would in turn have an obvious but quite interesting corollary: if, for a weight $W(x)$ with uniformly Lip 1 iterated logarithm, there is *even one* entire function $\Phi(z) \not\equiv 0$ of *some* (finite) *exponential type* making $W(x)\Phi(x)$ *bounded* on $\mathbb{R}$, there must be such functions $\varphi(z) \not\equiv 0$ of *arbitrarily small exponential type* having the *same property*. The example given in the first article will show that *even this corollary is false.*

The absence of an additional condition on *just* the weights with uniform Lip 1 iterated logarithms *would*, by the way, *imply* that absence for *all* weights meeting our (less stringent) local regularity requirement. Indeed, if $W(x) \geqslant 1$ fulfills the latter (with constants C, α and L), and

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2}\,dx \quad < \quad \infty,$$

we *know* from the proof of the first theorem in §B.1 that W *admits multipliers* (for which the last relation is at least *necessary*), *if and only if* the weight

$$W_1(x) \quad = \quad \exp\left\{\frac{4}{\pi\alpha}\int_{-\infty}^{\infty} \frac{L\log W(t)}{(x-t)^2+L^2}\,dt\right\}$$

also does. We see, however, that $|\,d\log W_1(x)/dx\,| \quad \leqslant \quad (1/L)\log W_1(x)$, i.e.,

$$\left|\frac{d\log\log W_1(x)}{dx}\right| \quad \leqslant \quad \frac{1}{L},$$

so W_1 *does* have a uniformly Lip 1 iterated logarithm.

Let us go on to the first example.

1. Example. Uniform Lip 1 condition on $\log\log W(x)$ not sufficient

Take the points

$$x_p \quad = \quad e^{p^{1/3}}, \quad p \quad = \quad 8, 9, 10, \ldots,$$

and put

$$\Delta_p \quad = \quad \begin{cases} x_8, & p = 8, \\ x_p - x_{p-1}, & p > 8. \end{cases}$$

Let then

$$F(z) \quad = \quad \prod_{p=8}^{\infty}\left(1 \;-\; \frac{z^2}{x_p^2}\right)^{[\Delta_p]},$$

where $[\Delta_p]$ denotes the largest integer $\leqslant \Delta_p$; it is not hard to see – is, indeed, a particular consequence of the following work – that the product is convergent, making $F(z)$ an entire function with a zero of order $[\Delta_p]$ at each of the points $\pm x_p$, $p \geqslant 8$, and no other zeros.

According to custom, we write $n(t)$ for the *number of zeros* of $F(z)$ in $[0, t]$ (counting multiplicities) when $t \geqslant 0$. Thus,

$$n(t) = 0 \quad \text{for } 0 \leqslant t < x_8$$

and

$$n(t) = [\Delta_8] + [\Delta_9] + \cdots + [\Delta_p] \qquad \text{for } x_p \leqslant t < x_{p+1}.$$

The *right side* of the last relation lies between

$$\Delta_8 + \Delta_9 + \cdots + \Delta_p - p = x_p - p$$

and

$$\Delta_8 + \Delta_9 + \cdots + \Delta_p = x_p,$$

so, since

$$p = (\log x_p)^3,$$

we have

$$t - \Delta_{p+1} - (\log x_p)^3 \leqslant n(t) \leqslant t \qquad \text{for } x_p \leqslant t < x_{p+1},$$

with the *second* inequality actually valid for *all* $t \geqslant 0$. Here,

$$\Delta_{p+1} = e^{(p+1)^{1/3}} - e^{p^{1/3}} = (\tfrac{1}{3}p^{-2/3} + O(p^{-5/3}))x_p$$

is

$$\sim \frac{x_p}{3(\log x_p)^2},$$

making

$$t\left(1 - \frac{1}{3(\log t)^2} - O\left(\frac{1}{\log t}\right)^5\right) \leqslant n(t) \leqslant t$$

for $x_p \leqslant t < x_{p+1}$. We thus certainly have

$$t - \frac{t}{(\log t)^2} \leqslant n(t) \leqslant t$$

for large values of t, with the *upper bound* in fact holding for all $t \geqslant 0$, as just noted.

We can write

$$\log|F(z)| \quad = \quad \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}n(t),$$

and the reader should now refer to problem 29 (§B.1, Chapter IX). Reasoning as in part (a) of that problem, one readily concludes that

$$\frac{\log|F(\mathrm{i}y)|}{y} \longrightarrow \pi \qquad \text{for } y \longrightarrow \infty,$$

since

$$\frac{n(t)}{t} \longrightarrow 1 \quad \text{as } t \longrightarrow \infty$$

by the previous relation. Clearly,

$$|F(z)| \leqslant F(\mathrm{i}|z|),$$

so our function $F(z)$ *is of exponential type* π.

To estimate $|F(x)|$ for real x, we refer to part (c) of the same problem, according to which

$$\log|F(x)| \quad \leqslant \quad 2n(x)\log\frac{1}{\lambda} \quad + \quad 2\int_0^\lambda \frac{\dfrac{n(xt)}{t} - tn\left(\dfrac{x}{t}\right)}{1 - t^2}\,\mathrm{d}t$$

for $x > 0$, where for λ we may take any number between 0 and 1. Assuming x *large*, we put

$$\lambda \quad = \quad 1 - \frac{1}{(\log x)^2}$$

and plug the above relation for $n(t)$ into the integral (using, of course, the upper bound with $n(xt)/t$ and the lower one with $tn(x/t)$). We thus find that

$$\log|F(x)| \quad \leqslant \quad 2n(x)\log\frac{1}{\lambda} \quad + \quad 2x\int_0^\lambda \frac{\mathrm{d}t}{(\log(x/t))^2(1-t^2)}$$

$$\leqslant \quad \text{const.}\frac{x}{(\log x)^2} \quad + \quad x\,\frac{\log 2 + 2\log\log x}{(\log x)^2} \quad \leqslant \quad C\,\frac{x\log\log x}{(\log x)^2}$$

for large values of x.

The quantity on the right is *increasing* when $x > 0$ is large enough, and satisfies

$$\int_e^\infty \frac{1}{x^2} C \frac{x \log\log x}{(\log x)^2} \mathrm{d}x \quad = \quad C \int_1^\infty \frac{\log u}{u^2} \mathrm{d}u \quad = \quad C \quad < \quad \infty.$$

Therefore, since $\log|F(x)|$ is *even* and bounded above by that quantity when x is large, we can conclude by the elementary Paley–Wiener multiplier theorem of Chapter X, §A.1 (obtained by a different method far back in §D of Chapter IV!) that there is, corresponding to any $\eta > 0$, a non-zero entire function $\psi(z)$ of exponential type $\leqslant \eta$ with $F(x)\psi(x)$ *bounded* on the real axis. The function $\psi(z)$ obtained in Chapter X is in fact of the form $\varphi(z + \mathrm{i})$, where

$$\varphi(z) \quad = \quad \prod_k \left(1 - \frac{z^2}{\lambda_k^2}\right)$$

is *even* and has only the *real* zeros $\pm\lambda_k$; it is thus clear that for $x \in \mathbb{R}$,

$$|\psi(x)| \quad = \quad |\varphi(x + \mathrm{i})| \quad \geqslant \quad |\varphi(x)|.$$

(Observe that for $\zeta \; = \; \xi + \eta\mathrm{i}$,

$$|1 - \zeta^2| \quad = \quad |1 + \zeta||1 - \zeta| \quad \geqslant \quad |1 + \xi||1 - \xi| \;!)$$

Hence,

$$|F(x)\varphi(x)| \quad \leqslant \quad \text{const.} \quad \text{on } \mathbb{R}$$

with an *even* function $\varphi(z)$ of exponential type $\leqslant$ η *having only real zeros.*

Fixing a constant $c > 0$ for which

$$c|F(x)\varphi(x)| \quad \leqslant \quad 1, \quad x \in \mathbb{R},$$

we put

$$\Psi(z) \quad = \quad cF(z)\varphi(z),$$

getting a certain *even* entire function Ψ, with *only real zeros*, having exponential type equal to a number B lying between π (the type of F) and $\pi + \eta$. For this function the Poisson representation

$$\log|\Psi(z)| \quad = \quad B\Im z \; + \; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log|\Psi(t)|}{|z - t|^2} \mathrm{d}t$$

from §G.1 of Chapter III is valid for $\Im z \; > \; 0$, the integral on the right being absolutely convergent. In particular,

$$\log\left|\frac{\mathrm{e}^B}{\Psi(x + \mathrm{i})}\right| \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{(x - t)^2 + 1} \log\frac{1}{|\Psi(t)|} \mathrm{d}t \qquad \text{for } x \in \mathbb{R},$$

where the integral is obviously $\geqslant$ 0.

