2.43. Taylor's series. An analytic function can be expanded in powers of its argument by a formula similar to Taylor's series for a real function.

Suppose that $f(z)$ is analytic on and inside a simple closed contour C, and let a be a point inside C. Then

$$f(z)=f(a)+(z-a)f'(a)+\ldots+\frac{(z-a)^n}{n!}f^{(n)}(a)+\ldots,$$

the series being convergent if $|z-a|<\delta$, where δ is the distance from a to the nearest point of C.

We start from Cauchy's formula

$$f(z)=\frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w-z}\,dw,$$

where we take Γ to be a circle with centre a and radius $\rho<\delta$. The formula holds if z lies inside this circle, i.e. if $|z-a|<\rho$.

Now
$$\frac{1}{w-z}=\frac{1}{w-a}+\frac{z-a}{(w-a)^2}+\ldots+\frac{(z-a)^n}{(w-a)^{n+1}}+\ldots,$$

the series being uniformly convergent on Γ. Hence we may multiply by $f(w)/2\pi i$ and integrate term by term round Γ. We obtain

$$f(z)=\frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w-a}\,dw+\frac{z-a}{2\pi i}\int_\Gamma \frac{f(w)}{(w-a)^2}\,dw+$$
$$+\ldots+\frac{(z-a)^n}{2\pi i}\int_\Gamma \frac{f(w)}{(w-a)^{n+1}}\,dw+\ldots,$$

and, by the formulae of § 2.41, this is the desired result. It is sometimes known as the Cauchy-Taylor theorem.

There is one difference to be noted between this proof and the corresponding investigation for functions of a real variable. In the real variable theory we obtain the first n terms of the expansion, and a remainder term, and a special investigation is required to see whether this term tends to zero. In the complex theory the fact that it tends to zero follows from our original hypotheses. This state of affairs is quite natural; for combining the above theorem with § 2.16, we see that *the necessary and sufficient condition that a function should be expansible in a power series is that it should be analytic in a region.* We cannot define an analytic function of a real variable except as one which can be expanded in a power series. If therefore we start from

other hypotheses, we may expect to meet with difficulties. For example, the radius of convergence of the series depends on the extent of the region where the function is analytic; and it may therefore be controlled by the existence of singularities off the real axis, of which, if we confine ourselves to real variables, we can have no knowledge.

Thus the expansion

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \dots$$

holds for $|x| < 1$ only. There seems to be nothing in the nature of the function, considered as a function of the *real* variable x, to account for this restriction. But if we make x complex, it is accounted for by the fact that the function is not analytic at the points $x = \pm i$.

2.5. Cauchy's inequality. *If*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (|z| < R),$$

and $M(r)$ is the upper bound of $|f(z)|$ on the circle $|z| = r$, $(r < R)$, then

$$|a_n| r^n \leqslant M(r)$$

for all values of n.

For
$$a_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{n+1}}\, dz,$$

and the theorem of § 2.31 gives at once

$$|a_n| \leqslant \frac{M(r)}{r^n}.$$

Cauchy's inequality may also be proved as follows. Let $\bar{a}_n$ be the conjugate of a_n. Then if $r < R$,

$$|f(z)|^2 = f(z)\overline{f(z)} = \sum_{m=0}^{\infty} a_m r^m e^{im\theta} \sum_{n=0}^{\infty} \bar{a}_n r^n e^{-in\theta}.$$

Both series being absolutely convergent, we may multiply by the usual rule (§ 1.65). The resulting series is uniformly convergent for $0 \leqslant \theta \leqslant 2\pi$, and we may therefore integrate term by term over this interval. On integration, all the terms for which $m \neq n$ vanish, and we obtain

$$\int_0^{2\pi} |f(z)|^2\, d\theta = \sum_{n=0}^{\infty} a_n \bar{a}_n r^{2n} \int_0^{2\pi} d\theta = 2\pi \sum_{n=0}^{\infty} |a_n|^2 r^{2n},$$

or
$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \frac{1}{2\pi} \int_0^{2\pi} |f(z)|^2 \, d\theta \leqslant \{M(r)\}^2.$$

The result clearly follows from this.

Example. Show that Cauchy's inequality reduces to an equality if, and only if, $f(z)$ is a constant multiple of a power of z.

2.51. Liouville's theorem. *A function which is analytic for all finite values of z, and is bounded, is a constant.*

We give two proofs of this important theorem.

First Proof. If $f(z)$ is analytic for all finite values of z, the Taylor's series
$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$
is convergent for all z. Also, if $|f(z)| \leqslant M$, then by Cauchy's inequality
$$|a_n| \leqslant M r^{-n}$$
for all values of n and r. Making $r \to \infty$, the right-hand side tends to zero if $n > 0$. Hence $a_n = 0$ for $n > 0$, and $f(z) = a_0$.

Second Proof. If z_1, z_2 are any two numbers,
$$f(z_1) - f(z_2) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_1} \, dz - \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_2} \, dz$$
$$= \frac{1}{2\pi i} \int_C \frac{z_1 - z_2}{(z - z_1)(z - z_2)} f(z) \, dz,$$
where C is a contour including both z_1 and z_2. Taking C to be a circle with centre the origin, and radius R greater than $|z_1|$ or $|z_2|$, we have
$$|f(z_1) - f(z_2)| \leqslant \frac{|z_1 - z_2| M R}{(R - |z_1|)(R - |z_2|)}.$$
The right-hand side tends to zero as $R \to \infty$. Hence $f(z_1) = f(z_2)$. Since this holds for all values of z_1 and z_2, $f(z)$ is a constant.

The same result holds if the function is bounded on any sequence of contours tending to infinity. This is clear from either proof.

2.52. The following is a more general result of the same kind. *If $f(z)$ is analytic for all finite values of z, and as $|z| \to \infty$*
$$f(z) = O(|z|^k),$$
then $f(z)$ is a polynomial of degree $\leqslant k$.

For by Cauchy's inequality

$$|a_n| \leqslant M(r)r^{-n} = O(r^{k-n}),$$

and the right-hand side tends to zero as $r \to \infty$ if $n > k$. Hence $a_n = 0$ for $n > k$, and the result follows.

2.53. The function $A(r)$. Let $A(r)$ denote the upper bound of the real part of $f(z)$ on $|z| = r$. We next prove an inequality similar to Cauchy's inequality, but involving $A(r)$ instead of $M(r)$.

We have

$$|a_n|r^n \leqslant \max\{4A(r), 0\} - 2\mathbf{R}\{f(0)\}$$

for all values of $n > 0$ and r.

Let $z = re^{i\theta}$, and

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = u(r,\theta) + iv(r,\theta),$$

$$a_n = \alpha_n + i\beta_n.$$

Then

$$u(r,\theta) = \sum_{n=0}^{\infty} (\alpha_n \cos n\theta - \beta_n \sin n\theta) r^n.$$

The series converges uniformly with respect to θ. Hence we may multiply by $\cos n\theta$ or $\sin n\theta$ and integrate term by term; and we obtain

$$\frac{1}{\pi}\int_0^{2\pi} u(r,\theta)\cos n\theta \, d\theta = \alpha_n r^n, \qquad \frac{1}{\pi}\int_0^{2\pi} u(r,\theta)\sin n\theta \, d\theta = -\beta_n r^n$$

for $n > 0$, while

$$\frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)\, d\theta = \alpha_0.$$

Hence

$$a_n r^n = (\alpha_n + i\beta_n) r^n = \frac{1}{\pi}\int_0^{2\pi} u(r,\theta) e^{-in\theta}\, d\theta \qquad (n > 0),$$

and

$$|a_n| r^n \leqslant \frac{1}{\pi}\int_0^{2\pi} |u(r,\theta)| d\theta.$$

Hence

$$|a_n| r^n + 2\alpha_0 \leqslant \frac{1}{\pi}\int_0^{2\pi} \{|u(r,\theta)| + u(r,\theta)\}\, d\theta.$$

Now $|u|+u$ is zero if $u<0$. Hence if $A(r)<0$ the right-hand side is zero. If $A(r)\geqslant 0$, the right-hand side does not exceed

$$\frac{1}{\pi}\int_0^{2\pi} 2A(r)\,d\theta = 4A(r).$$

This proves the theorem.

There are, of course, similar results involving the lower bound of $\mathbf{R}f(z)$, and the upper and lower bounds of $\mathbf{I}f(z)$.

2.54. The analogue of Liouville's theorem for $A(r)$. *If $f(z)$ is analytic for all finite values of z, and $A(r)$ is bounded as $r\to\infty$, then $f(z)$ is a constant. If $A(r)<Ar^k$, where A and k are constants, then $f(z)$ is a polynomial of degree $\leqslant k$.*

In the first case, it follows from the above theorem that $|a_n|r^n$ is bounded as $r\to\infty$ for every $n>0$. Hence $a_n=0$ for $n>0$, and $f(z)=a_0$. Similarly, in the second case, $a_n=0$ for $n>k$, and $f(z)$ is of degree $\leqslant k$. It is sufficient that the conditions should hold for some arbitrarily large values of r.

The first part of the theorem may also be proved directly as follows. Consider the function $\phi(z)=\exp\{f(z)\}$. Then

$$|\phi(z)| = e^{u(r,\theta)}.$$

Hence, if $u(r,\theta)\leqslant A$, then

$$|\phi(z)|\leqslant e^A.$$

Hence, by Liouville's theorem, $\phi(z)$ is a constant. Hence $f(z)$ is a constant.

2.6. The zeros of an analytic function. A zero of an analytic function $f(z)$ is a value of z such that $f(z)=0$. If $f(z)$ is analytic in the neighbourhood of $z=a$, then

$$f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n$$

for $|z-a|$ small enough; and if $z=a$ is a zero, one or more of the coefficients $a_0, a_1,\ldots$ vanish. If $a_n=0$ for $n<m$, but $a_m\neq 0$, then $f(z)$ is said to have a zero of the mth order. Thus every zero is of some definite integral order—a function cannot have a zero of fractional order inside a region where it is analytic.

At a zero of order m, we have

$$f(a)=f'(a)=\ldots=f^{(m-1)}(a)=0,$$

while $f^{(m)}(a)\neq 0$. This is clear from the form of Taylor's series.

The zeros of an analytic function are isolated points; that is to say, *if a function $f(z)$ is not identically zero, and is analytic in a region including $z=a$, then there is a circle $|z-a|=\rho$ $(\rho>0)$ inside which $f(z)$ has no zeros except possibly $z=a$ itself.*

The theorem may also be stated as follows:

Let $f(z)$ be a function analytic in a region D, and let $P_1, P_2, \ldots, P_n, \ldots$ be a set of points having a limit-point P inside D. Then if $f(z)=0$ at every point P_n, it follows that $f(z)=0$ at all points of D.

It may be supposed without loss of generality that P is $z=0$. Then $f(z)$ is analytic in a region including $z=0$, and hence

$$f(z)=\sum_{n=0}^{\infty} a_n z^n$$

for $|z|<R$, say. It will be proved that all the coefficients in this series are zero. If this is not so, there is a first coefficient which is not zero, say a_k. Then

$$f(z)=z^k(a_k+a_{k+1}z+\ldots) \qquad (|z|<R).$$

If $0<\rho<R$, the series is convergent for $z=\rho$, and so $a_n\rho^n$ is bounded, say $|a_n|\rho^n \leqslant K$. Hence

$$|f(z)| \geqslant |z|^k\left(|a_k|-\frac{K|z|}{\rho^{k+1}}-\frac{K|z|^2}{\rho^{k+2}}-\ldots\right)$$
$$=|z|^k\left\{|a_k|-\frac{K|z|}{\rho^k(\rho-|z|)}\right\},$$

and the right-hand side is positive if $|z|$ is sufficiently small, except at the point $z=0$ itself. This contradicts the hypothesis that $f(z)$ has zeros arbitrarily near to, but not coincident with, $z=0$. Hence in fact all the coefficients vanish. Hence $f(z)=0$ inside the circle of convergence of the above series.

We can now, however, repeat the process, starting from any point inside this circle; for the data now hold for any such point, by what has just been proved.* In this way the result may be extended to any point interior to D.

2.61. The theorem has the following obvious corollaries:

(i) If a function is analytic in a region, and vanishes at all points of any smaller region included in the given region, or along any arc of a continuous curve in the region, then it must vanish identically.

* For a more detailed discussion of such chains of circles see Ch. IV.

(ii) If two functions are analytic in a region, and have the same value at an infinity of points which have a limit-point in the region, then they must be equal throughout the region.

Examples. (i) The function $\sin z$ has zeros of the first order at the points $z = 0, \pm\pi, \pm 2\pi, \ldots$, and no other zeros. [The formula

$$|\sin(x+iy)| = \sqrt{(\sin^2 x + \sinh^2 y)},$$

P.M. § 233, ex. 2, shows that $\sin z$ has these zeros and no others.]

(ii) The function $\cos z$ has zeros of the first order at $z = \pm\frac{1}{2}\pi, \pm\frac{3}{2}\pi, \ldots$, and no other zeros.

(iii) If $f(z)$ and $g(z)$ are both analytic functions in a region D, and $f(z)g(z) = 0$ in D, then either $f(z) = 0$ throughout D, or $g(z) = 0$ throughout D.

2.7. Laurent series. *Let $f(z)$ be a function analytic in the ring-shaped region between two concentric circles C and C', of radii R and R' ($R' < R$), and centre a, and on the circles themselves.*

Then $f(z)$ can be expanded in a series of positive and negative powers of $z-a$, convergent at all points of the ring-shaped region.

We must remind the reader that all functions considered here are one-valued. This assumption excludes certain functions, any one value of which is analytic at all points of the ring. Consider, for example, the function

$$f(z) = z^p$$

where p is real. This is analytic except possibly at $z = 0$. Now, if $z = re^{i\theta}$,

$$f(z) = r^p e^{ip\theta}.$$

As we pass round a circle with centre the origin, starting at $\theta = 0$, say, $f(z)$ changes from r^p to $r^p e^{2ip\pi}$, and so does not return to its original value unless p is an integer.

To prove the theorem, let z be a point of the ring, and consider the integral

$$\frac{1}{2\pi i}\int \frac{f(w)}{w-z}\,dw$$

taken round the outer circle C in the positive direction, then along a radius vector (which we may suppose does not pass through z) to the inner circle C', then round C' in the negative direction, then back along the radius vector to the starting-point. This is a closed contour to which we may apply our previous results—the fact that part of it is described twice does not affect any of the arguments. The value of the integral is therefore $f(z)$.

Since $f(z)$ is one-valued, the two integrals along the radius vector joining the circles cancel, and we obtain

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\int_{C'} \frac{f(w)}{w-z}\,dw,$$

where now each integral is taken in the positive direction. As in the proof of the Cauchy-Taylor theorem

$$\frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\,dw = \sum_{n=0}^{\infty} a_n(z-a)^n,$$

where
$$a_n = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-a)^{n+1}}\,dw.$$

In this case, however, a_n is not in general equal to $f^{(n)}(a)/n!$, since $f(z)$ is not necessarily analytic throughout the interior of C.

Again

$$\frac{1}{z-w} = \frac{1}{z-a} + \frac{w-a}{(z-a)^2} + \ldots + \frac{(w-a)^{n-1}}{(z-a)^n} + \ldots,$$

this series being uniformly convergent on C'. Hence

$$-\frac{1}{2\pi i}\int_{C'} \frac{f(w)}{w-z}\,dw = \frac{1}{z-a}\cdot\frac{1}{2\pi i}\int_{C'} f(w)\,dw + \ldots +$$

$$+\frac{1}{(z-a)^n}\cdot\frac{1}{2\pi i}\int_{C'} (w-a)^{n-1}f(w)\,dw + \ldots = \sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n},$$

where
$$b_n = \frac{1}{2\pi i}\int_{C'} (w-a)^{n-1}f(w)\,dw.$$

These two series together form Laurent's expansion. They may be written together in the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n,$$

where
$$a_n = \frac{1}{2\pi i}\int \frac{f(w)}{(w-a)^{n+1}}\,dw$$

for all values of n, the integral being taken round any simple closed contour which passes round the ring.

In the particular case where $f(z)$ is analytic inside C', all the coefficients b_n are zero (by Cauchy's theorem), and the series reduces to Taylor's series.

Notice that the series of positive powers of $z-a$ converges,

not merely in the ring, but everywhere inside the circle C. Similarly the series of negative powers converges everywhere outside C'.

Examples. (i) Show that

$$e^{\frac{1}{2}c(z-\frac{1}{z})} = \sum_{n=-\infty}^{\infty} a_n z^n,$$

where
$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - c\sin\theta)\,d\theta.$$

(ii) Show that, if $c > 0$,

$$e^{z+\frac{c^3}{2z^3}} = \sum_{n=-\infty}^{\infty} a_n z^n$$

where
$$a_n = \frac{e^{-\frac{1}{2}c}}{2\pi c^n}\int_0^{2\pi} e^{c(\cos\theta+\cos^2\theta)}\cos\{c\sin\theta(1-\cos\theta)-n\theta\}\,d\theta.$$

2.71. Isolated singularities of an analytic function. Suppose that a function $f(z)$ is analytic throughout the neighbourhood of a point a, say for $|z-a| < R$, except at the point a itself. Then the point a is called an isolated singularity of the function.

Suppose that $f(z)$ is one-valued. We may then expand $f(z)$ in a Laurent series of powers of $z-a$, and the inner circle C' of § 2.7 may be taken as small as we please. Thus

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} b_n(z-a)^{-n} \qquad (0 < |z-a| < R).$$

There are now three possible cases. All the coefficients b_n may be zero. The function $f(z)$ is then equal to a function analytic for $|z-a| < R$, except at the point a; for example, we might define $f(z)$ to be 1 except at $z = a$, and $f(a) = 0$. This is a rather artificial sort of singularity, and of no further interest in the theory.

Secondly, the series of negative powers of $z-a$ may contain a finite number of terms only. Then $f(z)$ is said to have a *pole* at the point $z = a$. If b_m is the last coefficient which does not vanish, then

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{m} b_n(z-a)^{-n},$$

and the pole is said to be of order m, or to be a simple, double,... pole in the cases $m = 1, 2, \ldots$.

If $f(z)$ has a pole of order m, then plainly $(z-a)^m f(z)$ is analytic and not zero at $z=a$. Hence also

$$\phi(z)=\frac{1}{(z-a)^m f(z)}$$

is analytic and not zero at $z=a$. Hence

$$\frac{1}{f(z)}=(z-a)^m\phi(z),$$

so that $1/f(z)$ has a zero of order m at $z=a$.

Conversely, a similar argument shows that, if $f(z)$ has a zero of order m, then $1/f(z)$ has a pole of order m.

The finite series

$$\sum_{n=1}^{m} b_n(z-a)^{-n}$$

is called the *principal part* of $f(z)$ at $z=a$.

If $f(z)$ has a pole at $z=a$, then $|f(z)|\to\infty$ as $z\to a$. For

$$\left|\sum_{n=1}^{m} b_n(z-a)^{-n}\right|=|z-a|^{-m}\left|\sum_{n=1}^{m} b_n(z-a)^{m-n}\right|$$

$$\geqslant |z-a|^{-m}\left\{|b_m|-\sum_{n=1}^{m-1}|b_n||z-a|^{m-n}\right\},$$

and the expression in brackets tends to $|b_m|$, so that the whole tends to infinity—in fact the function is dominated by the last term in the principal part.

If $f(z)=O(|z-a|^{-k})$ as $|z-a|\to 0$, the singularity is at most a pole of order k; in particular, if $f(z)=O(1)$, there is no singularity (except of the trivial type first mentioned).

An argument similar to that of § 2.52, but with b_n, and r tending to 0, shows that $b_n=0$ for $n>k$. Again it is clearly sufficient that the data should hold on a sequence of contours tending to 0.

Examples. (i) The functions $\cot z$ and $\operatorname{cosec} z$ have simple poles at the points $z=0, \pm\pi, \pm 2\pi, \ldots$.

(ii) The functions $\tan z$ and $\sec z$ have simple poles at the points $z=\pm\frac{1}{2}\pi, \pm\frac{3}{2}\pi, \ldots$.

(iii) Find the poles of

$$\frac{1}{\sin z\pm\sin a},\qquad \frac{1}{\cos z\pm\cos a}.$$

(iv) The function $\operatorname{cosec} z^2$ has one double pole and an infinity of simple poles.

(v) Find the poles of the functions

$$\frac{1}{z^2+1}, \qquad \frac{1}{z^4+1}, \qquad \frac{1}{z^4+2z^2+1}.$$

2.72. Essential singularities. The third possibility is that in the expansion of $f(z)$ in powers of $z-a$, the series of negative powers may not terminate. The point $z=a$ is then called an *essential singularity* of $f(z)$. In this case

$$f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n},$$

where the last series does not terminate, but is convergent for all values of z except $z=a$.

The complicated behaviour of a function in the neighbourhood of an essential singularity is shown by the following theorem of Weierstrass.

Given any positive numbers ρ, ϵ, and any number c, there is a point z in the circle $|z-a|<\rho$ at which $|f(z)-c|<\epsilon$.

That is to say, $f(z)$ tends to any given limit as z tends to a through a suitable sequence of values.

We begin by proving that, if ρ and M are any positive numbers, then there are values of z in the circle $|z-a|<\rho$ at which $|f(z)|>M$. If this is not true, then $|f(z)|\leqslant M$ for $|z-a|<\rho$. Hence, if the radius of C' is R',

$$|b_n|=\left|\frac{1}{2\pi i}\int_{C'}(w-a)^{n-1}f(w)\,dw\right|\leqslant MR'^n,$$

by § 2.31. This holds for all positive values of n and R', and, making $R'\to 0$, we see that $b_n=0$ for $n\geqslant 1$. Hence there is no essential singularity, contrary to hypothesis.

Now consider any finite value of c. If $f(z)-c$ has zeros inside every circle $|z-a|=\rho$, the result follows at once. If not, we can choose ρ so small that $f(z)-c$ has no zero for $|z-a|<\rho$.

Then
$$\phi(z)=\frac{1}{f(z)-c}$$

is regular for $|z-a|<\rho$, except at $z=a$. The point $z=a$ is an essential singularity of $\phi(z)$; for

$$f(z)=\frac{1}{\phi(z)}+c,$$

and if $\phi(z)$ had a pole, $f(z)$ would be analytic; while, if $\phi(z)$ were analytic, $f(z)$ would be analytic or have a pole.

It now follows from the first part that there is a point z in the circle $|z-a| < \rho$ such that

$$|\phi(z)| > \frac{1}{\epsilon},$$

i.e.

$$|f(z)-c| < \epsilon,$$

and this is the result stated.

This theorem distinguishes clearly between poles and essential singularities. While at a pole $f(z)$ tends to infinity, at an essential singularity $f(z)$ has no unique limiting value, and in fact comes arbitrarily near to any assigned value an infinity of times.

Examples. (i) The functions

$$e^{1/z}, \qquad \sin\frac{1}{z}, \qquad \cos\frac{1}{z}$$

have isolated essential singularities at $z = 0$.

(ii) The function $\operatorname{cosec}(1/z)$ has a singularity at $z = 0$, but it is not an isolated singularity, being the limit-point of the poles at the points $z = 1/(n\pi)$. We call such a point an essential singularity also.

(iii) The function $e^{1/z}$ actually takes every value except 0 an infinity of times in the neighbourhood of $z = 0$; and it tends to the limit 0 as $z \to 0$ along the negative real axis.

2.73. The 'point at infinity'. We may consider 'infinity' as a point by making the substitution $z = 1/w$. Then the behaviour of $f(z)$ 'at infinity' depends on the behaviour of $f(1/w)$ at $w = 0$. We say that $f(z)$ is analytic, has a simple pole, etc., at infinity, if $f(1/w)$ has the same property at $w = 0$. Thus $f(z) = z^2$ has a double pole at infinity.

