Then, however, 0 *belongs* to our original open subset $\mathscr{O}$ of $\mathbb{R}$, and, given $\varepsilon > 0$, it is possible to take the $\delta > 0$ corresponding to it *small enough*, in our construction, so as to *still have*

$$0 \;\notin\; (\mathbb{R} \sim \mathscr{O}) \;+\; [-\delta, \delta] \;=\; E_\delta.$$

Doing so, we see that E_δ really is properly included in $\mathbb{R}$, making

$$\mathscr{D} \;=\; \mathbb{C} \sim E_\delta$$

a domain of the kind studied in Chapter VIII, §C, with $0 \in \mathscr{D}$. (We write $\mathscr{D}$ instead of the more logical $\mathscr{D}_\delta$ in order to *avoid* having to use subscripts of subscripts later on.)

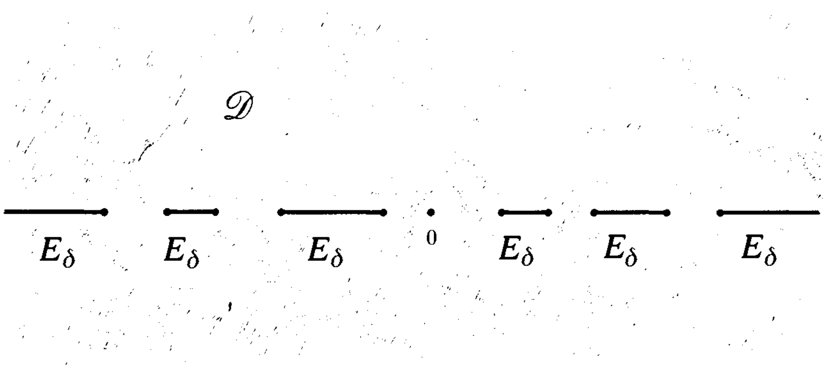

Figure 243

In the present circumstances, $\mathfrak{M}F_N$ is *harmonic* in $\mathscr{D}$ and *continuous* up to $\partial\mathscr{D}$, with

$$(\mathfrak{M}F_N)(t) \;\leqslant\; F_N(t) + \varepsilon \;=\; \log W_N(t) + \varepsilon \qquad \text{for } t \in \partial\mathscr{D},$$

and also, in view of the previous estimates,

$$(\mathfrak{M}F_N)(z) \;=\; O(1) - A|\Im z|$$

(everywhere). These facts make it possible for us to follow the procedure indicated at the very beginning of §C, Chapter VIII, and in that way carry out the harmonic estimation of $(\mathfrak{M}F_N)(z)$ in $\mathscr{D}$.

Let us, as in Chapter VIII, §C, denote the Phragmén–Lindelöf function for $\mathscr{D}$ by $Y_{\mathscr{D}}(z)$ and harmonic measure for that domain by $\omega_{\mathscr{D}}(\;\;, z)$.

Then, by the last relations we have

$$\begin{aligned}(\mathfrak{M}F_N)(0) \quad &\leqslant \quad \int_{\partial\mathscr{D}} (\mathfrak{M}F_N)(t)\,\mathrm{d}\omega_{\mathscr{D}}(t,\ 0) \quad - \quad AY_{\mathscr{D}}(0) \\ &\leqslant \quad \int_{\partial\mathscr{D}} (\log W_N(t) \ + \ \varepsilon)\,\mathrm{d}\omega_{\mathscr{D}}(t,\ 0) \quad - \quad AY_{\mathscr{D}}(0).\end{aligned}$$

(The first two members in this chain of inequalities are in fact equal, $\mathfrak{M}F_N$ being harmonic in $\mathscr{D}$.) Because $\log W_N(t) \leqslant \log W(t)$ and $\omega_{\mathscr{D}}(\partial\mathscr{D},\ 0) = 1$, the estimate just written implies that

$$\boxed{(\mathfrak{M}F_N)(0) \quad \leqslant \quad \varepsilon \ + \int_{\partial\mathscr{D}} \log W(t)\,\mathrm{d}\omega_{\mathscr{D}}(t,\ 0) \ - \ AY_{\mathscr{D}}(0);}$$

this, then, *must hold, whenever* $(\mathfrak{M}F_N)(0)$ *is not simply equal to* $\log W_N(0)$ *and hence* $\leqslant \log W(0)$.

In the boxed formula (derived, we remind the reader, *under the assumption* that $W(t) \longrightarrow \infty$ for $t \longrightarrow \pm\infty$), the *integrand* appearing in the right-hand integral *no longer depends on* N. But the *right side as a whole certainly involves* N (and ε as well!) *through the domain* $\mathscr{D}$, whose very construction depended on our knowing that $W(x) \geqslant N$ for $|x|$ sufficiently large! *In principle*, it does not generally seem possible to actually *know* $\mathscr{D}$ *precisely*, because such knowledge would depend on information about the function $(\mathfrak{M}F_N)(z)$ which we are in fact *trying to estimate* (really, *to find*) by using $\mathscr{D}$.

The formula is useful nevertheless, on account of the results found in §§C.4 and C.5 of Chapter VIII. As we saw there, when dealing with *certain kinds* of weights W, one can, by using quantities involving *only* W, express the *entire dependence of*

$$\int_{\partial\mathscr{D}} \log W(t)\,\mathrm{d}\omega_{\mathscr{D}}(t,\ 0)$$

on the domain $\mathscr{D}$ *in terms of* $Y_{\mathscr{D}}(0)$. That is the basis for the following applications.

2. Weight is the modulus of an entire function of exponential type

We come to one of the main results of this chapter – indeed, of the present book. The proof, based on the matters discussed above and in §C of Chapter VIII, uses also a refinement of the Riesz–Fejér factorization theorem which has never been explicitly formulated up to now, although it is essentially contained in the material of Chapters III and VI. Here it is:

Lemma. *Let $P(z)$, entire and of exponential type* 2B, *satisfy the condition*

$$\int_{-\infty}^{\infty} \frac{\log^+|P(x)|}{1+x^2}\,dx \quad < \quad \infty,$$

and suppose that $P(x) \geqslant 0$ on $\mathbb{R}$. Then there is an entire function $g(z)$ of exponential type B having all its zeros in $\Im z \leqslant 0$ and such that

$$g(z)\overline{g(\bar{z})} \;=\; P(z).$$

Proof. Except for the specification of the exponential type of g, this is just a restatement of the Riesz–Fejér result (the *third* theorem of §G.3, Chapter III). There is thus an entire function $g_0(z)$ having the stipulated properties, but we do not know its type.

In particular, $g_0(z)\overline{g_0(\bar{z})} = P(z)$. As long as λ is *real*, we then have

$$g_\lambda(z)\overline{g_\lambda(\bar{z})} \;=\; P(z)$$

for the function

$$g_\lambda(z) \;=\; e^{i\lambda z}g_0(z).$$

However, $\log|g_\lambda(iy)| = -\lambda y + \log|g_0(iy)|$, so we can evidently adjust the real parameter λ so as to make

$$\limsup_{y\to\infty}\frac{\log|g_\lambda(iy)|}{y} \qquad \text{and} \qquad \limsup_{y\to-\infty}\frac{\log|g_\lambda(iy)|}{|y|}$$

equal. Do this, and denote the common value of the two limsups by A, taking, then, $g(z)$ as $g_\lambda(z)$ for that particular choice of λ.

Since P is of exponential type $2B$, we have

$$\limsup_{y\to\infty}\frac{\log|P(iy)|}{y} \quad \leqslant \quad 2B.$$

At the same time, $|P(iy)| = |g(iy)||g(-iy)|$ for real y, so

$$\frac{\log|g(iy)|}{y} + \frac{\log|g(-iy)|}{y} = \frac{\log|P(iy)|}{y}.$$

On the real axis, $|g(x)|^2 = P(x)$, whence

$$\int_{-\infty}^{\infty}\frac{\log^+|g(x)|}{1+x^2}\,dx \quad < \quad \infty$$

by hypothesis. Also, $g(z)$, like $g_0(z)$, has all its zeros in $\Im z \leqslant 0$; the remark at the end of §G.1, Chapter III, thus applies to it, and we actually have

$$\frac{\log|g(\mathrm{i}y)|}{y} \quad \longrightarrow \quad A$$

as $y \longrightarrow \infty$. Now make $y \longrightarrow \infty$ through a sequence of values along which $\log|g(-\mathrm{i}y)|/|y|$ *also* tends to A; referring to the previous relation we see that

$$A + A \quad \leqslant \quad \limsup_{y\to\infty} \frac{\log|P(\mathrm{i}y)|}{y} \quad \leqslant \quad 2B,$$

so $A \leqslant B$.

By Chapter III, §E, we have

$$\log|g(z)| \quad \leqslant \quad A|\Im z| \;+\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|\log^+|g(t)|}{|z-t|^2}\,\mathrm{d}t,$$

and from this it follows easily as in the *fourth* theorem of §E.2, Chapter VI (see also §B.2 of that chapter), that *g is of exponential type A*. Since $g(z)\overline{g(\bar{z})} \;=\; P(z)$, P must be of exponential type $\leqslant 2A$, i.e., $B \leqslant A$. We have, however, just shown that $A \leqslant B$. Thus, $A = B$, and we are done.