We now take

$$W(x) \quad = \quad \frac{e^B}{|\Psi(x+i)|}, \qquad x \in \mathbb{R}.$$

Then $|W(x)| \geq 1$ and differentiation of the preceding formula immediately yields

$$\left|\frac{d \log W(x)}{dx}\right| \quad \leq \quad \log W(x),$$

making

$$|\log\log W(x) \ - \ \log\log W(x')| \quad \leq \quad |x - x'| \qquad \text{for } x, \ x' \in \mathbb{R}.$$

From the same formula we also see that

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2} dx \quad < \quad \infty.$$

It is claimed, however, that the weight W has no mutipliers of exponential type $< \ \pi$. Suppose, indeed, that there *is* a non-zero entire function $f(z)$ of exponential type $A' \ < \ \pi$ with $f(x)W(x)$ bounded on $\mathbb{R}$. According to the discussion following the *first* theorem of §B.1 we can, from f, obtain another non-zero entire function g of exponential type $A \leq A'$ (hence $A < \pi$ – A is in fact *equal* to A'), *having only real zeros*, with

$$|g(x)| \ \leq \ |f(x)| \quad \text{on } \mathbb{R},$$

so that $g(x)W(x)$ is also bounded for $x \in \mathbb{R}$. This function $g(z)$ (denoted by $C\psi(z)$ in the passage referred to) is a constant multiple of a product like

$$e^{bz} \prod_{\lambda'} \left(1 \ - \ \frac{z}{\lambda'}\right) e^{z/\lambda'}$$

formed from *real* numbers b and λ', and hence has the important property that

$$|g(\Re z)| \ \leq \ |g(z)|,$$

which we shall presently have occasion to use.

There is no loss of generality in our assuming that

$$|g(x)W(x)| \quad \leq \quad e^{-B} \qquad \text{for } x \in \mathbb{R}.$$

Referring to our definition of W, we see that this is the same as the relation

$$|g(x)| \quad \leq \quad e^{-2B}|\Psi(x+i)|, \qquad x \in \mathbb{R}.$$

To the function g, having all its zeros on the real axis (and surely *bounded* there!) we may apply the Poisson representation from §G.1 of Chapter III to get

$$\log|g(z)| \quad = \quad A\Im z \ + \ \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log|g(t)|}{|z-t|^2}\,dt, \qquad \Im z \ > \ 0.$$

In like manner,

$$\log|\Psi(z+i)| \quad = \quad B\Im z \ + \ \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log|\Psi(t+i)|}{|z-t|^2}\,dt, \qquad \Im z \ > \ 0,$$

so that, for $\Im z \ > \ 0$,

$$\log\left|\frac{g(z)}{\Psi(z+i)}\right| \quad = \quad (A-B)\Im z \ + \ \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log\left|\frac{g(t)}{\Psi(t+i)}\right| dt.$$

Since $A < \pi \leqslant B$, the right side in the last relation is $\leqslant \ -2B$ by the preceding inequality, so we have in particular

$$|g(x+i)| \quad \leqslant \quad e^{-2B}|\Psi(x+2i)|, \qquad x \in \mathbb{R}.$$

Let us note moreover that $e^{-2B}|\Psi(x+2i)| \ \leqslant \ 1$ on the real axis by the third Phragmén–Lindelöf theorem of Chapter III, §C, $\Psi(z)$ being of exponential type B and of modulus $\leqslant \ 1$ for real z. Another application of the same Phragmén–Lindelöf theorem thence shows that

$$e^{-2B}|\Psi(z+2i)| \quad \leqslant \quad e^{B|\Im z|}$$

This estimate will also be of use to us.

Our idea now is to show that $|g(x)|$ must get so small near the zeros $\pm x_p$ of our original function $F(z)$ as to make

$$\int_{-\infty}^{\infty} \frac{\log^-|g(x)|}{1+x^2}\,dx \quad = \quad \infty$$

and thus imply that $g(z) \ \equiv \ 0$ (*a contradiction!*) by §G.2 of Chapter III. We start by looking at $|g(x_p+i)|$, which a previous relation shows to be $\leqslant \ e^{-2B}|\Psi(x_p+2i)|$. The latter quantity we estimate by Jensen's formula.

For the moment, let us denote by $N(r, \ z_0)$ the *number of zeros of* $\Psi(z)$ inside any closed disk of the form $\{|z-z_0| \ \leqslant \ r\}$. Then we have, for any $R > 0$,

$$\log|e^{-2B}\Psi(x_p+2i)| \quad = \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|e^{-2B}\Psi(x_p+2i+Re^{i\vartheta})|\,d\vartheta$$
$$- \quad \int_0^R \frac{N(r, \ x_p+2i)}{r}\,dr.$$

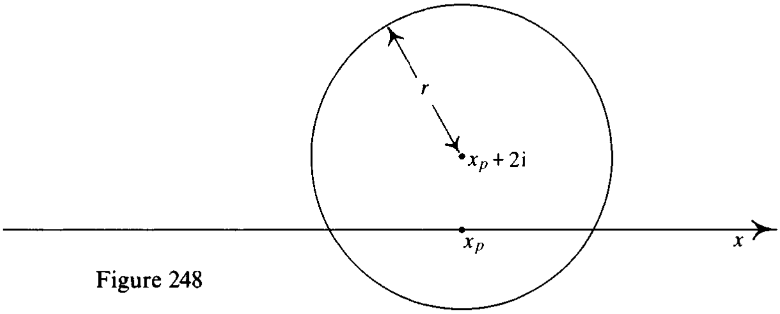

Figure 248

Substituting the *last* of the above relations involving Ψ into the first integral on the right and noting that $\Psi(z)$ has *at least a* $[\Delta_p]$*-fold zero at* x_p, we see that for $R > 2$,

$$\log|e^{-2B}\Psi(x_p+2i)| \;\leqslant\; \frac{2}{\pi}BR \;-\; [\Delta_p]\log\frac{R}{2}.$$

Write now $\nu(r, z_0)$ for the *number of zeros of* $g(z)$ in the closed disk $\{|z-z_0| \leqslant r\}$. Application of Jensen's formula to g then yields

$$\log|g(x_p+i)| \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi}\log|g(x_p+i+Re^{i\vartheta})|\,d\vartheta \;-\; \int_0^R \frac{\nu(r,\;x_p+i)}{r}\,dr.$$

Using the inequality $|g(x_p+i)| \leqslant e^{-2B}|\Psi(x_p+2i)|$ and referring to the preceding relation we find, after transposing, that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|g(x_p+i+Re^{i\vartheta})|\,d\vartheta$$

$$\leqslant\; \frac{2}{\pi}BR \;+\; \int_0^R\frac{\nu(r,\;x_p+i)}{r}\,dr \;-\; [\Delta_p]\log\frac{R}{2} \qquad \text{for } R > 2.$$

Since all the zeros of $g(z)$ *are real,* $\nu(r,\;x_p+i)$ *is certainly zero for* $r < 1$, whence

$$\int_0^R\frac{\nu(r,\;x_p+i)}{r}\,dr \;\leqslant\; \nu(R,\;x_p+i)\log R, \qquad R > 1,$$

which, together with the last, gives

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|g(x_p+i+Re^{i\vartheta})|\,d\vartheta$$

$$\leqslant\; \frac{2}{\pi}BR \;+\; \big(\nu(R,\;x_p+i) \;-\; [\Delta_p]\big)\log R \;+\; [\Delta_p]\log 2, \qquad R > 2.$$

We want to use this to show that for a certain R_p, $\int_{-R_p}^{R_p} \log|g(x_p + t)|\,dt$ comes out *very negative.*

To do that, we simply (trick!) plug the inequality $|g(\Re z)| \leqslant |g(z)|$ noted above into the left side of the last relation. We are, in other words, *flattening* the circle involved in Jensen's formula to its horizontal diameter which is then moved down to the real axis:

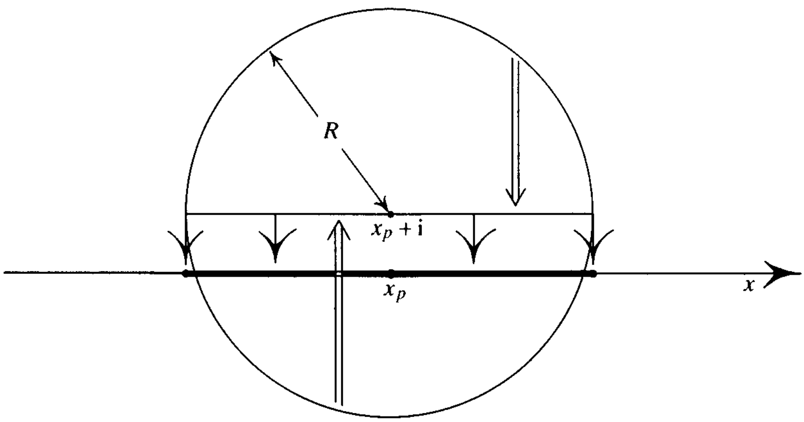

Figure 249

That causes $\log|g(x_p + i + Re^{i\vartheta})|$ to be replaced by $\log|g(x_p + R\cos\vartheta)|$ in the integral appearing in the relation in question; the resulting integral then becomes

$$\frac{1}{\pi}\int_{-R}^{R} \frac{\log|g(x_p + s)|}{\sqrt{(R^2 - s^2)}}\,ds$$

on making the substitution $R\cos\vartheta = s$. What we have just written is hence

$$\leqslant \quad \frac{2}{\pi}BR \; + \; \big(\nu(R,\; x_p + i) \; - \; [\Delta_p]\big)\log R \; + \; [\Delta_p]\log 2 \qquad \text{for } R > 2.$$

