

# PSEUDODIFFERENTIAL ARITHMETIC AND A REJECTION OF THE RIEMANN HYPOTHESIS

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**ABSTRACT.** The Weyl symbolic calculus of operators leads to the construction, if one takes for symbol a certain distribution in the plane decomposing over the set of zeros of the Riemann zeta function, of an operator with the following property: the Riemann hypothesis is equivalent to the validity of a collection of estimates involving this operator. Pseudodifferential arithmetic, a novel chapter of pseudodifferential operator theory, makes it possible to make the operator fully explicit. This leads in an unexpected way to a disproof of the conjecture in the following strong sense: zeros of zeta accumulate on the boundary of the critical strip.

## 1. INTRODUCTION

An understanding of the Riemann hypothesis cannot be dependent on complex analysis, in the style of the XIXth century, only. Though the analytic aspects of the Riemann zeta function must indeed contribute, arithmetic, especially prime numbers, must of necessity enter the discussion in a decisive role. The main part of the present paper consists in putting together two aspects of the same object – to wit, a certain hermitian form – one originating from its analytic nature and involving the zeros of the zeta function, and the other described in terms of congruence algebra. A complete understanding of the latter one is reached with the help of pseudodifferential arithmetic, a chapter of pseudodifferential operator theory adapted to the species of symbols needed here.

The central object of the proof to follow is the distribution

$$\mathfrak{T}_\infty(x, \xi) = \sum_{|j|+|k| \neq 0} a((j, k)) \delta(x - j) \delta(\xi - k) \quad (1.1)$$

in the plane, with  $(j, k) = \text{g.c.d.}(j, k)$  and  $a(r) = \prod_{p|r} (1 - p)$  for  $r = 1, 2, \dots$ . There is a collection  $(\mathfrak{E}_\nu)_{\nu \neq \pm 1}$  of so-called Eisenstein distributions,

$\mathfrak{E}_\nu$  homogeneous of degree  $-1 - \nu$ , such that, as an analytic functional,

$$\mathfrak{T}_\infty = 12 \delta_0 + \sum_{\zeta(\rho)=0} \text{Res}_{\nu=\rho} \left( \frac{\mathfrak{E}_{-\nu}}{\zeta(\nu)} \right), \quad (1.2)$$

where  $\delta_0$  is the unit mass at the origin of  $\mathbb{R}^2$ . This decomposition calls for using  $\mathfrak{T}_\infty$  in a possible approach to the zeros of the Riemann zeta function: only, to obtain a full benefit of this distribution, one must appeal to pseudodifferential analysis, more precisely to the Weyl symbolic calculus of operators [14]. This is the rule  $\Psi$  that associates to any distribution  $\mathfrak{S} \in \mathcal{S}'(\mathbb{R}^2)$  the so-called operator with symbol  $\mathfrak{S}$ , to wit the linear operator  $\Psi(\mathfrak{S})$  from  $\mathcal{S}(\mathbb{R})$  to  $\mathcal{S}'(\mathbb{R})$  weakly defined by the equation

$$(\Psi(\mathfrak{S}) u)(x) = \frac{1}{2} \int_{\mathbb{R}^2} \mathfrak{S} \left( \frac{x+y}{2}, \xi \right) e^{i\pi(x-y)\xi} u(y) dy d\xi. \quad (1.3)$$

It defines pseudodifferential analysis, which has been for more than half a century one of the main tools in the study of partial differential equations. However, the methods used in this regard in the present context do not intersect the usual ones (though familiarity with the more formal aspects of the Weyl calculus will certainly help) and may call for the denomination of “pseudodifferential arithmetic” (Sections 6 to 8).

Making use of the Euler operator  $2i\pi\mathcal{E} = 1 + x \frac{\partial}{\partial x} + \xi \frac{\partial}{\partial \xi}$  and of the collection of rescaling operators  $t^{2i\pi\mathcal{E}}$ , to wit

$$(t^{2i\pi\mathcal{E}} \mathfrak{S})(x, \xi) = t \mathfrak{S}(tx, t\xi), \quad t > 0. \quad (1.4)$$

we brought attention in [11] to the hermitian form  $(w | \Psi(Q^{2i\pi\mathcal{E}} \mathfrak{T}_\infty) w)$ . The following criterion for R.H. was obtained: that, for some  $\beta > 2$  and any function  $w \in C^\infty(\mathbb{R})$  supported in  $[0, \beta]$ , one should have for every  $\varepsilon > 0$

$$(w | \Psi(Q^{2i\pi\mathcal{E}} \mathfrak{T}_\infty) w) = O\left(Q^{\frac{1}{2}+\varepsilon}\right) \quad (1.5)$$

as  $Q \rightarrow \infty$  through squarefree integral values. The proof is based on a spectral-theoretic analysis of the operator  $\sum_{Q \text{ sqf}} Q^{-s+2i\pi\mathcal{E}}$  as a function of  $s \in \mathbb{C}$ : ultimately, if one assumes (1.5), a pole with no right to exist will be associated to any zero  $\rho$  of zeta such that  $\text{Re } \rho > \frac{1}{2}$ . We shall prove here a slightly different criterion, involving a pair  $v, u$  in place of  $w, w$ , in Section 5: the case when the supports of  $v$  and  $u$  do not intersect, nor do those of  $v$  and  $\overset{\vee}{u}$ , is especially important.

Eisenstein distributions will be described in detail in Section 3. They make up just a part of the domain of automorphic distribution theory,

which relates to the classical one of modular form theory but is more precise. The operator  $\pi^2 \mathcal{E}^2$  in the plane transfers under some map (the dual Radon transformation, in an arithmetic context) to the operator  $\Delta - \frac{1}{4}$  in the hyperbolic half-plane, where  $\Delta$  is the automorphic Laplacian. While this is crucial in other applications, it is another feature of automorphic distribution theory that will be essential here: the way it can cooperate with the Weyl symbolic calculus. Automorphic distribution theory has been developed in a series of books, ending with [10]. Its origin got some inspiration from the Lax-Phillips scattering theory of automorphic functions [5]: we shall develop this aspect in Section 12. Let us make it quite clear that the automorphy concepts will not be needed here, though both  $\mathfrak{T}_\infty$  and the Eisenstein distributions owe their existence to such considerations.

The following property of the operator  $\Psi(\mathfrak{E}_{-\nu})$  with symbol  $\mathfrak{E}_{-\nu}$  (shared by all operators with automorphic symbols) will be decisive in algebraic calculations: if  $v, u$  is a pair of  $C^\infty$  functions on the line such that  $0 < x^2 - y^2 < 8$  whenever  $\bar{v}(x)u(y) \neq 0$ , the hermitian form  $(v | \Psi(\mathfrak{E}_{-\nu}) u)$  depends only on the restriction of  $\bar{v}(x)u(y)$  to the set  $\{(x, y) : x^2 - y^2 = 4\}$ .

The first step towards a computation of the main hermitian form on the left-hand side of (1.5) (or its polarization) consists in transforming it into a finite-dimensional hermitian form. Given a positive integer  $N$ , one sets

$$\mathfrak{T}_N(x, \xi) = \sum_{j, k \in \mathbb{Z}} a((j, k, N)) \delta(x - j) \delta(\xi - k). \quad (1.6)$$

The distribution  $\mathfrak{T}_\infty$  is obtained as the limit, as  $N \nearrow \infty$  (a notation meant to express that  $N$  will be divisible by any given squarefree number, from a certain point on), of the distribution  $\mathfrak{T}_N^\times$  obtained from  $\mathfrak{T}_N$  by dropping the term corresponding to  $j = k = 0$ . If  $Q$  is squarefree, if the algebraic sum of supports of the functions  $v, u \in C^\infty(\mathbb{R})$  is contained in  $[0, 2\beta]$ , finally if  $N = RQ$  is a squarefree integer divisible by all primes less than  $\beta Q$ , one has

$$(v | \Psi(Q^{2i\pi\mathcal{E}} \mathfrak{T}_\infty) u) = (v | \Psi(Q^{2i\pi\mathcal{E}} \mathfrak{T}_N) u). \quad (1.7)$$

Now, the hermitian form on the right-hand side is amenable to an algebraic-arithmetic version. Indeed, transferring functions in  $\mathcal{S}(\mathbb{R})$  to functions on  $\mathbb{Z}/(2N^2)\mathbb{Z}$  under the linear map  $\theta_N$  defined by the equation

$$(\theta_N u)(n) = \sum_{\substack{n_1 \in \mathbb{Z} \\ n_1 \equiv n \pmod{2N^2}}} u\left(\frac{n_1}{N}\right), \quad n \pmod{2N^2}, \quad (1.8)$$

one obtains an identity

$$(v \mid \Psi(Q^{2i\pi\mathcal{E}} \mathfrak{T}_N) u) = \sum_{m, n \pmod{2N^2}} c_{R,Q}(\mathfrak{T}_N; m, n) \overline{\theta_N v(m)} (\theta_N u)(n). \quad (1.9)$$

The coefficients  $c_{R,Q}(\mathfrak{T}_N; m, n)$  are fully explicit, and the symmetric matrix defining this hermitian form has a Eulerian structure. The important point is to separate, so to speak, the  $R$ -factor and  $Q$ -factor in this expression.

Introduce the reflection  $n \mapsto \check{n}$  of  $\mathbb{Z}/(2N^2)\mathbb{Z}$  such that  $\check{n} \equiv n \pmod{R^2}$  and  $\check{n} \equiv -n \pmod{Q^2}$ . Then, defining  $\tilde{u}$  so that  $(\theta_N \tilde{u})(n) = (\theta_N u)(\check{n})$ , one has the identity

$$(v \mid \Psi(Q^{2i\pi\mathcal{E}} \mathfrak{T}_N) u) = \mu(Q) (v \mid \Psi(\mathfrak{T}_N) \tilde{u}), \quad (1.10)$$

which connects the purely arithmetic map  $n \mapsto \check{n}$  to the purely analytic rescaling operator  $Q^{2i\pi\mathcal{E}}$ .

The contributions that precede, to be developed in full in Sections 1 to 6, were for the most already published in [11]: specializing the parity of certain integers has led to a great simplification. It is at this point that, as a research paper, the present paper really starts. Under some strong support conditions relative to  $v, u$ , one obtains the identity

$$(v \mid \Psi(Q^{2i\pi\mathcal{E}} \mathfrak{T}_N) u) = \sum_{Q_1 Q_2 = Q} \mu(Q_1) \sum_{R_1 \mid R} \mu(R_1) \bar{v}\left(\frac{R_1}{Q_2} + \frac{Q_2}{R_1}\right) u\left(\frac{R_1}{Q_2} - \frac{Q_2}{R_1}\right). \quad (1.11)$$

In view of the role of this identity, we have given two quite different proofs of it. Nevertheless, just as what one would obtain from the use of the one-dimensional measure  $\sum_{k \neq 0} \mu(k) \delta(x - k)$ , it suffers from the presence of coefficients  $\mu(R_1)$ , with  $C_1 Q < R_1 < C_2 Q$ . However, if one constrains the support of  $u$  further, replacing for some  $\varepsilon > 0$   $u$  by  $u_Q$  such that

$u_Q(y) = Q^{\frac{\varepsilon}{2}} u(Q^\varepsilon y)$ , it follows from this identity that, if one assumes that  $v$  is supported in  $[2, \sqrt{8}]$  and that  $u$  is supported in  $[0, 1]$ , the function  $F_\varepsilon$  defined for  $\operatorname{Re} s$  large as the sum of the series

$$F_\varepsilon(s) = \sum_{Q \text{ sqf odd}} Q^{-s} (v \mid \Psi(Q^{2i\pi\varepsilon} \mathfrak{T}_\infty) u_Q) \quad (1.12)$$

is entire. This is a good substitute for our inability to prove the same about the function  $F_0(s)$  mentioned after (1.5): the function  $F_\varepsilon$  is much more tractable than  $F_0$ , and  $F_\varepsilon(s)$  goes to  $F_0(s)$  as  $\varepsilon \rightarrow 0$  if  $\operatorname{Re} s > 2$ . The whole question then depends on whether  $F_\varepsilon(s)$  has a limit as  $\varepsilon \rightarrow 0$  extends to the case when  $\operatorname{Re} s > \frac{3}{2}$ . Note the importance of having introduced two independent rescaling operators. But, doing so, we have destroyed the possibility to apply the criterion (1.5), and we must reconsider this question.

There are now two decompositions into homogeneous parts (of distributions in the plane or on the line) to be considered. Writing  $u = \frac{1}{i} \int_{\operatorname{Re} \mu = a} u^\mu d\mu$ , where  $u^\mu$  is homogeneous of degree  $-\frac{1}{2} - \mu$ , one obtains the formula

$$F_\varepsilon(s) = \frac{1}{2i\pi} \int_{\operatorname{Re} \nu = c} \frac{(1 - 2^{-\nu})^{-1}}{\zeta(\nu)} H_\varepsilon(s, \nu) d\nu, \quad c > 1, \quad (1.13)$$

with

$$H_\varepsilon(s, \nu) = \frac{1}{i} \int_{\operatorname{Re} \mu = a} f(s - \nu + \varepsilon\mu) (v \mid \Psi(\mathfrak{E}_{-\nu}) u^\mu) d\mu, \quad a < -\frac{3}{2}, \quad (1.14)$$

and

$$f(s) = (1 - 2^{-s})^{-1} \times \frac{\zeta(s)}{\zeta(2s)}. \quad (1.15)$$

Understanding the Riemann hypothesis then reduces to obtaining the largest half-plane in which  $F_\varepsilon(s)$  has a limit as  $\varepsilon \rightarrow 0$ . This depends on Cauchy-type analysis and, borrowing the language of scattering theory, on the decomposition of  $u^\mu$  into its ingoing and outgoing parts. The present proof diverges from what was expected to be a proof of the Riemann hypothesis only at the very end, leading to the fact that zeros of the Riemann zeta function accumulate on the boundary of the critical strip. Extending the result to the case of Dirichlet  $L$ -functions is straightforward.

In a last section, we shall recall how the concept of automorphic distribution relates to the Lax-Phillips automorphic scattering theory.

## 2. THE WEYL SYMBOLIC CALCULUS OF OPERATORS

In space-momentum coordinates, the Weyl calculus, or pseudodifferential calculus, depends on one free parameter with the dimension of action, called Planck's constant. In pure mathematics, even the more so when pseudodifferential analysis is applied to arithmetic, Planck's constant becomes a pure number: there is no question that the good such constant in "pseudodifferential arithmetic" is 2, as especially put into evidence [11, Chapter 6] in the pseudodifferential calculus of operators with automorphic symbols. To avoid encumbering the text with unnecessary subscripts, we shall denote as  $\Psi$  the rule denoted as  $\text{Op}_2$  in [11, (2.1.1)], to wit the rule that attaches to any distribution  $\mathfrak{S} \in \mathcal{S}'(\mathbb{R}^2)$  the linear operator  $\Psi(\mathfrak{S})$  from  $\mathcal{S}(\mathbb{R})$  to  $\mathcal{S}'(\mathbb{R})$  weakly defined as

$$(\Psi(\mathfrak{S})u)(x) = \frac{1}{2} \int_{\mathbb{R}^2} \mathfrak{S}\left(\frac{x+y}{2}, \xi\right) e^{i\pi(x-y)\xi} u(y) d\xi dy, \quad (2.1)$$

truly a superposition of integrals (integrate with respect to  $y$  first). The operator  $\Psi(\mathfrak{S})$  is called the operator with symbol  $\mathfrak{S}$ . Its integral kernel is the function

$$K(x, y) = \frac{1}{2} (\mathcal{F}_2^{-1} \mathfrak{S})\left(\frac{x+y}{2}, \frac{x-y}{2}\right), \quad (2.2)$$

where  $\mathcal{F}_2^{-1}$  denotes the inverse Fourier transformation with respect to the second variable.

Elementary cases of operators  $\Psi(\mathfrak{S})$  are the following. If  $\mathfrak{S}(x, \xi) = f(x)$ , the operator  $\Psi(\mathfrak{S})$  is the operator of multiplication by the function  $f$ . If  $\mathfrak{S}(x, \xi) = \delta(x)g(\xi)$ , one has  $[\Psi(\mathfrak{S})u](x) = (\mathcal{F}^{-1}g)(x)u(-x)$ .

If one defines the Wigner function  $\text{Wig}(v, u)$  of two functions in  $\mathcal{S}(\mathbb{R})$  as the function in  $\mathcal{S}(\mathbb{R}^2)$  such that

$$\text{Wig}(v, u)(x, \xi) = \int_{-\infty}^{\infty} \bar{v}(x+t) u(x-t) e^{2i\pi\xi t} dt, \quad (2.3)$$

one has

$$(v | \Psi(\mathfrak{S})u) = \langle \mathfrak{S}, \text{Wig}(v, u) \rangle, \quad (2.4)$$

with  $(v|u) = \int_{-\infty}^{\infty} \bar{v}(x) u(x) dx$ , while straight brackets refer to the bilinear operation of testing a distribution on a function. Immediately observe for future reference that if  $v$  and  $u$  are compactly supported, one can have  $\text{Wig}(v, u)(x, \xi) \neq 0$  only if  $2x$  lies in the algebraic sum of the supports of  $v$  and  $u$ .

Given two functions in  $\mathcal{S}(\mathbb{R})$ , the Wigner function of their Fourier transforms is

$$\text{Wig}(\widehat{v}, \widehat{u})(x, \xi) = \text{Wig}(v, u)(\xi, -x). \quad (2.5)$$

An incorrect proof (to make it correct, just use the definition of the Fourier transforms one at a time) goes as follows. The left-hand side of (2.5) is

$$\begin{aligned} \int_{\mathbb{R}^4} \overline{v}(r) u(s) \exp[2i\pi((x+t)r - (x-t)s)] e^{2i\pi t\xi} dr ds dt \\ = \int_{\mathbb{R}^2} \overline{v}(r) u(s) e^{2i\pi x(r-s)} \delta(r+s+\xi) dr ds, \end{aligned} \quad (2.6)$$

and setting  $r+s=\xi$ ,  $r-s=-\eta$  gives (2.5).

Another useful property of the calculus  $\Psi$  is expressed by the following two equivalent identities, obtained with the help of elementary manipulations of the Fourier transformation or with that of (2.2),

$$\Psi(\mathcal{F}^{\text{symp}} \mathfrak{S}) w = \Psi(\mathfrak{S}) \overset{\vee}{w}, \quad \mathcal{F}^{\text{symp}} \text{Wig}(v, u) = \text{Wig}(v, \overset{\vee}{u}), \quad (2.7)$$

where  $\overset{\vee}{w}(x) = w(-x)$  and the symplectic Fourier transformation in  $\mathbb{R}^2$  is defined in  $\mathcal{S}(\mathbb{R}^2)$  or  $\mathcal{S}'(\mathbb{R}^2)$  by the equation

$$(\mathcal{F}^{\text{symp}} \mathfrak{S})(x, \xi) = \int_{\mathbb{R}^2} \mathfrak{S}(y, \eta) e^{2i\pi(x\eta - y\xi)} dy d\eta. \quad (2.8)$$

Introduce the Euler operator

$$2i\pi\mathcal{E} = 1 + x \frac{\partial}{\partial x} + \xi \frac{\partial}{\partial \xi} \quad (2.9)$$

and, for  $t > 0$ , the operator  $t^{2i\pi\mathcal{E}}$  such that  $(t^{2i\pi\mathcal{E}} \mathfrak{S})(x, \xi) = t \mathfrak{S}(tx, t\xi)$ .

Denoting as  $U[2]$  the unitary rescaling operator such that  $(U[2] u)(x) = 2^{\frac{1}{4}} u(x\sqrt{2})$ , and setting  $\text{Resc} = 2^{-\frac{1}{2} + i\pi\mathcal{E}}$  or  $(\text{Resc} \mathfrak{S})(x, \xi) = \mathfrak{S}\left(2^{\frac{1}{2}}x, 2^{\frac{1}{2}}\xi\right)$ , one can connect the rule  $\Psi$  to the rule  $\text{Op}_1$  used in [11] and denoted as  $\text{Op}$  there by the equation [11, (2.1.14)]

$$U[2] \Psi(\mathfrak{S}) U[2]^{-1} = \text{Op}_1(\text{Resc} \mathfrak{S}). \quad (2.10)$$

This would enable us not to redo, in Section 6, the proof of the main result already written with another normalization in [11]: but, for self-containedness and simplicity (specializing certain parameters), we shall rewrite all proofs dependent on the symbolic calculus. The choice of the rule  $\Psi$  makes it possible to avoid splitting into cases, according to the parity

of the integers present there, the developments of pseudodifferential arithmetic in Section 6.

The following lemma will be useful later.

**Lemma 2.1.** *Given  $v, u \in \mathcal{S}(\mathbb{R})$ , one has*

$$(2i\pi\mathcal{E}) \operatorname{Wig}(v, u) = \operatorname{Wig}(v', xu) + \operatorname{Wig}(xv, u'). \quad (2.11)$$

*Proof.* One has  $\xi \frac{\partial}{\partial \xi} e^{2i\pi t \xi} = t \frac{\partial}{\partial t} e^{2i\pi t \xi}$ . The term  $\xi \frac{\partial}{\partial \xi} \operatorname{Wig}(v, u)$  is obtained from (2.3) with the help of an integration by parts, noting that the transpose of the operator  $t \frac{\partial}{\partial t}$  is  $-1 - t \frac{\partial}{\partial t}$ . Overall, using (2.10), one obtains

$$[(2i\pi\mathcal{E}) \operatorname{Wig}(v, u)](x, \xi) = \int_{-\infty}^{\infty} A(x, t) e^{2i\pi t \xi} dt, \quad (2.12)$$

with

$$\begin{aligned} A(x, t) &= x [\overline{v'}(x+t) u(x-t) + \overline{v}(x+t) u'(x-t)] \\ &\quad + t [-\overline{v'}(x+t) u(x-t) + \overline{v}(x+t) u'(x-t)] \\ &= (x-t) \overline{v'}(x+t) u(x-t) + (x+t) \overline{v}(x+t) u'(x-t). \end{aligned} \quad (2.13)$$

□

### 3. EISENSTEIN DISTRIBUTIONS

The objects in the present section can be found with more details in [11, Section 2.2]. Automorphic distributions are distributions in the Schwartz space  $\mathcal{S}'(\mathbb{R}^2)$ , invariant under the linear changes of coordinates associated to matrices in  $SL(2, \mathbb{Z})$ . It is the theory of automorphic and modular distributions, developed over a 20-year span, that led to the definition of the basic distribution  $\mathfrak{T}_\infty$  (4.5), and to that of Eisenstein distributions. However, the present disproof of R.H. depends strictly, and only, on pseudodifferential arithmetic and on the definition of Eisenstein distributions.



