value

$$\leqslant \quad \frac{1}{\tau^2}\int_{a\tau}^{b\tau} KCs\,\mathrm{d}s \quad \leqslant \quad \tfrac{1}{2}KCb^2.$$

The increasing function $\nu_\rho(\tau)$ thus has a *bounded derivative* in $(0, \infty)$.

We may at this point integrate

$$2\Re\int_0^\infty \frac{z^2}{z^2-\tau^2}\,\frac{\nu_\rho(\tau)}{\tau}\,\mathrm{d}\tau$$

by parts in the direction opposite to the one taken previously, to get

$$\int_0^\infty \log\left|1 - \frac{z^2}{\tau^2}\right| \mathrm{d}\nu_\rho(\tau);$$

this, then, is equal to

$$\int_a^b F(\xi z)\,\frac{\rho(\xi)}{\xi}\,\mathrm{d}\xi$$

by the above work. Making $y \longrightarrow 0$ now causes the first of these two integrals to tend to

$$\int_0^\infty \log\left|1 - \frac{x^2}{\tau^2}\right| \mathrm{d}\nu_\rho(\tau)$$

(that is especially easy to see here, where $0 \leqslant \mathrm{d}\nu_\rho(\tau) \leqslant \text{const.}\,\mathrm{d}\tau$). The integral just written is therefore equal to $F_\rho(x)$ according to what was observed initially. We are done.

Lemma. *Let $\nu(t)$ be odd and increasing, with $\nu(t)/t$ bounded on* $(0, \infty)$, *and put*

$$F(x) \quad = \quad \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\nu(t) \qquad \text{for } x \in \mathbb{R}.$$

Suppose that

$$\int_{-\infty}^\infty \frac{|F(x)|}{1+x^2}\,\mathrm{d}x \quad < \quad \infty.$$

Then, for the Hilbert transform

$$\tilde{F}(x) \quad = \quad \frac{1}{\pi}\int_{-\infty}^\infty \left(\frac{1}{x-t} + \frac{t}{t^2+1}\right) F(t)\,\mathrm{d}t$$

of F, we have, with a certain constant A,

$$\tilde{F}(x) \quad = \quad Ax \; - \; \pi\nu(x) \qquad \text{a.e.,} \quad x \in \mathbb{R}.$$

Proof. Write

$$F(z) \quad = \quad \int_0^\infty \log\left|1 \; - \; \frac{z^2}{t^2}\right| \mathrm{d}\nu(t)$$

for complex z. The given conditions on ν then make

$$F(z) \;\leqslant\; F(\mathrm{i}|z|) \;\leqslant\; \text{const.}|z|.$$

Based on this relation and on the property of $|F(x)|$ assumed in the hypothesis we can, by an argument like one made during the proof of the *second* theorem in §B.1, establish for F the representation from §G.1 of Chapter III:

$$F(z) \quad = \quad A|\Im z| \quad + \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|\,F(t)}{|z-t|^2}\,\mathrm{d}t.$$

A *harmonic conjugate* $G(z)$ for $F(z)$ in the upper half plane is therefore given by the formula

$$G(z) \quad = \quad -A\Re z \quad + \quad \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\Re z - t}{|z-t|^2} \; + \; \frac{t}{t^2+1}\right)F(t)\,\mathrm{d}t.$$

This, then, must, to within an additive constant, agree with the *obvious* harmonic conjugate for $\Im z > 0$ of the original logarithmic potential defining $F(z)$. In other words,

$$G(z) \quad = \quad C \; + \; \int_0^\infty \arg\left(1 \; - \; \frac{z^2}{t^2}\right)\mathrm{d}\nu(t)$$

where, for $\Im z \; > \; 0$, we take the determination of the argument having $\arg 1 \; = \; 0$ in order to ensure convergence of the integral on the right.

For $\Im z \; > \; 0$ and $t > 0$, $\;1 \; - \; (z^2/t^2)$ lies in the following domain:

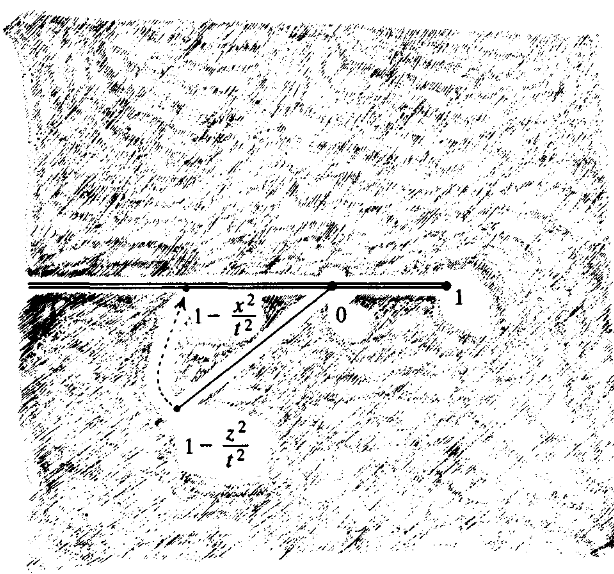

Figure 258

The branch of the argument we are using has in this domain a value between $-\pi$ and π. When z tends from the upper half plane to a given real $x > 0$, $\arg(1 - (z^2/t^2))$ tends to a *boundary value* equal to $-\pi$ for $0 < t < x$ (see the preceding figure) and to *zero* for $t > x$. As long, then, as $\nu(t)$ is *continuous* at such an x,

$$\int_0^\infty \arg\left(1 - \frac{z^2}{t^2}\right) \mathrm{d}\nu(t)$$

will tend to $-\pi\nu(x)$ as $z \longrightarrow x$ from the upper half plane. At the same time, $G(z)$ will, at almost every such x, tend to

$$-Ax + \tilde{F}(x)$$

as we know (see the scholium in §H.1 of Chapter III and problem 25 at the end of §C.1, Chapter VIII). Thus,

$$-Ax + \tilde{F}(x) = C - \pi\nu(x) \qquad \text{a.e.,} \quad x > 0.$$

On the negative real axis we find in the same way that

$$-Ax + \tilde{F}(x) = C + \pi\nu(|x|) \qquad \text{a.e.;}$$

the right side is, however, equal to $C - \pi\nu(x)$ there, ν being odd. Hence,

$$-Ax + \tilde{F}(x) = C - \pi\nu(x) \qquad \text{a.e.,} \quad x \in \mathbb{R}.$$

But $F(x)$ is even, making $\tilde{F}(x)$ odd, like $\nu(x)$. Therefore C must be zero, and

$$\tilde{F}(x) = Ax - \pi\nu(x) \qquad \text{a.e. on } \mathbb{R},$$

Q.E.D.

Lemma. *Let $F(x)$, even and $\mathscr{C}_2$, satisfy the condition*

$$\int_{-\infty}^{\infty} \frac{|F(x)|}{1+x^2}\,\mathrm{d}x < \infty,$$

and suppose that there is an increasing continuous odd function $\mu(x)$, $O(x)$ on $[0, \infty)$, such that $\tilde{F}(x) + \mu(x)$ is also increasing on $\mathbb{R}$, $\tilde{F}(x)$ being the Hilbert transform of F. Suppose, moreover, that $|\tilde{F}(x)/x|$ is bounded on $\mathbb{R}$. Then

$$F(x) = F(0) - \frac{1}{\pi}\int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\tilde{F}(t) \qquad \text{for } x \in \mathbb{R},$$

with the integral on the right (involving the signed *measure* $\mathrm{d}\tilde{F}(t)$ *) absolutely convergent.*

Remark. Boundedness of $\tilde{F}(x)/x$ on $\mathbb{R}$ actually *follows* from the rest of the hypothesis. That is the conclusion of the next lemma.

Proof of lemma. The given properties of $F(x)$ ensure* that $\tilde{F}(x)$ is at least $\mathscr{C}_1$, and that

$$f(z) \quad = \quad \frac{i}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{z-t} + \frac{t}{t^2+1}\right)F(t)\,dt,$$

analytic for $\Im z > 0$, is continuous up to the real axis, where it takes the boundary value

$$f(x) \quad = \quad F(x) + i\tilde{F}(x).$$

Since $F(x)$ is even, $\tilde{F}(x)$ is odd. Put

$$\nu(t) \quad = \quad \tilde{F}(t) + \mu(t),$$

then $\nu(t)$, like $\mu(t)$, is odd and continuous and, by hypothesis, increasing and $O(t)$ on $[0, \infty)$. For $\Im z \neq 0$ we can thus form the function

$$V(z) \quad = \quad \int_0^{\infty} \log\left|1 - \frac{z^2}{t^2}\right| d\nu(t);$$

it is harmonic in both the upper and the lower half planes. The same is true for

$$U(z) \quad = \quad \int_0^{\infty} \log\left|1 - \frac{z^2}{t^2}\right| d\mu(t).$$

These functions have, in $\Im z > 0$, the harmonic conjugates

$$\tilde{V}(z) \quad = \quad \int_0^{\infty} \arg\left(1 - \frac{z^2}{t^2}\right) d\nu(t),$$

$$\tilde{U}(z) \quad = \quad \int_0^{\infty} \arg\left(1 - \frac{z^2}{t^2}\right) d\mu(t)$$

* To show that $\tilde{F}(x)$ is $\mathscr{C}_1$ for $|x| < A$, say, take an even $\mathscr{C}_\infty$ function $\varphi(t)$ equal to 1 for $|t| \leqslant A$ and to 0 for $|t| \geqslant 2A$. Then, since $F(t)$ is also even, we have, for $|x| < A$,

$$\tilde{F}(x) \quad = \quad (2x/\pi)\int_A^{\infty}((1-\varphi(t))F(t)/(x^2-t^2))\,dt \quad + \quad (1/\pi)\int_{-2A}^{2A}(\varphi(t)F(t)/(x-t))\,dt.$$

The *first* expression on the right is clearly $\mathscr{C}_\infty$ in x for $|x| < A$. To the *second*, we apply the partial integration technique used often in this book, and get $(1/\pi)\int_{-2A}^{2A}\log|x-t|\;d(\varphi(t)F(t))$. Reason now as in the footnote to the theorem of §D.3. Since $d(\varphi(t)F(t))/dt$ is $\mathscr{C}_1$, the integral is also $\mathscr{C}_1$ (in x) for $|x| < A$.

(where the argument is determined so as to make $\arg 1 = 0$). Here, where $\nu(t)$ and $\mu(t)$ are continuous, we can argue as in the proof of the last lemma to show that $\tilde{V}(z)$ and $\tilde{U}(z)$ are continuous up to the real axis, where they take the boundary values

$$\tilde{V}(x) = -\pi\nu(x), \qquad \tilde{U}(x) = -\pi\mu(x).$$

Thus,

$$\tilde{V}(x) - \tilde{U}(x) = -\pi\tilde{F}(x).$$

Our assumptions on $\mu(x)$ are not strong enough to yield as much information about the behaviour of $U(z)$ (or of $V(z)$) at the points of $\mathbb{R}$. Consider, however, the *difference*

$$G(z) = V(z) - U(z).$$

Since $\tilde{F}(t) = \nu(t) - \mu(t)$, we can write

$$G(z) = \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}\tilde{F}(t)$$

for $\Im z \neq 0$; the integral on the right is, however, absolutely convergent *even when z is real*. To check this, take any $R > |z|$ and break up that integral into two pieces, the first over $[0, 2R]$ and the second over $[2R, \infty)$. Regarding the *first* portion, note that $\mathrm{d}\tilde{F}(t) = \tilde{F}'(t)\,\mathrm{d}t$ with $\tilde{F}'(t)$ continuous and hence *bounded* on finite intervals ($\tilde{F}(t)$ being $\mathscr{C}_1$); for the *second*, just use $|\mathrm{d}\tilde{F}(t)| \leqslant \mathrm{d}\nu(t) + \mathrm{d}\mu(t)$. In this way we also verify without difficulty that $G(z)$ is continuous up to (and *on*) $\mathbb{R}$, and takes there the boundary value

$$G(x) = \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\tilde{F}(t).$$

By this observation and the one preceding it we see that the function

$$g(z) = G(z) + \mathrm{i}(\tilde{V}(z) - \tilde{U}(z)),$$

analytic for $\Im z > 0$, is continuous up to the real axis where it has the boundary value

$$g(x) = G(x) - \mathrm{i}\pi\tilde{F}(x).$$

Bringing in now the function $f(z)$ described earlier, we can conclude that $\pi f(z) + g(z)$, analytic in $\Im z > 0$, is continuous up to $\mathbb{R}$ and assumes there the boundary value

$$\pi f(x) + g(x) = \pi F(x) + G(x).$$

The right side is obviously *real*, so we may use Schwarz reflection to continue $\pi f(z) + g(z)$ analytically across $\mathbb{R}$ and thus obtain an *entire function*. The latter's *real part*, $H(z)$, is hence *everywhere harmonic*, with

$$H(x) \quad = \quad \pi F(x) + G(x) \qquad \text{on } \mathbb{R}.$$

For $\Im z \neq 0$, we have $H(z) = H(\bar{z})$, so*

$$H(z) \quad = \quad \int_{-\infty}^{\infty} \frac{|\Im z| F(t)}{|z-t|^2} \mathrm{d}t \; + \; V(z) \; - \; U(z).$$

It is now claimed that $H(z)$ is a *linear function* of $\Re z$ and $\Im z$; this we verify by estimating the integrals $\int_{-\pi}^{\pi} (H(re^{i\vartheta}))^+ \mathrm{d}\vartheta$ for certain large values of r.

By the last relation, we have

$$(H(z))^+ \quad \leqslant \quad \int_{-\infty}^{\infty} \frac{|\Im z||F(t)|}{|z-t|^2} \mathrm{d}t \quad + \quad (V(z))^+ \quad + \quad (U(z))^-$$

for $\Im z \neq 0$. Consider first the *second* term on the right. Since $\nu(t)$ is increasing,

$$V(z) \quad \leqslant \quad \int_0^{\infty} \log\left(1 + \frac{|z|^2}{t^2}\right) \mathrm{d}\nu(t).$$

Here, $\nu(t) \leqslant$ const. t on $[0, \infty)$ by hypothesis, from which we deduce by the usual integration by parts that $V(z) \leqslant$ const.$|z|$, and thence that

$$\int_{-\pi}^{\pi} (V(re^{i\vartheta}))^+ \mathrm{d}\vartheta \quad \leqslant \quad \text{const.}\, r.$$

To estimate the circular means of the *third* term on the right we use the formula

$$\int_{-\pi}^{\pi} (U(re^{i\vartheta}))^- \mathrm{d}\vartheta \quad = \quad \int_{-\pi}^{\pi} (U(re^{i\vartheta}))^+ \mathrm{d}\vartheta \quad - \quad \int_{-\pi}^{\pi} U(re^{i\vartheta}) \mathrm{d}\vartheta$$

together with the inequality

$$U(z) \quad \leqslant \quad \text{const.}\, |z|,$$

analogous to the one for $V(z)$ just mentioned. For this procedure, a *lower* bound on $\int_{-\pi}^{\pi} U(re^{i\vartheta}) \mathrm{d}\vartheta$ is needed, and that quantity is indeed $\geqslant 0$, as we now show†. When $0 < t \leqslant r$,

* the *integral* in the next formula is just $\pi \Re f(z)$ when $\Im z > 0$

† one may also just refer to the subharmonicity of $U(z)$

$|1-(re^{i\vartheta}/t)^2|^2 = (1-r^2/t^2)^2 + 4(r/t)^2\sin^2\vartheta \geqslant |1-e^{2i\vartheta}|^2$, so, since $\mu(t)$ increases,

$$\begin{aligned} U(re^{i\vartheta}) &= \int_0^\infty \log\left|1 - \frac{r^2 e^{2i\vartheta}}{t^2}\right| d\mu(t) \\ &\geqslant \mu(r)\log|1-e^{2i\vartheta}| + \int_r^\infty \log\left|1 - \frac{r^2 e^{2i\vartheta}}{t^2}\right| d\mu(t). \end{aligned}$$

Integration of the two right-hand terms from $-\pi$ to π now presents no difficulty (Fubini's theorem being applicable to the second one), and both of the results are zero. Thus, $\int_{-\pi}^{\pi} U(re^{i\vartheta})d\vartheta \geqslant 0$ which, substituted with $(U(re^{i\vartheta}))^+ \leqslant \text{const.}\,r$ into the previous relation, yields

$$\int_{-\pi}^{\pi} (U(re^{i\vartheta}))^- d\vartheta \leqslant \text{const.}\,r.$$

Examination of the *first* right-hand term in the above inequality for $(H(z))^+$ remains. To estimate the circular means of that term – call it $P(z)$ – one argues as in the proof of the first theorem from §B.3, leaning heavily on the convergence of $\int_{-\infty}^{\infty}(|F(t)|/(1+t^2))dt$ (without which, it is true, $P(z)$ would be infinite!). In that way, one finds that

$$\int_{-\pi}^{\pi} P(r_n e^{i\vartheta})d\vartheta \leqslant \text{const.}\,r_n$$

for a certain sequence of r_n tending to ∞.

Combining our three estimates, we get

$$\int_{-\pi}^{\pi} (H(r_n e^{i\vartheta}))^+ d\vartheta \leqslant \text{const.}\,r_n,$$

and from this we can deduce as in the proof just referred to that

$$H(z) = H(0) + A\Re z + B\Im z,$$

thus verifying the above claim.

