(B-A-3\nu k) - \text{const}\,B - \text{const}\,k^2B^{1/2}].$$

To simplify this, let $(B + k)\delta = 1$, which can be regarded as a choice of B given δ, k and let $B - A = \frac{1}{2}\delta^{-1}$, which can be regarded as a choice of A. (Note that $B > A \geq 1$ for $k \geq 1$ and δ small.) Then the above estimate becomes

$$\begin{aligned}\int_S |I(\tfrac{1}{2} + it)|\, dt &\\ &\geq \text{const}\, \delta^{1/4}[2k(\tfrac{1}{2}\delta^{-1} - 3\nu k) - \text{const}\, \delta^{-1} - \text{const}\, k^2\delta^{-1/2}] \\ &= K_1 k\delta^{-3/4} - K_2 k^2 \nu \delta^{1/4} - K_3\delta^{-3/4} - K_4 k^2 \delta^{-1/4},\end{aligned}$$

where K_1, K_2, K_3, K_4 are positive constants. Note that the third term will be insignificant compared to the first if k is large enough and the fourth term will be insignificant compared to the first if δ is small enough. On the other hand

$$\begin{aligned}\int_S |I(\tfrac{1}{2} + it)|\, dt &\leq \int_A^B |I(\tfrac{1}{2} + it)|\, dt \\ &\leq \left(\int_A^B 1^2\, dt\right)^{1/2}\left[\int_A^B |I(\tfrac{1}{2} + it)|^2\, dt\right]^{1/2} \\ &\leq (B - A)^{1/2}\left[\frac{1}{i}\int_{(1/2)-i\infty}^{(1/2)+i\infty} |I(s)|^2\, ds\right]^{1/2} \\ &\leq \text{const}\, \delta^{-1/2}\left(\frac{K'k + \epsilon k^2}{\delta^{1/2}}\right)^{1/2} \\ &= K_5 \delta^{-3/4}(K'k + \epsilon k^2)^{1/2}.\end{aligned}$$

Thus

$$K_5\delta^{-3/4}(K'k + \epsilon k^2)^{1/2} \geq K_1 k \delta^{-3/4} - K_2 k^2 \nu \delta^{1/4} - K_3 \delta^{-3/4} - K_4 k^2 \delta^{-1/4},$$

$$\nu \geq \frac{K_1}{K_2} k^{-1}\delta^{-1} - \frac{K_3}{K_2} k^{-2}\delta^{-1} - \frac{K_4}{K_2}\delta^{-1/2} - \frac{K_5}{K_2}\delta^{-1}k^{-1}\left(\frac{K'}{k} + \epsilon\right)^{1/2}.$$

The coefficient of $\delta^{-1}k^{-1}$ on the right can be made positive by choosing ϵ sufficiently small and k sufficiently large. Therefore with this fixed value of k it has been shown that for all sufficiently small δ the number ν of roots ρ on the line segment from $\frac{1}{2}$ to $\frac{1}{2} + i\delta^{-1}$ is at least $K_6\delta^{-1} - K_7\delta^{-1/2}$ with $K_6 > 0$. Since the $\delta^{-1/2}$ term is insignificant for small δ this proves the theorem.

11.3 THERE ARE AT LEAST *KT* log *T* ZEROS ON THE LINE

The basic structure of Selberg's proof is the same as that of the Hardy–Littlewood proof in the preceding section, but the proof begins not with the transform equation

$$(1) \qquad \frac{2\xi(s)}{s(s-1)} = \int_0^\infty u^{-s}\left[G(u) - 1 - \frac{1}{u}\right] du \qquad (0 < \text{Re}\, s < 1)$$

but with a transform equation in which the left side is $2\xi(s)[s(s-1)]^{-1}\phi(s) \cdot\phi^*(s)$ with $\phi^*(s) = \overline{\phi(1 - \bar{s})}$ the "adjoint" of $\phi(s)$ and with $\phi(s)$ specially chosen. In essence $\phi(s)$ is chosen to be an approximation to $\zeta(s)^{-1/2}$. Loosely speaking, this has the effect of approximately canceling the zeros of $\xi(s)$ and smoothing it out in such a way that the estimates of $|I|$ can be sharpened. (See Selberg's 1946 paper [S2] for a discussion of the motivation for the choice of ϕ.)

Specifically $\phi(s)$ is defined as follows. The function $\zeta(s)^{-1/2}$ can for $\operatorname{Re} s > 1$ be written as the transform of an operator of the form $f(x) \mapsto \sum_{n=1}^{\infty} \alpha_n f(nx)$. For this it suffices to write

$$\begin{aligned}\zeta(s)^{-1/2} &= \prod_p \left(1 - \frac{1}{p^s}\right)^{1/2} \\ &= \prod_p \left(1 - \frac{1}{2} p^{-s} + \frac{(\frac{1}{2})(-\frac{1}{2})}{2} p^{-2s} - \cdots\right).\end{aligned}$$

If this product is expanded, there is exactly one term in ν^{-s} for every positive integer ν and its coefficient is given explicitly by

$$(2) \qquad (-1)^{n_1}\binom{\frac{1}{2}}{n_1}(-1)^{n_2}\binom{\frac{1}{2}}{n_2} \cdots (-1)^{n_k}\binom{\frac{1}{2}}{n_k},$$

where $\nu = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ is the prime factorization of ν. Let α_ν denote the coefficient (2). Then

$$|\alpha_\nu| \leq \left|\binom{-\frac{1}{2}}{n_1}\binom{-\frac{1}{2}}{n_2} \cdots \binom{-\frac{1}{2}}{n_k}\right| \leq 1,$$

so the series $\sum \alpha_n n^{-s}$ converges for $\operatorname{Re} s > 1$. Moreover the absolute convergence of the product for $\zeta(s)^{-1/2}$ shows that $\zeta(s)^{-1/2} = \sum \alpha_n n^{-s}$ for $\operatorname{Re} s > 1$. However, because $\zeta(s)$ has a simple pole at $s = 1$, this function $\zeta(s)^{-1/2}$ has a singularity at $s = 1$ and cannot be continued in any simple way over to the critical line $\operatorname{Re} s = \frac{1}{2}$. Selberg deals with this by using a sort of *convergence factor*, by introducing a large parameter X and setting

$$\beta_n = \begin{cases} \left(1 - \dfrac{\log n}{\log X}\right)n, & n \leq X, \\ 0, & n \geq X, \end{cases}$$

$$\phi(s) = \phi_X(s) = \sum_{n=1}^{\infty} \beta_n n^{-s}.$$

Since $\beta_n \sim \alpha_n$ for small values of n, the function $\phi(s)$ is in some sense an approximation to $\zeta(s)^{-1/2}$, at least in the halfplane $\operatorname{Re} s > 1$. On the other hand, the series defining ϕ is finite so $\phi(s)$ is defined and analytic for all s.

With this definition of $\phi(s)$ set

$$I(s) = \frac{1}{2\pi i}\int_{s-ik}^{s+ik} \frac{2\xi(s)}{s(s-1)} \phi(s)\phi^*(s) x^{s-1}\, ds$$

where $\phi^*(s) = \overline{\phi(1-\bar{s})} = \phi(1-s)$. Then I depends on three parameters,

k (half the length of the interval of integration), X [the large parameter measuring, roughly, the degree of approximation of $\phi(s)$ to $\zeta(s)^{-1/2}$], and x (a complex number on the unit circle $|x| = 1$ near $i^{-1/2}$ but above it, say $x = i^{-1/2}e^{i\delta}$, where δ is small and positive). The idea of the proof is to show that, when these parameters are suitably chosen, the modulus of $I(\frac{1}{2} + it)$ is on the average much less than

$$J(t) = \frac{1}{2\pi}\int_{t-k}^{t+k} \frac{2|\xi(\frac{1}{2}+iv)|}{v^2+\frac{1}{4}}\left|\phi\left(\frac{1}{2}+iv\right)\right|^2 e^{\pi v/4}e^{-v\delta}\,dv.$$

Since the modulus of $I(\frac{1}{2} + it)$ is equal to $J(t)$ unless $\xi(\frac{1}{2} + iv)$ changes sign in the interval $\{t - k \leq v \leq t + k\}$, this will show that on the average it is to be expected that $\xi(\frac{1}{2} + iv)$ does change sign and therefore that there is very often a root in the interval.

As before, the first step in the estimation of $|I|$ is to write $I(s)$ as the transform of an operator and to apply the Parseval formula. First write $2\xi(s) \cdot [s(s-1)]^{-1}\phi(s)\phi^*(s)$ as the transform of an operator by composing the operator with transform (1) with the operators

$$f(x) \mapsto \sum_{n=1}^{\infty} \beta_n f(nx), \qquad f(x) \mapsto \sum_{n=1}^{\infty} \frac{\beta_n}{n} f\left(\frac{x}{n}\right)$$

with transforms $\phi(s)$, $\phi^*(s)$, respectively, to find

$$\frac{2\xi(s)}{s(s-1)}\phi(s)\phi^*(s) = \int_0^\infty u^{-s} \sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty} \frac{\beta_\mu\beta_\nu}{\nu}\left[G\left(\frac{\mu u}{\nu}\right) - 1 - \frac{\nu}{\mu u}\right] du.$$

(The sums are actually finite, so there is no problem with termwise integration.) Put xu in place of u. Then the path of integration becomes the ray through x^{-1}, but the rapid vanishing of $G(xu) - 1$ as $u \to \infty$ on the real axis and the consequent (by the functional equation of G) rapid vanishing of $G(xu) - (xu)^{-1}$ as $u \downarrow 0$ makes it valid to replace this path of integration by the real axis. Since $G(u) - 1 = 2\sum_{n=1}^{\infty} \exp(-\pi n^2 u^2)$ and since

$$\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty} \frac{\beta_\mu\beta_\nu}{\mu} = \left(\sum_{\mu=1}^{\infty}\frac{\beta_\mu}{\mu}\right)\left(\sum_{\nu=1}^{\infty}\beta_\nu\right) = \phi(1)\phi(0),$$

this puts the formula in the form

$$\frac{2\xi(s)}{s(s-1)}\phi(s)\phi^*(s)x^{s-1} = \int_0^\infty u^{-s}\left[\Psi(xu) - \frac{\phi(1)\phi(0)}{xu}\right] du, \tag{3}$$

where

$$\Psi(xu) = 2\sum_{n=1}^{\infty}\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty} e^{-\pi n^2\mu^2x^2u^2/\nu^2}\frac{\beta_\mu\beta_\nu}{\nu}.$$

Then integration ds from $s - ik$ to $s + ik$ on both sides gives

$$I(s) = \int_0^\infty u^{-s}\frac{\sin(k\log u)}{\pi\log u}\left[\Psi(xu) - \frac{\phi(1)\phi(0)}{xu}\right] du;$$

so by the Parseval formula

$$(4)\quad \frac{1}{2\pi i}\int_{(1/2)-i\infty}^{(1/2)+i\infty} |I(s)|^2\,ds = \int_0^\infty \left(\frac{\sin(k\log u)}{\pi\log u}\right)^2 \left|\Psi(xu) - \frac{\phi(1)\phi(0)}{xu}\right|^2 du.$$

The left side of this equation is to be estimated by estimating the right side using the explicit formula for Ψ. Rather than estimating the right side directly, however, Selberg bases his estimates of it on an estimate of the integral

$$W(z,\theta) = \int_z^\infty |\Psi(xu)|^2\, u^{-\theta}\, du$$

instead. This estimate, which is proved in Section 11.4 below, is the following.

Lemma There exist constants K and δ_0 such that

$$W(z,\theta) \le \frac{K}{\delta^{1/2}\theta z^\theta \log X}$$

holds for all δ in the range $0 < \delta \le \delta_0$ (where δ enters the definition of W because $x = e^{-i\pi/4}e^{i\delta}$) provided the other parameters satisfy the restrictions $0 < \theta \le \frac{1}{2}$, $1 \le z \le \delta^{-1/15}$, and $1 \le X \le \delta^{-1/15}$ (where X enters the definition of W because the β's depend on X).