**Definition 3.1.** If  $\nu \in \mathbb{C}$ ,  $\operatorname{Re} \nu > 1$ , the Eisenstein distribution  $\mathfrak{E}_{-\nu}$  is defined by the equation, valid for every  $h \in \mathcal{S}(\mathbb{R}^2)$ ,

$$\langle \mathfrak{E}_{-\nu}, h \rangle = \sum_{|j|+|k| \neq 0} \int_0^\infty t^\nu h(jt, kt) dt. \quad (3.1)$$

It is immediate that the series of integrals converges if  $\operatorname{Re} \nu > 1$ , in which case  $\mathfrak{E}_{-\nu}$  is well defined as a tempered distribution. Indeed, writing  $|h(x, \xi)| \leq C(1 + |x| + |\xi|)^{-A}$  with  $A$  large, one has

$$\left| \int_0^\infty t^\nu h(jt, kt) dt \right| \leq C(|j| + |k|)^{-\operatorname{Re} \nu - 1} \int_0^\infty t^{\operatorname{Re} \nu} (1+t)^{-A} dt. \quad (3.2)$$

Obviously,  $\mathfrak{E}_{-\nu}$  is  $SL(2, \mathbb{Z})$ -invariant as a distribution, i.e., an automorphic distribution. It is homogeneous of degree  $-1 + \nu$ , i.e.,  $(2i\pi\mathcal{E}) \mathfrak{E}_{-\nu} = \nu \mathfrak{E}_{-\nu}$ : note that the transpose of  $2i\pi\mathcal{E}$  is  $-2i\pi\mathcal{E}$ . Its name stems from its relation (not needed here) with the classical notion of non-holomorphic Eisenstein series, as made explicit in [11, p.93]: it is, however, a more precise concept.

**Proposition 3.2.** *As a tempered distribution,  $\mathfrak{E}_{-\nu}$  extends as a meromorphic function of  $\nu \in \mathbb{C}$ , whose only poles are the points  $\nu = \pm 1$ : these poles are simple, and the residues of  $\mathfrak{E}_\nu$  there are*

$$\operatorname{Res}_{\nu=1} \mathfrak{E}_{-\nu} = 1 \quad \text{and} \quad \operatorname{Res}_{\nu=-1} \mathfrak{E}_{-\nu} = -\delta_0, \quad (3.3)$$

*the negative of the unit mass at the origin of  $\mathbb{R}^2$ . Recalling the definition (2.8) of the symplectic Fourier transformation  $\mathcal{F}^{\text{symp}}$ , one has, for  $\nu \neq \pm 1$ ,  $\mathcal{F}^{\text{symp}} \mathfrak{E}_{-\nu} = \mathfrak{E}_\nu$ .*

*Proof.* Denote as  $(\mathfrak{E}_{-\nu})_{\text{princ}}$  (resp.  $(\mathfrak{E}_{-\nu})_{\text{res}}$ ) the distribution defined in the same way as  $\mathfrak{E}_{-\nu}$ , except for the fact that in the integral (3.1), the interval of integration  $(0, \infty)$  is replaced by the interval  $(0, 1)$  (resp.  $(1, \infty)$ ), and observe that the distribution  $(\mathfrak{E}_{-\nu})_{\text{res}}$  extends as an entire function of  $\nu$ : indeed, it suffices to replace (3.2) by the inequality  $(1 + (|j| + |k|)t)^{-A} \leq C(1 + (|j| + |k|))^{-A}(1+t)^{-A}$  if  $t > 1$ . As a consequence of Poisson's formula,

one has when  $\operatorname{Re} \nu > 1$  the identity

$$\begin{aligned} \int_1^\infty t^{-\nu} \sum_{(j,k) \in \mathbb{Z}^2} (\mathcal{F}^{\text{symp}} h)(tk, tj) dt &= \int_1^\infty t^{-\nu} \sum_{(j,k) \in \mathbb{Z}^2} t^{-2} h(t^{-1}j, t^{-1}k) dt \\ &= \int_0^1 t^\nu \sum_{(j,k) \in \mathbb{Z}^2} h(tj, tk) dt, \end{aligned} \quad (3.4)$$

from which one obtains that

$$\langle \mathcal{F}^{\text{symp}}(\mathfrak{E}_\nu)_{\text{res}}, h \rangle = -\frac{(\mathcal{F}^{\text{symp}} h)(0, 0)}{1 - \nu} = \langle (\mathfrak{E}_{-\nu})_{\text{princ}}, h \rangle + \frac{h(0, 0)}{1 + \nu}. \quad (3.5)$$

From this identity, one finds the meromorphic continuation of the function  $\nu \mapsto \mathfrak{E}_\nu$ , including the residues at the two poles, as well as the fact that  $\mathfrak{E}_\nu$  and  $\mathfrak{E}_{-\nu}$  are the images of each other under  $\mathcal{F}^{\text{symp}}$ .  $\square$

So far as the zeta function is concerned, we recall its definition  $\zeta(s) = \sum_{n \geq 1} n^{-s} = \prod_p (1 - p^{-s})^{-1}$ , valid for  $\operatorname{Re} s > 1$ , and the fact that it extends as a meromorphic function in the entire complex plane, with a single simple pole, of residue 1, at  $s = 1$ ; also that, with  $\zeta^*(s) = \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s)$ , one has the functional equation  $\zeta^*(s) = \zeta^*(1 - s)$ .

**Lemma 3.3.** *One has if  $\nu \neq \pm 1$  the Fourier expansion*

$$\mathfrak{E}_{-\nu}(x, \xi) = \zeta(\nu) |\xi|^{\nu-1} + \zeta(1 + \nu) |x|^\nu \delta(\xi) + \sum_{r \neq 0} \sigma_{-\nu}(r) |\xi|^{\nu-1} \exp\left(2i\pi \frac{rx}{\xi}\right) \quad (3.6)$$

where  $\sigma_{-\nu}(r) = \sum_{1 \leq d|r} d^{-\nu}$ : the first two terms must be grouped when  $\nu = 0$ . Given  $b > \varepsilon > 0$ , the distribution  $(\nu - 1)\mathfrak{E}_{-\nu}$  remains in a bounded subset of  $\mathcal{S}'(\mathbb{R}^2)$  for  $\varepsilon \leq \operatorname{Re} \nu \leq b$ .

*Proof.* Isolating the term for which  $k = 0$  in (3.1), we write if  $\operatorname{Re} \nu > 1$ , after a change of variable,

$$\begin{aligned} \langle \mathfrak{E}_{-\nu}, h \rangle &= \zeta(1 + \nu) \int_{-\infty}^\infty |t|^\nu h(t, 0) dt + \frac{1}{2} \sum_{j \in \mathbb{Z}, k \neq 0} \int_{-\infty}^\infty |t|^\nu h(jt, kt) dt \\ &= \zeta(1 + \nu) \int_{-\infty}^\infty |t|^\nu h(t, 0) dt + \frac{1}{2} \sum_{j \in \mathbb{Z}, k \neq 0} \int_{-\infty}^\infty |t|^{\nu-1} (\mathcal{F}_1^{-1} h)\left(\frac{j}{t}, kt\right) dt, \end{aligned} \quad (3.7)$$

where we have used Poisson's formula at the end and denoted as  $\mathcal{F}_1^{-1}h$  the inverse Fourier transform of  $h$  with respect to the first variable. Isolating now the term such that  $j = 0$ , we obtain

$$\begin{aligned} \frac{1}{2} \sum_{j \in \mathbb{Z}, k \neq 0} \int_{-\infty}^{\infty} |t|^{\nu-1} (\mathcal{F}_1^{-1}h) \left( \frac{j}{t}, kt \right) dt &= \zeta(\nu) \int_{-\infty}^{\infty} |t|^{\nu-1} (\mathcal{F}_1^{-1}h) (0, t) dt \\ &+ \frac{1}{2} \sum_{jk \neq 0} \int_{-\infty}^{\infty} |t|^{\nu-1} (\mathcal{F}_1^{-1}h) \left( \frac{j}{t}, kt \right) dt, \end{aligned} \quad (3.8)$$

from which the main part of the lemma follows if  $\operatorname{Re} \nu > 1$  after we have made the change of variable  $t \mapsto \frac{t}{k}$  in the main term. The continuation of the identity uses also the fact that, thanks to the trivial zeros  $-2, -4, \dots$  of zeta, the product  $\zeta(\nu) |t|^{\nu-1}$  is regular at  $\nu = -2, -4, \dots$  and the product  $\zeta(1 + \nu) |t|^\nu$  is regular at  $\nu = -3, -5, \dots$ . That the sum  $\zeta(\nu) |\xi|^{\nu-1} + \zeta(1 + \nu) |x|^\nu \delta(\xi)$  is regular at  $\nu = 0$  follows from the facts that  $\zeta(0) = -\frac{1}{2}$  and that the residue at  $\nu = 0$  of the distribution  $|\xi|^{\nu-1} = \frac{1}{\nu} \frac{d}{d\xi} (|x|^\nu \operatorname{sign} \xi)$  is  $\frac{d}{d\xi} \operatorname{sign} \xi = 2\delta(\xi)$ .

The second assertion is a consequence of the Fourier expansion. The factor  $(\nu - 1)$  has been inserted so as to kill the pole of  $\mathfrak{E}_{-\nu}$ , or that of  $\zeta(\nu)$ , at  $\nu = 1$ . Bounds for the first two terms of the right-hand side of (3.6) are obtained from bounds for the zeta factors (cf. paragraph that precedes the lemma) and integrations by parts associated to powers of the operator  $\xi \frac{\partial}{\partial \xi}$  or  $x \frac{\partial}{\partial x}$ . For the main terms, we use the integration by parts associated to the equation

$$\left( 1 + \xi \frac{\partial}{\partial x} \right) \exp \left( 2i\pi \frac{rx}{\xi} \right) = (1 + 2i\pi r) \exp \left( 2i\pi \frac{rx}{\xi} \right). \quad (3.9)$$

□

Decompositions into homogeneous components of functions or distributions in the plane will be ever-present. Any function  $h \in \mathcal{S}(\mathbb{R}^2)$  can be decomposed in  $\mathbb{R}^2 \setminus \{0\}$  into homogeneous components according to the equations, in which  $c > -1$ ,

$$h = \frac{1}{i} \int_{\operatorname{Re} \nu = c} h_\nu d\nu, \quad h_\nu(x, \xi) = \frac{1}{2\pi} \int_0^\infty t^\nu h(tx, t\xi) dt. \quad (3.10)$$

Indeed, the integral defining  $h_\nu(x, \xi)$  is convergent for  $|x| + |\xi| \neq 0, \operatorname{Re} \nu > -1$ , and the function  $h_\nu$  so defined is  $C^\infty$  in  $\mathbb{R}^2 \setminus \{0\}$  and homogeneous of degree  $-1 - \nu$ ; it is also analytic with respect to  $\nu$ . Using twice the integration

by parts associated to Euler's equation  $-(1+\nu) h_\nu = \left(x \frac{\partial}{\partial x} + \xi \frac{\partial}{\partial \xi}\right) h_\nu(x, \xi)$ , one sees that the integral  $\frac{1}{i} \int_{\operatorname{Re} \nu=c} h_\nu(x, \xi) d\nu$  is convergent for  $c > -1$ : its value does not depend on  $c$ . Taking  $c = 0$  and setting  $t = e^{2\pi\tau}$ , one has for  $|x| + |\xi| \neq 0$

$$h_{i\lambda}(x, \xi) = \int_{-\infty}^{\infty} e^{2i\pi\tau\lambda} \cdot e^{2\pi\tau} h(e^{2\pi\tau}x, e^{2\pi\tau}\xi) d\tau, \quad (3.11)$$

and the Fourier inversion formula shows that  $\int_{-\infty}^{\infty} h_{i\lambda}(x, \xi) d\lambda = h(x, \xi)$ : this proves (3.10).

As a consequence, some automorphic distributions of interest (not all: so-called Hecke distributions are needed too in general) can be decomposed into Eisenstein distributions. A basic one is the “Dirac comb”

$$\mathfrak{D}(x, \xi) = 2\pi \sum_{|j|+|k| \neq 0} \delta(x-j) \delta(\xi-k) = 2\pi [\mathcal{D}ir(x, \xi) - \delta(x)\delta(\xi)] \quad (3.12)$$

where, as found convenient in some algebraic calculations, one introduces also the “complete” Dirac comb  $\mathcal{D}ir(x, \xi) = \sum_{j,k \in \mathbb{Z}} \delta(x-j) \delta(\xi-k)$ .

Noting the inequality  $|\int_0^\infty t^\nu h(tx, t\xi) dt| \leq C(|x| + |\xi|)^{-\operatorname{Re} \nu - 1}$ , one obtains if  $h \in \mathcal{S}(\mathbb{R}^2)$  and  $c > 1$ , pairing (3.12) with (3.10), the identity

$$\langle \mathfrak{D}, h \rangle = \frac{1}{i} \sum_{|j|+|k| \neq 0} \int_{\operatorname{Re} \nu=c} d\nu \int_0^\infty t^\nu h(tj, tk) dt. \quad (3.13)$$

It follows from (3.1) that, for  $c > 1$ ,

$$\mathfrak{D} = \frac{1}{i} \int_{\operatorname{Re} \nu=c} \mathfrak{E}_{-\nu} d\nu = 2\pi + \frac{1}{i} \int_{\operatorname{Re} \nu=0} \mathfrak{E}_{-\nu} d\nu, \quad (3.14)$$

the second equation being a consequence of the first in view of (3.3).

Integral superpositions of Eisenstein distributions, such as the one in (3.14), are to be interpreted in the weak sense in  $\mathcal{S}'(\mathbb{R}^2)$ , i.e., they make sense when tested on arbitrary functions in  $\mathcal{S}(\mathbb{R}^2)$ . Of course, pole-chasing is essential when changing contours of integration. But, as long as one satisfies oneself with weak decompositions in  $\mathcal{S}'(\mathbb{R}^2)$ , no difficulty concerning the integrability with respect to  $\operatorname{Im} \nu$  on the line ever occurs, because of the last assertion of Lemma 3.3 and of the identities

$$(b - \nu)^A \langle \mathfrak{E}_{-\nu}, W \rangle = \langle (b - 2i\pi\mathcal{E})^A \mathfrak{E}_{-\nu}, W \rangle = \langle \mathfrak{E}_{-\nu}, (b + 2i\pi\mathcal{E})^A W \rangle, \quad (3.15)$$

in which  $A = 0, 1, \dots$  may be chosen arbitrarily large and  $b$  is arbitrary.

**Theorem 3.4.** *For every tempered distribution  $\mathfrak{S}$  such that  $\mathfrak{S} \circ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \mathfrak{S}$ , especially every automorphic distribution, the integral kernel of the operator  $\Psi(\mathfrak{S})$  is supported in the set of points  $(x, y)$  such that  $x^2 - y^2 \in 4\mathbb{Z}$ . In particular, given  $\nu \in \mathbb{C}$ ,  $\nu \neq \pm 1$ , one has for every pair  $v, u$  in  $\mathcal{S}(\mathbb{R})$ ,*

$$(v \mid \Psi(\mathfrak{E}_{-\nu}) u) = \zeta(\nu) (|x|^{\nu-1} v \mid u) + \zeta(-\nu) (|x|^{-\nu-1} v \mid u) \\ + \sum_{r \neq 0} \sigma_{-\nu}(r) \int_{-\infty}^{\infty} \bar{v} \left( t + \frac{r}{t} \right) |t|^{\nu-1} u \left( t - \frac{r}{t} \right) dt. \quad (3.16)$$

Under the support condition that  $x > 0$  and  $0 < x^2 - y^2 < 8$  if  $v(x)u(y) \neq 0$ , one has

$$(v \mid \Psi(\mathfrak{E}_{-\nu}) u) = \int_0^{\infty} \bar{v}(t + t^{-1}) t^{\nu-1} u(t - t^{-1}) dt. \quad (3.17)$$

*Proof.* The invariance of  $\mathfrak{S}$  under the change  $(x, \xi) \mapsto (x, \xi + x)$  implies that the distribution  $(\mathcal{F}_2^{-1} \mathfrak{S})(x, z)$  is invariant under the multiplication by  $e^{2i\pi xz}$ . Applying (2.2), we obtain the first assertion.

Let us use the expansion (3.6), but only after we have substituted the pair  $(\xi, -x)$  for  $(x, \xi)$ , which does not change  $\mathfrak{E}_{-\nu}(x, \xi)$  for any  $\nu \neq \pm 1$  in view of (3.1): hence,

$$\mathfrak{E}_{-\nu}(x, \xi) = \zeta(\nu) |x|^{\nu-1} + \zeta(1+\nu) \delta(x) |\xi|^\nu + \sum_{r \neq 0} \sigma_{-\nu}(r) |x|^{\nu-1} \exp \left( -2i\pi \frac{r\xi}{x} \right). \quad (3.18)$$

The contributions to  $(v \mid \Psi(\mathfrak{E}_{-\nu}) u)$  of the first two terms of this expansion are obtained by what immediately followed (2.2). The sum of terms of the series for  $r \neq 0$ , to be designated as  $\mathfrak{E}_{-\nu}^{\text{trunc}}(x, \xi)$ , remains to be examined.

One has

$$(\mathcal{F}_2^{-1} \mathfrak{E}_{-\nu}^{\text{trunc}})(x, z) = \sum_{r \neq 0} \sigma_{-\nu}(r) |x|^{\nu-1} \delta \left( z - \frac{r}{x} \right). \quad (3.19)$$

Still using (2.2), the integral kernel of the operator  $\Psi(\mathfrak{S}_r)$ , with

$$\mathfrak{S}_r(x, \xi) = |x|^{\nu-1} \exp \left( -2i\pi \frac{r\xi}{x} \right) = \mathfrak{T}_r \left( x, \frac{\xi}{x} \right), \quad (3.20)$$

is half of

$$\begin{aligned} 2 K_r(x, y) &= (\mathcal{F}_2^{-1} \mathfrak{S}_r) \left( \frac{x+y}{2}, \frac{x-y}{2} \right) = \left| \frac{x+y}{2} \right| (\mathcal{F}_2^{-1} \mathfrak{S}_r) \left( \frac{x+y}{2}, \frac{x^2-y^2}{4} \right) \\ &= \left| \frac{x+y}{2} \right|^\nu \delta \left( \frac{x^2-y^2}{4} - r \right) = \left| \frac{x+y}{2} \right|^{\nu-1} \delta \left( \frac{x-y}{2} - \frac{2r}{x+y} \right). \end{aligned} \quad (3.21)$$

Making in the integral  $\int_{\mathbb{R}^2} K(x, y) \bar{v}(x) u(y) dx dy$  the change of variable which amounts to taking  $\frac{x+y}{2}$  and  $x-y$  as new variables, one obtains (3.16). Under the support assumptions made about  $v, u$  in the second part, only the term such that  $r = 1$  subsists.  $\square$

#### 4. A DISTRIBUTION DECOMPOSING OVER THE ZEROS OF ZETA

The distributions  $\mathfrak{T}_N$  and  $\mathfrak{T}_\infty$  to be introduced in this section, regarded as symbols in the Weyl calculus, are the basic ingredients of this approach to the Riemann hypothesis. We are primarily interested in scalar products  $(v | \Psi(\mathfrak{T}_N) u)$  and  $(v | \Psi(\mathfrak{T}_\infty) u)$  under support conditions compatible with those in Theorem 3.4 (to wit,  $x > 0$  and  $0 < x^2 - y^2 < 8$  when  $v(x) u(y) \neq 0$ ). The definition (4.2) of  $\mathfrak{T}_N$  as a measure is suitable for the discussion of the arithmetic side of the identity (4.13) crucial in this disproof of R.H., and the definition of  $\mathfrak{T}_\infty$  as an integral superposition of Eisenstein distributions is the one one must appeal to when discussing the analytic side of the identity.

Set for  $j \neq 0$

$$a(j) = \prod_{p|j} (1-p), \quad (4.1)$$

where  $p$ , in the role of defining the range of the subscript in a product, is always tacitly assumed to be prime. The distribution

$$\mathfrak{T}_N(x, \xi) = \sum_{j, k \in \mathbb{Z}} a((j, k, N)) \delta(x-j) \delta(\xi-k), \quad (4.2)$$

where the notation  $(j, k, N)$  refers to the g.c.d. of the three numbers, depends only on the “squarefree version” of  $N$ , defined as  $N_\bullet = \prod_{p|N} p$ . We also denote as  $\mathfrak{T}_N^\times$  the distribution obtained from  $\mathfrak{T}_N$  by discarding the term  $a(N) \delta(x) \delta(\xi)$ , in other words by limiting the summation to all pairs of integers  $j, k$  such that  $|j| + |k| \neq 0$ .