Our reasoning at this point is much like that in §D.1 of Chapter IX. Taking

$$R_p \quad = \quad \frac{\Delta_p}{2},$$

we multiply the preceding integral and the expression immediately following it by $R\,dR$ and integrate from $R_p/2$ to R_p. That yields

$$\frac{1}{\pi}\int_{R_p/2}^{R_p}\int_{-R}^{R}\frac{R\log|g(x_p+s)|}{\sqrt{(R^2-s^2)}}\,\mathrm{d}s\,\mathrm{d}R$$
$$\leqslant \quad \frac{7B}{12\pi}R_p^3 \quad + \quad \frac{3}{8}R_p^2\,\nu(R_p,\ x_p+\mathrm{i})\log R_p$$
$$- \quad \frac{3}{8}R_p^2[\Delta_p]\log\frac{R_p}{2} \quad + \quad \frac{3\log 2}{8}[\Delta_p]R_p^2.$$

An integral like the one on the left (involving $\log|\hat{\mu}(c+t)|$ instead of $\log|g(x_p+s)|$) has already figured in the proof of the theorem from the passage just referred to. Here, we may argue as in that proof (reversing the order of integration), for $\log|g(x)| \leqslant 0$ on the real axis, as follows from the inequalities $|g(x)|W(x) \leqslant \mathrm{e}^{-B}$ and $W(x) \geqslant 1$, valid thereon. In that way, one finds the left-hand integral to be

$$\geqslant \quad \frac{\sqrt{3}}{2\pi}R_p\int_{-R_p}^{R_p}\log|g(x_p+s)|\,\mathrm{d}s.$$

After dividing by $R_px_p^2$ and clearing out some coefficients, we see that

$$\frac{1}{x_p^2}\int_{-R_p}^{R_p}\log|g(x_p+s)|\,\mathrm{d}s$$
$$\leqslant \quad \frac{7B}{6\sqrt{3}}\frac{R_p^2}{x_p^2} \quad + \quad \frac{\sqrt{3}\,\pi}{4}\left\{\nu(R_p,\ x_p+\mathrm{i}) \ - \ [\Delta_p]\right\}\frac{R_p\log R_p}{x_p^2}$$
$$+ \quad \frac{\sqrt{3}\,\pi\log 2}{2}[\Delta_p]\frac{R_p}{x_p^2}$$

(the actual values of the numerical coefficients on the right are not so important).

The quantities $R_p = \Delta_p/2 = (x_p - x_{p-1})/2$ are *increasing* when $p > 8$ because $x_p = \exp(p^{1/3})$, and the latter function has a *positive second derivative* for $p > 8$. (*Now* we see why we use the sequence of points x_p beginning with x_8 !) The intervals $[x_p - R_p,\ x_p + R_p]$ *therefore do not overlap when* $p > 8$, so our desired conclusion, namely, that

$$\int_{-\infty}^{\infty}\frac{\log^-|g(x)|}{1+x^2}\,\mathrm{d}x \quad = \quad \infty,$$

will surely follow if we can establish that

$$\sum_{p>8}\frac{1}{x_p^2}\int_{-R_p}^{R_p}\log|g(x_p+s)|\,\mathrm{d}s \quad = \quad -\infty$$

with the help of the preceding relation.

Here, we are guided by a simple idea. Everything turns on the *middle term* figuring on the right side of our relation, for the sums of the *first* and *third* terms are readily seen to be *convergent*. To see how the middle term behaves, we observe that by *Levinson's theorem* (!), the function $g(z)$, bounded on the real axis and of exponential type A, should, *on the average*, have about

$$\frac{2}{\pi}AR_p \quad = \quad \frac{A}{\pi}\Delta_p$$

zeros on the interval $(x_p - R_p,\ x_p + R_p]$, for all of g's zeros are real. The quantity $\nu(R_p,\ x_p + \mathrm{i})$ is clearly *not more* than that number of zeros, so the factor in $\{\quad\}$ from our middle term should, *on the average*, be

$$\leqslant \quad -\frac{\pi - A}{\pi}\Delta_p$$

(approximately). Straightforward computation easily shows, however, that

$$\frac{\Delta_p R_p \log R_p}{x_p^2} \quad \sim \quad \frac{1}{18p}$$

for large values of p. It is thus quite plausible that the series

$$\sum_p \left\{\nu(R_p,\ x_p + \mathrm{i}) \quad - \quad [\Delta_p]\right\} \frac{R_p \log R_p}{x_p^2}$$

should diverge to $-\infty$. This inference is in fact *correct*, but for its justification we must resort to a technical device.

Picking a number $\gamma > 1$ *close* to 1 (the exact manner of choosing it will be described presently), we form the sequence

$$X_m \quad = \quad \gamma^m, \qquad m \ = \ 1, 2, 3, \ldots .$$

We think of $\{X_m\}$ as a *coarse* sequence of points, amongst which those of $\{x_p\}$ – regarded as a *fine* sequence – are interspersed:

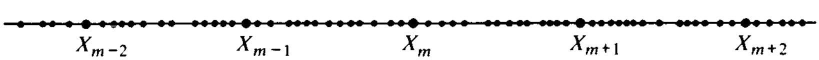

Figure 250

It is convenient to denote by $\nu(t)$ the *number of zeros* of $g(t)$ in $[0,\ t]$ for $t \geqslant 0$; then, as remarked above,

$$\nu(R_p,\ x_p + \mathrm{i}) \quad \leqslant \quad \nu(x_p + R_p) \quad - \quad \nu(x_p - R_p).$$

For any large value of m we thus have, recalling that $R_p = \Delta_p/2$,

$$\sum_{X_m < x_p \leqslant X_{m+1}} \nu(R_p,\ x_p + \mathrm{i}) \frac{R_p \log R_p}{x_p^2}$$

$$\leqslant \sum_{X_m < x_p \leqslant X_{m+1}} \left\{ \nu\left(x_p + \frac{\Delta_p}{2} \right) - \nu\left(x_p - \frac{\Delta_p}{2} \right) \right\} \cdot \sup_{X_m < x_p \leqslant X_{m+1}} \frac{\Delta_p \log \Delta_p}{2x_p^2}.$$

Denote by h_m the value of $\Delta_p/2$ corresponding to the *smallest* $x_p > X_m$, and by h'_{m+1} the value of that quantity corresponding to the *largest* $x_p \leqslant X_{m+1}$. Since, for $p > 8$, the intervals $[x_p - \frac{1}{2}\Delta_p,\ x_p + \frac{1}{2}\Delta_p]$ don't overlap, we have

$$\sum_{X_m < x_p \leqslant X_{m+1}} \left\{ \nu\left(x_p + \frac{\Delta_p}{2} \right) - \nu\left(x_p - \frac{\Delta_p}{2} \right) \right\} \leqslant \nu(X_{m+1} + h'_{m+1}) - \nu(X_m - h_m).$$

According to Levinson's theorem (the simpler version from §H.2 of Chapter III is adequate here), we have

$$\frac{A}{\pi}t - \mathrm{o}(t) \leqslant \nu(t) \leqslant \frac{A}{\pi}t + \mathrm{o}(t)$$

for t tending to ∞. Since, in our construction, $\Delta_p = \mathrm{o}(x_p)$ for large p, it follows that $h_m = \mathrm{o}(X_m)$ and $h'_{m+1} = \mathrm{o}(X_{m+1})$ for $m \longrightarrow \infty$; the preceding relation thus implies that

$$\nu(X_{m+1} + h'_{m+1}) - \nu(X_m - h_m) \leqslant \frac{A}{\pi}(X_{m+1} - X_m) + \mathrm{o}(X_{m+1})$$

when $m \longrightarrow \infty$. Hence, since $X_{m+1} - X_m = (1 - 1/\gamma)X_{m+1}$, we have

$$\sum_{X_m < x_p \leqslant X_{m+1}} \left\{ \nu\left(x_p + \frac{\Delta_p}{2} \right) - \nu\left(x_p - \frac{\Delta_p}{2} \right) \right\} \leqslant \left(\frac{A}{\pi} + \varepsilon \right)(X_{m+1} - X_m)$$

for any given $\varepsilon > 0$, as long as m is sufficiently large.

We require an estimate of $(\Delta_p \log \Delta_p)/x_p^2$ for $X_m < x_p \leqslant X_{m+1}$. As p tends to ∞,

$$\Delta_p \log \Delta_p \sim \tfrac{1}{3}p^{-2/3} e^{p^{1/3}} \log(\tfrac{1}{3}p^{-2/3} e^{p^{1/3}}) \sim \frac{x_p}{3p^{1/3}},$$

so

$$\frac{\Delta_p \log \Delta_p}{2x_p^2} \sim \frac{1}{6x_p \log x_p},$$

a *decreasing* function of p. Therefore, since $X_m = \gamma^m$,

$$\sup_{X_m < x_p \leqslant X_{m+1}} \frac{\Delta_p \log \Delta_p}{2x_p^2} \leqslant \left(\frac{1}{6} + o(1)\right) \frac{1}{mX_m \log \gamma}$$

for large values of m.

