so that
$$-\int_{\frac{1}{2}T}^{T} \log|h(2+it)|\,dt < \frac{T}{2X} = O(T^{l(\sigma_0)}).$$

Also we can apply the lemma of § 9.4 to $h(s)$, with $\alpha = 0$, $\beta \geqslant \frac{1}{2}$, $m \geqslant \frac{1}{2}$, and $M_{\sigma,t} = O(X^A T^A)$. We obtain
$$\arg h(s) = O(\log X+\log t)$$
for $\sigma \geqslant \frac{1}{2}$. Hence
$$\int_{\sigma_0}^{\frac{1}{2}} \{\arg h(\sigma+iT)-\arg h(\sigma+\tfrac{1}{2}iT)\}\,d\sigma = O(\log X+\log T) = O(\log T).$$

Hence
$$\int_{\sigma_0}^{2} \nu(\sigma, \tfrac{1}{2}T, T)\,d\sigma = O(T^{l(\sigma_0)}\log^m T).$$

Also
$$\int_{\sigma_0}^{2} \nu(\sigma, \tfrac{1}{2}T, T)\,d\sigma \geqslant \int_{\sigma_0}^{2} N(\sigma, \tfrac{1}{2}T, T)\,d\sigma \geqslant (\sigma_1-\sigma_0)N(\sigma_1, \tfrac{1}{2}T, T)$$
if $\sigma_0 < \sigma_1 \leqslant 2$. Taking $\sigma_1 = \sigma_0+1/\log T$, we have
$$T^{l(\sigma_0)} = T^{l(\sigma_1)+O(\sigma_1-\sigma_0)} = O(T^{l(\sigma_1)}).$$

Hence
$$N(\sigma_1, \tfrac{1}{2}T, T) = O(T^{l(\sigma_1)}\log^{m+1}T).$$

Replacing T by $\frac{1}{2}T$, $\frac{1}{4}T$,... and adding, the result follows.

9.17. The simplest application is

THEOREM 9.17. *For any fixed σ in $\frac{1}{2} < \sigma < 1$,*
$$N(\sigma, T) = O(T^{4\sigma(1-\sigma)+\epsilon}).$$

We use Theorem 4.11 with $x = T$, and obtain
$$f_X(s) = \sum_{m<T} \frac{1}{m^s} \sum_{n<X} \frac{\mu(n)}{n^s} - 1+O(T^{-\sigma}|M_X(s)|)$$
$$= \sum \frac{b_n(X)}{n^s}+O(T^{-\sigma}X^{1-\sigma}), \tag{9.17.1}$$
where, if $X < T$, $b_n(X) = 0$ for $n < X$ and for $n > XT$; and, as for a_n, $|b_n(X)| \leqslant d(n) = O(n^\epsilon)$. Hence
$$\int_{\frac{1}{2}T}^{T} \left|\sum \frac{b_n(X)}{n^s}\right|^2 dt = \tfrac{1}{2}T \sum \frac{|b_n(X)|^2}{n^{2\sigma}} + \sum\sum \frac{b_m b_n}{(mn)^\sigma} \int_{\frac{1}{2}T}^{T} \left(\frac{n}{m}\right)^{it} dt$$
$$= O\left(T \sum_{n\geqslant X} \frac{1}{n^{2\sigma-\epsilon}}\right) + O\left(\sum_{n<m<XT}\sum \frac{1}{(mn)^{\sigma-\epsilon}\log m/n}\right)$$
$$= O(TX^{1-2\sigma+\epsilon})+O\{(XT)^{2-2\sigma+\epsilon}\}$$

by (7.2.1). These terms are of the same order (apart from ϵ's) if $X = T^{2\sigma-1}$, and then
$$\int_{\frac{1}{2}T}^{T} \left|\sum \frac{b_n(X)}{n^s}\right|^2 dt = O(T^{4\sigma(1-\sigma)+\epsilon}).$$

The O-term in (9.17.1) gives
$$O(T^{1-2\sigma}X^{2-2\sigma}) = O(T^{1-2\sigma}X) = O(1).$$

The result therefore follows from Theorem 9.16.

9.18. The main instrument used in obtaining still better results for $N(\sigma, T)$ is the convexity theorem for mean values of analytic functions proved in § 7.8. We require, however, some slight extensions of the theorem. If the right-hand sides of (7.8.1) and (7.8.2) are replaced by finite sums
$$\sum C(T^a+1), \qquad \sum C'(T^b+1),$$
then the right-hand side of (7.8.3) is clearly to be replaced by
$$K\sum\sum (CT^a)^{(\beta-\sigma)/(\beta-\alpha)}(C'T^b)^{(\sigma-\alpha)/(\beta-\alpha)}.$$

In one of the applications a term $T^a\log^4 T$ occurs in the data instead of the above T^a. This produces the same change in the result. The only change in the proof is that, instead of the term
$$\int_0^\infty \left(\frac{u}{\delta}\right)^{a+2\alpha-1} e^{-2u}\,du = \frac{K}{\delta^{a+2\alpha-1}},$$
we obtain a term
$$\int_0^\infty \left(\frac{u}{\delta}\right)^{a+2\alpha-1} \log^4\frac{u}{\delta}\, e^{-2u}\,du$$
$$= \int_0^\infty \left(\frac{u}{\delta}\right)^{a+2\alpha-1} \left\{\log^4\frac{1}{\delta}+4\log^3\frac{1}{\delta}\log u+\ldots\right\} e^{-2u}\,du < \frac{K}{\delta^{a+2\alpha-1}}\log^4\frac{1}{\delta}.$$

THEOREM 9.18. *If $\zeta(\frac{1}{2}+it) = O(t^c\log^{c'}t)$, where $c' \leqslant \frac{3}{2}$, then*
$$N(\sigma, T) = O(T^{2(1+2c)(1-\sigma)}\log^5 T)$$
uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 1$.

If $0 < \delta < 1$,
$$\int_0^T |f_X(1+\delta+it)|^2\,dt = \sum_{m\geqslant X}\sum_{n\geqslant X} \frac{a_X(m)a_X(n)}{m^{1+\delta}n^{1+\delta}} \int_0^T \left(\frac{m}{n}\right)^{it} dt$$
$$= T\sum_{n\geqslant X} \frac{a_X^2(n)}{n^{2+2\delta}} + 2\sum_{X\leqslant m<n}\sum \frac{a_X(m)a_X(n)}{m^{1+\delta}n^{1+\delta}} \frac{\sin(T\log m/n)}{\log m/n}$$
$$\leqslant T\sum_{n\geqslant X} \frac{d^2(n)}{n^{2+2\delta}} + 2\sum_{X\leqslant m<n}\sum \frac{d(m)d(n)}{m^{1+\delta}n^{1+\delta}}.$$

Now† $\sum\limits_{n\leqslant x} d^2(n) < Ax\log^3 x, \qquad \sum\sum\limits_{m<n\leqslant x} \dfrac{d(m)d(n)}{(mn)^{\frac{1}{2}}\log n/m} < Ax\log^3 x.$

Hence

$$\sum_{n\geqslant X}\frac{d^2(n)}{n^{1+\xi}} = \sum_{n\geqslant X} d^2(n)\int\limits_n^\infty \frac{1+\xi}{x^{2+\xi}}\,dx = \int\limits_X^\infty \frac{1+\xi}{x^{2+\xi}}\sum_{X\leqslant n\leqslant x} d^2(n)\,dx$$

$$< \int\limits_X^\infty \frac{(1+\xi)A\log^3 x}{x^{1+\xi}}\,dx = \frac{A(1+1/\xi)}{X^\xi}\int\limits_1^\infty \frac{\log^3(Xy^{1/\xi})}{y^2}\,dy$$

(putting $x = Xy^{1/\xi}$) $\qquad < \dfrac{A}{\xi X^\xi}\left(\log X+\dfrac{1}{\xi}\right)^3.$

Hence $\qquad \sum\limits_{n\geqslant X}\dfrac{d^2(n)}{n^{2+2\delta}} < \dfrac{A\log^3 X}{X^{1+2\delta}} < \dfrac{A}{X\delta^3}$

since $X^{2\delta} = e^{2\delta\log X} > \frac{1}{6}(2\delta\log X)^3$.

Also, since $\qquad 1 < \log\lambda+\lambda^{-1} < \log\lambda+\lambda^{-\frac{1}{2}}$

for $\lambda > 1$,

$$\sum\sum_{X\leqslant m<n}\frac{d(m)d(n)}{(mn)^{1+\xi}\log n/m} < \sum\sum_{X\leqslant m<n}\frac{d(m)d(n)}{(mn)^{1+\xi}} + \sum\sum_{X\leqslant m<n}\frac{d(m)d(n)}{m^\xi n^{1+\xi}(mn)^{\frac{1}{2}}\log n/m}$$

$$< \left(\sum_{n=1}^\infty \frac{d(n)}{n^{1+\xi}}\right)^2 + \sum\sum_{1\leqslant m<n}\frac{d(m)d(n)}{(mn)^{\frac{1}{2}}\log n/m}\int\limits_n^\infty \frac{1+\xi}{x^{2+\xi}}\,dx$$

$$< \zeta^4(1+\xi)+\int\limits_1^\infty \frac{1+\xi}{x^{2+\xi}}\sum\sum_{m<n\leqslant x}\frac{d(m)d(n)}{(mn)^{\frac{1}{2}}\log n/m}\,dx$$

$$< \zeta^4(1+\xi)+\int\limits_1^\infty \frac{(1+\xi)A\log^3 x}{x^{1+\xi}}\,dx < \frac{A}{\xi^4}.$$

Hence $$\int\limits_0^T |f_X(1+\delta+it)|^2\,dt < A\left(\frac{T}{X}+1\right)\delta^{-4}. \tag{9.18.1}$$

For $\sigma = \frac{1}{2}$ we use the inequalities

$$|f_X|^2 \leqslant 2(|\zeta|^2|M_X|^2+1),$$

$$\int\limits_0^T |M_X(\tfrac{1}{2}+it)|^2\,dt \leqslant T\sum_{n<X}\frac{\mu^2(n)}{n}+2\sum\sum_{m<n<X}\frac{|\mu(m)\mu(n)|}{(mn)^{\frac{1}{2}}\log n/m}$$

$$\leqslant T\sum_{n<X}\frac{1}{n}+2\sum\sum_{m<n<X}\frac{1}{(mn)^{\frac{1}{2}}\log n/m}$$

$$< A(T+X)\log X,$$

by (7.2.1).

† The first result follows easily from (7.16.3); for the second, see Ingham (1); the argument of § 7.21, and the first result, give an extra $\log x$.

Hence $$\int\limits_0^T |f_X(\tfrac{1}{2}+it)|^2\,dt < AT^{2c}(T+X)\log^{2c'}(T+2)\log X. \tag{9.18.2}$$

The convexity theorem therefore gives

$$\int\limits_{\frac{1}{2}T}^T |f_X(\sigma+it)|^2\,dt$$

$$= O\left\{\left(\frac{T}{X}+1\right)\delta^{-4}\right\}^{(\sigma-\frac{1}{2})/(\frac{1}{2}+\delta)}\{T^{2c}(T+X)\log^{2c'}(T+2)\log X\}^{(1+\delta-\sigma)/(\frac{1}{2}+\delta)}$$

$$= O\left\{\frac{T+X}{\delta^4}\,\frac{T^{4c(1-\sigma)}}{X^{2\sigma-1}}(XT^{2c})^{\{(2\sigma-1)\delta\}/(\frac{1}{2}+\delta)}(\delta^4\log^3(T+2)\log X)^{(1+\delta-\sigma)/(\frac{1}{2}+\delta)}\right\}.$$

Taking $\delta = 1/\log(T+X)$, we obtain

$$O\{(T+X)T^{4c(1-\sigma)}X^{1-2\sigma}\log^4(T+X)\}.$$

If $X = T$, the result follows from Theorem 9.16.

For example, by Theorem 5.5 we may take $c = \frac{1}{6}$, $c' = \frac{3}{2}$. Hence

$$N(\sigma, T) = O(T^{\frac{3}{2}(1-\sigma)}\log^5 T). \tag{9.18.3}$$

This is an improvement on Theorem 9.17 if $\sigma > \frac{3}{4}$.

On the unproved Lindelöf hypothesis that $\zeta(\frac{1}{2}+it) = O(t^\epsilon)$, Theorem 9.18 gives $\qquad N(\sigma, T) = O(T^{2(1-\sigma)+\epsilon})$.

9.19. An improvement on Theorem 9.17 for all values of σ in $\frac{1}{2}<\sigma<1$ is effected by combining (9.18.3) with

THEOREM 9.19 (A). $N(\sigma, T) = O(T^{\frac{3}{2}-\sigma}\log^5 T)$.

We have

$$\int\limits_0^T |f_X(\tfrac{1}{2}+it)|^2\,dt < A\int\limits_0^T |\zeta(\tfrac{1}{2}+it)|^2|M_X(\tfrac{1}{2}+it)|^2\,dt+AT$$

$$< A\left\{\int\limits_0^T |\zeta(\tfrac{1}{2}+it)|^4\,dt\int\limits_0^T |M_X(\tfrac{1}{2}+it)|^4\,dt\right\}^{\frac{1}{2}}+AT.$$

Now $\qquad M_X^2(s) = \sum\limits_{n<X^2}\dfrac{c_n}{n^s}, \qquad |c_n| \leqslant d(n).$

Hence

$$\int\limits_0^T |M_X(\tfrac{1}{2}+it)|^4\,dt \leqslant T\sum_{n<X^2}\frac{d^2(n)}{n}+2\sum\sum_{m<n<X^2}\frac{d(m)d(n)}{(mn)^{\frac{1}{2}}\log n/m}$$

$$< AT\log^4 X+AX^2\log^3 X.$$

Hence $$\int\limits_0^T |f_X(\tfrac{1}{2}+it)|^2\,dt < AT^{\frac{1}{2}}(T+X^2)^{\frac{1}{2}}\log^2(T+2)\log^2 X. \tag{9.19.1}$$

From (9.18.1), (9.19.1), and the convexity theorem, we obtain

$$\int_{\frac{1}{2}T}^{T} |f_X(\sigma+it)|^2\,dt$$
$$= O\left\{\left(\frac{T}{X}+1\right)\delta^{-4}\right\}^{(\sigma-\frac{1}{2})/(\frac{1}{2}+\delta)}\{T^{\frac{1}{2}}(T+X^2)^{\frac{1}{2}}\log^2(T+2)\log^2 X\}^{(1+\delta-\sigma)/(\frac{1}{2}+\delta)}.$$

If $X = T^{\frac{1}{2}}$, $\delta = 1/\log(T+2)$, the result follows as before.

This is an improvement on Theorem 9.17 if $\frac{1}{2} < \sigma < \frac{3}{4}$.

Various results of this type have been obtained,† the most successful‡ being

THEOREM 9.19 (B). $N(\sigma, T) = O(T^{3(1-\sigma)/(2-\sigma)}\log^5 T)$.

This depends on a two-variable convexity theorem;§ if

$$J(\sigma,\lambda) = \left\{\int_0^T |f(\sigma+it)|^{1/\lambda}\,dt\right\}^{\lambda},$$

then $$J(\sigma, p\lambda+q\mu) = O\{J^p(\alpha,\lambda)J^q(\beta,\mu)\} \quad (\alpha<\sigma<\beta),$$

where $$p = \frac{\beta-\sigma}{\beta-\alpha}, \qquad q = \frac{\sigma-\alpha}{\beta-\alpha}.$$

We have

$$\int_0^T |f_X(\tfrac{1}{2}+it)|^{\frac{4}{3}}\,dt < A\int_0^T |\zeta(\tfrac{1}{2}+it)|^{\frac{4}{3}}|M_X(\tfrac{1}{2}+it)|^{\frac{4}{3}}\,dt + AT$$
$$< A\left\{\int_0^T |\zeta(\tfrac{1}{2}+it)|^4\,dt\right\}^{\frac{1}{3}}\left\{\int_0^T |M_X(\tfrac{1}{2}+it)|^2\,dt\right\}^{\frac{2}{3}} + AT$$
$$< A\{T\log^4(T+2)\}^{\frac{1}{3}}\{(T+X)\log X\}^{\frac{2}{3}} + AT$$
$$< A(T+X)\log^2(T+X). \tag{9.19.2}$$

In the two-variable convexity theorem, take $\alpha = \frac{1}{2}$, $\beta = 1+\delta$, $\lambda = \frac{3}{4}$, $\mu = \frac{1}{2}$, and use (9.18.1) and (9.19.2). We obtain

$$\int_0^T |f_X(\sigma+it)|^{1/K}\,dt$$
$$< A\{(T+X)\log^2(T+X)\}^{\frac{3}{2}(1-\sigma+\delta)/(1-\frac{1}{2}\sigma+\frac{3}{2}\delta)}\left\{\left(\frac{T}{X}+1\right)\delta^{-4}\right\}^{(\sigma-\frac{1}{2})/(1-\frac{1}{2}\sigma+\frac{3}{2}\delta)},$$

where $K = p\lambda+q\mu$ lies between $\frac{1}{2}$ and $\frac{3}{4}$. Taking $X = T$, $\delta = 1/\log T$, we obtain

$$\int_0^T |f_X(\sigma+it)|^{1/K}\,dt < AT^{3(1-\sigma)/(2-\sigma)}\log^4 T.$$

† Titchmarsh (5), Ingham (5), (6). ‡ Ingham (6). § Gabriel (1).

The result now follows from a modified form of Theorem 9.16, since

$$\log|1-f_X^2| \leqslant \log(1+|f_X|^2) < A|f_X|^{1/K}.$$

A. Selberg† has recently proved

THEOREM 9.19 (C). $N(\sigma, T) = O(T^{1-\frac{1}{4}(\sigma-\frac{1}{2})}\log T)$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 1$.

This is an improvement on the previous theorem if σ is a function of T such that $\sigma-\frac{1}{2}$ is sufficiently small.

9.20. The corresponding problems with σ equal or nearly equal to $\frac{1}{2}$ are naturally more difficult. Here the most interesting question is that of the behaviour of

$$\int_{\frac{1}{2}}^{1} N(\sigma, T)\,d\sigma \tag{9.20.1}$$

as $T\to\infty$. If the zeros of $\zeta(s)$ are $\beta+i\gamma$, this is equal to

$$\int_{\frac{1}{2}}^{1}\Big(\sum_{\beta>\sigma,0<\gamma\leqslant T} 1\Big)\,d\sigma = \sum_{\beta>\frac{1}{2},0<\gamma\leqslant T}\int_{\frac{1}{2}}^{\beta} d\sigma = \sum_{\beta>\frac{1}{2},0<\gamma\leqslant T}(\beta-\tfrac{1}{2}).$$

Hence an equivalent problem is that of the sum

$$\sum_{0<\gamma\leqslant T}|\beta-\tfrac{1}{2}|. \tag{9.20.2}$$

There are some immediate results.‡ If we apply the above argument, but use Theorem 7.2 (A) instead of Theorem 7.2, we obtain at once

$$\int_{\sigma_0}^{1} N(\sigma, T)\,d\sigma < AT\log\left\{\min\left(\log T, \log\frac{1}{\sigma_0-\frac{1}{2}}\right)\right\} \tag{9.20.3}$$

for $\frac{1}{2} \leqslant \sigma_0 \leqslant 1$; and in particular

$$\int_{\frac{1}{2}}^{1} N(\sigma, T)\,d\sigma = O(T\log\log T). \tag{9.20.4}$$

These, however, are superseded by the following analysis, due to A. Selberg (2), the principal result of which is that

$$\int_{\frac{1}{2}}^{1} N(\sigma, T)\,d\sigma = O(T). \tag{9.20.5}$$

We consider the integral

$$\int_T^{T+U} |\zeta(\tfrac{1}{2}+it)\psi(\tfrac{1}{2}+it)|^2\,dt,$$

† Selberg (5). ‡ Littlewood (4).

where $0 < U \leqslant T$ and ψ is a function to be specified later. We use the formulae of § 4.17. Since

$$e^{i\vartheta} = \{\chi(\tfrac{1}{2}+it)\}^{-\frac{1}{2}} = \left(\frac{t}{2\pi e}\right)^{\frac{1}{2}it} e^{-\frac{1}{8}\pi i}\left\{1+O\left(\frac{1}{t}\right)\right\},$$

we have

$$Z(t) = z(t)+\bar{z}(t)+O(t^{-\frac{1}{4}}), \tag{9.20.6}$$

where

$$z(t) = \left(\frac{t}{2\pi e}\right)^{\frac{1}{2}it} e^{-\frac{1}{8}\pi i} \sum_{n \leqslant x} n^{-\frac{1}{2}-it}$$

and $x = (t/2\pi)^{\frac{1}{2}}$. Let $T \leqslant t \leqslant T+U$, $\tau = (T/2\pi)^{\frac{1}{2}}$, $\tau' = \{(T+U)/2\pi\}^{\frac{1}{2}}$. Let

$$z_1(t) = \left(\frac{t}{2\pi e}\right)^{\frac{1}{2}it} e^{-\frac{1}{8}\pi i} \sum_{n \leqslant \tau} n^{-\frac{1}{2}-it}.$$

Proceeding as in § 7.3, we have

$$\begin{aligned}\int_T^{T+U} |z(t)-z_1(t)|^2\,dt &= O\left(U \sum_{\tau < n \leqslant \tau'} \frac{1}{n}\right)+O(T^{\frac{1}{2}}\log T) \\ &= O\left(U\frac{\tau'-\tau}{\tau}\right)+O(T^{\frac{1}{2}}\log T) \\ &= O(U^2/T)+O(T^{\frac{1}{2}}\log T).\end{aligned} \tag{9.20.7}$$

9.21. LEMMA 9.21. *Let m and n be positive integers, $(m,n) = 1$, $M = \max(m,n)$. Then*

$$\int_T^{T+U} z_1(t)\bar{z}_1(t)\left(\frac{n}{m}\right)^{it} dt = \frac{U}{(mn)^{\frac{1}{2}}} \sum_{r \leqslant \tau/M} \frac{1}{r} + O\{T^{\frac{1}{2}}M^2\log(MT)\}.$$

The integral is

$$\sum_{\mu \leqslant \tau} \sum_{\nu \leqslant \tau} \frac{1}{(\mu\nu)^{\frac{1}{2}}} \int_T^{T+U} \left(\frac{n\nu}{m\mu}\right)^{it} dt.$$

The terms with $m\mu = n\nu$ contribute

$$U \sum_{m\mu = n\nu}\sum \frac{1}{(\mu\nu)^{\frac{1}{2}}} = U \sum_{rn \leqslant \tau, rm \leqslant \tau} \frac{1}{(rn.rm)^{\frac{1}{2}}} = \frac{U}{(mn)^{\frac{1}{2}}} \sum_{r \leqslant \tau/M} \frac{1}{r}.$$

The remaining terms are

$$\begin{aligned}O\left\{\sum_{m\mu \neq n\nu}\sum \frac{1}{(\mu\nu)^{\frac{1}{2}}|\log(n\nu/m\mu)|}\right\} &= O\left\{\sum_{m\mu \neq n\nu}\sum \frac{M}{(m\mu n\nu)^{\frac{1}{2}}|\log(n\nu/m\mu)|}\right\} \\ &= O\left\{M \sum_{\kappa \leqslant M\tau, \lambda \leqslant M\tau}\sum \frac{1}{(\kappa\lambda)^{\frac{1}{2}}|\log\lambda/\kappa|}\right\} = O\{M^2\tau\log(M\tau)\},\end{aligned}$$

and the result follows.

