ORAL EXAM PAPER
SELF-ADJOINT OPERATORS AND ZEROS OF L-FUNCTIONS

KIM KLINGER-LOGAN

Abstract: Hilbert and Pólya raised the possibility of proving the Riemann Hypothesis by producing self-adjoint operators with eigenvalues $s(s-1)$ for zeros s of $\zeta(s)$. A spark of hope appeared when Haas (1977) numerically miscalculated eigenvalues for the automorphic Laplacian and obtained zeros of zeta in his list of s-values. In 1981, Hejhal identified the flaw and determined what Haas had actually computed – not genuine eigenvalues but parameters in an automorphic Helmholtz equation (the time-independent, stationary version of the wave equation). ColinDeVerdière speculated on a possible legitimization that languished for 30 years. Recent work of Bombieri and Garrett makes precise ColinDeVerdière's speculations and proves that, while the discrete spectrum (if any) must have s-values among zeros of corresponding zeta functions (expressed as periods of Eisenstein series), there is an operator-theoretic mechanism by which the regular behavior of zeta on the edge of the critical strip coerces discrete spectrum to be too regularly spaced to be compatible with Montgomery's pair-correlation conjecture. Similar mechanisms are shown to apply to more complicated non-compact periods of Eisenstein series producing L-functions. These operator-theoretic mechanisms also open up further possibilities bearing on the locations of zeros of L-functions.

Contents

Date: 11.06.2016.

Introduction

In this work we investigate the spectral theory for specific self-adjoint operators on Hilbert spaces of automorphic forms, with corollaries concerning zeros of L-functions that arise as periods of Eisenstein series. We first provide some historical context.

The Story. This work is an extension of the project begun by Bombieri and Garrett, originating in work of ColinDeVerdière and Hejhal in the 1980s. Their project was inspired by a story that begins in 1977 when Haas [26] miscomputed eigenvalues of the invariant Laplacian $\Delta = y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$ on $\Gamma\backslash\mathfrak{H}$ for $\Gamma = SL_2(\mathbb{Z})$ [see Section 3]. In his Master's thesis, Haas attempted to compute eigenvalues $\lambda_s = s(s-1)$ by numerical solution of the differential equation $(\Delta - \lambda_s)u = 0$. Since the invariant Laplacian Δ descends from the Casimir element of the universal enveloping algebra $U\mathfrak{sl}_2(\mathbb{R})$ and, furthermore, automorphic forms are eigenfunctions for such operators, solutions to such differential equations had a natural significance to number theorists. Thus, Haas' advisor Neunhoffer mailed the list of parameters s to A. Terras in San Diego but without detailed explanation of their derivation. In this list, Stark observed some zeros of $\zeta(s)$ and Hejhal noticed zeros of $L(s,\chi_{-3})$.[1]

Though it was not surprising that the values had interest to number theorists, it was unexpected that zeros of L-functions appeared as spectral parameters s. Given Hilbert and Pólya's remarks about the possibility of proving the Riemann Hypothesis by producing self-adjoint operators with eigenvalues $s(s-1)$ for zeros s of $\zeta(s)$, this finding was quite exciting (since Δ is a non-positive self-adjoint operator, this would mean that $\lambda_s \in \mathbb{R}$ and $\lambda_s \leq 0$ and hence either $s \in [0,1]$ or $\mathrm{Re}(s) = 1/2$). It follows then that *if* all zeros of $\zeta(s)$ are on this list of parameters of λ_s we have the Riemann Hypothesis. The mere numerical artifact does not suggest a *proof* and, in particular, there is no visible guarantee that *all* zeros of ζ are on this list of spectral parameters. However, given any strong correlation between zeros of $\zeta(s)$ spectral parameters for eigenvalues of a self-adjoint operator one might hope to retrieve a definite percentage using such a differential equation.

For instance, one hopeful route may be to show that all spectral parameters s appear as zeros of $\zeta(s)$. In this case, using the fact that the number $N(T)$ of zeros of ζ in the critical strip below height T is

$$N(T) = \frac{1}{2\pi}T\cdot\log\left(\frac{T}{2\pi e}\right) + O(\log T)$$

one might check what percentage of zeros of ζ are found on the critical line below any given height T. In the best-case scenario, asymptotically 100% of zeros would appear this way. However, any fraction over 40% would be progress and furthermore any correlation between spectral parameters for a self-adjoint operator and zeros of $\zeta(s)$ gives motivation for continued research [15].

Such an exciting prospect should naturally be met with skepticism. In roughly 1979-1981, Hejhal [29] recomputed the eigenvalues and found that all of these interesting parameters,

[1]This list only went up to height 20.45578 and was quite short due to the restricted computing power of the time. Only one zero of ζ was initially observed; however, there is only one nontrivial zero of ζ below height 21. See [26] or [29].

the zeros, were *missing*. He realized that Haas had allowed for some non-smoothness, misapplicating of the Henrici collocation method on the corners of the fundamental domain, and hence had found s-values that were solutions to the inhomogeneous equation

$$(\Delta - \lambda_s)u = \delta_\omega^{\mathrm{afc}} \tag{1}$$

where $\delta_\omega^{\mathrm{afc}} := \sum_{\gamma\in\Gamma} \delta_\omega^{\mathfrak{H}} \circ \gamma$ is the automorphic Dirac delta at the corner $\omega = e^{2\pi i/3}$ of the fundamental domain of $\Gamma\backslash\mathfrak{H}$, as opposed to the homogeneous equation $(\Delta - \lambda_s)u = 0$.

At first glance, this equation is unexpected and it may be difficult to explain the relationship between its spectral parameters and zeros of $\zeta(s)$. However, physicists such as Bethe and Peierls (1935) [4] had been studying similar *solvable models* since the 1930s with physically corroborated predictions. In number theory specifically, the automorphic *Green's function* (or resolvent kernel) has been studied by number theorists such as Neunhöffer [39] and Fay [17] in more classical terms and is exactly a solution to this differential equation. Further, Green, Miller, Russo and VanHove have examined similar models for the four-loop supergraviton behavior [19].

More relevant to the story of Haas and Hejhal, the implications of this new differential equation (1) toward RH are not as initially hoped since values $\lambda_s = s(s-1)$ admitting non-trivial solutions u are not genuine eigenfunctions for Δ. However, not all is lost. Bombieri and Garrett, building on work of Hejhal and ColinDeVerdière, have clarified and made precise ideas from [14] and proved some basic results showing the relevance of operator theory to location of zeros and other periods of Eisenstein series. More explicitly, Bombieri and Garrett make (*necessarily*) subtler operators related to Δ to better exploit the that fact the constant term of u_s is essentially θE_s .

The work of Bombieri and Garrett clarifies two promising observations initially made by ColinDeVerdière in 1982-3 [14]. The first was Lax and Phillips (1976) result [34] that for $a > 1$, if we define

$$L^2_a(\Gamma\backslash\mathfrak{H}) = \left\{ f \in L^2(\Gamma\backslash\mathfrak{H}) \;\middle|\; c_P f(x) = 0 \text{ for } y > a \right\}$$

for $c_P f(x) := \int_0^1 f(x+iy)\,dx$ then the Friedrichs' extension $\widetilde{\Delta}_a$ of Δ restricted to $C_c^\infty(\Gamma\backslash\mathfrak{H}) \cap L^2_a(\Gamma\backslash\mathfrak{H})$ has purely discrete spectrum. Furthermore, $\widetilde{\Delta}_a$ is self-adjoint so that the eigenvalues are real and for a distribution η_a at a defined by $\eta_a f = c_P f(ia)$

$$(\widetilde{\Delta}_a - \lambda_s)u = 0 \iff (\Delta - \lambda_s)u = c \cdot \eta_a \;\text{ and }\; \eta_a u = 0$$

for some constant c. If

$$(\widetilde{\Delta}_{z_0} - \lambda_s)u = 0 \quad\iff\quad (\Delta - \lambda_s)u = c \cdot \delta_{z_0}^{\mathrm{afc}} \;\text{ and }\; \delta_{z_0}^{\mathrm{afc}} u = 0 \quad \text{(for some c)}$$

a pseudo-Laplacians $\widetilde{\Delta}_{z_0}$ attached to the automorphic Dirac delta $\delta_{z_0}^{\mathrm{afc}}$ may contain spectral parameters relating to zeros of ζ. The issue then becomes that, in order to perform the Friedrichs extension attached to a distribution, that distribution must be contained in $H^{-1}(\Gamma\backslash\mathfrak{H})$. However, $\delta_\omega^{\mathrm{afc}} \in H^{-1-\epsilon}$ (see Section 5 for information about Sobolev spaces).

The second relevant observation made by ColinDeVerdière (1983) in [14] was (approximately) that projecting $\delta_\omega^{\mathrm{afc}}$ to the non-cuspidal spectrum would allow this new distribution θ be in

$H^{-3/4-\epsilon}(\Gamma\backslash\mathfrak{H}) \subseteq H^{-1}(\Gamma\backslash\mathfrak{H})$ by the moment bound of Hardy and Littlewood [27]. In 2011, Bombieri and Garrett made ColinDeVerdière's speculation precise and proved that the discrete spectrum $\lambda_s = s(s-1)$ (*if any*) of $\widetilde{\Delta}_\theta$ has parameters contained in the on-line zeros of $\zeta(s)L(s,\chi_{-3})$. There is no guarantee that the spectrum is non-empty.

In fact, the first purely new result of Bombieri and Garrett is their limitation of the fraction of zeros which could occur as s-values for discrete spectrum λ_s. They achieve this by showing a connection to the relatively regular behavior of $\zeta(s)$ on the edge of the critical strip, leading to conflict with Montgomery's pair correlation conjecture [38]. This provides a strong reason to believe that the most optimistic version of ColinDeVerdière's simplest formulation of a conjecture in the style of Hilbert-Pólya is false (barring significant failure of RH!). Further, the influence of the spectral theory of self-adjoint operators on spaces of automorphic forms is more complicated than a literal manifestation of Hilbert-Pólya. However, being that there is an overall lack of candidate operators that fit Hilbert and Pólya's suggestion, any promising suggestion in this direction progress. Furthermore, not all is lost – though this 'simple' case yields a negative result, the hope is that given this jumping-off point, we can recover the lost spectral parameters using more complicated boundary conditions. One virtue of this approach is that the same set-up and conclusions can apply to much broader contexts.

Generally, we would like to extend the results of Bombieri and Garrett about the interaction of zeros of L-functions with spectral parameters of self-adjoint operators to all self-dual L-functions. However, there are some is a subtle but important point to make in this direction. While the spectral theory does apply to Epstein zeta functions and finite real linear combinations of them, we should not expect for these results to generalize to any Dirichlet series since Epstein zeta functions have nontrivial off-line zeros [41]. Thus we must be cautious about the simplicity of the results used in this method. Based on the results of Bombieri and Garrett, it is fairly clear that there are no obstacles to the basic discussion for compact periods of Eisenstein series. We will investigate a case in which the period is non-compact. We will show that this period of an Eisenstein series factors over primes into an Euler product and we will identify the resulting L-function.

1. Example: Singular Potentials on $\mathbb{R}$

In the context of the larger project of Bombieri and Garrett, there are many analytical issues that arise when attempting to use spectral theory to solve differential equations on $\Gamma\backslash\mathfrak{H}$. Care must be taken to deal with issues function-valued like meromorphic continuation of solutions, and non-smooth eigenfunctions. Many of the techniques used to address these issues can be illustrated on simpler domains such as $\mathbb{R}$ or $\mathbb{R}/\mathbb{Z}$. In looking at examples on these elementary domains, we are able to lay bare the techniques used without getting caught-up in the physical issues of the modular curve $\Gamma\backslash\mathfrak{H}$. What follows in the next two sections is *not* intended to be an idiosynchratic reconstruction of standard material but rather an examination of analytical nuances that are underlying even the simplest cases. The idea is that the techniques used here extend to the automorphic case,whereas possibly simpler standard arguments may not extend.

In this section, I will address the issue of meromorphic continuation of solutions to differential equations. The meromorphic continuation of resolvents is not new and has been studied by Melrose [36], [37] and Zworski [51]. In fact, it is valuable to note that versions of the above differential equation have been studied as *solvable models* by physicists since the 1930s [4] with corroboration by physical experiments. However, given their subtlety, these situations were only apparently understood rigorously by physicists after Berezin and Faddeev's 1961 paper [3].[2] Of course the versions of this problem that were of interest to physicists at the time took place on the line $\mathbb{R}$ or $\mathbb{R}/\mathbb{Z}$ as opposed to an arithmetic quotient $\Gamma\backslash\mathfrak{H}$. Thus it is appropriate to begin by examining a simpler case of using spectral theory to solve a differential equation on $\mathbb{R}$.

We will use the Fourier transform to find solutions u to the differential equation

$$(\Delta + w^2)u = \delta \qquad \text{on } \mathbb{R} \tag{2}$$

where δ is the Dirac delta and then prove its meromorphic continuation as a vector-valued function. Even in this basic example, meromorphic continuation of solutions move out of the original Sobolev space. We will examine this particular case to provide precedent for similar but more complicated phenomena in the automorphic case.