A function which is analytic everywhere, including infinity, is a constant. For by Laurent's theorem, since $f(z)$ is regular for all finite values of z,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

$$f\left(\frac{1}{w}\right) = \sum_{n=0}^{\infty} \frac{a_n}{w^n}.$$

Since $f(1/w)$ is regular at $w = 0$, $a_n = 0$ for $n = 1, 2, \ldots$, so that $f(z) = a_0$.

A function which has no singularities other than poles is a rational function.

In the first place, there can only be a finite number of such poles; otherwise the poles would have a limit-point either at a finite point or at infinity, and at such a limit-point the function would not be analytic or have a pole, contrary to hypothesis.

Suppose, then, that the poles of $f(z)$ at finite points are at $a, b, \ldots, k$, with multiplicities $\alpha, \beta, \ldots, \kappa$. Then

$$g(z) = f(z)(z-a)^{\alpha} \ldots (z-k)^{\kappa}$$

is analytic except at infinity, where it has at most a pole. Hence

$$g(z) = \sum_{n=0}^{\infty} a_n z^n,$$

$$g(1/w) = \sum_{n=0}^{\infty} a_n w^{-n}.$$

Since the singularity of $g(1/w)$ at the origin, if there is one, is a pole, this series must terminate, i.e. $g(z)$ is a polynomial. Hence $f(z)$ is the quotient of two polynomials, i.e. a rational function.

Conversely, *a rational function has no singularities other than poles.*

2.8. Uniformly convergent series of analytic functions. *Suppose that*

(i) *each member of a sequence of functions*

$$u_1(z),\ u_2(z), \ldots, u_n(z), \ldots$$

is analytic inside a region D,

(ii) *the series*

$$\sum_{n=1}^{\infty} u_n(z)$$

is uniformly convergent throughout every region D' interior to D. Then the function

$$f(z) = \sum_{n=1}^{\infty} u_n(z)$$

is analytic inside D, and all its derivatives may be calculated by term-by-term differentiation.

Let C be a simple closed contour lying entirely inside D, and let z be a point inside C.

If we knew already that $f(z)$ was an analytic function, we should have

$$f(z)=\frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\,dw. \tag{1}$$

Actually we obtain this result from our data, and then use it to prove that $f(z)$ is analytic.

We have

$$u_n(z)=\frac{1}{2\pi i}\int_C \frac{u_n(w)}{w-z}\,dw$$

for each function $u_n(z)$. Hence

$$f(z)=\sum_{n=1}^{\infty} u_n(z)=\sum_{n=1}^{\infty}\frac{1}{2\pi i}\int_C \frac{u_n(w)}{w-z}\,dw.$$

But, since $\sum u_n(w)$ is uniformly convergent on C, we may multiply by $1/(w-z)$ and integrate term by term. Thus

$$\int_C\left\{\sum \frac{u_n(w)}{w-z}\right\}dw=\sum\int_C \frac{u_n(w)}{w-z}\,dw,$$

and we obtain

$$f(z)=\frac{1}{2\pi i}\int_C\left\{\sum \frac{u_n(w)}{w-z}\right\}dw=\frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\,dw,$$

i.e. we have proved (1).

We can now deduce from (1), as in § 2.41, that $f(z)$ has a derivative $f'(z)$, given by the formula

$$f'(z)=\frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z)^2}\,dw.$$

In this case the boundedness of $f(z)$ follows from the uniform convergence of the series. Hence $f(z)$ is analytic.

Also

$$\frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z)^2}\,dw=\frac{1}{2\pi i}\int_C \sum_{n=1}^{\infty} u_n(w)\,\frac{dw}{(w-z)^2}$$

$$=\sum_{n=1}^{\infty}\frac{1}{2\pi i}\int_C \frac{u_n(w)}{(w-z)^2}\,dw=\sum_{n=1}^{\infty} u_n'(z),$$

using the uniformity of convergence again. Hence the series may be differentiated term by term. Also the differentiated series is uniformly convergent in any region interior to D. For if D' is such a region, we can suppose that the curve C includes

D' in its interior, and that the least distance of points of D' from C is δ. Then for any point z of D'

$$\left|\sum_{n=N}^{N'} u_n'(z)\right| = \left|\sum_{N}^{N'} \frac{1}{2\pi i}\int_C \frac{u_n(w)}{(w-z)^2}\,dw\right|$$

$$= \left|\frac{1}{2\pi i}\int_C \sum_{N}^{N'} u_n(w)\,\frac{dw}{(w-z)^2}\right| \leqslant \frac{\epsilon l}{2\pi\delta^2},$$

where l is the length of C, and ϵ the maximum modulus of

$$\sum_{N}^{N'} u_n(w)$$

on C. Since the right-hand side is independent of z, and tends to zero when N and N' tend independently to infinity, the result follows.

The whole process may now be repeated, starting from the differentiated series, and the general result thus follows.

The theorem, in a slightly different form, is known as 'Weierstrass's double-series theorem'.*

2.81. Remarks on the above theorem.

(i) We have already pointed out (§ 2.37) the contrast between the conditions for term-by-term differentiation of real series, and of series of analytic functions. In the case of real series, we have to assume that the differentiated series is uniformly convergent. In the above theorem no such assumption is necessary; actually the differentiated series is uniformly convergent, but this is one of the conclusions of the theorem.

(ii) If we merely assumed that the given series was uniformly convergent on a certain closed curve C, we could prove as before that $f(z)$ was analytic at all points inside C.

(iii) Even if we assume that each $u_n(z)$ is analytic on the boundary of D, and that the series is uniformly convergent on the boundary, we cannot prove that $f(z)$ is analytic on the boundary, or that the differentiated series converges on the boundary. Consider, for example, the series

$$\sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

This is uniformly convergent for $|z| \leqslant 1$; but the differentiated

* See Knopp's *Infinite Series*, § 56.

series is not uniformly convergent in the neighbourhood of $z = 1$, nor is the function represented by the series analytic at $z = 1$. In fact

$$f'(z) = -\frac{\log(1-z)}{z}.$$

(iv) The theorem may be stated as a theorem on sequences of functions: if $f_n(z)$ is analytic in D for each value of n, and tends to $f(z)$ uniformly in any region interior to D, then $f(z)$ is analytic inside D, and $f'_n(z)$ tends to $f'(z)$ uniformly in any region interior to D.

Examples. (i) The function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is analytic for $\mathbf{R}(s) > 1$. [For the series is uniformly convergent in any finite region to the right of $\mathbf{R}(s) = 1$, see § 1.21, example.]

(ii) We have, for $\mathbf{R}(s) > 1$,

$$\zeta'(s) = -\sum_{n=2}^{\infty} n^{-s} \log n,$$

and generally

$$\zeta^{(k)}(s) = (-1)^k \sum_{n=2}^{\infty} n^{-s} \log^k n.$$

(iii) In what region does the series

$$\sum_{n=1}^{\infty} \frac{\sin nz}{2^n}$$

represent an analytic function?

(iv) The series

$$\sum_{n=1}^{\infty} \frac{\sin nz}{n^2}$$

is uniformly convergent on the real axis, but not in any *region* of the z-plane; so we can deduce nothing about the analytic character of the function which it represents.

2.82. Another proof of the theorem. We can also deduce Weierstrass's theorem from Morera's theorem (§ 2.42). For, since $\sum u_n(z)$ is uniformly convergent, we may integrate it term by term round any contour C. Thus

$$\int_C f(z)\, dz = \sum_{n=1}^{\infty} \int_C u_n(z)\, dz.$$

But, since each $u_n(z)$ is analytic, every term on the right is zero. Hence

$$\int_C f(z)\, dz = 0.$$

Hence, by Morera's theorem, $f(z)$ is analytic.

2.83. Definition of analytic functions by means of integrals. *Let $f(z,w)$ be a continuous function of the complex variables z and w, where z ranges over a region D, and w lies on a contour C. Let $f(z,w)$ be an analytic function of z in D, for every value of w on C. Then*

$$F(z) = \int_C f(z,w)\,dw$$

is an analytic function of z in D; and

$$F'(z) = \int_C \frac{\partial f}{\partial z}\,dw,$$

and similarly for higher derivatives.

We may suppose that the contour C consists of a single regular curve, on which $w = u+iv$, $u = u(t)$, $v = v(t)$, $t_0 \leqslant t \leqslant t_1$, and $u'(t)$ and $v'(t)$ are continuous.

Let Γ be a contour lying in D, on which $z = x+iy$, $x = x(s)$, $y = y(s)$, $s_0 \leqslant s \leqslant s_1$, and $x'(s)$ and $y'(s)$ are continuous. Let ζ be a point inside Γ. Then

$$f(\zeta,w) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z,w)}{z-\zeta}\,dz,$$

$$F(\zeta) = \frac{1}{2\pi i}\int_C dw \int_\Gamma \frac{f(z,w)}{z-\zeta}\,dz.$$

We may invert the order of these two integrations. For we can express each of these complex integrals as a sum of real integrals, as in § 2.3; and we clearly obtain an expression of the form

$$\int_{t_0}^{t_1} dt \int_{s_0}^{s_1} \{\phi(s,t)+i\psi(s,t)\}\,ds,$$

where ϕ and ψ are real continuous functions of s and t. Now we know that a repeated integral of this type may be inverted (§ 1.81). Hence

$$F(\zeta) = \frac{1}{2\pi i}\int_\Gamma \frac{dz}{z-\zeta}\int_C f(z,w)\,dw$$

$$= \frac{1}{2\pi i}\int_\Gamma \frac{F(z)}{z-\zeta}\,dz.$$

Thus $F(z)$ satisfies Cauchy's integral formula, and from this point the proof that $F(z)$ is analytic, and that we can differentiate under the integral sign, proceeds as in the theorem on uniformly convergent series.

Examples. (i) If $f(t)$ is continuous in (a, b), then

$$F(z) = \int_a^b \cos zt f(t)\, dt, \qquad G(z) = \int_a^b \sin zt f(t)\, dt$$

are analytic functions for all finite values of z.

(ii) Under the same conditions

$$\int_a^b \frac{f(t)}{z-t}\, dt$$

is analytic, except possibly when z is real and lies in the interval (a, b).

2.84. Infinite integrals. *Let C be a contour going to infinity, any bounded part of which is regular. Suppose that the conditions of the previous theorem are satisfied on any bounded part of C, and that*

$$\int_C f(z, w)\, dw$$

is uniformly convergent. Then the results of the previous theorem still hold.

Let C_n be the part of C inside the circle $|z| = n$, and let

$$F_n(z) = \int_{C_n} f(z, w)\, dw.$$

Then $F_n(z)$ is analytic for every n, by the theorem on finite integrals. Also

$$F_n(z) \to F(z)$$

uniformly as $n \to \infty$. Hence, by the theorem on uniformly convergent sequences, $F(z)$ is analytic. Finally

$$F'(z) = \lim_{n\to\infty} F_n'(z) = \lim_{n\to\infty} \int_{C_n} \frac{\partial f}{\partial z}\, dw = \int_C \frac{\partial f}{\partial z}\, dw.$$

2.85. Infinite integrals of the second kind. There is a similar theorem for the case of a finite contour C, at one end of which $f(z, w) \to \infty$. Such an integral represents an analytic function, provided that the convergence of the integral is uniform. The formal statement and the proof are practically the same as those of the previous theorem.

Examples. (i) The function

$$\Gamma(z) = \int_0^\infty e^{-w} w^{z-1}\, dw$$

is an analytic function for $\mathbf{R}(z) > 0$. [The uniform convergence of this integral has been discussed in § 1.51, ex. (i). It converges uniformly in

any finite region in which $\mathbf{R}(z) \geqslant a > 0$; and any point at which $\mathbf{R}(z) > 0$ is an internal point of such a region.]

(ii) In what regions do the integrals

$$\int_0^\infty e^{-zw^2}\,dw, \qquad \int_0^\infty \frac{\sin w}{w^z}\,dw, \qquad \int_0^\infty \frac{\cos w}{w^z}\,dw,$$

represent analytic functions?

(iii) The integral

$$\int_0^\infty \frac{\sin wz}{w}\,dw$$

converges uniformly in certain intervals of real values of z, but not in any *region*; so we cannot deduce anything about the analytic character of the function which it represents.

2.9. Remark on Laurent series. Suppose that we have obtained in any manner, or as the definition of $f(z)$, the formula

$$f(z) = \sum_{n=-\infty}^{\infty} A_n(z-a)^n \qquad (R' < |z| < R).$$

Is the series necessarily identical with the Laurent series of $f(z)$? Yes; for if C is the circle $|z-a| = \rho$, $R' < \rho < R$, the Laurent coefficient a_n is

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \sum_{m=-\infty}^{\infty} \frac{A_m}{2\pi i}\int_C \frac{(z-a)^m}{(z-a)^{n+1}}\,dz,$$

by uniform convergence; and the right-hand side is A_n, by § 2.3, exs. (iii) and (iv).

CHAPTER III

RESIDUES, CONTOUR INTEGRATION, ZEROS

3.1. The residue at a singularity. We know (§ 2.71) that, in the neighbourhood of an isolated singularity $z=a$, a one-valued analytic function $f(z)$ may be expanded in the form

$$f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n+\sum_{n=1}^{\infty} b_n(z-a)^{-n}.$$

The coefficient b_1 is of particular importance, and is called the *residue* of $f(z)$ at the point $z=a$. By the formulae of Laurent's expansion,

$$b_1=\frac{1}{2\pi i}\int_\gamma f(z)\,dz,$$

where γ is any circle with centre $z=a$, which excludes all other singularities of the function.

It is easily seen that, if $z=a$ is a simple pole,

$$b_1=\lim_{z\to a}(z-a)f(z).$$

3.11. The theorem of residues. *Let $f(z)$ be one-valued and analytic inside and on a simple closed contour C, except at a finite number of singularities $z_1, z_2,\ldots,z_n$. Let the residues of $f(z)$ at these points be $R_1, R_2,\ldots,R_n$. Then*

$$\int_C f(z)\,dz=2\pi i(R_1+R_2+\ldots+R_n).$$

Let $\gamma_1, \gamma_2,\ldots,\gamma_n$ be circles with centres $z_1, z_2,\ldots,z_n$, and radii so small that they lie entirely inside C and do not overlap. Then $f(z)$ is analytic in the region between C and these circles, so that, by Cauchy's theorem (see § 2.35),

$$\int_C f(z)\,dz=\int_{\gamma_1} f(z)\,dz+\ldots+\int_{\gamma_n} f(z)\,dz.$$

But
$$\int_{\gamma_1} f(z)\,dz=2\pi i R_1,$$

etc., and the result follows.

3.12. Contour integration. The theorem of residues may be used to evaluate a large number of real definite integrals. To do this we take a contour, part of which consists of the real axis, and the remaining part of which is usually made to tend to infinity. The process is called contour integration. It is best made clear by means of examples.

3.121. It is well known that

$$\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}.$$

To prove this by contour integration, consider the integral

$$\int \frac{dz}{1+z^2}$$

taken round the contour consisting of the real axis from $-R$ to R, with a semi-circle, on this line as diameter, above it. Since

$$\frac{1}{1+z^2} = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right),$$

the integrand has a pole at $z = i$, which is inside the contour if $R > 1$, with residue $1/2i$. Hence, by the theorem of residues, the integral is equal to π.

Now, on the semicircle, $|1+z^2| \geqslant R^2-1$, so that the integral round the semicircle does not exceed

$$\frac{\pi R}{R^2-1},$$

and so it tends to zero as $R \to \infty$. Hence

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{1+x^2} = \pi.$$

Since the integrand is an even function, the result now follows.

The integral of any even rational function which behaves suitably at infinity can be evaluated in a similar way.

Of course we know the indefinite integral of $1/(1+x^2)$, viz. $\arctan x$, and can evaluate the integral from this. The method shows to better advantage in cases where we do not know the indefinite integral.

3.122. It has been shown in § 1.76 that

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

To prove this by contour integration, consider the integral

$$\int \frac{e^{iz}}{z}\, dz$$

taken round the contour consisting of the real axis from $z=\rho$ to $z=R$, where $0<\rho<R$; a semicircle Γ of radius R above the real axis; the real axis again from $-R$ to $-\rho$; and finally a semicircle γ of radius ρ above the real axis. We take ρ small and R large. The small semicircle is necessary to avoid the singularity of the integrand at $z=0$, and the large semicircle is necessary to close up the contour.

The function e^{iz}/z has no singularity inside the contour, and the value of the integral is therefore zero. Thus

$$\int_{\rho}^{R} \frac{e^{ix}}{x}\,dx + \int_{\Gamma} \frac{e^{iz}}{z}\,dz + \int_{-R}^{-\rho} \frac{e^{-ix}}{-x}\,dx + \int_{\gamma} \frac{e^{iz}}{z}\,dz = 0.$$

The two integrals along the real axis are together equal to

$$\int_{\rho}^{R} \frac{e^{ix}-e^{-ix}}{x}\,dx = 2i\int_{\rho}^{R} \frac{\sin x}{x}\,dx.$$

The integral along Γ tends to 0 when $R\to\infty$. For

$$\left|\int_{\Gamma} \frac{e^{iz}}{z}\,dz\right| = \left|\int_{0}^{\pi} e^{iRe^{i\theta}}i\,d\theta\right| \leqslant \int_{0}^{\pi} e^{-R\sin\theta}\,d\theta$$

$$\leqslant \int_{0}^{\delta} d\theta + \int_{\delta}^{\pi-\delta} e^{-R\sin\delta}\,d\theta + \int_{\pi-\delta}^{\pi} d\theta < 2\delta + \pi e^{-R\sin\delta}.$$

We first take δ arbitrarily small, and then, having fixed δ, the second term may be made as small as we please by choosing R sufficiently large. Hence the integral along Γ tends to 0.

Finally,

$$\int_{\gamma} \frac{e^{iz}}{z}\,dz = \int_{\gamma} \frac{dz}{z} + \int_{\gamma} \frac{e^{iz}-1}{z}\,dz.$$

The integrand in the last integral is bounded as $\rho\to 0$, and so, by § 2.31, the integral tends to zero. Also

$$\int_{\gamma} \frac{dz}{z} = \int_{\pi}^{0} i\,d\theta = -i\pi.$$

Hence, making $\rho\to 0$ and $R\to\infty$, we obtain

$$2i\int_{0}^{\infty} \frac{\sin x}{x}\,dx - i\pi = 0$$

and the result follows.

Notice that the integral in the negative direction round the semicircle γ tends to $-i\pi$ into the residue at $z = 0$. It is easily verified that this is true of any *simple* pole, but not of a pole of higher order.

Notice also that we do not consider the integral

$$\int \frac{\sin z}{z}\, dz$$

because the integrand does not behave suitably at infinity.

3.123. *If* $0 < a < 1$,

$$\int_0^\infty \frac{x^{a-1}}{1+x}\, dx = \frac{\pi}{\sin a\pi}.$$

Consider the integral

$$\int \frac{z^{a-1}}{1+z}\, dz$$

taken along the real axis from $z = \rho$ to $z = R$; then in the positive direction along the circle Γ with centre the origin and radius R; then back along the real axis to $z = \rho$; and finally round the circle γ with centre the origin and radius ρ in the negative direction. This is a closed contour which excludes the origin. It is necessary to do so, because the function is not one-valued in a region which includes the origin, so that the theorem of residues would not apply to such a contour. The many-valued function z^{a-1} is taken to be real on the first part of the contour. It is then given at all other points by the formula $r^{a-1}e^{(a-1)i\theta}$, where $0 \leqslant \theta \leqslant 2\pi$.

There is one pole inside the contour, at $z = -1$, the residue there being $e^{(a-1)i\pi}$. Hence

$$\int_\rho^R \frac{x^{a-1}}{1+x}\, dx + \int_\Gamma \frac{z^{a-1}}{1+z}\, dz + \int_R^\rho \frac{(xe^{2i\pi})^{a-1}}{1+x}\, dx + \int_\gamma \frac{z^{a-1}}{1+z}\, dz$$
$$= 2\pi i\, e^{(a-1)i\pi}.$$

The two integrals along the real axis together give

$$(1 - e^{2i\pi(a-1)}) \int_\rho^R \frac{x^{a-1}}{1+x}\, dx = -2ie^{ia\pi} \sin a\pi \int_\rho^R \frac{x^{a-1}}{1+x}\, dx.$$

The other two integrals tend to 0. For on Γ

$$\left|\frac{z^{a-1}}{1+z}\right| \leqslant \frac{R^{a-1}}{R-1},$$

so that
$$\left|\int_\Gamma \frac{z^{a-1}}{1+z}dz\right| \leqslant \frac{R^{a-1}}{R-1}.2\pi R = \frac{2\pi R^a}{R-1},$$

which tends to zero since $a < 1$. Similarly

$$\left|\int_\gamma \frac{z^{a-1}}{1+z}\,dz\right| \leqslant \frac{2\pi\rho^a}{1-\rho},$$

which tends to zero since $a > 0$. The result therefore follows on making $R \to \infty$ and $\rho \to 0$.

3.124. The above result has an application to the theory of the Γ-function. Putting $y = 1-x$ in § 1.86 (4), we have

$$\Gamma(x)\Gamma(1-x) = \int_0^\infty \frac{u^{-x}}{1+u}\,du$$

$$= \frac{\pi}{\sin(1-x)\pi},$$

or
$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin x\pi}$$

where $0 < x < 1$.

3.125. *For* $n = 0, 1, 2, \ldots$

$$\int_0^\infty x^n e^{-x^{\frac{1}{4}}} \sin x^{\frac{1}{4}}\,dx = 0.$$

Putting $x = t^4$, the integral becomes

$$4\int_0^\infty t^{4n+3}e^{-t}\sin t\,dt.$$

Consider the integral
$$\int z^{4n+3}e^{(i-1)z}\,dz$$

taken along the real axis from 0 to R, then along a quadrant of a circle of radius R to the positive imaginary axis, and then back to the origin along the imaginary axis. On the arc of the circle
$$|e^{(i-1)z}| = e^{-R\cos\theta - R\sin\theta} \leqslant e^{-R},$$

so that
$$\left|\int z^{4n+3}e^{(i-1)z}\,dz\right| \leqslant \tfrac{1}{2}\pi R^{4n+4}e^{-R} \to 0.$$

Hence $$\int_0^\infty x^{4n+3}e^{(i-1)x}\,dx - \int_0^\infty (iy)^{4n+3}e^{(i-1)iy}i\,dy = 0,$$

or, replacing y by x in the last integral,

$$\int_0^\infty x^{4n+3}e^{-x}(e^{ix}-e^{-ix})\,dx = 0,$$

and the result follows.

3.126. *If $c > 0$, then*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{a^z}{z}\,dz = \begin{matrix} 1 & (a>1), \\ 0 & (0<a<1). \end{matrix}$$

If $a > 1$, i.e. $\log a > 0$, we consider the integral round the contour consisting of the line from $c-iR$ to $c+iR$, completed by a semicircle on the left. If R is sufficiently large, this contour includes the pole at $z = 0$, with residue 1; and it may be proved as in § 3.121 that the integral round the large semicircle tends to zero as $R \to \infty$.

If $a < 1$, we complete the contour by a semicircle on the right. There is now no pole in the contour, and the second result follows.

3.127. The Γ-function integral. We have

$$\int_0^\infty x^{p-1}e^{-ax}\,dx = \frac{\Gamma(p)}{a^p} \qquad (a>0,\ p>0).$$

If we could make the substitution $x = it$ in the integral, we should obtain

$$\int_0^\infty (it)^{p-1}e^{-ait}i\,dt = \frac{\Gamma(p)}{a^p},$$

and, multiplying by $e^{-\frac{1}{2}ip\pi}$ and separating real and imaginary parts, we find

$$\int_0^\infty t^{p-1}\begin{matrix}\cos\\ \sin\end{matrix} at\,dt = \frac{\Gamma(p)}{a^p}\begin{matrix}\cos\\ \sin\end{matrix}\tfrac{1}{2}p\pi. \tag{1}$$

The ordinary rules of integration by substitution, of course, do not cover a 'complex substitution' of this kind. The process is really an application of Cauchy's theorem. Consider the integral

$$\int z^{p-1}e^{-az}\,dz$$

taken round the contour consisting of the real axis from $z=\rho$ to $z=R$, the arc of $|z|=R$ to the imaginary axis, the imaginary axis from $z=iR$ to $z=i\rho$, and the arc of $|z|=\rho$ back to the starting-point. By Cauchy's theorem, the integral round this contour is zero. It may be proved as in previous cases that the integral along $|z|=\rho$ tends to 0 as $\rho\to 0$ if $p>0$, and that along $|z|=R$ tends to 0 as $R\to\infty$ if $p<1$. Hence the integral along the imaginary axis is minus that along the real axis, and on evaluating it we obtain (1) again, for $0<p<1$.

