Dirichlet Series

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Preface

In 2005, I taught a graduate course on Dirichlet series at Washington University. One of the students in the course, David Opěla, took notes and TeX'ed them up. We planned to turn these notes into a book, but the project stalled.

In 2015, I taught the course again, and revised the notes. I still intend to write a proper book, eventually, but until then I decided to make the notes available to anybody who is interested. The notes are not complete, and in particular lack a lot of references to recent papers.

Dirichlet series have been studied since the 19^{th} century, but as individual functions. Henry Helson in 1969 [**Hel69**] had the idea of studying function spaces of Dirichlet series, but this idea did not really take off until the landmark paper [**HLS97**] of Hedenmalm, Lindqvist and Seip that introduced a Hilbert space of Dirichlet series that is analogous to the Hardy space on the unit disk. This space, and variations of it, has been intensively studied, and the results are of great interest.

I would like to thank all the students who took part in the courses, and my two Ph.D. students, Brian Maurizi and Meredith Sargent, who did research on Dirichlet series. I would especially like to thank David Opěla for his work in rendering the original course notes into a legible draft. I would also like to thank the National Science Foundation, that partially supported me during the entire long genesis of this project, with grants DMS 0501079, DMS 0966845, DMS 1300280, DMS 1565243.

Notation

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$
$\mathbb{N}^+ = \{1, 2, 3, 4, \dots\}$
$\mathbb{Z}$ = integers
$\mathbb{Q}$ = rationals
$\mathbb{R}$ = reals
$\mathbb{C}$ = complex numbers
$\mathbb{P} = \{2, 3, 5, 7, \dots\} = \{p_1, p_2, p_3, p_4, \dots\}$
$\mathbb{P}_k = \{p_1, p_2, \dots, p_k\}$
$\mathbb{N}_k = \{n \in \mathbb{N}^+ : \text{ all prime factors of } n \text{ lie in } \mathbb{P}_k\}$
$s = \sigma + it,\ s \in \mathbb{C},\ \sigma, t \in \mathbb{R}$
$\Omega_\rho = \{s \in \mathbb{C};\ \mathrm{Re}\ s > \rho\}$
$\pi(x) = \#$ of primes $\leq x$
$\mu(n)$ = Möbius function
$d(k)$ = number of divisors of k
$d_j(k)$ = number of ways to factor k into exactly j factors
$\phi(n)$ = Euler totient function
$\Phi(s) = \sum_{p \in \mathbb{P}} \frac{\log p}{p^s}$
$\Theta(x) = \sum_{p \leq x} \log p$
σ_c = abscissa of convergence
σ_a = abscissa of absolute convergence
$\sigma_1 = \max(0, \sigma_c)$
σ_u = abscissa of uniform convergence
σ_b = abscissa of bounded convergence
$F(x)$ summatory function
$\displaystyle\fint_{-T}^{T}$ normalized integral
ε_n = Rademacher sequence
$\mathbb{E}$ = Expectation
$\mathbb{T}$ = torus
$z^{r(n)} := z_1^{t_1} \dots z_l^{t_l}$, where $n = p_1^{t_1} \cdots p_l^{t_l}$
$\mathcal{B} : \sum a_n z^{r(n)} \mapsto \sum a_n n^{-s}$
$\mathcal{Q} : \sum a_n n^{-s} \mapsto \sum a_n z^{r(n)}$
$\mathbb{T}^\infty$ = infinite torus

$\beta(x) = \sqrt{2}\sin(\pi x)$
$\mathcal{H}^2 = \{\sum_{n=1}^{\infty} a_n n^{-s} : \ \sum_n |a_n|^2 < \infty\}$
$\mathrm{Mult}\,(\mathcal{X}) = \{\varphi : \ \varphi f \in \mathcal{X}, \ \forall f \in \mathcal{X}\}$
$M_\varphi : \ f \mapsto \varphi f$
$\mathbb{D}^\infty$ = infinite polydisk
$E(\varepsilon, f)$ ε-translation numbers of f
$\mathcal{H}^2_w$ = weighted space of Dirichlet series
H^2_w = weighted space of power series
$Q_K : \left(\sum_{n=1}^{\infty} a_n n^{-s}\right) \mapsto \sum_{n \in \mathbb{N}_K} a_n n^{-s}$
ρ = Haar measure on $\mathbb{T}^\infty$
$\ell^2(G)$ = Hilbert space of square-summable functions on the group G
$\mathcal{X}_q$ = Dirichlet characters modulo q
$L(s, \chi)$ = Dirichlet L series
$H^p_\infty(\Omega_{1/2}) = \{g \in \mathrm{Hol}\ (\Omega_{1/2}) : \ [\sup_{\theta \in \mathbb{R}} \ \sup_{\sigma > 1/2} \int_\theta^{\theta+1} |g(\sigma + it)|^p \ dt]^{\frac{1}{p}} < \infty\}$
$\|f\|_{\mathcal{H}^p} = \left[\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(it)|^p \ dt\right]^{1/p}$
$\preccurlyeq$ The left-hand side is less than or equal to a constant times the right-hand side
$\approx$ Each side is $\preccurlyeq$ the other side
$\rho_{\mathcal{A}}(x, y) = \sup\{|\phi(y)\| \ : \ \phi(x) = 0, \ \|\phi\| \leq 1\}$
$\mathcal{H}^\infty \ = \ H^\infty(\Omega_0) \cap \mathbb{D}$
$\mathcal{E} \ : \ f \mapsto \langle f, g_i \rangle$
$g_i = k_{\lambda_i} / \|k_{\lambda_i}\|$

CHAPTER 1

Introduction

A *Dirichlet series* is a series of the form

$$\sum_{n=1}^{\infty} a_n n^{-s} =: f(s), \ \ s \in \mathbb{C}.$$

The most famous example is the *Riemann zeta function*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

NOTATION 1.1. By long-standing tradition, the complex variable in a Dirichlet series is denoted by s, and it is written as

$$s \ = \ \sigma + it.$$

We shall always use σ for $\Re(s)$ and t for $\Im(s)$.

NOTE 1.2. The Dirichlet series for $\zeta(s)$ converges if $\sigma > 1$; in fact, it converges absolutely for such s, since

$$|n^{-s}| = |e^{-(\sigma+it)\log n}| = |e^{-(\sigma+it)\log n}| = n^{-\sigma}.$$

Also, if $\sigma \leq 0$ or $0 < s \leq 1$, the series diverges, in the first case because the terms do not tend to zero, in the second by comparison with the harmonic series.

REMARK 1.3. Consider the power series $\sum_{n=1}^{\infty} z^n$; it converges to $\frac{1}{1-z}$, but only in the open unit disk. Nonetheless, it determines the analytic function $f(z) = \frac{1}{1-z}$ everywhere, since it has a unique analytic continuation to $\mathbb{C} \setminus \{1\}$. The Riemann zeta function can also be analytically continued outside of the region where it is defined by the series.

