XI

THE GENERAL DISTRIBUTION OF THE VALUES OF $\zeta(s)$

11.1. In the previous chapters we have been concerned almost entirely with the modulus of $\zeta(s)$, and the various values, particularly zero, which it takes. We now consider the problem of $\zeta(s)$ itself, and the values of s for which it takes any given value a.†

One method of dealing with this problem is to connect it with the famous theorem of Picard on functions which do not take certain values. We use the following theorem:‡

If $f(s)$ is regular and never 0 or 1 in $|s-s_0| \leqslant r$, and $|f(s_0)| \leqslant \alpha$, then $|f(s)| \leqslant A(\alpha,\theta)$ for $|s-s_0| \leqslant \theta r$, where $0 < \theta < 1$.

From this we deduce

THEOREM 11.1. *$\zeta(s)$ takes every value, with one possible exception, an infinity of times in any strip $1-\delta < \sigma \leqslant 1+\delta$.*

Suppose, on the contrary, that $\zeta(s)$ takes the distinct values a and b only a finite number of times in the strip, and so *never* above $t = t_0$, say. Let $T > t_0+1$, and consider the function $f(s) = \{\zeta(s)-a\}/(b-a)$ in the circles C, C', of radii $\frac{1}{2}\delta$ and $\frac{1}{4}\delta$ $(0 < \delta < 1)$, and common centre $s_0 = 1+\frac{1}{4}\delta+iT$. Then

$$|f(s_0)| \leqslant \alpha = \{\zeta(1+\tfrac{1}{4}\delta)+|a|\}/|b-a|,$$

and $f(s)$ is never 0 or 1 in C. Hence

$$|f(s)| < A(\alpha)$$

in C', and so $|\zeta(\sigma+iT)| < A(a,b,\alpha)$ for $1 \leqslant \sigma \leqslant 1+\frac{1}{8}\delta$, $T > t_0+1$. Hence $\zeta(s)$ is bounded for $\sigma > 1$, which is false, by Theorem 8.4(A). This proves the theorem.

We should, of course, expect the exceptional value to be 0.

If we assume the Riemann hypothesis, we can use a similar method inside the critical strip; but more detailed results independent of the Riemann hypothesis can be obtained by the method of Diophantine approximation. We devote the rest of the chapter to developments of this method.

† See Bohr (1)–(14), Bohr and Courant (1), Bohr and Jessen (1), (2), (5), Bohr and Landau (3), Borchsenius and Jessen (1), Jessen (1), van Kampen (1), van Kampen and Wintner (1), Kershner (1), Kershner and Wintner (1), (2), Wintner (1)–(4).

‡ See Landau's *Ergebnisse der Funktionentheorie*, § 24, or Valiron's *Integral Functions*, Ch. VI, § 3.

11.2. We restrict ourselves in the first place to the half-plane $\sigma > 1$; and we consider, not $\zeta(s)$ itself, but $\log\zeta(s)$, viz. the function defined for $\sigma > 1$ by the series

$$\log\zeta(s) = -\sum_p (p^{-s}+\tfrac{1}{2}p^{-2s}+\ldots).$$

We consider at the same time the function

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_p \log p(p^{-s}+p^{-2s}+\ldots).$$

We observe that both functions are represented by Dirichlet series, absolutely convergent for $\sigma > 1$, and capable of being written in the form

$$F(s) = f_1(p_1^{-s})+f_2(p_2^{-s})+\ldots,$$

where $f_n(z)$ is a power-series in z whose coefficients do not depend on s. In fact

$$f_n(z) = -\log(1-z), \qquad f_n(z) = z\log p_n/(1-z)$$

in the above two cases. In what follows $F(s)$ denotes either of the two functions.

11.3. We consider first the values which $F(s)$ takes on the line $\sigma = \sigma_0$, where σ_0 is an arbitrary number greater than 1. On this line

$$F(s) = \sum_{n=1}^{\infty} f_n(p_n^{-\sigma_0}e^{-it\log p_n}),$$

and, as t varies, the arguments $-t\log p_n$ are, of course, all related. But we shall see that there is an intimate connexion between the set U of values assumed by $F(s)$ on $\sigma = \sigma_0$ and the set V of values assumed by the function

$$\Phi(\sigma_0,\theta_1,\theta_2,\ldots) = \sum_{n=1}^{\infty} f_n(p_n^{-\sigma_0}e^{2\pi i\theta_n})$$

of an infinite number of independent real variables $\theta_1, \theta_2, \ldots$.

We shall in fact show that *the set U, which is obviously contained in V, is everywhere dense in V, i.e. that corresponding to every value v in V (i.e. to every given set of values $\theta_1, \theta_2, \ldots$) and every positive ϵ, there exists a t such that*

$$|F(\sigma_0+it)-v| < \epsilon.$$

Since the Dirichlet series from which we start is absolutely convergent for $\sigma = \sigma_0$, it is obvious that we can find $N = N(\sigma_0,\epsilon)$ such that

$$\left|\sum_{n=N+1}^{\infty} f_n(p_n^{-\sigma_0}e^{2\pi i\mu_n})\right| < \tfrac{1}{3}\epsilon \tag{11.3.1}$$

for any values of the μ_n, and in particular for $\mu_n = \theta_n$, or for

$$\mu_n = -(t\log p_n)/2\pi.$$

Now since the numbers $\log p_n$ are linearly independent, we can, by Kronecker's theorem, find a number t and integers $g_1, g_2,..., g_N$ such that

$$|-t\log p_n-2\pi\theta_n-2\pi g_n| < \eta \quad (n = 1, 2,..., N),$$

η being an assigned positive number. Since $f_n(p_n^{-\sigma_0}e^{2\pi i\theta})$ is, for each n, a continuous function of θ, we can suppose η so small that

$$\left|\sum_{n=1}^{N}\{f_n(p_n^{-\sigma_0}e^{2\pi i\theta_n})-f_n(p_n^{-\sigma_0}e^{-it\log p_n})\}\right| < \tfrac{1}{3}\epsilon. \tag{11.3.2}$$

The result now follows from (11.3.1) and (11.3.2).

11.4. We next consider the set W of values which $F(s)$ takes 'in the immediate neighbourhood' of the line $\sigma = \sigma_0$, i.e. the set of all values of w such that the equation $F(s) = w$ has, for every positive δ, a root in the strip $|\sigma-\sigma_0| < \delta$.

In the first place, it is evident that U is contained in W. Further, it is easy to see that U is everywhere dense in W. For, for sufficiently small δ (e.g. for $\delta < \frac{1}{2}(\sigma_0-1)$),

$$|F'(s)| < K(\sigma_0)$$

for all values of s in the strip $|\sigma-\sigma_0| < \delta$, so that

$$|F(\sigma_0+it)-F(\sigma_1+it)| < K(\sigma_0)|\sigma_1-\sigma_0| \quad (|\sigma_1-\sigma_0| < \delta). \tag{11.4.1}$$

Now each value w in W is assumed by $F(s)$ either on the line $\sigma = \sigma_0$, in which case it is a u, or at points σ_1+it arbitrarily near the line, in which case, in virtue of (11.4.1), we can find a u such that

$$|w-u| < K(\sigma_0)|\sigma_1-\sigma_0| < \epsilon.$$

We now proceed to prove that W *is identical with* V. Since U is contained in and is everywhere dense in both V and W, it follows that each of V and W is everywhere dense in the other. It is therefore obvious that W is contained in V, if V is closed.

We shall see presently that much more than this is true, viz. that V consists of all points of an area, including the boundary. The following direct proof that V is closed is, however, very instructive.

Let v^* be a limit-point of V, and let v_ν ($\nu = 1, 2,...$) be a sequence of v's tending to v^*. To each v_ν corresponds a point $P_\nu(\theta_{1,\nu}, \theta_{2,\nu},...)$ in the space of an infinite number of dimensions defined by $0 \leqslant \theta_{n,\nu} < 1$ ($n = 1, 2,...$), such that $\Phi(\sigma_0, \theta_{1,\nu},...) = v_\nu$.

Now since (P_ν) is a bounded set of points (i.e. all the coordinates are bounded), it has a limit-point P^* $(\theta_1^*, \theta_2^*,...)$, i.e. a point such that from (P_ν) we can choose a sequence (P_{ν_r}) such that each coordinate θ_{n,ν_r} of P_{ν_r} tends to the limit θ_n^* as $r \to \infty$.

It is now easy to prove that P^* corresponds to v^*, i.e. that

$$\Phi(\sigma_0, \theta_1^*,...) = v^*,$$

so that v^* is a point of V. For the series for v_{ν_r}, viz.

$$\sum_{n=1}^{\infty} f_n(p_n^{-\sigma_0}e^{2\pi i\theta_{n,\nu_r}}),$$

is uniformly convergent with respect to r, since (by Weierstrass's M-test) it is uniformly convergent with respect to all the θ's; further, the nth term tends to $f_n(p_n^{-\sigma_0}e^{2\pi i\theta_n^*})$ as $r \to \infty$. Hence

$$v^* = \lim_{r\to\infty} v_{\nu_r} = \lim_{r\to\infty}\sum_{n=1}^{\infty} f_n(p_n^{-\sigma_0}e^{2\pi i\theta_{n,\nu_r}}) = \Phi(\sigma_0, \theta_1^*,...),$$

which proves our result.

To establish the identity of V and W it remains to prove that V is contained in W. It is obviously sufficient (and also necessary) for this that W should be closed. But that W is closed does not follow, as might perhaps be supposed, from the mere fact that W is the set of values taken by a bounded analytic function in the immediate neighbourhood of a line. Thus e^{-z^2} is bounded and arbitrarily near to 0 in every strip including the real axis, but never actually assumes the value 0. The fact that W is closed (which we shall not prove directly) depends on the special nature of the function $F(s)$.

Let $v = \Phi(\sigma_0, \theta_1, \theta_2,...)$ be an arbitrary value contained in V. We have to show that v is a member of W, i.e. that, in every strip

$$|\sigma-\sigma_0| < \delta,$$

$F(s)$ assumes the value v.

Let
$$G(s) = \sum_{n=1}^{\infty} f_n(p_n^{-s}e^{2\pi i\theta_n}),$$

so that $G(\sigma_0) = v$. We choose a small circle C with centre σ_0 and radius less than δ such that $G(s) \neq v$ on the circumference. Let m be the minimum of $|G(s)-v|$ on C.

Kronecker's theorem enables us to choose t_0 such that, for every s in C,

$$|F(s+it_0)-G(s)| < m.$$

The proof is almost exactly the same as that used to show that U is everywhere dense in V. The series for $F(s)$ and $G(s)$ are uniformly convergent in the strip, and, for each fixed N, $\sum_1^N f_n(p_n^{-\sigma}e^{2\pi i\mu_n})$ is a continuous function of $\sigma, \mu_1,..., \mu_N$. It is therefore sufficient to show that we can choose t_0 so that the difference between the arguments of p_n^{-s} at $s = \sigma_0+it_0$ and $p_n^{-s}e^{2\pi i\theta_n}$ at $s = \sigma_0$, and consequently that between the respective arguments at every pair of corresponding points of the two circles is (mod 2π) arbitrarily small for $n = 1, 2,..., N$. The possibility of this choice follows at once from Kronecker's theorem.

We now have
$$F(s+it_0)-v = \{G(s)-v\}+\{F(s+it_0)-G(s)\},$$
and on the circumference of C
$$|F(s+it_0)-G(s)| < m \leqslant |G(s)-v|.$$
Hence, by Rouché's theorem, $F(s+it_0)-v$ has in C the same number of zeros as $G(s)-v$, and so at least one. This proves the theorem.

11.5. We now proceed to the study of the set V. Let V_n be the set of values taken by $f_n(p_n^{-s})$ for $\sigma = \sigma_0$, i.e. the set taken by $f_n(z)$ for $|z| = p_n^{-\sigma_0}$. Then V is the 'sum' of the sets of points $V_1, V_2,...$, i.e. it is the set of all values $v_1+v_2+...$, where v_1 is any point of V_1, v_2 any point of V_2, and so on. For the function $\log\zeta(s)$, V_n consists of the points of the curve described by $-\log(1-z)$ as z describes the circle $|z| = p_n^{-\sigma_0}$; for $\zeta'(s)/\zeta(s)$ it consists of the points of the curve described by
$$-(z\log p_n)/(1-z).$$

We begin by considering the function $\zeta'(s)/\zeta(s)$. In this case we can find the set V explicitly. Let
$$w_n = -\frac{z_n \log p_n}{1-z_n}.$$
As z_n describes the circle $|z_n| = p_n^{-\sigma_0}$, w_n describes the circle with centre
$$c_n = -\frac{p_n^{-2\sigma_0}\log p_n}{1-p_n^{-2\sigma_0}}$$
and radius
$$\rho_n = \frac{p_n^{-\sigma_0}\log p_n}{1-p_n^{-2\sigma_0}}.$$
Let
$$w_n = c_n+w_n' = c_n+\rho_n e^{i\phi_n},$$
and let
$$c = \sum_{n=1}^{\infty} c_n = \frac{\zeta'(2\sigma_0)}{\zeta(2\sigma_0)}.$$
Then V is the set of all the values of
$$c+\sum_{n=1}^{\infty}\rho_n e^{i\phi_n}$$
for independent $\phi_1, \phi_2,...$. The set V' of the values of $\sum\rho_n e^{i\phi_n}$ is the 'sum' of an infinite number of circles with centre at the origin, whose radii $\rho_1, \rho_2,...$ form, as it is easy to see, a decreasing sequence. Let V_n' denote the nth circle.

Then $V_1'+V_2'$ is the area swept out by the circle of radius ρ_2 as its centre describes the circle with centre the origin and radius ρ_1. Hence, since $\rho_2 < \rho_1$, $V_1'+V_2'$ is the annulus with radii $\rho_1-\rho_2$ and $\rho_1+\rho_2$.

The argument clearly extends to any finite number of terms. Thus $V_1'+...+V_N'$ consists of all points of the annulus
$$\rho_1-\sum_{n=2}^{N}\rho_n \leqslant |w| \leqslant \sum_{n=1}^{N}\rho_n,$$
or, if the left-hand side is negative, of the circle
$$|w| \leqslant \sum_{n=1}^{N}\rho_n.$$
It is now easy to see that

(i) *if $\rho_1 > \rho_2+\rho_3+...$, the set V' consists of all points w of the annulus*
$$\rho_1-\sum_{n=2}^{\infty}\rho_n \leqslant |w| \leqslant \sum_{n=1}^{\infty}\rho_n;$$

(ii) *if $\rho_1 \leqslant \rho_2+\rho_3+...$, V' consists of all points w for which*
$$|w| \leqslant \sum_{n=1}^{\infty}\rho_n.$$

For example, in case (ii), let w_0 be an interior point of the circle. Then we can choose N so large that
$$\sum_{N+1}^{\infty}\rho_n < \sum_{n=1}^{N}\rho_n-|w_0|.$$
Hence
$$w_1 = w_0-\sum_{N+1}^{\infty}\rho_n e^{i\phi_n}$$
lies within the circle $V_1'+...+V_N'$ for any values of the ϕ_n, e.g. for $\phi_{N+1} = ... = 0$. Hence
$$w_1 = \sum_{n=1}^{N}\rho_n e^{i\phi_n}$$
for some values of $\phi_1,..., \phi_n$, and so
$$w_0 = \sum_{n=1}^{\infty}\rho_n e^{i\phi_n}$$
as required. That V' also includes the boundary in each case is clear on taking all the ϕ_n equal.

The complete result is that there is an absolute constant $D = 2{\cdot}57...$, determined as the root of the equation
$$\frac{2^{-D}\log 2}{1-2^{-2D}} = \sum_{n=2}^{\infty}\frac{p_n^{-D}\log p_n}{1-p_n^{-2D}},$$

such that for $\sigma_0 > D$ we are in case (i), and for $1 < \sigma_0 \leqslant D$ we are in case (ii). The radius of the outer boundary of V' is

$$R = \frac{\zeta'(2\sigma_0)}{\zeta(2\sigma_0)} - \frac{\zeta'(\sigma_0)}{\zeta(\sigma_0)}$$

in each case; the radius of the inner boundary in case (i) is

$$r = 2\rho_1 - R = 2^{1-\sigma_0}\log 2/(1-2^{-2\sigma_0}) - R.$$

Summing up, we have the following results for $\zeta'(s)/\zeta(s)$.

THEOREM 11.5 (A). *The values which $\zeta'(s)/\zeta(s)$ takes on the line $\sigma = \sigma_0 > 1$ form a set everywhere dense in a region $R(\sigma_0)$. If $\sigma_0 > D$, $R(\sigma_0)$ is the annulus (boundary included) with centre c and radii r and R; if $\sigma_0 \leqslant D$, $R(\sigma_0)$ is the circular area (boundary included) with centre c and radius R; c, r, and R are continuous functions of σ_0 defined by*

$$c = \zeta'(2\sigma_0)/\zeta(2\sigma_0), \quad R = c - \zeta'(\sigma_0)/\zeta(\sigma_0), \quad r = 2^{1-\sigma_0}\log 2/(1-2^{-2\sigma_0}) - R.$$

Further, as $\sigma_0 \to \infty$,

$$\lim c = \lim r = \lim R = 0, \qquad \lim c/R = \lim (R-r)/R = 0;$$

as $\sigma_0 \to D$, $\lim r = 0$; and as $\sigma_0 \to 1$, $\lim R = \infty$, $\lim c = \zeta'(2)/\zeta(2)$.

THEOREM 11.5 (B). *The set of values which $\zeta'(s)/\zeta(s)$ takes in the immediate neighbourhood of $\sigma = \sigma_0$ is identical with $R(\sigma_0)$. In particular, since c tends to a finite limit and R to infinity as $\sigma_0 \to 1$, $\zeta'(s)/\zeta(s)$ takes all values infinitely often in the strip $1 < \sigma < 1+\delta$, for an arbitrary positive δ.*

The above results evidently enable us to study the set of points at which $\zeta'(s)/\zeta(s)$ takes the assigned value a. We confine ourselves to giving the result for $a = 0$; this is the most interesting case, since the zeros of $\zeta'(s)/\zeta(s)$ are identical with those of $\zeta'(s)$.

THEOREM 11.5 (C). *There is an absolute constant E, between 2 and 3, such that $\zeta'(s) \neq 0$ for $\sigma > E$, while $\zeta'(s)$ has an infinity of zeros in every strip between $\sigma = 1$ and $\sigma = E$.*

In fact it is easily verified that the annulus $R(\sigma_0)$ includes the origin if $\sigma_0 = 2$, but not if $\sigma_0 = 3$.

11.6. We proceed now to the study of $\log \zeta(s)$. In this case the set V consists of the 'sum' of the curves V_n described by the points

$$w_n = -\log(1-z_n)$$

as z_n describes the circle $|z_n| = p_n^{-\sigma_0}$.

In the first place, V_n is a convex curve. For if

$$u+iv = w = f(z) = f(x+iy),$$

and z describes the circle $|z| = r$, then

$$\frac{du}{dx} + i\frac{dv}{dx} = f'(z)\left(1+i\frac{dy}{dx}\right) = f'(z)\frac{x+iy}{iy}.$$

Hence $$\arctan\frac{dv}{du} = \arg\{zf'(z)\} - \tfrac{1}{2}\pi.$$

A sufficient condition that w should describe a convex curve as z describes $|z| = r$ is that the tangent to the path of w should rotate steadily through 2π as z describes the circle, i.e. that $\arg\{zf'(z)\}$ should increase steadily through 2π. This condition is satisfied in the case $f(z) = -\log(1-z)$; for $zf'(z) = z/(1-z)$ describes a circle enclosing the origin as z describes $|z| = r < 1$.

If $z = re^{i\theta}$, and $w = -\log(1-z)$, then

$$u = -\tfrac{1}{2}\log(1-2r\cos\theta+r^2), \qquad v = \arctan\frac{r\sin\theta}{1-r\cos\theta}.$$

The second equation leads to

$$r\cos\theta = \sin^2 v \pm \cos v(r^2-\sin^2 v)^{\frac{1}{2}}.$$

Hence, for real r and θ, $|v| \leqslant \arcsin r$. If $\cos\theta_1$ and $\cos\theta_2$ are the two values of $\cos\theta$ corresponding to a given v,

$$(1-2r\cos\theta_1+r^2)(1-2r\cos\theta_2+r^2) = (1-r^2)^2.$$

Hence if u_1 and u_2 are the corresponding values of u,

$$u_1+u_2 = -\log(1-r^2).$$

The curve V_n is therefore convex and symmetrical about the lines

$$u = -\tfrac{1}{2}\log(1-r^2) \quad \text{and} \quad v = 0.$$

Its diameters in the u and v directions are $\frac{1}{2}\log\{(1+r)/(1-r)\}$ and $\arcsin r$.

Let $$c_n = -\tfrac{1}{2}\log(1-p_n^{-2\sigma_0})$$

and $$w_n = c_n + w_n',$$

$$c = \sum_{n=1}^{\infty} c_n = \tfrac{1}{2}\log\zeta(2\sigma_0).$$

Then the points w_n' describe symmetrical convex figures with centre the origin. Let V' be the 'sum' of these figures.

It is now easy, by analogy with the previous case, to imagine the result. *The set V', which is plainly symmetrical about both axes, is either* (i) *the region bounded by two convex curves, one of which is entirely interior to the other, or* (ii) *the region bounded by a single convex curve. In each case the boundary is included as part of the region.*

This follows from a general theorem of Bohr on the 'summation' of a series of convex curves.

