through z'' – and then allowing ζ to approach z'', we see that the 60° opening of the sector $\widehat{z''zz'}$ *cannot be made larger* if the inequality in question is to hold for all $\zeta \in E_6$.)

Again, for $1 \leqslant k \leqslant 6$,

$$\int_{E_k} \log^+ \frac{\rho}{|z_k - \zeta|}\, d\mu(\zeta) \quad \leqslant \quad \int_E \log^+ \frac{\rho}{|z_k - \zeta|}\, d\mu(\zeta).$$

But the right-hand integral is $< \delta$ by choice of ρ since $z_k \in \bar{E}_k \subseteq E$! Thence, going back to the previous relations, we find that

$$\int_E \log^+ \frac{\rho}{|z - \zeta|}\, d\mu(\zeta) \quad < \quad 6\delta$$

as we set out to show.

As explained above, this implies that $U(z) < M' + 7\delta$ in a suitably small neighborhood of any $z_0 \in E$ and thus finally, that $U(z) \leqslant M'$ in $\mathbb{C} \sim E$, after squeezing δ. From that, however, our result follows as we saw at the beginning of this proof. We are done.

Problem 50

With K a *compact* subset of the open (*sic*!) unit disk Δ and μ a positive measure supported on K, put

$$V(z) \quad = \quad \int_K \log \left| \frac{1 - \bar{\zeta}z}{z - \zeta} \right| d\mu(\zeta), \qquad |z| \leqslant 1.$$

Suppose that V is *finite* at each point of K. Show then that if $W(z)$ is superharmonic and $\geqslant 0$ in Δ, and satisfies

$$W(z) \geqslant V(z)$$

for $z \in K$, we have $V(z) \leqslant W(z)$ in Δ.

Remark. The finiteness of V at the points of μ's support cannot be dispensed with here. Consider, for example,

$$V(z) \quad = \quad \log \frac{1}{|z|}$$

and

$$W(z) \quad = \quad \frac{1}{2} \log \frac{1}{|z|} \quad !$$

(Hint: Argue first as in the above proof to get, for any given $\varepsilon > 0$, a compact subset E of K with

$$\mu(K \sim E) \quad < \quad \varepsilon$$

and

$$\int_E \log^+ \frac{\rho}{|z-\zeta|}\, d\mu(\zeta) \longrightarrow 0$$

uniformly for $z \in E$ as $\rho \longrightarrow 0$.

Put

$$U(z) \quad = \quad \int_E \log\left|\frac{1-z\bar{\zeta}}{z-\zeta}\right| d\mu(\zeta);$$

here, $U(z) \leqslant V(z)$, so in particular $U(z) \leqslant W(z)$ on $E \subseteq K$. For any fixed $z \in \Delta \sim K$, $V(z) - U(z)$ is *small* if ε is, so it is enough to show that $U(z) \leqslant W(z)$ at each $z \in \Delta \sim E$.

The difference $W(z) - U(z)$ is superharmonic in $\Delta \sim E$; the last relation therefore holds (by a corollary from article 1) provided that

$$\liminf_{z \to z_0} (W(z) - U(z)) \quad \geqslant \quad 0$$

for each $z_0 \in \partial(\Delta \sim E)$.

When $|z_0| = 1$, this is manifest, W being $\geqslant 0$ in Δ with (here) $V(z)$ and $U(z)$ *continuous* and *zero* at z_0. It is hence only necessary to look at the behaviour near points $z_0 \in E$.

Fix any such z_0, and take any $\delta > 0$. Reasoning as in the above proof, show that

$$U(z) \quad < \quad U(z_0) \quad + \quad 7\delta$$

in a sufficiently small neighborhood of z_0. Since $W(z_0) \geqslant U(z_0)$, we therefore have

$$W(z) - U(z) \quad > \quad -8\delta$$

in such a neighborhood.)

We come now to the result about continuity spoken of at the beginning of this article.

Theorem (due independently to Evans and to Vasilesco). *Given a positive measure μ supported on a compact set K, put*

$$V(z) \quad = \quad \int_K \log \frac{1}{|z-\zeta|}\, d\mu(\zeta).$$

If the restriction of V to K *is continuous at a point $z_0 \in K$, $V(z)$ (as a function defined in $\mathbb{C}$) is continuous at z_0.*

Proof. Given $\varepsilon > 0$, there is an $\eta > 0$ (which we *fix*) such that

$$|V(z) - V(z_0)| \leqslant \varepsilon \quad \text{for } z \in K \text{ with } |z - z_0| \leqslant \eta.$$

Consider, on the *compact* set

$$K_\eta = K \cap \{|z - z_0| \leqslant \eta\}$$

the *continuous* functions

$$F_\rho(z) = \min\left\{V(z_0) - 2\varepsilon, \int_K \min\left(\log\frac{1}{|z-\zeta|}, \log\frac{1}{\rho}\right) d\mu(\zeta)\right\},$$

defined for each $\rho > 0$. When ρ *diminishes* towards 0, $F_\rho(z)$ *increases* for each fixed z, tending, moreover, to $\min(V(z_0) - 2\varepsilon, V(z))$, equal to the *constant* $V(z_0) - 2\varepsilon$ for $z \in K_\eta$. According to *Dini's theorem*, the convergence must then be *uniform* on K_η, so, for all sufficiently small $\rho > 0$, we have

$$\int_K \min\left(\log\frac{1}{|z-\zeta|}, \log\frac{1}{\rho}\right) d\mu(\zeta) > V(z_0) - 3\varepsilon, \qquad z \in K_\eta.$$

The integral on the left is, however,

$$\leqslant \int_K \log\frac{1}{|z-\zeta|}\, d\mu(\zeta),$$

which is in turn $\leqslant V(z_0) + \varepsilon$ for $z \in K_\eta$; subtraction thus yields

$$\int_K \log^+\frac{\rho}{|z-\zeta|}\, d\mu(\zeta) < 4\varepsilon, \qquad z \in K_\eta,$$

for $\rho > 0$ sufficiently small.

Fix any such $\rho < \eta/2$. We desire to use Maria's theorem so as to take advantage of the relation just found, but the appearance of $\log^+$ in the integrand instead of the logarithm gives rise to a slight difficulty.

Taking a new parameter λ with $1 < \lambda < 2$, we bring in the set

$$K_{\lambda\rho} = K \cap \{|z - z_0| \leqslant \lambda\rho\}.$$

Since $\lambda\rho < 2\rho < \eta$, we have $K_{\lambda\rho} \subseteq K_\eta$, so surely

$$\int_K \log^+\frac{\rho}{|z-\zeta|}\, d\mu(\zeta) < 4\varepsilon$$

for $z \in K_{\lambda\rho}$, whence, *a fortiori*,

$$\int_{K_{\lambda\rho}} \log\frac{\rho}{|z-\zeta|}\, d\mu(\zeta) \leqslant \int_{K_{\lambda\rho}} \log^+\frac{\rho}{|z-\zeta|}\, d\mu(\zeta) < 4\varepsilon$$

when z is in $K_{\lambda\rho}$.

Thence, applying Maria's result to the integral

$$\int_{K_{\lambda\rho}} \log \frac{\rho}{|z-\zeta|}\, d\mu(\zeta)$$

(which differs by but an additive constant from

$$\left. \int_{K_{\lambda\rho}} \log \frac{1}{|z-\zeta|}\, d\mu(\zeta) \right),$$

we see that it is in fact $\leqslant 4\varepsilon$ for *all* z. From this we will now deduce that

$$\int_{K_{\lambda\rho}} \log^+ \frac{\rho}{|z-\zeta|}\, d\mu(\zeta) \quad < \quad 5\varepsilon$$

(with $\log^+$ again and not log!) whenever z is sufficiently close to z_0, provided that $\lambda > 1$ is taken near enough to 1.