Use this estimate together with the preceding one in the above relation. It is found that

$$\sum_{X_m < x_p \leqslant X_{m+1}} v(R_p,\ x_p + i) \frac{R_p \log R_p}{x_p^2} \leqslant \frac{1}{6}\left(\frac{A}{\pi} + 2\varepsilon\right) \frac{\gamma - 1}{\log \gamma} \cdot \frac{1}{m}$$

for sufficiently large values of m, where $\varepsilon > 0$ is arbitrary.

We turn to the sum

$$\sum_{X_m < x_p \leqslant X_{m+1}} \frac{[\Delta_p]\, R_p \log R_p}{x_p^2},$$

for which a *lower bound* is needed. We have

$$\frac{[\Delta_p]\, R_p \log R_p}{x_p^2} \sim \frac{1}{2x_p^2} (\tfrac{1}{3} p^{-2/3} x_p)^2 \log(\tfrac{1}{6} p^{-2/3} x_p) \sim \frac{1}{18 p^{4/3}} \log x_p$$

$$= \frac{1}{18p} \qquad \text{for } p \longrightarrow \infty.$$

Thence, calling p_m the *smallest* value of p for which $x_p > X_m$ and p'_{m+1} the *largest* such value with $x_p \leqslant X_{m+1}$, the preceding sum works out to

$$\left(\frac{1}{18} + o(1)\right) \log \frac{p'_{m+1}}{p_m}$$

when m is large. In that case, $x_{p_m} = \exp(p_m^{1/3})$ is nearly $X_m = \gamma^m$, and $\exp((p'_{m+1})^{1/3})$ nearly γ^{m+1}. So then, p'_{m+1}/p_m is practically equal to $((m+1)/m)^3 \sim 1 + 3/m$, and

$$\log \frac{p'_{m+1}}{p_m} \sim \frac{3}{m}.$$

Thus,

$$\sum_{X_m < x_p \leqslant X_{m+1}} \frac{[\Delta_p]\, R_p \log R_p}{x_p^2} \geqslant \left(\frac{1}{6} - o(1)\right) \cdot \frac{1}{m}$$

for large values of m.

Now combine this result with the one obtained previously. One gets

$$\sum_{X_m < x_p \leqslant X_{m+1}} \{v(R_p,\ x_p + \mathrm{i}) \ - \ [\Delta_p]\} \frac{R_p \log R_p}{x_p^2}$$

$$\leqslant \ \frac{1}{6}\left\{\left(\frac{A}{\pi} \ + \ 2\varepsilon\right)\frac{\gamma - 1}{\log\gamma} \ - \ 1 \ + \ \varepsilon\right\}\cdot\frac{1}{m}$$

(with $\varepsilon > 0$ arbitrary) for large m. *We had, however,* $A/\pi < 1$. It is thus possible to *choose* $\gamma > 1$ *so as to make*

$$\frac{A}{\pi}\frac{\gamma - 1}{\log\gamma} \ < \ 1 \ - \ \delta,$$

say, where δ is a certain number > 0. *Fixing* such a γ, we can *then* take an $\varepsilon > 0$ small enough to ensure that

$$\frac{1}{6}\left\{\left(\frac{A}{\pi} \ + \ 2\varepsilon\right)\frac{\gamma - 1}{\log\gamma} \ - \ 1 \ + \ \varepsilon\right\} \ < \ -\frac{\delta}{7}.$$

The left-hand sum in the previous relation is then

$$\leqslant \ -\frac{\delta}{7m}$$

for sufficiently large m, so

$$\frac{\sqrt{3}\pi}{4}\sum_p \{v(R_p,\ x_p + \mathrm{i}) \ - \ [\Delta_p]\} \ \frac{R_p \log R_p}{x_p^2}$$

does diverge to $-\infty$.

Aside from the general term of this series, our upper bound on

$$\frac{1}{x_p^2}\int_{-R_p}^{R_p} \log|g(x_p + s)|\,\mathrm{d}s$$

involved *two other terms*, each of which is $\sim$ const.Δ_p^2/x_p^2 when p is large. But

$$\frac{\Delta_p^2}{x_p^2} \ \sim \ \frac{1}{9}p^{-4/3} \qquad \text{for } p \longrightarrow \infty,$$

so *the sum of those remaining terms is certainly convergent.* The divergence just established therefore does imply that

$$\sum_{p>8} \frac{1}{x_p^2}\int_{-R_p}^{R_p} \log|g(x_p + s)|\,\mathrm{d}s \ = \ -\infty,$$

and hence that

$$\int_{-\infty}^{\infty} \frac{\log^{-}|g(x)|}{1+x^2}\mathrm{d}x \quad = \quad \infty$$

as claimed, yielding finally our desired contradiction.

The weight

$$W(x) \quad = \quad \frac{\mathrm{e}^{B}}{|\Psi(x+\mathrm{i})|} \quad \geqslant \quad 1$$

constructed above thus *admits no multipliers f of exponential type* $< \pi$, *even though* it enjoys the regularity property

$$|\log\log W(x) \ - \ \log\log W(x')| \quad \leqslant \quad |x-x'|, \qquad x,\ x' \in \mathbb{R},$$

and satisfies the condition

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2}\mathrm{d}x \quad < \quad \infty.$$

2. Discussion

We see that our local regularity requirement and the convergence of the logarithmic integral do not, by themselves, ensure admittance of multipliers. Some other property of the weight is thus really involved.

For the weight W constructed in the example just given we actually had

$$|\log\log W(x) \ - \ \log\log W(x')| \quad \leqslant \quad |x-x'|$$

on $\mathbb{R}$. By this we are reminded that another regularity condition of similar appearance *has* previously been shown to *be* sufficient when combined with the requirement that $\int_{-\infty}^{\infty}(\log W(x)/(1+x^2))\mathrm{d}x < \infty$. The theorem proved in §C.1 of Chapter X (and reestablished by a different method at the end of §C.5 in this chapter) states that a weight W *does* admit multipliers if its logarithmic integral converges and

$$|\log W(x) \ - \ \log W(x')| \quad \leqslant \quad \text{const.}|x-x'|$$

on $\mathbb{R}$. A uniform Lipschitz condition on $\log W(x)$ thus *gives us* enough regularity, although such a requirement on $\log\log W(x)$ *does not.*

An *intermediate property* is in fact *already sufficient.* Consider a continuous weight W with $\log W(x) = O(x^2)$ near the origin (not a real restriction), and suppose, besides, that $W(x)$ is *even.* As remarked near the end of §B.1, that involves no loss of generality *either*, because $W(x)$ admits multipliers if and only if $W(x)W(-x)$ does. In these circumstances, convergence of the logarithmic integral is equivalent to the condition that

$$\int_0^\infty \frac{\log W(x)}{x^2} dx \quad < \quad \infty.$$

When this holds, we know, however, by the corollary at the end of §C.5 that $W(x)$, if it meets the local regularity requirement, *admits multipliers* as long as $\log W(x)/x$ belongs to the Hilbert space $\mathfrak{H}$ studied in §C.4, i.e., that

$$\| \log W(x)/x \|_E \quad < \quad \infty.$$

Problem 62 tells us on the other hand that an even weight $W(x)$ will *have* that property when the above integral is convergent and $\log W(x)$ uniformly Lip 1. These last conditions are thus *more stringent* than the *sufficient* ones furnished by the corollary of §C.5.

This fact leads us to believe, or at least to hope, that the intermediate property just spoken of could serve as basis for the formulation of *necessary and sufficient conditions for admittance of multipliers* by weights satisfying the local regularity requirement. But how one could set out to accomplish that is not immediately apparent, because pointwise behaviour *of the weight itself* seems at the same time *to be involved* and *not to be involved* in the matter.

Behaviour of the weight *itself* seems to *not* be directly involved (beyond the local regularity requirement), because, if $W(x)$ *admits multipliers, so does any weight* $W_1(x)$ *with* $1 \leqslant W_1(x) \leqslant W(x)$. Even when such a $W_1(x)$ meets the local regularity requirement, its behaviour may be very wild in comparison to whatever we may otherwise stipulate for $W(x)$. Were one, for instance to prescribe that $\| \log W(x)/x \|_E < \infty$, there would be weights W_1 *failing* to meet that criterion even though W *answered* to it – see the formula provided by the last theorem of §C.4.

Nevertheless, *some* condition on the behaviour of our weights *does* appear to be involved! Support for this point of view is obtained by putting the theorem on the multiplier (from §C.2) together with those of Szegő (from Chapter II!) and de Branges (Chapter VI, §F).