For our present purpose the following weaker but more obvious results will be sufficient. The set V' is included in the circle with centre the origin and radius

$$R = \sum_{n=1}^{\infty} \tfrac{1}{2}\log\frac{1+p_n^{-\sigma_0}}{1-p_n^{-\sigma_0}} = \tfrac{1}{2}\log\frac{\zeta^2(\sigma_0)}{\zeta(2\sigma_0)}.$$

If σ_0 is sufficiently large, V' lies entirely outside the circle of radius

$$\arcsin 2^{-\sigma_0} - \sum_{n=2}^{\infty} \tfrac{1}{2}\log\frac{1+p_n^{-\sigma_0}}{1-p_n^{-\sigma_0}} = \arcsin 2^{-\sigma_0} + \tfrac{1}{2}\log\frac{1+2^{-\sigma_0}}{1-2^{-\sigma_0}} - R.$$

If
$$\sum_{n=2}^{\infty} \arcsin p_n^{-\sigma_0} > \tfrac{1}{2}\log\frac{1+2^{-\sigma_0}}{1-2^{-\sigma_0}},$$

and so if σ_0 is sufficiently near to 1, V' includes all points inside the circle of radius

$$\sum_{n=1}^{\infty} \arcsin p_n^{-\sigma_0}.$$

In particular V' includes any given area, however large, if σ_0 is sufficiently near to 1.

We cannot, as in the case of circles, determine in all circumstances whether we are in case (i) or case (ii). It is not obvious, for example, whether there exists an absolute constant D' such that we are in case (i) or (ii) according as $\sigma_0 > D'$ or $1 < \sigma_0 \leqslant D'$. The discussion of this point demands a closer investigation of the geometry of the special curves with which we are dealing, and the question would appear to be one of considerable intricacy.

The relations between U, V, and W now give us the following analogues for $\log\zeta(s)$ of the results for $\zeta'(s)/\zeta(s)$.

THEOREM 11.6 (A). *On each line $\sigma = \sigma_0 > 1$ the values of* $\log\zeta(s)$ *are everywhere dense in a region $R(\sigma_0)$ which is either* (i) *the ring-shaped area bounded by two convex curves, or* (ii) *the area bounded by one convex curve. For sufficiently large values of σ_0 we are in case* (i), *and for values of σ_0 sufficiently near to* 1 *we are in case* (ii).

THEOREM 11.6 (B). *The set of values which* $\log\zeta(s)$ *takes in the immediate neighbourhood of $\sigma = \sigma_0$ is identical with $R(\sigma_0)$. In particular, since $R(\sigma_0)$ includes any given finite area when σ_0 is sufficiently near* 1, $\log\zeta(s)$ *takes every value an infinity of times in* $1 < \sigma < 1+\delta$.

As a consequence of the last result, we have

THEOREM 11.6 (C). *the function $\zeta(s)$ takes every value except* 0 *an infinity of times in the strip* $1 < \sigma < 1+\delta$.

This is a more precise form of Theorem 11.1.

11.7. We have seen above that $\log\zeta(s)$ takes any assigned value a an infinity of times in $\sigma > 1$. It is natural to raise the question *how often* the value a is taken, i.e. the question of the behaviour for large T of the number $M_a(T)$ of roots of $\log\zeta(s) = a$ in $\sigma > 1$, $0 < t < T$. This question is evidently closely related to the question as to how often, as $t \to \infty$, the point $(a_1 t, a_2 t,..., a_N t)$ of Kronecker's theorem, which, in virtue of the theorem, comes (mod 1) arbitrarily near every point in the N-dimensional unit cube, comes within a given distance of an assigned point $(b_1, b_2,..., b_N)$. The answer to this last question is given by the following theorem, which asserts that, roughly speaking, the point $(a_1 t,..., a_N t)$ comes near every point of the unit cube equally often, i.e. it does not give a preference to any particular region of the unit cube.

Let $a_1,..., a_N$ be linearly independent, and let γ be a region of the N-dimensional unit cube with volume Γ (in the Jordan sense). Let $I_\gamma(T)$ be the sum of the intervals between $t = 0$ and $t = T$ for which the point P $(a_1 t,..., a_N t)$ is (mod 1) *inside γ. Then*

$$\lim_{T\to\infty} I_\gamma(T)/T = \Gamma.$$

The region γ is said to have the volume Γ in the Jordan sense, if, given ϵ, we can find two sets of cubes with sides parallel to the axes, of volumes Γ_1 and Γ_2, included in and including γ respectively, such that

$$\Gamma_1 - \epsilon \leqslant \Gamma \leqslant \Gamma_2 + \epsilon.$$

If we call a point with coordinates of the form $(a_1 t,..., a_N t)$, mod 1, an 'accessible' point, Kronecker's theorem states that the accessible points are everywhere dense in the unit cube C. If now γ_1, γ_2 are two equal cubes with sides parallel to the axes, and with centres at accessible points P_1 and P_2, corresponding to t_1 and t_2, it is easily seen that

$$\lim I_{\gamma_1}(T)/I_{\gamma_2}(T) = 1.$$

For $(a_1 t,..., a_N t)$ will lie inside γ_2 when and only when $\{a_1(t+t_2-t_1),...\}$ lies inside γ_1.

Consider now a set of p non-overlapping cubes c, inside C, of side ϵ, each of which has its centre at an accessible point, and q of which lie inside γ; and a set of P overlapping cubes c', also centred on accessible points, whose union includes C and such that γ is included in a union of Q of them. Since the accessible points are everywhere dense, it is possible to choose the cubes such that q/P and Q/p are arbitrarily near to Γ. Now, denoting by $\sum_\gamma I_c(T)$ the sum of t-intervals in $(0, T)$ corresponding to the cubes c which lie in γ, and so on,

$$\sum_\gamma I_c(T) \Big/ \sum_C I_{c'}(T) \leqslant \frac{I_\gamma(T)}{T} \leqslant \sum_\gamma I_{c'}(T) \Big/ \sum_C I_c(T).$$

Making $T \to \infty$ we obtain

$$\frac{q}{P} \leqslant \overline{\lim_{T\to\infty}} \frac{I_\gamma(T)}{T} \leqslant \frac{Q}{p},$$

and the result follows.

11.8. We can now prove

THEOREM 11.8 (A). *If $\sigma = \sigma_0 > 1$ is a line on which $\log\zeta(s)$ comes arbitrarily near to a given number a, then in every strip $\sigma_0-\delta < \sigma < \sigma_0+\delta$ the value a is taken more than $K(a, \sigma_0, \delta)T$ times, for large T, in $0 < t < T$.*

To prove this we have to reconsider the argument of the previous sections, used to establish the existence of a root of $\log\zeta(s) = a$ in the strip, and use Kronecker's theorem in its generalized form. We saw that a sufficient condition that $\log\zeta(s) = a$ may have a root inside a circle with centre σ_0+it_0 and radius 2δ is that, for a certain N and corresponding numbers $\theta_1,\ldots, \theta_N$, and a certain $\eta = \eta(\sigma_0, \delta, \theta_1,\ldots, \theta_N)$

$$|-t_0\log p_n - 2\pi\theta_n - 2\pi g_n| < \eta \quad (n = 1, 2,\ldots, N).$$

From the generalized Kronecker's theorem it follows that the sum of the intervals between 0 and T in which t_0 satisfies this condition is asymptotically equal to $(\eta/2\pi)^N T$, and it is therefore greater than $\frac{1}{2}(\eta/2\pi)^N T$ for large T. Hence we can select more than $\frac{1}{8}(\eta/2\pi)^N T/\delta$ numbers t_0' in them, no two of which differ by less than 4δ. If now we describe circles with the points σ_0+it_0' as centres and radius 2δ, these circles will not overlap, and each of them will contain a zero of $\log\zeta(s)-a$. This gives the desired result.

We can also prove

THEOREM 11.8 (B). *There are positive constants $K_1(a)$ and $K_2(a)$ such that the number $M_a(T)$ of zeros of $\log\zeta(s)-a$ in $\sigma > 1$ satisfies the inequalities*

$$K_1(a)T < M_a(T) < K_2(a)T.$$

The lower bound follows at once from the above theorem. The upper bound follows from the more general result that *if b is any given constant, the number of zeros of $\zeta(s)-b$ in $\sigma > \frac{1}{2}+\delta$ $(\delta > 0)$, $0 < t < T$, is $O(T)$ as $T \to \infty$.*

The proof of this is substantially the same as that of Theorem 9.15 (A), the function $\zeta(s)-b$ playing the same part as $\zeta(s)$ did there. Finally the number of zeros of $\log\zeta(s)-a$ is not greater than the number of zeros of $\zeta(s)-e^a$, and so is $O(T)$.

11.9. We now turn to the more difficult question of the behaviour of $\zeta(s)$ in the critical strip. The difficulty, of course, is that $\zeta(s)$ is no longer represented by an absolutely convergent Dirichlet series. But by a device like that used in the proof of Theorem 9.17, we are able to obtain in the critical strip results analogous to those already obtained in the region of absolute convergence.

As before we consider $\log\zeta(s)$. For $\sigma \leqslant 1$, $\log\zeta(s)$ is defined, on each line $t =$ constant which does not pass through a singularity, by continuation along this line from $\sigma > 1$.

We require the following lemma.

LEMMA. *If $f(z)$ is regular for $|z-z_0| \leqslant R$, and*

$$\iint_{|z-z_0|\leqslant R} |f(z)|^2\, dxdy = H,$$

then

$$|f(z)| \leqslant \frac{(H/\pi)^{\frac{1}{2}}}{R-R'} \quad (|z-z_0| \leqslant R' < R).$$

For if $|z'-z_0| \leqslant R'$,

$$\{f(z')\}^2 = \frac{1}{2\pi i}\int_{|z-z'|=r} \frac{\{f(z)\}^2}{z-z'}dz = \frac{1}{2\pi}\int_0^{2\pi} \{f(z'+re^{i\theta})\}^2\, d\theta.$$

Hence

$$|f(z')|^2 \int_0^{R-R'} r\, dr \leqslant \frac{1}{2\pi}\int_0^{R-R'}\int_0^{2\pi} |f(z'+re^{i\theta})|^2 r\, drd\theta \leqslant \frac{H}{2\pi},$$

and the result follows.

THEOREM 11.9. *Let σ_0 be a fixed number in the range $\frac{1}{2} < \sigma \leqslant 1$. Then the values which $\log\zeta(s)$ takes on $\sigma = \sigma_0$, $t > 0$, are everywhere dense in the whole plane.*

Let

$$\zeta_N(s) = \zeta(s)\prod_{n=1}^{N}(1-p_n^{-s}).$$

This function is similar to the function $\zeta(s)M_X(s)$ of Chapter IX, but it happens to be more convenient here.

Let δ be a positive number less than $\frac{1}{2}(\sigma_0-\frac{1}{2})$. Then it is easily seen as in § 9.19 that for $N \geqslant N_0(\sigma_0, \epsilon)$, $T \geqslant T_0 = T_0(N)$,

$$\int_1^T |\zeta_N(\sigma+it)-1|^2\, dt < \epsilon T$$

uniformly for $\sigma_0-\delta \leqslant \sigma \leqslant \sigma_1+\delta$ $(\sigma_1 > 1)$. Hence

$$\int_1^T \int_{\sigma_0-\delta}^{\sigma_1+\delta} |\zeta_N(\sigma+it)-1|^2\, d\sigma dt < (\sigma_1-\sigma_0+2\delta)\epsilon T.$$

Hence

$$\int_{\nu-\frac{1}{2}}^{\nu+\frac{1}{2}} \int_{\sigma_0-\delta}^{\sigma_1+\delta} |\zeta_N(\sigma+it)-1|^2\, d\sigma dt < (\sigma_1-\sigma_0+2\delta)\sqrt{\epsilon}$$

for more than $(1-\sqrt{\epsilon})T$ integer values of ν. Since this rectangle contains the circle with centre $s = \sigma+it$, where $\sigma_0 \leqslant \sigma \leqslant \sigma_1$, $\nu-\frac{1}{2}+\delta \leqslant t \leqslant \nu+\frac{1}{2}-\delta$, and radius δ, it is easily seen from the lemma that we can choose δ and ϵ so that *given* $0 < \eta < 1$, $0 < \eta' < 1$, *we have*

$$|\zeta_N(\sigma+it)-1| < \eta \quad (\sigma_0 \leqslant \sigma \leqslant \sigma_1) \tag{11.9.1}$$

for a set of values of t of measure greater than $(1-\eta')T$, *and for*

$$N \geqslant N_0(\sigma, \eta, \eta'), \qquad T \geqslant T_0(N).$$

Let
$$R_N(s) = -\sum_{N+1}^{\infty} \mathrm{Log}(1-p_n^{-s}) \quad (\sigma > 1),$$

where Log denotes the principal value of the logarithm. Then

$$\zeta_N(s) = \exp\{R_N(s)\}.$$

We want to show that $R_N(s) = \mathrm{Log}\,\zeta_N(s)$, i.e. that $|\mathbf{I}R_N(s)| < \frac{1}{2}\pi$, for $\sigma \geqslant \sigma_0$ and the values of t for which (11.9.1) holds. This is true for $\sigma = \sigma_1$ if σ_1 is sufficiently large, since $|R_N(s)| \to 0$ as $\sigma_1 \to \infty$. Also, by (11.9.1), $\mathbf{R}\zeta_N(s) > 0$ for $\sigma_0 \leqslant \sigma \leqslant \sigma_1$, so that $\mathbf{I}R_N(s)$ must remain between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ for all values of σ in this interval. This gives the desired result.

We have therefore

$$|R_N(s)| = |\mathrm{Log}[1+\{\zeta_N(s)-1\}]| < 2\,|\zeta_N(s)-1| < 2\eta$$

for $\sigma_0 \leqslant \sigma \leqslant \sigma_1$, $N \geqslant N_0(\sigma_0, \eta, \eta')$, $T \geqslant T_0(N)$, in a set of values of t of measure greater than $(1-\eta')T$.

Now consider the function

$$F_N(\sigma_0+it) = -\sum_{n=1}^{N} \log(1-p_n^{-\sigma_0-it}),$$

and in conjunction with it the function of N independent variables

$$\Phi_N(\theta_1,\ldots,\theta_N) = -\sum_{n=1}^{N} \log(1-p_n^{-\sigma_0}e^{2\pi i\theta_n}).$$

Since $\sum p_n^{-\sigma_0}$ is divergent, it is easily seen from our previous discussion of the values taken by $\log\zeta(s)$ that the set of values of Φ_N includes any given finite region of the complex plane if N is large enough. In particular, if a is any given number, we can find a number N and values of the θ's such that

$$\Phi_N(\theta_1,\ldots,\theta_N) = a.$$

We can then, by Kronecker's theorem, find a number t such that $|F_N(\sigma_0+it)-a|$ is arbitrarily small. But this in itself is not sufficient to prove the theorem, since this value of t does not necessarily make $|R_N(s)|$ small. An additional argument is therefore required.

Let

$$\Phi_{M,N} = -\sum_{n=M+1}^{N} \log(1-p_n^{-\sigma_0}e^{2\pi i\theta_n}) = \sum_{n=M+1}^{N}\sum_{m=1}^{\infty} \frac{p_n^{-m\sigma_0}e^{2\pi im\theta_n}}{m}.$$

Then, expressing the squared modulus of this as the product of conjugates, and integrating term by term, we obtain

$$\int_0^1\int_0^1\cdots\int_0^1 |\Phi_{M,N}|^2\,d\theta_{M+1}\ldots d\theta_N = \sum_{n=M+1}^{N}\sum_{m=1}^{\infty}\frac{p_n^{-2m\sigma_0}}{m^2}$$
$$< \sum_{n=M+1}^{N} p_n^{-2\sigma_0}\sum_{m=1}^{\infty}\frac{1}{m^2} < A\sum_{n=M+1}^{\infty} p_n^{-2\sigma_0},$$

which can be made arbitrarily small, by choice of M, for all N. It therefore follows from the theory of Riemann integration of a continuous function that, given ϵ, we can divide up the $(N-M)$-dimensional unit cube into sub-cubes q_ν, each of volume λ, in such a way that

$$\lambda\sum_\nu \max_{q_\nu} |\Phi_{M,N}|^2 < \tfrac{1}{2}\epsilon^2.$$

Hence *for* $M \geqslant M_0(\epsilon)$ *and any* $N > M$, *we can find cubes of total volume greater than* $\frac{1}{2}$ *in which* $|\Phi_{M,N}| < \epsilon$.

We now choose our value of t as follows.

(i) Choose M so large, and give $\theta_1',\ldots,\theta_M'$ such values, that

$$\Phi_M(\theta_1',\ldots,\theta_M') = a.$$

It then follows from considerations of continuity that, given ϵ, we can find an M-dimensional cube with centre $\theta_1',\ldots,\theta_M'$ and side $d > 0$ throughout which

$$|\Phi_M(\theta_1,\ldots,\theta_M)-a| < \tfrac{1}{3}\epsilon.$$

(ii) We may also suppose that M has been chosen so large that, for any value of N, $|\Phi_{M,N}| < \frac{1}{3}\epsilon$ in certain $(N-M)$-dimensional cubes of total volume greater than $\frac{1}{2}$.

(iii) Having fixed M and d, we can choose N so large that, for $T > T_0(N)$, the inequality $|R_N(s)| < \frac{1}{3}\epsilon$ holds in a set of values of t of measure greater than $(1-\frac{1}{2}d^M)T$.

(iv) Let $I(T)$ be the sum of the intervals between 0 and T for which the point

$$\{-(t\log p_1)/2\pi,\ldots,-(t\log p_N)/2\pi\}$$

is (mod 1) inside one of the N-dimensional cubes, of total volume greater than $\frac{1}{2}d^M$, determined by the above construction. Then by the extended Kronecker's theorem, $I(T) > \frac{1}{2}d^MT$ if T is large enough. There are therefore values of t for which the point lies in one of these cubes, and for which at the same time $|R_N(s)| < \frac{1}{3}\epsilon$. For such a value of t

$$\begin{aligned}|\log\zeta(s)-a| &\leqslant |F_N(s)-a|+|R_N(s)|\\ &\leqslant |\Phi_M(\theta_1,\ldots,\theta_M)-a|+|\Phi_{M,N}|+|R_N(s)|\\ &< \tfrac{1}{3}\epsilon+\tfrac{1}{3}\epsilon+\tfrac{1}{3}\epsilon = \epsilon,\end{aligned}$$

and the result follows.

11.10. THEOREM 11.10. *Let $\frac{1}{2} < \alpha < \beta < 1$, and let a be any complex number. Let $M_{a,\alpha,\beta}(T)$ be the number of zeros of $\log\zeta(s)-a$ (defined as before) in the rectangle $\alpha < \sigma < \beta$, $0 < t < T$. Then there are positive constants $K_1(a,\alpha,\beta)$, $K_2(a,\alpha,\beta)$ such that*

$$K_1(a,\alpha,\beta)T < M_{a,\alpha,\beta}(T) < K_2(a,\alpha,\beta)T \quad (T > T_0).$$

We first observe that, for suitable values of the θ's, the series

$$-\sum_{n=1}^{\infty}\log(1-p_n^{-s}e^{2\pi i\theta_n})$$

is uniformly convergent in any finite region to the right of $\sigma = \frac{1}{2}$. This is true, for example, if $\theta_n = \frac{1}{2}n$ for sufficiently large values of n; for then

$$\sum_{n>n_0} p_n^{-s}e^{2\pi i\theta_n} = \sum_{n>n_0}(-1)^n p_n^{-s},$$

which is convergent for real $s > 0$, and hence uniformly convergent in any finite region to the right of the imaginary axis; and for any θ's $\sum |p_n^{-s}e^{2\pi i\theta_n}|^2 = \sum p_n^{-2\sigma}$ is uniformly convergent in any finite region to the right of $\sigma = \frac{1}{2}$.

If a is any given number, and the θ's have this property, we can choose n_1 so large that

$$\left|-\sum_{n=n_1+1}^{\infty}\log(1-p_n^{-s}e^{2\pi i\theta_n})\right| < \epsilon \quad (\sigma = \tfrac{1}{2}(\alpha+\beta)),$$

and at the same time so that the set of values of

$$-\sum_{n=1}^{n_1}\log(1-p_n^{-\frac{1}{2}\alpha-\frac{1}{2}\beta}e^{2\pi i\theta_n})$$

includes the circle with centre the origin and radius $|a|+|\epsilon|$. Hence by choosing first $\theta_{n_1+1},\ldots$, and then $\theta_1,\ldots,\theta_{n_1}$, we can find values of the θ's, say θ_1', $\theta_2',\ldots$, such that the series

$$G(s) = -\sum_{n=1}^{\infty}\log(1-p_n^{-s}e^{2\pi i\theta_n'})$$

is uniformly convergent in any finite region to the right of $\sigma = \frac{1}{2}$, and

$$G(\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta) = a.$$

We can then choose a circle C of centre $\frac{1}{2}\alpha+\frac{1}{2}\beta$ and radius $\rho < \frac{1}{4}(\beta-\alpha)$ on which $G(s) \neq a$.