3.128. Occasionally we use the converse process, and deduce the residue from the value of the integral.

If p is an even positive integer, the residue of $\tan^{p-1}\pi z$ *at* $z=\frac{1}{2}$ *is* $(-1)^{\frac{1}{2}p}/\pi$.

The residue is equal to

$$\frac{1}{2\pi i}\left\{\int_{-iR}^{1-iR}+\int_{1-iR}^{1+iR}+\int_{1+iR}^{iR}+\int_{iR}^{-iR}\right\}\tan^{p-1}\pi z\,dz,$$

and

$$\int_{1-iR}^{1+iR}=\int_{-iR}^{iR}=-\int_{iR}^{-iR},$$

since $\tan\pi z$ is periodic with period 1. Hence the residue is equal to

$$\frac{1}{2\pi i}\left\{\int_{-iR}^{1-iR}+\int_{1+iR}^{iR}\right\}\tan^{p-1}\pi z\,dz.$$

Now

$$\tan\pi z=\frac{1}{i}\frac{e^{2i\pi x-2\pi y}-1}{e^{2i\pi x-2\pi y}+1}\to\begin{matrix}-1/i\\+1/i\end{matrix}\qquad\left(y\to\begin{matrix}+\infty\\-\infty\end{matrix}\right).$$

Hence as $R\to\infty$

$$\int_{-iR}^{1-iR}\tan^{p-1}\pi z\,dz\to\left(\frac{1}{i}\right)^{p-1},$$

$$\int_{1+iR}^{iR}\tan^{p-1}\pi z\,dz\to-\left(-\frac{1}{i}\right)^{p-1}=\left(\frac{1}{i}\right)^{p-1},$$

and the residue is

$$\frac{1}{\pi i}\left(\frac{1}{i}\right)^{p-1}=\frac{1}{\pi i^p}=\frac{(-1)^{\frac{1}{2}p}}{\pi}.$$

3.13. *Consider the behaviour of the integral*

$$f(t) = \int_0^\infty e^{ixt - x^{\frac{1}{4}}} \sin x^{\frac{1}{4}}\, dx$$

as $t \to \infty$.

It is convenient, for reasons which will appear later, to begin by integrating by parts. Integrating the factor e^{ixt}, the integrated term vanishes at both limits, and we obtain

$$f(t) = \frac{i}{4t} \int_0^\infty e^{ixt - x^{\frac{1}{4}}} (\cos x^{\frac{1}{4}} - \sin x^{\frac{1}{4}}) x^{-\frac{3}{4}}\, dx.$$

As in previous examples, we replace the circular functions by exponentials, and consider, instead of $f(t)$, the function

$$\phi(t) = \tfrac{1}{4} \int_0^\infty e^{ixt - x^{\frac{1}{4}} + ix^{\frac{1}{4}}} x^{-\frac{3}{4}}\, dx.$$

Putting $x = u^4/t$, we obtain

$$\phi(t) = t^{-\frac{1}{4}} \int_0^\infty e^{iu^4 - (1-i)ut^{-\frac{1}{4}}}\, du.$$

Next, turn the line of integration through an angle λ, i.e. use Cauchy's theorem as in § 3.125. We obtain

$$\phi(t) = t^{-\frac{1}{4}} \int_0^\infty e^{iv^4 e^{4i\lambda} - (1-i)ve^{i\lambda} t^{-\frac{1}{4}}} e^{i\lambda}\, dv.$$

This process is valid if the real part of the coefficient of v^4 is negative for all values of λ through which the line of integration turns; i.e. if $\sin 4\lambda > 0$, or $\lambda < \frac{1}{4}\pi$.

Actually we take $\lambda = \frac{1}{8}\pi$. This has the effect of making the term in e^{-v^4} tend to zero as rapidly as possible. It gives

$$\phi(t) = t^{-\frac{1}{4}} e^{i\pi/8} \int_0^\infty e^{-v^4 - (1-i)ve^{i\pi/8} t^{-\frac{1}{4}}}\, dv.$$

When $t \to \infty$, this last integral, being uniformly convergent, tends to the limit

$$\int_0^\infty e^{-v^4}\, dv = \tfrac{1}{4} \int_0^\infty e^{-w} w^{-\frac{3}{4}}\, dw = \tfrac{1}{4}\Gamma(\tfrac{1}{4}).$$

Hence $$\phi(t) \sim \tfrac{1}{4}\Gamma(\tfrac{1}{4}) e^{i\pi/8} t^{-\frac{1}{4}}.$$

Similarly, if $$\psi(t) = \tfrac{1}{4}\int_0^\infty e^{ixt-x^{\frac{1}{2}}-ix^{\frac{1}{2}}}x^{-\frac{3}{4}}\,dx,$$

we obtain the same asymptotic formula for $\psi(t)$ as for $\phi(t)$. Hence finally

$$f(t) = \frac{i}{t}\left\{\frac{\phi(t)+\psi(t)}{2} - \frac{\phi(t)-\psi(t)}{2i}\right\}$$
$$\sim \tfrac{1}{4}i\Gamma(\tfrac{1}{4})e^{i\pi/8}t^{-\frac{5}{4}}.$$

A similar process might, of course, have been applied to the integral before integrating by parts. The reader may verify that it only leads to the result $f(t) = o(t^{-1})$.

3.2. Expansion of a meromorphic function. A function is said to be *meromorphic* in a region if it is analytic in the region except at a finite number of poles. The expression is used in contrast to *holomorphic*, which is sometimes used instead of analytic.

The simplest meromorphic functions are rational functions. We know that a rational function can be expressed in a simple way by means of partial fractions; we shall now obtain a similar expression for a more general class of meromorphic functions.

Let $f(z)$ be a function whose only singularities, except at infinity, are poles. We shall suppose for simplicity that all these poles are simple. Let them be $a_1, a_2, \ldots$, where

$$0 < |a_1| \leqslant |a_2| \leqslant |a_3| \leqslant \ldots,$$

and let the residues at the poles be $b_1, b_2, \ldots$ respectively. Suppose that there is a sequence of closed contours C_n, such that C_n includes $a_1, a_2, \ldots, a_n$, but no other poles; such that the minimum distance R_n of C_n from the origin tends to infinity with n, while L_n, the length of C_n, is $O(R_n)$; and such that, on C_n, $f(z) = o(R_n)$. This last condition will be satisfied if, for example, $f(z)$ is bounded on the system of contours C_n taken as a whole.

Under these conditions

$$f(z) = f(0) + \sum_{n=1}^{\infty} b_n\left(\frac{1}{z-a_n} + \frac{1}{a_n}\right)$$

for all values of z except the poles.

To prove this, consider the integral

$$I = \frac{1}{2\pi i} \int_{C_n} \frac{f(w)}{w(w-z)}\, dw,$$

where z is a point inside C_n. The integrand has poles at the points a_m, with residues $b_m/\{a_m(a_m-z)\}$; at $w=z$, with residue $f(z)/z$; and at $w=0$, with residue $-f(0)/z$. In particular cases these last two residues may of course vanish. Hence

$$I = \sum_{m=1}^{n} \frac{b_m}{a_m(a_m-z)} - \frac{f(0)}{z} + \frac{f(z)}{z}.$$

On the other hand

$$|I| \leqslant \frac{L_n}{2\pi R_n(R_n-|z|)} \max_{C_n} |f(w)|,$$

which tends to 0 as $n \to \infty$, under the conditions stated. Hence

$$\frac{f(z)}{z} = \frac{f(0)}{z} - \lim_{n\to\infty} \sum_{m=1}^{n} \frac{b_m}{a_m(a_m-z)},$$

and the result stated follows.

It is also obvious from the proof that *the series converges uniformly inside any closed contour such that all the poles are outside it.*

3.21. We leave to the reader the modifications which are necessary if $f(z)$ has poles of higher order than the first. A more important extension can be made to functions which do not satisfy the condition $f(z) = o(R_n)$ on C_n. Suppose now that this is not satisfied, but that there is a positive integer p such that $f(z) = O(R_n^p)$, or, more generally, $f(z) = o(R_n^{p+1})$, on C_n. Consider the integral

$$I = \frac{1}{2\pi i} \int_{C_n} \frac{f(w)}{w^{p+1}(w-z)}\, dw.$$

The calculations proceed as before, except that the residue at $w=0$ is now

$$-\frac{1}{z}\left\{\frac{f(0)}{z^p} + \frac{f'(0)}{z^{p-1}} + \ldots + \frac{f^{(p)}(0)}{p!}\right\}.$$

The integral again tends to 0 as $n \to \infty$, and we obtain

$$f(z) = f(0) + zf'(0) + \ldots + \frac{z^p f^{(p)}(0)}{p!} + \sum_{n=0}^{\infty} \frac{b_n z^{p+1}}{a_n^{p+1}(z-a_n)},$$

or

$$f(z)=\sum_{q=0}^{p}\frac{f^{(q)}(0)z^q}{q!}+\sum_{n=0}^{\infty}b_n\left(\frac{1}{z-a_n}+\frac{1}{a_n}+\frac{z}{a_n^2}+\dots+\frac{z^p}{a_n^{p+1}}\right).$$

3.22. Application to trigonometrical functions. Consider the function

$$f(z)=\operatorname{cosec} z-1/z \quad (z\neq 0), \qquad f(0)=0.$$

At the point $z=n\pi$, where n is any positive or negative integer, $\sin z$ has a simple zero, so that $f(z)$ has a simple pole. The residue is

$$\lim_{z\to n\pi}(z-n\pi)\left(\operatorname{cosec} z-\frac{1}{z}\right)=\lim_{\zeta\to 0}\frac{\zeta}{\sin(\zeta+n\pi)}=\lim_{\zeta\to 0}\frac{(-1)^n\zeta}{\sin\zeta}=(-1)^n.$$

But there is no singularity at $z=0$, since

$$\frac{z-\sin z}{z\sin z}=\frac{O(|z|^3)}{z^2+O(|z|^4)}=O(1).$$

Let C_n be the square with corners at the points

$$(n+\tfrac{1}{2})(\pm 1\pm i)\pi.$$

The function $1/z$ is obviously bounded on these squares. To prove that $\operatorname{cosec} z$ is bounded, consider separately the regions (i) $y>\frac{1}{2}\pi$, (ii) $-\frac{1}{2}\pi\leqslant y\leqslant\frac{1}{2}\pi$, (iii) $y<-\frac{1}{2}\pi$. In the first region

$$|\operatorname{cosec} z|=\left|\frac{2i}{e^{iz}-e^{-iz}}\right|\leqslant\frac{2}{e^{\frac{1}{2}\pi}-e^{-\frac{1}{2}\pi}},$$

and a similar result holds for the third region. Also $|\operatorname{cosec} z|$ is evidently bounded on the straight line joining $\frac{1}{2}(1-i)\pi$ to $\frac{1}{2}(1+i)\pi$, and so, since it has the period π, on all the lines $(n+\frac{1}{2}-\frac{1}{2}i)\pi$, $(n+\frac{1}{2}+\frac{1}{2}i)\pi$. Hence $\operatorname{cosec} z$ is bounded on the parts of C_n which lie in (ii), and so on the whole square.

The theorem of § 3.2 therefore gives

$$\operatorname{cosec} z-\frac{1}{z}=\sum_{n=-\infty}^{\infty}{}' (-1)^n\left(\frac{1}{z-n\pi}+\frac{1}{n\pi}\right),$$

the accent indicating that the term $n=0$ is omitted from the sum. Since, when we pass from C_{n-1} to C_n, we include the two poles $\pm n\pi$ together, we should, in the first place, bracket the corresponding residues together in the sum. However, the series with $n>0$ and $n<0$ converge separately, so that the brackets may be omitted.

If we add together the terms corresponding to $\pm n$, the expansion takes the form

$$\operatorname{cosec} z = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2\pi^2 - z^2}.$$

Examples. (i) Obtain the expansions

$$\sec z = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n(n+\frac{1}{2})}{(n+\frac{1}{2})^2\pi^2 - z^2},$$

$$\tan z = 2z \sum_{n=0}^{\infty} \frac{1}{(n+\frac{1}{2})^2\pi^2 - z^2},$$

and

$$\cot z = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2\pi^2}.$$

(ii) Obtain the corresponding expressions for the hyperbolic functions.

(iii) Prove that

$$\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 + 4n^2\pi^2}.$$

(iv) Prove that

$$\operatorname{cosec}^2 z = \sum_{n=-\infty}^{\infty} \frac{1}{(z - n\pi)^2}.$$

3.23. Expansion of an integral function as an infinite product. An integral function is a function which is analytic for all finite values of z. For example, e^z, $\cos z$, $\sin z$, are integral functions. An integral function may be regarded as a generalization of a polynomial; and, just as we can extend the partial fraction formula to certain meromorphic functions, so we can extend the expression of a polynomial as a product of factors to certain integral functions.

Let $f(z)$ be an integral function of z. Suppose that it has simple zeros at the points $a_1, a_2, \ldots$. In the neighbourhood of a_n,

$$f(z) = (z - a_n)g(z)$$

where $g(z)$ is analytic and not zero. Hence

$$\frac{f'(z)}{f(z)} = \frac{1}{z - a_n} + \frac{g'(z)}{g(z)},$$

and the last term is analytic at a_n. Hence $f'(z)/f(z)$ has a simple pole at $z = a_n$, with residue 1.

Suppose now that $f'(z)/f(z)$ is a function of the type considered in § 3.2. Then

$$\frac{f'(z)}{f(z)}=\frac{f'(0)}{f(0)}+\sum_{n=1}^{\infty}\left(\frac{1}{z-a_n}+\frac{1}{a_n}\right).$$

Integrating from 0 to z along a path not passing through any of the poles, we obtain

$$\log f(z)-\log f(0)=z\frac{f'(0)}{f(0)}+\sum_{n=1}^{\infty}\left\{\log(z-a_n)-\log(-a_n)+\frac{z}{a_n}\right\}.$$

The values of the logarithms will depend on the path chosen; but when we take exponentials all ambiguity disappears, and we obtain

$$f(z)=f(0)e^{z\frac{f'(0)}{f(0)}}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}}.$$

For example, the function $f(z)=\sin z/z$ satisfies our condition, and we obtain the well-known formula

$$\frac{\sin z}{z}=\prod_{n=-\infty}^{\infty}{}'\left(1-\frac{z}{n\pi}\right)e^{\frac{z}{n\pi}},$$

or

$$\sin z=z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right).$$

Similarly,

$$\cos z=\prod_{n=1}^{\infty}\left\{1-\frac{z^2}{(n-\frac{1}{2})^2\pi^2}\right\}.$$

If $f'(z)/f(z)$ satisfies the conditions of § 3.21, we obtain for $f(z)$ a product formula of the form

$$f(z)=f(0)e^{c_1z+\dots+c_{p+1}z^{p+1}}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\frac{z^2}{a_n^2}+\dots+\frac{1}{(p+1)}\frac{z^{p+1}}{a_n^{p+1}}}.$$

3.3. Summation of certain series. The method of contour integration is often effective in summing series of the form

$$\sum f(n)$$

where $f(z)$ is an analytic function of z of a fairly simple kind.

Let C be a closed contour including the points $m, m+1,\dots,n$, and suppose that $f(z)$ is analytic in this contour, except for poles at a finite number of points $a_1,\dots,a_k$, say simple poles with residues $b_1,\dots,b_k$. Consider the integral

$$\int_C \pi\cot\pi z f(z)\,dz.$$

The function $\pi \cot \pi z$ has in C simple poles at $z = m, m+1, \ldots, n$, with residue 1 at each pole. Hence $\pi \cot \pi z\, f(z)$ has the residues $f(m), f(m+1), \ldots, f(n)$. Including the residues due to poles of $f(z)$, we find that

$$\int_C \pi \cot \pi z f(z)\, dz = 2\pi i\{f(m)+f(m+1)+\ldots+f(n)+ \\ +b_1\pi \cot \pi a_1+\ldots+b_k\pi \cot \pi a_k\}.$$

Suppose, for example, that $f(z)$ is a rational function, none of whose poles are integers, and which is $O(|z|^{-2})$ at infinity. Take the contour C to be the square with corners $(n+\frac{1}{2})(\pm 1 \pm i)$. Then, as in § 3.22, the integral round C tends to zero as $n \to \infty$, and we have

$$\lim_{n\to\infty} \sum_{m=-n}^{n} f(m) = -\pi\{b_1 \cot \pi a_1+\ldots+b_k \cot \pi a_k\}.$$

Similarly, by using $\pi \operatorname{cosec} \pi z$ instead of $\pi \cot \pi z$, we can obtain expressions for sums of the form

$$\sum (-1)^m f(m).$$

Consider, for example, the series

$$\sum_{n=-\infty}^{\infty} \frac{1}{(a+n)^2}.$$

Here $f(z) = 1/(a+z)^2$ has a double pole at $z = -a$. By Taylor's theorem

$$\cot \pi z = \cot(-\pi a)+(\pi z+\pi a)\{-\operatorname{cosec}^2(-\pi a)\}+\ldots,$$

so that the residue of $\cot \pi z/(z+a)^2$ at $z = -a$ is $-\pi \operatorname{cosec}^2 \pi a$. Hence

$$\sum_{n=-\infty}^{\infty} \frac{1}{(a+n)^2} = \pi^2 \operatorname{cosec}^2 \pi a.$$

3.4. Poles and zeros of a meromorphic function. *If $f(z)$ is analytic inside and on a closed contour C, apart from a finite number of poles, and is not zero on the contour, then*

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)}\, dz = N-P$$

where N is the number of zeros inside the contour (a zero of order m being counted m times), and P is the number of poles (a pole of order m being counted m times).

Suppose that $z=a$ is a zero of order m. Then in the neighbourhood of this point

$$f(z)=(z-a)^m g(z),$$

where $g(z)$ is analytic and not zero. Hence

$$\frac{f'(z)}{f(z)}=\frac{m}{z-a}+\frac{g'(z)}{g(z)}.$$

The last term is analytic at $z=a$, so that $f'(z)/f(z)$ has a simple pole at $z=a$ with residue m. Hence the sum of the residues at the zeros of $f(z)$ is N.

Similarly the sum of the residues at the poles of $f(z)$ is $-P$ (we need merely change the sign of m).

It may be proved similarly that *if $\phi(z)$ is analytic in and on C and $f(z)$ has zeros at $a_1,\ldots,a_m$ and poles at $b_1,\ldots,b_n$, then*

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\phi(z)\,dz=\sum_{\mu=1}^{m}\phi(a_\mu)-\sum_{\nu=1}^{n}\phi(b_\nu).$$

3.41. *If $f(z)$ is analytic in C, then the above formula reduces to*

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz=N.$$

This result can also be expressed in another way. Since

$$\frac{d}{dz}\log\{f(z)\}=\frac{f'(z)}{f(z)},$$

we have

$$\int_C \frac{f'(z)}{f(z)}dz=\Delta_C\log\{f(z)\},$$

where Δ_C denotes the variation of $\log\{f(z)\}$ round the contour C. The value of the logarithm with which we start is clearly indifferent. Also

$$\log\{f(z)\}=\log|f(z)|+i\arg\{f(z)\},$$

and $\log|f|$ is one-valued. Hence the formula may be written

$$N=\frac{1}{2\pi}\Delta_C\arg\{f(z)\}.$$

3.42. Rouché's theorem. *If $f(z)$ and $g(z)$ are analytic inside and on a closed contour C, and $|g(z)|<|f(z)|$ on C, then $f(z)$ and $f(z)+g(z)$ have the same number of zeros inside C.*

In the first place it is clear that neither $f(z)$ nor $f(z)+g(z)$ has a zero on C. Hence, if N is the number of zeros of $f(z)$, and

N' the number of zeros of $f(z)+g(z)$,

$$2\pi N = \Delta_C \arg f,$$

$$2\pi N' = \Delta_C \arg(f+g) = \Delta_C \arg f + \Delta_C \arg\left(1+\frac{g}{f}\right).$$

To prove that $N = N'$, we have therefore to prove that

$$\Delta_C \arg\left(1+\frac{g}{f}\right) = 0.$$

Since $|g| < |f|$, the point $w = 1+g/f$ is always an interior point of the circle in the w-plane with centre 1 and radius 1; thus, if $w = \rho e^{i\phi}$, ϕ always lies between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$; and therefore $\arg(1+g/f) = \phi$ must return to its original value when z describes C—it cannot increase or decrease by a multiple of 2π. This proves the theorem.

Another proof is as follows. Let $\phi(z) = g(z)/f(z)$. Then

$$N' = \frac{1}{2\pi i}\int_C \frac{f'(z)+g'(z)}{f(z)+g(z)}\,dz = \frac{1}{2\pi i}\int_C \frac{f'+f'\phi+f\phi'}{f(1+\phi)}\,dz$$

$$= \frac{1}{2\pi i}\int_C \left(\frac{f'}{f}+\frac{\phi'}{1+\phi}\right)dz = N + \frac{1}{2\pi i}\int_C \frac{\phi'}{1+\phi}\,dz,$$

and the last integral is zero, as we see by expanding in powers of ϕ and integrating term by term.

3.43. The following is an example of the type of problem which can be solved by means of the above theorems.

In which quadrants do the roots of the equation

$$z^4+z^3+4z^2+2z+3 = 0$$

lie?

The equation has no real roots; for obviously it has no positive root: putting $z = -x$ it is

$$x^4-x^3+4x^2-2x+3 = 0.$$

For $0 < x < 1$ the first three terms together are positive, and so are the last two. For $x > 1$ the first two terms together are positive, and so are the last three.

Putting $z = iy$ the equation becomes

$$y^4-iy^3-4y^2+2iy+3 = 0,$$

and the real and imaginary parts of this do not vanish together. Hence there are no purely imaginary roots.

Now consider $\Delta \arg(z^4+\ldots+3)$ taken round the part of the

first quadrant bounded by $|z| = R$, where R is large. The variation along the real axis is zero. On the arc of the circle, $z = Re^{i\theta}$, and we have

$$\begin{aligned}\Delta \arg(z^4+\ldots) &= \Delta \arg(R^4 e^{4i\theta}) + \Delta \arg\{1+O(R^{-1})\} \\ &= 2\pi + O(R^{-1}).\end{aligned}$$

On the imaginary axis we have

$$\arg(z^4+\ldots) = \arctan\left(\frac{-y^3+2y}{y^4-4y^2+3}\right).$$

The numerator of the expression in brackets vanishes at $y = \sqrt{2}$, and the denominator at $y = \sqrt{3}$ and $y = 1$. Hence the rational fraction varies as follows as y varies from ∞ to 0:

$$\begin{array}{ccccccccc} y=\infty & & \sqrt{3} & & \sqrt{2} & & 1 & & 0 \\ & 0, & -, & \infty, & +, & 0, & -, & \infty, & +, \quad 0. \end{array}$$

Hence $\arctan(z^4+\ldots)$ decreases by 2π, and therefore the total variation of $\arg(z^4+\ldots)$ round the quadrant is zero, if R is large enough.

Hence there are no zeros in the first quadrant.

Since zeros occur in conjugate pairs, it follows that *there are no zeros in the fourth quadrant, and two in each of the second and third quadrants.*

Any algebraic equation may be treated in the same way.