9.22. LEMMA 9.22. *Defining m, n, M as before, and supposing*

$$T^{\frac{4}{5}} < U \leqslant T,$$

$$\int_T^{T+U} z_1^2(t)\left(\frac{n}{m}\right)^{it} dt = \frac{U}{(mn)^{\frac{1}{2}}} \sum_{\tau/m \leqslant r \leqslant \tau/n} \frac{1}{r} + O(MT^{\frac{1}{2}})+O(U^2/T)+O(T^{\frac{9}{10}}) \tag{9.22.1}$$

if $n \leqslant m$. If $m < n$, the first term on the right-hand side is to be omitted.

The left-hand side is

$$e^{-\frac{1}{4}\pi i} \sum_{\mu \leqslant \tau} \sum_{\nu \leqslant \tau} \frac{1}{(\mu\nu)^{\frac{1}{2}}} \int_T^{T+U} \left(\frac{t}{2\pi e}\frac{n}{\mu\nu m}\right)^{it} dt.$$

The integral is of the form considered in § 4.6, with

$$F(t) = t\log\frac{t}{ec}, \qquad c = \frac{2\pi\mu\nu m}{n}.$$

Hence by (4.6.5), with $\lambda_2 = (T+U)^{-1}$, $\lambda_3 = (T+U)^{-2}$, it is equal to

$$\begin{aligned}&(2\pi c)^{\frac{1}{2}} e^{\frac{1}{4}\pi i - ic} + O(T^{\frac{2}{5}}) + O\left\{\min\left(\frac{1}{|\log c/T|}, T^{\frac{1}{2}}\right)\right\} \\ &\qquad + O\left\{\min\left(\frac{1}{\log|(T+U)/c|}, T^{\frac{1}{2}}\right)\right\},\end{aligned} \tag{9.22.2}$$

with the leading term present only when $T \leqslant c \leqslant T+U$. We therefore obtain a main term

$$2\pi\left(\frac{m}{n}\right)^{\frac{1}{2}} \sum_{\mu \leqslant \tau} \sum_{\nu \leqslant \tau} e^{-2\pi i\mu\nu m/n} \tag{9.22.3}$$

where μ and ν also satisfy

$$\tau^2 n/m \leqslant \mu\nu \leqslant \tau'^2 n/m.$$

The double sum is clearly zero unless $n \leqslant m$, as we now suppose. The ν-summation runs over the range $\nu_1 \leqslant \nu \leqslant \nu_2$, where $\nu_1 = \tau^2 n/m\mu$ and $\nu_2 = \min(\tau'^2 n/m\mu, \tau)$, and μ runs over $\tau n/m \leqslant \mu \leqslant \tau$. The inner sum is therefore $\nu_2 - \nu_1 + O(n)$ if $n|\mu$, and $O(n)$ otherwise. The error term $O(n)$ contributes $O\{(mn)^{\frac{1}{2}}\tau\} = O(MT^{\frac{1}{2}})$ in (9.22.1). On writing $\mu = nr$ we are left with

$$2\pi\left(\frac{m}{n}\right)^{\frac{1}{2}} \sum_{\tau/m \leqslant r \leqslant \tau/n} (\nu_2 - \nu_1).$$

Let $\nu_3 = \tau'^2/mr$. Then $\nu_2 = \nu_3$ unless $r < \tau'^2/m\tau$. Hence the error on replacing v_2 by v_3 is

$$O\left\{\left(\frac{m}{n}\right)^{\frac{1}{2}} \sum_{\tau/m \leqslant r < \tau'^2/m\tau} \left(\frac{\tau'^2}{mr}-\tau\right)\right\} = O\left\{\left(\frac{m}{n}\right)^{\frac{1}{2}}\left(\frac{\tau'^2}{m\tau}-\frac{\tau}{m}+1\right)\left(\frac{\tau'^2}{\tau}-\tau\right)\right\}$$

$$= O\left\{(mn)^{-\frac{1}{2}}\left(\frac{\tau'^2-\tau^2}{\tau}\right)^2\right\}+O\left\{\left(\frac{m}{n}\right)^{\frac{1}{2}}\left(\frac{\tau'^2-\tau^2}{\tau}\right)\right\}$$

$$= O(U^2T^{-1})+O(M^{\frac{1}{2}}UT^{-\frac{1}{2}}).$$

Finally there remains

$$2\pi\left(\frac{m}{n}\right)^{\frac{1}{2}} \sum_{\tau/m \leqslant r \leqslant \tau/n} (v_3-v_1) = 2\pi\left(\frac{m}{n}\right)^{\frac{1}{2}} \sum_{\tau/m \leqslant r \leqslant \tau/n} \left(\frac{\tau'^2}{mr}-\frac{\tau^2}{mr}\right)$$

$$= \frac{U}{(mn)^{\frac{1}{2}}} \sum_{\tau/m \leqslant r \leqslant \tau/n} \frac{1}{r}.$$

Now consider the O-terms arising from (9.22.2). The term $O(T^{\frac{2}{5}})$ gives

$$O\left\{T^{\frac{2}{5}} \sum_{\mu \leqslant \tau} \sum_{\nu \leqslant \tau} \frac{1}{(\mu\nu)^{\frac{1}{2}}}\right\} = O(T^{\frac{2}{5}}\tau) = O(T^{\frac{9}{10}}).$$

Next

$$\sum_{\mu \leqslant \tau} \sum_{\nu \leqslant \tau} \frac{1}{(\mu\nu)^{\frac{1}{2}}} \min\left(\frac{1}{|\log(2\pi\mu\nu m/nT)|}, T^{\frac{1}{2}}\right)$$

$$= O\left\{T^{\epsilon} \sum_{r \leqslant \tau^2} \frac{1}{r^{\frac{1}{2}}} \min\left(\frac{1}{|\log(rm/n\tau^2)|}, T^{\frac{1}{2}}\right)\right\}.$$

Suppose, for example, that $n < m$. Then the terms with $r < \frac{1}{2}n\tau^2/m$ or $r > 2n\tau^2/m$ are

$$O\left(T^{\epsilon} \sum_{r \leqslant \tau^2} \frac{1}{r^{\frac{1}{2}}}\right) = O(T^{\epsilon}\tau) = O(T^{\frac{1}{2}+\epsilon}).$$

In the other terms, let $r = [n\tau^2/m]-r'$. We obtain

$$O\left\{T^{\epsilon} \sum_{r'} \frac{1}{(n\tau^2/m)^{\frac{1}{2}}} \frac{1}{|r'-\theta|/(n\tau^2/m)}\right\} \quad (|\theta| < 1)$$

$$= O\left\{T^{\epsilon}\left(\frac{n\tau^2}{m}\right)^{\frac{1}{2}}\log T\right\} = O(T^{\frac{1}{2}+\epsilon}),$$

omitting the terms $r' = -1$, 0, 1; and these are $O(T^{\frac{1}{2}+\epsilon})$.
A similar argument applies in the other cases.

9.23. LEMMA 9.23. *Let $(m,n) = 1$ with $m, n \leqslant X \leqslant T^{\frac{1}{5}}$. If $T^{\frac{14}{15}} \leqslant U \leqslant T$, then*

$$\int_T^{T+U} Z^2(t)\left(\frac{n}{m}\right)^{it} dt = \frac{U}{(mn)^{\frac{1}{2}}}\left\{\log\frac{T}{2\pi mn}+2\gamma\right\}+O(U^{\frac{3}{2}}T^{-\frac{1}{2}}\log T).$$

Let $Z(t) = z_1(t)+\overline{z_1(t)}+e(t)$. Then

$$\int_T^{T+U} \{z_1(t)+\overline{z_1(t)}\}^2\left(\frac{n}{m}\right)^{it} dt$$

$$= \int_T^{T+U} Z(t)^2\left(\frac{n}{m}\right)^{it} dt+O\left(\int_T^{T+U} |Z(t)e(t)|\,dt\right)+O\left(\int_T^{T+U} |e(t)|^2\,dt\right).$$

We have

$$\int_T^{T+U} |e(t)|^2\,dt = O(U^2/T)+O(T^{\frac{1}{2}}\log T) = O(U^2/T)$$

by (9.20.7), and

$$\int_T^{T+U} |Z(t)|^2\,dt = O(U\log T)+O(T^{\frac{1}{2}+\epsilon}) = O(U\log T),$$

by Theorem 7.4. Hence

$$\int_T^{T+U} |Z(t)e(t)|\,dt = O\{(U^2/T)^{\frac{1}{2}}(U\log T)^{\frac{1}{2}}\}$$

by Cauchy's inequality. It follows that

$$\int_T^{T+U} Z(t)^2\left(\frac{n}{m}\right)^{it} dt$$

$$= \int_T^{T+U} \{z_1(t)^2+\overline{z_1(t)^2}+2z_1(t)\overline{z_1(t)}\}\left(\frac{n}{m}\right)^{it} dt+O(U^{\frac{3}{2}}T^{-\frac{1}{2}}\log^{\frac{1}{2}}T).$$

By Lemmas 9.21 and 9.22 the main integral on the right is

$$\frac{U}{(mn)^{\frac{1}{2}}}\left(\sum_{r \leqslant \tau/n} \frac{1}{r}+\sum_{r \leqslant \tau/m} \frac{1}{r}\right)+O\{T^{\frac{1}{2}}X^2\log(XT)\}+O(XT^{\frac{1}{2}})+$$

$$+O(U^2/T)+O(T^{\frac{9}{10}})$$

whether $n \leqslant m$ or not. The result then follows, since

$$\sum_{r \leqslant \tau/n} \frac{1}{r} + \sum_{r \leqslant \tau/m} \frac{1}{r} = \log \frac{\tau^2}{mn} + 2\gamma + O\left(\frac{X}{\tau}\right),$$

and since the error terms $O\{T^{\frac{1}{2}}X^2 \log(XT)\}$, $O(XT^{\frac{1}{2}})$, $O(U^2/T)$, $O(T^{\frac{9}{10}})$ and $O(UXT^{-\frac{1}{2}})$ are all $O(U^{\frac{3}{2}}T^{-\frac{1}{2}} \log T)$.

9.24. THEOREM 9.24.

$$\int_{\frac{1}{2}}^{1} N(\sigma, T)\, d\sigma = O(T). \tag{9.24.1}$$

Consider the integral

$$I = \int_{T}^{T+U} |\zeta(\tfrac{1}{2}+it)\psi(\tfrac{1}{2}+it)|^2\, dt = \int_{T}^{T+U} Z^2(t)|\psi(\tfrac{1}{2}+it)|^2\, dt,$$

where
$$\psi(s) = \sum_{r<X} \delta_r r^{1-s}$$

and
$$\delta_r = \frac{\sum\limits_{\rho r<X} \mu(\rho r)\mu(\rho)/\phi(\rho r)}{\sum\limits_{\rho<X} \mu^2(\rho)/\phi(\rho)} = \frac{\mu(r)}{\phi(r)} \frac{\sum\limits_{\rho r<X, (\rho, r)=1} \mu^2(\rho)/\phi(\rho)}{\sum\limits_{\rho<X} \mu^2(\rho)/\phi(\rho)}.$$

Clearly
$$|\delta_r| \leqslant \frac{1}{\phi(r)}$$
for all values of r. Now

$$I = \sum_{q<X} \sum_{r<X} \delta_q \delta_r q^{\frac{1}{2}} r^{\frac{1}{2}} \int_{T}^{T+U} Z^2(t) \left(\frac{n}{m}\right)^{it} dt,$$

where $m = q/(q,r)$, $n = r/(q,r)$. Using Lemma 9.23, the main term contributes to this

$$\sum_{q<X} \sum_{r<X} \delta_q \delta_r q^{\frac{1}{2}} r^{\frac{1}{2}} \frac{U}{(mn)^{\frac{1}{2}}} \log \frac{Te^{2\gamma}}{2\pi mn} = U \sum_{q<X} \sum_{r<X} \delta_q \delta_r (q,r) \log \frac{Te^{2\gamma}(q,r)^2}{2\pi qr}$$
$$= U \log \frac{Te^{2\gamma}}{2\pi} \sum_{q<X} \sum_{r<X} \delta_q \delta_r (q,r) - 2U \sum_{q<X} \sum_{r<X} \delta_q \delta_r (q,r) \log q +$$
$$+ 2U \sum_{q<X} \sum_{r<X} \delta_q \delta_r (q,r) \log(q,r).$$

For a fixed $q < X$,

$$\sum_{r<X} (q,r)\, \delta_r = \left\{ \sum_{\rho<X} \frac{\mu^2(\rho)}{\phi(\rho)} \right\}^{-1} \sum_{r<X, \rho r<X} \sum \frac{(q,r)\mu(\rho r)\mu(\rho)}{\phi(\rho r)}.$$

Now
$$(q,r) = \sum_{\nu | (q,r)} \phi(\nu) = \sum_{\nu|q, \nu|r} \phi(\nu).$$

Hence the second factor on the right is

$$\sum_{r<X, \rho r<X} \sum \frac{\mu(\rho r)\mu(\rho)}{\phi(\rho r)} \sum_{\nu|q, \nu|r} \phi(\nu) = \sum_{\nu|q} \phi(\nu) \sum_{\substack{r<X, \rho r<X \\ \nu|r}} \sum \frac{\mu(\rho r)\mu(\rho)}{\phi(\rho r)}.$$

Put $\rho r = l$. Then $\rho\nu | \rho r$, $\rho\nu | l$, i.e. $\rho | (l/\nu)$. Hence we get

$$\sum_{\nu|q} \phi(\nu) \sum_{\substack{l<X \\ \nu|l}} \frac{\mu(l)}{\phi(l)} \sum_{\rho|(l/\nu)} \mu(\rho).$$

The ρ-sum is 0 unless $l = \nu$, when it is 1. Hence we get

$$\sum_{\nu|q} \phi(\nu) \frac{\mu(\nu)}{\phi(\nu)} = \sum_{\nu|q} \mu(\nu) = \begin{cases} 1 & (q=1), \\ 0 & (q>1). \end{cases}$$

Hence

$$\sum_{q<X} \sum_{r<X} \delta_q \delta_r (q,r) = \left\{ \sum_{\rho<X} \frac{\mu^2(\rho)}{\phi(\rho)} \right\}^{-1} \delta_1 = \left\{ \sum_{\rho<X} \frac{\mu^2(\rho)}{\phi(\rho)} \right\}^{-1}$$

and
$$\sum_{q<X} \sum_{r<X} \delta_q \delta_r (q,r) \log q = \left\{ \sum_{\rho<X} \frac{\mu^2(\rho)}{\phi(\rho)} \right\}^{-1} \delta_1 \log 1 = 0.$$

Let $\phi_a(n)$ be defined by

$$\sum_{n=1}^{\infty} \frac{\phi_a(n)}{n^s} = \frac{\zeta(s-a-1)}{\zeta(s)},$$

so that
$$\phi_a(n) = n^{1+a} \sum_{m|n} \frac{\mu(m)}{m^{1+a}} = n^{1+a} \prod_{p|n} \left(1 - \frac{1}{p^{1+a}}\right).$$

Let $\psi(n)$ be defined by
$$\sum_{n=1}^{\infty} \frac{\psi(n)}{n^s} = -\frac{\zeta'(s-1)}{\zeta(s)}.$$

Then
$$-\zeta'(s-1) = \zeta(s) \sum_{n=1}^{\infty} \frac{\psi(n)}{n^s},$$

and hence
$$n \log n = \sum_{d|n} \psi(d).$$

Hence
$$(q,r) \log(q,r) = \sum_{d|q, d|r} \psi(d)$$

and
$$\sum_{q<X} \sum_{r<X} \delta_q \delta_r (q,r) \log(q,r) = \sum_{d<X} \psi(d) \sum_{\substack{d|q, d|r \\ q<X, r<X}} \delta_q \delta_r$$
$$= \sum_{d<X} \psi(d) \Big(\sum_{d|q, q<X} \delta_q \Big)^2.$$

Now $$\psi(n) = \left[\frac{\partial}{\partial a}\phi_a(n)\right]_{a=0} = \phi(n)\left(\log n + \sum_{p|n}\frac{\log p}{p-1}\right),$$

$$\psi(n) \leqslant \phi(n)\left(\log n + \sum_{p|n}\log p\right) \leqslant 2\phi(n)\log n.$$

Also

$$\sum_{\substack{d|q\\ q<X}} \delta_q = \left\{\sum_{\rho<X}\frac{\mu^2(\rho)}{\phi(\rho)}\right\}^{-1}\sum_{\substack{\rho q<X\\ d|q}}\sum\frac{\mu(\rho q)\mu(\rho)}{\phi(\rho q)}$$

$$= \left\{\sum_{\rho<X}\frac{\mu^2(\rho)}{\phi(\rho)}\right\}^{-1}\sum_{\substack{n<X\\ d|n}}\frac{\mu(n)}{\phi(n)}\sum_{\rho|n/d}\mu(\rho) = \left\{\sum_{\rho<X}\frac{\mu^2(\rho)}{\phi(\rho)}\right\}^{-1}\frac{\mu(d)}{\phi(d)}.$$

Hence $$\sum_{q<X}\sum_{r<X}\delta_q\delta_r(q,r)\log(q,r) \leqslant 2\log X\left\{\sum_{\rho<X}\frac{\mu^2(\rho)}{\phi(\rho)}\right\}^{-1}.$$

Since

$$\sum_{n=1}^{\infty}\frac{\mu^2(n)}{\phi(n)n^s} = \prod_p\left(1+\frac{\mu^2(p)}{\phi(p)p^s}\right) = \prod_p\left(1+\frac{1}{(p-1)p^s}\right)$$

$$= \zeta(s+1)\prod_p\left(1-\frac{1}{p^{s+1}}\right)\left(1+\frac{1}{(p-1)p^s}\right),$$

we have $$\sum_{\rho<X}\frac{\mu^2(\rho)}{\phi(\rho)} \sim A\log X.$$

The contribution of all the above terms to I is therefore

$$O\left(U\frac{\log T}{\log X}\right)+O(U) = O(U)$$

on taking, say, $X = T^{\frac{1}{100}}$.

The O-term in Lemma 9.23 gives

$$O(U^{\frac{3}{2}}T^{-\frac{1}{2}}\log T)\sum_{q<X}\sum_{r<X}\frac{q^{\frac{1}{2}}r^{\frac{1}{2}}}{\phi(q)\phi(r)}$$

$$= O(U^{\frac{3}{2}}T^{-\frac{1}{2}}\log T)O(X)$$

$$= O(U^{\frac{3}{2}}T^{-\frac{49}{100}}\log T).$$

Taking say $U = T^{\frac{14}{15}}$, this is $O(U)$. Hence $I = O(U)$.

By an argument similar to that of § 9.16, it follows that

$$\int_{\frac{1}{2}}^{1}\{N(\sigma,T+U)-N(\sigma,T)\}\,d\sigma = O(U).$$

Replacing T by $T+U$ $T+2U$,... and adding, $O(T/U)$ terms, we obtain

$$\int_{\frac{1}{2}}^{1}\{N(\sigma,2T)-N(\sigma,T)\}\,d\sigma = O(T).$$

Replacing T by $\frac{1}{2}T$, $\frac{1}{4}T$,... and adding, the theorem follows.

It also follows that, if $\frac{1}{2}<\sigma\leqslant 1$,

$$N(\sigma,T) = \frac{2}{\sigma-\frac{1}{2}}\int_{\frac{1}{2}\sigma+\frac{1}{4}}^{\sigma}N(\sigma',T)\,d\sigma'$$

$$\leqslant \frac{2}{\sigma-\frac{1}{2}}\int_{\frac{1}{2}}^{1}N(\sigma,T)\,d\sigma = O\left(\frac{T}{\sigma-\frac{1}{2}}\right). \tag{9.24.2}$$

Lastly, *if $\phi(t)$ is positive and increases to infinity with t, all but an infinitesimal proportion of the zeros of $\zeta(s)$ in the upper half-plane lie in the region*

$$|\sigma-\tfrac{1}{2}| < \frac{\phi(t)}{\log t}.$$

The curved boundary of the region

$$\sigma = \tfrac{1}{2}+\frac{\phi(t)}{\log t}, \qquad T^{\frac{1}{2}}<t<T$$

lies to the right of $$\sigma = \sigma_1 = \tfrac{1}{2}+\frac{\phi(T^{\frac{1}{2}})}{\log T},$$

and $$N(\sigma_1,T) = O\left(\frac{T}{\sigma_1-\frac{1}{2}}\right) = O\left(\frac{T\log T}{\phi(T^{\frac{1}{2}})}\right) = o(T\log T).$$

Hence the number of zeros outside the region specified is $o(T\log T)$, and the result follows.

