where $f(t)$ belongs to L^p. Then

$$|F(x_\nu+h_\nu)-F(x_\nu)| = \left|\int_{x_\nu}^{x_\nu+h_\nu} f(t)\,dt\right|$$

$$\leqslant \left\{\int_{x_\nu}^{x_\nu+h_\nu} |f(t)|^p\,dt\right\}^{1/p}\left\{\int_{x_\nu}^{x_\nu+h_\nu} dt\right\}^{1-1/p} = h_\nu^{1-1/p}\left\{\int_{x_\nu}^{x_\nu+h_\nu} |f(t)|^p\,dt\right\}^{1/p}.$$

Hence

$$\sum |F(x_\nu+h_\nu)-F(x_\nu)|^p h_\nu^{1-p} \leqslant \sum \int_{x_\nu}^{x_\nu+h_\nu} |f(t)|^p\,dt \leqslant \int_a^b |f(t)|^p\,dt,$$

so that the condition is necessary. Since

$$\sum \int_{x_\nu}^{x_\nu+h_\nu} |f(t)|^p\,dt$$

tends to zero with $\sum h_\nu$, the alternative condition is also necessary.

Suppose now that the condition is satisfied, and let M be the upper bound of the given sums. Then, by Hölder's inequality for sums,

$$\sum |F(x_\nu+h_\nu)-F(x_\nu)| = \sum |F(x_\nu+h_\nu)-F(x_\nu)|h_\nu^{1/p-1}.h_\nu^{1-1/p}$$

$$\leqslant \{\sum |F(x_\nu+h_\nu)-F(x_\nu)|^p h_\nu^{1-p}\}^{1/p}(\sum h_\nu)^{1-1/p} \leqslant M^{1/p}(\sum h_\nu)^{1-1/p},$$

which tends to zero with $\sum h_\nu$. Hence $F(x)$ is absolutely continuous, and so is an integral, say

$$F(x) = F(a) + \int_a^x f(t)\,dt.$$

It remains to prove that $f(t)$ belongs to L^p. Consider a sequence of finite sets of points in the interval, the mth set being $x_{m,1}, x_{m,2},\ldots, x_{m,n}$, such that

$$\lim_{m\to\infty} \max_\nu (x_{m,\nu+1}-x_{m,\nu}) = 0.$$

For example, if the interval is $(0, 1)$ we may take $x_{m,\nu} = \nu/2^m$.

Let $$f_m(x) = \frac{F(x_{m,\nu+1})-F(x_{m,\nu})}{x_{m,\nu+1}-x_{m,\nu}}$$

in each interval $x_{m,\nu} \leqslant x < x_{m,\nu+1}$. If x is not one of the points

$x_{m,\nu}$, and $F'(x)$ exists, and $x_{m,\nu} < x < x_{m,\nu+1}$, then

$$f_m(x) = \frac{F(x_{m,\nu+1})-F(x)}{x_{m,\nu+1}-x}\,\frac{x_{m,\nu+1}-x}{x_{m,\nu+1}-x_{m,\nu}} + \\ + \frac{F(x_{m,\nu})-F(x)}{x_{m,\nu}-x}\,\frac{x-x_{m,\nu}}{x_{m,\nu+1}-x_{m,\nu}}$$

$$= \{F'(x)+\delta_1\}\frac{x_{m,\nu+1}-x}{x_{m,\nu+1}-x_{m,\nu}} + \{F'(x)+\delta_2\}\frac{x-x_{m,\nu}}{x_{m,\nu+1}-x_{m,\nu}}$$

$$= F'(x)+\delta_3,$$

where $|\delta_3| \leqslant |\delta_1|+|\delta_2|$, and δ_1 and δ_2 tend to zero as

$$x_{m,\nu+1}-x_{m,\nu} \to 0.$$

Hence
$$\lim_{m\to\infty} f_m(x) = F'(x) = f(x)$$

almost everywhere. Also

$$\int_a^b |f_m(x)|^p\,dx = \sum |F(x_{m,\nu+1})-F(x_{m,\nu})|^p |x_{m,\nu+1}-x_{m,\nu}|^{1-p} \leqslant M.$$

Hence, by Fatou's theorem (§ 10.81), $f(x)$ belongs to L^p, and

$$\int_a^b |f(x)|^p\,dx \leqslant \varliminf \int_a^b |f_m(x)|^p\,dx \leqslant M.$$

12.5. Mean convergence. If we are given a sequence of numbers, say s_n, we have usually to consider the behaviour of the difference s_n-s between s_n and a given number s. In dealing with a sequence of functions, say $f_n(x)$, and a given function $f(x)$, it is often not the difference but the mean or average value of the difference which is important. This can be defined in various ways. If the functions belong to the class L^p, where $p \geqslant 1$, we consider the integral

$$\int_a^b |f_n(x)-f(x)|^p\,dx. \tag{1}$$

If this integral tends to zero as $n \to \infty$, we say that $f_n(x)$ converges in mean (*en moyenne*, *im Mittel*), to $f(x)$, with index p.

If
$$\int_a^b |f_m(x)-f_n(x)|^p\,dx \tag{2}$$

tends to zero as m and n tend independently to infinity, we say that the sequence $f_n(x)$ converges in mean, with index p. Here the function $f(x)$ is not involved explicitly.

The fundamental theorem* of the subject is that *if the sequence $f_n(x)$ converges in mean, with index p, then there is a function $f(x)$ of the class L^p, defined uniquely apart from sets of measure zero, to which $f_n(x)$ converges in mean.*

The theorem is analogous to the 'general principle of convergence', that if $s_m - s_n \to 0$, then there is a number s to which s_n tends.

A word of explanation is necessary with regard to the 'uniqueness' of the limit-function $f(x)$. Suppose that we have found a function $f(x)$ which satisfies the given conditions. Then obviously any other function $g(x)$ which is equal to $f(x)$ almost everywhere also satisfies the conditions. So at any particular point the value of $f(x)$ is undetermined, though its general aggregate of values is in a sense determined. The function $f(x)$ should be regarded as a representative of a class of functions, any two of which are equal almost everywhere, and so all of which behave in the same way in integration.

The theorem for a finite interval and $p > 1$ may be proved as follows. To every integer ν corresponds a smallest positive integer n_ν such that

$$\int_a^b |f_m(x) - f_n(x)|^p \, dx < \frac{1}{3^\nu} \qquad (m \geqslant n_\nu,\ n \geqslant n_\nu).$$

In particular,

$$\int_a^b |f_{n_{\nu+1}}(x) - f_{n_\nu}(x)|^p \, dx < \frac{1}{3^\nu} \qquad (\nu = 1,\ 2,\ 3,\ldots).$$

If E_ν is the set where $|f_{n_{\nu+1}}(x) - f_{n_\nu}(x)| > 2^{-\nu/p}$, it follows that $m(E_\nu) < (\frac{2}{3})^\nu$. Hence the series

$$\sum_{\nu=1}^{\infty} |f_{n_{\nu+1}}(x) - f_{n_\nu}(x)|$$

is convergent, by comparison with $\sum 2^{-\nu/p}$, if, for some value of N, x does not belong to the set $E_{N+1} + E_{N+2} + \ldots$. Since the measure of this set tends to 0 as $N \to \infty$, it follows that the

* Fischer (1), F. Riesz (1), (2); W. H. and G. C. Young (2), where several alternative proofs are given; and Hobson (1).

above series is convergent for almost all values of x; hence so is

$$\sum_{\nu=1}^{\infty}\{f_{n_{\nu+1}}(x)-f_{n_\nu}(x)\},$$

i.e. there is a function $f(x)$ (defined almost everywhere) such that

$$\lim_{\nu\to\infty} f_{n_\nu}(x)=f(x)$$

almost everywhere.

This function $f(x)$ has the required property. For by Fatou's theorem

$$\varliminf_{\nu\to\infty}\int_a^b |f_{n_\mu}(x)-f_{n_\nu}(x)|^p\,dx \geqslant \int_a^b |f_{n_\mu}(x)-f(x)|^p\,dx;$$

but $$\int_a^b |f_{n_\mu}(x)-f_{n_\nu}(x)|^p\,dx<\epsilon \qquad (\mu>\nu_0,\ \nu>\nu_0).$$

Hence $$\int_a^b |f_{n_\mu}(x)-f(x)|^p\,dx\leqslant\epsilon \qquad (\mu>\nu_0),$$

i.e. $$\lim\int_a^b |f_{n_\mu}(x)-f(x)|^p\,dx=0,$$

i.e. the sub-sequence $f_{n_\mu}(x)$ converges in mean to $f(x)$.