Now we can give the

Theorem on the Multiplier (Beurling and Malliavin, 1961). *If $f(z)$, entire and of exponential type, is such that*

$$\int_{-\infty}^{\infty} \frac{\log^+|f(x)|}{1+x^2}\,\mathrm{d}x \quad < \quad \infty,$$

there are entire functions $\varphi(z) \not\equiv 0$ of arbitrarily small exponential type with

$$(1 \;+\; |f(x)|)\varphi(x)$$

bounded on the real axis.

Proof. Given $A > 0$, we wish to find a non-zero entire φ of exponential type $\leqslant A$ having the desired property. Our plan is to invoke the second theorem from §B.3, referring to the last theorem in §B.1. This involves our showing that $(\mathfrak{M}F)(0) < \infty$ where

$$F(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|}{|z-t|^2}\log W(t)\,\mathrm{d}t \;-\; A|\Im z|,$$

with $W(t) \geqslant 1$ an appropriate weight formed from f. For that purpose, the boxed formula at the end of the preceding article will be applied.

We proceed as at the beginning of §C.5, Chapter VIII, forming from f a new entire function $g_M(z)$ with $|g_M(x)| = |g_M(-x)| \geqslant 1$ on $\mathbb{R}$ and $g_M(0) = 1$. Our present construction differs slightly from the one made there.

Taking a large number M (whose value will depend on the type A of the multiplier we are seeking), we form the entire function

$$P_M(z) = \left(1 + \frac{z^2}{M^2}\right)\left(1 + \frac{z^2}{M^2}\left(f(z)\overline{f(\bar{z})} + f(-z)\overline{f(-\bar{z})}\right)\right),$$

the purpose of the first factor on the right being to ensure that

$$P_M(x) \longrightarrow \infty \quad \text{for } x \longrightarrow \pm\infty.$$

Given that $f(z)$ is of exponential type B, $P_M(z)$ will be entire, and of exponential type $\leqslant 2B$. It is clear that $P_M(z)$ is *even*, that

$$P_M(x) \geqslant 1 \quad \text{for } x \in \mathbb{R},$$

with

$$P_M(0) = 1,$$

and that

$$P_M(x) \geqslant |f(x)|^2/M^2 \qquad \text{for } |x| \geqslant 1.$$

From the hypothesis it follows also that

$$\int_{-\infty}^{\infty} \frac{\log P_M(x)}{1+x^2}\,dx < \infty;$$

we can, indeed, choose M so as to make

$$\int_{-\infty}^{\infty} \frac{\log P_M(x)}{x^2}\,dx \qquad (\textit{sic}\,!)$$

as small >0 as we like. Here, the behaviour of the integrand near the origin is alright, because (for large M)

$$|\log P_M(x)| \leqslant \text{const.}\ x^2/M^2 \qquad \text{for } |x| < 1$$

with a constant independent of M.

For any particular M, the lemma now gives us an entire function $g_M(z)$, of exponential type $\leqslant B$ (*half* that of P_M), having (here) all its zeros in the lower half plane, with

$$g_M(z)\overline{g_M(\bar{z})} = P_M(z).$$

Thence, in particular,

$$|g_M(x)| \;=\; \sqrt{P_M(x)} \;\geqslant\; |f(x)|/M$$

for real x of modulus $\geqslant 1$, so, since $P_M(x) \geqslant 1$ on $\mathbb{R}$,

$$1 + |f(x)| \;\leqslant\; C_M|g_M(x)|, \qquad x \in \mathbb{R},$$

with a constant C_M depending on M. Our result will thus be established if, for a suitable value of M, we can find an entire $\varphi(z) \not\equiv 0$ of exponential type A with $\varphi(x)g_M(x)$ bounded on $\mathbb{R}$. To do this, we follow the procedure explained in the last article.

Fixing a value of M (in a way to be described shortly) we take the weight $W(x) = |g_M(x)|$ and then use it in the formula written at the beginning of this proof so as to obtain a function F. According to the last theorem of §B.1 and the second one of §B.3, a function φ having the desired properties exists provided that $(\mathfrak{M}F)(0) < \infty$. It is now claimed that for proper choice of M *we in fact have*

$$(\mathfrak{M}F)(0) \;=\; 0.$$

To see this we verify (in the notation of the last article) that $(\mathfrak{M}F_N)(0) = 0$ for every $N \geqslant 1$. In the present circumstances, $W(0) = 1$, so for $N \geqslant 1$,

$$F_N(0) \;=\; F(0) \;=\; \log W(0) \;=\; 0,$$

and it is enough to show that assuming

$$(\mathfrak{M}F_N)(0) \;>\; \varepsilon$$

for some $\varepsilon > 0$ leads to a contradiction.

In case the last relation holds, it is certainly true that

$$(\mathfrak{M}F_N)(0) \;>\; \log W(0),$$

so the boxed formula from the end of the preceding article is valid, $W(x) = \sqrt{P_M(x)}$ having been ensured by our construction to tend to ∞ for $x \longrightarrow \pm\infty$. Thus,

$$(\mathfrak{M}F_N)(0) \;\leqslant\; \varepsilon + \int_{\partial\mathscr{D}} \log W(t)\,\mathrm{d}\omega_{\mathscr{D}}(t,\,0) \;-\; AY_{\mathscr{D}}(0),$$

where $\mathscr{D}$ is a certain (unknown) domain of the kind studied in §C of Chapter VIII.

Now *the second theorem of* §C.5 in Chapter VIII *can be used to estimate the quantity*

$$\int_{\partial\mathscr{D}} \log W(t)\,\mathrm{d}\omega_{\mathscr{D}}(t,\,0) \;=\; \int_{\partial\mathscr{D}} \log|g_M(t)|\,\mathrm{d}\omega_{\mathscr{D}}(t,\,0).$$

Our function g_M is of exponential type $\leqslant B$ and has otherwise the properties of the function G figuring in that theorem. Therefore,

$$\int_{\partial\mathscr{D}} \log|g_M(t)|\,d\omega_{\mathscr{D}}(t, 0) \quad\leqslant\quad Y_{\mathscr{D}}(0)\{J \;+\; \sqrt{(2eJ(J+\pi B/4))}\},$$

where

$$J \quad=\quad \int_0^\infty \frac{\log|g_M(x)|}{x^2}\,dx \quad=\quad \frac{1}{4}\int_{-\infty}^\infty \frac{\log P_M(x)}{x^2}\,dx.$$

As observed above, the right-hand integral will, for large enough M, *be as small as we like.* We can hence choose (and *fix*) *a value of M for which J* is *small enough to render*

$$J \;+\; \sqrt{(2eJ(J+\pi B/4))} \quad<\quad A.$$

This having been done, the previous relations yield

$$(\mathfrak{M}F_N)(0) \quad\leqslant\quad \varepsilon$$

(whatever N may be), contradicting our assumption that $(\mathfrak{M}F_N)(0) \;>\; \varepsilon$. Thus, $(\mathfrak{M}F_N)(0) \;=\; 0$, so, since this holds for every N, we have $(\mathfrak{M}F)(0) \;=\; 0$ as claimed, and the theorem is proved.

We are done.

Scholium. The multiplier $\varphi(z)$ of exponential type $\leqslant A$ obtained by going from the conclusion of the above argument to the *second* theorem in §B.3 and thence to the *last* one in §B.1 has real zeros only. This is immediately apparent on glancing at the description of the function φ appearing towards the end of the latter result's proof – *that* φ, by the way, is not the same as the multiplier we are talking about *here*, which, in the theorem referred to, was called ψ.

In their 1967 *Acta* paper, Beurling and Malliavin made the important observation that the zeros of the multiplier φ can also, in the present circumstances, be taken to be *uniformly separated*, in other words, *that any two of those zeros are distant by at least a certain amount $h > 0$.* This can be readily seen by putting together some of the above results and then using a simple measure-theoretic lemma.