3.44. The fundamental theorem of algebra. *Every polynomial of degree n has n zeros.*

In the first place, z^n has n zeros, all at the origin. Now consider any polynomial

$$a_0 + a_1 z + \ldots + a_n z^n,$$

where $a_n \neq 0$. Let

$$f(z) = a_n z^n, \qquad g(z) = a_0 + a_1 z + \ldots + a_{n-1} z^{n-1},$$

and take the contour C of Rouché's theorem to be a circle with centre the origin and radius $R > 1$. On C

$$|f(z)| = |a_n| R^n,$$

$$|g(z)| \leqslant |a_0| + |a_1| R + \ldots + |a_{n-1}| R^{n-1} \leqslant (|a_0| + \ldots + |a_{n-1}|) R^{n-1}.$$

Hence $|g| < |f|$ on C provided that

$$R > (|a_0| + \ldots + |a_{n-1}|)/|a_n|.$$

Hence, by Rouché's theorem, $f(z)+g(z)$ has n zeros in a circle with centre the origin whose radius R satisfies this condition.

The theorem can also be proved as follows. Suppose that the above polynomial has no zeros; then the function

$$\frac{1}{a_0+\ldots+a_n z^n}$$

is analytic for all values of z, since its only possible singularities are the zeros of the denominator; and it is bounded as $|z| \to \infty$. Hence, by Liouville's theorem, it is a constant. Hence the polynomial reduces to the single term a_0.

This proves only that the polynomial has one zero, and the fact that there are n has to be deduced by the familiar process of algebra.

3.45. A theorem of Hurwitz.* *Let $f_n(z)$ be a sequence of functions, each analytic in a region D bounded by a simple closed contour, and let $f_n(z) \to f(z)$ uniformly in D. Suppose that $f(z)$ is not identically zero. Let z_0 be an interior point of D. Then z_0 is a zero of $f(z)$ if, and only if, it is a limit-point of the set of zeros of the functions $f_n(z)$, points which are zeros for an infinity of values of n being counted as limit-points.*

This easily follows from Rouché's theorem. We can choose ρ so small that the circle $|z-z_0| = \rho$ lies entirely in D, and contains or has on it no zero of $f(z)$ except possibly the point z_0 itself. Then $|f(z)|$ has a positive lower bound on the circle, say $|f(z)| \geqslant m > 0$. Having fixed ρ and m, we can choose n_0 so large that

$$|f_n(z)-f(z)| < m \qquad (n > n_0)$$

on the circle. Since $f_n(z) = f(z)+\{f_n(z)-f(z)\}$, it follows from Rouché's theorem that, for $n > n_0$, $f_n(z)$ has the same number of zeros in the circle as $f(z)$; that is, if z_0 is a zero of $f(z)$, it has at least one, and otherwise it has none. This proves the theorem.

The example $f_n(z) = e^z/n$ shows that it is necessary to assume that $f(z)$ is not identically zero. The example in which $f_n(z) = 1-z^n/n$, and D is the unit circle, shows that the theorem does not apply to points on the boundary of D. For $f_n(z) \to 1$ uniformly in D and on the boundary, but every point of the boundary is a limit-point of zeros of the functions $f_n(z)$.

3.5. The functions $|f(z)|$, $\mathbf{R}\{f(z)\}$, $\mathbf{I}\{f(z)\}$. Let $f(z)$ be a function analytic in a given region, and let $u(x, y)$, $v(x, y)$ be its

* Hurwitz (1).

real and imaginary parts. We write

$$u_x = \frac{\partial}{\partial x} u(x, y), \quad u_y = \frac{\partial}{\partial y} u(x, y),$$

and similarly for derivatives of higher order.

We have already shown that the Cauchy-Riemann equations

$$u_x = v_y, \qquad u_y = -v_x,$$

hold at all points of the region.

Since $f''(z)$ exists, so do all the partial derivatives of u and v of the second order. Hence

$$u_{xx} = \frac{\partial}{\partial x}(v_y) = \frac{\partial}{\partial y}(v_x) = -u_{yy},$$

i.e. u satisfies the partial differential equation (Laplace's equation)

$$u_{xx} + u_{yy} = 0.$$

Similarly v satisfies the same equation.

A function which satisfies this differential equation is called a *harmonic function* or a *potential function*. The modulus $|f(z)|$ is not in general a harmonic function; but $\log|f(z)|$ is, since it is the real part of the function $\log\{f(z)\}$.

3.51. The loci $|f| = \text{const.}$, $\mathbf{R}\{f\} = \text{const.}$, $\mathbf{I}\{f\} = \text{const.}$, are curves in the z-plane.

If $|f(z)| = \text{constant}$ throughout a whole region where $f(z)$ is analytic, then $f(z) = \text{constant}$.

For if $|f(z)| = c$, then

$$u^2 + v^2 = c^2.$$

Hence

$$uu_x + vv_x = 0,$$
$$uu_y + vv_y = 0,$$

or, by the Cauchy-Riemann equations,

$$uu_x - vu_y = 0,$$
$$uu_y + vu_x = 0.$$

Eliminating u_y we obtain

$$(u^2 + v^2)u_x = 0.$$

Hence $u_x = 0$, and similarly u_y, v_x, and v_y are zero. Hence u and v are constants, i.e. $f(z)$ is constant.

If $u = c$ or $v = c$ the proof (which we leave to the reader) that $f(z)$ is a constant is even simpler.

3.52. *The zeros of $f(z)$ are the intersections of the curves $u=0$, $v=0$.* This is obvious.

At a simple zero, the curves $u=0$, $v=0$ intersect at right angles. This follows at once from the Cauchy-Riemann equations; or it may be seen by taking the zero to be at $z=0$, and writing

$$f(z)=ae^{i\alpha}z+O(|z|^2),$$

so that

$$u=ar\cos(\alpha+\theta)+O(r^2),$$
$$v=ar\sin(\alpha+\theta)+O(r^2).$$

Then the directions of the tangents to $u=0$, $v=0$, are given by $\theta=\frac{1}{2}\pi-\alpha$, $\theta=-\alpha$.

At a point where $f(z)$ is real, and $f'(z)=0$, the curve $v=0$ has a double point.

For at such a point $v=0$, $v_x=0$, $v_y=0$, which are the conditions for a double point.

The curves $|f(z)|=$ constant are called *level curves.*

Example. Prove that, at a double zero, each of the curves $u=0$, $v=0$, has a double point, and the two curves intersect at an angle $\frac{1}{4}\pi$.

3.53. *A level curve has a double point, if, and only if, it passes through a zero of $f'(z)$.*

The equation of a level curve is

$$u^2+v^2=c^2,$$

and this has a double point if, and only if,

$$uu_x+vv_x=0,$$
$$uu_y+vv_y=0.$$

Both these conditions are satisfied if $f'(z)=0$. Conversely, the second equation may be written

$$-uv_x+vu_x=0,$$

and squaring and adding we have

$$(u_x^2+v_x^2)(u^2+v^2)=0.$$

Hence $u_x=0$ and $v_x=0$, i.e. $f'(z)=0$.

3.54. The level curves and the zeros of $f(z)$. *If C is a simple closed level curve, and $f(z)$ is analytic inside and on it, then $f(z)$ has at least one zero inside C.*

Let $$f(z)=u+iv=ce^{i\phi}$$

on C, so that c is a constant. Then

$$c = \sqrt{(u^2+v^2)}, \qquad \phi = \arctan(v/u).$$

Let s be the length of C measured from some fixed point on it. Then

$$0 = \frac{dc}{ds} = \left(u\frac{du}{ds} + v\frac{dv}{ds}\right)\frac{1}{c}, \tag{1}$$

$$\frac{d\phi}{ds} = \left(u\frac{dv}{ds} - v\frac{du}{ds}\right)\frac{1}{c^2}. \tag{2}$$

Now $d\phi/ds$ cannot vanish on C. For if it did we should have, on squaring and adding the above equations,

$$(u^2+v^2)\left\{\left(\frac{du}{ds}\right)^2 + \left(\frac{dv}{ds}\right)^2\right\} = 0,$$

i.e.

$$\frac{du}{ds} = 0, \qquad \frac{dv}{ds} = 0.$$

Now

$$\frac{du}{ds} = u_x\frac{dx}{ds} + u_y\frac{dy}{ds},$$

$$\frac{dv}{ds} = v_x\frac{dx}{ds} + v_y\frac{dy}{ds} = -u_y\frac{dx}{ds} + u_x\frac{dy}{ds},$$

so that, squaring and adding,

$$(u_x^2+u_y^2)\left\{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right\} = 0.$$

The last factor is 1, so that $u_x = 0$, $u_y = 0$, i.e. $f'(z) = 0$. This is impossible on a level curve without double points.

It follows that $d\phi/ds$ has the same sign at all points of C, i.e. that ϕ increases or decreases steadily round the contour. Hence its variation round the contour is not zero.

But the variation of ϕ round the contour is equal to 2π multiplied by the number of zeros inside C. Hence there is at least one such zero.

3.55. *If $f(z)$ has n zeros inside C, then $f'(z)$ has $n-1$ zeros inside C.*

Let

$$f(z) = ce^{i\phi}$$

on C. Then

$$f'(z) = cie^{i\phi}\frac{d\phi}{dz}.$$

Hence

$$\arg\{f'(z)\} = \text{const.} + \phi + \arg\frac{d\phi}{dz}.$$

Hence, if Δ_C denotes variation round C,

$$\Delta_C \arg\{f'(z)\} = \Delta_C \arg\{f(z)\} + \Delta_C \arg \frac{d\phi}{dz}.$$

Let n' be the number of zeros of $f'(z)$. Then

$$2\pi n' = 2\pi n + \Delta_C \arg \frac{d\phi}{dz}. \tag{1}$$

Now
$$\frac{d\phi}{dz} = \frac{d\phi}{ds}\frac{ds}{dz},$$

and, as we have already seen, $d\phi/ds$ is real and of constant sign on C. Hence

$$\Delta_C \arg \frac{d\phi}{dz} = \Delta_C \arg \frac{ds}{dz},$$

Also
$$\frac{dz}{ds} = \frac{dx}{ds} + i\frac{dy}{ds} = \cos\psi + i\sin\psi = e^{i\psi},$$

where ψ is the angle the tangent to C makes with the x-axis. Hence

$$\Delta_C \arg \frac{ds}{dz} = -\Delta_C \psi = -2\pi,$$

so that on dividing (1) by 2π we obtain $n' = n-1$, the required result.

3.56. The following theorem sometimes gives useful information about the zeros of a function.*

Let C be a simple closed contour, inside and on which $f(z)$ is analytic. Then if $\mathbf{R}\{f(z)\}$ vanishes at $2k$ distinct points on C, $f(z)$ has at most k zeros inside C.

If $f(z) = u + iv$, the number n of zeros of $f(z)$ inside C is given by

$$n = \frac{1}{2\pi}\Delta_C\left(\arctan\frac{v}{u}\right).$$

Starting at a point where $u \neq 0$, we may take the initial value of $\arctan(v/u)$ to lie between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. We can only pass out of this range, say to $(\frac{1}{2}\pi, \frac{3}{2}\pi)$, if u vanishes, and only pass on to $(\frac{3}{2}\pi, \frac{5}{2}\pi)$, if u vanishes again. Thus, if u vanishes twice on C, $\Delta_C(\arctan v/u)$ is at most equal to 2π, and n is at most equal to 1. The general result obviously follows from the same argument.

* See, for example, Backlund (1).

3.6. Poisson's integral formula. *Let $f(z)$ be analytic in a region including the circle $|z| \leqslant R$, and let $u(r,\theta)$ be its real part. Then for $0 \leqslant r < R$*

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-r^2}{R^2-2Rr\cos(\theta-\phi)+r^2}\, u(R,\phi)\, d\phi.$$

There is a similar formula for the imaginary part $v(r,\theta)$ of $f(z)$.

These formulae are analogous to Cauchy's formula giving the value of $f(z)$ at any point inside a contour in terms of its values on the contour. They cannot, however, be obtained merely by separating Cauchy's formula into real and imaginary parts.

We shall give two proofs.

First Proof. We can suppose without loss of generality that $f(z) = \sum a_n z^n$, where all the coefficients a_n are real. For, in the general case, $a_n = \alpha_n + i\beta_n$, and

$$f(z) = \sum \alpha_n z^n + i \sum \beta_n z^n = f_1(z) + i f_2(z),$$

so that

$$\mathbf{R}(f) = \mathbf{R}(f_1) - \mathbf{I}(f_2).$$

Also, since $|\alpha_n| \leqslant |a_n|$, $|\beta_n| \leqslant |a_n|$, f_1 and f_2 are analytic for $|z| \leqslant R$. Hence the general result follows from the special case.

In the special case, if $f(re^{i\theta}) = u+iv$, then $f(re^{-i\theta}) = u-iv$.

Let z_1 be a point on the circle $|z| = R$, and let $f(z_1) = u_1 + iv_1$. Then, by Cauchy's formula,

$$u+iv = \frac{1}{2\pi i}\int \frac{u_1+iv_1}{z_1-z}\, dz_1 = \frac{1}{2\pi}\int_0^{2\pi} \frac{(u_1+iv_1)Re^{i\phi}\, d\phi}{Re^{i\phi}-re^{i\theta}}.$$

Since the point R^2/z is outside the circle, we have

$$0 = \frac{1}{2\pi i}\int \frac{u_1+iv_1}{z_1-R^2/z}\, dz_1 = \frac{1}{2\pi}\int_0^{2\pi} \frac{(u_1+iv_1)Re^{i\phi}\, d\phi}{Re^{i\phi}-R^2r^{-1}e^{-i\theta}}.$$

Replacing ϕ by $-\phi$, and so iv_1 by $-iv_1$, we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{(u_1-iv_1)Re^{-i\phi}\, d\phi}{Re^{-i\phi}-R^2r^{-1}e^{-i\theta}} = 0.$$

or

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{(u_1-iv_1)re^{i\theta}\, d\phi}{re^{i\theta}-Re^{i\phi}} = 0.$$

Subtracting this formula from the previous one, we obtain

$$u+iv=\frac{1}{2\pi}\int_0^{2\pi}\left\{u_1\frac{Re^{i\phi}+re^{i\theta}}{Re^{i\phi}-re^{i\theta}}+iv_1\right\}d\phi,$$

and now, taking real parts, the result follows.

SECOND PROOF. Let

$$f(z)=\sum_{n=0}^{\infty}(\alpha_n+i\beta_n)r^n e^{in\theta} \qquad (r\leqslant R).$$

Then, as in § 2.53,

$$\alpha_n R^n=\frac{1}{\pi}\int_0^{2\pi}u(R,\phi)\cos n\phi\,d\phi, \qquad \beta_n R^n=-\frac{1}{\pi}\int_0^{2\pi}u(R,\phi)\sin n\phi\,d\phi,$$

for $n>0$, while $\qquad \alpha_0=\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}u(R,\phi)\,d\phi.$

Hence

$$\begin{aligned}u(r,\theta)&=\sum_{n=0}^{\infty}(\alpha_n\cos n\theta-\beta_n\sin n\theta)r^n\\&=\frac{1}{2\pi}\int_0^{2\pi}u(R,\phi)\,d\phi+\\&\qquad+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{r^n}{R^n}\int_0^{2\pi}u(R,\phi)(\cos n\theta\cos n\phi+\sin n\theta\sin n\phi)\,d\phi\\&=\frac{1}{\pi}\int_0^{2\pi}u(R,\phi)\left\{\frac{1}{2}+\sum_{n=1}^{\infty}\cos n(\theta-\phi)\left(\frac{r}{R}\right)^n\right\}d\phi,\end{aligned}$$

the inversion being justified by uniform convergence. The result now follows on summing the series in brackets.

3.61. Jensen's theorem. *Let $f(z)$ be analytic for $|z|<R$. Suppose that $f(0)$ is not zero, and let $r_1, r_2,\ldots, r_n,\ldots$ be the moduli of the zeros of $f(z)$ in the circle $|z|<R$, arranged as a non-decreasing sequence. Then, if $r_n\leqslant r\leqslant r_{n+1}$,*

$$\log\frac{r^n|f(0)|}{r_1r_2\ldots r_n}=\frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta. \tag{1}$$

Here a zero of order p is counted p times. The interest of this formula is that it connects the modulus of the function with the moduli of the zeros.

It can be put in another form, in some ways more useful. Let $n(x)$ denote the number of zeros of $f(z)$ for $|z| \leqslant x$. Then, if $r_n \leqslant r \leqslant r_{n+1}$,

$$\log \frac{r^n}{r_1 \ldots r_n} = n \log r - \sum_{m=1}^{n} \log r_m$$
$$= \sum_{m=1}^{n-1} m(\log r_{m+1} - \log r_m) + n(\log r - \log r_n)$$
$$= \sum_{m=1}^{n-1} m \int_{r_m}^{r_{m+1}} \frac{dx}{x} + n \int_{r_n}^{r} \frac{dx}{x}.$$

Now $m = n(x)$ for $r_m \leqslant x < r_{m+1}$, $n = n(x)$ for $r_n \leqslant x < r$. Hence the right-hand side is equal to

$$\int_0^r \frac{n(x)}{x}\, dx,$$

and Jensen's formula takes the form

$$\int_0^r \frac{n(x)}{x}\, dx = \frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta - \log|f(0)|. \tag{2}$$

We shall give two proofs of the theorem.

First Proof. If $f(z)$ has no zero on $|z| = r$, then

$$n(r) = \frac{1}{2\pi i} \int \frac{f'(z)}{f(z)}\, dz = \frac{1}{2\pi} \int_0^{2\pi} \frac{f'(re^{i\theta})}{f(re^{i\theta})} re^{i\theta}\, d\theta. \tag{3}$$

Jensen's formula is obtained formally by dividing by r, integrating with respect to r, and taking real parts. This process is not obviously valid, owing to the infinities of the integrand. We therefore adopt a slightly different method.

In an interval between the moduli r_n, r_{n+1}, of two zeros, each side of Jensen's formula has a continuous derivative; the derivative of the left-hand side is n/r, and that of the right-hand side is

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{d}{dr} \{\log|f(re^{i\theta})|\}\, d\theta = \frac{1}{4\pi} \int_0^{2\pi} \frac{d}{dr} \{\log f(re^{i\theta}) + \log \bar{f}(re^{-i\theta})\}\, d\theta$$
$$= \frac{1}{4\pi} \int_0^{2\pi} \left\{ \frac{f'(re^{i\theta})}{f(re^{i\theta})} e^{i\theta} + \frac{\bar{f}'(re^{-i\theta})}{\bar{f}(re^{i\theta})} e^{-i\theta} \right\} d\theta = \mathbf{R} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{f'(re^{i\theta})}{f(re^{i\theta})} e^{i\theta}\, d\theta \right\},$$

which is also equal to n/r, by (3). Hence the two derivatives are equal in any such interval. Hence the two sides of Jensen's formula differ by a constant in any such interval.

Secondly, the two sides are obviously equal when $r = 0$.

Hence it is sufficient to prove that each side is continuous when r passes through a value r_n.

This is obvious in the case of the left-hand side. For the right-hand side, it will be sufficient to suppose that there is one zero of modulus r_n, and that its amplitude is zero. Then

$$\log|f(re^{i\theta})| = \log\left|1-\frac{r}{r_n}e^{i\theta}\right|+\psi(r,\theta),$$

where ψ is continuous in the neighbourhood of $r = r_n$. Hence it is sufficient to show that the integral

$$\int_0^{2\pi} \log\left|1-\frac{r}{r_n}e^{i\theta}\right| d\theta$$

is continuous at $r = r_n$. Now for $r/r_n < 2$

$$9 \geqslant \left|1-\frac{r}{r_n}e^{i\theta}\right|^2 = 1-2\frac{r}{r_n}\cos\theta+\frac{r^2}{r_n^2}$$
$$= \sin^2\theta+\left(\cos\theta-\frac{r}{r_n}\right)^2 \geqslant \sin^2\theta.$$

Hence, if $\delta < \pi$,

$$\left|\int_{-\delta}^{\delta} \log\left|1-\frac{r}{r_n}e^{i\theta}\right| d\theta\right| < \int_{-\delta}^{\delta} \left\{\log 3+\left|\log|\sin\theta|\right|\right\} d\theta$$
$$< \int_{-\delta}^{\delta} \left\{A+\left|\log|\theta|\right|\right\} d\theta < A\delta\log\frac{1}{\delta}.$$

We can choose δ so that this is arbitrarily small, for all values of r in the neighbourhood of r_n. Having fixed δ, the remainder of the integral is evidently continuous. Hence the whole integral is continuous.

Second Proof. We obtain the result in a number of stages.

(i) If $f(z)$ has no zeros for $|z| \leqslant r$, then $\log f(z)$ is analytic for $|z| \leqslant r$, and

$$\log f(0) = \frac{1}{2\pi i}\int_{|z|=r} \frac{\log f(z)}{z}\, dz = \frac{1}{2\pi}\int_0^{2\pi} \log\{f(re^{i\theta})\}\, d\theta,$$

and, taking real parts, we have the result.

(ii) If $a_1 = r_1 e^{i\theta_1}$, $0 < r_1 < r$, we have

$$\int_{|w|=1/r} \log(1-w\bar{a}_1)\frac{dw}{w} = 0$$

by Cauchy's theorem, the logarithm having its principal value. Hence, with suitable determinations of the logarithms,

$$\frac{1}{2\pi i}\int_{|w|=1/r} \log\left(1-\frac{1}{w\bar{a}_1}\right)\frac{dw}{w} = \frac{1}{2\pi i}\int_{|w|=1/r} \log\left(-\frac{1}{w\bar{a}_1}\right)\frac{dw}{w}$$

$$= \log\left(-\frac{1}{\bar{a}_1}\right) - \frac{1}{4\pi i}[\log^2 w]_{\arg w=0}^{\arg w=2\pi}$$

$$= \log\left(-\frac{1}{\bar{a}_1}\right) - \frac{1}{4\pi i}\left(\log\frac{1}{r}+2\pi i\right)^2 + \frac{1}{4\pi i}\log^2\frac{1}{r}.$$

Taking real parts,

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|1-\frac{r}{r_1}e^{i(\theta_1-\theta)}\right| d\theta = \log\frac{r}{r_1}.$$

This is Jensen's formula for

$$f(z) = 1-\frac{z}{a_1}.$$

(iii) The above result may be extended to the case $r = r_1$ by applying Cauchy's theorem to the circle $|w| = 1/r$ with a small circular indentation so that the point $w = 1/\bar{a}_1$ is excluded. The integral round the indentation tends to 0 with the radius, and the proof concludes as before.

(iv) In the general case

$$f(z) = \left(1-\frac{z}{a_1}\right)\left(1-\frac{z}{a_2}\right)\cdots\left(1-\frac{z}{a_n}\right)\phi(z),$$

where $\phi(z)$ is not zero for $|z| < r_{n+1}$, and $\phi(0) = f(0)$. The general result then follows by addition of the previous ones.

The theorem may be extended at once to a function which has poles as well as zeros. Let $f(z)$ satisfy the same conditions

as before, but now let it have zeros $a_1,\ldots,a_m$ and poles $b_1,\ldots,b_n$ with moduli not exceeding r. Then

$$\log\left\{\left|\frac{b_1\ldots b_n}{a_1\ldots a_m}f(0)\right|r^{m-n}\right\} = \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta. \tag{4}$$

For if $\quad f(z) = g(z)\Big/\left(1-\frac{z}{b_1}\right)\ldots\left(1-\frac{z}{b_n}\right) = g(z)/h(z),$

we have $$\log\frac{r^m g(0)}{|a_1\ldots a_m|} = \frac{1}{2\pi}\int_0^{2\pi}\log|g(re^{i\theta})|\,d\theta$$

and $$\log\frac{r^n}{|b_1\ldots b_n|} = \frac{1}{2\pi}\int_0^{2\pi}\log|h(re^{i\theta})|\,d\theta.$$

The result therefore follows on subtracting.