This lemma will be used not only in estimating $\int |I|^2$ but also in estimating $\int |J|^2$. Consider first the estimation of $\int |I|^2$. As in the previous case the symmetry of the integrand implies that the right side of (4) can be replaced by twice the integral from one to infinity. If this interval of integration is subdivided at the point h (in the previous argument h was $e^{\pi/k}$) and the usual inequality for $x^{-1}\sin x$ is used, then

$$\begin{aligned}
\frac{1}{2\pi i}\int_{(1/2)-i\infty}^{(1/2)+i\infty} |I(s)|^2\,ds &\le \frac{2k^2}{\pi^2}\int_1^h \left|\Psi(xu) - \frac{\phi(0)\phi(1)}{xu}\right|^2 du \\
&\quad + \frac{2}{\pi^2}\int_h^\infty (\log u)^{-2}\left|\Psi(xu) - \frac{\phi(0)\phi(1)}{xu}\right|^2 du \\
&\le \frac{4k^2}{\pi^2}\int_1^h |\Psi(xu)|^2\,du + \frac{4k^2}{\pi^2}|\phi(1)\phi(0)|^2\left(1 - \frac{1}{h}\right) \\
&\quad + \frac{4}{\pi^2}\int_h^\infty (\log u)^{-2}|\Psi(xu)|^2\,du \\
&\quad + \frac{4}{\pi^2}|\phi(1)\phi(0)|^2\int_h^\infty \frac{du}{u^2(\log u)^2}.
\end{aligned}$$

Assume $h \ge e$. (Later h will go to infinity.) The second and fourth terms combined are at most

$$\begin{aligned}
&|\phi(1)\phi(0)|^2\frac{4}{\pi^2}\left[k^2\left(1 - \frac{1}{h}\right) + \frac{1}{(\log h)^2}\int_h^\infty \frac{du}{u^2}\right] \\
&\le \left|\sum_\mu \frac{\beta_\mu}{\mu}\right|^2 |\sum_\nu \beta_\nu|^2 \frac{4}{\pi^2}\left(k^2 + \frac{1}{(\log h)^2}\right)
\end{aligned}$$

which, since $|\beta_j| \le 1$ and $\beta_j = 0$ for $j \ge X$, is at most a constant times $(\log X)^2 X^2[k^2 + (\log h)^{-2}]$. Therefore

$$\frac{1}{2\pi i}\int_{(1/2)+i\infty}^{(1/2)+i\infty} |I(s)|^2\,ds \le \frac{4k^2}{\pi^2}\int_1^h |\Psi(xu)|^2\,du + \frac{4}{\pi^2}\int_h^\infty (\log u)^{-2}|\Psi(xu)|^2\,du$$
$$+ K_1 k^2 X^2 (\log X)^2 + K_1 X^2 \frac{(\log X)^2}{(\log h)^2}.$$

The lemma can be used to estimate the two integrals on the right. Let K, δ_0 be as in the lemma, let $0 < \theta \le \frac{1}{2}$, $1 \le h \le \delta^{-1/15}$, and let $1 \le X \le \delta^{-1/15}$. Then the first integral above is at most

$$\begin{aligned}
&\frac{4k^2}{\pi^2}\int_1^h u^\theta u^{-\theta}|\Psi(xu)|^2\,du\\
&\le \frac{4k^2}{\pi^2}\int_1^h z^\theta\left[-\frac{\partial W}{\partial z}\right]dz\\
&= -\frac{4k^2}{\pi^2} z^\theta W(z,\theta)\Big|_1^h + \frac{4k^2\theta}{\pi^2}\int_1^h z^{\theta-1}W(z,\theta)\,dz\\
&\le \frac{4k^2}{\pi^2}\frac{K}{\delta^{1/2}\theta\log X} + \frac{4k^2\theta}{\pi^2}\int_1^h \frac{Kz^{\theta-1}}{\delta^{1/2}\theta z^\theta \log X}\,dz\\
&= \frac{4k^2 K}{\pi^2\delta^{1/2}\log X}\left(\frac{1}{\theta} + \log h\right).
\end{aligned}$$

If $\theta = \frac{1}{2}$ and h is very large, the second term dominates and the integral is at most a constant times $k^2\delta^{-1/2}(\log X)^{-1}(\log h)$. To estimate the second integral, use the identity

$$\int_0^{1/2} \theta u^{-\theta}\,d\theta = -\frac{1}{2u^{1/2}\log u} - \frac{1}{u^{1/2}(\log u)^2} + \frac{1}{(\log u)^2}$$

(integration by parts) and $h \ge e$ to find

$$\begin{aligned}
&\frac{4}{\pi^2}\int_h^\infty (\log u)^{-2}|\Psi(xu)|^2\,du\\
&= \frac{4}{\pi^2}\int_h^\infty\int_0^{1/2}\theta u^{-\theta}|\Psi(xu)|^2\,d\theta\,du\\
&\quad + \frac{4}{\pi^2}\int_h^\infty \frac{u^{-1/2}}{\log u}\left[\frac{1}{2} + \frac{1}{\log u}\right]|\Psi(xu)|^2\,du\\
&\le \frac{4}{\pi^2}\int_0^{1/2}\theta W(h,\theta)\,d\theta + \frac{6}{\pi^2}\int_h^\infty u^{-1/2}|\Psi(xu)|^2\,du\\
&\le \frac{4}{\pi^2}\int_0^{1/2}\frac{K\,d\theta}{\delta^{1/2}h^\theta\log X} + \frac{6}{\pi^2}\frac{K}{\delta^{1/2}\frac{1}{2}h^{1/2}\log X}\\
&= \frac{4}{\pi^2}\frac{K}{\delta^{1/2}\log X}\left(\frac{1}{\log h} - \frac{1}{h^{1/2}\log h}\right) + \frac{12K}{\pi^2\delta^{1/2}h^{1/2}\log X}.
\end{aligned}$$

If h is large, the dominant term is the first one, which is less than a constant times $\delta^{-1/2}(\log X)^{-1}(\log h)^{-1}$. Thus with the above restrictions on the parameters plus the condition that h be sufficiently large

$$(5)\qquad \frac{1}{2\pi i}\int_{(1/2)-i\infty}^{(1/2)+i\infty} |I(s)|^2\, ds \le \frac{K_2 k^2 \log h}{\delta^{1/2}\log X} + \frac{K_3}{\delta^{1/2}\log X \log h} + K_1 k^2 X^2 (\log X)^2 + K_1 X^2 \frac{(\log X)^2}{(\log h)^2}.$$

Choose X as large as possible, namely, $X = \delta^{-1/15}$. Then the third term is still less than $K_1 k^2 X^3 < K_1 k^2 \delta^{-1/5}$ which is insignificant compared to the first term (when δ is small and h large) and in the same way the fourth term is insignificant compared to the second. If the first two terms are to have the same order of magnitude, then $k^2(\log h)^2$ must have the order of magnitude 1; this motivates setting $h = e^{\pi/k}$ as in the previous proof. Since h is to become large, this implies $k \to 0$. Loosely speaking, the major shortcoming of the proof of Section 11.2 was the fact that in it k did not go to zero; this meant that the subdivision of the interval $\{A \le t \le B\}$ never became very fine. Thus $k \to 0$ is a major aspect of Selberg's proof. On the other hand—as is not surprising—it is essential that $k \to 0$ very, very slowly. For this reason set $k = (a \log \delta^{-1})^{-1}$, where a is a very small positive constant to be determined later. Finally, for notational convenience set $T = \delta^{-1}$. Then the parameters have been reduced to two, namely, T (large) and a (small) and the others have been related to these two by

$$(6)\qquad \delta = T^{-1}, \qquad x = e^{-i\pi/4} e^{i/T}, \qquad X = T^{1/15}, \qquad k = (a \log T)^{-1},$$

and by $h = e^{\pi/k}$ for the parameter h, which does not appear in the final result. What has been proved (except, of course, for the proof of the lemma) is that when the parameters of I are chosen in this way, then for every $a > 0$ the inequality

$$\frac{1}{2\pi i}\int_{(1/2)-i\infty}^{(1/2)+i\infty} |I(s)|^2\, ds \le K_4 \frac{k}{\delta^{1/2}\log X} = \frac{K_5 T^{1/2}}{a(\log T)^2}$$

holds for all sufficiently large T.

Consider now the estimation of the average value of $J(t)$. Let

$$F(v) = \frac{1}{\pi}\frac{|\xi(\frac{1}{2}+iv)|}{v^2+\frac{1}{4}} |\phi(\tfrac{1}{2}+iv)|^2 e^{\pi v/4} e^{-v\delta}$$

so that $J(t) = \int_{t-k}^{t+k} F(v)\, dv$. It is natural to begin by estimating the average magnitude of the positive function $F(v)$. It was shown in the preceding section that

$$\frac{|\xi(\frac{1}{2}+iv)|\, e^{\pi v/4}}{v^2+\frac{1}{4}} \ge \text{const}\, v^{-1/4} |\zeta(\tfrac{1}{2}+iv)|,$$

so $F(v) \ge \text{const}\, v^{-1/4} |\zeta(\frac{1}{2}+iv)|\, |\phi(\frac{1}{2}+iv)|^2 e^{-v\delta}$ and it will suffice to find a lower bound on the average magnitude of $\zeta(\frac{1}{2}+iv)[\phi(\frac{1}{2}+iv)]^2$. This can be done by considering the integral of $\zeta(s)\phi(s)^2$ around the boundary of a rectangle $\{\frac{1}{2} \leq \operatorname{Re} s \leq 2, A \leq \operatorname{Im} s \leq B\}$ where $B > A \geq 1$ as follows. The whole integral around the boundary is zero by Cauchy's theorem. By the definition of Lindelöf's μ-function, by $\mu(\frac{1}{2}) \leq \frac{1}{4}$, and by Lindelöf's theorem (see Section 9.2) $|\zeta(\sigma + it)|$ is less than a constant times $t^{1/4+\epsilon}$ for all $\sigma + it$ in the half-strip $\{\frac{1}{2} \leq \sigma \leq 2, t \geq 1\}$. On the other hand $|\phi(s)|$ in this half-strip is $|\sum \beta_n n^{-\sigma-it}| \leq \sum |\beta_n| n^{-\sigma} \leq \sum_{n\leq X} n^{-1/2}$, which by Euler–Maclaurin is about $\int_1^X u^{-1/2}\,du \leq 2X^{1/2}$ and which is therefore less than a constant times $X^{1/2}$. Therefore the integral of $\zeta(s)\phi(s)^2$ over the side $\operatorname{Im} s = A$ of the rectangle has modulus at most a constant times $A^{1/4+\epsilon}X$ and the integral over the side $\operatorname{Im} s = B$ modulus at most the same constant times $B^{1/4+\epsilon}X$. On the side $\operatorname{Re} s = 2$ of the rectangle, the integrand can be written in the form

$$\zeta(s)\phi(s)^2 = 1 + \sum_{n=2}^{\infty} \frac{a_n}{n^s},$$

where $|a_n|$ is less than the coefficient of n^{-s} in the expansion of $\zeta(s)^3$ (multiplication of absolutely convergent series); so

$$\int_{2+iA}^{2+iB} \zeta(s)\phi(s)^2\,ds = i(B - A) + \sum_{n=2}^{\infty} a_n \int_{2+iA}^{2+iB} \frac{ds}{n^s},$$

and the integral over this side differs from $i(B - A)$ by at most

$$\sum_{n=2}^{\infty} |a_n| \frac{2}{n^2 \log n} \leq \frac{2}{\log 2} \sum_{n=1}^{\infty} \frac{|a_n|}{n^2} \leq \frac{2}{\log 2}\zeta(2)^3 \leq \text{const.}$$

Therefore the integral over the side $\operatorname{Re} s = \frac{1}{2}$ differs from $i(B - A)$ by a quantity whose modulus is at most

$$\text{const} + \text{const}\, A^{1/4+\epsilon} + \text{const}\, B^{1/4+\epsilon}$$

which shows that

$$\left| \int_A^B \zeta(\tfrac{1}{2} + iv)\phi(\tfrac{1}{2} + iv)^2\,dv \right| \geq B - A - \text{const}\, B^{1/4+\epsilon}.$$

Therefore

$$\begin{aligned}
\int_A^B F(v)\,dv &\geq \text{const}\, B^{-1/4}e^{-B\delta} \int_A^B |\zeta(\tfrac{1}{2} + iv)||\phi(\tfrac{1}{2} + iv)|^2\,dv \\
&\geq \text{const}\, B^{-1/4}e^{-B\delta} \left| \int_A^B \zeta(\tfrac{1}{2} + iv)\phi(\tfrac{1}{2} + iv)^2\,dv \right| \\
&\geq \text{const}\, B^{-1/4}e^{-B\delta}(B - A) - \text{const}\, B^{\epsilon}, \\
\int_A^B J(t)\,dt &= \int_A^B \int_{t-k}^{t+k} F(v)\,dv\,dt \geq \int_{A+k}^{B-k} \int_{v-k}^{v+k} F(v)\,dt\,dv \\
&= \int_{A+k}^{B-k} 2kF(v)\,dv \\
&\geq \text{const}\, 2kB^{-1/4}e^{-B\delta}(B - A - 2k) - \text{const}\, 2kB^{\epsilon}.
\end{aligned}$$

As before, define A and B by $(B + k)\delta = 1$, $B - A = \frac{1}{2}\delta^{-1}$. In other words, set

$$B = T - k, \qquad A = \tfrac{1}{2}T - k. \tag{7}$$

Then since $k \longrightarrow 0$, it follows easily from the above that

$$\int_A^B J(t)\,dt \geq \text{const } 2kT^{3/4} = \frac{K_6 T^{3/4}}{a \log T} \tag{8}$$

for all sufficiently large T when a is fixed and the other parameters are determined as in (6) and (7). This estimate is less exact than the corresponding estimate in Section 11.2 because that estimate gave a lower bound for $\int_S J(t)\,dt$ where S is a *subset* of the interval $\{A \leq t \leq B\}$. However, this estimate can be made to serve a similar purpose by combining it with the following estimate of $\int_{-\infty}^{\infty} J(t)^2\,dt$.