For $x \in \mathbb{R}$, the last relation reduces to $\pi F(x) + G(x) = H(0) + Ax$. Here, $F(x)$ and $G(x)$ are both even, so $A = 0$, and

$$\pi F(x) = H(0) - \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\tilde{F}(t).$$

The integral on the right vanishes for $x = 0$, so $H(0) = \pi F(0)$, and finally

$$F(x) = F(0) - \frac{1}{\pi}\int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\tilde{F}(t)$$

for $x \in \mathbb{R}$, as required.

Lemma. *Let $F(x)$ be as in the preceding lemma, and suppose that for a certain continuous and increasing odd function $\mu(x)$,* $O(x)$ *on* $[0, \infty)$, *the sum $\tilde{F}(x) + \mu(x)$ is also increasing. Then $|\tilde{F}(x)/x|$ is bounded on* $\mathbb{R}$.

Proof. Is for the most part nothing but a crude version of the argument made in §H.2 of Chapter III.

It is really only the boundedness of $|\tilde{F}(x)/x|$ for *large* $x > 0$ that requires proof. That's because our assumptions on F make $\tilde{F}(x)$ odd and $\mathscr{C}_1$, and hence $\tilde{F}(x)/x$ even, and bounded near 0.

In order to see what happens for large values of x, we resort to Kolmogorov's theorem from Chapter III, §H.1, according to which

$$\int_{|\tilde{F}(x)| > \lambda} \frac{dx}{1 + x^2} \leqslant \frac{K}{\lambda} \qquad \text{for } \lambda > 0,$$

K being a certain constant depending on F. In this relation, put $\lambda = 5K \cdot 2^n$ with $n \geqslant 1$; then, since

$$\int_{2^n}^{2^{n+1}} \frac{dx}{1 + x^2} > \frac{1}{5 \cdot 2^n},$$

there must, in each open interval $(2^n, 2^{n+1})$, be a point x_n with

$$|\tilde{F}(x_n)| \leqslant 5K \cdot 2^n.$$

By hypothesis, the functions $\mu(x)$ and $\tilde{F}(x) + \mu(x)$ are increasing, so for $x_n \leqslant x \leqslant x_{n+1}$ we have

$$\begin{aligned} -5K \cdot 2^n + \mu(2^n) &\leqslant \tilde{F}(x_n) + \mu(x_n) \leqslant \tilde{F}(x) + \mu(x) \\ &\leqslant \tilde{F}(x_{n+1}) + \mu(x_{n+1}) \leqslant 5K \cdot 2^{n+1} + \mu(2^{n+2}), \end{aligned}$$

whence

$$-5K \cdot 2^n - \mu(x) \leqslant \tilde{F}(x) \leqslant 5K \cdot 2^{n+1} + \mu(2^{n+2}),$$

from which

$$-5K - \frac{\mu(x)}{x} \leqslant \frac{\tilde{F}(x)}{x} \leqslant 10K + 4\frac{\mu(2^{n+2})}{2^{n+2}}$$

in view of the relation $2^n < x_n < x_{n+1} < 2^{n+2}$.

It was also given that $\mu(t) \leqslant Ct$ on $[0, \infty)$. Thence,

$$\left|\frac{\tilde{F}(x)}{x}\right| \leqslant 10K + 4C$$

for $x_n \leqslant x \leqslant x_{n+1}$, and thus finally for all $x \geqslant x_1$.

Done.

We will need, finally, a simple extension of the Jensen formula for confocal ellipses derived in §C of Chapter IX.

Lemma. *Let $F(z)$ be subharmonic in and on a simply connected closed region $\bar{\Omega}$ containing the ellipse*

$$z = \frac{1}{2}\left(Re^{i\vartheta} + \frac{e^{-i\vartheta}}{R}\right), \qquad 0 \leqslant \vartheta \leqslant 2\pi,$$

in its interior, where $R > 1$. Suppose that μ is the positive measure on $\bar{\Omega}$ figuring in the Riesz representation of the superharmonic function $-F(z)$ in Ω, the interior of $\bar{\Omega}$, in other words, that

$$F(z) = \int_{\bar{\Omega}} \log|z-\zeta|\,d\mu(\zeta) + h(z) \qquad \text{for } z \in \Omega,$$

where $h(z)$ is harmonic *in Ω (see problem* 49, §A.2*). If, then, $M(r)$ denotes, for $1 < r \leqslant R$, the* total mass *μ has* inside or on the ellipse

$$z = \frac{1}{2}\left(re^{i\vartheta} + \frac{e^{-i\vartheta}}{r}\right),$$

we have

$$\int_1^R \frac{M(r)}{r}\,dr = \frac{1}{2\pi}\int_{-\pi}^{\pi} F\left(\frac{1}{2}\left(Re^{i\vartheta} + \frac{e^{-i\vartheta}}{R}\right)\right)d\vartheta - \frac{1}{\pi}\int_{-1}^{1} \frac{F(x)}{\sqrt{(1-x^2)}}\,dx.$$

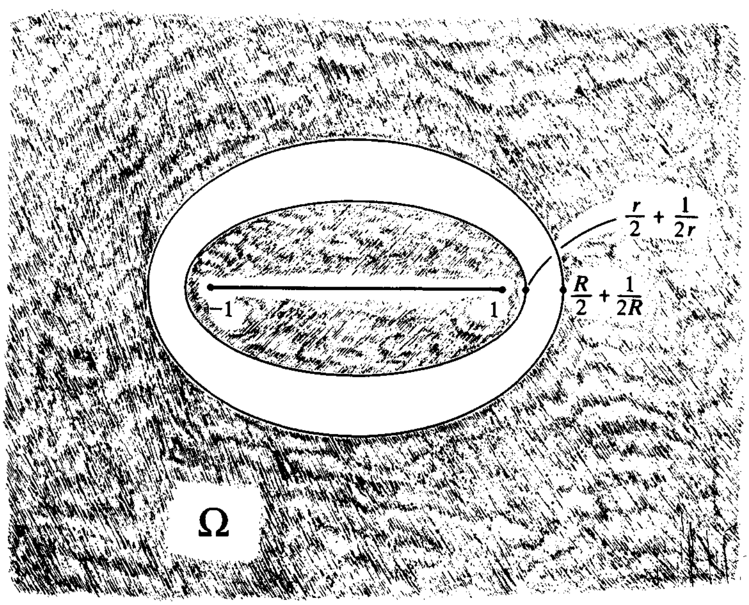

Figure 259

Proof. It is simplest to just *derive* this result from the theorem of Chapter IX, §C by double integration.

Fix, for the moment, any point $\zeta \in \bar{\Omega}$ with

$$\zeta = \frac{1}{2}\left(\rho e^{i\varphi} + \frac{e^{-i\varphi}}{\rho}\right)$$

where $\rho \geqslant 1$, and observe that (in case $\rho > 1$), we have

$$\rho = |\zeta + \sqrt{(\zeta^2 - 1)}|,$$

taking the proper determination of the square root for ζ outside the segment $[-1, 1]$.

Apply now the theorem referred to with the *analytic* function of z *equal to* $z - \zeta$ (!), getting

$$\int_{|\zeta+\sqrt{(\zeta^2-1)}|}^{R} \frac{dr}{r} = \int_{\rho}^{R} \frac{dr}{r} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|\frac{1}{2}\left(Re^{i\vartheta} + \frac{e^{-i\vartheta}}{R}\right) - \zeta\right| d\vartheta$$
$$- \frac{1}{\pi}\int_{-1}^{1} \frac{\log|x-\zeta|}{\sqrt{(1-x^2)}}\,dx,$$

the very first integral on the left being understood as zero for

$$|\zeta + \sqrt{(\zeta^2-1)}| \geqslant R.$$

Multiply the last relation by $d\mu(\zeta)$ and integrate over $\bar{\Omega}$. On the *left* we will get

$$\int_{\bar{\Omega}}\int_{|\zeta+\sqrt{(\zeta^2-1)}|}^{R} \frac{dr}{r}\,d\mu(\zeta) = \int_{1}^{R}\int_{1\leqslant|\zeta+\sqrt{(\zeta^2-1)}|\leqslant r} d\mu(\zeta)\,\frac{dr}{r} = \int_{1}^{R}\frac{M(r)}{r}\,dr,$$

and on the *right*, after changing the order of integration in both integrals,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi\left(\frac{1}{2}\left(Re^{i\vartheta} + \frac{e^{-i\vartheta}}{R}\right)\right)d\vartheta - \frac{1}{\pi}\int_{-1}^{1}\frac{\Phi(x)}{\sqrt{(1-x^2)}}\,dx,$$

where

$$\Phi(z) = \int_{\bar{\Omega}} \log|z-\zeta|\,d\mu(\zeta).$$

Our given subharmonic function $F(z)$ is equal to $\Phi(z) + h(z)$. Since $h(z)$ is harmonic in the *simply connected* region Ω, it has a harmonic conjugate $\tilde{h}(z)$ there, and the function

$$f(z) = \exp(h(z) + i\tilde{h}(z))$$

is *analytic and without zeros in* Ω. Apply the theorem of §C, Chapter IX, once more, this time to $f(z)$. Because $\log|f(z)| = h(z)$, we get

$$0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} h\left(\frac{1}{2}\left(Re^{i\vartheta} + \frac{e^{-i\vartheta}}{R}\right)\right)d\vartheta - \frac{1}{\pi}\int_{-1}^{1}\frac{h(x)}{\sqrt{(1-x^2)}}dx.$$

Adding the right side of this relation to the previous similar expression involving Φ equal, as we have seen, to

$$\int_1^R \frac{M(r)}{r}dr$$

now gives us the desired result.

Done.

2. Proof of the conjecture from §D.5

This book is coming to an end. Let us get on with the establishment of our conjecture which, after rephrasing, reads a bit more smoothly. One can, of course, write

$$\max(\tilde{\omega}'(x), K) - K$$

as

$$(\tilde{\omega}'(x) - K)^+.$$

Then the result we have in mind may be stated as follows:

Theorem. *Let $W(x) \geqslant 1$ be a weight meeting the local regularity requirement of* §B.1. *A necessary and sufficient condition for W to admit multipliers is that there exist an even $\mathscr{C}_\infty$ function $\omega(x)$, defined on $\mathbb{R}$, with*

$$\log W(x) \leqslant \omega(x)$$

there, such that

(i) $$\int_{-\infty}^{\infty}\frac{\omega(x)}{1+x^2}dx < \infty;$$

(ii) *the ($\mathscr{C}_\infty$) Hilbert transform $\tilde{\omega}(x)$ of ω has the following property:*

to any $\delta > 0$ there corresponds a K with

$$\tilde{\omega}'(x) \leqslant K$$

everywhere in $(0, \infty)$ outside a set of disjoint intervals (a_n, b_n), $a_n > 0$, such that

$$\sum_n \left(\frac{b_n - a_n}{a_n}\right)^2 \quad < \quad \infty,$$

for each of which

$$\int_{a_n}^{b_n} (\tilde{\omega}'(x) \;-\; K)^+ \, \mathrm{d}x \quad \leqslant \quad \delta(b_n - a_n).$$

Remark. For any $\mathscr{C}_\infty$ function $\omega(x) \geqslant \log W(x)$ (and thus $\geqslant 0$) satisfying (i), we have

$$\tilde{\omega}'(x) \quad = \quad \frac{1}{\pi}\int_0^\infty \frac{2\omega(x) - \omega(x+t) - \omega(x-t)}{t^2}\,\mathrm{d}t.$$

This is practically immediate, for then the functions

$$\omega(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z\, \omega(t)}{|z-t|^2}\,\mathrm{d}t,$$

$$\tilde{\omega}(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \left(\frac{\Re z - t}{|z-t|^2} \;+\; \frac{t}{t^2+1}\right)\omega(t)\,\mathrm{d}t,$$

both harmonic in $\Im z \;>\; 0$, have there partial derivatives *continuous up to* $\mathbb{R}$ (in the present circumstances). Thence, by the Cauchy–Riemann equations,

$$\tilde{\omega}'(x) \quad = \quad \lim_{y\to 0+} \frac{\partial\tilde{\omega}(x+\mathrm{i}y)}{\partial x} \quad = \quad -\lim_{y\to 0+} \frac{\partial\omega(x+\mathrm{i}y)}{\partial y}$$

$$= \quad \lim_{y\to 0+} \frac{\omega(x) - \omega(x+\mathrm{i}y)}{y},$$

and the last limit is clearly equal to the integral in question.

Proof of theorem

1⁰ The necessity. As we saw at the very beginning of §C, there is no loss of generality incurred in taking

$$W(x) \longrightarrow \infty \quad \text{for } x \longrightarrow \pm\infty\,;$$

this property we henceforth assume.

If $W(x)$ *admits multipliers* there is, corresponding to *any* $a > 0$, a non-zero entire function $f(z)$ of exponential type $\leqslant a$ with

$$W(x)|f(x)| \;\leqslant\; \text{const.} \quad \text{for } x \in \mathbb{R}.$$

If peradventure $f(0) = 0$, the quotient $f(x)/x$ satisfies the same kind of relation, for $W(x)$ must be bounded on finite intervals by the first lemma of §B.1. We can thus first divide $f(z)$ by whatever power of z is needed to get an entire function different from zero at the origin, and may therefore just as well *assume to begin with that*

$$f(0) = 1.$$

The *even* entire function $g(z) = f(z)f(-z)$ is then also 1 at the origin. It has exponential type $\leqslant 2a$, and we have

$$W(x)\,W(-x)\,|g(x)| \leqslant \text{const.}, \qquad x \in \mathbb{R}.$$

We know by the discussion following the first theorem in §B.1 that $g(z)$ may be taken to have *all its zeros real*, thus being of the form

$$g(z) = \prod_1^\infty \left(1 - \frac{z^2}{\lambda_k^2}\right),$$

where the λ_k are certain numbers > 0. The preceding relation then reads

$$\log W(x) + \log W(-x) + \sum_1^\infty \log\left|1 - \frac{x^2}{\lambda_k^2}\right| \leqslant \text{const.}, \qquad x \in \mathbb{R};$$

it is from the expressions

$$\log|g(x)| = \sum_1^\infty \log\left|1 - \frac{x^2}{\lambda_k^2}\right|$$

corresponding to entire functions g having smaller and smaller exponential type that we will construct a function $\omega(x)$ having the properties affirmed by our theorem.

Take, then, a sequence of entire functions $g_n(z)$, each of the form just indicated, such that

$$\log W(x) + \log W(-x) + \log|g_n(x)| \leqslant \Gamma_n, \qquad \text{say,}$$

for $x \in \mathbb{R}$, while the exponential type α_n of g_n tends monotonically to zero as $n \longrightarrow \infty$. By passing to a subsequence if necessary, we arrange matters so as to also have

$$\sum_n \alpha_n < \infty.$$

Denoting by $\mu_n(t)$ the *number of zeros of* $g_n(z)$ on the segment $[0, t]$

(which makes $\mu_n(t) = O(t)$ on $[0, \infty)$ for each n), we have

$$\log W(x) + \log W(-x) + \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\mu_n(t) \leqslant \Gamma_n, \qquad x \in \mathbb{R}.$$

Here, $\log W(-x) \longrightarrow \infty$ for $x \longrightarrow \pm\infty$, so for each n there is a number $A_n > 0$ with

$$\log W(-x) \geqslant \Gamma_n \qquad \text{for } |x| \geqslant A_n.$$

It follows then from the last relation that

$$\log W(x) + \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\mu_n(t) \leqslant 0 \qquad \text{for } |x| \geqslant A_n.$$

We can evidently choose the A_n successively so as to have

$$A_{n+1} > 2A_n;$$

this property will be assumed to hold from now on.

For each n, let

$$\nu_n(t) = \begin{cases} 0, & 0 \leqslant t < A_n/\sqrt{2}, \\ \mu_n(t), & t \geqslant A_n/\sqrt{2}; \end{cases}$$

like the $\mu_n(t)$, the $\nu_n(t)$ are each *increasing* and $O(t)$ on $[0, \infty)$. We then put

$$F_n(x) = -\int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\nu_n(t) \qquad (\textit{sic}\,!),$$

and claim in the first place that

$$F_n(x) \geqslant 0 \qquad \text{for } x \in \mathbb{R}.$$

This is true when $|x| \leqslant A_n$, for then $|1 - (x^2/t^2)| \leqslant 1$ for the values of t (all $\geqslant A_n/\sqrt{2}$) that are actually involved in the preceding integral. For $|x| > A_n$, use the evident formula

$$F_n(x) = \int_0^{A_n/\sqrt{2}} \left\{\log\left(\frac{x^2}{t^2} - 1\right) - \log\left(\frac{2x^2}{A_n^2} - 1\right)\right\} d\mu_n(t)$$
$$- \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\mu_n(t).$$

The *first* integral on the right is here clearly $\geqslant 0$, and the *second* $\geqslant \log W(x) \geqslant 0$ by the above inequality. This establishes the claim, and shows besides that

$$F_n(x) \geqslant \log W(x) \qquad \text{for } |x| \geqslant A_n.$$

In order to get the function $\omega(x)$, we first smooth out each of the $F_n(x)$ by multiplicative convolution, relying on the fulfillment by $W(x)$ of the local regularity requirement*. According to the latter, there are three constants C, L and $k \geqslant 0$, the first two > 0, such that, for any $x \in \mathbb{R}$,

$$\log W(x) \leqslant C \log W(t) + k$$

for the t belonging to a certain interval of length L containing the point x.