3.62. The Poisson-Jensen formula. *Let $f(z)$ have zeros at the points $a_1, a_2,\ldots, a_m$, and poles at $b_1, b_2,\ldots, b_n$, inside the circle $|z| \leqslant R$, and be analytic elsewhere inside and on the circle. Then*

$$\log|f(re^{i\theta})| = \frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-r^2}{R^2-2Rr\cos(\theta-\phi)+r^2}\log|f(Re^{i\phi})|\,d\phi -$$
$$-\sum_{\mu=1}^{m}\log\left|\frac{R^2-\bar{a}_\mu re^{i\theta}}{R(re^{i\theta}-a_\mu)}\right| + \sum_{\nu=1}^{n}\log\left|\frac{R^2-\bar{b}_\nu re^{i\theta}}{R(re^{i\theta}-b_\nu)}\right|.$$

This contains both Poisson's formula and Jensen's formula as particular cases. If there are no zeros or poles, it reduces to Poisson's formula for the real part of the function $\log f(z)$. On the other hand, if $r = 0$, we obtain the general Jensen formula

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\phi})|\,d\phi - \log\left\{\left|\frac{b_1b_2\ldots b_n}{a_1a_2\ldots a_m}\right|R^{m-n}\right\}.$$

(i) Let $f(z) = z-a$, where $|a| < R$. Then we have to prove that

$$\log|re^{i\theta}-a| = \frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-r^2}{R^2-2Rr\cos(\theta-\phi)+r^2}\log|Re^{i\phi}-a|\,d\phi -$$
$$-\log\left|\frac{R^2-\bar{a}re^{i\theta}}{R(re^{i\theta}-a)}\right|,$$

or

$$\log\left|R-\frac{\bar{a}re^{i\theta}}{R}\right|=\frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-r^2}{R^2-2Rr\cos(\theta-\phi)+r^2}\log|Re^{i\phi}-a|\,d\phi.$$

But this is equivalent to Poisson's formula for the real part of the function

$$\log\left(R-\frac{\bar{a}z}{R}\right),$$

which is analytic for $|z|\leqslant R$.

(ii) Similarly, if $f(z)=1/(z-b)$, the formula is equivalent to Poisson's formula for the real part of

$$\log\left(R-\frac{\bar{b}z}{R}\right).$$

(iii) If $f(z)$ is analytic and has no zeros or poles in $|z|\leqslant R$, the formula is Poisson's formula for the real part of $\log f(z)$.

The general case can now be obtained by addition of these particular cases.

3.7. In all the above theorems the region considered is a circle. We shall conclude the chapter by proving two theorems of the same general type as Jensen's theorem, but applying to a half-plane and a rectangle respectively.

Carleman's theorem.* *Let $f(z)$ be analytic for $|z|\geqslant\rho$, $-\frac{1}{2}\pi\leqslant\arg z\leqslant\frac{1}{2}\pi$, and suppose that it has the zeros $r_1e^{i\theta_1}, r_2e^{i\theta_2},\ldots, r_ne^{i\theta_n}$ inside the contour consisting of the semicircles $|z|=\rho$, $|z|=R$, $-\frac{1}{2}\pi\leqslant\arg z\leqslant\frac{1}{2}\pi$, and the parts of the imaginary axis joining them, and that it has no zeros on the contour. Then*

$$\sum_{\nu=1}^{n}\left(\frac{1}{r_\nu}-\frac{r_\nu}{R^2}\right)\cos\theta_\nu=\frac{1}{\pi R}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\log|f(Re^{i\theta})|\cos\theta\,d\theta+$$

$$+\frac{1}{2\pi}\int_\rho^R\left(\frac{1}{y^2}-\frac{1}{R^2}\right)\log|f(iy)f(-iy)|\,dy+O(1),$$

where $O(1)$ denotes a function of ρ and R which, for fixed ρ, is bounded as $R\to\infty$.

Consider the integral

$$I=\frac{1}{2\pi i}\int\log f(z)\left(\frac{1}{z^2}+\frac{1}{R^2}\right)dz$$

* Carleman (1).

taken round the contour in the positive direction, starting from the point $z = i\rho$ with a fixed determination of the logarithm. The integral along the small semicircle is bounded. On the negative imaginary axis $z = -iy$, and we obtain

$$\frac{1}{2\pi}\int_{\rho}^{R} \log\{f(-iy)\}\left(\frac{1}{y^2}-\frac{1}{R^2}\right) dy.$$

On the large semicircle, $z = Re^{i\theta}$, and we obtain

$$\frac{1}{2\pi i}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \log\{f(Re^{i\theta})\}\left(\frac{e^{-2i\theta}}{R^2}+\frac{1}{R^2}\right) iRe^{i\theta}\, d\theta$$

$$= \frac{1}{\pi R}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \log\{f(Re^{i\theta})\}\cos\theta\, d\theta.$$

The integral along the positive imaginary axis gives

$$\frac{1}{2\pi}\int_{\rho}^{R} \log\{f(iy)\}\left(\frac{1}{y^2}-\frac{1}{R^2}\right) dy,$$

and, taking the real part of I, we obtain the right-hand side of Carleman's formula.

Again, integrating by parts, we have

$$I = \frac{1}{2\pi i}\left[\log f(z)\left(\frac{z}{R^2}-\frac{1}{z}\right)\right] + \frac{1}{2\pi i}\int \frac{f'(z)}{f(z)}\left(\frac{1}{z}-\frac{z}{R^2}\right) dz.$$

As we describe the contour, $\log f(z)$ increases by $2\pi i n$. The integrated term is therefore purely imaginary. By the theorem of residues, the last integral is equal to

$$\sum_{\nu=1}^{n}\left(\frac{1}{r_\nu e^{i\theta_\nu}}-\frac{r_\nu e^{i\theta_\nu}}{R^2}\right),$$

and, taking real parts, the theorem follows.

The result is easily extended to the case where $f(z)$ has zeros on the imaginary axis; we make small indentations round these zeros, and proceed to the limit.

3.71. *Let $f(z)$ be analytic and bounded for $x \geqslant 0$, and let its zeros in the right half-plane be $r_1 e^{i\theta_1}$, $r_2 e^{i\theta_2}, \ldots$. Then the series*

$$\sum_{n=1}^{\infty} \frac{\cos\theta_n}{r_n}$$

is convergent.

Under the conditions stated, the right-hand side of Carleman's formula is bounded above, say $< M$. Hence

$$\sum_{\nu=1}^{n}\left(\frac{1}{r_\nu}-\frac{r_\nu}{R^2}\right)\cos\theta_\nu < M$$

for all values of R. Every term on the left is positive, and, if $r_\nu < \frac{1}{2}R$,

$$\frac{1}{r_\nu}-\frac{r_\nu}{R^2} > \frac{3}{4r_\nu}.$$

Hence

$$\sum_{r_\nu<\frac{1}{2}R}\frac{\cos\theta_\nu}{r_\nu} < \frac{4M}{3},$$

and the result follows.

It is easily seen that the theorem remains true if, instead of $f(z) = O(1)$, we suppose that $f(z) = O(e^{|z|^\alpha})$, where $\alpha < 1$. But if $\alpha = 1$ the theorem fails, as the example $f(z) = \cos z$ shows.

3.72. The above theorem may be used to prove the following result.

Let $f(z)$ be analytic for $x \geqslant 0$, and of the form $O(e^{-a|z|})$ as $z \to \infty$, where $a > 0$, uniformly for $|\arg z| \leqslant \frac{1}{2}\pi$. Then $f(z) = 0$ identically.

For consider the function $F(z) = f(z)\sin bz$, where $0 < b < a$. Here $F(z)$ is analytic and bounded for $x \geqslant 0$; it has zeros at the points $z = n\pi/b$, and $\sum b/n\pi$ is divergent. This is inconsistent with the result of the previous section, unless $F(z)$ is identically zero. Hence $F(z) = 0$, and so $f(z) = 0$.

A more complete form of this result will be obtained in § 5.8.

3.8. A theorem of Littlewood.* Let C denote the rectangle bounded by the lines $x = x_1$, $x = x_2$, $y = y_1$, $y = y_2$, where $x_1 < x_2$, $y_1 < y_2$. Let $f(z)$ be analytic and not zero on C, and meromorphic inside it. Let $F(z) = \log f(z)$, the logarithm being defined as follows: we start with a particular determination on $x = x_2$, and obtain the value at other points by continuous variation along $y =$ constant from $\log(x_2 + iy)$. If, however, this path would cross a zero or pole of $f(z)$, we take $F(z)$ to be $F(z \pm i0)$ according as we approach the path from above or below.

* Littlewood (4).

Let $\nu(x')$ denote the excess of the number of zeros of $f(z)$ over the number of poles in the part of the rectangle where $x > x'$. Then

$$\int_C F(z)\,dz = -2\pi i \int_{x_1}^{x_2} \nu(x)\,dx.$$

Consider first the function $f(z) = z - a$, where $a = \alpha + i\beta$ is a point of the rectangle. Let C' be the contour obtained by describing C in the positive direction from (x_2, y_1) as far as (x_1, β), then the straight line $y = \beta$ as far as $\alpha - \epsilon + i\beta$, then a circle of radius ϵ about $z = a$ described in the negative direction, and then returning along $y = \beta$ and the rest of C to the starting-point. Then $F(z)$ is analytic in C', so that

$$\int_{C'} F(z)\,dz = 0.$$

The integral round the small circle tends to zero with the radius, and it follows that

$$\int_C F(z)\,dz = -\int_{x_1}^{\alpha} \{F_1(z) - F_2(z)\}\,dx,$$

where F_1 and F_2 are the values of F on the two paths joining $x_1 + i\beta$ to $\alpha + i\beta$. Since we obtain F_2 from F_1 by passing in the negative direction round a simple zero of $f(z)$ at $z = a$, we have

$$F_2(z) = F_1(z) - 2\pi i.$$

Hence

$$\int_C F(z)\,dz = -2\pi i \int_{x_1}^{\alpha} dx = -2\pi i \int_{x_1}^{x_2} \nu(x)\,dx,$$

where $\nu(x) = 1$ $(x_1 < x < \alpha)$, 0 $(\alpha < x < x_2)$, i.e. $\nu(x)$ is the ν-function for the case considered.

The general theorem now easily follows by addition of terms corresponding to the various poles and zeros of $f(z)$.

MISCELLANEOUS EXAMPLES

1. Evaluate the integrals

$$\int_0^\infty \frac{dx}{x^4+1}, \qquad \int_0^\infty \frac{x^2\,dx}{x^4+1}, \qquad \int_0^\infty \frac{dx}{x^6+1}$$

by contour integration.

2. Evaluate the integrals

$$\int_0^\infty \frac{\cos x}{a^2+x^2}\,dx, \qquad \int_0^\infty \frac{\cos x}{(a^2+x^2)^2}\,dx, \qquad \int_0^\infty \frac{\sin^2 x}{x^2}\,dx, \qquad \int_0^\infty \frac{\sin^3 x}{x^3}\,dx$$

by contour integration.

3. Prove that, if $c > 0$,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{a^z}{z^2}\,dz = \begin{matrix}\log a & (a > 1),\\ 0 & (0 < a < 1).\end{matrix}$$

4. Prove that the integral

$$\int \frac{dz}{\sqrt{(4z^2+4z+3)}}$$

taken round the unit circle, starting with the positive value of the square root at $z = 1$, is equal to $i\pi$.

5. By integrating $\log^2 z/(1+z^2)$ round the usual semi-circular contour, prove that

$$\int_0^\infty \frac{\log^2 z}{1+z^2}\,dz = \frac{\pi^3}{8}.$$

6. By evaluating the integral

$$\frac{1}{2\pi i}\int \frac{dz}{(z-a)(z-1/a)}$$

round the unit circle, prove that, if $0 < a < 1$,

$$\int_0^{2\pi} \frac{d\theta}{1+a^2-2a\cos\theta} = \frac{2\pi}{1-a^2}.$$

What is the value of the integral if $a > 1$?

7. Prove that, if $b > a > -1$,

$$\int_0^{\frac{1}{2}\pi} \cos^a\theta \cos b\theta\, d\theta = \frac{\pi\Gamma(a+1)}{2^{a+1}\Gamma(\frac{1}{2}a+\frac{1}{2}b+1)\Gamma(\frac{1}{2}a-\frac{1}{2}b+1)}.$$

[Take the integral $\int (z+1/z)^a z^{b-1}\,dz$ round the right-hand half of the unit circle.]

8. By integrating

$$\int \frac{z\,dz}{a-e^{-iz}}$$

round the rectangle with corners at $-\pi$, π, $\pi+in$, $-\pi+in$, and making $n \to \infty$, show that

$$\int_0^\pi \frac{x\sin x\,dx}{1+a^2-2a\cos x} = \frac{\pi}{a}\log(1+a) \quad (0 < a < 1), \qquad \frac{\pi}{a}\log\frac{1+a}{a} \quad (a > 1).$$

[Lindelöf, *Calcul des Résidus*, pp. 48–9.]

9. Show that the function $f(x) = \operatorname{sech}\{x\sqrt{(\frac{1}{2}\pi)}\}$ satisfies the equation

$$f(t) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty f(x)\cos xt\, dx.$$

[Take the integral $\int \cos tz \operatorname{sech} az\, dz$ round the rectangle with corners at $\pm n$, $\pm n + i\pi/a$, and make $n \to \infty$.]

10. Show that the function

$$f(x) = \frac{1}{e^{x\sqrt{(2\pi)}}-1} - \frac{1}{x\sqrt{(2\pi)}}$$

satisfies the equation

$$f(t) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty f(x)\sin xt\, dx.$$

[Take the integral $$\int \frac{\sin zt}{e^{az}-1}\, dz,$$

where $a > 0$, round the rectangle with corners at 0, n, $n + 2i\pi/a$, $2i\pi/a$, and make $n \to \infty$.]

11. Prove that, if $0 < a < 1$ and $0 < c < 1$,

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{dz}{a^z \sin \pi z} = \frac{1}{\pi(1+a)}.$$

12. Prove that if $a > 0$, $-\frac{1}{2}\pi < a\lambda < \frac{1}{2}\pi$,

$$\int_0^\infty e^{-r^a \cos a\lambda} \begin{matrix}\cos\\ \sin\end{matrix}(r^a \sin a\lambda)\, dr = \begin{matrix}\cos\\ \sin\end{matrix}(\lambda)\, \frac{1}{a}\, \Gamma\left(\frac{1}{a}\right).$$

13. Sum the series

$$\sum_{n=1}^\infty \frac{1}{n^4+a^4}, \qquad \sum_{n=1}^\infty \frac{n^2}{n^4+a^4}.$$

14. Prove that if $-\pi < a < \pi$, and x is not an integer,

$$\sum_{n=1}^\infty (-1)^n \frac{n \sin na}{x^2-n^2} = \tfrac{1}{2}\pi \frac{\sin ax}{\sin \pi x}.$$

15. Prove that*

$$\frac{\coth \pi}{1^7} + \frac{\coth 2\pi}{2^7} + \frac{\coth 3\pi}{3^7} + \ldots = \frac{19\pi^7}{56700}.$$

16. Prove that

$$\sum_{n=1}^\infty \frac{1}{n^3 \sin n\pi\sqrt{2}} = -\frac{13\pi^3}{360\sqrt{2}}.$$

[Hardy. Consider the integral

* Ramanujan; see Watson (1).

$$\frac{1}{2\pi i}\int \frac{1}{\sin \pi z \sin \theta \pi z}\frac{dz}{z^3}$$

where $\theta = \sqrt{2}-1$. The series is convergent; for if m is the nearest integer to $n\sqrt{2}$,

$$|n\sqrt{2}-m| = \frac{|2n^2-m^2|}{n\sqrt{2}+m} \geqslant \frac{1}{n\sqrt{2}+m} > \frac{A}{n},$$

and hence $\operatorname{cosec} n\pi\sqrt{2} = O(n)$.]

17. Show that if $f(z) = \sum_{n=0}^{\infty} a_n/z^{n+1}$ $(|z| > 0)$, C is a closed contour including the origin, and $\phi(z)$ is regular in a sufficiently wide region, then

$$\frac{1}{2\pi i}\int_C f(w)\phi(z-w)\,dw = a_0\phi(z)-a_1\phi'(z)+\frac{a_2}{2!}\phi''(z)-\dots.$$

18. Prove that

$$e^{az}-e^{bz} = (a-b)ze^{\frac{1}{2}(a+b)z}\prod_{n=1}^{\infty}\left\{1+\frac{(a-b)^2z^2}{4n^2\pi^2}\right\}.$$

19. Show that, however small ρ is, all the zeros of the function

$$1+\frac{1}{z}+\frac{1}{2!z^2}+\dots+\frac{1}{n!z^n}$$

lie in the circle $|z| \leqslant \rho$, if n is sufficiently large.

20. If $a > e$, the equation $e^z = az^n$ has n roots inside the unit circle. [Take $f(z) = az^n$, $g(z) = e^z$, in Rouché's theorem.]

21. Show that, if α and β are real, the equation

$$z^{2n}+\alpha^2z^{2n-1}+\beta^2 = 0$$

has $n-1$ roots with positive real parts if n is odd, and n roots with positive real parts if n is even.

22. Prove that, if α is not an even integer,

$$\int_0^{\infty} e^{-x^\alpha}\cos xt\,dx \sim \frac{\Gamma(\alpha+1)\sin\frac{1}{2}\pi\alpha}{t^{\alpha+1}}$$

as $t \to \infty$ through real values.*

23. If $f(z) = u+iv$ is an analytic function of $z = x+iy$, and ψ is any function of x and y with differential coefficients of the first two orders, then

$$\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2 = \left\{\left(\frac{\partial\psi}{\partial u}\right)^2+\left(\frac{\partial\psi}{\partial v}\right)^2\right\}|f'(z)|^2,$$

and

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2} = \left(\frac{\partial^2\psi}{\partial u^2}+\frac{\partial^2\psi}{\partial v^2}\right)|f'(z)|^2.$$

[See Hardy (6), p. 270.]

* Pólya (1).

24. If $f(z) = u+iv$ is an analytic function of $z = x+iy$, show that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2|f(z)|^{p-2}|f'(z)|^2,$$

and that
$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|u|^p = p(p-1)|u|^{p-2}|f'(z)|^2.$$

25. Let $\phi(t)$ be a real integrable function in the interval (a, b), and let

$$f(z) = \int_a^b e^{zt}\phi(t)\,dt$$

have zeros at the points $r_1e^{i\theta_1}, r_2e^{i\theta_2},\ldots$. Then the series

$$\sum \frac{\cos\theta_n}{r_n}$$

is absolutely convergent.

[The function $e^{-bz}f(z)$ is bounded for $x \geqslant 0$, and $e^{az}f(z)$ is bounded for $x \leqslant 0$.]

CHAPTER IV

ANALYTIC CONTINUATION

4.1. General theory. It is natural to think of the aggregate of all values of z^2, say, or $\log z$, for all values of z, as a single entity, and each such aggregate we describe as *an analytic function*. We have, however, not yet encountered the general idea of an analytic function as a whole. What we have always been concerned with is the idea of a function associated with a region, and defined in that region by a formula. Thus

$$1+z+z^2+\ldots \qquad (|z|<1) \tag{1}$$

and

$$\int_0^\infty e^{-t(1-z)}\,dt \qquad (\mathrm{R}(z)<1) \tag{2}$$

appear as different functions, whose values happen to be the same for certain values of z. But it is obviously more natural to regard (1) as a part of (2), and (2) as part of the function defined for all values of z other than 1 as $1/(1-z)$.

This particular function is one-valued, i.e. has just one value for each value of z (except $z=1$). But it is also natural to regard the two values of $\sqrt{z}$ as parts of the same function, and our definition must include cases of this kind also.

To connect these new ideas with our previous theory, we require a process by which we can extend the definition of a function beyond a limited region in which it is originally defined. This process is called analytic continuation. It is characteristic of analytic functions of a complex variable, and has no counterpart in the theory of functions of a real variable.

4.11. Analytic continuation. Suppose that $f_1(z)$ and $f_2(z)$ are functions analytic in regions D_1 and D_2 respectively, and that D_1 and D_2 have a common part, throughout which $f_1(z)=f_2(z)$. Then we consider the aggregate of values of $f_1(z)$ and $f_2(z)$ at points interior to D_1 or D_2 as a single analytic function $f(z)$. Thus $f(z)$ is analytic in $D=D_1+D_2$, and $f(z)=f_1(z)$ in D_1, $f(z)=f_2(z)$ in D_2.

The function $f_2(z)$ may be considered as extending the domain in which $f_1(z)$ is defined, and it is called an analytic continuation of $f_1(z)$. Of course in the same way $f_1(z)$ is an analytic continua-

tion of $f_2(z)$. This process of extending the definition of a given function is known as analytic continuation.

For the process to have any value it is necessary that it should, under suitable conditions, give a unique result, and we shall show that this is so. Before giving the proof it may be interesting to note the difficulties which we encounter if we try to define a similar process for functions of a real variable.

It would be natural to suggest that if, say, $f(x) = \frac{1}{2}(\pi - x)$ for $0 < x < \pi$, then we should extend the definition of $f(x)$ to other values of x by using the same formula. The difficulty is that two formulae may represent the same function in one interval, but different functions in another interval, and there may be no obvious way of deciding which is the 'proper' formula. For example, the above function is also represented by the series

$$\frac{\sin x}{1} + \frac{\sin 2x}{2} + \dots$$

for $0 < x < \pi$; but if we define the function as the sum of this series, we find that its value in the interval $(-\pi, 0)$ is $-\frac{1}{2}(\pi + x)$.

This series is not uniformly convergent, but even if we restrict ourselves to uniformly convergent series, the same sort of thing happens. For example, the series

$$\frac{x \sin x}{1} + \frac{x \sin 2x}{2} + \dots$$

is uniformly convergent in an interval including $x = 0$; yet if we use it to continue its sum from positive to negative values of x, we obtain the undesirable conclusion that the continuation of $\frac{1}{2}x(\pi - x)$ is $-\frac{1}{2}x(\pi + x)$.

4.12. Uniqueness of analytic continuation. Suppose that we have a region D, overlapped by regions D_1 and D_2, which have a common part D_3, itself overlapping D. Let $f(z)$ be analytic in D, and let $f_1(z)$ be a continuation of $f(z)$ to D_1, and $f_2(z)$ a continuation of $f(z)$ to D_2. Then either of these functions provides a continuation of $f(z)$ to D_3. To show that the results of the two processes of continuation are the same, we have to show that $f_1(z) = f_2(z)$ throughout D_3. This follows from the theorem of § 2.6, which itself depends on the fact that an analytic function can be expressed as a power series. The function $f_1(z) - f_2(z)$ is analytic throughout D_3; it is zero in the part

of D_3 which overlaps D, since there $f_1(z) = f_2(z) = f(z)$. Hence it is zero throughout D_3.

The proof depends on the existence of a region common to D and D_3, and if there is no such region, the result no longer necessarily holds. We may now have $f_1(z) = f(z)$ in DD_1, $f_2(z) = f(z)$ in DD_2, but $f_1(z) \neq f_2(z)$ in D_3. This does not contradict the principle of uniqueness, since it only applies to regions throughout which the function is analytic; and now D, D_1, and D_2 may surround, without including, a point where the function is not analytic.*

4.13. In the second case considered above, where $f_1(z) \neq f_2(z)$ in D_3, we still consider the aggregate of values of $f_1(z)$ and $f_2(z)$ as a single analytic function of z, but now the function is not one-valued, and in fact is at least two-valued in D_3. In the same way, different methods of continuation may lead to many different results, and the function is then many-valued.

The reader of Hardy's *Pure Mathematics* is already familiar with the different values taken by the function $\log z$ (though, of course, there even the idea of a function analytic at a point does not appear). The properties of some other many-valued functions, such as $z^a = e^{a \log z}$, may be derived from those of $\log z$.