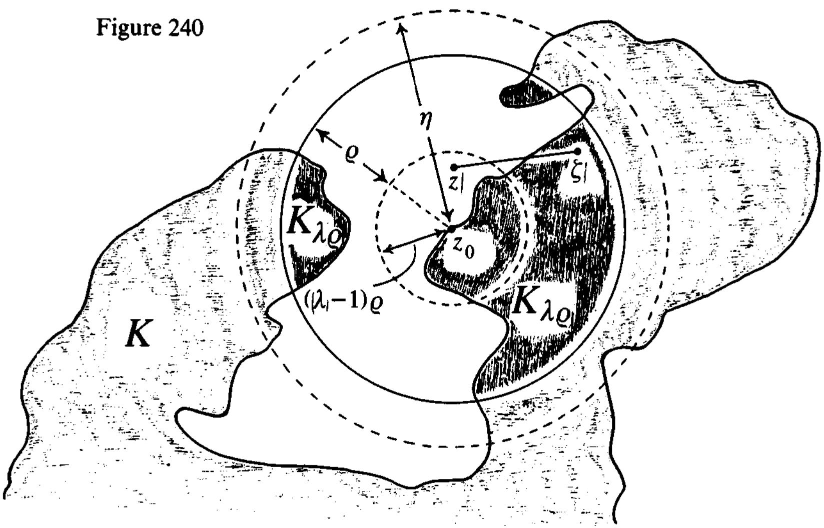
We have

$$\log^+ \frac{\rho}{|z-\zeta|} \quad = \quad \log \frac{\rho}{|z-\zeta|} \quad + \quad \log^- \frac{\rho}{|z-\zeta|}.$$

Here, when $\zeta \in K_{\lambda\rho}$ and $|z-z_0| \leqslant (\lambda-1)\rho$, we are assured that $|z-\zeta| \leqslant (2\lambda-1)\rho$, making

$$\log^- \frac{\rho}{|z-\zeta|} \quad \leqslant \quad \log(2\lambda-1).$$

Figure 240



Therefore, for $|z - z_0| \leqslant (\lambda - 1)\rho$,

$$\int_{K_{\lambda\rho}} \log^+ \frac{\rho}{|z-\zeta|}\, d\mu(\zeta) \quad \leqslant \quad \int_{K_{\lambda\rho}} \log \frac{\rho}{|z-\zeta|}\, d\mu(\zeta) \;+\; \mu(K_{\lambda\rho}) \log(2\lambda - 1).$$

By choosing (and then fixing) $\lambda > 1$ *close enough to* 1, we ensure that the second term on the right is $< \varepsilon$; since, then, the first is $\leqslant 4\varepsilon$ as we have seen, we get

$$\int_{K_{\lambda\rho}} \log^+ \frac{\rho}{|z-\zeta|}\, d\mu(\zeta) \quad < \quad 5\varepsilon \qquad \text{for } |z - z_0| \leqslant (\lambda - 1)\rho.$$

Now, when $|z - z_0| \leqslant (\lambda - 1)\rho$ and $\zeta \in K \sim K_{\lambda\rho}$, making $|\zeta - z_0| > \lambda\rho$, we have (see the preceding picture)

$$|z - \zeta| \;>\; \rho,$$

so

$$\log^+ \frac{\rho}{|z-\zeta|} \quad = \quad 0.$$

For $|z - z_0| \leqslant (\lambda - 1)\rho$, the integral in the last relation is thus equal to

$$\int_K \log^+ \frac{\rho}{|z-\zeta|}\, d\mu(\zeta),$$

which is hence $< 5\varepsilon$ then!

Let us return to $V(z)$, which can be expressed as

$$\int_K \min\left(\log\frac{1}{|z-\zeta|},\ \log\frac{1}{\rho}\right) d\mu(\zeta) \quad + \quad \int_K \log^+ \frac{\rho}{|z-\zeta|}\, d\mu(\zeta).$$

When z is close enough to z_0 the *second* term is $< 5\varepsilon$ as we have just shown; the *first* term, however, is *continuous* in z, and hence tends to

$$\int_K \min\left(\log\frac{1}{|z_0-\zeta|},\ \log\frac{1}{\rho}\right) d\mu(\zeta) \quad \leqslant \quad V(z_0)$$

as $z \longrightarrow z_0$. Therefore,

$$V(z) \;<\; V(z_0) \;+\; 6\varepsilon$$

for z sufficiently close to z_0.

At the same time, V is superharmonic, so by property (i) (!),

$$\liminf_{z \to z_0} V(z) \quad \geqslant \quad V(z_0).$$

Thus,

$$V(z) \longrightarrow V(z_0) \quad \text{as } z \longrightarrow z_0$$

since $\varepsilon > 0$ was arbitrary; the function V is thus continuous at z_0.

Q.E.D.

Corollary. *Let $U(z)$ be superharmonic in the unit disk, Δ, and* harmonic *in the open subset Ω thereof. If $z_0 \in \Delta \sim \Omega$ and the* restriction of U to $\Delta \sim \Omega$ *is continuous at z_0, $U(z)$ is continuous at z_0.*

Proof. Pick any r with $|z_0| < r < 1$; then, by the Riesz representation theorem from the preceding article,

$$U(z) \;=\; \int_{|\zeta|\leqslant r} \log\frac{1}{|z-\zeta|}\,d\mu(\zeta) \;+\; H(z)$$

for $|z| < r$, where $H(z)$ is harmonic for such z and μ is a positive measure. We know also from the *last* theorem of that article that

$$\mu(\Omega \cap \{|\zeta| < r\}) \;=\; 0;$$

taking, then, the compact set

$$K \;=\; (\{|\zeta| \leqslant r\} \cap \sim \Omega) \cup \{|\zeta| = r\},$$

we can write

$$U(z) \;=\; \int_K \log\frac{1}{|z-\zeta|}\,d\mu(\zeta) \;+\; H(z), \qquad |z| < r.$$

Here, since $|z_0| < r$, H is continuous at z_0, and the *restriction* of $U(z) - H(z)$ to K is also, by hypothesis. We thus arrive at the desired result by applying the theorem to

$$\int_K \log\frac{1}{|z-\zeta|}\,d\mu(\zeta).$$

Done.

Problem 51

Let μ be a positive measure supported on K, a compact subset of the open unit disk, and suppose that

$$V(z) \;=\; \int_K \log\left|\frac{1-\bar{\zeta}z}{z-\zeta}\right| d\mu(\zeta)$$

is finite at each point of K. Show that there is a sequence of positive measures μ_n supported on K for which:

(i) $d\mu_n(\zeta) \leqslant d\mu_{n+1}(\zeta) \leqslant d\mu(\zeta)$ for each n;

(ii) $\mu(K) - \mu_n(K) \xrightarrow[n]{} 0$;

(iii) each of the functions

$$V_n(z) = \int_K \log\left|\frac{1-\bar{\zeta}z}{z-\zeta}\right| d\mu_n(\zeta)$$

is continuous on $\bar{\Delta}$;

(iv) $V_n(z) \xrightarrow[n]{} V(z)$ for each $z \in \bar{\Delta}$.

(Hint: Start by arguing as in the proof of Maria's theorem, getting compact subsets K_n of K with $\mu(K \sim K_n) < 1/n$, on each of which the convergence of

$$\int_K \min\left(\log\left|\frac{1-\bar{\zeta}z}{z-\zeta}\right|, \log\frac{|1-\bar{\zeta}z|}{\rho}\right) d\mu(\zeta)$$

to $V(z)$ for ρ tending to zero is *uniform*. This makes the restriction of V to each K_n continuous thereon.

Arranging matters so as to have $K_n \subseteq K_{n+1}$ for each n, define μ_n by putting $\mu_n(E) = \mu(E \cap K_n)$ for $E \subseteq K$. Each of the differences $\sigma_n = \mu - \mu_n$ is also a positive measure on K.

We have (with V_n as in (iii)),

$$V_n(z) = V(z) - \int_K \log\left|\frac{1-\bar{\zeta}z}{z-\zeta}\right| d\sigma_n(\zeta),$$

where the integral on the right (*without* the $-$ sign) is superharmonic in Δ. Hence, since V, restricted to any of the K_n, is continuous thereon, we see that

$$\limsup_{\substack{z \to z_0 \\ z \in K_n}} V_n(z) \leqslant V_n(z_0) \quad \text{for } z_0 \in K_n.$$

On the other hand,

$$\liminf_{z \to z_0} V_n(z) \geqslant V_n(z_0).$$

The restriction of V_n to K_n, the support of μ_n, is thus continuous. Now apply the preceding theorem.

Observe finally that

$$d\mu_n(\zeta) = \chi_{K_n}(\zeta)\, d\mu(\zeta)$$

with, for each fixed $z_0 \in \bar{\Delta}$,

$$\log\left|\frac{1-\bar{\zeta}z_0}{z_0-\zeta}\right| \chi_{K_n}(\zeta) \longrightarrow \log\left|\frac{1-\bar{\zeta}z_0}{z_0-\zeta}\right| \quad \text{a.e. } (\mu)$$

on K as $n \longrightarrow \infty$. This makes $V_n(z_0) \xrightarrow[n]{} V(z_0)$ by monotone convergence.)

B. Relation of the existence of multipliers to the finiteness of a superharmonic majorant

1. Discussion of a certain regularity condition on weights

We return to the question formulated somewhat loosely at the beginning of §A in the last chapter, which was to find the conditions a weight $W(x) \geqslant 1$ must fulfill beyond the necessary one that

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2}\,dx \quad < \quad \infty,$$

in order to ensure the existence of entire functions $\varphi(z) \not\equiv 0$ of arbitrarily small exponential type making $W(x)\varphi(x)$ bounded (for instance) on $\mathbb{R}$. These must be conditions pertaining to the regularity of $W(x)$. Although an explicit minimal description of the needed regularity is not available as I write this, it seems likely that two separate requirements are involved.

One, not particularly bound up with the matter now under discussion, would serve to rule out the purely local idiosyncrasies in W's behaviour that could spoil existence of the above mentioned functions φ making $W(x)\varphi(x)$ bounded on $\mathbb{R}$ when such φ with, for example,

$$\int_{-\infty}^{\infty} |W(x)\varphi(x)|^p\,dx \quad < \quad \infty \qquad \text{(for some } p > 0\text{)}$$

were forthcoming. A very simple illustration helps to clarify this idea.