Choose, for each n, a small number $\eta_n > 0$ less than both of the quantities

$$\frac{A_n}{2A_{n+1}} \quad \text{and} \quad \frac{L}{4A_{n+1}};$$

it is convenient to also have the η_n tend monotonically towards 0 as $n \longrightarrow \infty$. Take then a sequence of *infinitely differentiable functions* $\rho_n(\xi) \geqslant 0$ with ρ_n supported on the interval $[1-\eta_n,\ 1+\eta_n]$, such that

$$\int_{1-\eta_n}^{1} \frac{\rho_n(\xi)}{\xi}\,d\xi = \int_{1}^{1+\eta_n} \frac{\rho_n(\xi)}{\xi}\,d\xi = 1.$$

When

$$0 \leqslant x \leqslant \frac{L}{2\eta_n},$$

the points ξx with $1-\eta_n \leqslant \xi \leqslant 1$ are included in the segment $[x - L/2,\ x]$ and the ones with $1 \leqslant \xi \leqslant 1+\eta_n$ in the segment $[x,\ x + L/2]$. One of those segments surely lies in the interval of length L containing x on which the preceding relation involving $\log W(t)$ does hold. By that relation and the specifications for ρ_n we thus have

$$\log W(x) \leqslant C \int_0^\infty \log W(\xi x) \frac{\rho_n(\xi)}{\xi}\,d\xi + k \qquad \text{for } 0 \leqslant x \leqslant \frac{L}{2\eta_n}.$$

If, however, x is also $\geqslant 2A_n$, $\log W(\xi x)$ will, by the inequality found above, be $\leqslant F_n(\xi x)$ for $1-\eta_n \leqslant \xi \leqslant 1+\eta_n$, since then

* This is not really needed here. See next footnote.

$\xi x \geqslant 2A_n - 2A_{n+1}\eta_n \geqslant A_n$. The *right side* of the last relation is therefore

$$\leqslant \quad C\int_0^\infty F_n(\xi x)\,\frac{\rho_n(\xi)}{\xi}\,\mathrm{d}\xi \quad + \quad k$$

for such x. This expression is hence $\geqslant \log W(x)$ for $2A_n \leqslant x \leqslant 2A_{n+1}$ because $2A_{n+1} \leqslant L/2\eta_n$. Exactly the same reasoning* can be used for negative real x, and we have

$$\log W(x) \quad \leqslant \quad C\int_0^\infty F_n(\xi x)\,\frac{\rho_n(\xi)}{\xi}\,\mathrm{d}\xi \quad + \quad k \qquad \text{for } 2A_n \leqslant |x| \leqslant 2A_{n+1}.$$

Put now

$$G_n(x) \quad = \quad 2C\int_0^\infty F_n(\xi x)\,\frac{\rho_n(\xi)}{\xi}\,\mathrm{d}\xi.$$

We have $G_n(0) = 0$, and

$$G_n(x) \geqslant 0, \qquad x \in \mathbb{R}.$$

Also, since $W(x) \longrightarrow \infty$ for $x \longrightarrow \pm\infty$, the preceding relation implies that

$$\log W(x) \leqslant G_n(x) \qquad \text{for } 2A_n \leqslant |x| \leqslant 2A_{n+1}$$

as long as n exceeds a certain number n_0. We can, of course, arrange to have this inequality hold for *all* n by simply *throwing away* the G_n and A_n with $n \leqslant n_0$ and re-indexing the *remaining* ones. *This we henceforth suppose done.*

By the first lemma of article 1,

$$G_n(x) \quad = \quad -\int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\sigma_n(t)$$

with an increasing function $\sigma_n(t)$, *infinitely differentiable* in $(0,\infty)$, given by the formula

$$\sigma_n(t) \quad = \quad 2C\int_0^\infty \nu_n(\xi t)\,\frac{\rho_n(\xi)}{\xi}\,\mathrm{d}\xi.$$

* As long as $\log W(x)$ is *uniformly continuous* on $[A_n, 3A_{n+1}]$, say, the argument just made will go through (with $C = 1$) for small enough η_n. The necessity proof now under way *therefore works whenever* $W(x) \geqslant 1$ *is continuous on* $\mathbb{R}$. For this, we do not even *need* the entire functions $g_n(z)$ but *only* a sequence of (not necessarily integer-valued) functions $\mu_n(t)$ increasing on $[0,\infty]$ and $O(t)$ there, with $\limsup_{t\to\infty}(\mu_n(t)/t) = \alpha_n/\pi$ going to zero as $n \longrightarrow \infty$ and $\log W(x) + \log W(-x) + \int_0^\infty \log|1 - (x^2/t^2)|\mathrm{d}\mu_n(t)$ bounded above on $\mathbb{R}$ for each n.

In the present circumstances,

$$\nu_n(t) \;=\; 0 \qquad \text{for } 0 \leqslant t < A_n/\sqrt{2},$$

so

$$\sigma_n(t) \;=\; 0 \qquad \text{for } 0 \leqslant t \leqslant A_n/(\sqrt{2}\,(1+\eta_n)),$$

ρ_n being supported on $[1-\eta_n,\ 1+\eta_n]$. Therefore, if we extend $\sigma_n(t)$ to *the whole real axis* by making it *odd*, we get a function which is actually $\mathscr{C}_\infty$ on $\mathbb{R}$. It follows also by the lemma referred to that $\sigma_n'(t)$ is, for each n, *bounded in* $(0, \infty)$, indeed, *bounded on* $\mathbb{R}$ after $\sigma_n(t)$ is extended in the way just mentioned.

For our function $\omega(x)$ we will take the *sum of the* $G_n(x)$ and an additive constant. In order to verify that that function enjoys the properties it should, we shall need some bounds on the $G_n(x)$ and the $\sigma_n(t)$. To obtain those bounds, we must go back and look again at the entire functions $g_n(z)$ of exponential type α_n with which we started.

Because $W(x) \geqslant 1$, we certainly have

$$|g_n(x)| \;\leqslant\; e^{\Gamma_n} \qquad \text{for } x \in \mathbb{R},$$

and Levinson's theorem from §H.2 of Chapter III can be applied to the g_n to yield

$$\frac{\mu_n(t)}{t} \longrightarrow \frac{\alpha_n}{\pi}, \qquad t \longrightarrow \infty.$$

(One could in fact make do with a less elaborate result here.) Hence, since $\nu_n(t) = \mu_n(t)$ for $t \geqslant A_n/\sqrt{2}$,

$$\frac{\nu_n(t)}{t} \longrightarrow \frac{\alpha_n}{\pi} \qquad \text{for } t \longrightarrow \infty$$

and thence, by definition of σ_n,

$$\limsup_{t\to\infty} \frac{\sigma_n(t)}{t} \;\leqslant\; \frac{4}{\pi}C(1+\eta_n)\alpha_n,$$

in view of ρ_n's vanishing outside of $[1-\eta_n,\ 1+\eta_n]$ and the condition that

$$\int_0^\infty \frac{\rho_n(\xi)}{\xi}\,d\xi \;=\; 2.$$

Let us now extend the definition of our function G_n to the whole complex plane by taking

$$G_n(z) \quad = \quad -\int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}\sigma_n(t);$$

this function is harmonic both in $\{\Im z > 0\}$ and in $\{\Im z < 0\}$, and, thanks to the smoothness of $\sigma_n(t)$, *continuous* up to the real axis. The previous relation for $\sigma_n(t)/t$ makes

$$G_n(z) \quad \geqslant \quad -4C(1+\eta_n)\alpha_n|z| \;-\; \mathrm{o}(|z|)$$

for large $|z|$, so, putting

$$\beta_n \quad = \quad \limsup_{y\to\infty}\left(\frac{-G_n(\mathrm{i}y)}{y}\right),$$

we have

$$\beta_n \quad \leqslant \quad 4C(1+\eta_n)\alpha_n .$$

As we know, $G_n(x) \geqslant 0$ on the real axis. This, together with the other properties of $G_n(z)$, ensures that an analogue of the representation from §G.1 of Chapter III is valid for that function, viz.,

$$G_n(z) \quad = \quad -\beta_n\Im z \quad + \quad \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z\, G_n(t)}{|z-t|^2}\mathrm{d}t, \qquad \Im z \;>\; 0.$$

(Cf. proof of the first lemma in §B.3. The more elaborate argument made during the proof of the second theorem in §B.1 is not needed here.)

According to the above observations about $\sigma_n(t)$, that function is certainly *zero* in a neighborhood of the origin. That, however, makes

$$G_n(z) \;=\; \mathrm{O}(|z|^2) \qquad \text{for } z \longrightarrow 0,$$

whence

$$\frac{G_n(\mathrm{i}y)}{y} \longrightarrow 0 \qquad \text{as } y \longrightarrow 0.$$

Referring to the preceding Poisson representation, we see from this that

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{G_n(t)}{t^2}\mathrm{d}t \quad = \quad \beta_n$$

for the *positive* function $G_n(t)$.

It is now easy to get a *uniform* bound on $\sigma_n(t)/t$. Since $G_n(x) \geqslant 0$ on the real axis and $G_n(z) = G_n(\bar{z})$, the above Poisson representation tells us in particular that $G_n(z) \geqslant -\beta_n|\Im z|$, in other words, that

$$\int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| d\sigma_n(t) \quad \leqslant \quad \beta_n|\Im z|.$$

Using this relation we can now show by a computation like the one involved in the proof of the *second* lemma from §C.5, Chapter VIII ($\sigma_n(t)$ being increasing) that

$$\frac{\sigma_n(t)}{t} \quad \leqslant \quad \frac{e\beta_n}{\pi} \qquad \text{for } t > 0.$$

We proceed to the construction and examination of our function $\omega(x)$. By the first lemma of §B.1, $\log W(x)$ is *bounded above* for $|x| \leqslant 2A_1$; we *fix* any such bound and use it as our value for $\omega(0)$. Then we put

$$\omega(x) \quad = \quad \omega(0) \; + \; \sum_1^\infty G_n(x);$$

because the $G_n(x)$ are $\geqslant 0$ on $\mathbb{R}$ with $\log W(x) \leqslant G_n(x)$ for $2A_n \leqslant |x| \leqslant 2A_{n+1}$, and $\log W(x) \leqslant \omega(0)$ for $|x| \leqslant 2A_1$, we certainly have

$$\log W(x) \;\leqslant\; \omega(x), \qquad x \in \mathbb{R}.$$

The function $\omega(x)$ is clearly even. Since $\beta_n \leqslant 4C(1+\eta_n)\alpha_n$ with $\eta_n \longrightarrow 0$ for $n \longrightarrow \infty$ and the sum of the α_n convergent, we have

$$\sum_1^\infty \beta_n \; < \; \infty,$$

whence, by the previous estimate of the integrals of the $G_n(t)/t^2$ and monotone convergence,

$$\int_{-\infty}^\infty \frac{\omega(x)}{1+x^2}\,dx \quad < \quad \infty.$$

Property (i) *thus holds for* ω.

To verify *infinite differentiability* for $\omega(x)$, we take

$$\sigma(t) \quad = \quad \sum_1^\infty \sigma_n(t)$$

and look at the increasing function $\sigma(t)$. The series on the right is surely convergent; we have indeed

$$\frac{\sigma(t)}{t} \quad \leqslant \quad \frac{\mathrm{e}}{\pi}\sum_{1}^{\infty}\beta_n,$$

a finite quantity, for $t > 0$, thanks to the uniform bounds on the ratios $\sigma_n(t)/t$ found above. The function $\sigma(t)$ is actually *zero* for $0 \leqslant t \leqslant A_1/(\sqrt{2}\,(1+\eta_1))$, since all of the $\sigma_n(t)$ are (here is where we use the property that the η_n *decrease*). The summand $\sigma_n(t)$ is moreover different from zero only when $|t| \geqslant A_n/(\sqrt{2}\,(1+\eta_n))$, *a large number for large n.* Hence, since for any *given* real x, $\log|1-(x^2/t^2)|$ is $\leqslant 0$ for $t \geqslant \sqrt{2}\,|x|$, we see – again by monotone convergence – that

$$\sum_{1}^{\infty} G_n(x) \quad = \quad -\int_0^{\infty} \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\sigma(t), \qquad x \in \mathbb{R}.$$

Therefore

$$\omega(x) \quad = \quad \omega(0) \; - \int_0^{\infty} \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\sigma(t).$$

Here, however, $\sigma(t)$ is infinitely differentiable on $\mathbb{R}$ and odd there; this is so because each individual $\sigma_n(t)$ is odd and $\mathscr{C}_\infty$ as noted above, and, on any given finite interval, *only finitely many* of the $\sigma_n(t)$ can be different from zero. We may therefore conclude that $\omega(x)$ is also $\mathscr{C}_\infty$ by invoking the result proved in the footnote to the theorem of §D.3.

Verification of property (ii) for the function $\omega(x)$ remains; that is more involved. What we have to do is look at the *size* of $\tilde{\omega}'(x)$ for $x > 0$.

We have

$$\omega(0) \; - \; \omega(x) \quad = \quad \int_0^{\infty} \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\sigma(t).$$

The ratio $\sigma(t)/t$ is bounded, and property (i) holds for ω. Therefore the *second* lemma from article 1 applies here, and tells us that

$$\tilde{\omega}(x) \quad = \quad \pi\sigma(x) \quad = \quad \pi\sum_{1}^{\infty}\sigma_n(x).$$

Thus,

$$\tilde{\omega}'(x) \quad = \quad \pi\sum_1^\infty \sigma_n'(x)$$

(with, on any *bounded* interval, only *finitely many terms* actually appearing in the sum on the right).

Let $\delta > 0$ be given. Corresponding to it, we have an N such that

$$\mathrm{e}\sum_N^\infty \beta_n \;<\; \delta;$$

fixing such an N, we form the increasing function

$$s(t) \quad = \quad \sum_N^\infty \sigma_n(t)$$

and investigate its behaviour for $t \geqslant 0$.

That function is, in the first place, *zero* for $0 \;\leqslant\; t \;\leqslant\; A_N/(\sqrt{2}\,(1+\eta_N))$. Also,

$$\frac{s(t)}{t} \quad \leqslant \quad \frac{\mathrm{e}}{\pi}\sum_N^\infty \beta_n \quad < \quad \frac{\delta}{\pi} \qquad \text{for } t > 0$$

by the above uniform estimate on the ratios $\sigma_n(t)/t$. Therefore, if $x > 0$ is *sufficiently small*, the ratio

$$\frac{s(t) \;-\; s(x)}{t \;-\; x}$$

will be *zero* for $0 \;\leqslant\; t \;\leqslant\; A_N/(\sqrt{2}\,(1+\eta_N)\,)$, and, for *larger* values of t,

$$\leqslant \quad \frac{A_N}{A_N \;-\; \sqrt{2}\,(1+\eta_N)x}\,\frac{s(t)}{t} \quad \leqslant \quad \frac{A_N}{A_N \;-\; \sqrt{2}\,(1+\eta_N)x}\cdot\frac{\mathrm{e}}{\pi}\sum_N^\infty \beta_n,$$

and hence $<\; \delta/\pi$. Doing, then, the F. Riesz construction on the graph of $s(x)$ vs. x for $x \geqslant 0$, and forming the open set

$$\mathcal{O} \quad = \quad \left\{x > 0\colon\; \frac{s(t)-s(x)}{t-x} \quad > \quad \frac{\delta}{\pi} \quad \text{for } some\ t > x\right\},$$

we see that $\mathcal{O}$ *can contain no points to the left of a certain* $a_0 \;>\; 0$.

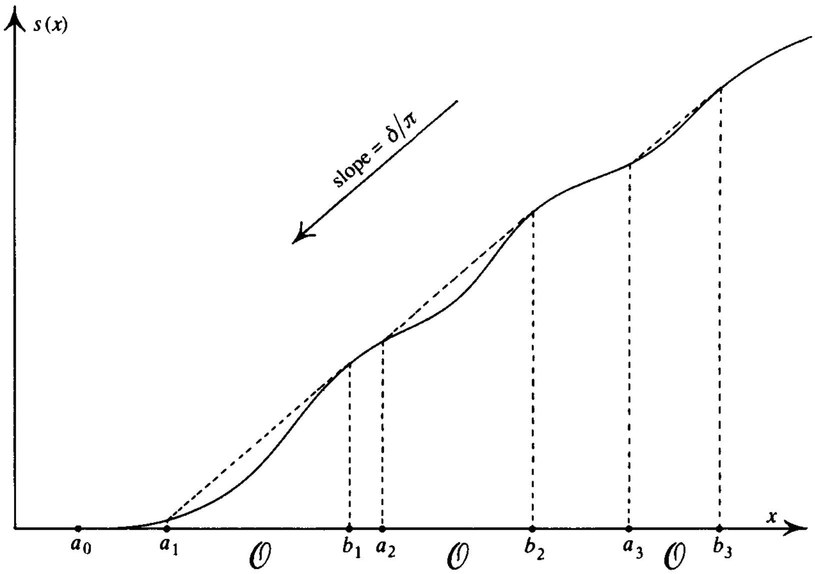

Figure 260

The set $\mathcal{O}$ is thus the union of a certain disjoint collection of intervals (a_k, b_k), $k = 1, 2, 3, \ldots$, with

$$a_k \geqslant a_0 > 0$$

for every k; these, of course, may be disposed in a much more complicated fashion than is shown in the diagram, there being no *a priori* lower limit to their lengths. *Every point $x > 0$ for which $s'(x) > \delta/\pi$ certainly belongs to $\mathcal{O}$*, so

$$s'(x) \leqslant \frac{\delta}{\pi} \quad \text{for } x \in [0, \infty) \sim \mathcal{O}.$$

For each $k \geqslant 1$,

$$\frac{s(b_k) - s(a_k)}{b_k - a_k} = \frac{\delta}{\pi}$$

as is clear from the figure.