Consider *any* continuous function $W(x) \geqslant 1$ tending to ∞ for

$x \longrightarrow \pm\infty$, and such that

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2} \mathrm{d}x \quad < \quad \infty.$$

Then the weight $\Omega(x) = (1+x^2)W(x)$ also has convergent logarithmic integral, so, by the version of Szegő's theorem set as problem 2 (in Chapter II), there is *no finite sum*

$$s(x) \quad = \quad \sum_{\lambda \geqslant 1} a_\lambda \mathrm{e}^{\mathrm{i}\lambda x}$$

which can make

$$\int_{-\infty}^{\infty} \frac{|1 \; - \; s(x)|}{\Omega(x)} \mathrm{d}x$$

smaller than a certain $\delta \; > \; 0$. Hence, for any such $s(x)$, we must have

$$\sup_{x \in \mathbb{R}} \frac{|1 \; - \; s(x)|}{W(x)} \quad \geqslant \quad \frac{\delta}{\pi}.$$

Given any $L > 0$, this holds a fortiori for sums $s(x)$ of the form

$$s(x) \quad = \sum_{1 \leqslant \lambda \leqslant 2L+1} a_\lambda \mathrm{e}^{\mathrm{i}\lambda x}.$$

Problem 65

(a) Show that in this circumstance there is, corresponding to any L, an entire function $\Phi(z)$ of exponential type L such that

$$\int_{-\infty}^{\infty} \frac{\log^+ |\Phi(x)|}{1+x^2} \mathrm{d}x \quad < \quad \infty$$

and

$$W(x_n) \;\leqslant\; |\Phi(x_n)|$$

at the points x_n of a *two-way* real sequence Λ with $x_n \neq x_m$ for $n \neq m$ and

$$\frac{n_\Lambda(t)}{t} \longrightarrow \frac{L}{\pi} \qquad \text{for } t \longrightarrow \pm\infty.$$

Here, $n_\Lambda(t)$ denotes (as usual) the number of points of Λ in $[0, \; t]$ when $t \geqslant 0$ and *minus* the number of such points in $[t, \; 0)$ when $t < 0$. (Hint: See §F.3, Chapter VI.)

(b) Show that for the sequence $\{x_n\} = \Lambda$ obtained in (a) we also have

$$\tilde{D}_{\Lambda_+} = \tilde{D}_{\Lambda_-} = \frac{L}{\pi}$$

for $\Lambda_+ = \Lambda \cap [0, \infty)$ and $\Lambda_- = (-\Lambda) \cap (0, \infty)$, with $\tilde{D}$ the Beurling–Malliavin effective density defined in §D.2 of chapter IX. (Hint: See the very end of §E.2, Chapter IX.)

(c) Show that for any given $A > 0$ there is a non-zero entire function $f(z)$ of exponential type $\leqslant A$, bounded on $\mathbb{R}$, with

$$W(x_n)|f(x_n)| \leqslant \text{const.}$$

at the points x_n of the sequence from part (a).

The result obtained in part (c) of this problem holds on the *mere assumptions that* $W(x) \geqslant 1$ is *continuous* and tends to ∞ for $x \longrightarrow \pm\infty$, and that

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2} dx < \infty.$$

The points x_n on which any of the products $W(x)f(x)$ is *bounded* behave, however, *rather closely* like the ones of the *arithmetic progression*

$$\frac{\pi}{L} n, \qquad n = 0, \pm 1, \pm 2, \ldots .$$

which, for large enough L, seem to 'fill out' the real axis. From this standpoint it appears to be plausible that *some regularity property* of the weight $W(x)$ would be both *necessary and sufficient* to ensure *boundedness of the products* $W(x)f(x)$ *on* $\mathbb{R}$.

These considerations illustrate our present difficulty, but also suggest a way out of it, which is to look for an additional condition *pertaining to a majorant of* $W(x)$ rather than *directly* to the latter. That such an approach is *reasonable* is shown by the first theorem of §B.1, according to which a weight $W(x) \geqslant 1$ meeting the local regularity requirement (with constants C, α and L) and satisfying $\int_{-\infty}^{\infty} (\log W(x)/(1+x^2))dx < \infty$ *admits multipliers if and only if a certain* $\mathscr{C}_\infty$ *majorant of it also does so.* For that majorant one may take the weight

$$\Omega(x) = M \exp\left\{\frac{4}{\pi\alpha}\int_{-\infty}^{\infty} \frac{L \log W(t)}{(x-t)^2 + L^2} dt\right\}$$

where M is a large constant, and then $|d(\log\log\Omega(x))/dx| \leqslant 1/L$ on $\mathbb{R}$.

This idea actually underlies much of what is done in §B.1. One may of course use the *even* $\mathscr{C}_\infty$ *majorant* $\Omega(x)\Omega(-x)$ *instead* – see the remark just preceding problem 52.

Let us try then to characterize a weight's admittance of multipliers by the *existence for it of some even majorant also admitting multipliers and having, in addition, some specific kind of regularity.* What we have in mind at present is essentially the regularity embodied in the *intermediate property* described earlier in this article. We think the criterion should be that $W(x)$ have an even $\mathscr{C}_\infty$ majorant $\Omega(x)$ with $\log^+\log\Omega(x)$ uniformly Lip 1, $\int_0^\infty(\log\Omega(x)/x^2)\,dx < \infty$, and $\|\log\Omega(x)/x\|_E < \infty$.

A minor hitch encountered at this point is easily taken care of. The trouble is that neither of the last two of the conditions on Ω is compatible with Ω's being a *majorant* of W when $W(x) > 1$ on a neighborhood of the origin. That, however, should not present a real problem because admittance of multipliers by a finite weight W meeting the local regularity requirement *does not depend* on the behaviour of $W(x)$ near 0 – according to the first lemma of §B.1, $W(x)$, if *not bounded* on finite intervals, would have to be *identically infinite* on one of length > 0. We can thus allow majorants $\Omega(x)$ which are merely $\geqslant W(x)$ *for* $|x|$ *sufficiently large*, instead of for *all* real x. In that way we arrive at a statement having (we hope) some chances of being true:

A finite weight $W(x) \geqslant 1$ *meeting the local regularity requirement admits multipliers if and only if there exists an even* $\mathscr{C}_\infty$ *function* $\Omega(x) \geqslant 1$ *with* $\Omega(0) = 1$ (*making* $\log\Omega(x) = O(x^2)$ *near* 0),

$\log^+\log\Omega(x)$ *uniformly* Lip 1 *on* $\mathbb{R}$,

$\Omega(x) \geqslant W(x)$ *whenever* $|x|$ *is sufficiently large,*

$$\int_0^\infty \frac{\log\Omega(x)}{x^2}\,dx < \infty,$$

and

$$\|\log\Omega(x)/x\|_E < \infty.$$

According to what we already know, the 'if' part of this proposition *is* valid, because a weight Ω with the stipulated properties *does* admit multipliers (it enjoys the *intermediate property*), and hence W must *also* do so. But the 'only if' part is still just a *conjecture.*

Support for believing 'only if' to be *correct* comes from a review of how the energy norm $\|\log W(x)/x\|_E$ entered into the argument of §C.5. There, as in §C.4 of Chapter VIII, that was through the use of Schwarz' inequality for the inner product $\langle\ ,\ \rangle_E$. This encourages us to look for a proof of the 'only if' part based on the Schwarz inequality's being best possible.

There is, on the other hand, nothing to prevent anyone's *doubting* the truth of 'only if'. We have again to choose between two approaches – to look for a proof or try constructing a counterexample. The *second* approach proves fruitful here.

In article 4 we give an example showing that the *existence* of an Ω having the properties enumerated above is *not necessary* for the admittance of multipliers by a weight W. The 'essential' condition we are seeking turns out to be more elusive than at first thought.

The reader who is still following the present discussion is urged not to lose patience with this §'s chain of seesaw arguments and interspersed seemingly artificial examples. By going on in such fashion we will arrive at a clear vision of the object of our search. See the first paragraph of article 5.

Our example's construction depends on an auxiliary result relating the norm $\|\ \|_E$ of a certain kind of Green potential to the same norm of a majorant for it. This we attend to in the next article.

3. Comparison of energies

The weight W to be presently constructed is similar to the one considered in article 1, being of the form

$$W(x) = \frac{\text{const.}}{\exp F(x+\mathrm{i})},$$

where $F(z)$, bounded above on the real axis, is given by the formula

$$F(z) = \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}\mu(t)$$

with $\mu(t)$ *increasing* and $O(t)$ (for both large and small values of t) on $[0, \infty)$. $W(x)$ is thus much like the *reciprocal* of the modulus of an entire function of exponential type.