NOTES FOR CHAPTER 9

9.25. The mean value of $S(t)$ has been investigated by Selberg (5). One has

$$\int_0^T |S(t)|^{2k}\,dt \sim \frac{(2k)!}{k!(2\pi)^{2k}}T(\log\log T)^k \tag{9.25.1}$$

for every positive integer k. Selberg's earlier conditional treatment (4) is discussed in §§ 14.20–24, the key feature used in (5) to deal with zeros off the critical line being the estimate given in Theorem 9.19(C). Selberg (5) also gave an unconditional proof of Theorem 14.19, which had previously been established on the Riemann hypothesis by Littlewood.

These results have been investigated further by Fujii [1], [2] and Ghosh [1], [2], who give results which are uniform in k.

It follows in particular from Fujii [1] that

$$\int_0^T |S(t+h)-S(t)|^2\,dt = \pi^{-2}T\log(3+h\log T)+O[T\{\log(3+h\log T)\}^{\frac{1}{2}}] \tag{9.25.2}$$

and

$$\int_0^T |S(t+h)-S(t)|^{2k}\,dt \ll T\{Ak^4\log(3+h\log T)\}^k \tag{9.25.3}$$

uniformly for $0 \leqslant h \leqslant \frac{1}{2}T$. One may readily deduce that

$$N_j(T) \ll N(T)\,e^{-A\sqrt{j}},$$

where $N_j(T)$ denotes the number of zeros $\beta+i\gamma$ of multiplicity exactly j, in the range $0<\gamma\leqslant T$. Moreover one finds that

$$\#\{n: 0<\gamma_n\leqslant T, \gamma_{n+1}-\gamma_n \geqslant \lambda/\log T\} \ll N(T)\exp\{-A\lambda^{\frac{1}{2}}(\log\lambda)^{-\frac{1}{4}}\},$$

uniformly for $\lambda \geqslant 2$, whence, in particular,

$$\sum_{0<\gamma_n\leqslant T}(\gamma_{n+1}-\gamma_n)^k \ll \frac{N(T)}{(\log T)^k}, \tag{9.25.4}$$

for any fixed $k \geqslant 0$. Fujii [2] also states that there exist constants $\lambda>1$ and $\mu<1$ such that

$$\frac{\gamma_{n+1}-\gamma_n}{2\pi/\log\gamma_n} \geqslant \lambda \tag{9.25.5}$$

and

$$\frac{\gamma_{n+1}-\gamma_n}{2\pi/\log\gamma_n} \leqslant \mu \tag{9.25.6}$$

each hold for a positive proportion of n (i.e. the number of n for which $0<\gamma_n\leqslant T$ is at least $AN(T)$ if $T\geqslant T_0$). Note that $2\pi/\log\gamma_n$ is the average spacing between zeros. The possibility of results such as (9.25.5) and (9.25.6) was first observed by Selberg [1].

9.26. Since the deduction of the results (9.25.5) and (9.25.6) is not obvious, we give a sketch. If M is a sufficiently large integer constant,

then (9.25.2) and (9.25.3) yield

$$\int_T^{2T} |S(t+h)-S(t)|^2 dt \gg T$$

and

$$\int_T^{2T} |S(t+h)-S(t)|^4\,dt \ll T$$

uniformly for

$$\frac{2\pi M}{\log T} \leqslant h \leqslant \frac{4\pi M}{\log T}.$$

By Hölder's inequality we have

$$\int_T^{2T} |S(t+h)-S(t)|^2 dt \leqslant \left(\int_T^{2T} |S(t+h)-S(t)|\,dt\right)^{\frac{2}{3}} \times\left(\int_T^{2T} |S(t+h)-S(t)|^4\,dt\right)^{\frac{1}{3}},$$

so that

$$\int_T^{2T} |S(t+h)-S(t)|\,dt \gg T.$$

We now observe that

$$S(t+h)-S(t) = N(t+h)-N(t)-\frac{h\log T}{2\pi}+O\left(\frac{1}{\log T}\right),$$

for $T\leqslant t\leqslant 2T$, whence

$$\int_T^{2T}\left|N(t+h)-N(t)-\frac{h\log T}{2\pi}\right|dt \gg T.$$

We proceed to write $h = 2\pi M\lambda/\log T$ and

$$\delta(t,\lambda) = N\left(t+\frac{2\pi\lambda}{\log T}\right)-N(t)-\lambda,$$

so that

$$N(t+h)-N(t)-\frac{h\log T}{2\pi}=\sum_{m=0}^{M-1}\delta\left(t+\frac{2\pi m\lambda}{\log T},\lambda\right).$$

Thus

$$T\ll\sum_{m=0}^{M-1}\int_{T+2\pi m\lambda/\log T}^{2T+2\pi m\lambda/\log T}|\delta(t,\lambda)|\,dt$$

$$=M\int_T^{2T}|\delta(t,\lambda)|\,dt+O(1),$$

and hence

$$\int_T^{2T}|\delta(t,\lambda)|\,dt\gg T \tag{9.26.1}$$

uniformly for $1\leqslant\lambda\leqslant 2$, since M is constant.

Now, if I is the subset of $[T,2T]$ on which $N\left(t+\frac{2\pi\lambda}{\log T}\right)=N(t)$, then

$$|\delta(t,\lambda)|\leqslant\begin{cases}\delta(t,\lambda)+2\lambda & (t\in I),\\ \delta(t,\lambda)+2\lambda-2 & (t\in[T,2T]-I),\end{cases}$$

so that (9.26.1) yields

$$T\ll\int_T^{2T}\delta(t,\lambda)\,dt+(2\lambda-2)T+2m(I),$$

where $m(I)$ is the measure of I. However

$$\int_T^{2T}\delta(t,\lambda)\,dt=O\left(\frac{T}{\log T}\right),$$

whence $m(I)\gg T$, if $\lambda>1$ is chosen sufficiently close to 1. Thus, if

$$S=\left\{n\colon T\leqslant\gamma_n\leqslant 2T,\ \gamma_{n+1}-\gamma_n\geqslant\frac{2\pi\lambda}{\log T}\right\},$$

then

$$T\ll m(I)\ll\sum_{n\in S}(\gamma_{n+1}-\gamma_n)+O(1),$$

so that

$$T^2\ll\left\{\sum_{n\in S}(\gamma_{n+1}-\gamma_n)\right\}^2\leqslant(\# S)\left(\sum_{n\in S}(\gamma_{n+1}-\gamma_n)^2\right)$$

$$\ll\# S\,\frac{T}{\log T},$$

by (9.25.4) with $k=2$. It follows that

$$\# S\gg N(T), \tag{9.26.2}$$

proving that (9.25.5) holds for a positive proportion of n.

Now suppose that μ is a constant in the range $0<\mu<1$, and put

$$U=\{n\colon T\leqslant\gamma_n\leqslant 2T\},$$

and

$$V=\left\{n\in U\colon\gamma_{n+1}-\gamma_n\leqslant\frac{2\pi\mu}{\log T}\right\},$$

whence $\# U=\frac{T}{2\pi}\log T+O(T)$. Then

$$T=\sum_{n\in U}(\gamma_{n+1}-\gamma_n)+O(1)$$

$$\geqslant\sum_{n\in U-V}(\gamma_{n+1}-\gamma_n)+O(1)$$

$$\geqslant\frac{2\pi\mu}{\log T}(\# U-\# V-\# S)+\frac{2\pi\lambda}{\log T}\,S+O(1)$$

$$=\frac{2\pi\mu}{\log T}\left(\frac{T}{2\pi}\log T-\# V\right)+\frac{2\pi(\lambda-\mu)}{\log T}\,\# S+O\left(\frac{T}{\log T}\right).$$

If the implied constant in (9.26.2) is η, it follows that $\# V\gg N(T)$, on taking $\mu=1-\nu$, with $0<\nu<\eta(\lambda-1)/(1-\eta)$. Thus (9.25.6) also holds for a positive proportion of n.

9.27. Ghosh [1] was able to sharpen the result of Selberg mentioned at the end of §9.10, to show that $S(t)$ has at least

$$T(\log T)\exp\left(-\frac{A\log\log T}{(\log\log\log T)^{\frac{1}{2}-\delta}}\right)$$

sign changes in the range $0\leqslant t\leqslant T$, for any positive δ, and $A=A(\delta)$, $T\geqslant T(\delta)$. He also proved (Ghosh [2]) that the asymptotic formula (9.25.1) holds for any positive real k, with the constant on the right hand side replaced by $\Gamma(2k+1)/\Gamma(k+1)(2\pi)^{2k}$. Moreover he showed (Ghosh [2]) that

$$\frac{|S(t)|}{\sqrt{(\log\log t)}} = f(t),$$

say, has a limiting distribution

$$P(\sigma) = 2\pi^{\frac{1}{2}} \int_0^\sigma e^{-\pi^2 z^2}\, dz,$$

in the sense that, for any $\sigma > 0$, the measure of the set of $t \in [0, T]$ for which $f(t) \leqslant \sigma$, is asymptotically $TP(\sigma)$. (A minor error in Ghosh's statement of the result has been corrected here.)

9.28. A great deal of work has been done on the 'zero-density estimates' of §§9.15–19, using an idea which originates with Halász [1]. However it is not possible to combine this with the method of §9.16, based on Littlewood's formula (9.9.1). Instead one argues as follows (Montgomery [1; Chapter 12]). Let

$$M_X(s)\zeta(s) = \sum_1^\infty a_n n^{-s},$$

so that $a_n = 0$ for $2 \leqslant n \leqslant X$. If $\zeta(\rho) = 0$, where $\rho = \beta + i\gamma$ and $\beta > \frac{1}{2}$, then we have

$$e^{-1/Y} + \sum_{n > X} a_n n^{-\rho} e^{-n/Y} = \sum_{n=1}^\infty a_n n^{-\rho} e^{-n/Y}$$

$$= \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} M_X(s+\rho)\zeta(s+\rho)\Gamma(s)\, Y^s\, ds,$$

by the lemma of §7.9. On moving the line of integration to $\mathbf{R}(s) = \frac{1}{2} - \beta$ this yields

$$M_X(1)\Gamma(1-\rho)\, Y^{1-\rho} +$$

$$+ \frac{1}{2\pi i} \int_{-\infty}^{\infty} M_X(\tfrac{1}{2}+it)\zeta(\tfrac{1}{2}+it)\Gamma(\tfrac{1}{2}-\beta+i(t-\gamma))\, Y^{\frac{1}{2}-\beta+i(t-\gamma)}\, dt,$$

since the pole of $\Gamma(s)$ at $s = 0$ is cancelled by the zero of $\zeta(s+\rho)$. If we now assume that $\log^2 T \leqslant \gamma \leqslant T$, and that $\log T \ll \log X$, $\log Y \ll \log T$,

then $e^{-1/Y} \gg 1$ and

$$M_X(1)\Gamma(1-\rho)\, Y^{1-\rho} = o(1),$$

whence either

$$\left| \sum_{n > X} a_n n^{-\rho} e^{-n/Y} \right| \gg 1$$

or

$$\int_{-\infty}^{\infty} |M_X(\tfrac{1}{2}+it)\zeta(\tfrac{1}{2}+it)\Gamma(\tfrac{1}{2}-\beta+i(t-\gamma))|\, dt \gg Y^{\beta-\frac{1}{2}}.$$

In the latter case one has

$$|M_X(\tfrac{1}{2}+it_\rho)\zeta(\tfrac{1}{2}+it_\rho)| \gg (\beta - \tfrac{1}{2})\, Y^{\beta-\frac{1}{2}}$$

for some t_ρ in the range $|t_\rho - \gamma| \leqslant \log^2 T$. The problem therefore reduces to that of counting discrete points at which one of the Dirichlet series $\Sigma a_n n^{-s} e^{-n/Y}$, $M_X(s)$, and $\zeta(s)$ is large. In practice it is more convenient to take finite Dirichlet polynomials approximating to these.

The methods given in §§9.17–19 correspond to the use of a mean-value bound. Thus Montgomery [1; Chapter 7] showed that

$$\sum_{r=1}^R \left| \sum_{n=1}^N a_n n^{-s_r} \right|^2 \ll (T+N)(\log N)^2 \sum_{n=1}^N |a_n|^2 n^{-2\sigma} \qquad (9.28.1)$$

for any points s_r satisfying

$$\mathbf{R}(s_r) \geqslant \sigma, \qquad |\mathbf{I}(s_r)| \leqslant T, \qquad \mathbf{I}(s_{r+1} - s_r) \geqslant 1, \qquad (9.28.2)$$

and any complex a_n. Theorems 9.17, 9.18, 9.19(A), and 9.19(B) may all be recovered from this (except possibly for worse powers of $\log T$). However one may also use Halász's lemma. One simple form of this (Montgomery [1; Theorem 8.2]) gives

$$\sum_{r=1}^R \left| \sum_{n=1}^N a_n n^{-s_r} \right|^2 \ll (N + RT^{\frac{1}{2}})(\log T) \sum_{n=1}^N |a_n|^2 n^{-2\sigma} \quad (9.28.3)$$

for any points s_r satisfying (9.28.2). Under suitable circumstances this implies a sharper bound for R than does (9.28.1). Under the Lindelöf hypothesis one may replace the term $RT^{\frac{1}{2}}$ in (9.28.3) by $RT^\epsilon N^{\frac{1}{2}}$, which is superior, since one invariably takes $N \leqslant T$ in applying the Halász lemma. (If $N \geqslant T$ it would be better to use (9.28.1).) Moreover Montgomery [1; Chapter 9] makes the conjecture (the Large Values

Conjecture):

$$\sum_{r=1}^{R}\left|\sum_{n=1}^{N} a_n n^{-s_r}\right|^2 \ll (N+RT^{\varepsilon})\sum_{n=1}^{N}|a_n|^2 n^{-2\sigma}$$

for points s_r satisfying (9.28.2). Using the Halász lemma with the Lindelöf hypothesis one obtains

$$N(\sigma, T) \ll T^{\varepsilon}, \qquad \tfrac{3}{4}+\varepsilon \leqslant \sigma \leqslant 1, \tag{9.28.4}$$

(Halász and Turán [1], Montgomery [1; Theorem 12.3]). If the Large Values Conjecture is true then the Lindelöf hypothesis gives the wider range $\frac{1}{2}+\varepsilon \leqslant \sigma \leqslant 1$ for (9.28.4)

9.29. The picture for unconditional estimates is more complex. At present it seems that the Halász method is only useful for $\sigma \geqslant \frac{3}{4}$. Thus Ingham's result, Theorem 9.19(B), is still the best known for $\frac{1}{2} < \sigma \leqslant \frac{3}{4}$. Using (9.28.3), Montgomery [1; Theorem 12.1] showed that

$$N(\sigma, T) \ll T^{2(1-\sigma)/\sigma}(\log T)^{14} \qquad (\tfrac{4}{5} \leqslant \sigma \leqslant 1),$$

which is superior to Theorem 9.19(B). This was improved by Huxley [1] to give

$$N(\sigma, T) \ll T^{3(1-\sigma)/(3\sigma-1)}(\log T)^{44} \quad (\tfrac{3}{4} \leqslant \sigma \leqslant 1). \tag{9.29.1}$$

Huxley used the Halász lemma in the form

$$R \ll \left\{NV^{-2}\sum_{n=1}^{N}|a_n|^2 n^{-2\sigma} + TNV^{-6}\left(\sum_{n=1}^{N}|a_n|^2 n^{-2\sigma}\right)^3\right\}(\log T)^2,$$

for points s_r satisfying (9.28.2) and the condition

$$\left|\sum_{n=1}^{N} a_n n^{-s_r}\right| \geqslant V.$$

In conjunction with Theorem 9.19(B), Huxley's result yields

$$N(\sigma, T) \ll T^{12/5(1-\sigma)}(\log T)^{44} \quad (\tfrac{1}{2} \leqslant \sigma \leqslant 1),$$

(c.f. (9.18.3)). A considerable number of other estimates have been given, for which the interested reader is referred to Ivic [3; Chapter 11]. We mention only a few of the most significant. Ivic [2] showed that

$$N(\sigma, T) \ll \begin{cases} T^{(3-3\sigma)/(7\sigma-4)+\varepsilon} & (\frac{3}{4} \leqslant \sigma \leqslant \frac{10}{13}) \\ T^{(9-9\sigma)/(8\sigma-2)+\varepsilon} & (\frac{10}{13} \leqslant \sigma \leqslant 1), \end{cases}$$

which supersede Huxley's result (9.29.1) throughout the range $\frac{3}{4} < \sigma < 1$. Jutila [1] gave a more powerful, but more complicated, result, which has a similar effect. His bounds also imply the 'Density hypothesis' $N(\sigma, T) \ll T^{2-2\sigma+\varepsilon}$, for $\frac{11}{14} \leqslant \sigma \leqslant 1$. Heath-Brown [6] improved this by giving

$$N(\sigma, T) \ll T^{(9-9\sigma)/(7\sigma-1)+\varepsilon} \quad (\tfrac{11}{14} \leqslant \sigma \leqslant 1).$$

When σ is very close to 1 one can use the Vinogradov–Korobov exponential sum estimates, as described in Chapter 6. These lead to

$$N(\sigma, T) \ll T^{A(1-\sigma)^{3/2}}(\log T)^{A'},$$

for suitable numerical constants A and A', (see Montgomery [1; Corollary 12.5], who gives $A = 1334$, after correction of a numerical error).

Selberg's estimate given in Theorem 9.19(C) has been improved by Jutila [2] to give

$$N(\sigma, T) \ll T^{1-(1-\delta)(\sigma-\frac{1}{2})}\log T$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 1$, for any fixed $\delta > 0$.

9.30. Of course Theorem 19.24 is an immediate consequence of Theorem 19.9(C), but the proof is a little easier. The coefficients δ_r used in §9.24 are essentially

$$\mu(r)r^{-1}\frac{\log X/r}{\log X},$$

and indeed a more careful analysis yields

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^2\left|\sum_{r \leqslant X}\mu(r)\frac{\log X/r}{\log X}r^{-\frac{1}{2}-it}\right|^2 dt \sim T\left(1+\frac{\log T}{\log X}\right).$$

Here one can take $X \leqslant T^{\frac{1}{2}-\varepsilon}$ using fairly standard techniques, or $X \leqslant T^{\frac{9}{17}-\varepsilon}$ by employing estimates for Kloosterman sums (see Balasubramanian, Conrey and Heath-Brown [1]). The latter result yields (9.24.1) with the implied constant 0·0845.

X

THE ZEROS ON THE CRITICAL LINE

10.1. General discussion. The memoir in which Riemann first considered the zeta-function has become famous for the number of ideas it contains which have since proved fruitful, and it is by no means certain that these are even now exhausted. The analysis which precedes his observations on the zeros is particularly interesting. He obtains, as in § 2.6, the formula

$$\Gamma(\tfrac{1}{2}s)\pi^{-\frac{1}{2}s}\zeta(s) = \frac{1}{s(s-1)}+\int\limits_1^\infty \psi(x)(x^{\frac{1}{2}s-1}+x^{-\frac{1}{2}-\frac{1}{2}s})\,dx,$$

where $$\psi(x) = \sum_{n=1}^\infty e^{-n^2\pi x}.$$

Multiplying by $\frac{1}{2}s(s-1)$, and putting $s = \frac{1}{2}+it$, we obtain

$$\Xi(t) = \tfrac{1}{2}-(t^2+\tfrac{1}{4})\int\limits_1^\infty \psi(x)x^{-\frac{3}{4}}\cos(\tfrac{1}{2}t\log x)\,dx. \tag{10.1.1}$$

Integrating by parts, and using the relation

$$4\psi'(1)+\psi(1) = -\tfrac{1}{2},$$

which follows at once from (2.6.3), we obtain

$$\Xi(t) = 4\int\limits_1^\infty \frac{d}{dx}\{x^{\frac{3}{2}}\psi'(x)\}x^{-\frac{1}{4}}\cos(\tfrac{1}{2}t\log x)\,dx. \tag{10.1.2}$$

Riemann then observes:

'Diese Function ist für alle endlichen Werthe von t endlich, und lässt sich nach Potenzen von tt in eine sehr schnell convergirende Reihe entwickeln. Da für einen Werth von s, dessen reeller Bestandtheil grösser als 1 ist, $\log\zeta(s) = -\sum\log(1-p^{-s})$ endlich bleibt, und von den Logarithmen der übrigen Factoren von $\Xi(t)$ dasselbe gilt, so kann die Function $\Xi(t)$ nur verschwinden, wenn der imaginäre Theil von t zwischen $\frac{1}{2}i$ und $-\frac{1}{2}i$ liegt. Die Anzahl der Wurzeln von $\Xi(t) = 0$, deren reeller Theil zwischen 0 und T liegt, ist etwa

$$= \frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi};$$

denn dass Integral $\int d\log\Xi(t)$ positive um den Inbegriff der Werthe von t erstreckt, deren imaginäre Theil zwischen $\frac{1}{2}i$ und $-\frac{1}{2}i$, und deren reeller Theil zwischen 0 und T liegt, ist (bis auf einen Bruchtheil von der Ordnung der Grösse $1/T$) gleich $\{T\log(T/2\pi)-T\}i$; dieses Integral aber ist gleich der Anzahl der in diesem Gebiet liegenden Wurzeln von $\Xi(t) = 0$, multiplicirt mit $2\pi i$. Man findet nun in der That etwa so viel reelle Wurzeln innerhalb dieser Grenzen, und es ist sehr wahrscheinlich, dass alle Wurzeln reelle sind.'

This statement, that all the zeros of $\Xi(t)$ are real, is the famous 'Riemann hypothesis', which remains unproved to this day. The memoir goes on:

'Hiervon wäre allerdings ein strenger Beweis zu wünschen; ich habe indess die Aufsuchung desselben nach einigen flüchtigen vergeblichen Versuchen vorläufig bei Seite gelassen, da er für den nächsten Zweck meiner Untersuchung [i.e. the explicit formula for $\pi(x)$] entbehrlich schien.'