**Proposition 4.1.** [11, Lemma 3.1.1] *For any squarefree integer  $N \geq 1$ , defining*

$$\zeta_N(s) = \prod_{p|N} (1 - p^{-s})^{-1}, \quad \text{so that } \frac{1}{\zeta_N(s)} = \sum_{1 \leq T|N} \mu(T) T^{-s}, \quad (4.3)$$

where  $\mu$  (also denoted as Möb when  $\mu$  has another role) is the Möbius indicator function ( $\mu(T) = 0$  unless  $T$  is squarefree, in which case it is 1 or  $-1$  according to the parity of its number of prime factors), one has

$$\mathfrak{T}_N^\times = \frac{1}{2i\pi} \int_{\operatorname{Re} \nu = c} \frac{1}{\zeta_N(\nu)} \mathfrak{E}_{-\nu} d\nu, \quad c > 1. \quad (4.4)$$

As  $N \nearrow \infty$ , a notation meant to convey that  $N \rightarrow \infty$  in such a way that any given finite set of primes constitutes eventually a set of divisors of  $N$ , the distribution  $\mathfrak{T}_N^\times$  converges weakly in the space  $\mathcal{S}'(\mathbb{R}^2)$  towards the distribution

$$\mathfrak{T}_\infty = \frac{1}{2i\pi} \int_{\operatorname{Re} \nu = c} \frac{\mathfrak{E}_{-\nu}}{\zeta(\nu)} d\nu, \quad c \geq 1. \quad (4.5)$$

*Proof.* Using the equation  $T^{-x} \frac{d}{dx} \delta(x - j) = \delta\left(\frac{x}{T} - j\right) = T \delta(x - Tj)$ , one has with  $\mathfrak{D}$  as introduced in (3.12)

$$\begin{aligned} \frac{1}{2\pi} \prod_{p|N} (1 - p^{-2i\pi\mathcal{E}}) \mathfrak{D}(x, \xi) &= \sum_{T|N} \mu(T) T^{-2i\pi\mathcal{E}} \sum_{|j|+|k| \neq 0} \delta(x - j) \delta(\xi - k) \\ &= \sum_{T|N} \mu(T) T \sum_{|j|+|k| \neq 0} \delta(x - Tj) \delta(\xi - Tk) \\ &= \sum_{T|N} \mu(T) T \sum_{\substack{|j|+|k| \neq 0 \\ j \equiv k \equiv 0 \pmod{T}}} \delta(x - j) \delta(\xi - k) \\ &= \sum_{\substack{|j|+|k| \neq 0 \\ T|(N, j, k)}} \mu(T) T \delta(x - j) \delta(\xi - k). \end{aligned} \quad (4.6)$$

Since  $\sum_{T|(N, j, k)} \mu(T) T = \prod_{p|(N, j, k)} (1 - p) = a((N, j, k))$ , one obtains

$$\frac{1}{2\pi} \prod_{p|N} (1 - p^{-2i\pi\mathcal{E}}) \mathfrak{D}(x, \xi) = \mathfrak{T}_N^\times(x, \xi). \quad (4.7)$$

One has

$$\begin{aligned} [\mathfrak{T}_N - \mathfrak{T}_N^\times](x, \xi) &= a(N) \delta(x) \delta(\xi) \\ &= \delta(x) \delta(\xi) \prod_{p|N} (1 - p) = \prod_{p|N} (1 - p^{-2i\pi\mathcal{E}}) (\delta(x) \delta(\xi)), \end{aligned} \quad (4.8)$$

so that, adding the last two equations,

$$\mathfrak{T}_N = \prod_{p|N} (1 - p^{-2i\pi\mathcal{E}}) \mathcal{D}ir. \quad (4.9)$$

Combining (4.7) with (3.14) and with  $(2i\pi\mathcal{E})\mathfrak{E}_{-\nu} = \nu \mathfrak{E}_{-\nu}$ , one obtains if  $c > 1$

$$\begin{aligned} \mathfrak{T}_N^\times &= \prod_{p|N} (1 - p^{-2i\pi\mathcal{E}}) \left[ \frac{1}{2i\pi} \int_{\operatorname{Re} \nu = c} \mathfrak{E}_{-\nu} d\nu \right] \\ &= \frac{1}{2i\pi} \int_{\operatorname{Re} \nu = c} \mathfrak{E}_{-\nu} \prod_{p|N} (1 - p^{-\nu}) d\nu, \end{aligned} \quad (4.10)$$

which is just (4.4). The  $d\nu$ -summability is guaranteed by (3.15). Equation (4.5) follows as well, taking the limit as  $N \nearrow \infty$ . Recall also (3.15).

In view of Hadamard's theorem, according to which  $\zeta(s)$  has no zero on the line  $\operatorname{Re} s = 1$ , to be completed by the estimate  $\zeta(1+iy) = O(\log |y|)$ , observing also that the pole of  $\zeta(\nu)$  at  $\nu = 1$  kills that of  $\mathfrak{E}_{-\nu}$  there, one can in (4.5) replace the condition  $c > 1$  by  $c \geq 1$ . □

Using (2.4), one obtains if  $v, u \in \mathcal{S}(\mathbb{R})$  the identity

$$(v | \Psi(\mathfrak{T}_N) u) = \sum_{j, k \in \mathbb{Z}} a((j, k, N)) \operatorname{Wig}(v, u)(j, k). \quad (4.11)$$

It follows from (2.3) that if  $v$  and  $u$  are compactly supported and the algebraic sum of the supports of  $v$  and  $u$  is supported in an interval  $[0, 2\beta]$  (this is the situation that will interest us),  $(v | \Psi(\mathfrak{T}_N) u)$  does not depend on  $N$  as soon as  $N$  is divisible by all primes  $< \beta$ . Taking the limit of this stationary sequence as  $N \nearrow \infty$ , one can in this case write also

$$(v | \Psi(\mathfrak{T}_\infty) u) = \sum_{j, k \in \mathbb{Z}} a((j, k)) \operatorname{Wig}(v, u)(j, k). \quad (4.12)$$



More generally, the following reduction, under some support conditions, of  $\mathfrak{T}_\infty$  to  $\mathfrak{T}_N$ , is immediate, and fundamental for our purpose. Assume that  $v$  and  $u \in C^\infty(\mathbb{R})$  are such that the algebraic sum of the supports of  $v$  and  $u$  is contained in  $[0, 2\beta]$ . Then, given a squarefree integer  $N = RQ$  (with  $R, Q$  integers) divisible by all primes  $< \beta Q$ , one has

$$(v \mid \Psi(Q^{2i\pi\mathcal{E}} \mathfrak{T}_\infty) u) = (v \mid \Psi(Q^{2i\pi\mathcal{E}} \mathfrak{T}_N) u). \quad (4.13)$$

Indeed, (4.12) and the fact that the transpose of  $2i\pi\mathcal{E}$  is  $-2i\pi\mathcal{E}$  yield

$$(v \mid \Psi(Q^{2i\pi\mathcal{E}} \mathfrak{T}_\infty) u) = Q^{-1} \sum_{j,k} a((j, k)) \text{Wig}(v, u) \left( \frac{j}{Q}, \frac{k}{Q} \right). \quad (4.14)$$

Next, from the observation that follows (2.3), one has  $0 < \frac{j}{Q} < \beta$ , or  $0 < j < \beta Q$ , for all nonzero terms of this sum, which implies that all prime divisors of  $j$  divide  $N$ . Note also that using here  $\mathfrak{T}_N$  or  $\mathfrak{T}_N^\times$  would not make any difference since  $\Psi(\delta_0) u = \underset{v}{u}$  and the interiors of the supports of  $v$  and  $\underset{u}{u}$  do not intersect.

*Remark 4.1.* In (4.5), introducing a sum of residues over all zeros of zeta with a real part above some large negative number, one can replace the line  $\text{Re } \nu = c$  with  $c > 1$  by a line  $\text{Re } \nu = c'$  with  $c'$  very negative: one cannot go further in the distribution sense. But [11, Theor. 3.2.2, Theor. 3.2.4], one can get rid of the integral if one agrees to interpret the identity in the sense of a certain analytic functional. Then, all zeros of zeta, non-trivial and trivial alike, enter the formula: the “trivial” part

$$\mathfrak{R}_\infty = 2 \sum_{n \geq 0} \frac{(-1)^{n+1}}{(n+1)!} \frac{\pi^{\frac{5}{2}+2n}}{\Gamma(\frac{3}{2}+n)\zeta(3+2n)} \mathfrak{E}_{2n+2} \quad (4.15)$$

has a closed expression [11, p.22] as a series of line measures. This will not be needed in the sequel.

We shall also need the distribution

$$\mathfrak{T}_{\frac{\infty}{2}}(x, \xi) = \sum_{|j|+|k| \neq 0} a((j, k, \frac{\infty}{2})) \delta(x-j) \delta(\xi-k), \quad (4.16)$$

where  $a((j, k, \frac{\infty}{2}))$  is the product of all factors  $1-p$  with  $p$  prime  $\neq 2$  dividing  $(j, k)$ . Since  $\prod_{p \neq 2} (1-p^{-\nu}) = \frac{(1-2^{-\nu})^1}{\zeta(\nu)}$ , one has

$$\mathfrak{T}_{\frac{\infty}{2}} = \frac{1}{2i\pi} \int_{\text{Re } \nu=c} (1-2^{-\nu})^{-1} \frac{\mathfrak{E}_{-\nu}}{\zeta(\nu)} d\nu, \quad c > 1. \quad (4.17)$$

The version of (4.13) we shall use, under the same support conditions about  $v, u$ , is

$$(v \mid \Psi(Q^{2i\pi\mathcal{E}}\mathfrak{T}_{\frac{\infty}{2}})u) = (v \mid \Psi(Q^{2i\pi\mathcal{E}}\mathfrak{T}_N)u) \quad (4.18)$$

if  $N$  and  $Q$  are squarefree odd and  $N$  is divisible by all odd primes  $< \beta Q$ .

In [11, Prop. 3.4.2 and 3.4.3], it was proved (with a minor difference due to the present change of  $\text{Op}_1$  to  $\Psi$ ) that, if for some  $\beta > 2$  and every function  $w \in C^\infty(\mathbb{R})$  supported in  $[0, \beta]$ , one has

$$(w \mid \Psi(Q^{2i\pi\mathcal{E}}\mathfrak{T}_\infty)w) = O\left(Q^{\frac{1}{2}+\varepsilon}\right) \quad (4.19)$$

as  $Q \rightarrow \infty$  through squarefree integral values, the Riemann hypothesis follows. Except for the change of the pair  $w, w$  to a pair  $v, u$  with specific support properties, this is the criterion to be discussed in the next section.

## 5. A CRITERION FOR THE RIEMANN HYPOTHESIS

Using (4.5) and the homogeneity of  $\mathfrak{E}_{-\nu}$ , one has

$$Q^{2i\pi\mathcal{E}}\mathfrak{T}_\infty = \frac{1}{2i\pi} \int_{\text{Re } \nu=c} Q^\nu \frac{\mathfrak{E}_{-\nu}}{\zeta(\nu)} d\nu, \quad c > 1. \quad (5.1)$$

From Lemma 3.3 and (3.15), the product by  $Q^{-1-\varepsilon}$  of the distribution  $Q^{2i\pi\mathcal{E}}\mathfrak{T}_\infty$  remains for every  $\varepsilon > 0$  in a bounded subset of  $\mathcal{S}'(\mathbb{R}^2)$  as  $Q \rightarrow \infty$ . If the Riemann hypothesis holds, the same is true after  $Q^{-1-\varepsilon}$  has been replaced by  $Q^{-\frac{1}{2}-\varepsilon}$ .

We shall state and prove some suitably modified version of the converse, involving for some choice of the pair  $v, u$  of functions in  $\mathcal{S}(\mathbb{R})$  the function

$$F_0(s) = \sum_{Q \in \text{Sq}^{\text{odd}}} Q^{-s} \left( v \mid \Psi(Q^{2i\pi\mathcal{E}}\mathfrak{T}_{\frac{\infty}{2}})u \right), \quad (5.2)$$

where we denote as  $\text{Sq}^{\text{odd}}$  the set of squarefree odd integers. We are discarding the prime 2 so as to make Theorem 7.2 below applicable later. Under the assumption, generalizing (4.19), that  $(v \mid \Psi(Q^{2i\pi\mathcal{E}}\mathfrak{T}_\infty)u) = O\left(Q^{\frac{1}{2}+\varepsilon}\right)$ , the function  $F_0(s)$  is analytic in the half-plane  $\text{Re } s > \frac{3}{2}$  and polynomially bounded in vertical strips in this domain, by which we mean, classically, that given  $[a, b] \subset ]\frac{3}{2}, \infty[$ , there exists  $M > 0$  such that, for some  $C > 0$ ,  $|F_0(\sigma + it)| \leq C(1 + |t|)^M$  when  $\sigma \in [a, b]$  and  $t \in \mathbb{R}$ .

First, we transform the series defining  $F_0(s)$  into a line integral of convolution type.

**Lemma 5.1.** *Given  $v, u \in \mathcal{S}(\mathbb{R})$ , the function  $F_0(s)$  introduced in (5.2) can be written for  $\operatorname{Re} s$  large*

$$F_0(s) = \frac{1}{2i\pi} \int_{\operatorname{Re} \nu=1} (1 + 2^{-s+\nu})^{-1} \frac{\zeta(s-\nu)}{\zeta(2(s-\nu))} \times \frac{(1 - 2^{-\nu})^{-1}}{\zeta(\nu)} \langle \mathfrak{E}_{-\nu}, \operatorname{Wig}(v, u) \rangle d\nu. \quad (5.3)$$

*Proof.* For  $\operatorname{Re} s > 1$ , one has the identity

$$\begin{aligned} \sum_{Q \in \operatorname{Sq}^{\text{odd}}} Q^{-s} &= \prod_{\substack{q \text{ prime} \\ q \neq 2}} (1 + q^{-s}) \\ &= (1 + 2^{-s})^{-1} \prod_q \frac{1 - q^{-2s}}{1 - q^{-s}} = (1 + 2^{-s})^{-1} \frac{\zeta(s)}{\zeta(2s)}. \end{aligned} \quad (5.4)$$

We apply this with  $s$  replaced by  $s - \nu$ , starting from (5.2). One has  $(v \mid \Psi(Q^{2i\pi\mathcal{E}} \mathfrak{E}_{-\nu}) u) = Q^\nu \langle \mathfrak{E}_{-\nu}, \operatorname{Wig}(v, u) \rangle$  according to (2.4). Using (5.1) and (5.4), one obtains (5.3) if  $\operatorname{Re} s > 2$ , noting that the denominator  $\zeta(\nu)$  takes care of the pole of  $\mathfrak{E}_{-\nu}$  at  $\nu = 1$ , which makes it possible to replace the line  $\operatorname{Re} \nu = c$ ,  $c > 1$  by the line  $\operatorname{Re} \nu = 1$ . Note that the integrability at infinity is taken care of by (3.15), together with Hadamard's theorem as recalled immediately after (4.10).  $\square$

**Lemma 5.2.** *Let  $\rho \in \mathbb{C}$  and consider a product  $h(\nu) f(s - \nu)$ , where the function  $f = f(z)$ , defined and meromorphic near the point  $z = 1$ , has a simple pole at this point, and the function  $h$ , defined and meromorphic near  $\rho$ , has at that point a pole of order  $\ell \geq 1$ . Then, the function  $s \mapsto \operatorname{Res}_{\nu=\rho} [h(\nu) f(s - \nu)]$  has at  $s = 1 + \rho$  a pole of order  $\ell$ .*

*Proof.* If  $h(\nu) = \sum_{j=1}^{\ell} a_j (\nu - \rho)^{-j} + O(1)$  as  $\nu \rightarrow \rho$ , one has for  $s$  close to  $1 + \rho$  but distinct from this point

$$\operatorname{Res}_{\nu=\rho} [h(\nu) f(s - \nu)] = \sum_{j=1}^{\ell} (-1)^{j-1} a_j \frac{f^{(j-1)}(s - \rho)}{(j-1)!}, \quad (5.5)$$

and the function  $s \mapsto f^{(j-1)}(s - \rho)$  has at  $s = 1 + \rho$  a pole of order  $j$ .  $\square$

*Remark 5.1.* An integral  $\int_{\gamma} h(\nu) f(s - \nu) d\nu$  over a finite part of a line cannot have a pole at  $s = s_0$  unless both  $h$  and the function  $\nu \mapsto g(s_0 - \nu)$  have poles at the same point  $\nu \in \gamma$ . This follows from the residue theorem together with the case  $\ell = 0$  of Lemma 5.2.

**Theorem 5.3.** *Assume that, for every given pair  $v, u$  of functions in  $\mathcal{S}(\mathbb{R})$ , the function  $F_0(s)$  defined in the integral version (5.3), initially defined and analytic for  $\operatorname{Re} s > 2$ , extends as an analytic function in the half-plane  $\{s : \operatorname{Re} s > \frac{3}{2}\}$ . Then, all zeros of zeta have real parts  $\leq \frac{1}{2}$ : in other words, the Riemann hypothesis does hold.*

*Proof.* Assume that a zero  $\rho$  of zeta with a real part  $> \frac{1}{2}$  exists: one may assume that the real part of  $\rho$  is the largest one among those of all zeros of zeta (if any other should exist !) with the same imaginary part and a real part  $> \frac{1}{2}$ . Choose  $\beta$  such that  $0 < \beta < \operatorname{Re}(\rho - \frac{1}{2})$ . Assuming that  $\operatorname{Re} s > 2$ , change the line  $\operatorname{Re} \nu = 1$  to a simple contour  $\gamma$  on the left of the initial line, enclosing the point  $\rho$  but no other point  $\rho$  with  $\zeta(\rho) = 0$ , coinciding with the line  $\operatorname{Re} \nu = 1$  for  $|\operatorname{Im} \nu|$  large, and such that  $\operatorname{Re} \nu > \operatorname{Re} \rho - \beta$  for  $\nu \in \gamma$ .

Let  $\Omega$  be the relatively open part of the half-plane  $\operatorname{Re} \nu \leq 1$  enclosed by  $\gamma$  and the line  $\operatorname{Re} \nu = 1$ . Let  $\mathcal{D}$  be the domain consisting of the numbers  $s$  such that  $s - 1 \in \Omega$  or  $\operatorname{Re} s > 2$ . When  $s \in \mathcal{D}$ , one has  $\operatorname{Re} s > 1 + \operatorname{Re} \rho - \beta > \frac{3}{2}$ .

Still assuming that  $\operatorname{Re} s > 2$ , one obtains the equation

$$F_0(s) = \frac{1}{2i\pi} \int_{\gamma} f(s - \nu) h_0(\nu) d\nu + \operatorname{Res}_{\nu=\rho} [h_0(\nu) f(s - \nu)], \quad (5.6)$$

with

$$\begin{aligned} h_0(\nu) &= (1 - 2^{-\nu})^{-1} \times \frac{\langle \mathfrak{E}_{-\nu}, \operatorname{Wig}(v, u) \rangle}{\zeta(\nu)}, \\ f(s - \nu) &= (1 + 2^{-s+\nu})^{-1} \times \frac{\zeta(s - \nu)}{\zeta(2(s - \nu))}. \end{aligned} \quad (5.7)$$

We show now that the integral term in (5.6) is holomorphic in the domain  $\mathcal{D}$ . The first point is that the numerator  $\zeta(s - \nu)$  of  $f(s - \nu)$  will not contribute singularities. Indeed, one can have  $s - \nu = 1$  with  $s \in \mathcal{D}$  and

$\nu \in \gamma$  only if  $s \in 1 + \Omega$ , since  $\operatorname{Re}(s - \nu) > 1$  if  $\operatorname{Re} s > 2$  and  $\nu \in \gamma$ . Then, the conditions  $s - 1 \in \Omega$  and  $s - 1 \in \gamma$  are incompatible because on one hand, the imaginary part of  $s - 1$  does not agree with that of any point of the two infinite branches of  $\gamma$ , while the rest of  $\gamma$  is a part of the boundary of  $\Omega$ . Finally, when  $s - 1 \in \Omega$  and  $\nu \in \gamma$ , that  $\zeta(2(s - \nu)) \neq 0$  follows from the inequalities  $\operatorname{Re}(s - \nu) \geq \operatorname{Re} \rho - \beta > \frac{1}{2}$ , since  $\operatorname{Re} s > 1 + (\operatorname{Re} \rho - \beta)$  and  $\operatorname{Re} \nu \leq 1$ .

Since  $F_0(s)$  is analytic for  $\operatorname{Re} s > \frac{3}{2}$ , it follows that the residue present in (5.6) extends as an analytic function of  $s$  in  $\mathcal{D}$ . But an application of Lemma 5.2, together with the first equation (5.7) and (3.17), shows that this residue is singular at  $s = 1 + \rho$  for some choice of the pair  $v, u$ . We have reached a contradiction.

*Remark 5.2.* This remark should prevent a possible misunderstanding. Though we are ultimately interested in a residue at  $\nu = \rho$  and in the continuation of  $F_0(s)$  near  $s = 1 + \rho$ , we have established (5.6) under the assumption that  $\operatorname{Re} s > 2$ , in which case  $s - 1$  does not lie in the domain covered by  $\nu$  between the line  $\operatorname{Re} \nu = 1$  and the line  $\gamma$ . We must therefore not add to the right-hand side of (5.6) the residue of the integrand at  $\nu = s - 1$  (the two residues would have killed each other). Separating the poles of the two factors has been essential. The conclusion resulted from analytic continuation and the assumption that  $F_0(s)$  extends as an analytic function for  $\operatorname{Re} s > \frac{3}{2}$ . □

**Corollary 5.4.** *Let  $\rho \in \mathbb{C}$  with  $\operatorname{Re} \rho > 0$  be given. Let  $v, u \in \mathcal{S}(\mathbb{R})$  be such that  $(v \mid \Psi(\mathfrak{E}_{-\rho}) u) = \langle \mathfrak{E}_{-\rho}, \operatorname{Wig}(v, u) \rangle \neq 0$ . If the function  $F_0(s)$ , defined for  $\operatorname{Re} s > 2$  by (5.2), can be continued analytically along a path connecting the point  $1 + \rho$  to a point with a real part  $> 2$ , the point  $\rho$  cannot be a zero of zeta. For any  $\rho$ , a pair  $v, u$  with  $v$  supported in  $[2, \sqrt{8}]$  and  $u$  supported in  $[0, 1]$ , such that  $(v \mid \Psi(\mathfrak{E}_{-\rho}) u) \neq 0$ , can be found.*

*Proof.* It follows the proof of Theorem 5.3. It suffices there to take  $\Omega$  containing the path in the assumption, and to observe that  $h_0(\nu)$ , as defined in (5.7), would under the nonvanishing condition  $\langle \mathfrak{E}_{-\rho}, \operatorname{Wig}(v, u) \rangle \neq 0$  have a pole at  $\rho$  if one had  $\zeta(\rho) = 0$ . □

The following proposition shows a way to reinforce the  $d\nu$ -summability in the integral (5.3) defining  $F_0(s)$  without losing the conclusion of Theorem 5.3.

