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The Convergence of Euler Products

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An interpretation of the Euler product is made in the theory of certain Hilbert spaces whose elements are entire functions. The main results are stated for those Dirichlet zeta-functions which are associated with nonprincipal characters. But they apply also in modified form to the classical zeta-function, which is associated with a principal character.

The Hilbert spaces of entire functions are derived from the Hardy space for the upper half-plane, which is here denoted $\mathscr{F}(1)$. The space contains those functions $F(z)$, which are analytic and of bounded type in the upper half-plane, which have square integrable boundary values on the real axis, and which satisfy an inequality of the form

$$\log|F(x+iy)| \leqslant \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{\log|F(t)|\, dt}{(t-x)^2+y^2}$$

for $y>0$. The norm of the space is defined by

$$\|F\|_{\mathscr{F}(1)}^2 = \int_{-\infty}^{+\infty} |F(t)|^2\, dt.$$

The reproducing kernel function for the space at a point w in the upper half-plane is

$$\frac{1}{2\pi i(\bar{w}-z)}.$$

Related Hilbert spaces of analytic functions are obtained using the concept of an analytic weight function. By an analytic weight function is meant any function $W(z)$ which is analytic and without zeros in the upper half-

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plane. If $W(z)$ is an analytic weight function, define the weighted Hardy space $\mathscr{F}(W)$ to be the set of functions $F(z)$, analytic in the upper half-plane, such that $F(z)/W(z)$ belongs to the Hardy space $\mathscr{F}(1)$. The norm of the space $\mathscr{F}(W)$ is defined so that multiplication by $W(z)$ is an isometric transformation of the Hardy space $\mathscr{F}(1)$ onto the space $\mathscr{F}(W)$. The reproducing kernel function of the space $\mathscr{F}(W)$ at a point w in the upper half-plane is

$$\frac{W(z)\,\bar{W}(w)}{2\pi i(\bar{w}-z)}.$$

A construction due to Beurling produces related Hilbert spaces of entire functions. Assume that $E(z)$ is a given entire function which satisfies the inequality

$$|E(x-iy)| < |E(x+iy)|$$

for $y > 0$. Since this condition implies that $E(z)$ has no zeros in the upper half-plane, a weighted Hardy space $\mathscr{F}(E)$ exists. Since $E^*(z) = \bar{E}(\bar{z})$ is an entire function which has the same modulus as $E(z)$ on the real axis, multiplication by $E^*(z)/E(z)$ is an isometric transformation of the weighted Hardy space $\mathscr{F}(E)$ into itself. Define $\mathscr{H}(E)$ to be the orthogonal complement in the space $\mathscr{F}(W)$ of the range of multiplication by $E^*(z)/E(z)$. Then $\mathscr{H}(E)$ is a Hilbert space of functions which admit analytic extensions to the complex plane. The reproducing kernel function of the space at any point w in the complex plane is

$$\frac{E(z)\,\bar{E}(w) - E^*(z)\,E(\bar{w})}{2\pi i(\bar{w}-z)}.$$

Any such space $\mathscr{H}(E)$ is easily verified to have the following properties:

(H1) Whenever $F(z)$ belongs to the space and has a nonreal zero w, the function $F(z)(z-\bar{w})/(z-w)$ belongs to the space and has the same norm as $F(z)$.

(H2) For any nonreal number w the linear functional defined on the space by $F(z)$ into $F(w)$ is continuous.

(H3) The function $F^*(z) = \bar{F}(\bar{z})$ belongs to the space whenever $F(z)$ belongs to the space, and it always has the same norm as $F(z)$.

It is not difficult to show [2] that these properties characterize such spaces. A Hilbert space whose elements are entire functions, which satisfies the axioms (H1), (H2), and (H3), and which contains a nonzero element, is isometrically equal to a space $\mathscr{H}(E)$ for some such entire function $E(z)$.

The theory of such Hilbert spaces of entire functions is relevant to the Riemann hypothesis because it is a theory of zeros of entire functions. If $E(z)$ is any entire function such that a space $\mathcal{H}(E)$ is defined, write $E(z) = A(z) - iB(z)$ where $A(z)$ and $B(z)$ are entire functions which are real for real z. In this notation the reproducing kernel function of the space $\mathcal{H}(E)$ reads

$$\frac{B(z)\,\bar{A}(w) - A(z)\,\bar{B}(w)}{\pi(z - \bar{w})}.$$

The positivity properties of reproducing kernel functions imply that the zeros of $A(z)$ and of $B(z)$ are real and are interlaced. The zeros are simple and are properly interlaced if $E(z)$ has no real zeros. The condition that $E(z)$ has no real zeros is that $F(z)/(z-w)$ belongs to the space $\mathcal{H}(E)$ whenever $F(z)$ is an element of the space which has a zero w. The condition is a consequence of the axiom (H1) when w is not real and is otherwise equivalent to the nonvanishing of $E(z)$ at w.

Some information about the Pólya class of entire functions is relevant to the particular spaces which appear in the Riemann hypothesis theory. The Pólya class axiomatizes a factorization theory which was developed by Hadamard for applications to zeta-functions.

An entire function $E(z)$ is said to be of Pólya class if it has no zeros in the upper half-plane, if it satisfies the inequality

$$|E(x - iy)| < |E(x + iy)|$$

for $y > 0$, and if $|E(x + iy)|$ is a nondecreasing function of $y > 0$ for each real x. A polynomial is of Pólya class if it has no zeros in the upper half-plane. A pointwise limit of entire functions of Pólya class is of Pólya class if it does not vanish identically. Every entire function of Pólya class is a limit, uniformly on compact sets, of polynomials which have no zeros in the upper half-plane.

Many of the defining functions of spaces $\mathcal{H}(E)$ which appear in classical analysis are of Pólya class. A reason for the pervasive quality of the Pólya class is its stability under bounded type perturbations [2]. Assume that $E_0(z)$ is an entire function of Pólya class and that $E(z)$ is an entire function, which has no zeros in the upper half-plane, such that the inequality

$$|E(x - iy)| \leqslant |E(x + iy)|$$

holds for $y > 0$. If the ratio $E(z)/E_0(z)$ is of bounded type in the upper half-plane, then $E(z)$ is of Pólya class.

Examples of such entire functions are obtained from Dirichlet zeta-functions. Let r be a given positive integer. A character modulo r is a function $\chi(n)$ of integers n which is periodic of period r, which satisfies the identity

$$\chi(mn) = \chi(m)\,\chi(n)$$

for all integers m and n, which vanishes at all integers which are not relatively prime to r, and which does not vanish identically.

A character is an even or an odd function. A character χ is said to be primitive modulo r if it is a character modulo r and if no character modulo s exists, s a proper divisor of r, which agrees with χ at points which are relatively prime to r. The principal character modulo r is the unique character modulo r whose values are nonnegative. It is primitive if, and only if, $r = 1$.

The Dirichlet zeta-function $\zeta_\chi(s)$ associated with a character χ modulo r is defined by

$$\zeta_\chi(s) = \sum_{n=1}^{\infty} \chi(n)\, n^{-s}.$$

The series converges in the half-plane $\operatorname{Re} s > 1$ and defines an analytic function of s in the half-plane. The Euler product

$$1/\zeta_\chi(s) = \prod (1 - \chi(p)\, p^{-s}),$$

which is taken over the primes p not dividing r, implies that the function $\zeta_\chi(s)$ has no zeros in the half-plane.

A theorem of Hadamard and de la Vallée Poussin (1896) states that the analytic continuation of the zeta-function has no zeros on the line $\operatorname{Re} s = 1$. The theorem was stated for the classical zeta-function $\zeta(s)$, which is the zeta-function $\zeta_\chi(s)$ associated with the principal character χ modulo one. But a similar argument applies to all character zeta-functions. A prior knowledge of these results is not assumed in the present work.

The functional identity for the classical zeta-function was discovered by Euler. The functions

$$\pi^{-(1/2)s}\Gamma(\tfrac{1}{2}s)\,\zeta(s)$$

and

$$\pi^{(1/2)s-(1/2)}\Gamma(\tfrac{1}{2}-\tfrac{1}{2}s)\,\zeta(1-s)$$

have analytic extensions to the complex plane except for simple poles at $s = 0$ and $s = 1$. The functional identity states that these extensions are equal. A similar functional identity is satisfied by the zeta-function $\zeta_\chi(s)$

when χ is a nonprincipal character provided that χ is a primitive character modulo r. If χ is even, the functions

$$(r/\pi)^{(1/2)s}\,\Gamma(\tfrac{1}{2}s)\,\zeta_\chi(s)$$

and

$$(r/\pi)^{(1/2)-(1/2)s}\,\Gamma(\tfrac{1}{2}-\tfrac{1}{2}s)\,\zeta_{\bar\chi}(1-s)$$

have entire extensions which are linearly dependent. If χ is odd, the functions

$$(r/\pi)^{(1/2)+(1/2)s}\,\Gamma(\tfrac{1}{2}+\tfrac{1}{2}s)\,\zeta_\chi(s)$$

and

$$(r/\pi)^{1-(1/2)s}\,\Gamma(1-\tfrac{1}{2}s)\,\zeta_\chi(1-s)$$

have entire extensions which are linearly dependent.

In 1859 Riemann conjectured that the classical zeta-function $\zeta(s)$ has no zeros in the half-plane $\operatorname{Re} s > \frac{1}{2}$. The present work supports the conjecture that the Dirichlet zeta-function $\zeta_\chi(s)$ has no zeros in the half-plane $\operatorname{Re} s > \frac{1}{2}$ when χ is a nonprincipal character modulo r. The results do not apply without modification to the principal character.

If χ is a character modulo r and is not the principal character modulo r, then a space $\mathscr{H}(E)$ exists

$$E(z) = (r/\pi)^{-(1/2)iz}\,\Gamma(\tfrac{1}{2}-\tfrac{1}{2}iz)\,\zeta_\chi(1-iz)$$

if χ is even and

$$E(z) = (r/\pi)^{-(1/2)iz}\,\Gamma(1-\tfrac{1}{2}iz)\,\zeta_\chi(1-iz)$$

if χ is odd. These spaces associated with zeta-functions appear in the scattering theory of modular forms [3]. The required inequality

$$|E(x-iy)| < |E(x+iy)|$$

is an expression of scattering phenomena.

In this notation the functional identity for the zeta-function states that $E(z-i)$ and $E^*(z)$ are linearly dependent when χ is a primitive character modulo r. The function $E(z)$ is of Pólya class. This concise summary of the Hadamard factorization can be derived from the stability of the Pólya class under bounded type perturbations. The required comparison Hilbert spaces of entire functions are the Sonine spaces, for which the Pólya class property has previously been verified [2].

A variant of these Hilbert spaces of entire functions is also used. It is convenient to replace z by $2z$ in the definition of $E(z)$ in order to facilitate applications of the theory of the gamma function. The new Hilbert spaces are obtained from the old ones by a homothetic expansion.

If χ is a character modulo r and is not the principal character modulo r, then a space $\mathscr{H}(E)$ exists,

$$E(z) = (r/\pi)^{\;iz}\, W(z)\, \zeta_\chi(1-2iz),$$

where

$$W(z) = \Gamma(\tfrac{1}{2} - iz)$$

if χ is even and

$$W(z) = \Gamma(1 - iz)$$

if χ is odd. The function $E(z)$ is of Pólya class. The functional identity states that $E(z - \frac{1}{2}i)$ and $E^*(z)$ are linearly dependent if χ is a primitive character modulo r. In this notation the Riemann hypothesis is the conjecture that the zeros of $E(z)$ lie on the line $z + \frac{1}{2}i = \bar{z}$.

A factorization theory for Hilbert spaces of analytic functions is relevant to the Riemann hypothesis. Another Hilbert space of functions analytic in the upper half-plane appears in the statement of the factorization theorem. If $\varphi(z)$ is a function which is analytic and has nonnegative real part in the upper half-plane, a unique Hilbert space $\mathscr{L}(\varphi)$ exists whose elements are functions analytic in the upper half-plane and which has the expression

$$\frac{\varphi(z) + \bar{\varphi}(w)}{\pi i(\bar{w} - z)}$$

as its reproducing kernel function at a point w in the upper half-plane.

The construction of the space $\mathscr{L}(\varphi)$ is made using the Poisson representation of functions which are analytic and have nonnegative real part in the upper half-plane. A unique nonnegative number p and an essentially unique nondecreasing function $\mu(x)$ of real x exist such that the identity

$$\frac{\varphi(z) + \bar{\varphi}(w)}{\pi i(\bar{w} - z)} = \frac{p}{\pi} + \frac{1}{\pi^2} \int_{-\infty}^{+\infty} \frac{d\mu(t)}{(t - z)(t - \bar{w})}$$

holds when z and w are in the upper half-plane. A partially isometric transformation of the space $L^2(\mu)$ into the space $\mathscr{L}(\varphi)$ is defined by taking $f(x)$ into

$$F(z) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{f(t)\, d\mu(t)}{t - z}.$$

The orthogonal complement of the range of the transformation consists of the constants which belong to the space $\mathcal{L}(\varphi)$. The space contains a nonzero constant if, and only if, p is nonzero.

A generalization of the notion of orthogonal complement [4] is used in the factorization theory. If a Hilbert space $\mathcal{P}$ is contained contractively in a Hilbert space $\mathcal{H}$, then a unique Hilbert space $\mathcal{Q}$ exists, which is contained contractively in $\mathcal{H}$, such that the inequality

$$\|c\|_{\mathcal{H}}^2 \leqslant \|a\|_{\mathcal{P}}^2 + \|b\|_{\mathcal{Q}}^2$$

holds whenever $c = a + b$ with a in $\mathcal{P}$ and b in $\mathcal{Q}$ and such that every element c of $\mathcal{H}$ admits some such decomposition for which equality holds. The space $\mathcal{Q}$ is called the complementary space to $\mathcal{P}$ in $\mathcal{H}$. The decomposition of an element c of $\mathcal{H}$ for which equality holds is unique. The element a of $\mathcal{P}$ is obtained from c under the adjoint of the inclusion of $\mathcal{P}$ in $\mathcal{H}$. The element b of $\mathcal{Q}$ is obtained from c under the adjoint of the inclusion of $\mathcal{Q}$ in $\mathcal{H}$.

A useful property of complementation is its preservation under onto partial isometries. Assume that $\mathcal{P}_0$ and $\mathcal{Q}_0$ are complementary subspaces of a Hilbert space $\mathcal{H}_0$ and that π is a partial isometry of $\mathcal{H}_0$ onto a Hilbert space $\mathcal{H}_1$. Then unique complementary spaces $\mathcal{P}_1$ and $\mathcal{Q}_1$ of $\mathcal{H}_1$ exist such that π acts as a partial isometry of $\mathcal{P}_0$ onto $\mathcal{P}_1$ and as a partial isometry of $\mathcal{Q}_0$ onto $\mathcal{Q}_1$.

A space $\mathcal{L}(\varphi_0)$ is contained contractively in a space $\mathcal{L}(\varphi)$ if, and only if, a space $\mathcal{L}(\varphi_1)$ exists such that

$$\varphi(z) = \varphi_0(z) + \varphi_1(z).$$

The spaces $\mathcal{L}(\varphi_0)$ and $\mathcal{L}(\varphi_1)$ are then complementary subspaces of the space $\mathcal{L}(\varphi)$. The minimal decomposition of

$$\frac{\varphi(z) + \bar{\varphi}(w)}{\pi i(\bar{w} - z)}$$

as an element of $\mathcal{L}(\varphi)$ is obtained with

$$\frac{\varphi_0(z) + \bar{\varphi}_0(w)}{\pi i(\bar{w} - z)}$$

as the element of $\mathcal{L}(\varphi_0)$ and with

$$\frac{\varphi_1(z) + \bar{\varphi}_1(w)}{\pi i(\bar{w} - z)}$$

as the element of $\mathcal{L}(\varphi_1)$ when w is in the upper half-plane.

A Hilbert space whose elements are pairs of functions analytic in the upper half-plane appears in the factorization theory. Assume that $W(z)$ is a given weight function and that $S(z)$ is a function analytic in the upper half-plane such that the function $S(z)/(z-\bar{w})$ belongs to the space $\mathscr{F}(W)$ when w is in the upper half-plane. Assume that $\mathscr{L}(\varphi)$ is a given space such that a contractive transformation of the space $\mathscr{F}(W)$ into the space $\mathscr{L}(\varphi)$ is defined by taking $F(z)$ into the function $f(z)$ defined by

$$\pi f(w) = \langle F(t), S(t)/(t-\bar{w})\rangle_{\mathscr{F}(W)}$$

when w is in the upper half-plane.

The condition placed on $\varphi(z)$ is computable. A space $\mathscr{L}(\varphi_0)$ exists such that the identity

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{S(t)\,\bar{S}(t)\,dt}{W(t)\,\bar{W}(t)(t-z)(t-\bar{w})} = \frac{\varphi_0(z)+\bar{\varphi}_0(w)}{i\bar{w}-iz}$$

holds when z and w are in the upper half-plane. A partially isometric transformation of the space $\mathscr{F}(W)$ into the space $\mathscr{L}(\varphi_0)$ is defined by taking $F(z)$ into the function $f(z)$ defined by

$$\pi f(w) = \langle F(t), S(t)/(t-\bar{w})\rangle_{\mathscr{F}(W)}$$

when w is in the upper half-plane. The condition is therefore that the space $\mathscr{L}(\varphi_0)$ is contained contractively in the space $\mathscr{L}(\varphi)$. An equivalent condition is that the identity

$$\varphi(z) = \varphi_0(z) + \varphi_1(z)$$

holds for a function $\varphi_1(z)$ such that a space $\mathscr{L}(\varphi_1)$ exists. The condition is then that the inequality

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{S(t)\,\bar{S}(t)\,dt}{W(t)\,\bar{W}(t)(t-z)(t-\bar{z})} \leqslant \frac{\varphi(z)+\bar{\varphi}(z)}{i\bar{z}-iz}$$

holds when z is in the upper half-plane.