Also, by Minkowski's inequality,

$$\left\{\int_a^b |f(x)-f_n(x)|^p\,dx\right\}^{1/p} \leqslant\left\{\int_a^b |f(x)-f_{n_\mu}(x)|^p\,dx\right\}^{1/p}+\left\{\int_a^b |f_{n_\mu}(x)-f_n(x)|^p\,dx\right\}^{1/p},$$

which tends to zero as n and n_μ tend to infinity, by what has just been proved and the original hypothesis. Hence the whole sequence $f_n(x)$ converges in mean to $f(x)$.

Finally, suppose that $f_n(x)$ converges in mean to $f(x)$ and also to $g(x)$. Then

$$\left(\int_a^b |f-g|^p\,dx\right)^{1/p}\leqslant\left(\int_a^b |f-f_n|^p\,dx\right)^{1/p}+\left(\int_a^b |g-f_n|^p\,dx\right)^{1/p}\to 0.$$

Hence the left-hand side is 0, and so $f(x)=g(x)$ almost everywhere.

If $p=1$ the proof is simpler, since it is not necessary to use Minkowski's inequality.

12.51. The proof applies almost unchanged to an infinite interval. We find that the above series are convergent almost everywhere in (a,b) for every b, i.e. almost everywhere in (a,∞); and Fatou's theorem holds for an infinite interval; for, taking the set E of § 10.81 to be the interval (a,b),

$$\int_a^b f(x)\,dx \leqslant \underline{\lim} \int_a^b f_n(x)\,dx \leqslant \underline{\lim} \int_a^\infty f_n(x)\,dx,$$

and, making $b\to\infty$, we obtain the required extension of Fatou's theorem. The proof for an infinite interval now follows.

12.52. *We have also (for a finite or infinite interval)*

$$\lim_{n\to\infty} \int |f_n(x)|^p\,dx = \int |f(x)|^p\,dx.$$

For by Minkowski's inequality

$$\left\{\int |f_n(x)|^p\,dx\right\}^{1/p} \leqslant \left\{\int |f(x)|^p\,dx\right\}^{1/p} + \left\{\int |f(x)-f_n(x)|^p\,dx\right\}^{1/p},$$

and also

$$\left\{\int |f(x)|^p\,dx\right\}^{1/p} \leqslant \left\{\int |f_n(x)|^p\,dx\right\}^{1/p} + \left\{\int |f(x)-f_n(x)|^p\,dx\right\}^{1/p}.$$

Hence $$\lim_{n\to\infty}\left\{\int |f_n(x)|^p\,dx\right\}^{1/p} = \left\{\int |f(x)|^p\,dx\right\}^{1/p},$$

and the result follows.

12.53. *If $f_n(x)$ converges in mean to $f(x)$ with index p, and $g(x)$ belongs to $L^{p/(p-1)}$, then*

$$\lim_{n\to\infty} \int f_n(x)g(x)\,dx = \int f(x)g(x)\,dx. \tag{1}$$

For

$$\left|\int \{f_n(x)-f(x)\}g(x)\,dx\right| \leqslant \left\{\int |f_n(x)-f(x)|^p\,dx\right\}^{1/p} \left\{\int |g(x)|^{p/(p-1)}\,dx\right\}^{1-1/p},$$

which tends to zero.

In particular

$$\lim_{n\to\infty} \int_a^x f_n(t)\,dt = \int_a^x f(t)\,dt \tag{2}$$

for all values of x in the interval considered.

For the function $g(t) = 1$ $(a<t<x)$, $=0$ $(t>x)$, belongs to $L^{p/(p-1)}$.

Examples. (i) If $f_n(x) \to f(x)$ boundedly over a finite interval, then $f_n(x)$ converges in mean to $f(x)$ with any index.

(ii) Consider the closed intervals $(0, \frac{1}{2})$, $(\frac{1}{2}, 1)$, $(0, \frac{1}{3})$, $(\frac{1}{3}, \frac{2}{3})$, $(\frac{2}{3}, 1)$, $(0, \frac{1}{4})$, etc. Let $f_n(x) = 1$ in the nth interval, and $f_n(x) = 0$ in the remainder of the interval $(0, 1)$. Then $f_n(x)$ converges in mean to zero in $(0, 1)$, with any index; but $f_n(x)$ does not tend to zero for any value of x.

(iii) If $f_n(x)$ converges in mean to $f(x)$, and $f_n(x) \to g(x)$ almost everywhere, then $f(x) = g(x)$ almost everywhere. [Use Egoroff's theorem.]

(iv) If $f_n(x)$ converges in mean to $f(x)$ with index p, and $g_n(x)$ to $g(x)$ with index $p/(p-1)$, then $\int f_n g_n\, dx \to \int fg\, dx$.

12.6. Repeated integrals. As in the elementary cases considered in § 1.8, the equation

$$\int_a^b dx \int_\alpha^\beta f(x,y)\, dy = \int_\alpha^\beta dy \int_a^b f(x,y)\, dx \tag{1}$$

is in general true in the Lebesgue theory. The general discussion of this, however, depends on the theory of the double integral

$$\int\int f(x,y)\, dxdy,$$

which in turn depends on the theory of two-dimensional sets of points. It would take us too far to carry this out in detail.

There is, however, a particular kind of repeated integral which includes many cases of interest, and which can be dealt with by the theory already developed.

Let $f(x)$ be integrable in the Lebesgue sense over (a, b), and $g(y)$ over (α, β), and let $k(x, y)$ be a continuous function of both variables, or, if it has discontinuities, let them be of the type described in § 1.82. *Then*

$$\int_a^b f(x)\, dx \int_\alpha^\beta g(y)k(x,y)\, dy = \int_\alpha^\beta g(y)\, dy \int_a^b f(x)k(x,y)\, dx. \tag{2}$$

Suppose first that $f(x)$ and $g(y)$ are bounded, say $|f(x)| \leqslant M$, $|g(y)| \leqslant M$. Let $|k(x,y)| \leqslant K$.

Let $\phi(x)$ be a continuous function satisfying 12.2 (1); and let $|\phi(x)| \leqslant M$. Let $\psi(y)$ be a continuous function related in the same way to $g(y)$.

Call the left-hand side of (2) I, and let

$$I' = \int_a^b \phi(x)\, dx \int_\alpha^\beta \psi(y)k(x,y)\, dy.$$

Then

$$I-I' = \int_a^b \{f(x)-\phi(x)\}\,dx \int_\alpha^\beta g(y)k(x,y)\,dy +$$

$$+\int_a^b \phi(x)\,dx \int_\alpha^\beta \{g(y)-\psi(y)\}k(x,y)\,dy,$$

and hence

$$|I-I'| \leqslant \int_a^b |f(x)-\phi(x)|(\beta-\alpha)MK\,dx + (b-a)M\eta$$

$$\leqslant MK(\beta-\alpha+b-a)\eta.$$

Similarly, if the right-hand side of (2) is J, and

$$J' = \int_\alpha^\beta \psi(y)\,dy \int_a^b \phi(x)k(x,y)\,dx,$$

then $|J-J'|$ tends to 0 with η.

But, by the theorem of § 1.81, $I' = J'$, since $\phi(x)\psi(y)k(x,y)$ is continuous, or has discontinuities of the restricted type.

Hence $|I-J|$ tends to 0 with η, and so $I = J$.

The extension to unbounded functions may be left to the reader; we suppose first that $f(x)$ and $g(x)$ are positive, and argue with $\{f(x)\}_n$ and $\{g(x)\}_n$ in the usual manner.

12.61. *If $f(x)$ is integrable over $(0, 1)$, and $g(x)$ over $(0, 2)$, then the integral*

$$\int_0^1 f(x)g(x+t)\,dx$$

exists for almost all values of t in $(0, 1)$, and represents an integrable function of t.

It is sufficient to consider the case where f and g are positive. Define $\{f(x)\}_n$ as usual, and let

$$F_n(t) = \int_0^1 \{f(x)\}_n\, g(x+t)\,dx.$$

This integral exists for all values of t, and, for a given n, $F_n(t)$ is bounded, and for each value of t it is a non-decreasing function of n. Also

$$\int_0^1 F_n(t)\,dt = \int_0^1 dt \int_0^1 \{f(x)\}_n\, g(x+t)\,dx = \int_0^1 \{f(x)\}_n\,dx \int_0^1 g(x+t)\,dt, \tag{1}$$

if we may invert the order of integration.