**Proposition 5.5.** *Given  $M = 1, 2, \dots$ , define*

$$F_{0,M}(s) = \frac{1}{2i\pi} \int_{\operatorname{Re} \nu=1} (1 + 2^{-s+\nu})^{-1} \frac{\zeta(s-\nu)}{\zeta(2(s-\nu))} \\ \times \nu^{-M} \frac{(1 - 2^{-\nu})^{-1}}{\zeta(\nu)} \langle \mathfrak{E}_{-\nu}, \operatorname{Wig}(v, u) \rangle d\nu. \quad (5.8)$$

*Theorem 5.3 remains valid if one substitutes  $F_{0,M}$  for  $F_0$ .*

*Proof.* It suffices to see how  $F_0$  can be rebuilt in terms of  $F_{0,M}$ , for which one observes the effect of multiplying by  $\nu^M$  an expression such as  $\langle \mathfrak{E}_{-\nu}, \operatorname{Wig}(v, u) \rangle$ . This is immediate since

$$\nu^M \langle \mathfrak{E}_{-\nu}, \operatorname{Wig}(v, u) \rangle = \langle \mathfrak{E}_{-\nu}, (-2i\pi\mathcal{E})^M \operatorname{Wig}(v, u) \rangle, \quad (5.9)$$

while the effect of applying  $(2i\pi\mathcal{E})^M$  on a Wigner function is provided by Lemma 2.1. □

## 6. PSEUDODIFFERENTIAL ARITHMETIC

As noted in (4.13), one can substitute for the analysis of the hermitian form  $(v | \Psi(Q^{2i\pi\mathcal{E}} \mathfrak{T}_{\frac{\infty}{2}}) u)$  that of  $(v | \Psi(Q^{2i\pi\mathcal{E}} \mathfrak{T}_N) u)$ , under the assumption that the algebraic sum of the supports of  $v$  and  $u$  is contained in  $[0, 2\beta]$ , provided that  $N$  is a squarefree odd integer divisible by all odd primes  $< \beta Q$ . The new hermitian form should be amenable to an algebraic treatment. In this section, we make no support assumptions on  $v, u$ , just taking them in  $\mathcal{S}(\mathbb{R})$ .

We consider operators of the kind  $\Psi(Q^{2i\pi\mathcal{E}} \mathfrak{S})$  with

$$\mathfrak{S}(x, \xi) = \sum_{j,k \in \mathbb{Z}} b(j, k) \delta(x - j) \delta(\xi - k), \quad (6.1)$$

under the following assumptions: that  $N$  is a squarefree integer decomposing as the product  $N = RQ$  of two positive integers, and that  $b$  satisfies the periodicity conditions

$$b(j, k) = b(j + N, k) = b(j, k + N). \quad (6.2)$$

A special case consists of course of the symbol  $\mathfrak{S} = \mathfrak{T}_N$ . The aim is to transform the hermitian form associated to the operator  $\Psi(Q^{2i\pi\mathcal{E}}\mathfrak{T}_N)$  to an arithmetic version.

The following proposition reproduces [11, Prop. 4.1.2, Prop. 4.1.3], the parameter denoted as  $\omega$  there being set to the value 2.

**Theorem 6.1.** *With  $N = RQ$  and  $b(j, k)$  satisfying the condition (6.2), define the function*

$$f_N(j, s) = \frac{1}{N} \sum_{k \bmod N} b(j, k) \exp\left(\frac{2i\pi ks}{N}\right), \quad j, s \in \mathbb{Z}/N\mathbb{Z}. \quad (6.3)$$

*Set, noting that the condition  $m - n \equiv 0 \bmod 2Q$  implies that  $m + n$  too is even,*

$$c_{R,Q}(\mathfrak{S}; m, n) = \text{char}(m+n \equiv 0 \bmod R, m-n \equiv 0 \bmod 2Q) f_N\left(\frac{m+n}{2R}, \frac{m-n}{2Q}\right). \quad (6.4)$$

*On the other hand, set, for  $u \in \mathcal{S}(\mathbb{R})$ ,*

$$(\theta_N u)(n) = \sum_{\ell \in \mathbb{Z}} u\left(\frac{n}{N} + 2\ell N\right), \quad n \bmod 2N^2. \quad (6.5)$$

*Then, one has*

$$(v \mid \Psi(Q^{2i\pi\mathcal{E}}\mathfrak{S}) u) = \sum_{m, n \in \mathbb{Z}/(2N^2)\mathbb{Z}} c_{R,Q}(\mathfrak{S}; m, n) \overline{\theta_N v(m)} (\theta_N u)(n). \quad (6.6)$$

*Proof.* There is no restriction here on the supports of  $v, u$ , and one can replace these two functions by  $v[Q], u[Q]$  defined as  $v[Q](x) = v(Qx)$  and  $u[Q](x) = u(Qx)$ . One has

$$(\theta_N u[Q])(n) = (\kappa u)(n) = \sum_{\ell \in \mathbb{Z}} u\left(\frac{n}{R} + 2QN\ell\right), \quad n \bmod 2N^2. \quad (6.7)$$

It just requires the definition (1.3) of  $\Psi$  and changes of variables amounting to rescaling  $x, y, \xi$  by the factor  $Q^{-1}$  to obtain  $(v[Q] \mid \Psi(Q^{2i\pi\mathcal{E}}\mathfrak{S}) u[Q]) = (v \mid \mathcal{B}u)$ , with

$$(\mathcal{B}u)(x) = \frac{1}{2Q^2} \int_{\mathbb{R}^2} \mathfrak{S}\left(\frac{x+y}{2}, \xi\right) u(y) \exp\left(\frac{i\pi}{Q^2}(x-y)\xi\right) dy d\xi. \quad (6.8)$$

The identity (6.6) to be proved is equivalent, with  $\kappa$  as defined in (6.7), to

$$(v | \mathcal{B}u) = \sum_{m,n \in \mathbb{Z}/(2N^2)\mathbb{Z}} c_{R,Q}(\mathfrak{S}; m, n) \overline{\kappa v(m)} (\kappa u)(n). \quad (6.9)$$

From (6.4), one has

$$\begin{aligned} (v | \mathcal{B}u) &= \frac{1}{2Q^2} \int_{-\infty}^{\infty} \bar{v}(x) dx \int_{\mathbb{R}^2} \mathfrak{S}\left(\frac{x+y}{2}, \xi\right) u(y) \exp\left(\frac{i\pi}{Q^2}(x-y)\xi\right) dy d\xi \\ &= \frac{1}{Q^2} \int_{-\infty}^{\infty} \bar{v}(x) dx \int_{\mathbb{R}^2} \mathfrak{S}(y, \xi) u(2y-x) \exp\left(\frac{2i\pi}{Q^2}(x-y)\xi\right) dy d\xi \\ &= \frac{1}{Q^2} \int_{-\infty}^{\infty} \bar{v}(x) dx \sum_{j,k \in \mathbb{Z}} b(j, k) u(2j-x) \exp\left(\frac{2i\pi}{Q^2}(x-j)k\right). \end{aligned} \quad (6.10)$$

Since  $b(j, k) = b(j, k + N)$ , one replaces  $k$  by  $k + N\ell$ , the new  $k$  lying in the interval  $[0, N - 1]$  of integers. One has (Poisson's formula)

$$\sum_{\ell \in \mathbb{Z}} \exp\left(\frac{2i\pi}{Q^2}(x-j)\ell N\right) = \sum_{\ell \in \mathbb{Z}} \exp\left(\frac{2i\pi}{Q}(x-j)\ell R\right) = \frac{Q}{R} \sum_{\ell \in \mathbb{Z}} \delta\left(x-j-\frac{\ell Q}{R}\right), \quad (6.11)$$

and, from (6.10),

$$\begin{aligned} &(\mathcal{B}u)(x) \\ &= \frac{1}{N} \sum_{\substack{j \in \mathbb{Z} \\ 0 \leq k < N}} b(j, k) \sum_{\ell \in \mathbb{Z}} u\left(j - \frac{\ell Q}{R}\right) \exp\left(\frac{2i\pi(x-j)k}{Q^2}\right) \delta\left(x-j-\frac{\ell Q}{R}\right) \\ &= \sum_{m \in \mathbb{Z}} t_m \delta\left(x - \frac{m}{R}\right), \end{aligned} \quad (6.12)$$

with  $m = Rj + \ell Q$  and  $t_m$  to be made explicit: we shall drop the summation with respect to  $\ell$  for the benefit of a summation with respect to  $m$ . Since, when  $x = j + \frac{\ell Q}{R} = \frac{m}{R}$ , one has  $\frac{x-j}{Q^2} = \frac{\ell}{N} = \frac{m-Rj}{NQ}$  and  $j - \frac{\ell Q}{R} = 2j - x =$



$2j - \frac{m}{R}$ , one has

$$\begin{aligned}
t_m &= \frac{1}{N} \sum_{\substack{j \in \mathbb{Z} \\ 0 \leq k < N}} b(j, k) \operatorname{char}(m \equiv Rj \bmod Q) u\left(2j - \frac{m}{R}\right) \exp\left(\frac{2i\pi k(m - Rj)}{NQ}\right) \\
&= \frac{1}{N} \sum_{\substack{0 \leq j < QN \\ 0 \leq k < N}} b(j, k) \operatorname{char}(m \equiv Rj \bmod Q) \\
&\quad \sum_{\ell_1 \in \mathbb{Z}} u\left(2(j + \ell_1 QN) - \frac{m}{R}\right) \exp\left(\frac{2i\pi k(m - Rj)}{QN}\right). \quad (6.13)
\end{aligned}$$

Recalling the definition (6.7) of  $\kappa u$ , one obtains

$$\begin{aligned}
t_m &= \frac{1}{N} \sum_{\substack{0 \leq j < QN \\ 0 \leq k < N}} b(j, k) \operatorname{char}(m \equiv Rj \bmod Q) \\
&\quad (\kappa u)(2Rj - m) \exp\left(\frac{2i\pi k(m - Rj)}{QN}\right). \quad (6.14)
\end{aligned}$$

The function  $\kappa u$  is  $(2N^2)$ -periodic, so one can replace the subscript  $0 \leq j < RQ^2$  by  $j \bmod RQ^2$ . Using (6.12), we obtain

$$\begin{aligned}
(v | \mathcal{B}u) &= \frac{1}{N} \sum_{j \bmod RQ^2} \sum_{0 \leq k < N} b(j, k) \sum_{\substack{m_1 \in \mathbb{Z} \\ m_1 \equiv Rj \bmod Q}} \\
&\quad \bar{v}\left(\frac{m_1}{R}\right) (\kappa u)(2Rj - m_1) \exp\left(\frac{2i\pi k(m_1 - Rj)}{QN}\right). \quad (6.15)
\end{aligned}$$

The change of  $m$  to  $m_1$  is just a change of notation.

Fixing  $k$ , we trade the set of pairs  $m_1, j$  with  $m_1 \in \mathbb{Z}, j \bmod RQ^2, m_1 \equiv Rj \bmod Q$  for the set of pairs  $m, n \in (\mathbb{Z}/(2N^2)\mathbb{Z}) \times (\mathbb{Z}/(2N^2)\mathbb{Z})$ , where  $m$  is the class mod  $2N^2$  of  $m_1$  and  $n$  is the class mod  $2N^2$  of  $2Rj - m_1$ . Of necessity,  $m + n \equiv 0 \bmod 2R$  and  $m - n \equiv 2(m - Rj) \equiv 0 \bmod 2Q$ . Conversely, given a pair of classes  $m, n \bmod 2N^2$  satisfying these conditions, the equation  $2Rj - m = n$  uniquely determines  $j \bmod \frac{2N^2}{2R} = RQ^2$ , as

it should. The sum  $\sum_{m_1 \equiv m \pmod{2N^2}} v\left(\frac{m_1}{R}\right)$  is just  $(\kappa v)(m)$ , and we have obtained the identity

$$(v | \mathcal{B}u) = \sum_{m, n \pmod{2N^2}} c_{R,Q}(\mathfrak{S}; m, n) \overline{(\kappa v)(m)} (\kappa u)(n), \quad (6.16)$$

provided we define

$$c_{R,Q}(\mathfrak{S}; m, n) = \frac{1}{N} \text{char}(m + n \equiv 0 \pmod{R}, m - n \equiv 0 \pmod{2Q}) \sum_{k \pmod{N}} b\left(\frac{m+n}{2R}, k\right) \exp\left(\frac{2i\pi k}{N} \frac{m-n}{2Q}\right), \quad (6.17)$$

which is just the way indicated in (6.3), (6.4).  $\square$

## 7. COMPUTATION OF THE ARITHMETIC SIDE OF THE MAIN IDENTITY

**Lemma 7.1.** *With the notation of Theorem 6.1, one has if  $N = RQ$  is squarefree odd*

$$c_{R,Q}(\mathfrak{T}_N; m, n) = \text{char}(m + n \equiv 0 \pmod{2R}) \text{char}(m - n \equiv 0 \pmod{2Q}) \sum_{\substack{R_1 R_2 = R \\ Q_1 Q_2 = Q}} \mu(R_1 Q_1) \text{char}\left(\frac{m+n}{R} \equiv 0 \pmod{R_1 Q_1}\right) \text{char}\left(\frac{m-n}{2Q} \equiv 0 \pmod{R_2 Q_2}\right). \quad (7.1)$$

*Proof.* We compute the function  $f_N(j, s)$  defined in (6.3). If  $N = N_1 N_2$  and  $cN_1 + dN_2 = 1$ , so that  $\frac{k}{N} = \frac{dk}{N_1} + \frac{ck}{N_2}$ , one identifies  $k \in \mathbb{Z}/N\mathbb{Z}$  with the pair  $(k_1, k_2) \in \mathbb{Z}/N_1\mathbb{Z} \times \mathbb{Z}/N_2\mathbb{Z}$  such that  $k_1 \equiv dk \pmod{N_1}$ ,  $k_2 \equiv ck \pmod{N_2}$ . On one hand,

$$\exp\left(\frac{2i\pi ks}{N}\right) = \exp\left(\frac{2i\pi k_1 s}{N_1}\right) \times \exp\left(\frac{2i\pi k_2 s}{N_2}\right). \quad (7.2)$$

On the other hand, as  $(d, N_1) = (c, N_2) = 1$ ,

$$a((j, k, N)) = a((j, k, N_1)) a((j, k, N_2)) = a((j, k_1, N_1)) a((j, k_2, N_2)). \quad (7.3)$$

It follows from (6.3) that  $f_N(j, s) = f_{N_1}(j, s) f_{N_2}(j, s)$ , and one has the Eulerian formula  $f_N = \otimes_{p|N} f_p$ , with

$$\begin{aligned} f_p(j, s) &= \frac{1}{p} \sum_{k \bmod p} (1 - p \operatorname{char}(j \equiv k \equiv 0 \bmod p)) \exp\left(\frac{2i\pi ks}{p}\right) \\ &= \frac{1}{p} \sum_{k \bmod p} \exp\left(\frac{2i\pi ks}{p}\right) - \operatorname{char}(j \equiv 0 \bmod p) \\ &= \operatorname{char}(s \equiv 0 \bmod p) - \operatorname{char}(j \equiv 0 \bmod p). \end{aligned} \quad (7.4)$$

Expanding the product,

$$f_N(j, s) = \sum_{N_1 N_2 = N} \mu(N_1) \operatorname{char}(s \equiv 0 \bmod N_2) \operatorname{char}(j \equiv 0 \bmod N_1). \quad (7.5)$$

The equation (7.1) follows from (6.4). □

**Theorem 7.2.** *Let  $N = RQ$  be squarefree odd. Let  $v, u \in C^\infty(\mathbb{R})$ , compactly supported, satisfying the conditions that  $x > 0$  and  $0 < x^2 - y^2 < 8$  when  $v(x)u(y) \neq 0$ . Then, if  $N$  is large enough,*

$$\begin{aligned} (v \mid \Psi(Q^{2i\pi\mathcal{E}} \mathfrak{T}_N) u) &= \sum_{Q_1 Q_2 = Q} \mu(Q_1) \\ &\quad \sum_{R_1 | R} \mu(R_1) \bar{v}\left(\frac{R_1}{Q_2} + \frac{Q_2}{R_1}\right) u\left(\frac{R_1}{Q_2} - \frac{Q_2}{R_1}\right). \end{aligned} \quad (7.6)$$

*Proof.* Characterize  $n \bmod 2N^2$  by  $n \in \mathbb{Z}$  such that  $-N^2 \leq n < N^2$ . The equation  $(\theta_N u)(n) = \sum_{\ell \in \mathbb{Z}} u\left(\frac{n}{N} + 2\ell N\right)$  imposes  $\ell = 0$ , so that  $(\theta_N u)(n) = u\left(\frac{n}{N}\right)$ . Similarly,  $(\theta_N v)(m) = v\left(\frac{m}{N}\right)$  if  $-N^2 \leq m < N^2$ .

The identity (6.6) and Lemma 7.1 yield

$$\begin{aligned} (v \mid \Psi(Q^{2i\pi\mathcal{E}} \mathfrak{T}_N) u) &= \sum_{\substack{R_1 R_2 = R \\ Q_1 Q_2 = Q}} \mu(R_1 Q_1) \operatorname{char}(m+n \equiv 0 \bmod 2RR_1 Q_1) \\ &\quad \operatorname{char}(m-n \equiv 0 \bmod 2QR_2 Q_2) \bar{v}\left(\frac{m}{N}\right) u\left(\frac{n}{N}\right). \end{aligned} \quad (7.7)$$

Set  $m+n = (2RR_1 Q_1) a$ ,  $m-n = (2QR_2 Q_2) b$ , with  $a, b \in \mathbb{Z}$ . Then,

$$\frac{m^2}{N^2} - \frac{n^2}{N^2} = \frac{4QR(R_1 R_2)(Q_1 Q_2) ab}{N^2} = 4ab. \quad (7.8)$$

For all nonzero terms of the last equation, one has  $0 < \frac{m^2}{N^2} - \frac{n^2}{N^2} < 8$ . On the other hand, the symbol  $(Q^{2i\pi\mathcal{E}}\mathfrak{T}_N)(x, \xi) = Q\mathfrak{T}_N(Qx, Q\xi)$ , just as the symbol  $\mathfrak{T}_N$ , is invariant under the change of  $x, \xi$  to  $x, x + \xi$ . It follows from Theorem 3.4 that its integral kernel  $K(x, y)$  is supported in the set of points  $(x, y)$  such that  $x^2 - y^2 \in 4\mathbb{Z}$ . With  $x = \frac{m}{N}$  and  $y = \frac{n}{N}$ , the only possibility is to take  $x^2 - y^2 = 4$ , hence  $ab = 1$ , finally  $a = b = 1$  since  $a > 0$ .

The equations  $m + n = (2RR_1Q_1)$ ,  $m - n = (2QR_2Q_2)$  give  $\frac{m}{N} = \frac{R_1}{Q_2} + \frac{Q_2}{R_1}$  and  $\frac{n}{N} = \frac{R_1}{Q_2} - \frac{Q_2}{R_1}$ . Be sure to note that, given  $N = RQ$  and  $R_1$ , there is no summation with respect to  $R_2 = \frac{R}{R_1}$ . □

## 8. ARITHMETIC SIGNIFICANCE OF LAST THEOREM

In view of the central role of Theorem 7.2 in the proof to follow of the Riemann hypothesis, we give now an independent proof of the major part of it. Reading this section is thus in principle unnecessary: but we find the equation (8.2) below illuminating.

**Theorem 8.1.** *Let  $N = RQ$  be a squarefree odd integer. Introduce the reflection  $n \mapsto \check{n}$  of  $\mathbb{Z}/(2N^2)\mathbb{Z}$  such that  $\check{n} \equiv n \pmod{R^2}$  and  $\check{n} \equiv -n \pmod{2Q^2}$ . Then, with the notation in Theorem 6.1, one has*

$$c_{R,Q}(\mathfrak{T}_N; m, n) = \mu(Q) c_{N,1}(\mathfrak{T}_N; m, \check{n}). \quad (8.1)$$

*If two functions  $u$  and  $\tilde{u}$  in  $\mathcal{S}(\mathbb{R})$  are such that  $(\theta_N \tilde{u})(n) = (\theta_N u)(\check{n})$  for every  $n \in \mathbb{Z}$ , one has*

$$(v \mid \Psi(Q^{2i\pi\mathcal{E}}\mathfrak{T}_N) u) = \mu(Q) (v \mid \Psi(\mathfrak{T}_N) \tilde{u}), \quad (8.2)$$

*Proof.* With the notation of Theorem 6.1, and making use of (7.4), one has the Eulerian decomposition  $f_N = \otimes f_p$ , in which, for every  $p$ ,  $f_p(j, s) = -f_p(s, j)$ : it follows that  $f_N(j, s) = \mu(N) f_N(s, j)$ . To prove (8.1), one may assume that  $R = 1$ ,  $N = Q$  since the  $R$ -factor is left unaffected by the

map  $n \mapsto \check{n}$ . Then,

$$\begin{aligned} c_{1,Q}(\mathfrak{S}; m, n) &= \text{char}(m - n \equiv 0 \pmod{2Q}) f\left(\frac{m+n}{2}, \frac{m-n}{2Q}\right), \\ c_{Q,1}(\mathfrak{S}; m, \check{n}) &= \text{char}(m + \check{n} \equiv 0 \pmod{2Q}) f\left(\frac{m+\check{n}}{2Q}, \frac{m-\check{n}}{2}\right) \\ &= \text{char}(m - n \equiv 0 \pmod{2Q}) f\left(\frac{m-n}{2Q}, \frac{m+n}{2}\right). \end{aligned} \quad (8.3)$$

That the first and third line are the same, up to the factor  $(\mu(Q))$ , follows from the set of equations (7.4)  $f_p(j, s) = -f_p(s, j)$ .

The equation (8.2) follows from (8.1) in view of (6.6).

In [11, p.62-64], we gave a direct proof of a more general result (involving a more general symbol  $\mathfrak{S}$  of type (6.1), not depending on Theorem 6.1.

□

It is not immediately obvious that, given  $u \in \mathcal{S}(\mathbb{R})$ , there exists  $\tilde{u} \in \mathcal{S}(\mathbb{R})$  such that  $(\theta_N \tilde{u})(n) = (\theta_N u)(\check{n})$  for every  $n \in \mathbb{Z}$ . Note that such a function  $\tilde{u}$  could not be unique, since any translate by a multiple of  $2N^2$  would do just as well. That such a function does exist [11, p.61] is obtained from an explicit formula, to wit

$$\tilde{u}(y) = \frac{1}{Q^2} \sum_{0 \leq \sigma, \tau < Q^2} u\left(y + \frac{2R\tau}{Q}\right) \exp\left(2i\pi \frac{\sigma(Nx + R^2\tau)}{Q^2}\right). \quad (8.4)$$

Indeed, with such a definition, one has

$$\begin{aligned} &(\theta_N \tilde{u})(n) \\ &= \frac{1}{Q^2} \sum_{\ell \in \mathbb{Z}} \sum_{0 \leq \sigma, \tau < Q^2} u\left(\frac{n}{N} + \frac{2R\tau}{Q} + \ell N\right) \exp\left(\frac{2i\pi\sigma}{Q^2} (n + \ell N^2 + R^2\tau)\right). \end{aligned} \quad (8.5)$$

Summing with respect to  $\sigma$ , this is the sme as

$$\sum_{\ell \in \mathbb{Z}} u\left(\frac{n + 2R^2\tau + \ell N^2}{N}\right), \quad (8.6)$$

where the integer  $\tau \in [0, Q^2[$  is characterized by the condition  $n + \ell N^2 + 2R^2\tau \equiv 0 \pmod{Q^2}$ , or  $n + 2R^2\tau \equiv 0 \pmod{Q^2}$ . Finally, as  $\ell \in \mathbb{Z}$ , the

number  $n + 2R^2\tau + \ell N^2$  runs through the set of integers  $n_2$  such that  $n_2 \equiv -n \pmod{Q^2}$  and  $n_2 \equiv n \pmod{2R^2}$ , in other words the set of numbers  $n_2$  such that  $n_2 \equiv \check{n} \pmod{2N^2}$ . But applying this formula leads to rather unpleasant (not published, though leading to the correct result) calculations, as we experienced.