Define the space $\mathscr{F}_S(W, \varphi)$ to be the set of pairs

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

of functions analytic in the upper half-plane such that $F(z)$ belongs to the space $\mathscr{F}(W)$ and such that

$$[i\varphi(z)\,F(z) + G(z)]/S(z)$$

belongs to the space $\mathcal{L}(\varphi)$. It will be seen that the space $\mathcal{F}_S(W, \varphi)$ is a Hilbert space when considered with a unique scalar product such that a partially isometric transformation of the space onto $\mathcal{F}(W)$ is defined by taking

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

into

$$\sqrt{2}\, F(z)$$

and such that a partially isometric transformation of the space onto $\mathcal{L}(\varphi)$ is defined by taking

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

into

$$\sqrt{2}[i\varphi(z)\, F(z) + G(z)]/S(z).$$

An isometric transformation of the space $\mathcal{F}(W)$ into the space $\mathcal{F}_S(W, \varphi)$ is defined by taking $F(z)$ into

$$\frac{1}{\sqrt{2}} \begin{pmatrix} F(z) \\ \tilde{F}(z) \end{pmatrix}$$

where $\tilde{F}(z)$ is defined by

$$\pi \tilde{F}(w) = \langle F(t), [S(t)\, \bar{S}(w) - \tfrac{1}{2} W(t)\, \bar{\varphi}(w)\, \bar{W}(w)]/(t - \bar{w}) \rangle_{\mathcal{F}(W)}$$

when w is in the upper half-plane. The orthogonal complement of the range of the transformation consists of the elements of the space whose upper coordinate vanishes identically. An isometric transformation of the orthogonal complement of the range of the transformation onto the space $\mathcal{L}(\varphi_1)$ is defined by taking

$$\begin{pmatrix} 0 \\ G(z) \end{pmatrix}$$

into

$$\sqrt{2}\, G(z)/S(z).$$

This orthogonal decomposition of the space $\mathcal{F}_S(W, \varphi)$ allows a construction of the space and a derivation of its properties. A straightforward calculation shows that the identity

$$\tilde{G}(z) = [\tilde{F}(z)\, S(w) - S(z)\, \tilde{F}(w)]/(z - w)$$

holds whenever

$$G(z) = [F(z)\, S(w) - S(z)\, F(w)]/(z - w)$$

for an element $F(z)$ of the space $\mathscr{F}(W)$ when w is in the upper half-plane. The pair

$$\begin{pmatrix} [F(z)\, S(w) - S(z)\, F(w)]/(z - w) \\ [G(z)\, S(w) - S(z)\, G(w)]/(z - w) \end{pmatrix}$$

belongs to the space $\mathscr{F}_S(W, \varphi)$ whenever

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

belongs to the space if w is in the upper half-plane.

A continuous transformation of the space $\mathscr{F}_S(W, \varphi)$ into two-dimensional Euclidean space is defined by taking

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

into

$$\begin{pmatrix} F(w) \\ G(w) \end{pmatrix}$$

when w is in the upper half-plane. The adjoint transformation is multiplication by the matrix-valued analytic function

$$\frac{W(z)\, \bar{W}(w)}{4\pi i(\bar{w} - z)} \begin{pmatrix} 1 \\ -i\varphi(z) \end{pmatrix} (1 i\bar{\varphi}(w)) - \frac{S(z)\, I\bar{S}(w)}{2\pi(z - \bar{w})},$$

where the notation I is used for the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This matrix-valued analytic function is the reproducing kernel function of the space $\mathscr{F}_S(W, \varphi)$ at a point w in the upper half-plane.

The space is used to formulate the main construction of the factorization theory.

THEOREM 1. *Assume that $\mathscr{H}(E)$ is a given space, that $W(z)$ is a given weight function, and that $S(z)$ is a function analytic in the upper half-plane such that multiplication by $S(z)$ is an isometric transformation of the space*

$\mathcal{H}(E)$ into the space $\mathcal{F}(W)$ and such that the space $\mathcal{F}(W)$ is the closed span of the functions $S(z)\,E(z)/(z-\bar{w})$ with w in the upper half-plane. Then a unique function $\varphi(z)$ exists such that a space $\mathcal{L}(\varphi)$ is defined, such that no nonzero element of the space is of the form $[1+\varphi(z)]\,F(z)/E^*(z)$ with $F(z)$ in $\mathcal{H}(E)$, such that a space $\mathcal{F}_S(W_\varphi, \varphi)$ exists,

$$W_\varphi(z) = \frac{W(z)}{A(z) - i\varphi(z)\,B(z)},$$

such that an isometric transformation of the space $\mathcal{F}_S(W_\varphi, \varphi)$ onto the space $\mathcal{L}(\varphi)$ is defined by taking

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

into

$$\sqrt{2}\,[i\varphi(z)\,F(z) + G(z)]/S(z),$$

and such that an isometric transformation of the space $\mathcal{F}_S(W_\varphi, \varphi)$ onto the orthogonal complement in the space $\mathcal{F}(W)$ of the image of $\mathcal{H}(E)$ is defined by taking

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

into

$$\sqrt{2}\,[A(z)\,F(z) + B(z)\,G(z)].$$

Proof of Theorem 1. Since multiplication by $S(z)$ maps the space $\mathcal{H}(E)$ into the space $\mathcal{F}(W)$, the function $S(z)\,E(z)/(z-\bar{w})$ belongs to the space $\mathcal{F}(W)$ when w is in the upper half-plane. A space $\mathcal{L}(\psi)$ exists such that the identity

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{T(t)\,\bar{T}(t)\,dt}{W(t)\,\bar{W}(t)(t-z)(t-\bar{w})} = \frac{\psi(z)+\bar{\psi}(w)}{i\bar{w}-iz}$$

holds when z and w are in the upper half-plane, $T(z) = S(z)\,E(z)$. The function $\psi(z)$ is unique within an added imaginary constant.

Since the space $\mathcal{F}(W)$ is assumed to be the closed span of the functions $T(z)/(z-\bar{w})$ with w in the upper half-plane, an isometric transformation of the space $\mathcal{F}(W)$ onto the space $\mathcal{L}(\psi)$ is obtained on taking $F(z)$ into the function $f(z)$ defined by

$$\pi f(w) = \langle F(t), T(t)/(t-\bar{w})\rangle_{\mathcal{F}(W)}$$

when w is in the upper half-plane. The transformation takes

$$[F(z)\,W(w) - W(z)\,F(w)]/(z-w)$$

into

$$W(w)[f(z) - f(w)]/(z-w)$$

when w is in the upper half-plane.

A space $\mathscr{F}_T(W, \psi)$ exists. For every element $F(z)$ of the space $\mathscr{F}(W)$ a unique function $\tilde{F}(z)$ analytic in the upper half-plane exists such that

$$\begin{pmatrix} F(z) \\ \tilde{F}(z) \end{pmatrix}$$

belongs to the space $\mathscr{F}_T(W, \psi)$. The identity

$$\pi\tilde{F}(w) = \langle F(t), [T(t)\,\bar{T}(w) - \tfrac{1}{2} W(t)\,\bar{\psi}(w)\,\bar{W}(w)]/(t-\bar{w})\rangle_{\mathscr{F}(W)}$$

holds when w is in the upper half-plane. The identity

$$\tilde{G}(z) = [\tilde{F}(z)\,T(w) - T(z)\,\tilde{F}(w)]/(z-w)$$

holds whenever

$$G(z) = [F(z)\,T(w) - T(z)\,F(w)]/(z-w)$$

for an element $F(z)$ of $\mathscr{F}(W)$ when w is in the upper half-plane. A straightforward calculation shows that the identity for difference-quotients

$$\begin{aligned}
&\langle [F(t)\,T(\beta) - T(t)\,F(\beta)]/(t-\beta), G(t)\,T(\alpha)\rangle_{\mathscr{F}(W)} \\
&\quad - \langle F(t)\,T(\beta), [G(t)\,T(\alpha) - T(t)\,G(\alpha)]/(t-\alpha)\rangle_{\mathscr{F}(W)} \\
&\quad - (\beta - \bar{\alpha})\langle [F(t)\,T(\beta) - T(t)\,F(\beta)]/(t-\beta), \\
&\qquad\qquad [G(t)\,T(\alpha) - T(t)\,G(\alpha)]/(t-\alpha)\rangle_{\mathscr{F}(W)} \\
&= \pi G(\alpha)^-\,\tilde{F}(\beta) - \pi\tilde{G}(\alpha)^-\,F(\beta)
\end{aligned}$$

holds for all elements $F(z)$ and $G(z)$ of the space $\mathscr{F}(W)$ when α and β are in the upper half-plane. An isometric transformation of the space $\mathscr{F}(W)$ onto the space $\mathscr{L}(\psi)$ is defined by taking $F(z)$ into

$$\sqrt{2}\,[i\varphi(z)\,F(z) + \tilde{F}(z)]/T(z).$$

For each element $F(z)$ of the space $\mathscr{H}(E)$ a unique function $\hat{F}(z)$ analytic in the upper half-plane exists such that

$$\begin{pmatrix} S(z) & F(z) \\ S(z) & \hat{F}(z) \end{pmatrix}$$

belongs to the space $\mathscr{F}_T(W, \psi)$. The identity

$$\hat{G}(z) = [\hat{F}(z)\, E(w) - E(z)\, \hat{F}(w)]/(z-w)$$

holds whenever

$$G(z) = [F(z)\, E(w) - E(z)\, F(w)]/(z-w)$$

for an element $F(z)$ of the space $\mathscr{H}(E)$ when w is in the upper half-plane. The identity for difference-quotients

$$\begin{aligned}
&\langle [F(t)\, E(\beta) - E(t)\, F(\beta)]/(t-\beta),\, G(t)\, E(\alpha)\rangle_{\mathscr{H}(E)} \\
&\quad - \langle F(t)\, E(\beta),\, [G(t)\, E(\alpha) - E(t)\, G(\alpha)]/(t-\alpha)\rangle_{\mathscr{H}(E)} \\
&\quad - (\beta - \bar{\alpha})\, \langle [F(t)\, E(\beta) - E(t)\, F(\beta)]/(t-\beta), \\
&\qquad\qquad [G(t)\, E(\alpha) - E(t)\, G(\alpha)]/(t-\alpha)\rangle_{\mathscr{H}(E)} \\
&= \pi G(\alpha)^-\, \hat{F}(\beta) - \pi \hat{G}(\alpha)^-\, F(\beta)
\end{aligned}$$

holds for all elements $F(z)$ and $G(z)$ of the space $\mathscr{H}(E)$ when α and β are in the upper half-plane.

It follows that the identity

$$G(\alpha)^-\, \hat{F}(\beta) - \hat{G}(\alpha)^-\, F(\beta) = 2iG(\alpha)^-\, F(\beta)$$

holds for all elements $F(z)$ and $G(z)$ of the space $\mathscr{H}(E)$ when α and β are in the upper half-plane. A number $h(w)$ exists such that the identity

$$\hat{F}(w) = iF(w) + h(w)\, F(w)$$

holds for every element $F(z)$ of the space $\mathscr{H}(E)$ when w is in the upper half-plane. Since the identity

$$h(\beta) = \bar{h}(\alpha)$$

holds when α and β are in the upper half-plane, $h = h(w)$ is real constant. Since $\psi(z)$ can be replaced by $\psi(z) - ih$ in the above constructions, it can be assumed without loss of generality that $h = 0$ and that the identity

$$\hat{F}(z) = iF(z)$$

holds for every element $F(z)$ of the space $\mathscr{H}(E)$.

A Hilbert space $\mathscr{T}$ will now be defined whose elements are pairs

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

of functions which are analytic in the upper half-plane except possibly at the zeros of $E^*(z)$. The space is defined so that multiplication by

$$\begin{pmatrix} A(z) & B(z) \\ -B(z) & A(z) \end{pmatrix}$$

is an isometric transformation of the space onto the orthogonal complement in the space $\mathscr{F}_T(W, \psi)$ of the elements of the form

$$\begin{pmatrix} S(z)\,F(z) \\ iS(z)\,F(z) \end{pmatrix}$$

with $F(z)$ in $\mathscr{H}(E)$.

Since the reproducing kernel function of the space $\mathscr{F}_T(W, \psi)$ is

$$\frac{W(z)\,\bar{W}(w)}{4\pi i(\bar{w}-z)}\begin{pmatrix} 1 \\ -i\psi(z) \end{pmatrix}(1 \;\; i\bar{\psi}(w)) - \frac{T(z)\,I\bar{T}(w)}{2\pi(z-\bar{w})}$$

and since the reproducing kernel function of the image of the space $\mathscr{H}(E)$ is

$$\frac{S(z)\,\bar{S}(w)}{2\pi(z-\bar{w})}\begin{pmatrix} A(z) & B(z) \\ -B(z) & A(z) \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
$$\times\begin{pmatrix} \bar{A}(w) & -\bar{B}(w) \\ \bar{B}(w) & \bar{A}(w) \end{pmatrix} - \frac{T(z)\,I\bar{T}(w)}{2\pi(z-\bar{w})},$$

the reproducing kernel function of the space $\mathscr{T}$ is

$$\frac{W_\varphi(z)\,\bar{W}_\varphi(w)}{4\pi i(\bar{w}-z)}\begin{pmatrix} 1 \\ -i\varphi(z) \end{pmatrix}(1 \;\; i\bar{\varphi}(w)) - \frac{S(z)\,I\bar{S}(w)}{2\pi(z-\bar{w})},$$

where $\varphi(z)$ is analytic in the upper half-plane except possibly at the zeros of $E^*(z)$ and

$$\psi(z) = \frac{\varphi(z)\,A(z) - iB(z)}{A(z) - i\varphi(z)\,B(z)}$$

and

$$W_\varphi(z) = \frac{W(z)}{A(z) - i\varphi(z)\,B(z)}.$$

The positivity properties of reproducing kernel functions imply that $\varphi(z)$ has a nonnegative real part at those points of the upper half-plane at which it is defined. Since the exceptional points are isolated, the condition implies that $\varphi(z)$ is analytic in the upper half-plane. A space $\mathcal{L}(\varphi)$ exists. A space $\mathcal{F}(W_\varphi)$ exists since $W_\varphi(z)$ is analytic and without zeros in the upper half-plane. The elements of the space $\mathcal{T}$ are pairs of functions analytic in the upper half-plane.

Multiplication by

$$[1+\psi(z)]/E(z) = [1+\varphi(z)]/[A(z) - i\varphi(z)\, B(z)]$$

is an isometric transformation of the space $\mathcal{H}(E)$ into the space $\mathcal{L}(\varphi)$ by construction. The reproducing kernel function of the orthogonal complement of the range of the transformation is equal to

$$\frac{E^*(z)}{A(z)-i\varphi(z)\,B(z)}\,\frac{\varphi(z)+\bar{\varphi}(w)}{\pi i(\bar{w}-z)}\,\frac{E(\bar{w})}{\bar{A}(w)+i\bar{\varphi}(w)\,\bar{B}(w)}.$$

It follows that multiplication by

$$E^*(z)/[A(z)-i\varphi(z)\,B(z)]$$

is an isometric transformation of the space $\mathcal{L}(\varphi)$ onto the orthogonal complement in the space $\mathcal{L}(\psi)$ of the image of the space $\mathcal{H}(E)$. No nonzero element of the space $\mathcal{L}(\varphi)$ is of the form

$$[1+\varphi(z)]\,F(z)/E^*(z)$$

with $F(z)$ in the space $\mathcal{H}(E)$.

Since an isometric transformation of the space $\mathcal{T}$ onto the orthogonal complement in the space $\mathcal{L}(\psi)$ of the image of the space $\mathcal{H}(E)$ is defined by taking

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

into

$$\sqrt{2}\,[i\psi(z)\,A(z)\,F(z) + i\psi(z)\,B(z)\,G(z) - B(z)\,F(z) + A(z)\,G(z)]/T(z)$$
$$= \sqrt{2}\,[A(z)+i\psi(z)\,B(z)]/E(z) \times [i\varphi(z)\,F(z)+G(z)]/S(z)$$

where the identity

$$[A(z)-i\varphi(z)\,B(z)][A(z)+i\psi(z)\,B(z)] = E(z)\,E^*(z)$$

is satisfied, an isometric transformation of the space $\mathcal{T}$ onto the space $\mathcal{L}(\varphi)$ is defined by taking

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

into

$$\sqrt{2}\,[i\varphi(z)\,F(z)+G(z)]/S(z).$$

The existence of a partial isometry of the space $\mathscr{T}$ onto the space $\mathscr{F}(W_\varphi)$ which takes

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

into

$$\sqrt{2}\,F(z)$$

is a consequence of the computation of the reproducing kernel function of the space $\mathscr{T}$. A space $\mathscr{F}_S(W_\varphi, \varphi)$ therefore exists and is isometrically equal to $\mathscr{T}$.

Since multiplication by

$$\begin{pmatrix} A(z) & B(z) \\ -B(z) & A(z) \end{pmatrix}$$

is an isometric transformation of the space $\mathscr{F}_S(W_\varphi, \varphi)$ onto the orthogonal complement in the space $\mathscr{F}_T(W, \psi)$ of the image of the space $\mathscr{H}(E)$, the transformation which takes

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

into

$$\sqrt{2}\,[A(z)\,F(z)+B(z)\,G(z)]$$

is an isometry of the space $\mathscr{F}_S(W_\varphi, \varphi)$ onto the orthogonal complement in the space $\mathscr{F}(W)$ of the image of the space $\mathscr{H}(E)$.

Uniqueness of the function $\varphi(z)$ with the desired properties will now be verified. Assume that any function $\varphi(z)$ is given such that a space $\mathscr{L}(\varphi)$ is defined, such that no nonzero element of the space is of the form $[1+\varphi(z)]\,F(z)/E^*(z)$ for an element $F(z)$ of the space $\mathscr{H}(E)$, such that a space $\mathscr{F}_S(W_\varphi, \varphi)$ exists,

$$W_\varphi(z)=\frac{W(z)}{A(z)-i\varphi(z)\,B(z)},$$

such that an isometric transformation of the space $\mathscr{F}_S(W_\varphi, \varphi)$ onto the space $\mathscr{L}(\varphi)$ is defined by taking

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

into

$$\sqrt{2}\,[i\varphi(z)\,F(z) + G(z)]/S(z),$$

and such that an isometric transformation of the space $\mathcal{F}_S(W_\varphi, \varphi)$ onto the orthogonal complement in $\mathcal{F}(W)$ of the image of $\mathcal{H}(E)$ is defined by taking

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

into

$$\sqrt{2}\,[A(z)\,F(z) + B(z)\,G(z)].$$

These conditions imply that no nonzero element

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

of the space $\mathcal{F}_S(W_\varphi, \varphi)$ exists such that

$$[A(z)\,F(z) + B(z)\,G(z)]/S(z)$$

belongs to $\mathcal{H}(E)$. And no nonzero element

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

of the space $\mathcal{F}_S(W_\varphi, \varphi)$ exists such that

$$\begin{pmatrix} A(z) & B(z) \\ -B(z) & A(z) \end{pmatrix} \begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

is of the form

$$\begin{pmatrix} S(z)\,H(z) \\ iS(z)\,H(z) \end{pmatrix}$$

for an element $H(z)$ of the space $\mathcal{H}(E)$.