Consider any weight $W_0(x) \geqslant 1$ for which a non-zero entire function φ_0 of exponential type $A < \pi$ with $W_0(x)|\varphi_0(x)| \leqslant 1$ on $\mathbb{R}$ is known to exist.

Unless $W_0(x)$ is already bounded (a case without interest for us here!), $\varphi_0(z)$ must differ from a pure exponential and hence have infinitely many zeros (coming from its Hadamard factorization). Dividing out any two of those then gives us a new non-zero entire function φ, also of exponential type A, for which $W_0(x)|\varphi(x)| \leqslant \text{const.}/(1+x^2)$, $x \in \mathbb{R}$, so that

$$\int_{-\infty}^{\infty} W_0(x)|\varphi(x)|\,dx \quad < \quad \infty.$$

Taking the new weight

$$W(x) \quad = \quad |\sin \pi x|^{-1/2} W_0(x)$$

which becomes infinite at each integer, we still have

$$\int_{-\infty}^{\infty} W(x)|\varphi(x)|\,dx \quad < \quad \infty.$$

There is, however, no longer any entire $\psi(z) \not\equiv 0$ of exponential type $< \pi$ for which $W(x)\psi(x)$ is bounded on $\mathbb{R}$. Indeed, such a function ψ would have to be bounded on $\mathbb{R}$ and hence satisfy $|\psi(z)| \leqslant \text{const.}\exp(B|\Im z|)$, with $B < \pi$, by the third Phragmén–Lindelöf theorem in §C of Chapter III. In the present circumstances, ψ would also have to vanish at each integer, but then the usual application of Jensen's formula would show that it must vanish identically. Starting with a weight W_0 for which entire $\varphi_0 \not\equiv 0$ of *arbitrarily small exponential type* $A > 0$ with $\varphi_0(x)W_0(x)$ bounded on $\mathbb{R}$ are available, we thus find ourselves in a situation where – adopting the language of §A, Chapter X – the related weight $W(x)$ *admits multipliers in* $L_1(\mathbb{R})$ *but not in* $L_\infty(\mathbb{R})$.

By such rather artificial and almost trivial constructions one obtains various weights W from the original W_0 that admit multipliers in some spaces $L_p(\mathbb{R})$ but not in others. This seems to have nothing to do with the real reason (whatever it may be) for W_0 to have admitted multipliers (in $L_\infty(\mathbb{R})$) to begin with. That must also be the reason why the weights W admit multipliers in certain of the $L_p(\mathbb{R})$, and thus probably involves *some property of behaviour common to* $W_0(x)$ *and all of the* $W(x)$, independent of the special irregularities introduced in passing from the former to the latter. If this is so, it is natural to think of that behaviour property as *the essential one* governing admittance of multipliers, and the *second regularity condition* for weights would be *that they possess it*. By the *first regularity condition*, weights like $|\sin \pi x|^{-1/2}W_0(x)$ would be ruled out.

From this point of view, a search for the presumed essential second condition appears to be of primary importance. In order to be unhindered in that search, one is motivated to start by imposing on the weights W some imperfect version of the first condition, stronger than needed*, rather than seeking to express the latter in minimal form. That is how we will proceed here.

Such a version of the first condition should be both simple and sufficiently general. One, given in Beurling and Malliavin's 1962 paper, is very mild but rather elaborate. Discussion of it is postponed to the

* even at the cost of then arriving at a less than fully general version of the second condition

scholium at the end of this article. The following simpler variant seems adequate for most purposes; it is easy to work with and still applicable to a broad class of weights.

Regularity requirement. *There are three strictly positive constants, L, C and α such that, for each $x \in \mathbb{R}$, one has a real interval J_x of length L containing x with*

$$W(t) \geq C(W(x))^{\alpha} \qquad \text{for } t \in J_x.$$

(Unless $W(x)$ is bounded – a case without interest for us here – the parameter α figuring in the condition must obviously be ≤ 1.) *Much of the work in the present chapter will be limited to the weights W that meet this requirement.**

What our condition does is impose a weak kind of *uniform semicontinuity* on $\log^+ \log W(x)$. It implies, for instance, a certain boundedness property on finite intervals.

Lemma. *A weight $W(x)$ meeting the regularity requirement is either identically infinite on some interval of length L or else bounded above on every finite interval.*

Proof. Suppose that $-M \leq x_n \leq M$ and $W(x_n) \xrightarrow[n]{} \infty$. Wlog, let

$$x_n \xrightarrow[n]{} x_0.$$

To each x_n is associated an interval J_{x_n} of length L containing it, on which

$$W(x) \geq C(W(x_n))^{\alpha}.$$

For infinitely many values of n, J_{x_n} must extend *to the same side of* x_n (either to the right or to the left) by a distance $\geq L/2$. Assuming, wlog, that we have infinitely many such intervals extending by that amount *to the right* of the corresponding points x_n, we see that

$$x_0' = x_0 + \frac{L}{4}$$

lies in *infinitely many of them.* The preceding relation therefore makes $W(x_0') = \infty$. Then, however, $W(x) = \infty$ for the x belonging to the interval $J_{x_0'}$ of length L.

* Regarding its partial elimination, see Remark 5 near the end of §E.2.

Here are some of the ways in which weights fulfilling the regularity requirement arise.

Lemma. *If* $\Omega(t) \geqslant 0$, *the average*

$$W(x) \quad = \quad \frac{1}{2L}\int_{-L}^{L} \Omega(x+t)\,\mathrm{d}t$$

satisfies the requirement with parameters L, $C = 1/2$ *and* $\alpha = 1$.

Proof. Given any x, we have

$$\frac{1}{2L}\int_J \Omega(t)\,\mathrm{d}t \quad \geqslant \quad \frac{1}{2}W(x)$$

for an interval J equal to *one* of the two segments $[x-L,\ x]$, $[x,\ x+L]$. Taking that interval J as J_x, we then have

$$W(\xi) \quad = \quad \frac{1}{2L}\int_{-L}^{L} \Omega(\xi+t)\,\mathrm{d}t \quad \geqslant \quad \frac{1}{2L}\int_{J_x} \Omega(s)\,\mathrm{d}s \quad \geqslant \quad \frac{1}{2}W(x)$$

for each $\xi \in J_x$.

In like manner, one verifies:

Lemma. *If* $\Omega(t) \geqslant 0$ *and* $p > 0$ *(sic!)*,

$$W(x) \quad = \quad \left(\frac{1}{2L}\int_{-L}^{L} (\Omega(x+t))^p\,\mathrm{d}t\right)^{1/p}$$

satisfies the requirement with parameters L, $C = 2^{-1/p}$ *and* $\alpha = 1$.

Lemma. *If* $\Omega(t) \geqslant 1$, *the weight*

$$W(x) \quad = \quad \exp\left\{\frac{1}{2L}\int_{-L}^{L} \log\Omega(x+t)\,\mathrm{d}t\right\}$$

satisfies the requirement with parameters L, $C = 1$ *and* $\alpha = 1/2$.

Weights meeting the requirement are also obtained by use of the Poisson kernel:

Lemma. *Let* $\Omega(t) \geqslant 1$ *be such that*

$$\int_{-\infty}^{\infty} \frac{\log\Omega(t)}{1+t^2}\,\mathrm{d}t \quad < \quad \infty.$$

Then, for fixed $y > 0$, the weight

$$W(x) \quad = \quad \exp\left\{\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y \log \Omega(t)}{(x-t)^2 + y^2}\,\mathrm{d}t\right\}$$

fulfills the requirement with parameters L, $C = 1$ and $\alpha \;=\; \mathrm{e}^{-L/2y}$.

Proof. Since $\log \Omega(t) \;\geqslant\; 0$, we have (Harnack!)

$$\left|\frac{\mathrm{d} \log W(x)}{\mathrm{d}x}\right| \quad \leqslant \quad \frac{1}{y} \log W(x),$$

so that

$$\log W(\xi) \quad \geqslant \quad (\log W(x))\mathrm{e}^{-|\xi - x|/y}.$$

Take $J_x \;=\; [x - L/2,\ x + L/2]$.

A weight meeting the regularity requirement and also admitting multipliers has a $\mathscr{C}_\infty$ *majorant* with the same properties.

Theorem. *Let $W(x) \;\geqslant\; 1$ fulfill the requirement with parameters L, C, and α, and suppose that*

$$\int_{-\infty}^{\infty} \frac{\log W(t)}{1+t^2}\,\mathrm{d}t \quad < \quad \infty.$$

There is then an infinitely differentiable weight $W_1(x) \;\geqslant\; W(x)$ also meeting the requirement such that, corresponding to any entire function $\varphi(z) \;\not\equiv\; 0$ of exponential type $\leqslant\; A$ making $W(x)|\varphi(x)| \;\leqslant\; 1$ on $\mathbb{R}$, one has an entire $\psi(z) \;\not\equiv\; 0$ of exponential type $\leqslant\; mA$ with $W_1(x)|\psi(x)| \;\leqslant$ const., $\quad x \in \mathbb{R}$. Here, for m we can take any integer $\geqslant\; 4/\alpha$.