Parseval's equation applied to the transform equation (3) gives

$$\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \frac{4|\xi(s)|^2}{|s|^2|s-1|^2}|\phi(s)|^2|\phi(1-s)|^2|x^{s-1}|^2\,ds$$
$$= \int_0^\infty \left|\Psi(xu) - \frac{\phi(1)\phi(0)}{xu}\right|^2 du.$$

When the formula for $F(v)$ and the familiar symmetry of the integral on the right are used, this becomes

$$2\pi\int_{-\infty}^{\infty} F(v)^2\,dv = 2\int_1^\infty \left|\Psi(xu) - \frac{\phi(1)\phi(0)}{xu}\right|^2 du$$
$$\leq 4\int_1^\infty |\Psi(xu)|^2\,du + 4|\phi(1)\phi(0)|^2\int_1^\infty u^{-2}\,du.$$

The second integral on the right is $4|\phi(1)\phi(0)|^2$ which is less than a constant times $X^2(\log X)^2$ which is less than $T^{1/5}$. The first integral on the right is $W(1, 0)$, but since the lemma does not apply when $\theta = 0$, an estimate of $W(1, 0)$ has yet to be made. Now

$$\int_1^\infty |\Psi(xu)|^2\,du = \int_1^{T^2} |\Psi(xu)|^2\,du + \int_{T^2}^\infty |\Psi(xu)|^2\,du$$
$$\leq e^2\int_1^{T^2} |\Psi(xu)|^2 e^{-\log u/\log T}\,du + \int_{T^2}^\infty |\Psi(xu)|^2\,du$$
$$\leq e^2 W\left(1, \frac{1}{\log T}\right) + \int_{T^2}^\infty |\Psi(xu)|^2\,du$$
$$\leq e^2 15KT^{1/2} + \int_{T^2}^\infty |\Psi(xu)|^2\,du$$

by the lemma, so it will suffice to show that $\int_{T^2}^\infty |\Psi(xu)|^2\,du$ is less than $T^{1/2}$ in order to find an upper estimate of $\int F^2\,dv$. Expand $|\Psi(xu)|^2$ as a sextuple sum

$$\sum_{m,n,\mu,\nu,\kappa,\lambda} \frac{\beta_\kappa\beta_\lambda\beta_\mu\beta_\nu}{\lambda\nu} e^{-\pi m^2\kappa^2 u^2 x^2/\lambda^2} e^{-\pi n^2\mu^2 u^2 \bar{x}^2/\nu^2}.$$

For fixed m, n the sum over $\mu, \nu, \kappa, \lambda$ consists of at most X^4 terms the largest of which has modulus at most $\exp[-\pi m^2 X^{-2} u^2 \operatorname{Re} x^2 - \pi n^2 X^{-2} u^2 \operatorname{Re} x^2] = \exp[-\pi(m^2 + n^2) X^{-2} u^2 \sin 2\delta]$, so $|\Psi(xu)|^2$ is at most X^4 times the double sum of this exponential over all m, n. But this is less than

$$\begin{aligned} X^4 &\int_0^\infty \int_0^\infty e^{-\pi(x^2+y^2)X^{-2}u^2 \sin 2\delta}\, dx\, dy \\ &= X^4 \int_0^{\pi/2} \int_0^\infty e^{-\pi r^2 X^{-2} u^2 \sin 2\delta} r\, dr\, d\theta \\ &= \frac{\pi}{2} X^4 \cdot \frac{1}{2} \int_0^\infty e^{-\pi v X^{-2} u^2 \sin 2\delta}\, dv \le \text{const}\, \frac{X^2}{u^2 \delta}, \end{aligned}$$

so its integral du from T^2 to ∞ is at most a constant times $(X^2/\delta)T^{-2} = X^2/T \le T^{-13/15}$. Thus $\int_{-\infty}^\infty F(v)^2\, dv$ is less than a constant times $T^{1/2}$, from which it follows that

$$\begin{aligned} \int_{-\infty}^\infty J(t)^2\, dt &= \int_{-\infty}^\infty \left[\int_{t-k}^{t+k} F(v)\, dv\right]^2 dt \\ &\le \int_{-\infty}^\infty \left(\int_{t-k}^{t+k} 1^2\, dv\right)\left(\int_{t-k}^{t+k} F(v)^2\, dv\right) dt \\ &= 2k \int_{-\infty}^\infty \int_{t-k}^{t+k} F(v)^2\, dv\, dt \\ &= 2k \int_{-\infty}^\infty \int_{v-k}^{v+k} F(v)^2\, dt\, dv \\ &= 4k^2 \int_{-\infty}^\infty F(v)^2\, dv \\ &\le \text{const}\, k^2 T^{1/2} = \frac{K_7 T^{1/2}}{a^2 (\log T)^2}. \end{aligned}$$

Now let ν be the number of zeros of $\xi(\frac{1}{2} + it)$ in the interval $\{0 \le t \le B + k\}$. Let the entire real axis be divided into intervals of length k, and for each of the ν zeros strike out the interval which contains it and the two intervals which adjoin this one. Let S be the subset of $\{A \le t \le B\}$ consisting of points which do not lie in stricken intervals and let $\tilde{S}$ be the subset of $\{A \le t \le B\}$ consisting of those which do. Then the total length of the intervals of $\tilde{S}$ is at most $3\nu k$ and $|I(\frac{1}{2} + it)| = J(t)$ on S so

$$\begin{aligned} \int_S |I(\tfrac{1}{2} + it)|\, dt &= \int_S J(t)\, dt = \int_A^B J(t)\, dt - \int_{\tilde{S}} J(t)\, dt \\ &\ge \frac{K_6 T^{3/4}}{a \log T} - \left(\int_{\tilde{S}} 1^2\, dt\right)^{1/2} \left(\int_{\tilde{S}} J(t)^2\, dt\right)^{1/2} \\ &\ge \frac{K_6 T^{3/4}}{a \log T} - (3\nu k)^{1/2} \left[\int_{-\infty}^\infty J(t)^2\, dt\right]^{1/2} \\ &\ge \frac{K_6 T^{3/4}}{a \log T} - (3\nu k)^{1/2} \frac{K_7^{1/2} T^{1/4}}{a \log T}. \end{aligned}$$

On the other hand

$$\begin{aligned}\int_S |I(\tfrac{1}{2}+it)|\,dt &\le \left(\int_S 1^2\right)^{1/2}\left(\int_S |I(\tfrac{1}{2}+it)|^2\,dt\right)^{1/2}\\ &\le \left(\int_A^B dt\right)^{1/2}\left(\frac{1}{i}\int_{1/2-i\infty}^{1/2+i\infty}|I(s)|^2\,ds\right)^{1/2}\\ &\le \text{const } T^{1/2}\frac{T^{1/4}}{a^{1/2}\log T}\\ &= \frac{K_8 T^{3/4}}{a^{1/2}\log T}.\end{aligned}$$

Thus

$$(3\nu k)^{1/2} \ge K_7^{-1/2}(K_6 T^{1/2} - a^{1/2}K_8 T^{1/2}).$$

For fixed a sufficiently small the right side is a positive constant times $T^{1/2}$; hence

$$3\nu k \ge K_9 T, \qquad \nu \ge \frac{K_9 T}{3k} = \frac{K_9 a}{3} T \log T,$$

and Selberg's theorem is proved.

11.4 PROOF OF A LEMMA

The core of Selberg's proof—and the only part of the proof which makes use of the special choice of the function $\phi(s)$—is the estimate of the integral $W(z, \theta)$ stated in the lemma of the preceding section. This section is devoted to the proof of this estimate.

Recall the following definitions from Section 11.3. There δ is a small positive number; $x = e^{-i\pi/4}e^{i\delta}$; X is a large positive number; α_n is the coefficient of n^{-s} in the Dirichlet series expansion of $\zeta(s)^{-1/2}$; β_n is zero for $n \ge X$ and $(\log X)^{-1}\log(X/n)\alpha_n$ if $n \le X$; $\Psi(u)$ is defined by

$$\Psi(u) = 2\sum_{n=1}^{\infty}\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty} e^{-\pi n^2\mu^2u^2/\nu^2}\frac{\beta_\mu\beta_\nu}{\nu};$$

and $W(z, \theta)$ is defined by

$$W(z, \theta) = \int_z^\infty u^{-\theta}|\Psi(xu)|^2\,du,$$

z, θ being positive real numbers. The lemma to be proved states that there exist positive constants K, δ_0 such that $W(z, \theta) \le K(\delta^{1/2}\theta z^\theta \log X)^{-1}$ whenever the parameters lie in the ranges

$$0 < \delta \le \delta_0, \qquad 0 < \theta \le \tfrac{1}{2}, \qquad 1 \le X \le \delta^{-1/15}, \qquad 1 \le z \le \delta^{-1/15}.$$

Substitution of the definition of Ψ into the definition of W gives an expression of W as the integral of a sextuple sum of terms of the form

$$4u^{-\theta}\frac{\beta_\kappa\beta_\lambda\beta_\mu\beta_\nu}{\lambda\nu}e^{-\pi m^2\kappa^2u^2x^2/\lambda^2}e^{-\pi n^2\mu^2u^2\bar{x}^2/\nu^2}$$
$$= 4u^{-\theta}\frac{\beta_\kappa\beta_\lambda\beta_\mu\beta_\nu}{\lambda\nu}$$
$$\times \exp\left[-\pi u^2\left(\frac{m^2\kappa^2}{\lambda^2}+\frac{n^2\mu^2}{\nu^2}\right)\operatorname{Re}x^2 - i\pi u^2\left(\frac{m^2\kappa^2}{\lambda^2}-\frac{n^2\mu^2}{\nu^2}\right)\operatorname{Im}x^2\right].$$

Let Σ_1 denote the integral of the sum of the terms in which the imaginary part of the exponential is zero, that is, those terms in which $(m\kappa/\lambda) = (n\mu/\nu)$, and let Σ_2 denote the integral of the sum of the remaining terms. The difficult part of the proof is the estimation of Σ_1, after which it is comparatively easy to show that Σ_2 is much smaller than the estimate which is obtained for Σ_1.