To justify this, approximate to g by a continuous function ψ, as in the previous proof, and let

$$\chi(t) = \int_0^1 \{f(x)\}_n \psi(x+t)\,dx.$$

Then
$$\int_0^1 \chi(t)\,dt = \int_0^1 \{f(x)\}_n\,dx \int_0^1 \psi(x+t)\,dt, \tag{2}$$

this inversion being justified by the above theorem. Now

$$|F_n(t)-\chi(t)| \leqslant \int_0^1 \{f(x)\}_n |g(x+t)-\psi(x+t)|\,dx < n\eta,$$

so that the left-hand side of (1) differs from that of (2) by less than $n\eta$. Similarly the right-hand sides differ by less than $n\eta$. Hence, making $\eta \to 0$, we obtain (1).

Hence
$$\int_0^1 F_n(t)\,dt \leqslant \int_0^1 f(x)\,dx \int_0^2 g(y)\,dy.$$

Hence, as $n \to \infty$, $F_n(t)$ tends to a finite limit for almost all values of t (§ 10.82). The result now follows from the theorem of § 10.82.

12.62. Repeated infinite integrals. *If $f(x)$, $g(y)$, and $k(x,y)$ are positive, and the conditions of* § 12.6 *are satisfied for all values of $b > a$ and $\beta > \alpha$, then*

$$\int_a^\infty f(x)\,dx \int_\alpha^\infty g(y)k(x,y)\,dy = \int_\alpha^\infty g(y)\,dy \int_a^\infty f(x)k(x,y)\,dx \tag{1}$$

provided that either side is convergent.

The theorem is similar to that of § 1.85, but the supplementary conditions which appear there are now a consequence of the main hypothesis.

Suppose that the right-hand side of (1) is convergent. Since

$$\int_a^X f(x)k(x,y)\,dx \leqslant \int_a^\infty f(x)k(x,y)\,dx, \tag{2}$$

and the left-hand side of (2) is a measurable (in fact a continuous) function of y, it follows that

$$\int_\alpha^\infty g(y)\,dy \int_a^X f(x)k(x,y)\,dx$$

is convergent. Hence

$$\lim_{n\to\infty}\int_\alpha^n g(y)\,dy\int_a^X f(x)k(x,y)\,dx=\lim_{n\to\infty}\int_a^X f(x)\,dx\int_\alpha^n g(y)k(x,y)\,dy$$

is finite. Also

$$F_n(x)=f(x)\int_\alpha^n g(y)k(x,y)\,dy$$

is a non-decreasing function of n for each value of x. It therefore follows from § 10.82 that $F_n(x)$ tends to a finite limit, as $n\to\infty$, for almost all values of x in (a, X); i.e.

$$\int_\alpha^\infty g(y)k(x,y)\,dy$$

is convergent for almost all values of x in (a, X); and by § 10.82

$$\int_a^X f(x)\,dx\int_\alpha^\infty g(y)k(x,y)\,dy=\lim_{n\to\infty}\int_a^X f(x)\,dx\int_\alpha^n g(y)k(x,y)\,dy$$

$$=\lim_{n\to\infty}\int_\alpha^n g(y)\,dy\int_a^X f(x)k(x,y)\,dx=\int_\alpha^\infty g(y)\,dy\int_a^X f(x)k(x,y)\,dx. \quad (3)$$

By (2), the right-hand side of (3) is bounded as $X\to\infty$; hence so is the left-hand side, and therefore the left-hand side of (1) is convergent.

We can now prove in a similar way that the order of integration in

$$\int_\alpha^Y g(y)\,dy\int_a^\infty f(x)k(x,y)\,dx$$

may be inverted. The final result then follows as in § 1.85.

MISCELLANEOUS EXAMPLES.

1. If $f(x)$ is integrable over (a, b), and $a=x_0<x_1<x_2<\dots<x_n=b$, then

$$\sum_{\nu=0}^{n-1}\left|\int_{x_\nu}^{x_{\nu+1}} f(t)\,dt\right|\to\int_a^b |f(t)|\,dt$$

as the greatest partial interval tends to zero.

[The proof is elementary for continuous functions; and then the general result may be deduced by means of the theorem of § 12.2.]

2. If $F(x)$ is absolutely continuous in (a, b), its total variation in the interval is

$$\int_a^b |F'(x)|\,dx.$$

[Use the result of the previous example.]

3. Show that, if $f(x)$ and $g(x)$ belong to L^2,

$$\int \{f(x)\}^2\,dx \int \{g(x)\}^2\,dx - \left\{\int f(x)g(x)\,dx\right\}^2$$
$$= \tfrac{1}{2}\int dy \int \{f(x)g(y)-f(y)g(x)\}^2\,dx,$$

and hence obtain another proof of Schwarz's inequality.

4. We use $\log' x$ to denote $\log x$ if $x \geqslant e$, and $\log' x = 1$ if $x < e$.
Show that if $\{f(x)\}^2 \log' f(x)$ and $\{g(x)\}^2/\log' g(x)$ are integrable over (a, b), then $f(x)g(x)$ is integrable over (a, b).

[Let E be the set where $f \leqslant g/\log' g$. Then

$$\int_E fg\,dx \leqslant \int_E \frac{\{g(x)\}^2}{\log' g(x)}\,dx.$$

In CE, $g \leqslant f \log' g$. If $g \leqslant e$ this gives

$$g \leqslant f \leqslant f \log' f.$$

If $g > e$, $\sqrt{g} < Ag/\log' g < Af$, $\log g < A \log f$, and hence again, $g < Af \log' f$. Hence

$$\int_{CE} fg\,dx \leqslant \int_{CE} \{f(x)\}^2 \log' f(x)\,dx.]$$

5. If $f^p(\log' f)^q$ and $g^{p/(p-1)}(\log' g)^{-q/(p-1)}$ are integrable, then $f(x)g(x)$ is integrable.

6. Prove that
$$uv \leqslant u \log u + e^{v-1} \qquad (u > 1,\ v > 1).$$

Deduce that if $f(x)\log' f(x)$ and $e^{g(x)}$ are integrable, so is $f(x)g(x)$.

[W. H. Young (4). The inequality may be verified by putting $u = e^x$, $v = y+1$.]

7. If $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\alpha+\beta+\gamma = 1$,

$$\left|\int fgh\,dx\right| \leqslant \left(\int |f|^{1/\alpha}\,dx\right)^{\alpha}\left(\int |g|^{1/\beta}\,dx\right)^{\beta}\left(\int |h|^{1/\gamma}\,dx\right)^{\gamma}.$$

8. If $\lambda > 0$, $\mu > 0$, $\lambda\mu < 1$, and $f(x)$ and $g(x)$ belong to suitable L-classes, then

$$\left|\int fg\,dx\right|^{(1+\lambda)(1+\mu)/(1-\lambda\mu)}$$
$$\leqslant \int |f|^{1+\lambda}|g|^{1+\mu}\,dx \left(\int |f|^{1+\lambda}\,dx\right)^{\mu(1+\lambda)/(1-\lambda\mu)} \left(\int |g|^{1+\mu}\,dx\right)^{\lambda(1+\mu)/(1-\lambda\mu)}$$

[W. H. Young (2); the result may be obtained by suitable substitutions in ex. 7.]

9. If $F(x)$ is the integral of a function of the class L^p, where $p > 1$, then as $h \to 0$
$$F(x+h) - F(x) = o\,(h^{1-1/p}).$$

[If $F(x)$ is the integral of $f(x)$,

$$|F(x+h)-F(x)| = \left|\int_x^{x+h} f(t)\,dt\right|$$

$$\leqslant \left\{\int_x^{x+h} |f(t)|^p\,dt\right\}^{1/p}\left\{\int_x^{x+h} dt\right\}^{1-1/p} = h^{1-1/p}\left\{\int_x^{x+h} |f(t)|^p\,dt\right\}^{1-1/p},$$

and the last factor tends to zero with h.]

10. If $f(x)$ belongs to $L^p(0, \infty)$, where $p > 1$, the integral

$$\int_0^\infty f(x)\,\frac{\sin xy}{x}\,dx$$

is uniformly convergent in any finite interval.