Let us look again at the *least superharmonic majorant* $(\mathfrak{M}F)(z)$ of the function

$$F(z) \quad=\quad \frac{1}{\pi}\int_{-\infty}^\infty \frac{|\Im z|\log|g_M(t)|}{|z-t|^2}\,dt \;-\; A|\Im z|$$

formed from the weight $W(x) = |g_M(x)|$ used in the preceding proof. Here, $|g_M(t)| = |g_M(-t)|$, so $F(z) = F(-z)$ and therefore $(\mathfrak{M}F)(z) = (\mathfrak{M}F)(-z)$ (cf. beginning of proof of first lemma, §B.3). We know that $\mathfrak{M}F$ is finite, but here, since $(\mathfrak{M}F)(0) = F(0) = \log|g_M(0)| = 0$, we cannot affirm that $(\mathfrak{M}F)(z)$ is harmonic in a neighborhood of 0 and thus are not able to directly apply the first theorem from §B.3. An analogous result is nevertheless available by problem 57. In the present circumstances, with $(\mathfrak{M}F)(z)$ *even*, that result takes the form

$$(\mathfrak{M}F)(z) \;=\; C - \int_0^1 \log|z^2 - t^2|\,\mathrm{d}\rho(t) - \int_1^\infty \log\left|1 - \frac{z^2}{t^2}\right|\mathrm{d}\rho(t),$$

where ρ is a certain *positive* measure on $[0, \infty)$ with

$$\frac{\rho([0,t])}{t} \longrightarrow \frac{A}{\pi} \qquad \text{for } t \longrightarrow \infty.$$

Because $(\mathfrak{M}F)(0) < \infty$, we actually have

$$\int_0^1 \log(t^2)\,\mathrm{d}\rho(t) \;>\; -\infty,$$

so, after changing the value of the constant C, we can just as well write

$$(\mathfrak{M}F)(z) \;=\; C - \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right|\mathrm{d}\rho(t).$$

By the first lemma of §C.5, Chapter VIII (where the function corresponding to our present $g_M(z)$ was denoted by $G(z)$), we have

$$\log|g_M(z)| \;=\; \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right|\mathrm{d}\nu(t) \qquad \text{for } \Im z \geq 0,$$

$\nu(t)$ being a certain absolutely continuous (and *smooth*) increasing function. Taking the function $g_M(z)$ to be of exponential type exactly *equal* to B (so as not to bring in more letters !), we also have

$$\log|g_M(z)| \;=\; B\Im z \;+\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log|g_M(t)|}{|z-t|^2}\,\mathrm{d}t$$

for $\Im z > 0$ by §G.1 of Chapter III.* Referring to the above formula for $F(z) = F(\bar{z})$, we see from the last two relations that

$$F(z) \;=\; \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right|\mathrm{d}\nu(t) \;-\; (A+B)|\Im z|.$$

* see also end of proof of lemma at beginning of this article

$(\mathfrak{M}F)(z)$ is, however, a *majorant of* $F(z)$. Hence,

$$F(z) \;-\; (\mathfrak{M}F)(z) \;=\; \int_0^\infty \log\left|1-\frac{z^2}{t^2}\right|(\mathrm{d}\nu(t)+\mathrm{d}\rho(t))$$
$$-\;\;(A+B)|\Im z| \;-\; C \qquad \text{is} \;\leqslant\; 0.$$

Our statement about the zeros of $\varphi(z)$ will follow from this inequality.

The real line is the union of two disjoint subsets, an *open* one, Ω, on which

$$F(x) \;-\; (\mathfrak{M}F)(x) \;<\; 0,$$

and the *closed* set $E \;=\; \mathbb{R} \sim \Omega$, on which

$$F(x) \;-\; (\mathfrak{M}F)(x) \;=\; 0.$$

According to the *third* lemma of §B.2, $(\mathfrak{M}F)(z)$ is *harmonic* in a neighborhood of each $x_0 \in \Omega$, so the measure involved in the Riesz representation of $(\mathfrak{M}F)(z)$ can have no mass in such a neighborhood (last theorem, §A.2). This means that

$$\mathrm{d}\rho(t) \;=\; 0 \qquad \text{in } \Omega \cap [0,\infty).$$

It is now claimed that

$$\mathrm{d}\nu(t) \;+\; \mathrm{d}\rho(t) \;\leqslant\; \frac{A+B}{\pi}\mathrm{d}t \qquad \text{on } E \cap [0,\infty).$$

Once this is established, the separation of the zeros of our multiplier $\varphi(z)$ is immediate. That function is gotten by dividing out any two zeros from the even entire function $\varphi_1(z)$ given by

$$\log|\varphi_1(z)| \;=\; \int_0^\infty \log\left|1-\frac{z^2}{t^2}\right|\mathrm{d}[\rho(t)]$$

(as in the proof of the last theorem, §B.1). Because $\mathrm{d}\nu(t) \;\geqslant\; 0$, the preceding two relations will certainly make

$$\mathrm{d}\rho(t) \;\leqslant\; \frac{A+B}{\pi}\mathrm{d}t \qquad \text{for } t \;\geqslant\; 0,$$

and thus *any two zeros of* $\varphi_1(z)$ *will be distant by at least*

$$\frac{\pi}{A+B}$$

units, in conformity with Beurling and Malliavin's observation.

Verification of the claim remains, and it is there that we resort to the

Lemma. *Let μ be a finite positive measure on $\mathbb{R}$ without point masses. Then the derivative $\mu'(t)$ exists* (finite or infinite) *for all t save those belonging to a Borel set E_0 with $\mu(E_0) + |E_0| = 0$. If E is any compact subset of* $\mathbb{R}$,

$$|E| \quad = \quad \int_E \frac{1}{\mu'(t) + 1}(\mathrm{d}\mu(t) + \mathrm{d}t).$$

Proof. The initial statement is like that of Lebesgue's differentiation theorem which, however, only asserts the existence of a (finite) derivative $\mu'(t)$ almost everywhere (with respect to Lebesgue measure). The present result can nonetheless be deduced from the latter one by making a change of variable. Lest the reader feel that he or she is being hoodwinked by the juggling of notation, let us proceed somewhat carefully.

Put, as usual, $\mu(t) = \int_0^t \mathrm{d}\mu(\tau)$, making the standard interpretation of the integral for $t < 0$. By hypothesis, $\mu(t)$ is bounded, increasing and without jumps, so

$$S(t) \quad = \quad \mu(t) \; + \; t$$

is a *continuous, strictly increasing* map of $\mathbb{R}$ onto itself. S therefore has a *continuous* (and also strictly increasing) *inverse* which we denote by T:

$$T(\mu(t) + t) \quad = \quad t.$$

If $\varphi(s)$ is continuous and of compact support we have the elementary substitution formula

$$(*) \qquad \int_{-\infty}^{\infty} \varphi(s)\,\mathrm{d}s \quad = \quad \int_{-\infty}^{\infty} \varphi(S(t))\,(\mathrm{d}\mu(t) + \mathrm{d}t)$$

which is easily checked by looking at Riemann sums. The dominated convergence theorem shows that $(*)$ is valid as well for any function φ everywhere equal to the *pointwise limit* of a *bounded* sequence of continuous ones with *fixed* compact support. That is the case, in particular, for $\varphi = \chi_F$, the characteristic function of a compact set F, and we thus have

$$|F| \quad = \quad \int_{-\infty}^{\infty} \chi_F(s)\,\mathrm{d}s \quad = \quad \int_{-\infty}^{\infty} \chi_F(S(t))(\mathrm{d}\mu(t) + \mathrm{d}t) \quad = \quad \mu(T(F)) \; + \; |T(F)|.$$

The quantity $|T(F)|$ is, of course, nothing other than the Lebesgue–Stieltjes measure $\int_F \mathrm{d}T(s)$ generated by the increasing function $T(s)$ in the usual way. The last relation shows that

$$|T(F)| \;\leqslant\; |F|$$

for compact sets F; the measure on the left is thus *absolutely continuous* with respect to Lebesgue measure, and indeed

$$0 \leqslant \frac{T(s+h)-T(s)}{h} \leqslant 1 \qquad \text{for } h \neq 0.$$

By the theorem of Lebesgue already referred to, we know that the derivative

$$T'(s) = \lim_{h\to 0} \frac{T(s+h)-T(s)}{h}$$

exists for all s outside some Borel set F_0 with $|F_0| = 0$. The image $E_0 = T(F_0)$ is also Borel (T being one-one and continuous both ways) and, for any *compact subset* C of E_0, the previous identity yields

$$\mu(C) + |C| = |S(C)| \leqslant |F_0| = 0,$$

since $C = T(S(C))$ and $S(C) \subseteq S(E_0) = F_0$. Therefore

$$\mu(E_0) + |E_0| = 0.$$

Suppose that $t \notin E_0$. Then $s = \mu(t) + t$ (for which $T(s) = t$) cannot lie in F_0, T being one-one, and thus $T'(s)$ exists. For $\delta \neq 0$ and any such t (and corresponding s), write

$$h(s,\delta) = \mu(t+\delta) - \mu(t) + \delta.$$

We have $s + h(s,\delta) = \mu(t+\delta) + t + \delta$, so by definition of T, $T(s+h(s,\delta)) = t+\delta$, and

$$(\dagger) \qquad \frac{T(s+h(s,\delta)) - T(s)}{h(s,\delta)} = \frac{\delta}{\mu(t+\delta)-\mu(t)+\delta}.$$

The function $\mu(t)$ is in any event continuous, so at each s (and corresponding t),

$$h(s,\delta) \longrightarrow 0 \quad \text{as } \delta \longrightarrow 0.$$

Therefore, when $t \notin E_0$, $\lim_{\delta\to 0}\big((\mu(t+\delta)-\mu(t))/\delta + 1\big)^{-1}$ must, by ($\dagger$), *exist* and equal $T'(s)$. This shows that $\mu'(t)$ exists for such t (being infinite in case $T'(s) = 0$).