From $\mu(t)$ one can, as in §C.5 of Chapter VIII, form another increasing function $\nu(t)$, this one defined* and infinitely differentiable on $\mathbb{R}$, $O(|t|)$

* by the formula $\nu'(t) = (1/\pi)\int_0^\infty \{((t+s)^2+1)^{-1} + ((t-s)^2+1)^{-1}\}\,\mathrm{d}\mu(t)$; see next article, about 3/4 of the way through.

there and *odd*, such that

$$F(x+\mathrm{i}) \;-\; F(\mathrm{i}) \;=\; \int_0^\infty \log\left|1 \;-\; \frac{x^2}{t^2}\right| \mathrm{d}\nu(t)$$

for $x \in \mathbb{R}$. The right-hand integral can in turn be converted to

$$-x\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\left(\frac{\nu(t)}{t}\right),$$

and our weight $W(x)$ thereby expressed in the form

$$\text{const.} \;+\; x\int_{-\infty}^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\left(\frac{\nu(t)}{t}\right).$$

The reader should take care to distinguish between this representation and the one which has frequently been used in this book for certain entire functions $G(z)$ of exponential type. The latter also involves a function $\nu(t)$, increasing and $O(t)$ on $[0, \infty)$, but reads

$$\log|G(x)| \;=\; -x\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\left(\frac{\nu(t)}{t}\right)$$

with a *minus sign* in front of the integral. It will eventually become clear that this *difference in sign* is very important for the matter under discussion.

The weight W we will be working with in the next article is closely related to the *Green potentials* studied in §C.4, since

$$\frac{1}{x}\log\left(\frac{W(x)}{W(0)}\right) \;=\; \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\left(\frac{\nu(t)}{t}\right).$$

We will want to be able to affirm that this expression belongs to the Hilbert space $\mathfrak{H}$ considered in §C.4 provided that there is some even $\Omega(x) \geqslant 1$ with $\log\Omega(x)/x$ in $\mathfrak{H}$ and $\int_0^\infty(\log\Omega(x)/x^2)\mathrm{d}x$ finite, such that $W(x) \leqslant \Omega(x)$ for all x of sufficiently large modulus.

This kind of comparison is well known for the simpler circumstance involving *pure potentials*. Those are the potentials

$$U_\rho(x) \;=\; \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t)$$

corresponding to *positive* measures ρ. *Cartan's lemma* says that if for two of them, U_ρ and U_σ, we have $U_\rho(x) \leqslant U_\sigma(x)$ for $x \geqslant 0$, then

$\|U_\rho\|_E \leqslant \|U_\sigma\|_E$. Proof:

$$\|U_\rho\|_E^2 = \int_0^\infty U_\rho(x)\mathrm{d}\rho(x) \leqslant \int_0^\infty U_\sigma(x)\mathrm{d}\rho(x) = \int_0^\infty U_\rho(x)\mathrm{d}\sigma(x)$$
$$\leqslant \int_0^\infty U_\sigma(x)\mathrm{d}\sigma(x) = \|U_\sigma\|_E^2 \quad !$$

The result obviously depends greatly on the positivity of ρ and σ.

For our weight W, $(1/x)\log(W(x)/W(0))$ is of the form $U_\rho(x)$, but the measure ρ is *not positive*. Instead,

$$\mathrm{d}\rho(t) = \mathrm{d}\left(\frac{v(t)}{t}\right) = \frac{\mathrm{d}v(t)}{t} - \frac{v(t)}{t}\frac{\mathrm{d}t}{t},$$

and all that the properties of our v give us is the relation

$$\mathrm{d}\rho(t) \geqslant -\text{const.}\frac{\mathrm{d}t}{t}.$$

As we know,

$$\int_0^\infty \log\left|\frac{x+t}{x-t}\right|\frac{\mathrm{d}t}{t} = \frac{\pi^2}{2} \qquad \text{for } x > 0;$$

the measure σ on $(0,\infty)$ with $\mathrm{d}\sigma(t) = \mathrm{d}t/t$ thus *just misses* having finite energy. *Finiteness* of $\|U_\rho\|_E$, *if realized*, must hence be due to *interference* between $\mathrm{d}v(t)/t$ and $v(t)\,\mathrm{d}t/t^2$. A version of Cartan's result is nevertheless still available in this situation.

In order to deal with the measure $\mathrm{d}t/t$ we will use the following two elementary lemmas.

Lemma. *For $A > 0$, we have:*

$$\int_0^A\int_A^\infty \log\left|\frac{x+t}{x-t}\right|\frac{\mathrm{d}t}{t}\frac{\mathrm{d}x}{x} = 2 + \frac{2}{3^3} + \frac{2}{5^3} + \cdots;$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_A^\infty \log\left|\frac{x+t}{x-t}\right|\frac{\mathrm{d}t}{t} = \frac{1}{x}\log\left|\frac{x+A}{x-A}\right| \qquad \text{for } x>0,\ x\neq A;$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^A \log\left|\frac{x+t}{x-t}\right|\frac{\mathrm{d}t}{t} = -\frac{1}{x}\log\left|\frac{x+A}{x-A}\right| \qquad \text{for } x>0,\ x\neq A.$$

Proof. To establish the first relation, make the changes of variable

$$\xi = \frac{x}{A}, \quad \tau = \frac{t}{A}$$

and expand the logarithm in powers of ξ/τ, then integrate term by term.

For the last two relations, we use a different change of variable, putting $s = t/x$. Then the left side of the *second* relation becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{A/x}^{\infty} \log\left|\frac{1+s}{1-s}\right| \frac{\mathrm{d}s}{s},$$

and this may be worked out for $x \neq A$ by the fundamental theorem of calculus. The third relation follows in like manner.

Lemma. *Let* $\rho(t) = \int_0^t \mathrm{d}\rho(\tau)$ *be bounded for* $0 \leqslant t < \infty$. *Then, for* $A > 0$, *the two expressions*

$$\int_0^A \int_A^{\infty} \log\left|\frac{x+t}{x-t}\right| \frac{\mathrm{d}t}{t}\, \mathrm{d}\rho(x),$$

$$\int_0^A \int_A^{\infty} \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t)\, \frac{\mathrm{d}x}{x},$$

are bounded in absolute value by quantities independent of A.

Proof. Considering the *second* expression, we have, for large $M > A$ and any $M' > M$,

$$\int_M^{M'} \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t)$$

$$= \quad \rho(M')\log\left|\frac{1+(x/M')}{1-(x/M')}\right| \quad - \quad \rho(M)\log\left|\frac{1+(x/M)}{1-(x/M)}\right| \quad + \quad 2x\int_M^{M'} \frac{\rho(t)}{t^2-x^2}\,\mathrm{d}t$$

whenever $0 < x < A$. Because $|\rho(t)|$ is bounded, the right side is equal to x *times a quantity uniformly small for* $0 < x < A$ when M and M' are both large. The second expression is therefore equal to the *limit*, for $M \longrightarrow \infty$, of the double integrals

$$\int_0^A \int_A^M \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t)\, \frac{\mathrm{d}x}{x}.$$

Any one of these is equal to

$$\int_A^M \left(\int_0^A \log\left|\frac{x+t}{x-t}\right| \frac{\mathrm{d}t}{t}\right) \mathrm{d}\rho(x);$$

here we use partial integration on the *outer* integral and refer to the third formula provided by the preceding lemma. In that way we get

$$\rho(M)\int_0^A \log\left|\frac{M+t}{M-t}\right|\frac{dt}{t} \quad - \quad \rho(A)\int_0^A \log\left|\frac{A+t}{A-t}\right|\frac{dt}{t}$$

$$+ \quad \int_A^M \log\left|\frac{x+A}{x-A}\right|\frac{\rho(x)}{x}dx.$$

Remembering that

$$\int_0^\infty \log\left|\frac{x+t}{x-t}\right|\frac{dt}{t} \quad = \quad \frac{\pi^2}{2} \qquad \text{for } x > 0,$$

we see that the last expression is $\leqslant 3\pi^2 K/2$ in absolute value if $|\rho(t)| \leqslant K$ on $[0, \infty)$. Thence,

$$\left|\int_0^A \int_A^\infty \log\left|\frac{x+t}{x-t}\right| d\rho(t)\frac{dx}{x}\right| \quad \leqslant \quad \frac{3\pi^2}{2}K$$

independently of $A > 0$.

Treatment of the *first* expression figuring in the lemma's statement is similar (and easier). We are done.

Now we are ready to give our version of Cartan's lemma. So as not to obscure its main idea with fussy details, we avoid insisting on more generality than is needed for the next article. An alternative formulation is furnished by problem 68 below.