It is now claimed that

$$\sum_1^\infty \left(\frac{b_k - a_k}{a_k}\right)^2 < \infty;$$

this we will show by an argument essentially the same as the one made in §D.1 of Chapter IX, using, however, the fifth lemma from the preceding article in place of the theorem of §C in Chapter IX. We work with the *subharmonic* function

$$U(z) \quad = \quad -\sum_N^\infty G_n(z) \quad = \quad \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}s(t);$$

the two right-hand expressions are proved equal by what amounts to the reasoning used above in checking the analogous formula for $\omega(x)$ in terms of σ (monotone convergence).

Since the $G_n(x)$ are all $\geqslant 0$ on $\mathbb{R}$, we have

$$U(x) \quad \geqslant \quad \omega(0) - \omega(x), \qquad x \in \mathbb{R},$$

so, because ω has property (i),

$$\int_{-\infty}^{\infty} \frac{U(x)}{1+x^2} \mathrm{d}x \quad > \quad -\infty.$$

Writing

$$\beta \quad = \quad \sum_N^\infty \beta_n,$$

and recalling that $G_n(z) \geqslant -\beta_n |\Im z|$, we see, moreover, that

$$U(z) \leqslant \beta |\Im z|.$$

The convergence of the series $\sum_k ((b_k - a_k)/a_k)^2$ will be deduced from the last two inequalities involving U and the fact that $\beta < \delta/\mathrm{e}$ due to our choice of N. (It would in fact be enough if we merely had $\beta < \delta$; our having been somewhat crude in the estimation of the $\sigma_n(t)/t$ has required us to work with an extra margin of safety expressed by the factor $1/\mathrm{e}$.)

Fixing our attention on any *particular* interval (a_k, b_k), let us denote its *midpoint* by c and its *length* by 2Δ, so as to have

$$(a_k, b_k) \quad = \quad (c - \Delta, \ c + \Delta).$$

The following discussion, corresponding to the one in §D.1 of Chapter IX, is actually quite simple; it may, however, at first appear complicated because of the changes of variable involved in it.

We take a certain quantity $R > 1$ (the same, in fact, for each of the intervals (a_k, b_k) – its exact *size* will be specified presently) and then, choosing a value for the parameter l,

$$\frac{2\Delta}{R + \dfrac{1}{R}} \quad < \quad l \quad \leqslant \quad \Delta,$$

apply the fifth lemma of article 1 to the subharmonic function

$$F(z) \quad = \quad U(lz + c).$$

Fix, for the moment, any number A large enough to ensure that

$$lA \gg \Delta,$$

and let us look at the Riesz representation for $F(z)$ (obtained by putting a minus sign in front of the one for the *superharmonic* function $-F(z)$!) in the disk $\{|z| < A\}$. In terms of the variable $w = lz + c$, we have this picture:

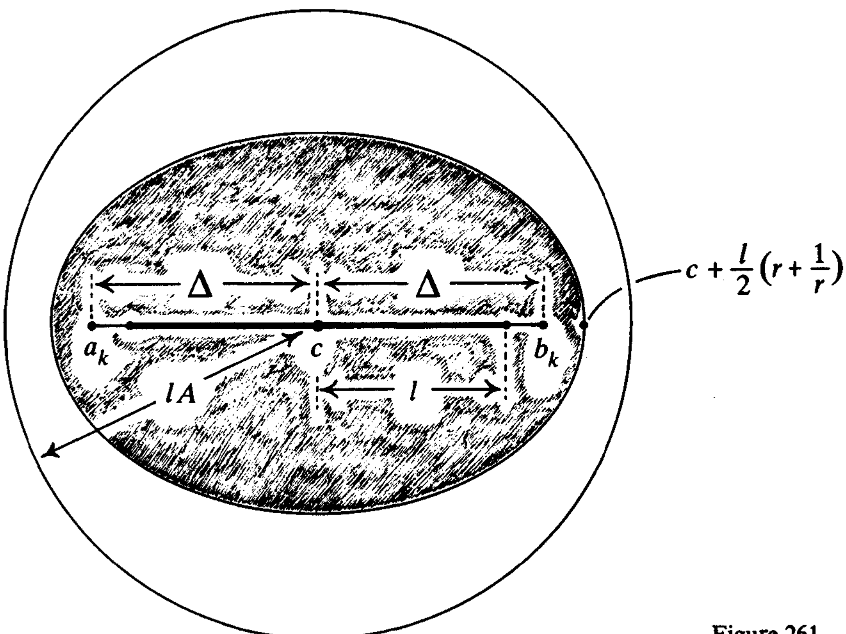

Figure 261

Because

$$U(w) \quad = \quad \int_0^{\infty} \log\left|1 - \frac{w^2}{t^2}\right| \mathrm{d}s(t),$$

we can, after making the change of variable $t = l\tau + c$, write

$$F(z) = U(lz+c) = \int_{-A}^{A} \log|z-\tau|\,ds(l\tau+c) + h(z),$$

with $h(z)$ a certain function *harmonic for* $|z| < A$. If, then, $r > 1$ and $\frac{1}{2}(r + 1/r) < A$, the *closed region* of the z-plane *bounded by the ellipse*

$$z = \frac{1}{2}\left(re^{i\vartheta} + \frac{e^{-i\vartheta}}{r}\right)$$

(whose image in the w-plane is shown in the above figure) *has mass* $M(r)$ *equal to*

$$\int_{-\frac{1}{2}(r+1/r)}^{\frac{1}{2}(r+1/r)} ds(l\tau+c) = s\left(c + \frac{l}{2}\left(r+\frac{1}{r}\right)\right) - s\left(c - \frac{l}{2}\left(r+\frac{1}{r}\right)\right)$$

assigned to it by the measure associated with the Riesz representation for $F(z)$ just given. By the fifth lemma of the last article we thus have

$$\int_1^R \frac{s(c + \frac{1}{2}l(r+r^{-1})) - s(c - \frac{1}{2}l(r+r^{-1}))}{r}\,dr = \int_1^R \frac{M(r)}{r}\,dr$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} U\left(c + \frac{l}{2}\left(Re^{i\vartheta} + \frac{e^{-i\vartheta}}{R}\right)\right)d\vartheta - \frac{1}{\pi}\int_{-1}^{1} \frac{U(lx+c)}{\sqrt{(1-x^2)}}\,dx.$$

As in §D.1 of Chapter IX, it is convenient to now write

$$R = e^{\gamma}$$

(thus making γ a certain fixed quantity > 0), and to take a number $\varepsilon > 0$, considerably *smaller* than γ (corresponding to the quantity denoted by η in the passage referred to). If the parameter l is actually

$$\geqslant \frac{\Delta}{\cosh\varepsilon},$$

we will have

$$s\left(c + \frac{l}{2}\left(r+\frac{1}{r}\right)\right) - s\left(c - \frac{l}{2}\left(r+\frac{1}{r}\right)\right) \geqslant s(b_k) - s(a_k)$$

$$\text{for } r \geqslant e^{\varepsilon}$$

(see once more the preceding figure). By construction of our intervals (a_k, b_k), the quantity on the right is equal to

$$\frac{\delta}{\pi}(b_k - a_k) = \frac{2\delta}{\pi}\Delta.$$

The previous relation thus yields

$$\frac{2\delta}{\pi}\Delta\int_{e^{\varepsilon}}^{e^{\gamma}}\frac{dr}{r} \;\leqslant\; \int_1^R \frac{M(r)}{r}dr \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} U(c \;+\; l\cosh(\gamma+i\vartheta))d\vartheta$$

$$-\;\frac{1}{\pi}\int_{-1}^{1}\frac{U(lx+c)}{\sqrt{(1-x^2)}}dx \qquad \text{for } \frac{\Delta}{\cosh\varepsilon} \leqslant l \leqslant \Delta.$$

Using the inequality

$$U(c \;+\; l\cosh(\gamma+i\vartheta)) \;\leqslant\; \beta l\,|\Im\cosh(\gamma+i\vartheta)| \;=\; \beta l\sinh\gamma\,|\sin\vartheta|$$

to estimate the next-to-the last integral on the right and making the change of variable $\xi \;=\; lx$ in the last one, one finds, after rearrangement, that

$$\int_{-l}^{l}\frac{U(\xi+c)}{\sqrt{(l^2-\xi^2)}}d\xi \;\leqslant\; 2\beta l\sinh\gamma \;-\; 2\delta(\gamma-\varepsilon)\Delta$$

$$\leqslant\; 2\sinh\gamma\left(\beta \;-\; \delta\frac{\gamma-\varepsilon}{\sinh\gamma}\right)\Delta, \qquad \text{for } \frac{\Delta}{\cosh\varepsilon} \leqslant l \leqslant \Delta.$$

Recall that in our present construction, we have

$$\beta \;<\; \frac{\delta}{e};$$

it is therefore certainly possible (and by far!) to choose $\gamma > 0$ *small enough* to make

$$\delta\frac{\gamma}{\sinh\gamma} \;>\; \beta$$

(*thus* is the value of $R \;=\; e^{\gamma}$ finally specified!), and then to fix an $\varepsilon > 0$ *yet smaller*, so as to *still have*

$$\delta\,\frac{\gamma-\varepsilon}{\sinh\gamma} \;>\; \beta.$$

These choices having been made, we write

$$a \;=\; \delta\,\frac{\gamma-\varepsilon}{\sinh\gamma} \;-\; \beta,$$

so that $a > 0$, and the above inequality becomes

$$\int_{-l}^{l}\frac{U(\xi+c)}{\sqrt{(l^2-\xi^2)}}d\xi \;\leqslant\; -2a\Delta\sinh\gamma, \qquad \frac{\Delta}{\cosh\varepsilon} \leqslant l \leqslant \Delta.$$

This relation is now multiplied by $l\,dl$, and both sides integrated over the range

$$\frac{\Delta}{\cosh\varepsilon} \leqslant l \leqslant \Delta.$$

In our circumstances, $U(t) \leqslant 0$ on the real axis, and the computation at this point is practically identical to the one in §D.1 of Chapter IX. It therefore suffices to merely give the result, which reads

$$\int_{a_k}^{b_k} U(t)\,dt \;=\; \int_{-\Delta}^{\Delta} U(\xi + c)\,d\xi \;\leqslant\; -a\Delta^2 \sinh\gamma\,\tanh\varepsilon$$

$$= -\frac{a}{4}(\sinh\gamma\,\tanh\varepsilon)(b_k - a_k)^2.$$

The last inequality, involving the quantities a, γ, and ε, now *fixed* (and > 0), holds for *any* of the intervals (a_k, b_k), $k \geqslant 1$. Since the a_k are $\geqslant a_0 > 0$ and $U(t) \leqslant 0$ on $\mathbb{R}$, we see from it that

$$\int_{a_k}^{b_k} \frac{U(t)}{t^2+1}\,dt \;\leqslant\; -\frac{aa_0^2}{4(a_0^2+1)}\sinh\gamma\tanh\varepsilon\left(\frac{b_k - a_k}{b_k}\right)^2.$$

Finally (using again the fact that $U(t) \leqslant 0$ on $\mathbb{R}$), we get

$$\sum_{k=1}^{\infty}\left(\frac{b_k - a_k}{b_k}\right)^2 \;\leqslant\; -\frac{4(a_0^2+1)}{aa_0^2\sinh\gamma\tanh\varepsilon}\int_0^{\infty}\frac{U(t)}{t^2+1}\,dt \;<\; \infty,$$

and our claim that $\sum_k((b_k - a_k)/a_k)^2 < \infty$ is thereby established*.

From this result, *property* (ii) *for the function* $\omega(x)$ readily follows. As we know,

$$\tilde{\omega}'(x) \;=\; \pi\sum_1^{\infty}\sigma_n'(x) \;=\; \pi\sum_1^{N-1}\sigma_n'(x) \;+\; \pi s'(x).$$

It has already been noted that *each one* of the derivatives $\sigma_n'(x)$ is *bounded* for $x > 0$; there is thus a number K such that

$$\pi\sum_1^{N-1}\sigma_n'(x) \;\leqslant\; K - \delta, \qquad x > 0.$$

The derivative $\pi s'(x)$ is, at the same time, $\leqslant \delta$ for all the $x > 0$ *outside*

$$\mathcal{O} \;=\; \bigcup_{k\geqslant 1}(a_k, b_k).$$

* for the preceding displayed relation implies in particular that the ratios b_k/a_k are *bounded* – cf. discussion, top of p. 81

Therefore

$$\tilde{\omega}'(x) - K = \pi\sum_{1}^{N-1}\sigma_n'(x) - (K-\delta) + \pi s'(x) - \delta$$

is $\leqslant 0$ for $x \in (0,\infty) \sim \mathcal{O}$.

On any of the components (a_k, b_k) of $\mathcal{O}$, we have

$$(\tilde{\omega}'(x) - K)^+ \leqslant \left(\pi\sum_{1}^{N-1}\sigma_n'(x) - (K-\delta)\right)^+ + (\pi s'(x) - \delta)^+$$

$$\leqslant 0 + \pi s'(x)$$

($s(x)$ being increasing!), so

$$\int_{a_k}^{b_k}(\tilde{\omega}'(x) - K)^+\,\mathrm{d}x \leqslant \pi\int_{a_k}^{b_k}s'(x)\,\mathrm{d}x = \pi(s(b_k) - s(a_k)) = \delta(b_k - a_k).$$

Property (ii) *therefore holds*, the sum $\sum_k((b_k - a_k)/a_k)^2$ being convergent.

The *necessity* of our condition is thus proved.

*2° **The sufficiency**.* Suppose that there *is* an even $\mathscr{C}_\infty$ function

$$\omega(x) \geqslant \log W(x)$$

having the properties enumerated in the theorem's statement; we must show that W admits multipliers. Let, then, $\delta > 0$ be given; corresponding to it we have a K and an open subset $\mathcal{O}$ of $(0,\infty)$ with

$$\tilde{\omega}'(x) \leqslant K \quad \text{for } x \in (0,\infty) \sim \mathcal{O},$$

and, if $\mathcal{O}$ is the union of the disjoint intervals (a_k, b_k),

$$\sum_k\left(\frac{b_k - a_k}{a_k}\right)^2 < \infty,$$

while

$$\int_{a_k}^{b_k}(\tilde{\omega}'(x) - K)^+\,\mathrm{d}x \leqslant \delta(b_k - a_k)$$

for each k.

We start by expressing $\tilde{\omega}(x)$ as the difference of two functions, each continuous and *increasing* on $[0,\infty)$. The given properties of $\tilde{\omega}'(x)$ make it possible for us to define a bounded measurable function $p(x)$ on $[0,\infty)$

with $0 \leqslant p(x) \leqslant \delta$ by taking

$$p(x) = \delta \quad \text{for } x \in [0, \infty) \sim \mathcal{O}$$

and then, on each of the interval components (a_k, b_k) of $\mathcal{O}$, having $p(x)$ assume the constant value needed to make

$$\int_{a_k}^{b_k} \{(\tilde{\omega}'(x) - K)^+ + p(x)\} \, dx = \delta(b_k - a_k).$$

Put now

$$\pi v_1(x) = \int_0^x \{(\tilde{\omega}'(t) - K)^+ + p(t)\} \, dt \qquad \text{for } x \geqslant 0$$

and

$$\pi v_2(x) = \int_0^x \{(K - \tilde{\omega}'(t))^+ + p(t)\} \, dt, \qquad x \geqslant 0.$$

We have $\tilde{\omega}(0) = 0$, for, since $\omega(x)$ is $\mathscr{C}_\infty$ and even, $\tilde{\omega}(x)$ is $\mathscr{C}_\infty$ and odd. Therefore, when $x \geqslant 0$,

$$\tilde{\omega}(x) - Kx = \int_0^x (\tilde{\omega}'(t) - K) \, dt$$

$$= \int_0^x \{(\tilde{\omega}'(t) - K)^+ - (\tilde{\omega}'(t) - K)^-\} \, dt = \pi v_1(x) - \pi v_2(x);$$

i.e.,

$$\pi v_1(x) + Kx - \pi v_2(x) = \tilde{\omega}(x), \qquad x \geqslant 0,$$

with $v_1(x)$ and $v_2(x)$ both *increasing* and *continuous* for $x \geqslant 0$.

The ratio $v_1(x)/x$ is *bounded* for $x > 0$. Indeed, if $x \in [0, \infty) \sim \mathcal{O}$, $\pi v_1(x) = \delta x$ by the definition of our function $p(x)$. And if $a_k < x < b_k$ for some k,

$$\frac{\pi v_1(x)}{x} < \frac{\pi v_1(b_k)}{a_k} = \delta \frac{b_k}{a_k}.$$

The ratios b_k/a_k are, however, *bounded above*, since $\sum_k ((b_k - a_k)/a_k)^2$ is convergent. Hence

$$\frac{v_1(x)}{x} \leqslant \text{const.} \qquad \text{for } x > 0.$$

The *odd* continuous increasing function $\mu(x)$ equal to $\pi\nu_1(x) + Kx$ for $x \geqslant 0$ is thus $O(x)$ on $[0, \infty)$, and $\tilde{\omega}(x) - \mu(x)$, *also odd* and equal, for $x \geqslant 0$, to $-\pi\nu_2(x)$ by the above formula, is *decreasing* on $\mathbb{R}$. The *fourth* lemma of article 1 can therefore be invoked (with $F(x) = -\omega(x)$, $\omega(x)$ being $\mathscr{C}_\infty$, $\geqslant 0$ and enjoying property (i) by hypothesis). This yields

$$|\tilde{\omega}(x)| \leqslant \text{const.}|x| \qquad \text{for } x \in \mathbb{R},$$

which in turn makes

$$\nu_2(x) \leqslant \text{const.}\, x \qquad \text{for } x \geqslant 0$$

in view of the preceding estimate on $\nu_1(x)$ and the formula just referred to.