First, recall that the Fourier transform can be used to solve (2) for $\mathrm{Re}(w) > 0$:
Taking the Fourier transform of both sides of (2) yields

$$\left((-it)^2 + w^2\right)\hat{u} = 1 \qquad \text{for } \mathrm{Re}(w) > 0$$

and solving for $\hat{u}$ we get

$$\hat{u}_w(t) = \frac{1}{(-it)^2 + w^2}.$$

Now, using Fourier inversion, in $\mathrm{Im}(w) > 0$,

$$u_w(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{itx}\hat{u}_w(t)\, dt = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{itx}}{(-it)^2 + w^2}\, dt = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{itx}}{w^2 - t^2}\, dt.$$

It is routine to directly evaluate this integral using residue calculus as follows:

$$\int_{\mathbb{R}} \frac{e^{itx}}{w^2 - t^2}\, dt = \lim_{R\to\infty}\left(\int_{S_R} f(z)\, dz - \int_{A_R} f(z)\, dz\right)$$

for $f(z) = \dfrac{e^{izx}}{w^2 - z^2}$, S_R the semi-circle of radius R in the upper half plane, and A_R the arc of radius R in the upper half plane. Note that e^{izx} is an entire function so $f(z)$ has simple pole at $z = w$ inside S_R (for $|R| > |w|$). In the case,

$$\int_{S_R} f(z)\, dz = 2\pi i \cdot \mathrm{Res}_{z=w} f(z) = \frac{-\pi i e^{iw|x|}}{w}$$

[2] Of course the tools given by Sobolev, Friedrichs and Schwartz had all been available to mathematicians by the 1950s.

for $|R| > |w|$ (and 0 otherwise). Taking the limit as $R \to \infty$, it is easy to show that $\int_{A_R} f(z)\,dz \to 0$ and so for $\mathrm{Im}(w) > 0$

$$u_w(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{itx}}{w^2 - t^2}\,dt = \frac{e^{iw|x|}}{2iw}.$$

It is convenient to be able to examine the features of such an explicit and elementary function in contrast to automorphic forms. From the explicit formula for u_w, we know $u_w(x)$ has meromorphic continuation in w. However, for $\mathrm{Im}(w) = 0$, the meromorphic continuation is not in $L^2(\mathbb{R})$ (with respect to x). Furthermore, for $\mathrm{Im}(w) < 0$, the meromorphic continuation grows exponentially in x. Although direct evaluation is possible by first evaluating the residues, we want *qualitative* meromorphic continuation this solution to $\mathrm{Im}(w) < 0$.

Explicitly, an argument involving residues as described above will not be extendable to the automorphic case. Instead, we will investigate the meromorphic continuation of solutions as vector-valued functions. After some set-up, this will involve a regularization argument applied to the spectral integral above. Though cumbersome in the elementary case, this is a necessary feature of such a meromorphic continuation argument to be extendable to the automorphic case as the space has large continuous spectrum.

Instead of computing using residues, consider for $\mathrm{Im}(w) > 0$,

$$u_w(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{itx}}{(-it)^2 + w^2}\,dt = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\cos(tx)}{(-it)^2 + w^2}\,dt$$
$$= \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\cos(tx) - \cos(wx)}{(-it)^2 + w^2}\,dt + \cos(wx)\cdot\frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{(-it)^2 + w^2}\,dt$$
$$= \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\cos(tx) - \cos(wx)}{(-it)^2 + w^2}\,dt + \frac{1}{2iw}\cos(wx).$$

Since the integral is not compactly supported, it is not easy to show that it is continuous as a function of w. Instead, we want to view the analytic continuation in w as a function (of x)-valued function in order to determine the space in which the analytic continuation of u exists. In order to do this we will need that the functions of x are of moderate growth. In the automorphic case, sup-norm estimates of the continuous spectrum would imply the Lindelöf Hypothesis. These spaces of moderate growth are described as follows.

Let $b \in \mathbb{R}$, define the seminorm ν_b on $C_c(\mathbb{R})$ defined for $f \in C_c(\mathbb{R})$ by

$$\nu_b(f) := \sup_{x\in\mathbb{C}} e^{-b|x|}\cdot|f(x)|$$

and let Mod^b be the completion of $C_c^\infty(\mathbb{R})$ with respect to ν_b. Note that Mod^b consists of continuous functions f so that $\lim_{x\to\infty}|f(x)|e^{-b|x|} = 0$. Let Mod^∞ be the ascending union $\bigcup_b \mathrm{Mod}^b$.

Theorem 1. *$H^1 \subset Mod^0$ where this inclusion is continuous.*

Proof. Let $f \in C_c^\infty(\mathbb{R})$. By the Fundamental theorem of Calculus and Cauchy-Schwarz,

$$|f(x) - f(y)| = \left| \int_y^x f'(t)\ dt \right| = \left| \int_y^x f'(t) \cdot 1\ dt \right| \leq \int_y^x |f'(t) \cdot 1|\ dt$$

$$\leq |f'|_{L^2} \cdot \left| \int_y^x 1\ dt \right|^{1/2} = |f'|_{L^2} \cdot |x-y|^{1/2}.$$

Now let $N = |f'|_{L^2}$. Suppose the supremum of f occurs at x_o so $f(x_o) \geq 0$ and let $m = \sup_x |f(x)|$. Without loss of generality, assume $x_o = 0$. Letting $y = x_o = 0$, the inequality above becomes $|f(x) - m| \leq N\sqrt{|x|}$ and since m is the supremum of f, we have $m - f(x) \leq N\sqrt{|x|}$ or

$$f(x) \geq m - N\sqrt{|x|}.$$

This describes a positive interval $\left[-\frac{m^2}{N^2}, \frac{m^2}{N^2}\right]$ so

$$|f|_{L^2}^2 \geq \int_{-\frac{m^2}{N^2}}^{\frac{m^2}{N^2}} (m - N\sqrt{|x|})^2\ dx = m^2 \cdot 2\frac{m^2}{N^2} - \frac{8}{3}mN\frac{m^3}{N^3} + N^2 \cdot \frac{m^4}{N^4} = (2 - \frac{8}{3} + 1)\frac{m^4}{N^2} = \frac{2}{3}\frac{m^4}{N^2}.$$

Thus

$$\frac{3}{2} \cdot |f|_{L^2}^2 |f'|_{L^2}^2 \geq \sup_x |f(x)|^4$$

and since $(a^2 + b^2)^2 \geq 4a^2b^2$,

$$(|f|_{L^2}^2 + |f'|_{L^2}^2)^2 \geq \sup_x |f(x)|^4$$

and so $|f|_{H^1} \geq \sup_x |f(x)|$. Thus an H^1-Cauchy sequence of C_c^∞-functions converges in sup-norm so it converges to a continuous function. The continuous functions F occurring as sup-norm limits of test function $\{f_j\}$ do go to 0 at infinity: Given $\epsilon > 0$ choose i_o so that $\sup_x |f_i(x) - f_j(x)| < \epsilon$ for all $i, j \geq i_o$. Then $\sup_x |f_{i_o}(x) - F(x)| \leq \epsilon$ so $|F(x)| \leq \epsilon$ off the compact support of f_{i_o}.

$\square$

One notably observes that $\mathrm{Mod}^0 = C_0^0(\mathbb{R})$ namely continuous functions with vanish at $\pm\infty$. There are simpler arguments for the containment in the previous proof; however, the value in the proof given above that it does not depend on harmonic analysis and is easily extendable to $\Gamma\backslash\mathfrak{H}$.

We need one more preliminary lemma before we can prove u_w has meromorphic continuation as we are considering the case of function (of x)-valued functions. Though a holomorphic function of two variables is holomorphic in each respective variable, the assertion here is somewhat different that that.

For $U \subset \mathbb{C} - \{0\}$ open, let $\mathrm{Hol}^b(U) := \{f \in \mathrm{Mod}^b(U) \mid f \text{ is holomorphic}\}$. Here the topology is given by the norm

$$\sigma_K(f) := \sup_{x \in K} |f(x)|_{\mathrm{Mod}^b}$$

for $K \subset U$ compact. Again let $\mathrm{Hol}^\infty(U)$ be the ascending union $\bigcup_b \mathrm{Hol}^b(U)$.

Let B be a Banach space with norm $|\cdot|_B$. Let $\mathrm{Hol}^B(U)$ be the space of holomorphic B-valued functions on an open subset $U \subset \mathbb{C}^n$ with seminorm $\nu_K(f) = \sup_{x\in K} |f(x)|_B$ as K ranges over compact subsets of U.

Lemma 2. *Let $F(t,w)$ be holomorphic B-valued on an open set $U_1 \times U_2$ in $\mathbb{C}^2$. Then $f(t) = (w \mapsto F(t,w))$ is a continuous $Hol^B(U_2)$-valued function on U_1.*

Proof. Let $\epsilon > 0$ and compact $K_2 \subset U_2$ and $t \in U_1$. Let $t \in U_1' \subset U_1$ so that U_1' is open with compact closure K_1. Since F is uniformly continuous on $K_1 \times K_2$, there is an $r > 0$ such that $|F(t,w) - F(t',w')|_B < \epsilon$ for all $|t-t'| < r$ and $|w-w'| < r$ with (t,w) and (t',w') in $K_1 \times K_2$. Hence $|F(t,w) - F(t',w')|_B < \epsilon$ for all $|t-t'| < r$ and for all $w \in K_2$. In other words,

$$\sup_{w\in K_2} |F(t,w) - F(t',w)| < \epsilon$$

for $t' \in U_1$ and $|t-t'| < r$. □

We can then establish the meromorphic continuation of the solution u_w as follows. Recall that our goal is to have u_w meromorphically continued as a vector-valued function. In this case the vectors are also functions in Mod^∞. We will employ a regularity argument in order to establish the meromorphic continuation of vector-valued functions.

Theorem 3. *For all $w \neq 0$, the solution*

$$u_w(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\cos(tx) - \cos(wx)}{(-it)^2 + w^2}\, dt + \frac{1}{2iw}\cos(wx)$$

described above exists as a Mod^∞-valued function and has meromorphic continuation as a Mod^∞-valued function.

Proof. For the solution

$$u_w(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\cos(tx) - \cos(wx)}{(-it)^2 + w^2}\, dt + \frac{1}{2iw}\cos(wx)$$

where $\dfrac{\cos(wx)}{2iw} \in \mathrm{Mod}^\infty$ away from $w = 0$.

For $T \gg |w|$, we break up the integral as

$$\int_{\mathbb{R}} \frac{\cos(tx) - \cos(wx)}{(-it)^2 + w^2}\, dt = \int_{|t|\le T} \frac{\cos(tx) - \cos(wx)}{(-it)^2 + w^2}\, dt + \int_{|t|>T} \frac{\cos(tx) - \cos(wx)}{(-it)^2 + w^2}\, dt$$

$$(3) \quad = \int_{|t|\le T} \frac{\cos(tx) - \cos(wx)}{(-it)^2 + w^2}\, dt + \int_{|t|>T} \frac{\cos(tx)}{(-it)^2 + w^2}\, dt - \cos(wx)\int_{|t|>T} \frac{1}{(-it)^2 + w^2}\, dt.$$

We will refer to the terms in (3) as the first, second and third integral respectively. The first integrand is holomorphic in t and w away from $w = 0$. For w in an open set U with compact closure the integrand takes values in a fixed Banach space $\mathrm{Mod}^b \subset \mathrm{Mod}^\infty$.

By Lemma 2 above, $t \to (w \to \frac{\cos tx - \cos wx}{(-it)^2 + w^2})$ is a continuous $\mathrm{Hol}^b(U)$-valued function of t. Restricting t to $[-T,T]$, this function has compact support. Gelfand-Pettis integrals of compactly supported functions which take values in quasi-complete, locally convex spaces exist (see Section 7.1). Thus for $w \in U$ (where U is open and has compact closure), the function

$$w \to \int_{|t|\leq T} \frac{\cos(tx) - \cos(wx)}{(-it)^2 + w^2} \, dt$$

is a holomorphic Mod^b-valued function with b depending on U.

Consider the second integral in (3). For fixed $|w| \ll T$,

$$\int_{|t|>T} \frac{\cos(tx)}{(-it)^2 + w^2} \, dt$$

is the Fourier inversion map. For $f \in C_c^\infty(\mathbb{R})$ define the Hilbert space norm

$$|f|_s = \int_{|t|>T} |f(t)|^2 \cdot (1 + t^2) \, dt$$

and let V^s be the completion of $C_c^\infty(\mathbb{R})$ with respect to $|\cdot|_s$. By extending the Plancherel theorem, Fourier inversion is an isometry from V^s to the Sobolev space $H^s(\mathbb{R})$ (see Section 5). Let $X = \{t \in \mathbb{R} \mid |t| > T\}$ and V_T^s be the completion of $C_c^\infty(X)$ with respect to $|\cdot|_s$. Then the Fourier inversion integral is an isometry from V_T^s to a closed subspace of H^s. Thus in order to show that the Fourier inversion integral is a holomorphic $H^{3/2-\epsilon}$-valued function of w, it suffices to show that

$$w \mapsto \frac{1}{-t^2 + w^2}$$

is a holomorphic $V_T^{3/2-\epsilon}$-valued function of w. Since we are avoiding $w = 0$, it suffices to examine the function

$$\lambda \to \frac{1}{t^2 - \lambda}$$

for $\lambda = -w^2$. The error in estimating the difference quotient by the derivative is

$$\begin{aligned}
&\frac{1}{h}\left(\int_{|t|>T} \frac{1}{t^2 - (\lambda + h)} \, dt - \int_{|t|>T} \frac{1}{t^2 - \lambda} \, dt\right) - \int_{|t|>T} \frac{1}{(t^2 - \lambda)^2} \, dt \\
&= \int_{|t|>T} \frac{1}{(t^2 - (\lambda + h))(t^2 - \lambda)} \, dt - \int_{|t|>T} \frac{1}{(t^2 - \lambda)^2} \, dt \\
&= h \cdot \int_{|t|>T} \frac{1}{(t^2 - (\lambda + h))(t^2 - \lambda)^2} \, dt.
\end{aligned}$$

As $h \to 0$ this difference also goes to 0 uniformly for $|w| \leq T/2$, proving the complex-differentiability in w.