Theorem. *Let $\omega(x)$, even and tending to ∞ for $x \longrightarrow \pm\infty$, be given by a formula*

$$\omega(x) \quad = \quad -\int_0^\infty \log\left|1 \; - \; \frac{x^2}{t^2}\right| d\nu(t),$$

where $\nu(t)$, odd and increasing, is $\mathscr{C}_\infty$ on $\mathbb{R}$, with $\nu(t)/t$ bounded there. Suppose there is an even function $\Omega(x) \geqslant 1$, with

$$\int_0^\infty \frac{\log\Omega(x)}{x^2}dx \quad < \quad \infty$$

and $\log\Omega(x)/x$ in the Hilbert space $\mathfrak{H}$ of §C.4, *such that*

$$\omega(x) \;\leqslant\; \log\Omega(x)$$

for all x of sufficiently large absolute value. Then $\omega(x)/x$ also belongs to

$\mathfrak{H}$, *and*

$$\int_0^\infty \frac{\omega(x)}{x^2}\,d\nu(x) \quad < \quad \infty.$$

Proof. If there is an Ω meeting the stipulated conditions, there is an L such that

$$\omega(x) \;\leqslant\; \log\Omega(x) \qquad \text{for } x \geqslant L.$$

Because $\omega(x) \longrightarrow \infty$ for $x \longrightarrow \infty$, we can take (and *fix*) L large enough to also make

$$\omega(x) \;\geqslant\; 0 \qquad \text{for } x \geqslant L.$$

The given properties of $\nu(t)$ make $\omega(0) = 0$ and $\omega(x)$ infinitely differentiable* on $\mathbb{R}$. Therefore, since $\omega(x)$ is even, we have $\omega(x) = O(x^2)$ near 0, and, having chosen L, we can find an M such that

$$-x^2M \;\leqslant\; \omega(x) \;\leqslant\; x^2M \qquad \text{for } 0 \leqslant x \leqslant L.$$

According to the first lemma of §B.4, Chapter VIII, the given formula for $\omega(x)$ can be rewritten

$$\omega(x) \quad = \quad x\int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu(t)}{t}\right), \qquad x > 0.$$

We put

$$\rho(t) \quad = \quad \frac{\nu(t)}{t},$$

making $\rho(t) \geqslant 0$ and bounded by hypothesis, with

$$d\rho(t) \;\geqslant\; -\frac{\nu(t)}{t}\frac{dt}{t} \;\geqslant\; -C\,\frac{dt}{t}.$$

* To check infinite differentiability of $\omega(x)$ in $(-A, A)$, say, take any *even* $\mathscr{C}_\infty$ function $\varphi(t)$ equal to 1 for $|t| \leqslant A$ and to 0 for $|t| \geqslant 2A$. Then, since $\nu'(t)$ is also even, we have

$$\omega(x) \;=\; \int_A^\infty \log|1 - x^2/t^2|(1-\varphi(t))\nu'(t)\,dt \;+\; \int_{-2A}^{2A} \log|x-t|\,\varphi(t)\nu'(t)\,dt$$
$$-\int_{-2A}^{2A} \log|t|\,\varphi(t)\nu'(t)\,dt.$$

The *first* integral on the right is clearly $\mathscr{C}_\infty$ in x for $|x| < A$. When $|x| < A$, the *second* one can be rewritten as $\int_{-3A}^{3A}\varphi(x-s)\nu'(x-s)\log|s|\,ds$, and this, like φ and ν', is $\mathscr{C}_\infty$ (in x), since $\log|s| \in L_1(-3A, 3A)$.

In order to keep our notation simple, let us, without real loss of generality, assume that $C = 1$, i.e., that

$$0 \leqslant \rho(t) \leqslant 1$$

and

$$\mathrm{d}\rho(t) \geqslant -\frac{\mathrm{d}t}{t}.$$

We now consider the Green potentials

$$U_A(x) = \int_0^A \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t),$$

where $A > L$. Since $\nu(t)$ is $\mathscr{C}_\infty$ and $\rho(t) = \nu(t)/t$ bounded, it is readily verified with the help of l'Hôpital's rule that $\rho(t)$ (taken as $\nu'(0)$ for $t = 0$) is *differentiable right down to the origin*, and that

$$\rho'(t) = \frac{t\nu'(t) - \nu(t)}{t^2}$$

stays bounded as $t \longrightarrow 0$. The quantity $|\rho'(t)|$ is thus bounded on each of the finite segments $[0, A]$, and the double integrals

$$\int_0^A \int_0^A \log\left|\frac{x+t}{x-t}\right| |\mathrm{d}\rho(t)|\,|\mathrm{d}\rho(x)|$$

hence finite. Each of the potentials U_A therefore belongs to $\mathfrak{H}$. We proceed to obtain an upper bound on $\|U_A\|_E$ which, for $A > L$, is *independent* of A.

By the absolute convergence just noted we have, according to §C.4,

$$\|U_A\|_E^2 = \int_0^A U_A(x)\,\mathrm{d}\rho(x).$$

In view of the above formula for $\omega(x)$, we can write

$$U_A(x) = \frac{\omega(x)}{x} - \int_A^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t).$$

When $A \longrightarrow \infty$, the integral on the right tends to zero *uniformly* for $0 \leqslant x \leqslant L$ (see beginning of the proof of the *second* of the above lemmas). Therefore, $|\rho'(x)|$ being bounded for $0 \leqslant x \leqslant L$,

$$\int_0^L U_A(x)\,\mathrm{d}\rho(x) = \int_0^L \frac{\omega(x)}{x}\rho'(x)\,\mathrm{d}x + o(1)$$

for $A \longrightarrow \infty$. Referring to one of the initial inequalities for $\omega(x)$, we see that

$$\int_0^L U_A(x)\,\mathrm{d}\rho(x) \quad \leqslant \quad \int_0^L xM|\rho'(x)|\,\mathrm{d}x \; + \; \mathrm{o}(1)$$

for large A.

Our main work is with the integral $\int_L^A U_A(x)\,\mathrm{d}\rho(x)$. Since $\mathrm{d}\rho(t) \;\geqslant\; -\mathrm{d}t/t$, it is convenient to put

$$\mathrm{d}\rho(t) + \frac{\mathrm{d}t}{t} \quad = \quad \mathrm{d}\sigma(t),$$

getting a *positive* measure σ on $[0,\infty)$. The above relation connecting $U_A(x)$ and $\omega(x)/x$ then gives us

$$U_A(x) \quad \leqslant \quad \frac{\omega(x)}{x} \quad + \quad \int_A^\infty \log\left|\frac{x+t}{x-t}\right| \frac{\mathrm{d}t}{t}, \qquad x > 0;$$

$$U_A(x) \quad \geqslant \quad \frac{\omega(x)}{x} \quad - \quad \int_A^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\sigma(t), \qquad x > 0.$$

Thence, by our initial relations for $\omega(x)$,

$$U_A(x) \quad \leqslant \quad \frac{\log\Omega(x)}{x} \quad + \quad \int_A^\infty \log\left|\frac{x+t}{x-t}\right| \frac{\mathrm{d}t}{t}, \qquad x > L;$$

$$U_A(x) \quad \geqslant \quad -\int_A^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\sigma(t), \qquad x \geqslant L.$$

Since $\log\Omega(x) \;\geqslant\; 0$ by hypothesis and $\log|(x+t)/(x-t)| \;\geqslant\; 0$ for x and $t \;\geqslant\; 0$, the first of these inequalities yields

$$\int_L^A U_A(x)\,\mathrm{d}\sigma(x) \quad \leqslant \quad \int_0^A \frac{\log\Omega(x)}{x}\mathrm{d}\sigma(x) \quad + \quad \int_0^A \int_A^\infty \log\left|\frac{x+t}{x-t}\right| \frac{\mathrm{d}t}{t}\,\mathrm{d}\sigma(x)$$

$$\leqslant \quad \int_0^\infty \frac{\log\Omega(x)}{x^2}\mathrm{d}x \quad + \quad \int_0^A \frac{\log\Omega(x)}{x}\mathrm{d}\rho(x) \quad + \quad \int_0^A \int_A^\infty \log\left|\frac{x+t}{x-t}\right| \frac{\mathrm{d}t}{t}\,\mathrm{d}\rho(x)$$

$$+ \quad \int_0^A \int_A^\infty \log\left|\frac{x+t}{x-t}\right| \frac{\mathrm{d}t}{t}\frac{\mathrm{d}x}{x}.$$

Similarly, from the second inequality,

$$-\int_L^A U_A(x)\frac{\mathrm{d}x}{x} \quad \leqslant \quad \int_0^A \int_A^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\sigma(t)\frac{\mathrm{d}x}{x}$$

$$= \quad \int_0^A \int_A^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t)\frac{\mathrm{d}x}{x} \quad + \quad \int_0^A \int_A^\infty \log\left|\frac{x+t}{x-t}\right| \frac{\mathrm{d}t}{t}\frac{\mathrm{d}x}{x}.$$

Putting these results together and then adding on the one obtained previously, we find that for large $A > L$,

$$\|U_A\|_E^2 \;=\; \int_0^A U_A(x)\,\mathrm{d}\rho(x)$$

$$\leqslant\; \mathrm{o}(1) + \int_0^L xM|\rho'(x)|\,\mathrm{d}x + \int_0^\infty \frac{\log\Omega(x)}{x^2}\,\mathrm{d}x + \int_0^A \frac{\log\Omega(x)}{x}\,\mathrm{d}\rho(x)$$

$$+ \int_0^A\int_A^\infty \log\left|\frac{x+t}{x-t}\right| \frac{\mathrm{d}t}{t}\,\mathrm{d}\rho(x) + \int_0^A\int_A^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t)\,\frac{\mathrm{d}x}{x}$$

$$+ 2\int_0^A\int_A^\infty \log\left|\frac{x+t}{x-t}\right| \frac{\mathrm{d}t}{t}\,\frac{\mathrm{d}x}{x}.$$

It is part of our hypothesis that

$$\int_0^\infty \frac{\log\Omega(x)}{x^2}\,\mathrm{d}x \;<\; \infty.$$

Because $0 \leqslant \rho(t) \leqslant 1$, the *fourth* and *fifth* of the right-hand integrals are *bounded* (by quantities independent of A) according to the second of the above lemmas. By the first of those lemmas, the *sixth* integral is equal to a finite constant independent of A. We thus have a constant c independent of A such that

$$\|U_A\|_E^2 \;\leqslant\; c \;+\; \int_0^A \frac{\log\Omega(x)}{x}\,\mathrm{d}\rho(x)$$

for large $A > L$.