In the approximate formula for $N(T)$, Riemann's $1/T$ may be a mistake for $\log T$; for, since $N(T)$ has an infinity of discontinuities at least equal to 1, the remainder cannot tend to zero. With this correction, Riemann's first statement is Theorem 9.4, which was proved by von Mangoldt many years later.

Riemann's second statement, on the real zeros of $\Xi(t)$, is more obscure, and his exact meaning cannot now be known. It is, however, possible that anyone encountering the subject for the first time might argue as follows. We can write (10.1.2) in the form

$$\Xi(t) = 2\int\limits_0^\infty \Phi(u)\cos ut\,du, \tag{10.1.3}$$

where $$\Phi(u) = 2\sum_{n=1}^\infty (2n^4\pi^2e^{\frac{9}{2}u}-3n^2\pi e^{\frac{5}{2}u})e^{-n^2\pi e^{2u}}. \tag{10.1.4}$$

This series converges very rapidly, and one might suppose that an approximation to the truth could be obtained by replacing it by its first term; or perhaps better by

$$\Phi^*(u) = 2\pi^2\cosh\tfrac{9}{2}u\,e^{-2\pi\cosh 2u},$$

since this, like $\Phi(u)$, is an even function of u, which is asymptotically equivalent to $\Phi(u)$. We should thus replace $\Xi(t)$ by

$$\Xi^*(t) = 4\pi^2\int\limits_0^\infty \cosh\tfrac{9}{2}u\,e^{-2\pi\cosh 2u}\cos ut\,du.$$

The asymptotic behaviour of $\Xi^*(t)$ can be found by the method of steepest descents. To avoid the calculation we shall quote known Bessel-function formulae. We have†

$$K_z(a) = \int\limits_0^\infty e^{-a\cosh u}\cosh zu\,du,$$

and hence $$\Xi^*(t) = \pi^2\{K_{\frac{9}{4}+\frac{1}{2}u}(2\pi)+K_{\frac{9}{4}-\frac{1}{2}u}(2\pi)\}.$$

For fixed z, as $\nu\to\infty$

$$I_\nu(z) \sim (\tfrac{1}{2}z)^\nu/\Gamma(\nu+1).$$

† Watson, *Theory of Bessel Functions*, 6.22 (5).

Hence

$$I_{-\frac{9}{4}-\frac{1}{2}it}(2\pi) \sim \frac{\pi^{-\frac{9}{4}-\frac{1}{2}it}}{\Gamma(-\frac{5}{4}-\frac{1}{2}it)} \sim \frac{1}{\pi\sqrt{2}} e^{\frac{1}{4}\pi t}\left(\frac{t}{2\pi}\right)^{\frac{7}{4}}\left(\frac{t}{2\pi e}\right)^{\frac{1}{2}it} e^{-\frac{7}{8}i\pi},$$

$$I_{\frac{9}{4}+\frac{1}{2}it}(2\pi) \sim \frac{\pi^{\frac{9}{4}+\frac{1}{2}it}}{\Gamma(\frac{13}{4}+\frac{1}{2}it)} = O(e^{\frac{1}{4}\pi t}t^{-\frac{11}{4}}),$$

$$K_{\frac{9}{4}+\frac{1}{2}it}(2\pi) = \tfrac{1}{2}\pi \operatorname{cosec} \pi(\tfrac{9}{4}+\tfrac{1}{2}it)\{I_{-\frac{9}{4}-\frac{1}{2}it}(2\pi)-I_{\frac{9}{4}+\frac{1}{2}it}(2\pi)\}$$

$$\sim \frac{1}{\sqrt{2}} e^{-\frac{1}{4}\pi t}\left(\frac{t}{2\pi}\right)^{\frac{7}{4}}\left(\frac{t}{2\pi e}\right)^{\frac{1}{2}it} e^{\frac{7}{8}i\pi}.$$

Hence $$\Xi^*(t) \sim \pi^{\frac{1}{4}}2^{-\frac{5}{4}}t^{\frac{7}{4}}e^{-\frac{1}{4}\pi t}\cos\left(\tfrac{1}{2}t\log\frac{t}{2\pi e}+\tfrac{7}{8}\pi\right).$$

The right-hand side has zeros at

$$\tfrac{1}{2}t\log\frac{t}{2\pi e}+\tfrac{7}{8}\pi = (n+\tfrac{1}{2})\pi,$$

and the number of these in the interval $(0, T)$ is

$$\frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi}+O(1).$$

The similarity to the formula for $N(T)$ is indeed striking.

However, if we try to work on this suggestion, difficulties at once appear. We can write

$$\Xi(t)-\Xi^*(t) = \int_{-\infty}^{\infty} \{\Phi(u)-\Phi^*(u)\}e^{iut}\,du.$$

To show that this is small compared with $\Xi(t)$ we should want to move the line of integration into the upper half-plane, at least as far as $\mathbf{I}(u) = \frac{1}{4}\pi$; and this is just where the series for $\Phi(u)$ ceases to converge. Actually

$$|\Xi(t)| > At^{\frac{7}{4}}e^{-\frac{1}{4}\pi t}|\zeta(\tfrac{1}{2}+it)|,$$

and $|\zeta(\frac{1}{2}+it)|$ is unbounded, so that the suggestion that $\Xi^*(t)$ is an approximation to $\Xi(t)$ is false, at any rate if it is taken in the most obvious sense.

10.2. Although every attempt to prove the Riemann hypothesis, that all the complex zeros of $\zeta(s)$ lie on $\sigma = \frac{1}{2}$, has failed, it is known *that $\zeta(s)$ has an infinity of zeros on $\sigma = \frac{1}{2}$*. This was first proved by Hardy in 1914. We shall give here a number of different proofs of this theorem.

First method.† We have

$$\Xi(t) = -\tfrac{1}{2}(t^2+\tfrac{1}{4})\pi^{-\frac{1}{4}-\frac{1}{2}it}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}it)\zeta(\tfrac{1}{2}+it),$$

where $\Xi(t)$ is an even integral function of t, and is real for real t. A zero of $\zeta(s)$ on $\sigma = \frac{1}{2}$ therefore corresponds to a real zero of $\Xi(t)$, and it is a question of proving that $\Xi(t)$ has an infinity of real zeros.

† Hardy (1).

Putting $x = -i\alpha$ in (2.16.2), we have

$$\frac{2}{\pi}\int_0^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}}\cosh\alpha t\,dt = e^{-\frac{1}{2}i\alpha}-2e^{\frac{3}{2}i\alpha}\psi(e^{2i\alpha})$$
$$= 2\cos\tfrac{1}{2}\alpha-2e^{\frac{3}{2}i\alpha}\{\tfrac{1}{2}+\psi(e^{2i\alpha})\}. \tag{10.2.1}$$

Since $\zeta(\frac{1}{2}+it) = O(t^A)$, $\Xi(t) = O(t^Ae^{-\frac{1}{4}\pi t})$, and the above integral may be differentiated with respect to α any number of times provided that $\alpha < \frac{1}{4}\pi$. Thus

$$\frac{2}{\pi}\int_0^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}}t^{2n}\cosh\alpha t\,dt = \frac{(-1)^n\cos\frac{1}{2}\alpha}{2^{2n-1}}-2\left(\frac{d}{d\alpha}\right)^{2n}e^{\frac{3}{2}i\alpha}\{\tfrac{1}{2}+\psi(e^{2i\alpha})\}.$$

We next prove that the last term tends to 0 as $\alpha \to \frac{1}{4}\pi$, for every fixed n. The equation (2.6.3) gives at once the functional equation

$$x^{-\frac{1}{4}}-2x^{\frac{1}{4}}\psi(x) = x^{\frac{1}{4}}-2x^{-\frac{1}{4}}\psi\left(\frac{1}{x}\right),$$

or $$\psi(x) = x^{-\frac{1}{2}}\psi\left(\frac{1}{x}\right)+\tfrac{1}{2}x^{-\frac{1}{2}}-\tfrac{1}{2}.$$

Hence
$$\psi(i+\delta) = \sum_1^\infty e^{-n^2\pi(i+\delta)} = \sum_1^\infty (-1)^n e^{-n^2\pi\delta}$$
$$= 2\psi(4\delta)-\psi(\delta)$$
$$= \frac{1}{\sqrt{\delta}}\psi\left(\frac{1}{4\delta}\right)-\frac{1}{\sqrt{\delta}}\psi\left(\frac{1}{\delta}\right)-\frac{1}{2}.$$

It is easily seen from this that $\frac{1}{2}+\psi(x)$ and all its derivatives tend to zero as $x \to i$ along any route in an angle $|\arg(x-i)| < \frac{1}{2}\pi$.

We have thus proved that

$$\lim_{\alpha\to\frac{1}{4}\pi}\int_0^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}}t^{2n}\cosh\alpha t\,dt = \frac{(-1)^n\pi\cos\frac{1}{8}\pi}{2^{2n}}. \tag{10.2.2}$$

Suppose now that $\Xi(t)$ were ultimately of one sign, say, for example, positive for $t \geqslant T$. Then

$$\lim_{\alpha\to\frac{1}{4}\pi}\int_T^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}}t^{2n}\cosh\alpha t\,dt = L,$$

say. Hence $$\int_T^{T'} \frac{\Xi(t)}{t^2+\frac{1}{4}}t^{2n}\cosh\alpha t\,dt \leqslant L$$

for all $\alpha < \frac{1}{4}\pi$ and $T' > T$. Hence, making $\alpha \to \frac{1}{4}\pi$,

$$\int_{T}^{T'} \frac{\Xi(t)}{t^2+\frac{1}{4}} t^{2n} \cosh \tfrac{1}{4}\pi t \, dt \leqslant L.$$

Hence the integral $$\int_{0}^{\infty} \frac{\Xi(t)}{t^2+\frac{1}{4}} t^{2n} \cosh \tfrac{1}{4}\pi t \, dt$$

is convergent. The integral on the left of (10.2.2) is therefore uniformly convergent with respect to α for $0 \leqslant \alpha \leqslant \frac{1}{4}\pi$, and it follows that

$$\int_{0}^{\infty} \frac{\Xi(t)}{t^2+\frac{1}{4}} t^{2n} \cosh \tfrac{1}{4}\pi t \, dt = \frac{(-1)^n \pi \cos \frac{1}{8}\pi}{2^{2n}}$$

for every n.

This, however, is impossible; for, taking n odd, the right-hand side is negative, and hence

$$\int_{T}^{\infty} \frac{\Xi(t)}{t^2+\frac{1}{4}} t^{2n} \cosh \tfrac{1}{4}\pi t \, dt < -\int_{0}^{T} \frac{\Xi(t)}{t^2+\frac{1}{4}} t^{2n} \cosh \tfrac{1}{4}\pi t \, dt$$
$$< KT^{2n},$$

where K is independent of n. But by hypothesis there is a positive $m = m(T)$ such that $\Xi(t)/(t^2+\frac{1}{4}) \geqslant m$ for $2T \leqslant t \leqslant 2T+1$. Hence

$$\int_{T}^{\infty} \frac{\Xi(t)}{t^2+\frac{1}{4}} t^{2n} \cosh \tfrac{1}{4}\pi t \, dt \geqslant \int_{2T}^{2T+1} m t^{2n} \, dt \geqslant m(2T)^{2n}.$$

Hence $$m 2^{2n} < K,$$

which is false for sufficiently large n. This proves the theorem.

10.3. A variant of the above proof depends on the following theorem of Fejér:†

Let n be any positive integer. Then the number of changes in sign in the interval $(0, a)$ of a continuous function $f(x)$ is not less than the number of changes in sign of the sequence

$$f(0), \quad \int_{0}^{a} f(t)\, dt, \quad \ldots, \quad \int_{0}^{a} f(t) t^n \, dt. \qquad (10.3.1)$$

We deduce this from the following theorem of Fekete:‡

† Fejér (1). ‡ Fekete (1).

The number of changes in sign in the interval $(0, a)$ of a continuous function $f(x)$ is not less than the number of changes in sign of the sequence

$$f(a), \quad f_1(a), \quad \ldots, \quad f_n(a), \qquad (10.3.2)$$

where

$$f_\nu(x) = \int_{0}^{x} f_{\nu-1}(t)\, dt \qquad (\nu = 1, 2, \ldots, n), \qquad f_0(x) = f(x).$$

To prove Fekete's theorem, suppose first that $n = 1$. Consider the curve $y = f_1(x)$. Now $f_1(0) = 0$, and, if $f(a)$ and $f_1(a)$ have opposite signs, y is positive decreasing or negative increasing at $x = a$. Hence $f(x)$ has at least one zero.

Now assume the theorem for $n-1$. Suppose that there are k changes of sign in the sequence $f_1(x), \ldots, f_n(x)$. Then $f_1(x)$ has at least k changes of sign. We have then to prove that

(i) if $f(a)$ and $f_1(a)$ have the same sign, $f(x)$ has at least k changes of sign,

(ii) if $f(a)$ and $f_1(a)$ have opposite signs, $f(x)$ has at least $k+1$ changes of sign.

Each of these cases is easily verified by considering the curve $y = f_1(x)$. This proves Fekete's theorem.

To deduce Fejér's theorem, we have

$$f_\nu(x) = \frac{1}{(\nu-1)!} \int_{0}^{x} (x-t)^{\nu-1} f(t)\, dt,$$

and hence

$$f_\nu(a) = \frac{1}{(\nu-1)!} \int_{0}^{a} (a-t)^{\nu-1} f(t)\, dt = \frac{1}{(\nu-1)!} \int_{0}^{a} f(a-t) t^{\nu-1}\, dt.$$

We may therefore replace the sequence (10.3.2) by the sequence

$$f(a), \quad \int_{0}^{a} f(a-t)\, dt, \quad \ldots, \quad \int_{0}^{a} f(a-t) t^{n-1}\, dt. \qquad (10.3.3)$$

Since the number of changes of sign of $f(t)$ is the same as the number of changes of sign of $f(a-t)$, we can replace $f(t)$ by $f(a-t)$. This proves Fejér's theorem.

To prove that there are an infinity of zeros of $\zeta(s)$ on the critical line, we prove as before that

$$\lim_{\alpha \to \frac{1}{4}\pi} \int_{0}^{\infty} \frac{\Xi(t)}{t^2+\frac{1}{4}} t^{2n} \cosh \alpha t \, dt = \frac{(-1)^n \pi \cos \frac{1}{8}\pi}{2^{2n}}.$$

Hence $$\int_{0}^{a} \frac{\Xi(t)}{t^2+\frac{1}{4}} t^{2n} \cosh \alpha t \, dt$$

has the same sign as $(-1)^n$ for $n = 0, 1,..., N$, if $a = a(N)$ is large enough and $\alpha = \alpha(N)$ is near enough to $\frac{1}{4}\pi$. Hence $\Xi(t)$ has at least N changes of sign in $(0, a)$, and the result follows.†

10.4. Another method‡ is based on Riemann's formula (10.1.2). Putting $x = e^{2u}$ in (10.1.2), we have

$$\Xi(t) = 4\int\limits_0^\infty \frac{d}{du}\{e^{3u}\psi'(e^{2u})\}e^{-\frac{1}{2}u}\cos ut\,du$$

$$= 2\int\limits_0^\infty \Phi(u)\cos ut\,du,$$

say. Then, by Fourier's integral theorem,

$$\Phi(u) = \frac{1}{\pi}\int\limits_0^\infty \Xi(t)\cos ut\,dt,$$

and hence also

$$\Phi^{(2n)}(u) = \frac{(-1)^n}{\pi}\int\limits_0^\infty \Xi(t)t^{2n}\cos ut\,dt.$$

Since $\psi(x)$ is regular for $\mathbf{R}(x) > 0$, $\Phi(u)$ is regular for $-\frac{1}{4}\pi < \mathbf{I}(u) < \frac{1}{4}\pi$. Let

$$\Phi(iu) = c_0 + c_1u^2 + c_2u^4 + ... \quad (|u| < \tfrac{1}{4}\pi).$$

Then
$$(2n)!\,c_n = (-1)^n\Phi^{(2n)}(0) = \frac{1}{\pi}\int\limits_0^\infty \Xi(t)t^{2n}\,dt.$$

Suppose now that $\Xi(t)$ is of one sign, say $\Xi(t) > 0$, for $t > T$. Then $c_n > 0$ for $n > n_0$, since

$$\int\limits_0^\infty \Xi(t)t^{2n}\,dt > \int\limits_{T+1}^{T+2} \Xi(t)t^{2n}\,dt - \int\limits_0^T |\Xi(t)|t^{2n}\,dt$$

$$> (T+1)^{2n}\int\limits_{T+1}^{T+2} \Xi(t)\,dt - T^{2n}\int\limits_0^T |\Xi(t)|\,dt.$$

It follows that $\Phi^{(n)}(iu)$ increases steadily with u if $n > 2n_0$. But in fact $\Phi(u)$ and all its derivatives tend to 0 as $u \to \frac{1}{4}i\pi$ along the imaginary axis, by the properties of $\psi(x)$ obtained in § 10.2. The theorem therefore follows again.

10.5. The above proofs of Hardy's theorem are all similar in that they depend on the consideration of 'moments' $\int f(t)t^n\,dt$. The following

† Fekete (2). ‡ Pólya (3).

method† depends on a contrast between the asymptotic behaviour of the integrals

$$\int\limits_T^{2T} Z(t)\,dt, \qquad \int\limits_T^{2T} |Z(t)|\,dt,$$

where $Z(t)$ is the function defined in § 4.17. If $Z(t)$ were ultimately of one sign, these integrals would be ultimately equal, apart possibly from sign. But we shall see that in fact they behave quite differently.

Consider the integral

$$\int \{\chi(s)\}^{-\frac{1}{2}}\zeta(s)\,ds,$$

where the integrand is the function which reduces to $Z(t)$ on $\sigma = \frac{1}{2}$, taken round the rectangle with sides $\sigma = \frac{1}{2}$, $\sigma = \frac{5}{4}$, $t = T$, $t = 2T$. This integral is zero, by Cauchy's theorem. Now

$$\int\limits_{\frac{1}{2}+iT}^{\frac{1}{2}+2iT} \{\chi(s)\}^{-\frac{1}{2}}\zeta(s)\,ds = i\int\limits_T^{2T} Z(t)\,dt.$$

Also by (4.12.3)

$$\{\chi(s)\}^{-\frac{1}{2}} = \left(\frac{t}{2\pi}\right)^{\frac{1}{2}\sigma-\frac{1}{4}+\frac{1}{2}it} e^{-\frac{1}{2}it-\frac{1}{8}i\pi}\left\{1+O\left(\frac{1}{t}\right)\right\}.$$

Hence, by (5.1.2) and (5.1.4),

$$\{\chi(s)\}^{-\frac{1}{2}}\zeta(s) = O(t^{\frac{1}{2}\sigma-\frac{1}{4}}.t^{\frac{1}{2}-\frac{1}{2}\sigma+\epsilon}) = O(t^{\frac{1}{4}+\epsilon}) \quad (\tfrac{1}{2} \leqslant \sigma \leqslant 1),$$

$$= O(t^{\frac{1}{2}\sigma-\frac{1}{4}+\epsilon}) = O(t^{\frac{3}{8}+\epsilon}) \quad (1 < \sigma \leqslant \tfrac{5}{4}).$$

The integrals along the sides $t = T$, $t = 2T$ are therefore $O(T^{\frac{3}{8}+\epsilon})$.

The integral along the right-hand side is

$$\int\limits_T^{2T} \left(\frac{t}{2\pi}\right)^{\frac{3}{8}+\frac{1}{2}it} e^{-\frac{1}{2}it-\frac{1}{8}i\pi}\left\{1+O\left(\frac{1}{t}\right)\right\}\zeta(\tfrac{5}{4}+it)i\,dt.$$

The contribution of the O-term is

$$\int\limits_T^{2T} O(t^{-\frac{5}{8}})\,dt = O(T^{\frac{3}{8}}).$$

The other term is a constant multiple of

$$\sum_{n=1}^\infty n^{-\frac{5}{4}}\int\limits_T^{2T} \left(\frac{t}{2\pi}\right)^{\frac{3}{8}+\frac{1}{2}it} e^{-\frac{1}{2}it-it\log n}\,dt.$$

Now
$$\frac{d^2}{dt^2}\left(\tfrac{1}{2}t\log\frac{t}{2\pi}-\tfrac{1}{2}t-t\log n\right) = \frac{1}{2t}.$$

Hence, by Lemma 4.5, the integral in the above sum is $O(T^{\frac{7}{8}})$, uniformly with respect to n, so that the whole sum is also $O(T^{\frac{7}{8}})$.

† See Landau, *Vorlesungen*, ii. 78–85.

Combining all these results, we obtain

$$\int_T^{2T} Z(t)\,dt = O(T^{\frac{1}{4}}). \tag{10.5.1}$$

On the other hand,

$$\int_T^{2T} |Z(t)|\,dt = \int_T^{2T} |\zeta(\tfrac{1}{2}+it)|\,dt \geqslant \left|\int_T^{2T} \zeta(\tfrac{1}{2}+it)\,dt\right|.$$

But

$$i\int_T^{2T} \zeta(\tfrac{1}{2}+it)\,dt = \int_{\frac{1}{2}+iT}^{\frac{1}{2}+2iT} \zeta(s)\,ds = \int_{\frac{1}{2}+iT}^{2+iT} + \int_{2+iT}^{2+2iT} + \int_{2+2iT}^{\frac{1}{2}+2iT}$$

$$= \left[s - \sum_{n=2}^{\infty} \frac{1}{n^s \log n}\right]_{2+iT}^{2+2iT} + \int_{\frac{1}{2}}^{2} O(T^{\frac{1}{2}})\,d\sigma = iT + O(T^{\frac{1}{2}}).$$

Hence

$$\int_T^{2T} |Z(t)|\,dt > AT. \tag{10.5.2}$$

Hardy's theorem now follows from (10.5.1) and (10.5.2).