Let
$$m = \min_{s \text{ on } C}|G(s)-a|.$$

Now let
$$\Phi_{M,N}(s) = -\sum_{n=M+1}^{N}\log(1-p_n^{-s}e^{2\pi i\theta_n}).$$

Then, as in the previous proof,

$$\int_0^1 \cdots \int_0^1 \iint_{|s-\frac{1}{2}\alpha-\frac{1}{2}\beta|\leqslant\frac{1}{4}(\beta-\alpha)} |\Phi_{M,N}(s)|^2\, d\theta_{M+1}\ldots d\theta_N\, d\sigma dt < A\sum_{M+1}^{\infty} p_n^{-2\alpha}.$$

Hence for $M \geqslant M_0(\epsilon)$ and any $N > M$ we can find cubes of total volume greater than $\frac{1}{2}$ in which

$$\iint_{|s-\frac{1}{2}\alpha-\frac{1}{2}\beta|\leqslant\frac{1}{4}(\beta-\alpha)} |\Phi_{M,N}(s)|^2\, d\sigma dt < \epsilon$$

and so in which (by the lemma of § 11.9)

$$|\Phi_{M,N}(s)| < 2(\epsilon/\pi)^{\frac{1}{2}}(\beta-\alpha)^{-\frac{1}{2}} \quad (|s-\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta| \leqslant \tfrac{1}{4}(\beta-\alpha)).$$

We also want a little more information about $R_N(s)$, viz. that $R_N(s)$ is regular, and $|R_N(s)| < \eta$, throughout the rectangle

$$|\sigma-\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta| \leqslant \tfrac{1}{4}(\beta-\alpha), \quad t_0-\tfrac{1}{2} \leqslant t \leqslant t_0+\tfrac{1}{2},$$

for a set of values of t_0 of measure greater than $(1-\eta')T$. As before it is sufficient to prove this for $\zeta_N(s)-1$, and by the lemma it is sufficient to prove that

$$\phi(t_0) = \int_{\alpha}^{\beta} d\sigma \int_{t_0-1}^{t_0+1} |\zeta_N(s)-1|^2\, dt < \epsilon$$

for such t_0, by choice of N. Now

$$\begin{aligned}\int_1^T \phi(t_0)\, dt &= \int_{\alpha}^{\beta} d\sigma \int_1^T dt_0 \int_{t_0-1}^{t_0+1} |\zeta_N(s)-1|^2\, dt\\ &\leqslant \int_{\alpha}^{\beta} d\sigma \int_1^{T+1} |\zeta_N(s)-1|^2\, dt \int_{t-1}^{t+1} dt_0 = 2\int_{\alpha}^{\beta} d\sigma \int_1^{T+1} |\zeta_N(s)-1|^2\, dt < \epsilon T\end{aligned}$$

by choice of N as before. Hence the measure of the set where $\phi(t_0) > \sqrt{\epsilon}$ is less than $\sqrt{\epsilon}T$, and the desired result follows.

It now follows as before that there is a set of values of t_0 in $(0, T)$, of measure greater than KT, such that for $|s-\frac{1}{2}\alpha-\frac{1}{2}\beta| \leqslant \frac{1}{4}(\beta-\alpha)$

$$\left|\sum_{n=1}^{M} \log(1-p_n^{-s}e^{2\pi i\theta_n'}) - \sum_{n=1}^{M} \log(1-p_n^{-s-it_0})\right| < \tfrac{1}{4}m,$$

$$|\Phi_{M,N}(s)| < \tfrac{1}{4}m,$$

and also
$$|R_N(s+it_0)| < \tfrac{1}{4}m.$$

At the same time we can suppose that M has been taken so large that

$$\left|G(s)+\sum_{n=1}^{M} \log(1-p_n^{-s}e^{2\pi i\theta_n'})\right| < \tfrac{1}{4}m \quad (\sigma \geqslant \alpha).$$

Then
$$|\log\zeta(s)-G(s)| < m$$

on the circle with centre $\frac{1}{2}\alpha+\frac{1}{2}\beta+it_0$ and radius ρ. Hence, as before, $\log\zeta(s)-a$ has at least one zero in such a circle. The number of such circles for $0 < t_0 < T$ which do not overlap is plainly greater than KT. The lower bound for $M_{a,\alpha,\beta}(T)$ therefore follows; the upper bound holds by the same argument as in the case $\sigma > 1$.

It has been proved by Bohr and Jensen, by a more detailed study of the situation, that there is a $K(a,\alpha,\beta)$ such that

$$M_{a,\alpha,\beta}(T) \sim K(a,\alpha,\beta)T.$$

An immediate corollary of Theorem 11.10 is that, *if $N_{a,\alpha,\beta}(T)$ is the number of points in the rectangle $\frac{1}{2} < \alpha < \sigma < \beta < 1$, $0 < t < T$ where $\zeta(s) = a$ $(a \neq 0)$, then*

$$N_{a,\alpha,\beta}(T) > K(a,\alpha,\beta)T \quad (T > T_0).$$

For $\zeta(s) = a$ if $\log\zeta(s) = \log a$, any one value of the right-hand side being taken. This result, in conjunction with Theorem 9.17, shows that the value 0 of $\zeta(s)$, if it occurs at all in $\sigma > \frac{1}{2}$, is at any rate quite exceptional, zeros being infinitely rarer than a-values for any value of a other than zero.

NOTES FOR CHAPTER 11

11.11. Theorem 11.9 has been generalized by Voronin [1], [2], who obtained the following 'universal' property for $\zeta(s)$. Let D_r be the closed disc of radius $r < \frac{1}{4}$, centred at $s = \frac{3}{4}$, and let $f(s)$ be any function continuous and non-vanishing on D_r, and holomorphic on the interior of D_r. Then for any $\varepsilon > 0$ there is a real number t such that

$$\max_{s \in D_r} |\zeta(s+it) - f(s)| < \varepsilon. \tag{11.11.1}$$

It follows that the curve

$$\gamma(t) = \left(\zeta(\sigma+it), \zeta'(\sigma+it), \ldots, \zeta^{(n-1)}(\sigma+it)\right)$$

is dense in $\mathbb{C}^n$, for any fixed σ in the range $\frac{1}{2} < \sigma < 1$. (In fact Voronin [1] establishes this for $\sigma = 1$ also.) To see this we choose a point $z = (z_0, z_1, \ldots, z_{n-1})$ with $z_0 \neq 0$, and take $f(s)$ to be a polynomial for which $f^{(m)}(\sigma) = z_m$ for $0 \leqslant m < n$. We then fix an R such that $0 < R < \frac{1}{4} - |\sigma - \frac{3}{4}|$, and such that $f(s)$ is nonvanishing on the closed disc $|s-\sigma| \leqslant R$. Thus, if $r = R + |\sigma - \frac{3}{4}|$, the disc D_r contains the circle $|s-\sigma| = R$, and hence (11.11.1) in conjunction with Cauchy's inequality

$$|g^{(m)}(z_0)| \leqslant \frac{m!}{R^m} \max_{|z-z_0|=R} |g(z)|,$$

yields

$$|\zeta^{(m)}(\sigma+it) - z_m| \leqslant \frac{m!}{R^m}\varepsilon \quad (0 \leqslant m < n).$$

Hence $\gamma(t)$ comes arbitrarily close to z. The required result then follows, since the available z are dense in $\mathbb{C}^n$.

Voronin's work has been extended by Bagchi [1] (see also Gonek [1]) so that D_r may be replaced by any compact subset D of the strip $\frac{1}{2} < \mathbf{R}(s) < 1$, whose complement in $\mathbb{C}$ is connected. The condition on f is then that it should be continuous and non-vanishing on D, and holomorphic on the interior (if any) of D. From this it follows that if Φ is any continuous function, and $h_1 < h_2 < \ldots < h_m$ are real constants, then $\zeta(s)$ cannot satisfy the differential–difference equation

$$\Phi\{\zeta(s+h_1), \zeta'(s+h_1), \ldots, \zeta^{(n_1)}(s+h_1), \zeta(s+h_2), \zeta'(s+h_2), \ldots,$$
$$\zeta^{(n_2)}(s+h_2), \ldots\} = 0$$

unless Φ vanishes identically. This improves earlier results of Ostrowski [1] and Reich [1].

11.12. Levinson [6] has investigated further the distribution of the solutions $\rho_a = \beta_a + i\gamma_a$ of $\zeta(s) = a$. The principal results are that

$$\#\{\rho_a : 0 \leqslant \gamma_a \leqslant T\} = \frac{T}{2\pi}\log T + O(T)$$

and

$$\#\{\rho_a : 0 \leqslant \gamma_a \leqslant T, |\beta_a - \tfrac{1}{2}| \geqslant \delta\} = O_\delta(T) \quad (\delta > 0).$$

Thus (c.f. § 9.15) *all but an infinitesimal proportion of the zeros of $\zeta(s) - a$ lie in the strip $\frac{1}{2} - \delta < \sigma < \frac{1}{2} + \delta$, however small δ may be.*

In reviewing this work Montgomery (Math. Reviews 53 # 10737) quotes an unpublished result of Selberg, namely

$$\sum_{\substack{0 \leqslant \gamma_a \leqslant T \\ \beta_a \geqslant \frac{1}{2}}} (\beta_a - \tfrac{1}{2}) \sim \frac{1}{4\pi^{\frac{3}{2}}} T(\log\log T)^{\frac{1}{2}}. \tag{11.12.1}$$

This leads to a stronger version of the above principle, in which the infinite strip is replaced by the region

$$|\sigma - \tfrac{1}{2}| < \frac{\phi(t)(\log\log t)^{\frac{1}{2}}}{\log t},$$

where $\phi(t)$ is any positive function which tends to infinity with t. It should be noted for comparison with (11.12.1) that the estimate

$$\sum_{0 \leqslant \gamma_a \leqslant T} (\beta_a - \tfrac{1}{2}) = O(\log T)$$

is implicit in Levinson's work. It need hardly be emphasized that despite this result the numbers ρ_a are far from being symmetrically distributed about the critical line.

11.13. The problem of the distribution of values of $\zeta(\frac{1}{2}+it)$ is rather different from that of $\zeta(\sigma+it)$ with $\frac{1}{2}<\sigma<1$. In the first place it is not known whether the values of $\zeta(\frac{1}{2}+it)$ are everywhere dense, though one would conjecture so. Secondly there is a difference in the rates of growth with respect to t. Thus, for a fixed $\sigma > \frac{1}{2}$, Bohr and Jessen (1), (2) have shown that there is a continuous function $F(z;\sigma)$ such that

$$\frac{1}{2T} m\{t \in [-T, T]: \log\zeta(\sigma+it) \in R\} \to \iint_R F(x+iy;\sigma)\,dx\,dy \quad (T\to\infty)$$

for any rectangle $R \subset \mathbb{C}$ whose sides are parallel to the real and imaginary axes. Here, as usual, m denotes Lebesgue measure, and $\log\zeta(s)$ is defined by continuous variation along lines parallel to the real axis, using (1.1.9) for $\sigma > 1$. By contrast, the corresponding result for $\sigma = \frac{1}{2}$ states that

$$\frac{1}{2T} m\left\{t \in [-T, T]: \frac{\log\zeta(\frac{1}{2}+it)}{\sqrt{[\frac{1}{2}\{\log\log(3+|t|)\}]}} \in R\right\} \to \frac{1}{2\pi}\iint_R e^{-(x^2+y^2)/2}\,dx\,dy$$
$$(T\to\infty).$$

(The right hand side gives a 2-dimensional distribution with mean 0 and variance 1.) This is an unpublished theorem of Selberg, which may be obtained via the method of Ghosh [2].

By using a different technique, based on the mean-value bounds of §7.23, Jutila [4] has obtained information on 'large deviations' of $\log|\zeta(\frac{1}{2}+it)|$. Specifically, he showed that there is a constant $A > 0$ such that

$$m\{t \in [0, T]: |\zeta(\tfrac{1}{2}+it)| \geqslant V\} \ll T\exp\left(-A\frac{\log^2 V}{\log\log T}\right),$$

uniformly for $1 \leqslant V \leqslant \log T$.

XII

DIVISOR PROBLEMS

12.1. THE divisor problem of Dirichlet is that of determining the asymptotic behaviour as $x \to \infty$ of the sum

$$D(x) = \sum_{n \leqslant x} d(n),$$

where $d(n)$ denotes, as usual, the number of divisors of n. Dirichlet proved in an elementary way that

$$D(x) = x\log x + (2\gamma - 1)x + O(x^{\frac{1}{2}}). \tag{12.1.1}$$

In fact

$$\begin{aligned} D(x) &= \sum_{mn \leqslant x} \sum 1 = \sum_{m \leqslant \sqrt{x}} \sum_{n \leqslant \sqrt{x}} 1 + 2 \sum_{m \leqslant \sqrt{x}} \sum_{\sqrt{x} < n \leqslant x/m} 1 \\ &= [\sqrt{x}]^2 + 2 \sum_{m \leqslant \sqrt{x}} \left(\left[\frac{x}{m}\right] - [\sqrt{x}]\right) \\ &= 2 \sum_{m \leqslant \sqrt{x}} \left[\frac{x}{m}\right] - [\sqrt{x}]^2 \\ &= 2 \sum_{m \leqslant \sqrt{x}} \left\{\frac{x}{m} + O(1)\right\} - \{\sqrt{x} + O(1)\}^2 \\ &= 2x\{\log \sqrt{x} + \gamma + O(x^{-\frac{1}{2}})\} + O(\sqrt{x}) - \{x + O(\sqrt{x})\}, \end{aligned}$$

and (12.1.1) follows. Writing

$$D(x) = x\log x + (2\gamma - 1)x + \Delta(x)$$

we thus have

$$\Delta(x) = O(x^{\frac{1}{2}}). \tag{12.1.2}$$

Later researches have improved this result, but the exact order of $\Delta(x)$ is still undetermined.

The problem is closely related to that of the Riemann zeta-function. By (3.12.1) with $a_n = d(n)$, $s = 0$, $T \to \infty$, we have

$$D(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta^2(w) \frac{x^w}{w}\, dw \quad (c > 1),$$

provided that x is not an integer. On moving the line of integration to the left, we encounter a double pole at $w = 1$, the residue being $x\log x + (2\gamma - 1)x$, by (2.1.16). Thus

$$\Delta(x) = \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \zeta^2(w) \frac{x^w}{w}\, dw \quad (0 < c' < 1).$$

The more general problem of

$$D_k(x) = \sum_{n \leqslant x} d_k(n),$$

where $d_k(n)$ is the number of ways of expressing n as a product of k factors, was also considered by Dirichlet. We have

$$D_k(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta^k(w) \frac{x^w}{w}\, dw \quad (c > 1).$$

Here there is a pole of order k at $w = 1$, and the residue is of the form $xP_k(\log x)$, where P_k is a polynomial of degree $k-1$. We write

$$D_k(x) = xP_k(\log x) + \Delta_k(x), \tag{12.1.3}$$

so that $\Delta_2(x) = \Delta(x)$.

The classical elementary theorem† of the subject is

$$\Delta_k(x) = O(x^{1-1/k} \log^{k-2} x) \quad (k = 2, 3, \ldots,). \tag{12.1.4}$$

We have already proved this in the case $k = 2$. Now suppose that it is true in the case $k-1$. We have

$$\begin{aligned} D_k(x) &= \sum_{n_1 n_2 \ldots n_k \leqslant x} 1 = \sum_{mn \leqslant x} d_{k-1}(n) \\ &= \sum_{m \leqslant x^{1/k}} \sum_{n \leqslant x/m} d_{k-1}(n) + \sum_{x^{1/k} < m \leqslant x} \sum_{n \leqslant x/m} d_{k-1}(n) \\ &= \sum_{m \leqslant x^{1/k}} \sum_{n \leqslant x/m} d_{k-1}(n) + \sum_{n \leqslant x^{1-1/k}} d_{k-1}(n) \sum_{x^{1/k} < m \leqslant x/n} 1 \\ &= \sum_{m \leqslant x^{1/k}} D_{k-1}\left(\frac{x}{m}\right) + \sum_{n \leqslant x^{1-1/k}} \left\{\frac{x}{n} - x^{1/k} + O(1)\right\} d_{k-1}(n) \\ &= \sum_{m \leqslant x^{1/k}} D_{k-1}\left(\frac{x}{m}\right) + x \sum_{n \leqslant x^{1-1/k}} \frac{d_{k-1}(n)}{n} - x^{1/k} D_{k-1}(x^{1-1/k}) + \\ &\qquad + O\{D_{k-1}(x^{1-1/k})\}. \end{aligned}$$

Let us denote by $p_k(z)$ a polynomial in z, of degree $k-1$ at most, not always the same one. Then

$$\sum_{m \leqslant \xi} \frac{\log^{k-2} m}{m} = p_k(\log \xi) + O\left(\frac{\log^{k-2} \xi}{\xi}\right).$$

Hence

$$\sum_{m \leqslant x^{1/k}} \frac{x}{m} P_{k-1}\left(\frac{x}{m}\right) = xp_k(\log x) + O(x^{1-1/k} \log^{k-2} x).$$

Also

$$\begin{aligned} \sum_{m \leqslant x^{1/k}} \Delta_{k-1}\left(\frac{x}{m}\right) &= O\left\{x^{1-1/(k-1)} \log^{k-3} x \sum_{m \leqslant x^{1/k}} \frac{1}{m^{1-1/(k-1)}}\right\} \\ &= O\{x^{1-1/(k-1)} \log^{k-3} x \,.\, x^{1/(k(k-1))}\} = O(x^{1-1/k} \log^{k-3} x). \end{aligned}$$

† See e.g. Landau (5).

The next term is

$$x \sum_{n \leqslant x^{1-1/k}} \frac{D_{k-1}(n)-D_{k-1}(n-1)}{n} = x \sum_{n \leqslant x^{1-1/k}} \frac{D_{k-1}(n)}{n(n+1)} + \frac{xD_{k-1}(N)}{N+1},$$

where $N = [x^{1-1/k}]$. Now

$$x \sum_{n \leqslant x^{1-1/k}} \frac{P_{k-1}(\log n)}{n+1} + x\frac{NP_{k-1}(\log N)}{N+1} = xp_k(\log x)+O(x^{1/k}\log^{k-2}x)$$

and

$$x \sum_{n \leqslant x^{1-1/k}} \frac{\Delta_{k-1}(n)}{n(n+1)} + \frac{x\Delta_{k-1}(N)}{N+1} = Cx - x\sum_{n > x^{1-1/k}} \frac{\Delta_{k-1}(n)}{n(n+1)} + \frac{x\Delta_{k-1}(N)}{N+1}$$

$$= Cx - x\sum_{n > x^{1-1/k}} O\left\{\frac{\log^{k-3} n}{n^{1+1/(k-1)}}\right\} + O(xN^{-1/(k-1)}\log^{k-3}N)$$

$$= Cx+O(x^{1-1/k}\log^{k-3}x).$$

Finally

$$x^{1/k}D_{k-1}(x^{1-1/k}) = x^{1/k}\{x^{1-1/k}P_{k-1}(\log x^{1-1/k})+O(x^{(1-1/k)(1-1/(k-1))}\log^{k-3}x)\}$$

$$= xp_{k-1}(\log x)+O(x^{1-1/k}\log^{k-3}x).$$

This proves (12.1.4).

We may define the order α_k of $\Delta_k(x)$ as the least number such that

$$\Delta_k(x) = O(x^{\alpha_k+\epsilon})$$

for every positive ϵ. Thus it follows from (12.1.4) that

$$\alpha_k \leqslant \frac{k-1}{k} \quad (k = 2, 3,...). \tag{12.1.5}$$

The exact value of α_k has not been determined for any value of k.

12.2. The simplest theorem which goes beyond this elementary result is

THEOREM 12.2.†

$$\alpha_k \leqslant \frac{k-1}{k+1} \quad (k = 2, 3, 4,...).$$

Take $a_n = d_k(n)$, $\psi(n) = n^\epsilon$, $\alpha = k$, $s = 0$, and let x be half an odd integer, in Lemma 3.12. Replacing w by s, this gives

$$D_k(x) = \frac{1}{2\pi i}\int\limits_{c-iT}^{c+iT} \zeta^k(s)\frac{x^s}{s}\,ds + O\left(\frac{x^c}{T(c-1)^k}\right)+O\left(\frac{x^{1+\epsilon}}{T}\right) \quad (c > 1).$$

† Voronoi (1), Landau (5).