All the conditions for application of the *third* lemma in article 1 (again with $F(x) = -\omega(x)$) are now verified. By that result we get, for $x \in \mathbb{R}$,

$$\begin{aligned}
\omega(x) &= \omega(0) - \frac{1}{\pi}\int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\tilde{\omega}(t) \\
&= \omega(0) + \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\nu_2(t)
\end{aligned}$$

$$- \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\nu_1(t) - \frac{K}{\pi}\int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}t.$$

The very last integral on the right is of course zero, so we have

$$\boxed{\begin{aligned}
&\omega(x) + \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\nu_1(t) \\
&\quad = \omega(0) + \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\nu_2(t), \qquad x \in \mathbb{R}.
\end{aligned}}$$

The rest of our work here is based mainly on this formula.

Before looking more closely at the increasing function $\nu_1(t)$ and the expression

$$\int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\nu_1(t)$$

corresponding to it, we should attend to a detail regarding the *location* of the open set $\mathcal{O}$. We can, namely, arrange to *ensure that* $\mathcal{O} \subseteq (1, \infty)$

(which will turn out to be convenient later on) *by merely taking K large enough* to begin with. In the present circumstances, $\tilde{\omega}(x)$ (like $\omega(x)$) is $\mathscr{C}_\infty$ on $\mathbb{R}$,* so $\tilde{\omega}'(x)$ is *bounded on any finite interval.* Hence, if K is chosen large enough in the first place, we will have

$$\tilde{\omega}'(x) \quad \leqslant \quad K \qquad \text{for } 0 \leqslant x \leqslant 1,$$

so that *any component* (a_k, b_k) of the set $\mathcal{O}$ corresponding to this K which *lies entirely* in $(0, 1)$ may be simply *thrown away* (and $p(x) = \pi v_1'(x)$ just taken *equal* to δ thereon) without in any way affecting the properties of $v_1(x)$ and $v_2(x)$ used up to now. There may, however, *still* be a component (a_l, b_l) of $\mathcal{O}$ with $a_l < 1 < b_l$. In that event, b_l is certainly finite, and there is thus a $K' \geqslant K$ with

$$\tilde{\omega}'(x) \quad \leqslant \quad K' \qquad \text{for } 0 \leqslant x \leqslant b_l.$$

Then, if we *also throw away* (a_l, b_l), what *remains* of $\mathcal{O}$ will be a certain open set $\mathcal{O}' \subseteq (1, \infty)$ composed of the intervals (a_k, b_k) from $\mathcal{O}$ that *do not intersect* $[0, 1]$. We will have

$$\tilde{\omega}'(x) \quad \leqslant \quad K' \qquad \text{for } x \in (0, \infty) \sim \mathcal{O}',$$

and for each of the (a_k, b_k) making up $\mathcal{O}'$,

$$\int_{a_k}^{b_k} (\tilde{\omega}'(x) - K')^+ \, dx \quad \leqslant \quad \delta(b_k - a_k),$$

since the same relation holds with $K \leqslant K'$ standing in place of K'. *By increasing K to K'*, we thus ensure that *none of the intervals* (a_k, b_k) *appearing in our construction intersect with* $[0, 1]$. This merely amounts to *choosing a larger value initially for the number K corresponding to our given* δ, which *we henceforth assume as having been done.* The intervals (a_k, b_k) involved in the formation of $v_1(x)$ and $v_2(x)$ are in such fashion guaranteed to *all lie in* $(1, \infty)$.

Having seen to this matter, we turn our attention to the behaviour of $\pi v_1(t)$ for $t \geqslant 0$. As we have already noted, $\pi v_1(t) = \delta t$ for $t \geqslant 0$ lying *outside* all the intervals (a_k, b_k). When $a_k < t < b_k$, we have, since $v_1(t)$ increases,

$$\delta a_k \leqslant \pi v_1(t) \leqslant \delta b_k.$$

At the same time,

$$\delta a_k < \delta t < \delta b_k,$$

* cf. initial footnote to third lemma of article 1

so

$$|\pi v_1(t) - \delta t| \leqslant \delta(b_k - a_k) \qquad \text{for } a_k < t < b_k.$$

Thence,

$$\int_{a_k}^{b_k} \frac{|\pi v_1(t) - \delta t|}{t^2} \mathrm{d}t \leqslant \delta\left(\frac{b_k - a_k}{a_k}\right)^2$$

which, with the preceding observation, implies that

$$\int_0^\infty \frac{|\pi v_1(t) - \delta t|}{t^2} \mathrm{d}t < \infty$$

on account of the convergence of $\sum_k((b_k - a_k)/a_k)^2$; *it is here that we have made crucial use of that hypothesis.*

Let us, in the usual fashion, extend the increasing function $v_1(t)$ to all of $\mathbb{R}$ by making it *odd* there. Then the function

$$\Delta(t) = v_1(t) - \frac{\delta}{\pi}t$$

is *also* odd and, moreover, *zero* for $-1 < t < 1$ due to our having arranged that none of the intervals (a_k, b_k) intersect with $[0, 1]$. According to what we have just seen,

$$\int_{-\infty}^{\infty} \frac{|\Delta(t)|}{t^2} \mathrm{d}t < \infty;$$

$\Delta(t)$ thus satisfies the hypothesis of the initial lemma in §B.2, *Chapter* X, with δ/π playing the rôle of the number D figuring there. That result gives us a function $q(t)$, *zero* for $-1 < t < 1$, having the other properties of the one there denoted by $\delta(t)$, corresponding to a value δ/π of the parameter η. (Here we write $q(t)$ instead of $\delta(t)$ because the letter δ is already in service.) Since our present function $\Delta(t)$ is *odd*, the one furnished by the lemma referred to may be taken to be *odd also*, and

$$\lambda(t) = \frac{\delta}{\pi}t + q(t)$$

is then *odd*, besides being *increasing* on $\mathbb{R}$. We have

$$\lambda(t) = \frac{\delta}{\pi}t \qquad \text{for } -1 < t < 1$$

and moreover,

$$\frac{\lambda(t)}{t} \longrightarrow \frac{\delta}{\pi} \quad \text{as } t \longrightarrow \pm\infty.$$

One may now apply the *first* theorem of §B.2, Chapter X, to the present functions $\Delta(t)$ and $q(t)$, and then do a calculation like the one used to prove the lemma of §B.1 there. On account of the oddness of $\nu_1(t)$ and $\lambda(t)$, that computation simplifies quite a bit*, and the final result is that

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} \left| \int_0^{\infty} \log\left|1 - \frac{x^2}{t^2}\right| d(\nu_1(t) + \lambda(t)) \right| dx \quad < \quad \infty.$$

Used with the previous boxed formula and the assumption (in the hypothesis) that $\omega(x) \geqslant 0$ enjoys property (i), this implies that

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} \left| \int_0^{\infty} \log\left|1 - \frac{x^2}{t^2}\right| d(\nu_2(t) + \lambda(t)) \right| dx \quad < \quad \infty.$$

Put now

$$V(z) \quad = \quad \int_0^{\infty} \log\left|1 - \frac{z^2}{t^2}\right| d(\nu_2(t) + \lambda(t));$$

the last relation can then be written

$$\int_{-\infty}^{\infty} \frac{|V(x)|}{1+x^2} dx \quad < \quad \infty.$$

We have

$$\nu_2(t) \ + \ \lambda(t) \quad \leqslant \quad \text{const.}\, t \qquad \text{for } t \geqslant 0,$$

so V also satisfies an inequality of the form

$$V(z) \quad \leqslant \quad \text{const.}\, |z|.$$

These two properties of V imply that

$$V(z) \quad = \quad B|\Im z| \quad + \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|\, V(t)}{|z-t|^2}\, dt$$

with a suitable constant $B \geqslant 0$, according to a version of the result from §G.1, Chapter III – the use of such a version here can be justified by an

* The usual partial integration is carried out with $\Delta(t) + q(t)$ playing the rôle of $\nu(t)$, then the relation $\int_{-\infty}^{\infty} \log|1 - (x^2/t^2)|\, dt = 0$ is used.

argument like one made while proving the second theorem of §B.1.* From this formula and the first of the two relations for V preceding it, we get

$$\int_{-\infty}^{\infty} \frac{|V(x+\mathrm{i})|}{1+x^2}\,\mathrm{d}x \quad < \quad \infty$$

in the usual way.

We desire to apply the *Theorem on the Multiplier* at this point, and for that an *entire function of exponential type* is needed. (It is not true here that $V(x) \geqslant 0$ on $\mathbb{R}$, so we are unable to directly adapt the *proof* of that theorem given in §C.2 to the function V.) Take, then, the entire function φ of exponential type given by the formula

$$\log|\varphi(z)| \quad = \quad \int_0^{\infty} \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}[\nu_2(t) + \lambda(t)].$$

By the lemma in §A.1, Chapter X,

$$\log|\varphi(x+\mathrm{i})| \quad \leqslant \quad V(x+\mathrm{i}) \; + \; \log^+|x| \qquad \text{for } x \in \mathbb{R},$$

whence, by the preceding inequality,

$$\int_{-\infty}^{\infty} \frac{\log^+|\varphi(x+\mathrm{i})|}{1+x^2}\,\mathrm{d}x \quad < \quad \infty.$$

The theorem on the multiplier thus gives us a non-zero entire function $\psi(z)$, of exponential type $\delta' \leqslant \delta$, bounded on $\mathbb{R}$ and with $|\varphi(x+\mathrm{i})\psi(x)|$ bounded on $\mathbb{R}$ as well. We may, of course, get such a ψ with $\delta' = \delta$ by simply multiplying the initial one by $\cos(\delta - \delta')z$.

We can also take $\psi(z)$ to be *even*, since φ is even, and, of course, can have $\psi(0) \neq 0$. The discussion following the first theorem of §B.1 shows furthermore that we can take $\psi(z)$ to have *real zeros only*, and thus be given in the form

$$\log|\psi(z)| \quad = \quad \int_0^{\infty} \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}\sigma(t),$$

with $\sigma(t)$ increasing, integer-valued, zero near the origin, and satisfying

$$\frac{\sigma(t)}{t} \longrightarrow \frac{\delta}{\pi} \qquad \text{for } t \longrightarrow \infty$$

* By its definition, $\nu_2(t)$ is absolutely continuous with $\nu_2'(t)$ bounded on finite intervals; $\lambda(t)$, on the other hand, has a graph similar to the one shown in fig. 226 (Chapter X, §B.2). These properties make $(V(z))^+$ continuous at the points of $\mathbb{R}$, and the arguments from §§E and G.1 of Chapter III may be used.

(by Levinson's theorem). By first dividing out four of the zeros of ψ if need be (it *has* infinitely many, being of exponential type $\delta > 0$ and bounded on $\mathbb{R}$!) we can finally ensure that in fact

$$|\varphi(x+\mathrm{i})\psi(x)| \quad \leqslant \quad \frac{\text{const.}}{(x^2+1)^2} \qquad \text{for } x \in \mathbb{R}$$

with (perhaps another) ψ of the kind described.

A relation between $V(x+\mathrm{i})$ and $\log|\varphi(x+\mathrm{i})|$ *opposite in sense to the above one* is now called for. To get it, observe that

$$V(z) \quad = \quad \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}(\min(\nu_2(t)+\lambda(t),\ 1))$$
$$+ \quad \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}(\nu_2(t)+\lambda(t)-1)^+.$$

Since $\lambda(t) = \delta t/\pi$ for $0 \leqslant t \leqslant 1$ and $\nu_2'(t)$ is certainly *bounded* there, the first integral on the right is

$$\leqslant \quad 2\log^+|z| \ + \ \text{const.}.$$

Therefore, when $x \in \mathbb{R}$,

$$V(x+\mathrm{i}) \ \leqslant \ 2\log^+|x| \ + \ \text{const.} \ + \ \int_0^\infty \log\left|1 - \frac{(x+\mathrm{i})^2}{t^2}\right| \mathrm{d}(\nu_2(t)+\lambda(t)-1)^+.$$

However, $(\nu_2(t)+\lambda(t)-1)^+ \leqslant [\nu_2(t)+\lambda(t)]$ for $t \geqslant 0$, so, by reasoning identical to that used in proving the lemma of §A.1, Chapter X, we find that the last right-hand integral is

$$\leqslant \quad \log|\varphi(x+\mathrm{i})| \ + \ \log^+|x|.$$

Thus,

$$V(x+\mathrm{i}) \quad \leqslant \quad \log|\varphi(x+\mathrm{i})| \ + \ 3\log^+|x| \ + \ \text{const.}, \qquad x \in \mathbb{R}.$$

Referring to the previous relation involving $\varphi(x+\mathrm{i})$ and $\psi(x)$, we thence obtain

$$V(x+\mathrm{i}) \ + \ \log|\psi(x)| \quad \leqslant \quad \text{const.}, \qquad x \in \mathbb{R}.$$

Clearly, $V(x) \leqslant V(x+\mathrm{i})$, so we have

$$V(x) \quad + \quad \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\sigma(t) \quad \leqslant \quad \text{const.} \qquad \text{for } x \in \mathbb{R}$$

by our formula for $\log|\psi(z)|$.

Now by the previous boxed formula and our definition of the function V,

$$V(x) \quad = \quad \omega(x) \;-\; \omega(0) \;+\; \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d(\nu_1(t) + \lambda(t)).$$

Combination of this with the preceding thus yields

$$\omega(x) \quad + \quad \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d(\nu_1(t) + \lambda(t) + \sigma(t)) \quad \leqslant \quad \text{const.}, \qquad x \in \mathbb{R}$$

and hence, since $\log W(x) \leqslant \omega(x)$,

$$\log W(x) \quad + \quad \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d(\nu_1(t) + \lambda(t) + \sigma(t)) \quad \leqslant \quad \text{const.}, \quad x \in \mathbb{R}.$$

Here, $\lambda(t)/t$ and $\sigma(t)/t$ both tend to δ/π for $t \longrightarrow \infty$ as we have already noted. Again, since the ratios b_k/a_k corresponding to the intervals $(a_k,\ b_k)$ used in the construction of $\nu_1(t)$ must tend to 1 for $k \longrightarrow \infty$, we also have*

$$\frac{\nu_1(t)}{t} \longrightarrow \frac{\delta}{\pi} \qquad \text{for } t \longrightarrow \infty$$

(look again at the above discussion of the behaviour of ν_1). For the increasing function

$$\rho(t) \quad = \quad \nu_1(t) \;+\; \lambda(t) \;+\; \sigma(t)$$

it is thus true that

$$\frac{\rho(t)}{t} \longrightarrow \frac{3\delta}{\pi} \qquad \text{as } t \longrightarrow \infty,$$

and that

$$\log W(x) \quad + \quad \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\rho(t) \quad \leqslant \quad \text{const.}, \qquad x \in \mathbb{R}.$$

The quantity $\delta > 0$ was, however, arbitrary. Therefore, since $W(x)$, by hypothesis, meets the local regularity requirement of §B.1, *it admits multipliers* according to the *second theorem* of that §, and *sufficiency is now established.*

Our result is completely proved.

* although a_k need not $\longrightarrow\ \infty$ for $k \longrightarrow \infty$, all *sufficiently large* a_k certainly *do* have *arbitrarily large* indices k.

Remark 1 (added in proof). In the *sufficiency* proof, fulfilment of our local regularity requirement is only used at the end; in the absence of that requirement one still gets functions $\rho(t)$, increasing and $O(t)$ on $[0,\infty)$, with $\limsup_{t\to\infty}(\rho(t)/t)$ arbitrarily small and

$$\log W(x) \quad + \quad \int_0^\infty \log|1 \; - \; (x^2/t^2)|\mathrm{d}\rho(t)$$

bounded above on $\mathbb{R}$. The *necessity* proof, on the other hand, actually goes through – see the footnotes to its first part – whenever $W(x) \geqslant 1$ is *continuous* and such $\rho(t)$ exist. *The existence of a majorant $\omega(x)$ having the properties specified by the theorem is therefore equivalent to the existence of such increasing functions ρ for continuous weights W.* Our theorem thus holds, in particular, for continuous weights meeting the milder regularity requirement from the scholium at the end of §B.1. Continuity, indeed, need not even be assumed for such weights; that is evident after a little thought about the abovementioned footnotes and the passage they refer to.

Remark 2. The proof for the *necessity* shows that if $W(x)$ *does* admit multipliers, a majorant $\omega(x)$ for $\log W(x)$ having the properties asserted by the theorem exists, with

$$\omega(x) \quad = \quad \omega(0) \quad - \quad \int_0^\infty \log\left|1 \; - \; \frac{x^2}{t^2}\right|\mathrm{d}\sigma(t),$$

where $\sigma(t)$ is *increasing* on $[0,\infty)$, *zero* for t close enough to 0, and $O(t)$ for $t \longrightarrow \infty$. Now look again at the example in §D.4 and the discussion in §D.5!

Remark 3. It was by thinking about the above result that I came upon the method explained in §§B.2, B.3 and used in §C, being led to it by way of the construction in problem 55 (near end of §B.2).