A unique Hilbert space $\mathcal{T}$ exists, whose elements are pairs

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

of functions analytic in the upper half-plane, such that an isometric transformation of $\mathcal{H}(E)$ into $\mathcal{T}$ is defined by taking $F(z)$ into

$$\sqrt{2}\begin{pmatrix} S(z)\,F(z) \\ iS(z)\,F(z) \end{pmatrix}$$

and such that multiplication by

$$\begin{pmatrix} A(z) & B(z) \\ -B(z) & A(z) \end{pmatrix}$$

is an isometric transformation of the space $\mathscr{F}_S(W_\varphi, \varphi)$ onto the orthogonal complement in $\mathscr{T}$ of the image of the space $\mathscr{H}(E)$.

A space $\mathscr{L}(\psi)$ exists such that the identity

$$\psi(z) = \frac{\varphi(z)\, A(z) - iB(z)}{A(z) - i\varphi(z)\, B(z)}$$

holds when z is in the upper half-plane. Multiplication by

$$[1 + \psi(z)]/E(z) = [1 + \varphi(z)]/[A(z) - i\varphi(z)\, B(z)]$$

is a contractive transformation of the space $\mathscr{H}(E)$ into the space $\mathscr{L}(\varphi)$. Multiplication by

$$E^*(z)/[A(z) - i\varphi(z)\, B(z)] = [A(z) + i\psi(z)\, B(z)]/E(z)$$

is a contractive transformation of the space $\mathscr{L}(\varphi)$ into the space $\mathscr{L}(\psi)$. The ranges of the contractions become complementary subspaces of $\mathscr{L}(\varphi)$ when considered with the unique scalar products such that the transformations are isometries. Since no nonzero element of the space $\mathscr{L}(\varphi)$ is of the form

$$[1 + \varphi(z)]\, F(z)/E^*(z)$$

with $F(z)$ in the space $\mathscr{H}(E)$, each of these transformations is an isometry into the space $\mathscr{L}(\psi)$. The ranges of these transformations are orthogonal complements in the space $\mathscr{L}(\varphi)$.

Since an isometric transformation of the space $\mathscr{F}_S(W_\varphi, \varphi)$ onto the space $\mathscr{L}(\varphi)$ is defined by taking

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

into

$$\sqrt{2}\, [i\varphi(z)\, F(z) + G(z)]/S(z)$$

by hypothesis, an isometric transformation of the space $\mathscr{T}$ onto the space $\mathscr{L}(\psi)$ is defined by taking

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

into

$$\sqrt{2}\,[i\psi(z)\,F(z)+G(z)]/T(z),$$

where $T(z)=S(z)\,E(z)$.

Since an isometric transformation of the space $\mathcal{F}_S(W_\varphi, \varphi)$ onto the orthogonal complement in $\mathcal{F}(W)$ of the image of $\mathcal{H}(E)$ is defined by taking

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

into

$$\sqrt{2}\,[A(z)\,F(z)+B(z)\,G(z)],$$

an isometric transformation of the space $\mathcal{T}$ into the space $\mathcal{F}(W)$ is defined by taking

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

into

$$\sqrt{2}\,F(z).$$

It has been shown that a space $\mathcal{F}_T(W, \psi)$ exists and that it is isometrically equal to the space $\mathcal{T}$. Since an isometric transformation of the space $\mathcal{F}(W)$ into the space $\mathcal{L}(\psi)$ is obtained by taking $F(z)$ into the function $f(z)$ defined by

$$\pi f(w)=\langle F(t),\, T(t)/(t-\bar{w})\rangle_{\mathcal{F}(W)},$$

when w is in the upper half-plane, it follows that the identity

$$\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{T(t)\,\bar{T}(t)\,dt}{W(t)\,\bar{W}(t)(t-z)(t-\bar{w})}=\frac{\psi(z)+\bar{\psi}(w)}{i\bar{w}-iz}$$

holds when z and w are in the upper half-plane. An argument earlier in the proof shows that the function $\psi(z)$ is now uniquely determined by the condition that

$$\begin{pmatrix} S(z)\,F(z) \\ iS(z)\,F(z) \end{pmatrix}$$

belongs to the space $\mathcal{F}_T(W, \psi)$ whenever $F(z)$ belongs to the space $\mathcal{H}(E)$. Uniqueness of $\varphi(z)$ follows from the uniqueness of $\psi(z)$.

This completes the proof of the theorem.

A bar is used to denote the conjugate transpose of a rectangular matrix. Assume that $S(z)$ is a given entire function which does not vanish identically and that

$$M(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$$

is a matrix of entire functions, which are real for real z, such that the identity

$$A(z)\, D(z) - B(z)\, C(z) = S(z)\, S^*(z)$$

is satisfied and such that the matrices

$$\frac{M(z)\, I\bar{M}(z) - S(z)\, I\bar{S}(z)}{z - \bar{z}}$$

and

$$\frac{M(z)\, I\bar{M}(z) - S^*(z)\, IS(\bar{z})}{z - \bar{z}}$$

are nonnegative when z is in the upper half-plane.

These conditions imply that the function

$$W(z) = A(z) - iB(z)$$

has no zeros in the upper half-plane, the function

$$\varphi(z) = \frac{D(z) + iC(z)}{A(z) - iB(z)}$$

has nonnegative real part in the half-plane, and the inequalities

$$|S(z)/W(z)|^2 \leqslant \operatorname{Re} \varphi(z)$$

and

$$|S^*(z)/W(z)|^2 \leqslant \operatorname{Re} \varphi(z)$$

are satisfied in the half-plane. Equality holds in both inequalities on the real axis.

An application can now be made of the Poisson representation of functions which are analytic and have nonnegative real part in the upper half-plane. A nonnegative number p exists such that the identity

$$\frac{\varphi(z) + \bar{\varphi}(w)}{\pi i(\bar{w} - z)} = \frac{p}{\pi} + \frac{1}{\pi^2} \int_{-\infty}^{+\infty} \frac{S(t)\, \bar{S}(t)\, dt}{W(t)\, \bar{W}(t)(t - z)(t - \bar{w})}$$

holds when z and w are in the upper half-plane. Spaces $\mathscr{F}_S(W, \varphi)$ and $\mathscr{F}_{S^*}(W, \varphi)$ exist.

An isometric transformation of the space $\mathscr{F}(W)$ into the space $\mathscr{F}_S(W, \varphi)$ is defined by taking $F(z)$ into

$$\frac{1}{\sqrt{2}}\begin{pmatrix} F(z) \\ \tilde{F}(z) \end{pmatrix},$$

where $\tilde{F}(z)$ is defined by

$$\pi\tilde{F}(w) = \langle F(t), [S(t)\,\bar{S}(w) - \tfrac{1}{2}W(t)\,\bar{\varphi}(w)\,\bar{W}(w)]/(t-\bar{w})\rangle_{\mathscr{F}(W)}$$

when w is in the upper half-plane. The orthogonal complement of the range of the transformation consists of the constant multiples of the pair

$$\begin{pmatrix} 0 \\ S(z) \end{pmatrix}$$

which belong to the space $\mathscr{F}_S(W, \varphi)$. The pair belongs to the space if, and only if, p is nonzero. The square of the norm of this element of the space is then equal to π/p.

The space $\mathscr{H}_S(M)$ is defined to be the set of elements

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

of the space $\mathscr{F}_S(W, \varphi)$ such that $F(z)$ belongs to the space $\mathscr{H}(E)$. For such elements the identity

$$\pi\tilde{F}(w) = \langle F(t), [S(t)\,\bar{S}(w) - A(t)\,\bar{D}(w) + B(t)\,\bar{C}(w)]/(t-\bar{w})\rangle_{\mathscr{F}(W)}$$

holds when w is in the upper half-plane. The space $\mathscr{H}_S(M)$ is a Hilbert space which is contained isometrically in the space $\mathscr{F}_S(W, \varphi)$. An isometric transformation of the space $\mathscr{H}_S(M)$ onto the space $\mathscr{H}_{S^*}(M)$ is defined by taking

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

into

$$\begin{pmatrix} F^*(z) \\ G^*(z) \end{pmatrix}.$$

The pair

$$\begin{pmatrix} [F(z)\,S(w) - S(z)\,F(w)]/(z-w) \\ [G(z)\,S(w) - S(z)\,G(w)]/(z-w) \end{pmatrix}$$

belongs to the space $\mathscr{H}_S(M)$ for all complex numbers w whenever

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

belongs to the space. The identity for difference-quotients

$$\begin{aligned}
&\left\langle \begin{pmatrix} [F(t)\, S(\beta) - S(t)\, F(\beta)]/(t-\beta) \\ [G(t)\, S(\beta) - S(t)\, G(\beta)]/(t-\beta) \end{pmatrix}, \begin{pmatrix} F(t) & S(\alpha) \\ G(t) & S(\alpha) \end{pmatrix} \right\rangle_{\mathscr{H}_S(M)} \\
&\quad - \left\langle \begin{pmatrix} F(t) & S(\beta) \\ G(t) & S(\beta) \end{pmatrix}, \begin{pmatrix} [F(t)\, S(\alpha) - S(t)\, F(\alpha)]/(t-\alpha) \\ [G(t)\, S(\alpha) - S(t)\, G(\alpha)]/(t-\alpha) \end{pmatrix} \right\rangle_{\mathscr{H}_S(M)} \\
&\quad - (\beta - \bar{\alpha}) \left\langle \begin{pmatrix} [F(t)\, S(\beta) - S(t)\, F(\beta)]/(t-\beta) \\ [G(t)\, S(\beta) - S(t)\, G(\beta)]/(t-\beta) \end{pmatrix}, \right. \\
&\qquad\qquad \left. \begin{pmatrix} [F(t)\, S(\alpha) - S(t)\, F(\alpha)]/(t-\alpha) \\ [G(t)\, S(\alpha) - S(t)\, G(\alpha)]/(t-\alpha) \end{pmatrix} \right\rangle_{\mathscr{H}_S(M)} \\
&= 2\pi F(\alpha)^- \, G(\beta) - 2\pi G(\alpha)^- \, F(\beta)
\end{aligned}$$

holds for all elements

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

of the space for all complex numbers α and β. The reproducing kernel function of the space $\mathscr{H}_S(M)$ at a point w of the complex plane is

$$\frac{M(z)\, I\bar{M}(w) - S(z)\, I\bar{S}(w)}{2\pi(z - \bar{w})}.$$

A factorization theorem can now be stated for Hilbert spaces of entire functions.

THEOREM 2. *If $S(a, b, z)$ is an entire function such that multiplication by $S(a, b, z)$ is an isometric transformation of a space $\mathscr{H}(E(a))$ into a space $\mathscr{H}(E(b))$ and if $E(a, z)$ has no real zeros, then the identity*

$$(A(b, z),\, B(b, z)) = (A(a, z),\, B(a, z))\, M(a, b, z)$$

holds for a unique matrix-valued entire function

$$M(a, b, z) = \begin{pmatrix} A(a, b, z) & B(a, b, z) \\ C(a, b, z) & D(a, b, z) \end{pmatrix}$$

such that a space $\mathcal{H}_{S(a,b)}(M(a,b))$ *exists and such that an isometric transformation of the space onto the orthogonal complement in* $\mathcal{H}(E(b))$ *of the image of* $\mathcal{H}(E(a))$ *is defined by taking*

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

into

$$\sqrt{2}\,[A(a,z)\,F(z) + B(a,z)\,G(z)].$$

Proof of Theorem 2. Define

$$S(a,z) = E(a,z).$$

The hypotheses of the theorem imply that the function

$$S(b,z) = S(a,z)\,S(a,b,z)$$

is associated with the space $\mathcal{H}(E(b))$ in the sense that

$$[F(z)\,S(b,w) - S(b,z)\,F(w)]/(z-w)$$

belongs to the space for every complex number w whenever $F(z)$ belongs to the space.

By the proof of the previous theorem an entire function $\tilde{F}(z)$ can be associated with every element $F(z)$ of $\mathcal{H}(E(b))$ with these properties: The identity for difference-quotients

$$\begin{aligned} &\langle [F(t)\,S(b,\beta) - S(b,t)\,F(\beta)]/(t-\beta),\ G(t)\,S(b,\alpha)\rangle_{\mathcal{H}(E(b))} \\ &\quad - \langle F(t)\,S(b,\beta),\ [G(t)\,S(b,\alpha) - S(b,t)\,G(\alpha)]/(t-\alpha)\rangle_{\mathcal{H}(E(b))} \\ &\quad - (\beta - \bar{\alpha})\langle [F(t)\,S(b,\beta) - S(b,t)\,F(\beta)]/t-\beta), \\ &\qquad\qquad [G(t)\,S(b,\alpha) - S(b,t)\,G(\alpha)]/(t-\alpha)\rangle_{\mathcal{H}(E(b))} \\ &= \pi G(\alpha)^{-}\,\tilde{F}(\beta) - \pi\tilde{G}(\alpha)^{-}\,F(\beta) \end{aligned}$$

holds for all elements $F(z)$ and $G(z)$ of $\mathcal{H}(E(b))$ for all complex numbers α and β. The identity

$$\tilde{G}(z) = [\tilde{F}(z)\,S(b,w) - S(b,z)\,\tilde{F}(w)]/(z-w)$$

holds whenever

$$G(z) = [F(z)\,S(b,w) - S(b,z)\,F(w)]/(z-w)$$

for an element $F(z)$ of $\mathcal{H}(E(b))$ for some complex number w. A continuous linear functional on $\mathcal{H}(E(b))$ is defined by taking $F(z)$ into $F(w)$ for every complex number w.

The transformation can be constructed in a unique way such that the identity

$$\tilde{F}(z) = iF(z)$$

holds for every element $F(z)$ of $\mathcal{H}(E(b))$ in the image of $\mathcal{H}(E(a))$. A space $\mathcal{H}_{S(b)}(M(b))$ exists such that an isometric transformation of $\mathcal{H}(E(b))$ onto $\mathcal{H}_{S(b)}(M(b))$ is defined by taking $F(z)$ into

$$\frac{1}{\sqrt{2}}\begin{pmatrix} F(z) \\ \tilde{F}(z) \end{pmatrix}.$$

Write $E(b, z) = A(b, z) - iB(b, z)$ where $A(b, z)$ and $B(b, z)$ are entire functions which are real for real z. The choice of $M(b, z)$ can be made in a unique way such that

$$M(b, z) = \begin{pmatrix} A(b, z) & B(b, z) \\ C(b, z) & D(b, z) \end{pmatrix}$$

for entire functions $C(b, z)$ and $D(b, z)$ which are real for real z.

A space $\mathcal{H}_{S(a)}(M(a))$ exists,

$$M(a, z) = \begin{pmatrix} A(a, z) & B(a, z) \\ -B(a, z) & A(a, z) \end{pmatrix}.$$

An isometric transformation of the space $\mathcal{H}(E(a))$ onto the space $\mathcal{H}_{S(a)}(M(a))$ is defined by taking $F(z)$ into

$$\frac{1}{\sqrt{2}}\begin{pmatrix} F(z) \\ iF(z) \end{pmatrix}.$$

Multiplication by $S(a, b, z)$ is an isometric transformation of the space $\mathcal{H}_{S(a)}(M(a))$ into the space $\mathcal{H}_{S(b)}(M(b))$.

A unique Hilbert space $\mathcal{H}$ exists, whose elements are pairs

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

of analytic functions, such that multiplication by $M(a, z)$ is an isometric transformation of the space onto the orthogonal complement in $\mathcal{H}_{S(b)}(M(b))$ of the image of $\mathcal{H}_{S(a)}(M(a))$. If

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

belongs to the space, then

$$E(a, z)[F(z) + iG(z)]$$

and

$$E^*(a, z)[F(z) - iG(z)]$$

are entire functions. The function

$$F(z) + iG(z)$$

is analytic outside of the set of zeros of $E(a, z)$. The function

$$F(z) - iG(z)$$

is analytic outside of the set of zeros of $E^*(a, z)$. Since $E(a, z)$ has no real zeros by hypothesis, the functions $F(z)$ and $G(z)$ are analytic across the real axis.

A matrix

$$M(a, b, z) = \begin{pmatrix} A(a, b, z) & B(a, b, z) \\ C(a, b, z) & D(a, b, z) \end{pmatrix}$$

of analytic functions which are real for real z is defined by the factorization

$$M(b, z) = M(a, z)\, M(a, b, z).$$

The entries of the matrix are analytic outside the set of zeros of $E(a, z)\, E^*(a, z)$. The identity

$$A(a, b, z)\, D(a, b, z) - B(a, b, z)\, C(a, b, z) = S(a, b, z)\, S^*(a, b, z)$$

is satisfied. The expression

$$\frac{M(a, b, z)\, I\bar{M}(a, b, w) - S(a, b, z)\, I\bar{S}(a, b, w)}{2\pi(z - \bar{w})}$$

is the reproducing kernel function of the space at any point w which is not a zero of $E(a, z)\, E^*(a, z)$.

These conditions imply that the functions

$$E(a, z)[A(a, b, z) + iC(a, b, z)],$$

$$E^*(a, z)[A(a, b, z) - iC(a, b, z)],$$

$$E(a, z)[D(a, b, z) - iB(a, b, z)],$$

$$E^*(a, z)[D(a, b, z) + iB(a, b, z)]$$

are entire. The functions

$$A(a, b, z) + iC(a, b, z)$$

and

$$D(a, b, z) - iB(a, b, z)$$

are analytic outside of the zeros of $E(a, z)$. The functions

$$A(a, b, z) - iC(a, b, z)$$

and

$$D(a, b, z) + iB(a, b, z)$$

are analytic outside of the set of zeros of $E^*(a, z)$.