For this continuation, it can be shown that $\zeta(-2n) = 0$, for all $n \in \mathbb{N}^+$ and that there are no other zeros outside of the strip $0 \leq \mathrm{Re}\ s \leq 1$. The *Riemann hypothesis*, proposed by Bernhard Riemann in 1859, is one of the most famous unanswered conjectures in mathematics. It states that all the zeros other than the even negative integers have real part equal to $\frac{1}{2}$.

We shall prove in Theorem 2.19 that the zeta function has no zeroes on the line $\{\Re s = 1\}$.

The importance of the Riemann zeta function and the Riemann hypothesis lies in their intimate connection with prime numbers and their distribution. On the simplest level, this can be explained by the Euler Product formula below.

Recall that an infinite product $\prod_{n=1}^{\infty} a_n$ is said to *converge*, if the partial products tend to a non-zero finite number (or if one of the a_n's is zero). This is equivalent to the requirement that $\sum_{n=1}^{\infty} \log a_n$ converges (or $a_n = 0$, for some $n \in \mathbb{N}^+$). See *e.g.* [**Gam01**, XIII.3].

NOTATION 1.4. We shall let $\mathbb{P}$ denote the set of primes, and when convenient we shall write

$$\mathbb{P} \; = \; \{p_1, p_2, p_3, p_4, \dots\} \; = \; \{2, 3, 5, 7, \dots\}$$

to label the primes in increasing order. We shall let $\mathbb{P}_k$ denote the first k primes.

THEOREM 1.5. **(Euler Product formula)** *For $\sigma > 1$,*

$$\prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1} = \zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

Formal proof:

$$\begin{aligned} \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1} \; &= \; \left(1 - \frac{1}{2^s}\right)^{-1} \left(1 - \frac{1}{3^s}\right)^{-1} \left(1 - \frac{1}{5^s}\right)^{-1} \dots \\ &= \; \left(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \frac{1}{2^{3s}} + \dots\right) \times \\ & \quad \; \left(1 + \frac{1}{3^s} + \frac{1}{3^{2s}} + \frac{1}{3^{3s}} + \dots\right) \dots \end{aligned}$$

If we formally multiply out this infinite product, we can only obtain a non-zero product by choosing 1 from all but finitely many brackets. This product will be $\frac{1}{q_1^{r_1 s} q_2^{r_2 s} \dots q_k^{r_k s}} = \frac{1}{n^s}$. For each $n \in \mathbb{N}^+$, the term $\frac{1}{n^s}$ will appear exactly once, by the existence and uniqueness of prime factoring.

For a *rigorous proof* assume that Re $s > 1$, and fix $k \in \mathbb{N}^+$. Then

$$\begin{aligned}\prod_{p\in\mathbb{P}_k}\left(1-\frac{1}{p^s}\right)^{-1} &= \prod_{p\in\mathbb{P}_k}\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\frac{1}{p^{3s}}+\dots\right)\\ &= \sum_{n=p_1^{r_1}\dots p_k^{r_k}}\frac{1}{n^s}, \end{aligned}\tag{1.6}$$

where the last equality holds by a variation of the formal argument above and convergence is not a problem, since we are multiplying finitely many absolutely convergent series.

Using (1.6), we have, for Re $s > 1$,

$$\left|\zeta(s)-\prod_{p\in\mathbb{P}_k}\left(1-\frac{1}{p^s}\right)^{-1}\right| = \left|\sum_{\{n;\ p_l|n,\ l>k\}}\frac{1}{n^s}\right| \le \sum_{n\ge p_{k+1}}\frac{1}{n^\sigma} \to 0, \text{ as } k\to\infty.$$

Thus the product converges to $\zeta(s)$.

To see that the limit is non-zero, we have

$$\begin{aligned}\left|1-(1-\frac{1}{p^s})^{-1}\right| &\le \frac{1}{p^\sigma}\frac{1}{p^\sigma-1}\\ &\le \frac{2}{p^\sigma} \quad \text{for } p \text{ large.}\end{aligned}$$

Since $\sigma > 1$, this means that the infinite product converges absolutely, and therefore $\sum \log(1-\frac{1}{p^s})^{-1}$ converges absolutely. □

NOTATION 1.7. We shall let Ω_ρ denote the open half-plane

$$\Omega_\rho = \{s : \Re(s) > \rho\}.$$

COROLLARY 1.8. *$\zeta(s)$ has no zeros in Ω_1.*

Proof: For $s \in \Omega_1$, $\zeta(s)$ is given by an absolutely convergent product. Thus, it can only be zero if one of the terms is zero. But $\left(1-\frac{1}{p^s}\right)^{-1} = 0$ if and only if $p^s = 0$, which never happens. □

THEOREM 1.9. $\sum_{p\in\mathbb{P}}\frac{1}{p} = \infty$.

Proof: Suppose not, then $\sum_{p\in\mathbb{P}}\frac{1}{p}$ converges. By the Taylor expansion of $\log(1-x)$, for x close enough to 0,

$$-x \le \log(1-x) \le -\frac{x}{2},$$

so we conclude that $\sum_{p\in\mathbb{P}} \log\left(1-\frac{1}{p}\right)$ also converges. Since

$$\log\left(1-\frac{1}{p}\right) \;<\; \log\left(1-\frac{1}{p^\sigma}\right),$$

for all $\sigma > 1$ and $p \in \mathbb{P}$, we get

$$\begin{aligned} -\infty &< \sum_{p\in\mathbb{P}} \log\left(1-\frac{1}{p}\right) \\ &< \lim_{\sigma\to 1+} \sum_{p\in\mathbb{P}} \log\left(1-\frac{1}{p^\sigma}\right) \\ &= -\lim_{\sigma\to 1+} \log\frac{1}{\zeta(\sigma)} \\ &= -\infty, \end{aligned}$$

a contradiction. □

The following discrete version of integration by parts is often useful when working with Dirichlet series. In it, integrals are replaced by sums, and derivatives by differences. (In the familiar formula $\int_m^n u dv = u(n)v(n) - u(m)v(m) - \int_m^n v du$, we let u correspond to b, v to A and thus dv to a.)

In fact, one can prove integration by parts for Riemann integrals using the definition (via Riemann sums) and Lemma 1.10.