Remark 4. It seems possible to tie the theorem's property (ii) more closely to the *local* behaviour of $\omega(x)$. Referring to the remark following the statement of the theorem, we see that

$$\tilde{\omega}'(x) \quad = \quad \frac{1}{\pi}\int_0^{Y(x)} \frac{2\omega(x) - \omega(x+t) - \omega(x-t)}{t^2}\,\mathrm{d}t$$

$$+ \quad \frac{1}{\pi}\int_{Y(x)}^\infty \frac{2\omega(x) - \omega(x+t) - \omega(x-t)}{t^2}\,\mathrm{d}t,$$

where for $Y(x)$ we can take *any* positive quantity, depending on x in any way we want.

Because $\omega \geqslant 0$, the *second* of the two integrals on the right is

$$\leqslant \quad \frac{2}{\pi Y(x)}\omega(x);$$

it is, on the other hand,

$$\geqslant \quad -\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\omega(t)}{(x-t)^2+(Y(x))^2}\,\mathrm{d}t.$$

For the present purpose this last expression's behaviour is adequately described by the 1967 lemma of Beurling and Malliavin given in §E.2 of Chapter IX. That result shows that for any given $\eta > 0$, the integral in question will lie between $-\eta$ and 0 for a function $Y(x) > 0$ with

$$\int_{-\infty}^{\infty}\int_0^{Y(x)}\frac{\mathrm{d}y\mathrm{d}x}{1+x^2+y^2} \quad < \quad \infty;$$

such a function is hence *not too large.*

Once a function $Y(x)$ is at hand, the set of $x > 0$ on which $\tilde{\omega}'(x)$ exceeds some large K seems to essentially be determined by the behaviour of $\omega(x)/Y(x)$ and of the integral

$$\frac{1}{\pi}\int_0^{Y(x)}\frac{2\omega(x)-\omega(x+t)-\omega(x-t)}{t^2}\,\mathrm{d}t.$$

Both of these expressions involve *local behaviour* of ω.

I think an investigation along this line is worth trying, but have no time to undertake it now. *This book must go to press.*

Remark 5 (added in proof). We have been dealing with the notion of multiplier adopted in §B.1, using that term to desiquate a non-zero entire function of exponential type whose product with a given weight is *bounded* on $\mathbb{R}$. This specification of boundedness is largely responsible for our having had to introduce a local regularity requirement in §B.1.

Such requirements become to a certain extent irrelevant if we return to the broader interpretation of the term accepted in Chapter X and permit its use whenever the product in question belongs to some $L_p(\mathbb{R})$. This observation, already made by Beurling and Malliavin at the end of their 1962 article, is based on the following analogue of the second theorem in §B.2:

Lemma. *Let $\Omega(x) \geqslant 1$ be Lebesgue measurable. Suppose, given $A > 0$, that there is a function $\rho(t)$, increasing and $\mathrm{O}(t)$ on $[0, \infty)$, with*

$$\limsup_{t\to\infty}(\rho(t)/t) \quad \leqslant \quad A/\pi$$

and

$$\log \Omega(x) \quad + \quad \int_0^\infty \log|1 \; - \; (x^2/t^2)| \, \mathrm{d}\rho(t) \quad \leqslant \quad \mathrm{O}(1) \qquad \text{a.e.}$$

on $\mathbb{R}$. *Then, if* $0 < p < \infty$, *there is a non-zero entire function* $\psi(z)$ *of exponential type* $\leqslant \; 4(p+2)A$ *such that*

$$\int_{-\infty}^\infty |\Omega(x)\psi(x)|^p \mathrm{d}x \quad < \quad \infty.$$

Proof. We consider the case $p = 1$; treatment for the other values of p is similar.

Take, then, the increasing function $\rho(t)$ furnished by the hypothesis and put

$$\nu(t) \;=\; 4\rho(t),$$

making

$$4\log \Omega(x) \quad + \quad \int_0^\infty \log|1 \; - \; (x^2/t^2)| \, \mathrm{d}\nu(t) \quad \leqslant \quad C \qquad \text{a.e., } x \in \mathbb{R}.$$

Since $\limsup_{t\to\infty} (\nu(t)/t) \;\leqslant\; 4A/\pi$, the entire function $\varphi(z)$ given by the formula

$$\log|\varphi(z)| \quad = \quad \int_0^\infty \log|1 \; - \; (z^2/t^2)| \, \mathrm{d}[\nu(t)]$$

is of exponential type $\leqslant\; 4A$; this may be checked by using partial integration to estimate $\log \varphi(|z|)$.

Putting

$$U(z) \quad = \quad \int_0^\infty \log|1 \; - \; (z^2/t^2)| \, \mathrm{d}\nu(t),$$

we have

$$(\Omega(x))^4 \exp U(x) \quad \leqslant \quad C \qquad \text{a.e., } x \in \mathbb{R}.$$

The idea behind our proof is that $|\varphi(x)|$ cannot be too much larger than $\exp U(x)$.

The usual integration by parts yields

$$\log|\varphi(x)| \; - \; U(x) \quad = \quad \int_0^\infty \log|1 \; - \; (x^2/t^2)| \, \mathrm{d}\big([\nu(t)] - \nu(t)\big)$$

$$= \quad \int_0^\infty \frac{2x^2}{x^2 - t^2} \cdot \frac{[\nu(t)] - \nu(t)}{t} \mathrm{d}t$$

at every $x \in \mathbb{R}$ where $v'(x)$ exists and is finite, and hence almost everywhere (see the lemma in §B.1 of Chapter X). After extending v from $[0, \infty)$ to $\mathbb{R}$ by making it *odd* (which poses no problem, $v(t)$ being $O(t)$ for $t \geqslant 0$), we can rewrite the last integral as

$$\int_{-\infty}^{\infty} \frac{x}{x-t} \cdot \frac{[v(t)] - v(t)}{t} \mathrm{d}t \quad = \quad \int_{-\infty}^{\infty} \left(\frac{1}{x-t} + \frac{1}{t} \right) ([v(t)] - v(t)) \, \mathrm{d}t$$

$$= \quad b \quad + \quad \int_{-\infty}^{\infty} \left(\frac{1}{x-t} + \frac{t}{t^2+1} \right) ([v(t)] - v(t)) \, \mathrm{d}t,$$

where the quantity

$$b \quad = \quad \int_{-\infty}^{\infty} \frac{[v(t)] - v(t)}{t(t^2+1)} \mathrm{d}t$$

is finite. Hence, aside from the additive constant b, $\log|\varphi(x)| - U(x)$ is just the Hilbert transform of $\pi([v(x)] - v(x))$ which is, however, *bounded by* π *in absolute value*. Referring now to problem 45(*c*) (Chapter X, §F), we see that

$$\int_{-\infty}^{\infty} \frac{|\varphi(x)|^{1/4} \mathrm{e}^{-U(x)/4}}{1+x^2} \mathrm{d}x \quad < \quad \infty.$$

From this and the above relation involving $\Omega(x)$ and $U(x)$ we have, finally

$$\int_{-\infty}^{\infty} \frac{\Omega(x)|\varphi(x)|^{1/4}}{1+x^2} \mathrm{d}x \quad < \quad \infty.$$

Write

$$\psi(z) \quad = \quad \left(\frac{\sin Az}{z} \right)^8 \varphi(z);$$

$\psi(z)$ is entire, of exponential type $\leqslant 12A$, with $|\psi(x)| \leqslant \text{const.}|\varphi(x)|/(x^2+1)^4$ on the real axis. It thence follows by the preceding inequality that

$$\int_{-\infty}^{\infty} \Omega(x)|\psi(x)|^{1/4} \mathrm{d}x \quad < \quad \infty.$$

In order to conclude from this that

$$\int_{-\infty}^{\infty} \Omega(x)|\psi(x)| \mathrm{d}x \quad < \quad \infty$$

(thus proving the lemma in the case $p = 1$), it is enough to show that $\psi(x)$ is *bounded* on $\mathbb{R}$.

For that purpose, we note that $\int_{-\infty}^{\infty} |\psi(x)|^{1/4} \mathrm{d}x < \infty$ since $\Omega(x) \geqslant 1$, so surely

$$\int_{-\infty}^{\infty} \frac{\log^+ |\psi(x)|}{1 + x^2} \mathrm{d}x < \infty.$$

This gives us the right to use the theorem from §G.1 of Chapter III (the easier one in that chapter's §E would do just as well) to get

$$\log|\psi(x + \mathrm{i})| \leqslant 12A + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log|\psi(t)|}{(x-t)^2 + 1} \mathrm{d}t$$

for $x \in \mathbb{R}$. By the inequality between arithmetic and geometric means, the integral on the right is

$$\leqslant 4\log\left(\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\psi(t)|^{1/4}}{(x-t)^2 + 1}\mathrm{d}t\right) \leqslant 4\log\left(\frac{1}{\pi}\int_{-\infty}^{\infty} |\psi(t)|^{1/4}\mathrm{d}t\right)$$

which, as we just observed, is finite. Therefore $\log|\psi(x + \mathrm{i})| \leqslant$ const., $x \in \mathbb{R}$. One can now conclude that $\psi(x)$ is bounded on $\mathbb{R}$, either by appealing to the third Phragmén–Lindelöf theorem from §C of Chapter III or by simply noting that $|\psi(x)| \leqslant |\psi(x + \mathrm{i})|$ on $\mathbb{R}$ for our function ψ (which has only real zeros). The proof is complete.

Let us now refer to Remark 1, and once more to the *sufficiency* proof for the above theorem. The argument made there furnished, for each $A > 0$, a function $\rho(t)$ satisfying the hypothesis of the lemma with the weight $\Omega(x) = \exp\omega(x)$; comparison of $\omega(x)$ with $\log W(x)$ did not take place until the very end. We can thereby conclude that *the existence, for* $\log W(x)$, *of an* a.e. *majorant* $\omega(x)$ *having the other properties enumerated in the theorem implies, for each* $p < \infty$, *the existence of entire functions* $\psi(z) \not\equiv 0$ *of arbitrarily small exponential type with*

$$\int_{-\infty}^{\infty} |W(x)\psi(x)|^p \mathrm{d}x < \infty.$$

The function $\omega(x)$ with the stipulated properties does not even need to be an actual *majorant* of $\log W(x)$; as long as

$$\int_{-\infty}^{\infty} (\mathrm{e}^{-\omega(x)}W(x))^{r_0} \mathrm{d}x < \infty$$

for some $r_0 > 0$, *we will still, for each* $r < r_0$, *have entire functions* ψ *of the kind described with*

$$\int_{-\infty}^{\infty} |W(x)\psi(x)|^r \,\mathrm{d}x \quad < \quad \infty.$$

This also follows from the lemma; it suffices to take $\Omega(x) = \exp\omega(x)$ and $p = r_0/(r_0 - r)$, and then use Hölder's inequality.

The first of these results should be confronted with one going in the opposite direction that was already pointed out in Remark 1. That says that, *for a continuous weight* $W(x) \geqslant 1$, *the existence of entire functions* $\psi(z) \not\equiv 0$ *of arbitrarily small exponential type making* $W(x)\psi(x)$ *bounded on* $\mathbb{R}$ *implies existence of a majorant* $\omega(x)$ *for* $\log W(x)$ *with the properties specified by the theorem.* Thus, insofar as *continuous* weights are concerned, our theorem's majorant criterion is at the same time a *necessary* condition for the admittance of multipliers (in the narrow L_∞ sense) and a *sufficient* one, albeit in the broader L_p sense only. No additional regularity of the weight (beyond continuity) is involved here.

A very similar observation can be made about the last theorem in §B.3. *Any continuous weight* $W(x) \geqslant 1$ *will admit multipliers in the* L_p *sense* (*with* $p < \infty$) *provided that, for each* $A > 0$, *the smallest superharmonic majorant of*

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|\log W(t)}{|z-t|^2}\,\mathrm{d}t \quad - \quad A|\Im z|$$

is finite. This finiteness is, on the other hand, *necessary for the admittance of multipliers in the* L_∞ *sense by the weight* W. It is worthwhile in this connection to note, finally, the following fact: *for continuous weights* W, *finiteness of the smallest superharmonic majorants just mentioned is equivalent to the existence of an* $\omega(x)$ *enjoying all the properties described by the theorem.* That is an immediate consequence of the next-to-the-last theorem in §B.3 and Remark 1.

Scholium. One way of looking at the theorem on the multiplier is to view it as a *guarantee* of *admittance of multipliers* by smooth even weights $W(x) = e^{\omega(x)} \geqslant 1$ with

$$\int_{-\infty}^{\infty} \frac{\omega(x)}{1+x^2}\,\mathrm{d}x \quad < \quad \infty$$

under the subsidiary condition that $\tilde{\omega}(x) - Kx$ be *decreasing* on $\mathbb{R}$ for some K, i.e., that

$$\tilde{\omega}'(x) \;\leqslant\; K.$$

As long as the *growth* of $\tilde{\omega}(x)$ is *thus limited, convergence of the logarithmic* *integral* of W is *in itself sufficient.** Referring, however, to the very elementary Paley–Wiener multiplier theorem from §A.1, Chapter X, we see that the convergence is *also sufficient* subject to a *similar requirement* on $\omega(x)$ *itself*, namely that $\omega(x)$ be *increasing* for $x \geqslant 0$.

Part of what this article's theorem does is to *generalize* the first result. As long as $W(x)$ meets the local regularity requirement, *more growth* of $\tilde{\omega}(x)$ is in fact *permissible*; the theorem tells us exactly *how much*. Could not then the Paley–Wiener result be generalized in the same way, so as to allow for a *certain amount of decrease* in $\omega(x)$ for $x \geqslant 0$?

What comes to mind is that perhaps an *analogous generalization* of the second result would carry over. In that way one is led to consider the following conjecture:

Let $W(x) = e^{\omega(x)}$ *with* $\omega(x) \geqslant 0$, $\mathscr{C}_\infty$ *and even. Suppose that*

$$\int_{-\infty}^{\infty} \frac{\omega(x)}{1+x^2}\,dx \quad < \quad \infty,$$

and that for a certain K,

$$\omega'(x) \quad \geqslant \quad -K$$

for all $x > 0$ *outside a set of disjoint intervals* $(a_k,\ b_k) \subseteq (0, \infty)$ *with*

$$\sum_k \left(\frac{b_k - a_k}{a_k}\right)^2 \quad < \quad \infty,$$

for each of which

$$\int_{a_k}^{b_k} (\omega'(x))^- \,dx \quad \leqslant \quad K(b_k - a_k).$$

Then $W(x)$ *admits multipliers.*

This conjecture is *true*. To prove it, one constructs a positive function $w(x)$, *uniformly* Lip 1 *on* $\mathbb{R}$, such that

$$w(x) \quad \geqslant \quad \omega(x)$$

* Without imposition of any local regularity requirement. Indeed, putting $Kt - \tilde{\omega}(t) = \pi\nu(t)$ and then $U(z) = \omega(0) + \int_0^\infty \log|1-(z/t)^2|d\nu(t)$, we have $\omega(x) = U(x) \leqslant U(x+i)$ (see p. 503 and the lemmas, p. 516 and 521). If $\varphi(z)$ is the entire function given by $\log|\varphi(z)| = \int_0^\infty \log|1-(z/t)^2|d[\nu(t)]$, $|\varphi(x+i)|$ *admits* multipliers in the present circumstances (see lemma, p. 521 and then pp. 546–7). But then $\exp U(x+i)$ does also (see p. 548), and so, finally, does $W(x) = \exp U(x)$.

there, and

$$\int_{-\infty}^{\infty} \frac{w(x)}{1+x^2}\,dx \quad < \quad \infty.$$

By the result in §C, Chapter X, it is *known* that $\exp w(x)$ *admits multipliers*. Hence $W(x) = \exp \omega(x)$ *must also*. The construction of $w(x)$ is outlined in the following two problems.

We may, first of all, ensure that all the intervals $(a_k,\ b_k)$ lie in $(1, \infty)$ by taking K large enough to begin with (see discussion in first half of the proof of sufficiency for the above theorem). This detail being settled, we take a function $\varphi(x) \geqslant 0$ defined on $[0, \infty)$ as follows:

$$\varphi(x) \quad = \quad K \; - \; (\omega'(x))^- \qquad \text{for} \quad x \in [0, \infty) \;\sim\; \bigcup_k (a_k, b_k);$$

$$\varphi(x) \quad = \quad K \; - \; \frac{1}{b_k - a_k}\int_{a_k}^{b_k} (\omega'(t))^-\,dt \qquad \text{for } a_k \;<\; x \;<\; b_k.$$

We then put

$$P(x) \quad = \quad \int_0^x \{(\omega'(t))^+ \; + \; \varphi(t)\}\,dt$$

and

$$N(x) \quad = \quad \int_0^x \{(\omega'(t))^- \; + \; \varphi(t)\}\,dt$$

getting, in this way, two continuous functions $P(x)$ and $N(x)$, both *increasing* on $[0, \infty]$, with

$$\omega(x) \quad = \quad P(x) \; - \; N(x), \qquad x \geqslant 0.$$

Note that

$$N(x) \;=\; Kx \qquad \text{for } x \in [0, \infty) \;\sim\; \bigcup_k (a_k, b_k);$$

in particular, $N(x) = Kx$ for $0 \leqslant x \leqslant 1$.