Finally, for the third integral in (3), notice that the factor $\cos wx$ is a Mod^∞ function of x. We see that the integral itself is a holomorphic scalar-valued function of w by considering the relevant difference quotient. As above, it is sufficient to prove holomorphy for $\lambda = -w^2$: suppressing constants, the error in estimating the difference quotient by the derivative as above we see

$$\frac{1}{h}\left(\int_{|t|>T} \frac{1}{t^2 - (\lambda + h)} \, dt - \int_{|t|>T} \frac{1}{t^2 - \lambda} \, dt\right) - \int_{|t|>T} \frac{1}{(t^2 - \lambda)^2} \, dt$$

$$= \int_{|t|>T} \frac{1}{(t^2-(\lambda+h))(t^2-\lambda)}\, dt - \int_{|t|>T} \frac{1}{(t^2-\lambda)^2}\, dt$$
$$= h \cdot \int_{|t|>T} \frac{1}{(t^2-(\lambda+h))(t^2-\lambda)^2}\, dt.$$

As $h \to 0$ this difference also goes to 0 uniformly for w in compact sets away from 0 and $T \gg |w|$, proving the complex-differentiability in w.

$\square$

Now we have a qualitative meromorphic continuation of the spectral expansion of a solution to $(\Delta + w^2)u = \delta$ as a continuous function of controlled growth. Arguments similar to those above will be useful in meromorphically continuing solutions to $(\Delta - \lambda_s)u = \delta_\omega^{\mathrm{afc}}$ on $\Gamma\backslash\mathfrak{H}$.

The solutions to this differential equation $(\Delta + w^2)u = \delta$ happen *never* to be eigenfunctions for the Friedrichs' extension. Furthermore, these function $u_w(x)$ have corners. We call solutions with such property which are also eigenfunctions "exotic eigenfunctions."

2. Expansions in Exotic Eigenfunctions

In the automorphic case certain eigenfunctions have exotic characteristics such as certain *failures of smoothness*. Specifically, this happens when we restrict the domain of Δ to $L^2_a(\Gamma\backslash\mathfrak{H}) \cap C_c^\infty(\Gamma\backslash\mathfrak{H})$ and take the Friedrichs extensions. When this is done the spectrum becomes purely discrete but given that we have truncated the Eisenstein series not all of the eigenfunctions are smooth. Though it is atypical and possibly counter-intuitive to have non-smooth eigenfunctions this is not an anomaly.

In order to warm-up to these characteristics we will investigate a more elementary case where exotic eigenfunctions arise as a basis for $L^2(\mathbb{R}/2\pi\mathbb{Z})$. Specifically, we will examine the convergence of spectral expansions in terms of these non-smooth eigenfunctions. Furthermore, to use spectral theory in the automorphic case, it will be necessary for us to write spectral expansions not just for functions in $L^2(\Gamma\backslash\mathfrak{H})$ but distributions as well. We must be careful to say in what sense and where such expansions converge. Thus we will examine the convergence of spectral expansions for distributions in $H^{-1}(\mathbb{R}/2\pi\mathbb{Z})$ in terms of exotic (i.e. non-smooth) eigenfunctions.

The most familiar basis of $L^2(\mathbb{R}/2\pi\mathbb{Z})$ is the collection of eigenfunctions for $\Delta = \frac{d^2}{dx^2}$ namely
$$\mathcal{E} := \{e^{imx} \mid m \in \mathbb{Z}\}.$$
These functions are smooth (i.e. do not have corners) and form an orthogonal basis for $L^2(\mathbb{R}/2\pi\mathbb{Z})$. Though Fourier series provide nice spectral expansions for functions in $L^2(\mathbb{R}/2\pi\mathbb{Z})$, at certain points in the automorphic case, we will want to use eigenfunction expansions of functions and distributions in terms of non-smooth ('exotic') eigenfunctions. We can similarly derive a basis of exotic eigenfunctions for $L^2(\mathbb{R}/2\pi\mathbb{Z})$. The collection
$$\mathcal{S} := \left\{ \sin\left(\frac{nx}{2}\right) \mid n = 1, 2, \ldots \right\}$$
is an orthonormal basis of $L^2(\mathbb{R}/2\pi\mathbb{Z})$ which are eigenfunctions for Δ (in the interior) subject to the boundary condition $u(0) = u(2\pi) = 0$. These elements of $\mathcal{S}$ can also be also realized

as eigenfunctions for the Friedrichs extension $\widetilde{S}$ of $S = \Delta|_{\mathbb{C}^\infty(\mathbb{R}/2\pi\mathbb{Z})\cap \ker \delta_{2\pi\mathbb{Z}}}$. To see this note that

$$\left(\Delta + \left(\frac{n}{2}\right)^2\right) \cdot \sin\left(\frac{nx}{2}\right) = c \cdot \delta_{2\pi\mathbb{Z}} \quad \& \quad \delta_{2\pi\mathbb{Z}}\left(\sin\left(\frac{nx}{2}\right)\right) = 0$$

for some constant $c \neq 0$ (see Appendix 7.3).

This is not surprising since L^2 spaces do not distinguish whether the endpoints 0 and 2π have been identified. However, what is surprising is the fact that the eigenfunctions in $\mathcal{S}$ are not smooth and have corners i.e. $\sin\left(\frac{nx}{2}\right)$ vanish at 0 and for n odd they are not smooth at 0. However, they are genuine eigenfunctions. Our point of concern then is where L^2 expansions of such eigenfunctions converge.

Note that half of the eigenfunctions in $\mathcal{E}$ and $\mathcal{S}$ are shared. Further we can express elements of $\mathcal{E}$ as infinite sums of elements of $\mathcal{S}$ which converge in L^2 but not necessarily (and often *definitely* not) in H^1. To see this, observe that

$$\langle \cos mx, \sin\left(\frac{nx}{2}\right)\rangle_{L^2(\mathbb{R}/2\pi\mathbb{Z})} = \frac{n}{\left(\frac{n}{2}\right)^2 + m^2}.$$

Thus for fixed m we have that

$$\cos(mx) = \frac{1}{\pi}\sum_{\substack{n\geq 1\\ \text{odd}}} \frac{n}{\left(\frac{n}{2}\right)^2 + m^2} \cdot \sin\left(\frac{nx}{2}\right)$$

converges in L^2 since $\frac{n}{\left(\frac{n}{2}\right)^2+m^2} \sim 1/n$. However, for $m = 0$, we get that $\cos 0 = 1$ and

$$\frac{1}{\pi}\sum_{\substack{n\geq 1\\ \text{odd}}} \frac{n}{\left(\frac{n}{2}\right)^2} \cdot \sin\left(\frac{nx}{2}\right) = \frac{1}{\pi}\sum_{\substack{n\geq 1\\ \text{odd}}} \frac{2}{n} \cdot \sin\left(\frac{nx}{2}\right)$$

which does not converge in H^1. (This is easiest to see using the H^1-norm on the circle: $||f||_{H^1}^2 = \sum_n (1 + |n|^2)|\hat{f}(n)|^2$.)

On the other hand, exotic eigenfunction expansions of $\mathcal{S}$ in terms of the standard ones in $\mathcal{E}$ *all* do converge in H^1 (not merely L^2) since elements of $\mathcal{S}$ *are* in the H^1-span of the standard eigenfunctions. Furthermore, for fixed n we have that

$$\sin\left(\frac{nx}{2}\right) = \frac{1}{\pi}\sum_{m\geq 1} \frac{n}{\left(\frac{n}{2}\right)^2 + m^2} \cdot \cos(mx)$$

which converges in L^2 since $\frac{n}{\left(\frac{n}{2}\right)^2+m^2} \sim 1/m^2$.

In order to solve differential equations using spectral expansions, we will also need to be able to express distributions such as $\delta_a \in H^{-1}$ supported at $a \neq 0$ in terms of the exotic eigenfunctions in $\mathcal{S}$. A reasonable guess for this expansion is

$$\delta_a = \frac{1}{\pi}\sum_{n=1}^{\infty} \langle \delta_a, \sin\left(\frac{nx}{2}\right)\rangle \cdot \sin\left(\frac{nx}{2}\right).$$

The question is in what sense this expansion converges. Resolving this technical issue will be the work of the rest of this section. To see the issue, compare the expansions of the solution

u and the distributions δ_a as follows:

Suppose that u_λ is an eigenfunction for the Friedrichs extension of $T = \Delta|_{\ker \delta_a}$ (Δ restricted to $C_c^\infty(\mathbb{R}/2\pi\mathbb{Z}) \cap \ker \delta_a$). This means that u_λ is a solution to

$$\begin{cases} (\Delta - \lambda)u_\lambda = \delta_a \\ \delta_a u_\lambda = 0 \end{cases} \qquad \text{(in a distributional sense)}$$

(see Appendix 7.3). We can then write

$$(\Delta - \lambda)u_\lambda = \frac{1}{\pi}\sum_{n=1}^{\infty} \langle \delta_a, \sin\left(\frac{nx}{2}\right)\rangle \cdot \sin\left(\frac{nx}{2}\right)$$

and *dividing* gives us

$$u_\lambda = \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\langle \delta_a, \sin\left(\frac{nx}{2}\right)\rangle}{-(\frac{n}{2})^2 - \lambda} \cdot \sin\left(\frac{nx}{2}\right).$$

From the boundary condition we also have that $\delta_a u_\lambda = 0$ which is true iff

$$0 = \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{|\langle \delta_a, \sin\left(\frac{nx}{2}\right)\rangle|^2}{-(\frac{n}{2})^2 - \lambda}.$$

The problem is that if the above were true then checking our result *seems* to give

$$\delta_a = (\Delta - \lambda)u_\lambda = (\Delta - \lambda) \cdot \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\langle \delta_a, \sin\left(\frac{nx}{2}\right)\rangle}{-(\frac{n}{2})^2 - \lambda} \cdot \sin\left(\frac{nx}{2}\right)$$

$$= \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\langle \delta_a, \sin\left(\frac{nx}{2}\right)\rangle}{-(\frac{n}{2})^2 - \lambda} \cdot (\Delta - \lambda)\sin\left(\frac{nx}{2}\right)$$

$$= \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\langle \delta_a, \sin\left(\frac{nx}{2}\right)\rangle}{-(\frac{n}{2})^2 - \lambda} \cdot \left(\left(-\left(\frac{n}{2}\right)^2 - \lambda\right)\sin\left(\frac{nx}{2}\right) + n\delta_{2\pi\mathbb{Z}}\right) \tag{4}$$

$$= \frac{1}{\pi}\sum_{n=1}^{\infty} \langle \delta_a, \sin\left(\frac{nx}{2}\right)\rangle \cdot \sin\left(\frac{nx}{2}\right) + \left(\frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\langle \delta_a, \sin\left(\frac{nx}{2}\right)\rangle}{-(\frac{n}{2})^2 - \lambda}\right)\delta_{2\pi n\mathbb{Z}}$$

$$= \delta_a - \left(\frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\langle \delta_a, \sin\left(\frac{nx}{2}\right)\rangle}{\lambda + (\frac{n}{2})^2}\right)\delta_{2\pi n\mathbb{Z}}$$

However this sum does not converge, and in fact, we want the outcome to be δ_a, *not* δ_a plus a multiple of $\delta_{2\pi\mathbb{Z}}$. The resolution of this inconsistency is suppression of the distribution $\delta_{2\pi\mathbb{Z}}$. However, this requires some effort: we consider a map to a quotient in which $\delta_{2\pi\mathbb{Z}}$ goes to 0 and in which the image of this series converges.

The Friedrichs extension $\widetilde{S}$ maps to a quotient space in which $\delta_{2\pi\mathbb{Z}}$ is mapped to 0 (but δ_a is not). Since $\widetilde{S}$ maps its domain to $L^2(\mathbb{R}/2\pi\mathbb{Z})$ there are no solutions to $(\widetilde{S} - \lambda)u = \delta_a$. However, extending $\widetilde{S}$ to a genuine distributional differential operator brings the undesirable appearance of $\delta_{2\pi\mathbb{Z}}$. Thus we need an extension $\widetilde{S}^\#$ of $\widetilde{S}$ that has an *image* of δ_a in its image.

Let $\mathfrak{B}^1$ be the H^1-closure of dom $\widetilde{S}$ and $\mathfrak{B}^{-1}$ be its dual $(\mathfrak{B}^1)^*$. Define $\widetilde{S}^\# : \mathfrak{B}^1 \to (\operatorname{dom} \widetilde{S})^* \cong \mathfrak{B}^{-1}$ by

$$(\widetilde{S}^\# u)(v^c) = \langle u, \widetilde{S}v \rangle_{L^2(\mathbb{R}/2\pi\mathbb{Z})}$$

for $u \in \mathfrak{B}^1$ and $v \in \operatorname{dom} \widetilde{S}$ where $v \mapsto v^c$ is complex conjugation. Observe that by symmetry of $\widetilde{S}$ that $\widetilde{S}^\#$ agrees with $\widetilde{S}$ on $\operatorname{dom} \widetilde{S}$. For clarity we have the following diagram (which is only commutative when the map $H^1 \hookrightarrow L^2$ is removed):

$$\begin{tikzcd}
H^1 \arrow[r] \arrow[rr, bend left, "\Delta"] & L^2 \arrow[r] & H^{-1} \arrow[d, "j^*"] \\
\mathfrak{B}^1 \arrow[u, "j"] \arrow[rr, bend right, "\widetilde{S}^\#"'] & & \mathfrak{B}^{-1} \arrow[d, "="] \\
\operatorname{dom} \widetilde{S} \arrow[u, "\mathrm{inc}"] \arrow[uur, "\widetilde{S}"] & & (\operatorname{dom} \widetilde{S})^*
\end{tikzcd}$$

Lemma 4. *dom* $\widetilde{S} = \{x \in \mathfrak{B}^1 \mid \widetilde{S}^\# x \in L^2(\mathbb{R}/2\pi\mathbb{Z})\}$

Proof. Let U be $\widetilde{S}^\#$ restricted to $\{x \in \mathfrak{B}^1 \mid \widetilde{S}^\# x \in L^2(\mathbb{R}/2\pi\mathbb{Z})\}$ so that $\operatorname{dom} \widetilde{S} \subseteq \operatorname{dom} U$. Also for $y \in \operatorname{dom} \widetilde{S}$ by definition we have $\langle Ux, y \rangle = \langle x, \widetilde{S}y \rangle$. This gives us that $\operatorname{dom} U \subset \operatorname{dom}(\widetilde{S}^*) = \operatorname{dom} \widetilde{S}$ since $\widetilde{S}$ is self-adjoint. ☐

Lemma 5. *The eigenfunctions for $\widetilde{S}$ remain orthogonal in $\mathfrak{B}^1$. In particular, for v an $\widetilde{S}$-eigenfunction with eigenvalue λ,*

$$\langle u, v \rangle_{H^1} = (1 - \lambda) \cdot \langle u, v \rangle_{L^2}$$

for $u \in$ dom $\widetilde{S}$.