LEMMA 1.10. **(Abel's Summation by parts formula)** *Let $A_n = \sum_{k=1}^n a_k$, then*

$$\sum_{k=m}^{n} a_k b_k \;=\; A_n b_n - A_{m-1} b_m + \sum_{k=m}^{n-1} A_k (b_k - b_{k+1}).$$

Proof: Since $a_k = A_k - A_{k-1}$, we have

$$\begin{aligned} \sum_{k=m}^{n} a_k b_k &= \sum_{k=m}^{n} [A_k - A_{k-1}] b_k \\ &= \sum_{k=m}^{n} A_k b_k - \sum_{k=m}^{n} A_{k-1} b_k \\ &= \sum_{k=m}^{n-1} A_k [b_k - b_{k+1}] - A_{m-1} b_m + A_n b_n. \end{aligned}$$

□

NOTATION 1.11. For $x > 0$, we let $\pi(x)$ denote the number of primes less than or equal to x.

The prime number theorem (see Chapter 2) is an estimate of how big $\pi(n)$ is for large n. We can use the Euler product formula to relate π and the Riemann zeta function.

THEOREM 1.12. *For $\sigma > 1$,*

$$\log \zeta(s) = s \int_2^\infty \frac{\pi(x)}{x(x^s - 1)} \, dx \, .$$

Proof: In the following calculation we use the fact that $[\pi(k) - \pi(k-1)]$ is equal to 1 if k is a prime, and 0 if k is composite; the equality $\sum_{k=1}^n [\pi(k) - \pi(k-1)] = \pi(n)$; and summation by parts.

$$\begin{aligned}
\log \zeta(s) &= -\sum_{p \in \mathbb{P}} \log\left(1 - \frac{1}{p^s}\right) \\
&= -\sum_{k=2}^{\infty} [\pi(k) - \pi(k-1)] \log\left(1 - \frac{1}{k^s}\right) \\
&= -\lim_{L\to\infty} \sum_{k=2}^{L} [\pi(k) - \pi(k-1)] \log\left(1 - \frac{1}{k^s}\right) \\
&= -\lim_{L\to\infty} \quad \left\{ \sum_{k=2}^{L-1} \pi(k) \left[\log\left(1 - \frac{1}{k^s}\right) - \log\left(1 - \frac{1}{(k+1)^s}\right)\right] \right. \\
&\quad \left. + \ \pi(1) \log\left(1 - \frac{1}{2^s}\right) - \pi(L) \log\left(1 - \frac{1}{L^s}\right) \right\}
\end{aligned}$$

The penultimate term vanishes, since $\pi(1) = 0$. As for the last term, the trivial bound $\pi(L) \leq L$ gives

$$\left| \pi(L) \log\left(1 - \frac{1}{L^s}\right) \right| \leq L \cdot L^{-\sigma} \to 0 \text{ as } L \to \infty.$$

We let $L \to \infty$, and use the fact that $\frac{d}{dx} \log(1 - \frac{1}{x^s}) = \frac{s}{x^{s+1} - x}$, to get:

$$\begin{aligned}
\log \zeta(s) &= -\sum_{k=2}^{\infty} \pi(k) \left[\log\left(1 - \frac{1}{k^s}\right) - \log\left(1 - \frac{1}{(k+1)^s}\right)\right] \\
&= -\sum_{k=2}^{\infty} \pi(k) \int_k^{k+1} \frac{-s}{x^{s+1} - x} \, dx \\
&= s \int_2^\infty \frac{\pi(x)}{x^{s+1} - x} \, dx \, .
\end{aligned}$$

$\square$

NOTATION 1.13. The *Möbius function* is helpful when working with the Riemann zeta function. It is given as follows:

$$\mu(n) = \begin{cases} 1, \; n = 1, \\ (-1)^k, \; \text{if } n \text{ is the product of } k \text{ distinct primes}, \\ 0, \; \text{otherwise.} \end{cases}$$

Its values for the first few positive integer are in the table below:

n	1	2	3	4	5	6	7	8	9	10	11	12
$\mu(n)$	1	-1	-1	0	-1	1	-1	0	0	1	-1	0

THEOREM 1.14. *For* $\sigma > 1$,

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \mu(n) n^{-s}.$$

Proof: We only present a formal proof — convergence can be checked in the same way as was done for the Euler product formula.

$$\begin{aligned} \frac{1}{\zeta(s)} &= \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right) \\ &= \left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{5^s}\right)\cdots \\ &= 1 - \sum_{p \in \mathbb{P}} p^{-s} + \sum_{p,q \in \mathbb{P},\; p \neq q} p^{-s} q^{-s} - \ldots \\ &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \end{aligned}$$

$\square$

It is obvious that the Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ converges (converges absolutely, respectively) for all $s \in \Omega_\rho$ if and only if the series $\sum_{n=1}^{\infty} (a_n n^{-\rho}) n^{-s}$ converges (conv. abs., resp.) for all $s \in \Omega_0$. This ability to translate the Dirichlet series horizontally often allows one to simplify calculations. (It is analogous to working with power series and assuming the center is at 0). The proof of the proposition below is a typical example of this.

The following "uniqueness-of-coefficients" theorem will be used frequently.

PROPOSITION 1.15. *Suppose that* $\sum_{n=1}^{\infty} a_n n^{-s}$ *converges absolutely to* $f(s)$ *in some half-plane* Ω_ρ *and* $f(s) \equiv 0$ *in* Ω_ρ*. Then* $a_n = 0$ *for all* $n \in \mathbb{N}^+$.

Proof: As remarked, we may assume that $\rho < 0$, so in particular, $\sum |a_n| < \infty$. Suppose all the a_n's are not 0, and let n_0 be the smallest natural number such that $a_{n_0} \neq 0$.
Claim: $\lim_{\sigma \to \infty} f(\sigma) n_0^\sigma = a_{n_0}$.

To prove the claim note that

$$\begin{aligned} 0 &\leq n_0^\sigma \left| \sum_{n > n_0} a_n n^{-\sigma} \right| \\ &\leq \sum_{n > n_0} |a_n| \left(\frac{n_0}{n} \right)^\sigma \\ &\leq \left(\frac{n_0}{n_0 + 1} \right)^\sigma \sum_{n > n_0} |a_n|, \end{aligned}$$

and the last term tends to 0 as $\sigma \to \infty$, since $\sum |a_n|$ converges. As

$$f(\sigma) n_0^\sigma = a_{n_0} + n_0^\sigma \sum_{n > n_0} a_n n^{-\sigma},$$

the claim is proved.

The proof is also finished, because the limit in the claim is obviously 0, a contradiction. $\square$

Recall that the Cauchy product formula for the product of power series states that

$$\left(\sum a_n z^n \right) \left(\sum b_m z^m \right) = \sum_{k=0}^{\infty} \left(\sum_{0 \leq n \leq k} a_n b_{k-n} \right) z^k,$$

if at least one of the sums on the left-hand side converges absolutely. The Dirichlet series analogue below involves the sum over all divisors of a given integer. The multiplicative structure of the natural numbers is far more complex than their additive structure. Indeed, as an additive semigroup $\mathbb{N}^+$ is singly generated, while as a multiplicative semigroup it is not finitely generated — the smallest set of generators is $\mathbb{P}$. This is one of the reasons why the theory of Dirichlet series is more complicated than the theory of power series. Now, we state the Dirichlet series analogue of the Cauchy product formula. The proof is immediate.