11. If $f(x)$ belongs to L^p, where $p > 1$, and $\phi(y)$ is the integral defined in the previous example, then as $h \to 0$

$$\phi(y+h)-\phi(y) = o\,(h^{1/p}).$$

[For
$$\phi(y+h)-\phi(y) = \int_0^\infty \frac{f(x)}{x}\,[\sin\{x(y+h)\} - \sin xy]\,dx$$

$$= 2\int_0^\infty \frac{f(x)}{x}\sin \tfrac{1}{2}xh \cos x(y+\tfrac{1}{2}h)\,dx.$$

Hence

$$|\phi(y+h)-\phi(y)| \leqslant 2\int_0^\infty \left|f(x)\,\frac{\sin\frac{1}{2}xh}{x}\right|dx$$

$$\leqslant 2\left\{\int_0^\infty |f(x)|^p\,dx\right\}^{1/p}\left\{\int_0^\infty \left|\frac{\sin\frac{1}{2}xh}{x}\right|^{p/(p-1)} dx\right\}^{1-1/p}.$$

The first factor is a constant, and the second factor, on putting $x = \xi/h$, is seen to be a multiple of $h^{1/p}$. This gives the required result with O instead of o. If, however, we apply the above argument to the integrals over $(0, \delta)$ and (Δ, ∞), where δ is arbitrarily small and Δ arbitrarily large, and notice that

$$\left|\int_\delta^\Delta \frac{f(x)}{x}\sin\tfrac{1}{2}xh\cos x(y+\tfrac{1}{2}h)\,dx\right| \leqslant \int_\delta^\Delta \frac{|f(x)|}{x}\,|\sin\tfrac{1}{2}xh|\,dx = O(h)$$

for fixed δ and Δ, the required result follows.]

12. Show that the integral

$$\phi(y) = \int_0^\infty f(x) \frac{\sin xy}{\sqrt{x}}\, dx$$

is absolutely convergent if $f(x)$ belongs to L^p, where $1 < p < 2$; and that, as $y \to 0$,

$$\phi(y) = o\,(y^{1/p-\frac{1}{2}}).$$

13. If $f(x)$ is uniformly continuous over $(0, \infty)$, and belongs to a class $L^p(0, \infty)$, then $f(x) \to 0$ as $x \to \infty$.

14. If $f(x)$ belongs to $L^p(0, \infty)$, where $p > 1$, so do the functions

$$\phi(x) = \frac{1}{x} \int_0^x f(t)\, dt, \qquad \psi(x) = \int_x^\infty \frac{f(t)}{t}\, dt.$$

[Hardy (17) and (19). Consider $\phi(x)$, for example. It is bounded except as $x \to 0$ or $x \to \infty$. Hence

$$\int_a^b |\phi(x)|^p\, dx$$

exists for $0 < a < b < \infty$; and it is sufficient to prove that this integral remains bounded as $a \to 0$ and $b \to \infty$.

We may suppose without loss of generality that $f(t) \geqslant 0$. Let

$$f_1(x) = \int_0^x f(t)\, dt.$$

Then $x^{1-p}\{f_1(x)\}^p$ tends to zero, both as $x \to 0$ and as $x \to \infty$; for

$$\{f_1(x)\}^p \leqslant \int_0^x \{f(t)\}^p\, dt \left(\int_0^x dt\right)^{p-1} = x^{p-1} \int_0^x \{f(t)\}^p\, dt,$$

whence the result for $x = 0$ follows. Again, if $x > \xi$, a similar argument shows that

$$f_1(x) \leqslant \int_0^\xi f(t)\, dt + \left\{x^{p-1} \int_\xi^x \{f(t)\}^p\, dt\right\}^{1/p},$$

and we can choose ξ so large that the last factor is arbitrarily small, for all $x > \xi$. This gives the result for $x \to \infty$.

We write
$$\int_a^b \{\phi(x)\}^p\, dx = \int_a^b \{f_1(x)\}^p x^{-p}\, dx,$$

and integrate by parts, obtaining

$$\left[\frac{\{f_1(x)\}^p x^{1-p}}{1-p}\right]_a^b + \frac{p}{p-1} \int_a^b \{f_1(x)\}^{p-1} f(x) x^{1-p}\, dx$$

$$= o\,(1) + \frac{p}{p-1} \int_a^b \{\phi(x)\}^{p-1} f(x)\, dx.$$

Hence

$$\int_a^b \{\phi(x)\}^p\,dx \leqslant o(1)+\frac{p}{p-1}\left\{\int_a^b \{\phi(x)\}^p\,dx\right\}^{1-1/p}\left\{\int_a^b \{f(x)\}^p\,dx\right\}^{1/p},$$

and dividing by the factor

$$\left\{\int_a^b \{\phi(x)\}^p\,dx\right\}^{1-1/p},$$

and making $a \to 0$, $b \to \infty$, we obtain

$$\left\{\int_0^\infty \{\phi(x)\}^p\,dx\right\}^{1/p} \leqslant \frac{p}{p-1}\left\{\int_0^\infty \{f(x)\}^p\,dx\right\}^{1/p}.$$

We leave the corresponding process for $\psi(x)$ to the reader.]

15. Prove that, with the hypotheses of the previous example, the integrals

$$\int_0^\infty |\phi(x)|^q x^{q/p-1}\,dx, \qquad \int_0^\infty |\psi(x)|^q x^{q/p-1}\,dx$$

are convergent for $q \geqslant p$.

16. If $f(x)$ belongs to $L^p(0, \infty)$, where $p > 1$, and

$$\phi(x) = \int_0^\infty e^{-xy} f(y)\,dy,$$

then $y^{1-2/p}\phi(y)$ belongs to L^p.

[For

$$|\phi(x)| \leqslant \int_0^{1/x} |f(y)|\,dy + \int_{1/x}^\infty \frac{|f(y)|}{xy}\,dy,$$

and the result follows from ex. 14.]

17. If $f(x)$ belongs to $L^p(a, b)$, there is a continuous function $g(x)$ such that

$$\int_a^b |f(x)-g(x)|^p\,dx < \epsilon.$$

[The result for bounded $f(x)$ follows at once from § 12.2, and the general result may then be deduced from this.]

18. If $f(x)$ belongs to L^p over an interval including (a, b), then

$$\lim_{h\to 0}\int_a^b |f(x+h)-f(x)|^p\,dx = 0.$$

[The result is immediate for continuous functions. For the general result use the previous example.]

19. If $f(x)$ belongs to $L^p(-\infty, \infty)$, then

$$\lim_{h\to 0}\int_{-\infty}^\infty |f(x+h)-f(x)|^p\,dx = 0.$$

20. If $f(x)$ belongs to $L^p(-\infty, \infty)$, and $g(x)$ to $L^{p/(p-1)}(-\infty, \infty)$, then

$$F(t) = \int_{-\infty}^{\infty} f(x+t)\,g(x)\,dx$$

is a continuous function of t.

[For $|F(t+h)-F(t)|$

$$\leqslant \left\{\int_{-\infty}^{\infty} |f(x+t+h)-f(x+t)|^p\,dx\right\}^{1/p} \left\{\int_{-\infty}^{\infty} |g(x)|^{p/(p-1)}\,dx\right\}^{1-1/p}.]$$

21. The function $F(t)$ of the previous example tends to zero at infinity.

[Write $\int_{-\infty}^{\infty} = \int_{-\infty}^{-\frac{1}{2}t} + \int_{-\frac{1}{2}t}^{\infty}$.]

22. If $f(x)$ belongs to $L^p(-\infty, \infty)$, then

$$F(x) = \int_{-\infty}^{\infty} \frac{f(t)}{1+(x-t)^2}\,dt$$

is continuous, and belongs to $L^p(-\infty, \infty)$.

[Use the inequality

$$|F(x)|^p \leqslant \int_{-\infty}^{\infty} \frac{|f(t)|^p\,dt}{\{1+(x-t)^2\}^{\frac{1}{2}p}} \left\{\int_{-\infty}^{\infty} \frac{dt}{\{1+(x-t)^2\}^{\frac{1}{2}p/(p-1)}}\right\}^{p-1}.]$$

23. If $f(x)$ is integrable, the integral

$$f_\alpha(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\,dt \qquad (\alpha > 0)$$

exists almost everywhere, and $f_\alpha(x)$ is integrable.

[The function $f_\alpha(x)$ is the integral of $f(x)$ of order α; for some properties of such integrals see Hardy (12) and Hardy and Littlewood (5).]

24. If $f(x)$ belongs to $L^p (p > 1)$, show by the method of ex. 20 that $f_\alpha(x)$ is continuous if $\alpha > 1/p$.

25. If $f_{\alpha,\beta}(x)$ denotes the integral of order β of $f_\alpha(x)$, then

$$f_{\alpha,\beta}(x) = f_{\alpha+\beta}(x) \qquad (\alpha > 0,\ \beta > 0),$$

wherever the right-hand side exists.