Now take the integral round the rectangle $-a-iT$, $c-iT$, $c+iT$, $-a+iT$, where $a > 0$. We have, by (5.1.1) and the Phragmén-Lindelöf principle,

$$\zeta(s) = O(t^{(a+\frac{1}{2})(c-\sigma)/(a+c)})$$

in the rectangle. Hence

$$\int\limits_{-a+iT}^{c+iT} \zeta^k(s)\frac{x^s}{s}\,ds = O\left(\int\limits_{-a}^{c} T^{k(a+\frac{1}{2})(c-\sigma)/(a+c)-1}x^\sigma\,d\sigma\right)$$

$$= O(T^{k(a+\frac{1}{2})-1}x^{-a})+O(T^{-1}x^c),$$

since the integrand is a maximum at one end or the other of the range of integration. A similar result holds for the integral over

$$(-a-iT, c-iT).$$

The residue at $s = 1$ is $xP_k(\log x)$, and the residue at $s = 0$ is

$$\zeta^k(0) = O(1).$$

Finally

$$\int\limits_{-a-iT}^{-a+iT} \zeta^k(s)\frac{x^s}{s}\,ds = \int\limits_{-a-iT}^{-a+iT} \chi^k(s)\zeta^k(1-s)\frac{x^s}{s}\,ds$$

$$= \sum_{n=1}^{\infty} d_k(n) \int\limits_{-a-iT}^{-a+iT} \frac{\chi^k(s)}{n^{1-s}}\frac{x^s}{s}\,ds$$

$$= ix^{-a}\sum_{n=1}^{\infty} \frac{d_k(n)}{n^{1+a}} \int\limits_{-T}^{T} \frac{\chi^k(-a+it)}{-a+it}(nx)^{it}\,dt.$$

For $1 \leqslant t \leqslant T$,

$$\chi(-a+it) = Ce^{-it\log t+it\log 2\pi+it}t^{a+\frac{1}{2}}+O(t^{a-\frac{1}{2}})$$

and

$$\frac{1}{-a+it} = \frac{1}{it}+O\left(\frac{1}{t^2}\right).$$

The corresponding part of the integral is therefore

$$-iC^k\int\limits_{1}^{T} e^{ikt(-\log t+\log 2\pi+1)}(nx)^{it}t^{(a+\frac{1}{2})k-1}\,dt+O(T^{(a+\frac{1}{2})k-1}),$$

provided that $(a+\frac{1}{2})k > 1$. This integral is of the form considered in Lemma 4.5, with

$$F(t) = kt(-\log t+\log 2\pi+1)+t\log nx.$$

Since

$$F''(t) = -\frac{k}{t} \leqslant -\frac{k}{T},$$

the integral is

$$O(T^{(a+\frac{1}{2})k-\frac{1}{2}}),$$

uniformly with respect to n and x. A similar result holds for the integral over $(-T,-1)$, while the integral over $(-1,1)$ is bounded. Hence

$$\Delta_k(x) = O\left(\frac{x^c}{T(c-1)^k}\right)+O\left(\frac{x^{1+\epsilon}}{T}\right)+O\left(\frac{T^{(a+\frac{1}{2})k-1}}{x^a}\right)+$$
$$+x^{-a}\sum_{n=1}^{\infty}\frac{d_k(n)}{n^{1+a}}O(T^{(a+\frac{1}{2})k-\frac{1}{2}})$$
$$= O\left(\frac{x^c}{T(c-1)^k}\right)+O\left(\frac{x^{1+\epsilon}}{T}\right)+O\left(\frac{T^{(a+\frac{1}{2})k-\frac{1}{2}}}{x^a}\right).$$

Taking $c = 1+\epsilon$, $a = \epsilon$, the terms are of the same order, apart from ϵ's, if

$$T = x^{2/(k+1)}.$$

Hence $$\Delta_k(x) = O(x^{(k-1)/(k+1)+\epsilon}).$$

The restriction that x should be half an odd integer is clearly unnecessary to the result.

12.3. By using some of the deeper results on $\zeta(s)$ we can obtain a still better result for $k \geqslant 4$.

THEOREM 12.3.† $$\alpha_k \leqslant \frac{k-1}{k+2} \quad (k = 4, 5,\ldots).$$

We start as in the previous theorem, but now take the rectangle as far as $\sigma = \frac{1}{2}$ only. Let us suppose that

$$\zeta(\tfrac{1}{2}+it) = O(t^\lambda).$$

Then $$\zeta(s) = O(t^{\lambda(c-\sigma)/(c-\frac{1}{2})})$$

uniformly in the rectangle. The horizontal sides therefore give

$$O\left(\int_{\frac{1}{2}}^{c} T^{k\lambda(c-\sigma)/(c-\frac{1}{2})-1}x^\sigma\,d\sigma\right) = O(T^{k\lambda-1}x^{\frac{1}{2}})+O(T^{-1}x^c).$$

Also $$\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\zeta^k(s)\frac{x^s}{s}\,ds = O(x^{\frac{1}{2}})+O\left(x^{\frac{1}{2}}\int_1^T|\zeta(\tfrac{1}{2}+it)|^k\frac{dt}{t}\right).$$

Now $$\int_1^T|\zeta(\tfrac{1}{2}+it)|^k\frac{dt}{t} \leqslant \max_{1\leqslant t\leqslant T}|\zeta(\tfrac{1}{2}+it)|^{k-4}\int_1^T|\zeta(\tfrac{1}{2}+it)|^4\frac{dt}{t}$$
$$= O\left\{T^{(k-4)\lambda}\int_1^T|\zeta(\tfrac{1}{2}+it)|^4\frac{dt}{t}\right\}.$$

† Hardy and Littlewood (4).

Also $$\phi(T) = \int_1^T|\zeta(\tfrac{1}{2}+it)|^4\,dt = O(T^{1+\epsilon}),$$

by (7.6.1), so that

$$\int_1^T|\zeta(\tfrac{1}{2}+it)|^4\frac{dt}{t} = \int_1^T\phi'(t)\frac{dt}{t} = \left[\frac{\phi(t)}{t}\right]_1^T+\int_1^T\frac{\phi(t)}{t^2}\,dt$$
$$= O(T^\epsilon)+O\left(\int_1^T\frac{1}{t^{1-\epsilon}}\,dt\right) = O(T^\epsilon).$$

Hence $$\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\zeta^k(s)\frac{x^s}{s}\,ds = O(x^{\frac{1}{2}})+O(x^{\frac{1}{2}}T^{(k-4)\lambda+\epsilon}).$$

Altogether we obtain

$$\Delta_k(x) = O(T^{-1}x^c)+O(x^{\frac{1}{2}}T^{k\lambda-1})+O(x^{\frac{1}{2}}T^{(k-4)\lambda+\epsilon}).$$

The middle term is of smaller order than the last if $\lambda \leqslant \frac{1}{4}$. Taking $c = 1+\epsilon$, the other two terms are of the same order, apart from ϵ's, if

$$T = x^{1/(2(k-4)\lambda+2)}.$$

This gives $$\Delta_k(x) = O(x^{\{2(k-4)\lambda+1\}/\{2(k-4)\lambda+2\}+\epsilon}).$$

Taking $\lambda = \frac{1}{6}+\epsilon$ (Theorems 5.5, 5.12) the result follows. Further slight improvements for $k \geqslant 5$ are obtained by using the results stated in § 5.18.

12.4. The above method does not give any new result for $k = 2$ or $k = 3$. For these values slight improvements on Theorem 12.2 have been made by special methods.

THEOREM 12.4.† $$\alpha_2 \leqslant \frac{27}{82}.$$

The argument of § 12.2 shows that

$$\Delta(x) = \frac{1}{2\pi i}\sum_{n=1}^{\infty}d(n)\int_{-a-iT}^{-a+iT}\frac{\chi^2(s)}{n^{1-s}}\frac{x^s}{s}\,ds+O\left(\frac{T^{2a}}{x^a}\right)+O\left(\frac{x^c}{T}\right) \tag{12.4.1}$$

where $a > 0$, $c > 1$. Let $T^2/(4\pi^2x) = N+\frac{1}{2}$, where N is an integer, and consider the terms with $n > N$. As before, the integral over $1 \leqslant t \leqslant T$ is of the form

$$\frac{1}{x^a n^{1+a}}\int_1^T e^{iF(t)}\{t^{2a}+O(t^{2a-1})\}\,dt, \tag{12.4.2}$$

† van der Corput (4).

where
$$F(t) = 2t(-\log t+\log 2\pi+1)+t\log nx,$$
$$F'(t) = \log\frac{4\pi^2 nx}{t^2}.$$

Hence $F'(t) \geqslant \log\dfrac{n}{N+\frac{1}{2}}$, and (12.4.2) is
$$\frac{1}{x^a n^{1+a}}\left\{O\left(\frac{T^{2a}}{\log\{n/(N+\frac{1}{2})\}}\right)+O(T^{2a})\right\}.$$
For $n \geqslant 2N$ this contributes to (12.4.1)
$$O\left\{\frac{T^{2a}}{x^a}\sum_{n=2N}^{\infty}\frac{d(n)}{n^{1+a}}\right\} = O(N^{\epsilon}),$$
and for $N < n < 2N$ it contributes
$$O\left\{\frac{T^{2a}}{x^a}\sum_{n=N+1}^{2N}\frac{d(n)}{n^{1+a}\log\{n/(N+\frac{1}{2})\}}\right\} = O\left(N^{\epsilon}\sum_{m=1}^{N}\frac{1}{m}\right) = O(N^{\epsilon}).$$
Similarly for the integral over $-T \leqslant t \leqslant -1$; and the integral over $-1 < t < 1$ is clearly $O(x^{-a})$.

If $n \leqslant N$, we write
$$\int_{-a-iT}^{-a+iT} = \int_{-i\infty}^{i\infty} - \left(\int_{iT}^{i\infty}+\int_{-i\infty}^{-iT}+\int_{-iT}^{-a-iT}+\int_{-a+iT}^{iT}\right).$$
The first term is
$$\frac{1}{n}\int_{-i\infty}^{i\infty} 2^{2s}\pi^{2s-2}\sin^2\tfrac{1}{2}s\pi\,\Gamma^2(1-s)\frac{(nx)^s}{s}\,ds$$
$$= -\frac{1}{n\pi^2}\int_{1-i\infty}^{1+i\infty}\cos^2\tfrac{1}{2}w\pi\,\Gamma(w)\Gamma(w-1)\{2\pi\sqrt{(nx)}\}^{2-2w}\,dw$$
$$= -4i\sqrt{\left(\frac{x}{n}\right)}[K_1\{4\pi\sqrt{(nx)}\}+\tfrac{1}{2}\pi Y_1\{4\pi\sqrt{(nx)}\}]$$
in the usual notation of Bessel functions.†

The first integral in the bracket is
$$\int_T^{\infty} e^{iF(t)}\left(A+\frac{A'}{t}+O(t^{-2})\right)dt = O\left\{\frac{1}{\log\{(N+\frac{1}{2})/n\}}\right\},$$
which gives
$$\sum_{n=1}^{N}\frac{d(n)}{n\log\{(N+\frac{1}{2})/n\}} = O(N^{\epsilon})$$

† See, e.g., Titchmarsh, *Fourier Integrals*, (7.9.8), (7.9.11).

as before; and similarly for the second integral. The last two give
$$O\left\{\sum_{n=1}^{N}\frac{d(n)}{n}\int_{-a}^{0}\left(\frac{nx}{T^2}\right)^{\sigma}d\sigma\right\} = O\left\{\sum_{n=1}^{N}\frac{d(n)}{n}\left(\frac{T^2}{nx}\right)^{a}\right\} = O\left\{\left(\frac{T^2}{x}\right)^{a}\right\}.$$
Altogether we have now proved that
$$\Delta(x) = -\frac{2\sqrt{x}}{\pi}\sum_{n=1}^{N}\frac{d(n)}{\sqrt{n}}[K_1\{4\pi\sqrt{(nx)}\}+\tfrac{1}{2}\pi Y_1\{4\pi\sqrt{(nx)}\}]+O\left(\frac{T^{2a}}{x^a}\right)+O\left(\frac{x^c}{T}\right). \tag{12.4.3}$$
By the usual asymptotic formulae† for Bessel functions, this may be replaced by
$$\Delta(x) = \frac{x^{\frac{1}{4}}}{\pi\sqrt{2}}\sum_{n=1}^{N}\frac{d(n)}{n^{\frac{3}{4}}}\cos\{4\pi\sqrt{(nx)}-\tfrac{1}{4}\pi\}+O(x^{-\frac{1}{4}})+O\left(\frac{T^{2a}}{x^a}\right)+O\left(\frac{x^c}{T}\right). \tag{12.4.4}$$
Now
$$\sum_{n=1}^{N} d(n)e^{4\pi i\sqrt{(nx)}} = 2\sum_{m\leqslant\sqrt{N}}\sum_{n\leqslant N/m} e^{4\pi i\sqrt{(mnx)}} - \sum_{m\leqslant\sqrt{N}}\sum_{n\leqslant\sqrt{N}} e^{4\pi i\sqrt{(mnx)}}. \tag{12.4.5}$$
Consider the sum
$$\sum_{\frac{1}{2}N/m<n\leqslant N/m} e^{4\pi i\sqrt{(mnx)}}.$$
We apply Theorem 5.13, with $k = 5$, and
$$f(n) = 2\sqrt{(mnx)}, \qquad f^{(5)}(n) = A(mx)^{\frac{1}{2}}n^{-\frac{9}{2}}.$$
Hence the sum is
$$O\left\{\frac{N}{m}\left(\frac{(mx)^{\frac{1}{2}}}{(N/m)^{\frac{9}{2}}}\right)^{\frac{1}{30}}\right\}+O\left\{\left(\frac{N}{m}\right)^{\frac{7}{8}}\left(\frac{(N/m)^{\frac{9}{2}}}{(mx)^{\frac{1}{2}}}\right)^{\frac{1}{30}}\right\}$$
$$= O\{(N/m)^{\frac{17}{20}}(mx)^{\frac{1}{60}}\}+O\{(N/m)^{\frac{41}{40}}(mx)^{-\frac{1}{60}}\}.$$
Replacing N by $\frac{1}{2}N$, $\frac{1}{4}N$,..., and adding, the same result holds for the sum over $1 \leqslant n \leqslant N/m$. Hence the first term on the right of (12.4.5) is
$$O\left(N^{\frac{17}{20}}x^{\frac{1}{60}}\sum_{m\leqslant\sqrt{N}}m^{-\frac{5}{6}}\right)+O\left(N^{\frac{41}{40}}x^{-\frac{1}{60}}\sum_{m\leqslant\sqrt{N}}m^{-\frac{25}{24}}\right) = O(N^{\frac{14}{15}}x^{\frac{1}{60}})+O(N^{\frac{41}{40}}x^{-\frac{1}{60}}).$$
Similarly the second inner sum is
$$O\{(\sqrt{N})^{\frac{17}{20}}(mx)^{\frac{1}{60}}\}+O\{(\sqrt{N})^{\frac{41}{40}}(mx)^{-\frac{1}{60}}\},$$
and the whole sum is
$$O\left(N^{\frac{17}{40}}x^{\frac{1}{60}}\sum_{m\leqslant\sqrt{N}}m^{\frac{1}{60}}\right)+O\left(N^{\frac{41}{80}}x^{-\frac{1}{60}}\sum_{m\leqslant\sqrt{N}}m^{-\frac{1}{60}}\right)$$
$$=O(N^{\frac{14}{15}}x^{\frac{1}{60}})+O(N^{\frac{241}{240}}x^{-\frac{1}{60}}).$$

† Watson, *Theory of Bessel Functions*, §§ 7.21, 7.23.

Hence, multiplying by $e^{-\frac{1}{4}i\pi}$ and taking the real part,

$$\sum_{n=1}^{N} d(n)\cos\{4\pi\sqrt{(nx)}-\tfrac{1}{4}\pi\} = O(N^{\frac{14}{15}}x^{\frac{1}{60}})+O(N^{\frac{41}{40}}x^{-\frac{1}{60}}).$$

Using this and partial summation, (12.4.4) gives

$$\begin{aligned}\Delta(x) &= O(N^{\frac{14}{15}-\frac{3}{4}}x^{\frac{1}{4}+\frac{1}{60}})+O(N^{\frac{41}{40}-\frac{3}{4}}x^{\frac{1}{4}-\frac{1}{60}})+O(N^{a})+O(N^{-\frac{1}{2}}x^{c-\frac{1}{2}}) \\ &= O(N^{\frac{11}{60}}x^{\frac{4}{15}})+O(N^{\frac{11}{40}}x^{\frac{7}{30}})+O(N^{a})+O(N^{-\frac{1}{2}}x^{c-\frac{1}{2}}).\end{aligned}$$

Taking $a = \epsilon$, $c = 1+\epsilon$, the first and last terms are of the same order, apart from ϵ's, if

$$N = [x^{\frac{14}{41}}].$$

Hence

$$\Delta(x) = O(x^{\frac{27}{82}+\epsilon}),$$

the result stated.

A similar argument may be applied to $\Delta_3(x)$. We obtain

$$\Delta_3(x) = \frac{x^{\frac{1}{3}}}{\pi\sqrt{3}} \sum_{n<T^3/(8\pi^3 x)} \frac{d_3(n)}{n^{\frac{2}{3}}}\cos\{6\pi(nx)^{\frac{1}{3}}\}+O\left(\frac{x^{1+\epsilon}}{T}\right), \tag{12.4.6}$$

and deduce that

$$\alpha_3 \leqslant \tfrac{37}{75}.$$

The detailed argument is given by Atkinson (3).

If the series in (12.4.4) were absolutely convergent, or if the terms more or less cancelled each other, we should deduce that $\alpha_2 \leqslant \frac{1}{4}$; and it may reasonably be conjectured that this is the real truth. We shall see later that $\alpha_2 \geqslant \frac{1}{4}$, so that it would follow that $\alpha_2 = \frac{1}{4}$. Similarly from (12.4.6) we should obtain $\alpha_3 = \frac{1}{3}$; and so generally it may be conjectured that

$$\alpha_k = \frac{k-1}{2k}.$$

12.5. *The average order of* $\Delta_k(x)$. We may define β_k, the average order of $\Delta_k(x)$, to be the least number such that

$$\frac{1}{x}\int_0^x \Delta_k^2(y)\,dy = O(x^{2\beta_k+\epsilon})$$

for every positive ϵ. Since

$$\frac{1}{x}\int_0^x \Delta_k^2(y)\,dy = \frac{1}{x}\int_0^x O(y^{2\alpha_k+\epsilon})\,dy = O(x^{2\alpha_k+\epsilon}),$$

we have $\beta_k \leqslant \alpha_k$ for each k. In particular we obtain a set of upper bounds for the β_k from the above theorems.

As usual, the problem of average order is easier than that of order, and we can prove more about the β_k than about the α_k. We shall first prove the following theorem.†

† Titchmarsh (22).

THEOREM 12.5. *Let γ_k be the lower bound of positive numbers σ for which*

$$\int_{-\infty}^{\infty} \frac{|\zeta(\sigma+it)|^{2k}}{|\sigma+it|^2}\,dt < \infty. \tag{12.5.1}$$

Then $\beta_k = \gamma_k$; and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|\zeta(\sigma+it)|^{2k}}{|\sigma+it|^2}\,dt = \int_0^{\infty} \Delta_k^2(x)x^{-2\sigma-1}\,dx \tag{12.5.2}$$

provided that $\sigma > \beta_k$.

We have

$$D_k(x) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT} \frac{\zeta^k(s)}{s}x^s\,ds \quad (c>1).$$

Applying Cauchy's theorem to the rectangle $\gamma-iT$, $c-iT$, $c+iT$, $\gamma+iT$, where γ is less than, but sufficiently near to, 1, and allowing for the residue at $s = 1$, we obtain

$$\Delta_k(x) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} \frac{\zeta^k(s)}{s}x^s\,ds. \tag{12.5.3}$$

Actually (12.5.3) holds for $\gamma_k < \gamma < 1$. For† $\zeta^k(s)/s \to 0$ uniformly as $t \to \pm\infty$ in the strip. Hence if we integrate the integrand of (12.5.3) round the rectangle $\gamma'-iT$, $\gamma-iT$, $\gamma+iT$, $\gamma'+iT$, where

$$\gamma_k < \gamma' < \gamma < 1,$$

and make $T \to \infty$, we obtain the same result with γ' instead of γ.

If we replace x by $1/x$, (12.5.3) expresses the relation between the Mellin transforms

$$f(x) = \Delta_k(1/x), \qquad \mathfrak{F}(s) = \zeta^k(s)/s,$$

the relevant integrals holding also in the mean-square sense. Hence Parseval's formula for Mellin transforms‡ gives

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|\zeta(\gamma+it)|^{2k}}{|\gamma+it|^2}\,dt = \int_0^{\infty} \Delta_k^2\left(\frac{1}{x}\right)x^{2\gamma-1}\,dx = \int_0^{\infty} \Delta_k^2(x)x^{-2\gamma-1}\,dx \tag{12.5.4}$$

provided that $\gamma_k < \gamma < 1$.

It follows that, if $\gamma_k < \gamma < 1$,

$$\int_{\frac{1}{2}X}^{X} \Delta_k^2(x)x^{-2\gamma-1}\,dx < K = K(k,\gamma),$$

$$\int_{\frac{1}{2}X}^{X} \Delta_k^2(x)\,dx < KX^{2\gamma+1},$$

† By an application of the lemma of § 11.9.

‡ See Titchmarsh, *Theory of Fourier Integrals*, Theorem 71.

and, replacing X by $\frac{1}{2}X$, $\frac{1}{4}X$,..., and adding,

$$\int_1^X \Delta_k^2(x)\,dx < KX^{2\gamma+1}.$$

Hence $\beta_k \leqslant \gamma$, and so $\beta_k \leqslant \gamma_k$.