Proof. Since $\lambda \in \mathbb{R}$,

$$(1 - \lambda)^{-1} \cdot \langle u, v \rangle_{H^1} = \langle u, (1 - \widetilde{S})^{-1} v \rangle_{H^1} = \langle u, v \rangle_{L^2}$$

by characterization of the Friedrichs extension. ☐

It is important to note that for $u \in H^1$ but not in $\mathfrak{B}^1$ we cannot compute its H^1-inner product with $v \in \operatorname{dom} \widetilde{S}$ in the same way. The self-adjointness of $\widetilde{S}$ *entails* that it is the *maximal* symmetric extension of S. Consider $u = \cos mx \in H^1$ but not in $\mathfrak{B}^1$ and for $\widetilde{S}$-eigenfunction v with eigenvalue $-\left(\frac{n}{2}\right)^2 \neq m^2$,

$$\langle (1 - \Delta)u, v \rangle_{L^2} = (1 + m^2) \cdot \langle u, v \rangle_{L^2} \neq (1 + \left(\frac{n}{2}\right)^2) \cdot \langle u, v \rangle_{L^2} = \langle u, (1 - \widetilde{S})v \rangle_{L^2}$$

since $\langle u, v \rangle_{L^2} = \frac{n}{\left(\frac{n}{2}\right)^2 - m^2} \neq 0$. The left half of this computation takes advantage of the fact that Δ maps $\cos mx$ to L^2. The correction of computing H^1 inner products resolves the fact that the L^2-expansion of $\cos mx$ in terms of $\sin\left(\frac{nx}{2}\right)$ does not converge in H^1. Instead,

Corollary 6. *The L^2-expansion of $\cos mx$ in terms of eigenfunctions for $\widetilde{S}$ is not an H^1-expansion of $\cos mx$. Instead, the orthogonal projection in H^1 of u to the H^1-closure $\mathfrak{B}^1$ of the span of eigenfunctions is*

$$\begin{aligned} \textit{projection}(\cos mx) &= \sum_{\substack{n\geq 1\\ n \text{ odd}}} \langle \cos mx, \sin\left(\frac{nx}{2}\right)\rangle_{H^1} \cdot \frac{\sin\left(\frac{nx}{2}\right)}{|\sin\left(\frac{nx}{2}\right)|_{H^1}^2} \\ &= (1+m^2) \sum_{\substack{n\geq 1\\ n \text{ odd}}} \langle \cos mx, \sin\left(\frac{nx}{2}\right)\rangle_{L^2} \cdot \frac{\sin\left(\frac{nx}{2}\right)}{(1+n^2)\cdot|\sin\left(\frac{nx}{2}\right)|_{L^2}^2} \\ &= \frac{(1+m^2)}{\pi} \sum_{\substack{n\geq 1\\ n \text{ odd}}} \frac{n}{\left(\frac{n}{2}\right)^2 - m^2} \cdot \frac{1}{(1+n^2)} \sin\left(\frac{nx}{2}\right) \end{aligned}$$

where the coefficients are $O(1/n^3)$ so this converges in H^1.

Theorem 7. *The $\widetilde{S}$ eigenfunctions are an orthogonal basis for $\mathfrak{B}^1 \subset H^1$.*

Proof. As noted above, the $\widetilde{S}$ eigenfunctions are orthogonal in $\mathfrak{B}^1$. To show that this collection is a basis for $\mathfrak{B}^1$ we use a continuity argument since $\mathfrak{B}^1$ is the H^1-closure of $\operatorname{dom}\widetilde{S}$: For $v \in \mathfrak{B}^1$, let $v = \lim_i v_i$ and H^1-limit with $v_i \in \operatorname{dom}\widetilde{S}$. For u a λ-eigenfunction for $\widetilde{S}$,

$$\begin{aligned} \langle v, u\rangle_{H^1} = \langle \lim_i v_i, u\rangle_{H^1} = \lim_i (1-\lambda)\langle v_i, u\rangle_{L^2} = (1-\lambda)\lim_i \langle v_i, u\rangle_{L^2} \\ = (1-\lambda)\lim_i \langle v_i - v, u\rangle_{L^2} + (1-\lambda)\langle v, u\rangle_{L^2} = (1-\lambda)\langle \lim_i v_i - v, u\rangle_{L^2} + (1-\lambda)\langle v, u\rangle_{L^2} \\ = (1-\lambda)\langle 0, u\rangle_{L^2} + (1-\lambda)\langle v, u\rangle_{L^2} = (1-\lambda)\langle v, u\rangle_{L^2}. \end{aligned}$$

Since the eigenfunctions u are an orthogonal basis for L^2, they are an orthogonal basis for their H^1-closure $\mathfrak{B}^1$. $\square$

Thus distributions in H^{-1} can only be expressed in terms of $\mathcal{S}$ eigenfunctions if such a distribution belongs to $\mathfrak{B}^{-1}$. Let $j : \mathfrak{B}^1 \to H^1$ be the inclusion map with dual $j^* : H^{-1} \to \mathfrak{B}^{-1}$. Since $\operatorname{dom}\widetilde{S} \subset \mathfrak{B}^1 \subset \ker \delta_{2\pi\mathbb{Z}}$ we have that $j^*\delta_{2\pi\mathbb{Z}} = 0$. Also since $j^*\delta_a \in \mathfrak{B}^{-1}$ it now makes sense to look for solutions to the equation

$$(\widetilde{S}^{\#} - \lambda)u = j^*\delta_a$$

for $u \in \mathfrak{B}^1$. Of course, to make use of this we want $(\Delta - \lambda)u = \delta_a$ to imply $(\widetilde{S}^{\#} - \lambda)u = j^*\delta_a$. More precisely, we want

$$j^* \circ \Delta|_{H^1} \circ j = \widetilde{S}^{\#}.$$

Thus we need

Lemma 8. *The restriction U of $j^* \circ \Delta|_{H^1} \circ j$ to $\operatorname{dom} U := \{u \in \mathfrak{B}^1 \mid (j^* \circ \Delta|_{H^1} \circ j)u \in L^2\} \subset \mathfrak{B}^1$ is symmetric and extends $\widetilde{S}$. Hence $U = \widetilde{S}$.*

Proof. We have $\operatorname{dom}\widetilde{S} \subset \operatorname{dom} U$ since $\operatorname{dom}\widetilde{S} = \{u \in H^1 \mid \Delta u \in L^2 + \mathbb{C}\cdot\delta_{2\pi\mathbb{Z}} \;\&\; \delta_{2\pi\mathbb{Z}} u = 0\}$. For $u, v \in \operatorname{dom}\widetilde{S}$ with $\widetilde{S}u = \Delta u + a\cdot\delta_{2\pi\mathbb{Z}}$ and $\widetilde{S}v = \Delta v + b\cdot\delta_{2\pi\mathbb{Z}}$, we see that U agrees with $\widetilde{S}$ on $\operatorname{dom}\widetilde{S}$:
Tracking $j : \mathfrak{B}^1 \to H^1$,

$$\langle Uu, v\rangle_{L^2} = \langle (j^* \circ \Delta \circ j)u, v\rangle_{L^2} = \langle (j^* \circ \Delta \circ j)u, v\rangle_{\mathfrak{B}^{-1}\times\mathfrak{B}^1} = \langle j^*(\widetilde{S}u - a\delta_{2\pi\mathbb{Z}}), v\rangle_{\mathfrak{B}^{-1}\times\mathfrak{B}^1}$$

$$= \langle \widetilde{S}u - a\delta_{2\pi\mathbb{Z}}, jv\rangle_{H^{-1}\times H^1} = \langle \widetilde{S}u, jv\rangle_{L^2} - a\langle \delta_{2\pi\mathbb{Z}}, jv\rangle_{H^{-1}\times H^1} = \langle \widetilde{S}u, jv\rangle_{L^2} + 0$$
$$= \langle \widetilde{S}u, v\rangle_{L^2}.$$

Similarly to show the symmetry of U:
For $u, v \in \operatorname{dom} U$, using the distributional symmetry of Δ,

$$\langle Uu, v\rangle_{L^2} = \langle (j^* \circ \Delta \circ j)u, v\rangle_{\mathfrak{B}^{-1}\times\mathfrak{B}^1} = \langle \Delta(ju), jv\rangle_{H^{-1}\times H^1} = \langle ju, \Delta(jv)\rangle_{H^1\times H^{-1}}$$
$$= \langle u, (j^* \circ \Delta \circ j)v\rangle_{\mathfrak{B}^1\times\mathfrak{B}^{-1}} = \langle u, Uv\rangle_{L^2}.$$

Since $\widetilde{S}$ is self-adjoint, it is maximal symmetric so $U = \widetilde{S}$. $\square$

Similarly, have the desired result:

Theorem 9. $\left(j^* \circ \Delta\Big|_{H^1} \circ j\right) = U^\# = \widetilde{S}^\# : \mathfrak{B}^1 \to \mathfrak{B}^{-1}$

Proof. Let $u \in \mathfrak{B}^1$ and $v \in \operatorname{dom}\widetilde{S} = \operatorname{dom} U$ using the distributional symmetry of Δ,

$$((j^* \circ \Delta \circ j)u)\,(v^c) = \langle (j^* \circ \Delta \circ j)u, v\rangle_{\mathfrak{B}^{-1}\times\mathfrak{B}^1} = \langle \Delta(ju), jv\rangle_{H^{-1}\times H^1} = \langle ju, \Delta(jv)\rangle_{H^1\times H^{-1}}$$
$$= \langle u, (j^* \circ \Delta \circ j)v\rangle_{\mathfrak{B}^1\times\mathfrak{B}^{-1}} = \langle u, \widetilde{S}v\rangle_{L^2} = (\widetilde{S}^\# u)(v^c)$$

since $j^* \circ \Delta \circ j$ is $\widetilde{S}$ on $\operatorname{dom}\widetilde{S}$. Since $\mathfrak{B}^1$ is the H^1-closure of $\operatorname{dom}\widetilde{S}$ this gives $j^* \circ \Delta \circ j$ is $\widetilde{S}$ on $\operatorname{dom}\widetilde{S}$. Since $\mathfrak{B}^1$ is the H^1-closure of $\operatorname{dom}\widetilde{S}$ we have that $j^* \circ \Delta \circ j = \widetilde{S}^\#$ $\square$

Using this we get the desired Corollary

Corollary 10. *If $(\Delta - \lambda)u = \delta_a$ for $u \in \mathfrak{B}^1$, then $(\widetilde{S}^\# - \lambda)u = j^*\delta_a$.*

Proof. With the inclusion $j : \mathfrak{B}^1 \to H^1$, we have $(\Delta - \lambda)(ju) = \delta_{2\pi\mathbb{Z}}$ gives $j^*(\Delta - \lambda)(ju) = j^*\delta_a$. Invoking Theorem 9, our result follows. $\square$

Using j^* to send δ_a to its image in $\mathfrak{B}^{-1}$ we can write an expansion for $j^*\delta_a$ in terms of exotic eigenfunctions and our desired result follows directly from this Corollary.

Theorem 11. *For $j^*\delta_a \in \mathfrak{B}^{-1}$*

$$j^*\delta_a = \frac{1}{\pi}\sum_{n=1}^{\infty}\langle j^*\delta_a, \sin\left(\frac{nx}{2}\right)\rangle \cdot \sin\left(\frac{nx}{2}\right)$$

converges in $\mathfrak{B}^1$.

Proof. Since $j^*\delta_{2\pi\mathbb{Z}} = 0$, there is no appearance of $\delta_{2\pi\mathbb{Z}}$ in equation (4) and the result follows from the previous Corollary. $\square$

Notice that with this expansion we do not run into the same trouble that we did when trying the write out an expansion for an arbitrary $\delta_a \in H^{-1}$ (but not necessarily in $\mathfrak{B}^{-1}$), namely: $j^*\delta_a = (\widetilde{S}^\# - \lambda)u$ (as opposed to $(\Delta - \lambda)u$) and so no extra $\delta_{2\pi\mathbb{Z}}$ term appear when taking $(\widetilde{S}^\# - \lambda)$ on u_λ as in line (4) of experimentation with the initial attempted expansion of δ_a.

3. Simplest Automorphic Case

For the spectral decomposition of automorphic forms, we recall the space on which these forms live as well as the relevant players in the decomposition. In [16], L. Faddeev used functional analysis to provide the full spectral theory of automorphic forms which had been used by Selberg. We will explain the material related to results in the next two sections.