[We have to invert the repeated integral

$$\int_0^x (x-t)^{\beta-1}\,dt \int_0^t (t-u)^{\alpha-1} f(u)\,du$$

and use Ch. I, ex. 18. The integral $\int_\delta^x \int_0^{t-\delta}$ may be inverted by § 12.6. We can then make $\delta \to 0$ and use the theorem of § 10.82.]

CHAPTER XIII

FOURIER SERIES

13.1. Trigonometrical series and Fourier series. A trigonometrical series is a series of the form

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx), \qquad (1)$$

where the coefficients a_0, a_1, b_1,... are independent of x. The problem of representing a given function $f(x)$ by a series of this form was first encountered by Fourier in a problem of the conduction of heat. Subsequently it was found that these series play an important part in the theory of functions of a real variable, and it is from this point of view that we shall consider them here.

We naturally begin by trying to find formulae for the coefficients a_n, b_n, in terms of the given function $f(x)$. Suppose that the series converges uniformly, or even boundedly, to $f(x)$; we may then multiply by $\cos mx$, where m is a positive integer, and integrate term by term over the interval $(0, 2\pi)$. Since

$$\int_0^{2\pi}\cos mx\cos nx\,dx=\pi\ (n=m), \qquad =0\ (n\neq m)$$

and

$$\int_0^{2\pi}\cos mx\sin nx\,dx=0$$

for all values of n, we obtain the result

$$a_m=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos mx\,dx. \qquad (2)$$

The same formula also gives a_0; and similarly, multiplying by $\sin mx$ and integrating term by term,

$$b_m=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin mx\,dx. \qquad (3)$$

The formulae (2) and (3) are known as the Euler-Fourier formulae for the coefficients.

There is, however, no *a priori* reason for supposing that a given function can be expanded in a boundedly convergent trigonometrical series. The above process is therefore not a proof that the coefficients necessarily have the above form. What it really suggests is that we should adopt a different point of view. Instead of starting with the series, and assuming that it has a certain property, we start from the function, and define the coefficients by the above formulae. We then consider the properties of the series so formed.

Suppose, then, that we are given a function $f(x)$, integrable in the sense of Lebesgue over the interval $(0, 2\pi)$. Then the integrals (2) and (3) exist, and the numbers a_m, b_m defined by them are called the *Fourier coefficients of* $f(x)$. The trigonometrical series of the form (1), with these coefficients, is called the Fourier series of $f(x)$.

The scheme of the chapter is as follows. We first try to determine conditions under which the Fourier series converges to $f(x)$. A number of these conditions are found, but they are all rather special ones (§§ 13.11–13.25). We next consider a generalized kind of convergence (summability $(C, 1)$), and find that it enables us to put the theory into a more systematic form (§§ 13.3–13.35). In the following sections we consider some problems of term-by-term integration; and this leads us to consider properties of the Fourier coefficients themselves, apart from the Fourier series. In §§ 13.8–13.86 we return to the question of the relation between Fourier series in particular and trigonometrical series in general. Lastly we give some of the corresponding theory of Fourier integrals.

13.11. The convergence problem. The first problem which we have to consider is whether the series formed in the above manner converges, and, if it does, whether its sum is $f(x)$.

At the time when Fourier series first came into use, there seemed to many mathematicians to be something paradoxical in saying that an 'arbitrary' function could be represented by a series of functions, each of which is continuous and periodic. The reader who has examined the peculiarities of some of the series in Chapter I is perhaps prepared to believe that even this is possible; and we shall show that the series does, substantially, do what is required of it. We must not, however, expect too much.

In the first place, every term of the series has the period 2π; hence the sum of the series, if there is one, also has the period 2π. We therefore define the function $f(x)$ first in the interval

$0 \leqslant x < 2\pi$; outside this interval we define it by periodicity, i.e. by the equation

$$f(x+2\pi) = f(x).$$

Secondly, it is impossible that, whatever $f(x)$ is, the series should converge to the sum $f(x)$ for every value of x. Consider, for example, two functions $f(x)$ and $g(x)$ which differ at one point only. They have the same Fourier series, so that it cannot represent both functions at every point. More generally, two 'equivalent' functions, i.e. functions which are equal almost everywhere, have the same Fourier series, which therefore cannot represent them both if they differ anywhere.

Actually we shall see that the series does represent the function, provided that the function is not too complicated; and even in the most complicated cases, the series still represents in some sense the main features of the function.

13.12. Fourier series and Laurent series. There is a close formal connexion between a Fourier series and a Laurent series. Let $F(z)$ be a one-valued analytic function, regular for $R' < |z| < R$. Then

$$F(z) = \sum_{n=-\infty}^{\infty} c_n z^n,$$

where $$c_n = \frac{1}{2\pi i} \int\limits_{|z|=r} \frac{F(z)}{z^{n+1}}\, dz \qquad (R' < r < R).$$

Putting $z = re^{i\theta}$, we have

$$F(re^{i\theta}) = \sum_{n=-\infty}^{\infty} A_n e^{in\theta},$$

where $$A_n = \frac{1}{2\pi} \int\limits_0^{2\pi} F(re^{i\phi}) e^{-in\phi}\, d\phi.$$

The expansion may also be written

$$F(re^{i\theta}) = A_0 + \sum_{n=1}^{\infty} \{(A_n + A_{-n})\cos n\theta + i(A_n - A_{-n})\sin n\theta\},$$

where

$$A_0 = \frac{1}{2\pi} \int\limits_0^{2\pi} F(re^{i\phi})\, d\phi, \qquad A_n + A_{-n} = \frac{1}{\pi} \int\limits_0^{2\pi} F(re^{i\phi}) \cos n\phi\, d\phi,$$

$$i(A_n - A_{-n}) = \frac{1}{\pi} \int\limits_0^{2\pi} F(re^{i\phi}) \sin n\phi\, d\phi.$$

We have thus expressed the Laurent series in the form of a Fourier series. The fact that in this case the series represents the function, and indeed converges uniformly to it, follows from the theory of analytic functions. In general we assume much less about the function than that it is analytic, and the problem requires quite different methods.

13.2. Dirichlet's integral. Let $0 \leqslant x < 2\pi$, and let

$$s_n = s_n(x) = \tfrac{1}{2}a_0 + \sum_{m=1}^{n} (a_m \cos mx + b_m \sin mx). \qquad (1)$$

This partial sum can be represented as a definite integral. We have

$$s_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt +$$

$$+\frac{1}{\pi}\sum_{m=1}^{n}\left\{\cos mx \int_0^{2\pi} f(t)\cos mt\,dt + \sin mx \int_0^{2\pi} f(t)\sin mt\,dt\right\}$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left\{\tfrac{1}{2} + \sum_{m=1}^{n}\cos m(x-t)\right\}f(t)\,dt = \frac{1}{2\pi}\int_0^{2\pi}\frac{\sin(n+\frac{1}{2})(x-t)}{\sin\frac{1}{2}(x-t)}f(t)\,dt.$$

Putting $t = x+u$, this becomes

$$s_n = \frac{1}{2\pi}\int_{-x}^{2\pi-x}\frac{\sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u}f(x+u)\,du,$$

or, since the integrand has the period 2π, and so takes the same values in $(2\pi-x, 2\pi)$ as in $(-x, 0)$,

$$s_n = \frac{1}{2\pi}\int_0^{2\pi}\frac{\sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u}f(x+u)\,du. \qquad (2)$$

This formula is known as *Dirichlet's integral*. It may also be written in the form

$$s_n = \frac{1}{2\pi}\int_0^{\pi}\frac{\sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u}\{f(x+u)+f(x-u)\}\,du. \qquad (3)$$

This is obtained by writing $u = -v$ in the range $(\pi, 2\pi)$, so that this part of (2) becomes

$$\int_{-2\pi}^{-\pi} \frac{\sin(n+\frac{1}{2})v}{\sin\frac{1}{2}v} f(x-v)\, dv = \int_{0}^{\pi} \frac{\sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u} f(x-u)\, du$$

by periodicity.

Suppose, in particular, that $f(x) = 1$ for all values of x. Then $a_0 = 2$, and all the rest of the Fourier coefficients are zero, so that $s_n = 1$ for $n > 0$. In this case the above formula becomes

$$1 = \frac{1}{2\pi} \int_{0}^{\pi} \frac{\sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u} 2\, du.$$

Multiplying this by s, and subtracting from (3), we have

$$s_n - s = \frac{1}{2\pi} \int_{0}^{\pi} \frac{\sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u} \{f(x+u)+f(x-u)-2s\}\, du. \tag{4}$$

A necessary and sufficient condition that the series should converge to the sum s is, therefore, that this integral should tend to zero. The 'convergence problem' is the problem of determining under what conditions the integral tends to zero, and, when it does so, whether $s = f(x)$. We may consider the convergence problem for one particular value of x, for all values of x, or for almost all values of x; or for some other set of values of x. We begin by considering one particular value of x.