Remark. As we know, the integral condition on $\log W$ follows from the existence of just *one* entire function φ having the properties in question.

Proof of theorem. Any entire function φ satisfying the conditions of the hypothesis must in particular have modulus $\leqslant\; 1$ on the real axis, so, by the second theorem of §G.2, Chapter III,

$$\log|\varphi(z)| \quad \leqslant \quad A\Im z \quad + \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log|\varphi(t)|}{|z-t|^2}\,\mathrm{d}t$$

for $\Im z \;>\; 0$. Adding to both sides the finite quantity

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log W(t)}{|z-t|^2}\,\mathrm{d}t$$

we see, remembering the given relation

$$\log|\varphi(t)| + \log W(t) \leqslant 0, \qquad t \in \mathbb{R},$$

that

$$\log|\varphi(z)| + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log W(t)}{|z-t|^2}\,dt \leqslant A\Im z, \qquad \Im z > 0.$$

Put now $z = x + iL$, and use the fact that

$$\log W(t) \geqslant \alpha \log W(x) + \log C$$

for t belonging to an interval of length L containing the point x. Since $\log W(t) \geqslant 0$, the integral on the left comes out

$$\geqslant \frac{1}{4}(\alpha \log W(x) + \log C),$$

and we find that

$$\frac{4}{\alpha}\log|\varphi(x+iL)| + \log W_1(x) \leqslant \text{const.}, \qquad x \in \mathbb{R},$$

where

$$W_1(x) = C^{-1/\alpha}\exp\left\{\frac{4}{\pi\alpha}\int_{-\infty}^{\infty}\frac{L\log W(t)}{(x-t)^2+L^2}\,dt\right\}$$

is certainly $\geqslant W(x)$. This function is, on the other hand, infinitely differentiable, and it satisfies the regularity requirement by the last lemma.

At the same time,

$$W_1(x)|\varphi(x+iL)|^{4/\alpha} \leqslant \text{const.}, \qquad x \in \mathbb{R}.$$

Because φ is bounded on the real axis, we know by the third Phragmén–Lindelöf theorem of §C, Chapter III that $\varphi(x+iL)$ is also bounded for $x \in \mathbb{R}$. Hence, taking any integer $m \geqslant 4/\alpha$, we have

$$W_1(x)|\psi(x)| \leqslant \text{const.}, \qquad x \in \mathbb{R},$$

with the entire function

$$\psi(z) = (\varphi(z+iL))^m,$$

obviously of exponential type $\leqslant mA$.

Done.

The elementary result just proved permits us to restrict our attention to *infinitely differentiable weights* when searching for the form of the 'essential' second condition that those meeting the regularity requirement must satisfy in order to admit multipliers. This observation will play a rôle in the last two §§ of the present chapter. But the main service rendered by the requirement is to make the property of admitting multipliers reduce to a more general one, easier to work with, for weights fulfilling it.

In order to explain what is meant by this, let us first consider the situation where an entire function $\varphi(z) \not\equiv 0$ of exponential type $\leqslant A$ with $W(x)|\varphi(x)| \leqslant$ const. on $\mathbb{R}$ is *known to exist*. If the weight $W(x)$ is *even*, some details of the following discussion may be skipped, making it *shorter* (although not really *easier*). One can in fact stick to just even weights (and even functions $\varphi(z)$) and still *get by* – see the remark following the last theorem in this article – and the reader is invited to make this simplification if he or she wants to. We treat the general case here in order to show that such investigations do not become *that much harder* when *evenness* is *abandoned*.

Assume that $W(x) \geqslant 1$ is either *continous*, or fulfills the *regularity requirement* (of course, one property does not imply the other). Then, since $\varphi(z) \not\equiv 0$, $W(x)$ cannot be identically infinite on any interval of length > 0. By the first of the above lemmas, this means that $W(x)$ is *bounded on finite intervals* under the second assumption. The same is of course true in the event of the first assumption.

The function $W(x)$ is, in particular, bounded near the origin, so if $\varphi(z)$ has a zero there – of order k, say – the product $W(x)\varphi(x)/x^k$ will still be bounded on $\mathbb{R}$. We can, in other words, assume wlog that $\varphi(0) \neq 0$, and hence that φ has a Hadamard factorization of the form

$$\varphi(z) \quad = \quad Ce^{\gamma z}\prod_{\lambda}\left(1-\frac{z}{\lambda}\right)e^{z/\lambda}.$$

Following a procedure already familiar to us, we construct from the product on the right a new entire function $\psi(z)$ *having only real zeros* (cf. §H.3 of Chapter III and the first half of the proof of the second Beurling–Malliavin theorem, §B.3, Chapter X).

Denote by Λ the set of zeros λ figuring in the above product with $\Re\lambda \neq 0$. For each $\lambda \in \Lambda$ we put

$$\frac{1}{\lambda'} \quad = \quad \Re\left(\frac{1}{\lambda}\right);$$

this gives us real numbers λ' with $|\lambda'| \geqslant |\lambda|$. (It is understood here that each λ' is to be taken with a multiplicity equal to the number of times that the corresponding $\lambda \in \Lambda$ figures as a zero of φ.) The *number* $N(r)$ of points λ' having modulus $\leqslant r$ (counting multiplicities) is thus *at most* equal to the *total numbers of zeros with such modulus that φ has* (again counting multiplicities). The latter quantity has, however, *asymptotic behaviour for large r governed by Levinson's theorem* (§H.3, Chapter III), because $\varphi(x)$ must be *bounded* on $\mathbb{R}$, $W(x)$ being $\geqslant 1$. In this way we see that that quantity is $\sim 2A'r/\pi$ for $r \longrightarrow \infty$, where A' is some positive number $\leqslant$ the *type* of φ, and hence $\leqslant A$.

We therefore have

$$\frac{N(r)}{r} \leqslant \frac{2A'}{\pi} + o(1)$$

for large r.

This being so, the product

$$e^{z\Re\gamma} \prod_{\lambda\in\Lambda} \left(1 - \frac{z}{\lambda'}\right) e^{z/\lambda'}$$

(with each factor repeated according to the multiplicity of the corresponding λ') is *convergent* (see §A, Chapter III) and hence equal to some entire function $\psi(z)$. We know from §B of Chapter III, however, that the preceding relation involving $N(r)$ is insufficient to ensure ψ's being of exponential type. In order to show that, we resort to an indirect argument (cf. §H.3, Chapter III).

What the condition on $N(r)$ does give is the estimate $\log|\psi(z)| \leqslant O(|z|\log|z|)$, valid for large $|z|$ (§B, Chapter III); we thus have

$$\log|\psi(z)| \leqslant O(|z|^{1+\varepsilon})$$

(with arbitrary $\varepsilon > 0$) for z with large modulus. At the same time, $\psi(x)$ is bounded on the real axis. Indeed, for $\lambda \in \Lambda$,

$$\left|\left(1 - \frac{x}{\lambda}\right)e^{x/\lambda}\right| \geqslant \left|1 - \frac{x}{\lambda'}\right| e^{x/\lambda'}, \qquad x \in \mathbb{R},$$

whereas, for any *purely imaginary zero λ of φ*,

$$\left|\left(1 - \frac{x}{\lambda}\right)e^{x/\lambda}\right| = \sqrt{\left(1 + \frac{x^2}{|\lambda|^2}\right)} \geqslant 1, \qquad x \in \mathbb{R}.$$

Comparison of the above product equal to $\psi(z)$ with the Hadamard representation for φ thus shows at once that $|C\psi(x)| \leqslant |\varphi(x)|$ for $x \in \mathbb{R}$, yielding

$$|\psi(x)| \leqslant \text{const.}, \qquad x \in \mathbb{R},$$

since such a relation holds for $\varphi(x)$.

On the *imaginary axis*, the above estimate on $\log|\psi(z)|$ can be improved. We have:

$$\log|\psi(\mathrm{i}y)| \;=\; \frac{1}{2}\sum_{\lambda\in\Lambda}\log\left(1+\frac{y^2}{(\lambda')^2}\right) \;=\; \frac{1}{2}\int_0^\infty \log\left(1+\frac{y^2}{r^2}\right)\mathrm{d}N(r)$$

$$=\; |y|\int_0^\infty \frac{|y|}{t^2+y^2}\,\frac{N(r)}{r}\,\mathrm{d}r$$

(note that $N(r) = 0$ for $r > 0$ close to zero). Plugging the above inequality for $N(r)$ into the last integral, we see immediately that

$$\limsup_{y\to\pm\infty}\frac{\log|\psi(\mathrm{i}y)|}{|y|} \;\leqslant\; A'.$$

Use this relation together with the two previous estimates on ψ to make a Phragmén–Lindelöf argument in each of the quadrants I, II, III and IV. One finds as in §H.3 of Chapter III that

$$|\psi(z)| \;\leqslant\; \text{const.}e^{A'|\Im z|}.$$

Thus, since $A' \leqslant A$, $\psi(z)$ is *of exponential type* $\leqslant A$ (as our original function φ was).

This argument has been given at length because it will be used again later on. Then we will simply *refer* to it, omitting the details.