For each fixed quadruple $(\kappa, \lambda, \mu, \nu)$ there is an infinite sequence of terms of Σ_1, namely, the terms corresponding to (m, n) where $m/n = (\lambda\mu/\kappa\nu)$. Let a/b be the expression of $(\lambda\mu/\kappa\nu)$ in lowest terms; then the pairs (m, n) are (a, b), $(2a, 2b)$, $(3a, 3b)$, . . . and therefore

$$\Sigma_1 = \int_z^\infty 4u^{-\theta}\sum_{\kappa\lambda\mu\nu}\frac{\beta_\kappa\beta_\lambda\beta_\mu\beta_\nu}{\lambda\nu}\sum_{r=1}^\infty \exp\left[-2\pi u^2\frac{r^2a^2\kappa^2}{\lambda^2}\sin 2\delta\right]du,$$

where a depends on $(\kappa, \lambda, \mu, \nu)$ as above. This expression can be made more symmetrical by defining q to be the greatest common divisor of $\lambda\mu$ and $\kappa\nu$ (the factor canceled when $\lambda\mu/\kappa\nu$ is reduced to lowest terms) so that $a = \lambda\mu/q$, $b = \kappa\nu/q$, and the above becomes

$$\Sigma_1 = 4\int_z^\infty u^{-\theta}\sum_{\kappa\lambda\mu\nu}\frac{\beta_\kappa\beta_\lambda\beta_\mu\beta_\nu}{\lambda\nu}\sum_{r=1}^\infty \exp[-2\pi u^2r^2\mu^2\kappa^2q^{-2}\sin 2\delta]\,du.$$

The sum over $\kappa\lambda\mu\nu$ is finite because β_j is zero for $j \geq X$. Therefore this sum can be taken out from under the integral sign, and Σ_1 becomes a finite sum of terms of the form

$$\text{const}\int_z^\infty u^{-\theta}\sum_{r=1}^\infty e^{-u^2r^2\eta^2}\,du$$

where $\eta = \kappa\mu q^{-1}(2\pi\sin 2\delta)^{1/2}$. Such a term can be estimated as follows:

$$\begin{aligned}\int_z^\infty u^{-\theta}\sum_{r=1}^\infty e^{-u^2r^2\eta^2}\,du &= \sum_{r=1}^\infty\int_z^\infty u^{1-\theta}e^{-u^2r^2\eta^2}\,d\log u\\ &= \sum_{r=1}^\infty\int_{zr\eta}^\infty\left(\frac{y}{r\eta}\right)^{1-\theta}e^{-y^2}\,d\log y\\ &= \eta^{\theta-1}\sum_{r=1}^\infty r^{\theta-1}\int_{zr\eta}^\infty y^{-\theta}e^{-y^2}\,dy\\ &= \eta^{\theta-1}\int_{z\eta}^\infty\Big(\sum_{1\le r\le y/z\eta} r^{\theta-1}\Big)y^{-\theta}e^{-y^2}\,dy.\end{aligned}$$

The sum in the integrand can be estimated by Euler–Maclaurin summation

$$\sum_{1\le r\le N} r^{\theta-1} = \int_1^N v^{\theta-1}\,dv + \frac{1}{2}[1^{\theta-1} + N^{\theta-1}] + \int_1^N \bar{B}_1(v)(\theta-1)v^{\theta-2}\,dv$$
$$= \frac{N^\theta}{\theta} - \frac{1}{\theta} + \frac{1}{2} + \frac{1}{2}N^{\theta-1} + \int_1^\infty \bar{B}_1(v)(\theta-1)v^{\theta-2}\,dv$$
$$- \int_N^\infty \bar{B}_1(v)(\theta-1)v^{\theta-2}\,dv.$$

Let N be the smallest integer greater than $y/z\eta$. Then subtracting $N^{\theta-1}$ from both sides gives

$$\sum_{1\le r\le y/z\eta} r^{\theta-1} = \frac{N^\theta}{\theta} - \frac{1}{\theta} + \frac{1}{2} - \frac{1}{2}N^{\theta-1} + \int_1^\infty \bar{B}_1(v)(\theta-1)v^{\theta-2}\,dv$$
$$- \int_N^\infty \bar{B}_1(v)(\theta-1)v^{\theta-2}\,dv.$$

Since $\theta - 1 < 0$, this differs from

$$\frac{1}{\theta}\left(\frac{y}{z\eta}\right)^\theta - \frac{1}{\theta} + \frac{1}{2} + \int_1^\infty \bar{B}_1(v)(\theta-1)v^{\theta-2}\,dv$$

by at most

$$\left|\frac{N^\theta}{\theta} - \frac{1}{\theta}\left(\frac{y}{z\eta}\right)^\theta - \frac{1}{2}N^{\theta-1} - (\theta-1)\int_N^\infty \bar{B}_1(v)v^{\theta-2}\,dv\right|$$
$$\le \int_{y/z\eta}^N u^{\theta-1}\,du + \frac{1}{2}\left(\frac{y}{z\eta}\right)^{\theta-1}$$
$$+ |\theta-1|\int_0^{1/2}\left|u-\frac{1}{2}\right|(N+u)^{\theta-2}\,du$$
$$\le \frac{3}{2}\left(\frac{y}{z\eta}\right)^{\theta-1} + \frac{1}{2}\cdot\frac{1}{2}N^{\theta-2}$$
$$< \frac{3}{2}\left(\frac{y}{z\eta}\right)^{\theta-1} + \frac{1}{4}N^{\theta-1} < 2\left(\frac{y}{z\eta}\right)^{\theta-1}.$$

Substitution of this estimate into the original integral shows that

$$\text{(1)}\qquad \int_z^\infty u^{-\theta}\sum_{r=1}^\infty e^{-u^2r^2\eta^2}\,du$$

differs from

$$\text{(2)}\quad \eta^{\theta-1}\int_{z\eta}^\infty\left[\frac{1}{\theta}\left(\frac{y}{z\eta}\right)^\theta - \frac{1}{\theta} + \frac{1}{2} + \int_1^\infty \bar{B}_1(v)(\theta-1)v^{\theta-2}\,dv\right]y^{-\theta}e^{-y^2}\,dy$$

by at most

$$2\eta^{\theta-1}(z\eta)^{1-\theta}\int_{z\eta}^\infty y^{\theta-1}y^{-\theta}e^{-y^2}\,dy$$
$$= 2z^{1-\theta}\int_{z\eta}^1 y^{-1}e^{-y^2}\,dy + 2z^{1-\theta}\int_1^\infty y^{-1/2}e^{-y^2}\,dy$$
$$\le \text{const}\; z^{1-\theta}\,|\log z\eta| + \text{const}\; z^{1-\theta}.$$

If the integral (2) is extended to the interval $\{0 \le y < \infty\}$, it becomes

$$\eta^{-1}z^{-\theta}\frac{1}{\theta}\int_0^\infty e^{-y^2}\,dy + \eta^{\theta-1}\left(-\frac{1}{\theta}+\frac{1}{2}+\int_1^\infty \bar{B}_1(v)(\theta-1)v^{\theta-2}\,dv\right)$$
$$\times\left(\int_0^\infty y^{-\theta}e^{-y^2}\,dy\right)$$
$$= \frac{\pi^{1/2}}{2\eta\theta z^\theta} + \frac{A(\theta)}{\theta}\eta^{\theta-1},$$

where $A(\theta)$ is a bounded function of $\{0 \le \theta \le \frac{1}{2}\}$, and this differs from the original integral (2) by at most

$$\eta^{-1}z^{-\theta}\frac{1}{\theta}\int_0^{z\eta} e^{-y^2}\,dy + \eta^{\theta-1}\left(-\frac{1}{\theta}+\frac{1}{2}+\int_1^\infty \bar{B}_1(v)(\theta-1)v^{\theta-2}\,dv\right)$$
$$\times\left(\int_0^{z\eta} y^{-\theta}e^{-y^2}\,dy\right)$$
$$\le \frac{z^{1-\theta}}{\theta} + \eta^{\theta-1}\,\mathrm{const}\,\frac{A(\theta)}{\theta}\int_0^{z\eta} y^{-\theta}\,dy \le \mathrm{const}\,\frac{z^{1-\theta}}{\theta}$$

because $A(\theta)$ is bounded. Therefore the original integral (1) differs from

$$\frac{\pi^{1/2}}{2\eta\theta z^\theta} + \frac{A(\theta)}{\theta}\eta^{\theta-1}$$

by at most

$$\mathrm{const}\, z^{1-\theta}\,|\log z\eta| + \mathrm{const}\, z^{1-\theta}/\theta.$$

Finally, using this estimate of (1) in Σ_1 shows that Σ_1 differs from

$$(3)\qquad 4\sum_{\kappa\lambda\mu\nu}\frac{\beta_\kappa\beta_\lambda\beta_\mu\beta_\nu}{\lambda\nu}$$
$$\times\left[\frac{\pi^{1/2}q}{2\kappa\mu(2\pi\sin 2\delta)^{1/2}\theta z^\theta} + \frac{A(\theta)}{\theta}(2\pi\sin 2\delta)^{(\theta-1)/2}\left(\frac{q}{\kappa\mu}\right)^{1-\theta}\right]$$

by at most a constant times

$$(4)\qquad \sum_{\kappa\lambda\mu\nu}\frac{|\beta_\kappa\beta_\lambda\beta_\mu\beta_\nu|}{\lambda\nu}\left\{\frac{z^{1-\theta}}{\theta} + z^{1-\theta}\left|\log\left[\frac{z\kappa\mu}{q}(2\pi\sin 2\delta)^{1/2}\right]\right|\right\}.$$

In the nonzero terms $\kappa, \lambda, \mu, \nu$ are all at most X, so $\kappa\mu/q$ lies between X^{-2} and X^2. On the other hand $\sin 2\delta$ is of the order of magnitude of δ; so, since X and z are at most $\delta^{-1/15}$, the logarithm in the error estimate (4) is at most a constant times $\log(1/\delta)$ in absolute value, and therefore grows rather slowly as $\delta \to 0$. Specifically

$$\frac{z\kappa\mu}{q}(2\pi\sin 2\delta)^{1/2} \ge \frac{1\cdot 1\cdot 1}{X^2}(2\pi)^{1/2}\delta^{1/2} \ge \mathrm{const}\,\delta^{2/15}\delta^{1/2} \ge \delta,$$
$$\frac{z\kappa\mu}{q}(2\pi\sin 2\delta)^{1/2} \le \frac{\delta^{-1/15}X^2}{1}(2\pi)^{1/2}(2\delta)^{1/2} \le \mathrm{const}\,\delta^{-1/5}\delta^{1/2} \le 1,$$
$$\log\delta \le \log\frac{z\kappa\mu}{q}(2\pi\sin 2\delta)^{1/2} \le 0,$$

for all sufficiently small δ. Since

$$\sum_{\kappa\lambda\mu\nu} \frac{|\beta_\kappa\beta_\lambda\beta_\mu\beta_\nu|}{\lambda\nu} \leq \sum_{\kappa\lambda\mu\nu} \frac{1}{\nu\lambda} = \Big(\sum_\kappa 1\Big)\Big(\sum_\lambda \frac{1}{\lambda}\Big)\Big(\sum_\mu 1\Big)\Big(\sum_\nu \frac{1}{\nu}\Big) \leq X^4,$$

this shows that the error estimate (4) is at most a constant times

$$\begin{aligned} X^4 \frac{z^{1-\theta}}{\theta} \log\Big(\frac{1}{\delta}\Big) &\leq \Big(\frac{1}{\delta}\Big)^{4/15}\Big(\frac{1}{\delta}\Big)^{1/15} \frac{1}{\theta z^\theta} \log\Big(\frac{1}{\delta}\Big) \\ &\leq \frac{\delta^{-1/3}\delta^{1/2} \log(1/\delta) \log X}{\delta^{1/2}\theta z^\theta \log X} \\ &\leq \frac{\delta^{1/6}\cdot\frac{1}{15}[\log(1/\delta)]^2}{\delta^{1/2}\theta z^\theta \log X} \end{aligned}$$

for all sufficiently small δ. Since this is smaller than the desired estimate $K(\delta^{1/2}\theta z^\theta \log X)^{-1}$ for small δ, it can be neglected and the desired estimate of Σ_1 is equivalent to the statement that the approximating quantity (3) is less than a constant times $(\delta^{1/2}\theta z^\theta \log X)^{-1}$ for all sufficiently small δ.

Consider now the approximating quantity (3). With the new notation

$$S(\sigma) = \sum_{\kappa\lambda\mu\nu} \Big(\frac{q}{\kappa\mu}\Big)^\sigma \frac{\beta_\kappa\beta_\lambda\beta_\mu\beta_\nu}{\lambda\nu}$$

(where for a given quadruple κ, λ, μ, ν the integer q is defined to be the greatest common divisor of $\kappa\nu$ and $\lambda\mu$), this quantity can be written

$$\text{(3}'\text{)} \qquad \frac{\sqrt{2}}{\theta z^\theta (\sin 2\delta)^{1/2}} S(1) + 4\frac{A(\theta)}{\theta}(2\pi \sin 2\delta)^{(\theta-1)/2} S(1-\theta).$$

Since $A(\theta)$ is bounded and since $\sin 2\delta$ is essentially equal to 2δ, in order to estimate this quantity it will suffice to estimate the quadruple sum $S(\sigma)$. This will be done by first reducing the estimation of $S(\sigma)$ to the estimation of a double sum, by then reducing this to the estimation of a single sum, and by finally estimating the single sum.

The first step in the estimation of $S(\sigma)$ makes use of a generalization of Euler's famous identity

$$\text{(5)} \qquad n = \sum_{m|n} \phi(m)$$

where ϕ is the Euler ϕ-function. For present purposes† the most convenient definition of ϕ is

$$\phi(n) = n \sum_{m|n} \frac{\mu(m)}{m} = n \prod_{p|n} \Big(1 - \frac{1}{p}\Big).$$

In order to prove that the function ϕ so defined has property (5), let $Q(x)$

†It is easily shown that $\phi(n)$ is equal to the number of integers between 0 and n relatively prime to n, and this property is the usual definition of ϕ.