13.21. The Riemann-Lebesgue theorem. The following theorem is fundamental in the theory.

If $f(x)$ is integrable over (a, b), then as $\lambda \to \infty$

$$\int_{a}^{b} f(x)\cos\lambda x\, dx \to 0, \qquad \int_{a}^{b} f(x)\sin\lambda x\, dx \to 0.$$

Consider, for example, the cosine integral. If $f(x)$ is an integral, we may integrate by parts, and obtain

$$\int_{a}^{b} f(x)\cos\lambda x\, dx = \left[f(x)\frac{\sin\lambda x}{\lambda}\right]_{a}^{b} - \frac{1}{\lambda}\int_{a}^{b} f'(x)\sin\lambda x\, dx.$$

The last integral is bounded, so that the whole is $O(1/\lambda)$.

In the general case, given ϵ, we can (§ 12.2) define an absolutely continuous function $\phi(x)$ such that

$$\int_a^b |f(x)-\phi(x)|\,dx < \epsilon.$$

Then $$\left|\int_a^b \{f(x)-\phi(x)\}\cos\lambda x\,dx\right| \leqslant \int_a^b |f(x)-\phi(x)|\,dx < \epsilon$$

for all values of λ; and, by the first part,

$$\left|\int_a^b \phi(x)\cos\lambda x\,dx\right| < \epsilon \qquad (\lambda > \lambda_0).$$

Hence $$\left|\int_a^b f(x)\cos\lambda x\,dx\right| < 2\epsilon \qquad (\lambda > \lambda_0),$$

the required result. A similar proof applies to the sine integral.

There is an alternative proof on the lines of § 13.72, using the example of § 12.2.

13.22. The Riemann-Lebesgue theorem has the following important consequences:

The Fourier coefficients of any integrable function tend to zero.

This is the particular case of the theorem where $\lambda = n$, and the limits are 0 and 2π.

The behaviour of the Fourier series for a particular value of x depends on the behaviour of the function in the immediate neighbourhood of this point only.

Let δ be a positive number less than π, and let $g(t) = f(t)$ in the interval $x-\delta < t < x+\delta$, and $g(t) = 0$ in the rest of the interval $(x-\pi, x+\pi)$. Let the partial sums of the Fourier series of $g(t)$ be denoted by S_n. Then

$$S_n = \frac{1}{2\pi}\int_0^\pi \frac{\sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u}\{g(x+u)+g(x-u)\}\,du$$

$$= \frac{1}{2\pi}\int_0^\delta \frac{\sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u}\{f(x+u)+f(x-u)\}\,du.$$

Hence

$$s_n - S_n = \frac{1}{2\pi}\int_\delta^\pi \frac{\sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u}\{f(x+u)+f(x-u)\}\,du.$$

Now the function

$$\operatorname{cosec} \tfrac{1}{2}u\{f(x+u)+f(x-u)\}$$

is integrable over (δ, π) if $\delta > 0$; and hence, by the Riemann-Lebesgue theorem,
$$s_n - S_n \to 0.$$

Hence, however small δ may be, the behaviour of s_n depends on the nature of $f(t)$ in the interval $(x-\delta, x+\delta)$ only, and is not affected by the values which it takes outside this interval.

It is this property which makes it possible for the series to represent an arbitrary function; but the series only represents the function at the point x as a sort of limit of its average value over the interval $(x-\delta, x+\delta)$, and this will be equal to $f(x)$ only if the behaviour of the function is sufficiently simple. As we have already remarked in § 13.1, the value of $f(t)$ at the point $t = x$ itself does not determine or affect in any way the sum of the series.

13.23. Convergence tests. We first put the 'necessary and sufficient condition for convergence to the sum s' into a more convenient form. Let

$$\phi(u) = f(x+u)+f(x-u)-2s.$$

Then the condition, by § 13.2 (4), is

$$\lim_{n\to\infty} \int_0^\pi \frac{\sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u}\,\phi(u)\,du = 0. \tag{1}$$

We may replace this by

$$\lim_{n\to\infty} \int_0^\delta \frac{\sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u}\,\phi(u)\,du = 0, \tag{2}$$

where $0 < \delta \leqslant \pi$; for, by the Riemann-Lebesgue theorem, the difference between the integrals in (1) and (2) tends to zero. Next we may replace (2) by

$$\lim_{n\to\infty} \int_0^\delta \frac{\sin(n+\frac{1}{2})u}{u}\,\phi(u)\,du = 0; \tag{3}$$

for $(\operatorname{cosec}\frac{1}{2}u-2/u)\phi(u)$ is integrable over $(0,\delta)$, and so, by the Riemann-Lebesgue theorem,

$$\lim_{n\to\infty}\int_0^\delta \sin(n+\tfrac{1}{2})u\left\{\frac{1}{\sin\frac{1}{2}u}-\frac{2}{u}\right\}\phi(u)\,du = 0.$$

We are now in a position to state some tests for convergence.

13.231. Dini's test. *If $\phi(u)/u$ is integrable over $(0,\delta)$, then the series converges to the sum s.*

This theorem is at once obvious from the above formula (3) and the Riemann-Lebesgue theorem. It should of course be remembered that the integrability of $\phi(u)/u$ in the Lebesgue sense implies 'absolute integrability'. The existence of

$$\lim_{\epsilon\to 0}\int_\epsilon^\delta \frac{\phi(u)}{u}\,du$$

is not a sufficient condition for convergence.

Examples. (i) At any point where $f(x)$ is differentiable, the series converges to the sum $f(x)$.

[At such a point, $\phi(u)/u$ is bounded.]

(ii) More generally, if $f(x)$ satisfies the 'Lipschitz condition' of order α, i.e.

$$f(x+h)-f(x) = O(|h|^\alpha) \qquad (0<\alpha<1),$$

then the series converges to the sum $f(x)$.

13.232. Jordan's test. *If $f(t)$ is of bounded variation in the neighbourhood of $t = x$, then the series converges to the sum*

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\}.$$

Since 'bounded variation' means 'bounded variation over an interval', this condition is really one for convergence over an interval.

We know that, if $f(x)$ is of bounded variation, the limits $f(x+0)$ and $f(x-0)$ exist. Hence

$$\phi(u) = f(x+u)+f(x-u)-f(x+0)-f(x-0)$$

is of bounded variation in an interval to the right of $u = 0$, and $\phi(u)\to 0$ as $u \to 0$. Hence we may write

$$\phi(u) = \phi_1(u)-\phi_2(u),$$

where ϕ_1 and ϕ_2 are positive increasing functions of u; each of these functions tends to the same limit as $u\to 0$; and we may, by subtracting a constant from each function, arrange that this limit shall be zero.

Suppose that δ is so small that $\phi(u)$ is of bounded variation in the interval $(0,\delta)$. Then

$$\begin{aligned}\int_0^\delta \frac{\sin(n+\frac{1}{2})u}{u}\,\phi(u)\,du \\ &= \int_0^\delta \frac{\sin(n+\frac{1}{2})u}{u}\,\phi_1(u)\,du - \int_0^\delta \frac{\sin(n+\frac{1}{2})u}{u}\,\phi_2(u)\,du \\ &= J_1 - J_2,\end{aligned}$$

say. Consider the integral J_1. Given ϵ, choose η so small that $\phi_1(\eta) < \epsilon$. Then, by the second mean-value theorem,

$$\begin{aligned}\int_0^\eta \frac{\sin(n+\frac{1}{2})u}{u}\,\phi_1(u)\,du &= \phi_1(\eta)\int_\xi^\eta \frac{\sin(n+\frac{1}{2})u}{u}\,du \quad (0<\xi<\eta) \\ &= \phi_1(\eta)\int_{(n+\frac{1}{2})\xi}^{(n+\frac{1}{2})\eta} \frac{\sin v}{v}\,dv.\end{aligned}$$

The last integral is bounded for all values of n, ξ, and η, so that

$$\left|\int_0^\eta \frac{\sin(n+\frac{1}{2})u}{u}\,\phi_1(u)\,du\right| < A\epsilon.$$

Having fixed η, by the Riemann-Lebesgue theorem

$$\left|\int_\eta^\delta \frac{\sin(n+\frac{1}{2})u}{u}\,\phi_1(u)\,du\right| < \epsilon \qquad (n>n_0).$$

Hence $J_1 \to 0$; and similarly $J_2 \to 0$. This proves the theorem.