Let us return to our weight $W(x)$. Since, as we have seen, $|\psi(x)| \leqslant \text{const.}|\varphi(x)|$ on $\mathbb{R}$, it is true that

$$W(x)|\psi(x)| \;\leqslant\; \text{const.}, \qquad x\in\mathbb{R}.$$

Knowing, then, of the existence of *any* entire function $\varphi(z) \not\equiv 0$ having exponential type $\leqslant A$ and satisfying this relation, *we can construct a new one*, $\psi(z)$, *with only real zeros, that also satisfies it.* Moreover, as the above work shows, we can get such a ψ with $\psi(0) = 1$.

We now rewrite the last relation using a Stieltjes integral. As in §B of Chapter X, it is convenient to introduce an increasing function $n(t)$, equal, for $t > 0$, to the *number of zeros* λ' *of* ψ (counting multiplicities) in $[0, t]$, and, for $t < 0$, to the *negative of the number of such* λ' in $[t, 0)$. This function $n(t)$ (N.B. *it should not be confounded with* $N(r)$!) is *integer-valued* and, since $\psi(0) = 1$, *identically zero in a neighborhood of the origin.* Application of the Levinson theorem from §H.2 of Chapter III to the entire function $\psi(z)$ shows that the limits of $n(t)/t$ for $t \longrightarrow \pm\infty$ exist, both being equal to a number $\leqslant A/\pi$. Thus,

$$\frac{n(t)}{t} \;\leqslant\; \frac{A}{\pi} + \mathrm{o}(1) \qquad \text{for } t \longrightarrow \pm\infty.$$

Writing γ instead of $\Re\gamma$, the product representation for $\psi(z)$ can be put in the form

$$\log|\psi(z)| \quad = \quad \gamma\Re z \quad + \quad \int_{-\infty}^{\infty}\left(\log\left|1 - \frac{z}{t}\right| + \frac{\Re z}{t}\right)\mathrm{d}n(t).$$

The relation involving W and ψ can hence be expressed thus:

$$\gamma x + \int_{-\infty}^{\infty}\left(\log\left|1-\frac{x}{t}\right| + \frac{x}{t}\right)\mathrm{d}n(t) + \log W(x) \leqslant \text{const.}, \qquad x \in \mathbb{R}.$$

The existence of our original multiplier φ for W, of exponential type $\leqslant A$, has in this way enabled us to get an increasing integer-valued function $n(t)$ having the above properties and fulfilling the last relation.

If, on the other hand, one *has* an integer-valued increasing function $n(t)$ meeting these conditions, it is easy to construct an entire function ψ of exponential type $\leqslant A$ making $W(x)|\psi(x)| \leqslant$ const. on $\mathbb{R}$. All one need do is put

$$\psi(z) \quad = \quad \mathrm{e}^{\gamma z}\prod_{\lambda'}\left(1 - \frac{z}{\lambda'}\right)\mathrm{e}^{z/\lambda'}$$

with λ' running through the *discontinuities* of $n(t)$, each taken a number of times equal to the corresponding jump in $n(t)$. The boundedness of the product $W(x)\psi(x)$ then follows directly, and the Phragmén–Lindelöf argument used previously shows $\psi(z)$ to be of exponential type $\leqslant A$. *The existence of our multiplier* φ is, in other words, *equivalent to that of an increasing integer-valued function* $n(t)$ *satisfying the conditions just enumerated.*

Our regularity requirement is of course not *needed* for this equivalence, which holds for any weight bounded in a neighborhood of the origin. *What that requirement does is permit us,* when dealing with weights subject to it, *to drop* from the last statement *the condition that* $n(t)$ *be integer-valued.* The cost of this is that one ends with a multiplier φ of exponential type *several times larger than* A instead of one with type $\leqslant A$.

Some version of the lemma from §A.1 of Chapter X is needed for this reduction. If $W(x)$ were known to be *even* (with the increasing function involved odd!), the lemma could be used as it stands, and the proof of the next theorem made shorter (regarding this, the reader is again directed to the remark following the second of the next two theorems). The general situation requires a more elaborate form of that result. As in §B.2 of Chapter X, it is convenient to use $[p]$ to denote *the least integer* $\geqslant p$ *when* *p is negative, while maintaining the usual meaning of that symbol for $p \geqslant 0$.* The following variant of the lemma is then sufficient for our purposes:

Lemma. *Let $\nu(t)$ be increasing on $\mathbb{R}$, zero on $(-a, a)$, where $a > 0$, and $O(t)$ for $t \longrightarrow \pm\infty$. Then, for $\Im z \neq 0$, we have*

$$c\Re z + \int_{-\infty}^{\infty}\left(\log\left|1-\frac{z}{t}\right| + \frac{\Re z}{t}\right)(\mathrm{d}[\nu(t)] - \mathrm{d}\nu(t))$$

$$\leqslant \log^+\left|\frac{\Re z}{\Im z}\right| + \log\left|1 + \frac{|\Re z| + \mathrm{i}|\Im z|}{a}\right|,$$

c being a certain real constant depending on ν.

A proof of this estimate was already carried out for $a = 1$ and $\Im z = 1$ while establishing the *Little Multiplier Theorem* in §B.2 of Chapter X. The argument for the general case is not different from the one made there.*

We are now able to establish the promised reduction.

Theorem. *Let the weight $W(x) \geqslant 1$ meet our regularity requirement, with parameters L, C and α. Suppose there is an increasing function $\rho(t)$, zero on a neighborhood of the origin, with*

$$\frac{\rho(t)}{t} \leqslant \frac{A}{\pi} + o(1) \qquad \text{for } t \longrightarrow \pm\infty$$

and

$$\gamma x + \int_{-\infty}^{\infty}\left(\log\left|1-\frac{x}{t}\right| + \frac{x}{t}\right)\mathrm{d}\rho(t) + \log W(x) \leqslant \text{const.}$$

on the real axis, where γ is a real constant. Then there is a non-zero entire function $\psi(z)$ of exponential type $\leqslant 4A/\alpha$ with $W(x)\psi(x)$ bounded on $\mathbb{R}$.

Remark. The number 4 could be replaced by any other > 2 by refining one point in the following argument.

Proof of theorem. Put

$$U(z) = \gamma\Re z + \int_{-\infty}^{\infty}\left(\log\left|1-\frac{z}{t}\right| + \frac{\Re z}{t}\right)\mathrm{d}\rho(t);$$

our conditions on $\rho(t)$ make the right-hand integral have unambiguous

* By following the procedure indicated in the footnote on p. 186, one can, noting that $[\nu(t)] - \nu(t) \geqslant 0$ for $t < 0$, improve the upper bound furnished by the lemma to $\log|z/\Im z|$; this is *independent* of the size of the interval $(-a, a)$ on which $\nu(t)$ is known to vanish.

meaning for *all* complex z, taking, perhaps, the value $-\infty$ for some of these.* The *lack of evenness* of $W(x)$ and $U(z)$ will necessitate our attention to certain details.

$U(z)$ is *subharmonic* in the complex plane; it is, in other words, equal there to *the negative of a superharmonic function* having the properties taken up near the beginning of §A.1. According to the *first* of those we have in particular

$$\limsup_{z \to x_0} U(z) \quad \leqslant \quad U(x_0) \qquad \text{for } x_0 \in \mathbb{R}.$$

Our hypothesis, however, is that $U(x_0) + \log W(x_0) \leqslant K$, say, on $\mathbb{R}$, with $\log W(x_0) \geqslant 0$ there. Hence

$$\limsup_{z \to x_0} U(z) \quad \leqslant \quad K, \qquad x_0 \in \mathbb{R}.$$

Starting from this relation, one now *repeats* for $U(z)$ the Phragmén–Lindelöf argument made above for $\log|\psi(z)|$, using the properties of $\rho(t)$ in place of those of $N(r)$. In that way, it is found that

$$U(z) \quad \leqslant \quad K + A|\Im z|.$$

The function $U(z)$ is actually *harmonic*† for $\Im z > 0$, and we proceed to establish for it the Poisson representation

$$U(z) \quad = \quad A'\Im z \quad + \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \, U(t)}{|z-t|^2} \mathrm{d}t$$

in that half plane, with

$$A' \quad = \quad \limsup_{y \to \infty} \frac{U(\mathrm{i}y)}{y} \quad \leqslant \quad A.$$

(This step could be avoided if $W(x)$ were known to be continuous; such continuity is, however, superfluous here.) Our *formula* for $U(z)$ shows $U(\mathrm{i}y)$ to be $\geqslant 0$ for $y > 0$, so the quantity A' is certainly $\geqslant 0$. That it does not *exceed* A is guaranteed by the estimate on $U(z)$ just found. That estimate and the *fourth* theorem of §C, Chapter III, now show that in fact

$$U(z) \quad \leqslant \quad K + A'\Im z \qquad \text{for } \Im z > 0;$$

the function $U(z) - K - A'\Im z$ is thus *harmonic and* $\leqslant 0$ in the upper half plane.