Instead, we shall give arithmetic the priority, starting from a manageable expression of the map  $n \mapsto \check{n}$ . Recall (4.18) that if  $v, u$  are compactly supported and the algebraic sum of their supports is contained in some interval  $[0, 2\beta]$ , and if one interests oneself in  $\left(v \mid \Psi\left(Q^{2i\pi\mathcal{E}}\mathfrak{T}_{\frac{\infty}{2}}\right)u\right)$ , one can replace  $\mathfrak{T}_{\frac{\infty}{2}}$  by  $\mathfrak{T}_N$  provided that  $N = RQ$  is divisible by all odd primes  $< \beta Q$ . With this in mind, the following lemma will make it possible to assume without loss of generality that  $R \equiv 1 \pmod{2Q^2}$ .

**Lemma 8.2.** *Let  $Q$  be a squarefree odd positive integer and let  $\beta > 0$  be given. There exists  $R > 0$ , with  $N = RQ$  squarefree odd divisible by all odd primes  $< \beta Q$ , such that  $R \equiv 1 \pmod{2Q^2}$ .*

*Proof.* Choose  $R_1$  positive, odd and squarefree, relatively prime to  $Q$ , divisible by all odd primes  $< \beta Q$  relatively prime to  $Q$ , and  $\bar{R}_1$  such that  $\bar{R}_1 R_1 \equiv 1 \pmod{2Q^2}$  and  $\bar{R}_1 \equiv 1 \pmod{R_1}$ . Since  $[R_1, 2Q] = 1$ , there exists  $x \in \mathbb{Z}$  such that  $x \equiv 1 \pmod{R_1}$  and  $x \equiv \bar{R}_1 \pmod{2Q^2}$ . Choosing (Dirichlet's theorem) a prime  $r$  such that  $r \equiv x \pmod{2R_1 Q^2}$ , the number  $R = R_1 r$  satisfies the desired condition.  $\square$

With such a choice of  $R$ , we can make the map  $n \mapsto \check{n}$  from  $\mathbb{Z}/(2N^2)\mathbb{Z}$  to  $\mathbb{Z}/(2N^2)\mathbb{Z}$  explicit. Indeed, the solution of a pair of congruences  $x \equiv \lambda \pmod{R^2}$ ,  $x \equiv \mu \pmod{2Q^2}$  is given as  $x \equiv (1 - R^2)\lambda + \mu R^2 \pmod{2N^2}$ . In particular, taking  $\lambda = n$ ,  $\mu = -n$ , one obtains  $\check{n} \equiv n(1 - 2R^2) \pmod{2N^2}$ . Then, defining for instance  $\tilde{u}(y) = u(y(1 - 2R^2))$ , one has indeed  $(\theta_N \tilde{u})(n) = (\theta_N u)(\check{n})$ . Of course, the support of  $\tilde{u}$  is not the same as that of  $u$ , but no support conditions are necessary for (8.2) to hold.

We can now make a quick partial verification of the identity (7.6), starting with the case for which  $Q = 1$ . The point of Theorem 8.1 is that the general case can be reduced to this special one.

**Lemma 8.3.** *Let  $v, u$  be two functions in  $\mathcal{S}(\mathbb{R})$ . One has for every square-free integer  $N$  the identity*

$$(v \mid \Psi(\mathfrak{T}_N) u) = \sum_{T \mid N} \mu(T) \sum_{j, k \in \mathbb{Z}} \bar{v} \left( Tj + \frac{k}{T} \right) u \left( Tj - \frac{k}{T} \right). \quad (8.7)$$

*Proof.* Together with the operator  $2i\pi\mathcal{E}$ , let us introduce the operator  $2i\pi\mathcal{E}^\natural = r \frac{\partial}{\partial r} - s \frac{\partial}{\partial s}$  when the coordinates  $(r, s)$  are used on  $\mathbb{R}^2$ . One has if  $\mathcal{F}_2^{-1}$  denotes the inverse Fourier transformation with respect to the second variable  $\mathcal{F}_2^{-1}[(2i\pi\mathcal{E})\mathfrak{S}] = (2i\pi\mathcal{E}^\natural)\mathcal{F}_2^{-1}\mathfrak{S}$  for every tempered distribution  $\mathfrak{S}$ . From the relation (4.11) between  $\mathfrak{T}_N$  and the Dirac comb, and Poisson's formula, one obtains

$$\mathcal{F}_2^{-1}\mathfrak{T}_N = \prod_{p \mid N} \left(1 - p^{-2i\pi\mathcal{E}^\natural}\right) \mathcal{F}_2^{-1}\mathcal{D}ir = \sum_{T \mid N} \mu(T) T^{-2i\pi\mathcal{E}^\natural} \mathcal{D}ir, \quad (8.8)$$

explicitly

$$(\mathcal{F}_2^{-1}\mathfrak{T}_N)(r, s) = \sum_{T \mid N} \mu(T) \sum_{j, k \in \mathbb{Z}} \delta\left(\frac{r}{T} - j\right) \delta(Ts - k). \quad (8.9)$$

The integral kernel of the operator  $\Psi(\mathfrak{T}_N)$  is (2.2)

$$\begin{aligned} K(x, y) &= \frac{1}{2} (\mathcal{F}_2^{-1}\mathfrak{T}_N) \left( \frac{x+y}{2}, \frac{x-y}{2} \right) \\ &= \sum_{T \mid N} \mu(T) \sum_{j, k \in \mathbb{Z}} \delta\left(x - Tj - \frac{k}{T}\right) \delta\left(y - Tj + \frac{k}{T}\right). \end{aligned} \quad (8.10)$$

The equation (8.7) follows.  $\square$

We now take advantage of (8.2) and Lemma 8.2, after the proof of which we have seen that one could choose  $\tilde{u}(y) = u(y(1 - 2R^2) + 2aN^2)$ , with any  $a \in \mathbb{Z}$ , to obtain a quick verification of the main feature of (7.6). As a particular case of the conditions in Theorem 7.2, we assume that  $0 < x^2 - y^2 < 8$  when  $v(x)u(y) \neq 0$ , say that  $v$  is supported in  $[2, \sqrt{8}]$  and  $u$  in  $[0, 1]$ , and that  $N$  is large enough. From (8.7), one obtains

$$\begin{aligned} (v \mid \Psi(Q^{2i\pi\mathcal{E}}\mathfrak{T}_N) u) &= \mu(Q) \sum_{T \mid N} \mu(T) \sum_{j, k \in \mathbb{Z}} \bar{v} \left( Tj + \frac{k}{T} \right) \tilde{u} \left( Tj - \frac{k}{T} \right) \\ &= \mu(Q) \sum_{T \mid N} \mu(T) \sum_{j, k \in \mathbb{Z}} \bar{v} \left( Tj + \frac{k}{T} \right) u \left( \left( Tj - \frac{k}{T} \right) (1 - 2R^2) + 2aN^2 \right). \end{aligned} \quad (8.11)$$

Let  $x$  and  $y$  be the arguments of  $\bar{v}$  and  $u$  in the last expression. Note that that of  $u$  is truly defined mod  $2N^2$ , while no such proviso is made about  $v$ . One has  $u(y) \neq 0$  for at most one value of  $a$ , which we choose so as to have  $x^2 - y^2 = 4$ , not only  $x^2 - y^2 \equiv 4 \pmod{2N^2}$ . Then,

$$\begin{aligned}
1 &= \frac{x^2 - y^2}{4} = \frac{x - y}{2} \times \frac{x + y}{2} \\
&= \frac{1}{2} \left[ Tj + \frac{k}{T} - \left( Tj - \frac{k}{T} \right) (1 - 2R^2) - 2aN^2 \right] \\
&\quad \times \frac{1}{2} \left[ Tj + \frac{k}{T} + \left( Tj - \frac{k}{T} \right) (1 - 2R^2) + 2aN^2 \right] \\
&= \left[ \frac{k}{T} + R^2 \left( Tj - \frac{k}{T} \right) - aN^2 \right] \left[ Tj - R^2 \left( Tj - \frac{k}{T} \right) + aN^2 \right]. \quad (8.12)
\end{aligned}$$

With  $\alpha = \frac{x-y}{2}$ ,  $\beta = \frac{x+y}{2}$ , one has, as a congruence mod  $2N^2$ ,

$$Q\beta \equiv Q \left[ (1 - R^2)Tj + \frac{R^2k}{T} \right] \equiv (1 - R^2)QTj + \frac{NRk}{T}, \quad (8.13)$$

an integer since  $T|N$ . Writing  $R^2 = 1 + 2\lambda Q^2$ , so that

$$\alpha \equiv \frac{(1 - R^2)k}{T} + R^2Tj = -\frac{2\lambda Q^2k}{T} + R^2Tj, \quad (8.14)$$

one sees in the same way that  $R\alpha \in \mathbb{Z}$ .

Since  $\alpha\beta \equiv 1$ , the numbers  $m = Q\beta$  and  $n = \frac{R}{\beta}$  are integers. Setting  $m = m_1m_2$  and  $n = n_1n_2$  with  $m_1, n_1|R$  and  $m_2, n_2|Q$ , the equation  $mn = QR$  yields  $m_1n_1 = \pm R$ ,  $m_2n_2 = \pm Q$  and  $\beta = \frac{R}{n} = \pm \frac{m_1n_1}{n_1n_2} = \pm \frac{m_1}{n_2}$ . In other words,  $\beta = \frac{R_1}{Q_2}$  with  $R_1|R$  and  $Q_2|Q$ . This is an equality, not just a congruence, since adding to  $R_1$  a multiple of  $2N^2$  would put  $\beta$  out of the interval where it must lie to contribute a nonzero term.

This is only a quick verification of Theorem 7.2 (under the condition  $R \equiv 1 \pmod{2Q^2}$ ), not a complete second proof since recovering the integers  $T, j, k$  from the set  $\{R, Q, R_1, Q_2\}$  looks like a complicated task. In particular, we have not verified, here, that the coefficient of  $\bar{v} \left( \frac{R_1}{Q_2} + \frac{Q_2}{R_1} \right) u \left( \frac{R_1}{Q_2} - \frac{Q_2}{R_1} \right)$  is  $\mu(R_1Q_1) = \mu(Q)\mu(R_1Q_2)$ . Theorem 7.2, based on (6.4), gives the coefficients in full.



9. A SERIES  $F_\varepsilon(s)$  AND ITS INTEGRAL VERSION

**Definition 9.1.** Let  $\varepsilon > 0$  be fixed. We associate to any pair  $v, u$  of functions in  $\mathcal{S}(\mathbb{R})$  the function

$$F_\varepsilon(s) := \sum_{Q \in \text{Sq}^{\text{odd}}} Q^{-s} \left( v \mid \Psi \left( Q^{2i\pi\varepsilon} \mathfrak{T}_{\frac{\infty}{2}} \right) u_Q \right), \quad (9.1)$$

with  $u_Q(y) = Q^{\frac{\varepsilon}{2}} u(Q^\varepsilon y)$  (we might have denoted it as  $u_{Q^\varepsilon}$ ).

**Theorem 9.2.** Let  $v, u \in C^\infty(\mathbb{R})$ , with  $v$  supported in  $[2, \sqrt{8}]$  and  $u$  supported in  $[0, 1]$ . Then, the function  $F_\varepsilon(s)$  is entire. For  $\text{Re } s > 2$ ,  $F_\varepsilon(s)$  converges as  $\varepsilon \rightarrow 0$  towards the function  $F_0(s)$  introduced in (5.2).

*Proof.* When  $v(x)u(y) \neq 0$  one has  $0 < x^2 - y^2 < 8$ , which will make it possible, in the proof of Proposition 9.3, to use the simple integral expression (3.18) of  $(v \mid \Psi(\mathfrak{E}_{-\nu}) u)$ .

We use Theorem 7.2. Under the given support assumptions, one can rewrite (7.6) as

$$\begin{aligned} \left( v \mid \Psi(Q^{2i\pi\varepsilon} \mathfrak{T}_{\frac{\infty}{2}}) u \right) &= \sum_{Q_1 Q_2 = Q} \mu(Q_1) \\ &\quad \sum_{\substack{R_1 \in \text{Sq}^{\text{odd}} \\ (R_1, Q) = 1}} \mu(R_1) \bar{v} \left( \frac{R_1}{Q_2} + \frac{Q_2}{R_1} \right) u \left( \frac{R_1}{Q_2} - \frac{Q_2}{R_1} \right). \end{aligned} \quad (9.2)$$

Indeed, given  $Q$ ,  $\left( v \mid \Psi(Q^{2i\pi\varepsilon} \mathfrak{T}_{\frac{\infty}{2}}) u \right)$  coincides according to (4.18) with  $\left( v \mid \Psi(Q^{2i\pi\varepsilon} \mathfrak{T}_N) u \right)$  for  $N$  odd divisible, for some  $\beta > 0$  depending on the supports of  $v$  and  $u$ , by all odd primes  $< \beta Q$ : we shall take  $\beta = \frac{1+\sqrt{8}}{2}$  and fix  $N$  accordingly. Now, given  $R_1$  squarefree odd, the condition that  $R_1$  divides some number  $R$  with the property that  $N = RQ$  is squarefree odd implies the condition  $(R_1, Q) = 1$ . In the reverse direction, the condition  $0 < \frac{R_1}{Q_2} - \frac{Q_2}{R_1} < 1$  implies  $R_1 < \frac{1+\sqrt{5}}{2} Q$ . This implies, since  $R_1$  is squarefree, that  $R_1 \mid N$  or, if  $(R_1, Q) = 1$ , that  $R_1 \mid R$ .

Then, for  $\operatorname{Re} s > 2$ ,

$$\begin{aligned} F_\varepsilon(s) &= \sum_{Q \in \operatorname{Sq}^{\text{odd}}} Q^{-s} \left( v \mid \Psi \left( Q^{2i\pi\mathcal{E}} \mathfrak{T}_{\frac{\infty}{2}} \right) u_Q \right) = \sum_{Q \in \operatorname{Sq}^{\text{odd}}} Q^{-s+\frac{\varepsilon}{2}} \sum_{Q_1 Q_2 = Q} \mu(Q_1) \\ &\quad \sum_{\substack{R_1 \in \operatorname{Sq}^{\text{odd}} \\ (R_1, Q) = 1}} \mu(R_1) \bar{v} \left( \frac{R_1}{Q_2} + \frac{Q_2}{R_1} \right) u \left( Q^\varepsilon \left( \frac{R_1}{Q_2} - \frac{Q_2}{R_1} \right) \right). \end{aligned} \quad (9.3)$$

For all nonzero terms of this series, one has for some absolute constant  $C$

$$\left| \frac{R_1}{Q_2} - \frac{Q_2}{R_1} \right| \leq Q^{-\varepsilon}, \quad \left| \frac{R_1}{Q_2} - 1 \right| \leq C Q^{-\varepsilon}, \quad |R_1 - Q_2| \leq C Q_2 Q^{-\varepsilon}. \quad (9.4)$$

The number of available  $R_1$  is at most  $C Q_2 Q^{-\varepsilon}$ . On the other hand,

$$\left| \frac{R_1}{Q_2} + \frac{Q_2}{R_1} - 2 \right| \leq \left| \frac{R_1}{Q_2} - 1 \right| + \left| \frac{Q_2}{R_1} - 1 \right| \leq C Q^{-\varepsilon}. \quad (9.5)$$

Since  $v(x)$  is flat at  $x = 2$ , one has for every  $M \geq 1$  and some constant  $C_M$

$$\begin{aligned} |(v \mid \Psi(Q^{2i\pi\mathcal{E}} \mathfrak{T}_N) u_Q)| &\leq C C_M Q^{\frac{\varepsilon}{2}} \sum_{Q_1 Q_2 = Q} (Q_2 Q^{-\varepsilon}) Q^{-M\varepsilon} \\ &\leq C C_M C_2 Q^{\frac{\varepsilon}{2} + 1 - (M+1)\varepsilon + \varepsilon'} = C C_M C_2 Q^{1 - (M+\frac{1}{2})\varepsilon + \varepsilon'}, \end{aligned} \quad (9.6)$$

where  $C_2 Q^{\varepsilon'}$  is a bound ( $\varepsilon'$  can be taken arbitrarily small) for the number of divisors of  $Q$ . The theorem follows.  $\square$

We wish now to compute  $F_\varepsilon(s)$  by analytic methods. The novelty, in comparison to the analysis made in Section 5, is that we have replaced  $u$  by  $u_Q$ : in just the same way as the distribution  $\mathfrak{T}_{\frac{\infty}{2}}$  was decomposed into Eisenstein distributions, it is  $u$  that we must now decompose into homogeneous components. Let us recall (Mellin transformation, or Fourier transformation up to a change of variable) that functions on the line decompose into generalized eigenfunctions of the self-adjoint operator  $i \left( \frac{1}{2} + y \frac{d}{dy} \right)$ , according to the decomposition (analogous to (3.10))

$$u = \frac{1}{i} \int_{\operatorname{Re} \mu = 0} u^\mu d\mu, \quad (9.7)$$

with

$$u^\mu(y) = \frac{1}{2\pi} \int_0^\infty \theta^{\mu-\frac{1}{2}} u(\theta y) d\theta. \quad (9.8)$$

The function  $u^\mu$  is homogeneous of degree  $-\frac{1}{2} - \mu$ . Note that  $\mu$  is in the superscript position, in order not to confuse  $u^\mu$  with a case of  $u_Q$ . A justification of (9.7), (9.8) is as follows.

Given  $u = u(y)$  in  $\mathcal{S}(\mathbb{R})$  and  $y \neq 0$ , consider the function  $f(t) = e^{\pi t} u(e^{2\pi t} y)$ . The functions  $f$  and  $\hat{f}$  are integrable. Define

$$u^{i\lambda}(y) := \hat{f}(-\lambda) = \frac{1}{2\pi} \int_0^\infty \theta^{i\lambda - \frac{1}{2}} u(\theta y) d\theta. \quad (9.9)$$

Then,  $u(y) = f(0) = \int_{-\infty}^\infty \hat{f}(-\lambda) d\lambda = \frac{1}{i} \int_{\operatorname{Re} \mu = 0} u^\mu d\mu$ .

**Proposition 9.3.** *Let  $v, u \in C^\infty(\mathbb{R})$  satisfy the support conditions in Theorem 9.2. For every  $\nu \in \mathbb{C}$ , the function*

$$\mu \mapsto \Phi(v, u; \nu, \mu) := \int_0^\infty t^{\nu-1} \bar{v}(t + t^{-1}) u^\mu(t - t^{-1}) dt, \quad (9.10)$$

*initially defined for  $\operatorname{Re} \mu > -\frac{1}{2}$ , extends as an entire function, rapidly decreasing in vertical strips. One has*

$$(v \mid \Psi(\mathfrak{E}_{-\nu}) u) = \frac{1}{i} \int_{\operatorname{Re} \mu = 0} \Phi(v, u; \nu, \mu) d\mu. \quad (9.11)$$

*Proof.* When  $\bar{v}(t + t^{-1}) \neq 0$ ,  $t$  is bounded and bounded away from zero. The last factor of the integrand of (9.10) has a singularity at  $t = 1$ , taken care of by the fact that  $(t - t^{-1})^2 \geq C^{-1}(t + t^{-1} - 2)$  for  $t^{\pm 1}$  bounded, while  $v = v(x)$  is flat at  $x = 2$ . Powers of  $(1 + |\mu|)^{-1}$  can be gained with the help of repeated integrations by parts associated to the identity

$$\left(-\frac{1}{2} - \mu\right) u^\mu(t - t^{-1}) = (t + t^{-1})^{-1} t \frac{d}{dt} (u^\mu(t - t^{-1})). \quad (9.12)$$

Let us prove (9.11). Using (9.7), one has

$$\frac{1}{i} \int_{\operatorname{Re} \mu = 0} \Phi(v, u; \nu, \mu) d\mu = \int_0^\infty t^{\nu-1} \bar{v}(t + t^{-1}) u(t - t^{-1}) dt. \quad (9.13)$$

The right-hand side is the same as  $(v \mid \Psi(\mathfrak{E}_{-\nu}) u)$  according to Theorem 3.4.

The equations (9.7) and (9.11) do not imply, however, that  $\Phi(v, u; \nu, \mu)$  is the same as  $(v \mid \Psi(\mathfrak{E}_{-\nu}) u^\mu)$ , because the pair  $v, u^\mu$  does not satisfy the

support conditions in the second part of Theorem 3.4. Actually, as seen from the proof of Theorem 3.4,

$$\Phi(v, u; \nu, \mu) = (v \mid \Psi(\mathfrak{S}_1) u^\mu), \quad (9.14)$$

where  $\mathfrak{S}_1(x, \xi) = |x|^{\nu-1} \exp\left(-\frac{2i\pi\xi}{x}\right)$  is the term corresponding to the choice  $r = 1$  in the Fourier expansion (3.18) of  $\mathfrak{E}_{-\nu}$ . Such a reduction has been made possible by the demands made on the supports of  $v, u$   $\square$

We shall also need to prove that, with a loss tempered by a power of  $|\mu|$ , the function  $\nu \mapsto \Phi(v, u; \nu, \mu)$  is integrable on lines  $\operatorname{Re} \nu = c$  with  $c > 1$ .