Take now any continuous function $\psi(s)$ of compact support. Because of the *absolute continuity* of the measure $\int_F \mathrm{d}T(s)$ already noted, we have

$$\int_{-\infty}^{\infty} \psi(s)\,\mathrm{d}T(s) = \int_{-\infty}^{\infty} \psi(s)\,T'(s)\,\mathrm{d}s.$$

Here,

$$T'(s) \quad = \quad \lim_{\delta \to 0} \frac{T(s + h(s,\delta)) \; - \; T(s)}{h(s,\delta)} \qquad \text{a.e.}$$

where $h(s,\delta)$ is the quantity introduced above. The difference quotients on the right lie, however, between 0 and 1. Hence $\int_{-\infty}^{\infty} \psi(s) T'(s)\,\mathrm{d}s$ equals the limit, for $\delta \longrightarrow 0$, of

$$\int_{-\infty}^{\infty} \psi(s) \, \frac{T(s + h(s,\delta)) \; - \; T(s)}{h(s,\delta)} \, \mathrm{d}s$$

by dominated convergence. In this last expression the *integrand* is continuous and of compact support when $\delta \neq 0$. We may therefore use $(*)$ to make the substitution $s \; = \; \mu(t) + t$ therein; with the help of $(\dagger)$, that gives us

$$\int_{-\infty}^{\infty} \psi(\mu(t) + t) \, \frac{\delta}{\mu(t + \delta) - \mu(t) + \delta} \, (\mathrm{d}\mu(t) \; + \; \mathrm{d}t).$$

The quantity $\delta/(\mu(t + \delta) - \mu(t) + \delta)$ lies between 0 and 1 and, as we have just seen, tends to $1/(\mu'(t) + 1)$ for every t outside E_0 when $\delta \longrightarrow 0$, where $\mu(E_0) \; + \; |E_0| \; = \; 0$. Another application of the dominated convergence theorem thus shows the integral just written to tend to $\int_{-\infty}^{\infty} \psi(\mu(t) + t)(\mu'(t) + 1)^{-1}(d\mu(t) \; + \; \mathrm{d}t)$ as $\delta \longrightarrow 0$. In this way, we see that

$$\int_{-\infty}^{\infty} \psi(s)\,\mathrm{d}T(s) \quad = \quad \int_{-\infty}^{\infty} \frac{\psi(\mu(t) + t)}{\mu'(t) + 1} \, (\mathrm{d}\mu(t) \; + \; \mathrm{d}t)$$

when ψ is continuous and of compact support.

Extension of this formula to functions ψ of the form χ_F with F compact now proceeds as at the beginning of the proof. Given, then, any compact E, we put $F \; = \; S(E)$, making $T(F) \; = \; E$ and $\chi_F(\mu(t) + t) \; = \; \chi_F(S(t)) \; = \; \chi_E(t)$; using $\psi(s) \; = \; \chi_F(s)$ we thus find that

$$|E| \quad = \quad |T(F)| \quad = \quad \int_{-\infty}^{\infty} \chi_F(s)\,\mathrm{d}T(s) \quad = \quad \int_{-\infty}^{\infty} \chi_E(t) \, \frac{\mathrm{d}\mu(t) + \mathrm{d}t}{\mu'(t) + 1}.$$

The lemma is established.

We proceed to the claim.

Problem 58

(a) Show that in our present situation, neither $\nu(t)$ nor $\rho(t)$ can have any point masses. (Hint: Concerning $\rho(t)$, recall that $(\mathfrak{M}F)(x)$ is continuous!)

(b) We take $\nu(0) = \rho(0) = 0$ and then extend the increasing *functions* $\nu(t)$ and $\rho(t)$ from $[0, \infty)$ to $\mathbb{R}$ by making them *odd*. Show that for $x \in E$ (the set on which $F(x) - (\mathfrak{M}F)(x) = 0$), we have

$$\frac{1}{2}\int_{-\infty}^{\infty}\log\left(1 + \frac{y^2}{(x-t)^2}\right)(\mathrm{d}\nu(t) + \mathrm{d}\rho(t)) - (A+B)y \leqslant 0 \quad \text{for } y > 0.$$

(c) Writing $\mu(t) = \nu(t) + \rho(t)$, show that for fixed $y > 0$,

$$\frac{1}{2}\int_{-\infty}^{\infty}\log\left(1 + \frac{y^2}{(x-t)^2}\right)\mathrm{d}\mu(t) = \int_0^{\infty}\frac{y^2}{y^2+\tau^2}\,\frac{\mu(x+\tau)-\mu(x-\tau)}{\tau}\,\mathrm{d}\tau.$$

(Hint: Since $\mu(\{x\}) = 0$, the left hand integral is the limit, for $\delta \longrightarrow 0$, of

$$\frac{1}{2}\int_{|t-x|\geqslant\delta}\log\left(1 + \frac{y^2}{(t-x)^2}\right)\mathrm{d}\mu(t).$$

Here we may integrate by parts to get

$$\int_{\delta}^{\infty}\frac{y^2}{y^2+\tau^2}\,\frac{\mu(x+\tau)-\mu(x-\tau) - (\mu(x+\delta)-\mu(x-\delta))}{\tau}\,\mathrm{d}\tau.$$

Now make $\delta \longrightarrow 0$ and use monotone convergence.)

(d) Hence show that *for each* $x \in E$ *where* $\mu'(x)$ *exists*, we have

$\pi\mu'(x) \leqslant A + B.$

(Hint: Refer to (b).)

(e) Show then that if F is any compact subset of E,

$\pi\mu(F) \leqslant (A+B)|F|,$

whence

$$\nu(F) + \rho(F) \leqslant \frac{A+B}{\pi}|F|.$$

(Hint: Apply the lemma.)

The reader who prefers a more modern treatment yielding the result of part (e) may, in place of (d), establish that

$$\pi(\underline{D}\mu)(x) \leqslant A + B \quad \text{for } x \in E,$$

where

$$(\underline{D}\mu)(x) = \liminf_{\Delta x \to 0}\frac{\mu(x+\Delta x) - \mu(x)}{\Delta x}.$$

Then, instead of using the lemma to get part (e), a suitable version of Vitali's covering theorem can be applied.

Remark. The original proof of the theorem of Beurling and Malliavin is different from the one given in this article, and the reader interested in working seriously on the subject of the present chapter should study it.

The first exposition of that proof is contained in a famous (and very rare) Stanford University preprint written by Malliavin in 1961, and the final version is in his joint *Acta* paper with Beurling, published in 1962. Other presentations can be found in Kahane's Seminaire Bourbaki lecture for 1961–62, and in de Branges' book. But the clearest explanation of the proof's *idea* is in a much later paper of Malliavin appearing in the 1979 *Arkiv*. Although some details are omitted in that paper, it is probably the best place to start reading.

3. A quantitative version of the preceding result.

Theorem. *Let $\Phi(z)$ be entire and even, of exponential type B, with $\Phi(x) \geqslant 0$ on the real axis and*

$$\int_{-\infty}^{\infty} \frac{\log^+ \Phi(x)}{1+x^2}\,dx \quad < \quad \infty.$$

For $M > 0$, denote by J_M the quantity

$$\int_0^{\infty} \frac{1}{x^2} \log\left(1 \; + \; x^2 \frac{\Phi(x)}{M}\right) dx.$$

Suppose that for some given $A > 0$, M is large enough to make

$$J_M \quad + \quad \frac{1}{\sqrt{\pi}} \sqrt{(J_M(J_M \; + \; \pi B))} \quad < \quad A.$$

Then there is an even entire function $\varphi(z)$ of exponential type A with

$$\varphi(0) \;=\; 1$$

and

$$|\varphi(x)|\,\Phi(x) \quad \leqslant \quad 2e^2(A+B)^2 M \qquad \text{for } x \in \mathbb{R}.$$

Remark. There are actually functions φ having all the stipulated properties and satisfying a relation like the last one with the coefficient 2 replaced by any number > 1 – hence indeed by 1, as follows by a normal family argument. By that kind of argument one also sees that there are such φ

corresponding to a value of M *for which*

$$J_M + \frac{1}{\sqrt{\pi}}\sqrt{(J_M(J_M + \pi B))} = A.$$

Such improvements are not very significant.

Proof of theorem. We argue as in the last article, working this time with the auxiliary entire function

$$P(z) = \left(1 + \frac{z^2}{R^2}\right)\left(1 + z^2\frac{\Phi(z)}{M}\right)$$

which involves a large constant R as well as the parameter M. An extra factor has again been introduced on the right in order to make sure that

$$P(x) \longrightarrow \infty \quad \text{for } x \longrightarrow \pm\infty.$$

Like $\Phi(z)$, the function $P(z)$ is even, entire, and of exponential type B; it is, moreover, $\geqslant 1$ on the real axis and we can use it as a *weight* thereon. As long as M fulfills the condition in the hypothesis,

$$J = \int_0^\infty \frac{1}{x^2}\log P(x)\,dx$$

will satisfy the relation

$$J + \frac{1}{\sqrt{\pi}}\sqrt{(J(J + \pi B))} \leqslant A$$

for large enough R; we *choose and fix* such a value of that quantity.