The positivity property of the reproducing kernel function of the space $\mathscr{H}$ implies that the number

$$\begin{pmatrix} u \\ v \end{pmatrix}^{-} M(a, b, w) \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \bar{M}(a, b, w) \begin{pmatrix} u \\ v \end{pmatrix}$$

is nonnegative whenever w is in the upper half-plane and u and v are complex numbers such that

$$\begin{pmatrix} u \\ v \end{pmatrix}^{-} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

is nonnegative. This means that the number

$$\frac{\bar{A}(a, b, w)\,\lambda + \bar{C}(a, b, w)}{\bar{B}(a, b, w)\,\lambda + \bar{D}(a, b, w)}$$

is in the upper half-plane whenever λ is in the upper half-plane, and the number

$$\frac{D(a, b, w)\,\lambda - C(a, b, w)}{-B(a, b, w)\,\lambda + A(a, b, w)}$$

is in the upper half-plane whenever λ is in the upper half-plane. It follows that the inequalities

$$|A(a, b, w) - iC(a, b, w)| \leqslant |A(a, b, w) + iC(a, b, w)|$$

and

$$|D(a, b, w) + iB(a, b, w)| \leqslant |D(a, b, w) - iB(a, b, w)|$$

are satisfied when w is in the upper half-plane.

Since the functions

$$A(a, b, z) + iC(a, b, z)$$

and

$$D(a, b, z) - iB(a, b, z)$$

are analytic in the upper half-plane, the functions

$$A(a, b, z) - iC(a, b, z)$$

and

$$D(a, b, z) + iB(a, b, z)$$

are analytic in the upper half-plane. It follows that each of the function $A(a, b, z)$, $B(a, b, z)$, $C(a, b, z)$, and $D(a, b, z)$ is analytic in the upper half-plane. Since each of the functions is analytic across the real axis and has real values on the real axis, each is entire. Since these functions determine the reproducing kernel function of the space $\mathscr{H}$, the elements of the space are entire functions.

The space $\mathscr{H}$ is isometrically equal to a space $\mathscr{H}_{S(a,b)}(M(a, b))$ which is easily seen to have the desired properties.

This completes the proof of the theorem.

The simplest nontrivial example of a factorization is obtained when a space $\mathscr{H}_S(M)$ has dimension one. Such spaces exist when the function $S(z)$ is linear. Assume that $S(z) = z - \lambda$ for a complex number λ. If u and v are complex numbers, not both zero such that

$$\lambda - \bar{\lambda} = u\bar{v} - v\bar{u},$$

then a space $\mathscr{H}_S(M)$ exists with

$$M(z) = \begin{pmatrix} z - \lambda + u\bar{v} & -u\bar{u} \\ v\bar{v} & z - \lambda - v\bar{u} \end{pmatrix}.$$

Note that the identity

$$\frac{M(z)\, I\bar{M}(w) - S(z)\, I\bar{S}(w)}{z - \bar{w}} = \begin{pmatrix} u \\ v \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}^-$$

is satisfied. The space is spanned by the element

$$\begin{pmatrix} u \\ v \end{pmatrix}$$

the square of whose norm is equal to 2π.

Consider any space $\mathcal{H}(E)$ such that the functions $E(z-i)$ and $E^*(z)$ are linearly dependent. Then the zeros of $E(z)$ are symmetrically placed about the line $z+i=\bar{z}$. But polynomial examples show that the zeros of $E(z)$ need not lie on that line. If however $E(z)$ is of Pólya class, a related entire function exists whose zeros lie on the desired line.

THEOREM 3. *If $E(z)$ is an entire function of Pólya class such that $E(z-i)$ and $E^*(z)$ are linearly dependent and if $E(z)$ is not a constant multiple of* $\exp(\pi n z)$ *for an integer n, then the zeros of $E(z+i)-E(z-i)$ are simple and lie in the line $z+i=\bar{z}$.*

Proof of Theorem 3. The conclusion of the theorem is valid when $E(z)$ is a constant multiple of $\exp(hz)$ for a real number h which is not an integral multiple of π. It can therefore be assumed that $E(z)$ is not a constant multiple of $\exp(hz)$ for a real number h. Since $E(z)$ can be multiplied by a constant of absolute value one without changing the hypotheses or the conclusion of the theorem, it can be assumed that $E(z-i)=E^*(z)$.

It will first be shown that the functions $E(z+i)$ and $E^*(z)$ are linearly independent. Argue by contradiction assuming that they are linearly dependent. Then $E(z+i)$ and $E(z-i)$ are linearly dependent, a condition which implies that $E(z)$ has no zeros. But an entire function of Pólya class which has no zeros is the exponential of a quadratic polynomial. The condition that $E(z+i)$ and $E(z-i)$ are linearly dependent implies that the polynomial is linear. The identity $E(z-i)=E^*(z)$ now implies that $E(z)$ is a constant multiple of $\exp(hz)$ for a real number h, which is contrary to construction.

This completes the proof that $E(z+i)$ and $E^*(z)$ are linearly independent. Since $E(z)$ is of Pólya class, the function $E_1(z)=E(z+\frac{1}{2}i)$ is of Pólya class. Since $E(z+i)$ and $E^*(z)$ are linearly independent, the functions $E_1(z)$ and $E_1^*(z)$ are linearly independent. A space $\mathcal{H}(E_1)$ therefore exists. Write

$$E_1(z)=A_1(z)-iB_1(z)$$

where $A_1(z)$ and $B_1(z)$ are entire functions which are real for real z. The inequality

$$|E_1(x-iy)|<|E_1(x+iy)|$$

for $y>0$ now implies that $A_1(z)$ and $B_1(z)$ have only real zeros. Since $E_1(z)$ has no real zeros, the zeros of $A_1(z)$ and $B_1(z)$ are simple and are properly interlaced. Since the function

$$E(z+\tfrac{1}{2}i)-E^*(z-\tfrac{1}{2}i)=-2iB_1(z)$$

has only real simple zeros, the function

$$E(z+i)-E^*(z)=E(z+i)-E(z-i)$$

has only simple zeros, and they lie on the line $z+i=\bar{z}$.

This completes the proof of the theorem.

A positivity condition can now be stated for a space $\mathscr{H}(E)$ which implies that the zeros of $E(z)$ which are symmetric about the line lie on the line.

THEOREM 4. *Let $\mathscr{H}(E)$ be a given space such that $E(z)$ is of Pólya class. Assume that every element of the space is of the form $F(z)+F(z+i)$ for an element $F(z)$ of the space such that $F(z+i)$ belongs to the space and that the real part of*

$$\langle F(t+i), F(t)\rangle_{\mathscr{H}(E)}$$

is nonnegative for every such element $F(z)$. If λ is a zero of $E(z)$ such that $\bar{\lambda}-i$ is also a zero of $E(z)$, then $\lambda+i=\bar{\lambda}$.

Proof of Theorem 4. Consider the adjoint of the transformation which takes $F(z)$ into $F(z+i)$ whenever $F(z)$ and $F(z+i)$ belong to $\mathscr{H}(E)$. If $F(z)$ and $G(z)$ are elements of $\mathscr{H}(E)$ such that the pair $(F(z), G(z))$ belongs to the graph of the adjoint transformation, then the real part of $\langle F(t), G(t)\rangle$ is nonnegative. If the real part of $\langle F(t), G(t)\rangle$ is zero, then the identity

$$\langle F(t), G_0(t)\rangle_{\mathscr{H}(E)}+\langle G(t), F_0(t)\rangle_{\mathscr{H}(E)}=0$$

holds for every pair of elements $F_0(z)$ and $G_0(z)$ of $\mathscr{H}(E)$ such that $(F_0(z), G_0(z))$ belongs to the graph of the adjoint transformation. But an element of the graph of the adjoint transformation is obtained with $F(z)=K(w,z)$ and $G(z)=K(w+i,z)$ for every complex number w, where

$$K(w,z)=\frac{E(z)\bar{E}(w)-E^*(z)E(\bar{w})}{2\pi i(\bar{w}-z)}$$

is the reproducing kernel function of the space $\mathscr{H}(E)$ at w. It follows that the real part of

$$K(w,w+i)=\langle K(w,t), K(w+i,t)\rangle_{\mathscr{H}(E)}$$

is nonnegative. If the real part of $K(w,w+i)$ is zero, the identity

$$K(w,z+i)+K(w+i,z)=0$$

holds for all complex numbers z and w.

Argue by contradiction assuming that $E(z)$ admits a zero λ such that $\bar{\lambda} - i$ is a zero of $E(z)$ distinct from λ. Since the function

$$K(\lambda, z) = \frac{E^*(z)\, E(\bar{\lambda})}{2\pi i(z - \bar{\lambda})}$$

vanishes at $\lambda + i$ and since the function

$$K(\lambda + i, z) = \frac{E(z)\, \bar{E}(\lambda + i)}{2\pi i(\bar{\lambda} - i - z)}$$

vanishes at λ, the identity

$$E^*(z + i)\, E(\bar{\lambda}) = E(z)\, \bar{E}(\lambda + i)$$

is satisfied. Since λ and $\bar{\lambda} - i$ are unequal zeros of $E(z)$, the coefficients $E(\bar{\lambda})$ and $\bar{E}(\lambda + i)$ are nonzero. The identity states that $E(z - i)$ and $E^*(z)$ are linearly dependent. It follows that the identity

$$K(w, z + i) + K(w + i, z)$$
$$= \frac{[E(z + i) - E(z - i)]\, \bar{E}(w) + E(z)[\bar{E}(w + i) - \bar{E}(w - i)]}{2\pi i(\bar{w} - i - z)}$$

is satisfied.

Since the expression is nonnegative when z and w are equal, the ratio

$$[E(z + i) - E(z - i)]/E(z)$$

has nonnegative real part in the half-plane $i\bar{z} - iz > -1$. The numerator cannot vanish identically because that condition implies that $E(z)$ has no zeros. Since $E(z)$ is of Pólya class, the numerator has only simple zeros, which lie on the line $z + i = \bar{z}$. It follows that the denominator has only simple zeros, which lie on the same line.

This completes the proof of the theorem.

A generalization of the Riemann hypothesis has been verified under a positivity condition. Examples will now be given of spaces which satisfy the condition. A fundamental example is the Hardy space $\mathscr{F}(1)$. A theorem of Paley and Wiener states that the elements of the space are Fourier transforms

$$F(z) = \int_0^\infty f(t)\, e^{2\pi i t z}\, dt$$

of square integrable functions $f(x)$ of real x, which vanish for negative arguments,

$$\|F\|^2_{\mathscr{F}(1)} = \int_0^\infty |f(t)|^2\, dt.$$

A consequence of the theorem is that a nonnegative and contractive transformation in the space $\mathcal{F}(1)$ is defined by taking $F(z)$ into $F(z+ik)$ for every positive number k.

The theorem has a consequence which can be stated in a more flexible form. Assume that $W_+(z)$ and $W_-(z)$ are weight functions such that

$$W_+(z) = W_-(z+ik)$$

for a positive number k. Then a contractive transformation of the space $\mathcal{F}(W_-)$ into the space $\mathcal{F}(W_+)$ is defined by taking $F(z)$ into $F(z+ik)$. The adjoint transformation takes $F(z)$ into

$$F(z+ik)\, W_-(z)/W_+(z+ik).$$

The desired positivity condition in the space $\mathcal{F}(W)$ is equivalent to properties of the defining weight function $W(z)$.

THEOREM 5. *Let $W(z)$ be a given weight function. Every element of the space $\mathcal{F}(W)$ is of the form $F(z)+F(z+i)$ for an element $F(z)$ of the space such that $F(z+i)$ belongs to the space and the real part of*

$$\langle F(t+i), F(t)\rangle_{\mathcal{F}(W)}$$

is nonnegative for every such element $F(z)$ if, and only if, $W(z)$ has an analytic extension to the half-plane $i\bar{z}-iz > -1$ such that the real part of $W(z)/W(z+i)$ is nonnegative in the half-plane.

Proof of Theorem 5. Assume that every element of the space is of the form $F(z)+F(z+i)$ for an element $F(z)$ of the space such that $F(z+i)$ belongs to the space and that the real part of

$$\langle F(t+i), F(t)\rangle_{\mathcal{F}(W)}$$

is always nonnegative. Consider the adjoint of the transformation which takes $F(z)$ into $F(z+i)$ whenever $F(z)$ and $F(z+i)$ belong to the space. This is a transformation with the property that the real part of $\langle F(t), G(t)\rangle_{\mathcal{F}(W)}$ is nonnegative whenever $(F(z), G(z))$ belongs to its graph. The pair $(F(z), G(z))$ belongs to the graph when $F(z) = K(w, z)$ and $G(z) = K(w+i, z)$ for a point w in the upper half-plane, where

$$K(w, z) = \frac{W(z)\,\bar{W}(w)}{2\pi i(\bar{w}-z)}$$

is the reproducing kernel function of the space $\mathcal{F}(W)$.

Positive-definiteness of the expression

$$K(w+i, z) + K(w, z+i)$$

can now be obtained. If $w_1, \ldots, w_r$ are points in the upper half-plane and if $c_1, \ldots, c_r$ are corresponding complex numbers, then the expression

$$\sum c_\beta^- [K(w_\alpha + i, w_\beta) + K(w_\alpha, w_\beta + i)]\, c_\alpha$$

is nonnegative. This is true because it is equal to

$$\left\langle \sum K(w_\alpha + i, t)\, c_\alpha, \sum K(w_\beta, t)\, c_\beta \right\rangle_{\mathcal{F}(W)}$$
$$+ \left\langle \sum K(w_\alpha, t)\, c_\alpha, \sum K(w_\beta + i, t)\, c_\beta \right\rangle_{\mathcal{F}(W)}$$

which is nonnegative because an element $(F(z), G(z))$ of the graph of the adjoint transformation is given by

$$F(z) = \sum K(w_\alpha, z)\, c_\alpha$$

and

$$G(z) = \sum K(w_\beta + i, z)\, c_\beta.$$

The positive-definiteness inequality in the case of one point implies that $W(z)/W(z+i)$ has nonnegative real part in the upper half-plane. In proving analyticity of $W(z)$ in the half-plane $i\bar{z} - iz > -1$ it can be assumed that $W(z+i)$ is not a constant multiple of $W(z)$ since these functions are otherwise analytic in the complex plane and the desired inequality is true trivially. Since the real part of $W(z)/W(z+i)$ is then positive in the upper half-plane, a function $B(z)$ which is analytic and bounded by one in the half-plane is defined by

$$B(z) = \frac{W(z) - W(z+i)}{W(z) + W(z+i)}.$$

The positive-definiteness of the expression

$$K(w+i, z) + K(w, z+i)$$

in the upper half-plane implies positive-definiteness of the expression

$$\frac{1 - B(z)\, \bar{B}(w)}{2\pi i(\bar{w} - i - z)}$$

in the half-plane.

Use is now made of the Hilbert space $\mathcal{H}$ whose elements are analytic functions in the half-plane $i\bar{z} - iz > -1$ and which has the expression

$$\frac{1}{2\pi i(\bar{w} - i - z)}$$

as its reproducing kernel function at a point w of the half-plane. An isometric transformation of the Hardy space $\mathcal{F}(1)$ onto the space $\mathcal{H}$ is defined by taking $F(z)$ into $F(z + \frac{1}{2}i)$.

The positive-definiteness inequality implies that a contractive transformation of the space $\mathcal{H}$ into itself exists which takes

$$\frac{1}{2\pi i(\bar{w} - i - z)}$$

into

$$\frac{\bar{B}(w)}{2\pi i(\bar{w} - i - z)}$$

for every point w in the half-plane. Since the adjoint transformation coincides with multiplication by $B(z)$, the function $B(z)$ has an analytic extension which is bounded by one in the half-plane $i\bar{z} - iz > -1$. These properties of $B(z)$ imply that $W(z)$ has an analytic extension to the half-plane such that the real part of $W(z)/W(z + i)$ is nonnegative in the half-plane.

Assume that the weight function $W(z)$ has an analytic extension to the half-plane $i\bar{z} - iz > -1$ such that the real part of $W(z)/W(z + i)$ is nonnegative in the half-plane. Then the expression

$$\frac{W(z)/W(z + i) + \bar{W}(w)/\bar{W}(w + i)}{2\pi i(\bar{w} - i - z)}$$

is a positive-definite function of z and w in the half-plane by the Poisson representation of functions which are analytic and have nonnegative real part in a half-plane. An equivalent assertion is that the expression

$$\frac{W(z)\,\bar{W}(w + i) + W(z + i)\,\bar{W}(w)}{2\pi i(\bar{w} - i - z)}$$

is a positive-definite function of z and w in the half-plane.

It has been shown that the expression

$$K(w, z + i) + K(w + i, z)$$

is a positive-definite function of z and w in the upper half-plane. Consider the transformation which takes $F(z)$ into $F(z+i)$ whenever $F(z)$ and $F(z+i)$ belong to the space. Then the adjoint relation has the property that the real part of

$$\langle F(t), G(t)\rangle_{\mathcal{F}(W)}$$

is nonnegative whenever $(F(z), G(z))$ belongs to its graph. The set of elements of the space which are of the form $F(z)+G(z)$ with $(F(z), G(z))$ in the graph of the adjoint relation is closed. Fundamental examples of such elements are obtained when $F(z)=K(w, z)$ and $G(z)=K(w+i, z)$ with w in the upper half-plane. The orthogonal complement of the set of such elements is the set of elements $F(z)$ of the space such that $F(z)+F(z+i)=0$.

In order to complete the proof of the theorem it must be shown that no nonzero element $F(z)$ of the space exists such that $F(z)+F(z+i)=0$. It then follows that every element of the space is of the form $F(z)+G(z)$ with $(F(z), G(z))$ in the graph of the adjoint relation. And it follows that every element of the space is of the form $F(z)+F(z+i)$ for an element $F(z)$ of the space such that $F(z+i)$ belongs to the space, and that the real part of

$$\langle F(t+i), F(t)\rangle_{\mathcal{F}(W)}$$

is nonnegative for every such element $F(z)$.

Consider any element $F(z)$ of the space such that $F(z)+F(z+i)=0$. Then every zero of $F(z)$ is repeated periodically with period i. Since $F(z)/W(z)$ is of bounded type in the upper half-plane, its zeros z_n in the half-plane are close to the real axis in the sense that the sum

$$\sum (i/z_n - i/\bar{z}_n)$$

is finite if does not vanish identically. Since the sum is infinite when there is a zero which is repeated periodically with period i, the existence of a zero of $F(z)$ implies that $F(z)$ vanishes identically. A consequence is that the set of elements $F(z)$ of the space such that $F(z)+F(z+i)=0$ has dimension zero or one.