4.14. Definition of an analytic function as a whole. An analytic function is usually defined originally in some restricted region of the plane. The principle of continuation enables us to define *an analytic function*, without reference to any particular region in which it is defined. It consists of the original function, and all continuations thereof, and all continuations of these continuations, and so on. In this way we may succeed in defining the function $f(z)$ for all values of z, or everywhere except at certain special points; or only in some restricted region of the plane beyond which we are unable to pass. In the last case the region is referred to as the *region of existence* of the function, and its boundary as a *natural boundary* of the function. In the case of many-valued functions we shall obtain many values of the function for some or all values of z.

* This case may be illustrated by a figure in which the regions D, D_1, and D_2 are circles with centres at the vertices of an equilateral triangle, and each radius just exceeds half the side of the triangle. The function may not be analytic at the centre of the triangle.

The definition depends *prima facie* on the particular definition of the function from which we start. Since, however, the relation between two functions which are continuations of each other is reciprocal, all these processes may be reversed; and it will appear from the general theory that the definition is really independent of any particular starting-point.

4.15. The standard method of continuation. The standard method of continuation is the method of power series. Suppose that we start with the series

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$$

convergent in a circle $|z-a| < R$. Taking any point b in this circle other than a, we calculate the value of the function $f(b)$ and the derivatives $f'(b), f''(b),\ldots$, and so obtain the expansion of the function in powers of $z-b$. This series will certainly converge in any circle, centre b, which lies in the original circle, and it may converge in a larger circle, and so provide an analytic continuation of the function. So the whole function may be constructed by means of power series. Each of the power series, or, what comes to the same thing, each set of values $f(a), f'(a), f''(a),\ldots$, is called an *element* of the function.

The adoption of this particular method as a standard is justified by the following theorem: *All values of the function obtained by any method of continuation can also be obtained by means of power series.*

Let C be a contour joining two points $z = a$ and $z = b$, along which we have continued the function $f(z)$ by any means; that is, we have a sequence of formulae which define $f(z)$ in a sequence of regions D_n, such that (i) every point of C is an interior point of one or more D_n's, and (ii) consecutive D_n's overlap, and the different definitions of $f(z)$ agree in the common parts.

We now attempt to carry out the same process by means of power series; i.e. we try to find a sequence of points $z_1, z_2,\ldots$ on C such that the circle of convergence about each of them includes the next, such that the values found from the power series are the same as those found in the other way, and such that we reach b in this way in a finite number of steps.

With each point z on C is associated a positive radius of

convergence ρ, and ρ *is a continuous function of* z. For take two neighbouring points z, $z+h$, and let ρ and ρ' be the corresponding radii of convergence. Let $|h|<\rho$. Since $f(z)$ is regular in the circle with centre $z+h$ and radius $\rho-|h|$, it follows from the Cauchy-Taylor theorem that

$$\rho' \geqslant \rho-|h|. \tag{1}$$

If $|h|<\rho'$ we can use the same argument with z and $z+h$ interchanged, so that

$$\rho \geqslant \rho'-|h|,$$

i.e.

$$\rho' \leqslant \rho+|h|. \tag{2}$$

Since the alternative to $|h|<\rho'$ is $\rho' \leqslant |h|$, (2) holds in any case. But (1) and (2) together show that $\rho' \to \rho$ as $h \to 0$, which is what is required.

Since ρ is continuous it attains its lower bound, and so, since it is always positive, its lower bound is positive. Let the lower bound be δ.

We now start at $z=a$ with a power series. Let z_1 be the point at distance $\frac{1}{2}\delta$ along the contour. It lies inside the circle of convergence about a, so that we can expand in powers of $z-z_1$. The new radius of convergence is at least δ, so that we can go on to the point z_2 distant δ from a along the curve. Proceeding in this way we plainly reach $z=b$ in a finite number of steps. The fact that we obtain the same value at b in this way as in the other way follows from the general uniqueness theorem.

4.16. Branches of a many-valued function. We have defined an analytic function as the aggregate of all values which can be obtained by continuation from any element of the function. In general the function will be many-valued, i.e. starting from z_0, say, we can, by taking suitable paths, arrive at z_1 with many different values of $f(z_1)$. We may, however, make this impossible by restricting ourselves to the interior of some particular region. We then say that there is a *branch* of the function in this region. Consider, for example, the function $\sqrt{z}$. The system of values defined by $\sqrt{r}e^{\frac{1}{2}i\theta}$ $(-\pi<\theta<\pi)$ is a branch in the plane cut along the negative real axis from the origin to infinity; and the system $-\sqrt{r}e^{\frac{1}{2}i\theta}$ is another branch in the same region. Similarly the function $\log z$ has in this region an infinity

of branches defined by

$$\log r + i(\theta + 2n\pi) \qquad (-\pi < \theta < \pi),$$

every integer n giving a branch.

It should be understood that there is no unique way of dividing up a function into branches; for example, we might, in the above cases, cut the plane along any other line from the origin to infinity. But, however we do it, we obtain a definite number of branches, e.g. $\sqrt{z}$ has two. The question of the number of branches will be considered again later.

4.2. Singularities of an analytic function. The only singularities which we have so far defined are isolated singularities of functions analytic and one-valued in a given region, or limit-points of such singularities. These were classified as poles and essential singularities. This classification now proves to be inadequate.

We shall now say that a one-valued analytic function is *regular* at any point which is interior to one of the circles used in continuation from the original element; and that it is singular at any limit-point of regular points which is not a regular point. A point where the function is singular is called a singular point or singularity. This definition includes the poles and essential singularities which we considered before; but there may also be singularities which are not isolated. In § 4.7 we shall construct functions for which every point of the unit circle is singular. A point of this kind is usually called an essential singularity also.

The expression 'regular', as we have used it here, means more than 'analytic'. A function may be analytic at a point, in accordance with the definition of § 2.14, without being regular there; for example, let $f(z) = e^{-1/z}$ for $-\frac{1}{4}\pi \leqslant \arg z \leqslant \frac{1}{4}\pi, |z| > 0$, and let $f(z) = 0$ elsewhere. It is easily seen that this function is analytic at $z = 0$, and $f'(0) = 0$. Consider, however, the contour consisting of the triangle with vertices at 0 and $1 \pm \frac{1}{2}i$. The function is analytic everywhere inside and on the contour, but it is evidently not regular at $z = 0$. The distinction is, however, not very important, since it has to be made only for somewhat artificial functions like the one considered.

In the theory of many-valued functions we have another kind of singularity, known as a *branch-point*. Suppose that, on continuing the function $f(z)$ round any sufficiently small circle with

centre z_0, we return to the starting-point with a value of the function different from the one with which we started. Then z_0 is said to be a branch-point of $f(z)$. For example, if we continue $\sqrt{z} = \sqrt{r}e^{\frac{1}{2}i\theta}$ round the circle of centre 0 and radius r from $\theta = 0$ to $\theta = 2\pi$, the value of the function changes from $\sqrt{r}$ to $-\sqrt{r}$. Hence $z = 0$ is a branch-point of $\sqrt{z}$. Similarly it is a branch-point of $1/\sqrt{z}$ and $\log z$.

Notice that a branch-point is not necessarily an 'infinity' of the function.

A branch of a many-valued function may, of course, have poles and essential singularities; and a point may be a singularity for one branch of the function but not for another. For example, the point $z = 1$ is a pole of the branch of $1/\log z$ corresponding to the value of $\log z$ which is zero at $z = 1$, but not for any other value of the logarithm. A general definition of regular and singular points is not quite so simple for many-valued functions as for one-valued functions, and it is usually sufficient to consider particular branches separately. We define a regular point of a branch in the same way as for a one-valued function; but a singularity such as a branch-point cannot be assigned to one particular branch.

Examples. (i) The function z^a, defined as $e^{a \log z}$, has an infinity of values unless a is real and rational, when it has a finite number of values.

(ii) The function $z^{\frac{1}{3}}(1-z)^{\frac{1}{2}}$ has six values.

(iii) One branch of the function

$$\frac{1}{z}\log\frac{1}{1-z}$$

is given for $|z| < 1$ by the series

$$\frac{1}{z}(z+\tfrac{1}{2}z^2+\tfrac{1}{3}z^3+\ldots),$$

and $z = 0$ is a regular point for this branch; but it is a pole for every other branch.

(iv) The function $\{\log 1/(1-z)\}^{\frac{1}{2}}$ has singularities at $z = 0$ and $z = 1$; $z = 0$ is a branch-point for one determination of the logarithm.

(v) Consider the singularities of the function $\log\log z$.

4.21. *If the radius of convergence of the series*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is finite, $f(z)$ has at least one singularity on the circle of convergence.

Let C be the circle of convergence, of radius R, and C' a

concentric circle of radius $R' < R$. Let ρ be the radius of convergence of the power series about a point z on C'. As in § 4.15, ρ is a continuous function of z. Also $\rho \geqslant R-R'$. Let δ be the lower bound of ρ for z on C'. Then $\delta \geqslant R-R'$.

If $\delta > R-R'$, the circles of convergence about points on C' together cover the region $|z| < R'+\delta$, and so $f(z)$ is regular in this larger circle. Hence, by the Cauchy-Taylor theorem, the radius of convergence of $\sum a_n z^n$ is greater than R, contrary to hypothesis.

It follows that $\delta = R-R'$. Since a continuous function attains its lower bound, there is a point $R'e^{i\alpha}$, say, on C', at which $\rho = R-R'$. Then $Re^{i\alpha}$ is a singularity of $f(z)$. For if it were a regular point, $f(z)$ would be regular in a circle with centre $z = Re^{i\alpha}$, and then the radius of convergence about $R'e^{i\alpha}$ would be greater than $R-R'$.

Since we have established the existence of a singularity on the circle of convergence, we may speak of it as the singularity nearest to the origin, or one of the nearest. We may then say that the circle of convergence passes through the nearest singularity of the function to the origin.

4.22. *If we continue an analytic function $f(z)$ along two different routes from z_0 to z_1, and obtain two different values of $f(z_1)$, then $f(z)$ must have a singularity somewhere between the two routes.*

We construct two chains of regions, say $D_1, \ldots, D_m$ and $D'_1, \ldots, D'_n$, such that two consecutive regions of either chain overlap, D_1 and D'_1 include z_0, D_m and D'_n include z_1; $f_k(z)$ is analytic in D_k, and $g_k(z)$ in D'_k; $f_k(z) = f_{k-1}(z)$ in the common part of D_k and D_{k-1}, and $f_1(z) = g_1(z)$ in the common part of D_1 and D'_1.

We have then to prove that, if we can continue the function to every point between the two routes, then $f_m(z_1) = g_n(z_1)$.

If δ is small enough, we can construct a polygonal line, starting at a point a in $D_1D'_1$, and ending at b in $D_mD'_n$, with vertices at points $(p\delta, q\delta)$, such that the circles of radius 2δ with these points as centres lie entirely in the first chain, and each contains the centre of the following one. This chain of circles can be substituted for the first chain of regions. A similar chain of circles, with the same δ, can be substituted for the second chain of regions.

We can now replace the first route by a succession of new routes, consisting of circles of radius 2δ with centres at points $(p\delta, q\delta)$, such that each circle of each route overlaps the previous route, and the circles of the same route on each side of it, without leaving any space uncovered. No circle has a radius smaller than 2δ, or the previous theorem would show the existence of a singularity. It follows from the general principle of uniqueness of continuation that, with each such route, we arrive at z_1 with the same value of $f(z_1)$. Also, in a finite number of steps we pass from one of our original routes to the other; since the function is regular at every point between the two routes, the process of continuation is never stopped.

4.3. Riemann surfaces. The function $\sqrt{z}$ is a two-valued function of z; but, if we put $z = re^{i\theta}$ and distinguish between equal values of z arising from different values of θ, it is possible to represent it as a one-valued function. Suppose we consider the values of z corresponding to $\pi < \theta < 3\pi$ as distinct from those corresponding to $-\pi < \theta < \pi$; but those corresponding to $3\pi < \theta < 5\pi$ as the same again, and so on. This is equivalent to replacing the ordinary z-plane by two planes. We may think of them as superposed, each of them being cut along the negative real axis, and the planes being joined cross-wise along the cut. The configuration thus obtained is called a Riemann surface.

If now we pass along a path encircling the origin, starting on the upper plane from the negative real axis, we pass round the upper plane once, then cross to the lower plane, pass round it once, and then return to the upper plane.

This corresponds to the way in which we obtain the two different values of $\sqrt{z}$. On the upper plane, say with $-\pi < \theta < \pi$, we have $\sqrt{z} = \sqrt{r}e^{\frac{1}{2}i\theta}$ $(-\frac{1}{2}\pi < \frac{1}{2}\theta < \frac{1}{2}\pi)$, and on the lower plane $\sqrt{z} = \sqrt{r}e^{\frac{1}{2}i\theta}$ $(\frac{1}{2}\pi < \frac{1}{2}\theta < \frac{3}{2}\pi)$; and if θ is increased further we return to the upper plane again, and the values are repeated. Thus $\sqrt{z}$ is a one-valued function on the Riemann surface.

We represent the function $\log z$ in a similar way upon an infinity of superposed planes, each cut along the negative real axis and joined to the opposite edge of the one below. In this case there is no return to the starting-point.

For a function such as $\sqrt{\{(z-a)(z-b)\}}$ we may make a cut on

each plane along the straight line joining the points $z = a$ and $z = b$, and join cross-wise along the cut.

The number of branches of a many-valued function may be defined as the least number of planes which are required to form a Riemann surface on which the function is one-valued.

Considerable ingenuity is required in constructing Riemann surfaces for more complicated functions. They are of great importance in the general theory of many-valued functions, but it is beyond the scope of this chapter to pursue the subject further.

4.4. Integrals containing a complex parameter. We know that if z is real and positive, then

$$\int_0^\infty e^{-zt}\,dt = \frac{1}{z}. \tag{1}$$

Now the integral is uniformly convergent in any finite region to the right of the imaginary axis, and therefore represents an analytic function of z, regular for $\mathbf{R}(z) > 0$. Hence the function

$$F(z) = \int_0^\infty e^{-zt}\,dt - \frac{1}{z}$$

is regular for $\mathbf{R}(z) > 0$, and $F(z) = 0$ on the real axis. Hence $F(z) = 0$ wherever it is regular, i.e. (1) holds for complex values of z whose real part is positive. Thus we may put $z = x+iy$ $(x > 0)$ and separate real and imaginary parts, and obtain the well-known results

$$\int_0^\infty e^{-xt}\cos yt\,dt = \frac{x}{x^2+y^2}, \qquad \int_0^\infty e^{-xt}\sin yt\,dt = \frac{y}{x^2+y^2}. \tag{2}$$

Examples. (i) Prove that

$$\int_0^\infty e^{-zt^2}\,dt = \frac{\sqrt{\pi}}{2\sqrt{z}} \qquad (\mathbf{R}(z) > 0).$$

[Assuming the result for real values of z, the general result may be obtained either by analytic continuation, or by using Cauchy's theorem to turn the line of integration through an angle $-\frac{1}{2}\arg z$.]

(ii) Prove that $$\int_0^\pi \frac{dt}{1-z\cos t} = \frac{\pi}{\sqrt{(1-z^2)}}$$

except when z is real and $z \geqslant 1$ or $z \leqslant -1$.

4.41. The Gamma-function. The formula

$$\Gamma(z) = \int_0^\infty e^{-w} w^{z-1}\, dw \tag{1}$$

defines $\Gamma(z)$ as an analytic function, regular for $\mathbf{R}(z) > 0$ (§ 2.85). As it stands it tells us nothing about $\Gamma(z)$ on the imaginary axis or to the left of it.

Consider, however, the function

$$f(z) = \int_C e^{-w}(-w)^{z-1}\, dw, \tag{2}$$

where C consists of the real axis from ∞ to δ, the circle $|w| = \delta$ described in the positive direction, and the real axis from δ to ∞ again. The many-valued function $(-w)^{z-1} = e^{(z-1)\log(-w)}$ is made definite by taking $\log(-w)$ to be real at $w = -\delta$. The contour integral is uniformly convergent in any finite region of the z-plane, for the question of convergence now arises at infinity only, a case already discussed in § 1.51. Hence $f(z)$ is regular for all finite values of z.

If $w = \rho e^{i\phi}$, then $\log w = \log\rho + i(\phi - \pi)$ on the contour. The integrals along the real axis therefore give

$$\int_\delta^\infty \{-e^{-\rho+(z-1)(\log\rho - i\pi)} + e^{-\rho+(z-1)(\log\rho+i\pi)}\}\, d\rho$$

$$= -2i\sin z\pi \int_\delta^\infty e^{-\rho}\rho^{z-1}\, d\rho.$$

On the circle of radius δ

$$|(-w)^{z-1}| = |e^{(z-1)\{\log\delta + i(\phi-\pi)\}}| = e^{(x-1)\log\delta - y(\phi-\pi)} = O(\delta^{x-1}).$$

The integral round the circle is therefore $O(\delta^x) = o(1)$ as $\delta \to 0$ if $x > 0$. Hence, making $\delta \to 0$, we obtain

$$f(z) = -2i\sin z\pi \int_0^\infty e^{-\rho}\rho^{z-1}\, d\rho = -2i\sin z\pi\, \Gamma(z) \quad (\mathbf{R}(z) > 0).$$

Now the function $\frac{1}{2}if(z)\operatorname{cosec} z\pi$ is regular for all values of z except possibly at the poles of $\operatorname{cosec} z\pi$, viz. $z = 0, \pm 1, \pm 2, \ldots$; and it equals $\Gamma(z)$ for $\mathbf{R}(z) > 0$. We can therefore take this function as a continuation of $\Gamma(z)$ over the whole z-plane. But we know already that $\Gamma(z)$ is regular at $z = 1, 2, \ldots$. Hence the only possible poles are at $z = 0, -1, -2, \ldots$.

These points are actually poles of $\Gamma(z)$; for if z is a negative integer, $(-w)^{z-1}$ is one-valued, and the integral (2) can be evaluated by the calculus of residues. We obtain

$$f(-n) = -2\pi i/n!,$$

and the residue of $\Gamma(z)$ is

$$\lim_{z\to -n} \frac{2\pi i}{n!} \frac{z+n}{2i \sin z\pi} = \frac{(-1)^n}{n!}.$$

All the gamma-function formulae can now be extended to general complex values of z. For example, the functional equation

$$\Gamma(z)\Gamma(1-z) = \pi \operatorname{cosec} z\pi,$$

proved on the assumption that z is real and $0 < z < 1$, holds for all non-integral values of z.

A consequence of this formula is that $1/\Gamma(z)$ is an integral function. For in the above formula the poles of $\Gamma(1-z)$ are all cancelled by zeros of $\sin z\pi$.

We can now prove for $\Gamma(z)$ formulae similar to those of §§ 3.22–3. By § 1.86 (4)

$$\frac{\Gamma(z-h)\Gamma(h)}{\Gamma(z)} = \frac{1}{h} + \int_0^1 \{(1-t)^{z-h-1}-1\}t^{h-1}\,dt \qquad (0 < h < x)$$

$$= \frac{1}{h} + \int_0^1 \{(1-t)^{z-1}-1\}t^{-1}\,dt + o(1)$$

as $h \to 0$. The left-hand side is

$$\frac{1}{\Gamma(z)}\{\Gamma(z)-h\Gamma'(z)+\ldots\}\left\{\frac{1}{h}+A+\ldots\right\},$$

where A is a constant. Equating the constant terms, we obtain

$$\frac{\Gamma'(z)}{\Gamma(z)} = \int_0^1 \{1-(1-t)^{z-1}\}\frac{dt}{t} - A \qquad (x > 0).$$

Putting $1/t = \sum (1-t)^n$ and integrating term by term,

$$\frac{\Gamma'(z)}{\Gamma(z)} = \sum_{n=0}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+z}\right)-A.$$

The process is justified by § 1.77 if $z > 1$; the result holds by analytic continuation for any z except a negative integer.

It is easily seen that the formula can be rearranged as

$$\frac{\Gamma'(z)}{\Gamma(z)}+\frac{1}{z}=\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+z}\right)-C,$$

where C is another constant. Integrating and taking exponentials,

$$\frac{1}{\Gamma(z)}=e^{Cz}z\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}.$$

Putting $z=1$, $\qquad 1=e^{C}\prod_{1}^{\infty}\left(1+\frac{1}{n}\right)e^{-1/n}.$

Hence

$$C=-\log\prod_{1}^{\infty}\left(1+\frac{1}{n}\right)e^{-1/n}$$

$$=\lim_{N\to\infty}\left(1+\frac{1}{2}+\ldots+\frac{1}{N}-\log N\right)=\gamma,$$

γ being Euler's constant.

4.42. Stirling's formula for complex values of z. The formula of the previous section gives

$$\log\Gamma(z)=\sum_{n=1}^{\infty}\left\{\frac{z}{n}-\log\left(1+\frac{z}{n}\right)\right\}-\gamma z-\log z, \tag{1}$$

each logarithm having its principal value. Now it is easily verified that

$$\int_0^N\frac{[u]-u+\frac{1}{2}}{u+z}\,du=\sum_{n=0}^{N-1}\int_n^{n+1}\left(\frac{n+\frac{1}{2}+z}{u+z}-1\right)du$$

$$=\sum_{n=1}^{N-1}\left\{\frac{z}{n}-\log\left(1+\frac{z}{n}\right)\right\}-\log\{(N-1)!\}-z\left(1+\frac{1}{2}+\ldots+\frac{1}{N-1}\right)-$$
$$-(z+\tfrac{1}{2})\log z+(N-\tfrac{1}{2}+z)\log(N+z)-N.$$

Using 1.87 (1), and the relations

$$1+\frac{1}{2}+\ldots+\frac{1}{N-1}=\log N+\gamma+o(1),$$

$$\log(N+z) = \log N + \frac{z}{N} + O\left(\frac{1}{N^2}\right),$$

and making $N \to \infty$, it follows that

$$\log\Gamma(z) = (z-\tfrac{1}{2})\log z - z + \tfrac{1}{2}\log 2\pi + \int_0^\infty \frac{[u]-u+\frac{1}{2}}{u+z}\,du. \quad (2)$$

Writing $\phi(u) = \int_0^u ([v]-v+\frac{1}{2})\,dv$, $\phi(u)$ is bounded, since clearly $\phi(n+1) = \phi(n)$ if n is an integer. Hence the last term in (2) is

$$\int_0^\infty \frac{\phi'(u)}{u+z}\,du = \int_0^\infty \frac{\phi(u)}{(u+z)^2}\,du = O\left\{\int_0^\infty \frac{du}{u^2+r^2-2ur\cos\delta}\right\} = O\left(\frac{1}{r}\right)$$

uniformly for $-\pi+\delta \leqslant \arg z \leqslant \pi-\delta$. This is the extension of Stirling's formula to complex values of z.

Examples. (i) For any constant a

$$\log\Gamma(z+a) = (z+a-\tfrac{1}{2})\log z - z + \tfrac{1}{2}\log 2\pi + O(1/|z|)$$

as $|z| \to \infty$, uniformly for $-\pi+\delta \leqslant \arg z \leqslant \pi-\delta$.

(ii) For any fixed value of x, as $y \to \pm\infty$

$$|\Gamma(x+iy)| \sim e^{-\frac{1}{2}\pi|y|}|y|^{x-\frac{1}{2}}\sqrt{(2\pi)}.$$

(iii) Show that the series $\phi(u) = \sum_{\nu=1}^{\infty}(1-\cos 2\nu\pi u)/(2\pi^2\nu^2)$ can be inserted in the above formula and integrated term by term. Hence prove that the integral in (2) is

$$\frac{1}{12z} + O\left(\frac{1}{|z|^2}\right).$$

This process can be carried to any number of terms by repeated partial integrations.