In particular *if $f(x)$ has only a finite number of maxima and minima and a finite number of discontinuities in the interval $(0, 2\pi)$, its Fourier series is convergent for all values of x to the sum* $\frac{1}{2}\{f(x+0)+f(x-0)\}$. For such a function is of bounded variation in the whole interval. These conditions are known as *Dirichlet's conditions*. They are, of course, satisfied in many cases; but they have the disadvantage that the sum of two functions which satisfy them does not itself necessarily satisfy them.

In connexion with Jordan's test, it is interesting to note that *if $f(x)$ is a function of bounded variation over $(0, 2\pi)$, its Fourier series is boundedly convergent.*

For let $0 \leqslant x < \pi$, and write Dirichlet's integral for $s_n(x)$ in the form

$$\frac{1}{2\pi} \int_{-\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{\sin(n+\frac{1}{2})(x-t)}{\sin\frac{1}{2}(x-t)} f(t)\,dt.$$

Since

$$\frac{1}{\sin\frac{1}{2}(x-t)} - \frac{1}{\frac{1}{2}(x-t)}$$

is bounded for $-\frac{3}{2}\pi < x-t < \frac{3}{2}\pi$, this differs by a bounded function from

$$\frac{1}{\pi} \int_{-\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{\sin(n+\frac{1}{2})(x-t)}{x-t} f(t)\,dt.$$

Let $f(t) = f_1(t) - f_2(t)$, where f_1 and f_2 are positive non-decreasing in $(-\frac{1}{2}\pi, \frac{3}{2}\pi)$. Then, by the second mean-value theorem,

$$\int_{-\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{\sin(n+\frac{1}{2})(x-t)}{x-t} f_1(t)\,dt = f_1(\tfrac{3}{2}\pi) \int_{\xi}^{\frac{3}{2}\pi} \frac{\sin(n+\frac{1}{2})(x-t)}{x-t}\,dt \qquad (-\tfrac{1}{2}\pi < \xi < \tfrac{3}{2}\pi),$$

which is bounded for all n and x, as in the above proof. A similar result holds for f_2. Hence the series is boundedly convergent over $(0, \pi)$, and similarly over $(\pi, 2\pi)$.

13.233. de la Vallée-Poussin's test.* *If the function*

$$\psi(t) = \frac{1}{t} \int_0^t \phi(u)\,du$$

is of bounded variation in an interval to the right of $t = 0$, then the series is convergent. If s is so chosen that $\psi(t) \to 0$ as $t \to 0$, the sum of the series is s.

For $$\phi(t) = \frac{d}{dt}\{t\psi(t)\} = \psi(t) + t\psi'(t).$$

Since $\psi(t)$ is of bounded variation and tends to zero, the part of the integral § 13.23 (3) involving it tends to zero, as in Jordan's test; and since $\psi'(t)$ is integrable (§ 11.54), the part involving $t\psi'(t)$ tends to zero, as in Dini's test.

* The same test was given previously by du Bois-Reymond, but of course with Riemann integrals.

13.24. Relations between the above tests.* Consider the function

$$f(x) = \frac{1}{\log 1/x} \quad (0 < x < \pi), \qquad = 0 \quad (\pi \leqslant x \leqslant 2\pi).$$

This function is bounded and monotonic in the neighbourhood of $x = 0$, so that Jordan's condition is satisfied, and the series converges. But Dini's condition is not satisfied, since the integral

$$\int_0^\delta \frac{dt}{t \log 1/t}$$

is divergent. Thus *Dini's condition does not include Jordan's.*

On the other hand, *Jordan's condition does not include Dini's.* For consider the function

$$f(x) = x^\alpha \sin 1/x \quad (0 < x < \pi), \qquad = 0 \quad (\pi \leqslant x \leqslant 2\pi),$$

where $0 < \alpha < 1$. Then Dini's condition for convergence at $x = 0$ is obviously satisfied. But the function is not of bounded variation (Ch. XI, ex. 5), i.e. Jordan's condition is not satisfied.

Lastly, *de la Vallée-Poussin's test includes both Dini's and Jordan's,* i.e. if either Dini's or Jordan's condition is fulfilled, then so is de la Vallée-Poussin's.

We first remark that *if* $g(x)$ *is of bounded variation in* $(0, \delta)$, *then so is*

$$G(x) = \frac{1}{x} \int_0^x g(t)\, dt.$$

For $g(x) = g_1(x) - g_2(x)$, where $g_1(x)$ and $g_2(x)$ are positive, non-decreasing, and bounded; and

$$G(x) = \frac{1}{x} \int_0^x g_1(x)\, dx - \frac{1}{x} \int_0^x g_2(x)\, dx = G_1(x) - G_2(x),$$

say, and it is easily seen that $G_1(x)$ and $G_2(x)$ are both positive, non-decreasing, and bounded. Hence $G(x)$ is of bounded variation.

The relation between Jordan's test and de la Vallée-Poussin's test follows at once; if $\phi(t)$ is of bounded variation, so is $\psi(t)$.

* For a detailed discussion of this question see Hardy (13).

Now consider Dini's test. If $\phi(u)/u$ is integrable,

$$\chi(t)=\int_0^t \frac{\phi(u)}{u}\,du$$

is a function of bounded variation; and

$$\psi(t)=\frac{1}{t}\int_0^t u\frac{d}{du}\{\chi(u)\}\,du=\chi(t)-\frac{1}{t}\int_0^t \chi(u)\,du,$$

which is also of bounded variation, by the above remark. Hence de la Vallée-Poussin's condition is satisfied.

13.25. Convergence throughout an interval. If one of the above conditions is fulfilled at all points of an interval, of course the series converges throughout the interval; and if the condition is fulfilled uniformly, the convergence is uniform. The simplest case is as follows.

The Fourier series of $f(x)$ converges uniformly to $f(x)$ in any interval interior to an interval where $f(x)$ is continuous and of bounded variation.

For in such an interval we can write $f(x)=f_1(x)-f_2(x)$, where $f_1(x)$ and $f_2(x)$ are continuous and non-decreasing. Then, by the property of uniform continuity, we can find η so that

$$|f_1(x+h)-f_1(x)|<\epsilon \qquad (|h|<\eta),$$

the choice of η depending only on ϵ and not on the value of x in the interval. It will be seen on referring to the proof of Jordan's test that this implies the uniform convergence of the integral dealt with in proving the test. We have also to show that the parts of Dirichlet's integral which have been shown to tend to 0, actually tend uniformly to 0; the reader should have no difficulty in verifying this.

The property of uniform convergence is, however, not so important as might be expected in the case of Fourier series, because questions of term-by-term integration can be dealt with under much more general conditions (§ 13.5).

No simple restriction on $f(x)$ which ensures that the Fourier series shall be convergent *almost everywhere*, without obviously proving more than this, appears to be known. It might, for example, be conjectured that continuity would be such a condition; but no result of the kind suggested has been proved. On the other hand, a condition bearing not on the function itself, but on the Fourier coefficients, has been given: *the Fourier series is convergent almost everywhere if the series*

$$\sum (a_n^2+b_n^2)\log n$$

*is convergent.**

13.3. Summation of series by arithmetic means. If a series $u_1+u_2+...$ is not convergent, i.e. if $s_n = u_1+...+u_n$ does not tend to a limit, it is sometimes possible to associate with the series a 'sum' in a less direct way. The simplest such method is 'summation by arithmetic means'. We take the arithmetic mean

$$\sigma_n = \frac{s_1+s_2+...+s_n}{n}$$

of the partial sums of the given series. If $s_n \to s$, then also $\sigma_n \to s$; for if $s_n = s+\delta_n$, then

$$\sigma_n = s+\frac{\delta_1+\delta_2+...+\delta_n}{n},$$

and the last term tends to zero if $\delta_n \to 0$, by the lemma of § 1.23.

But σ_n may tend to a limit even though s_n does not. Consider, for example, the series

$$1-1+1-1+....$$

Here the partial sums s_n are alternately 1 and 0, and it is easily seen that $\sigma_n \to \frac{1}{2}$.

A series for which σ_n tends to a limit is said to be summable by arithmetic means, or by Cesàro's means of the first order, or $(C, 1)$.

Examples. (i) The series $1+0-1+1+0-1+...$ is summable $(C, 1)$ to the sum $\frac{2}{3}$.

(ii) The series $\sin x + \sin 2x + \sin 3x + ...$ is summable $(C, 1)$ for all values of x; the sum is $\frac{1}{2}\cot\frac{1}{2}x$ if x is not an even multiple of π, and otherwise is 0.