Using the weight $W(x) = P(x)$ and the number A, we then form the function $F(z)$ and the sequence of $F_N(z)$ corresponding to it as in the previous two articles, and set out to show that $(\mathfrak{M}F)(0) = 0$.

This is done as before, by verifying that $(\mathfrak{M}F_N)(0) \leqslant 0$ for each N. Assuming, on the contrary, that some $(\mathfrak{M}F_N)(0)$ is *strictly larger* than some $\varepsilon > 0$, we have

$$(\mathfrak{M}F_N)(0) \leqslant \varepsilon + \int_{\partial\mathscr{D}} \log P(t)\,d\omega_{\mathscr{D}}(t, 0) - AY_{\mathscr{D}}(0)$$

with a domain $\mathscr{D}$ of the sort considered in §C of Chapter VIII, since here

$$F_N(t) \leqslant F(t) = \log P(t), \quad t \in \mathbb{R},$$

where $\log P(t) \longrightarrow \infty$ for $t \longrightarrow \pm\infty$. We proceed to estimate the integral on the right.

The entire function $P(z)$ is real and $\geqslant 1$ on R, so, by the lemma from the last article, we can get an entire function $g(z)$ of exponential type $B/2$ (*half* that of P), having all its zeros in the lower half plane, and such that

$$g(z)\overline{g(\bar{z})} = P(z).$$

For the entire function

$$G(z) = (g(z))^2$$

of exponential type B with all its zeros in $\Im z < 0$ we then have $|G(x)| = P(x)$ on $\mathbb{R}$, so that

$$\int_{\partial\mathscr{D}} \log P(t)\,\mathrm{d}\omega_{\mathscr{D}}(t,\ 0) = \int_{\partial\mathscr{D}} \log|G(t)|\,\mathrm{d}\omega_{\mathscr{D}}(t,\ 0).$$

Here, $G(z)$ satisfies the hypothesis of the second theorem in §C.5 of Chapter VIII, and that result can be used to get an upper bound for the last integral. We can, however, do somewhat better by first improving the theorem, using, at the very end of its proof, the estimate furnished by problem 28(c) in place of the one applied there. The effect of this is to replace the term $\sqrt{(2\mathrm{e}\,J(J+\pi B/4))}$ figuring in the theorem's conclusion by

$$\frac{1}{\sqrt{\pi}}\sqrt{(J(J + \pi B))}$$

with

$$J = \int_0^\infty \frac{1}{x^2}\log|G(x)|\,\mathrm{d}x = \int_0^\infty \frac{1}{x^2}\log P(x)\,\mathrm{d}x,$$

and in that way one finds that

$$\int_{\partial\mathscr{D}} \log|G(t)|\,\mathrm{d}\omega_{\mathscr{D}}(t,\ 0) \leqslant Y_{\mathscr{D}}(0)\left\{J + \frac{1}{\sqrt{\pi}}\sqrt{(J(J+\pi B))}\right\}.$$

Substituted into the above relation, this yields

$$(\mathfrak{M}F_N)(0) \leqslant \varepsilon,$$

a contradiction, thanks to our initial assumption about M and our choice of R. It follows that $(\mathfrak{M}F_N)(0) \leqslant 0$ for every N and thus that $(\mathfrak{M}F)(0) = 0$.

Knowing that, we can, since $P(0) = 1$, apply the *corollary* to the second theorem of §B.3. That gives us, corresponding to any $\eta > 0$, an increasing function $\rho(t)$, *zero* on a neighborhood of the origin, such that

$$\frac{\rho(t)}{t} \longrightarrow \frac{A}{\pi} \qquad \text{for } t \longrightarrow \pm\infty$$

and that

$$\log P(x) \quad + \quad \gamma x \quad + \quad \int_{-\infty}^{\infty} \left(\log\left|1 - \frac{x}{t}\right| + \frac{x}{t} \right) \mathrm{d}\rho(t) \quad \leqslant \quad \eta$$

on $\mathbb{R}$, γ being a certain real constant. In the present circumstances $P(x) = P(-x)$, so, taking the increasing function

$$\nu(t) \quad = \quad \tfrac{1}{2}(\rho(t) \; - \; \rho(-t))$$

(also zero on a neighborhood of the origin), we have simply

$$\log P(x) \quad + \quad \int_0^{\infty} \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\nu(t) \quad \leqslant \quad \eta \qquad \text{for } x \in \mathbb{R}.$$

Our given function Φ is $\geqslant 0$ on the real axis. Therefore $P(x) \geqslant x^2\Phi(x)/M$ there, and our last relation certainly implies that

$$\log\left(x^2 \frac{\Phi(x)}{M} \right) \quad + \quad \int_0^{\infty} \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\nu(t) \quad \leqslant \quad \eta$$

on $\mathbb{R}$. Denote for the moment

$$\log\left| z^2 \frac{\Phi(z)}{M} \right| \quad + \quad \int_0^{\infty} \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}\nu(t)$$

by $U(z)$; this function is *subharmonic*, and, since Φ is of exponential type B while

$$\frac{\nu(t)}{t} \longrightarrow \frac{A}{\pi} \qquad \text{for } t \longrightarrow \infty,$$

we have

$$U(z) \quad \leqslant \quad (B + A)|z| \; + \; \mathrm{o}(|z|)$$

for large $|z|$. Because $U(x) \leqslant \eta$ on $\mathbb{R}$, we see by the third Phragmén–Lindelöf theorem of §C, Chapter III, that

$$U(z) \quad \leqslant \quad \eta \; + \; (A + B)\Im z \qquad \text{for } \Im z \; \geqslant \; 0.$$

To the integral $\int_0^{\infty} \log|1 - (z^2/t^2)| \mathrm{d}\nu(t)$ we now apply the lemma of

§A.1, Chapter X, according to which

$$\int_0^\infty \log\left|1-\frac{z^2}{t^2}\right|(\mathrm{d}[\nu(t)]-\mathrm{d}\nu(t)) \quad\leqslant\quad \log\left\{\frac{\max(|x|,|y|)}{2|y|}+\frac{|y|}{2\max(|x|,|y|)}\right\}$$

(where, as usual, $z = x+\mathrm{i}y$). Used together with the preceding inequality for $U(z)$, this yields

$$\log\left|z^2\frac{\Phi(z)}{M}\right| \quad+\quad \int_0^\infty \log\left|1-\frac{z^2}{t^2}\right|\mathrm{d}[\nu(t)]$$

$$\leqslant\quad \eta \;+\; (A+B)y \;+\; \log\left\{\frac{\max(|x|,y)}{2y}+\frac{y}{2\max(|x|,y)}\right\}$$

for $\Im z = y > 0$.

There is clearly an entire function $\varphi(z)$ with

$$\log|\varphi(z)| \quad=\quad \int_0^\infty \log\left|1-\frac{z^2}{t^2}\right|\mathrm{d}[\nu(t)];$$

φ is *even* and $\varphi(0) = 1$. Moreover, in view of the asymptotic behaviour of $\nu(t)$ for large t, *$\varphi(z)$ is of exponential type A*. In terms of φ, the preceding relation becomes

$$|\Phi(z)\varphi(z)| \quad\leqslant\quad M\mathrm{e}^{(A+B)y+\eta}\,\frac{\{\max(|x|,y)/y \;+\; y/\max(|x|,y)\}}{2(x^2 \;+\; y^2)}, \qquad y>0.$$

The fraction on the right is just

$$\frac{1}{2y^2}\cdot\frac{\max(\xi,1)\;+\;1/\max(\xi,1)}{\xi^2\;+\;1}$$

with $\xi = |x|/y$, and hence is $\leqslant 1/y^2$. Thus, putting $z = x+\mathrm{i}h$ with $h>0$, we see that

$$|\Phi(x+\mathrm{i}h)\varphi(x+\mathrm{i}h)| \quad\leqslant\quad \frac{M}{h^2}\mathrm{e}^{(A+B)h+\eta} \qquad \text{for } x\in\mathbb{R}.$$

Applying once more the third Phragmén–Lindelöf theorem of Chapter III, §C, this time to $\Phi(z)\varphi(z)$ (of exponential type $A+B$) in the half plane $\{\Im z \leqslant h\}$, we get finally

$$|\Phi(x)\varphi(x)| \quad\leqslant\quad \frac{M}{h^2}\mathrm{e}^{2(A+B)h+\eta}, \qquad x\in\mathbb{R},$$

and, putting $h = 1/(A+B)$, we have

$$\Phi(x)|\varphi(x)| \quad\leqslant\quad \mathrm{e}^{\eta}\mathrm{e}^2(A+B)^2M \qquad \text{on } \mathbb{R}.$$

The quantity $\eta > 0$ was arbitrary, so the desired result is established. We are done.