Another variant of this method is obtained by starting again from (10.2.1). Putting $\alpha = \frac{1}{4}\pi - \delta$, we obtain

$$\int_0^{\infty} \frac{\Xi(t)}{t^2+\frac{1}{4}} \cosh\{(\tfrac{1}{4}\pi-\delta)t\}\,dt = O(1) + O\Big\{\sum_{n=1}^{\infty} \exp(-n^2\pi i e^{-2i\delta})\Big\}$$

$$= O(1) + O\Big(\sum_{n=1}^{\infty} e^{-n^2\pi \sin 2\delta}\Big) = O(1) + O\Big(\int_0^{\infty} e^{-x^2\pi \sin 2\delta}\,dx\Big) = O(\delta^{-\frac{1}{2}})$$

as $\delta \to 0$. If, for example, $\Xi(t) > 0$ for $t > t_0$, it follows that for $T > t_0$

$$\int_T^{2T} |Z(t)|\,dt = \left|\int_T^{2T} Z(t)\,dt\right| < A\int_T^{2T} \frac{\Xi(t)}{t^2+\frac{1}{4}} t^{\frac{1}{4}} e^{\frac{1}{4}\pi t}\,dt$$

$$< AT^{\frac{1}{4}} \int_T^{2T} \frac{\Xi(t)}{t^2+\frac{1}{4}} e^{\frac{1}{4}\pi t - \frac{1}{2}t/T}\,dt < AT^{\frac{1}{4}} \int_{t_0}^{\infty} \frac{\Xi(t)}{t^2+\frac{1}{4}} \cosh\left\{\left(\frac{1}{4}\pi - \frac{1}{2T}\right)t\right\} dt$$

$$= O(T^{\frac{1}{4}}.T^{\frac{1}{2}}) = O(T^{\frac{3}{4}}).$$

This is inconsistent with (10.5.2), so that the theorem again follows.

10.6. Still another method† depends on the formula (4.17.4), viz.

$$Z(t) = 2\sum_{n \leqslant x} \frac{\cos(\vartheta - t\log n)}{\sqrt{n}} + O(t^{-\frac{1}{4}}),$$

† Titchmarsh (11).

where $x = \sqrt{(t/2\pi)}$. Here $\vartheta = \vartheta(t)$ is defined by

$$\chi(\tfrac{1}{2}+it) = e^{-2i\vartheta(t)},$$

so that

$$\vartheta'(t) = -\frac{1}{2}\frac{\chi'(\frac{1}{2}+it)}{\chi(\frac{1}{2}+it)} = -\frac{1}{2}\left\{\log\pi - \frac{1}{2}\frac{\Gamma'(\frac{1}{4}-\frac{1}{2}it)}{\Gamma(\frac{1}{4}-\frac{1}{2}it)} - \frac{1}{2}\frac{\Gamma'(\frac{1}{4}+\frac{1}{2}it)}{\Gamma(\frac{1}{4}+\frac{1}{2}it)}\right\}$$

$$= -\tfrac{1}{2}\log\pi + \tfrac{1}{4}\log(\tfrac{1}{16}+\tfrac{1}{4}t^2) - \frac{1}{1+4t^2} - \mathbf{R}\int_0^{\infty} \frac{u\,du}{\{u^2+(\frac{1}{4}+\frac{1}{2}it)^2\}(e^{2\pi u}-1)},$$

and we have

$$\vartheta'(t) = \tfrac{1}{2}\log t - \tfrac{1}{2}\log 2\pi + O(1/t),$$

$$\vartheta(t) \sim \tfrac{1}{2}t\log t, \qquad \vartheta''(t) \sim \frac{1}{2t}.$$

The function $\vartheta(t)$ is steadily increasing for $t \geqslant t_0$. If ν is any positive integer ($\geqslant \nu_0$), the equation $\vartheta(t) = \nu\pi$ therefore has just one solution, say t_ν, and $t_\nu \sim 2\pi\nu/\log\nu$. Now

$$Z(t_\nu) = 2(-1)^\nu \sum_{n \leqslant x} \frac{\cos(t_\nu \log n)}{\sqrt{n}} + O(t_\nu^{-\frac{1}{4}}).$$

The sum

$$g(t_\nu) = \sum_{n \leqslant x} \frac{\cos(t_\nu \log n)}{\sqrt{n}} = 1 + \frac{\cos(t_\nu \log 2)}{\sqrt{2}} + \ldots$$

consists of the constant term unity and oscillatory terms; and the formula suggests that $g(t_\nu)$ will usually be positive, and hence that $Z(t)$ will usually change sign in the interval $(t_\nu, t_{\nu+1})$.

We shall prove

THEOREM 10.6. *As* $N \to \infty$

$$\sum_{\nu=\nu_0}^{N} Z(t_{2\nu}) \sim 2N, \qquad \sum_{\nu=\nu_0}^{N} Z(t_{2\nu+1}) \sim -2N.$$

It follows at once that $Z(t_{2\nu})$ is positive for an infinity of values of ν, and that $Z(t_{2\nu+1})$ is negative for an infinity of values of ν; and the existence of an infinity of real zeros of $Z(t)$, and so of $\Xi(t)$, again follows.

We have

$$\sum_{\nu=M+1}^{N} g(t_{2\nu}) = \sum_{\nu=M+1}^{N} \sum_{n \leqslant \sqrt{(t_{2\nu}/2\pi)}} \frac{\cos(t_{2\nu}\log n)}{\sqrt{n}}$$

$$= N - M + \sum_{2 \leqslant n \leqslant \sqrt{(t_{2N}/2\pi)}} \frac{1}{\sqrt{n}} \sum_{\tau \leqslant t_{2\nu} \leqslant t_{2N}} \cos(t_{2\nu}\log n),$$

where $\tau = \max(t_{2M+2}, 2\pi n^2)$. The inner sum is of the form

$$\sum \cos\{2\pi\phi(\nu)\},$$

where
$$\phi(\nu) = \frac{t_{2\nu}\log n}{2\pi}.$$

We may define t_ν for all $\nu \geqslant \nu_0$ (not necessarily integral) by $\vartheta(t_\nu) = \nu\pi$. Then
$$\phi'(\nu) = \frac{\log n}{2\pi}\frac{dt_{2\nu}}{d\nu}, \qquad \vartheta'(t_{2\nu})\frac{dt_{2\nu}}{d\nu} = 2\pi,$$

so that
$$\phi'(\nu) = \frac{\log n}{\vartheta'(t_{2\nu})}.$$

Hence $\phi'(\nu)$ is positive and steadily decreasing, and, if ν is large enough,
$$\phi''(\nu) = -2\pi\log n\frac{\vartheta''(t_{2\nu})}{\{\vartheta'(t_\nu)\}^3} \sim -\frac{8\pi\log n}{t_{2\nu}\log^3 t_{2\nu}} < -A\frac{\log n}{t_{2N}\log^3 t_{2N}}.$$

Hence, by Theorem 5.9,
$$\sum_{\tau\leqslant t_{2\nu}\leqslant t_{2N}} \cos(t_{2\nu}\log n) = O\left(t_{2N}\frac{\log^{\frac12} n}{t_{2N}^{\frac12}\log^{\frac32} t_{2N}}\right) + O\left(\frac{t_{2N}^{\frac12}\log^{\frac32} t_{2N}}{\log^{\frac12} n}\right)$$
$$= O(t_{2N}^{\frac12}\log^{\frac32} t_{2N}).$$

Hence
$$\sum_{2\leqslant n\leqslant\sqrt{(t_{2N}/2\pi)}} \frac{1}{\sqrt{n}} \sum_{\tau\leqslant t_{2\nu}\leqslant t_{2N}} \cos(t_{2\nu}\log n) = O(t_{2N}^{\frac34}\log^{\frac32} t_{2N})$$
$$= O(N^{\frac34}\log^{\frac34} N).$$

Hence
$$\sum_{\nu=M+1}^{N} Z(t_{2\nu}) = 2N + O(N^{\frac34}\log^{\frac34} N),$$

and a similar argument applies to the other sum.

10.7. We denote by $N_0(T)$ the number of zeros of $\zeta(s)$ of the form $\frac12+it$ $(0 < t \leqslant T)$. The theorem already proved shows that $N_0(T)$ tends to infinity with T. We can, however, prove much more than this.

THEOREM 10.7.† *$N_0(T) > AT$.*

Any of the above proofs can be put in a more precise form so as to give results in this direction. The most successful method is similar in principle to that of § 10.5, but is more elaborate. We contrast the behaviour of the integrals
$$I = \int_t^{t+H} \Xi(u)\frac{e^{\frac14\pi u}}{u^2+\frac14}e^{-u/T}\,du, \qquad J = \int_t^{t+H} |\Xi(u)|\frac{e^{\frac14\pi u}}{u^2+\frac14}e^{-u/T}\,du,$$

where $T \leqslant t \leqslant 2T$ and $T \to \infty$.

† Hardy and Littlewood (3).

We use the theory of Fourier transforms. Let $F(u), f(y)$ be functions related by the Fourier formulae
$$F(u) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f(y)e^{iyu}\,dy, \qquad f(y) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} F(u)e^{-iyu}\,du.$$

Integrating over $(t, t+H)$, we obtain
$$\int_t^{t+H} F(u)\,du = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f(y)\frac{e^{iyH}-1}{iy}e^{iyt}\,dy,$$

so that
$$\int_t^{t+H} F(u)\,du, \qquad f(y)\frac{e^{iyH}-1}{iy}$$

are Fourier transforms. Hence the Parseval formula gives
$$\int_{-\infty}^{\infty}\left|\int_t^{t+H} F(u)\,du\right|^2 dt = \int_{-\infty}^{\infty} |f(y)|^2\frac{4\sin^2\frac12 Hy}{y^2}\,dy.$$

If $F(u)$ is real, $|f(y)|$ is even, and we have
$$\int_{-\infty}^{\infty}\left|\int_t^{t+H} F(u)\,du\right|^2 dt = 2\int_0^{\infty} |f(y)|^2\frac{4\sin^2\frac12 Hy}{y^2}\,dy$$
$$\leqslant 2H^2\int_0^{1/H} |f(y)|^2\,dy + 8\int_{1/H}^{\infty}\frac{|f(y)|^2}{y^2}\,dy. \tag{10.7.1}$$

Now (2.16.2) may be written
$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Xi(t)}{t^2+\frac14}e^{i\xi t}\,dt = \tfrac12 e^{\frac12\xi} - e^{-\frac12\xi}\psi(e^{-2\xi}).$$

Putting $\xi = -i(\frac14\pi-\frac12\delta)-y$, it is seen that we may take
$$F(t) = \frac{1}{\sqrt{(2\pi)}}\frac{\Xi(t)}{t^2+\frac14}e^{(\frac14\pi-\frac12\delta)t}, \qquad f(y) = \tfrac12 e^{-\frac12 i(\frac14\pi-\frac12\delta)-\frac12 y} -$$
$$- e^{\frac12 i(\frac14\pi-\frac12\delta)+\frac12 y}\psi(e^{i(\frac12\pi-\delta)+2y}).$$

Let $H \geqslant 1$. The contribution of the first term in $f(y)$ to (10.7.1) is clearly $O(H)$. Putting $y = \log x$, $G = e^{1/H}$, we therefore obtain
$$\int_{-\infty}^{\infty}\left|\int_t^{t+H} F(u)\,du\right|^2 dt = O\left\{H^2\int_1^G |\psi(e^{i(\frac12\pi-\delta)}x^2)|^2\,dx\right\} +$$
$$+ O\left\{\int_G^{\infty} |\psi(e^{i(\frac12\pi-\delta)}x^2)|^2\frac{dx}{\log^2 x}\right\} + O(H). \tag{10.7.2}$$

Now

$$|\psi(e^{i(\frac12\pi-\delta)}x^2)|^2 = \left|\sum_{n=1}^{\infty} e^{-n^2\pi x^2(\sin\delta+i\cos\delta)}\right|^2$$

$$= \sum_{n=1}^{\infty} e^{-2n^2\pi x^2\sin\delta} + \sum_{m\neq n}\sum e^{-(m^2+n^2)\pi x^2\sin\delta+i(m^2-n^2)\pi x^2\cos\delta}.$$

As in § 10.5, the first sum is $O(x^{-1}\delta^{-\frac12})$, and its contribution to (10.7.2) is therefore

$$O\left(H^2\int_1^G x^{-1}\delta^{-\frac12}\,dx\right)+O\left(\int_G^\infty \frac{\delta^{-\frac12}\,dx}{x\log^2 x}\right)$$

$$= O\{H^2(G-1)\delta^{-\frac12}\}+O(\delta^{-\frac12}/\log G) = O(H\delta^{-\frac12}).$$

The sum with $m \neq n$ contributes to the second term in (10.7.2) terms of the form

$$\int_G^\infty e^{-(m^2+n^2)\pi x^2\sin\delta+i(m^2-n^2)\pi x^2\cos\delta}\frac{dx}{\log^2 x} = O\left\{\frac{e^{-(m^2+n^2)\pi G^2\sin\delta}}{|m^2-n^2|\,G\log^2 G}\right\}$$

$$= O\left(\frac{H^2e^{-(m^2+n^2)\pi\sin\delta}}{|m^2-n^2|}\right)$$

by Lemma 4.3. Hence the sum is

$$O\left(H^2\sum_{m=2}^{\infty}\sum_{n=1}^{m-1}\frac{e^{-(m^2+n^2)\pi\sin\delta}}{m^2-n^2}\right) = O\left(H^2\sum_{m=2}^{\infty}\frac{e^{-m^2\pi\sin\delta}}{m}\sum_{n=1}^{m-1}\frac{1}{m-n}\right)$$

$$= O\left(H^3\sum_{m=2}^{\infty}\frac{\log m}{m}e^{-m^2\pi\sin\delta}\right) = O\left\{H^2\left(\sum_{m\leqslant 1/\delta}\frac{\log m}{m}+\sum_{m>1/\delta}e^{-m^2\pi\sin\delta}\right)\right\}$$

$$= O\left(H^2\log^2\frac{1}{\delta}\right) = O(H\delta^{-\frac12})$$

for $\delta < \delta_0(H)$. The first integral in (10.7.2) may be dealt with in the same way. Hence

$$\int_{-\infty}^{\infty}\left|\int_t^{t+H} F(u)\,du\right|^2 dt = O(H\delta^{-\frac12}).$$

Taking $\delta = 1/T$ and $T > T_0(H)$, it follows that

$$\int_T^{2T} |I|^2\,dt = O(HT^{\frac12}). \tag{10.7.3}$$

10.8. We next prove that

$$J > (AH+\Psi)T^{-\frac14}, \tag{10.8.1}$$

where

$$\int_T^{2T} |\Psi|^2\,dt = O(T) \quad (0 < H < T). \tag{10.8.2}$$

We have, if $s = \frac12+it$, $T \leqslant t \leqslant 2T$,

$$T^{\frac14}\,|\Xi(t)|\frac{e^{\frac14\pi t}}{t^2+\frac14} > A|\zeta(\tfrac12+it)|.$$

Hence

$$T^{\frac14}J > A\int_t^{t+H}|\zeta(\tfrac12+iu)|\,du > A\left|\int_t^{t+H}\zeta(\tfrac12+iu)\,du\right|$$

$$= A\left|\int_t^{t+H}\left\{\sum_{n<AT}\frac{1}{n^{\frac12+iu}}+O(T^{-\frac12})\right\}du\right|$$

$$= AH+O\left\{\left|\int_t^{t+H}\sum_{2\leqslant n<AT}\frac{1}{n^{\frac12+iu}}\,du\right|\right\}+O(HT^{-\frac12})$$

$$= AH+O\left\{\left|\sum_{2\leqslant n<AT}\left(\frac{1}{n^{\frac12+i(t+H)}\log n}-\frac{1}{n^{\frac12+it}\log n}\right)\right|\right\}+O(HT^{-\frac12}).$$

It is now sufficient to prove that

$$\int_T^{2T}\left|\sum_{2\leqslant n<AT}\frac{1}{n^{\frac12+it}\log n}\right|^2 dt = O(T),$$

and the calculations are similar to those of § 7.3, but with an extra factor $\log m\log n$ in the denominator.

To prove Theorem 10.7, let S be the sub-set of the interval $(T, 2T)$ where $I = J$. Then

$$\int_S |I|\,dt = \int_S J\,dt.$$

Now

$$\int_S |I|\,dt \leqslant \int_T^{2T}|I|\,dt \leqslant \left(T\int_T^{2T}|I|^2\,dt\right)^{\frac12} < AH^{\frac12}T^{\frac34}$$

by (10.7.3); and by (10.8.1) and (10.8.2)

$$\int_S J\,dt > T^{-\frac14}\int_S (AH+\Psi)\,dt$$

$$> AT^{-\frac14}Hm(S)-T^{-\frac14}\int_T^{2T}|\Psi|\,dt$$

$$> AT^{-\frac14}Hm(S)-T^{-\frac14}\left(T\int_T^{2T}|\Psi|^2\,dt\right)^{\frac12}$$

$$> AT^{-\frac14}Hm(S)-AT^{\frac34},$$

where $m(S)$ is the measure of S. Hence, for $H \geqslant 1$ and $T > T_0(H)$,

$$m(S) < ATH^{-\frac12}.$$

Now divide the interval $(T, 2T)$ into $[T/2H]$ pairs of abutting intervals j_1, j_2, each, except the last j_2, of length H, and each j_2 lying to the right of the corresponding j_1. Then either j_1 or j_2 contains a zero of $\Xi(t)$ unless j_1 consists entirely of points of S. Suppose that the latter occurs for ν j_1's. Then

$$\nu H \leqslant m(S) < ATH^{-\frac{1}{2}}.$$

Hence there are, in $(T, 2T)$, at least

$$[T/2H]-\nu > \frac{T}{H}\left(\frac{1}{3}-\frac{A}{\sqrt{H}}\right) > \frac{T}{4H}$$

zeros if H is large enough. This proves the theorem.

10.9. For many years the above theorem of Hardy and Littlewood, that $N_0(T) > AT$, was the best that was known in this direction. Recently it has been proved by A. Selberg (2) that $N_0(T) > AT \log T$. This is a remarkable improvement, since it shows that a finite proportion of the zeros of $\zeta(s)$ lie on the critical line. On the Riemann hypothesis, of course,

$$N_0(T) = N(T) \sim \frac{1}{2\pi} T \log T.$$

The numerical value of the constant A in Selberg's theorem is very small.†

The essential idea of Selberg's proof is to modify the series for $\zeta(s)$ by multiplying it by the square of a partial sum of the series for $\{\zeta(s)\}^{-\frac{1}{2}}$. To this extent, it is similar to the proofs given in Chapter IX of theorems about the general distribution of the zeros.

We define α_ν by

$$\frac{1}{\sqrt{\zeta(s)}} = \sum_{\nu=1}^{\infty} \frac{\alpha_\nu}{\nu^s} \quad (\sigma > 1), \qquad \alpha_1 = 1.$$

It is seen from the Euler product that $\alpha_\mu \alpha_\nu = \alpha_{\mu\nu}$ if $(\mu, \nu) = 1$. Since the series for $(1-z)^{\frac{1}{2}}$ is majorized by that for $(1-z)^{-\frac{1}{2}}$, we see that, if

$$\sqrt{\zeta(s)} = \sum_{\nu=1}^{\infty} \frac{\alpha'_\nu}{\nu^s}, \qquad \alpha'_1 = 1,$$

then $|\alpha_\nu| \leqslant \alpha'_\nu \leqslant 1$.

Let $$\beta_\nu = \alpha_\nu\left(1-\frac{\log \nu}{\log X}\right) \quad (1 \leqslant \nu < X).$$

Then $$|\beta_\nu| \leqslant 1.$$

† It was calculated in an Oxford dissertation by S. H. Min.

All sums involving β_ν run over $[1, X]$ (or we may suppose $\beta_\nu = 0$ for $\nu \geqslant X$). Let

$$\phi(s) = \sum \frac{\beta_\nu}{\nu^s}.$$

10.10. Let†

$$\Phi(z) = \frac{1}{4\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(\tfrac{1}{2}s)\pi^{-\frac{1}{2}s}\zeta(s)\phi(s)\phi(1-s)z^s \, ds$$

where $c > 1$. Moving the line of integration to $\sigma = \frac{1}{2}$, and evaluating the residue at $s = 1$, we obtain

$$\Phi(z) = \tfrac{1}{2}z\phi(1)\phi(0) + \frac{1}{4\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma(\tfrac{1}{2}s)\pi^{-\frac{1}{2}s}\zeta(s)\phi(s)\phi(1-s)z^s \, ds$$

$$= \tfrac{1}{2}z\phi(1)\phi(0) - \frac{z^{\frac{1}{2}}}{2\pi} \int_{-\infty}^{\infty} \frac{\Xi(t)}{t^2+\frac{1}{4}} |\phi(\tfrac{1}{2}+it)|^2 z^{it} \, dt.$$

On the other hand,

$$\Phi(z) = \frac{1}{4\pi i} \sum_{n=1}^{\infty} \sum_{\mu} \sum_{\nu} \beta_\mu \beta_\nu \int_{c-i\infty}^{c+i\infty} \Gamma(\tfrac{1}{2}s)\pi^{-\frac{1}{2}s} \frac{z^s}{n^s \mu^s \nu^{1-s}} \, ds$$

$$= \sum_{n=1}^{\infty} \sum_{\mu} \sum_{\nu} \frac{\beta_\mu \beta_\nu}{\nu} \exp\left(-\frac{\pi n^2 \mu^2}{z^2 \nu^2}\right).$$

Putting $z = e^{-i(\frac{1}{4}\pi-\frac{1}{2}\delta)-y}$, it follows that the functions

$$F(t) = \frac{1}{\sqrt{(2\pi)}} \frac{\Xi(t)}{t^2+\frac{1}{4}} |\phi(\tfrac{1}{2}+it)|^2 e^{(\frac{1}{4}\pi-\frac{1}{2}\delta)t},$$

$$f(y) = \tfrac{1}{2}z^{\frac{1}{2}}\phi(1)\phi(0) - z^{-\frac{1}{2}} \sum_{n=1}^{\infty} \sum_{\mu} \sum_{\nu} \frac{\beta_\mu \beta_\nu}{\nu} \exp\left(-\frac{\pi n^2 \mu^2}{z^2 \nu^2}\right)$$

are Fourier transforms. Hence, as in § 10.7,

$$\int_{-\infty}^{\infty} \left\{\int_t^{t+h} F(u) \, du\right\}^2 dt \leqslant 2h^2 \int_0^{1/h} |f(y)|^2 \, dy + 8 \int_{1/h}^{\infty} |f(y)|^2 y^{-2} \, dy \qquad (10.10.1)$$

where $h \leqslant 1$ is to be chosen later.