The upper half-plane $\mathfrak{H}$ is acted upon by $G = SL_2(\mathbb{R})$ by linear fractional transformations. Furthermore, $K = SO(2)$ is the maximal compact subgroup of G as well as the isotropy group of $i \in \mathfrak{H}$. Hence $\mathfrak{H} \cong G/K$.

An obvious choice of discrete subgroup of G is $\Gamma = SL_2(\mathbb{Z})$. Topologically, the quotient $\Gamma\backslash\mathfrak{H}$ is a sphere with a point removed and the usual fundamental domain for Γ on $\mathfrak{H}$ is

$$F = \{z \in \mathfrak{H} \mid |z| \geq 1 \ \& \ |\mathrm{Re}(z)| \leq 1/2\}.$$

We describe the spectral decomposition of $L^2(\Gamma\backslash\mathfrak{H})$ with respect to the invariant Laplacian Δ descended from the Casimir element of the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ of G. Let $U\mathfrak{g}$ be the universal enveloping algebra. The Casimir element Ω of $U\mathfrak{g}$ is the image of $1_{\mathfrak{g}} \in \mathrm{End}(\mathfrak{g})$ under the chain of G-equivariant maps

$$\mathrm{End}_{\mathbb{C}}(\mathfrak{g}) \to \mathfrak{g} \otimes \mathfrak{g}^* \to \mathfrak{g} \otimes \mathfrak{g} \to A\mathfrak{g} \to U\mathfrak{g}$$

where $A\mathfrak{g}$ is often denoted $\bigotimes^\bullet \mathfrak{g}$. Since the Casimir element is in the center of the enveloping algebra, it commutes with the G-action. Writing Ω in coordinates gives $\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$.

The simplest Δ-eigenfunctions are $x + iy \to y^s$ for $s \in \mathbb{C}$ and indeed $\Delta y^s = s(s-1)y^s$.

Since Δ is a symmetric operator on $L^2(\Gamma\backslash\mathfrak{H})$ (albeit unbounded), it is natural to examine the spectral decomposition of $L^2(\Gamma\backslash\mathfrak{H})$ with respect to Δ.

4. Spectral Decomposition of $\Gamma\backslash\mathfrak{H}$

Let $\Gamma = SL_2(\mathbb{Z})$ and $\mathfrak{H}$ the upper half plane. If $\Gamma\backslash\mathfrak{H}$ were compact the spectrum would be discrete. However, $\Gamma\backslash\mathfrak{H}$ is not compact but is fiite-volume and there is both discrete and continuous spectrum in the spectral decomposition of $L^2(\Gamma\backslash\mathfrak{H})$. The spectral decomposition with respect to the invariant Laplacian $\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ is

$$L^2(\Gamma\backslash\mathfrak{H}) = L^2_{\mathrm{cusp}}(\Gamma\backslash\mathfrak{H}) \oplus \mathbb{C} \oplus L^2_{\mathrm{Eis}}(\Gamma\backslash\mathfrak{H})$$

(i.e. cuspforms and pseudo-Eisenstein series–integrals of Eisenstein series plus residue). Explicitly, for $f \in L^2(\Gamma\backslash\mathfrak{H})$,

$$f = \sum_{F \text{ cusp}} \langle f, F\rangle_{\Gamma\backslash\mathfrak{H}} \cdot F + \frac{\langle f, 1\rangle_{\Gamma\backslash\mathfrak{H}} \cdot 1}{\langle 1, 1\rangle_{\Gamma\backslash\mathfrak{H}}} + \frac{1}{4\pi i}\int_{(1/2)} \langle f, E_s\rangle_{\Gamma\backslash\mathfrak{H}} \cdot E_s ds$$

converges in $L^2(\Gamma\backslash\mathfrak{H})$ where the first sum is taken over an orthonormal basis of cuspforms F (for the full computation see Appendix 7.2).

There are a few things to note about this spectral decomposition. First, Eisenstein series $E_s(z) = 1/2 \sum_{c,d \text{ coprime}} \frac{y^s}{|cz+d|^{2s}}$ are not bounded and not in $L^2(\Gamma\backslash\mathfrak{H})$. In fact, since

$$E_s(\omega) = \frac{\zeta(s)L(s,\chi_{-3})}{\zeta(2s)}$$

for $\omega = e^{2\pi i/3}$ if we knew the asymptotic behavior in s, we would be able to prove results that are stronger than we should reasonably expect to prove. Specifically, this gives the period of the Eisenstein series is $E_s(\omega) = \frac{\zeta_{\mathbb{Q}(\sqrt{-3})}(s)}{\zeta(2s)}$ and so $E_{1/2+it}(\omega) = \frac{\zeta_{\mathbb{Q}(\sqrt{-3})}(1/2+it)}{\zeta(1+2it)}$ where the denominator is on the edge of the strip and $\zeta_{\mathbb{Q}(\sqrt{-3})}(s)$ is a Dedekind zeta function which can be identified with $\zeta(s)L(s,\chi_{-3})$. Because we are on $\Re(s) = 1/2$, if we know that these periods were $O(t^\epsilon)$, we would be able to prove the Lindelöf Hypothesis (for $\epsilon > 0$, $\zeta(1/2+it)$ is $\mathcal{O}(t^\epsilon)$). However, this result would be unreasonably strong as a starting assumption. We avoid direct discussion of pointwise convergence. Thus, taking a somewhat indirect approach. We can write pseudo-Eisenstein series for test functions $\varphi \in C_0^\infty((0,\infty))$ as

$$\psi_\varphi = \sum_{\gamma \in P\cap\Gamma\backslash\Gamma} \varphi(\mathrm{Im}(\gamma z)) = \frac{1}{4\pi i}\int_{(1/2)} \langle \psi_\varphi, E_s\rangle \cdot E_s \, ds + \mathcal{M}\varphi(1) \cdot \mathrm{res}_{s=1} E_s$$

and as we would do with the case in $\mathbb{R}$, prove Plancherel for ψ_φ then extend to all of $L^2(\Gamma\backslash\mathfrak{H})$ by denseness as in the case with the spectral synthesis for Schwartz functions. Note that the proof of automorphic Plancherel does *not* refer to sup norms.

5. Global Automorphic Spaces

While spectral expansions converge to their function in L^2, these L^2 expansions are not guaranteed to be *classically* differentiable (or even continuous). We are guaranteed weak differentiability but this property is often not strong enough for us to be able to use spectral theory to solve differential equations. Sobolev spaces allow us to refine the notion of differentiability. We sketch the relevant background here.

From the spectral expansion in Section 4, Plancherel is

$$||f||^2 = \sum_{F \text{ cusp}} |\langle f, F\rangle|^2 + \frac{|\langle f, 1\rangle|^2}{\langle 1, 1\rangle} + \frac{1}{2\pi}\int_0^\infty |\langle f, E_{\frac{1}{2}+it}\rangle|^2 dt.$$

For ease of notation let

$$\Xi := \{\text{orthonormal basis of cuspforms}\} \cup \{1\} \cup \left(\frac{1}{2} + i[0,\infty)\right)$$

where the half-line parametrizes Eisenstein series $E_{\frac{1}{2}+it}$. The spectral measure on Ξ gives each cuspform and constant point-mass measure 1 and gives the half-line a constant multiple of Lebesgue measure. For $\xi \in \Xi$, let

$$\Phi_\xi = \begin{cases} F & \text{for } \xi = F \\ 1/\langle 1, 1\rangle^{1/2} & \text{for } \xi = 1 \\ E_s & \text{for } \xi = s = \frac{1}{2} + it \end{cases}.$$

Then we can write the spectral decomposition as

$$f = \int_{\Xi} \langle f, \Phi_\xi \rangle \cdot \Phi_\xi \; d\xi$$

and Plancherel as

$$||f||^2 = \int_{\Xi} |\langle f, \Phi_\xi \rangle|^2 \; d\xi.$$

Plancherel asserts that the integrals on test functions extend to an isometry $\mathcal{F} : L^2(X) \to L^2(\Xi)$.

For $f \in C_c^\infty(X)$ and λ_ξ the eigenvalue of Δ on Φ_ξ we have

$$\langle \Delta f, \Phi_\xi \rangle = \int_X f \Delta \overline{\Phi}_\xi = \lambda_\xi \cdot \int_X f \overline{\Phi}_\xi.$$

Hence for test functions,

$$\Delta f = \int_{\Xi} \langle \Delta f, \Phi_\xi \rangle \Phi_\xi \; d\xi = \int_{\Xi} \lambda_\xi \langle f, \Phi_\xi \rangle \Phi_\xi \; d\xi.$$

Thus, Δ moves through the integral:

$$\Delta f = \mathcal{F}^{-1} \mathcal{F} \Delta f = \mathcal{F}^{-1} \lambda_\xi \mathcal{F} f.$$

Thus the derivative makes sense on test functions. We now extend Δ to a more general collection of functions.

For the general differentiability of functions on $X = \Gamma \backslash G / K$, it is sufficient to use the right action of the Lie algebra $\mathfrak{g}$ of G on $\Gamma \backslash G$ since the measure and integral on $\Gamma \backslash G / K$ are that of $\Gamma \backslash G$ restricted to right K-invariant functions. Thus

$$||f||^2 = \int_{\Gamma \backslash G} |f|^2 \qquad \text{and} \qquad \langle f, \varphi \rangle = \int_{\Gamma \backslash G} f \overline{\varphi}$$

for $f, \varphi \in C_c^\infty(\Gamma \backslash G)$. The Sobolev space $H^\ell(X)$ is the completion of test functions $f \in C_c^\infty(X)$ with respect to the norm

$$|f|_{H^\ell}^2 = \langle (1 - \Delta)^\ell f, f \rangle_{L^2}.$$

Observe that the differential operator Δ on test functions gives a map

$$\Delta : H^\ell \cap C_c^\infty(\Gamma \backslash G) \to H^{\ell - 2} \cap C_c^\infty(\Gamma \backslash G)$$

which is continuous by design. Since Δ commutes with K, Δ extends to a continuous linear map

$$\Delta : H^\ell(X) \to H^{\ell-2}(X).$$

As expected, the standard

Theorem 12. *$\Delta : H^\ell \to H^{\ell-2}$ continuously.*

By expressing Sobolev norm and differentiation using spectral transforms $\mathcal{F}$, it is true that $\mathcal{F} H^\ell$ is contained in

$$V^\ell = \{ v \mid (1 - \lambda_\xi)^{\ell/2} v \in L^2(\Xi) \}$$

for $\ell \geq 0$. The Hilbert space structure of V^ℓ is given by the norm

$$|v|_{V^\ell}^2 = \int_\Xi (1-\lambda_\xi)^\ell |v|^2.$$

Furthermore, we have the following standard result:

Theorem 13. *$\mathcal{F} : H^\ell \to V^\ell$ is a Hilbert space isomorphism.*

Proof. Note that the Fourier transform is a linear map and that

$$|v|_{H^\ell}^2 = |\mathcal{F}v|_{V^\ell}^2$$

by properties of the Fourier transform and Plancherel. The challenging part of the argument is establishing surjectivity. First we will show that

$$(1-\Delta)\mathcal{F}^{-1}v = \mathcal{F}^{-1}\left((1-\lambda_\xi)v\right)$$

for $(1-\lambda_\xi)v \in L^2(\Xi)$.

For $\varphi \in C_c^\infty$,

$$\big((1-\Delta)\mathcal{F}^{-1}v\big)(\varphi) = \mathcal{F}^{-1}v\left((1-\Delta)\varphi\right) = v\left(\mathcal{F}(1-\Delta)\varphi\right)$$

by Plancherel for $L^2(X)$ and $L^2(\Xi)$. Again by L^2 Plancherel,

$$= \left((1-\lambda_\xi)\mathcal{F}\varphi\right) = (1-\lambda_\xi)v(\mathcal{F}\varphi) = \big(\mathcal{F}^{-1}\left((1-\lambda_\xi)v\right)\big)(\varphi).$$

This shows that the distribution $(1-\Delta)\mathcal{F}^{-1}v$ is given by integration against the L^2 function $\mathcal{F}^{-1}\left((1-\lambda_\xi)v\right)$.

Now to show that $\mathcal{F}(H^{2\ell}) = V^{2\ell}$ note that when $2\ell = 0$ this is justified by Plancherel. For $2\ell > 0$, let $v \in V^{2\ell}$ and for $k \leq \ell$,

$$(1-\Delta)^k\mathcal{F}^{-1}v = \mathcal{F}^{-1}(1-\lambda_\xi)^k v \in \mathcal{F}^{-1}V^{2\ell-2k} \subset \mathcal{F}^{-1}L^2 = L^2.$$

In other words the distributional derivatives $(1-\Delta)^k\mathcal{F}^{-1}v$ are in L^2 and for $\mathcal{F}^{-1}v \in H^2$. $\square$

Clearly, we do not define negative index Sobolev spaces in terms of negative-order differential operators but in terms of duals. For $\ell \geq 0$, $H^{-\ell}$ is the Hilbert-space dual to H^ℓ.

The continuous L^2 differentiation $\Delta : H^\ell \to H^{\ell-2}$ for $\ell \geq 2$ described above gives an adjoint (still denoted Δ)

$$\Delta : H^{-(\ell-2)} \to H^{-\ell}$$

for $\ell \geq 2$.