The inverse Mellin formula is

$$\frac{\zeta^k(s)}{s} = \int_0^\infty \Delta_k\left(\frac{1}{x}\right)x^{s-1}\,dx = \int_0^\infty \Delta_k(x)x^{-s-1}\,dx. \qquad (12.5.5)$$

The right-hand side exists primarily in the mean-square sense, for $\gamma_k < \sigma < 1$. But actually the right-hand side is uniformly convergent in any region interior to the strip $\beta_k < \sigma < 1$; for

$$\int_{\frac{1}{2}X}^{X} |\Delta_k(x)|x^{-\sigma-1}\,dx \leqslant \left\{\int_{\frac{1}{2}X}^{X} \Delta_k^2(x)\,dx \int_{\frac{1}{2}X}^{X} x^{-2\sigma-2}\,dx\right\}^{\frac{1}{2}}$$
$$= \{O(X^{2\beta_k+1+\epsilon})O(X^{-2\sigma-1})\}^{\frac{1}{2}} = O(X^{\beta_k-\sigma+\epsilon}),$$

and on putting $X = 2, 4, 8,\ldots$, and adding we obtain

$$\int_1^\infty |\Delta_k(x)|x^{-\sigma-1}\,dx < K.$$

It follows that the right-hand side of (12.5.5) represents an analytic function, regular for $\beta_k < \sigma < 1$. The formula therefore holds by analytic continuation throughout this strip. Also (by the argument just given) the right-hand side of (12.5.4) is finite for $\beta_k < \gamma < 1$. Hence so is the left-hand side, and the formula holds. Hence $\gamma_k \leqslant \beta_k$, and so, in fact, $\gamma_k = \beta_k$. This proves the theorem.

12.6. THEOREM 12.6 (A).†

$$\beta_k \geqslant \frac{k-1}{2k} \quad (k = 2, 3,\ldots).$$

If $\frac{1}{2} < \sigma < 1$, by Theorem 7.2

$$C_\sigma T < \int_{\frac{1}{2}T}^{T} |\zeta(\sigma+it)|^2\,dt \leqslant \left\{\int_{\frac{1}{2}T}^{T} |\zeta(\sigma+it)|^{2k}\,dt\right\}^{1/k}\left(\int_{\frac{1}{2}T}^{T} dt\right)^{1-1/k}.$$

Hence
$$\int_{\frac{1}{2}T}^{T} |\zeta(\sigma+it)|^{2k}\,dt \geqslant 2^{k-1}C_\sigma^k T.$$

† Titchmarsh (22).

Hence, if $0 < \sigma < \frac{1}{2}$, $T > 1$,

$$\int_{-\infty}^{\infty} \frac{|\zeta(\sigma+it)|^{2k}}{|\sigma+it|^2}\,dt > \int_{\frac{1}{2}T}^{T} \frac{|\zeta(\sigma+it)|^{2k}}{|\sigma+it|^2}\,dt > \frac{C'}{T^2}\int_{\frac{1}{2}T}^{T} |\zeta(\sigma+it)|^{2k}\,dt$$
$$> C''T^{k(1-2\sigma)-2}\int_{\frac{1}{2}T}^{T} |\zeta(1-\sigma-it)|^{2k}\,dt \quad \text{(by the functional equation)}$$
$$\geqslant C''2^{k-1}C_{1-\sigma}^k T^{k(1-2\sigma)-1}.$$

This can be made as large as we please by choice of T if $\sigma < \frac{1}{2}(k-1)/k$.

Hence
$$\gamma_k \geqslant \frac{k-1}{2k}$$

and the theorem follows.

THEOREM 12.6 (B).†

$$\alpha_k \geqslant \frac{k-1}{2k} \quad (k = 2, 3,\ldots).$$

For $\alpha_k \geqslant \beta_k$.

Much more precise theorems of the same type are known. Hardy proved first that both

$$\Delta(x) > Kx^{\frac{1}{4}}, \qquad \Delta(x) < -Kx^{\frac{1}{4}}$$

hold for some arbitrarily large values of x, and then that $x^{\frac{1}{4}}$ may in each case be replaced by

$$(x\log x)^{\frac{1}{4}}\log\log x.$$

12.7. We recall that (§ 7.9) the numbers σ_k are defined as the lower bounds of σ such that

$$\frac{1}{T}\int_1^T |\zeta(\sigma+it)|^{2k}\,dt = O(1).$$

We shall next prove

THEOREM 12.7. *For each integer $k \geqslant 2$, a necessary and sufficient condition that*

$$\beta_k = \frac{k-1}{2k} \qquad (12.7.1)$$

is that

$$\sigma_k \leqslant \frac{k+1}{2k}. \qquad (12.7.2)$$

Suppose first that (12.7.2) holds. Then by the functional equation

$$\int_1^T |\zeta(\sigma+it)|^{2k}\,dt = O\left\{T^{k(1-2\sigma)}\int_1^T |\zeta(1-\sigma-it)|^{2k}\,dt\right\} = O(T^{k(1-2\sigma)+1})$$

† Hardy (2).

for $\sigma < \frac{1}{2}(k-1)/k$. It follows from the convexity of mean values that

$$\int_1^T |\zeta(\sigma+it)|^{2k}\,dt = O(T^{1+(\frac{1}{2}+1/2k+\epsilon/2k-\sigma)k})$$

for
$$\frac{k-1-\epsilon}{2k} < \sigma < \frac{k+1+\epsilon}{2k}.$$

The index of T is less than 2 if

$$\sigma > \frac{k-1+\epsilon}{2k}.$$

Then
$$\int_{\frac{1}{2}T}^T \frac{|\zeta(\sigma+it)|^{2k}}{|\sigma+it|^2}\,dt = O(T^{-\delta}) \quad (\delta > 0).$$

Hence (12.5.1) holds. Hence $\gamma_k \leqslant \frac{1}{2}(k-1)/k$. Hence $\beta_k \leqslant \frac{1}{2}(k-1)/k$, and so, by Theorem 12.6 (A), (12.7.1) holds.

On the other hand, if (12.7.1) holds, it follows from (12.5.2) that

$$\int_1^T |\zeta(\sigma+it)|^{2k}\,dt = O(T^2)$$

for $\sigma > \frac{1}{2}(k-1)/k$. Hence by the functional equation

$$\int_1^T |\zeta(\sigma+it)|^{2k}\,dt = O(T^{k(1-2\sigma)+2})$$

for $\sigma < \frac{1}{2}(k+1)/k$. Hence, by the convexity theorem, the left-hand side is $O(T^{1+\epsilon})$ for $\sigma = \frac{1}{2}(k+1)/k$; hence, in the notation of § 7.9, $\sigma'_k \leqslant \frac{1}{2}(k+1)/k$, and so (12.7.2) holds.

12.8. THEOREM 12.8.†

$$\beta_2 = \tfrac{1}{4}, \qquad \beta_3 = \tfrac{1}{3}, \qquad \beta_4 \leqslant \tfrac{3}{7}.$$

By Theorem 7.7, $\sigma_k \leqslant 1-1/k$. Since

$$1-\frac{1}{k} \leqslant \frac{k+1}{2k} \quad (k \leqslant 3)$$

it follows that $\beta_2 = \frac{1}{4}$, $\beta_3 = \frac{1}{3}$.

The available material is not quite sufficient to determine β_4. Theorem 12.6 (A) gives $\beta_4 \geqslant \frac{3}{8}$. To obtain an upper bound for it, we observe that, by Theorem 5.5. and (7.6.1),

$$\int_1^T |\zeta(\tfrac{1}{2}+it)|^8\,dt = O\left(T^{\frac{2}{3}+\epsilon}\int_1^T |\zeta(\tfrac{1}{2}+it)|^4\,dt\right) = O(T^{\frac{5}{3}+\epsilon}),$$

† The value of β_2 is due to Hardy (3), and that of β_3 to Cramér (4); for β_4 see Titchmarsh (22).

and, since $\sigma_4 \leqslant \frac{7}{10}$ by Theorem 7.10,

$$\int_1^T |\zeta(\tfrac{3}{10}+it)|^8\,dt = O\left(T^{\frac{8}{5}}\int_1^T |\zeta(\tfrac{7}{10}-it)|^8\,dt\right) = O(T^{\frac{13}{5}+\epsilon}).$$

Hence by the convexity theorem

$$\int_1^T |\zeta(\sigma+it)|^8\,dt = O(T^{4-\frac{14}{3}\sigma+\epsilon})$$

for $\frac{3}{10} < \sigma < \frac{1}{2}$. It easily follows that $\gamma_4 \leqslant \frac{3}{7}$, i.e. $\beta_4 \leqslant \frac{3}{7}$.

NOTES FOR CHAPTER 12

12.9. For large k the best available estimates for α_k are of the shape $\alpha_k \leqslant 1 - Ck^{-\frac{2}{3}}$, where C is a positive constant. The first such result is due to Richert [**2**]. (See also Karatsuba [**1**], Ivic [**3**; Theorem 13.3] and Fujii [**3**].) These results depend on bounds of the form (6.19.2).

For the range $4 \leqslant k \leqslant 8$ one has $\alpha_k \leqslant \frac{3}{4} - 1/k$ (Heath-Brown [**8**]) while for intermediate values of k a number of estimates are possible (see Ivic [**3**; Theorem 13.2]). In particular one has $\alpha_9 \leqslant \frac{35}{54}$, $\alpha_{10} \leqslant \frac{41}{60}$, $\alpha_{11} \leqslant \frac{7}{10}$, and $\alpha_{12} \leqslant \frac{5}{7}$.

12.10. The following bounds for α_2 have been obtained.

$\frac{33}{100} = 0{\cdot}330000\ldots$	van der Corput (**2**),
$\frac{27}{82} = 0{\cdot}329268\ldots$	van der Corput (**4**),
$\frac{15}{46} = 0{\cdot}326086\ldots$	Chih [**1**], Richert [**1**],
$\frac{12}{37} = 0{\cdot}324324\ldots$	Kolesnik [**1**],
$\frac{346}{1067} = 0{\cdot}324273\ldots$	Kolesnik [**2**],
$\frac{35}{108} = 0{\cdot}324074\ldots$	Kolesnik [**4**],
$\frac{139}{429} = 0{\cdot}324009\ldots$	Kolesnik [**5**].

In general the methods used to estimate α_2 and $\mu(\frac{1}{2})$ are very closely related. Suppose one has a bound

$$\sum_{M<m\leqslant M_1}\ \sum_{N<n\leqslant N_1} \exp\left[2\pi i\{x(mn)^{\frac{1}{2}} + cx^{-1}(mn)^{\frac{3}{2}}\}\right] \ll (MN)^{\frac{3}{4}}x^{2\vartheta-\frac{1}{2}}, \tag{12.10.1}$$

for any constant c, uniformly for $M < M_1 \leqslant 2M$, $N < N_1 \leqslant 2N$, and $MN \leqslant x^{2-4\vartheta}$. It then follows that $\mu(\frac{1}{2}) \leqslant \frac{1}{2}\vartheta$, $\alpha_2 \leqslant \vartheta$, and $E(T) \ll T^{\vartheta+\epsilon}$ (for $E(T)$ as in §7.20). In practice those versions of the van der Corput method used to tackle $\mu(\frac{1}{2})$ and α_2 also apply to (12.10.1), which explains the similarity between the table of estimates given above and that presented in §5.21 for $\mu(\frac{1}{2})$. This is just one manifestation of the close similarity exhibited by the functions $E(T)$ and $\Delta(x)$, which has its origin in the formulae (7.20.6) and (12.4.4). The classical lattice-point problem for the circle falls within the same area of ideas. Thus, if the bound (12.10.1) holds, along with its analogue in which the summation condition $m \equiv 1 \pmod 4$ is imposed, then one has

$$\#\{(m, n) \in \mathbb{Z}^2 \colon m^2 + n^2 \leqslant x\} = \pi x + O(x^{9+\varepsilon}).$$

Jutila [3] has taken these ideas further by demonstrating a direct connection between the size of $\Delta(x)$ and that of $\zeta(\frac{1}{2}+it)$ and $E(T)$. In particular he has shown that if $\alpha_2 = \frac{1}{4}$ then $\mu(\frac{1}{2}) \leqslant \frac{3}{20}$ and $E(T) \ll T^{\frac{5}{16}+\varepsilon}$.

Further work has also been done on the problem of estimating α_3. The best result at present is $\alpha_3 \leqslant \frac{43}{96}$, due to Kolesnik [3]. For α_4, however, no sharpening of the bound $\alpha_4 \leqslant \frac{1}{2}$ given by Theorem 12.3 has yet been found. This result, dating from 1922, seems very resistant to any attempt at improvement.

12.11. The Ω-results attributed to Hardy in §12.6 may be found in Hardy [1]. However Hardy's argument appears to yield only

$$\Delta(x) = \Omega_+((x \log x)^{\frac{1}{4}} \log\log x), \tag{12.11.1}$$

and not the corresponding Ω_- result. The reason for this is that Dirichlet's Theorem is applicable for Ω_+, while Kronecker's Theorem is needed for the Ω_- result. By using a quantitative form of Kronecker's Theorem, Corrádi and Kátai [1] showed that

$$\Delta(x) = \Omega_-\left\{x^{\frac{1}{4}} \exp\left(c \frac{(\log\log x)^{\frac{1}{4}}}{(\log\log\log x)^{\frac{3}{4}}}\right)\right\},$$

for a certain positive constant c. This improved earlier work of Ingham [1] and Gangadharan [1]. Hardy's result (12.11.1) has also been sharpened by Hafner [1] who obtained

$$\Delta(x) = \Omega_+\left[(x \log x)^{\frac{1}{4}} (\log\log x)^{\frac{1}{4}(3+2\log 2)} \exp\{-c(\log\log\log x)^{\frac{1}{2}}\}\right]$$

for a certain positive constant c. For $k \geqslant 3$ he also showed [2] that, for a suitable positive constant c, one has

$$\Delta_k(x) = \Omega_*\left[(x \log x)^{(k-1)/2k} (\log\log x)^a \exp\{-c(\log\log\log x)^{\frac{1}{2}}\}\right],$$

where

$$a = \frac{k-1}{2k}(k \log k + k + 1)$$

and Ω_* is Ω_+ for $k = 3$ and $\Omega_\pm$ for $k \geqslant 4$.

12.12. As mentioned in §7.2[illegible] we now have $\sigma_4 \leqslant \frac{5}{8}$, whence $\beta_4 = \frac{3}{8}$, (Heath-Brown [8]). For $k = 2$ and 3 one can give asymptotic formulae for

$$\int_0^x \Delta_k(y)^2 \, dy.$$

Thus Tong [1] showed that

$$\int_0^x \Delta_k(y)^2 \, dy = \frac{x^{(2k-1)/k}}{(4k-2)\pi^2} \sum_{n=1}^{\infty} d_k(n)^2 n^{-(k+1)/k} + R_k(x)$$

with $R_2(x) \ll x(\log x)^5$ and

$$R_k(x) \ll x^{c_k+\varepsilon}, \qquad c_k = 2 - \frac{3 - 4\sigma_k}{2k(1-\sigma_k) - 1}, \qquad (k \geqslant 3).$$

Taking $\sigma_3 \leqslant \frac{7}{12}$ (see §7.22) yields $c_3 \leqslant \frac{14}{9}$. However the available information concerning σ_k is as yet insufficient to give $c_k < (2k-1)/k$ for any $k \geqslant 4$. It is perhaps of interest to note that Hardy's result (12.11.1) implies $R_2(x) = \Omega\{x^{\frac{3}{4}}(\log x)^{-\frac{1}{4}}\}$, since any estimate $R_2(x) \ll F(x)$ easily leads to a bound $\Delta_2(x) \ll \{F(x) \log x\}^{\frac{1}{3}}$, by an argument analogous to that given for the proof of Lemma α in §14.13.

Ivic [3; Theorems 13.9 and 13.10] has estimated the higher moments of $\Delta_2(x)$ and $\Delta_3(x)$. In particular his results imply that

$$\int_0^x \Delta_2(y)^8 \, dy \ll x^{3+\varepsilon}.$$

For $\Delta_3(x)$ his argument may be modified slightly to yield

$$\int_0^x |\Delta_3(y)|^3 \, dy \ll x^{2+\varepsilon}.$$

These results are readily seen to contain the estimates $\alpha_2 \leqslant \frac{1}{3}$, $\beta_2 \leqslant \frac{1}{4}$ and $\alpha_3 \leqslant \frac{1}{2}$, $\beta_3 \leqslant \frac{1}{3}$ respectively.

XIII

THE LINDELÖF HYPOTHESIS

13.1. The Lindelöf hypothesis is that

$$\zeta(\tfrac{1}{2}+it) = O(t^{\epsilon})$$

for every positive ϵ; or, what comes to the same thing, that

$$\zeta(\sigma+it) = O(t^{\epsilon})$$

for every positive ϵ and every $\sigma \geqslant \frac{1}{2}$; for either statement is, by the theory of the function $\mu(\sigma)$, equivalent to the statement that $\mu(\sigma) = 0$ for $\sigma \geqslant \frac{1}{2}$. The hypothesis is suggested by various theorems in Chapters V and VII. It is also the simplest possible hypothesis on $\mu(\sigma)$, for on it the graph of $y = \mu(\sigma)$ consists simply of the two straight lines

$$y = \tfrac{1}{2}-\sigma \quad (\sigma \leqslant \tfrac{1}{2}), \qquad y = 0 \quad (\sigma \geqslant \tfrac{1}{2}).$$

We shall see later that the Lindelöf hypothesis is true if the Riemann hypothesis is true. The converse deduction, however, cannot be made —in fact (Theorem 13.5) the Lindelöf hypothesis is equivalent to a much less drastic, but still unproved, hypothesis about the distribution of the zeros.

In this chapter we investigate the consequences of the Lindelöf hypothesis. Most of our arguments are reversible, so that we obtain necessary and sufficient conditions for the truth of the hypothesis.

13.2. Theorem 13.2.† *Alternative necessary and sufficient conditions for the truth of the Lindelöf hypothesis are*

$$\frac{1}{T}\int_{1}^{T} |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt = O(T^{\epsilon}) \quad (k = 1, 2,...); \tag{13.2.1}$$

$$\frac{1}{T}\int_{1}^{T} |\zeta(\sigma+it)|^{2k}\,dt = O(T^{\epsilon}) \quad (\sigma > \tfrac{1}{2},\ k = 1, 2,...); \tag{13.2.2}$$

$$\frac{1}{T}\int_{1}^{T} |\zeta(\sigma+it)|^{2k}\,dt \sim \sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^{2\sigma}} \quad (\sigma > \tfrac{1}{2},\ k = 1, 2,...). \tag{13.2.3}$$

The equivalence of the first two conditions follows from the convexity theorem (§ 7.8), while that of the last two follows from the analysis of § 7.9. It is therefore sufficient to consider (13.2.1).

† Hardy and Littlewood (5).

The necessity of the condition is obvious. To prove that it is sufficient, suppose that $\zeta(\frac{1}{2}+it)$ is not $O(t^{\epsilon})$. Then there is a positive number λ, and a sequence of numbers $\frac{1}{2}+it_\nu$, such that $t_\nu \to \infty$ with ν, and

$$|\zeta(\tfrac{1}{2}+it_\nu)| > Ct_\nu^{\lambda} \quad (C > 0).$$

On the other hand, on differentiating (2.1.4) we obtain, for $t \geqslant 1$,

$$|\zeta'(\tfrac{1}{2}+it)| < Et,$$

E being a positive absolute constant. Hence

$$|\zeta(\tfrac{1}{2}+it)-\zeta(\tfrac{1}{2}+it_\nu)| = \left|\int_{t_\nu}^{t} \zeta'(\tfrac{1}{2}+iu)\,du\right| < 2E|t-t_\nu|t_\nu < \tfrac{1}{2}Ct_\nu^{\lambda}$$

if $|t-t_\nu| \leqslant t_\nu^{-1}$ and ν is sufficiently large. Hence

$$|\zeta(\tfrac{1}{2}+it)| > \tfrac{1}{2}Ct_\nu^{\lambda} \quad (|t-t_\nu| \leqslant t_\nu^{-1}).$$

Take $T = \frac{2}{3}t_\nu$, so that the interval $(t_\nu-t_\nu^{-1}, t_\nu+t_\nu^{-1})$ is included in $(T, 2T)$ if ν is sufficiently large. Then

$$\int_{T}^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt > \int_{t_\nu-t_\nu^{-1}}^{t_\nu+t_\nu^{-1}} (\tfrac{1}{2}Ct_\nu^{\lambda})^{2k}\,dt = 2(\tfrac{1}{2}C)^{2k}t_\nu^{2k\lambda-1},$$

which is contrary to hypothesis if k is large enough. This proves the theorem.

We could plainly replace the right-hand side of (13.2.1) by $O(T^{A})$ without altering the theorem or the proof.

13.3. Theorem 13.3. *A necessary and sufficient condition for the truth of the Lindelöf hypothesis is that, for every positive integer k and $\sigma > \frac{1}{2}$,*

$$\zeta^k(s) = \sum_{n \leqslant t^{\delta}} \frac{d_k(n)}{n^s} + O(t^{-\lambda}) \quad (t > 0), \tag{13.3.1}$$

where δ is any given positive number less than 1, *and $\lambda = \lambda(k, \delta, \sigma) > 0$.*

We may express this roughly by saying that, on the Lindelöf hypothesis, the behaviour of $\zeta(s)$, or of any of its positive integral powers, is dominated, throughout the right-hand half of the critical strip, by a section of the associated Dirichlet series whose length is less than any positive power of t, however small. The result may be contrasted with what we can deduce, without unproved hypothesis, from the approximate functional equation.