$= \sum_{n<x} n$ and let $\Phi(x) = \sum_{n<x} \phi(n)$ with the usual adjustment for integral values of x. Then the definition of ϕ is tantamount to

$$\Phi(x) = \sum_{m=1}^{\infty} \mu(m) Q\left(\frac{x}{m}\right)$$

[at any integer $x = n$ both sides jump by $\phi(n) = \sum_{m|n} \mu(m)n/m$] which by Möbius inversion is equivalent to

$$Q(x) = \sum_{m=1}^{\infty} \Phi\left(\frac{x}{m}\right)$$

which implies the desired conclusion (5) [because at $x = n$ the left side jumps by n and the right side by $\sum_{m|n} \phi(m)$]. In exactly the same way it follows that the function $\phi_a(n)$ defined by

$$\phi_a(n) = n^{1+a} \sum_{m|n} \frac{\mu(m)}{m^{1+a}} = n^{1+a} \prod_{p|n} \left(1 - \frac{1}{p^{1+a}}\right)$$

satisfies

$$n^{1+a} = \sum_{m|n} \phi_a(m).$$

This identity and the fact that q is the greatest common divisor of $\kappa\nu$ and $\lambda\mu$ imply that $S(\sigma)$ can be rewritten in the form

$$\begin{aligned} S(\sigma) &= \sum_{\kappa\lambda\mu\nu} \left[\sum_{\rho|\kappa\nu, \rho|\lambda\mu} \phi_{\sigma-1}(\rho) \right] \frac{\beta_\kappa \beta_\lambda \beta_\mu \beta_\nu}{\kappa^\sigma \mu^\sigma \lambda\nu} \\ &= \sum_{\rho} \phi_{\sigma-1}(\rho) \sum_{\rho|\kappa\nu, \rho|\lambda\mu} \frac{\beta_\kappa \beta_\lambda \beta_\mu \beta_\nu}{\kappa^\sigma \mu^\sigma \lambda\nu} \\ &= \sum_{\rho} \phi_{\sigma-1}(\rho) \left[\sum_{\rho|\kappa\nu} \frac{\beta_\kappa \beta_\nu}{\kappa^\sigma \nu} \right]^2, \end{aligned} \tag{6}$$

where for given ρ the inner sum is over all pairs of positive integers κ, ν such that ρ divides $\kappa\nu$. Since $\phi_{\sigma-1}(\rho)$ is easy to estimate, this reduces the estimation of $S(\sigma)$ to the estimation of a double sum.

Now let ρ be given. For each pair κ, ν with $\rho|\kappa\nu$ let d denote the smallest factor of κ such that κ/d is relatively prime to ρ, and let d_1 denote the smallest factor of ν such that ν/d_1 is relatively prime to ρ. In other words, let d and d_1, respectively, be the product of all prime factors of κ and ν, respectively, which divide ρ. Set $\kappa' = \kappa/d$, $\nu' = \nu/d_1$. Since $\rho|\kappa\nu$ implies $\rho|dd_1$ and since, given d, d_1, the range of κ', ν' is all integers relatively prime to ρ,

$$\sum_{\rho|\kappa\nu} \frac{\beta_\kappa \beta_\nu}{\kappa^\sigma \nu} = \sum_{\rho|dd_1} \left(\sum_{\kappa'} \frac{\beta_{d\kappa'}}{(d\kappa')^\sigma}\right)\left(\sum_{\nu'} \frac{\beta_{d_1\nu'}}{d_1\nu'}\right),$$

where d, d_1 range over positive integers all of whose prime factors divide ρ, and κ', ν' range over positive integers none of whose prime factors divide ρ. Now since d, κ' are relatively prime, formula (2) of Section 11.3 for α_ν gives

$\alpha_{d\kappa'} = \alpha_d \alpha_{\kappa'}$; hence

$$\sum_{\kappa'} \frac{\beta_{d\kappa'}}{(d\kappa')^\sigma} = \sum_{d\kappa' \le X} \frac{\alpha_d \alpha_{\kappa'}}{\log X} \log\left(\frac{X}{d\kappa'}\right)(d\kappa')^{-\sigma}$$
$$= \frac{\alpha_d}{d^\sigma \log X} \sum_{\kappa' \le X/d} \frac{\alpha_{\kappa'}}{(\kappa')^\sigma} \log \frac{X}{d\kappa'},$$

and therefore

(7)
$$\sum_{\rho | \kappa\nu} \frac{\beta_\kappa \beta_\nu}{\kappa^\sigma \nu} = \frac{1}{(\log X)^2} \sum_{\rho | dd_1} \frac{\alpha_d \alpha_{d_1}}{d^\sigma d_1} \sum_{\kappa' \le X/d} \frac{\alpha_{\kappa'}}{(\kappa')^\sigma} \times \log \frac{X}{d\kappa'} \sum_{\nu' \le X/d_1} \frac{\alpha_{\nu'}}{\nu'} \log \frac{X}{d_1 \nu'},$$

where ρ is given, d and d_1 range over positive integers whose prime factors all divide ρ, and κ' and ν' range over positive integers relatively prime to ρ.

The single sum

(8)
$$\sum_{\kappa' \le Y} \frac{\alpha_{\kappa'}}{(\kappa')^\sigma} \log\left(\frac{Y}{\kappa'}\right),$$

where κ' ranges over positive integers relatively prime to ρ, can be estimated as follows. For $\sigma > 1$

$$\sum_{\kappa'} \frac{\alpha_{\kappa'}}{(\kappa')^\sigma} = \prod_{p \nmid \rho} \left(1 - \frac{1}{p^\sigma}\right)^{1/2} = \frac{1}{[\zeta(\sigma)]^{1/2}} \prod_{p | \rho} \left(1 - \frac{1}{p^\sigma}\right)^{-1\ 2}$$

virtually by the definition of $\alpha_{\kappa'}$. This is not applicable in the range $\{\frac{1}{2} \le \sigma \le 1\}$ which is under consideration ($\sigma = 1 - \theta$ or $\sigma = 1$), but it shows that in terms of Fourier analysis the problem of estimating (8) can be restated: Let the operator whose transform is $\zeta(s)^{-1/2} \prod_{p|\rho} (1 - p^{-s})^{-1/2}$ in $\operatorname{Re} s > 1$, namely,

$$f(x) \mapsto \sum_{\kappa'} \alpha_{\kappa'} f(\kappa' x),$$

be modified to

$$f(x) \mapsto \sum_{\kappa' \le Y} \alpha_{\kappa'} \log \frac{Y}{\kappa'} f(\kappa' x)$$

(for a large parameter Y) so as to be finite and hence to have a transform defined throughout the s-plane. Estimate the value of the transform of the modified operator at points of the line segment $\{\frac{1}{2} \le \sigma \le 1, t = 0\}$ of the s-plane. This can be done as follows.

Integration by parts in the basic formula of Section 3.3 gives

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} x^s \frac{ds}{s^2} = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} (\log x) x^s \frac{ds}{s} = \begin{cases} 0, & 0 < x \le 1, \\ \log x, & 1 \le x < \infty. \end{cases}$$

Thus termwise integration gives, for any series $\sum b_n$,

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \left\{ \sum_{n=1}^{\infty} b_n n^{-s-\sigma} \right\} Y^s \frac{ds}{s^2} = \sum_{n \le Y} b_n \left(\log \frac{Y}{n}\right) n^{-\sigma}$$

provided a is such that $\sum b_n n^{-a-\sigma}$ converges absolutely. This formula expresses the transform of the modified operator in terms of the transform of the unmodified one and in the case at hand shows that

$$\sum_{\kappa' \le Y} \frac{\alpha_{\kappa'}}{(\kappa')^{\sigma}} \log \frac{Y}{\kappa'} \tag{9}$$
$$= \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{1}{[\zeta(s+\sigma)]^{1/2}} \prod_{p|\rho} \left(1 - \frac{1}{p^{s+\sigma}}\right)^{-1/2} Y^s \frac{ds}{s^2}$$

for any a such that $a + \sigma > 1$, that is, any a such that $a > 1 - \sigma$. If $\sigma \neq 1$, then the singularity of the integrand at $s = 1 - \sigma$ is not serious and the line of integration can be moved over to $a = 1 - \sigma$. To prove a precise statement to this effect, it is necessary to have an estimate of $|\zeta(s+\sigma)^{-1/2}|$ in the half-plane $\operatorname{Re} s \geq 1 - \sigma$ or, what is the same, an estimate of $|\zeta(s)|^{-1}$ in the half-plane $\operatorname{Re} s \geq 1$. A simple estimate of this sort is

$$\left|\frac{1}{\zeta(s)}\right| \leq A|s-1| \qquad (\operatorname{Re} s \geq 1) \tag{10}$$

for some positive constant A. This estimate is proved below. Using it, it is easy to prove that the integral (9) converges for $a = 1 - \sigma$, that the equation still holds, and that the value of the left side is therefore at most

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} A^{1/2}|t|^{1/2} \prod_{p|\rho} \left(1 - \frac{1}{p}\right)^{-1/2} Y^{1-\sigma} \frac{dt}{[(1-\sigma)^2 + t^2]}$$
$$= \left[\prod_{p|\rho} \left(1 + \frac{1}{p}\right)\left(1 - \frac{1}{p^2}\right)^{-1}\right]^{1/2} Y^{1-\sigma} \frac{A^{1/2}}{\pi}$$
$$\times \int_0^{\infty} \frac{t^{3/2}}{t^2 + (1-\sigma)^2} \, d \log t$$
$$\leq \prod_{p|\rho} \left(1 + \frac{1}{p}\right)^{1/2} \left[\prod_{p} \left(1 - \frac{1}{p^2}\right)^{-1}\right]^{1/2} Y^{1-\sigma} \frac{A^{1/2}}{\pi}$$
$$\times \int_0^{\infty} \frac{[(1-\sigma)u]^{3/2}}{(1-\sigma)^2(u^2+1)} \, d \log u$$
$$= \text{const} \prod_{p|\rho} \left(1 + \frac{1}{p}\right)^{1/2} Y^{1-\sigma}(1-\sigma)^{-1/2}$$

in absolute value when $\sigma < 1$. If $\sigma = 1$, then the line of integration cannot be moved over to $a = 1 - \sigma$ because of the factor s^2 in the denominator of the integrand. Instead, take as the path of integration the line from $-i\infty$ to $-iB$, the semicircle in $\operatorname{Re} s \geq 0$ from $-iB$ to $+iB$, and the line from iB to $i\infty$, where B is a constant to be determined later. Using Cauchy's theorem and the estimate (10), it is easily seen that formula (9) holds with this new path of integration in place of the line $\operatorname{Re} s = a$. Along the semicircle $dz/z = i\,d\theta$, and the integral along this portion has modulus at most

$$\frac{1}{2\pi}(AB)^{1/2} \prod_{p|\rho} \left(1 - \frac{1}{p}\right)^{-1/2} Y^B \int_{-\pi}^{\pi} \frac{d\theta}{B}$$

[using the assumption $Y > 1$ which is justified by the fact that the quantity (8) to be estimated is zero if $Y \leq 1$] which is less than a constant times

$$B^{-1/2} \prod_{p|\rho} \left(1 + \frac{1}{p}\right)^{1/2} e^{B \log Y}.$$

The integral over the portion of the path of integration along the imaginary axis has modulus at most

$$\frac{2}{2\pi} \int_B^\infty A^{1/2} t^{1/2} \prod_{p|\rho} \left(1 - \frac{1}{p}\right)^{-1/2} \frac{dt}{t^2} \leq \text{const} \prod_{p|\rho} \left(1 + \frac{1}{p}\right)^{1/2} B^{-1/2}.$$

These two estimates are of the same order of magnitude when $B \log Y$ is of the order of magnitude of 1. Therefore assume $Y > 1$ and set $B = (\log Y)^{-1}$. It follows that when $\sigma = 1$ the quantity (9) in question is at most a constant times $(\log Y)^{1/2} \prod_{p|\rho} (1 + p^{-1})^{1/2}$. This estimate extends from $\sigma = 1$ to $1 - \frac{1}{2}(\log Y)^{-1} \leq \sigma \leq 1$ if the path of integration is taken to be the line $\operatorname{Re} s = 1 - \sigma$ with a detour around the right side of the circle $|s| = (\log Y)^{-1}$ where it intercepts the line. The result is again that the quantity (9) to be estimated has modulus at most a constant times $(\log Y)^{1/2} \prod_{p|\rho} (1 + p^{-1})^{1/2}$. But for $\frac{1}{2} \leq \sigma \leq 1 - \frac{1}{2}(\log Y)^{-1}$ the previous estimate shows it has modulus at most a constant times $\prod_{p|\rho} (1 + p^{-1})^{1/2} Y^{1-\sigma} (\log Y)^{1/2}$. Therefore in any case (even $Y \leq 1$) there is a constant K' such that

$$\left| \sum_{\kappa' \leq Y} \frac{\alpha_{\kappa'}}{(\kappa')^\sigma} \log \frac{Y}{\kappa'} \right| \leq K' \prod_{p|\rho} \left(1 + \frac{1}{p}\right)^{1/2} Y^{1-\sigma} |\log Y|^{1/2}. \tag{11}$$