Putting $y = \log x$, $G = e^{1/h}$, the first integral on the right is equal to

$$\int_1^G \left| \frac{e^{-i(\frac{1}{4}\pi-\frac{1}{2}\delta)}}{2x} \phi(1)\phi(0) - \sum_{n=1}^{\infty} \sum_{\mu} \sum_{\nu} \frac{\beta_\mu \beta_\nu}{\nu} \exp\left(-\frac{\pi n^2 \mu^2}{\nu^2} e^{i(\frac{1}{2}\pi-\delta)} x^2\right) \right|^2 dx.$$

† Titchmarsh (26).

Calling the triple sum $g(x)$, this is not greater than

$$2\int_1^G \frac{|\phi(1)\phi(0)|^2}{4x^2}\,dx+2\int_1^G |g(x)|^2\,dx < \tfrac{1}{2}|\phi(1)\phi(0)|^2+2\int_1^G |g(x)|^2\,dx.$$

Similarly the second integral in (10.10.1) does not exceed

$$\frac{|\phi(1)\phi(0)|^2}{2G\log^2 G}+2\int_G^\infty \frac{|g(x)|^2}{\log^2 x}\,dx.$$

10.11. We have to obtain upper bounds for these integrals as $\delta\to 0$, but it is more convenient to consider directly the integral

$$J(x,\theta)=\int_x^\infty |g(u)|^2u^{-\theta}\,du \quad (0<\theta\leqslant\tfrac{1}{2},\ x\geqslant 1).$$

This is equal to

$$\sum_{m=1}^\infty\sum_{n=1}^\infty\sum_{\kappa\lambda\mu\nu}\frac{\beta_\kappa\beta_\lambda\beta_\mu\beta_\nu}{\lambda\nu}\int_x^\infty \exp\left\{-\pi\left(\frac{m^2\kappa^2}{\lambda^2}+\frac{n^2\mu^2}{\nu^2}\right)u^2\sin\delta+\right.$$
$$\left.+i\pi\left(\frac{m^2\kappa^2}{\lambda^2}-\frac{n^2\mu^2}{\nu^2}\right)u^2\cos\delta\right\}\frac{du}{u^\theta}.$$

Let Σ_1 denote the sum of those terms in which $m\kappa/\lambda=n\mu/\nu$, and Σ_2 the remainder. Let $(\kappa\nu,\lambda\mu)=q$, so that

$$\kappa\nu=aq,\qquad \lambda\mu=bq,\qquad (a,b)=1.$$

Then, in Σ_1, $ma=nb$, so that $n=ra$, $m=rb$ $(r=1,2,\ldots)$. Hence

$$\Sigma_1=\sum_{\kappa\lambda\mu\nu}\frac{\beta_\kappa\beta_\lambda\beta_\mu\beta_\nu}{\lambda\nu}\sum_{r=1}^\infty\int_x^\infty \exp\left(-2\pi\frac{r^2\kappa^2\mu^2}{q^2}u^2\sin\delta\right)\frac{du}{u^\theta}.$$

Now

$$\sum_{r=1}^\infty\int_x^\infty e^{-r^2u^2\eta}\frac{du}{u^\theta}=\eta^{\frac{1}{2}\theta-\frac{1}{2}}\sum_{r=1}^\infty\frac{1}{r^{1-\theta}}\int_{xr\sqrt\eta}^\infty e^{-y^2}\frac{dy}{y^\theta}$$
$$=\eta^{\frac{1}{2}\theta-\frac{1}{2}}\int_{x\sqrt\eta}^\infty\frac{e^{-y^2}}{y^\theta}\left(\sum_{r\leqslant y/(x\sqrt\eta)}\frac{1}{r^{1-\theta}}\right)dy.$$

The last r-sum is of the form

$$\frac{1}{\theta}\left(\frac{y}{x\sqrt\eta}\right)^\theta-\frac{1}{\theta}+K(\theta)+O\left\{\left(\frac{y}{x\sqrt\eta}\right)^{\theta-1}\right\},$$

where $K(\theta)$, and later $K_1(\theta)$, are bounded functions of θ. Hence we obtain

$$\frac{1}{\theta x^\theta\eta^{\frac{1}{2}}}\left\{\int_0^\infty e^{-y^2}\,dy+O(x\sqrt\eta)\right\}-\frac{\eta^{\frac{1}{2}\theta-\frac{1}{2}}}{\theta}\left[\int_0^\infty e^{-y^2}y^{-\theta}\,dy+O\{(x\sqrt\eta)^{1-\theta}\}\right]+$$
$$+\eta^{\frac{1}{2}\theta-\frac{1}{2}}K(\theta)\left[\int_0^\infty e^{-y^2}y^{-\theta}\,dy+O\{(x\sqrt\eta)^{1-\theta}\}\right]+O\{x^{1-\theta}\log(2+\eta^{-1})\}$$
$$=\frac{\sqrt\pi}{2\theta x^\theta\eta^{\frac{1}{2}}}+\frac{K_1(\theta)\eta^{\frac{1}{2}\theta-\frac{1}{2}}}{\theta}+O\left\{\frac{x^{1-\theta}}{\theta}\log(2+\eta^{-1})\right\}.$$

Putting $\eta=2\pi\kappa^2\mu^2q^{-2}\sin\delta$, it follows that

$$\Sigma_1=\frac{S(0)}{2(2\sin\delta)^{\frac{1}{2}}\theta x^\theta}+\frac{K_1(\theta)}{\theta}(2\pi\sin\delta)^{\frac{1}{2}\theta-\frac{1}{2}}S(\theta)+$$
$$+O\left\{\frac{x^{1-\theta}\log(2+\eta^{-1})}{\theta}\sum_{\kappa\lambda\mu\nu}\frac{|\beta_\kappa\beta_\lambda\beta_\mu\beta_\nu|}{\lambda\nu}\right\}, \tag{10.11.1}$$

where
$$S(\theta)=\sum_{\kappa\lambda\mu\nu}\left(\frac{q}{\kappa\mu}\right)^{1-\theta}\frac{\beta_\kappa\beta_\lambda\beta_\mu\beta_\nu}{\lambda\nu}.$$

Defining $\phi_a(n)$ as in § 9.24, we have

$$q^{1-\theta}=\sum_{\rho|q}\phi_{-\theta}(\rho)=\sum_{\rho|\kappa\nu,\rho|\lambda\mu}\phi_{-\theta}(\rho).$$

Hence
$$S(\theta)=\sum_{\rho<X^2}\phi_{-\theta}(\rho)\left(\sum_{\rho|\kappa\nu}\frac{\beta_\kappa\beta_\nu}{\kappa^{1-\theta}\nu}\right)^2.$$

Let d and d_1 denote positive integers whose prime factors divide ρ. Let $\kappa=d\kappa'$, $\nu=d_1\nu'$, where $(\kappa',\rho)=1$, $(\nu',\rho)=1$. Then

$$\sum_{\rho|\kappa\nu}\frac{\beta_\kappa\beta_\nu}{\kappa^{1-\theta}\nu}=\sum_{\rho|dd_1}\frac{1}{d^{1-\theta}d_1}\sum_{\kappa'}\frac{\beta_{d\kappa'}}{\kappa'^{1-\theta}}\sum_{\nu'}\frac{\beta_{d_1\nu'}}{\nu'}.$$

Now, for $(\kappa',\rho)=1$, $\qquad \beta_{d\kappa'}=\dfrac{\alpha_d\alpha_{\kappa'}}{\log X}\log\dfrac{X}{d\kappa'}.$

Hence the above sum is equal to

$$\frac{1}{\log^2 X}\sum_{\rho|dd_1}\frac{\alpha_d\alpha_{d_1}}{d^{1-\theta}d_1}\sum_{\kappa'\leqslant X/d}\frac{\alpha_{\kappa'}}{\kappa'^{1-\theta}}\log\frac{X}{d\kappa'}\sum_{\nu'\leqslant X/d_1}\frac{\alpha_{\nu'}}{\nu'}\log\frac{X}{d_1\nu'}.$$

10.12. Lemma 10.12. *We have*

$$\sum_{\kappa'\leqslant X/d}\frac{\alpha_{\kappa'}}{\kappa'^{1-\theta}}\log\frac{X}{d\kappa'}=O\left\{\left(\frac{X}{d}\right)^\theta\log^{\frac{1}{2}}\frac{X}{d}\prod_{p|\rho}\left(1+\frac{1}{p}\right)^{\frac{1}{2}}\right\} \tag{10.12.1}$$

uniformly with respect to θ.

We may suppose that $X \geqslant 2d$, since otherwise the lemma is trivial.

Now
$$\frac{1}{2\pi i}\int\limits_{1-i\infty}^{1+i\infty}\frac{x^s}{s^2}\,ds = 0 \quad (0<x\leqslant 1), \quad \log x \quad (x>1).$$

Also
$$\sum_{(\kappa',\rho)=1}\frac{\alpha_{\kappa'}}{\kappa'^{1-\theta+s}} = \prod_{(p,\rho)=1}\left(1-\frac{1}{p^{1-\theta+s}}\right)^{\frac12} = \sum_{p|\rho}\left(1-\frac{1}{p^{1-\theta+s}}\right)^{-\frac12}\frac{1}{\sqrt{\zeta(1-\theta+s)}}.$$

Hence the left-hand side of (10.12.1) is equal to
$$\frac{1}{2\pi i}\int\limits_{1-i\infty}^{1+i\infty}\frac{1}{s^2}\left(\frac{X}{d}\right)^s\prod_{p|\rho}\left(1-\frac{1}{p^{1-\theta+s}}\right)^{-\frac12}\frac{ds}{\sqrt{\zeta(1-\theta+s)}}. \qquad (10.12.2)$$

There are singularities at $s=0$ and $s=\theta$. If $\theta \geqslant \{\log(X/d)\}^{-1}$, we can take the line of integration through $s=\theta$, the integral round a small indentation tending to zero. Now
$$\left|\frac{1}{\zeta(1+it)}\right| < A|t|$$
for all t (large or small). Also
$$\prod_{p|\rho}\left(1-\frac{1}{p^{1-\theta+s}}\right)^{-1} = O\left\{\left|\prod_{p|\rho}\left(1+\frac{1}{p^{1-\theta+s}}\right)\right|\right\} = O\left\{\prod_{p|\rho}\left(1+\frac1p\right)\right\}.$$

Hence (10.12.2) is
$$O\left\{\left(\frac{X}{d}\right)^\theta\prod_{p|\rho}\left(1+\frac1p\right)^{\frac12}\int\limits_{-\infty}^{\infty}\frac{|t|^{\frac12}\,dt}{\theta^2+t^2}\right\} = O\left\{\left(\frac{X}{d}\right)^\theta\prod_{p|\rho}\left(1+\frac1p\right)^{\frac12}\frac{1}{\theta^{\frac12}}\right\},$$
and the result stated follows.

If $\theta < \{\log(X/d)\}^{-1}$, we take the same contour as before modified by a detour round the right-hand side of the circle $|s| = 2\{\log(X/d)\}^{-1}$. On this circle
$$|(X/d)^s| \leqslant e^2,$$
the p-product goes as before, and
$$|\zeta(1-\theta+s)| > A\log(X/d).$$

Hence the integral round the circle is
$$O\left\{\log^{-\frac12}\frac{X}{d}\prod_{p|\rho}\left(1+\frac1p\right)^{\frac12}\int\left|\frac{ds}{s^2}\right|\right\} = O\left\{\log^{\frac12}\frac{X}{d}\prod_{p|\rho}\left(1+\frac1p\right)^{\frac12}\right\}.$$

The integral along the part of the line $\sigma=\theta$ above the circle is
$$O\left\{\left(\frac{X}{d}\right)^\theta\prod_{p|\rho}\left(1+\frac1p\right)^{\frac12}\int\limits_{A(\log X/d)^{-1}}^{\infty}\frac{dt}{t^{\frac32}}\right\} = O\left\{\left(\frac{X}{d}\right)^\theta\log^{\frac12}\frac{X}{d}\prod_{p|\rho}\left(1+\frac1p\right)^{\frac12}\right\}.$$

The lemma is thus proved in all cases.

10.13. Lemma 10.13.
$$\sum_{\rho|dd_1}\frac{|\alpha_d\alpha_{d_1}|}{dd_1} = O\left\{\frac1\rho\prod_{p|\rho}\left(1+\frac1p\right)\right\}.$$

Defining α'_d as in § 10.9, we have
$$\sum_{\rho|dd_1}\frac{|\alpha_d\alpha_{d_1}|}{dd_1}\leqslant\sum_{\rho|dd_1}\frac{\alpha'_d\alpha'_{d_1}}{dd_1} = \sum_{\rho|D}\frac1D,$$
where D is a number of the same class as d or d_1,
$$= \frac1\rho\prod_{p|\rho}\left(1-\frac1p\right)^{-1} = O\left\{\frac1\rho\prod_{p|\rho}\left(1+\frac1p\right)\right\}.$$

10.14. Lemma 10.14.
$$S(\theta) = O\left(\frac{X^{2\theta}}{\log X}\right)$$
uniformly with respect to θ. In particular
$$S(0)=O\left(\frac{1}{\log X}\right).$$

By the formulae of § 10.11, and the above lemmas,
$$\sum_{\rho|\kappa\nu}\frac{\beta_\kappa\beta_\nu}{\kappa^{1-\theta}\nu} = O\left\{\frac{1}{\log^2X}\sum_{\rho|dd_1}\frac{|\alpha_d\alpha_{d_1}|}{d^{1-\theta}d_1}\left(\frac{X}{d}\right)^\theta\log^{\frac12}\frac{X}{d}\log^{\frac12}\frac{X}{d_1}\prod_{p|\rho}\left(1+\frac1p\right)\right\}$$
$$= O\left\{\frac{X^\theta}{\log X}\prod_{p|\rho}\left(1+\frac1p\right)\sum_{\rho|dd_1}\frac{|\alpha_d\alpha_{d_1}|}{dd_1}\right\}$$
$$= O\left\{\frac{X^\theta}{\rho\log X}\prod_{p|\rho}\left(1+\frac1p\right)^2\right\}.$$

Hence
$$S(\theta) = O\left\{\frac{X^{2\theta}}{\log^2X}\sum_{\rho\leqslant X^2}\frac{\phi_{-\theta}(\rho)}{\rho^2}\prod_{p|\rho}\left(1+\frac1p\right)^4\right\}$$
$$= O\left\{\frac{X^{2\theta}}{\log^2X}\sum_{\rho\leqslant X^2}\frac{1}{\rho^{1+\theta}}\prod_{p|\rho}\left(1+\frac1p\right)^4\right\}$$
$$= O\left\{\frac{X^{2\theta}}{\log^2X}\sum_{\rho\leqslant X^2}\frac{1}{\rho^{1+\theta}}\sum_{n|\rho}\frac{1}{n^{\frac12}}\right\},$$
since
$$\prod_{p|\rho}\left(1+\frac1p\right)^4 = O\left\{\prod_{p|\rho}\left(1+\frac4p\right)\right\} = O\left\{\prod_{p|\rho}\left(1+\frac{1}{p^{\frac12}}\right)\right\} = O\left(\sum_{n|\rho}\frac{1}{n^{\frac12}}\right).$$

Hence

$$S(\theta) = O\left\{\frac{X^{2\theta}}{\log^2 X}\sum_{n\leqslant X^2}\sum_{\rho_1\leqslant X^2/n}\frac{1}{(n\rho_1)^{1+\theta}n^{\frac{1}{2}}}\right\}$$

$$= O\left\{\frac{X^{2\theta}}{\log^2 X}\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}+\theta}}\sum_{\rho_1\leqslant X^2/n}\frac{1}{\rho_1^{1+\theta}}\right\}$$

$$= O\left\{\frac{X^{2\theta}}{\log^2 X}\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}\sum_{\rho_1\leqslant X^2}\frac{1}{\rho_1}\right\}$$

$$= O\left(\frac{X^{2\theta}}{\log X}\right).$$

10.15. Estimation of Σ_1. From (10.11.1), Lemma 10.14, and the inequality $|\beta_\nu| \leqslant 1$, we obtain

$$\Sigma_1 = O\left(\frac{1}{\delta^{\frac{1}{2}}\theta x^\theta \log X}\right) + O\left\{\frac{(\delta^{\frac{1}{2}}xX^2)^\theta}{\delta^{\frac{1}{2}}\theta x^\theta \log X}\right\} + O\left\{\frac{x^{1-\theta}\log(X/\delta)}{\theta}X^2\log^2 X\right\}.$$

We shall ultimately take $X = \delta^{-c}$ and $h = (a\log X)^{-1}$, where a and c are suitable positive constants. Then $G = X^a = \delta^{-ac}$. If $x \leqslant G$, the last two terms can be omitted in comparison with the first if $GX^2 = O(\delta^{-\frac{1}{4}})$, i.e. if $(a+2)c \leqslant \frac{1}{4}$. We then have

$$\Sigma_1 = O\left(\frac{1}{\delta^{\frac{1}{2}}\theta x^\theta \log X}\right). \tag{10.15.1}$$

10.16. Estimation of Σ_2. If P and Q are positive, and $x \geqslant 1$,

$$\int_x^\infty e^{-Pu^2+iQu^2}\frac{du}{u^\theta} = \frac{1}{2}\int_{x^2}^\infty \frac{e^{-Pv}}{v^{\frac{1}{2}\theta+\frac{1}{2}}}e^{iQv}\,dv = O\left(\frac{e^{-P}}{x^\theta Q}\right),$$

e.g. by applying the second mean-value theorem to the real and imaginary parts. Hence

$$\Sigma_2 = O\left[\frac{1}{x^\theta}\sum_{\kappa\lambda\mu\nu}\frac{1}{\lambda\nu}\sum_{mn}{}'\left|\frac{m^2\kappa^2}{\lambda^2}-\frac{n^2\mu^2}{\nu^2}\right|^{-1}\exp\left\{-\pi\left(\frac{m^2\kappa^2}{\lambda^2}+\frac{n^2\mu^2}{\nu^2}\right)\sin\delta\right\}\right].$$

The terms with $m\kappa/\lambda > n\mu/\nu$ contribute to the m,n sum

$$O\left\{\sum_{m=1}^{\infty}e^{-\pi m^2\kappa^2\lambda^{-2}\sin\delta}\sum_{n<m\kappa\nu/\lambda\mu}\left(\frac{m^2\kappa^2}{\lambda^2}-\frac{n^2\mu^2}{\nu^2}\right)^{-1}\right\}.$$

Now $$\frac{m^2\kappa^2}{\lambda^2}-\frac{n^2\mu^2}{\nu^2} \geqslant \frac{m\kappa}{\lambda}\left(\frac{m\kappa}{\lambda}-\frac{n\mu}{\nu}\right) = \frac{m\kappa(m\kappa\nu-n\lambda\mu)}{\lambda^2\nu},$$

and $$\sum_n \frac{1}{m\kappa\nu-n\lambda\mu} \leqslant 1+\frac{1}{\lambda\mu}+\frac{1}{2\lambda\mu}+\ldots = 1+O\left(\frac{\log mX}{\lambda\mu}\right).$$

Hence the m,n sum is

$$O\left\{\frac{\lambda^2\nu}{\kappa}\sum_{m=1}^{\infty}\left(\frac{1}{m}+\frac{\log(mX)}{m\lambda\mu}\right)e^{-\pi m^2\kappa^2\lambda^{-2}\sin\delta}\right\}$$

$$= O\left\{\frac{\lambda^2\nu}{\kappa}\left(1+\frac{\log X}{\lambda\mu}\right)\log\frac{X^2}{\delta}+\frac{\lambda\nu}{\kappa\mu}\log^2\frac{X^2}{\delta}\right\}$$

$$= O\left(\frac{\lambda^2\nu}{\kappa}\log\frac{1}{\delta}\right)+O\left(\frac{\lambda\nu}{\kappa\mu}\log^2\frac{1}{\delta}\right),$$

since, as in §10.15, we have $X = \delta^{-c}$, with $0 < c \leqslant \frac{1}{8}$. The remaining terms may be treated similarly. Hence

$$\Sigma_2 = O\left\{\frac{1}{x^\theta}\sum_{\kappa\lambda\mu\nu}\left(\frac{\lambda}{\kappa}\log\frac{1}{\delta}+\frac{1}{\kappa\mu}\log^2\frac{1}{\delta}\right)\right\} = O\left(\frac{X^4}{x^\theta}\log^2\frac{1}{\delta}\right). \tag{10.16.1}$$

10.17. Lemma 10.17. *Under the assumptions of* § 10.15

$$\int_{-\infty}^{\infty}\left|\int_t^{t+h}F(u)\,du\right|^2 dt = O\left(\frac{h}{\delta^{\frac{1}{2}}\log X}\right). \tag{10.17.1}$$

By (10.15.1) and (10.16.1),

$$J(x,\theta) = O\left(\frac{1}{\delta^{\frac{1}{2}}\theta x^\theta \log X}\right) \tag{10.17.2}$$

uniformly with respect to θ. Hence

$$\int_1^G |g(x)|^2\,dx = -\int_1^G x^\theta\frac{\partial J}{\partial x}\,dx = [-x^\theta J]_1^G+\theta\int_1^G x^{\theta-1}J\,dx$$

$$= O\left(\frac{1}{\delta^{\frac{1}{2}}\theta\log X}\right)+O\left(\theta\int_1^G\frac{dx}{\delta^{\frac{1}{2}}\theta x\log X}\right) = O\left(\frac{\log G}{\delta^{\frac{1}{2}}\log X}\right),$$

taking, for example, $\theta = \frac{1}{2}$. Also

$$\int_0^{\frac{1}{2}}\theta J(G,\theta)\,d\theta = \int_G^\infty |g(x)|^2\,dx\int_0^{\frac{1}{2}}\theta x^{-\theta}\,d\theta$$

$$= \int_G^\infty |g(x)|^2\left(\frac{1}{\log^2 x}-\frac{1}{2x^{\frac{1}{2}}\log x}-\frac{1}{x^{\frac{1}{2}}\log^2 x}\right)dx$$

$$\geqslant \int_G^\infty \frac{|g(x)|^2}{\log^2 x}\,dx-\frac{3}{2}\int_G^\infty\frac{|g(x)|^2}{x^{\frac{1}{2}}}\,dx$$

since $G = e^{1/h} \geqslant e$. Hence

$$\int\limits_G^\infty \frac{|g(x)|^2}{\log^2 x}\,dx \leqslant \int\limits_0^{\frac{1}{2}} \theta J(G,\theta)\,d\theta + \tfrac{3}{2}J(G,\tfrac{1}{2})$$

$$= O\left(\int\limits_0^{\frac{1}{2}} \frac{d\theta}{\delta^{\frac{1}{2}} G^\theta \log X}\right) + O\left(\frac{1}{\delta^{\frac{1}{2}} G^{\frac{1}{2}} \log X}\right) = O\left(\frac{1}{\delta^{\frac{1}{2}} \log G \log X}\right).$$

Also $\phi(0) = O(X)$, $\phi(1) = O(\log X)$. The result therefore follows from the formulae of § 10.10.