Taking $a_n = d_k(n)$ in Lemma 3.12, we have (if x is half an odd integer)

$$\sum_{n<x} \frac{d_k(n)}{n^s} = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \zeta^k(s+w)\frac{x^w}{w}\,dw + O\left(\frac{x^c}{T(\sigma+c-1)^k}\right)$$

where $c > 1-\sigma+\epsilon$. Now let $0 < t < T-1$, and integrate round the rectangle $\frac{1}{2}-\sigma-iT$, $c-iT$, $c+iT$, $\frac{1}{2}-\sigma+iT$. We have

$$\frac{1}{2\pi i}\int_{\text{rectangle}} \zeta^k(s+w)\frac{x^w}{w}\,dw = \zeta^k(s)+\frac{x^{1-s}}{1-s}P\left(\frac{1}{1-s},\,\log x\right)$$
$$= \zeta^k(s)+O(x^{1-\sigma+\epsilon}t^{-1+\epsilon}),$$

P being a polynomial in its arguments. Also

$$\left(\int_{\frac{1}{2}-\sigma-iT}^{c-iT}+\int_{c+iT}^{\frac{1}{2}-\sigma+iT}\right)\zeta^k(s+w)\frac{x^w}{w}\,dw = O(x^cT^{-1+\epsilon})$$

by the Lindelöf hypothesis; and

$$\int_{\frac{1}{2}-\sigma-iT}^{\frac{1}{2}-\sigma+iT}\zeta^k(s+w)\frac{x^w}{w}\,dw = O\left\{x^{\frac{1}{2}-\sigma}\int_{-T}^{T}\frac{|\zeta^k(\frac{1}{2}+it+iv)|}{|\frac{1}{2}+iv|}\,dv\right\}$$
$$= O(x^{\frac{1}{2}-\sigma}T^{\epsilon})$$

by the Lindelöf hypothesis. Hence

$$\zeta^k(s) = \sum_{n<x}\frac{d_k(n)}{n^s}+O\left\{\frac{x^c}{T(\sigma+c-1)^k}\right\}+O(x^{1-\sigma+\epsilon}t^{\epsilon-1})+$$
$$+O(x^cT^{-1+\epsilon})+O(x^{\frac{1}{2}-\sigma}T^{\epsilon}),$$

and (13.3.1) follows on taking $x = [t^{\delta}]+\frac{1}{2}$, $c = 2$, $T = t^3$.

Conversely, the condition is clearly sufficient, since it gives

$$\zeta^k(s) = O\Big(\sum_{n\leqslant t^{\delta}}n^{\epsilon-\sigma}\Big)+O(t^{-\lambda}) = O(t^{\delta(1-\epsilon-\sigma)}),$$

where δ is arbitrarily small.

The result may be used to prove the equivalence of the conditions of the previous section, without using the general theorems quoted.

13.4. Another set of conditions may be stated in terms of the numbers α_k and β_k of the previous chapter.

THEOREM 13.4. *Alternative necessary and sufficient conditions for the truth of the Lindelöf hypothesis are*

$$\alpha_k \leqslant \tfrac{1}{2}\qquad (k = 2, 3,...), \tag{13.4.1}$$

$$\beta_k \leqslant \tfrac{1}{2}\qquad (k = 2, 3,...), \tag{13.4.2}$$

$$\beta_k = \frac{k-1}{2k}\qquad (k = 2, 3,...). \tag{13.4.3}$$

As regards sufficiency, we need only consider (13.4.2), since the other

conditions are formally more stringent. Now (13.4.2) gives $\gamma_k \leqslant \frac{1}{2}$, and so

$$\int_{\frac{1}{2}T}^{T}\frac{|\zeta(\sigma+it)|^{2k}}{|\sigma+it|^2}\,dt = O(1)\quad (\sigma > \tfrac{1}{2}),$$

$$\int_{\frac{1}{2}T}^{T}|\zeta(\sigma+it)|^{2k}\,dt = O(T^2)\quad (\sigma > \tfrac{1}{2}).$$

The truth of the Lindelöf hypothesis follows from this, as in § 13.2.

Now suppose that the Lindelöf hypothesis is true. We have, as in § 12.2,

$$D_k(x) = \frac{1}{2\pi i}\int_{2-iT}^{2+iT}\zeta^k(s)\frac{x^s}{s}\,ds+O\left(\frac{x^2}{T}\right).$$

Now integrate round the rectangle with vertices at $\frac{1}{2}-iT$, $2-iT$, $2+iT$, $\frac{1}{2}+iT$. We have

$$\int_{\frac{1}{2}\pm iT}^{2\pm iT}\zeta^k(s)\frac{x^s}{s}\,ds = O(x^2T^{\epsilon-1}),$$

$$\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\zeta^k(s)\frac{x^s}{s}\,ds = O\left\{x^{\frac{1}{2}}\int_{-T}^{T}|\tfrac{1}{2}+it|^{\epsilon-1}\,dt\right\} = O(x^{\frac{1}{2}}T^{\epsilon}).$$

The residue at $s = 1$ accounts for the difference between $D_k(x)$ and $\Delta_k(x)$. Hence

$$\Delta_k(x) = O(x^{\frac{1}{2}}T^{\epsilon})+O(x^2T^{\epsilon-1}).$$

Taking $T = x^2$, it follows that $\alpha_k \leqslant \frac{1}{2}$. Hence also $\beta_k \leqslant \frac{1}{2}$. But in fact $\sigma_k \leqslant \frac{1}{2}$ on the Lindelöf hypothesis, so that, by Theorem 12.7, (13.4.3) also follows.

13.5. *The Lindelöf hypothesis and the zeros.*

THEOREM 13.5.† *A necessary and sufficient condition for the truth of the Lindelöf hypothesis is that, for every $\sigma > \frac{1}{2}$,*

$$N(\sigma, T+1)-N(\sigma, T) = o(\log T).$$

The necessity of the condition is easily proved. We apply Jensen's formula

$$\log\frac{r^n}{r_1...r_n} = \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta-\log|f(0)|,$$

where $r_1,...$ are the moduli of the zeros of $f(s)$ in $|s| \leqslant r$, to the circle with centre $2+it$ and radius $\frac{3}{2}-\frac{1}{4}\delta$, $f(s)$ being $\zeta(s)$. On the Lindelöf

† Backlund (4).

hypothesis the right-hand side is less than $o(\log t)$; and, if there are N zeros in the concentric circle of radius $\frac{3}{2}-\frac{1}{2}\delta$, the left-hand side is greater than

$$N\log\{(\tfrac{3}{2}-\tfrac{1}{4}\delta)/(\tfrac{3}{2}-\tfrac{1}{2}\delta)\}.$$

Hence the number of zeros in the circle of radius $\frac{3}{2}-\frac{1}{2}\delta$ is $o(\log t)$; and the result stated, with $\sigma = \frac{1}{2}+\delta$, clearly follows by superposing a number (depending on δ only) of such circles.

To prove the converse,† let C_1 be the circle with centre $2+iT$ and radius $\frac{3}{2}-\delta$ $(\delta > 0)$, and let Σ_1 denote a summation over zeros of $\zeta(s)$ in C_1. Let C_2 be the concentric circle of radius $\frac{3}{2}-2\delta$. Then for s in C_2

$$\psi(s) = \frac{\zeta'(s)}{\zeta(s)} - \sum\nolimits_1 \frac{1}{s-\rho} = O\left(\frac{\log T}{\delta}\right).$$

This follows from Theorem 9.6 (A), since for each term which is in one of the sums

$$\sum\nolimits_1 \frac{1}{s-\rho}, \qquad \sum_{|t-\gamma|<1} \frac{1}{s-\rho},$$

but not in the other, $|s-\rho| \geqslant \delta$; and the number of such terms is $O(\log T)$.

Let C_3 be the concentric circle of radius $\frac{3}{2}-3\delta$, C the concentric circle of radius $\frac{1}{2}$. Then $\psi(s) = o(\log T)$ for s in C, since each term is $O(1)$, and by hypothesis the number of terms is $o(\log T)$. Hence Hadamard's three-circles theorem gives, for s in C_3,

$$|\psi(s)| < \{o(\log T)\}^{\alpha}\{O(\delta^{-1}\log T)\}^{\beta}$$

where $\alpha+\beta = 1$, $0 < \beta < 1$, α and β depending on δ only. Thus in C_3

$$\psi(s) = o(\log T),$$

for any given δ.

Now

$$\int\limits_{\frac{1}{2}+3\delta}^{2} \psi(s)\,d\sigma = \log\zeta(2+it) - \log\zeta(\tfrac{1}{2}+3\delta+it) -$$
$$- \sum\nolimits_1 \{\log(2+it-\rho) - \log(\tfrac{1}{2}+3\delta+it-\rho)\}$$
$$= O(1) - \log\zeta(\tfrac{1}{2}+3\delta+it) + o(\log T) +$$
$$+ \sum\nolimits_1 \log(\tfrac{1}{2}+3\delta+it-\rho),$$

since $\sum_1$ has $o(\log T)$ terms. Also, if $t = T$, the left-hand side is $o(\log T)$. Hence, putting $t = T$ and taking real parts,

$$\log|\zeta(\tfrac{1}{2}+3\delta+iT)| = o(\log T) + \sum\nolimits_1 \log|\tfrac{1}{2}+3\delta+iT-\rho|.$$

Since $|\frac{1}{2}+3\delta+iT-\rho| < A$ in C_1, it follows that

$$\log|\zeta(\tfrac{1}{2}+3\delta+iT)| < o(\log T),$$

i.e. the Lindelöf hypothesis is true.

† Littlewood (4).

13.6. THEOREM 13.6 (A).† *On the Lindelöf hypothesis*

$$S(t) = o(\log t).$$

The proof is the same as Backlund's proof (§ 9.4) that, without any hypothesis, $S(t) = O(\log t)$, except that we now use $\zeta(s) = O(t^{\epsilon})$ where we previously used $\zeta(s) = O(t^{A})$.

THEOREM 13.6 (B).‡ *On the Lindelöf hypothesis*

$$S_1(t) = o(\log t).$$

Integrating the real part of (9.6.3) from $\frac{1}{2}$ to $\frac{1}{2}+3\delta$,

$$\int\limits_{\frac{1}{2}}^{\frac{1}{2}+3\delta} \log|\zeta(s)|\,d\sigma = \sum_{|\gamma-t|<1} \int\limits_{\frac{1}{2}}^{\frac{1}{2}+3\delta} \log|s-\rho|\,d\sigma + O(\delta\log t),$$

where $\rho = \beta+i\gamma$ runs through zeros of $\zeta(s)$. Now

$$\int\limits_{\frac{1}{2}}^{\frac{1}{2}+3\delta} \log|s-\rho|\,d\sigma = \frac{1}{2}\int\limits_{\frac{1}{2}}^{\frac{1}{2}+3\delta} \log\{(\sigma-\beta)^2+(\gamma-t)^2\}\,d\sigma \leqslant \frac{3\delta}{2}\log 2$$

and

$$\geqslant \int\limits_{\frac{1}{2}}^{\frac{1}{2}+3\delta} \log|\sigma-\beta|\,d\sigma \geqslant \int\limits_{\frac{1}{2}}^{\frac{1}{2}+3\delta} \log|\sigma-\tfrac{1}{2}-\tfrac{3}{2}\delta|\,d\sigma = 3\delta(\log\tfrac{3}{2}\delta-1).$$

Hence

$$\int\limits_{\frac{1}{2}}^{\frac{1}{2}+3\delta} \log|\zeta(s)|\,d\sigma = \sum_{|\gamma-t|<1} O\left(\delta\log\frac{1}{\delta}\right) + O(\delta\log t)$$
$$= O(\delta\log 1/\delta\,.\log t).$$

Also, as in the proof of Theorem 13.5,

$$\log\zeta(s) = \sum\nolimits_1 \log(s-\rho) + o(\log t) \quad (\tfrac{1}{2}+3\delta \leqslant \sigma \leqslant 2).$$

Hence

$$\int\limits_{\frac{1}{2}+3\delta}^{2} \log|\zeta(s)|\,d\sigma = \sum\nolimits_1 \int\limits_{\frac{1}{2}+3\delta}^{2} \log|s-\rho|\,d\sigma + o(\log t)$$
$$= \sum\nolimits_1 O(1) + o(\log t)$$
$$= o(\log t).$$

Hence, by Theorem 9.9,

$$S_1(t) = \frac{1}{\pi}\int\limits_{\frac{1}{2}}^{2} \log|\zeta(s)|\,d\sigma + O(1)$$
$$= O(\delta\log 1/\delta\,.\log t) + o(\log t) + O(1),$$

and the result follows on choosing first δ and then t.

† Cramér (1), Littlewood (4). ‡ Littlewood (4).

NOTES FOR CHAPTER 13

13.7. Since the proof of Theorem 13.6(A) is not quite straightforward we give the details. Let

$$g(z) = \tfrac{1}{2}\{\zeta(z+2+iT)+\zeta(z+2-iT)\}$$

and define $n(r)$ to be the number of zeros of $g(z)$ in the disc $|z| \leqslant r$. As in § 9.4 one finds that $S(T) \ll n(\tfrac{3}{2})+1$. Moreover, by Jensen's Thorem, one has

$$\int_0^R \frac{n(r)}{r}\,dr = \frac{1}{2\pi}\int_0^{2\pi} \log|g(Re^{i\vartheta})|\,d\vartheta - \log|g(0)|. \tag{13.7.1}$$

With our choice of $g(z)$ we have $\log|g(0)| = \log|\mathbf{R}\zeta(2+iT)| = O(1)$. We shall take $R = \tfrac{3}{2}+\delta$. Then, on the Lindelöf Hypothesis, one finds that

$$|\zeta(Re^{i\vartheta}+2\pm iT)| \leqslant T^{\varepsilon}$$

for $\cos\vartheta \geqslant -3/(2R)$ and T sufficiently large. The remaining range for ϑ is an interval of length $O(\delta^{\frac{1}{2}})$. Here we write $\mathbf{R}(Re^{i\vartheta}+2) = \sigma$, so that $\tfrac{1}{2}-\delta \leqslant \sigma \leqslant \tfrac{1}{2}$. Then, using the convexity of the μ function, together with the facts that $\mu(0) = \tfrac{1}{2}$ and, on the Lindelöf Hypothesis, that $\mu(\tfrac{1}{2}) = 0$, we have $\mu(\sigma) \leqslant \delta$. It follows that

$$|\zeta(Re^{i\vartheta}+2\pm iT)| \leqslant T^{\delta+\varepsilon}$$

for $\cos\vartheta \leqslant -3/2R$, and large enough T. We now see that the right-hand side of (13.7.1) is at most

$$O(\varepsilon\log T) + O\{\delta^{\frac{1}{2}}(\delta+\varepsilon)\log T\}.$$

Since

$$\frac{\delta}{R}\,n(\tfrac{3}{2}) \leqslant \int_0^R \frac{n(r)}{r}\,dr$$

we conclude that

$$n(\tfrac{3}{2}) = O\left\{\left(\frac{\varepsilon}{\delta}+\delta^{-\frac{1}{2}}(\delta+\varepsilon)\right)\log T\right\},$$

and on taking $\delta = \varepsilon^{\frac{2}{3}}$ we obtain $n(\tfrac{3}{2}) = O(\varepsilon^{\frac{1}{3}}\log T)$, from which the result follows.

13.8. It has been observed by Ghosh and Goldston (in unpublished

work) that the converse of Theorem 13.6(B) follows from Lemma 21 of Selberg (5).

THEOREM 13.8. *If $S_1(t) = o(\log t)$, then the Lindelöf hypothesis holds.*

We reproduce the arguments used by Selberg and by Ghosh and Goldston here. Let $\tfrac{1}{2} \leqslant \sigma \leqslant 2$, and consider the integral

$$\frac{1}{2\pi i}\int_{5-i\infty}^{5+i\infty} \frac{\log\zeta(s+iT)}{4-(s-\sigma)^2}\,ds.$$

Since $\log\zeta(s+iT) \ll 2^{-\mathbf{R}(s)}$ the integral is easily seen to vanish, by moving the line of integration to the right. We now move the line of integration to the left, to $\mathbf{R}(s) = \sigma$, passing a pole at $s = 2+\sigma$, with residue $-\tfrac{1}{4}\log\zeta(2+\sigma+iT) = O(1)$. We must make detours around $s = 1-iT$, if $\sigma < 1$, and around $s = \rho - iT$, if $\sigma < \beta$. The former, if present, will produce an integral contributing $O(T^{-2})$, and the latter, if present, will be

$$-\int_0^{\beta-\sigma} \frac{du}{4-\{u+i(\gamma-T)\}^2}.$$

It follows that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\log\zeta(\sigma+it+iT)}{4+t^2}\,dt - \sum_{\beta>\sigma}\int_0^{\beta-\sigma} \frac{du}{4-\{u+i(\gamma-T)\}^2} = O(1),$$

for $T \geqslant 1$. We now take real parts and integrate for $\tfrac{1}{2} \leqslant \sigma \leqslant 2$. Then by Theorem 9.9 we have

$$\tfrac{1}{2}\int_{-\infty}^{\infty} \frac{S_1(t+T)}{4+t^2}\,dt = \sum_{\beta>\frac{1}{2}}\int_0^{\beta-\frac{1}{2}} (\beta-\tfrac{1}{2}-u)\,\mathbf{R}\left(\frac{1}{4-\{u+i(\gamma-T)\}^2}\right)du + O(1). \tag{13.8.1}$$

By our hypothesis the integral on the left is $o(\log T)$. Moreover

$$\mathbf{R}\left(\frac{1}{4-\{u+i(\gamma-T)\}^2}\right) \geqslant \begin{cases} A\ (>0) & \text{if } |\gamma-T| \leqslant 1,\\ 0, & \text{otherwise.}\end{cases}$$

If $\sigma > \tfrac{1}{2}$ is given, then each zero counted by $N(\sigma, T+1) - N(\sigma, T)$ contributes at least $\tfrac{1}{2}(\sigma-\tfrac{1}{2})^2 A$ to the sum on the right of (13.8.1), whence $N(\sigma, T+1) - N(\sigma, T) = o(\log T)$. Theorem 13.8 therefore follows from Theorem 13.5.

XIV

CONSEQUENCES OF THE RIEMANN HYPOTHESIS

14.1. In this chapter we assume the truth of the unproved Riemann hypothesis, that all the complex zeros of $\zeta(s)$ lie on the line $\sigma = \frac{1}{2}$. It will be seen that a perfectly coherent theory can be constructed on this basis, which perhaps gives some support to the view that the hypothesis is true. A proof of the hypothesis would make the 'theorems' of this chapter essential parts of the theory, and would make unnecessary much of the tentative analysis of the previous chapters.

The Riemann hypothesis, of course, leaves nothing more to be said about the 'horizontal' distribution of the zeros. From it we can also deduce interesting consequences both about the 'vertical' distribution of the zeros and about the order problems. In most cases we obtain much more precise results with the hypothesis than without it. But even a proof of the Riemann hypothesis would not by any means complete the theory. The finer shades in the behaviour of $\zeta(s)$ would still not be completely determined.

On the Riemann hypothesis, the function $\log\zeta(s)$, as well as $\zeta(s)$, is regular for $\sigma > \frac{1}{2}$ (except at $s = 1$). This is the basis of most of the analysis of this chapter.

We shall not repeat the words 'on the Riemann hypothesis', which apply throughout the chapter.

14.2. THEOREM 14.2.† *We have*

$$\log\zeta(s) = O\{(\log t)^{2-2\sigma+\epsilon}\} \tag{14.2.1}$$

uniformly for $\frac{1}{2} < \sigma_0 \leqslant \sigma \leqslant 1$.

Apply the Borel–Carathéodory theorem to the function $\log\zeta(z)$ and the circles with centre $2+it$ and radii $\frac{3}{2}-\frac{1}{2}\delta$ and $\frac{3}{2}-\delta$ $(0 < \delta < \frac{1}{2})$. On the larger circle

$$\mathbf{R}\{\log\zeta(z)\} = \log|\zeta(z)| < A\log t.$$

Hence, on the smaller circle,

$$\begin{aligned} |\log\zeta(z)| &\leqslant \frac{3-2\delta}{\frac{1}{2}\delta}A\log t + \frac{3-\frac{3}{2}\delta}{\frac{1}{2}\delta}|\log|\zeta(2+it)|| \\ &< A\delta^{-1}\log t. \end{aligned} \tag{14.2.2}$$

† Littlewood (1).