4.43. The Zeta-function. The function $\zeta(z)$, defined originally by the series

$$\zeta(z) = \frac{1}{1^z} + \frac{1}{2^z} + \dots \qquad (\mathbf{R}(z) > 1), \quad (1)$$

has been shown (§ 1.78 (ii)) to be also given by the formula

$$\zeta(z) = \frac{1}{\Gamma(z)}\int_0^\infty \frac{w^{z-1}}{e^w-1}\,dw \qquad (\mathbf{R}(z) > 1). \quad (2)$$

We can use this formula to continue $\zeta(z)$ across the line $x=1$, in the same way that we continued $\Gamma(z)$ across $x=0$. In fact we can prove in precisely the same way as before that, if $\mathbf{R}(z)>1$,

$$\zeta(z) = -\frac{1}{2i\sin z\pi\,\Gamma(z)}\int_C \frac{(-w)^{z-1}}{e^w-1}\,dw, \tag{3}$$

where, as before, the contour C comes from positive infinity and encircles the origin once in the positive direction. The only difference is that C must now exclude all the poles of $1/(e^w-1)$ other than $w=0$, viz. the points $w=\pm 2i\pi, \pm 4i\pi, \ldots$.

Using the functional equation for the Γ-function, we may write the result in the form

$$\zeta(z) = \frac{i\Gamma(1-z)}{2\pi}\int_C \frac{(-w)^{z-1}}{e^w-1}\,dw.$$

As in the case of the Γ-function, the contour integral is an integral function of z. This formula therefore provides the continuation of $\zeta(z)$ over the whole plane. The only possible singularities are at the poles of $\Gamma(1-z)$, viz. at $z=1, 2, \ldots$. But we know already that $\zeta(z)$ is regular at $z=2, 3, \ldots$. Hence the only possible pole is at $z=1$. This is actually a simple pole, with residue 1. For at $z=1$ the contour integral is equal to

$$\int_C \frac{dw}{e^w-1} = 2\pi i$$

by the theorem of residues, and $\Gamma(1-z)$ has a simple pole with residue -1, whence the result.

Again, it is well known that

$$\frac{1}{e^w-1} = \frac{1}{w} - \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}B_n w^{2n-1}}{(2n)!}$$

where the coefficients B_n (Bernoulli's numbers) are rational numbers. Hence we can evaluate $\zeta(-n)$, where n is any positive integer, by the theorem of residues. We find that

$$\zeta(0) = -\tfrac{1}{2},$$

$$\zeta(-2m) = 0 \qquad (m = 1, 2, \ldots),$$

$$\zeta(-2m-1) = \frac{(-1)^{m+1}B_{m+1}}{2m+2} \qquad (m = 0, 1, \ldots).$$

4.44. The functional equation for the ζ-function. The ζ-function satisfies the functional equation

$$\zeta(1-z) = 2^{1-z}\pi^{-z}\cos\tfrac{1}{2}\pi z\,\Gamma(z)\zeta(z).$$

To prove this, we take the formula (3) of the previous section, where now z may have any value, and deform the contour into the contour C_n consisting of the square with centre the origin and sides parallel to the axes, length of side $(4n+2)\pi$, together with the positive real axis from $(2n+1)\pi$ to infinity. In so doing we pass over poles of the integrand at the points $w = 2i\pi, 4i\pi, \ldots, 2ni\pi$, and $-2i\pi, \ldots, -2ni\pi$. The residue at $2\nu i\pi$ $(\nu > 0)$ is

$$e^{(z-1)(\log 2\nu\pi - \frac{1}{2}i\pi)} = (2\nu\pi)^{z-1} i e^{-\frac{1}{2}i\pi z},$$

and at $-2\nu i\pi$ it is

$$e^{(z-1)(\log 2\nu\pi + \frac{1}{2}i\pi)} = -(2\nu\pi)^{z-1} i e^{\frac{1}{2}i\pi z}.$$

The sum of these two residues is

$$(2\nu\pi)^{z-1} 2\sin\tfrac{1}{2}\pi z.$$

Hence § 4.43 (3) gives

$$\sin\pi z\,\Gamma(z)\zeta(z) = -\frac{1}{2i}\int_{C_n}\frac{(-w)^{z-1}}{e^w-1}\,dw + 2\pi\sin\tfrac{1}{2}\pi z\sum_{\nu=1}^{n}(2\nu\pi)^{z-1}.$$

Suppose now that $\mathbf{R}(z) < 0$. On the square

$$|(-w)^{z-1}| = e^{(x-1)\log|w| - y\arg(-w)} = O(n^{x-1}),$$

and $|e^w - 1| > A$, while the length of the square is $O(n)$. Hence this part of the integral is $O(n^x)$, and so it tends to zero. The remaining part of the integral plainly tends to zero also. Hence, making $n \to \infty$, we obtain

$$\begin{aligned}\sin\pi z\,\Gamma(z)\zeta(z) &= 2\pi\sin\tfrac{1}{2}\pi z(2\pi)^{z-1}\sum_{\nu=1}^{\infty}\nu^{z-1}\\ &= 2\pi\sin\tfrac{1}{2}\pi z(2\pi)^{z-1}\zeta(1-z),\end{aligned}$$

which is equivalent to the result stated. This proves the functional equation for $\mathbf{R}(z) < 0$, and so, by § 4.42, for all values of z.

4.45. An alternative proof. The following proof* proceeds on quite different lines. Let

$$f(x) = \sum_{n=0}^{\infty}\frac{\sin(2n+1)x}{2n+1}. \tag{1}$$

* Hardy (15).

This series is boundedly convergent, and $f(x) = (-1)^m \frac{1}{4}\pi$ for $m\pi < x < (m+1)\pi$, $m = 0, 1, 2, \ldots$; for

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n} - \sum_{n=1}^{\infty} \frac{\sin 2nx}{2n},$$

and the results easily follow from those of § 1.76, ex. (ii). We may therefore multiply (1) by x^{p-1} $(0 < p < 1)$ and integrate term by term over any finite interval $(0, X)$. Thus

$$\int_0^X x^{p-1} f(x)\, dx = \sum_{n=0}^{\infty} \frac{1}{2n+1} \int_0^X x^{p-1} \sin(2n+1)x\, dx. \tag{2}$$

We may then replace X by ∞ provided that

$$\lim_{X\to\infty} \sum_{n=0}^{\infty} \frac{1}{2n+1} \int_X^{\infty} x^{p-1} \sin(2n+1)x\, dx = 0. \tag{3}$$

Integrating by parts, the integral is

$$X^{p-1} \frac{\cos(2n+1)X}{2n+1} + \frac{p-1}{2n+1} \int_X^{\infty} x^{p-2} \cos(2n+1)x\, dx$$

$$= O\left(\frac{X^{p-1}}{2n+1}\right) + O\left(\frac{1}{2n+1} \int_X^{\infty} x^{p-2}\, dx\right) = O\left(\frac{X^{p-1}}{2n+1}\right),$$

and (3) clearly follows.

Inserting the value of $f(x)$, and evaluating the integrals on the right-hand side by § 3.127, we therefore obtain

$$\tfrac{1}{4}\pi \sum_{m=0}^{\infty} (-1)^m \int_{m\pi}^{(m+1)\pi} x^{p-1}\, dx = \Gamma(p) \sin \tfrac{1}{2}p\pi \sum_{n=0}^{\infty} \frac{1}{(2n+1)^{p+1}}.$$

The series on the right-hand side converges to

$$(1-2^{-p-1})\zeta(p+1).$$

That on the left is

$$\frac{\pi^p}{p}\left[1 + \sum_{m=1}^{\infty} (-1)^m \{(m+1)^p - m^p\}\right].$$

This series is convergent for $p < 1$, and, as a little consideration of the above argument shows, uniformly convergent for $\mathbf{R}(p) \leqslant 1-\delta < 1$. Its sum is therefore an analytic function of p, regular for $\mathbf{R}(p) < 1$. But for $p < 0$ it is

$$2(1^p - 2^p + 3^p - \ldots) = 2(1-2^{p+1})\zeta(-p).$$

By the theory of analytic continuation its sum is the same analytic function of p for $\mathbf{R}(p) < 1$.

Hence, for $0 < p < 1$,

$$\frac{\pi^{p+1}}{2p}(1-2^{p+1})\zeta(-p) = \Gamma(p)\sin\tfrac{1}{2}p\pi(1-2^{-p-1})\zeta(p+1),$$

and putting $p = z-1$ we have the same functional equation as before. The proof holds for $1 < z < 2$ only, but the result, proved for these values of z, holds for all values by analytic continuation.

4.5. The principle of reflection. *Let $f(z)$ be an analytic function, regular in a region D intersected by the real axis, and real on the real axis. Then $f(z)$ takes conjugate values for conjugate values of z.*

Let z_0 be an interior point of D on the real axis. Then

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

for sufficiently small values of $|z-z_0|$.

All the coefficients a_n are real; for

$$a_0 = f(z_0), \qquad a_1 = f'(z_0), \ldots .$$

Clearly a_0 is real. a_1 may be calculated as the limit of

$$\frac{f(z)-f(z_0)}{z-z_0}$$

as $z \to z_0$ by real values. Hence a_1 is real. So they are all real.

The result now follows inside the circle of convergence of the above series. The general result then follows by continuation, since the power series about conjugate points will always have conjugate coefficients.

4.51. A method of obtaining the analytic continuation of certain functions is given by the Riemann-Schwarz 'principle of reflection'. This is contained in the following theorem, which is a sort of converse of the previous one.

Suppose that a region D of the z-plane has as part of its boundary a segment l of a straight line; and that $w = f(z)$ is an analytic function, regular in D and continuous on l, and such that, as z describes l, w describes a straight line λ in the w-plane. Let z be a point of D, z_1 its reflection in l, and let w_1 be the reflection of w in λ. Then $w_1 = w_1(z_1)$ is an analytic continuation of w.

In the first place, w_1 is an analytic function of z_1; for it is easily seen from a figure that, if w' corresponds to z', and w_1', z_1' are their reflections,

$$|z_1'-z_1| = |z'-z|, \qquad |w_1'-w_1| = |w'-w|;$$

and

$$\arg(z_1'-z_1) = 2\alpha-\arg(z'-z), \qquad \arg(w_1'-w_1) = 2\beta-\arg(w'-w),$$

where α and β are the angles between l and λ respectively and the real axis. Now when $z' \to z$, the limit

$$\lim \frac{w'-w}{z'-z}$$

exists, i.e. the limits

$$\lim \left|\frac{w'-w}{z'-z}\right|, \qquad \lim\{\arg(w'-w)-\arg(z'-z)\}$$

exist. Hence the limits

$$\lim \left|\frac{w_1'-w_1}{z_1'-z_1}\right|, \qquad \lim\{\arg(w_1'-w_1)-\arg(z_1'-z_1)\}$$

exist, and so

$$\lim \frac{w_1'-w_1}{z_1'-z_1}.$$

exists, i.e. w_1 is an analytic function of z_1.

Secondly, it is clear that, on l, $w_1 = w$.

To prove that the two functions are analytic continuations of each other, take any point of the line l, and describe round it a circle C so small that it lies entirely inside D and its reflection D_1. Let c be the boundary of the part of C in D, c_1 of the part in D_1. Let $\phi(z) = w$ in the part of D inside C, and $\phi(z) = w_1$ in the remainder of C. Then $\phi(z)$ is continuous, and it is sufficient to prove that $\phi(z)$ is an analytic function of z.

Let z_0 be a point inside C and D. Then*

$$\phi(z_0) = \frac{1}{2\pi i}\int_c \frac{\phi(z)}{z-z_0}\,dz,$$

and, since $\phi(z)/(z-z_0)$ is regular in D_1,

$$0 = \frac{1}{2\pi i}\int_{c_1} \frac{\phi(z)}{z-z_0}\,dz.$$

* See the end of § 2.35.

Adding, we obtain

$$\phi(z_0) = \frac{1}{2\pi i} \int_C \frac{\phi(z)}{z-z_0}\, dz, \tag{1}$$

the integrals along l cancelling.

We clearly obtain the same formula if z_0 is any point inside C and D_1; and also, since each side of (1) is continuous, if z_0 is inside C and on l. But, as in § 2.8, the right-hand side of (1) is an analytic function of z_0, regular inside C. This proves the theorem.

The method of proof also gives the following general theorem: *if two functions $f(z)$, $f_1(z)$, are analytic and regular in regions D and D_1 separated by a contour C, and continuous on C, and $f(z) = f_1(z)$ along C, then the two functions are analytic continuations of each other.*

4.6. Hadamard's multiplication theorem.* The following problem, which was considered by Hadamard, is a good example of the principles of analytic continuation. Suppose that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is convergent for $|z| < R$, and

$$g(z) = \sum_{n=0}^{\infty} b_n z^n$$

is convergent for $|z| < R'$, and that the singularities of $f(z)$ and $g(z)$ are known. What can be said about the singularities of the function

$$F(z) = \sum_{n=0} a_n b_n z^n, \tag{1}$$

whose coefficients are the products of those in the given series?

The general result is that, *if $f(z)$ has singularities at $\alpha_1, \alpha_2, \ldots$, and $g(z)$ at $\beta_1, \beta_2, \ldots$, then the singularities of $F(z)$ are to be found among the points $\alpha_m \beta_n$.*

Let us suppose, to take the simplest possible case, that $f(z)$ has just one singularity, $z = \alpha$, and $g(z)$ just one singularity, $z = \beta$.

In the first place, $F(z)$ is regular for sufficiently small values of z, and in fact for $|z| < RR'$. For if $\epsilon > 0$

$$|a_n(R-\epsilon)^n| < K, \qquad |b_n(R'-\epsilon)^n| < K,$$

so that

$$|a_n b_n| < \frac{K}{\{(R-\epsilon)(R'-\epsilon)\}^n},$$

* Hadamard (3).

and by taking ϵ small enough we see that the radius of convergence of the series for $F(z)$ is at least equal to RR'.

Hadamard's theorem depends on the following representation of $F(z)$ as an integral:

$$F(z) = \frac{1}{2\pi i} \int_C f(w) g\left(\frac{z}{w}\right) \frac{dw}{w}, \tag{2}$$

where C is a contour, including the origin, on which $|w| < R$, $|z/w| < R'$. To prove this, write

$$g\left(\frac{z}{w}\right) = \sum_{n=0}^{\infty} b_n \left(\frac{z}{w}\right)^n$$

in the integral, and integrate term by term, as we may by uniform convergence. We obtain

$$\begin{aligned} \frac{1}{2\pi i} \int_C f(w) g\left(\frac{z}{w}\right) \frac{dw}{w} &= \sum_{n=0}^{\infty} \frac{b_n z^n}{2\pi i} \int_C \frac{f(w)}{w^{n+1}} dw \\ &= \sum_{n=0}^{\infty} a_n b_n z^n, \end{aligned}$$

the required result. In order that the inequalities $|w| < R$, $|z/w| < R'$, should be consistent, it is, of course, necessary that $|z| < RR'$. If this condition is satisfied, C may, for example, be any circle between $|w| = R$ and $|w| = |z|/R'$.

In the case where each function has just one singularity, $R = |\alpha|$ and $R' = |\beta|$.

We next continue the function $F(z)$ beyond the circle $|z| = RR'$ by deforming the contour C. As long as C remains fixed, z may, in (2), take any value such that z/β remains inside the contour C. For, by § 2.83, the right-hand side of (2) is an analytic function of z for all such values of z, and the continuation of $F(z)$ to all such values follows at once.

Suppose on the other hand that we deform C into another contour C_1, which includes $z = 0$ and excludes $z = \alpha$. Let

$$F_1(z) = \frac{1}{2\pi i} \int_{C_1} f(w) g\left(\frac{z}{w}\right) \frac{dw}{w}. \tag{3}$$

Then by Cauchy's theorem $F_1(z) = F(z)$, provided that the point $w = z/\beta$ lies within both C and C_1; for then the integrand is an analytic function of w, regular between C and C_1.

The formula (3) therefore provides the continuation of $F(z)$ to all values of z such that z/β lies within C_1.

The only restriction on z, therefore, is that z/β must lie within a contour which excludes $z=\alpha$. But we can choose such a contour for every z except $z=\alpha\beta$.

Thus $F(z)$ is regular except at $z=\alpha\beta$. The proof, however, applies only to what we may call the principal branch of $F(z)$, viz. that obtained by continuation from the original element without encircling any of the points $\alpha\beta$. In the above example the contour C_1 cannot have a loop going round α and enclosing the origin again on the other side, since the integrand has a singularity at $w=0$. Hence if $z/\beta \to 0$ along a path encircling the point α, it is impossible to choose C_1 appropriately. The point $z=0$ may therefore be a singularity of other branches of $F(z)$.

In the general case the details of the proof are, of course, more complicated, but the general method is the same.

Examples. (i) If

$$f(z)=\frac{1}{a-z}, \qquad g(z)=\frac{1}{b-z},$$

then

$$F(z)=\frac{1}{ab-z}.$$

(ii) If

$$f(z)=\frac{1}{1-z^2}, \qquad g(z)=\frac{z}{1-z^2},$$

then $F(z)=0$, so that the points $\alpha\beta$ are not *necessarily* singularities of $F(z)$.

4.7. Functions with natural boundaries.

Let

$$f(z)=\sum_{n=0}^{\infty} z^{n!}.$$

Then $f(z)$ is an analytic function, regular for $|z|<1$. Let

$$z=re^{2p\pi i/q},$$

and consider the behaviour of $f(z)$ as $r\to 1$ through real values.

Now

$$f(z)=\sum_{n=0}^{q-1} z^{n!}+\sum_{n=q}^{\infty} z^{n!}=f_1(z)+f_2(z),$$

say. Then $f_1(z)$ is a polynomial, and tends to a finite limit as $r\to 1$. When $n\geqslant q$, q is a divisor of $n!$, and so

$$z^{n!}=r^{n!} \qquad (n\geqslant q).$$

Hence

$$f_2(z)=\sum_{n=q}^{\infty} r^{n!},$$

which tends to infinity as $r\to 1$. Hence $f(z)\to\infty$, and so $z=e^{2p\pi i/q}$ is a singularity of $f(z)$. But points of this kind are

dense everywhere round the unit circle, so that there is no arc, however small, on which $f(z)$ is regular. It is therefore impossible to continue $f(z)$ across the unit circle, and so the unit circle is a natural boundary of the function.

A similar result holds for the function

$$f(z) = \sum_{n=0}^{\infty} z^{2^n}$$

—we put $z = re^{2p\pi i/2^s}$ and proceed as before.

4.71. Lambert's series.* Let

$$f(z) = \sum_{n=1}^{\infty} d(n) z^n \qquad (|z| < 1),$$

where $d(n)$ denotes the number of divisors of n. We shall prove that the unit circle is a natural boundary of this function.

Consider the double series

$$\sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} z^{\mu\nu}.$$

If we arrange it as a single power series we obtain $f(z)$, and if we sum it by rows we obtain

$$f(z) = \sum_{\mu=1}^{\infty} \frac{z^\mu}{1-z^\mu} \qquad (|z| < 1).$$

The double series is absolutely convergent for $|z| < 1$, so that the transformation is justified.

Let $$z = re^{2p\pi i/q}$$

where p and q are positive integers, $p > 0$, $q > 1$, and p is prime to q. Then we shall prove that, as $r \to 1$,

$$(1-r)f(z) \to \infty.$$

For let $$f(z) = \sum\nolimits_1 \frac{z^\mu}{1-z^\mu} + \sum\nolimits_2 \frac{z^\mu}{1-z^\mu},$$

where, in $\sum_1$, μ takes all values $\equiv 0 \pmod q$, in $\sum_2$ all other values. In $\sum_1$, putting $\mu = mq$,

$$z^\mu = (re^{2\pi ip/q})^{mq} = r^{mq},$$

so that

$$(1-r)\sum\nolimits_1 = (1-r)\sum_{m=1}^{\infty} \frac{r^{mq}}{1-r^{mq}}$$

$$= \frac{1-r}{1-r^q} \sum_{m=1}^{\infty} \frac{1-r^q}{1-r^{mq}} r^{mq}$$

* See Knopp (1).

$$= \frac{1}{1+r+\ldots+r^{q-1}} \sum_{m=1}^{\infty} \frac{r^{mq}}{1+r^q+\ldots+r^{(m-1)q}}$$

$$\geqslant \frac{1}{q} \sum_{m=1}^{\infty} \frac{r^{mq}}{m} = \frac{1}{q} \log \frac{1}{1-r^q} \to \infty.$$

On the other hand, if $\mu \not\equiv 0 \pmod{q}$,

$$|1-z^\mu|^2 = |1-r^\mu e^{2\pi i p\mu/q}|^2 = (1-r^\mu)^2 + 4r^\mu \sin^2 \frac{p\mu\pi}{q}$$

$$\geqslant 4r^\mu \sin^2 \frac{\pi}{q}.$$

Hence

$$|(1-r)\sum_2| \leqslant \frac{(1-r)}{2\sin \pi/q} \sum_{\mu=0}^{\infty} r^{\frac{1}{2}\mu} = \frac{1+\sqrt{r}}{2\sin \pi/q} \leqslant \frac{1}{\sin \pi/q}.$$

Hence, as in the previous cases, the unit circle is a natural boundary of $f(z)$.

MISCELLANEOUS EXAMPLES

1. The power series $\quad z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \ldots$

may be continued to a wider region by means of the series

$$\log 2 - \frac{1-z}{2} - \frac{(1-z)^2}{2.2^2} - \frac{(1-z)^3}{3.2^3} - \ldots.$$

2. The power series $\quad z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \ldots$

and $\quad i\pi - (z-2) + \frac{1}{2}(z-2)^2 - \frac{1}{3}(z-2)^3 + \ldots$

have no common region of convergence, but are analytic continuations of the same function.

3. The functions defined by the series

$$1 + az + a^2z^2 + \ldots$$

and

$$\frac{1}{1-z} - \frac{(1-a)z}{(1-z)^2} + \frac{(1-a)^2z^2}{(1-z)^3} - \ldots$$

are analytic continuations of each other.

4. If $f(z)$ and $g(z)$ are integral functions, the integral

$$\frac{1}{2\pi i} \int \left\{ \frac{f(w)}{w-z} + \frac{zg(1/w)}{zw-w^2} \right\} dw,$$

taken round the unit circle, represents $f(z)$ inside the circle and $g(1/z)$ outside it.

5. If $F(\alpha, \beta; \gamma; z)$ denotes the series

$$1 + \frac{\alpha\beta}{1.\gamma} z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2\,\gamma(\gamma+1)} z^2 + \ldots,$$

show that the series $\quad f(z) = F(a, 1; c; z)$

and
$$g(z) = \frac{1}{1-z} F\left(c-a,\, 1\,;\, c\,;\, \frac{z}{z-1}\right)$$
have a common region of convergence; that in this region they both satisfy the differential equation
$$z(1-z)\frac{d^2u}{dz^2} + \{c-(a+2)z\}\frac{du}{dz} - au = 0;$$
that $f(0) = g(0)$ and $f'(0) = g'(0)$; and hence that the two functions are analytic continuations of each other.

6. The function
$$\frac{1}{\sqrt{(2-z)}+1}$$
has two power series about $z = 0$, with radii of convergence 1 and 2 respectively.

7. Consider the singularities of the functions
$$\exp\left\{\frac{1}{\sqrt{(2-z)}+1}\right\}, \qquad \log\left\{\frac{1}{\sqrt{(2-z)}+1}\right\}.$$

8. Show that the formulae (2) of § 4.4, which were proved there for real values of x and y, hold also for all complex values such that $|\mathbf{I}(y)| < \mathbf{R}(x)$.

9. Prove that
$$\Gamma(z) = \int_1^\infty e^{-w} w^{z-1}\, dw + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(z+n)},$$
and hence give another proof of the analytic properties of $\Gamma(z)$.

10. Prove that
$$\int_0^\infty t^{z-1}\cos t\, dt = \Gamma(z)\cos \tfrac{1}{2}\pi z$$
if $0 < \mathbf{R}(z) < 1$; and that
$$\int_0^\infty t^{z-1}\sin t\, dt = \Gamma(z)\sin \tfrac{1}{2}\pi z$$
if $-1 < \mathbf{R}(z) < 1$.

11. Prove that if $0 < \mathbf{R}(z) < 1$
$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^\infty w^{z-1}\left(\frac{1}{e^w-1} - \frac{1}{w}\right) dw,$$
and that if $-1 < \mathbf{R}(z) < 0$
$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^\infty w^{z-1}\left(\frac{1}{e^w-1} - \frac{1}{w} + \frac{1}{2}\right) dw.$$

[Consider the corresponding contour integrals as in § 4.43.]