* any such z must be *real* – $U(z)$ is *finite* for $\Im z \neq 0$

† and, in particular, *finite* (see preceding footnote) – the integral in the following Poisson representation is thus surely *convergent*.

By §F.1 of Chapter III we therefore have

$$U(z) - K - A'\Im z = -b\Im z - \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z\,\mathrm{d}\sigma(t)}{|z-t|^2}$$

for $\Im z > 0$, with a constant $b \geqslant 0$ and a certain *positive measure* σ on $\mathbb{R}$. It is readily verified that b must equal zero. Our desired Poisson representation for $U(z)$ will now follow from an argument like the one in §G.1 of Chapter III *if we verify absolute continuity of* σ.

For this purpose, it is enough to show that when $y \longrightarrow 0$,

$$\int_{-M}^{M} |U(x+\mathrm{i}y) - U(x)|\,\mathrm{d}x \quad\longrightarrow\quad 0$$

for each finite M. Given such an M, we can write

$$U(z) = \gamma\Re z + \left(\int_{-2M}^{2M} + \int_{|t|>2M}\right)\left(\log\left|1-\frac{z}{t}\right| + \frac{\Re z}{t}\right)\mathrm{d}\rho(t).$$

The *second* of the two integrals involved here clearly tends *uniformly* to

$$\int_{|t|>2M}\left(\log\left|1-\frac{x}{t}\right| + \frac{x}{t}\right)\mathrm{d}\rho(t)$$

as $z = x+\mathrm{i}y$ tends to x, when $-M \leqslant x \leqslant M$. Hence, since $\rho(t)$ is zero on a neighborhood of the origin, the matter at hand boils down to checking that

$$\int_{-M}^{M}\left|\int_{-2M}^{2M}\left(\log|x+\mathrm{i}y-t| - \log|x-t|\right)\mathrm{d}\rho(t)\right|\mathrm{d}x \quad\longrightarrow\quad 0$$

as $y \longrightarrow 0$. The inner integrand is already positive here, so the left-hand expression is just

$$\int_{-2M}^{2M}\int_{-M}^{M}\left(\log|x+\mathrm{i}y-t| - \log|x-t|\right)\mathrm{d}x\,\mathrm{d}\rho(t).$$

In this last, however, the inner integral is easily seen – by direct calculation, if need be – to tend to zero uniformly for $-2M \leqslant t \leqslant 2M$ as $y \longrightarrow 0$. (Incidentally, $\int_{-M}^{M}\log|w-x|\,\mathrm{d}x$ is the negative of a logarithmic potential generated by a *bounded* linear density on a finite segment, and therefore continuous everywhere in w.) The preceding relation therefore holds, so σ is absolutely continuous, giving us the desired Poisson representation for $U(z)$.

Once that representation is available, we have, for $\Im z > 0$,

$$U(z) + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log W(t)}{|z-t|^2}\,dt$$

$$= A'\Im z + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z\,(U(t)+\log W(t))}{|z - t|^2}\,dt.$$

Since, however, $U(t) + \log W(t) \leqslant K$ on $\mathbb{R}$, the right side of this relation must be $\leqslant K + A'\Im z$, so we have

$$\gamma\Re z + \int_{-\infty}^{\infty}\left(\log\left|1-\frac{z}{t}\right| + \frac{\Re z}{t}\right)d\rho(t)$$

$$+ \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log W(t)}{|z-t|^2}\,dt \leqslant K + A'\Im z, \qquad \Im z > 0.$$

(Putting $z = i$, we see by the way that $\int_{-\infty}^{\infty}(\log W(t)/(1+t^2))\,dt < \infty$.)

By hypothesis, W meets our regularity requirement with parameters L, C, and α; this means that

$$\log W(t) \geqslant \alpha\log W(x) + \log C$$

for $t \in J_x$, an interval of length L containing x. Therefore, if

$$z = x + iL,$$

the *second* integral on the left in the preceding relation is $\geqslant (\alpha/4)\log W(x) + (1/4)\log C$. After multiplying the latter through by $4/\alpha$ we thus find, recalling that $A' \leqslant A$,

$$\frac{4\gamma}{\alpha}x + \int_{-\infty}^{\infty}\left(\log\left|1-\frac{x+iL}{t}\right| + \frac{x}{t}\right)d(4\rho(t)/\alpha)$$

$$+ \log W(x) \leqslant K', \qquad x \in \mathbb{R},$$

where

$$K' = \frac{4K + 4AL - \log C}{\alpha}.$$

It is at this point that we apply the last lemma, with

$$\nu(t) = \frac{4}{\alpha}\rho(t)$$

and $z = x + iL$. If $\rho(t)$, and hence $\nu(t)$, vanishes on the neighborhood $(-a, a)$ of the origin, we see on combining that lemma with the preceding relation that

$$\beta x \quad + \quad \int_{-\infty}^{\infty}\left(\log\left|1-\frac{x+\mathrm{i}L}{t}\right| + \frac{x}{t}\right)\mathrm{d}[\nu(t)] \quad + \quad \log W(x)$$

$$\leqslant \quad K' + \log^+\left|\frac{x}{L}\right| + \log\left|1 + \frac{|x|+\mathrm{i}L}{a}\right|$$

on $\mathbb{R}$, with a certain constant β. From this we have, *a fortiori*,

$$\beta x \quad + \quad \int_{-\infty}^{\infty}\left(\log\left|1-\frac{x}{t}\right| + \frac{x}{t}\right)\mathrm{d}[\nu(t)] \quad + \quad \log W(x)$$

$$\leqslant \quad K'' + 2\log^+|x| \qquad \text{for } x\in\mathbb{R},$$

K'' being a new constant. The first two terms on the left add up, however, to $\log|\varphi(x)|$, where

$$\varphi(z) \quad = \quad \mathrm{e}^{\beta z}\prod_{\lambda}\left(1-\frac{z}{\lambda}\right)\mathrm{e}^{z/\lambda}$$

is the Hadamard product formed from the discontinuities λ of $[\nu(t)]$, each one taken with multiplicity equal to the height of the jump in that function corresponding to it. Since

$$\frac{\nu(t)}{t} \quad = \quad \frac{4\rho(t)}{\alpha t} \quad \leqslant \quad \frac{4A}{\alpha\pi} + o(1)$$

for $t \longrightarrow \pm\infty$ (hypothesis!), that product is certainly convergent in the complex plane, and φ is an entire function. In terms of it, the previous relation can be rewritten as

$$W(x)|\varphi(x)| \quad \leqslant \quad \text{const.}(x^2+1), \qquad x\in\mathbb{R}.$$

It is now claimed that $\varphi(z)$ *must have infinitely many zeros* λ, *unless* $W(x)$ *is already bounded on* $\mathbb{R}$ (in which case our theorem is trivially true). Because those λ are the discontinuities of $[\nu(t)] = [4\rho(t)/\alpha]$, the presence of infinitely many of them is equivalent to the *unboundedness* of $\rho(t)$ (either above or below). It is thus enough to show that *if* $|\rho(t)|$ *is bounded*, $W(x)$ *is also bounded.*

We do this by proving that if $|\rho(t)|$ is *bounded, the function* $U(z)$ *used above must be equal to zero.* For real y, we have

$$U(\mathrm{i}y) \quad = \quad \frac{1}{2}\int_{-\infty}^{\infty}\log\left|1 + \frac{y^2}{t^2}\right|\mathrm{d}\rho(t) \quad = \quad \int_{-\infty}^{\infty}\frac{y^2}{y^2+t^2}\,\frac{\rho(t)}{t}\,\mathrm{d}t.$$

Here, $\rho(t)$ vanishes for $|t| < a$, so, if $|\rho(t)|$ is also *bounded*, the ratio $\rho(t)/t$

appearing in the last integral tends to *zero* for $t \longrightarrow \pm\infty$, besides being bounded on $\mathbb{R}$. That, however, makes

$$\int_{-\infty}^{\infty} \frac{y}{y^2+t^2}\frac{\rho(t)}{t}\,dt \quad\longrightarrow\quad 0 \qquad \text{for } y \longrightarrow \pm\infty,$$

as one readily sees on breaking up the integral into two appropriate pieces. We thus have

$$\frac{U(iy)}{|y|} \quad\longrightarrow\quad 0 \qquad \text{for } y \longrightarrow \pm\infty,$$

and *the quantity* A' figuring in the above examination of $U(z)$ *is equal to zero.* By the estimate obtained there, we must then have

$$U(z) \;\leqslant\; K$$

for $\Im z \geqslant 0$, and exactly the same reasoning (or the evident equality of $U(\bar{z})$ and $U(z)$) shows this to also hold for $\Im z \leqslant 0$. The subharmonic function $U(z)$ is, in other words, *bounded above in the complex plane if* $|\rho(t)|$ *is bounded.*

Such a subharmonic function is, however, necessarily constant. That is a general proposition, set below as problem 52. In the present circumstances, we can arrive at the same conclusion by a simple *ad hoc* argument. Since $\rho(t)/t \geqslant 0$, the previous formula for $U(iy)$ yields, for $y > 0$,

$$U(iy) \quad\geqslant\quad \int_{-y}^{y} \frac{y^2}{y^2+t^2}\frac{\rho(t)}{t}\,dt \quad\geqslant\quad \frac{1}{2}\int_{-y}^{y}\frac{\rho(t)}{t}\,dt.$$