If c is a positive number, consider the transformation which takes $F(z)$ into $cF(z+i)$ whenever $F(z)$ and $F(z+i)$ belong to the space. Then the real part of

$$\langle F(t), G(t)\rangle_{\mathcal{F}(W)}$$

is nonnegative whenever $(F(z), G(z))$ belongs to the graph of the adjoint relation. The set of elements of the space of the form $F(z)+G(z)$ with

$(F(z), G(z))$ in the graph of the adjoint relation is a closed vector subspace whose orthogonal complement is the set of elements $F(z)$ of this space such that $F(z) + cF(z + i) = 0$. The set of such elements $F(z)$ has dimension zero or one. The dimension of the set of such elements $F(z)$ is equal to the dimension of the set elements of $F(z)$ of the space such that $F(z) + F(z + i) = 0$.

This information is used when

$$c = \exp(-h)$$

for a positive number h. Since the function $\exp(ihz)$ is analytic and bounded by one in the upper half-plane and since it has absolute value one on the real axis, multiplication by $\exp(ihz)$ is an isometric transformation of the space into itself. If $G(z)$ is an element of the space such that $G(z) + cG(z + i) = 0$, then

$$F(z) = \exp(ihz)\, G(z)$$

is an element of this space such that $F(z) + F(z + i) = 0$.

Consider any element $F(z)$ of the space such that $F(z) + F(z + i) = 0$. It has been shown that

$$\exp(-ihz)\, F(z)$$

belongs to the space for every positive number h and has the same norm as $F(z)$. The condition implies that $F(z)$ vanishes identically.

This completes the proof of the theorem.

A weight function which appears on the theory of the Hankel transformation of order ν is

$$W(z) = \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - iz).$$

The space $\mathscr{F}(W)$ has the desired positivity properties when ν is nonnegative.

THEOREM 6. *If*

$$W(z) = \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - iz),$$

where $\nu \geqslant 0$ *and if* $0 < k \leqslant 1$, *then the function*

$$W(z)/W(z + ik)$$

has positive real part in the half-plane $i\bar{z} - iz > -k$.

Proof of Theorem 6. It is sufficient by continuity to give a proof when ν is positive. The desired conclusion follows from the recurrence relation for the gamma-function when $k = 1$. Otherwise write $\nu = 2h - 1$ where h is positive and apply the Euler representation

$$\int_0^1 t^{h-iz-1}(1-t)^{k-1}\,dt = \frac{\Gamma(h-iz)\,\Gamma(k)}{\Gamma(h+k-iz)},$$

where $\Gamma(k)$ is a positive number. The representation can be written

$$\int_0^\infty f(t)\,e^{itz}\,dt = \frac{\Gamma(h-iz)\,\Gamma(k)}{\Gamma(h+k-iz)},$$

where

$$f(x) = e^{-hx}(1-e^{-x})^{k-1}$$

is a function with nonnegative values whose first derivative has nonpositive values and whose second derivatives has nonnegative values. Iterated integration by parts gives the representation

$$\int_0^\infty \frac{1+itz-e^{itz}}{z^2}\,f''(t)\,dt = \frac{\Gamma(h-iz)\,\Gamma(k)}{\Gamma(h+k-iz)}$$

where the expression

$$\frac{1+itz-e^{itz}}{z^2}$$

has positive real part in the upper half-plane for every positive number t. This property of the function is established using the Phragmén–Lindelöf principle since the real part of the function has nonnegative values on the real axis.

This completes the proof of the theorem.

Related Hilbert spaces of entire functions appear in the theory of the Hankel transformation of order ν. These spaces are defined when $\nu > -1$. The set of entire functions $F(z)$ such that $c^{iz}F(z)$ and $c^{iz}F^*(z)$ belong to the space $\mathscr{F}(W)$ is a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) when considered with the unique scalar product such that multiplication by c^{iz} is an isometric transformation of the space into the space $\mathscr{F}(W)$.

A nontrivial property of the space of entire functions is that it contains a nonzero element. This information results from properties of the Hankel transformation of order ν which were observed in 1880 by N. Sonine. For this reason the space is called the Sonine space of order ν and parameter c. The space is a space $\mathscr{H}(E)$ for an entire function $E(z)$ of Pólya class whose computation is an interesting problem. A solution of the problem was obtained by the author [1] in the case $\nu = 0$ and by James and Virginia Rovnyak [5] in the case $\nu = 1$.

The Sonine spaces of order ν have a completeness property [2]. Let $E(c, z)$ be an entire function defining the space with parameter c. The space $\mathcal{H}(E(a))$ is contained isometrically in the space $\mathcal{H}(E(b))$ when $a \leqslant b$. Consider any Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) and which is contained isometrically in the space $\mathcal{H}(E(b))$ for some parameter b. If $F(z)/(z-w)$ belongs to the space whenever $F(z)$ belongs to the space and has a zero w, and if the space contains a nonzero element, then the space is isometrically equal to the space $\mathcal{H}(E(a))$ for some parameter a, $a \leqslant b$.

The Sonine spaces of nonnegative order satisfy the desired positivity condition.

THEOREM 7. *Assume that $\mathcal{H}(E)$ is the Sonine space of nonnegative order ν and parameter c. If k is a given number, $0 < k \leqslant 1$, then every element of the space is of the form $F(z) + F(z+ik)$ for an element $F(z)$ of the space such that $F(z+ik)$ belongs to the spaces and the real part of*

$$\langle F(t+ik), F(t) \rangle_{\mathcal{H}(E)}$$

is nonnegative for every such element $F(z)$.

Proof of Theorem 7. Define

$$W(z) = \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - iz).$$

It has been shown that $W(z)/W(z+ik)$ has positive real part in the half-plane $i\bar{z} - iz > -k$. It follows that every element of the space $\mathcal{F}(W)$ is of the form $F(z) + F(z+ik)$ for an element $F(z)$ of the space such that $F(z+ik)$ belongs to the space and that the real part of

$$\langle F(t+ik), F(t) \rangle_{\mathcal{F}(W)}$$

is nonnegative for every such element $F(z)$. It also follows that every element of the space is of the form $F(z) + c^k F(z+ik)$ for an element $F(z)$ of the space such that $F(z+ik)$ belongs to the space.

Since multiplication by c^{iz} takes the space $\mathcal{H}(E)$ into the space $\mathcal{F}(W)$, every element of the space is of the form $F(z) + F(z+ik)$ for a function $F(z)$ such that $c^{iz}F(z)$ and $c^{iz}F(z+ik)$ belong to the space $\mathcal{F}(W)$. These conditions imply that $F(z)$ is an entire function. The theorem is proved by showing that $F(z)$ belongs to the space $\mathcal{H}(E)$. The nonnegativity of the real part of the scalar product

$$\langle c^{it}F(t+ik), c^{it}F(t) \rangle_{\mathcal{F}(W)}$$

then implies the nonnegativity of the real part of the scalar product

$$\langle F(t+ik), F(t) \rangle_{\mathscr{H}(E)}.$$

Since the space $\mathscr{H}(E)$ satisfies the axiom (H3), the function $F^*(z) + F^*(z-ik)$ belongs to the space. It has been seen that

$$F^*(z) + F^*(z-ik) = G(z) + G(z+ik)$$

for an entire function $G(z)$ such that $c^{iz}G(z)$ and $c^{iz}G(z+ik)$ belong to the space $\mathscr{F}(W)$. It will be shown that

$$F^*(z) = G(z+ik)$$

and that

$$F^*(z-ik) = G(z).$$

The desired conclusion that $F(z)$ belongs to the space $\mathscr{H}(E)$ then follows from the definition of the space.

It remains to show that the entire function

$$F^*(z) - G(z+ik) = G(z) - F^*(z-ik)$$

vanishes identically. Since the function changes sign when z is replaced by $z-ik$, it is sufficient to show that the function is a constant. This conclusion will be obtained using Liouville's theorem. It must be shown that the function is bounded in the complex plane. Since the modulus of the function is periodic of period ik, it is sufficient to show that the function is bounded in the strip

$$-2k < i\bar{z} - iz < 0.$$

Since $F_1(z) = c^{iz}F(z)$ belongs to the space $\mathscr{F}(W)$, the inequality

$$|F(z)|^2 \leqslant \|F_1\|^2_{\mathscr{F}(W)} \frac{c^{i\bar{z}-iz}W(z)\,\bar{W}(z)}{2\pi(i\bar{z}-iz)}$$

holds when z is in the upper half-plane. It follows that the inequality

$$|F^*(z)|^2 \leqslant \|F_1\|^2_{\mathscr{F}(W)} \frac{c^{iz-i\bar{z}}W^*(z)\,W(\bar{z})}{2\pi(iz-i\bar{z})}$$

holds when z is in the lower half-plane.

Since $G_1(z) = c^{iz}G(z)$ belongs to the space $\mathscr{F}(W)$, the inequality

$$|G(z)|^2 \leqslant \|G_1\|^2_{\mathscr{F}(W)} \frac{c^{i\bar{z}-iz}W(z)\,\bar{W}(z)}{2\pi(i\bar{z}-iz)}$$

holds when z is in the upper half-plane. It follows that the inequality

$$|G(z+ik)|^2 \leqslant \|G_1\|^2_{\mathscr{F}(W)} \frac{e^{i\bar{z}+2k-iz} W(z+ik)\, \bar{W}(z+ik)}{2\pi(i\bar{z}+2k-iz)}$$

holds when z is in the half-plane $i\bar{z}-iz > -2k$.

The inequality

$$\begin{aligned}|F^*(z) - G(z-ik)|^2 &\leqslant (\|F_1\|^2_{\mathscr{F}(W)} + \|G_1\|^2_{\mathscr{F}(W)}) \\ &\times \left[\frac{e^{iz-i\bar{z}} W^*(z)\, W(\bar{z})}{2\pi(iz-i\bar{z})} + \frac{e^{i\bar{z}+2k-iz} W(z+ik)\, \bar{W}(z+ik)}{2\pi(i\bar{z}+2k-iz)}\right]\end{aligned}$$

holds when z is in the strip

$$-2k < i\bar{z} - iz < 0.$$

Since the reciprocal of the gamma-function is an entire function of Pólya class which is real for real z, the estimate implies that the function

$$|F^*(z) - G(z-ik)|^2 \min(iz-i\bar{z}, i\bar{z}+2k-iz)$$

is bounded in the strip. Since the modulus of the function $F^*(z) - G(z-ik)$ is periodic of period ik and since the logarithm of its modulus is subharmonic, the function is bounded in the complex plane.

This completes the proof of the theorem.

A computation of adjoints will now be made. If v is a given positive number, define

$$W_0(z) = \Gamma(\tfrac{1}{2}v - iz)$$

and

$$W_1(z) = \Gamma(\tfrac{1}{2}v + \tfrac{1}{2} - iz).$$

A contractive transformation of the space $\mathscr{F}(W_0)$ into the space $\mathscr{F}(W_1)$ is defined by taking $F(z)$ into $F(z+\frac{1}{2}i)$. The adjoint is a contractive transformation of the space $\mathscr{F}(W_1)$ into the space $\mathscr{F}(W_0)$ which takes $F(z)$ into $F(z+\frac{1}{2}i)/(\frac{1}{2}v - iz)$. A related computation of adjoints holds for Sonine spaces.

THEOREM 8. *If v and c are given positive numbers, multiplication by $\frac{1}{2}v - iz$ is an isometric transformation of the Sonine space $\mathscr{H}(E_0)$ of order $v-1$ and parameter c onto the set of elements of the Sonine space $\mathscr{H}(E_2)$*

of order $\nu+2$ and parameter c which vanish at $-\frac{1}{2}i\nu$. The choice of $E_0(z)$ can be made equal to a constant multiple of the reproducing kernel function of the space $\mathscr{H}(E_2)$ at $\frac{1}{2}i\nu$. If $E_0(z)$ is chosen with a positive value at $\frac{1}{2}i\nu$, then $E_0(z)$ has positive values on the upper half of the imaginary axis and satisfies the identity

$$E_0^*(z) = E_0(-z).$$

The adjoint of the transformation of the space $\mathscr{H}(E_0)$ into the Sonine space $\mathscr{H}(E_1)$ of order ν and parameter c which takes $F(z)$ into $F(z+\frac{1}{2}i)$ is the transformation of the space $\mathscr{H}(E_1)$ into the space $\mathscr{H}(E_0)$ which takes $F(z)$ into

$$[F(z+\tfrac{1}{2}i) - F(\tfrac{1}{2}i - \tfrac{1}{2}i\nu)\, E_0^*(z)/E_0^*(-\tfrac{1}{2}i\nu)]/(\tfrac{1}{2}\nu - iz).$$

Proof of Theorem 8. Define

$$W_0(z) = \Gamma(\tfrac{1}{2}\nu - iz)$$

and

$$W_1(z) = \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - iz)$$

and

$$W_2(z) = \Gamma(\tfrac{1}{2}\nu + 1 - iz).$$

Then multiplication by $\frac{1}{2}\nu - iz$ is an isometric transformation of the space $\mathscr{F}(W_0)$ onto the space $\mathscr{F}(W_2)$. The adjoint of the transformation of the space $\mathscr{F}(W_0)$ into the space $\mathscr{F}(W_1)$ which takes $F(z)$ into $F(z+\frac{1}{2}i)$ is the composition of the transformation of the space $\mathscr{F}(W_1)$ into the space $\mathscr{F}(W_2)$ which takes $F(z)$ into $F(z+\frac{1}{2}i)$ and the inverse of multiplication by $\frac{1}{2}\nu - iz$ as a transformation of the space $\mathscr{F}(W_0)$ onto the space $\mathscr{F}(W_2)$.

It is immediate from the definition of the Sonine spaces that multiplication by $\frac{1}{2}\nu - iz$ is an isometry of the space $\mathscr{H}(E_0)$ onto the set of elements of the space $\mathscr{H}(E_2)$ which vanishes at $-\frac{1}{2}i\nu$. The choice of $E_2(z)$ can be made so as to vanish at $-\frac{1}{2}i\nu$ and so as to have a positive value at $\frac{1}{2}i\nu$. The choice of $E_0(z)$ can be made so that

$$E_0(z) = E_2(z)/(\tfrac{1}{2}\nu - iz).$$

Then $E_0(z)$ is a constant multiple of the reproducing kernel function of the space $\mathscr{H}(E_2)$ at the point $\frac{1}{2}i\nu$, and it has a positive value at $\frac{1}{2}i\nu$.

Every Sonine space is symmetric about the origin in the sense that an isometry of the space onto itself is defined by taking $F(z)$ into $F(-z)$. This condition implies that

$$E_2^*(z) = E_2(-z)$$

and that

$$E_0^*(z) = E_0(-z).$$

Since $E_0(z)$ has real values on the imaginary axis, since it has no zeros in the upper half-plane, and since it has a positive value at $\frac{1}{2}iv$, it has positive values on the upper half of the imaginary axis.

Note that $E_0^*(z)$ is a constant multiple of the reproducing kernel function of the space $\mathscr{H}(E_2)$ at the point $-\frac{1}{2}iv$. The orthogonal projection of the space $\mathscr{H}(E_2)$ onto the subspace of functions which vanish at $-\frac{1}{2}iv$ takes $F(z)$ into

$$F(z) - F(-\tfrac{1}{2}iv)\, E_0^*(z)/E_0^*(-\tfrac{1}{2}iv).$$

The computation of adjoints is now made using the fact that multiplication by e^{iz} is an isometric transformation of the space $\mathscr{H}(E_0)$ into the space $\mathscr{F}(W_0)$, of the space $\mathscr{H}(E_1)$ into the space $\mathscr{F}(W_1)$, and of the space $\mathscr{H}(E_2)$ into the space $\mathscr{F}(W_2)$. The adjoint of the transformation of the space $\mathscr{H}(E_0)$ into the space $\mathscr{H}(E_1)$ which takes $F(z)$ into $F(z + \frac{1}{2}i)$ is obtained by taking an element $F(z)$ of the space $\mathscr{H}(E_1)$ into the element $F(z + \frac{1}{2}i)$ of the space $\mathscr{H}(E_2)$, projecting it into the element

$$F(z + \tfrac{1}{2}i) - F(\tfrac{1}{2}i - \tfrac{1}{2}iv)\, E_0^*(z)/E_0^*(-\tfrac{1}{2}iv)$$

of the space $\mathscr{H}(E_2)$ which vanishes at $-\frac{1}{2}iv$, and dividing by $\frac{1}{2}v - iz$ to obtain the desired element of the space $\mathscr{H}(E_0)$.

This completes the statement of the theorem.

The defining function $E(z)$ of the Sonine space of order v and parameter c can be chosen without zeros in the half-plane $i\bar{z} - iz > -1$ when v is positive.

THEOREM 9. *Let v and c be given positive numbers. If the defining function $E_0(z)$ of the Sonine space of order $v - 1$ and parameter c is chosen to be a constant multiple of the reproducing kernel function of the Sonine space of order $v + 1$ and parameter c at the point $\frac{1}{2}iv$, then the defining function $E(z)$ of the Sonine space of order v and parameter c can be chosen to be a constant multiple of $E_0(z + \frac{1}{2}i)$.*

Proof of Theorem 9. Let $\mathscr{H}$ denote the Sonine space of order v and parameter c. It will first be shown that the range of the transformation of the space $\mathscr{H}(E_0)$ into the space $\mathscr{H}$ which takes $F(z)$ into $F(z + \frac{1}{2}i)$ is dense in the space $\mathscr{H}$. Since the adjoint transformation has a kernel of dimension zero or one, the closure of the range of the transformation has deficiency zero or one in $\mathscr{H}$. But the closure of the range is easily seen to be a Hilbert space of entire function which satisfies the axioms (H1), (H2), and (H3) in the scalar product of $\mathscr{H}$. The closure of the range contains $F(z)/(z-w)$ whenever it contains an element $F(z)$ with a zero w. Since the closure of the range is itself a Sonine space of order ν and some parameter, it must be the Sonine space of order ν and parameter c. This completes the verification that the transformation has a dense range in the space $\mathscr{H}$.