Similarly, $V^{-\ell}$ is the Hilbert space dual to V^ℓ by integration

$$\langle v, w\rangle = \int_X v(\xi)w(\xi)\, d\xi.$$

Furthermore, the density of $V^{\ell+1}$ in V^ℓ and the identity $V^0 = L^2$ yields

$$\cdots \subset V^2 \subset V^1 \subset V^0 \subset V^{-1} \subset V^{-2} \subset \ldots.$$

The density of $V^{\ell+1}$ in V^ℓ for every $\ell \in \mathbb{Z}$ is converted to density of $H^{\ell+1}$ in H^ℓ by the Fourier transform. Thus we have the diagram

$$\begin{tikzcd}
\dots \arrow[r, "\text{inc}"] & H^2 \arrow[r, "\text{inc}"] \arrow[d, "\mathcal{F}"] & H^1 \arrow[r] \arrow[d, "\mathcal{F}"] & H^0 \arrow[r, "\text{inc}"] \arrow[d, "\mathcal{F}"] & H^{-1} \arrow[r, "\text{inc}"] \arrow[d, "\mathcal{F}"] & H^{-2} \arrow[r, "\text{inc}"] \arrow[d, "\mathcal{F}"] & \dots \\
\dots \arrow[r, "\text{inc}"] & V^2 \arrow[r, "\text{inc}"] & V^1 \arrow[r, "\text{inc}"] & V^0 \arrow[r, "\text{inc}"] & V^{-1} \arrow[r, "\text{inc}"] & V^{-2} \arrow[r, "\text{inc}"] & \dots
\end{tikzcd}$$

6. The Project

We will begin by computing the period of an Eisenstein series and identify it as a self-dual L-function. Let k be a field and $\tilde{k}$ a quadratic extension of k. We will take an Eisenstein series E_s on $G = \mathrm{Res}_k^{\tilde{k}}(GL_2(\tilde{k}))$, restrict and integrate it against a cuspform f on $H = GL_2(k)$.

As usual, for $g \in G$ and $P_k = \left\{ \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \right\}$ the standard parabolic in G, the Eisenstein series is

$$E_s(g) = \sum_{\gamma \in P_k \backslash G_k} \varphi(\gamma \cdot g)$$

where φ is the locally spherical vector

$$\varphi\left(\begin{bmatrix} a & * \\ 0 & d \end{bmatrix} \cdot K \right) = \left| \frac{a}{d} \right|^s$$

for K in a maximal compact subgroup of $SL_2(\mathbb{R})$. When restricted to $H_\mathbb{A}$, E_s breaks up as a sum of H_k-invariant functions

$$E_s|_{H_\mathbb{A}}(h) = \sum_{\xi \in P_k \backslash G_k / H_k} \left(\sum_{\delta \in (\xi^{-1} P_k \xi \cap H_k) \backslash H_\mathbb{A}} \varphi(\xi \delta h) \right).$$

Thus our period initially looks like

$$\int_{Z_\mathbb{A} H_k \backslash H_\mathbb{A}} E_s(h) f(h) \, dh.$$

After unwinding, we will show that this period factors into local integrals which we'll identify as Euler factors of an L-function together with an auxiliary period.

Denote the subspace of $L^2(Z_\mathbb{A} G_k \backslash G_\mathbb{A})$ generated by Eisenstein series as $\mathfrak{E}$ and let $F \in \mathfrak{E}$. This means that F is of the form $F = \int A_s E_s$ where A_s is some constant.

Let θ be a distribution so that $\theta : F \to \int_{Z_\mathbb{A} H_k \backslash H_\mathbb{A}} f \cdot F$. The key lemma for our argument will be to show that

$$\begin{aligned} \theta F = \theta \left(\int_{(1/2)} A_s E_s \right) &= \int_{(1/2)} A_s \cdot \theta E_s \\ &= \int_{(1/2)} A_s \int_{Z_\mathbb{A} H_k \backslash H_\mathbb{A}} f \cdot E_s \end{aligned}$$

where we have found that $\int_{Z_{\mathbb{A}} H_k \backslash H_{\mathbb{A}}} f \cdot E_s$ is a L-function. In the compact case this is easily justified by Gelfand-Pettis but more work must be done here. We also need that $\theta \in H^{-1}$ which can be established using a second-moment bound on θE_s. Finally we will show that for $\mathfrak{E}$ the subspace of $L^2(Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}})$ generated by Eisenstein series and $S = \widetilde{\Delta}_{\mathfrak{E}}$, the discrete spectrum of S (if any) is a subset of the zeros of θE_s. Furthermore, similar pair correlation results can be used to show that at most a proper fraction of the zeros of θE_s can appear as spectral parameters.

7. Appendices

7.1. **Vector-valued Integrals.** There is at least one technical point to address. We will need a bit of machinery introduced by Gelfand (1936) [25] and Pettis (1938) [40]. Their construction produces integrals of continuous vector-valued functions with compact support. These integrals are not *constructed* using limits, in contrast to Bochner integrals, but instead are *characterized* by the desired property that they commute with linear functionals.

Let V be a complex topological vector space. Let f be a measurable V-valued function on a measure space X. A *Gelfand-Pettis integral* of f is a vector $I_f \in V$ so that

$$\alpha(I_f) = \int_X \alpha \circ f$$

for all $\alpha \in V^*$. Assuming that it exists and is unique, the vector I_f is denoted $I_f = \int_X f$.

Uniqueness and linearity of the integral follow from the fact that V^* separates points by Hahn-Baach. Establishing the existence of Gelfand-Pettis integrals is more delicate.

Theorem 14. *Let X be a compact Hausdorff topological space with a finite positive regular Borel measure. Let V be a quasi-complete, locally convex topological vectorspace. Then continuous compactly-supported V-values functions f on X have Gelfand-Pettis integrals.*

The importance of the characterization of the Gelfand-Pettis integral is exhibited in the following corollary.

Corollary 15. *Let $T : V \to W$ be a continuous linear map of locally convex quasi-complete topological vector spaces and f a continuous V-valued function on X. Then*

$$T\left(\int_X f\right) = \int_X T \circ f.$$

Proof. Since W^* separates points, it suffices to show that

$$\mu\left(T\left(\int_X f\right)\right) = \mu\left(\int_X T \circ f\right).$$

Since $\mu \circ T \in V^*$, the characterization of Gelfand-Pettis integrals gives

$$\mu\left(T\left(\int_X f\right)\right) = (\mu \circ T)\left(\int_X f\right) = \int_X \mu(T \circ f) = \mu\left(\int_X T \circ f\right).$$

□

7.2. **Determination of the Spectral Decomposition.** We will now exhibit the computation of this spectral decomposition. Let N be the upper-triangular unipotent matrices in $G = SL_2(\mathbb{R})$, A the diagonal matrices, A^+ the diagonal matrices with positive diagonal entries, $P = NA$ the parabolic subgroup of upper-triangular matrices, $P^+ = NA^+$, $\Gamma = SL_2(\mathbb{Z})$ and $\Gamma_\infty = P^+ \cap \Gamma = N \cap \Gamma$. For simplicity, normalize the total measure of K to 1 rather than 2π.

Pseudo-Eisenstein Series. Pseudo-Eisenstein series are solutions to the *adjunction problem*: given $\varphi \in C_c^\infty(N\backslash G)$, we want to find $\Psi_\varphi \in C_c^\infty$ such that

$$\langle c_P f, \varphi\rangle_{N\backslash G} = \langle f, \Psi_\varphi\rangle_{\Gamma\backslash G}$$

for f on $\Gamma\backslash G$ and $\langle f, F\rangle_{\Gamma\backslash G} = \int_{\Gamma\backslash G} f \cdot F$.

We can compute the canonical expression for Ψ_φ from this desired equality using the left N-invariance of φ and the left Γ-invariance of f as follows: (Note that $P \cap \Gamma$ differs from $N \cap \Gamma$ only by $\pm 1_2$ which act trivially on $\mathfrak{H} \cong G/K$.)

$$\begin{aligned}
\langle c_P f, \varphi\rangle_{N\backslash\mathfrak{H}} &= \int_{N\backslash\mathfrak{H}} c_P f(z)\varphi(\mathrm{Im}(z))\ \frac{dx\ dy}{y^2} = \int_{N\backslash\mathfrak{H}} \left(\int_{N\cap\Gamma\backslash N} f(nz)\ dn\right)\varphi(\mathrm{Im}(z))\ \frac{dx\ dy}{y^2} \\
&= \int_{\Gamma_\infty\backslash\mathfrak{H}} f(z)\varphi(\mathrm{Im}(z))\ \frac{dx\ dy}{y^2} = \int_{\Gamma\backslash\mathfrak{H}} \sum_{\gamma\in P\cap\Gamma\backslash\Gamma} f(\gamma z)\varphi(\mathrm{Im}(\gamma z))\ \frac{dx\ dy}{y^2} \\
&= \int_{\Gamma\backslash\mathfrak{H}} f(z)\left(\sum_{\gamma\in P\cap\Gamma\backslash\Gamma} \varphi(\mathrm{Im}(\gamma z))\right)\ \frac{dx\ dy}{y^2} = \langle f, \Psi_\varphi\rangle_{\Gamma\backslash\mathfrak{H}}
\end{aligned}$$

Thus we define the *pseudo-Eisenstein series* as $\Psi_\varphi(z) = \displaystyle\sum_{\gamma\in P\cap\Gamma\backslash\Gamma} \varphi(\mathrm{Im}(\gamma z))$. Note that the pseudo-Eisenstein series is absolutely and uniformly convergent for $z \in C$ where C is a compact subset of G. Furthermore, $\Psi_\varphi \in C_c^\infty(\Gamma\backslash G)$.

Now, it is a corollary of the above characterization of psuedo-Eisenstein series that the square integrable cuspforms are the orthogonal complement of the (closed) subspace of $L^2(\Gamma\backslash\mathfrak{H})$ spanned by pseudo-Eisenstein series with $\varphi \in C_0^\infty(N\backslash\mathfrak{H}) \cong C_0^\infty(0,\infty)$. Thus we have

$$L^2(\Gamma\backslash\mathfrak{H}) = L^2_{\text{cusp}}(\Gamma\backslash\mathfrak{H}) \oplus L^2_{\text{p-Eis}}(\Gamma\backslash\mathfrak{H}).$$

7.2.1. *Decomposition of Pseudo-Eisenstein Series.* We further decompose $L^2(\Gamma\backslash\mathfrak{H})$ by examining the pseudo-Eisenstein series Ψ_φ. The spectral decomposition of the data φ induces a spectral decomposition for Ψ_φ. Identifying $N\backslash\mathfrak{H} \cong N\backslash G/K \cong A^+$, Mellin inversion gives

$$\varphi(\mathrm{Im}\ z) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}\varphi(s)(\mathrm{Im}\ z)^s\ ds$$

for any real σ. This decomposition of φ is achieved as follows.

Replacing ξ by $\xi/2\pi$ in Fourier inversion we get

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)e^{-it\xi}\ dt\right)e^{i\xi x}\ d\xi.$$

Fourier transforms on $\mathbb{R}$ put into multiplicative coordinates are *Mellin transforms*:
For $\varphi \in C_c^\infty(0,\infty)$, take $f(x) = \varphi(e^x)$. Let $y = e^x$ and $r = e^t$ (the exponential in the implied inner integral) and rewrite Fourier inversion as

$$f(x) = \varphi(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_0^{\infty} \varphi(r)r^{-i\xi}\ \frac{dr}{r}\right)y^{i\xi}\ d\xi$$

since $dt = \frac{dr}{r}$. Note that this integral converges as a C^∞-function-valued function.

The Fourier transform (inner integral) in these coordinates is Mellin transform. For compactly supported φ, the integral definition extends to all $s \in \mathbb{C}$ as $\mathcal{M}\varphi(s) = \int_0^\infty \varphi(r) r^{-s} \frac{dr}{r}$. Mellin inversion is

$$\varphi(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{M}\varphi(i\xi) y^{i\xi} \, d\xi.$$

With ξ the imaginary part of a complex variable s, we can rewrite this as a complex path integral

$$\varphi(y) = \frac{1}{2\pi i} \int_{0-i\infty}^{0+i\infty} \mathcal{M}\varphi(s) y^s \, ds$$

since $d\xi = -i \, ds$. For $f \in C_c^\infty(\mathbb{R})$, $\hat{f}(\xi)$ converges nicely for all complex values of ξ so it extends to an entire function in ξ of rapid decay on horizontal lines (Payley-Wiener Theorem). This extension property applies to φ allowing us to move the contour as above.

Thus the pseudo-Eisenstein series is

$$\Psi_\varphi(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \varphi(\mathrm{Im}(\gamma z)) = \frac{1}{2\pi i} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \int_{\sigma - i\infty}^{\sigma + i\infty} \mathcal{M}\varphi(s) \cdot (\mathrm{Im}(\gamma z))^s \, ds.$$

It would be natural to take $\sigma = 0$ however, at $\sigma = 0$ the double integral would not be absolutely convergent and the two integrals cannot be interchanged. For $\sigma > 1$ the double integral is absolutely convergent and (using Fubini) we have,

$$\Psi_\varphi(z) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \mathcal{M}\varphi(s) \cdot E_s(z) \, ds$$

for $E_s(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (\mathrm{Im}(\gamma z))^s$ the Eisenstein series.