THEOREM 1.16. *Assume that* $\sum_{n=1}^{\infty} a_n n^{-s}$ *and* $\sum_{m=1}^{\infty} b_m m^{-s}$ *converge absolutely. Then*

$$\left(\sum_{n=1}^{\infty} a_n n^{-s} \right) \left(\sum_{m=1}^{\infty} b_m m^{-s} \right) = \sum_{k=1}^{\infty} \left(\sum_{n | k} a_n b_{k/n} \right) k^{-s},$$

with absolute convergence.

Corollary 1.17. *For $\sigma > 1$,*

$$\zeta^2(s) = \sum_{k=1}^{\infty} d(k) k^{-s},$$

where $d(k)$ denotes the number of divisors of k. More generally,

$$\zeta^j(s) = \sum_{k=1}^{\infty} d_j(k) k^{-s},$$

where $d_j(k)$ denotes the number of ways to factor k into exactly j factors. Here, 1 is allowed to be a factor and two factorings that differ only by the order of the factors are considered to be distinct.

Proof: We shall prove the first formula. Using Theorem 1.16, we have, for $\sigma > 1$,

$$\begin{aligned}\zeta^2(s) &= \left(\sum_{n=1}^{\infty} n^{-s}\right)\left(\sum_{m=1}^{\infty} m^{-s}\right) \\ &= \sum_{k=1}^{\infty}\left(\sum_{n|k} 1\right) k^{-s} \\ &= \sum_{k=1}^{\infty} d(k) k^{-s}.\end{aligned}$$

The proof of the second formula is analogous. □

The formula for $\frac{1}{\zeta(s)}$ implies the following identity for the Möbius function. (It can also be proved directly.)

Corollary 1.18. $\sum_{n|k} \mu(n) = 0$*, for all $k \geq 2$.*

Proof: For $\sigma > 1$, write

$$\begin{aligned}1 &= \zeta(s)\zeta^{-1}(s) \\ &= \left(\sum_{m=1}^{\infty} \frac{1}{m^s}\right)\left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}\right) \\ &= \sum_{k=1}^{\infty}\left(\sum_{n|k} \mu(n)\right) \frac{1}{k^s}.\end{aligned}$$

Comparing the coefficients of the outer-most Dirichlet series completes the proof. □

PROPOSITION 1.19. **(Möbius inversion formula)** *Let f, g be functions on $\mathbb{N}^+$. If*

$$g(q) = \sum_{n|q} f(n), \qquad \text{then} \qquad f(q) = \sum_{d|q} \mu(q/p) g(d).$$

Proof:

$$\begin{aligned}
\sum_{d|q} \mu(q/p) g(d) &= \sum_{d|q} \mu(q/p) \sum_{n|d} f(n) \\
&= \sum_{n|q} \left(\sum_{d|q,\ \frac{q}{d} | \frac{q}{n}} \mu(q/d) \right) f(n) \\
&= \sum_{n|q} \left(\sum_{s | \frac{q}{n}} \mu(s) \right) f(n) \\
&= f(q),
\end{aligned}$$

since, by the preceding corollary, the bracket is non-zero only when $q/n = 1$. $\square$

DEFINITION 1.20. The *Euler totient function* $\phi(n)$ is defined as $\#\{1 \leq k \leq n;\ \gcd(n, k) = 1\}$.

Clearly, $\phi(p) = p - 1$, iff p is a prime. In fact, one can express $\phi(n)$ in terms of the prime factors of n.

LEMMA 1.21. *If $n = q_1^{r_1} \dots q_k^{r_k}$ with $r_j > 0$, then*

$$\phi(n) = n \prod_{j=1}^{k} \left(1 - \frac{1}{q_j} \right).$$

Proof: First, note that $\gcd(n, m) \neq 1$, if and only if, $q_j \big| m$, for some $1 \leq j \leq k$. Consider the uniform probability distribution on $\{1, \dots, n\}$. Let E_j be the event that q_j divides a randomly chosen number in $\{1, \dots, n\}$. For any l that divides n, there are exactly n/l numbers in $\{1, \dots, n\}$ divisible by l. Thus, the events $\{E_j\}_{j=1}^{k}$ are independent and hence so are their complements. Hence, $\phi(n)/n$, the probability that a randomly chosen number is not divisible by any q_j, is equal to the product of the probabilities that it is not divisible by q_j, that is $\prod_j (1 - 1/q_j)$. $\square$

THEOREM 1.22. *For $\sigma > 2$,*

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\phi(n)}{n^s}.$$

Proof: Again, we will only prove it formally, since turning it into a rigorous proof is routine, but renders the proof harder to read. By the Euler product formula, we have

$$\begin{aligned}
\frac{\zeta(s-1)}{\zeta(s)} &= \prod_{p\in\mathbb{P}} \frac{\left(1-\frac{1}{p^s}\right)}{\left(1-\frac{1}{p^{s-1}}\right)} \\
&= \prod_{p\in\mathbb{P}} \left(1-\frac{1}{p^s}\right)\left[1+\frac{p}{p^s}+\frac{p^2}{p^{2s}}+\dots\right] \\
&= \prod_{p\in\mathbb{P}} \left(\left[1+\frac{p}{p^s}+\frac{p^2}{p^{2s}}+\dots\right]-\left[\frac{1}{p^s}+\frac{p}{p^{2s}}+\frac{p^2}{p^{3s}}+\dots\right]\right) \\
&= \prod_{p\in\mathbb{P}} \left[1+\left(1-\frac{1}{p}\right)\left(\frac{p}{p^s}+\frac{p^2}{p^{2s}}+\dots\right)\right] \\
&= \sum_{n=1}^{\infty} a_n n^{-s},
\end{aligned}$$

where

$$a_n = \prod_{j=1}^{k}\left(1-\frac{1}{q_j}\right) q_j^{r_j} = n\prod_{j=1}^{k}\left(1-\frac{1}{q_j}\right) = \phi(n),$$

for $n = q_1^{r_1}\dots q_k^{r_k}$. □

1.1. Exercises

1. Prove that if $\chi : \mathbb{N}^+ \to \mathbb{T}\cup\{0\}$ is a quasi-character, which means $\chi(mn) = \chi(m)\chi(n)$, the same argument that proved the Euler product formula (Theorem 1.5) shows that

$$\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{-s}} = \prod_{p\in\mathbb{P}}\left(\frac{1}{1-\chi(p)p^{-s}}\right).$$

1.2. Notes

For a thorough treatment of the Riemann zeta function, see [**Tit86**]. The material in this chapter comes from the first few pages of Titchmarsh's magisterial book.