The proof makes use of the fact that the adjoint of the transformation of the space $\mathscr{H}(E_0)$ into itself which takes $F(z)$ into

$$[F(z) - E_0(z)\, F(w)/E_0(w)]/(z-w)$$

is the transformation which takes $G(z)$ into

$$[G(z) - E_0^*(z)\, G(\bar{w})/E_0^*(\bar{w})]/(z-\bar{w})$$

whenever $E_0(w)$ is not zero. An equivalent assertion is that the identity

$$\begin{aligned}\langle [F(t)\, E_0(w) - E_0(t)\, F(w)]/(t-w),\, G(t)\, \bar{E}_0(w)\rangle_{\mathscr{H}(E_0)}\\ = \langle F(t)\, E_0(w),\, [G(t)\, \bar{E}_0(w) - E_0^*(t)\, G(\bar{w})]/(t-\bar{w})\rangle_{\mathscr{H}(E_0)}\end{aligned}$$

holds for all elements $F(z)$ and $G(z)$ of the space $\mathscr{H}(E_0)$. Let

$$K_0(w, z) = \frac{E_0(z)\, \bar{E}_0(w) - E_0^*(z)\, E_0(\bar{w})}{2\pi i(\bar{w} - z)}$$

be the reproducing kernel function of the space $\mathscr{H}(E_0)$. Since the finite linear combinations of reproducing kernel functions are dense in the space $\mathscr{H}(E_0)$, it is sufficient to verify the identity when

$$F(z) = K_0(\alpha, z)$$

and

$$G(z) = K_0(\beta, z)$$

for some complex numbers α and β. By the defining property of reproducing kernel functions the desired identity reads

$$\begin{aligned}[K_0(\alpha, \beta)\, E_0(w) - E_0(\beta)\, K_0(\alpha, \omega)]/(\beta - w)\\ = [K_0(\alpha, \beta)\, E_0(w) - E_0(\bar{\alpha})\, K_0(\bar{w}, \beta)]/(\bar{\alpha} - w).\end{aligned}$$

The identity is a consequence of the form of the reproducing kernel function.

The entire function

$$S(z) = E_0(z + \tfrac{1}{2}i)$$

has the property that

$$[F(z)\,S(w) - S(z)\,F(w)]/(z-w)$$

belongs to the space $\mathscr{H}$ for every complex number w whenever $F(z)$ belongs to the space. It will be shown that the adjoint of the transformation of the space $\mathscr{H}$ into itself which takes $F(z)$ into

$$[F(z) - S(z)\,F(w)/S(w)]/(z-w)$$

is the transformation which takes $G(z)$ into

$$[G(z) - S^*(z)\,G(\bar{w})/S^*(\bar{w})]/(z-\bar{w})$$

whenever $S(w)$ is not zero.

It needs to be shown that the identity

$$\begin{aligned}\langle [F(t) - S(t)\,F(w)/S(w)]/(t-w), G(t)\rangle_{\mathscr{H}} \\ = \langle F(t), [G(t) - S^*(t)\,G(\bar{w})/S^*(\bar{w})]/(t-\bar{w})\rangle_{\mathscr{H}}\end{aligned}$$

holds for all elements $F(z)$ and $G(z)$ of the space $\mathscr{H}$. By continuity it can be assumed that

$$F(z) = F_0(z + \tfrac{1}{2}i)$$

for an element $F_0(z)$ of the space $\mathscr{H}(E_0)$. The scalar products on each side can be replaced by scalar products in the space $\mathscr{H}(E_0)$ by the computation of the adjoint of the shift transformation of the space $\mathscr{H}(E_0)$ into the space $\mathscr{H}$. Note that

$$S^*(z + \tfrac{1}{2}i) = E_0^*(z).$$

Define

$$G_0(z) = [G(z + \tfrac{1}{2}i) - E_0^*(z)\,G(\tfrac{1}{2}i - \tfrac{1}{2}iv)/E_0^*(-\tfrac{1}{2}iv)]/(\tfrac{1}{2}v - iz)$$

and

$$H(z) = [G(z) - S^*(z)\,G(\bar{w})/S^*(\bar{w})]/(z-\bar{w}).$$

The desired conclusion follows from the identity

$$\begin{aligned}[H(z + \tfrac{1}{2}i) - E_0^*(z)\,H(\tfrac{1}{2}i - \tfrac{1}{2}iv)/E_0^*(-\tfrac{1}{2}iv)]/(\tfrac{1}{2}v - iz) \\ = [G_0(z) - E_0^*(z)\,G_0(\bar{w} - \tfrac{1}{2}i)/E_0^*(\bar{w} - \tfrac{1}{2}i)]/(z + \tfrac{1}{2}i - \bar{w})\end{aligned}$$

and the computation of adjoints in the space $\mathscr{H}(E_0)$.

Since the space $\mathscr{H}$ is isometrically equal to a space $\mathscr{H}(E)$, its reproducing kernel function is of the form

$$K(w, z) = \frac{E(z)\,\bar{E}(w) - E^*(z)\,E(\bar{w})}{2\pi i(\bar{w} - z)}.$$

When

$$F(z) = K(\alpha, z)$$

and

$$G(z) = K(\beta, z)$$

for complex numbers α and β, the identity

$$\begin{aligned}&\langle [F(t)\,S(w) - S(t)\,F(w)]/(t - w),\, G(t)\,\bar{S}(w)\rangle_{\mathscr{H}(E)}\\ &\quad = \langle F(t)\,S(w),\, [G(t)\,\bar{S}(w) - S^*(t)\,G(\bar{w})]/(t - \bar{w})\rangle_{\mathscr{H}(E)}\end{aligned}$$

reads

$$\begin{aligned}&[K(\alpha, \beta)\,S(w) - S(\beta)\,K(\alpha, w)]/(\beta - w)\\ &\quad = [K(\alpha, \beta)\,S(w) - S(\bar{\alpha})\,K(\bar{w}, \beta)]/(\bar{\alpha} - w).\end{aligned}$$

The identity implies that $S(z)$ is a linear combination of $E(z)$ and $E^*(z)$.

One of three conditions must now hold. Either $S(z)$ and $S^*(z)$ are linearly dependent, or the inequality

$$|S(x - iy)| < |S(x + iy)|$$

holds for $y > 0$, or the reverse inequality holds for $y > 0$. Argue by contradiction assuming that the inequality

$$|S(x + iy)| \leqslant |S(x - iy)|$$

holds for $y > 0$.

The inequality implies that $E_0(z)$ and $S(z)$ have no zeros since

$$S(z) = E_0(z + \tfrac{1}{2}i),$$

where

$$|E_0(x - iy)| < |E_0(x + iy)|$$

for $y > 0$. The entire functions

$$E_0^*(z)/E_0(z)$$

and

$$S(z)/S^*(z)$$

have no zeros, are bounded by one in the upper half-plane, and have absolute value one on the real axis. It follows that they are exponentials of linear functions. The condition implies that

$$E_0(z+i)/E_0(z) = E_0^*(z)/E_0(z)\, S(z+\tfrac{1}{2}i)/S^*(z+\tfrac{1}{2}i)$$

is bounded by one in the upper half-plane and is the exponential of a linear function.

Consider the weight function

$$W_0(z) = \Gamma(\tfrac{1}{2}\nu - iz).$$

Then the analytic functions

$$E_0(z)/W_0(z)$$

and

$$E_0(z+i)/W_0(z+i)$$

are of bounded type and of equal mean type in the upper half-plane. Since the linear function

$$W_0(z+i)/W_0(z) = \tfrac{1}{2}\nu - iz$$

is of bounded type and of zero mean type in the upper half-plane, the entire function

$$E_0(z+i)/E_0(z)$$

is of bounded type and of zero mean type in the upper half-plane. Since it is the exponential of a linear function, it is a constant. A contradiction is now obtained since the condition implies that

$$E_0^*(z)/E_0(z)$$

is a constant.

A space $\mathscr{H}(S)$ has now been shown to exist. Since the reproducing kernel functions

$$\frac{S(z)\,\bar{S}(w) - S^*(z)\,S(\bar{w})}{2\pi i(\bar{w}-z)}$$

of the space $\mathcal{H}(S)$ and

$$\frac{E(z)\,\bar{E}(w) - E^*(z)\,E(\bar{w})}{2\pi i(\bar{w} - z)}$$

of the space $\mathcal{H}(E)$ are linearly dependent, this choice of $E(z)$ can be made so that $E(z)$ and $S(z)$ are linearly dependent.

This completes the proof of the theorem.

The defining function of a Sonine space satisfies a recurrence relation which is a generalization of the recurrence relation for the gamma-function. The recurrence relation is a consequence of a pair of contiguous relations for the defining function of the Sonine space of positive order ν and parameter c with the defining function of the Sonine space of order $\nu - 1$ and parameter c.

THEOREM 10. *Let ν and c be positive numbers. Assume that the defining function $E_0(z)$ of the Sonine space of order $\nu - 1$ and parameter c is chosen to be a positive multiple of the reproducing kernel function of the Sonine space of order $\nu + 1$ and parameter c at the point $\frac{1}{2}i\nu$. Assume that the defining function $E(z)$ of the Sonine space of order ν and parameter c is chosen to be a positive multiple of $E_0(z + \frac{1}{2}i)$. Then a positive number P and a real number Q exist such that*

$$P^2 - Q^2 = c$$

and such that the identities

$$E_0(z + \tfrac{1}{2}i) = PE(z)$$

and

$$[PE_0(z) - QE_0^*(z)]\,E_0(\tfrac{1}{2}i\nu) = 2\pi PK(\tfrac{1}{2}i\nu - \tfrac{1}{2}i, z + \tfrac{1}{2}i)$$

are satisfied, where

$$K(w, z) = \frac{E(z)\,\bar{E}(w) - E^*(z)\,E(\bar{w})}{2\pi i(\bar{w} - z)}$$

is the reproducing kernel function of the space $\mathcal{H}(E)$.

Proof of Theorem 10. The argument continues the proof of the previous theorem. Since the adjoint of the transformation of the space $\mathcal{H}(E_0)$ into the space $\mathcal{H}(E)$ which takes $F(z)$ into $F(z + \frac{1}{2}i)$ takes $K(w, z)$ into $K_0(w + \frac{1}{2}i, z)$ for every complex number w, the identity

$$K_0(w + \tfrac{1}{2}i, z) = \frac{K(w, z + (1/2)i) - K(w, (1/2)i - (1/2)i\nu)\,E_0^*(z)/E_0^*(-(1/2)\,i\nu)}{(1/2)\,\nu - iz}$$

is satisfied. An equivalent identity is

$$\begin{aligned}
&E_0(z)\,\bar{E}_0(w+\tfrac{1}{2}i) - E_0^*(z)\,E_0(\bar{w}-\tfrac{1}{2}i)\\
&=\frac{E(z+(1/2)i) - E((1/2)i-(1/2)\,iv)\,E_0^*(z)/E_0^*(-(1/2)\,iv)}{(1/2)\,v - iz}\,\bar{E}(w)\\
&\quad-\frac{E^*(z+(1/2)i) - E^*((1/2)i-(1/2)\,iv)\,E_0^*(z)/E_0^*(-(1/2)\,iv)}{(1/2)\,v - iz}\,E(\bar{w})\\
&\quad+i\,\frac{E((1/2)i-(1/2)\,iv)\,\bar{E}(w) - E((1/2)\,iv-(1/2)i)\,E(\bar{w})}{\bar{w}-(1/2)i-(1/2)\,iv}\;\frac{E_0^*(z)}{E_0^*(-(1/2)\,iv)}.
\end{aligned}$$

Since the functions $E(z)$ and $E^*(z)$ are linearly independent, complex constants P, Q, R, and S exist such that the identities

$$\begin{aligned}
&\frac{E(z+(1/2)i) - E((1/2)i-(1/2)\,iv)\,E_0^*(z)/E_0^*(-(1/2)\,iv)}{(1/2)\,v - iz}\\
&\qquad= E_0(z)\,P - E_0^*(z)\,Q
\end{aligned}$$

and

$$\begin{aligned}
&\frac{E^*(z+(1/2)i) - E((1/2)\,iv-(1/2)i)\,E_0^*(z)/E_0^*(-(1/2)\,iv)}{(1/2)v - iz}\\
&\qquad= -E_0(z)\,R + E_0^*(z)\,S
\end{aligned}$$

are satisfied. Since $E_0(z)$ and $E_0^*(z)$ are linearly independent, the identities

$$E_0(z+\tfrac{1}{2}i) = \bar{P}E(z) + \bar{R}E^*(z)$$

and

$$\begin{aligned}
&E_0^*(z+\tfrac{1}{2}i)\,E_0(\tfrac{1}{2}iv)\\
&\quad= \bar{S}E^*(z)\,E_0(\tfrac{1}{2}iv) + \bar{Q}E(z)\,E_0(\tfrac{1}{2}iv)\\
&\qquad-\frac{E(z)\,E((1/2)i-(1/2)\,iv) - E^*(z)\,E((1/2)\,iv-(1/2)i)}{(1/2)v-(1/2)+iz}
\end{aligned}$$

are satisfied.

Since the functions $E_0(z+\frac{1}{2}i)$ and $E(z)$ are linearly dependent and since the functions $E(z)$ and $E^*(z)$ are linearly independent, $R=0$. The choice of $E(z)$ and of $E_0(z+\frac{1}{2}i)$ has been made in such a way that P is positive.

Since the identity

$$\frac{E^*(z+(1/2)i) - E((1/2)\,iv+(1/2)i)\,E_0^*(z)/E_0^*(-(1/2)\,iv)}{(1/2)\,v - iz} = E_0^*(z)\,S$$

is satisfied and since the functions $E^*(z+\frac{1}{2}i)$ and $E_0^*(z)$ are linearly dependent, $S=0$.

The number Q is real because $E_0(z)$ and $E(z)$ have real values on the imaginary axis. These equations imply the functional identity

$$\begin{aligned} &P^2E^*(z+i)\, E(\tfrac{1}{2}iv-\tfrac{1}{2}i) \\ &\quad = PQE(z)\, E(\tfrac{1}{2}iv-\tfrac{1}{2}i) \\ &\qquad -\frac{E(z)\, E((1/2)i-(1/2)iv)-E^*(z)\, E((1/2)\, iv-(1/2)i)}{(1/2)v-(1/2)+iz}. \end{aligned}$$

The identity

$$\begin{aligned} &P^2E(z+i)\, E(\tfrac{1}{2}iv-\tfrac{1}{2}i)^2-P^2(\tfrac{1}{2}v+\tfrac{1}{2}-iz) \\ &\qquad \times E(z)(P^2-Q^2)\, E(\tfrac{1}{2}iv-\tfrac{1}{2}i)^2 \\ &\quad = PQE(z)\, E(\tfrac{1}{2}iv-\tfrac{1}{2}i)\, E(\tfrac{1}{2}i-\tfrac{1}{2}iv) \\ &\qquad +[PQ(\tfrac{1}{2}v+\tfrac{1}{2}-iz)\, E(\tfrac{1}{2}iv-\tfrac{1}{2}i)-E(\tfrac{1}{2}i-\tfrac{1}{2}iv)] \\ &\qquad \times\frac{E(z)\, E((1/2)i-(1/2)\, iv)-E^*(z)\, E((1/2)iv-(1/2)i)}{(1/2)v-(1/2)+iz} \end{aligned}$$

is obtained on taking the star of each side of the functional identity, replacing z by $z+i$, and eliminating the terms in $E^*(z+i)$ using the functional identity. It follows that

$$P^2-Q^2=\lim\frac{E(z+i)}{((1/2)v+(1/2)-iz)\, E(z)}$$

where the limit is taken as z goes to infinity on the upper half of the imaginary axis.

Since the recurrence relation for the gamma-function can be written

$$W(z+i)=(\tfrac{1}{2}v+\tfrac{1}{2}-iz)\, W(z),$$

the identity can be written

$$P^2-Q^2=\lim\frac{E(z+i)/W(z+i)}{E(z)/W(z)},$$

where the limit is taken in the same sense. Since the function

$$c^{iz}E(z)/W(z)$$

is of bounded type and of zero mean type in the upper half-plane, it follows [2] that

$$c = \lim \frac{E(z+i)/W(z+i)}{E(z)/W(z)}.$$

The identity

$$P^2 - Q^2 = c$$

follows.

This completes the proof of the theorem.

The Sonine spaces of order ν satisfy differential equations which are given by a general structure theory for Hilbert spaces of entire functions [2]. The equations satisfied by the Sonine spaces have previously been studied by the author [1] and by James and Virginia Rovnyak [5]. The absolute continuity of quantities which appear in the functional identity have been verified in this previous work.

The results apply to the Sonine spaces of order ν where $\nu > -1$. Assume that the defining function $E(t, z)$ of the Sonine space of order ν and parameter t is chosen in any way such that

$$E^*(t, z) = E(t, -z)$$

and so that $E(t, z)$ is an absolutely continuous function of t for every complex number z. This is the case for example if $E(t, z)$ is chosen to have value one at the origin, but other choices of $E(t, z)$ are advantageous for calculations. The differential equations

$$B'(t, z) = zA(t, z)\,\alpha'(t) - \rho(t)\,B(t, z)$$

and

$$-A'(t, z) = zB(t, z)\,\gamma'(t) - \rho(t)\,A(t, z)$$

are satisfied for absolutely continuous and nondecreasing functions $\alpha(t)$ and $\gamma(t)$ such that

$$t\alpha'(t)\,t\gamma'(t) = 1$$

and for a locally integrable function $\rho(t)$ of t with real values. The function $\rho(t)$ vanishes identically in the case that $E(t, z)$ is chosen with value one at the origin.

The coefficients in the recurrence relations for the Sonine spaces satisfy differential equations when considered as functions of the parameter.