Now apply Hadamard's three-circles theorem to the circles C_1, C_2, C_3 with centre σ_1+it $(1 < \sigma_1 \leqslant t)$, passing through the points $1+\eta+it$, $\sigma+it$, $\frac{1}{2}+\delta+it$. The radii are thus

$$r_1 = \sigma_1-1-\eta, \qquad r_2 = \sigma_1-\sigma, \qquad r_3 = \sigma_1-\tfrac{1}{2}-\delta.$$

If the maxima of $|\log\zeta(z)|$ on the circles are M_1, M_2, M_3, we obtain

$$M_2 \leqslant M_1^{1-a}M_3^a,$$

where

$$\begin{aligned} a = \log\frac{r_2}{r_1}\Big/\log\frac{r_3}{r_1} &= \log\left(1+\frac{1+\eta-\sigma}{\sigma_1-1-\eta}\right)\Big/\log\left(1+\frac{\frac{1}{2}+\eta-\delta}{\sigma_1-1-\eta}\right) \\ &= \frac{1+\eta-\sigma}{\frac{1}{2}+\eta-\delta}+O\left(\frac{1}{\sigma_1}\right) = 2-2\sigma+O(\delta)+O(\eta)+O\left(\frac{1}{\sigma_1}\right). \end{aligned}$$

By (14.2.2), $M_3 < A\delta^{-1}\log t$; and, since

$$\log\zeta(s) = \sum_{n=2}^{\infty}\frac{\Lambda_1(n)}{n^s} \quad (\Lambda_1(n) \leqslant 1), \tag{14.2.3}$$

$$M_1 \leqslant \max_{x\geqslant 1+\eta}\left|\sum_{n=2}^{\infty}\frac{\Lambda_1(n)}{n^z}\right| \leqslant \sum_{n=2}^{\infty}\frac{1}{n^{1+\eta}} < \frac{A}{\eta}.$$

Hence

$$|\log\zeta(\sigma+it)| < \left(\frac{A}{\eta}\right)^{1-a}\left(\frac{A\log t}{\delta}\right)^a < \frac{A}{\eta^{1-a}\delta^a}(\log t)^{2-2\sigma+O(\delta)+O(\eta)+O(1/\sigma_1)}.$$

The result stated follows on taking δ and η small enough and σ_1 large enough. More precisely, we can take

$$\sigma_1 = \frac{1}{\delta} = \frac{1}{\eta} = \log\log t;$$

since $\qquad (\log t)^{O(\delta)} = e^{O(\delta\log\log t)} = e^{O(1)} = O(1),$

etc., we obtain

$$\log\zeta(s) = O\{\log\log t(\log t)^{2-2\sigma}\} \quad \left(\frac{1}{2}+\frac{1}{\log\log t} \leqslant \sigma \leqslant 1\right). \tag{14.2.4}$$

Since the index of $\log t$ in (14.2.1) is less than unity if ϵ is small enough, it follows that (with a new ϵ)

$$-\epsilon\log t < \log|\zeta(s)| < \epsilon\log t \quad (t > t_0(\epsilon)),$$

i.e. we have both

$$\zeta(s) = O(t^\epsilon), \tag{14.2.5}$$

$$\frac{1}{\zeta(s)} = O(t^\epsilon) \tag{14.2.6}$$

for every $\sigma > \frac{1}{2}$. In particular, *the truth of the Lindelöf hypothesis follows from that of the Riemann hypothesis.*

It also follows that *for every fixed* $\sigma > \frac{1}{2}$, *as* $T \to \infty$

$$\int_1^T \frac{dt}{|\zeta(\sigma+it)|^2} \sim \frac{\zeta(2\sigma)}{\zeta(4\sigma)} T.$$

For $\sigma > 1$ this follows from (7.1.2) and (1.2.7). For $\frac{1}{2} < \sigma \leqslant 1$ it follows from (14.2.6) and the analysis of § 7.9, applied to $1/\zeta(s)$ instead of to $\zeta^k(s)$.

14.3. The function† $\nu(\sigma)$. For each $\sigma > \frac{1}{2}$ we define $\nu(\sigma)$ as the lower bound of numbers a such that

$$\log \zeta(s) = O(\log^a t).$$

It is clear from (14.2.3) that $\nu(\sigma) \leqslant 0$ for $\sigma > 1$; and from (14.2.2) that $\nu(\sigma) \leqslant 1$ for $\frac{1}{2} < \sigma \leqslant 1$; and in fact from (14.2.1) that $\nu(\sigma) \leqslant 2-2\sigma$ for $\frac{1}{2} < \sigma \leqslant 1$.

On the other hand, since $\Lambda_1(2) = 1$, (14.2.3) gives

$$|\log \zeta(s)| \geqslant \frac{1}{2^\sigma} - \sum_{n=3}^{\infty} \frac{\Lambda_1(n)}{n^\sigma},$$

and hence $\nu(\sigma) \geqslant 0$ if σ is so large that the right-hand side is positive. Since

$$\sum_{n=3}^{\infty} \frac{\Lambda_1(n)}{n^\sigma} \leqslant \sum_{n=3}^{\infty} \frac{1}{n^\sigma} < \int_2^\infty \frac{dx}{x^\sigma} = \frac{2^{1-\sigma}}{\sigma-1}$$

this is certainly true for $\sigma \geqslant 3$. Hence $\nu(\sigma) = 0$ for $\sigma \geqslant 3$.

Now let $\frac{1}{2} < \sigma_1 < \sigma < \sigma_2 \leqslant 4$, and suppose that

$$\log \zeta(\sigma_1+it) = O(\log^a t), \qquad \log \zeta(\sigma_2+it) = O(\log^b t).$$

Let
$$g(s) = \log \zeta(s)\{\log(-is)\}^{-k(s)},$$

where $k(s)$ is the linear function of s such that $k(\sigma_1) = a$, $k(\sigma_2) = b$, viz.

$$k(s) = \frac{(s-\sigma_1)b+(\sigma_2-s)a}{\sigma_2-\sigma_1}.$$

Here
$$\{\log(-is)\}^{-k(s)} = e^{-k(s)\log\log(-is)},$$

where
$$\log(-is) = \log(t-i\sigma), \qquad \log\log(-is) \ (t > e)$$

denote the branches which are real for $\sigma = 0$. Thus

$$\log(-is) = \log t + \log\left(1-\frac{i\sigma}{t}\right) = \log t + O\left(\frac{1}{t}\right),$$

$$\log\log(-is) = \log\log t + \log\left\{1+O\left(\frac{1}{t\log t}\right)\right\}$$
$$= \log\log t + O(1/t).$$

† Bohr and Landau (3), Littlewood (5).

Hence
$$|\{\log(-is)\}^{-k(s)}| = e^{-\mathbf{R}\{k(s)\log\log(-is)\}} = e^{-k(\sigma)\log\log t+O(1/t)}$$
$$= (\log t)^{-k(\sigma)}\{1+O(1/t)\}.$$

Hence $g(s)$ is bounded on the lines $\sigma = \sigma_1$ and $\sigma = \sigma_2$; and it is $O(\log^K t)$ for some K uniformly in the strip. Hence, by the theorem of Phragmén and Lindelöf, it is bounded in the strip. Hence

$$\log \zeta(s) = O\{(\log t)^{k(\sigma)}\},$$

i.e.
$$\nu(\sigma) \leqslant k(\sigma) = \frac{(\sigma-\sigma_1)b+(\sigma_2-\sigma)a}{\sigma_2-\sigma_1}. \tag{14.3.1}$$

Taking $\sigma = 3$, $\sigma_2 = 4$, $\nu(3) = 0$, $b = 0$, we obtain $a \geqslant 0$. Hence $\nu(\sigma) \geqslant 0$ for $\sigma > \frac{1}{2}$. Hence $\nu(\sigma) = 0$ for $\sigma > 1$.

Since $\nu(\sigma)$ is finite for every $\sigma > \frac{1}{2}$, we can take $a = \nu(\sigma_1)+\epsilon$, $b = \nu(\sigma_2)+\epsilon$ in (14.3.1). Making $\epsilon \to 0$, we obtain

$$\nu(\sigma) \leqslant \frac{(\sigma-\sigma_1)\nu(\sigma_2)+(\sigma_2-\sigma)\nu(\sigma_1)}{\sigma_2-\sigma_1},$$

i.e. $\nu(\sigma)$ *is a convex function of* σ. Hence it is continuous, and it is non-increasing since it is ultimately zero.

We can also show that $\zeta'(s)/\zeta(s)$ *has the same* ν*-function as* $\log \zeta(s)$. Let $\nu_1(s)$ be the ν-function of $\zeta'(s)/\zeta(s)$. Since

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{2\pi i} \int\limits_{|s-z|=\delta} \frac{\log \zeta(z)}{(s-z)^2}\, dz = O\left\{\frac{1}{\delta}(\log t)^{\nu(\sigma-\delta)+\epsilon}\right\},$$

we have
$$\nu_1(\sigma) \leqslant \nu(\sigma-\delta)$$

for every positive δ; and since $\nu(\sigma)$ is continuous it follows that

$$\nu_1(\sigma) \leqslant \nu(\sigma).$$

We can show, as in the case of $\nu(\sigma)$, that $\nu_1(\sigma)$ is non-increasing, and is zero for $\sigma \geqslant 3$. Hence for $\sigma < 3$

$$\log \zeta(s) = -\int_\sigma^3 \frac{\zeta'(x+it)}{\zeta(x+it)}\, dx - \log \zeta(3+it)$$

$$= O\left\{\int_\sigma^3 (\log t)^{\nu_1(x)+\epsilon}\, dx\right\} + O(1)$$

$$= O\{(\log t)^{\nu_1(\sigma)+\epsilon}\},$$

i.e.
$$\nu(\sigma) \leqslant \nu_1(\sigma).$$

The exact value of $\nu(\sigma)$ is not known for any value of σ less than 1. All we know is

THEOREM 14.3. *For* $\frac{1}{2} < \sigma < 1$,

$$1-\sigma \leqslant \nu(\sigma) \leqslant 2(1-\sigma).$$

The upper bound follows from Theorem 14.2 and the lower bound from Theorem 8.12. The same lower bound can, however, be obtained in another and in some respects simpler way, though this proof, unlike the former, depends essentially on the Riemann hypothesis. For the proof we require some new formulae.

14.4. THEOREM 14.4.† *As* $t \to \infty$,

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} e^{-\delta n} + \sum_{\rho} \delta^{s-\rho}\Gamma(\rho-s) + O(\delta^{\sigma-\frac{1}{2}} \log t), \tag{14.4.1}$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant \frac{9}{8}$, $e^{-\sqrt{t}} \leqslant \delta \leqslant 1$.

Taking $a_n = \Lambda(n)$, $f(s) = -\zeta'(s)/\zeta(s)$ in the lemma of § 7.9, we have

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} e^{-\delta n} = -\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma(z-s) \frac{\zeta'(z)}{\zeta(z)} \delta^{s-z}\, dz. \tag{14.4.2}$$

Now, by Theorem 9.6 (A),

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{|t-\gamma|<1} \frac{1}{s-\frac{1}{2}-i\gamma} + O(\log t),$$

and there are $O(\log t)$ terms in the sum. Hence

$$\frac{\zeta'(s)}{\zeta(s)} = O(\log t)$$

on any line $\sigma \neq \frac{1}{2}$. Also

$$\frac{\zeta'(s)}{\zeta(s)} = O\left(\frac{\log t}{\min|t-\gamma|}\right) + O(\log t)$$

uniformly for $-1 \leqslant \sigma \leqslant 2$. Since each interval $(n, n+1)$ contains values of t whose distance from the ordinate of any zero exceeds $A/\log n$, there is a t_n in any such interval for which

$$\frac{\zeta'(s)}{\zeta(s)} = O(\log^2 t) \quad (-1 \leqslant \sigma \leqslant 2,\ t = t_n).$$

† Littlewood (5), to the end of § 14.8.

By the theorem of residues,

$$\frac{1}{2\pi i}\left(\int_{2-it_n}^{2+it_n} + \int_{2+it_n}^{\frac{1}{4}+it_n} + \int_{\frac{1}{4}+it_n}^{\frac{1}{4}-it_n} + \int_{\frac{1}{4}-it_n}^{2-it_n}\right)\Gamma(z-s)\frac{\zeta'(z)}{\zeta(z)}\delta^{s-z}\, dz$$

$$= \frac{\zeta'(s)}{\zeta(s)} + \sum_{-t_n<\gamma<t_n} \Gamma(\rho-s)\delta^{s-\rho} - \Gamma(1-s)\delta^{s-1}.$$

The integrals along the horizontal sides tend to zero as $n \to \infty$, so that

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} e^{-\delta n} = -\frac{1}{2\pi i} \int_{\frac{1}{4}-i\infty}^{\frac{1}{4}+i\infty} \Gamma(z-s)\frac{\zeta'(z)}{\zeta(z)}\delta^{s-z}\, dz -$$

$$-\frac{\zeta'(s)}{\zeta(s)} - \sum_{\rho} \Gamma(\rho-s)\delta^{s-\rho} + \Gamma(1-s)\delta^{s-1}.$$

Since $\Gamma(z-s) = O(e^{-A|y-t|})$, the integral is

$$O\left\{\int_{-\infty}^{\infty} e^{-A|y-t|} \log(|y|+2)\delta^{\sigma-\frac{1}{4}}\, dy\right\}$$

$$= O\left\{\int_{0}^{2t} e^{-A|y-t|} \log(|2t|+2)\delta^{\sigma-\frac{1}{4}}\, dy\right\} +$$

$$+ O\left\{\left(\int_{-\infty}^{0} + \int_{2t}^{\infty}\right) e^{-\frac{1}{2}A|y|} \log(|y|+2)\delta^{\sigma-\frac{1}{4}}\, dy\right\}$$

$$= O(\delta^{\sigma-\frac{1}{4}} \log t) + O(\delta^{\sigma-\frac{1}{4}}) = O(\delta^{\sigma-\frac{1}{4}} \log t).$$

Also

$$\Gamma(1-s)\delta^{s-1} = O(e^{-At}\delta^{\sigma-1}) = O(e^{-At}\delta^{-\frac{1}{2}})$$

$$= O(e^{-At+\frac{1}{2}\sqrt{t}}) = O(e^{-At}) = O(\delta^{\sigma-\frac{1}{4}} \log t).$$

This proves the theorem.

14.5. We can now prove more precise results about $\zeta'(s)/\zeta(s)$ and $\log \zeta(s)$ than those expressed by the inequality $\nu(\sigma) \leqslant 2-2\sigma$.

THEOREM 14.5. *We have*

$$\frac{\zeta'(s)}{\zeta(s)} = O\{(\log t)^{2-2\sigma}\}, \tag{14.5.1}$$

$$\log \zeta(s) = O\left\{\frac{(\log t)^{2-2\sigma}}{\log\log t}\right\}, \tag{14.5.2}$$

uniformly for $\frac{1}{2} < \sigma_0 \leqslant \sigma \leqslant \sigma_1 < 1$.

We have

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| \leqslant \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma} e^{-\delta n} + \delta^{\sigma-\frac{1}{2}} \sum_{\rho} |\Gamma(\rho-s)| + O(\delta^{\sigma-\frac{1}{4}} \log t).$$

Now

$$\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{\sigma}}e^{-\delta n} = -\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Gamma(z-\sigma)\frac{\zeta'(z)}{\zeta(z)}\delta^{\sigma-z}\,dz = O(\delta^{\sigma-1}),$$

since we may move the line of integration to $\mathbf{R}(z) = \frac{1}{4}$, and the leading term is the residue at $z = 1$. Also

$$|\Gamma(\rho-s)| < Ae^{-A|\gamma-t|}$$

uniformly for σ in the above range. Hence

$$\sum|\Gamma(\rho-s)| < A\sum_{\gamma}e^{-A|t-\gamma|} = A\sum_{n=1}^{\infty}\sum_{n-1\leqslant|t-\gamma|<n}e^{-A|t-\gamma|}.$$

The number of terms in the inner sum is

$$O\{\log(t+n)\} = O(\log t)+O\{\log(n+1)\}.$$

Hence we obtain

$$O\Big[\sum_{n=1}^{\infty}e^{-An}\{\log t+\log(n+1)\}\Big] = O(\log t).$$

Hence $$\frac{\zeta'(s)}{\zeta(s)} = O(\delta^{\sigma-1})+O(\delta^{\sigma-\frac{1}{2}}\log t)+O(\delta^{\sigma-\frac{1}{4}}\log t),$$

and taking $\delta = (\log t)^{-2}$ we obtain the first result.

Again for $\sigma_0 \leqslant \sigma \leqslant \sigma_1$

$$\log\zeta(s) = \log\zeta(\sigma_1+it)-\int_{\sigma}^{\sigma_1}\frac{\zeta'(x+it)}{\zeta(x+it)}\,dx$$

$$= O\{(\log t)^{2-2\sigma_1+\epsilon}\}+O\Big\{\int_{\sigma}^{\sigma_1}(\log t)^{2-2x}\,dx\Big\}$$

$$= O\{(\log t)^{2-2\sigma_1+\epsilon}\}+O\Big\{\frac{(\log t)^{2-2\sigma}}{\log\log t}\Big\}.$$

If $\sigma \leqslant \sigma_2 < \sigma_1$ and $\epsilon < 2(\sigma_1-\sigma_2)$, this is of the required form; and since σ_1 and so σ_2 may be as near to 1 as we please, the second result (with σ_2 for σ_1) follows.

14.6. To obtain the alternative proof of the inequality $\nu(\sigma) \geqslant 1-\sigma$ we require an approximate formula for $\log\zeta(s)$.

THEOREM 14.6. *For fixed α and σ such that $\frac{1}{2} < \alpha < \sigma \leqslant 1$, and $e^{-\sqrt{t}} \leqslant \delta \leqslant 1$,*

$$\log\zeta(s) = \sum_{n=1}^{\infty}\frac{\Lambda_1(n)}{n^{s}}e^{-\delta n}+O\{\delta^{\sigma-\alpha}(\log t)^{\nu(\alpha)+\epsilon}\}+O(1).$$

Moving the line of integration in (14.4.2) to $\mathbf{R}(w) = \alpha$, we have

$$\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{s}}e^{-\delta n} = -\frac{\zeta'(s)}{\zeta(s)}-\Gamma(1-s)\,\delta^{s-1}-\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\Gamma(z-s)\frac{\zeta'(z)}{\zeta(z)}\delta^{s-z}\,dz.$$

Since $\zeta'(s)/\zeta(s)$ has the ν-function $\nu(\sigma)$, the integral is of the form

$$O\Big\{\delta^{\sigma-\alpha}\int_{-\infty}^{\infty}e^{-A|y-t|}\{\log(|y|+2)\}^{\nu(\alpha)+\epsilon}\,dy\Big\} = O\{\delta^{\sigma-\alpha}(\log t)^{\nu(\alpha)+\epsilon}\};$$

and $\Gamma(1-s)\delta^{s-1}$ is also of this form, as in § 14.4. Hence

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{s}}e^{-\delta n}+O\{\delta^{\sigma-\alpha}(\log t)^{\nu(\alpha)+\epsilon}\}.$$

This result holds uniformly in the range $[\sigma, \frac{9}{8}]$, and so we may integrate over this interval. We obtain

$$\log\zeta(s)-\sum_{n=1}^{\infty}\frac{\Lambda_1(n)}{n^{s}}e^{-\delta n}+O\{\delta^{\sigma-\alpha}(\log t)^{\nu(\alpha)+\epsilon}\}$$

$$= \log\zeta(\tfrac{9}{8}+it)-\sum_{n=1}^{\infty}\frac{\Lambda_1(n)}{n^{\frac{9}{8}+it}}e^{-\delta n} = O(1),$$

as required.

14.7. Proof that $\nu(\sigma) \geqslant 1-\sigma$. Theorem 14.6 enables us to extend the method of Diophantine approximation, already used for $\sigma > 1$, to values of σ between $\frac{1}{2}$ and 1. It gives

$$\log|\zeta(s)| = \sum_{n=1}^{\infty}\frac{\Lambda_1(n)}{n^{\sigma}}\cos(t\log n)e^{-\delta n}+O\{\delta^{\sigma-\alpha}(\log t)^{\nu(\alpha)+\epsilon}\}+O(1),$$

$$= \sum_{n=1}^{N}\frac{\Lambda_1(n)}{n^{\sigma}}\cos(t\log n)e^{-\delta n}+O\Big(\sum_{n=N+1}^{\infty}e^{-\delta n}\Big)+O\{\delta^{\sigma-\alpha}(\log t)^{\nu(\alpha)+\epsilon}\}+O(1)$$

for all values of N. Now by Dirichlet's theorem (§ 8.2) there is a number t in the range $2\pi \leqslant t \leqslant 2\pi q^{N}$, and integers $x_1,\ldots, x_N$, such that

$$\left|t\frac{\log n}{2\pi}-x_n\right| \leqslant \frac{1}{q} \quad (n = 1, 2,\ldots, N).$$

Let us assume for the moment that this number t satisfies the condition of Theorem 14.6 that $e^{-\sqrt{t}} \leqslant \delta$. It gives

$$\sum_{n=1}^{N}\frac{\Lambda_1(n)}{n^{\sigma}}\cos(t\log n)e^{-\delta n} \geqslant \sum_{n=1}^{N}\frac{\Lambda_1(n)}{n^{\sigma}}\cos\frac{2\pi}{q}e^{-\delta n}$$

$$= \sum_{n=1}^{N}\frac{\Lambda_1(n)}{n^{\sigma}}e^{-\delta n}+O\Big(\frac{1}{q}\Big)\sum_{n=1}^{N}\frac{1}{n^{\sigma}}.$$

Now

$$\sum_{n=1}^{N}\frac{\Lambda_1(n)}{n^\sigma}e^{-\delta n}\geqslant\frac{1}{\log N}\sum_{n=1}^{N}\frac{\Lambda(n)}{n^\sigma}e^{-\delta n}$$

$$\geqslant\frac{1}{\log N}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^\sigma}e^{-\delta n}+O\Big(\sum_{n=N+1}^{\infty}e^{-\delta n}\Big)$$

$$>\frac{K(\sigma)\delta^{\sigma-1}}{\log N}+O\Big(\frac{e^{-\delta N}}{\delta}\Big)$$

as in § 14.5. Hence

$$\log|\zeta(s)|>\frac{K(\sigma)\delta^{\sigma-1}}{\log N}+O\Big(\frac{e^{-\delta N}}{\delta}\Big)+O\Big(\frac{N^{1-\sigma}}{q}\Big)+O\{\delta^{\sigma-\alpha}(\log t)^{\nu(\alpha)+\epsilon}\}+O(1).$$

Take $q=N=[\delta^{-a}]$, where $a>1$. The second and third terms on the right are then bounded. Also

$$\log t\leqslant N\log q+\log 2\pi\leqslant\frac{a}{\delta^a}\log\frac{1}{\delta}+\log 2\pi,$$

so that $$\delta\leqslant K(\log t)^{-1/a+\epsilon}.$$

Hence $$\log|\zeta(s)|>K(\log t)^{1-\sigma-\eta}+O\{(\log t)^{\alpha-\sigma+\nu(\alpha)+\eta'}\},$$

where η and η' are functions of a which tend to zero as $a\to 1$.