(iii) The series $\frac{1}{2}+\cos x+\cos 2x+\cos 3x+...$ is summable $(C, 1)$ to the sum zero if x is not an even multiple of π.

(iv) If $\sum u_n$ is summable $(C, 1)$, $s_n = o(n)$.

[For $s_n = n\sigma_n-(n-1)\sigma_{n-1}$.]

(v) Let $t_n = u_1+2u_2+...+nu_n$. If $\sum u_n$ is summable $(C, 1)$, a necessary and sufficient condition that it should be convergent is $t_n = o(n)$.

[For $t_n = (n+1)s_n-n\sigma_n$.]

* Plessner (2).

(vi) If $\sum u_n$ is summable $(C, 1)$, and $u_n = o(1/n)$, then $\sum u_n$ is convergent.

[For $t_n = o(n)$, by the lemma of § 1.23. The result is analogous to Tauber's theorem.]

(vii) A necessary and sufficient condition that $\sum u_n$ should be summable $(C, 1)$ is that

$$\sum \frac{t_n}{n(n+1)}$$

should be convergent.

$$\left[\text{For} \qquad \sum_{n=1}^{N} \frac{t_n}{n(n+1)} = \frac{N}{N+1}\sigma_N.\right]$$

(viii) If $\sum u_n$ is summable $(C, 1)$, and $u_n = O(1/n)$, then $\sum u_n$ is convergent.

[Hardy; this is analogous to Littlewood's extension of Tauber's theorem. If $\sum u_n$ is not convergent, then $t_N > A_1 N$, or $t_N < -A_1 N$, for an infinity of values of N—say e.g. the former. Since

$$t_{n+1} = t_n + (n+1)u_n > t_n - A_2,$$

we have
$$t_{N+\nu} > \tfrac{1}{2}A_1 N \qquad (0 \leqslant \nu < \tfrac{1}{2}NA_1/A_2).$$

Hence
$$\sum_{n=N}^{N+\frac{1}{2}NA_1/A_2} \frac{t_n}{n(n+1)} > A,$$

and by (vii) the series is not summable $(C, 1)$.]

(ix) A series of positive terms is summable $(C, 1)$ only if it is convergent.

[If $s_n \to \infty$, then $\sigma_n \to \infty$.]

13.31. Summability of Fourier series. It was discovered by Fejér* that the method of summation by arithmetic means applies particularly well to Fourier series. We write

$$\sigma_n = \frac{s_0 + s_1 + \ldots + s_{n-1}}{n},$$

where s_n is given by § 13.21 (3). Hence

$$\sigma_n = \frac{1}{2n\pi}\int_0^{\pi} \frac{\sin\frac{1}{2}u + \sin\frac{3}{2}u + \ldots + \sin(n-\frac{1}{2})u}{\sin\frac{1}{2}u}\{f(x+u)+f(x-u)\}\,du$$

$$= \frac{1}{2n\pi}\int_0^{\pi} \frac{\sin^2\frac{1}{2}nu}{\sin^2\frac{1}{2}u}\{f(x+u)+f(x-u)\}\,du. \tag{1}$$

This formula is known as *Fejér's integral*. Its importance is due to the fact that the factor $\sin^2\frac{1}{2}nu/\sin^2\frac{1}{2}u$ is positive. This makes

* Fejér (1).

it much easier to deal with Fejér's integral than with Dirichlet's, in which the corresponding factor, $\sin(n+\frac{1}{2})u/\sin\frac{1}{2}u$, oscillates between positive and negative values.

In the particular case where $f(x) = 1$, the formula becomes

$$1 = \frac{1}{2n\pi}\int_0^\pi \frac{\sin^2\frac{1}{2}nu}{\sin^2\frac{1}{2}u}\, 2\, du,$$

since now $\sigma_n = 1$ for $n > 0$. Hence, multiplying by s and subtracting,

$$\sigma_n - s = \frac{1}{2n\pi}\int_0^\pi \frac{\sin^2\frac{1}{2}nu}{\sin^2\frac{1}{2}u}\{f(x+u)+f(x-u)-2s\}\, du. \tag{2}$$

A necessary and sufficient condition that the series should be summable $(C, 1)$ to the sum s is, therefore, that the integral (2) should tend to zero.

As in the convergence problem, we can simplify the condition. We write

$$\phi(u) = f(x+u)+f(x-u)-2s$$

as before. Then, if δ is any positive number less than π, a necessary and sufficient condition that the series should be summable $(C, 1)$ to s is

$$\lim_{n\to\infty}\frac{1}{n}\int_0^\delta \frac{\sin^2\frac{1}{2}nu}{\sin^2\frac{1}{2}u}\phi(u)\, du = 0; \tag{3}$$

for

$$\left|\frac{1}{n}\int_\delta^\pi \frac{\sin^2\frac{1}{2}nu}{\sin^2\frac{1}{2}u}\phi(u)\, du\right| \leqslant \frac{1}{n}\int_\delta^\pi \frac{|\phi(u)|}{\sin^2\frac{1}{2}u}\, du,$$

which plainly tends to zero. Finally, the condition may be put in the form

$$\lim_{n\to\infty}\frac{1}{n}\int_0^\delta \frac{\sin^2\frac{1}{2}nu}{u^2}\phi(u)\, du = 0; \tag{4}$$

for

$$\left|\frac{1}{n}\int_0^\delta \sin^2\tfrac{1}{2}nu\left\{\frac{1}{\sin^2\frac{1}{2}u}-\frac{1}{(\frac{1}{2}u)^2}\right\}\phi(u)\, du\right|$$
$$\leqslant \frac{1}{n}\int_0^\delta \left\{\frac{1}{\sin^2\frac{1}{2}u}-\frac{1}{(\frac{1}{2}u)^2}\right\}|\phi(u)|\, du,$$

which tends to zero.

13.32. Fejér's theorem. *The Fourier series of $f(x)$ is summable $(C, 1)$ to the sum*

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\}$$

for every value of x for which this expression has a meaning. In particular, the series is summable $(C, 1)$ to the sum $f(x)$ at every point where $f(x)$ is continuous.

We now put $s=\frac{1}{2}\{f(x+0)+f(x-0)\}$ in the above formulae. Then $\phi(u)\to 0$ with u, and we have to prove that 13.31 (4) is true. Suppose that $|\phi(u)|\leqslant\epsilon$ for $u\leqslant\eta$. Then

$$\left|\frac{1}{n}\int\limits_0^\delta \frac{\sin^2\frac{1}{2}nu}{u^2}\phi(u)\,du\right| \leqslant \frac{1}{n}\int\limits_0^\eta \frac{\sin^2\frac{1}{2}nu}{u^2}\epsilon\,du+\frac{1}{n}\int\limits_\eta^\delta \frac{\sin^2\frac{1}{2}nu}{u^2}|\phi(u)|\,du$$

$$\leqslant \frac{\epsilon}{n}\int\limits_0^\eta \frac{\sin^2\frac{1}{2}nu}{u^2}\,du+\frac{1}{n}\int\limits_\eta^\delta \frac{|\phi(u)|}{u^2}\,du$$

$$=I_1+I_2,$$

say. Now

$$\frac{1}{n}\int\limits_0^\eta \frac{\sin^2\frac{1}{2}nu}{u^2}\,du=\frac{1}{2}\int\limits_0^{\frac{1}{2}n\eta}\frac{\sin^2 v}{v^2}\,dv<\frac{1}{2}\int\limits_0^\infty \frac{\sin^2 v}{v^2}\,dv,$$

which is a constant. Hence $I_1<A\epsilon$. Having fixed η, it is clear that $I_2\to 0$ as $n\to\infty$. This proves the theorem.

13.33. Summability throughout an interval. The following theorem is an almost immediate consequence of Fejér's theorem.

The Fourier series of $f(x)$ is uniformly summable in any interval included in an interval where $f(x)$ is continuous.

For $f(x)$ is uniformly continuous in any such interval, and so, in the above proof, the choice of η depends only on ϵ and not on x. The result follows at once from this.

Weierstrass's approximation theorem. *If $f(x)$ is continuous in (a, b), and ϵ is a given positive number, there is a polynomial $p(x)$ such that*

$$|f(x)-p(x)|<\epsilon \qquad (a\leqslant x\leqslant b).$$

We can make a preliminary transformation so that the interval considered lies within $(0, 2\pi)$. Then, by the above theorem, there is a 'trigonometrical polynomial' $\sigma_n(x)$ such that