CHAPTER 2

The Prime Number Theorem

2.1. Statement of the Prime number theorem

We have defined $\pi(n)$ to be the number of primes less than or equal to n. Euclid's proof that there are an infinite number of primes says that $\lim_{n\to\infty} \pi(n) = \infty$; but how fast does it grow? By Theorem 1.9 and Abel's summation by parts formula we know

$$\begin{aligned} \infty &= \sum_{p\in\mathbb{P}} \frac{1}{p} \\ &= \sum_{n=2}^{\infty} [\pi(n) - \pi(n-1)] \frac{1}{n} \\ &\approx \sum_{n=2}^{\infty} \pi(n) \frac{1}{n^2}, \end{aligned}$$

so $\pi(n)$ cannot be $O(n^\alpha)$ for any $\alpha < 1$.

Gauss conjectured that

$$\pi(x) \sim \frac{x}{\log x}, \tag{2.1}$$

where the asymptotic symbol $\sim$ in (2.1) means that the ratio of the quantities on either side tends to 1 as $x \to \infty$. Tchebyshev proved that

$$.93 \frac{x}{\log x} \le \pi(x) \le 1.1 \frac{x}{\log x}$$

for x large, and also showed that if

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x}$$

exists, it must be 1. The full prime number theorem was proved in 1896 by de la Vallée Poussin and Hadamard.

Theorem 2.2. **[Prime Number Theorem]**

$$\pi(x) \sim \frac{x}{\log x}.$$

Looking at some examples, we see that

$$\left.\begin{aligned} \pi(10^6) &= 78{,}498 \\ \frac{10^6}{\log(10^6)} &\approx 72{,}382 \end{aligned}\right\} \implies \frac{\pi(10^6)}{\frac{10^6}{\log 10^6}} \approx 1.08$$

and

$$\left.\begin{aligned} \pi(10^9) &= 50{,}847{,}478 \\ \frac{10^9}{\log(10^9)} &\approx 48{,}254{,}942 \end{aligned}\right\} \implies \frac{\pi(10^9)}{\frac{10^9}{\log 10^9}} \approx 1.05$$

Definition 2.3. For $s \in \Omega_1$, we define

$$\Phi(s) := \sum_{p \in \mathbb{P}} \frac{\log p}{p^s}.$$

It is easy to see that this Dirichlet series converges absolutely in Ω_1.

Definition 2.4. For $x \in \mathbb{R}$, define

$$\Theta(x) := \sum_{p \in \mathbb{P},\ p \le x} \log p.$$

The key to proving the Prime number theorem is establishing the estimate $\Theta(x) \sim x$ (Proposition 2.27).

Say more here?

2.2. Proof of the Prime number theorem

We will now prove the Prime number theorem in a series of steps.

Lemma 2.5.

$$\Theta(x) = O(x) \text{ as } x \to \infty, \text{i.e., } \limsup_{x \to \infty} \frac{\Theta(x)}{x} < \infty.$$

Proof: Note that

$$\binom{2n}{n} \ge \prod_{n < p \le 2n,\ p \in \mathbb{P}} p.$$

Indeed, the LHS is a positive integer that is divisible by the RHS. Note that we have not yet proved that there are any primes between n and $2n$, so the RHS may be an empty product (we interpret empty products as having the value 1).

Thus,

$$\binom{2n}{n} \ge \prod_{n < p \le 2n} p = e^{\Theta(2n) - \Theta(n)}.$$

Now, by the binomial theorem,

$$2^{2n} = (1+1)^{2n} = \binom{2n}{0} + \cdots + \binom{2n}{n} + \cdots + \binom{2n}{2n},$$

Thus

$$2^{2n} \geq \binom{2n}{n} \implies e^{n\log 4} \geq \binom{2n}{n} \geq e^{\Theta(2n)-\Theta(n)},$$

and consequently

$$\Theta(2n) - \Theta(n) \leq n\log 4.$$

For $x \in \mathbb{R}, x \geq 1$, we have

$$\begin{aligned}
\Theta(2x) - \Theta(x) &\leq \Theta(\lfloor 2x \rfloor) - \Theta(\lfloor x \rfloor) \\
&\leq \Theta(2\lfloor x \rfloor + 1) - \Theta(\lfloor x \rfloor) \\
&\leq \log(\lfloor 2x \rfloor + 1) + \Theta(\lfloor 2x \rfloor) - \Theta(\lfloor x \rfloor) \\
&\leq cx.
\end{aligned}$$

Now fix x and choose $n \in \mathbb{N}$ such that $\frac{x}{2^{n+1}} \leq 1 \leq \frac{x}{2^n}$. Then, by telescoping,

$$\begin{aligned}
\Theta(x) - \Theta(1) &= \sum_{j=0}^{n} \Theta(\frac{x}{2^j}) - \Theta(\frac{x}{2^{j+1}}) \\
&\leq \sum_{j=0}^{n} c\frac{x}{2^{j+1}} \\
&= cx.
\end{aligned}$$

Since $\Theta(1) = 0$, we conclude that

$$\Theta(x) = O(x) \tag{2.6}$$

□

Recall that $\Phi(s) = \sum_{p\in\mathbb{P}} \frac{\log p}{p^s}$. Since $\sum_{p\in\mathbb{P}} \frac{1}{p} = \infty$ (Theorem 1.9) we conclude that $\Phi(s)$ has a pole at 1.

Lemma 2.7. *The function $\Phi(s) - \frac{1}{s-1}$ is holomorphic in $\overline{\Omega_1}$.*

Proof: In Ω_1,

$$\zeta(s) = \prod_{p\in\mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1} = \prod_{p\in\mathbb{P}} \frac{1}{1-p^s}$$

By logarithmic differentiation we obtain

$$\begin{aligned}\frac{\zeta(s)}{\zeta'(s)} &= -\sum_{p\in\mathbb{P}} \frac{\frac{\partial}{\partial s}(1-p^{-s})}{1-p^{-s}} \\ &= -\sum_{p\in\mathbb{P}} (p^{-s}\log p)\frac{1}{1-p^{-s}} \\ &= -\sum_{p\in\mathbb{P}} \frac{\log p}{p^s-1} \qquad (2.8)\end{aligned}$$

Now

$$\frac{1}{p^s-1} = \frac{1}{p^s} + \frac{1}{p^s(p^s-1)} \qquad (2.9)$$

Combining (2.8) and (2.9), we obtain, for $s \in \Omega_1$,

$$\frac{-\zeta'(s)}{\zeta(s)} = \sum_{p\in\mathbb{P}} \frac{\log p}{p^s} + \sum_{p\in\mathbb{P}} \frac{\log p}{p^s(p^s-1)} \qquad (2.10)$$

Note that we can rearrange the terms since the series converge absolutely in Ω_1. Thus, for $s \in \Omega_1$,

$$\frac{-\zeta'(s)}{\zeta(s)} = \Phi(s) + \sum_{p\in\mathbb{P}} \frac{\log p}{p^s(p^s-1)} \qquad (2.11)$$

The second term on the RHS defines an analytic function in $\Omega_{1/2}$ as the series converges there absolutely. Thus in $\Omega_{1/2}$, any information about the analyticity of $\frac{-\zeta'(s)}{\zeta(s)}$ translates into the analyticity of $\Phi(s)$.