If ever $\rho(t)$ is different from zero, there must be some k and y_0, both > 0, with either $\rho(t) \geqslant k$ for $y \geqslant y_0$ or $\rho(t) \leqslant -k$ for $y \leqslant -y_0$, and in both cases the last right-hand integral will be

$$\geqslant \quad \frac{k}{2}\log\frac{y}{y_0}$$

for $y \geqslant y_0$. This, however, would make $U(iy) \longrightarrow \infty$ for $y \longrightarrow \infty$, *contradicting the boundedness of* $U(z)$, so we must have $\rho(t) \equiv 0$. But then

$$\gamma x \quad=\quad U(x) \quad\leqslant\quad K \;-\; \log W(x), \qquad x \in \mathbb{R},$$

which contradicts our assumption that $W(x) \geqslant 1$ (either for $x \longrightarrow \infty$ or for $x \longrightarrow -\infty$) *unless* $\gamma = 0$. Finally, then, the boundedness of $\rho(t)$ forces $U(x)$ to *reduce to zero*, whence

$$\log W(x) \quad=\quad U(x) \;+\; \log W(x) \quad\leqslant\quad K, \qquad x \in \mathbb{R},$$

i.e., $W(x)$ *is bounded*, as we claimed.

Thus, except for the latter trivial situation, $|\rho(t)|$ is unbounded and the entire function $\varphi(z)$ has infinitely many zeros. Dividing it by the factors $1 - z/\lambda$ corresponding to *any two such zeros*, we obtain a *new entire function*, $\psi(z)$, such that*

$$W(x)|\psi(x)| \leqslant \text{const.}, \qquad x \in \mathbb{R}.$$

We now repeat the Phragmén–Lindelöf argument applied previously to another function $\psi(z)$ and then, in the course of the present proof, to $U(z)$. Since

$$\frac{[\nu(t)]}{t} \leqslant \frac{\nu(t)}{t} = \frac{4\rho(t)}{\alpha t} \leqslant \frac{4A}{\alpha\pi} + o(1)$$

for $t \longrightarrow \pm\infty$, we find in that way that

$$|\psi(z)| \leqslant \text{const.}\, e^{4A|\Im z|/\alpha};$$

ψ is thus of exponential type $\leqslant 4A/\alpha$. We have $\psi(0) = 1$, so $\psi(z) \not\equiv 0$. Referring to the previous relation involving W and ψ, we see that the theorem is proved.

Let us now settle on a definite meaning for the notion of admitting multipliers, hitherto understood somewhat loosely, and agree to henceforth employ that term *only when actual boundedness on* $\mathbb{R}$ *is involved.*

Definition. A weight $W(x) \geqslant 1$ will be said to admit multipliers if there are entire functions $\varphi(z) \not\equiv 0$ of arbitrarily small exponential type for which

$$W(x)|\varphi(x)| \leqslant \text{const.}, \qquad x \in \mathbb{R}.$$

Combining the last theorem with the conclusion of the discussion preceding it, we then have the

Corollary. *A weight $W(x) \geqslant 1$ fulfilling our regularity requirement admits multipliers iff, corresponding to any $A > 0$, there is an increasing function $\rho(t)$, zero on some neighborhood of the origin, with*

$$\frac{\rho(t)}{t} \leqslant \frac{A}{\pi} + o(1) \qquad \text{for } t \longrightarrow \pm\infty$$

* $W(x)$ must be *bounded* in the neighborhood of each of the two zeros of φ just removed. Otherwise W would be identically infinite on an interval of length L by the first lemma in this article, and then the Poisson integral of $U(t) \leqslant K - \log W(t)$ would diverge. That, however, cannot happen, as we have already remarked in a footnote near the beginning of this proof.

and at the same time

$$\gamma x \quad + \quad \int_{-\infty}^{\infty} \left(\log \left| 1 - \frac{x}{t} \right| \; + \; \frac{x}{t} \right) \mathrm{d}\rho(t) \quad + \quad \log W(x) \quad \leqslant \quad \text{const.}$$

on $\mathbb{R}$ *for some real constant* γ.

In the case where $W(x)$ is equal to $|F(x)|$ for some *entire function* $F(z)$ of *exponential type*, the results just given hold *without any additional special assumption about the regularity of W.*

Theorem. *Let* $F(z)$ *be entire and of exponential type, with* $|F(x)| \geqslant 1$ *on* $\mathbb{R}$. *Suppose there is an increasing function* $\rho(t)$, *zero on a neighborhood of the origin, such that*

$$\frac{\rho(t)}{t} \quad \leqslant \quad \frac{A}{\pi} \; + \; \mathrm{o}(1) \qquad \text{for } t \longrightarrow \pm\infty$$

and

$$\gamma x \quad + \quad \int_{-\infty}^{\infty} \left(\log \left| 1 - \frac{x}{t} \right| \; + \; \frac{x}{t} \right) \mathrm{d}\rho(t) \quad + \quad \log |F(x)| \quad \leqslant \quad \text{const.}$$

on $\mathbb{R}$ *for some real constant* γ. *Then there is an entire function* $\psi(z) \not\equiv 0$ *of exponential type* $\leqslant A$ *(sic!) with*

$$|F(x)\psi(x)| \quad \leqslant \quad \text{const.}, \qquad x \in \mathbb{R}.$$

Proof. Writing $|F(x)| = W(x)$, one starts out and proceeds as in the demonstration of the proceeding theorem, up to the point where the relation

$$U(z) \quad + \quad \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\Im z \log W(t)}{|z-t|^2} \mathrm{d}t \quad \leqslant \quad K \; + \; A \Im z$$

is obtained for $\Im z > 0$, with

$$U(z) \quad = \quad \gamma \Re z \quad + \quad \int_{-\infty}^{\infty} \left(\log \left| 1 - \frac{z}{t} \right| \; + \; \frac{\Re z}{t} \right) \mathrm{d}\rho(t).$$

From this one sees in particular* that

$$\int_{-\infty}^{\infty} \frac{\log |F(t)|}{1+t^2} \mathrm{d}t \quad = \quad \int_{-\infty}^{\infty} \frac{\log W(t)}{1+t^2} \mathrm{d}t \quad < \quad \infty,$$

which enables us to use some results from Chapter III.

* cf. footnotes near beginning of proof of the preceding theorem.

We can, in the first place, assume that *all the zeros of $F(z)$ lie in the lower half plane*, according to the *second* theorem of §G.3 in Chapter III. Then, however, by §G.1 of that chapter,

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log W(t)}{|z-t|^2}dt \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log|F(t)|}{|z-t|^2}dt$$

$$= \quad \log|F(z)| \; - \; B\Im z \qquad \text{for } \Im z \; > \; 0,$$

where

$$B \quad = \quad \limsup_{y\to\infty} \frac{\log|F(\mathrm{i}y)|}{y}.$$

Our previous relation involving U and W thus becomes

$$U(z) \; + \; \log|F(z)| \quad \leqslant \quad K \; + \; (A+B)\Im z, \qquad \Im z \; > \; 0.$$

In this we put $z \; = \; x+\mathrm{i}$, getting

$$\gamma x \quad + \quad \int_{-\infty}^{\infty}\left(\log\left|1-\frac{x+\mathrm{i}}{t}\right| \; + \; \frac{x}{t}\right)d\rho(t)$$

$$+ \quad \log|F(x+\mathrm{i})| \quad \leqslant \quad \text{const.}, \qquad x\in\mathbb{R}.$$

Apply now the lemma used in the proof of the last theorem, but this time with

$$\nu(t) \quad = \quad \rho(t).$$

In that way one sees that

$$\beta x \quad + \quad \int_{-\infty}^{\infty}\left(\log\left|1-\frac{x+\mathrm{i}}{t}\right| \; + \; \frac{x}{t}\right)d[\rho(t)]$$

$$+ \quad \log|F(x+\mathrm{i})| \quad \leqslant \quad 2\log^+|x| \; + \; O(1), \qquad x\in\mathbb{R},$$

with a new real constant β. There is as before a certain entire function φ with $\log|\varphi(x+\mathrm{i})|$ equal to the *sum of the first two terms on the left*, and we have

$$|F(x+\mathrm{i})\varphi(x+\mathrm{i})| \quad \leqslant \quad \text{const.}(x^2+1), \qquad x\in\mathbb{R}.$$

It now follows as previously that *$\varphi(z)$ has infinitely many zeros*, unless $|F(x)|$ is *itself* bounded, in which case there is nothing to prove. Dividing out from $\varphi(z)$ the linear factors corresponding to *two* of those zeros gives us an entire function $\psi(z) \; \not\equiv \; 0$ with

$$|F(x+\mathrm{i})\psi(x+\mathrm{i})| \quad \leqslant \quad \text{const.}, \qquad x\in\mathbb{R}.$$

Here, our initial assumption that $|F(x)| \geqslant 1$ on $\mathbb{R}$ and the Poisson representation for $\log|F(z)|$ in $\{\Im z > 0\}$ already used imply that

$$|F(x+\mathrm{i})| \geqslant \text{const.} > 0 \qquad \text{for } x \in \mathbb{R},$$

so by the preceding relation we have in particular

$$|\psi(x+\mathrm{i})| \leqslant \text{const.}, \qquad x \in \mathbb{R}.$$

By hypothesis, we also have

$$\frac{[\rho(t)]}{t} \leqslant \frac{\rho(t)}{t} \leqslant \frac{A}{\pi} + o(1)$$

for $t \longrightarrow \pm\infty$, permitting us to use once again the Phragmén–Lindelöf argument made three times already in this article. In that way we see that

$$|\psi(z+\mathrm{i})| \leqslant \text{const.}\, \mathrm{e}^{A|\Im z|},$$

meaning that ψ is of exponential type $\leqslant A$. The product $F(z+\mathrm{i})\psi(z+\mathrm{i})$ is then also of exponential type. Since that product is by the above relation *bounded for real z*, we have by the third theorem of §C in Chapter III, that

$$|F(x)\psi(x)| \leqslant \text{const.} \qquad \text{for } x \in \mathbb{R}.$$

Our function ψ thus has all the properties claimed by the theorem, and we are done.