7.2.2. *Eisenstein Series.* We have a decomposition of Ψ_φ in terms of φ. We now want to rewrite this piece of the decomposition so as to refer only to Ψ_φ not φ. Note that E_s has meromorphic continuation on the entire complex plane. Thus we can move the line of integration to the left and choose $\sigma = 1/2$ to achieve

$$\Psi_\varphi = \frac{1}{2\pi i} \int_{1/2 - i\infty}^{1/2 + i\infty} \mathcal{M}\varphi(s) E_s \, ds + \sum_{s_0} \mathrm{res}_{s=s_0}(\mathcal{M}\varphi(s) E_s).$$

As with the pseudo-Eisenstein series, the Eisenstein series E_s fits into an adjunction relation

$$\langle E_s, f \rangle_{\Gamma \backslash \mathfrak{H}} = \langle y^s, c_P f \rangle_{\Gamma_\infty \backslash \mathfrak{H}}$$

for f on $\Gamma \backslash \mathfrak{H}$. Notice that

$$\begin{aligned} \langle y^s, c_P f \rangle_{A^+} &= \int_{N \backslash \mathfrak{H}} c_P f(z) \cdot y^s \, \frac{dx \, dy}{y^2} = \int_{N \backslash \mathfrak{H}} \left(\int_{\Gamma_\infty \backslash N} f(nz) \, dn \right) \cdot y^s \, \frac{dx \, dy}{y^2} \\ &= \int_{\Gamma_\infty \backslash \mathfrak{H}} f(z) \cdot y^s \, \frac{dx \, dy}{y^2} = \int_{P \cap \Gamma \backslash \mathfrak{H}} f(z) \cdot y^s \, \frac{dx \, dy}{y^2} \\ &= \int_{\Gamma \backslash \mathfrak{H}} \sum_{\gamma \in P \cap \Gamma \backslash \Gamma} f(\gamma z) \cdot \mathrm{Im}(\gamma z)^s \, \frac{dx \, dy}{y^2} \end{aligned}$$

$$= \int_{\Gamma\backslash\mathfrak{H}} f(z) \cdot \sum_{\gamma \in P\cap\Gamma\backslash\Gamma} \mathrm{Im}(\gamma z)^s \ \frac{dx\ dy}{y^2} = \langle E_s, f\rangle_{\Gamma\backslash\mathfrak{H}}.$$

Thus

$$\langle E_s, f\rangle_{\Gamma\backslash\mathfrak{H}} = \int_0^\infty c_P f(iy) y^s \frac{dy}{y^2} = \int_0^\infty c_P f(iy) y^{-(1-s)}\ \frac{dy}{y} = \mathcal{M}(c_P f)(1-s).$$

On the other hand, since $c_P E_s = y^s + c_s y^{1-s}$ for $c_s = \dfrac{\xi(2s-1)}{\xi(2s)}$, we have

$$\langle E_s, \Psi_\varphi\rangle_{\Gamma\backslash\mathfrak{H}} = \langle c_P E_s, \varphi\rangle_{\Gamma_\infty\backslash\mathfrak{H}} = \langle y^s + c_s y^{1-s}, \varphi\rangle_{\Gamma_\infty\backslash\mathfrak{H}} = \int_0^\infty (y^s + c_s y^{1-s}) \cdot \varphi(y)\ \frac{dy}{y^2}$$

$$= \int_0^\infty (y^{-(1-s)} + c_s y^{-s}) \cdot \varphi(y)\ \frac{dy}{y} = \mathcal{M}\varphi(1-s) + c_s \mathcal{M}\varphi(s).$$

So we have the identity,

$$\mathcal{M}(c_P \Psi_\varphi)(s) = \langle E_{1-s}, \Psi_\varphi\rangle_{\Gamma\backslash\mathfrak{H}} = \mathcal{M}\varphi(s) + c_{1-s}\mathcal{M}\varphi(1-s).$$

Using this and returning to our equation for Ψ_φ above, we get

$$\Psi_\varphi - \sum_{s_0} \mathrm{res}_{s=s_0}(\mathcal{M}\varphi(s)E_s) = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \mathcal{M}\varphi(s) E_s\ ds$$

$$= \frac{1}{2\pi i}\int_{1/2-i0}^{1/2+i\infty} \mathcal{M}\varphi(s)E_s + \mathcal{M}\varphi(1-s)E_{1-s}\ ds$$

$$= \frac{1}{2\pi i}\int_{1/2-i0}^{1/2+i\infty} \mathcal{M}\varphi(s)E_s + c_{1-s}\mathcal{M}\varphi(1-s)E_s\ ds$$

$$= \frac{1}{2\pi i}\int_{1/2-i0}^{1/2+i\infty} \mathcal{M}c_p\Psi_\varphi(s)E_s\ ds$$

from the functional equation $E_{1-s} = c_{1-s}E_s$ and our identity above

$$= \frac{1}{4\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \langle \Psi_\varphi, E_s\rangle_{\Gamma\backslash\mathfrak{H}} \cdot E_s\ ds.$$

7.2.3. *Residue.* Finally, let us examine the residue. For $\Gamma = SL_2(\mathbb{Z})$, the only pole of E_s in the half plane $\mathrm{Re}(s) \geq 1/2$ is at $s_0 = 1$. This pole is simple and the residue is a constant function. Thus we can compute the residue as follows:

$$\sum_{s_0} \mathrm{res}_{s=s_0}(\mathcal{M}\varphi(s)E_s) = \mathcal{M}\varphi(1) \cdot \mathrm{res}_{s=1}E_s$$

where

$$\mathcal{M}\varphi(1) = \int_0^\infty \varphi(y) y^{-1}\frac{dy}{y} = \int_0^\infty \varphi(y)\frac{dy}{y^2} = \int_{N\backslash\mathfrak{H}} \varphi(\mathrm{Im} z)\frac{dx\ dy}{y^2}$$

$$= \int_{N\backslash\mathfrak{H}}\int_{\Gamma_\infty\backslash N} \varphi(\mathrm{Im}(nz))\ dn\ \frac{dx\ dy}{y^2} = \int_{N\backslash\mathfrak{H}} \varphi(\mathrm{Im}(nz)) \left(\int_{\Gamma_\infty\backslash N} 1\ dn\right) \frac{dx\ dy}{y^2}$$

$$= \int_{\Gamma_\infty\backslash\mathfrak{H}} \varphi(\mathrm{Im} z)\ \frac{dx\ dy}{y^2}$$

since the volume of $\Gamma_\infty\backslash N$ is 1 and φ is left N-invariant

$$= \int_{\Gamma\backslash\mathfrak{H}} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \varphi(\mathrm{Im} z)\ \frac{dx\ dy}{y^2} = \int_{\Gamma\backslash\mathfrak{H}} \Psi_\varphi(z)\ \frac{dx\ dy}{y^2} = \langle\Psi_\varphi, 1\rangle_{\Gamma\backslash\mathfrak{H}}.$$

7.3. **Friedrichs Extensions.** The crux of the project pursued by Bomberi and Garrett involves somewhat non-intuitive self-adjoint extensions of restrictions of symmetric operators. The technique alluded to by ColinDeVerdière in [14] involves exploiting somewheat unexpected behavior of certain self-adjoint extensions of symmetric operators. Typical densely-defined symmetric unbounded operators are not self-adjoint. In 1929 vonNeumann classified self-adjoint extensions of symmetric operators [47]. In 1934, Friedrichs gave a canonical self-adjoint extension of a symmetric semi-bounded operator [18].

A semibounded symmetric operator is one which is satisfies $\langle Sv, v\rangle \geq c\cdot\langle v, v\rangle$ or $\langle Sv, v\rangle \leq c\cdot\langle v, v\rangle$ for some constant $c > 0$. The operator $1-\Delta$ is an canonical example of such an operator since $\langle \Delta f, g\rangle \geq 0$. We can construct the Friedrichs' extension of a densely-defined, symmetric operator S as follows:

Without loss of generality, consider a densely-defined, symmetric operator S with domain D_S with

$$\langle Sv, v\rangle \geq \langle v, v\rangle$$

for all $v \in D_S$. (Note that any semibounded operator can be exhibited this way by multiplying by a constant and adding or subtracting a constant.)

Define the inner product $\langle,\rangle_1$ on D_S by

$$\langle v, w\rangle_1 := \langle Sv, w\rangle$$

for $v, w \in D_S$ and let V^1 be the completion of D_S with respect to the metric induced by $\langle,\rangle_1$. Since $\langle v, v\rangle_1 \geq \langle v, v\rangle$, the inclusion map $D_S \hookrightarrow V$ extends to a continuous map $V^1 \hookrightarrow V$. Furthermore, since D_S is dense in V, we have that V^1 is also dense in V.

For $w \in V$, the functional $v \mapsto \langle v, w\rangle$ is a continuous linear functional on V^1 with norm

$$\sup_{|v|_1\leq 1} |\langle v, w\rangle| \leq \sup_{|v|_1\leq 1} |v|\cdot|w| \leq \sup_{|v|_1\leq 1} |v|_1\cdot|w| = |w|.$$

By the Riesz-Fisher Theorem on V^1, there is a $w' \in V^1$ so that

$$\langle v, w'\rangle_1 = \langle v, w\rangle$$

for all $v \in V^1$ and $w \in V$ with norm bounded by the norm of $v \mapsto \langle v, w\rangle$; explicitly, $|w'|_1 \leq |w|$. The map $A : V \to V^1$ defined by $w \mapsto w'$ is linear. The inverse of A will be a self-adjoint extension $\widetilde{S}$ of S. This is the *Friedrichs extension*. We now show that $\widetilde{S}$ is in fact self-adjoint and an extension of S.

First note that since $|Aw|_1 = |w'|_1 \leq |w|$ from above, the operator norm is $\sup_{|w|\leq 1}|Aw|_1 \leq 1$ and so A is continuous.

Also observe that for $w' \in D_S$ and all $v \in V^1$,

$$\langle v, w'\rangle_1 = \langle v, Sw'\rangle = \langle v, A(Sw')\rangle_1$$

so $A(Sw') = w'$ for each $w' \in D_S$. Hence $AV \subset V^1$ contains the domain D_S of S.

We also see that A is injective since $\ker A = 0$: since V^1 is dense in V, if $0 = \langle v, Aw\rangle_1 = \langle v, w\rangle$ for all $v \in V^1$ then $w = 0$. Thus the inverse $\widetilde{S}$ of A is defined on $D_{\widetilde{S}} = AV \subset V^1$. Hence $\widetilde{S}$ is injective and is surjective for $D_{\widetilde{S}} \to V$.

Now to show that $\widetilde{S}$ is an extension of S, it remains to show that $A(\widetilde{S}w) = A(Sw)$ for $w \in D_S$. For $v, w \in D_S \subset D_{\widetilde{S}}$,

$$\langle v, \widetilde{S}\rangle = \langle v, A(\widetilde{S}w)\rangle_1 = \langle v, w\rangle_1 = \langle Sv, w\rangle = \langle v, Sw\rangle.$$

Since D_s is dense in V we have that $\widetilde{S}w = Sw$.

We also must show that $\widetilde{S}$ is symmetric. First note that A is symmetric: for $w' = Aw \in AV$ since $\langle v, Aw\rangle_1 = \langle v, w\rangle$ we have $\langle v, w'\rangle_1 = \langle v, \widetilde{S}w'\rangle$ and

$$\langle Av, w\rangle = \langle Av, \widetilde{S}Aw\rangle = \langle Av, Aw\rangle_1$$

which is symmetric in v and w. Thus since $D_{\widetilde{S}} = AV$ and

$$\langle \widetilde{S}Av, Aw\rangle = \langle v, Aw\rangle = \langle Av, w\rangle = \langle Av, Aw\rangle_1 = \langle Av, \widetilde{S}Aw\rangle$$

and so $\widetilde{S}$ is symmetric.

Furthermore, this extension $\widetilde{S}$ remains semibounded i.e. $\langle \widetilde{S}v, v\rangle \geq \langle v, v\rangle$ for all $v = Aw \in D_{\widetilde{S}} = AV$ since

$$\langle \widetilde{S}v, v\rangle = \langle \widetilde{S}Aw, v\rangle = \langle w, v\rangle = \langle Aw, v\rangle_1 = \langle v, v\rangle_1 \geq \langle v, v\rangle.$$

It remains to show that $\widetilde{S}$ is self-adjoint. Note that any proper extension $T \supset \widetilde{S}$ is not injective since $\widetilde{S}$ surjects to V. So if S^* were a proper extension of $\widetilde{S}$ there would be $v \in D_{\widetilde{S}}$ so that for all $w \in D_{\widetilde{S}}$,

$$0 = \langle \widetilde{S}^* v, w\rangle = \langle v, \widetilde{S}w\rangle = \langle v, \widetilde{S}w\rangle.$$

Since $\widetilde{S}$ surjects to V, there is a $w \in D_{\widetilde{S}}$ such that $\widetilde{S}w = v$. Hence $v = 0$ and $\widetilde{S}^*$ cannot be a proper extension of $\widetilde{S}$. Thus $\widetilde{S}$ is self-adjoint.

This construction serves as a proof for the following theorem of Friedrichs [18].

Theorem 16. *A positive, densely-defined, symmetric operator S with domain D_S has a positive self-adjoint extension with the same lower bound.*

This extension has useful properties of particular interest to our project. An alternative characterization of the extension makes this more clear.

Assume that V has a $\mathbb{C}$-conjugate-linear complex conjugation $v \to v^c$ with the properties: $(v^c)^c = v$ and $\langle v^c, w^c\rangle = \overline{\langle v, w\rangle}$. Further let S commute with conjugation so that

$(Sv)^c = S(v^c)$. Let V^{-1} be the dual of V^1 so that $V^1 \subset V \subset V^{-1}$.

Note that given this small specification, there is an alternate characterization of the Friedrichs extension. To specify it, define a continuous, complex-linear map $S^\# : V^1 \to V^{-1}$ by

$$(S^\# v)(w) = \langle v, w^c \rangle_1$$

for $v, w \in V^1$.