THEOREM 11. *Let ν be a given positive number. Assume that the defining function $E_0(t, z)$ of the Sonine space of order $\nu - 1$ and parameter t is chosen to be a positive multiple of the reproducing kernel function of the Sonine space of order $\nu + 1$ and parameter t at the point $\frac{1}{2}i\nu$. Assume that the defining function $E(t, z)$ of the Sonine space of order ν and parameter t is chosen to be a positive multiple of $E_0(t, z + \frac{1}{2}i)$. Let $P(t)$ and $Q(t)$ be the unique real numbers such that*

$$P(t)^2 - Q(t)^2 = t$$

and such that the contiguous relations

$$E_0(t, z + \tfrac{1}{2}i) = P(t)\, E(t, z)$$

and

$$\begin{aligned} &E_0^*(t, z + \tfrac{1}{2}i)\, E_0(t, \tfrac{1}{2}i\nu) \\ &\quad = P(t)\, Q(t)\, E(t, z)\, E(t, \tfrac{1}{2}i\nu - \tfrac{1}{2}i) \\ &\qquad - \frac{E(t, z)\, E(t, (1/2)i - (1/2)i\nu) - E^*(t, z)\, E(t, (1/2)\, i\nu - (1/2)\, i)}{(1/2)\, \nu - (1/2) + iz} \end{aligned}$$

are satisfied. Then $E_0(t, z)$ and $E(t, z)$ are absolutely continuous functions of t for every complex number z. The differential equations

$$B'(t, z) = t^{-1} z A(t, z) - \rho(t)\, B(t, z)$$

and

$$-A'(t, z) = t^{-1} z B(t, z) - \rho(t)\, A(t, z)$$

are satisfied with

$$\rho(t) = 2t^{-2} Q(t)/P(t).$$

The quantities $P(t)$ and $Q(t)$ are absolutely continuous functions of t which satisfy the equations

$$\begin{aligned} P'(t) &= \tfrac{1}{2} t^{-1} P(t) + \nu t^{-2} P(t)\, Q(t)^2 \\ &\quad - 2t^{-2} Q(t)\, E(t, \tfrac{1}{2}i - \tfrac{1}{2}i\nu)/E(t, \tfrac{1}{2}i\nu - \tfrac{1}{2}i) \end{aligned}$$

and

$$\begin{aligned} Q'(t) &= \tfrac{1}{2} t^{-1} Q(t) + \nu t^{-2} P(t)^2\, Q(t) \\ &\quad - 2t^{-2} P(t)\, E(t, \tfrac{1}{2}i - \tfrac{1}{2}i\nu)/E(t, \tfrac{1}{2}i\nu - \tfrac{1}{2}i). \end{aligned}$$

The differential equations

$$B_0'(t, z) = zA_0(t, z)\, \alpha_0'(t) - \rho_0(t)\, B_0(t, z)$$

and

$$-A_0'(t, z) = zB_0(t, z)\, \gamma_0'(t) - \rho_0(t)\, A_0(t, z)$$

are satisfied with

$$t\alpha_0'(t) = [P(t) - Q(t)]/[P(t) + Q(t)]$$

and

$$t\gamma_0'(t) = [P(t) + Q(t)]/[P(t) - Q(t)]$$

and

$$\rho_0(t) = vt^{-2} P(t)\, Q(t).$$

Proof of Theorem 11. The present calculations redo those of previous work in a more convenient notation. It has previously been shown that the expression

$$E_0(t, z)/E_0(t, \tfrac{1}{2}iv)$$

is an absolutely continuous function of t for every complex number z. It follows that the expression

$$E(t, z)/E(t, \tfrac{1}{2}iv - \tfrac{1}{2}i)$$

is an absolutely continuous function of t for every complex number z. So the expressions

$$A(t, z)/E(t, \tfrac{1}{2}iv - \tfrac{1}{2}i)$$

and

$$B(t, z)/E(t, \tfrac{1}{2}iv - \tfrac{1}{2}i)$$

are absolutely continuous functions of t for every complex number z. But it has previously been shown that the expression

$$B(t, z)\, \bar{A}(t, w) - A(t, z)\, \bar{B}(t, w)$$

is an absolutely continuous function of t for all complex numbers z and w. It therefore follows that $E(t, \tfrac{1}{2}iv - \tfrac{1}{2}i)$ is an absolutely continuous function of t. The expressions $A(t, z)$ and $B(t, z)$ are then absolutely continuous functions of t which satisfy the stated differential equations for some choice of $\alpha(t)$, $\gamma(t)$, and $\rho(t)$. It is also known from previous work that the Sonine spaces have infinite dimension. The functions $A(t, z)$ and $B(t, z)$ satisfy no nontrivial linear relation with polynomial coefficients.

Differentiate each side of the first contiguous relation with respect to t and eliminate the derivatives of $E(t, z)$ and $E_0(t, z+\frac{1}{2}i)$ using the differential equations satisfied by these functions. Eliminate functions of $z+\frac{1}{2}i$ in terms of functions of z using the contiguous relations. Compare coefficients of $E(t, z)$ and $E^*(t, z)$. The two identities which result are

$$\begin{aligned}
&-i(z+\tfrac{1}{2}i)[\alpha_0'(t)+\gamma_0'(t)]\, P(t)^2\, E(t, \tfrac{1}{2}iv-\tfrac{1}{2}i)\\
&\qquad +[2\rho_0(t)-i(z+\tfrac{1}{2}i)\,\alpha_0'(t)+i(z+\tfrac{1}{2}i)\,\gamma_0'(t)]\\
&\qquad \times [P(t)\,Q(t)\,E(t, \tfrac{1}{2}iv-\tfrac{1}{2}i)-E(t, \tfrac{1}{2}i-\tfrac{1}{2}iv)/(\tfrac{1}{2}v-\tfrac{1}{2}+iz)]\\
&\quad =[2P'(t)-izP(t)\,\alpha'(t)-izP(t)\,\gamma'(t)]\, P(t)\, E(t, \tfrac{1}{2}iv-\tfrac{1}{2}i)
\end{aligned}$$

and

$$\begin{aligned}
&[2\rho_0(t)-i(z+\tfrac{1}{2}i)\,\alpha_0'(t)+i(z+\tfrac{1}{2}i)\,\gamma_0'(t)]/(\tfrac{1}{2}v-\tfrac{1}{2}+iz)\\
&\quad =[2\rho(t)-iz\alpha'(t)+iz\gamma'(t)]\, P(t)^2.
\end{aligned}$$

These equations imply that

$$\alpha'(t)=1/t=\gamma'(t)$$

and that

$$2\rho_0(t)+\tfrac{1}{2}v\alpha_0'(t)-\tfrac{1}{2}v\gamma_0'(t)=0$$

and that

$$2\rho_0(t)=vP(t)^2\,\rho(t)$$

and that

$$[P(t)+Q(t)]\,\alpha_0'(t)+[P(t)-Q(t)]\,\gamma_0'(t)=2P(t)/t$$

and that

$$\begin{aligned}
2P'(t)-P(t)/t&=\tfrac{1}{2}vQ(t)[\gamma_0'(t)-\alpha_0'(t)]\times[\alpha_0'(t)-\gamma_0'(t)]\\
&\qquad \times P(t)^{-1}\, E(t, \tfrac{1}{2}i-\tfrac{1}{2}iv)/E(t, \tfrac{1}{2}iv-\tfrac{1}{2}i).
\end{aligned}$$

Differentiate each side of the second contiguous relation with respect to t and eliminate the derivatives of $E(t, z)$ and $E_0(t, z+\frac{1}{2}i)$ using the differential equations satisfied by these functions. Again eliminate functions of

$z+\frac{1}{2}i$ in terms of functions of z using the contiguous relations. Compare coefficients of $E(t,z)$ and $E^*(t,z)$. The two identities which result are

$$\begin{aligned}
& i\left(z+\frac{1}{2}i\right) P(t)\,Q(t)\,E\left(t,\frac{1}{2}iv-\frac{1}{2}i\right)\frac{1}{2}\,[\alpha_0'(t)+\gamma_0'(t)] \\
& \quad -i\left(z+\frac{1}{2}i\right)\frac{E(t,(1/2)i-(1/2)iv)}{(1/2)v-(1/2)+iz}\,\frac{1}{2}\,[\alpha_0'(t)+\gamma_0'(t)] \\
& \quad +i\left(z+\frac{1}{2}i\right) P(t)^2\,E\left(t,\frac{1}{2}iv-\frac{1}{2}i\right) \\
& \quad \times\frac{1}{2}\,[\alpha_0'(t)-\gamma_0'(t)]+\rho_0(t)\,P(t)^2\left(t,\frac{1}{2}iv-\frac{1}{2}i\right) \\
& \quad +[Q(t)\,it^{-1}z-Q'(t)]\,P(t)\,E\left(t,\frac{1}{2}iv-\frac{1}{2}i\right) \\
& \quad =\frac{E(t,(1/2)i-(1/2)iv)}{(1/2)v-(1/2)+iz}\,\frac{[P(t)\,E(t,(1/2)iv-(1/2)i]'}{P(t)\,E(t,(1/2)iv-(1/2)i)} \\
& \quad +t^{-1}E\left(t,\frac{1}{2}i-\frac{1}{2}iv\right)
\end{aligned}$$

and

$$\begin{aligned}
& i\left(z+\frac{1}{2}i\right)\frac{E(t,(1/2)\,iv-(1/2)\,i)}{(1/2)\,v-(1/2)+iz}\,\frac{1}{2}\,[\alpha_0'(t)+\gamma_0'(t)] \\
& \quad -\rho(t)\,P(t)\,Q(t)\,E\left(t,\frac{1}{2}iv-\frac{1}{2}i\right) \\
& \quad =-\frac{E(t,(1/2)\,iv-(1/2)\,i)}{(1/2)\,v-(1/2)+iz}\,\frac{[P(t)\,E(t,(1/2)\,iv-(1/2)\,i)]'}{P(t)\,E(t,(1/2)\,iv-(1/2)\,i)} \\
& \quad +t^{-1}E\left(t,\frac{1}{2}iv-\frac{1}{2}i\right).
\end{aligned}$$

These equations imply that

$$\begin{aligned}
2P'(t)-P(t)/t=vP(t)^2\,Q(t)\,\rho(t)-2P(t)& \\
\times\rho(t)\,E(t,\tfrac{1}{2}i-\tfrac{1}{2}iv)/E(t,\tfrac{1}{2}iv-\tfrac{1}{2}i)&
\end{aligned}$$

and that

$$\rho(t)=2t^{-2}Q(t)/P(t)$$

and that

$$2Q'(t) - Q(t)/t = 2\nu t^{-2} P(t)^2\, Q(t) \\ \qquad - 4t^{-2} P(t)\, E(t, \tfrac{1}{2}i - \tfrac{1}{2}i\nu)/E(t, \tfrac{1}{2}i\nu - \tfrac{1}{2}i).$$

The stated differential equations are now easily obtained.

This completes the proof of the theorem.

These results can advantageously be restated in another notation. Let ν be a given positive number. The defining function $E(t, z)$ of the Sonine space of order ν and parameter t and the defining function $E_0(t, z)$ of the Sonine space of order $\nu - 1$ and parameter t can be chosen with positive values on the upper half of the imaginary axis so that the identities

$$t^{1/2} E(t, z) = E_0(t, z + \tfrac{1}{2}i) + \sigma(t)\, E_0^*(t, z + \tfrac{1}{2}i)$$

and

$$E^*(t, z) - \sigma_0(t)\, E(t, z) = (\tfrac{1}{2}\nu - \tfrac{1}{2} + iz)\, t^{1/2} E_0^*(t, z + \tfrac{1}{2}i)$$

hold for real numbers $\sigma(t)$ and $\sigma_0(t)$. The quantities $E(t, z)$ and $E_0(t, z)$ are then absolutely continuous functions of t for every complex number z. The differential equations

$$B'(t, z) = t^{-1} z A(t, z) - 2t^{-2} \sigma(t)\, B(t, z)$$

and

$$-A'(t, z) = t^{-1} z B(t, z) - 2t^{-2} \sigma(t)\, A(t, z)$$

and

$$B_0'(t, z) = t^{-1} z A_0(t, z) - 2t^{-2} \sigma_0(t)\, B_0(t, z)$$

and

$$-A_0'(t, z) = t^{-1} z B_0(t, z) - 2t^{-2} \sigma_0(t)\, A_0(t, z)$$

are satisfied. The coefficients $\sigma(t)$ and $\sigma_0(t)$ are absolutely continuous functions of t which satisfy the differential equations

$$\sigma'(t) = \nu t^{-1} \sigma(t) - 2t^{-2} \sigma(t)[1 - \sigma(t)^2]$$

and

$$\sigma_0'(t) = -(\nu - 1)\, t^{-1} \sigma_0(t) + 2t^{-2} \sigma(t)[1 - \sigma_0(t)^2].$$

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The inequalities $-1 < \sigma(t) < 1$ hold for all positive numbers t. The inequality $\sigma_0(t) > 1$ holds for all positive numbers t when $\nu < 1$. A simplification occurs when $\nu = 1$ because the identity $\sigma_0(t) = 1$ then holds for all positive numbers t. The determination of the function $\sigma(t)$ is an unsolved problem already in the case $\nu = 1$.

A computation of asymptotic behavior is a consequence of the differential equations satisfied by the Sonine spaces.

THEOREM 12. *Assume that the defining function $E(c, z)$ of the Sonine space of order ν and parameter c is chosen so that $E(c, z - \frac{1}{2}i)$ is a positive multiple of the reproducing kernel function of the Sonine space of order $\nu + 1$ and parameter c at the point $\frac{1}{2}i\nu$. Then the identity*

$$W(z) = \lim t^{iz} E(t, z)$$

holds uniformly on compact subsets of the upper half-plane, where

$$W(z) = \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - iz).$$

If ν is nonnegative, then the identity

$$W(z) = \lim t^{iz} E(t, z)$$

holds uniformly on compact subsets of the half-plane $i\bar{z} - iz > -1$, and the inequality

$$|\log[t^{iz} E(t, z)/W(z)]| \leqslant 2/t$$

holds in the upper half-plane.

Proof of Theorem 12. The argument begins with the case that ν is nonnegative. It will be shown that the limit of $t^{iz} E(t, z)$ exists uniformly on compact subsets of the upper half-plane and that the limit is an analytic weight function. It will then be shown that the weight function $W(z)$ so obtained is equal to $\Gamma(\frac{1}{2}\nu + \frac{1}{2} - iz)$.

Use is made of the differential equations

$$B'(t, z) = t^{-1} z A(t, z) - \rho(t)\, B(t, z)$$

and

$$-A'(t, z) = t^{-1} z B(t, z) - \rho(t)\, A(t, z)$$

which are satisfied with

$$\rho(t) = 2t^{-2} Q(t)/P(t)$$

when ν is positive and with

$$\rho(t) = 2t^{-2}$$

when ν is zero. Since $P(t)$ and $Q(t)$ are real numbers such that

$$P(t)^2 - Q(t)^2 = t,$$

the inequalities

$$-2t^{-2} \leqslant \rho(t) \leqslant 2t^{-2}$$

are satisfied.

Since the differential equation

$$\frac{d}{dt} \log[t^{iz} E(t, z)] = \rho(t)\, E^*(t, z)/E(t, z)$$

holds in the upper half-plane, and since the inequality

$$|E^*(t, z)| < |E(t, z)|$$

holds in the half-plane, the inequality

$$\left| \frac{d}{dt} \log[t^{iz} E(t, z)] \right| \leqslant 2/t^2$$

holds in the half-plane. It follows that the inequality

$$|\log[b^{iz} E(b, z)] - \log[a^{iz} E(a, z)]| \leqslant 2/a - 2/b$$

holds in the half-plane when $a \leqslant b$. The functions

$$\log[t^{iz} E(t, z)]$$

converge uniformly in the upper half-plane to an analytic function which can be written

$$\log W(z)$$

for an analytic weight function $W(z)$. The inequality

$$|\log[t^{iz} E(t, z)/W(z)]| \leqslant 2/t$$

holds in the upper half-plane for every positive number t. The function $W(z)$ has a continuous extension to the closed half-plane. It has no zeros in the closed half-plane, and the inequality holds in the closed half-plane.

If an entire function $F(z)$ belongs to the space $\mathcal{H}(E(a))$, then it also belongs to the space $\mathcal{H}(E(b))$ when $a \leqslant b$, and the identity

$$\|F\|^2_{\mathcal{H}(E(a))} = \int_{-\infty}^{+\infty} |F(t)/E(b, t)|^2 \, dt$$

is satisfied. The identity

$$\|F\|^2_{\mathcal{H}(E(a))} = \int_{-\infty}^{+\infty} |F(t)/W(t)|^2 \, dt$$

is obtained in the limit of large b. The Sonine space $\mathcal{H}(E(a))$ therefore coincides with the set of entire functions $F(z)$ such that $a^{iz}F(z)$ and $a^{iz}F^*(z)$ belong to the space $\mathcal{F}(W)$. Multiplication by a^{iz} is an isometric transformation of the space $\mathcal{H}(E(a))$ into the space $\mathcal{F}(W)$.

The same conclusion holds by the definition of the Sonine spaces when $W(z)$ is replaced by $\Gamma(\frac{1}{2}\nu + \frac{1}{2} - iz)$. By the uniqueness of the norm-determining measure for the Sonine spaces, the identity

$$W(z)\, W^*(z) = \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - iz)\, \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} + iz)$$

holds for all real numbers z. The function

$$W(z)/\Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - iz)$$

is analytic and without zeros in the upper half-plane, and it has a continuous extension to the closed half-plane. Since it is of bounded type and of zero mean type in the half-plane, and since it has absolute value one on the real axis, it is a constant of absolute value one. Since $E(a, z)$ has positive values on the upper half of the imaginary axis for every parameter a, $W(z)$ has positive values on the upper half of the imaginary axis. It follows that

$$W(z) = \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - iz).$$

Since multiplication by a^{iz} is an isometric transformation of the space $\mathcal{H}(E(a))$ into the space $\mathcal{F}(W)$, the reproducing kernel function of the image of the space $\mathcal{H}(E(a))$ in the space $\mathcal{F}(W)$ is

$$a^{iz - i\bar{w}} \frac{E(a, z)\, \bar{E}(a, w) - E^*(a, z)\, E(a, \bar{w})}{2\pi i(\bar{w} - z)}.$$

The image of the Sonine space of order ν and parameter a is contained in the image of the Sonine space of order ν and parameter b when $a < b$. Since the union of the images of the Sonine spaces of order ν is dense in the space

$\mathcal{F}(W)$, these reproducing kernel functions converge to the reproducing kernel function

$$\frac{W(z)\,\bar{W}(w)}{2\pi i(\bar{w}-z)}$$

of the space $\mathcal{F}(W)$. It follows that the ratio

$$E^*(a, z)/E(a, z)$$

converges to zero uniformly on compact subsets of the upper half-plane.

Assume that ν is positive and that the defining function $E_0(t, z)$ of the Sonine space of order $\nu - 1$ and parameter t is chosen so that $E_0(t, z - \frac{1}{2}i)$ is a positive multiple of the reproducing kernel function of the Sonine space of order ν and parameter t at the point $\frac{1}{2}i\nu - \frac{1}{2}i$. Then the identities

$$t^{1/2}E(t, z) = E_0(t, z + \tfrac{1}{2}i) + \sigma(t)\,E_0^*(t, z + \tfrac{1}{2}i)$$

and

$$E^*(t, z) - \sigma_0(t)\,E(t, z) = (\tfrac{1}{2}\nu - \tfrac{1}{2} + iz)\,t^{1/2}E_0^*(t, z + \tfrac{1}{2}i)$$

hold for real numbers $\sigma(t)$ and $\sigma_0(t)$, where

$$\rho(t) = 2t^{-2}\sigma(t)$$

and

$$\sigma_0(t) = E(t, \tfrac{1}{2}i - \tfrac{1}{2}i\nu)/E(t, \tfrac{1}{2}i\nu - \tfrac{1}{2}i).$$

It has been shown that the identity

$$\lim \sigma_0(t) = 0$$

holds when $\nu > 1$. It follows that the identity

$$\lim \sigma(t) = 0$$

holds when $\nu > 0$.