This completes the estimate of the single sum except for the proof of the needed estimate (10) of $|\zeta(s)|^{-1}$. Since this estimate implies $\zeta(1 + it) \neq 0$, its proof cannot be expected to be entirely elementary. The trigonometric inequality $3 + 4 \cos \theta + \cos 2\theta \geq 0$ of Section 5.2 combined with the formula $\log \zeta(s) = \int_0^\infty x^{-s}\, dJ(x)$ of Section 1.11 gives

$$\begin{aligned} &\operatorname{Re}\{3 \log \zeta(\sigma) + 4 \log \zeta(\sigma + it) + \log \zeta(\sigma + 2it)\} \\ &\quad = \int_0^\infty x^{-\sigma} \operatorname{Re}\{3 + 4x^{-it} + x^{-2it}\}\, dJ(x) \geq 0, \end{aligned}$$

$$4 \log |\zeta(\sigma + it)| \geq -3 \log |\zeta(\sigma)| - \log |\zeta(\sigma + 2it)|$$

$$|\zeta(\sigma + it)| \geq |\zeta(\sigma)|^{-3/4} |\zeta(\sigma + 2it)|^{-1/4},$$

for all $\sigma > 1$. Since $(s - 1)\zeta(s)$ is bounded on the interval $1 \leq s \leq 2$ and since $|\zeta(\sigma + 2it)|$ is less than a constant times $\log(2t)$ for $\sigma \geq 1, 2t \geq 1$ (see Section 9.2), this shows that there is a positive constant K such that

$$|\zeta(\sigma + it)| \geq K(\sigma - 1)^{3/4} (\log t)^{-1/4}$$

for all $t \geq 1$, $1 < \sigma \leq 2$. This will be used to find a lower estimate of

$|\zeta(1+it)|$ by combining it with the fundamental theorem of calculus $\zeta(\sigma+it)-\zeta(1+it)=\int_1^\sigma \zeta'(u+it)\,du$ and the following estimate of $\zeta'(s)$ by Euler–Maclaurin summation:

$$\begin{aligned}
-\zeta'(s) &= \sum_{n=1}^{\infty} \frac{\log n}{n^s} \\
&= \sum_{n=1}^{N-1} \frac{\log n}{n^s} + \int_N^\infty \frac{\log u}{u^s}\,du + \frac{1}{2}\frac{\log N}{N^s} \\
&\quad + \int_N^\infty \bar{B}_1(u)\left(\frac{1}{u^{s+1}} - \frac{s\log u}{u^{s+1}}\right) du \\
&= \sum_{n=1}^{N-1} \frac{\log n}{n^s} + \frac{\log N}{(s-1)N^{s-1}} + \frac{1}{(s-1)^2 N^{s-1}} + \frac{1}{2}\frac{\log N}{N^s} \\
&\quad + \int_N^\infty \bar{B}_1(u)\frac{1-s\log u}{u^{s+1}}\,du
\end{aligned}$$

holds at first for $\operatorname{Re} s > 1$ but then by analytic continuation holds for the entire halfplane $\operatorname{Re} s > 0$ where the integral on the right converges. With $s = \sigma + it$, $N = [t]$ this gives

$$\begin{aligned}
|\zeta'(\sigma+it)| \le \sum_{n=1}^{N-1} \frac{\log t}{n} &+ \frac{\log t}{[(\sigma-1)^2+t^2]^{1/2}} + \frac{1}{(\sigma-1)^2+t^2} \\
&+ \frac{\log t}{2} + \frac{1}{2} + |\sigma+it| \int_N^\infty \bar{B}_1(u)\frac{\log u}{u^{s+1}}\,du
\end{aligned}$$

for $\sigma \ge 1$; hence $|\zeta'(\sigma+it)|$ is less than a constant times $(\log t)^2$ for $t \ge 2$, which gives

$$\begin{aligned}
|\zeta(1+it)| &\ge |\zeta(\sigma+it)| - \int_1^\sigma |\zeta'(u+it)|\,du \\
&\ge K_1\frac{(\sigma-1)^{3/4}}{(\log t)^{1/4}} - K_2(\sigma-1)(\log t)^2
\end{aligned}$$

for $1 < \sigma < 2$, $t \ge 2$. These two terms are of the same order of magnitude when $(\sigma-1)^{1/4}(\log t)^{9/4}$ is of the order of magnitude of 1, so choose σ by $(\sigma-1) = c(\log t)^{-9}$ to find

$$|\zeta(1+it)| \ge (K_1c^{3/4} - K_2c)(\log t)^{-7}$$

which gives $|\zeta(1+it)| \ge K_3(\log t)^{-7}$ when c is sufficiently small. Therefore $|\zeta(1+it)^{-1}t^{-1}| \le K_3^{-1}(\log t)^7 t^{-1} \le K_4$ for $t \ge 2$, so the function $[\zeta(s)(s-1)]^{-1}$ is bounded on the line segment $\{\operatorname{Re} s = 1, \operatorname{Im} s \ge 2\}$. Since it is also bounded on the rectangle $\{1 \le \operatorname{Re} s \le 2, |\operatorname{Im} s| \le 2\}$ and on the halfplane $\operatorname{Re} s \ge 2$, it follows from Lindelöf's theorem (Section 9.2) that it is bounded on the strip $\{1 \le \operatorname{Re} s \le 2\}$; hence it is bounded on the halfplane $\operatorname{Re} s \ge 1$ and the estimate (10) follows.

The estimate (11) of the single sum can be used to find that the double sum (7) is at most

$$\frac{1}{(\log X)^2}\sum_{\rho|dd_1,d<X,d_1<X}\frac{|\alpha_d||\alpha_{d_1}|}{d^\sigma d_1}K'\prod_{p|\rho}\left(1+\frac{1}{p}\right)^{1/2}\left(\frac{X}{d}\right)^{1-\sigma}\left(\log\frac{X}{d}\right)^{1/2}$$
$$\times K'\prod_{p|\rho}\left(1+\frac{1}{p}\right)^{1/2}\left(\log\frac{X}{d_1}\right)^{1/2}$$

which, since $[\log(X/d)\log(X/d_1)]^{1/2}\le\log X$, is at most a constant times

$$\frac{1}{\log X}\prod_{p|\rho}\left(1+\frac{1}{p}\right)X^{1-\sigma}\sum_{\rho|dd_1,d<X,d_1<X}\frac{|\alpha_d||\alpha_{d_1}|}{dd_1},$$

where ρ and X are given and where d, d_1 range over integers all of whose prime factors divide ρ. Let D represent numbers all of whose prime factors divide ρ. Then

$$\sum_{\rho|dd_1,d<X,d_1<X}\frac{|\alpha_d||\alpha_{d_1}|}{dd_1}=\sum_D\sum_{\substack{dd_1=\rho D\\ d<X,d_1<X}}\frac{|\alpha_d||\alpha_{d_1}|}{\rho D}\tag{12}$$

and for any fixed D the terms of the sum corresponding to D are at most

$$\frac{1}{\rho D}\sum_{dd_1=\rho D}|\alpha_d||\alpha_{d_1}|,$$

where $dd_1=\rho D$ ranges over all possible factorizations of ρD. Now since

$$\left|\binom{+\frac{1}{2}}{n}\right|\le(-1)^n\binom{-\frac{1}{2}}{n},$$

the terms of the expansion of $(1-p^{-s})^{-1/2}$ as a Dirichlet series, which are all positive, dominate in absolute value the corresponding terms of the expansion of $(1-p^{-s})^{1/2}$. Consequently the terms of the expansion of $\prod_p(1-p^{-s})^{-1/2}=\zeta(s)^{1/2}$ dominate in absolute value the corresponding terms of the expansion of $\prod_p(1-p^{-s})^{1/2}=\zeta(s)^{-1/2}$. Since these latter terms have absolute value $|\alpha_n|/n^s$, squaring both sides shows that $\sum_{jk=n}|\alpha_j||\alpha_k|$ is less than the coefficient of n^{-s} in $[\zeta(s)^{1/2}]^2=\zeta(s)$ which is one. Therefore (12) is at most

$$\rho^{-1}\sum_D D^{-1}=\rho^{-1}\prod_{p|\rho}\left(1-\frac{1}{p}\right)^{-1}=\rho^{-1}\prod_{p|\rho}\left(1-\frac{1}{p^2}\right)^{-1}\left(1+\frac{1}{p}\right)$$
$$\le\rho^{-1}\zeta(2)\prod_{p|\rho}\left(1+\frac{1}{p}\right)$$

and the double sum (7) to be estimated is at most a constant times

$$\frac{1}{\log X}\prod_{p|\rho}\left(1+\frac{1}{p}\right)^2X^{1-\sigma}\frac{1}{\rho}.$$

Therefore the quadruple sum $S(\sigma)$ has, by (6), modulus at most a constant times

$$\sum_{\rho\le X^2} |\phi_{\sigma-1}(\rho)| \left[\prod_{p|\rho}\left(1+\frac{1}{p}\right)^2 \frac{X^{1-\sigma}}{\rho\log X}\right]^2$$

because $\rho > X^2$ and $\rho\,|\,\kappa\nu$ imply $\beta_\kappa\beta_\nu = 0$. Now $|\phi_{\sigma-1}(\rho)| = \rho^\sigma \prod_{p|\rho}(1-p^{-\sigma}) < \rho^\sigma$ and $\prod_{p|\rho}[1+(1/p)]^4$ is less than a constant times $\prod_{p|\rho}(1+p^{-1/2})$. [To prove the latter fact, it suffices to note that $(1+x)^4 < 1 + x^{1/2}$ for all sufficiently small x, so the quotient $(1+p^{-1})^4/(1+p^{-1/2})$ is less than one for all but a finite number of primes p.] Therefore $|S(\sigma)|$ is at most a constant times

$$\begin{aligned}\frac{X^{2-2\sigma}}{(\log X)^2}\sum_{\rho\le X^2}\rho^{\sigma-2}\prod_{p|\rho}\left(1+\frac{1}{p^{1/2}}\right) &\le \frac{X^{2-2\sigma}}{(\log X)^2}\sum_{\rho\le X^2}\rho^{\sigma-2}\sum_{n|\rho} n^{-1/2}\\ &= \frac{X^{2-2\sigma}}{(\log X)^2}\sum\sum_{mn\le X^2}(mn)^{\sigma-2}n^{-1/2}\\ &\le \frac{X^{2-2\sigma}}{(\log X)^2}\sum_{n=1}^{\infty} n^{-3/2}\sum_{m\le X^2} m^{-1}\end{aligned}$$

which is at most a constant times $X^{2-2\sigma}(\log X)^{-1}$. Finally, this estimate of $S(\sigma)$ shows that (3′) has modulus at most a constant times

$$\frac{1}{\theta z^\theta(\sin 2\delta)^{1/2}}\frac{1}{\log X} + \frac{(\sin 2\delta)^{\theta/2}}{\theta(\sin 2\delta)^{1/2}}\frac{X^{2\theta}}{\log X}.$$

Since $(\sin 2\delta)^{\theta/2}$ is less than a constant times $\delta^{\theta/2}$, while $X^{2\theta} \le \delta^{-2\theta/15}$, and $z^\theta \le \delta^{-\theta/15}$, the second term is less than or equal to a constant times the first. Therefore (3) and hence Σ_1 are less than a constant times $(\delta^{1/2}\theta z^\theta \log X)^{-1}$ for δ sufficiently small. This completes the estimate of Σ_1.