10.18. So far the integrals considered have involved $F(t)$. We now turn to the integrals involving $|F(t)|$. The results about such integrals are expressed in the following lemmas.

LEMMA 10.18. $$\int\limits_{-\infty}^{\infty} |F(t)|^2\,dt = O\left(\frac{\log 1/\delta}{\delta^{\frac{1}{2}} \log X}\right).$$

By the Fourier transform formulae, the left-hand side is equal to

$$2\int\limits_0^\infty |f(y)|^2\,dy = 2\int\limits_1^\infty \left|\frac{e^{-i(\frac{1}{4}\pi - \frac{1}{2}\delta)}}{2x}\phi(1)\phi(0) - g(x)\right|^2 dx$$

$$\leqslant 4\int\limits_1^\infty |g(x)|^2\,dx + O(X^2 \log^2 X).$$

Taking $x = 1$, $\theta = \{\log(1/\delta)\}^{-1}$ in (10.17.2), we have

$$\int\limits_1^\infty |g(u)|^2 e^{-\log u/(\log 1/\delta)}\,du = O\left(\frac{\log 1/\delta}{\delta^{\frac{1}{2}} \log X}\right).$$

Hence $$\int\limits_1^{\delta^{-2}} |g(u)|^2\,du = O\left(\frac{\log 1/\delta}{\delta^{\frac{1}{2}} \log X}\right).$$

We can estimate the integral over (δ^{-2}, ∞) in a comparatively trivial manner. As in § 10.11, this is less than

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \sum_{\kappa\lambda\mu\nu} \frac{|\beta_\kappa \beta_\lambda \beta_\mu \beta_\nu|}{\lambda\nu} \int\limits_{\delta^{-2}}^\infty \exp\left\{-\pi\left(\frac{m^2\kappa^2}{\lambda^2} + \frac{n^2\mu^2}{\nu^2}\right)u^2 \sin\delta\right\} du.$$

Using, for example, $\kappa^2\lambda^{-2}\sin\delta > AX^{-2}\delta > A\delta^2$ (since $X = \delta^{-c}$ with $c < \frac{1}{2}$), and $|\beta_\nu| \leqslant 1$, this is

$$O\left\{X^2 \log^2 X \sum_{m=1}^\infty \sum_{n=1}^\infty \int\limits_{\delta^{-2}}^\infty e^{-A(m^2+n^2)\delta^2 u^2}\,du\right\}$$

$$= O\left(X^2 \log^2 X \int\limits_{\delta^{-2}}^\infty e^{-A\delta^2 u^2}\,du\right) = O(X^2 \log^2 X\, e^{-A/\delta^2}),$$

which is of the required form.

10.19. LEMMA 10.19.

$$\int\limits_{-\infty}^\infty \left\{\int\limits_t^{t+h} |F(u)|\,du\right\}^2 dt = O\left(\frac{h^2 \log 1/\delta}{\delta^{\frac{1}{2}} \log X}\right).$$

For the left-hand side does not exceed

$$\int\limits_{-\infty}^\infty \left\{h\int\limits_t^{t+h} |F(u)|^2\,du\right\} dt = h\int\limits_{-\infty}^\infty |F(u)|^2\,du \int\limits_{u-h}^u dt = h^2 \int\limits_{-\infty}^\infty |F(u)|^2\,du,$$

and the result follows from the previous lemma.

10.20. LEMMA 10.20. *If* $\delta = 1/T$,

$$\int\limits_0^T |F(t)|\,dt > AT^{\frac{3}{4}}.$$

We have

$$\left(\int\limits_{\frac{1}{2}+i}^{2+i} + \int\limits_{2+i}^{2+iT} + \int\limits_{2+iT}^{\frac{1}{2}+iT} + \int\limits_{\frac{1}{2}+iT}^{\frac{1}{2}+i}\right)\zeta(s)\phi^2(s)\,ds = 0.$$

Since $\phi(s) = O(X^{\frac{1}{2}})$ for $\sigma \geqslant \frac{1}{2}$, the first term is $O(X)$, and the third is $O(XT^{\frac{1}{4}})$. Also

$$\zeta(s)\phi^2(s) = 1 + \sum_{n=2}^\infty \frac{a_n}{n^s},$$

where $|a_n| \leqslant d_3(n)$. Hence

$$\int\limits_{2+i}^{2+iT} \zeta(s)\phi^2(s)\,ds = i(T-1) + \sum_{n=2}^\infty a_n \int\limits_{2+i}^{2+iT} \frac{ds}{n^s}$$

$$= i(T-1) + O\left(\sum_{n=2}^\infty \frac{d_3(n)}{n^2 \log n}\right)$$

$$= iT + O(1).$$

It follows that $$\int\limits_0^T \zeta(\tfrac{1}{2}+it)\phi^2(\tfrac{1}{2}+it)\,dt \sim T.$$

Hence
$$\int_0^T |F(t)|\,dt > A\int_0^T t^{-\frac{1}{4}}|\zeta(\tfrac{1}{2}+it)\phi^2(\tfrac{1}{2}+it)|\,dt$$
$$> AT^{-\frac{1}{4}}\int_{\frac{1}{2}T}^{T} |\zeta(\tfrac{1}{2}+it)\phi^2(\tfrac{1}{2}+it)|\,dt$$
$$> AT^{-\frac{1}{4}}\left|\int_{\frac{1}{2}T}^{T} \zeta(\tfrac{1}{2}+it)\phi^2(\tfrac{1}{2}+it)\,dt\right|$$
$$> AT^{\frac{3}{4}}.$$

10.21. LEMMA 10.21.
$$\int_0^T dt\int_t^{t+h} |F(u)|\,du > AhT^{\frac{3}{4}}.$$

The left-hand side is equal to
$$\int_0^{T+h} |F(u)|\,du \int_{\max(0,u-h)}^{\min(T,u)} dt \geqslant \int_h^T |F(u)|\,du\int_{u-h}^{u} dt = h\int_h^T |F(u)|\,du,$$
and the result follows from the previous lemma.

10.22. THEOREM 10.22.
$$N_0(T) > AT\log T.$$

Let E be the sub-set of $(0,T)$ where
$$\int_t^{t+h} |F(u)|\,du > \left|\int_t^{t+h} F(u)\,du\right|.$$
For such values of t, $F(u)$ must change sign in $(t,t+h)$, and hence so must $\Xi(u)$, and hence $\zeta(\frac{1}{2}+iu)$ must have a zero in this interval.

Since the two sides are equal except in E,
$$\int_E dt\int_t^{t+h}|F(u)|\,du \geqslant \int_E\left\{\int_t^{t+h}|F(u)|\,du-\left|\int_t^{t+h}F(u)\,du\right|\right\}dt$$
$$= \int_0^T\left\{\int_t^{t+h}|F(u)|\,du-\left|\int_t^{t+h}F(u)\,du\right|\right\}dt$$
$$> AhT^{\frac{3}{4}}-\int_0^T\left|\int_t^{t+h}F(u)\,du\right|dt.$$
The left-hand side is not greater than
$$\left\{\int_E dt\int_E\left(\int_t^{t+h}|F(u)|\,du\right)^2 dt\right\}^{\frac{1}{2}} \leqslant \left\{m(E)\int_{-\infty}^{\infty}\left(\int_t^{t+h}|F(u)|\,du\right)^2 dt\right\}^{\frac{1}{2}}$$
$$< A\{m(E)\}^{\frac{1}{2}}hT^{\frac{1}{4}}\left(\frac{\log T}{\log X}\right)^{\frac{1}{2}}$$

by Lemma 10.19 with $\delta = 1/T$. The second term on the right is not greater than
$$\left\{\int_0^T dt\int_0^T\left|\int_t^{t+h}F(u)\,du\right|^2 dt\right\}^{\frac{1}{2}} < \frac{Ah^{\frac{1}{2}}T^{\frac{3}{4}}}{\log^{\frac{1}{2}}X}$$
by Lemma 10.17. Hence
$$\{m(E)\}^{\frac{1}{2}} > A_1T^{\frac{1}{2}}\left(\frac{\log X}{\log T}\right)^{\frac{1}{2}}-A_2\frac{T^{\frac{1}{2}}}{h^{\frac{1}{2}}\log^{\frac{1}{2}}T},$$
where A_1 and A_2 denote the particular constants which occur. Since $X = T^c$ and $h = (a\log X)^{-1} = (ac\log T)^{-1}$,
$$\{m(E)\}^{\frac{1}{2}} > A_1c^{\frac{1}{2}}T^{\frac{1}{2}}-A_2(ac)^{\frac{1}{2}}T^{\frac{1}{2}}.$$
Taking a small enough, it follows that
$$m(E) > A_3T.$$
Hence, of the intervals $(0,h)$, $(h,2h)$,... contained in $(0,T)$, at least $[A_3T/h]$ must contain points of E. If $(nh,(n+1)h)$ contains a point t of E, there must be a zero of $\zeta(\frac{1}{2}+iu)$ in $(t,t+h)$, and so in $(nh,(n+2)h)$. Allowing for the fact that each zero might be counted twice in this way, there must be at least
$$\tfrac{1}{2}[A_3T/h] > AT\log T$$
zeros in $(0,T)$.

10.23. In this section we return to the function $\Xi^*(t)$ mentioned in § 10.1. In spite of its deficiencies as an approximation to $\Xi(t)$, it is of some interest to note that *all the zeros of* $\Xi^*(t)$ *are real.*†

A still better approximation to $\Phi(u)$ is
$$\Phi^{**}(u) = \pi(2\pi\cosh\tfrac{9}{2}u-3\cosh\tfrac{5}{2}u)e^{-2\pi\cosh 2u}.$$
This gives
$$\Xi^{**}(t) = 2\int_0^{\infty}\Phi^{**}(u)\cos ut\,du,$$
and we shall also prove that *all the zeros of* $\Xi^{**}(t)$ *are real.*

The function $K_z(a)$ is, for any value of a, an even integral function of z. We begin by proving that *if a is real all its zeros are purely imaginary.*

It is known that $w = K_z(a)$ satisfies the differential equation
$$\frac{d}{da}\left(a\frac{dw}{da}\right) = \left(a+\frac{z^2}{a}\right)w.$$
This is equivalent to the two equations
$$\frac{dw}{da} = \frac{W}{a},\qquad \frac{dW}{da} = \left(a+\frac{z^2}{a}\right)w.$$

† Pólya (1), (2), (4).

These give $$\frac{d}{da}(W\bar{w}) = \frac{1}{a}\{|W|^2+(a^2+z^2)|w|^2\}.$$

It is also easily verified that w and W tend to 0 as $a \to \infty$. It follows that, if w vanishes for a certain z and $a = a_0 > 0$, then

$$\int_{a_0}^{\infty} \{|W|^2+(a^2+z^2)|w|^2\}\frac{da}{a} = 0.$$

Taking imaginary parts,

$$2ixy \int_{a_0}^{\infty} \frac{|w|^2}{a}\,da = 0.$$

Here the integral is not 0, and $K_z(a)$ plainly does not vanish for z real, i.e. $y = 0$. Hence $x = 0$, the required result.

We also require the following lemma.

Let c be a positive constant, $F(z)$ an integral function of genus 0 or 1, which takes real values for real z, and has no complex zeros and at least one real zero. Then all the zeros of

$$F(z+ic)+F(z-ic) \tag{10.23.1}$$

are also real.

We have $$F(z) = Cz^q e^{\alpha z}\prod_{n=1}^{\infty}\left(1-\frac{z}{\alpha_n}\right)e^{z/\alpha_n},$$

where C, α, α_1,... are real constants, $\alpha_n \neq 0$ for $n = 1, 2,...,$ $\sum \alpha_n^{-2}$ is convergent, q a non-negative integer. Let z be a zero of (10.23.1). Then

$$|F(z+ic)| = |F(z-ic)|,$$

so that

$$1 = \left|\frac{F(z-ic)}{F(z+ic)}\right|^2 = \left\{\frac{x^2+(y-c)^2}{x^2+(y+c)^2}\right\}^q \prod_{n=1}^{\infty}\frac{(x-\alpha_n)^2+(y-c)^2}{(x-\alpha_n)^2+(y+c)^2}.$$

If $y > 0$, every factor on the right is < 1; if $y < 0$, every factor is > 1. Hence in fact $y = 0$.

The theorem that the zeros of $\Xi^*(t)$ are all real now follows on taking

$$F(z) = K_{\frac{1}{2}iz}(2\pi), \qquad c = \tfrac{9}{2}.$$

10.24. For the discussion of $\Xi^{**}(t)$ we require the following lemma.

Let $|f(t)| < Ke^{-|t|^{2+\delta}}$ for some positive δ, so that

$$F(z) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f(t)e^{izt}\,dt$$

is an integral function of z. Let all the zeros of $F(z)$ be real. Let $\phi(t)$ be an integral function of t of genus 0 or 1, real for real t. Then the zeros of

$$G(z) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f(t)\phi(it)e^{izt}\,dt$$

are also all real.

We have $$\phi(t) = Ct^q e^{\alpha t}\prod_{m=1}^{\infty}\left(1-\frac{t}{\alpha_m}\right)e^{t/\alpha_m},$$

where the constants are all real, and $\sum \alpha_m^{-2}$ is convergent. Let

$$\phi_n(t) = Ct^q e^{\alpha t}\prod_{m=1}^{n}\left(1-\frac{t}{\alpha_m}\right)e^{t/\alpha_m}.$$

Then $\phi_n(t) \to \phi(t)$ uniformly in any finite interval, and (as in my *Theory of Functions*, § 8.25) $$|\phi_n(t)| < Ke^{|t|^{2+\epsilon}}$$

uniformly with respect to n. Hence

$$G(z) = \lim_{n\to\infty}\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f(t)\phi_n(it)e^{izt}\,dt = \lim_{n\to\infty} G_n(z),$$

say. It is therefore sufficient to prove that, for every n, the zeros of $G_n(z)$ are real.

Now it is easily verified that $F(z)$ is an integral function of order less than 2. Hence, if its zeros are real, so are those of

$$(D-\alpha)F(z) = e^{\alpha z}\frac{d}{dz}\{e^{-\alpha z}F(z)\}$$

for any real α. Applying this principle repeatedly, we see that all the zeros of

$$H(z) = D^q(D-\alpha_1)...(D-\alpha_n)F(z) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f(t)(it)^q(it-\alpha_1)...(it-\alpha_n)e^{izt}\,dt$$

are real. Since

$$G_n(z) = \frac{(-1)^n C}{\alpha_1...\alpha_n}H\left(z+\alpha+\frac{1}{\alpha_1}+...+\frac{1}{\alpha_n}\right)$$

the result follows.

Taking $$f(t) = 4\sqrt{(2\pi)}e^{-2\pi\cosh 2t}$$

we obtain $$F(z) = K_{\frac{1}{2}iz}(2\pi),$$

all of whose zeros are real. If

$$\phi(t) = \tfrac{1}{2}\pi^2\cos\tfrac{9}{2}t,$$

then $G(z) = \Xi^*(z)$, and it follows again that all the zeros of $\Xi^*(z)$ are real. If
$$\phi(t) = \tfrac{1}{2}\pi^2\left(\cos\frac{9}{2}t - \frac{3}{2\pi}\cos\frac{5}{2}t\right),$$
then $G(z) = \Xi^{**}(z)$. Hence all the zeros of $\Xi^{**}(z)$ are real.

10.25. By way of contrast to the Riemann zeta-function we shall now construct a function which has a similar functional equation, and for which the analogues of most of the theorems of this chapter are true; but which has no Euler product, and for which the analogue of the Riemann hypothesis is false.

We shall use the simplest properties of Dirichlet's L-functions (mod 5). These are defined for $\sigma > 1$ by
$$L_0(s) = \sum_{n=1}^{\infty} \frac{\chi_0(n)}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{6^s} + \dots,$$
$$L_1(s) = \sum_{n=1}^{\infty} \frac{\chi_1(n)}{n^s} = \frac{1}{1^s} + \frac{i}{2^s} - \frac{i}{3^s} - \frac{1}{4^s} + \frac{1}{6^s} + \dots,$$
$$L_2(s) = \sum_{n=1}^{\infty} \frac{\chi_2(n)}{n^s} = \frac{1}{1^s} - \frac{i}{2^s} + \frac{i}{3^s} - \frac{1}{4^s} + \frac{1}{6^s} + \dots,$$
$$L_3(s) = \sum_{n=1}^{\infty} \frac{\chi_3(n)}{n^s} = \frac{1}{1^s} - \frac{1}{2^s} - \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{6^s} + \dots.$$
Each $\chi(n)$ has the period 5. It is easily verified that in each case
$$\chi(m)\chi(n) = \chi(mn)$$
if m is prime to n; and hence that
$$L(s) = \prod_p \left\{1 - \frac{\chi(p)}{p^s}\right\}^{-1} \quad (\sigma > 1).$$
It is also easily seen that
$$L_0(s) = \left(1 - \frac{1}{5^s}\right)\zeta(s),$$
so that $L_0(s)$ is regular except for a simple pole at $s = 1$. The other three series are convergent for any real positive s, and hence for $\sigma > 0$. Hence $L_1(s)$, $L_2(s)$, and $L_3(s)$ are regular for $\sigma > 0$.

Now consider the function
$$\begin{aligned} f(s) &= \tfrac{1}{2}\sec\theta\{e^{-i\theta}L_1(s) + e^{i\theta}L_2(s)\} \\ &= \frac{1}{1^s} + \frac{\tan\theta}{2^s} - \frac{\tan\theta}{3^s} - \frac{1}{4^s} + \frac{1}{6^s} + \dots \\ &= \frac{1}{5^s}\{\zeta(s,\tfrac{1}{5}) + \tan\theta\,\zeta(s,\tfrac{2}{5}) - \tan\theta\,\zeta(s,\tfrac{3}{5}) - \zeta(s,\tfrac{4}{5})\}, \end{aligned}$$
where $\zeta(s,a)$ is defined as in § 2.17.

By (2.17) $f(s)$ is an integral function of s, and for $\sigma < 0$ it is equal to
$$\frac{2\Gamma(1-s)}{5^s(2\pi)^{1-s}}\left\{\sin\tfrac{1}{2}\pi s \times\right.$$
$$\times \sum_{m=1}^{\infty}\left(\cos\frac{2m\pi}{5} + \tan\theta\cos\frac{4m\pi}{5} - \tan\theta\cos\frac{6m\pi}{5} - \cos\frac{8m\pi}{5}\right)\frac{1}{m^{1-s}} +$$
$$\left. + \cos\tfrac{1}{2}\pi s\sum_{m=1}^{\infty}\left(\sin\frac{2m\pi}{5} + \tan\theta\sin\frac{4m\pi}{5} - \tan\theta\sin\frac{6m\pi}{5} - \sin\frac{8m\pi}{5}\right)\frac{1}{m^{1-s}}\right\}$$
$$= \frac{4\Gamma(1-s)\cos\frac{1}{2}\pi s}{5^s(2\pi)^{1-s}}\sum_{m=1}^{\infty}\left(\sin\frac{2m\pi}{5} + \tan\theta\sin\frac{4m\pi}{5}\right)\frac{1}{m^{1-s}}.$$
If
$$\sin\frac{4\pi}{5} + \tan\theta\sin\frac{8\pi}{5} = \tan\theta\left(\sin\frac{2\pi}{5} + \tan\theta\sin\frac{4\pi}{5}\right), \tag{10.25.1}$$
this is equal to
$$\frac{4\Gamma(1-s)\cos\frac{1}{2}\pi s}{5^s(2\pi)^{1-s}}\left(\sin\frac{2\pi}{5} + \tan\theta\sin\frac{4\pi}{5}\right)f(1-s).$$
The equation (10.25.1) reduces to
$$\sin 2\theta = 2\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{2},$$
and we take θ to be the root of this between 0 and $\frac{1}{4}\pi$. We obtain
$$\tan\theta = \frac{\sqrt{(10-2\sqrt{5})}-2}{\sqrt{5}-1},$$
$$\sin\frac{2\pi}{5} + \tan\theta\sin\frac{4\pi}{5} = \frac{\sqrt{5}}{2},$$
and $f(s)$ satisfies the functional equation
$$f(s) = \frac{2\Gamma(1-s)\cos\frac{1}{2}s\pi}{5^{s-\frac{1}{2}}(2\pi)^{1-s}}f(1-s).$$

There is now no difficulty in extending the theorems of this chapter to $f(s)$. We can write the above equation as
$$\left(\frac{5}{\pi}\right)^{\frac{1}{2}s}\Gamma(\tfrac{1}{2}+\tfrac{1}{2}s)f(s) = \left(\frac{5}{\pi}\right)^{\frac{1}{2}-\frac{1}{2}s}\Gamma(1-\tfrac{1}{2}s)f(1-s),$$
and putting $s = \frac{1}{2}+it$ we obtain an even integral function of t analogous to $\Xi(t)$.