12. Deduce the functional equation for the zeta-function from the first formula of ex. 11, and the formula of Ch. III, Misc. Ex. 10.

13. Deduce the functional equation for the zeta-function from the second formula of ex. 11, and ex. (iii) of § 3.22.

14. The function $L(z)$ is defined for $\mathbf{R}(z) > 1$ by the series

$$L(z) = \frac{1}{1^z} - \frac{1}{3^z} + \frac{1}{5^z} - \dots .$$

Show that $L(z)$ is an integral function of z, and that it satisfies the functional equation

$$L(1-z) = 2^z \pi^{-z} \sin \tfrac{1}{2}\pi z \; \Gamma(z) L(z).$$

15. A function $f(z)$ is defined for $|z| < 1$ by the series

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)^s} \qquad (s > 0).$$

Show that
$$f(z) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - z}\, dt,$$

and hence that $f(z)$ is regular except possibly on the positive real axis.

By deforming the line of integration into a suitable curve, show that the principal branch of $f(z)$ is regular except at the point $z = 1$.

16. Show that the singularities of the principal branch of

$$\sum_{n=0}^{\infty} \frac{a_n z^n}{(n+1)^s}$$

are the same for all real values of s.

17. Two functions $f(x)$, $g(x)$, are connected by the formulae

$$f(x) = \int_{-\infty}^{\infty} e^{ixt} g(t)\, dt, \qquad g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixt} f(t)\, dt,$$

for real values of x. Show that there cannot be a finite interval outside which both $f(x) = 0$ and $g(x) = 0$, unless both functions are everywhere 0.

[If $f(x) = 0$ for $x < a$ and $x > b$, $g(x)$ is analytic.]

18. If
$$f(z) = \sum_{n=0}^{\infty} z^{2^n},$$

show that
$$f(z) = f(z^2) + z,$$

and deduce from this that $|z| = 1$ is a natural boundary of the function.

19. If α is a real irrational number, the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n (z - e^{2in\alpha\pi})}$$

represents two different analytic functions, one inside the unit circle and one outside it, the unit circle being a natural boundary of each. If α is

rational, the series represents a single rational function. [In the first case the function is unbounded as $z \to e^{2im\alpha\pi}$ along the radius vector.]

20. The function
$$\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2+n}\right)$$
has the line $\mathbf{R}(z) = 1$ as a natural boundary.

[Every point of this line is a limit-point of zeros.]

21. Let
$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (|z| < 1)$$
and
$$\phi(z) = \sum_{n=0}^{\infty} \frac{a_n z^n}{n!}.$$
Then the integral
$$F(z) = \int_0^{\infty} e^{-t}\phi(zt)\,dt$$
provides the continuation of $f(z)$ across any arc of the circle of convergence where $f(z)$ is regular.

[This is Borel's method of continuation—see his *Leçons sur les séries divergentes*, p. 94. We have $F(z) = f(z)$ wherever the series for $f(z)$ is convergent; for by § 1.79 we may then insert the series for $\phi(zt)$ and integrate term by term. But if $f(z)$ can be continued at all, $F(z)$ exists in a wider region. Let z be a regular point of $f(z)$. As in § 4.6 we have
$$\phi(zt) = \frac{1}{2\pi i}\int_C f(w)e^{zt/w}\frac{dw}{w},$$
where C is a contour including the origin, and excluding the singularities of $f(w)$. Hence
$$|\phi(zt)| < Ke^{Mt}$$
where K is independent of z and t, and M is the maximum of $\mathbf{R}(z/w)$ for values of w on C. To prove the integral for $F(z)$ convergent, we must have $M < 1$. Now $\mathbf{R}(z/w) < 1$ if w lies outside the circle on Oz as diameter. We therefore take C to be a concentric circle of slightly larger radius—say $\frac{1}{2}|z|+\delta$; on it $M = |z|/(|z|+\delta)$. Also C must exclude all singularities of $f(w)$, and it does so if z lies in a region D formed as follows: Through each singular point of $f(w)$, draw a line perpendicular to the line joining the point to the origin. The unit circle is included in a polygon D formed by these lines. It is easily seen from a figure that the conditions are fulfilled if z is inside D and δ is small enough. It now follows without difficulty that Borel's integral gives the continuation of $f(z)$ to all points inside D.]

22. Verify Borel's theorem for the functions
$$\frac{1}{1-z}, \qquad \frac{1}{1-z^2}, \qquad \frac{1}{1-z^4}.$$

CHAPTER V

THE MAXIMUM-MODULUS THEOREM

5.10 The maximum-modulus theorem. Let $f(z)$ be an analytic function, regular in a region D and on its boundary C, which we take to be a simple closed contour. Then $|f(z)|$ is continuous in D, since

$$\big||f(z+h)|-|f(z)|\big| \leqslant |f(z+h)-f(z)|,$$

which tends to zero with h. Hence $|f(z)|$ has a maximum value, which is attained at one or more points. The fundamental theorem of the chapter is that $|f(z)|$ reaches its maximum on the boundary C, and not at any interior point of D. We may express it by saying that *if $|f(z)| \leqslant M$ on C, then the same inequality holds at all points of D.*

A more precise form of the theorem is as follows:

If $|f(z)| \leqslant M$ on C, then $|f(z)| < M$ at all interior points of D, unless $f(z)$ is a constant (when of course $|f(z)| = M$ everywhere).

We shall give a number of different proofs of this theorem.

5.11. First proof. This depends on the lemma that *if $\phi(x)$ is continuous, $\phi(x) \leqslant k$, and*

$$\frac{1}{b-a}\int_a^b \phi(x)\,dx \geqslant k, \tag{1}$$

then $\phi(x) = k$. For if $\phi(\xi) < k$, there is an interval $(\xi-\delta, \xi+\delta)$ in which $\phi(x) \leqslant k-\epsilon$, say; and

$$\int_a^b \phi(x)\,dx \leqslant 2\delta(k-\epsilon)+(b-a-2\delta)k = (b-a)k-2\delta\epsilon,$$

contradicting (1).

To prove the theorem, suppose that, at an interior point z_0 of D, $|f(z)|$ has a value at least equal to its value anywhere else. Let Γ be a circle with centre z_0 lying entirely in D. Then

$$f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-z_0}\,dz. \tag{2}$$

Putting $z-z_0 = re^{i\theta}$, $f(z)/f(z_0) = \rho e^{i\phi}$, so that ρ and ϕ are functions of θ, we may write (2) as

$$1 = \frac{1}{2\pi}\int_0^{2\pi} \rho e^{i\phi}\,d\theta. \tag{3}$$

Hence
$$1 \leqslant \frac{1}{2\pi}\int_0^{2\pi} \rho\, d\theta.$$

But by hypothesis $\rho \leqslant 1$. Hence, by the lemma, $\rho = 1$ for all values of θ.

Taking the real part of (3) we now obtain
$$1 = \frac{1}{2\pi}\int_0^{2\pi} \cos\phi\, d\theta,$$
so that, by the lemma, $\cos\phi = 1$. Hence $f(z) = f(z_0)$ on Γ, and so everywhere; that is, $f(z)$ is a constant.

5.12. Second proof. This is similar in principle to the first proof, but, instead of Cauchy's integral, we use the fact (§ 2.5) that, if
$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n,$$
then
$$\frac{1}{2\pi}\int_0^{2\pi} |f(z_0+re^{i\theta})|^2\, d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$
Under the same hypotheses as before, the left-hand side does not exceed $|f(z_0)|^2$, i.e. $|a_0|^2$. Hence
$$|a_0|^2 + |a_1|^2 r^2 + |a_2|^2 r^4 + \ldots \leqslant |a_0|^2$$
for a positive value of r. Hence $a_1 = a_2 = \ldots = 0$, and $f(z)$ is a constant.

5.13. Third proof. If z_0 is an interior point of D, we may expand $f(z)$ in a series of powers of $z-z_0$,
$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n,$$
with a positive radius of convergence. Putting
$$z-z_0 = re^{i\theta}, \qquad a_n = A_n e^{i\alpha_n},$$
this is
$$f(z) = \sum_{n=0}^{\infty} A_n r^n e^{i(\alpha_n+n\theta)}.$$
Hence
$$|f(z)|^2 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_m A_n r^{m+n} e^{i(\alpha_m+m\theta-\alpha_n-n\theta)}. \tag{1}$$

Suppose first that $a_0 \neq 0$. Since the series is absolutely convergent, we may rearrange it as a power series in r with a positive radius of convergence. Let k be the smallest positive value of

n for which $a_n \neq 0$. Then

$$|f(z)|^2 = A_0^2 + 2A_0A_kr^k\cos(\alpha_0-\alpha_k-k\theta) + \sum_{n=k+1}^{\infty} c_nr^n, \qquad (2)$$

where $|c_n| < c^n$ for some value of c. Hence

$$\left|\sum_{n=k+1}^{\infty} c_nr^n\right| < \sum_{n=k+1}^{\infty} c^nr^n = c^{k+1}r^{k+1}/(1-cr),$$

which is less than $A_0A_kr^k$ if r is small enough. For such a value of r, $|f(z)|^2-A_0^2$ takes both positive and negative values as θ varies between 0 and 2π, the middle term on the right of (2) varying between $-2A_0A_kr^k$ and $2A_0A_kr^k$. Hence A_0 is neither a maximum nor a minimum of $|f(z)|$.

The proof breaks down if there is no a_n $(n > 0)$ which is not zero. But then $f(z) = a_0$ for all values of z.

Finally, if $a_0 = 0$, $|f(z_0)| = 0$, which cannot be a maximum, but is a minimum.

This proves the theorem. We have also shown incidentally that $|f(z)|$ *cannot have, in D, a minimum other than* 0. This may also be proved by applying the general theorem to the function $1/f(z)$.

5.14. Harmonic functions. The corresponding theorem for harmonic functions is that *a function which is harmonic in a region cannot have a maximum at an interior point of the region.* For let u be the real part of $f(z)$. If u is a maximum at an interior point, so is e^u; i.e. so is $|F(z)| = |e^{f(z)}|$. This has been shown to be impossible.

The theorem can also be proved by an argument similar to that of § 5.13. Without going into complete detail, this may be seen in a general way as follows. Let $u(x,y)$ be the real part of an analytic function $f(z) = \sum a_nz^n$, regular at $z = 0$. Then

$$u(x,y) = \mathbf{R}\sum a_n(x+iy)^n,$$

and we obtain for $u(x,y)$ a double series of powers of x and y. The coefficients being those of Taylor's theorem, we have

$$u(x,y)-u(0,0) = u_xx+u_yy+\tfrac{1}{2}(u_{xx}x^2+2u_{xy}xy+u_{yy}y^2)+\dots.$$

A necessary condition for a maximum is that $u_x = u_y = 0$. But since, for a harmonic function, $u_{xx}+u_{yy} = 0$, u_{xx} and u_{yy} have opposite signs, and we can make $u(x,y)-u(0,0)$ either positive or negative by taking $x = 0$ and y small, or $y = 0$ and x small.

Examples. (i) If $|f(z)| > m$ on $|z| = a$, $f(z)$ is regular for $|z| < a$, and $|f(0)| < m$, then $f(z)$ has at least one zero in $|z| < a$.

[For $|f(z)|$ has a minimum inside the circle, and the minimum value must be zero.]

(ii) Use the previous example to show that every algebraic equation has a root.

5.15. The maximum-modulus theorem is also true for a function $f(z)$ which is regular but not one-valued in a region, provided that $|f(z)|$ is one-valued (e.g. $\sqrt{z}$ in a ring-shaped region surrounding the origin); for the above proofs hold for any branch of the function.

5.16. Let $f(z)$ be an analytic function regular for $|z| < R$, and let $M(r)$ denote the maximum of $|f(z)|$ on $|z| = r$. *Then $M(r)$ is a steadily increasing function of r for $r < R$.* For it follows at once from the above theorem that $M(r_1) \leqslant M(r_2)$ if $r_1 < r_2$, and $M(r_1)$ can only be equal to $M(r_2)$ if $f(z)$ is a constant.

Similarly the function $A(r)$, defined in § 2.53 as the maximum of $\mathbf{R}\{f(z)\}$, is an increasing function of r. For

$$e^{A(r)} = \max_{|z|=r} |e^{f(z)}|.$$

5.2. Schwarz's lemma. *If $f(z)$ is an analytic function, regular for $|z| \leqslant R$, and $|f(z)| \leqslant M$ for $|z| = R$, and $f(0) = 0$, then*

$$|f(re^{i\theta})| \leqslant \frac{Mr}{R} \qquad (0 \leqslant r \leqslant R).$$

Let $\phi(z) = f(z)/z$. Then $\phi(z)$ is regular for $|z| \leqslant R$, and

$$|\phi(z)| \leqslant M/R$$

on the circle $|z| = R$. The same inequality therefore holds inside the circle also, and since $|\phi(z)| = |f(z)|/r$ the result follows.

5.21. Vitali's convergence theorem.* *Let $f_n(z)$ be a sequence of functions, each regular in a region D; let*

$$|f_n(z)| \leqslant M$$

for every n and z in D; and let $f_n(z)$ tend to a limit, as $n \to \infty$, at a set of points having a limit-point inside D. Then $f_n(z)$ tends uniformly to a limit in any region bounded by a contour interior to D, the limit being, therefore, an analytic function of z.

It is sufficient to consider the case where D is a circle, and the limit-point is its centre. For then, returning to the general

* This proof is given by Jentzsch (1).

case, we can prove uniform convergence in a circle with centre the limit-point interior to D. Then we can repeat the process with any point of this circle; and so, by the method used in analytic continuation, extend the domain of uniform convergence to any region bounded by a contour interior to D.

We may take the limit-point as origin. Let the radius of the circle D be R. Let

$$f_n(z) = a_{0,n} + a_{1,n}z + \dots \qquad (|z| \leqslant R). \tag{1}$$

Then $$|f_n(z) - f_n(0)| \leqslant |f_n(z)| + |f_n(0)| \leqslant 2M.$$

But $f_n(z) - f_n(0)$ is zero at $z = 0$. Hence, by Schwarz's lemma,

$$|f_n(z) - f_n(0)| \leqslant \frac{2M|z|}{R} \qquad (|z| \leqslant R).$$

Let z' ($\neq 0$) be a point where the sequence converges. Then

$$\begin{aligned} |f_n(0) - f_{n+m}(0)| &\leqslant |f_n(0) - f_n(z')| + |f_n(z') - f_{n+m}(z')| + \\ &\qquad + |f_{n+m}(z') - f_{n+m}(0)| \\ &\leqslant 4M|z'|/R + |f_n(z') - f_{n+m}(z')|. \end{aligned}$$

We can choose z' so that the first term is arbitrarily small, and then, since $f_n(z')$ tends to a limit, we can choose n so large that the second term is arbitrarily small for all positive values of m. Hence $f_n(0)$, i.e. $a_{0,n}$, tends to a limit, say a_0.

Next consider the function

$$g_n(z) = \{f_n(z) - a_{0,n}\}/z = a_{1,n} + a_{2,n}z + \dots.$$

This also tends to a limit at z', since, as we have just proved, $a_{0,n}$ tends to a limit. Also

$$|g_n(z)| \leqslant 2M/R$$

for $|z| = R$, and so also for $|z| < R$. Thus $g_n(z)$ satisfies the same conditions as $f_n(z)$ (except for the value of its upper bound), and hence $a_{1,n}$ tends to a limit, say a_1. Similarly $a_{\nu,n}$ tends to a limit for all values of ν.

Finally, the convergence of (1) is uniform with respect to n and z for $|z| \leqslant R - \epsilon$, since, by Cauchy's inequality, $|a_{\nu,n}| \leqslant M/R^\nu$. So, since every term tends to a limit, the sum tends to a limit uniformly for $|z| \leqslant R - \epsilon$. This proves the theorem.

5.22. *From any sequence of functions regular and bounded in D, in the sense of the above theorem, we can select a sub-sequence which converges uniformly in any region interior to D.*

Let $f_n(z)$ be the sequence of functions, and let $|f_n(z)| \leqslant M$ in D. Let $z_1, z_2, \ldots$ be a sequence of points having a limit-point inside D. Then the points $w_n = f_n(z_1)$ all lie inside the circle $|w| \leqslant M$ in the w-plane. Hence they have at least one limit-point; i.e. there is a sequence of values of n, say $n_1, n_2, \ldots$, such that the sequence of functions

$$f_{n_1}(z), f_{n_2}(z), \ldots \tag{1}$$

converges at the point z_1.

Similarly from this sequence of functions we can select a sub-sequence

$$f_{p_1}(z), f_{p_2}(z), \ldots \tag{2}$$

which converges at z_2; and then from this a sub-sequence

$$f_{q_1}(z), f_{q_2}(z), \ldots \tag{3}$$

which converges at z_3; and so on.

Now consider the sequence

$$f_{n_1}(z), f_{p_2}(z), f_{q_3}(z), \ldots$$

formed by taking the diagonal terms of the above double array. Each of these functions belongs to the sequence (1), and so the sequence converges at z_1; each function after the first belongs to (2), and so the sequence converges at z_2; and so on. Therefore the sequence converges at each of the points $z_1, z_2, \ldots$, and so, by Vitali's theorem, uniformly in any region interior to D.

5.23. Montel's theorem.* *Let $f(z)$ be an analytic function of z, regular in the half-strip S defined by $a < x < b$, $y > 0$. If $f(z)$ is bounded in S, and tends to a limit l, as $y \to \infty$, for a certain fixed value ξ of x between a and b, then $f(z)$ tends to this limit l on every line $x = x_0$ in S, and indeed $f(z) \to l$ uniformly for $a+\delta \leqslant x_0 \leqslant b-\delta$.*

Consider the sequence of functions $f_n(z) = f(z+in)$, where $n = 0, 1, 2, \ldots$, in the rectangle R defined by $a < x < b$, $0 < y < 2$. Then $f_n(z) \to l$ at every point of the line $x = \xi$. Hence, by Vitali's theorem, $f_n(z) \to l$ uniformly in any region interior to R, and in particular in the rectangle $a+\delta \leqslant x \leqslant b-\delta$, $\frac{1}{2} \leqslant y \leqslant \frac{3}{2}$. This proves the theorem.

The result may be generalized by means of conformal transformations. For example, let $z = i \log w$. Then the strip in the z-plane becomes an angle in the w-plane, and the theorem states

* Montel (1), Hardy (18), Bohr (4).

that if $\phi(w)$ is bounded in the angle $\alpha < \arg w < \beta$, and $\phi(w) \to l$ as $w \to \infty$ along any line $\arg w =$ constant between α and β, then $\phi(w) \to l$ uniformly in any angle $\alpha+\delta \leqslant \arg w \leqslant \beta-\delta$.

5.24. The following theorem illustrates another way in which the maximum-modulus theorem can be applied.

Let $f(z)$ be regular, and $|f(z)| \leqslant M$, in the circle $|z-a| \leqslant R$, and suppose that $f(a) \neq 0$. Then the number of zeros of $f(z)$ in the circle $|z-a| \leqslant \frac{1}{3}R$ does not exceed $A \log\{M/|f(a)|\}$.

We may suppose that $a = 0$. Let $z_1, z_2, \ldots, z_n$ be the zeros of $f(z)$ in $|z| \leqslant \frac{1}{3}R$, and let

$$g(z) = f(z) \Big/ \prod_{m=1}^{n} \left(1 - \frac{z}{z_m}\right).$$

Then $g(z)$ is regular for $|z| \leqslant R$, and on $|z| = R$ we have $|z/z_m| \geqslant 3$ for $m = 1, 2, \ldots, n$. Hence

$$|g(z)| \leqslant M \Big/ \prod_{m=1}^{n} (3-1) = 2^{-n} M$$

for $|z| = R$, and so also for $|z| < R$. In particular this is true for $z = 0$. Since $g(0) = f(0)$, it follows that

$$|f(0)| \leqslant 2^{-n} M,$$

and hence
$$n \leqslant \frac{1}{\log 2} \log \frac{M}{|f(0)|},$$
the desired result.

The factor $\frac{1}{3}$ can clearly be replaced by any number less than $\frac{1}{2}$. Actually a more complete result can be obtained from Jensen's theorem (§ 3.61). If the zeros in $|z| \leqslant R$ are $r_1, r_2, \ldots, r_N$, then

$$\log \frac{R^N}{r_1 r_2 \ldots r_N} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| \, d\theta - \log |f(0)|$$
$$\leqslant \log M - \log |f(0)|.$$

Let the zeros in the circle $|z| \leqslant \delta R$, where $0 < \delta < 1$, be $r_1, r_2, \ldots, r_n$. Then the left-hand side is not less than

$$\log \frac{R^n}{r_1 r_2 \ldots r_n} \geqslant \log \left(\frac{1}{\delta}\right)^n = n \log \frac{1}{\delta}.$$

Hence
$$n \leqslant \frac{1}{\log 1/\delta} \log \frac{M}{|f(0)|}.$$

5.3. Hadamard's three-circles theorem. *Let $f(z)$ be an analytic function, regular for $r_1 \leqslant |z| \leqslant r_3$. Let $r_1 < r_2 < r_3$, and let M_1, M_2, M_3 be the maxima of $|f(z)|$ on the three circles $|z| = r_1, r_2, r_3$ respectively. Then*

$$M_2^{\log(r_3/r_1)} \leqslant M_1^{\log(r_3/r_2)} M_3^{\log(r_2/r_1)}. \tag{1}$$

Let $\phi(z) = z^\lambda f(z)$, where λ is a constant to be determined later. Then $\phi(z)$ is regular in the ring-shaped region between $|z| = r_1$ and $|z| = r_3$, and $|\phi(z)|$ is one-valued. Hence the maximum of $|\phi(z)|$ occurs on one of the bounding circles, i.e.

$$|\phi(z)| \leqslant \max(r_1^\lambda M_1, r_3^\lambda M_3).$$

Hence on $|z| = r_2$

$$|f(z)| \leqslant \max(r_1^\lambda r_2^{-\lambda} M_1, r_3^\lambda r_2^{-\lambda} M_3). \tag{2}$$

We have now to choose λ to the best advantage, and this is done by making the two expressions in the bracket equal. We therefore define λ by the equation

$$r_1^\lambda M_1 = r_3^\lambda M_3.$$

Thus $$\lambda = -\{\log(M_3/M_1)\}/\{\log(r_3/r_1)\}.$$

With this value of λ, (2) gives

$$M_2 \leqslant (r_2/r_1)^{-\lambda} M_1,$$

and hence $$M_2^{\log(r_3/r_1)} \leqslant (r_2/r_1)^{\log(M_3/M_1)} M_1^{\log(r_3/r_1)}$$
$$= M_1^{\log(r_3/r_2)} M_3^{\log(r_2/r_1)},$$

the required result.

Notice that the case of equality can occur only if $\phi(z)$ is a constant, i.e. if $f(z)$ is a constant multiple of a power of z.

5.31. Convex functions. A function $\phi(x)$ of a real variable x is said to be *convex downwards*, or simply *convex*, if the curve $y = \phi(x)$ between x_1 and x_2 always lies below the chord joining the points $\{x_1, \phi(x_1)\}$ and $\{x_2, \phi(x_2)\}$. Analytically the condition is

$$\phi(x) < \frac{x_2 - x}{x_2 - x_1}\phi(x_1) + \frac{x - x_1}{x_2 - x_1}\phi(x_2) \qquad (x_1 < x < x_2). \tag{1}$$

The function is said to be convex in the wide sense if the sign of equality can also occur.

A convex function is continuous. For if we make $x \to x_1$ in (1), we obtain $\phi(x_1 + 0) \leqslant \phi(x_1)$; and if we make $x_2 \to x$ we obtain $\phi(x) \leqslant \phi(x+0)$. Hence $\phi(x) = \phi(x+0)$ for all values of x. Simi-