The function $\zeta(s)$ has a pole at 1 with residue 1 and so $\zeta(s) - \frac{1}{s-1}$ is analytic near 1, and consequently, $\zeta'(s) + \frac{1}{(s-1)^2}$ is analytic near 1. Thus $\frac{\zeta'(s)}{\zeta(s)} - \frac{-\frac{1}{(s-1)^2}}{\frac{1}{s-1}} = \frac{\zeta'(s)}{\zeta(s)} - \frac{1}{(s-1)}$ is analytic near 1.

Thus

$$\Phi(s) = \frac{\zeta'(s)}{\zeta(s)} - \sum_{p\in\mathbb{P}} \frac{\log p}{p^s(p^s-1)} \qquad (2.12)$$

is holomorphic in $\Omega_{1/2} \cap \{s : \ \zeta(s) \neq 0\}$.

It remains to prove that $\zeta(s) \neq 0$ if $\operatorname{Re} s \geq 1$.

Definition 2.13. The *von Mangoldt function* $\Lambda : \mathbb{N}_0 \to \mathbb{R}$, is defined as

$$\Lambda(m) = \begin{cases} \log p, \ \text{if } m = p^k, \\ 0, \ \text{else.} \end{cases} \qquad (2.14)$$

PROPOSITION 2.15. *For $s \in \Omega_1$*

$$\frac{-\zeta'(s)}{\zeta(s)} = \sum_{n\geq 2} \frac{\Lambda(n)}{n^s} = \sum_{p\in\mathbb{P}} \frac{\log p}{p^s - 1} \tag{2.16}$$

holds.

PROOF: We have $\zeta(s) = \prod_{p\in\mathbb{P}}(1 - \frac{1}{p^s})^{-1}$ and thus

$$\begin{aligned} \frac{\zeta'(s)}{\zeta(s)} &= -\sum_{p\in\mathbb{P}} \log p \frac{p^{-s}}{1 - \frac{1}{p^s}} \\ &= -\sum_{p\in\mathbb{P}} \frac{\log p}{p^s - 1} \end{aligned}$$

which proves that the first and last term in the statement of the proposition are equal. For Re $s > 1$, $\|1/p^s\| < 1$, so the first equality above yields

$$\begin{aligned} \frac{\zeta'(s)}{\zeta(s)} &= -\sum_{p\in\mathbb{P}} (\log p) p^{-s} (1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots) \\ &= -\sum_{p\in\mathbb{P}} \sum_{k=1}^{\infty} \log p (p^k)^{-s}. \end{aligned}$$

This double summation goes over exactly those numbers $n = p^k$ for which $\Lambda(n)$ does not vanish and thus, for $s \in \Omega_1$,

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{p\in\mathbb{P}} \sum_{k=1}^{\infty} \log p (p^k)^{-s} = \sum_{n\geq 2} \frac{\Lambda(n)}{n^s} \tag{2.17}$$

□

LEMMA 2.18. *Let $x_0 \in \mathbb{R}$ and assume F is holomorphic in a neighborhood of x_0, $F(x_0) = 0$ and $F \neq 0$. Then there exists $\varepsilon > 0$ such that*

$$Re \left(\frac{F'(x)}{F(x)} \right) > 0$$

for $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$.

PROOF: Write $F(x) = a_k(s - x_0)^k + a_{k+1}(s - x_0)^{k+1} + \ldots$ where $k > 0$. Then

$$\begin{aligned}
\frac{F'(x)}{F(x)} &= \frac{ka_k(x-x_0)^{k-1} + (k+1)a_{k+1}(x-x_0)^k + \ldots}{a_k(x-x_0)^k + a_{k+1}(x-x_0)^{k+1} + \ldots} \\
&= \frac{k + \frac{(k+1)a_{k+1}}{a_k}(x-x_0) + \ldots}{(x-x_0) + \frac{a_{k+1}}{a_k}(x-x_0)^2 + \ldots} \\
&\approx \frac{k}{x-x_0} > 0,
\end{aligned}$$

for $x \in (x_0, x_0 + \varepsilon)$. □

THEOREM 2.19. *The Riemann ζ function does not vanish on the line $\{\Re(s) = 1\}$.*

PROOF: Suppose that $\zeta(1 + it_0) = 0$, for $t_0 \in \mathbb{R} \setminus \{0\}$. Define

$$F(s) := \zeta^3(s)\zeta^4(s + it_0)\zeta(s + 2it_0). \tag{2.20}$$

At $s = 1$, we see that ζ^3 has a pole of order 3, and $\zeta^4(s+it_0)$ vanishes to order 4, so $F(1) = 0$. Thus, in a neighborhood of 1, F is holomorphic. Using Lemma 2.18, Re $\left(\frac{F'(x)}{F(x)}\right) > 0$ for $x \in (1, 1 + \varepsilon)$. Computing

$$\begin{aligned}
\frac{F'(x)}{F(x)} &= 3\frac{\zeta'(x)}{\zeta(x)} + 4\frac{\zeta'(x+it_0)}{\zeta(x+it_0)} + \frac{\zeta'(x+2it_0)}{\zeta(x+2it_0)} \\
&= \sum_{n\geq 2} \Lambda(n)\left[-3n^{-x} - 4n^{-x}e^{-it_0 \log n} - n^{-x}e^{-2it_0 \log n}\right],
\end{aligned}$$

thus,

$$\begin{aligned}
\text{Re}\,\frac{F'(x)}{F(x)} &= \sum_{n\geq 2} -\Lambda(n)n^{-x}\left[3 + 4\cos(t_0 \log n) + \cos(2t_0 \log n)\right] \\
&= \sum_{n\geq 2} -\Lambda(n)n^{-x}\left[2 + 4\cos(t_0 \log n) + 2\cos(t_0 \log n)\right]
\end{aligned}$$

We observe that $-\Lambda(n)n^{-x} \leq 0$ for every $n \geq 2$ while the term in the square bracket is always non-negative, since it the square of

$$\sqrt{2}[1 + \cos(t_0 \log n)],$$

a contradiction with Lemma 2.18. □

LEMMA 2.21. *Let $f(t) : [0, \infty) \to \mathbb{C}$ be bounded and suppose that*

$$g(s) = \int_0^\infty f(t)e^{-st}\,dt \tag{2.22}$$

extends to a holomorphic function in $\overline{\Omega_0}$. Then $\int_0^\infty f(t)\, dt$ exists and equals $g(0)$.