**Proposition 9.4.** *Let  $v = v(x)$  and  $u = u(y)$  be two functions satisfying the conditions in Theorem 9.2. Defining the operators, to be applied to  $v$ , such that*

$$D_{-1}^\mu v = v'', \quad D_0^\mu = -2\mu \left(xv' + \frac{v}{2}\right), \quad D_1^\mu v = \left(\frac{1}{2} + \bar{\mu}\right) \left(\frac{3}{2} + \bar{\mu}\right) x^2 v, \quad (9.15)$$

one has if  $\operatorname{Re} \mu < \frac{1}{2}$

$$\left(\frac{1}{2} + \nu^2\right) \Phi(v, u; \nu, \mu) = \sum_{j=-1,0,1} \Phi\left(D_j^\mu v, y^{-2j} u; \nu, \mu + 2j\right). \quad (9.16)$$

As a function of  $\nu$  on any line  $\operatorname{Re} \nu = c$  with  $c > 1$ , the function  $\Phi(v, u; \nu, \mu)$  is a  $O((1+|\operatorname{Im} \nu|)^{-2})$ , with a loss of uniformity relative to  $\mu$  bounded by  $|\mu|^2$ .

*Proof.* The equation (9.10) and the identity  $\nu^2 t^\nu = (t \frac{d}{dt})^2 t^\nu$  give (noting that the operator  $t \frac{d}{dt}$  is the negative of its transpose if one uses the measure  $\frac{dt}{t}$ )

$$\nu^2 \Phi(v, u; \nu, \mu) = \int_0^\infty t^{\nu-1} \left(t \frac{d}{dt}\right)^2 (\bar{v}(t+t^{-1}) u^\mu(t-t^{-1})) dt. \quad (9.17)$$

To facilitate the calculations which follow, observe, setting  $|s|_1^\alpha = |s|^\alpha \operatorname{sign} s$ , that  $\frac{d}{ds} |s|^\alpha = \alpha |s|_1^{\alpha-1}$  and  $\frac{d}{ds} |s|_1^\alpha = \alpha |s|^{\alpha-1}$ , finally  $t \frac{d}{dt} v(t+t^{-1}) = (t-t^{-1}) v'(t+t^{-1})$  and  $t \frac{d}{dt} u(t-t^{-1}) = (t+t^{-1}) u'(t-t^{-1})$ .

One has

$$\begin{aligned} & t \frac{d}{dt} \left[ \bar{v}(t+t^{-1}) |t-t^{-1}|^{-\mu-\frac{1}{2}} \right] \\ &= \bar{v}'(t+t^{-1}) |t-t^{-1}|^{-\mu+\frac{1}{2}} - \left(\mu + \frac{1}{2}\right) \bar{v}(t+t^{-1}) (t+t^{-1}) |t-t^{-1}|^{-\mu-\frac{3}{2}}. \end{aligned} \quad (9.18)$$

Next,

$$\begin{aligned} & \left(t \frac{d}{dt}\right)^2 \left[ \bar{v}(t+t^{-1}) |t-t^{-1}|^{-\mu-\frac{1}{2}} \right] \\ &= \bar{v}''(t+t^{-1}) |t-t^{-1}|^{-\mu+\frac{3}{2}} - 2\mu (t+t^{-1}) \bar{v}'(t+t^{-1}) |t-t^{-1}|^{-\mu-\frac{1}{2}} \\ &\quad - \left(\mu + \frac{1}{2}\right) \bar{v}(t+t^{-1}) |t-t^{-1}|^{-\mu-\frac{1}{2}} \\ &\quad + \left(\mu + \frac{1}{2}\right) \left(\mu + \frac{3}{2}\right) (t+t^{-1})^2 \bar{v}(t+t^{-1}) |t-t^{-1}|^{-\mu-\frac{5}{2}}. \end{aligned} \quad (9.19)$$

Now, one has  $y^2 u^\mu = (y^2 u)^{\mu-2}$ ,  $y^{-2} u^\mu = (y^{-2} u)^{\mu+2}$ . If  $u$  is even, so that  $u^\mu(y)$  is a multiple of  $|y|^{-\mu-\frac{1}{2}}$ , using these identities with  $y = t-t^{-1}$  leads to the identity (9.16). Just exchanging the signed and unsigned versions of the power function gives the same result if  $u$  is odd. The loss by a factor  $(1+|\mu|)^2$  is insignificant in view of Proposition 9.3.  $\square$

**Proposition 9.5.** *Let  $v, u$  satisfy the conditions of Theorem 9.2. Assuming  $c > 1$  and  $\operatorname{Re} s$  large, one has the identity*

$$F_\varepsilon(s) = \frac{1}{2i\pi} \int_{\operatorname{Re} \nu=c} \frac{(1-2^{-\nu})^{-1}}{\zeta(\nu)} H_\varepsilon(s, \nu) d\nu, \quad (9.20)$$

with

$$H_\varepsilon(s, \nu) = \frac{1}{i} \int_{\operatorname{Re} \mu=0} f(s-\nu+\varepsilon\mu) \Phi(v, u; \nu, \mu) d\mu \quad (9.21)$$

and  $f(s) = (1+2^{-s})^{-1} \frac{\zeta(s)}{\zeta(2s)}$ .

*Proof.* Using (9.1) and (4.17), one has for  $c > 1$

$$F_\varepsilon(s) = \frac{1}{2i\pi} \sum_{Q \in \operatorname{Sq}^{\text{odd}}} Q^{-s} \int_{\operatorname{Re} \nu=c} \frac{(1-2^{-\nu})^{-1}}{\zeta(\nu)} Q^\nu (v | \Psi(\mathfrak{E}_{-\nu}) u_Q) d\nu. \quad (9.22)$$

One has  $(u_Q)^\mu = (u^\mu)_Q = Q^{-\varepsilon\mu} u^\mu$ , and, according to (9.10) and (9.11),

$$(v \mid \Psi(\mathfrak{E}_{-\nu}) u_Q) = \frac{1}{i} \int_{\operatorname{Re} \mu=0} Q^{-\varepsilon\mu} \Phi(v, u; \nu, \mu) d\mu. \quad (9.23)$$

Recall (Proposition 9.3) that  $\Phi(v, u; \nu, \mu)$  is a rapidly decreasing function of  $\mu$  in vertical strips.

To insert this equation into (9.22), we use Proposition 9.4, which provides the  $d\nu$ -summability, at the price of losing at most the factor  $(1+|\mu|)^2$ . We obtain if  $c > 1$  and  $\operatorname{Re} s$  is large enough

$$F_\varepsilon(s) = -\frac{1}{2\pi} \sum_{Q \in \operatorname{Sq}^{\text{odd}}} \int_{\operatorname{Re} \nu=c} Q^{-s+\nu} \frac{(1-2^{-\nu})^{-1}}{\zeta(\nu)} d\nu \int_{\operatorname{Re} \mu=0} Q^{-\varepsilon\mu} \Phi(v, u; \nu, \mu) d\mu \quad (9.24)$$

or, using (5.4),

$$F_\varepsilon(s) = -\frac{1}{2\pi} \int_{\operatorname{Re} \nu=c} \frac{(1-2^{-\nu})^{-1}}{\zeta(\nu)} d\nu \int_{\operatorname{Re} \mu=0} f(s-\nu+\varepsilon\mu) \Phi(v, u; \nu, \mu) d\mu, \quad (9.25)$$

which is the announced proposition.  $\square$

A resolvent of the self-adjoint operator  $i \left( y \frac{d}{dy} + \frac{1}{2} \right)$  in  $L^2(\mathbb{R})$  is given by the equations

$$\left[ \left( y \frac{d}{dy} + \frac{1}{2} + \mu \right)^{-1} u \right] (y) = \begin{cases} \int_0^1 \theta^{\mu-\frac{1}{2}} u(\theta y) d\theta & \text{if } \operatorname{Re} \mu > 0, \\ -\int_1^\infty \theta^{\mu-\frac{1}{2}} u(\theta y) d\theta & \text{if } \operatorname{Re} \mu < 0. \end{cases} \quad (9.26)$$

Let  $u \in \mathcal{S}(\mathbb{R})$  be flat at 0 and let  $\mu \in \mathbb{C}$ . Starting from (9.8), setting  $\delta = y \frac{d}{dy}$  and using the integration by parts associated to the identity  $\theta \frac{d}{d\theta} \theta^{\mu+\frac{1}{2}} = (\mu + \frac{1}{2}) \theta^{\mu+\frac{1}{2}}$ , one obtains for  $N = 0, 1, \dots$  and  $y \neq 0$  the identity

$$u^\mu(y) = \frac{(-1)^N}{(\mu + \frac{1}{2})^N} \cdot \frac{1}{2\pi} \int_0^\infty \theta^{\mu-\frac{1}{2}} (\delta^N u)(\theta y) d\theta. \quad (9.27)$$

We define

$$\begin{aligned} u_+^{\mu,N}(y) &= \frac{(-1)^N}{(\mu + \frac{1}{2})^N} \cdot \frac{1}{2\pi} \int_0^1 \theta^{\mu-\frac{1}{2}} (\delta^N u)(\theta y) d\theta, \\ u_-^{\mu,N}(y) &= \frac{(-1)^N}{(\mu + \frac{1}{2})^N} \cdot \frac{1}{2\pi} \int_1^\infty \theta^{\mu-\frac{1}{2}} (\delta^N u)(\theta y) d\theta, \quad y \neq 0. \end{aligned} \quad (9.28)$$

One has if  $\operatorname{Re} \mu > -\frac{1}{2}$

$$\left| u_+^{\mu,N}(y) \right| \leq \frac{1}{2\pi} |\mu + \frac{1}{2}|^{-N-1} \sup |\delta^N u| \quad (9.29)$$

and, under the assumption that  $\operatorname{Re} \mu < -\frac{1}{2}$ , the same inequality holds after one has substituted  $u_-^{\mu,N}$  for  $u_+^{\mu,N}$ .

We decompose accordingly the function in (9.10), setting

$$\Phi_N^\pm(v, u; \nu, \mu) = \int_0^\infty \bar{v}(t + t^{-1}) t^{\nu-1} u_\pm^{\mu,N}(t - t^{-1}) dt. \quad (9.30)$$

The sum  $\Phi(v, u; \nu, \mu) = \Phi_N^+(v, u; \nu, \mu) + \Phi_N^-(v, u; \nu, \mu)$  does not depend on  $N$ .

**Proposition 9.6.** *Let  $v, u$  satisfy the assumptions of Theorem 9.2. For every pair  $C_1, C_2$  of non-negative integers, the function*

$$(1 + |\operatorname{Im} \nu|)^{C_2} (1 + |\mu|)^{C_1} \Phi_N^+(v, u; \nu, \mu) \quad (9.31)$$

*is bounded if  $N+1 \geq C_1 + C_2$ , in a uniform way relative to  $\operatorname{Re} \mu \geq \alpha > -\frac{1}{2}$  if  $|\operatorname{Re} \nu|$  is bounded. The function*

$$(1 + |\operatorname{Im} \nu|)^{C_2} (1 + |\mu|)^{C_1} \Phi_N^-(v, u; \nu, \mu) \quad (9.32)$$

*is bounded under the same condition, in a uniform way relative to  $\operatorname{Re} \mu \leq \beta < -\frac{1}{2}$  if  $|\operatorname{Re} \nu|$  is bounded.*

*Proof.* The way  $\Phi_N^\pm(v, u; \nu, \mu)$  behaves in terms of  $\mu$  as  $|\operatorname{Im} \mu| \rightarrow \infty$  is a consequence of the similar estimates relative to  $u_\pm^{\mu,N}(t - t^{-1})$ . Following the proof of Proposition 9.4, one can improve the bounds by powers of  $(1 + |\operatorname{Im} \nu|)^{-1}$ , losing the corresponding power of  $1 + |\mu|$ , and the claimed uniformity is preserved. □

*Remark 9.1.* Anticipating on the application in the next section, we could dispense with factors of the kind  $(1 + |\operatorname{Im} \nu|)^{C_2}$ , relying instead on Proposition 5.5 to improve the  $d\nu$ -convergence at infinity. Indeed, after the contour change in Theorem 10.2 has produced the residue in (10.8), an extra factor such as  $\nu^{-M}$  in (10.4) becomes  $(s-1+\varepsilon\mu)^{-M}$ , just what is needed, as shown in Lemma 10.1, to take care of the bound at infinity of  $(\zeta(s-1+\varepsilon\mu))^{-1}$ . Taking  $N = 1$  would then suffice to ensure  $d\mu$ -summability.

## 10. A REFUTATION OF THE RIEMANN HYPOTHESIS

This section is entirely based on Cauchy-type analysis. Note the following benefit – not the only one – of using the approximation  $F_\varepsilon(s)$  of  $F_0(s)$ . As soon as Theorem 10.3 has been established, the whole task consists in establishing bounds, uniform with respect to  $\varepsilon$ , for integrals the convergence of which is not in question. On the other hand, with one exception in the proof of Theorem 10.4, all changes of contour are made under the assumption that  $\operatorname{Re} s$  is large: it is only, as explained in Remark 5.2, the results of the changes that must be analyzed for  $s$  in the domains of interest.

**Lemma 10.1.** *If  $d \in ]\frac{1}{2}, 1[$  is such that, for some  $\delta > 0$ , zeta has no zero in the strip  $\{s: d - \delta < \operatorname{Re} s < d + \delta\}$ , one has for some pair  $C, M$  the estimate  $|\zeta(d + i\tau)|^{-1} \leq C(1 + |\tau|)^M$  for every  $\tau \in \mathbb{R}$ .*

*Proof.* Recall first the Borel-Caratheodory lemma: if a function  $f(w)$  is holomorphic and satisfies the condition  $\operatorname{Re} f(w) \leq A$  for  $|w| < \delta$ , finally if  $f(0) = 0$ , one has for  $|w| \leq \frac{\delta}{2}$  the estimate  $|f(w)| \leq 4A$ . With  $s_0 = d + i\tau$ , let  $f(w) = \log \frac{\zeta(d+i\tau+w)}{\zeta(d+i\tau)}$ . The Lindelöf convexity inequality [7, p.201], considerably more precise than the estimate  $|\zeta(d + i\tau)| = O(|\tau|)$  for  $|\tau| \rightarrow \infty$ , sufficient for our purpose, gives for  $|\tau|$  large

$$\operatorname{Re} f(w) = \log |\zeta(d + i\tau + w)| - \log |\zeta(d + i\tau)| \leq C \log |\tau|, \quad (10.1)$$

for some  $C > 0$ , and it follows [7, p.225] from the Borel-Caratheodory lemma that one has if  $|\tau| \geq 2$  the estimate

$$|\zeta(d + i\tau)|^{-1} \leq |\tau|^{4C} |\zeta(d + i\tau + w)|^{-1}. \quad (10.2)$$

Then, we use the following result, due to Valiron [13], which provides a sequence of heights at which crossing the critical strip is reasonably safe.



It is the fact that there exists a sequence  $(T_k)_{k \geq 1}$ ,  $T_k \in [k, k+1[$ , and a pair  $B, M_1$  such that

$$\inf_{0 \leq \sigma \leq 1} |\zeta(\sigma + iT_k)| \geq B^{-1} k^{-M_1}. \quad (10.3)$$

A modern proof, which I owe to Gerald Tenenbaum, is to be found as [9, Lemma 4.2].

From  $d + i\tau$ , one can reach a point  $d + iT_k$  by adding consecutively a number of the order of  $\frac{2}{\delta}$  of increments  $w$  with  $|w| \leq \frac{\delta}{2}$ . The lemma follows.  $\square$

**Theorem 10.2.** *Let  $v, u \in C^\infty(\mathbb{R})$  satisfy the assumptions of Theorem 9.2, here recalled:  $v$  is supported in  $[2, \sqrt{8}]$  and  $u$  in  $[0, 1]$ . For  $c > 1$  and  $2 < \operatorname{Re} s < c + 1$ , one has  $F_\varepsilon(s) = E_\varepsilon(s) - i G_\varepsilon(s)$ , with*

$$E_\varepsilon(s) = -\frac{1}{2\pi} \int_{\operatorname{Re} \mu=0} \int_{\operatorname{Re} \nu=c} \frac{(1 - 2^{-\nu})^{-1}}{\zeta(\nu)} f(s - \nu + \varepsilon\mu) \Phi(v, u; \nu, \mu) d\mu d\nu \quad (10.4)$$

and

$$G_\varepsilon(s) = \frac{4}{\pi^2} \int_{\operatorname{Re} \mu=0} \frac{(1 - 2^{-s+1-\varepsilon\mu})^{-1}}{\zeta(s-1+\varepsilon\mu)} \Phi(v, u; s-1+\varepsilon\mu, \mu) d\mu. \quad (10.5)$$

We recall that  $f(s) = (1 + 2^{-s})^{-1} \frac{\zeta(s)}{\zeta(2s)}$  and that the function  $\Phi(v, u; \nu, \mu)$  was defined in Proposition 9.3.

Set  $\sigma_0 = \sup \{\operatorname{Re} \rho : \zeta(\rho) = 0\}$ . The function  $G_\varepsilon(s)$ , as defined by (10.5) for  $\operatorname{Re}(s-1) > \sigma_0$ , extends analytically to the half-plane  $\operatorname{Re}(s-1) > \frac{\sigma_0}{2}$ , and the decomposition  $F_\varepsilon(s) = E_\varepsilon(s) - i G_\varepsilon(s)$  is valid for  $c + \frac{\sigma_0}{2} < \operatorname{Re} s < c + 1$ .

*Proof.* We start from Proposition 9.5 here recalled,

$$F_\varepsilon(s) = -\frac{1}{2\pi} \int_{\operatorname{Re} \nu=c} \int_{\operatorname{Re} \mu=0} \frac{(1 - 2^{-\nu})^{-1}}{\zeta(\nu)} f(s - \nu + \varepsilon\mu) \Phi(v, u; \nu, \mu) d\nu d\mu, \quad (10.6)$$

an identity valid if  $c > 1$  and  $\operatorname{Re} s > c + 1$ . The double integral is absolutely convergent in view of Propositions 9.3 and 9.4. The same integral converges for  $c + \frac{\sigma_0}{2} < \operatorname{Re} s < c + 1$ . Indeed, in that case, the function  $f(s - \nu + \varepsilon\mu)$  is still non-singular, since the numerator of its expression just recalled is non-singular and the denominator is nonzero. The bound  $\zeta(s - \nu + \varepsilon\mu) = O(|\operatorname{Im}(s - \nu + \varepsilon\mu)|^{\frac{1}{2}})$  since  $\operatorname{Re}(s - \nu) > 0$  [7, p.201] completes

the estimates.

However, the convergence in the cases for which  $\operatorname{Re} s > c + 1$  and  $c + \frac{\sigma_0}{2} < \operatorname{Re} s < c + 1$  holds for incompatible reasons. To make the jump possible, we choose  $c_1$  such that  $1 < c_1 < c$  and observe that one can replace in (10.6) the integral sign  $\int_{\operatorname{Re} \nu=c}$  by  $\int_{\operatorname{Re} \nu=c_1}$ , enlarging the domain of validity of the new identity to the half-plane  $\operatorname{Re} s > c_1 + 1$ . In the case when  $c_1 + 1 < \operatorname{Re} s < c + 1$ , one has for  $\operatorname{Re} \mu = 0$

$$\begin{aligned} & \frac{1}{2i\pi} \int_{\operatorname{Re} \nu=c_1} \frac{(1-2^{-\nu})^{-1}}{\zeta(\nu)} f(s-\nu+\varepsilon\mu) \Phi(v, u; \nu, \mu) d\nu \\ &= \frac{1}{2i\pi} \int_{\operatorname{Re} \nu=c} \frac{(1-2^{-\nu})^{-1}}{\zeta(\nu)} f(s-\nu+\varepsilon\mu) \Phi(v, u; \nu, \mu) d\nu \\ & - \operatorname{Res}_{\nu=s-1+\varepsilon\mu} \left[ \frac{(1-2^{-\nu})^{-1}}{\zeta(\nu)} f(s-\nu+\varepsilon\mu) \Phi(v, u; \nu, \mu) \right]. \quad (10.7) \end{aligned}$$

The  $d\mu$ -integral of the left-hand side coincides with  $F_\varepsilon(s)$  for  $\operatorname{Re} s > c_1 + 1$  according to Proposition 9.5, hence extends as an entire function. As already observed, the  $d\nu$ -integral which is the first term on the right-hand side is analytic in the domain  $\{s: c + \frac{\sigma_0}{2} < \operatorname{Re} s < c + 1\}$  and so is its  $d\mu$ -integral  $E_\varepsilon(s)$ .

The residue of the function  $f(s) = (1+2^{-s})^{-1} \frac{\zeta(s)}{\zeta(2s)}$  at  $s = 1$  is  $\frac{4}{\pi^2}$ . The second term on the right-hand side of (10.7) is thus

$$\frac{4}{\pi^2} \frac{(1-2^{-s+1-\varepsilon\mu})^{-1}}{\zeta(s-1+\varepsilon\mu)} \Phi(v, u; s-1+\varepsilon\mu, \mu), \quad (10.8)$$

and  $G_\varepsilon(s)$  is for  $\operatorname{Re} s > 1 + \sigma_0$  the  $d\mu$ -integral of this expression.

One has by definition  $G_\varepsilon(s) = i(F_\varepsilon(s) - E_\varepsilon(s))$  if  $c_1 + 1 < \operatorname{Re} s < c + 1$ . The right-hand side of this identity is analytic for  $c + \frac{\sigma_0}{2} < \operatorname{Re} s < c + 1$ . The function  $G_\varepsilon(s)$ , in the definition of which  $c > 1$  is not present, thus extends analytically for  $\operatorname{Re} s > 1 + \frac{\sigma_0}{2}$ , and the identity that defines it extends for  $c + \frac{\sigma_0}{2} < \operatorname{Re} s < c + 1$ . □

**Theorem 10.3.** *Let  $Z = \{\operatorname{Re} \rho: 0 < \operatorname{Re} \rho < 1, \zeta(\rho) = 0\}$ , and let  $\sigma_0 = \sup Z$ . There is no interval  $[a, b]$  with  $\frac{\sigma_0}{2} < a < \sigma_0 < b$  such that for every pair  $v, u$  satisfying the support conditions in Theorem 10.2 and every compact subset  $K$  of the strip  $\{s: a < \operatorname{Re}(s-1) < b\}$ , the function  $G_\varepsilon(s)$  is*

bounded in  $K$ , in a uniform way relative to  $\varepsilon > 0$ .

*Proof.* The condition  $a > \frac{\sigma_0}{2}$  ensures that  $G_\varepsilon(s)$  is well-defined for  $\operatorname{Re}(s-1) \geq a$ . Let us assume that the interval  $(a, b)$  satisfies the conditions the impossibility of which is to be proven. In view of Theorem 5.3 and Corollary 5.4, it suffices to show that, under the given assumption, the function  $F_0(s)$  could be continued to the half-plane  $\operatorname{Re}(s-1) > a$ .