It remains to show that Σ_2 is insignificant compared to $(\delta^{1/2}\theta z^\theta \log X)^{-1}$. Now Σ_2 is the integral of a sum of terms of the form

$$4u^{-\theta}\frac{\beta_\kappa\beta_\lambda\beta_\mu\beta_\nu}{\lambda\nu}\exp(-Pu^2 - iQu^2),$$

where

$$P = \pi\left(\frac{m^2\kappa^2}{\lambda^2} + \frac{n^2\mu^2}{\nu^2}\right)\sin 2\delta > 0,$$

$$Q = \pi\left(\frac{m^2\kappa^2}{\lambda^2} - \frac{n^2\mu^2}{\nu^2}\right)\cos 2\delta \ne 0.$$

Every term with $Q > 0$ is the complex conjugate of a term with $Q < 0$, so when the finite sum over $\kappa\lambda\mu\nu$ is taken outside the integral sign, Σ_2 is twice the real part of

$$\sum_{\kappa\lambda\mu\nu}\frac{4\beta_\kappa\beta_\lambda\beta_\mu\beta_\nu}{\lambda\nu}\int_z^\infty \sum_{m=1}^{\infty}\sum_{n\mu/\nu<m\kappa/\lambda} u^{-\theta}e^{-Pu^2-iQu^2}\,du,$$

where P, Q are defined in terms of the sextuple κ, λ, μ, ν, m, n as above. The sum over m, n can be integrated termwise because it is dominated by the integral of $\sum e^{-P}$ so $|\Sigma_2|$ is at most a constant times

$$\sum_{\kappa\lambda\mu\nu mn, Q>0} \left| \int_z^\infty u^{-\theta} e^{-Pu^2 - iQu^2}\, du \right|.$$

A typical term of this sextuple sum can be estimated by writing

$$\begin{aligned}\int_z^\infty & u^{1-\theta} e^{-Pu^2 - iQu^2}\, d\log u \\ &= \int_{Qz^2}^\infty e^{-Pv/Q} e^{-iv} (v/Q)^{(1-\theta)/2} \tfrac{1}{2}\, d\log v \\ &= Q^{(\theta-1)/2} \tfrac{1}{2} \int_{Qz^2}^\infty e^{-Pv/Q} (\cos v - i \sin v) v^{-(\theta+1)/2}\, dv.\end{aligned}$$

By the lemma of Section 9.7 the real and imaginary parts of this integral have absolute value at most

$$Q^{(\theta-1)/2} e^{-Pz^2} (Qz^2)^{-(\theta+1)/2} = Q^{-1} e^{-Pz^2} z^{-1-\theta} \le Q^{-1} e^{-P} z^{-1-\theta}$$

so Σ_2 has modulus at most a constant times

$$\frac{1}{z^{1+\theta}} \sum_{\kappa\lambda\mu\nu mn, Q>0} e^{-P} Q^{-1}. \tag{13}$$

For fixed κ, λ, μ, ν, m the sum over n is at most $z^{-1-\theta}$ times

$$\begin{aligned}\frac{1}{\pi \cos 2\delta} & \sum_{n\mu/\nu < m\kappa/\lambda} e^{-\pi m^2 \kappa^2 \lambda^{-2} \sin 2\delta} \left(\frac{m\kappa}{\lambda} - \frac{n\mu}{\nu}\right)^{-1} \left(\frac{m\kappa}{\lambda} + \frac{n\mu}{\nu}\right)^{-1} \\ &\le \frac{e^{-\pi m^2 X^{-2} \sin 2\delta}}{(\pi \cos 2\delta)(m\kappa/\lambda)} \sum_{n < m\kappa\nu/\lambda\mu} \left(\frac{m\kappa\nu - n\lambda\mu}{\lambda\nu}\right)^{-1} \\ &\le \frac{\lambda^2 \nu e^{-\pi m^2 X^{-2} \sin 2\delta}}{m\kappa\pi \cos 2\delta} \sum_{n < m\kappa\nu/\lambda\mu} \frac{1}{m\kappa\nu - n\lambda\mu}.\end{aligned}$$

The sum here is a sum of reciprocal positive integers spaced at intervals of $\mu\lambda$; the largest term of the sum is at most one and the other terms can be estimated using the inequality

$$\frac{1}{k} \le \frac{1}{\mu\lambda} \int_{k-\mu\lambda}^{k} \frac{dv}{v} \qquad (k > \mu\lambda)$$

to find that the above sum is at most

$$\begin{aligned}\frac{\lambda^2 \nu e^{-\pi m^2 X^{-2} \sin 2\delta}}{m\kappa\pi \cos 2\delta} & \left(1 + \frac{1}{\mu\lambda} \int_1^{m\kappa\nu - \lambda\mu} \frac{dv}{v}\right) \\ &\le \frac{X^3 e^{-\pi m^2 X^{-2} \sin 2\delta}}{m\pi \cos 2\delta} [1 + \log(m\kappa\nu)] \\ &\le \frac{X^3 e^{-\pi m^2 X^{-2} \sin 2\delta}}{m\pi \cos 2\delta} [1 + \log(mX^2)].\end{aligned}$$

This is independent of $\kappa, \lambda, \mu, \nu$ so summation over these variables multiplies the estimate by X^4 at most and the sextuple sum (13) is at most

$$\frac{X^7}{z^{1+\theta}\pi\cos 2\delta}\sum_{m=1}^{\infty} e^{-\pi m^2 X^{-2}\sin 2\delta}\left(\frac{1}{m}+\frac{\log(mX^2)}{m}\right).$$

For $m \geq e$ the function $m^{-1}[1+\log(mX^2)]$ is less than $2m^{-1}\log(mX^2)$ and this function decreases, so this is at most

$$\frac{X^7}{z^{1+\theta}\pi\cos 2\delta}\left[\sum_{m=1}^{3} e^{-\pi m^2 X^{-2}\sin 2\delta}\left(\frac{1}{m}+\frac{\log(mX^2)}{m}\right)\right.$$
$$\left.+\,2\int_3^\infty e^{-\pi u^2 X^{-2}\sin 2\delta}\frac{\log(uX^2)}{u}\,du\right].$$

The first term in brackets is at most $3[1+\log(3X^2)] \leq \text{const} + \text{const}\log X \leq \text{const}\log(1/\delta)$. If the integral in brackets is split at the point where the exponent is $-\pi$, that is, where† $u = X(\sin 2\delta)^{-1/2}$, it is seen to be at most 2 times

$$\int_3^{X(\sin 2\delta)^{-1/2}} \log(uX^2)\,d\log(uX^2) + \int_1^\infty e^{-\pi v^2}\log\frac{vX^3}{(\sin 2\delta)^{1/2}}\,d\log v$$
$$\leq \frac{1}{2}\left[\log\left(\frac{X^3}{(\sin 2\delta)^{1/2}}\right)\right]^2 + \log\frac{X^3}{(\sin 2\delta)^{1/2}}\int_1^\infty e^{-\pi v^2}\,d\log v$$
$$+\int_1^\infty e^{-\pi v^2}\log v\,d\log v.$$

Since $X^3(\sin 2\delta)^{-1/2} \leq \delta^{-3/15}\,\delta^{-1/2} \leq \delta^{-1}$, this is less than a constant times $[\log(1/\delta)]^2$ as $\delta \downarrow 0$. Therefore Σ_2 is less than a constant times

$$\frac{X^7}{z^{1+\theta}}\left(\log\frac{1}{\delta}\right)^2.$$

Since this quantity multiplied by $\delta^{1/2}\theta z^\theta \log X$ is at most $\frac{1}{2}z^{-1}\delta^{-7/15}\delta^{1/2}$ $\cdot[\log(1/\delta)]^2(1/15)[\log(1/\delta)]$ and because this quantity approaches zero as $\delta \downarrow 0$, this completes the proof of the lemma.

Note added in second printing: The theorems of this chapter have now been superseded by Levinson's theorem that *a third of the zeros lie on the line*, that is, $(NT) > \frac{1}{3}(T/2\pi)\log(T/2\pi)$ for all sufficiently large T. See his article "More than one third of zeros of Riemann's zeta-function are on $\sigma = \frac{1}{2}$," *Advances in Mathematics*, **13** (1974) pp. 383–436.

†Note that this point lies to the right of $u = 3$ when δ is sufficiently small.

Chapter 12

Miscellany

12.1 THE RIEMANN HYPOTHESIS AND THE GROWTH OF $M(x)$

Let dM be the Stieltjes measure such that the formula

(1) $$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \qquad (\operatorname{Re} s > 1)$$

[(1) of Section 5.6] takes the form

$$\frac{1}{\zeta(s)} = \int_0^\infty x^{-s}\, dM(x) \qquad (\operatorname{Re} s > 1).$$

Then $M(x) = \int_0^x dM$ is a step function which is zero at $x = 0$, which is constant except at positive integers, and which has a jump of $\mu(n)$ at n. As usual, the value of M at a jump is by definition $\frac{1}{2}[M(n-\varepsilon) + M(n+\varepsilon)] = \sum_{j=1}^{n-1} \mu(j) + \frac{1}{2}\mu(n)$. Integration by parts gives for $\operatorname{Re} s > 1$

$$\begin{aligned}\frac{1}{\zeta(s)} &= \int_0^\infty d[x^{-s}M(x)] - \int_0^\infty M(x)\, d(x^{-s})\\ &= \lim_{X\to\infty}\left[X^{-s}M(X) + s\int_0^X M(x)x^{-s-1}\, dx\right]\\ &= s\int_0^\infty M(x)x^{-s-1}\, dx\end{aligned}$$

because the obvious inequality $|M(x)| \le x$ implies that $x^{-s}M(x) \to 0$ as $x \to \infty$ and that $\int_0^\infty M(x)x^{-s-1}\, dx$ converges, both provided $\operatorname{Re} s > 1$. Now if $M(x)$ grows less rapidly than x^a for some $a > 0$, then this integral for $1/\zeta(s)$ converges for all s in the halfplane $\{\operatorname{Re}(a - s) < 0\} = \{\operatorname{Re} s > a\}$, and therefore, by analytic continuation, the function $1/\zeta(s)$ is analytic in this halfplane. Since $1/\zeta(s)$ has poles on the line $\operatorname{Re} s = \frac{1}{2}$, this shows that $M(x)$ *does not* *grow less rapidly than x^a for any $a < \frac{1}{2}$.* Moreover, it shows that *in order to prove the Riemann hypothesis, it would suffice to prove that $M(x)$ grows less rapidly than $x^{(1/2)+\varepsilon}$ for all $\varepsilon > 0$.* Littlewood in his 1912 note [L12] on the three circles theorem proved that this sufficient condition for the Riemann hypothesis is also necessary; that is, he proved the following theorem.

Theorem The Riemann hypothesis is equivalent to the statement that for every $\varepsilon > 0$ the function $M(x)x^{-(1/2)-\varepsilon}$ approaches zero as $x \to \infty$.

Proof It was shown above that the second statement implies the Riemann hypothesis. Assume now that the Riemann hypothesis is true. Then Backlund's proof in Section 9.4 shows [using the Riemann hypothesis to conclude that $F(s) = \zeta(s)$] that for every $\varepsilon > 0$, $\delta > 0$, and $\sigma_0 > 1$ there is a T_0 such that $|\log \zeta(\sigma + it)| < \delta \log t$ whenever $t \geq T_0$ and $\frac{1}{2} + \varepsilon \leq \sigma \leq \sigma_0$. Since $|\log \zeta(s)|$ is bounded on the halfplane $\{\text{Re } s \geq \sigma_0\}$, this implies that on the quarterplane $\{s = \sigma + it\colon \sigma = \frac{1}{2} + \varepsilon, t \geq T_0\}$ there is a constant K such that $|1/\zeta(s)| \leq Kt^\delta$. This is the essential step of the proof. Littlewood omits the remainder of the proof, stating merely that it follows from known theorems. One way of completing the proof is as follows.

The estimates (2) and (3) of Section 3.3 show that the error in the approximation

$$\begin{aligned} M(x) = \sum_{n<x} \mu(n) &\sim \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \left[\sum_{n=1}^{\infty} \mu(n)\left(\frac{x}{n}\right)^s\right] \frac{ds}{s} \\ &= \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \frac{x^s}{\zeta(s)} \frac{ds}{s}, \end{aligned}$$

for x not an integer is at most

$$\begin{aligned} &\sum_{n<x} \frac{|\mu(n)|(x/n)^2}{\pi T \log(x/n)} + \sum_{n>x} \frac{|\mu(n)|(x/n)^2}{\pi T \log(n/x)} \\ &\qquad \leq \frac{x^2}{\pi T}\left[\sum_{n<x} \frac{1}{n^2 \log(x/n)} + \sum_{n>x} \frac{1}{n^2 \log(n/x)}\right]. \end{aligned}$$

The first sum in brackets is at most

$$\sum_{n \leq x/2} \frac{1}{n^2 \log(x/n)} + \sum_{x/2<n<x} \frac{1}{n^2 \log\{1 + [(x-n)/n]\}}$$

assuming, of course, that x is not an integer. Since $\log(1 + y) \geq \frac{1}{2}y$ for $0 \leq y \leq 1$ this is at most†

$$\begin{aligned} \frac{1}{\log 2} \sum_{n \leq x/2} \frac{1}{n^2} + \sum_{x/2<n<x} \frac{1}{n^2} \cdot \frac{2n}{x-n} &\leq \frac{\zeta(2)}{\log 2} + \frac{4}{x} \sum_{0<x-n<x/2} \frac{1}{x-n} \\ &\leq \frac{\zeta(2)}{\log 2} + \frac{4}{x} \cdot \frac{1}{x-[x]} + \frac{4}{x} \sum_{0<j<x/2} \frac{1}{j}. \end{aligned}$$

†As usual, $[x]$ denotes the largest integer less than x.