Proof: Let

$$g_T(s) = \int_0^T f(t) e^{-st}\, dt \ . \tag{2.23}$$

Then g_T is an entire function by Morera's theorem, and $g_T(0) = \int_0^T f(t)\, dt$. We want to show that $\lim_{t\to\infty} g_T(0) = g(0)$.

insert image around here

For $R,\ \delta > 0$ let $U_{R,\delta} := \mathbb{D}(0,R) \cap \Omega_{-\delta}$. For any $R > 0$ there is $\delta > 0$ such that g is holomorphic in $\overline{U_{R,\delta}}$, since by hypothesis g is holomorphic in a neighborhood of $\overline{\Omega_0}$. Let $C := \partial U_{R,\delta}$ and $C_+ = C \cap \Omega_0$ and $C_- = C \setminus \overline{\Omega_0}$. By Cauchy's theorem:

$$g(0) - g_T(0) = \frac{1}{2\pi i} \int_C \left[g(s) - g_t(s)\right] e^{sT} \left(1 + \frac{s^2}{R^2}\right) \frac{ds}{s}, \tag{2.24}$$

since $e^{st}(1 + \frac{s^2}{R^2})$ has value 1 at 0 and is holomophic everywhere in our contour. Let $h(s) := \left[g(s) - g_t(s)\right] e^{sT} \left(1 + \frac{s^2}{R^2}\right)$. In Ω_0, we have

$$\begin{aligned}
|g(s) - g_T(s)| &= \left| \int_T^\infty f(t) e^{-st}\, dt \right| \\
&\leq M \left| \int_T^\infty e^{-st}\, dt \right| \\
&= M \left| \int_T^\infty e^{-(\text{Re } s)t}\, dt \right| \\
&= M \frac{1}{\text{Re } s} e^{-\text{Re } s\ T}
\end{aligned}$$

Thus

$$\begin{aligned}
\left| \int_{C_+} h(s) \frac{ds}{s} \right| &\leq \int_{C_+} \left| f(t) e^{-st}\, dt \right| \\
&\leq M \int_{C_+} \frac{e^{-\text{Re } sT}}{\text{Re } s} \left| \frac{e^{sT}}{s} \left(1 + \frac{s^2}{R^2}\right) \right| |\ ds\ |
\end{aligned}$$

For $s \in C_+$, we have $|s| = R$ and so

$$\left(1 + \frac{s^2}{R^2}\right) \frac{1}{s} = \frac{R^2 + s^2}{R^2 s} = \frac{|s|^2 + s}{R^2 s} = \frac{\overline{s} + s}{R^2}$$

Thus,

$$
\begin{aligned}
\left|\int_{C+} h(s)\frac{ds}{s}\right| &\leq \frac{M}{2\pi}\int_{C_+}\frac{e^{-\text{Re } sT}}{\text{Re } s}\frac{e^{\text{Re } sT}}{s}\frac{2\text{Re } s}{R^2}|\ ds\ | \\
&\leq \frac{M}{\pi R^2}\pi R \\
&= \frac{M}{R}
\end{aligned}
$$

We conclude $\int_{C_+} h(s)\frac{ds}{s} \to 0$ as $R \to 0$.

For C_-, we will show that both

$$
I_1(T) \ := \int_{C_-} |g(s)|\, e^{sT}\left(1+\frac{s^2}{R^2}\right)\frac{ds}{s}
$$

and

$$
I_2(T) \ := \int_{C_-} |g_T(s)|\, e^{sT}\left(1+\frac{s^2}{R^2}\right)\frac{ds}{s}
$$

tend to 0 as R tends to 0.

We start with I_1:

$$
\begin{aligned}
|g_T(s)| &= \left|\int_0^T f(t)e^{-st}\ dt\ \right| \\
&\leq M\int_0^T e^{-(\text{Re } s)t}\ dt \\
&\leq M\int_{-\infty}^T e^{-(\text{Re } s)t}\ dt \\
&= \frac{M}{\text{Re } s}e^{-\text{Re } st}
\end{aligned}
$$

Therefore,

$$
I_1(T) \leq \int_{C_-}\frac{Me^{-(\text{Re } s)t}}{|\text{Re } s|}\left|e^{sT}\left(1+\frac{s^2}{R^2}\right)\frac{1}{s}\right|\ |\ ds\ |
$$

and since g_T is an entire function, we can integrate over the semicircle Γ_- instead of C_- and use the same estimates as in Ω_0 to get

$$
I_1(T) \leq \frac{M}{R}.
$$

Now

$$
I_2(T) = \int_{C_-}\left[g(s)\left(1+\frac{s^2}{R^2}\right)\frac{1}{s}\right]e^{sT}\ ds
$$

and the expression in square bracket is independent of T and holomorphic in a neighborhood of C_- while $e^{sT} \to 0$ as $T \to \infty$. Using dominated convergence theorem, we conclude that $I_2 \to 0$ as $T \to \infty$.

Thus,

$$\begin{aligned} |g(0) - g(T)| &\leq \left| \int_{C_+} h(s) \frac{ds}{s} \right| + |I_1(T)| + |I_2(T)| \\ &\leq \frac{M}{R} + \frac{M}{R} + I_2(T) \to \frac{2M}{R} \end{aligned}$$

Taking the limit as $R \to \infty$ implies that $g(0) = \lim_{T\to\infty} g_T(0)$ □

LEMMA 2.25. *The integral* $\int_1^\infty \frac{\Theta(x)-x}{x^2}\ dx$ *converges.*

PROOF: For Re $s > 1$,

$$\Phi(s) = \sum_{p \in pri} \frac{\log p}{p^s} = \int_1^\infty \frac{d\Theta(x)}{dx}$$

We are using the Stiltjes integral in the last expression because $\Theta(x)$ is a step function.