Theorem 17. *Let $X = \{v \in V^1 \mid S^\# v \in V\}$. Then the Friedrichs extension of S is $\widetilde{S} = S^\#|_X$ with domain $D_{\widetilde{S}} = X$.*

Proof. Let $T = S^\#|_X$. Let $: V \to V^1$ be the inverse of $\widetilde{S}$ defined by $\langle Av, w\rangle_1 = \langle v, w\rangle$ for all $w \in V^1$ and $v \in V$ from the Riesz-Fischer Theorem. Then

$$\langle TAv, w\rangle = \langle Av, w\rangle_1 = \langle v, w\rangle$$

for $v \in V$ and $w \in V^1$. Also,

$$\langle ATv, w\rangle_1 = \langle Tv, w\rangle = \langle v, w\rangle_1$$

for $v \in X$ and $w \in V^1$. This $T = A^{-1} = \widetilde{S}$. $\square$

Extensions of Restrictions. Using the latter characterization of the Friedrichs extension we will now explain how the construction of the extension works with the case of restricted operators. Assume that S and the related terms are defined as above. Let $\Theta \subset D_S$ be a S-stable subspace. Let the orthogonal complement to Θ in V be

$$\ker \Theta = \{v \in V \mid \langle v, \theta\rangle = 0 \text{ for all } \theta \in \Theta\}.$$

For our purposes, given an operator S as above, we will want to define $T = S|_{D_S \cap \ker \Theta}$ so that $D_T = D_S \cap \ker \Theta$.

Note that for $v \in D_T$ and $\theta \in \Theta$,

$$\langle Tv, \theta\rangle = \langle Sv, \theta\rangle = \langle v, S\theta\rangle \in \{\langle v, \theta'\rangle \mid \theta' \in \Theta\} = \{0\}$$

and so $T(D_T) \subset \ker \Theta$. Furthermore since T is a restriction of S, symmetry and $\langle Tv, v\rangle \geq \langle v, v\rangle$ are inherited from S.

In contrast with these inherited properties, it is nontrivial to give a simple condition to ensure that D_T is dense in $\ker \Theta$ and that the V^1-closure of D_T is $V^1 \cap \ker \Theta$. This delicacy is demonstrated in Lax and Phillips [34]. For cut-off above height $a > 1$, an argument using the geometry of the fundamental domain for Γ shows that $\Theta \cap V^1$ is dense in V.[3] For this reason we will *assume* $D_T = D_S \cap \ker \Theta$ is V-dense in $\ker \Theta$ and V^1-dense in $V^1 \cap \ker \Theta =: W^1$.

Let W^{-1} be the dual of W so we have $W^1 \subset \ker \Theta \subset W^{-1}$. Define $S^\# : V^1 \to V^{-1}$ by

$$(S^\# v)(w) := \langle v, w^c\rangle_1,$$

for $v, w \in V^1$.

[3]For $a < 1$, there are serious complications so we do not address this case.

Theorem 18. *Let $\Theta^{\#}$ be the V^{-1}-completion of Θ. The Friedrichs extension $\widetilde{T}$ of T has domain $D_{\widetilde{T}} = \{v \in W^1 \mid S^{\#}v \in V + \Theta^{\#}\}$ and is characterized by*

$$\widetilde{T}v = w \quad \Longleftrightarrow \quad S^{\#}v \in w + \Theta^{\#}$$

for $v \in D_{\widetilde{T}}$ and $w \in ker\,\Theta$.

Proof. Define $T^{\#} : W^1 \to W^{-1}$ by

$$(T^{\#}v)(w) := \langle v, w^c\rangle_1$$

we can then define the the domain of the Friedrichs extension $\widetilde{T}$ as

$$D_{\widetilde{T}} = \{w \in W^1 \mid T^{\#}w \in \ker\Theta\}$$

so that $\widetilde{T} = T^{\#}|_{D_{\widetilde{T}}}$. With the inclusion $j : W^1 \to V^1$ for all $x, y \in W^1$

$$(T^{\#}x)(y) = \langle jx, (jy)^c\rangle_1 = (S^{\#}jx)(jy) = \big((j^* \circ S^{\#} \circ j)x\big)\,(y)$$

and so $T^{\#} = j^* \circ S^{\#} \circ j$ and

$$D_{\widetilde{T}} = \{w \in W^1 \mid j^*(S^{\#}(jw)) = 0\}.$$

The orthogonal compliment $\ker\Theta$ to Θ in V is a closed subspace of V^1 and the dual W^{-1} of W^1 is

$$W^{-1} = (V^1 \cap \ker\Theta)^* \cong V^{-1}/\Theta^{\#}.$$

$\square$

The Friedrichs extension makes the following diagram commute

$$\begin{tikzcd}
V^1 \arrow[r] \arrow[rr, bend left, "S^{\#}"] \arrow[d, "j^{-1}"] & V \arrow[r] & V^{-1} \arrow[d, "j^*"] \\
W^1 \arrow[r, "\widetilde{T}"] \arrow[rr, bend right, "T^{\#}"'] & \ker\Theta \arrow[r] & W^{-1}
\end{tikzcd}$$

References

[1] R.J. Backlund, *Sur les zéros de la fonction $\zeta(s)$ de Riemann*, C.R. **158** (1914), 1979-81.

[2] R.J. Backlund, *Über die Nullstellen der Riemannschen Zetafunktion*, Dissertation, Helsingfors, 1916. *Über die Nullstellen der Riemannschen Zetafunktion*, Acta Math. **41** (1918), 345-75.

[3] F.A. Berezin, L.D. Faddeev, *Remarks on Schrödinger's equation with a singular potential*, Soviet Math. Dokl **2** (1961), 372-375.

[4] H. Bethe, R. Peierls, *Quantum theory of the diplon*, Proc. Royal Soc. London **148a** (1935), 146-156.

[5] E. Bombieri, D. Hejhal, *Sur les zéros des fonctions zêta d'Epstein*, C. R. Acad. Sci. Paris Sér. I Math. **304** (1987), 213-217.

[6] E. Bombieri, D. Hejhal, *On the distribution of zeros of linear combinations of Euler products*, Duke Math. J. **80** (1995), 821-862.

[7] A. Borel, *Admissible representations of a semi-simple group over a local field with vectors fixed under an Iwahori subgroup*, Inv. Math 35 (1976), pp. 233-259.

[8] P. Cartier, *Some numerical computations relating to automorphic functions*, in *Computers in Number Theory*, ed., A.O.L. Atkin and B.J. Birch, Academic Press, 1971, 37-48.

[9] P. Cartier, D. Hejhal, *Analyse numérique d'un problème de valeurs propres à haute précision (application aux fonctions automorphes)*, preprint I.H.E.S. (1978).

[10] P. Cartier, D. Hejhal, *Sur les zéros de la fonction zêta de Selberg*, preprint I.H.E.S. (1979).

[11] P. Cartier, *Comment l'hypothèse de Riemann ne fut pas prouvée [extraits de deux lettres de P. Cartier à A. Weil, datées du 12 août et du 15 septembre 1979]*, Séminaire Delang-Pisot-Poitou (Théorie des Nombres, Paris), 1980-81.

[12] W. Casselman, *The unramified principal series of p-adic groups I: the spherical function*, Comp. Math vol 40 fasc. 2 (1980), pp. 387-406.

[13] Y. ColinDeVerdière, *Une nouvelle démonstration du prolongement méromorphe des séries d'Eisenstein*, C. R. Acad. Sci. Paris Sér. I Math. **293** (1981), no. 7, 361-363.

[14] Y. ColinDeVerdière, *Pseudo-laplaciens, I, II*, Ann. Inst. Fourier (Grenoble) **32** (1982, 1983) no. 3, xiii, 275-286, no. 2, 87-113.

[15] J. B. Conrey, *At least two fifths of the zeros of the Riemann zeta function are on the critical line*, Bull. of the Amer. Math. Soc.(1989) Vol 20, 79-81

[16] L. Faddeev. *Expansions in eigenfunctions of the Laplace operator on the fundamental domain of a discrete group on the Lobachevskii plane*, Trans. of the Moscow Math. Soc. (1967) Vol 17, 357-386.

[17] J. D. Fay, *Fourier coefficients of the resolvent for a Fuchsian group.* Journal für die reine und angewandte Mathematik 293/294 (1977), 143-203.

[18] K.O. Friedrichs, *Spektraltheorie halbbeschrankter Operatoren*, Math. Ann. **109, 110** (1934, 1935), 465-487, 685-713, 777-779.

[19] M. Green, S. Miller, J. Russo, P Vanhove, *Eisenstein series for higher-rank group and string theory amplitudes*, Comm. in Number Theory and Physics (2010) Vol 4 No 3, 551-596.

[20] P. Garrett, *Euler factorization of global integrals*, in *Euler Products and Eisenstein Series*, Proc. Symp. Pure Math. vol. 66, 1999.
http://math.umn.edu/~garrett/m/v/euler_factorization.pdf

[21] P. Garrett, *Vector-Valued Integrals* www.math.umn.edu/~garrett/m/fun/Notes/07_vv_integrals.pdf (accessed 12 Mar 2015).

[22] P. Garrett, *Colin de Verdiere's meromorphic continuation of Eisenstein series* (2012), http://www.math.umn.edu/~garrett/m/v/quad_rec.pdf (accessed 14 June 2015).

[23] P. Garrett, *Automorphic Levi-Sobolev spaces, boundary-value problems, and self-adjoint operators*, video, IAS/Princeton, Jan 2013\hk https://video.ias.edu/1213/jointiaspu/0124-PaulGarrett

[24] P. Garrett (2013) *Self-adjoint operators on spaces of automorphic forms*, video, ICERM/Brown, https://icerm.brown.edu/video_archive/#/play/228

[25] I. M. Gelfand, *Sur un lemme de la theorie des espaces lineaires*, Comm. Inst. Sci. Math. de Kharkoff, no. 4, vol. 13, 1936, pp. 35-40.

[26] H. Haas, *Numerische Berechnung der Eigenwerte der Differentialgleichung $y^2\Delta u + \lambda u = 0$ für ein unendliches Gebiet im* $\mathbb{R}^2$, Diplomarbeit, Universität Heidelberg (1977), 155 pp.

[27] G. H. Hardy, J. E. Littlewood, *Contributions to the theory of the Riemann zeta-function and the theory of the distributions of primes*, Acta Math. **41** (1918), 119-196.
[28] D. Hejhal, *The Selberg trace formula and the Riemann zeta function*, Duke J. Math. **43** (1976), 441-482.
[29] D. Hejhal, *Some observations concerning eigenvalues of the Laplacian and Dirichlet L-series*, in *Recent Progress in Analytic Number Theory*, ed. H. Halberstam and C. Hooley, vol. 2, Academic Press, NY, 1981, 95-110.
[30] D. Hejhal *Zeros of Epstein Zeta functions and Supercomputers*, Proc. of the Int. Congress of Mathematicians (1986).
[31] D. Hejhal, *On the triple correlation of zeros of the zeta function*, Int. Math. Res. Notices **7** (1994), 293-302.
[32] D. Hejhal, private correspondence, 14 March 2015.
[33] G. Lachaud. *Spectral analysis and the Riemann hypothesis.* J. Comput. Appl. Math. (2003) 160: 175-190.
[34] P. Lax, R. Phillips, *Scattering theory for automorphic functions*, Annals of Math. Studies, Princeton, 1976.
[35] H. Matsumoto, *Analyse Harmonique dans les systems de Tits bornologiques de type affine*, Lecture Notes in Math. no. 590, Springer-Verlag, 1977.
[36] R. Mazzeo, R. Melrose, *Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature*, J. Funct. Anal. 75 (1987), 260-310.
[37] R. Melrose, *Gemrtric Scattering Theory*, Stanford Lectures: Distinguished Visiting Lectures in Mathematics, Cambridge Uiversity Press, 1995.
[38] H.L. Montgomery, *The pair correlation of zeros of the zeta-function*, Proc. Symp. Pure Math. 24, AMS, Providence R.I., 1973, 181-193.
[39] H. Neunhöffer, *Uber die analytische Fortsetzung von Poincaréreihen* , S.-B. Heidelberger Akad. Wiss. Math.-Natur. Kl. (1973), 33-90.
[40] B. J. Pettis, *On integration in vectorspaces*, Trans. AMS **44**, 1938, 277-304.
[41] H. Potter, E. Titchmarsh, *Zeros of Epstein's Zeta-functions*, Proc. London Math. Soc. (1935) s2-39 (1): 372-384.
[42] Z. Rudnick and P. Sarnak, *Zeros of principal L-functions and random matrix theory*, Duke Math. J. **81** (1996), 269-322.
[43] H. Stark, *On the zeros of Epstein's zeta functions*, Mathematika **14** (1967), 47-55.
[44] L.H. Thomas, *The interaction between a neutron and a proton and the structure of* H^3, Phys. Rev. **47** (1935), 903-909.
[45] E.C. Titchmarsh, *The theory of the Riemann zeta-function*, second edition, revised by D.H. Heath-Brown, Oxford Univ. Press, 1986. (First edition, 1951, successor to *The Zeta-Function of Riemann*, 1930.)
[46] A. Venkov, *Spectral theory of automorphic functions, the Selberg zeta-function, and some problems of analytic number theory and mathematical physics*, Russian Math. Surveys **34** (1979) no. 3, 79-153.
[47] J. von Neumann, *Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren*, Math. Ann. **102** (1929), 49-131.
[48] S.M. Voronin, *On the zeros of zeta-functions of quadratic forms*, Trudy Mat. Inst. Steklov. **142** (1976), 135-147.
[49] D. Zagier, *Eisenstein series and the Riemann zeta function*, in 1979 Bombay Colloquium on *Automorphic Forms, Representation Theory, and Arithmetic*, Tata Inst. Fund. Research, 1981.
[50] J. Zorbas, *Perturbation of self-adjoint operators by Dirac distributions*, J. Math. Phys. **21** no. 4 (1980), 840-847.
[51] M. Zworski, *Mathematical Study of Scattering Resonances* (2016) https://math.berkeley.edu/∼zworski/revres.pdf (accessed 19 Oct 2016).