Scholium. Let us try to understand the rôle played by the parameter M in the result just proved. As long as

$$\int_{-\infty}^{\infty} \frac{\log^+ \Phi(x)}{1+x^2} dx \quad < \quad \infty,$$

it is surely true that with

$$J_M \quad = \quad \int_0^{\infty} \frac{1}{x^2} \log\left(1 \;+\; x^2\frac{\Phi(x)}{M}\right) dx,$$

the expression

$$J_M \quad + \quad \frac{1}{\sqrt{\pi}}\sqrt{(J_M(J_M \;+\; \pi B))}$$

eventually becomes less than any given $A > 0$ when M increases without limit; we *cannot*, however, tell *how large* M must be taken for that to happen *if only the value of the former integral and the type* B *of* Φ *are known*. Our result thus does not enable us to determine, *using that information alone*, how small $\sup_{x\in\mathbb{R}}\Phi(x)|\varphi(x)|$ can be rendered by taking a suitable even entire function φ of exponential type A with $\varphi(0) = 1$.

This is even the case for *polynomials* Φ (special kinds of functions of exponential type zero!).

Problem 59

Show that for the polynomials

$$\Phi_N(z) \quad = \quad \left(\frac{z^2}{2N^2} \;-\; 1\right)^{2N}$$

one has

$$\int_0^{\infty} \frac{1}{x^2} \log^+ \Phi_N(x)\, dx \quad \leqslant \quad \text{const.},$$

but that for $J > 0$ small enough, there is no value of M which will make

$$\int_0^{\infty} \frac{1}{x^2} \log\left(1 \;+\; x^2\frac{\Phi_N(x)}{M}\right) dx \quad \leqslant \quad J$$

for all N simultaneously.

$$\left(\text{Hint: Look at the values of } \int_0^\infty \frac{1}{x^2}\log^+\left(\frac{\Phi_N(x)}{7^N}\right)\mathrm{d}x.\right)$$

The parameter M, made to depend on A by requiring that

$$J_M \; + \; \frac{1}{\sqrt{\pi}}\sqrt{(J_M(J_M \; + \; \pi B))}$$

be $<$ A (or simply *equal to* A; see the remark following our theorem's statement) does nevertheless seem to be the *main factor governing how small*

$$\sup_{x\in\mathbb{R}} \Phi(x)\,|\varphi(x)|$$

can be for even entire functions φ of exponential type A with $\varphi(0) = 1$. The evidence for this is especially convincing when *entire functions* Φ of exponential type zero are concerned. Then the discrepancy between the above result and any best possible one essentially involves nothing more than *a constant factor affecting the type A of the multiplier* φ in question.

Problem 60

Suppose that $W(x) \geqslant 1$ is even and that there is an even entire function φ of exponential type A with $\varphi(0) = 1$ and

$$W(x)|\varphi(x)| \quad \leqslant \quad K \quad \text{for } x\in\mathbb{R}.$$

(a) Show that then

$$\int_{x_0}^\infty \frac{1}{x\sqrt{(x^2-x_0^2)}}\log\left(\frac{W(x)}{K}\right)\mathrm{d}x \quad \leqslant \quad \frac{\pi}{2}A$$

for any $x_0 > 0$. (Hint: Use harmonic estimation in

$$\mathscr{D} \quad = \quad \mathbb{C} \sim (-\infty, -x_0] \sim [x_0, \infty)$$

to get an upper bound for $\log|\varphi(0)|$. Note that $\omega_{\mathscr{D}}(\;\;, z)$ and $Y_{\mathscr{D}}(z)$ are explicitly available for this domain; for the latter, see, for instance, §A.2 of Chapter VIII).

Suppose now that φ, A and K are as in (a) *and that there is in addition an $x_0 > 0$ such that $W(x) \leqslant K$ for $|x| \leqslant x_0$ while $W(x) \geqslant K$ for $|x| \geqslant x_0$. Note that $W(x)$ need not be an increasing function of $|x|$ for this to hold for certain values of K*:

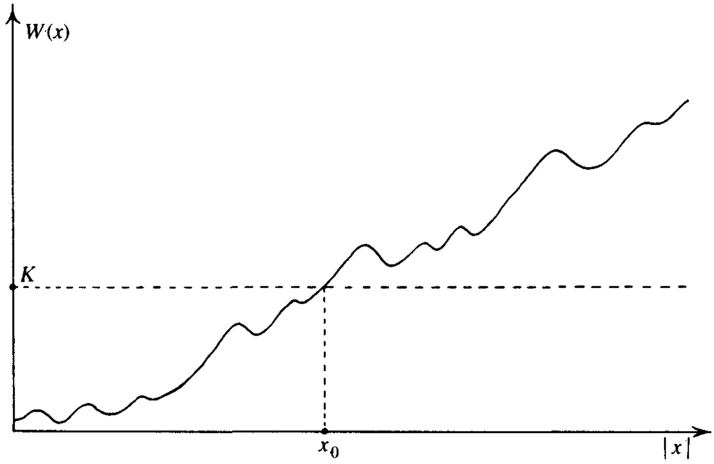

Figure 244

(b) Show that then, given any $\eta > 0$, there is a constant C depending only on η such that

$$\int_0^\infty \frac{1}{x^2} \log\left(1 + \frac{A^2x^2}{C}\frac{W(x)}{K}\right) dx \;<\; \left(\frac{\pi}{2} + \eta\right)A.$$

(Hint: Observe that

$$\log\left(1 + \frac{A^2x^2}{C}\frac{W(x)}{K}\right) \;\leqslant\; \log\left(1 + \frac{A^2x^2}{C}\right) \;+\; \log^+\left(\frac{W(x)}{K}\right).$$

Refer to part (a).)

By this problem we see in particular that if $W(x)$ *is the restriction to* $\mathbb{R}$ *of an entire function of exponential type zero* having, for some given K, the behaviour described therein, then, for

$$M \;=\; CK/A^2,$$

the integral J_M corresponding to W *satisfies the condition of our theorem pertaining to multipliers of exponential type*

$$A' \;=\; \frac{\sqrt{\pi}+1}{\sqrt{\pi}}\left(\frac{\pi}{2} + \eta\right)A$$

(rather than to those of exponential type A). For suitable choice of the constant C, the right side is

$$<\;\; 2.5A$$

Thus, subject to the above proviso regarding $W(x)$ and K, *the theorem will furnish* an even entire function ψ with $\psi(0) = 1$, *of exponential type* $2.5A$, for which

$$W(x)|\psi(x)| \quad \leqslant \quad 12.5\mathrm{e}^2 CK \quad \text{on } \mathbb{R}$$

whenever the existence of such an entire φ, of *exponential type* A, with

$$W(x)|\varphi(x)| \quad \leqslant \quad K \quad \text{on } \mathbb{R}$$

is otherwise known.

There may be certain even entire functions Φ, $\geqslant 1$ on $\mathbb{R}$ and of exponential type zero, such that, for some *arbitrarily large values* of x_0, $\Phi(x) \leqslant \Phi(x_0)$ for $|x| \leqslant x_0$ and $\Phi(x) \geqslant \Phi(x_0)$ for $|x| \geqslant x_0$ with, in addition, *the graph of* $\log\Phi(x)$ vs $|x|$ *having a sizeable hump immediately to the right of each abscissa* x_0:

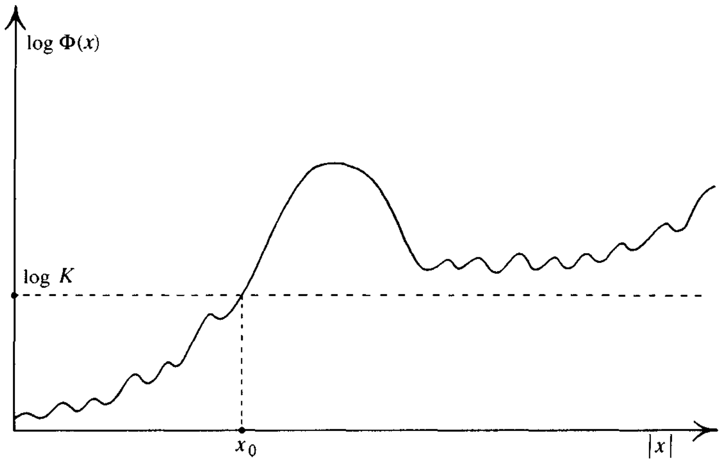

Figure 245

If we *have* such a function Φ and use the weight $W(x) = \Phi(x)$, the condition

$$\int_{x_0}^{\infty} \frac{1}{x\sqrt{(x^2 - x_0^2)}} \log\left(\frac{\Phi(x)}{K}\right) \mathrm{d}x \quad \leqslant \quad \frac{\pi}{2} A$$

obtained in part (a) of the problem would, for $K = \Phi(x_0)$ with any of the x_0 just described, give us

$$\int_0^{\infty} \frac{1}{x^2} \log^+\left(\frac{\Phi(x)}{K}\right) \mathrm{d}x \quad \leqslant \quad cA$$

where c is a number *definitely smaller* than $\pi/2$, and thus make it possible to *bring down the bound found in part* (b) *from* $(\pi/2 + \eta)A$ *to* $(c+\eta)A$. It is conceivable that one could construct such an entire function Φ with humps large enough to make

$$c \leqslant \frac{\sqrt{\pi}}{\sqrt{\pi}+1}$$

for a sequence of values of K tending to ∞ and values of A corresponding to them (through the *first* of the above two integral inequalities) tending to zero. Denoting the first sequence by $\{K_n\}$ and the second by $\{A_n\}$, we see that *for the function* Φ (if there is one!), the *upper bound provided by the theorem* would, for $A = A_n$, *be proportional to* K_n and hence *exceed the actual value in question by at most a constant factor* (for such A). Although I do not think the value of c can be diminished that much, the construction is perhaps worth *trying*. I have no time for that now; this book must go to press.