We use integration by parts with $u := x^{-s}$ and $dv := d\Theta(x)$. Then $du = -sx^{-(s+1)}\ dx$ and $v(x) = \Theta(x)$, giving

$$\Phi(x) = x^{-s}\Theta(x)\Big|_1^\infty + s\int_1^\infty \frac{\Theta(x)}{x^{s+1}}\ dx\ .$$

The first term vanishes since $\Theta(x) = O(x)$ as $x \to \infty$. We conclude that

$$\Phi(s) = s\int_1^\infty \frac{\Theta(x)}{x^{s+1}}\ dx\ .$$

Now let us use the substitution, $x = e^t$ to get

$$\Phi(s) = s\int_0^\infty \Theta(e^t)e^{-ts}\ dt\ .$$

We want apply Lemma 2.21 to $f(t) := \Theta(e^t)e^{-t} - 1$ and $g(s) = \frac{\Theta(s+1)}{s+1} - \frac{1}{s}$. By Lemma 2.5, we get that $f(t)$ is bounded and by Lemma 2.7, we know that $\frac{\Theta(s+1)}{s+1} - \frac{1}{s}$ is holomorphic in $\overline{\Omega_0}$. In order to apply Lemma 2.21, we need to check that $g(s)$ is the Laplace transform of $f(t)$.

We have

$$\int_0^\infty \Theta(e^t)e^{-t}e^{-ts}\ dt\ = \int_0^\infty \Theta(e^t)e^{-t(s+1)}\ dt$$

and

$$\int_0^\infty 1e^{-ts}\ dt\ = \frac{1}{s}$$

and thus $g(s)$ is the Laplace transform of $f(t)$, and we can apply Lemma 2.21 to conclude that $\int_0^\infty f(t)\, dt$ exists.

$$\begin{aligned}
\int_0^\infty f(t)dt &= \int_0^\infty \left[\Theta(e^t)e^{-t} - 1\right] \, dt \\
&= \int_1^\infty \left[\Theta(x)\frac{1}{x} - 1\right] \frac{dx}{x} \\
&= \int_1^\infty \left[\frac{\Theta(x) - x}{x^2}\right] \, dx
\end{aligned}$$

which concludes the proof. □

NOTE 2.26. See [**Fol99**, p. 107] for information on integration by parts in the context of the Stieltjes integrals.

PROPOSITION 2.27. *$\lim_{x\to\infty} \frac{\Theta(x)}{x} = 1$, that is, $\Theta(x) \sim x$.*

PROOF: We will proceed by contraction. There are two cases.

First assume that $\limsup_{x\to\infty} \frac{\Theta(x)}{x} > 1$. Thus, there exists $\lambda > 1$ and a sequence $\{x_n\}$ with $x_n \to \infty$ such that $\Theta(x_n) > \lambda x_n$. Then, since Θ is non-decreasing,

$$\int_{x_n}^{\lambda x_n} \frac{\Theta(t) - t}{t^2} \, dt \;\geq \int_{x_n}^{\lambda x_n} \frac{\lambda x_n - t}{t^2} \, dt \;=: c_\lambda.$$

We integrate the two pieces,

$$\int_{x_n}^{\lambda x_n} \frac{\lambda x_n}{t^2} \, dt \;= \lambda x_n \left(-\frac{1}{t}\Big|_{x_n}^{\lambda x_n} \right) = \lambda - 1$$

and

$$\int_{x_n}^{\lambda x_n} \frac{dt}{t} = \log(\lambda x_n) - \log x_n = \log\lambda$$

to conclude that $c_\lambda = \lambda - 1 - \log\lambda > 0$ by a well-known inequality for log. This implies that $\int_1^\infty \frac{\Theta(x)-x}{x^2} \, dx$ does not converge, a contradiction.

The second case is that $\liminf_{x\to\infty} \frac{\Theta(x)}{x} < 1$, so there is $\lambda < 1$ and a sequence $\{x_n\}$ with $x_n \to \infty$ and $\frac{\Theta(x_n)}{x_n} < \lambda$. As before,

$$\int_{\lambda x_n}^{x_n} \frac{\Theta(t) - t}{t^2} \, dt \;\leq \int_{x_n}^{\lambda x_n} \frac{\lambda x_n - t}{t^2} \, dt \;= -c_\lambda = -(\lambda - 1 - \log\lambda) < 0$$

and we reach a contraction as in the first case. □

Proof of Theorem 2.2. We can estimate

$$\begin{aligned}\Theta(x) &= \sum_{p\le x}\log p\\ &\le \sum_{p\le x}\log x\\ &= \pi(x)\log x.\end{aligned}$$

By Proposition 2.27, $\Theta(x)\sim x$ and thus we have the bound

$$\liminf_{x\to\infty}\ \pi(x)\frac{\log x}{x}\ \ge\ 1.$$

For the other bound, let $\varepsilon>0$, and write

$$\begin{aligned}\Theta(x) &\ge \sum_{x^{1-\varepsilon}\le p\le x}\log p\\ &\ge \sum_{x^{1-\varepsilon}\le p\le x}\log x^{1-\varepsilon}\\ &= [\pi(x)-\pi(x^{1-\varepsilon})](1-\varepsilon)\log x\\ &= (1-\varepsilon)\log x\left[\pi(x)+O(x^{1-\varepsilon})\right]\end{aligned}$$

where the last equality come from Lemma 2.5.

We have

$$\pi(x)\log x\le\frac{1}{1-\varepsilon}\Theta(x)+O(x^{1-\varepsilon}\log x)$$

and hence

$$\frac{\pi(x)\log x}{x}\le\frac{1}{1-\varepsilon}\frac{\Theta(x)}{x}+O(x^{-\varepsilon}\log x).$$

Using Proposition 2.27 again, we get

$$\limsup_{x\to\infty}\ \pi(x)\frac{\log x}{x}\ \le\ \frac{1}{1-\varepsilon}$$

for every $\varepsilon>0$. Taking $\varepsilon\to 0$ yields

$$\limsup_{x\to\infty}\ \pi(x)\frac{\log x}{x}\ \le\ 1$$

which concludes the proof. □

2.3. Historical Notes

The *offset logarithmic integral function*, $Li(x):=\int_2^x\frac{dt}{\log t}$ satisfies $Li(x)\approx\frac{x}{\log x}\approx\pi(x)$ but is a better approximation to $\pi(x)$.

Gauss conjectured that $\pi(n)\le Li(n)$. This was disproved by E. Littlewood in 1914.

During the proof of the Prime Number Theorem, we used the fact that $\zeta(s)$ does not vanish for Re $s \geq 1$. More precise estimates showing that the zeros of $\zeta(s)$ must lie "close to" the critical line $\{\text{Re } s = 1/2\}$ yield estimates on the error $|\pi(x) - Li(x)|$.

The Riemann hypothesis is equivalent to the error estimate

$$\pi(x) = Li(x) + O(\sqrt{x}\log x).$$

The best known estimate is of the error is

$$\pi(x) = Li(x) + O(xe^{-\frac{A(\log x)^{3/5}}{(\log\log x)^{1/5}}}).$$