By a compactness argument (Montel's theorem), replacing the set  $\{\varepsilon: \varepsilon < \varepsilon_0\}$  by an appropriate sequence going to zero, one may assume that  $G_\varepsilon(s)$  converges in the strip  $a < \operatorname{Re}(s-1) < b$ , as  $\varepsilon \rightarrow 0$ , towards an analytic function  $G_0^\sharp(s)$ .

On the other hand, starting from (10.5) and using Proposition 9.3 and Lemma 10.1, finally (9.11), one sees that, for  $\operatorname{Re}(s-1) > \sigma_0$ ,  $G_\varepsilon(s)$  converges as  $\varepsilon \rightarrow 0$  towards the function

$$\begin{aligned} G_0(s) &= \frac{4}{\pi^2} \int_{\operatorname{Re} \mu=0} \frac{(1-2^{-s+1})^{-1}}{\zeta(s-1)} \Phi_N(v, u; s-1, \mu) d\mu \\ &= \frac{4i}{\pi^2} \frac{(1-2^{-s+1})^{-1}}{\zeta(s-1)} (v | \Psi(\mathfrak{E}_{1-s}) u), \end{aligned} \quad (10.9)$$

which must coincide with  $G_0^\sharp(s)$  for  $\sigma_0 < \operatorname{Re}(s-1) < b$ . We have thus obtained a continuation of the function  $G_0(s)$  to the half-plane  $\operatorname{Re}(s-1) > a$ . Next, the function  $E_\varepsilon(s)$ , which coincides as has been seen in Theorem 10.2 with the sum  $F_\varepsilon(s) + i G_\varepsilon(s)$  in the strip  $c-1 + \frac{\sigma_0}{2} < \operatorname{Re}(s-1) < c$ , has there a limit  $E_0(s)$ , obtained by taking  $\varepsilon = 0$  on the right-hand side of (10.4). We choose  $c$  such that  $1 < c < a+1 - \frac{\sigma_0}{2}$ .

We sum up what precedes. As  $\varepsilon \rightarrow 0$ , the function  $G_\varepsilon(s)$  has a locally uniform limit  $G_0(s)$  for  $\operatorname{Re}(s-1) > a$ , and  $E_\varepsilon(s)$  has a locally uniform limit  $E_0(s)$  for  $c-1 + \frac{\sigma_0}{2} < \operatorname{Re}(s-1) < c$ : one has  $F_\varepsilon(s) = E_\varepsilon(s) - i G_\varepsilon(s)$  in the second interval so that, since  $a > c-1 + \frac{\sigma_0}{2}$ , the function  $F_\varepsilon(s)$  has a locally uniform limit for  $a < \operatorname{Re}(s-1) < c$ . This limit is  $F_0(s)$  when  $\operatorname{Re}(s-1) > 1$ , as seen for instance from Theorem 9.2, so that it is for  $\operatorname{Re}(s-1) > a$  a continuation of  $F_0(s)$ . □

Let us consider the factors of the integral (10.5) for  $G_\varepsilon(s)$ . In view of Lemma 10.1, for  $\operatorname{Re}(s-1)$  lying in a closed subinterval of an open interval

in which there are no real parts of zero, the factor  $(\zeta(s-1+\varepsilon\mu))^{-1}$  is bounded by  $C(1+|\operatorname{Im}(\varepsilon\mu)|)^M$  as long as  $\operatorname{Im}(s-1)$  is bounded. On the other hand, the  $\Phi$ -function is rapidly decreasing as a function of  $\operatorname{Im} \mu$ , and this will remain true when the line  $\operatorname{Re} \mu = 0$  has been changed to another one. However, there is no uniformity in the latter set of estimates with respect to the real part of  $\mu$ , as will be necessary presently when letting  $\varepsilon$  go to 0: to depend on it, we shall have to use Proposition 9.6, which demands cutting the  $\Phi$ -function into two pieces.

**Theorem 10.4.** *There cannot exist any pair  $a, \eta$  with  $\frac{\sigma_0}{2} < a < a + \eta < \sigma_0 < a + 2\eta$  such that the function zeta has no zero with a real part in the interval  $[a, a + \eta]$ .*

*Proof.* Let  $v, u$  be a pair of functions satisfying the support conditions in Theorem 10.2. Recall from Theorem 10.2 that  $G_\varepsilon(s)$ , as defined by (10.5) for  $\operatorname{Re}(s-1) > \sigma_0$ , extends analytically to the half-plane  $\operatorname{Re}(s-1) > \frac{\sigma_0}{2}$ . Our first task will consist in building an integral representation of  $G_\varepsilon(s)$  in the half-plane  $\operatorname{Re}(s-1) > a$ . We use the decompositions (9.30)  $\Phi(v, u; \nu, \mu) = \Phi_N^+(v, u; \nu, \mu) + \Phi_N^-(v, u; \nu, \mu)$ , and  $G_{\varepsilon, N}^+(s) + G_{\varepsilon, N}^-(s)$ , with

$$G_{\varepsilon, N}^\pm(s) = \frac{4}{\pi^2} \int_{\operatorname{Re} \mu = 0} \frac{(1 - 2^{-s+1-\varepsilon\mu})^{-1}}{\zeta(s-1+\varepsilon\mu)} \Phi_N^\pm(v, u; s-1+\varepsilon\mu, \mu) d\mu \quad (10.10)$$

if  $\operatorname{Re} s > 1 + \sigma_0$ . Given a pair contradicting the claim of the theorem, we shall show that either term of the decomposition (10.10) can be continued to the half-plane  $\{s: \operatorname{Re} s > a\}$ , proving at the same time the required bounds independent of  $\varepsilon > 0$  for  $a < \operatorname{Re}(s-1) < a + 2\eta$ . The sum of the continuations will of course coincide with the restriction of  $G_\varepsilon(s)$  to this half-plane and, as  $\frac{\sigma_0}{2} < a < \sigma_0 < a + 2\eta$ , the contradiction will be a consequence of Theorem 10.3. We analyze the two terms separately.

Choose  $\beta$  such that  $\frac{1}{2\varepsilon} < \beta = O(\frac{1}{\varepsilon})$ . For  $\operatorname{Re}(s-1) > \frac{\sigma_0}{2}$ , even the more so for  $\operatorname{Re}(s-1) > a$ , one has  $\operatorname{Re}(s-1+\varepsilon\beta) > \operatorname{Re}(s-\frac{1}{2}) > \frac{1+\sigma_0}{2} \geq \sigma_0$ . We may thus write for  $\operatorname{Re}(s-1) > \frac{\sigma_0}{2}$ , after a contour change,

$$G_{\varepsilon, N}^+(s) = \frac{4}{\pi^2} \int_{\operatorname{Re} \mu = \beta} \frac{(1 - 2^{-s+1-\varepsilon\mu})^{-1}}{\zeta(s-1+\varepsilon\mu)} \Phi_N^+(v, u; s-1+\varepsilon\mu, \mu) d\mu, \quad (10.11)$$

benefitting from Proposition 9.3 to ensure convergence. On one hand, the real part of  $\nu = s-1+\varepsilon\mu$  is bounded as  $\varepsilon \rightarrow 0$ , in opposition to that

of  $\mu$ . On the other hand, even though  $\beta \rightarrow \infty$  as  $\varepsilon \rightarrow 0$ , the estimate of  $\Phi_N^+(v, u; \nu, \mu)$ , for any given  $C$ , by a  $O((1 + |\operatorname{Im} \mu|)^{-C})$  is uniform in view of (9.31). The part of the analysis concerning  $G_{\varepsilon, N}^+(s)$  follows. Recall that the change of contour is made under the assumption that  $\operatorname{Re} s$  is large, in this case  $\operatorname{Re}(s-1) > \sigma_0$  (cf. Remark 5.2): after the change has been made, we use (10.11) to define the sought-after continuation of  $G_{\varepsilon, N}^+(s)$ .

Some more care is needed when dealing with  $G_{\varepsilon, N}^-(s)$ . Besides the function

$$G_{\varepsilon, N}^-(s) = \frac{4}{\pi^2} \int_{\operatorname{Re} \mu=0} \frac{(1 - 2^{-s+1-\varepsilon\mu})^{-1}}{\zeta(s-1+\varepsilon\mu)} \Phi_N^-(v, u; s-1+\varepsilon\mu, \mu) d\mu, \quad (10.12)$$

as defined for  $\operatorname{Re}(s-1) > \sigma_0$ , we introduce the function  $K_{\varepsilon, N}^-(s)$  seemingly defined by exactly the same integral, but regarded in the domain  $\{s: a < \operatorname{Re}(s-1) < a+\eta\}$ , where the integral converges in view of Lemma 10.1. As  $a+\eta < \sigma_0$ , the function  $K_{\varepsilon, N}^-$  may not (not yet, actually) be regarded as a continuation of the function  $G_{\varepsilon, N}^-$ : an intermediary will be required.

Choose  $\eta'$  such that  $0 < \eta' < \eta$  and  $a+\eta+\eta' > \sigma_0$ , and consider with  $\alpha = -\frac{\eta'}{\varepsilon}$  the integral

$$H_{\varepsilon, N}^-(s) := \frac{4}{\pi^2} \int_{\operatorname{Re} \mu=\alpha} \frac{(1 - 2^{-s+1-\varepsilon\mu})^{-1}}{\zeta(s-1+\varepsilon\mu)} \Phi_N^-(v, u; s-1+\varepsilon\mu, \mu) d\mu. \quad (10.13)$$

It is convergent if  $a < \operatorname{Re}(s-1-\eta') < a+\eta$ , i.e.,  $a+\eta' < \operatorname{Re}(s-1) < a+\eta+\eta'$ . In the non-void domain  $\{s: a+\eta' < \operatorname{Re}(s-1) < a+\eta\}$ , the integrals for  $K_{\varepsilon, N}^-(s)$  and  $H_{\varepsilon, N}^-(s)$  are both convergent, and the second one is the result of the application to the first one of a change of the  $\mu$ -contour. Beware that in this instance, one cannot rely on Remark 5.2 since there is an upper limit on  $\operatorname{Re}(s-1)$ : but, during the move,  $t = \operatorname{Re}(\varepsilon\mu)$  stays between  $-\eta'$  and 0, so that  $a < \operatorname{Re}(s-1+t) < a+\eta$  if  $a+\eta' < \operatorname{Re}(s-1) < a+\eta$ , which justifies the change of contour. Hence,  $K_{\varepsilon, N}^-(s) = H_{\varepsilon, N}^-(s)$  when  $a+\eta' < \operatorname{Re}(s-1) < a+\eta$  and one obtains a continuation of this function to the union  $\{s: a < \operatorname{Re}(s-1) < a+\eta+\eta'\} = \{s: a < \operatorname{Re}(s-1) < a+\eta\} \cup \{s: a+\eta' < \operatorname{Re}(s-1) < a+\eta+\eta'\}$  when piecing the two definitions.

On the other side, the identity (10.13) is valid for  $a+\eta' < \operatorname{Re}(s-1) < a+\eta+\eta'$ . As  $a+\eta' < a+\eta < \sigma_0 < a+\eta+\eta'$ , (10.13) is valid for

$\sigma_0 < \operatorname{Re}(s-1) < a + \eta + \eta'$ , a subdomain of the half-plane  $\operatorname{Re}(s-1) > \sigma_0$  in which (10.12) is valid: we have obtained, using  $H_{\varepsilon,N}^-(s)$  as an intermediary, a continuation of  $G_{\varepsilon,N}^-(s)$  to the half-plane  $\{s: \operatorname{Re}(s-1) > a\}$ , given when  $a < \operatorname{Re}(s-1) < a + \eta$  as  $K_{\varepsilon,N}^-(s)$  and when  $a + \eta' < \operatorname{Re}(s-1) < a + \eta + \eta'$  as  $H_{\varepsilon,N}^-(s)$ .

As zeta has no zero with a real part in  $[a, a + \eta]$ , it follows from the integral representation of  $K_{\varepsilon,N}^-(s)$  that this function is bounded in a locally uniform way in the strip  $\{s: a < \operatorname{Re}(s-1) < a + \eta\}$ , in a uniform way relative to  $\varepsilon > 0$ . Again, the estimate of  $\Phi_N^-(v, u; s-1+\varepsilon\mu, \mu)$  by a rapidly decreasing function of  $\operatorname{Im} \mu$  is uniform relative to  $\varepsilon$  as  $\operatorname{Re} \mu \rightarrow -\infty$ , and it follows from (10.13) that  $H_{\varepsilon,N}^-(s)$  is bounded in a locally uniform way in the strip  $\{s: a + \eta' < \operatorname{Re}(s-1) < a + \eta + \eta'\}$ , in a uniform way relative to  $\varepsilon > 0$ .

This concludes the proof of Theorem 10.4. □

*Remarks 10.1* (i) In Theorem 10.4, the condition  $a + 2\eta > \sigma_0$  is exactly what is needed, in the sense that it could not be replaced, for any  $\lambda > 1$ , by  $a + (1 + \lambda)\eta > \sigma_0$ . Indeed, the integral (10.13) converges in the domain  $\{s: a - \varepsilon\alpha < \operatorname{Re}(s-1) < a + \eta - \varepsilon\alpha\}$ . But, under the new condition, this is useful only if  $a + \eta - \varepsilon\alpha \geq a + (1 + \lambda)\eta$ , in other words  $\alpha \leq -\frac{\lambda\eta}{\varepsilon}$ . Then,  $a - \varepsilon\alpha \geq a + \lambda\eta > a + \eta$ , the two intervals  $[a, a + \eta]$  and  $[a - \varepsilon\alpha, a + \eta - \varepsilon\alpha]$  are disjoint and Theorem 10.3 is unapplicable.

(ii) Another look at the equations (9.28) and (9.30) shows that, in Proposition 9.6, one could, when dealing with  $\Phi_N^+$ , replace the claim “is bounded if  $N + 1 \geq C_1 + C_2$  in a uniform way ...” by “goes to zero as  $\operatorname{Re} \mu \rightarrow \infty$ ”. Something similar goes when  $\Phi_N^-$  is considered. This implies that  $G_{\varepsilon,N}^+(s)$  goes to zero as  $\varepsilon \rightarrow 0$  for  $a < \operatorname{Re}(s-1) < a + 2\eta$  and, using (10.13), that  $G_{\varepsilon,N}^-(s)$  goes to zero as  $\varepsilon \rightarrow 0$  for  $a + \eta' < \operatorname{Re}(s-1) < a + \eta' + \eta$ . Using the Montel argument in the proof of Theorem 10.3, one sees that, for some sequence  $(\varepsilon_j)$  going to zero,  $G_{\varepsilon_j}$  has a (locally uniform) limit for  $a < \operatorname{Re}(s-1) < a + 2\eta$ : of necessity, this limit is zero since it is analytic and zero in the subdomain  $\{s: a + \eta' < \operatorname{Re}(s-1) < a + \eta + \eta'\}$ . This is of no consequence as, in order that the function  $G_0^\sharp(s) = \lim_{j \rightarrow \infty} G_{\varepsilon_j}(s)$  (with the notation in the proof of Theorem 10.3) should agree with  $G_0(s)$  in some non-void domain, we would have to assume that  $a + 2\eta > a + \eta + \eta' > \sigma_0$ . But it is precisely the result of Theorem 10.4 that this is to be excluded.

As seen in the proof of Theorem 10.3, the function  $G_0^\sharp$  has no significance unless its domain of analyticity intersects that of  $G_0$ .

**Theorem 10.5.** *One has  $\sigma_0 = \sup Z \geq \frac{4}{7}$ . In particular, the Riemann hypothesis does not hold. The set  $Z$  of real parts of non-trivial zeros is infinite.*

*Proof.* By definition, zeta has no zero with a real part in  $] \sigma_0, 1 ]$ ; as a consequence of the functional equation, it does not have any zero with a real part in  $[0, 1 - \sigma_0[$ . Applying Theorem 10.4, one obtains a contradiction if one can find a pair  $a, \eta$  such that

$$\frac{\sigma_0}{2} < a < a + \eta < 1 - \sigma_0, \quad a + 2\eta > \sigma_0. \quad (10.14)$$

This is indeed the case if  $\sigma_0 < \frac{4}{7}$ . For, then, one can find  $\eta$  such that  $2\sigma_0 - 1 < \eta < \frac{\sigma_0}{4}$ . Choosing  $a$  such that  $\sigma_0 - 2\eta < a < 1 - \sigma_0 - \eta$ , as is possible since  $\eta > 2\sigma_0 - 1$ , we are done: all that remains to be done is noting that  $a > \frac{\sigma_0}{2}$ , or  $\sigma_0 - 2\eta > \frac{\sigma_0}{2}$ , a consequence of  $\eta < \frac{\sigma_0}{4}$ .

Now, assume that the set  $Z$  is finite, let  $\sigma_1$  be the point of  $Z$  immediately inferior to the largest point  $\sigma_0$ . Choose  $\eta$  such that  $\frac{\sigma_0 - \sigma_1}{2} < \eta < \sigma_0 - \sigma_1$ , then  $a$  such that  $\sigma_1 < a < \sigma_0 - \eta$ . Then,  $a + \eta < \sigma_0$  and  $a + 2\eta > \sigma_0$ : this contradicts the result of Theorem 10.4.

□

In [1], Bombieri (almost) proved that if the Riemann hypothesis does not hold, the set of zeros of zeta is infinite. Concentrating on the real parts of zeros has proved decisive.

*Remark 10.2.* The inequality  $\sigma_0 \geq \frac{4}{7}$  can be improved to  $\sigma_0 \geq \frac{2}{3}$  in the following way. In the definition (10.4) of  $E_\varepsilon(s)$ , replace the condition  $c > 1$  by  $c > \sigma_0$ . Then, in Theorem 10.4, one can replace the condition  $a > \frac{\sigma_0}{2}$  by  $a > \frac{\sigma_0}{2} + c - 1$ , or  $a > \frac{3\sigma_0}{2} - 1$ . The set of conditions (10.14) leads to the desired improvement.

Actually, this improvement has an only temporary role, since we shall prove that  $\sigma_0 = 1$ . To obtain this, we shall drop in the main series (5.2) or (9.1) the assumption that the summation variable  $Q$  is squarefree.

11. ZEROS OF ZETA ACCUMULATE ON THE LINE  $\operatorname{Re} s = 1$ 

In Theorem 10.5 and Remark 10.2, we have reached, with the inequality  $\sigma_0 \geq \frac{2}{3}$ , the limit of the method. If one replaces in the series (5.2) the sum over all squarefree odd integers by the sum over all odd integers, one can reach, as we shall see, the correct value  $\sigma_0 = 1$ .

We must first answer the natural question “why did we not do this in the first place?”. The answer is that when, years ago, we embarked on this program, it was under the conviction that the Riemann hypothesis did hold. In a version of the criterion (4.19), in which R.H. appears as a consequence of an estimate regarding  $(v \mid \Psi(Q^{2i\pi\mathcal{E}} \mathfrak{T}_{\frac{\infty}{2}}) u)$ , putting more constraints on  $Q$  led to a situation which we regarded as more favorable. As it turns out, R.H. is disproved in the present paper, and a more favorable situation is obtained, especially in Theorem 9.2, if one lowers the constraint on the summation variable  $Q$  in the series there.

Yes, a slightly shorter proof would have been obtained if we had started in the right direction from the beginning. But we do not regret this very small détournement, one point of which is being satisfied that the present proof is an in-depth analysis of the Riemann hypothesis, not just the consequence of a favorable accident. Also, in this direction, Remark 11.1 below will relate our understanding of the Riemann hypothesis to an arithmetic question.

First, we generalize Theorem 7.2, recalling (paragraph following (4.2)) that  $Q_{\bullet}$  is the squarefree version of  $Q$ .