Our result seems farther from the truth when functions Φ of exponential type $B > 0$ are in question. For those, the condition on J_M figuring in the statement is essentially of the form

$$J_M \leqslant \text{const.}\ A^2$$

when A is small.

It is not terribly difficult to build even functions Φ of exponential type > 0 whose graphs (for real x) contain infinitely many *very long and practically flat plateaux*, e.g.,

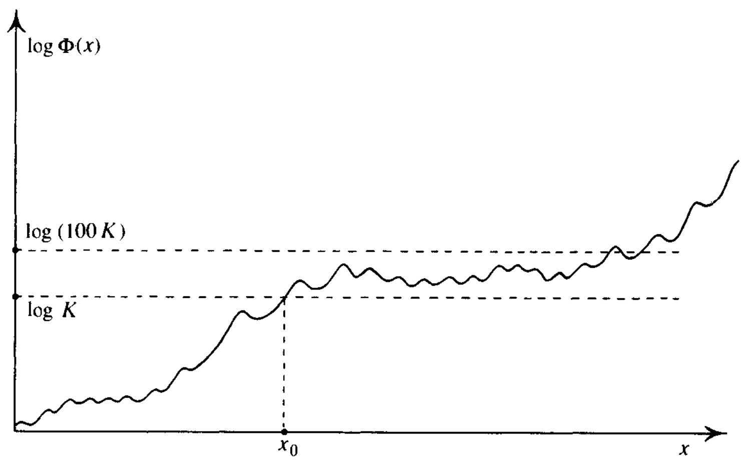

Figure 246

For this kind of function Φ one has arbitrarily large values of K (and of x_0 corresponding to them) such that

$$\int_0^\infty \frac{1}{x^2}\log^+\left(\frac{\Phi(x)}{100K}\right)dx$$

(say) *is exceedingly small* in comparison to

$$\int_{x_0}^\infty \frac{1}{x\sqrt{(x^2-x_0^2)}}\log\left(\frac{\Phi(x)}{K}\right)dx;$$

putting the *latter* integral *equal* to $(\pi/2)A$, we can thus (for values of A corresponding to these particular ones of K) have the *former* integral $\leqslant$ const. A^2, and even much smaller. What brings

$$\int_0^\infty \frac{1}{x^2}\log\left(1 \;+\; A^2x^2\frac{\Phi(x)}{100CK}\right)dx$$

back up to a constant multiple of A in this situation is *not* the presence of $\Phi(x)/K$ in the integrand, but rather that of x^2 ! In order to reduce this last integral to a multiple of A^2, the A^2 figuring in the integrand must be replaced by A^4, making M *a constant multiple* of

$$K/A^4$$

if J_M is to satisfy the condition in the theorem. We thus find a *discrepancy involving the factor* $1/A^4$ between the *upper bound*

$$2e^2(A+B)^2M \quad \cong \quad \text{const}\, K/A^4$$

furnished by our result (for *small* $A > 0$) and the *correct value, at least equal to* K when

$$A \quad = \quad \frac{2}{\pi}\int_{x_0}^\infty \frac{1}{x\sqrt{(x^2-x_0^2)}}\log\left(\frac{\Phi(x)}{K}\right)dx.$$

It frequently turns out in actual examples that the K related to A in this way (and such that $\Phi(x) \leqslant K$ for $|x| < x_0$ while $\Phi(x) \geqslant K$ for $|x| \geqslant x_0$) goes to infinity quite rapidly as $A \longrightarrow 0$; one commonly finds that $K \sim \exp(\text{const.}/A)$. Compared with such behaviour, a few factors of $1/A$ more or less are practically of no account. Considering especially the approximate nature of the bound on $\int_{\partial\mathscr{D}}\log|G(t)|d\omega_{\mathscr{D}}(t,\ 0)$ that we have been using, it hardly seems possible to attain greater precision by the present method.

4. Still more about the energy. Description of the Hilbert space $\mathfrak{H}$ used in Chapter VIII, §C.5

Beginning with §B.5 of Chapter VIII, we have been denoting

$$\int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t)\mathrm{d}\rho(x)$$

by $E(\mathrm{d}\rho(t),\ \mathrm{d}\rho(t))$ when dealing with real signed measures ρ on $[0, \infty)$ without point mass at the origin making the double integral absolutely convergent. In work with such measures ρ it is also convenient to write $U_\rho(x)$ for the Green potential

$$\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t),$$

at least in cases where the integral is well defined for $x > 0$. In the latter circumstance $U_\rho(x)$ *cannot*, as remarked at the end of §C.3, Chapter VIII, *be identically zero on* $(0, \infty)$ (or, for that matter, vanish a.e. with respect to $|\mathrm{d}\rho|$ there) *unless the measure ρ vanishes.* It thus makes sense to regard

$$\sqrt{(E(\mathrm{d}\rho(t),\ \mathrm{d}\rho(t)))}$$

as a *norm*, $\|U_\rho\|_E$, for the functions $U_\rho(x)$ arising in such fashion. This norm comes from a *bilinear form* $\langle\ ,\ \rangle_E$ on those functions U_ρ, defined by putting

$$\langle U_\rho,\ U_\sigma\rangle_E \quad = \quad \int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t)\mathrm{d}\sigma(x)$$

for any two of them, U_ρ and U_σ; the form's *positive definiteness* is a direct consequence of the results in §B.5, Chapter VIII. Since $\|U_\rho\|_E = \sqrt{(\langle U_\rho,\ U_\rho\rangle_E)}$, we obtain a certain *real Hilbert space* $\mathfrak{H}$ by forming the (abstract) *completion of the collection of functions U_ρ in the norm* $\|\ \ \|_E$.

The space $\mathfrak{H}$ was already used in the proof of the second theorem of §C.5, Chapter VIII. There, merely the *existence* of $\mathfrak{H}$ was needed, and we did not require any *concrete description* of its *elements*. One can indeed *make do* with just that existence and still proceed quite far. *Specific knowledge of* $\mathfrak{H}$ is, however, really necessary if one is to fully understand (and appreciate) the remaining work of this chapter. The present article is provided for that purpose.

It is actually better to use a wider collection of Green potentials U_ρ in forming the space $\mathfrak{H}$. One starts by showing that if ρ is a signed measure on $[0, \infty)$ without point mass at 0 making

$$\int_0^{\infty}\int_0^{\infty} \log\left|\frac{x+t}{x-t}\right| |d\rho(t)||d\rho(x)| \quad < \quad \infty,$$

the integral,

$$\int_0^{\infty} \log\left|\frac{x+t}{x-t}\right| d\rho(t)$$

is absolutely convergent at least *almost everywhere* (but perhaps not everywhere!) for $0 < x < \infty$. In these more general circumstances we will continue to denote that integral by $U_\rho(x)$; we will also have occasion to use the *extension* of that function to the *complex plane* given by the formula

$$U_\rho(z) \quad = \quad \int_0^{\infty} \log\left|\frac{z+t}{z-t}\right| d\rho(t).$$

It turns out to be true for these functions U_ρ that $E(d\rho(t), d\rho(t))$ is *determined* when $U_\rho(x)$ is *specified a.e.* on $\mathbb{R}$ (indeed, on $(0, \infty)$); we will in fact obtain a *formula* for the former quantity involving just the function $U_\rho(x)$. This will justify our writing

$$\|U_\rho\|_E \quad = \quad \sqrt{(E(d\rho(t),\ d\rho(t)))};$$

the space $\mathfrak{H}$ *will then be taken as the completion of the present class of functions* U_ρ *in the norm* $\|\quad\|_E$. It will follow from our work* that the Hilbert space $\mathfrak{H}$ thus defined coincides with the one initially referred to in this article which, *a priori*, could be a proper subspace of it. That fact is pointed out now; we shall not, however, insist on it during our discussion for *as such* it will not be used.

We shall see in a moment that our space $\mathfrak{H}$ consists of *actual Lebesgue measurable odd functions defined* a.e. *on* $\mathbb{R}$; those will need to be *characterized.*

Let's get down to work.

Lemma. *If ρ is a real signed measure on $[0, \infty)$ without point mass at 0, such that*

$$\int_0^{\infty}\int_0^{\infty} \log\left|\frac{x+t}{x-t}\right| |d\rho(t)||d\rho(x)| \quad < \quad \infty,$$

* see the last theorem in this article and the remark following it

the integral

$$\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t)$$

is absolutely convergent for almost all real x, and equal a.e. on $\mathbb{R}$ *to* an odd Lebesgue measurable function which is locally L_1.