Fix now any number $M > K$ and consider the open set

$$\Omega \quad = \quad \Big\{x > 0\colon\; \frac{N(x) - N(\xi)}{x - \xi} \;>\; M(x - \xi)$$

$$\text{for some positive } \xi < x \ (sic!)\Big\}.$$

Ω can be obtained by shining light *up from underneath* the graph of $N(x)$ vs x *from the left*, in a direction of slope M:

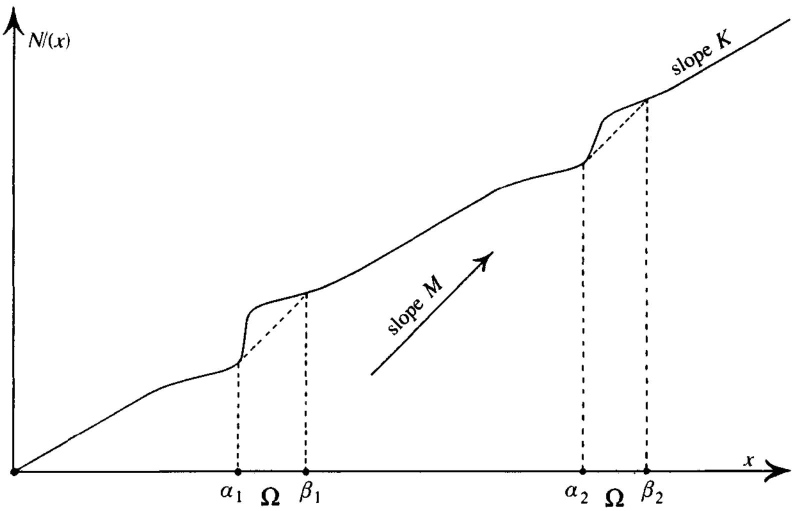

Figure 262

Ω is a *disjoint union* of certain open intervals $(\alpha_l, \beta_l) \subseteq (0, \infty)$ (not to be confounded with the given intervals (a_k, b_k)), and for $x \in (0, \infty) \sim \Omega$, $N'(x) \leqslant M$.

Problem 69

(a) Show that

$$\int_0^\infty \frac{|N(x) - Kx|}{x^2} dx < \infty.$$

(Hint: cf. the examination of $\pi v_1(t)$ in the proof of sufficiency for the above theorem.)

(b) Show that the intervals (α_l, β_l) actually lie in $(1, \infty)$.

For the rest of this problem, we make the following construction. Considering any one of the intervals (α_l, β_l), denote by $\mathscr{L}_l$ the line of slope M through the points $(\alpha_l, N(\alpha_l))$ and $(\beta_l, N(\beta_l))$. Then denote by γ_l the *abscissa* of the point where $\mathscr{L}_l$ and the *line of slope K through the origin intersect* (cf. proof of *third* lemma in §D.2, Chapter IX). Note that γ_l may well coincide with α_l or β_l, or even lie *outside* $[\alpha_l, \beta_l]$.

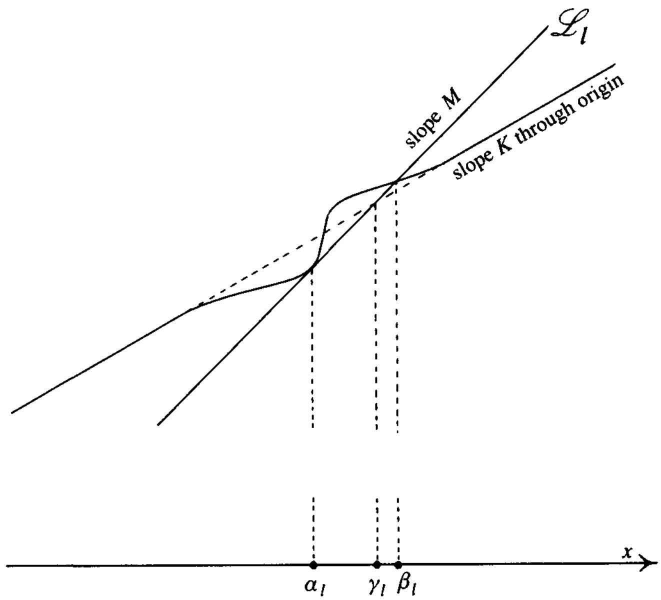

Figure 263

Let R be the set of *indices* l for which γ_l *lies to the right of the midpoint of* $(\alpha_l,\ \beta_l)$, and S the set of those indices for which γ_l lies *to the left* of that midpoint.

(c) Show that $\sum_{l\in S}((\beta_l - \alpha_l)/\alpha_l)^2 < \infty$. (Hint: cf. proof of third lemma, Chapter IX, §D.2. Note that the *difference* between our present construction and the one used there is that *left and right have exchanged rôles*, as have *above and below*!)

(d) Show that if $\eta > 0$, there cannot be infinitely many indices l in R for which $\beta_l - \alpha_l > \eta\alpha_l$. (Hint: It is enough to consider η with

$$0 < \frac{M-K}{2K}\eta < 1.$$

If $(\alpha_l,\ \beta_l)$ is any interval corresponding to an $l \in R$ with $\beta_l - \alpha_l > \eta\alpha_l$, write

$$\alpha_l' = \left(1 - \frac{M-K}{2K}\eta\right)\alpha_l$$

and then estimate

$$\int_{\alpha_l'}^{\alpha_l} \frac{Kx - N(x)}{x^2}\,dx$$

from below. Note that if this situation arises for infinitely many l in R, there must still be infinitely many of those indices for which the intervals (α_l', α_l) are *disjoint.*)

(e*) Show that $\sum_{l\in R}((\beta_l - \alpha_l)/\alpha_l)^2 < \infty$. (Hint: by (d) we may wlog suppose that $((M-K)/2K)(\beta_l - \alpha_l) < \frac{1}{2}\alpha_l$ for all $l \in R$. For those l we then put

$$\alpha_l^* = \alpha_l - \frac{M-K}{2K}(\beta_l - \alpha_l)$$

and estimate each of the integrals

$$\int_{\alpha_l^*}^{\alpha_l} \frac{Kx - N(x)}{x^2}\,\mathrm{d}x$$

from below. Starting, then, with an arbitrarily large *finite* subset R' of R, we go *first* to the *rightmost* of the (α_l, β_l) with $l \in R'$, and then make a covering argument like the one in the proof of the third lemma, §D.2, Chapter IX (used when considering the sums *over* S' figuring there), moving, however, *back towards the left instead of towards the right*, and working with the intervals (α_l^*, α_l). This gives a bound on

$$\sum_{l\in R'} \left(\frac{\beta_l - \alpha_l}{\alpha_l}\right)^2$$

independent of the size of R'.)

To finish this problem, we define a function $N_0(x)$ by putting

$$N_0(x) = N(x) \quad \text{for} \quad x \in [0, \infty) \sim \bigcup_l (\alpha_l, \beta_l)$$

and

$$N_0(x) = N(\alpha_l) + M(x - \alpha_l) \quad \text{for} \quad \alpha_l < x < \beta_l.$$

This makes

$$N_0(x) \leqslant N(x) \quad \text{for} \quad x \geqslant 0$$

and

$$N_0'(x) \leqslant M.$$

(f) Show that

$$\int_0^\infty \frac{N(x) - N_0(x)}{x^2}\,\mathrm{d}x < \infty.$$

Carrying through the steps of the last problem has given us the increasing functions $P(x)$, $N(x)$ and $N_0(x)$, having the properties indicated above.

Let now

$$w_0(x) \quad = \quad P(x) \;-\; N_0(x) \qquad \text{for } x \geqslant 0.$$

Then

$$w_0(x) \quad \geqslant \quad P(x) \;-\; N(x) \quad = \quad \omega(x), \qquad x \geqslant 0$$

while

$$w_0'(x) \quad \geqslant \quad -N_0'(x) \quad \geqslant \quad -M.$$

At the same time, since

$$\int_0^\infty \frac{\omega(x)}{1+x^2}\,\mathrm{d}x \quad < \quad \infty,$$

we have

$$\int_0^\infty \frac{w_0(x)}{1+x^2}\,\mathrm{d}x \quad < \quad \infty$$

by part (f) of the problem, since

$$w_0(x) \;-\; \omega(x) \quad = \quad N(x) \;-\; N_0(x).$$

Problem 70

Denote by $w(x)$ the *smallest majorant* of $w_0(x)$ on $[0, \infty)$ having the property that

$$|w(x) - w(x')| \;\leqslant\; M|x - x'| \qquad \text{for } x \text{ and } x' \;\geqslant\; 0.$$

The object of this problem is to prove that

$$\int_0^\infty \frac{w(x)}{1+x^2}\,\mathrm{d}x \quad < \quad \infty.$$

(a) Given $\eta > 0$, show that one cannot have $w_0(x) \; > \; \eta x$ for arbitrarily large x. (Hint: Given any such $x > 0$, estimate

$$\int_x^{(1+(\eta/2M))x} \frac{w_0(t)}{t^2}\,\mathrm{d}t$$

from below. Cf. problem 69(d).)

(b) Hence show that $w(x) \; < \; \infty$ for $x \geqslant 0$ and that in $(0, \infty)$, $w(x) > w_0(x)$ on a certain set of disjoint *bounded* open intervals lying therein.

Continuing with this problem we take *just the intervals from* (b) *that lie in* $(1, \infty)$, and denote them by $(A_n,\; B_n)$, with $n \;=\; 1, 2, 3, \ldots$. In order to verify the desired property of $w(x)$, it is enough to show that

$$\int_{A_0}^{\infty} \frac{w(x)}{x^2}\,dx \quad < \quad \infty,$$

where $A_0 \;=\; \inf_{n\geqslant 1} A_n$, a quantity $\geqslant$ 1. In

$$(A_0,\ \infty) \sim \bigcup_{n=1}^{\infty} (A_n,\ B_n)$$

we have $w(x) \;=\; w_0(x)$, where, as we know

$$\int_{1}^{\infty} \frac{w_0(x)}{x^2}\,dx \quad < \quad \infty.$$

It is therefore only necessary for us to prove that

$$\sum_{n\geqslant 1} \int_{A_n}^{B_n} \frac{w(x)}{x^2}\,dx \quad < \quad \infty.$$

Note that for each $n \geqslant 1$, we have

$$w(A_n) \;=\; w_0(A_n),$$
$$w(B_n) \;=\; w_0(B_n)$$

and

$$w(x) \;=\; w_0(A_n) \;+\; M(x - A_n) \qquad \text{for } A_n \leqslant x \leqslant B_n.$$

(c) Show that $B_n/A_n \longrightarrow 1$ as $n \longrightarrow \infty$. (Hint: If $\eta > 0$ and there are infinitely many n with $B_n/A_n \geqslant 1+\eta$, the corresponding A_n must tend to ∞ since the $(A_n,\ B_n)$ are disjoint. Observe that for such n, since $w_0(x) \geqslant \omega(x) \geqslant 0$,

$$w_0(B_n) \;\geqslant\; M\eta B_n/(1+\eta).$$

Refer to part (a).)

(d) For each $n \geqslant 1$, write

$$B_n^* \;=\; B_n \;+\; (B_n - A_n).$$

Show then that

$$\int_{A_n}^{B_n} \frac{w(x)}{x^2}\,dx \quad \leqslant \quad \left(\frac{B_n^*}{A_n}\right)^2 \int_{B_n}^{B_n^*} \frac{w_0(x)}{x^2}\,dx.$$

Hint:

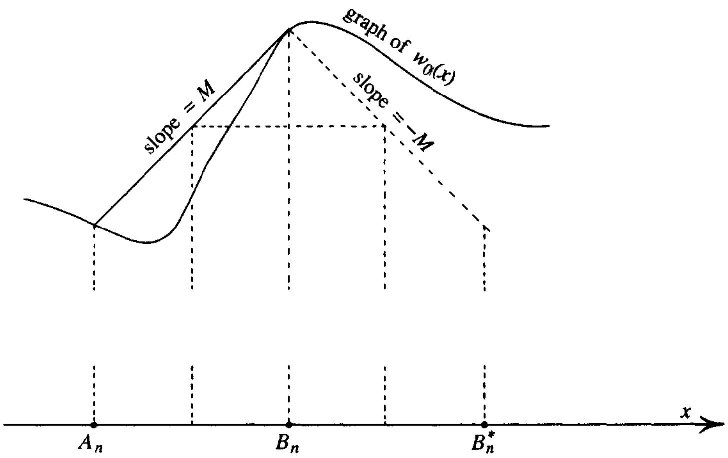

Figure 264

(e) Let us agree to call an interval (A_n, B_n) *special* if

$$w(A_n) \;\geqslant\; M(B_n - A_n).$$

Show then that if $(A_n,\ B_n)$ is special,

$$\int_{A_n}^{B_n} \frac{w(x)}{x^2}\,dx \;\leqslant\; 3\left(\frac{B_n}{A_n}\right)^2 \int_{A_n}^{B_n} \frac{w_0(x)}{x^2}\,dx.$$

Hint:

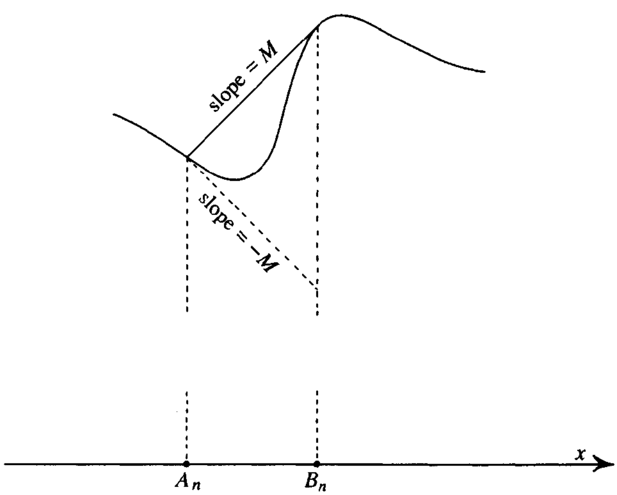

Figure 265

(f) Given any *finite* set T of integers $\geqslant 1$, obtain an upper bound *independent of* T on

$$\sum_{n\in T}\int_{A_n}^{B_n}\frac{w(x)}{x^2}\,dx,$$

hence showing that

$$\sum_{n\geqslant 1}\int_{A_n}^{B_n}\frac{w(x)}{x^2}\,dx \quad < \quad \infty.$$

(Procedure: Reindex the $(A_n,\ B_n)$ with $n\in T$ so as to have *n increase* from 1 up to some finite value as those intervals go *towards the right*. By (c), the ratios B_n^*/A_n must be bounded above by a quantity independent of T. Use then the result from (d) to estimate

$$\int_{A_1}^{B_1}\frac{w(x)}{x^2}\,dx.$$

Show next that *any interval* $(A_n,\ B_n)$ *entirely contained in* $(B_1,\ B_1^*)$ *must be special*. For such intervals, the result from (e) may be used to estimate

$$\int_{A_n}^{B_n}\frac{w(x)}{x^2}\,dx.$$

If there is an interval $(A_m,\ B_m)$ *intersecting* with $(B_1,\ B_1^*)$ but *not* lying *therein* $(m\in T)$, $(B_1,\ B_1^*)$ and $(B_m,\ B_m^*)$ are certainly *disjoint*, and we may again use the result of (d) to estimate

$$\int_{A_m}^{B_m}\frac{w(x)}{x^2}\,dx.$$

Then look to see if there are any $(A_n,\ B_n)$ entirely contained in $(B_m,\ B_m^*)$ and keep on going in this fashion, *moving steadily towards the right*, until all the $(A_n,\ B_n)$ with $n\in T$ are accounted for.)

The function $w(x)$ furnished by the constructions of these two problems is finally extended from $[0,\infty)$ to all of $\mathbb{R}$ by making it even. Then we will have

$$|w(x)-w(x')| \;\leqslant\; M|x-x'| \qquad \text{for } x \text{ and } x' \text{ in } \mathbb{R},$$

$$w(x) \;\geqslant\; \omega(x) \qquad \text{on } \mathbb{R},$$

and

$$\int_{-\infty}^{\infty}\frac{w(x)}{1+x^2}\,dx \quad < \quad \infty,$$

this last by problem 70. Our w thus has the properties we needed, and $W(x) = \exp\omega(x)$ admits multipliers, as explained at the beginning of this scholium.

One might hope to turn *around* the result just obtained and somehow show, in parallel to the necessity part of this article's theorem, that, for admittance of multipliers by a weight $W(x) \geqslant 1$ meeting the local regularity requirement, existence of a $\mathscr{C}_\infty$ even ω with

$$e^{\omega(x)} \quad \geqslant \quad W(x) \qquad \text{on } \mathbb{R}$$

enjoying the other properties enumerated in the conjecture is *necessary*.

Problem 71

Show that such a proposition would be *false*. (Hint: Were such an ω to exist, the preceding constructions would give us an even uniformly Lip 1 $w(x) \geqslant \omega(x)$ for which

$$\int_{-\infty}^{\infty} \frac{w(x)}{1+x^2}\,dx \quad < \quad \infty.$$

Modify $w(x)$ in smooth fashion near 0 so as to obtain a new uniformly Lip 1 even function $w_1(x) \geqslant 0$, equal to zero at the origin and $O(x^2)$ *near* there, agreeing with $w(x)$ for $|x| \geqslant 1$, say. Then

$$\int_0^{\infty} \frac{w_1(x)}{x^2}\,dx \quad < \quad \infty.$$

Refer to problem 62 (end of §C.4) and then to the example of §D.4.)

August, 1983 – March 1986.
Manuscript completed on
March 2, 1986, in Outremont.
Deeply affected by the
assassination of Olof Palme,
prime minister of Sweden, on
the Friday, February 28th
preceding.