**Theorem 11.1.** *Let  $N = RQ$ , where we assume that  $(R, Q) = 1$ , that  $R$  is squarefree and that both factors are odd. Let  $v, u \in C^\infty(\mathbb{R})$ , compactly supported, satisfying the conditions that  $x > 0$  and  $0 < x^2 - y^2 < 8$  when  $v(x)u(y) \neq 0$ . Then, if  $N$  is large enough,*

$$(v \mid \Psi(Q^{2i\pi\mathcal{E}} \mathfrak{T}_N) u) = \sum_{\substack{R_1 R_2 = R \\ Q_1 Q_2 = Q_{\bullet}}} \mu(R_1 Q_1) \bar{v} \left( \frac{R_1}{Q_2} \frac{Q_{\bullet}}{Q} + \frac{Q_2}{R_1} \frac{Q}{Q_{\bullet}} \right) u \left( \frac{R_1}{Q_2} \frac{Q_{\bullet}}{Q} - \frac{Q_2}{R_1} \frac{Q}{Q_{\bullet}} \right). \quad (11.1)$$



*Proof.* Theorem 6.1 remains valid if one drops the assumption that  $N$  is squarefree: the only point that matters here is that  $R$  and  $Q$  should be relatively prime. We thus compute, with  $b(j, k) = a((j, k, N)) = 1 - p \operatorname{char}(j \equiv k \equiv N \equiv 0 \pmod{p})$ , the sum  $f_N(j, s)$  defined in (6.3). The Eulerian reduction (7.4) is still valid if  $(N_1, N_2) = 1$  and one can reduce the computation to the case when  $N = Q = p^\gamma$  for some prime  $p$ : there is no difference with the earlier situation so far as the  $R$ -factor is concerned. One has

$$\begin{aligned}
 f_N(j, s) &= \frac{1}{p^\gamma} \sum_{k \pmod{p^\gamma}} [1 - p \operatorname{char}(j \equiv k \equiv 0 \pmod{p})] \exp\left(\frac{2i\pi ks}{p^\gamma}\right) \\
 &= \frac{1}{p^\gamma} \sum_{k \pmod{p^\gamma}} \exp\left(\frac{2i\pi ks}{p^\gamma}\right) - \operatorname{char}(j \equiv 0 \pmod{p}) \frac{1}{p^{\gamma-1}} \sum_{k_1 \pmod{p^{\gamma-1}}} \exp\left(\frac{2i\pi k_1 s}{p^{\gamma-1}}\right) \\
 &= \operatorname{char}(s \equiv 0 \pmod{p^\gamma}) - \operatorname{char}(j \equiv 0 \pmod{p}) \operatorname{char}(s \equiv 0 \pmod{p^{\gamma-1}}) \\
 &= \operatorname{char}(s \equiv 0 \pmod{p^{\gamma-1}}) \times \left[ \operatorname{char}\left(\frac{s}{p^{\gamma-1}} \equiv 0 \pmod{p}\right) - \operatorname{char}(j \equiv 0 \pmod{p}) \right]. \tag{11.2}
 \end{aligned}$$

If  $Q = \prod_p p^{\gamma_p}$ , one has  $\prod_p p^{\gamma_p-1} = \frac{Q}{Q_\bullet}$ . Piecing together the equations (11.2) for all values of  $p$  dividing  $N$ , one obtains

$$\begin{aligned}
 f_N(j, s) &= \operatorname{char}\left(s \equiv 0 \pmod{\frac{Q}{Q_\bullet}}\right) \times \sum_{M_1 M_2 = N_\bullet} \mu(M_1) \\
 &\quad \operatorname{char}\left(\frac{s}{Q/Q_\bullet} \equiv 0 \pmod{M_2}\right) \operatorname{char}(j \equiv 0 \pmod{M_1}). \tag{11.3}
 \end{aligned}$$

The transformation  $\theta_N$  is defined as in (6.5). Just as in the proof of Theorem 7.2, one has  $(\theta_N v)(m) = v\left(\frac{m}{N}\right)$  and  $(\theta_N u)(n) = u\left(\frac{n}{N}\right)$  if  $N$  is large. Applying the recipe in Theorem 6.1 and setting  $M_2 = R_2 Q_2$ ,  $M_1 =$

$R_1 Q_1$ , one finds

$$(v \mid \Psi(Q^{2i\pi\mathcal{E}} \mathfrak{T}_N) u) = \sum_{\substack{R_1 R_2 = R \\ Q_1 Q_2 = Q_\bullet}} \mu(R_1 Q_1) \bar{v} \left( \frac{m}{N} \right) u \left( \frac{n}{N} \right) \\ \text{char} \left( m - n \equiv 0 \pmod{2Q \left( \frac{Q}{Q_\bullet} \right) (R_2 Q_2)} \right) \text{char}(m + n \equiv 0 \pmod{2RR_1 Q_1}). \quad (11.4)$$

Setting

$$m + n = 2R(R_1 Q_1) a, \quad m - n = 2Q \left( \frac{Q}{Q_\bullet} \right) (R_2 Q_2) b \quad (11.5)$$

with  $a, b \in \mathbb{Z}$  and  $a > 0$ , one has

$$\frac{m^2 - n^2}{N^2} = \frac{R(R_1 Q_1) \times Q \left( \frac{Q}{Q_\bullet} \right) R_2 Q_2}{N^2} \times 4ab = 4ab, \quad (11.6)$$

so that  $a = b = 1$ . One has

$$\frac{RR_1 Q_1}{N} = \frac{R_1 Q_1}{Q} = \frac{R_1}{Q_2} \times \frac{Q_\bullet}{Q}, \quad \frac{Q}{N} \left( \frac{Q}{Q_\bullet} \right) R_2 Q_2 = \frac{1}{R} \frac{Q R_2 Q_2}{Q_\bullet} = \frac{Q_2}{R_1} \frac{Q}{Q_\bullet} \quad (11.7)$$

and, finally,

$$\frac{m}{N} = \frac{R_1}{Q_2} \frac{Q_\bullet}{Q} + \frac{Q_2}{R_1} \frac{Q}{Q_\bullet}, \quad \frac{n}{N} = \frac{R_1}{Q_2} \frac{Q_\bullet}{Q} - \frac{Q_2}{R_1} \frac{Q}{Q_\bullet}. \quad (11.8)$$

The equation (11.1) follows.  $\square$

**Theorem 11.2.** *Zeros of  $\zeta(s)$  accumulate on the line  $\text{Re } s = 1$ . For every  $\delta < 1$ , there exists a zero of zeta with a real part in  $[\delta, \frac{1+\delta}{2}]$ .*

*Proof.* First, Theorem 9.2 generalizes as follows. Still defining  $u_Q(y) = Q^{\frac{\varepsilon}{2}} u(Q^\varepsilon y)$ , set

$$\tilde{F}_\varepsilon(s) := \sum_{Q \text{ odd}} Q^{-s} \left( v \mid \Psi \left( Q^{2i\pi\mathcal{E}} \mathfrak{T}_{\frac{\infty}{2}} \right) u_Q \right). \quad (11.9)$$

Assuming that  $v$  is supported in  $[2, \sqrt{8}]$  and  $u$  in  $[0, 1]$ , the function  $\tilde{F}_\varepsilon(s)$  is entire for every  $\varepsilon > 0$ ; for  $\text{Re } s > 1 + \sigma_0$ , it converges as  $\varepsilon \rightarrow 0$  towards the function

$$\tilde{F}_0(s) = \sum_{Q \text{ odd}} Q^{-s} \left( v \mid \Psi \left( Q^{2i\pi\mathcal{E}} \mathfrak{T}_{\frac{\infty}{2}} \right) u \right). \quad (11.10)$$

The proof of this follows the proof of Theorem 9.2, after one has noted that (4.18) ( a reduction of  $Q^{2i\pi\mathcal{E}}\mathfrak{T}_{\infty}$  to  $Q^{2i\pi\mathcal{E}}\mathfrak{T}_N$ ) extends without modification if dropping the assumption that  $Q$  is squarefree.

From this point on, the proof follows that of Theorem 10.5. The function  $(1 + 2^{-s})^{-1} \frac{\zeta(s)}{\zeta(2s)}$  must be replaced by  $\tilde{f}(s) = \sum_{Q \text{ odd}} Q^{-s} = (1 - 2^{-s})\zeta(s)$  and  $f(s - \nu + \varepsilon\mu)$  by  $\tilde{f}(s - \nu + \varepsilon\mu)$ . Half-zeros of zeta do no longer enter the picture, and the condition  $a > \frac{\sigma_0}{2}$  in the proof of Theorem 10.5 can be dropped.

That, for every  $\delta \in ]\frac{1}{2}, 1[$ , there exists a zero of zeta with a real part in  $[\delta, \frac{1+\delta}{2}]$ , is a consequence of Theorem 10.4.  $\square$

*Remark 11.1.* Just so as to answer a possibly natural question, if one replaces for some  $\kappa = 1, 2, \dots$ , in the definition of  $F_0(s)$ , the summation over all squarefree odd integers by that over all odd integers in the decompositions of which all primes are taken to powers with exponents  $\leq \kappa$ , the result obtained by a generalization of the proof of Theorem 10.5 is that  $\sigma_0 \geq \frac{2}{3 + \frac{1}{\kappa+1}}$ . It suffices to substitute for the function  $f(s) = (1 + 2^{-s})^{-1} \frac{\zeta(s)}{\zeta(2s)}$  the function  $(1 + 2^{-s} + \dots + 2^{-\kappa s})^{-1} \frac{\zeta(s)}{\zeta((\kappa+1)s)}$ . Replacing in (10.4) the condition  $c > 1$  by  $c > \sigma_0$ , as done in Remark 10.2, one can improve this to  $\sigma_0 \geq \frac{3}{4 + \frac{1}{\kappa+1}}$ , or  $\sigma_0 \geq \frac{3}{4}$  if using all values of  $\kappa$ : this is still less than the result of Theorem 11.2.

Generalizing the result to the case of Dirichlet  $L$ -series does not present any new difficulty.

**Theorem 11.3.** *Let  $\chi$  be a Dirichlet character mod  $M$  ( $M = 2, 3, \dots$ ), to wit a function  $\chi$  on  $\mathbb{Z}$  such that  $\chi(n) = 0$  if  $(n, M) > 1$  while, if  $(n, M) = 1$ ,  $\chi(n)$  coincides with the value on the class of  $n$  of some character of the group  $(\mathbb{Z}/M\mathbb{Z})^\times$ . We assume that  $\chi$  is primitive, i.e., not induced by a Dirichlet character mod  $M'$ , in which  $M'$  would be a divisor of  $M$  distinct from  $M$ . Consider the associated Dirichlet  $L$ -series*

$$L(s, \chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} = \sum_{n \geq 1} \frac{\chi(n)}{n^s}, \quad (11.11)$$

and recall that the function  $L(s, \chi)$  is entire unless the character  $\chi$  is trivial [7, p.331]. One has  $\sigma_0 = \sup \{ \operatorname{Re} \rho : L(\rho, \chi) = 0 \} = 1$ .

*Proof.* It is an adaptation of the proof just given for the Riemann zeta function. Setting, for  $r \neq 0$ ,  $a^\chi(r) = \prod_{p|r} (1 - p\chi(p))$ , we introduce for every squarefree integer  $N$  the distribution

$$\mathfrak{T}_N^\chi(x, \xi) = \sum_{j, k \in \mathbb{Z}} a^\chi((j, k, N)) \delta(x - j) \delta(\xi - k), \quad (11.12)$$

its truncation  $(\mathfrak{T}_N^\chi)^\times$  obtained by dropping the term for which  $j = k = 0$ , and the weak limit as  $N \nearrow \infty$

$$\mathfrak{T}_\infty^\chi = \frac{1}{2i\pi} \int_{\operatorname{Re} \nu = c} \frac{\mathfrak{E}_{-\nu}}{L(\nu, \chi)} d\nu, \quad c \geq 1. \quad (11.13)$$

Discarding the prime 2 in the definition as a product of  $L(s, \chi)$ , we define in a similar way  $\mathfrak{T}_{\frac{\infty}{2}}^\chi$ .

The novelties in the analogue of Theorem 6.1 are the following. We define now, as in (6.3),

$$f_N^\chi(j, s) = \frac{1}{N} \sum_{k \bmod N} b^\chi(j, k) \exp\left(\frac{2i\pi ks}{N}\right) \quad (11.14)$$

with, this time,  $b^\chi(j, k) = a^\chi((j, k, N)) = \prod_{p|(j, k, N)} (1 - p\chi(p))$ . Then,  $f_N^\chi = \otimes_{p|N} f_p^\chi$ , with (following the proof of (7.4))

$$\begin{aligned} f_p^\chi(j, s) &= \frac{1}{p} \sum_{k \bmod p} (1 - p\chi(p) \operatorname{char}(j \equiv k \equiv 0 \bmod p)) \exp\left(\frac{2i\pi ks}{p}\right) \\ &= \operatorname{char}(s \equiv 0 \bmod p) - \chi(p) \operatorname{char}(j \equiv 0 \bmod p). \end{aligned} \quad (11.15)$$

It follows that

$$f_N^\chi(j, s) = \sum_{N_1 N_2 = N} \mu(N_1) \chi(N_1) \operatorname{char}(s \equiv 0 \bmod N_2) \operatorname{char}(j \equiv 0 \bmod N_1), \quad (11.16)$$

which leads to a generalization of Theorem 6.1. In terms of the reflection map, introducing the distribution  $\mathfrak{T}_N^{\bar{\chi}}$ , one has

$$c_{R, Q}(\mathfrak{T}_N^\chi, m n) = \mu(Q) \chi(Q) c_{N, 1}(\mathfrak{T}_N^{\bar{\chi}}; m, \overset{\vee}{n}). \quad (11.17)$$

One can conclude with the method used in Section 10 or the one used in the present section.

Be sure to use the series  $\sum Q^{-s+2i\pi\mathcal{E}}$  as in the case of the Riemann zeta function, certainly not the series  $\sum \chi(Q) Q^{-s+2i\pi\mathcal{E}}$ : the pole of  $\zeta(s)$  at  $s = 1$  played a crucial role as soon as in the proof of Theorem 5.3. Only, in (10.4), it is the function  $\frac{(1-\chi(2)2^{-\nu})^{-1}}{L(\nu, \chi)}$  that takes the place of  $\frac{(1-2^{-\nu})^{-1}}{\zeta(\nu)}$ .  $\square$

*Remark 11.1.* It looks very unlikely that one will “ever” find an explicit non-trivial zero of zeta not on the critical line. It is even hard to see how one could find a number  $\tau > 0$ , in the spirit of Skewes’ number (a huge bound on the location of the first sign change in the difference  $\pi(n) - \text{li}(n)$ ), such that one could be assured that there exist non-trivial zeros with a positive imaginary part less than  $\tau$ . The developments in the present section were conclusive because we concentrated on the real parts, not the imaginary parts of zeros. In this respect, short of finding a spectral interpretation of the real parts of zeros, we shall build in Section 2.1 a family  $(ds_{\Sigma}^{(\rho)})$  of automorphic one-dimensional objects in the hyperbolic half-plane with the following property: assuming that  $\rho$  is real,  $ds_{\Sigma}^{(\rho)}$  misses some Eisenstein series  $E_{\frac{1-i\lambda}{2}}^*$  in its spectral decomposition if and only if  $\frac{\rho}{2}$  is the real part of a zero of zeta.

## 12. THE ROLE OF THE LAX-PHILLIPS AUTOMORPHIC SCATTERING THEORY

The theory of automorphic distributions could have been born from several approaches: the Lax-Phillips automorphic scattering theory [5], representation-theoretic facts such as the way [2] the principal series of representations can be realized by means of functions on the hyperbolic half-plane or by distributions in the plane, or the theory of the Radon transformation [3] in one of its simplest examples. As a matter of fact, it grew out of a cooperation between the Lax-Phillips theory and pseudodifferential operator theory. In the course of the two decades between its introduction [8] and its application ([12] and the present book) to important problems, we were led to stressing the pseudodifferential and representation-theoretic aspects, and we almost forgot the Lax-Phillips origin of the construction. We feel that recovering this aspect of the theory may contribute to a good understanding of the whole rich situation.

In the eighties, with the aim of teaching ourselves some basics of harmonic analysis and modular form theory, we developed generalizations of

the Weyl pseudodifferential calculus of operators in which domains such as the hyperbolic half-plane  $\mathbb{H}$  could serve as so-called phase spaces (the place where symbols live). The space of functions serving as arguments of the operators could be taken as any of the spaces in the holomorphic discrete series of representations of  $SL(2, \mathbb{R})$  or its continuation  $(\mathcal{D}_{\tau+1})_{\tau > -1}$  (Knapp's notation). While representation-theoretic and Hilbert-type facts were quite satisfactory, the symbolic calculus of operators was not: the explicit formulas for the composition of symbols (the notion corresponding to the composition of operators) were not as easy as the ones of the Weyl calculus. We soon found out [8, theor. 9.9] that, in order to save the situation, we had to use for symbols pairs of functions on  $\mathbb{H}$ , and let the two parts of the calculus act on a pair of spaces associated to parameters  $\tau, \tau + 1$ : the case when  $\tau = -\frac{1}{2}$  is the one that would coincide, after some changes of variables, to the Weyl calculus.

This necessity to use pairs of functions on  $\mathbb{H}$  brought our interest to the Lax-Phillips automorphic scattering theory. It starts with the consideration of the cone

$$C = \{\eta = (\eta_0, \eta_1, \eta_2) \in \mathbb{R}^3 : \eta_0 > 0, \eta_0^2 - \eta_1^2 - \eta_2^2 > 0\}, \quad (12.1)$$

its boundary  $\partial C$  (the forward light-cone in  $(1+2)$ -dimensional spacetime), and the “mass hyperboloid”  $\mathcal{H} = \{\eta : \eta_0 > 0, \eta_0^2 - \eta_1^2 - \eta_2^2 = 1\}$ . As is well-known, the hyperbolic half-plane  $\mathbb{H}$  and  $\mathcal{H}$  are equivalent models of the symmetric space  $G/K$ , with  $G = SL(2, \mathbb{R})$ ,  $K = SO(2)$ . One enriches the correspondence between the two by means of rescalings, so as to fill up  $C$ , obtaining the map

$$(t, z) \mapsto \begin{pmatrix} \eta_0 + \eta_1 & \eta_2 \\ \eta_2 & \eta_0 - \eta_1 \end{pmatrix} = e^t \begin{pmatrix} \frac{|z|^2}{y} & \frac{x}{y} \\ \frac{x}{y} & \frac{1}{y} \end{pmatrix} \quad (12.2)$$

from  $\mathbb{R} \times \mathbb{H}$  to  $C$ . In  $C$ , one takes interest in the “d'Alembertian” operator

$$\square = \frac{\partial^2}{\partial \eta_0^2} - \frac{\partial^2}{\partial \eta_1^2} - \frac{\partial^2}{\partial \eta_2^2}. \quad (12.3)$$

On  $\mathbb{H}$ , one disposes of the Laplacian  $\Delta$ . Now [5, p.11], under the map (12.2) and the gauge transformation  $u \mapsto W = e^{-\frac{t}{2}}u$ , the equation  $\square W = 0$  inside  $C$  is equivalent to the wave equation

$$\frac{\partial^2 u}{\partial t^2} + \left(\Delta - \frac{1}{4}\right)u = 0. \quad (12.4)$$

On one hand, one considers the problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + (\Delta - \frac{1}{4}) u = 0 \\ u(0, z) = f_0(z) \\ \frac{\partial u}{\partial t}(0, z) = f_1(z), \end{cases} \quad (12.5)$$

in which a solution of (12.4) is characterized by a pair of Cauchy data. On the other hand,  $\partial C$  is totally characteristic for the operator  $\square$  and, in a strikingly different way, just one datum on  $\partial C$  is needed to characterize  $W$ .

Indeed, given a solution of the equation  $\square W = 0$  inside  $C$ , extend it by zero outside  $C$ , obtaining a function  $\check{W}$  in  $\mathbb{R}^3$ . The function  $W$  is characterized by the density (a function on  $\partial C$ ) of  $\square \check{W}$  (taken in the distribution sense) with respect to the measure  $\frac{d\eta_1 d\eta_2}{\eta_0}$ . To make the correspondence from the pair  $(f_0, f_1)$  of Cauchy data to this density, we prefer, having pseudodifferential analysis in mind, to substitute even functions  $h$  on  $\mathbb{R}^2$  for functions on  $\partial C$  by means of the map  $h \mapsto Qh$  such that

$$h(x, \xi) = (Qh) \left( \frac{x^2 + \xi^2}{2}, \frac{x^2 - \xi^2}{2}, x\xi \right). \quad (12.6)$$

Then, define as  $\widetilde{Qh}$  the measure on  $\mathbb{R}^3$ , supported in  $\partial C$ , the density of which with respect to the measure  $\frac{d\eta_1 d\eta_2}{\eta_0}$  is the function  $Qh$ .

Let  $Z_2$  be the fundamental solution of  $\square$  supported in the closure of  $C$ , as provided by M.Riesz' theory [6], given as

$$Z_2(\eta) = \frac{1}{2\pi} (\eta_0^2 - \eta_1^2 - \eta_2^2)^{-\frac{1}{2}}, \quad \eta \in C. \quad (12.7)$$

The function  $\check{W} = Z_2 * \widetilde{Qh}$ , supported in the closure of  $C$ , satisfies the (distribution) equation  $\square(Z_2 * \widetilde{Qh}) = \widetilde{Qh}$  in  $\mathbb{R}^3$ : in particular  $\square W = 0$  inside  $C$ . Just one function (the function  $h$ ) is thus needed to build a solution  $W$  of  $\square W = 0$  inside  $C$ .

All solutions can be obtained in this way, under some explicit regularity conditions. To prove this, we must make the correspondence between the pair  $(f_0, f_1)$  in the Cauchy problem (12.5) and the function  $h$ , when  $u$  and  $W$  are linked by the Lax-Phillips transformation, explicit. To this effect, we introduce the Radon transformation  $V$  from functions in  $\mathbb{H}$  to even functions in  $\mathbb{R}^2$  which is the adjoint of the map  $V^*$ , the operator from functions, or distributions in  $\mathbb{R}^2$  to functions in  $\mathbb{H}$  defined by the equation,

when convergent,

$$(V^* h)(g \cdot i) = \int_K h((gk) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}) dk, \quad (12.8)$$

where  $g \in SL(2, \mathbb{R})$  and  $K = SO(2)$ . This is the simplest case of an extensive theory developed by Helgason [3]. On the other hand, we introduce the operator  $T$  on even functions in the plane, the function in the spectral-theoretic sense of  $2i\pi\mathcal{E}$  given as [8, (4.10)]

$$T = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{\Gamma(\frac{1}{2} - i\pi\mathcal{E})}{\Gamma(-1\pi\mathcal{E})} = \pi^{-\frac{1}{2}} (-i\pi\mathcal{E}) \int_0^\infty t^{-\frac{1}{2}} (1+t)^{-1+i\pi\mathcal{E}} dt. \quad (12.9)$$

After some calculations [8, p.29 and p.195-197] based on the use of the Radon transformation and the M.Riesz transform, one ends up with the explicit formula

$$h = 2^{\frac{1}{2}} (TV f_1 + (i\pi\mathcal{E}) TV f_0). \quad (12.10)$$

What is especially important for us in the whole construction is the question of finding and using a square root of the operator  $\Delta - \frac{1}{4}$ . The answer provided by operators on  $L^2(\mathbb{H})$  the integral kernels of which are functions of the point-pair invariant  $\cosh d(z, w)$  leads to the spectral theory of the Laplacian, in the open space  $\mathbb{H}$  or in other situations, including the automorphic case. There is another answer, which consists in replacing the space  $L^2(\mathbb{H})$  by its square, and taking the matrix operator  $\begin{pmatrix} 0 & I \\ -\Delta + \frac{1}{4} & 0 \end{pmatrix}$ . While it may first look as a joke, it is far from being one, and it is the main object of interest in [5]. Under the correspondence from pairs  $(f_0, f_1)$  of functions in  $\mathbb{H}$  to even functions  $h$  in the plane, this transfers to the operator  $i\pi\mathcal{E}$  [8, p.195-197], an operator which we have used consistently in this paper.

The two “square roots” of  $\Delta - \frac{1}{4}$  are totally distinct, and have totally distinct applications. The first is the useful one in analysis in  $L^2(\mathbb{H})$  and is by essence a non-negative self-adjoint operator. The operator  $2i\pi\mathcal{E}$  differs from  $-2i\pi\mathcal{E}$  and the two operators give rise to a notion close (but simpler) to that of ingoing and outgoing waves which is the main subject of automorphic scattering theory. The terminology is justified by the equation

$$g^+(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\lambda}{(1+i\lambda)^2} \int_0^1 \theta^{1+i\lambda} (\delta^2 g)(\theta y, \theta \eta) \frac{d\theta}{\theta}, \quad (12.11)$$



and the similar one regarding  $g^-(y)$ . The splitting  $g = g^+ + g^-$  plays an essential role in [12, prop.8.2], on the way to a proof of the Ramanujan-Petersson conjecture for Maass forms.

A similar role is played, in the present paper, in relation to the one-dimensional operator  $i(t\frac{d}{dt} + \frac{1}{2})$ , by the decomposition (9.28).

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