Remark. Suppose that we know of an increasing function $\rho(t)$, zero on a neighborhood of the origin, satisfying the conditions assumed for the above results with some number $A > 0$ and a weight $W(x) \geqslant 1$. For the increasing function $\mu(t) = \rho(t) - \rho(-t)$, also zero on a neighborhood of the origin, we then have

$$\frac{\mu(t)}{t} \leqslant \frac{2A}{\pi} + o(1) \qquad \text{for } t \longrightarrow \infty,$$

as well as

$$\int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\mu(t) + \log\{W(x)W(-x)\} \leqslant \text{const.}$$

for $x \in \mathbb{R}$. In this relation, both terms appearing on the left are *even*; that enables us to simplify the argument made in proving the first of the preceding two theorems when applying it in the present situation.

If the weight $W(x)$ meets our regularity requirement* with parameters

* see also Remark 5 near the end of §E.2.

L, C, and α, we do have

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{L\log\{W(t)W(-t)\}}{(x-t)^2+L^2}\,dt \;\geqslant\; \frac{\alpha}{4}\log\{W(x)W(-x)\} + \frac{\log C}{2} \quad \text{for } x\in\mathbb{R};$$

this one sees by writing the logarithm figuring in the left-hand member as a sum and then dealing separately with the two integrals thus obtained. The behaviour of the even subharmonic function

$$V(z) \;=\; \int_0^{\infty}\log\left|1-\frac{z^2}{t^2}\right|d\mu(t)$$

is easier to investigate than that of the function $U(z)$ used in the above proofs (cf. §B of Chapter III). When $V(x+iL)$ has made its appearance, one may apply directly the lemma from §A.1 of Chapter X instead of resorting to the latter's more complicated variant given above.

By proceeding in this manner, one obtains an *even* entire function $\Psi(z)$ with

$$W(x)W(-x)|\Psi(x)| \;\leqslant\; \text{const.}, \qquad x\in\mathbb{R},$$

and thus, since $W(-x) \geqslant 1$ (!),

$$W(x)|\Psi(x)| \;\leqslant\; \text{const.}, \qquad x\in\mathbb{R}.$$

The function $\Psi(z)$ is of exponential type, but here that type turns out to be bounded above by $8A/\alpha$ rather than by $4A/\alpha$ as we found for the function $\psi(z)$ obtained previously.

Insofar as W's *admitting of multipliers* is concerned, the *extra factor of two* is of no importance. The reader may therefore prefer this approach (involving a preliminary reduction to the even case) which bypasses some fussy details of the one followed above, but yields less precise estimates for the exponential types of the multipliers obtained. Anyway, according to the remark following the statement of the first of the above two theorems, the estimate $4A/\alpha$ on the type of $\psi(z)$ is not very precise.

Problem 52

Show that a function $V(z)$ *superharmonic* in the whole complex plane and bounded *below* there is constant. (Hint: Referring to the first theorem of §A.2, take the means $(\Phi_r V)(z)$ considered there. Assuming wlog that $V(z) \not\equiv \infty$, each of those means is also superharmonic and bounded below in $\mathbb{C}$, and it is enough to establish the result *for them*. The $\Phi_r V$ are also $\mathscr{C}_\infty$, so we may as well assume to begin with that $V(z)$ is $\mathscr{C}_\infty$.

That reduction made, observe that if $V(z)$ is actually *harmonic* in $\mathbb{C}$, the desired result boils down to Liouville's theorem, so it suffices to *establish* this harmonicity. For that purpose, fix any z_0 and look at the means

$$V_r(z_0) \quad = \quad \frac{1}{2\pi}\int_0^{2\pi} V(z_0 + re^{i\vartheta})\,d\vartheta.$$

Consult the proof of the *second* lemma in §A.2, and then show that

$$\frac{\partial V_r(z_0)}{\partial \log r}$$

is a *decreasing* function of r, so that $V_r(z_0)$ *either remains constant for all* $r > 0$ – and hence equal to $V(z_0)$ – *or else tends to* $-\infty$ *as* $r \longrightarrow \infty$. In the second case, V could not be bounded below in $\mathbb{C}$. Apply Gauss' theorem from §A.1.)

Scholium. The regularity requirement for weights given in the 1962 paper of Beurling and Malliavin is much less stringent than the one we have been using. A relaxed version of the former can be stated thus:

There are four constants $C > 0$, $\alpha > 0$, $\beta < 1$ and $\gamma < 1$ such that, to each $x \in \mathbb{R}$ corresponds an interval I_x of length $e^{-|x|^\gamma}$ (sic!) containing x with

$$W(t) \quad \geqslant \quad C e^{-|x|^\beta}(W(x))^\alpha \qquad \text{for } t \in I_x.$$

The point we wish to make here is that the exponentials in $|x|^\gamma$ and $|x|^\beta$ are *in a sense* red herrings; a close analogue of the first of the above two theorems, *with practically the same proof*, is valid for weights meeting the more general condition. The only new ingredient needed is the elementary Paley–Wiener multiplier theorem.

Problem 53

Suppose that $W(x) \geqslant 1$ fulfills the condition just formulated, and that there is an increasing function $\rho(t)$, zero on a neighborhood of the origin, with

$$\frac{\rho(t)}{t} \quad \leqslant \quad \frac{A}{\pi} + o(1) \qquad \text{for } t \longrightarrow \pm\infty$$

and

$$cx \quad + \quad \int_{-\infty}^{\infty}\left(\log\left|1 - \frac{x}{t}\right| + \frac{x}{t}\right)d\rho(t) \quad + \quad \log W(x) \quad \leqslant \quad \text{const.}$$

on $\mathbb{R}$, where c is a certain real constant. Show that for any $\eta > 0$ there is an entire function $\psi(z) \not\equiv 0$ of exponential type $< 4A/\alpha + \eta$ making

$$W(x)|\psi(x)| \leqslant \text{const.}, \qquad x \in \mathbb{R}.$$

(Hint: Follow exactly the proof of the result referred to until arriving at the relation

$$U(z) + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log W(t)}{|z-t|^2}\,dt \leqslant K + A'\Im z, \qquad \Im z > 0.$$

In this, substitute $z = x + ie^{-|x|^\gamma}$ (!) and invoke the condition, finding, for that value of z,

$$\frac{4c}{\alpha}x + \int_{-\infty}^{\infty}\left(\log\left|1 - \frac{z}{t}\right| + \frac{x}{t}\right) d(4\rho(t)/\alpha)$$

$$+ \log W(x) \leqslant K' + \frac{4}{\alpha}|x|^\beta$$

with a new constant K'. Using the lemma (with z as above!) and continuing as before, we get an entire function $\varphi(z) \not\equiv 0$ such that

$$\frac{\log|\varphi(iy)|}{|y|} \leqslant \frac{4A}{\alpha} + o(1) \qquad \text{for } y \longrightarrow \pm\infty,$$

$$\log|\varphi(z)| \leqslant O(|z|^{1+\varepsilon})$$

for large $|z|$ ($\varepsilon > 0$ being arbitrary), and finally

$$W(x)|\varphi(x)| \leqslant \text{const.}(x^2+1)\exp\left(|x|^\gamma + \frac{4}{\alpha}|x|^\beta\right)$$

on the real axis. To the right side of the last relation, apply the theorem from §A.1 of Chapter X (and §D of Chapter IV!).).

2. The smallest superharmonic majorant

According to the results from the latter part of the preceding article (beginning with the second theorem therein), a weight $W(x) \geqslant 1$ having any one of various regularity properties *admits multipliers* if and only if, corresponding to any $A > 0$, there exists an increasing function $\rho(t)$, zero on some neighborhood of the origin, such that

$$\frac{\rho(t)}{t} \leqslant \frac{A}{\pi} + o(1) \qquad \text{for } t \longrightarrow \pm\infty$$