Bibliography for volume II

Adamian, V.M., Arov, D.Z. and Krein, M.G. O beskonie͡chnykh Gankeli͡ovykh matritsakh i obobschonnykh zadachakh Karateodori–Feĭera i F. Rissa. *Funkts. analiz i i͡evo prilozhenia*, 2:**1** (1968), 1–19. Infinite Hankel matrices and generalized problems of Carathéodory–Fejér and F. Riesz. *Functl. Analysis and Appl.*, **2** (1968), 1–18.

Adamian, V.M., Arov, D.Z. and Krein, M.G. Beskonie͡chnye Gankeli͡ovy matritsy i obobshchonnye zadachi Karateodori–Feĭera i I. Shura. *Funkts. analiz i ievo prilozhenia* 2:**4** (1968), 1–17. Infinite Hankel matrices and generalized problems of Carathéodory–Fejér and I. Schur. *Functl. Analysis and Appl.*, **2** (1968), 269–81.

Adamian, V.M., Arov, D.Z. and Krein, M.G. Analiticheskie svoĭstva par Shmidta Gankeli͡ova operatora i obobshchonnai͡a zadacha Shura–Takagi. *Mat. Sbornik*, **86** (128), (1971), 34–75. Analytic properties of Schmidt pairs for a Hankel operator and the generalized Schur–Takagi problem. *Math. U.S.S.R. Sbornik*, **15** (1971), 31–73.

Adamian, V.M., Arov, D.Z., and Krein, M.G. Beskonie͡chnye blochno-Gankeli͡ovy matritsy i svi͡azannye s nimi problemy prodolzhenii͡a. *Izvestia Akad. Nauk Armi͡an. S.S.R. Ser. Mat.*, **6** (1971), 87–112. Infinite Hankel block matrices and related extension problems. *A.M.S. Translations* (2), **111** (1978), 133–56.

Ahlfors, L.V. *Conformal Invariants: Topics in Geometric Function Theory.* McGraw-Hill, New York, 1973.

Ahlfors, L.V. and Beurling, A. Conformal invariants and function-theoretic nullsets. *Acta Math.* **83** (1950), 101–29.

Akhiezer, N.I. and Levin, B.Ia. Obobshchenie ni͡eravenstva S.N. Bernshte͡ina dli͡a proizvodnykh ot tselykh funktsiĭ. *Issledovanii͡a po sovremennym problemam teorii funktsiĭ kompleksnovo peremennovo*, edited by A.I. Markushevich. Gosfizmatizdat., Moscow, 1960, pp 111–65. Généralisation de l'inégalité de S.N. Bernstein pour les dérivées des fonctions entières. *Fonctions d'une variable complexe. Problèmes contemporains*, edited by A.I. Marcouchevitch, translated by L. Nicolas. Gauthier-Villars, Paris, 1962, pp 109–61.

Arocena, R. and Cotlar, M. Generalized Herglotz–Bochner theorem and L^2-weighted inequalities with finite measures. *Conference on Harmonic Analysis* *in Honor of Antoni Zygmund*, edited by W. Beckner *et al.* Wadsworth, Belmont, 1983. Volume I, pp 258–69.

Arocena, R., Cotlar, M. and Sadosky, C. Weighted inequalities in L^2 and lifting properties. *Mathematical Analysis and Applications*, Part A. Advances in Math. Supplemental Studies, 7a, edited by L. Nachbin. Academic Press, New York, 1981, pp 95–128.

Bateman Manuscript Project. *Tables of Integral Transforms*, Volume II, edited by A. Erdélyi. McGraw-Hill, New York, 1954.

Bernstein, V. *Leçons sur les progrès récents de la théorie des séries de Dirichlet.* Gauthier-Villars, Paris, 1933.

Beurling, A. *Etudes sur un problème de majoration.* Thesis, Uppsala, 1933.

Beurling, A. and Malliavin, P. On the zeros of entire functions of exponential type (I). Preprint, 1961.

Beurling, A. and Malliavin, P. On Fourier transforms of measures with compact support. *Acta Math.*, **107** (1962), 291–309.

Beurling, A. and Malliavin, P. On the closure of characters and the zeros of entire functions. *Acta Math.*, **118** (1967), 79–93.

Boas, R.P. *Entire Functions.* Academic press, New York, 1954.

Borichev, A.A. and Volberg, A.L. Teoremy i͡edinstvennosti dli͡a pochti analiticheskikh funktsiĭ. *Algebra i analiz*, **1** (1989), 146–77.

Carleson, L. *Selected Problems on Exceptional Sets.* Van Nostrand, Princeton, 1967.

Carleson, L. and Jones, P. Weighted norm inequalities and a theorem of Koosis. *Institut Mittag-Leffler*, Report no 2, 1981.

Coifman, R. A real variable characterization of H^p. *Studia Math.* **51** (1974), 269–74.

Cotlar, M. and Sadosky, C. On some L^p versions of the Helson–Szegő theorem. *Conference on Harmonic Analysis in Honor of Antoni Zygmund*, edited by W. Beckner, *et al.* Wadsworth, Belmont, 1983. Volume I, pp 306–17.

De Branges, L. The α-local operator problem. *Canadian J. Math.* **11** (1959), 583–92.

De Branges, L. *Hilbert Spaces of Entire Functions.* Prentice-Hall, Englewood Cliffs, 1968

Duren, P. *Theory of H^p Spaces.* Academic Press, New York, 1970.

Erdélyi, A. *et al.*—see under *Bateman Manuscript Project.*

Frostman, O. *Potential d'équilibre et capacité des ensembles avec quelques applications à la théorie des fonctions.* Thesis, Lund, 1935.

Fuchs, W. On the growth of functions of mean type. *Proc. Edinburgh Math. Soc.* Ser. 2, **9** (1954), 53–70.

Fuchs, W. *Topics in the Theory of Functions of one Complex Variable.* Van Nostrand, Princeton, 1967.

Gamelin, T. *Uniform Algebras and Jensen Measures.* L.M.S. lecture note series, 32. Cambridge Univ. Press, Cambridge, 1978.

García-Cuerva, J. and Rubio de Francia, J.L. *Weighted Norm Inequalities and Related Topics.* North-Holland, Amsterdam, 1985.

Garnett, J. *Analytic Capacity and Measure.* Lecture notes in math., 297. Springer, Berlin, 1972.

Garnett, J. *Bounded Analytic Functions.* Academic Press, New York, 1981.

Grötzsch, H. A series of papers, all in the *Berichte der Sächs. Akad. zu Leipzig.* Here are some of them:
Extremalprobleme der konformen Abblidung, in Volume 80 (1928), pp 367–76.
Über konforme Abbildung unendlich vielfach zusammenhängender schlichter Bereiche mit endlich vielen Häufungsrandkomponenten, in Volume 81 (1929), pp 51–86.
Zur konformen Abbildung mehrfach zusammenhängender schlichter Bereiche, in Volume 83 (1931), pp 67–76.
Zum Parallelschlitztheorem der konformen Abbildung schlichter unendlich-vielfach zusammenhängender Bereiche, in Volume 83 (1931), pp 185–200.
Über die Verzerrung bei schlichter konformer Abbildung mehrfach zusammenhängender schlichter Bereiche, in Volume 83 (1931), pp 283–97.

Haliste, K. Estimates of harmonic measures. *Arkiv för mat.* **6** (1967), 1–31.

Heins, M. *Selected Topics in the Classical Theory of Functions of a Complex Variable.* Holt, Rinehart and Winston, New York, 1962.

Helms, L. *Introduction to Potential Theory.* Wiley-Interscience, New York, 1969.

Helson, H. and Sarason, D. Past and future. *Math. Scand.* **21** (1967), 5–16.

Helson, H. and Szegő, G. A problem in prediction theory. *Annali mat. pura ed appl., ser.* 4, *Bologna.* **51** (1960), 107–38.

Hersch, J. Longueurs extrémales et théorie des fonctions. *Comm. Math. Helv.* **29** (1955), 301–37.

Herz, C. Bounded mean oscillation and regulated martingales. *Trans. A.M.S.*, **193** (1974), 199–215.

Hruščev, S.V. (Khrushchëv) and Nikolskiĭ, N.K. Funktsionalnai͡a model i ni͡ekotorye zadachi spektralnoĭ teorii funktsiĭ. *Trudy mat. inst. im. Steklova,* **176** (1987), 97–210. A function model and some problems in the spectral theory of functions. *Proc. Steklov Inst. of Math.*, A.M.S., Providence, **176** (1988), 101–214.

Hruščev, S.V. (Khrushchëv), Nikolskiĭ, N.K. and Pavlov, B.S. Unconditional bases of exponentials and of reproducing kernels. *Complex Analysis and Spectral Theory,* edited by V.P. Havin and N.K. Nikolskiĭ, Lecture notes in math., 864. Springer, Berlin, 1981, pp 214–335.

Kahane, J.P. Sur quelques problèmes d'unicité et de prolongement, relatifs aux fonctions approachables par des sommes d'exponentielles. *Annales Inst. Fourier Grenoble,* **5** (1955), 39–130.

Kahane, J.P. Travaux de Beurling et Malliavin. *Séminaire Bourbaki,* 1961/62, fasc 1, exposé no 225, 13 pages. Secrétariat mathématique, Inst. H. Poincaré, Paris, 1962. Reprinted by Benjamin, New York, 1966.

Kahane, J.P. and Salem, R. *Ensembles parfaits et séries trigonométriques.* Hermann, Paris, 1963.

Kellogg, O.D. *Foundations of Potential Theory.* Dover, New York, 1953.

Koosis, P. Sur la non-totalité de certaines suites d'exponentielles sur des intervalles assez longs. *Annales Ecole Norm. Sup.*, Sér. 3. **75** (1958), 125–52.

Koosis, P. Sur la totalite des systèmes d'exponentielles imaginaires. *C.R. Acad. Sci. Paris.* **250** (1960), 2102–3.

Koosis, P. Weighted quadratic means of Hilbert transforms. *Duke Math. J.*, **38** (1971), 609–34.

Koosis, P. Moyennes quadratiques de transformées de Hilbert et fonctions de type exponential. *C.R. Acad. Sci. Paris*, Sér. A, **276** (1973), 1201–4.

Koosis, P. Harmonic estimation in certain slit regions and a theorem of Beurling and Malliavin. *Acta Math.*, **142** (1979), 275–304.

Koosis, P. *Introduction to H_p Spaces.* L.M.S. lecture note series, 40. Cambridge Univ. Press, Cambridge, 1980.

Koosis, P. Moyennes quadratiques pondérées de fonctions périodiques et de leurs conjuguées harmoniques. *C.R. Acad. Sci. Paris*, Sér A, **291** (1980), 255–7.

Koosis, P. Entire functions of exponential type as multipliers for weight functions. *Pacific J. Math.*, **95** (1981), 105–23.

Koosis, P. Fonctions entières de type exponentiel comme multiplicateurs. Un exemple et une condition nécessaire et suffisante. *Annales Ecole Norm. Sup.*, Sér 4, **16** (1983), 375–407.

Koosis, P. La plus petite majorante surharmonique at son rapport avec l'existence des fonctions entières de type exponentiel jouant le rôle de multiplicateurs. *Annales Inst. Fourier Grenoble*, **33** (1983), 67–107.

Kriete, T. On the structure of certain $H^2(\mu)$ spaces. *Indiana Univ. Math. J.*, **28** (1979), 757–73.

Landkof, N.S. *Osnovy sovremennoĭ teorii potentsiala.* Nauka, Moscow, 1966. *Foundations of Modern Potential Theory.* Springer, New York, 1972.

Lebedev, N.A. *Printisip ploshchadeĭ v teorii odnolistnykh funktsiĭ.* Nauka, Moscow, 1975.

Leontiev, A.F. *Rī͡ady eksponent.* Nauka, Moscow, 1976.

Levinson, N. *Gap and Density Theorems.* Amer. Math. Soc. (Colloq. Publ., Volume 26), New York, 1940, reprinted 1968.

Lindelöf, E. *Le calcul des résidus et ses applications á la théorie des fonctions.* Gauthier-Villars, Paris, 1905.

Malliavin, P. Sur la croissance radiale d'une fonction méromorphe. *Illinois J. of Math.*, **1** (1957), 259–96.

Malliavin, P. The 1961 preprint. See Beurling, A. and Malliavin, P.

Malliavn, P. On the multiplier theorem for Fourier transforms of measures with compact support. *Arkiv för Mat.*, **17** (1979), 69–81.

Malliavin, P. and Rubel, L. On small entire functions of exponential type with given zeros. *Bull. Soc. Math. de France*, **89** (1961), 175–206.

Mandelbrojt, S. *Dirichlet Series.* Rice Institute Pamphlet, Volume 31, Houston, 1944.

Mandelbrojt, S. *Séries adhérentes. Régularisation des suites. Applications.* Gauthier-Villars, Paris, 1952.

Mandelbrojt, S. *Séries de Dirichlet – Principes et méthodes.* Gauthier-Villars, Paris, 1969. *Dirichlet Series. Principles and Methods.* Reidel, Dordrecht, 1972.

Markushevich, A.I. *Teorī͡a analiticheskikh funktsiĭ.* Gostekhizdat, Moscow, 1950. Second augmented and corrected edition, Volumes I, II. Nauka, Moscow, 1967–8. *Theory of Functions of a Complex Variable*, Volumes I–III. Prentice-Hall, Englewood Cliffs, 1965–7. Second edition in one vol., Chelsea, New York, 1977.

Nagy, Béla Sz. and Foiaş, C. *Harmonic Analysis of Operators on Hilbert Space.* North-Holland, Amsterdam, 1970.

Nehari, Z. *Conformal Mapping.* McGraw-Hill, New York, 1952.

Neuwirth, J. and Newman, D.J. Positive $H^{1/2}$ functions are constant. *Proc. A.M.S.*, **18** (1967), 958.

Nevanlinna, R. *Eindeutige analytische Funktionen.* Second edition, Springer, Berlin, 1953. *Analytic Functions.* Springer, New York, 1970.

Nikolskiĭ, N.K. *Lektsii ob operatore sdviga.* Nauka, Moscow, 1980. *Treatise on the Shift Operator. Spectral Function Theory.* With appendix by S.V. Hruščev and V.V. Peller. Springer, Berlin, 1986.

Ohtsuka, M. *Dirichlet Problem, Extremal Length and Prime Ends.* Van Nostrand-Reinhold, New York, 1970.

Paley, R. and Wiener, N. *Fourier Transforms in the Complex Domain.* Amer. Math. Soc. (Colloq. Publ., Volume 19), Providence, 1934, reprinted 1960.

Pfluger, A. Extremallängen und Kapazität. *Comm. Math. Helv.* **29** (1955), 120–31.

Phelps, R. *Lectures on Choquet's Theorem.* Van Nostrand, Princeton, 1966.

Pólya, G. Untersuchungen über Lücken und Singularitäten von Potenzreihen. *Math. Zeitschr.*, **29** (1929), 549–640.

Proceedings of Symposia in Pure Mathematics, Volume VII. Convexity. Amer. Math. Soc., Providence, 1963.

Redheffer, R. On even entire functions with zeros having a density. *Trans. A.M.S.*, **77** (1954), 32–61.

Redheffer, R. Interpolation with entire functions having a regular distribution of zeros. *J. Analyse Math.*, **20** (1967), 353–70.

Redheffer, R. Eine Nevanlinna-Picardsche Theorie en miniature. *Arkiv för Mat.*, **7** (1967), 49–59.

Redheffer, R. Elementary remarks on completeness. *Duke Math. J.*, **35** (1968), 103–16.

Redheffer, R. Two consequences of the Beurling-Malliavin theory. *Proc. A.M.S.*, **36** (1972), 116–22.

Redheffer, R. Completeness of sets of complex exponentials. *Advances in Math.*, **24** (1977), 1–62.

Rubel, L. Necessary and sufficient conditions for Carlson's theorem on entire functions. *Trans. A.M.S.*, **83** (1956), 417–29.

Rubio de Francia, J.L. Boundedness of maximal functions and singular integrals in weighted L^p spaces. *Proc. A.M.S.*, **83** (1981), 673–9.

Sarason, D. *Function Theory on the Unit Circle.* Virginia Polytechnic Inst., Blacksburg, 1978.

Schwartz, L. *Etude des sommes d'exponentielles.* Hermann, Paris, 1959.

Titchmarsh, E.C. *The Theory of Functions.* Second edition, Oxford Univ. Press, Oxford, 1939. Reprinted 1952.

Treil, S.R. Geometrischeskiĭ podkhod k vesovym otsenkam preobrazovaniía Gilberta. *Funkts. analiz i íevo prilozh.*, 17:**4** (1983), 90–1. A geometric approach to the weighted estimates of Hilbert transforms. *Funct. Analysis and Appl.*, 17:**4** (1983), 319–21, Plenum Publ. Corp.

Treil, S.R. Operatornyĭ podkhod k vesovym otsenkam singulíarnykh integralov. *Akad. Nauk S.S.S.R., Mat. Inst. im. Steklova, Leningrad. Zapiski nauchnykh seminarov L.O.M.I.*, **135** (1984), 150–74.

Tsuji, M. *Potential Theory in Modern Function Theory.* Maruzen, Tokyo, 1959. Reprinted by Chelsea, New York, 1975.

Zalcman, L. *Analytic Capacity and Rational Approximation.* Lecture notes in math., 50. Springer, Berlin, 1968.

Index