If the first term on the right is of larger order than the second, it follows at once that $\nu(\sigma)\geqslant 1-\sigma$. Otherwise

$$\alpha-\sigma+\nu(\alpha)\geqslant 1-\sigma,$$

and making $\alpha\to\sigma$ the result again follows.

We have still to show that the t of the above argument satisfies $e^{-\sqrt{t}}\leqslant\delta$. Suppose on the contrary that $\delta<e^{-\sqrt{t}}$ for some arbitrarily small values of δ. Now, by (8.4.4),

$$|\zeta(s)|\geqslant\Big(\cos\frac{2\pi}{q}-2N^{1-\sigma}\Big)\zeta(\sigma)>\frac{A}{\sigma-1}(\tfrac{1}{2}-2N^{1-\sigma})$$

for $\sigma>1$, $q\geqslant 6$. Taking $\sigma=1+\log 8/\log N$,

$$|\zeta(s)|>\frac{A}{\sigma-1}=A\log N>A\log\frac{1}{\delta}>At^{\frac{1}{2}}.$$

Since $|\zeta(s)|\to\infty$ and $t\geqslant 2\pi$, $t\to\infty$, and the above result contradicts Theorem 3.5. This completes the proof.

14.8. The function $\zeta(1+it)$. We are now in a position to obtain fairly precise information about this function. We shall first prove

THEOREM 14.8. *We have*

$$|\log\zeta(1+it)|\leqslant\log\log\log t+A. \tag{14.8.1}$$

In particular

$$\zeta(1+it)=O(\log\log t), \tag{14.8.2}$$

$$\frac{1}{\zeta(1+it)}=O(\log\log t). \tag{14.8.3}$$

Taking $\sigma=1$, $\alpha=\frac{3}{4}$ in Theorem 14.6, we have

$$|\log\zeta(1+it)|\leqslant\sum_{n=1}^{\infty}\frac{\Lambda_1(n)}{n}e^{-\delta n}+O(\delta^{\frac{1}{4}}\log t)+O(1)$$

$$\leqslant\sum_{n=1}^{N}\frac{\Lambda_1(n)}{n}+\sum_{n=N+1}^{\infty}e^{-\delta n}+O(\delta^{\frac{1}{4}}\log t)+O(1)$$

$$\leqslant\log\log N+O(e^{-\delta N}/\delta)+O(\delta^{\frac{1}{4}}\log t)+O(1)$$

by (3.14.4). Taking $\delta=\log^{-4}t$, $N=1+[\log^5 t]$, the result follows.

Comparing this result with Theorems 8.5 and 8.8, we see that, as far as the order of the functions $\zeta(1+it)$ and $1/\zeta(1+it)$ is concerned, the result is final. It remains to consider the values of the constants involved in the inequalities.

14.9. We define a function $\beta(\sigma)$ as

$$\beta(\sigma)=\frac{\nu(\sigma)}{2-2\sigma}.$$

By the convexity of $\nu(\sigma)$ we have, for $\frac{1}{2}<\sigma<\sigma'<1$,

$$\nu(\sigma')\leqslant\frac{(1-\sigma')\nu(\sigma)+(\sigma'-\sigma)\nu(1)}{1-\sigma}=\frac{1-\sigma'}{1-\sigma}\nu(\sigma),$$

i.e. $$\beta(\sigma')\leqslant\beta(\sigma).$$

Thus $\beta(\sigma)$ is non-increasing in $(\frac{1}{2},1)$. We write

$$\beta(\tfrac{1}{2})=\lim_{\sigma\to\frac{1}{2}+0}\beta(\sigma),\qquad\beta(1)=\lim_{\sigma\to 1-0}\beta(\sigma).$$

Then by Theorem 14.3, for $\frac{1}{2}<\sigma<1$,

$$\tfrac{1}{2}\leqslant\beta(1)\leqslant\beta(\sigma)\leqslant\beta(\tfrac{1}{2})\leqslant 1.$$

We shall now prove†

THEOREM 14.9. *As* $t\to\infty$

$$|\zeta(1+it)|\leqslant 2\beta(1)e^\gamma\{1+o(1)\}\log\log t, \tag{14.9.1}$$

$$\frac{1}{|\zeta(1+it)|}\leqslant 2\beta(1)\frac{6e^\gamma}{\pi^2}\{1+o(1)\}\log\log t. \tag{14.9.2}$$

† Littlewood (6).

We observe that the $O(1)$ in Theorem 14.6 is actually $o(1)$ if $\delta \to 0$. Also, taking $\sigma = 1$,

$$\delta^{1-\alpha}(\log t)^{\nu(\alpha)+\epsilon} = o(1)$$

if
$$\delta = (\log t)^{-2\beta(\alpha)-\eta} \quad (\eta > 0).$$

Hence, for such δ,

$$\log\zeta(1+it) = \sum_{n=1}^{\infty}\frac{\Lambda_1(n)}{n^{1+it}}e^{-\delta n}+o(1)$$
$$= \sum_{p,m}\frac{e^{-\delta p^m}}{mp^{m(1+it)}}+o(1)$$
$$= \sum_{p,m}\frac{e^{-\delta mp}}{mp^{m(1+it)}}+\sum_p\sum_{m>1}\frac{e^{-\delta p^m}-e^{-\delta mp}}{mp^{m(1+it)}}+o(1).$$

Now the modulus of the second double sum does not exceed

$$\sum_p\sum_{m>1}\frac{e^{-\delta p^m}-e^{-\delta mp}}{p^m}.$$

This is evidently uniformly convergent for $\delta \geqslant 0$, the summand being less than p^{-m}. Since each term tends to zero with δ the sum is $o(1)$. Hence

$$\log\zeta(1+it) = \sum_{p,m}\frac{e^{-\delta mp}}{mp^{m(1+it)}}+o(1)$$
$$= -\sum_p\log\left(1-\frac{e^{-\delta p}}{p^{1+it}}\right)+o(1)$$
$$= -\sum_{p\leqslant \varpi}\log\left(1-\frac{e^{-\delta p}}{p^{1+it}}\right)+O\left(\sum_{\varpi+1}^{\infty}e^{-\delta n}\right)+o(1).$$

The second term is $O(e^{-\delta\varpi}/\delta) = o(1)$ if $\varpi = [\delta^{-1-\epsilon}]$. Also

$$1-\frac{1}{p} \leqslant \left|1-\frac{e^{-\delta p}}{p^{1+it}}\right| \leqslant 1+\frac{1}{p}.$$

Hence, by (3.15.2),

$$\log|\zeta(1+it)| \leqslant -\sum_{p\leqslant\varpi}\log\left(1-\frac{1}{p}\right)+o(1)$$
$$= \log\log\varpi+\gamma+o(1),$$

or
$$|\zeta(1+it)| \leqslant e^{\gamma+o(1)}\log\varpi.$$

Now
$$\log\varpi \leqslant (1+\epsilon)\log\frac{1}{\delta} = (1+\epsilon)\{2\beta(\alpha)+\eta\}\log\log t,$$

and taking α arbitrarily near to 1, we obtain (14.9.1). Similarly, by (3.15.3),

$$\log\frac{1}{|\zeta(1+it)|} \leqslant \sum_{p\leqslant\varpi}\log\left(1+\frac{1}{p}\right)+o(1)$$
$$= \log\log\varpi+\log\frac{6e^{\gamma}}{\pi^2}+o(1),$$

and (14.9.2) follows from this.

Comparing Theorem 14.9 with Theorems 8.9 (A) and (B), we see that, since we know only that $\beta(1) \leqslant 1$, in each problem a factor 2 remains in doubt. It is possible that $\beta(1) = \frac{1}{2}$, and if this were so each constant would be determined exactly.

14.10. The function $S(t)$. We shall next discuss the behaviour of this function on the Riemann hypothesis.

If $\frac{1}{2} < \alpha < \sigma < \beta$, $T < t < T'$, we have

$$\log\zeta(s) = \frac{1}{2\pi i}\left(\int_{\beta+iT}^{\beta+iT'}+\int_{\beta+iT'}^{\alpha+iT'}+\int_{\alpha+iT'}^{\alpha+iT}+\int_{\alpha+iT}^{\beta+iT}\right)\frac{\log\zeta(z)}{z-s}\,dz.$$

Let $\beta > 2$. By (14.2.2),

$$\int_{\alpha+iT}^{2+iT}\frac{\log\zeta(z)}{z-s}\,dz = O\left\{\frac{1}{t-T}\int_{\alpha}^{2}|\log\zeta(x+iT)|\,dx\right\} = O\left(\frac{\log T}{t-T}\right).$$

Also
$$\int_{2+iT}^{\beta+iT}\frac{\log\zeta(z)}{z-s}\,dz = \sum_{n=2}^{\infty}\Lambda_1(n)\int_{2+iT}^{\beta+iT}\frac{n^{-z}}{z-s}\,dz.$$

Now

$$\int_{2+iT}^{\beta+iT}\frac{n^{-z}}{z-s}\,dz = \left[\frac{-n^{-z}}{(z-s)\log n}\right]_{2+iT}^{\beta+iT}-\frac{1}{\log n}\int_{2+iT}^{\beta+iT}\frac{n^{-z}}{(z-s)^2}\,dz$$
$$= O\left\{\frac{1}{n^2(t-T)}\right\}+O\left\{\frac{1}{n^2}\int_{-\infty}^{\infty}\frac{dx}{(x-\sigma)^2+(t-T)^2}\right\} = O\left\{\frac{1}{n^2(t-T)}\right\}.$$

Hence
$$\int_{2+iT}^{\beta+iT}\frac{\log\zeta(z)}{z-s}\,dz = O\left(\frac{1}{t-T}\right),$$

and hence
$$\int_{\alpha+iT}^{\beta+iT}\frac{\log\zeta(z)}{z-s}\,dz = O\left(\frac{\log T}{t-T}\right)$$

uniformly with respect to β. Similarly for the integral over $(\beta+iT', \alpha+iT')$.

Also $$\int_{\beta+iT}^{\beta+iT'} \frac{\log\zeta(z)}{z-s}\,dz = O\left(\frac{T'-T}{\beta-\sigma}\right).$$

Making $\beta\to\infty$, it follows that

$$\log\zeta(s) = \frac{1}{2\pi i}\int_{\alpha+iT}^{\alpha+iT'} \frac{\log\zeta(z)}{s-z}\,dz + O\left(\frac{\log T}{t-T}\right) + O\left(\frac{\log T'}{T'-t}\right). \tag{14.10.1}$$

A similar argument shows that, if $\mathbf{R}(s') < \frac{1}{2}$,

$$0 = \frac{1}{2\pi i}\int_{\alpha+iT}^{\alpha+iT'} \frac{\log\zeta(z)}{s'-z}\,dz + O\left(\frac{\log T}{t-T}\right) + O\left(\frac{\log T'}{T'-t}\right). \tag{14.10.2}$$

Taking $s' = 2\alpha-\sigma+it$, so that

$$s'-z = 2\alpha-\sigma+it-(\alpha+iy) = \alpha-iy-(\sigma-it),$$

and replacing (14.10.2) by its conjugate, we have

$$0 = \frac{1}{2\pi i}\int_{\alpha+iT}^{\alpha+iT'} \frac{\log|\zeta(z)|-i\arg\zeta(z)}{z-s}\,dz + O\left(\frac{\log T}{t-T}\right) + O\left(\frac{\log T'}{T'-t}\right). \tag{14.10.3}$$

From (14.10.1) and (14.10.3) it follows that

$$\log\zeta(s) = \frac{1}{\pi i}\int_{\alpha+iT}^{\alpha+iT'} \frac{\log|\zeta(z)|}{s-z}\,dz + O\left(\frac{\log T}{t-T}\right) + O\left(\frac{\log T'}{T'-t}\right) \tag{14.10.4}$$

and $$\log\zeta(s) = \frac{1}{\pi}\int_{\alpha+iT}^{\alpha+iT'} \frac{\arg\zeta(z)}{s-z}\,dz + O\left(\frac{\log T}{t-T}\right) + O\left(\frac{\log T'}{T'-t}\right). \tag{14.10.5}$$

14.11. We can now show that each of the functions

$$\max\{\log|\zeta(s)|, 0\}, \qquad \max\{-\log|\zeta(s)|, 0\},$$
$$\max\{\arg\zeta(s), 0\}, \qquad \max\{-\arg\zeta(s), 0\}$$

has the same ν-function as $\log\zeta(s)$. Consider, for example,

$$\max\{\arg\zeta(s), 0\},$$

and let its ν-function be $\nu_1(\sigma)$. Since

$$|\arg\zeta(s)| \leqslant |\log\zeta(s)|$$

we have at once $$\nu_1(\sigma) \leqslant \nu(\sigma).$$

Also (14.10.5) gives

$$\arg\zeta(s) = \frac{1}{\pi}\int_T^{T'} \frac{\sigma-\alpha}{(\sigma-\alpha)^2+(t-y)^2}\arg\zeta(\alpha+iy)\,dy + O\left(\frac{\log T}{t-T}\right) + O\left(\frac{\log T'}{T'-t}\right) \tag{14.11.1}$$

$$< A(\log T')^{\nu_1(\alpha)+\epsilon}\int_T^{T'} \frac{\sigma-\alpha}{(\sigma-\alpha)^2+(t-y)^2}\,dy + O\left(\frac{\log T}{t-T}\right) + O\left(\frac{\log T'}{T'-t}\right)$$

$$< A(\log t)^{\nu_1(\alpha)+\epsilon} + O(t^{-1}\log t),$$

taking, for example, $T = \frac{1}{2}t$, $T' = 2t$.

It is clear from this that $\nu_1(\sigma)$ is non-increasing. Also the Borel–Carathéodory inequality, applied to circles with centre $2+it$ and radii $2-\alpha-\delta$, $2-\alpha-2\delta$, gives

$$|\log\zeta(\alpha+\delta+it)| < \frac{A}{\delta}\left\{(\log t)^{\nu_1(\alpha)+\epsilon} + \frac{\log t}{t}\right\} + \frac{A}{\delta}|\log|\zeta(2+it)||.$$

If $\alpha+\delta < 1$, so that $\nu(\alpha+\delta) > 0$, it follows that

$$\nu(\alpha+\delta) \leqslant \nu_1(\alpha)+\epsilon.$$

Since ϵ and δ may be as small as we please, and $\nu(\sigma)$ is continuous, it follows that $$\nu(\alpha) \leqslant \nu_1(\alpha).$$

Hence $$\nu_1(\sigma) = \nu(\sigma) \quad (\tfrac{1}{2} < \sigma < 1).$$

Similarly all the ν-functions are equal.

14.12. *Ω-results† for $S(t)$ and $S_1(t)$.*

THEOREM 14.12 (A). *Each of the inequalities*

$$S(t) > (\log t)^{\frac{1}{2}-\epsilon}, \tag{14.12.1}$$
$$S(t) < -(\log t)^{\frac{1}{2}-\epsilon} \tag{14.12.2}$$

has solutions for arbitrarily large values of t.

Making $\alpha\to\frac{1}{2}$ in (14.11.1), by bounded convergence

$$\arg\zeta(s) = \int_{\frac{1}{2}t}^{2t} \frac{\sigma-\frac{1}{2}}{(\sigma-\frac{1}{2})^2+(t-y)^2}S(y)\,dy + O\left(\frac{\log t}{t}\right) \quad (\sigma > \tfrac{1}{2}). \tag{14.12.3}$$

If $S(t) < \log^{\alpha} t$ for all large t, this gives

$$\arg\zeta(s) < A\log^{\alpha} t\int_{\frac{1}{2}t}^{2t} \frac{\sigma-\frac{1}{2}}{(\sigma-\frac{1}{2})^2+(t-y)^2}\,dy + O\left(\frac{\log t}{t}\right)$$

$$< A\log^{\alpha} t + O(t^{-1}\log t).$$

† Landau (1), Bohr and Landau (3), Littlewood (5).

The above analysis shows that this is false if $a < \nu(\sigma)$, which is satisfied if $a < \frac{1}{2}$ and σ is near enough to $\frac{1}{2}$. This proves the first result, and the other may be proved similarly.

Theorem 14.12 (B).

$$S_1(t) = \Omega\{(\log t)^{\frac{1}{2}-\epsilon}\}.$$

From (14.10.5) with $\alpha \to \frac{1}{2}$ we have

$$\log \zeta(s) = i \int_{\frac{1}{2}t}^{2t} \frac{S(y)}{s-\frac{1}{2}-iy}\, dy + O(1)$$

$$= i\left[\frac{S_1(y)}{s-\frac{1}{2}-iy}\right]_{\frac{1}{2}t}^{2t} + \int_{\frac{1}{2}t}^{2t} \frac{S_1(y)}{(s-\frac{1}{2}-iy)^2}\, dy + O(1)$$

$$= \int_{\frac{1}{2}t}^{2t} \frac{S_1(y)}{(s-\frac{1}{2}-iy)^2}\, dy + O(1) \tag{14.12.4}$$

since $S_1(y) = O(\log y)$. The result now follows as before.

In view of the result of Selberg stated in § 9.9, this theorem is true independently of the Riemann hypothesis. In the case of $S(t)$, Selberg's method gives only an index $\frac{1}{3}$ instead of the index $\frac{1}{2}$ obtained on the Riemann hypothesis.

14.13. We now turn to results of the opposite kind.† We know that without any hypothesis

$$S(t) = O(\log t), \qquad S_1(t) = O(\log t),$$

and that on the Lindelöf hypothesis, and *a fortiori* on the Riemann hypothesis, each O can be replaced by o. On the Riemann hypothesis we should expect something more precise. The result actually obtained is

Theorem 14.13.

$$S(t) = O\left(\frac{\log t}{\log\log t}\right), \tag{14.13.1}$$

$$S_1(t) = O\left(\frac{\log t}{(\log\log t)^2}\right). \tag{14.13.2}$$

We first prove three lemmas.

Lemma α. *Let*

$$\phi(t) = \max_{1 \leqslant u \leqslant t} |S_1(u)|,$$

so that $\phi(t)$ is non-decreasing, and $\phi(t) = O(\log t)$. Then

$$S(t) = O[\{\phi(2t)\log t\}^{\frac{1}{2}}].$$

† Landau (11), Cramér (1), Littlewood (4), Titchmarsh (3).

This is independent of the Riemann hypothesis. We have

$$N(t) = L(t) + R(t),$$

where $L(t)$ is defined by (9.3.1), and $R(t) = S(t) + O(1/t)$. Now

$$N(T+x) - N(T) \geqslant 0 \quad (0 < x < T).$$

Hence

$$R(T+x) - R(T) \geqslant -\{L(T+x) - L(T)\} > -Ax\log T.$$

Hence

$$\int_T^{T+x} R(t)\, dt = xR(T) + \int_0^x \{R(T+u) - R(T)\}\, du$$

$$> xR(T) - A\int_0^x u\log T\, du$$

$$> xR(T) - Ax^2\log T.$$

Hence

$$R(T) < \frac{1}{x}\int_T^{T+x} R(t)\, dt + Ax\log T$$

$$= \frac{S_1(T+x) - S_1(T)}{x} + O\left(\frac{1}{T}\right) + Ax\log T$$

$$= O\left\{\frac{\phi(2T)}{x}\right\} + O\left(\frac{1}{T}\right) + Ax\log T.$$

Taking $x = \{\phi(2T)/\log T\}^{\frac{1}{2}}$, the upper bound for $S(T)$ follows. Similarly by considering integrals over $(T-x, T)$ we obtain the lower bound.

Lemma β. *Let $\sigma \leqslant 1$, and let*

$$F(T) = \max|\log \zeta(s)| + \log^{\frac{1}{4}} T \quad \left(\sigma - \tfrac{1}{2} \geqslant \frac{1}{\log\log T}, \quad 4 \leqslant t \leqslant T\right).$$

Then

$$\log \zeta(s) = O\{F(T+1)e^{-A(\sigma-\frac{1}{2})\log\log T}\}$$

$$\left(\frac{1}{2} + \frac{1}{\log\log T} \leqslant \sigma \leqslant 2, \quad 4 \leqslant t \leqslant T\right).$$

We apply Hadamard's three-circles theorem as in § 14.2, but now take

$$\sigma_1 = \frac{3}{2} + \frac{1}{\log\log T}, \quad \eta = \tfrac{1}{4}, \quad \delta = \frac{1}{\log\log T}, \quad \sigma \leqslant \tfrac{5}{4}.$$

We obtain

$$M_2 < AM_3^a = AM_3(1/M_3)^{1-a},$$

where

$$M_3 \leqslant F(T+1),$$

and

$$1 - a = \log\frac{r_3}{r_2}\Big/\log\frac{r_3}{r_1} = \log\left(1 + \frac{\sigma-\frac{1}{2}-\delta}{\sigma_1-\sigma}\right)\Big/\log\left(\frac{\sigma_1-\frac{1}{2}-\delta}{\sigma_1-1-\eta}\right)$$

$$> A(\sigma-\tfrac{1}{2}-\delta).$$