Proof. For x and $t > 0$, $\log|(x+t)/(x-t)| > 0$, and the left-hand expression is simply changed to its *negative* if, in it, x is replaced by $-x$. The whole lemma thus follows if we verify that

$$\int_0^a \int_0^\infty \log\left|\frac{x+t}{x-t}\right| |\mathrm{d}\rho(t)|\, \mathrm{d}x \quad < \quad \infty$$

for each finite a. We will use Schwarz' inequality for this purpose.

Fixing $a > 0$, we take the *restriction* λ *of ordinary Lebesgue measure to* $[0, a]$, and easily verify by direct calculation that

$$\int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\lambda(t)\,\mathrm{d}\lambda(x) \quad < \quad \infty.$$

According, then, to the remark at the end of §B.5, Chapter VIII, the previous expression, nothing other than

$$E(|\mathrm{d}\rho(t)|,\ \mathrm{d}\lambda(t))$$

in the notation of that §, is

$$\leqslant \quad \sqrt{(E(|\mathrm{d}\rho(t)|,\ |\mathrm{d}\rho(t)|)\cdot E(\mathrm{d}\lambda(t),\ \mathrm{d}\lambda(t)))},$$

a finite quantity (by the hypothesis). Done.

By almost the same reasoning we can show that *the Hilbert space* $\mathfrak{H}$ must consist of Lebesgue measurable and locally integrable functions on $\mathbb{R}$. In the logical development of the present material, that statement should come somewhat later. Let us, however, strike while the iron is hot:

Theorem. *Suppose that the signed measures* ρ_n *on* $[0, \infty)$, *each without point mass at the origin, are such that*

$$\int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| |\mathrm{d}\rho_n(t)|\,|\mathrm{d}\rho_n(x)| \quad < \quad \infty$$

and that furthermore,

$$E\big(\mathrm{d}\rho_n(t) - \mathrm{d}\rho_m(t),\ \mathrm{d}\rho_n(t) - \mathrm{d}\rho_m(t)\big) \quad \xrightarrow[n,m]{} \quad 0.$$

Then, for each compact subset K of $\mathbb{R}$, the functions

$$U_n(x) \quad = \quad \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho_n(t)$$

(each defined a.e. by the lemma) form a Cauchy sequence in $L_1(K)$.

Proof. It is again sufficient to check this for sets $K = [0, a]$, where $a > 0$. Fixing any such a and focussing our attention on *some particular pair* (n, m), we take the function

$$\varphi(x) \quad = \quad \begin{cases} \operatorname{sgn}(U_n(x) - U_m(x)), & 0 \leqslant x \leqslant a, \\ 0 \text{ otherwise}. \end{cases}$$

Then,

$$\int_0^a |U_n(x) - U_m(x)|\,\mathrm{d}x \quad = \quad \int_0^\infty (U_n(x) - U_m(x))\,\varphi(x)\,\mathrm{d}x.$$

We have, however,

$$\int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right| |\varphi(t)|\,\mathrm{d}t\,|\varphi(x)|\,\mathrm{d}x \quad \leqslant \quad \int_0^a\int_0^a \log\left|\frac{x+t}{x-t}\right| \mathrm{d}t\,\mathrm{d}x$$

with the right side finite, as already noted. Thence, by the remark at the end of §B.5, Chapter VIII,

$$\int_0^\infty (U_n(x) - U_m(x))\,\varphi(x)\,\mathrm{d}x \quad = \quad E(\mathrm{d}\rho_n(t) - \mathrm{d}\rho_m(t),\ \varphi(t)\,\mathrm{d}t)$$
$$\leqslant \quad \sqrt{(E(\mathrm{d}\rho_n(t) - \mathrm{d}\rho_m(t),\ \mathrm{d}\rho_n(t) - \mathrm{d}\rho_m(t)) \cdot E(\varphi(t)\mathrm{d}t,\ \varphi(t)\mathrm{d}t))}.$$

Since $\log|(x+t)/(x-t)| > 0$ for x and $t > 0$,

$$E(\varphi(t)\,\mathrm{d}t,\ \varphi(t)\,\mathrm{d}t) \quad \leqslant \quad \int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right| |\varphi(t)||\varphi(x)|\,\mathrm{d}t\,\mathrm{d}x$$

which, as we have just seen, is bounded above by a finite quantity – call it C_a – *depending on a but completely independent of n and m* ! The preceding relation thus boils down to

$$\int_0^a |U_n(x) - U_m(x)|\,\mathrm{d}x \quad \leqslant \quad \sqrt{(C_a E(\mathrm{d}\rho_n(t) - \mathrm{d}\rho_m(t),\ \mathrm{d}\rho_n(t) - \mathrm{d}\rho_m(t)))},$$

and the theorem is proved.

Corollary. *Under the hypothesis of the theorem, a* subsequence *of the $U_n(x)$ converges a.e. to a locally integrable odd function $U(x)$ defined a.e. on $\mathbb{R}$. For* *any bounded measurable function φ of compact support in $[0, \infty)$, we have*

$$E(\mathrm{d}\rho_n(t),\ \varphi(t)\,\mathrm{d}t) \quad \underset{n}{\longrightarrow} \quad \int_0^\infty U(x)\,\varphi(x)\,\mathrm{d}x.$$

Proof. The first part of the statement follows by elementary measure theory from the theorem. A standard application of Fatou's lemma then shows that

$$\int_0^a |U(x) - U_n(x)|\,\mathrm{d}x \quad \underset{n}{\longrightarrow} \quad 0$$

for each finite a. Since the *left-hand* member of the limit relation to be proved is just

$$\int_0^\infty U_n(x)\,\varphi(x)\,\mathrm{d}x,$$

we are done.

Remark. Later on, an important generalization of the corollary will be given.

If we *only knew* that the measures $\varphi(t)\,\mathrm{d}t$ formed from *bounded* φ of *compact support* in $[0, \infty)$ were $\sqrt{(E(\ \ ,\ \ ))}$ *dense in the collection of signed measures* $\mathrm{d}\rho(t)$ satisfying the hypothesis of the above lemma, *it would follow* from the results just proved that *any element of that collection's abstract completion in said norm is determined by the measurable function $U(x)$ associated to the element in the way described by the corollary.* The density in question is indeed not too hard to verify; we will not, however, proceed in this manner. Instead, the statement just made will be established as a consequence of a formula to be derived below which, for other reasons, is needed in our work.

Given a measure ρ satisfying the hypothesis of our lemma, we will write

$$U_\rho(x) \quad = \quad \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t).$$

The function $U_\rho(x)$ is thus *odd*, and defined at least a.e. on $\mathbb{R}$. Concerning extension of the function U_ρ to the *complex plane*, we observe that the integral

$$\int_0^\infty \log\left|\frac{z+t}{z-t}\right| \mathrm{d}\rho(t)$$

converges *absolutely and uniformly* for z ranging over any compact subset of $\{\Im z > 0\}$ or of $\{\Im z < 0\}$.

It is sufficient to consider compact subsets K of the half-open quadrant

$$\{z:\ \Re z \geqslant 0\ (sic!)\text{ and } \Im z > 0\}.$$

We have, by the lemma,

$$\int_0^\infty \log\left|\frac{x_0+t}{x_0-t}\right| |d\rho(t)| \quad < \quad \infty$$

for almost all $x_0 > 0$; *fixing any one of them* gives us a number C_K corresponding to the compact subset K such that

$$\log\left|\frac{z+t}{z-t}\right| \quad \leqslant \quad C_K \log\left|\frac{x_0+t}{x_0-t}\right| \qquad \text{for } t > 0 \text{ and } z \in K.$$

The affirmed uniform convergence is now manifest.

The integral

$$\int_0^\infty \log\left|\frac{z+t}{z-t}\right| d\rho(t)$$

is thus very well defined when z lies *off the real axis*; we denote that expression by $U_\rho(z)$. The uniform convergence just established makes $U_\rho(z)$ *harmonic* in both *the upper* and *the lower half planes*. It is, moreover, *odd*, and *vanishes on the imaginary axis*. At real points x where the integral used to define $U_\rho(x)$ is absolutely convergent, we have

$$U_\rho(x) \quad = \quad \lim_{y\to 0} U_\rho(x+iy),$$

so *on* $\mathbb{R}$, the function U_ρ can be regarded as the *boundary data* (existing a.e.) for the *harmonic function* $U_\rho(z)$ defined in *either* of the *half planes* bounded by $\mathbb{R}$.

We turn to the proof of the formula mentioned above which, for measures ρ meeting the conditions of the lemma, enables us to express $E(d\rho(t),\ d\rho(t))$ in terms of ρ's Green potential $U_\rho(x)$. We have the good fortune to *already know what that formula should be*, for, if the *behaviour* of

$$\rho(t) \quad = \quad \int_0^t d\rho(\tau)$$

is *nice enough*, problem 23(a), from the beginning of §B.8, Chapter VIII,