Recalling, however, that

$$U_A(x) \;=\; \int_0^A \log\left|\frac{x+t}{x-t}\right|\mathrm{d}\rho(t)$$

and that $\log\Omega(x)/x$ is in $\mathfrak{H}$ by hypothesis, we see from the fifth lemma of §C.4 that

$$\int_0^A \frac{\log\Omega(x)}{x}\,\mathrm{d}\rho(x) \;=\; \left\langle \frac{\log\Omega(x)}{x},\; U_A(x)\right\rangle_E \;\leqslant\; \left\|\frac{\log\Omega(x)}{x}\right\|_E \|U_A\|_E.$$

The preceding relation thus becomes

$$\|U_A\|_E^2 \;\leqslant\; c \;+\; \|\log\Omega(x)/x\|_E\,\|U_A\|_E.$$

Knowing, then, that $\|U_A\|_E < \infty$, we get by 11th grade algebra (!) that

$$\|U_A\|_E \;\leqslant\; \frac{1}{2}\left(\|\log\Omega(x)/x\|_E + \sqrt{(\|\log\Omega(x)/x\|_E^2 + 4c)}\right)$$

for large $A > L$, *with the bound on the right independent of* A.

Now it is easy to show that $\omega(x)/x$ belongs to $\mathfrak{H}$. Since $\omega(x)/x$ is *odd*, we need, according to the last theorem of §C.4, merely check that $\|\omega(x)/x\|_E < \infty$ where, for $\|\ \ \|_E$, the general definition adopted towards the middle of §C.4 is taken. As observed earlier,

$$U_A(x) \longrightarrow \frac{\omega(x)}{x} \qquad \text{u.c.c. in } [0, \infty)$$

for $A \longrightarrow \infty$. Thence, by the second theorem of §C.4 and Fatou's lemma,

$$\|\omega(x)/x\|_E^2 \quad \leqslant \quad \liminf_{A \to \infty} \|U_A\|_E^2.$$

(Cf. the discussion of how $\mathfrak{H}$ is formed, about half way into §C.4.) The result just found therefore implies that

$$\|\omega(x)/x\|_E \quad \leqslant \quad \frac{1}{2}\left(\|\log \Omega(x)/x\|_E \ + \ \sqrt{(\|\log \Omega(x)/x\|_E^2 \ + \ 4c)}\right),$$

making $\omega(x)/x \in \mathfrak{H}$. (Appeal to the *last* theorem of §C.4 can be avoided here. A sequence of the U_A with $A \longrightarrow \infty$ certainly *converges weakly* to *some* element, say U, of $\mathfrak{H}$. Some convex linear combinations of those U_A then *converge in norm* $\|\ \ \|_E$ to U, which then can be easily identified with $\omega(x)/x$, reasoning as in the discussion towards the middle of §C.4.)

Once it is known that $\omega(x)/x \in \mathfrak{H}$, the rest of the theorem is almost immediate. The relations for $\omega(x)$ given near the beginning of this proof make

$$\int_0^\infty \frac{|\omega(x)|}{x^2}\,dx \quad \leqslant \quad \int_0^L M\,dx \ + \ \int_L^\infty \frac{\log \Omega(x)}{x^2}\,dx$$

$$\leqslant \quad ML \ + \ \int_0^\infty \frac{\log \Omega(x)}{x^2}\,dx \quad < \quad \infty,$$

so, since (here) $0 \leqslant \nu(x)/x \leqslant 1$, we have

$$\int_0^\infty \frac{|\omega(x)|}{x}\,\frac{\nu(x)}{x^2}\,dx \quad < \quad \infty.$$

It is thus enough to verify that

$$\int_0^\infty \frac{\omega(x)}{x}\,d\left(\frac{\nu(x)}{x}\right) \quad < \quad \infty$$

in order to show that

$$\int_0^\infty \frac{\omega(x)}{x^2}\,d\nu(x)$$

is finite. Since $|\mathrm{d}(v(x)/x)/\mathrm{d}x|$ is bounded for $0 \leqslant x \leqslant L$ with $|\omega(x)/x^2| \leqslant M$ there, while $\omega(x) \geqslant 0$ for $x > L$, the *first* of these integrals is *perfectly unambiguous* according to the observation just made, and equal to the limit, for $A \longrightarrow \infty$, of

$$\int_0^A \frac{\omega(x)}{x}\,\mathrm{d}\left(\frac{v(x)}{x}\right).$$

Here we may again resort to the fifth lemma of §C.4, according to which any one of the last integrals, identical with $\int_0^A (\omega(x)/x)\,\mathrm{d}\rho(x)$, is just the inner product

$$\left\langle \frac{\omega(x)}{x},\ U_A(x) \right\rangle_E \quad \leqslant \quad \left\| \frac{\omega(x)}{x} \right\|_E \|U_A\|_E.$$

Plugging in the bounds found above, we see that for large $A > L$,

$$\int_0^A \frac{\omega(x)}{x}\,\mathrm{d}\left(\frac{v(x)}{x}\right) \quad \leqslant \quad \frac{1}{4}\left(\left\| \frac{\log \Omega(x)}{x} \right\|_E + \sqrt{\left(\left\| \frac{\log \Omega(x)}{x} \right\|_E^2 + 4c \right)} \right)^2,$$

and, making $A \longrightarrow \infty$, we arrive at the desired conclusion.

The theorem is proved.

The variant of this result referred to earlier, to be given in problem 68, applies to functions $\omega(x)$ of the form

$$\omega(x) \quad = \quad x\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t),$$

where the *only* assumptions on the measure ρ are that it is absolutely continuous, with $\rho'(t)$ bounded on each finite interval, and that

$$\mathrm{d}\rho(t) \quad \geqslant \quad -C\frac{\mathrm{d}t}{t} \qquad \text{for } t \geqslant 1.$$

That generalization is related to some material of independent interest taken up in problems 66 and 67.

Let us, as usual, write

$$U_\rho(x) \quad = \quad \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t).$$

Under our assumption on ρ, the integral on the right is certainly unambiguously defined because

$$\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \min(1,\ 1/t)\,\mathrm{d}t$$

is finite for $x > 0$ and, if K is large enough,

$$\mathrm{d}\sigma(t) \quad = \quad \mathrm{d}\rho(t) \; + \; K\min(1,\; 1/t)\,\mathrm{d}t$$

is $\geqslant 0$ for $t \geqslant 0$.* The preceding integral is indeed $O(x\log(1/x))$ for small values of $x > 0$, so, by applying Fubini's theorem separately to

$$\int_0^A \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\sigma(t)\frac{\mathrm{d}x}{x}$$

and to the similar expression with $\mathrm{d}\sigma(t) - \mathrm{d}\rho(t)$ standing in place of $\mathrm{d}\sigma(t)$, we see that for each finite $A > 0$, $\int_0^A (U_A(x)/x)\,\mathrm{d}x$ is well defined and equal to

$$\int_0^\infty \int_0^A \log\left|\frac{x+t}{x-t}\right| \frac{\mathrm{d}x}{x}\,\mathrm{d}\rho(t).$$

By writing $\mathrm{d}\rho(t)$ one more time as the difference of the two positive measures $\mathrm{d}\sigma(t)$ and $K\min(1,\; 1/t)\mathrm{d}t$, one verifies that the last expression is in turn equal to

$$\lim_{M\to\infty} \int_0^M \int_0^A \log\left|\frac{x+t}{x-t}\right| \frac{\mathrm{d}x}{x}\,\mathrm{d}\rho(t).$$

Problem 66

In this problem, we suppose that the above assumptions on the measure ρ hold, and that in addition the integrals

$$\int_0^A \frac{U_\rho(x)}{x}\,\mathrm{d}x$$

are *bounded* as $A \longrightarrow \infty$. The object is to then obtain a preliminary grip on the magnitude of $|\rho(t)|$.

(a) Show that for each M and A.

$$\int_0^M \int_0^A \log\left|\frac{x+t}{x-t}\right| \frac{dx}{x}\,\mathrm{d}\rho(t) \quad = \quad \rho(M)\int_0^A \log\left|\frac{x+M}{x-M}\right| \frac{\mathrm{d}x}{x}$$
$$+ \quad \int_0^M \log\left|\frac{t+A}{t-A}\right| \frac{\rho(t)}{t}\,\mathrm{d}t.$$

(Hint: cf. proof of second lemma, beginning of this article.)

* Only *this* property of ρ is used in problems 66 and 67; absolute continuity of that measure plays no rôle in them (save that $\rho(t)$ should be replaced by $\rho(t) - \rho(0+)$ throughout if ρ has point mass at the origin).