This shows that it is bounded for large x provided $x - [x]$ is not too small, say, for example, if x is a half integer (half an odd integer). The second sum in brackets can be estimated in a similar way to arrive at the conclusion that for half-integer values of x the error in the approximation

$$M(x) \sim \frac{1}{2\pi i}\int_{2-iT}^{2+iT} \frac{x^s}{\zeta(s)}\frac{ds}{s}$$

is less than a constant times x^2/T as $x \longrightarrow \infty$. But by Cauchy's theorem (and the Riemann hypothesis) the integral on the right is equal to

$$\frac{1}{2\pi i}\int_{2-iT}^{(1/2)+\varepsilon-iT} \frac{x^s}{\zeta(s)}\frac{ds}{s} + \frac{1}{2\pi i}\int_{(1/2)+\varepsilon-iT}^{(1/2)+\varepsilon+iT} \frac{x^s}{\zeta(s)}\frac{ds}{s} + \frac{1}{2\pi i}\int_{(1/2)+\varepsilon+iT}^{2+iT} \frac{x^s}{\zeta(s)}\frac{ds}{s}.$$

The estimate $|1/\zeta(s)| \leq Kt^\delta$ shows that the first integral and the third integral are each less than a constant times $x^2 \cdot KT^{\delta-1}$ while the middle integral is at most

$$\left|\frac{1}{2\pi i}\int_{(1/2)+\varepsilon-iT_0}^{(1/2)+\varepsilon+iT_0} \frac{x^s}{\zeta(s)}\frac{ds}{s}\right| + \frac{2}{2\pi}\int_{T_0}^{T} \frac{x^{(1/2)+\varepsilon}Kt^\delta}{t}\,dt$$
$$\leq x^{(1/2)+\varepsilon}\,\text{const} + \frac{x^{(1/2)+\varepsilon}KT^\delta}{\pi\delta}.$$

Setting $T = x^2$ then shows that $M(x)$ is less than a constant times $x^{(1/2)+\varepsilon+2\delta}$ for all large half integers x. Since $M(x)$ changes by at most ± 1 between half integers, the same is true for all values of x and, since $\varepsilon > 0$ and $\delta > 0$ were arbitrary, it follows that the Riemann hypothesis implies $M(x)$ grows less rapidly than $x^{1/2+\varepsilon}$ for all $\varepsilon > 0$ as desired.

Corollary If the Riemann hypothesis is true, then the series (1) converges throughout the halfplane $\{\text{Re } s > \frac{1}{2}\}$ to the function $1/\zeta(s)$.

Proof

$$\sum_{n<x} \mu(n)/n^s = \int_0^x u^{-s}\,dM(u) = x^{-s}M(x) + s\int_0^x M(u)u^{-s-1}\,du.$$

The theorem shows that if the Riemann hypothesis is true and if Re $s > \frac{1}{2}$, then the limit of this expression as $x \longrightarrow \infty$ exists and is equal to $s\int_0^\infty M(u)$ $u^{-s-1}\,du$. This limit is $1/\zeta(s)$ for Re $s > 1$ and hence by analytic continuation for Re $s > \frac{1}{2}$ as well.

Stieltjes wrote to Hermite in 1885 that he had succeeded in proving the even stronger statement that $M(x) = O(x^{1/2})$—that is, $M(x)/x^{1/2}$ remains bounded as $x \longrightarrow \infty$—and he observed that this implies the Riemann hypothesis. This letter [S6] from Stieltjes to Hermite is what lies behind Hadamard's startling statement in his paper [H2] that he is publishing his proof that the zeta function has no zeros on the line $\{\text{Re } s = 1\}$ only because Stieltjes' proof that it has no zeros in the halfplane $\{\text{Re } s > \frac{1}{2}\}$ has not yet been published and is probably much more difficult!

In retrospect it seems very unlikely that Stieltjes had actually proved the Riemann hypothesis. Although in his time this was quite new territory—Stieltjes was among the first to penetrate the mysteries of Riemann's paper—enough work has now been done on the Riemann hypothesis to justify extreme skepticism about any supposed proof, and one must be very skeptical indeed in view of the fact that Stieltjes himself was unable in later years to reconstruct his proof. All he says about it is that it was very difficult, that it was based on arithmetic arguments concerning $\mu(n)$, and that he put it aside hoping to find a simpler proof of the Riemann hypothesis based on the theory of the zeta function rather than on arithmetic. Moreover, even *assuming* the Riemann hypothesis, Stieltjes' stronger claim $M(x) = O(x^{1/2})$ has never been proved. All in all, except to remember that a first-rate mathematician once believed that the most fruitful approach to the Riemann hypothesis was through a study of the growth of $M(x)$ as $x \longrightarrow \infty$, the incident is probably best forgotten.

12.2 THE RIEMANN HYPOTHESIS AND FAREY SERIES

It was shown in the preceding section that the Riemann hypothesis is equivalent to the arithmetic statement "$M(x) = o(x^{(1/2)+\varepsilon})$ for all $\varepsilon > 0$." Similarly, it was shown in Section 5.5 that the Riemann hypothesis is equivalent to the arithmetic statement "$\psi(x) - x = o(x^{(1/2)+\varepsilon})$ for all $\varepsilon > 0$." A third arithmetic statement equivalent to the Riemann hypothesis was found by Franel and Landau [F1] in the 1920s. Theirs deals with *Farey series.*

For a given real number $x > 1$ consider the rational numbers which, when expressed in lowest terms, have the denominators less than x. (For the sake of convenience assume x is not an integer.) The *Farey series*† corresponding to x is a complete set of representatives modulo 1 of these rational numbers, namely, the positive rationals less than or equal to 1 which can be expressed with denominators less than x. For example, for $x = 7\frac{1}{2}$ the Farey series is

$$\tfrac{1}{7}, \tfrac{1}{6}, \tfrac{1}{5}, \tfrac{1}{4}, \tfrac{2}{7}, \tfrac{1}{3}, \tfrac{2}{5}, \tfrac{3}{7}, \tfrac{1}{2}, \tfrac{4}{7}, \tfrac{3}{5}, \tfrac{2}{3}, \tfrac{5}{7}, \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{5}{6}, \tfrac{6}{7}, 1.$$

†On the history of the name see Hardy and Wright [H7]. Note that the Farey series is not a series at all but a finite sequence.

A fascinating property of this sequence—and in fact the property which first attracted attention to it—is that if p/q, r/s are successive terms of the sequence, then $qr - ps = 1$. This property will not, however, play a role here. Let $A(x)$ denote the number of terms in the Farey series corresponding to x, for example, $A(7\frac{1}{2}) = 18$. Since the $A(x)$ terms of the Farey series are unequally spaced through the interval from 0 to 1, they will in general differ from the the equally spaced points $1/A(x), 2/A(x), \ldots, A(x)/A(x) = 1$. For $\nu = 1, 2, \ldots, A(x)$ let δ_ν denote the amount by which the νth term of the Farey series differs from $\nu/A(x)$; for example, when $x = 7\frac{1}{2}$, $\delta_5 = (2/7) - (5/18) = 1/126$. The theorem of Franel and Landau is that *the Riemann hypothesis is equivalent to the statement that* $|\delta_1| + |\delta_2| + \cdots + |\delta_{A(x)}| = o(x^{(1/2)+\varepsilon})$ *for all* $\varepsilon > 0$ as $x \longrightarrow \infty$.

The connection between Farey series and the zeta function, so surprising at first, can be deduced from the following formula. Let f be a real-valued function defined on the interval [0, 1]—or, perhaps better, let f be a *periodic* real-valued function of a real variable with period one—and let $r_1, r_2, \ldots, r_{A(x)} = 1$ denote the terms of the Farey series corresponding to x. Then†

$$\sum_{\nu=1}^{A(x)} f(r_\nu) = \sum_{k=1}^{\infty} \sum_{j=1}^{k} f\left(\frac{j}{k}\right) M\left(\frac{x}{k}\right). \tag{1}$$

Thus the rather irregular operation of summing f over the Farey series can be expressed more regularly using the function M and a double sum. Note that the right side of (1) is defined for all x and that it gives a natural extension of the left side to integer values $x = n$, namely, the mean of the value for $n + \varepsilon$ and the value for $n - \varepsilon$ or, what is the same, the sum of f over all positive rationals less than or equal to 1 with denominators less than or equal to n, counting those with denominator exactly n with weight $\frac{1}{2}$.

Formula (1) can be proved as follows. Let $D(x)$ be the function

$$D(x) = \begin{cases} 1 & \text{for } x > 1, \\ \frac{1}{2} & \text{for } x = 1, \\ 0 & \text{for } x < 1. \end{cases}$$

Then the definition of M gives $M(x) = \sum \mu(n)D(x/n)$. But by Möbius inversion this is equivalent to $D(x) = \sum M(x/n)$. Now for any fraction in lowest terms p/q ($0 < p \leq q$, p and q relatively prime integers) the term $f(p/q) = f(2p/2q) = f(3p/3q) = \cdots$ occurs on the right side of (1) with the coefficient $M(x/q) + M(x/2q) + M(x/3q) + \cdots = D(x/q)$ which is one if $q < x$ and zero if $q > x$, which is its coefficient on the left side of (1). This completes the proof of (1).

†Note that the sum on the right is finite because all terms with $k > x$ are zero.

Now formula (1) applied to $f(u) = e^{2\pi iu}$ gives

$$\sum_{\nu=1}^{A(x)} e^{2\pi i r_\nu} = \sum_{k=1}^{\infty} \sum_{j=1}^{k} e^{2\pi i j/k} M\left(\frac{x}{k}\right).$$

But since $\sum_{j=1}^{k} e^{2\pi i j/k}$ is a sum of k complex numbers equally spaced around the unit circle, it is zero unless $k = 1$ in which case it is 1. Thus the right side is simply $M(x)$. Hence, with $A(x)$ abbreviated to A,

$$\begin{aligned} M(x) &= \sum_{\nu=1}^{A} e^{2\pi i r_\nu} = \sum_{\nu=1}^{A} e^{2\pi i[(\nu/A)+\delta_\nu]} \\ &= \sum_{\nu=1}^{A} e^{2\pi i\nu/A}(e^{2\pi i\delta_\nu} - 1) + \sum_{\nu=1}^{A} e^{2\pi i\nu/A} \\ |M(x)| &\le \sum_{\nu=1}^{A} |e^{2\pi i\delta_\nu} - 1| + 0 = \sum_{\nu=1}^{A} |e^{\pi i\delta_\nu} - e^{-\pi i\delta_\nu}| \\ &= 2\sum |\sin \pi\delta_\nu| \le 2\pi \sum_{\nu=1}^{A} |\delta_\nu| \end{aligned}$$

which proves one half of the Franel–Landau theorem, namely, the half which states that $\sum|\delta_\nu| = o(x^{(1/2)+\epsilon})$ implies the Riemann hypothesis.

The key step in the proof of the converse half is to apply formula (1) to the function† $\bar{B}_1(u) = u - [u] + \frac{1}{2}$ of Section 6.2. The technique used in Section 6.2 to prove $B_n(2u) = 2^{n-1}[B_n(u) + B_n(u + \frac{1}{2})]$ gives immediately the identity

$$B_n(ku) = k^{n-1}\left[B_n(u) + B_n\left(u + \frac{1}{k}\right) + \cdots + B_n\left(u + \frac{k-1}{k}\right)\right].$$

The same identity applies to $\bar{B}_n$ because by the periodicity of $\bar{B}_n$ it suffices to consider the case $0 \le u < 1/k$, in which all the values of $\bar{B}_n$ in the identity coincide with those of B_n. Thus

$$(2) \qquad \bar{B}_1\left(u + \frac{1}{k}\right) + \bar{B}_1\left(u + \frac{2}{k}\right) + \cdots + \bar{B}_1(u + 1) = \bar{B}_1(ku)$$

and

$$\begin{aligned} \sum_{\nu=1}^{A} \bar{B}_1(u + r_\nu) &= \sum_{k=1}^{\infty} \sum_{j=1}^{k} \bar{B}_1\left(u + \frac{j}{k}\right) M\left(\frac{x}{k}\right) \\ &= \sum_{k=1}^{\infty} \bar{B}_1(ku) M\left(\frac{x}{k}\right). \end{aligned}$$

Let G denote this function. The two expressions for G lead to two different ways of evaluating the definite integral

$$I = \int_0^1 [G(u)]^2 \, du$$

†For the sake of neatness one should stipulate $\bar{B}_1(0) = 0$ so that the value at the jump is the middle value. This is not necessary here.