We conclude that $f(s)$ has an infinity of zeros on the line $\sigma = \frac{1}{2}$, and that the number of such zeros between 0 and T is greater than AT.

On the other hand, we shall now prove that $f(s)$ *has an infinity of zeros in the half-plane* $\sigma > 1$.

If p is a prime, we define $\alpha(p)$ by

$$\alpha(p) = \tfrac{1}{2}(1+i)\chi_1(p)+\tfrac{1}{2}(1-i)\chi_2(p),$$

so that $$\alpha(p) = \pm 1 \quad \text{or} \quad \pm i.$$

For composite n, we define $\alpha(n)$ by the equation

$$\alpha(n_1 n_2) = \alpha(n_1)\alpha(n_2).$$

Thus $|\alpha(n)|$ is always 0 or 1. Let

$$M(s,\chi) = \sum_{n=1}^{\infty} \frac{\alpha(n)\chi(n)}{n^s} = \prod_p \left(1-\frac{\alpha(p)\chi(p)}{p^s}\right)^{-1},$$

where χ denotes either χ_1 or χ_2. Let

$$N(s) = \tfrac{1}{2}\{M(s,\chi_1)+M(s,\chi_2)\}.$$

Now
$$\alpha(p)\chi_1(p) = \tfrac{1}{2}(1+i)\chi_1^2+\tfrac{1}{2}(1-i)\chi_1\chi_2,$$
$$\alpha(p)\chi_2(p) = \tfrac{1}{2}(1+i)\chi_1\chi_2+\tfrac{1}{2}(1-i)\chi_2^2,$$

and these are conjugate since $\chi_1^2 = \bar{\chi}_2^2$ and χ_1^2 and $\chi_1\chi_2$ are real. Hence $M(s,\chi_1)$ and $M(s,\chi_2)$ are conjugate for real s, and $N(s)$ is real.

Let s be real, greater than 1, and $\to 1$. Then

$$\log M(s,\chi_1) = \sum_p \frac{\alpha(p)\chi_1(p)}{p^s}+O(1)$$
$$= \tfrac{1}{2}(1+i)\sum_p \frac{\chi_1^2(p)}{p^s}+\tfrac{1}{2}(1-i)\sum_p \frac{\chi_1(p)\chi_2(p)}{p^s}+O(1).$$

Now $\chi_1^2 = \chi_3$ and $\chi_1\chi_2 = \chi_0$. Hence

$$\sum_p \frac{\chi_1^2(p)}{p^s} = \sum_p \frac{\chi_3(p)}{p^s} = \log L_3(s)+O(1) = O(1),$$

$$\sum_p \frac{\chi_1(p)\chi_2(p)}{p^s} = \sum_p \frac{\chi_0(p)}{p^s} = \log L_0(s)+O(1) = \log\frac{1}{s-1}+O(1).$$

Hence
$$\log M(s,\chi_1) = \tfrac{1}{2}(1-i)\log\frac{1}{s-1}+O(1),$$

$$N(s) = \mathbf{R}M(s,\chi_1) = \frac{1}{\sqrt{(s-1)}}\cos\left(\tfrac{1}{2}\log\frac{1}{s-1}\right)e^{O(1)}.$$

It is clear from this formula that $N(s)$ *has a zero at each of the points* $s = 1+e^{-(2m+1)\pi}$ $(m = 1, 2,...)$.

Now for $\sigma \geqslant 1+\delta$, and $\chi = \chi_1$ or χ_2,

$$\log L(s+i\tau,\chi)-\log M(s,\chi)$$
$$= \sum_{p\leqslant P}\left\{\log\left(1-\frac{\alpha(p)\chi(p)}{p^s}\right)-\log\left(\frac{1-p^{-i\tau}\chi(p)}{p^s}\right)\right\}+O\left(\frac{1}{P^\delta}\right)$$
$$= O\left\{\sum_{\substack{p\leqslant P\\ p\neq 5}} \frac{|\alpha(p)-p^{-i\tau}|}{p^\sigma}\right\}+O\left(\frac{1}{P^\delta}\right).$$

Let $\alpha(p) = e^{2\pi i\beta(p)}$. By Kronecker's theorem, given q, there is a number τ and integers x_p such that

$$\left|\tau\frac{\log p}{2\pi}+\beta(p)-x_p\right| \leqslant \frac{1}{q} \quad (p \leqslant P).$$

Then $$|\alpha(p)-p^{-i\tau}| = |e^{2\pi i(\beta(p)+(\tau\log p)/2\pi)}-1| \leqslant e^{2\pi/q}-1.$$

Hence $$\log L(s+i\tau,\chi)-\log M(s,\chi) = O\left(\frac{\log P}{q}\right)+O\left(\frac{1}{P^\delta}\right),$$

and we can make this as small as we please by choosing first P and then q. Using this with χ_1 and χ_2, it follows that, given $\epsilon > 0$ and $\delta > 0$, there is a τ such that

$$|f(s+i\tau)-N(s)| < \epsilon \quad (\sigma \geqslant 1+\delta).$$

Let $s_1 > 1$ be a zero of $N(s)$. For any $\eta > 0$ there exists an η_1 with $0 < \eta_1 < \eta$, $\eta_1 < s_1-1$, such that $N(s) \neq 0$ for $|s-s_1| = \eta_1$. Let

$$\epsilon = \min_{|s-s_1|=\eta_1} |N(s)|$$

and $\delta < s_1-\eta_1-1$. Then, by Rouché's theorem, $N(s)$ and

$$N(s)-\{N(s)-f(s+i\tau)\}$$

have the same number of zeros inside $|s-s_1| = \eta_1$, and so at least one. Hence $f(s)$ has at least one zero inside the circle $|s-s_1-i\tau| = \eta_1$.

A slight extension of the argument shows that the number of zeros of $f(s)$ in $\sigma > 1$, $0 < t \leqslant T$, exceeds AT as $T\to\infty$. For by the extension of Dirichlet's theorem (§ 8.2) the interval $(t_0, mq^P t_0)$ contains at least m values of t, differing by at least t_0, such that

$$\left|t\frac{\log p}{2\pi}-x_p'\right| \leqslant \frac{1}{q} \qquad (p \leqslant P).$$

The above argument then shows the existence of a zero in the neighbourhood of each point $s_1+i(\tau+t)$.

The method is due to Davenport and Heilbronn (1), (2); they proved that a class of functions, of which an example is

$$\sum_{m,n\neq 0,0}\sum \frac{1}{(m^2+5n^2)^s},$$

has an infinity of zeros for $\sigma > 1$. It has been shown by calculation† that this particular function has a zero in the critical strip, not on the critical line. The method throws no light on the general question of the occurrence of zeros of such functions in the critical strip, but not on the critical line.

NOTES FOR CHAPTER 10

10.26. In § 10.1 Titchmarsh's comment on Riemann's statement about the approximate formula for $N(T)$ is erroneous. It is clear that Riemann meant that the relative error $\{N(T)-L(T)\}/N(T)$ is $O(T^{-1})$.

10.27. Further work has been done on the problem mentioned at the end of § 10.25. Davenport and Heilbronn (1), (2) showed in general that if Q is any positive definite integral quadratic form of discriminant d, such that the class number $h(d)$ is greater than 1, then the Epstein Zeta-function

$$\zeta_Q(s) = \sum_{\substack{x,y=-\infty \\ (x,y)\neq(0,0)}}^{\infty} Q(x,y)^{-s} \qquad (\sigma > 1)$$

has zeros to the right of $\sigma = 1$. In fact they showed that the number of such zeros up to height T is at least of order T (and hence of exact order T). This result has been extended to the critical strip by Voronin [3], who proved that, for such functions $\zeta_Q(s)$, the number of zeros up to height T, for $\frac{1}{2} < \sigma_1 \leqslant \mathrm{I}(s) \leqslant \sigma_2 < 1$, is also of order at least T (and hence of exact order T). This answers the question raised by Titchmarsh at the end of § 10.25.

10.28. Much the most significant result on $N_0(T)$ is due to Levinson [2], who showed that

$$N_0(T) \geqslant \alpha N(T) \tag{10.28.1}$$

for large enough T, with $\alpha = 0{\cdot}342$. The underlying idea is to relate the distribution of zeros of $\zeta(s)$ to that of the zeros of $\zeta'(s)$. To put matters in

† Potter and Titchmarsh (1).

their proper perspective we first note that Berndt [1] has shown that

$$\#\{s=\sigma+it\colon 0<t\leqslant T, \zeta'(s)=0\} = \frac{T}{2\pi}\left(\log\frac{T}{4\pi}-1\right)+O(\log T),$$

and that Speiser (1) has proved that the Riemann Hypothesis is equivalent to the non-vanishing of $\zeta'(s)$ for $0<\sigma<\frac{1}{2}$. This latter result is related to the unconditional estimate

$$\begin{aligned}\#\{s=\sigma+it\colon -1<\sigma<\tfrac{1}{2}, T_1<t\leqslant T_2, \zeta'(s)=0\}\\ = \#\{s=\sigma+it\colon 0<\sigma<\tfrac{1}{2}, T_1<t\leqslant T_2, \zeta(s)=0\}\\ +O(\log T_2),\end{aligned} \tag{10.28.2}$$

zeros being counted according to multiplicity. This is due to Levinson and Montgomery [1], who also gave a number of other interesting results on the distribution of the zeros of $\zeta'(s)$.

We sketch the proof of (10.28.2). We shall make frequent reference to the logarithmic derivative of the functional equation (2.6.4), which we write in the form

$$\begin{aligned}\frac{\zeta'(s)}{\zeta(s)}+\frac{\zeta'(1-s)}{\zeta(1-s)} &= \log\pi-\tfrac{1}{2}\left(\frac{\Gamma'(\frac{1}{2}s)}{\Gamma(\frac{1}{2}s)}+\frac{\Gamma'(\frac{1}{2}-\frac{1}{2}s)}{\Gamma(\frac{1}{2}-\frac{1}{2}s)}\right)\\ &= -F(s),\end{aligned} \tag{10.28.3}$$

say. We note that $F(\frac{1}{2}+it)$ is always real, and that

$$F(s) = \log(t/2\pi)+O(1/t) \tag{10.28.4}$$

uniformly for $t \geqslant 1$ and $|\sigma| \leqslant 2$. To prove (10.28.2) it suffices to consider the case in which the numbers T_j are chosen so that $\zeta(s)$ and $\zeta'(s)$ do not vanish for $t = T_j$, $-1 \leqslant \sigma \leqslant \frac{1}{2}$. We examine the change in argument in $\zeta'(s)/\zeta(s)$ around the rectangle with vertices $\frac{1}{2}-\delta+iT_1$, $\frac{1}{2}-\delta+iT_2$, $-1+iT_2$, and $-1+iT_1$, where δ is a small positive number. Along the horizontal sides we apply the ideas of § 9.4 to $\zeta(s)$ and $\zeta'(s)$ separately. We note that $\zeta(s)$ and $\zeta'(s)$ are each $O(t^A)$ for $-3 \leqslant \sigma \leqslant 1$. Moreover we also have $|\zeta(-1+iT_j)| \gg T_j^{\frac{3}{2}}$, by the functional equation, and hence also

$$|\zeta'(-1+iT_j)| \gg T_j^{\frac{3}{2}}\left|\frac{\zeta'(-1+iT_j)}{\zeta(-1+iT_j)}\right| \gg T_j^{\frac{3}{2}}\log T_j,$$

by (10.28.3) and (10.28.4). The method of § 9.4 therefore shows that $\arg\zeta(s)$ and $\arg\zeta'(s)$ both vary by $O(\log T_2)$ on the horizontal sides of the rectangle. On the vertical side $\sigma = -1$ we have

$$\frac{\zeta'(s)}{\zeta(s)} = \log\left(\frac{t}{2\pi}\right) + O(1),$$

by (10.28.3) and (10.28.4), so that the contribution to the total change in argument is $O(1)$. For the vertical side $\sigma = \frac{1}{2} - \delta$ we first observe from (10.28.3) and (10.28.4) that

$$\mathbf{R}\left(-\frac{\zeta'(\frac{1}{2}+it)}{\zeta(\frac{1}{2}+it)}\right) \geqslant 1 \tag{10.28.5}$$

if $t \geqslant T_1$ with T_1 sufficiently large. It follows that

$$\mathbf{R}\left(-\frac{\zeta'(\frac{1}{2}-\delta+it)}{\zeta(\frac{1}{2}-\delta+it)}\right) \geqslant \tfrac{1}{2} \tag{10.28.6}$$

for $T_1 \leqslant t \leqslant T_2$, if $\delta = \delta(T_2)$ is small enough. To see this, it suffices to examine a neighbourhood of a zero $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$. Then

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{m}{s-\rho} + m' + O(|s-\rho|),$$

where $m \geqslant 1$ is the multiplicity of ρ. The choice $s = \frac{1}{2} + it$ with $t \to \gamma$ therefore yields $\mathbf{R}(m') \geqslant 1$, by (10.28.5). Hence, on taking $s = \frac{1}{2} - \delta + it$, we find that

$$\mathbf{R}\left(-\frac{\zeta'(s)}{\zeta(s)}\right) = \frac{m\delta}{|s-\rho|^2} + \mathbf{R}(m') + O(|s-\rho|) \geqslant \tfrac{1}{2}$$

for $|s-\rho|$ small enough. The inequality (10.28.6) now follows. We therefore see that $\arg \zeta'(s)/\zeta(s)$ varies by $O(1)$ on the vertical side $\mathbf{R}(s) = \frac{1}{2} - \delta$ of our rectangle, which completes the proof of (10.28.2).

If we write N for the quantity on the left of (10.28.2) it follows that

$$N_0(T_2) - N_0(T_1) = \{N(T_2) - N(T_1)\} - 2N + O(\log T_2), \tag{10.28.7}$$

so that we now require an upper bound for N. This is achieved by applying the 'mollifier method' of §§9.20–24 to $\zeta'(1-s)$. Let $\nu(\sigma, T_1, T_2)$ denote the number of zeros of $\zeta'(1-s)$ in the rectangle $\sigma \leqslant \mathbf{R}(s) \leqslant 2$, $T_1 < \mathbf{I}(s) < T_2$. The method produces an upper bound for

$$\int_u^2 \nu(\sigma, T_1, T_2)\, d\sigma, \tag{10.28.8}$$

which in turn yields an estimate $N \leqslant c\{N(T_2) - N(T_1)\}$ for large T_2. The constant c in this latter bound has to be calculated explicitly, and must

be less than $\frac{1}{2}$ for (10.28.7) to be of use. This is in contrast to (9.20.5), in which the implied constant was not calculated explicitly, and would have been relatively large. It is difficult to have much feel in advance for how large the constant c produced by the method will be. The following very loose argument gives one some hope that c will turn out to be reasonably small, and so it transpires in practice.

In using (10.28.8) to obtain a bound for N we shall take

$$u = \tfrac{1}{2} - a/\log T_2,$$

where a is a positive constant to be chosen later. The zeros $\rho' = \beta' + i\gamma'$ of $\zeta'(1-s)$ have an asymmetrical distribution about the critical line. Indeed Levinson and Montgomery [1] showed that

$$\sum_{0<\gamma'\leqslant T} (\tfrac{1}{2} - \beta') \sim \frac{T}{2\pi} \log\log T,$$

whence β' is $\frac{1}{2} - (\log\log \gamma')/\log \gamma'$ on average. Thus one might reasonably hope that a fair proportion of such zeros have $\beta' < u$, thereby making the integral (10.28.8) rather small.

We now look in more detail at the method. In the first place, it is convenient to replace $\zeta'(1-s)$ by

$$\zeta(s) + \frac{\zeta'(s)}{F(s)} = G(s),$$

say. If we write $h(s) = \pi^{-\frac{1}{2}s}\,\Gamma(\frac{1}{2}s)$ then (10.28.3), together with the functional equation (2.6.4), yields

$$\zeta'(1-s) = -\frac{F(s)\,h(s)\,G(s)}{h(1-s)},$$

so that $G(s)$ and $\zeta'(1-s)$ have the same zeros for t large enough. Now let

$$\psi(s) = \sum_{n<y} b_n n^{-s} \tag{10.28.9}$$

be a suitable 'mollifier' for $G(s)$, and apply Littlewood's formula (9.9.1) to the function $G(s)\psi(s)$ and the rectangle with vertices $u + iT_1$, $2 + iT_2$, $2 + iT_1$, $u + iT_2$. Then, as in §9.16, we find that

$$N \leqslant \frac{\log T_2}{a} \int_u^2 \nu(\sigma, T_1, T_2)\, d\sigma$$

$$\leqslant \frac{\log T_2}{2\pi a} \int_{T_1}^{T_2} \log |G(u+it)\,\psi(u+it)|\, dt + O(\log T_2).$$

Moreover, as in §9.16 we have

$$\int_{T_1}^{T_2} \log |G(u+it)\psi(u+it)|\,dt$$

$$\leqslant \tfrac{1}{2}(T_2-T_1)\log\left(\frac{1}{T_2-T_1}\int_{T_1}^{T_2}|G(u+it)\psi(u+it)|^2\,dt\right).$$

Hence, if we can show that

$$\int_{T_1}^{T_2}|G(u+it)\psi(u+it)|^2\,dt \sim c(a)\,(T_2-T_1) \qquad (10.28.10)$$

for suitable T_1, T_2, we will have

$$N \leqslant \left(\frac{\log c(a)}{2a}+o(1)\right)\{N(T_2)-N(T_1)\}, \qquad (10.28.11)$$

whence

$$N_0(T_2)-N_0(T_1) \geqslant \left(1-\frac{\log c(a)}{a}+o(1)\right)\{N(T_2)-N(T_1)\}$$

by (10.28.7).

The computation of the mean value (10.28.10) is the most awkward part of Levinson's argument. In [2] he takes $y = T_2^{\frac{1}{2}-\varepsilon}$ and

$$b_n = \mu(n)\,n^{u-\frac{1}{2}}\frac{\log y/n}{\log y}.$$

This leads eventually to (10.28.10) with

$$c(a) = e^{2a}\left(\frac{1}{2a^3}+\frac{1}{24a}\right)-\frac{1}{2a^3}-\frac{1}{a^2}-\frac{25}{24a}+\frac{7}{12}-\frac{a}{12}.$$

The optimal choice of a is roughly $a = 1{\cdot}3$, which produces (10.28.1) with $= 0{\cdot}342$.

The method has been improved slightly by Levinson [4], [5], Lou [1] and Conrey [1] and the best constant thus far is $\alpha = 0{\cdot}3658$ (Conrey [1]). The principal restriction on the method is that on the size of y in (10.28.9). The above authors all take $y = T_2^{\frac{1}{2}-\varepsilon}$, but there is some scope for improvement via the ideas used in the mean-value theorems (7.24.5), (7.24.6), and (7.24.7).

10.29. An examination of the argument just given reveals that the right hand side of (10.28.11) gives an upper bound for $N+N^*$, where

$$N^* = \#\{s=\tfrac{1}{2}+it\colon T_1<t\leqslant T_2,\ \zeta'(s)=0\},$$

(zeros being counted according to multiplicities). However it is clear from (10.28.3) and (10.28.4) that $\zeta'(\frac{1}{2}+it)$ can only vanish if $\zeta(\frac{1}{2}+it)$ does. Consequently, if we write $N^{(r)}$ for the number of zeros of $\zeta(s)$ of multiplicity r, on the line segment $s=\frac{1}{2}+it$, $T_1<t\leqslant T_2$, we will have

$$N^* = \sum_{r=2}^{\infty}(r-1)\,N^{(r)}.$$

Thus (10.28.7) may be replaced by

$$N^{(1)}-\sum_{r=3}^{\infty}(r-2)\,N^{(r)} = \{N(T_2)-N(T_1)\}-2(N+N^*)+O(\log T_2).$$

If we now define $N^{(r)}(T)$ in analogy to $N^{(r)}$, but counting zeros $\frac{1}{2}+it$ with $0<t\leqslant T$, we may deduce that

$$N^{(1)}(T)-\sum_{r=3}^{\infty}(r-2)\,N^{(r)}(T) \geqslant \alpha N(T), \qquad (10.29.1)$$

for large enough T, and $\alpha = 0{\cdot}342$. In particular at least a third of the non-trivial zeros of $\zeta(s)$ not only lie on the critical line, but are simple. This observation is due independently to Heath-Brown [5] and Selberg (unpublished). The improved constants α mentioned above do not all allow this refinement. However it has been shown by Anderson [1] that (10.29.1) holds with $\alpha = 0{\cdot}3532$.

10.30. Levinson's method can be applied equally to the derivatives $\xi^{(m)}(s)$ of the function $\xi(s)$ given by (2.1.12). One can show that the zeros of these functions lie in the critical strip, and that the number of them, $N_m(T)$ say, for $0<t\leqslant T$, is $N(T)+O_m(\log T)$. If the Riemann hypothesis holds then all these zeros must lie on the critical line. Thus it is of some interest to give unconditional estimates for

$$\liminf_{T\to\infty} N_m(T)^{-1}\,\#\{t\colon 0<t\leqslant T,\ \xi^{(m)}(\tfrac{1}{2}+it)=0\} = \alpha_m,$$

say. Levinson [3], [5] showed that $\alpha_1 \geqslant 0{\cdot}71$, and Conrey [1] improved and extended the method to give $\alpha_1 \geqslant 0{\cdot}8137$, $\alpha_2 \geqslant 0{\cdot}9584$ and in general $\alpha_m = 1+O(m^{-2})$.