Define

$$W_0(z) = \Gamma(\tfrac{1}{2}\nu - iz).$$

A previous argument shows that the expression

$$t^{iz-i\bar{w}}[E_0(t, z)\,\bar{E}_0(t, w) - E_0^*(t, z)\,E_0(t, \bar{w})]$$

converges to

$$W_0(z)\,\bar{W}_0(w)$$

uniformly for z and w in any compact subset of the upper half-plane.

Use is now made of the identity

$$t^{1/2}E(t, z) = E_0(t, z + \tfrac{1}{2}i) + \sigma(t)\,E_0^*(t, z + \tfrac{1}{2}i).$$

Since the identity

$$W(z) = \lim t^{iz}E(t, z)$$

holds uniformly on compact subsets of the upper half-plane, since the inequality

$$|E_0^*(t, z)| < |E_0(t, z)|$$

holds in the upper half-plane, and since

$$\lim \sigma(t) = 0,$$

the identity

$$W_0(z) = \lim t^{1/2}E_0(t, z)$$

holds uniformly on compact subsets of the half-plane $i\bar{z} - iz > 1$.

It follows that the ratios

$$E_0^*(t, z)/E_0(t, z)$$

converge to zero uniformly on compact subsets of the same half-plane. Since the ratios are analytic and bounded by one in the upper half-plane, they converge to zero uniformly on compact subsets of the upper half-plane.

Since the expression

$$t^{iz - i\bar{w}}E_0(t, z)\,\bar{E}_0(t, w)$$

converges to

$$W_0(z)\,\bar{W}_0(w)$$

uniformly on compact subsets of the upper half-plane, and since $E_0(t, z)$ and $W(z)$ have positive values on the upper half of the imaginary axis, the identity

$$W_0(z) = \lim t^{iz}E_0(t, z)$$

holds uniformly on compact subsets of the upper half-plane.

Another use is now made of the identity

$$t^{1/2}E(t, z) = E_0(t, z + \tfrac{1}{2}i) + \sigma(t)\, E_0^*(t, z + \tfrac{1}{2}i).$$

It follows that the identity

$$W(z) = \lim t^{iz} E(t, z)$$

holds uniformly on compact subsets of the half-plane $i\bar{z} - iz > -1$.

The same conclusion will now be obtained when ν is zero. An equivalent assertion is that the identity

$$W_0(z) = \lim t^{iz} E_0(t, z)$$

holds uniformly on compact subsets of the half-plane $i\bar{z} - iz > -1$ when $\nu = 1$. The result is a consequence of the identity

$$E^*(t, z) - E(t, z) = izt^{1/2} E_0^*(t, z + \tfrac{1}{2}i)$$

which can be written

$$E(t, z) - E^*(t, z) = -izt^{1/2} E_0(t, z - \tfrac{1}{2}i).$$

Since

$$W(z) = -izW_0(z - \tfrac{1}{2}i)$$

and since

$$E^*(t, z)/E(t, z)$$

converges to zero uniformly on compact subsets of the upper half-plane, the identity

$$W(z) = \lim t^{iz} E(t, z)$$

uniformly on compact subsets of the upper half-plane implies the identity

$$W_0(z) = \lim t^{iz} E_0(t, z)$$

uniformly on compact subsets of the half-plane $i\bar{z} - iz > -1$.

The desired conclusions have now been obtained when ν is positive or zero. The conclusions which have been obtained for the Sonine spaces of order $\nu - 1$ when ν is positive contain the desired conclusions when ν is negative.

This completes the proof of the theorem.

A construction of Hilbert spaces of entire functions associated with Dirichlet zeta-functions is made from the Sonine spaces of entire functions.

THEOREM 13. *Assume that χ is a nonprincipal character modulo r. Let ν be a given positive number. Consider the space $\mathcal{H}(E)$ corresponding to the entire function*

$$E(z) = (r/\pi)^{-iz}\, W(z)\, \zeta_\chi(1 + \nu - 2iz),$$

where

$$W(z) = \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - iz)$$

if χ is even and

$$W(z) = \Gamma(\tfrac{1}{2}\nu + 1 - iz)$$

if χ is odd. For each positive integer n, which is divisible by r, such that n/r is relatively prime to r and is not divisible by the square of a prime, define

$$S_n(a, b, z) = \prod (p^{-iz} - \chi(p)\, p^{-1-\nu+iz}),$$

where the product is taken over the prime divisors p of n which are not divisors of r. For each such index n define $\mathcal{H}(E_n(b))$ to be the Sonine space of order ν and parameter n/π if χ is even and to be the Sonine space of order $1 + \nu$ and parameter n/π if χ is odd. Then the set $\mathcal{H}(E)$ becomes a space $\mathcal{H}(E_n(a))$ when considered with the unique scalar product such that multiplication by $S_n(a, b, z)$ is an isometric transformation of the space $\mathcal{H}(E_n(a))$ into the space $\mathcal{H}(E_n(b))$. If λ is a given zero of $E(z)$, then $E_n(a, z)$ can be chosen to have a zero at λ. The identity

$$E(z)\, \bar{E}(w) = \lim E_n(a, z)\, \bar{E}_n(a, w)$$

then holds uniformly for z and w in any compact subset of the complex plane. Multiplication by

$$(n/\pi)^{iz}\, S_n(a, b, z)$$

is an isometric transformation of the space $\mathcal{H}(E_n(a))$ into the space $\mathcal{F}(W)$ for every index n. The expression

$$\frac{W(z)\, \bar{W}(\bar{\lambda})}{2\pi i(\lambda - z)} - (n/\pi)^{iz - i\lambda}\, \frac{S_n(a, b, z)\, E_n(a, z)\, \bar{E}_n(a, \bar{\lambda})\, \bar{S}_n(a, b, \bar{\lambda})}{2\pi i(\lambda - z)}$$

is an element of the space $\mathcal{F}(W)$ which is orthogonal to the image of the space $\mathcal{H}(E_n(a))$ and the square of whose norm is less than or equal to

$$\frac{|W(\bar{\lambda})|^2}{2\pi i(\lambda - \bar{\lambda})} \times \left[1 - \prod (1 - p^{-1-\nu})^2 \prod (1 - p^{-1-\nu+i\bar{\lambda}-i\lambda})^2 \right],$$

where the products are taken over the primes p which are not divisors of n.

Proof of Theorem 13. Since the inequalities

$$\prod (1-p^{-1-\nu}) \leqslant |\zeta_\chi(1+\nu-2iz)| \leqslant \prod (1-p^{-1-\nu})^{-1}$$

hold in the upper half-plane, where the products are taken over the primes p which are not divisors of r, the space $\mathcal{H}(E)$ coincides as a set with the Sonine space $\mathcal{H}(E_r(b))$. A bound for the inclusion of the space $\mathcal{H}(E)$ in the space $\mathcal{H}(E_r(b))$ and for the inclusion of the space $\mathcal{H}(E_r(b))$ in the space $\mathcal{H}(E)$ is

$$\prod (1-p^{-1-\nu})^{-1}$$

where the products are taken over the primes p which are not divisors of r.

For every index n a Hilbert space exists, which coincides as a set with $\mathcal{H}(E)$, such that multiplication by $S_n(a, b, z)$ is an isometric transformation of the space into the Sonine space $\mathcal{H}(E_n(b))$. Since the space satisfies the axioms (H1), (H2), and (H3) and contains a nonzero element, it is isometrically equal to a space $\mathcal{H}(E_n(a))$.

A bound for the inclusion of the space $\mathcal{H}(E_n(a))$ in the space $\mathcal{H}(E)$ and for the inclusion of the space $\mathcal{H}(E)$ in the space $\mathcal{H}(E_n(a))$ is given by

$$\prod (1-p^{-1-\nu})^{-1},$$

where the product is taken over the primes p which are not divisors of n. The notation

$$K(w, z) = \frac{E(z)\,\bar{E}(w) - E^*(z)\,E(\bar{w})}{2\pi i(\bar{w}-z)}$$

is used for the reproducing kernel function of the space $\mathcal{H}(E)$. The notation

$$K_n(a, w, z) = \frac{E_n(a, z)\,\bar{E}_n(a, w) - E_n^*(a, z)\,E_n(a, \bar{w})}{2\pi i(\bar{w}-z)}$$

is used for the reproducing kernel functions of the space $\mathcal{H}(E_n(a))$. The expressions

$$K(w, z) - K_n(a, w, z) \prod (1-p^{-1-\nu})^2$$

and

$$K_n(a, w, z) - K(w, z) \prod (1-p^{-1-\nu})^2$$

are positive-definite functions of z and w in the complex plane, where the products are taken over the primes p which are not divisors of n. The identity

$$K(w, z) = \lim K_n(a, w, z)$$

holds uniformly for z and w in any compact subset of the complex plane.

If λ is a zero of $E(z)$ and if $E_n(a, z)$ is always chosen to have λ as a zero, then the identity

$$E(z)\, \bar{E}(\bar{\lambda}) = \lim E_n(a, z)\, \bar{E}_n(a, \bar{\lambda})$$

holds uniformly in any compact subset of the complex plane. It follows that the identity

$$E(z)\, \bar{E}(w) = \lim E_n(a, z)\, \bar{E}_n(a, w)$$

holds uniformly for z and w in any compact subset of the complex plane.

Since multiplication by $S_n(a, b, z)$ is an isometric transformation of the space $\mathscr{H}(E_n(a))$ into the space $\mathscr{H}(E_n(b))$ by construction and since multiplication by $(n/\pi)^{iz}$ is an isometric transformation of the space $\mathscr{H}(E_n(b))$ into the space $\mathscr{F}(W)$ by definition, multiplication by $(n/\pi)^{iz}\, S_n(a, b, z)$ is an isometric transformation of the space $\mathscr{H}(E_n(a))$ into the space $\mathscr{F}(W)$.

The expression

$$W(z)\, \bar{W}(w)/[2\pi i(\bar{w} - z)] - (n/\pi)^{iz - iw}\, S_n(a, b, z)\, \bar{S}_n(a, b, w)\, K_n(a, w, z)$$

is the reproducing kernel function for the orthogonal complement in the space $\mathscr{F}(W)$ of the image of the space $\mathscr{H}(E_n(a))$. This computation requires w to lie in the upper half-plane. It is used when $w = \bar{\lambda}$, which is in the upper half-plane because λ is a zero of $E(z)$. The square of the norm of the expression is then

$$|W(\bar{\lambda})|^2/[2\pi i(\lambda - \bar{\lambda})] - (n/\pi)^{i\bar{\lambda} - i\lambda}\, |S_n(a, b, \bar{\lambda})|^2\, K_n(a, \bar{\lambda}, \bar{\lambda})$$

by the reproducing kernel property.

It has been seen that the inequality

$$K_n(a, \bar{\lambda}, \bar{\lambda}) \geqslant K(\bar{\lambda}, \bar{\lambda}) \prod (1 - p^{-1-\nu})^2$$

is satisfied, where the product is taken over the primes p which are not divisors of n. Since the identity

$$(n/r)^{i\bar{\lambda}}\, S_n(a, b, \bar{\lambda})\, \zeta_\chi(1 + \nu - 2i\bar{\lambda}) = \prod (1 - p^{-1-\nu+2i\bar{\lambda}})$$

is satisfied, where the product is taken over the primes p which are not divisors of n, the inequality

$$(n/r)^{i\bar{\lambda} \quad i\lambda} \, |S_n(a, b, \bar{\lambda}) \, \zeta_\chi(1 + \nu - 2i\bar{\lambda})|^2 \leqslant \prod (1 - p^{\quad 1 \quad \nu + i\bar{\lambda} \quad i\lambda})^2$$

is satisfied, where the product is again taken over the primes p which are not divisors of n. The desired norm estimate is now obtained from the identity

$$E(\bar{\lambda}) = (r/\pi)^{\quad i\bar{\lambda}} \, \zeta_\chi(1 + \nu - 2i\bar{\lambda}) \, W(\bar{\lambda}).$$

This completes the proof of the theorem.

The construction of Hilbert spaces of entire functions associated with Dirichlet zeta-functions creates a context in which the factorization theorems apply. The given Dirichlet zeta-functions are then approximated by functions having zero free half-planes. The results are an application of the theory of asymptotic behavior for the Sonine spaces.

THEOREM 14. *Let*

$$W(z) = \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - iz)$$

be the weight function associated with the Sonine spaces of nonnegative order ν. *Let* $\mathscr{H}(E)$ *be a given space such that* $E(z)$ *has no real zeros. Assume that* $S(z)$ *is an entire function with no zeros in the upper half-plane such that multiplication by* $S(z)$ *is an isometric transformation of the space* $\mathscr{H}(E)$ *into the space* $\mathscr{F}(W)$ *and such that* $S(z)\,E(z)/W(z)$ *is of zero mean type in the upper half-plane. Let* $\varphi(z)$ *be the unique function such that a space* $\mathscr{L}(\varphi)$ *is defined, such that no nonzero element of the space is of the form* $[1 + \varphi(z)]\,F(z)/E^*(z)$ *with* $F(z)$ *in* $\mathscr{H}(E)$, *such that a space* $\mathscr{F}_S(W_\varphi, \varphi)$ *exists,*

$$W_\varphi(z) = \frac{W(z)}{A(z) - i\varphi(z)\,B(z)},$$

such that an isometric transformation of the space $\mathscr{F}_S(W_\varphi, \varphi)$ *into the space* $\mathscr{L}(\varphi)$ *is defined by taking*

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

into

$$\sqrt{2}[i\varphi(z)\,F(z) + G(z)]/S(z),$$

and such that an isometric transformation of the space $\mathcal{F}_S(W_\varphi, \varphi)$ onto the orthogonal complement in the space $\mathcal{F}(W)$ of the image of the space $\mathcal{H}(E)$ is defined by taking

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

into

$$\sqrt{2}\,[A(z)\,F(z) + B(z)\,G(z)].$$

If multiplication by $b^{iz}S(z)$ is an isometric transformation of the space $\mathcal{H}(E)$ into the Sonine space $\mathcal{H}(E(b))$ of order ν and parameter b for some positive number b, then $\varphi(z)$ has an analytic extension to the half-plane $i\bar{z} - iz > -1$, the function $A(z) - i\varphi(z)\,B(z)$ has no zeros in the half-plane, and the expression

$$\frac{\varphi(z) + \bar{\varphi}(w)}{\pi i(\bar{w} - z)} - 2\,\frac{S(z)/W_\varphi(z)\,\bar{S}(w)/\bar{W}_\varphi(w)}{\pi i(\bar{w} - z)}$$

is a positive-definite function of z and w in the half-plane.

Proof of Theorem 14. A parameter b is said to be admissible if multiplication by

$$S(a, b, z) = b^{iz}S(z)$$

is an isometric transformation of the space $\mathcal{H}(E) = \mathcal{H}(E(a))$ into the Sonine space $\mathcal{H}(E(b))$ of order ν and parameter b. An admissible parameter exists by hypothesis. Any parameter which is less than admissible parameter is admissible. There is a smallest admissible parameter. It is assumed that a is a positive number less than the smallest admissible parameter.

If b is an admissible parameter, the defining function $E(b, z)$ of the Sonine space of order ν and parameter b is chosen so as to have no zeros in the half-plane $i\bar{z} - iz > -1$ and so that the identity

$$W(z) = \lim b^{iz}E(b, z)$$

holds uniformly on compact subsets of the half-plane. Let $M(a, b, z)$ be the unique matrix-valued entire function such that a space $\mathcal{H}_{S(a,b)}(M(a, b))$ exists, such that the identity

$$(A(b, z), B(b, z)) = (A(a, z), B(a, z))\,M(a, b, z)$$

is satisfied, and such that an isometric transformation of the space onto the orthogonal complement in $\mathscr{H}(E(b))$ of this image of $\mathscr{H}(E(a))$ is defined by taking

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

into

$$\sqrt{2}\,[A(a, z)\,F(z) + B(a, z)\,G(z)].$$

Assume that $S(a, z)$ is an entire function without zeros in the upper half-plane such that

$$[F(z)\,S(a, w) - S(a, z)\,F(w)]/(z - w)$$

belongs to the space $\mathscr{H}(E(a))$ for all complex numbers w whenever $F(z)$ belongs to $\mathscr{H}(E(a))$ and such that $S(a, z)/E(a, z)$ is of zero mean type in the upper half-plane. An example of such a function $S(a, z)$ is $E(a, z)$. Then entire functions $C(a, z)$ and $D(a, z)$ exist, which are real for real z, such that a space $\mathscr{H}_{S(a)}(M(a))$ exists,

$$M(a, z) = \begin{pmatrix} A(a, z) & B(a, z) \\ C(a, z) & D(a, z) \end{pmatrix},$$

and such that no nonzero element

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

of the space exists with $F(z) = 0$. In the case $S(a, z) = E(a, z)$ such functions are given by

$$C(a, z) = -B(a, z)$$

and

$$D(a, z) = A(a, z).$$

Define

$$S(b, z) = S(a, z)\,S(a, b, z)$$

and

$$M(b, z) = M(a, z)\,M(a, b, z).$$

A space $\mathcal{H}_{S(b)}(M(b))$ exists. The space decomposes into complementary spaces $\mathcal{P}$ and $\mathcal{Q}$ with these properties: Multiplication by $S(a, b, z)$ is an isometric transformation of the space $\mathcal{H}_{S(a)}(M(a))$ onto $\mathcal{P}$. Multiplication by $M(a, z)$ is an isometric transformation of the space $\mathcal{H}_{S(a,b)}(M(a, b))$ onto $\mathcal{Q}$.

It will now be verified that the space $\mathcal{H}_{S(b)}(M(b))$ contains no nonzero element

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

with $F(z) = 0$. Any such element can be written

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix} = S(a, b, z) \begin{pmatrix} F_0(z) \\ G_0(z) \end{pmatrix} + M(a, z) \begin{pmatrix} F_1(z) \\ G_1(z) \end{pmatrix},$$

where

$$\begin{pmatrix} F_0(z) \\ G_0(z) \end{pmatrix}$$

belongs to the space $\mathcal{H}_{S(a)}(M(a))$ and

$$\begin{pmatrix} F_1(z) \\ G_1(z) \end{pmatrix}$$

belongs to the space $\mathcal{H}_{S(a,b)}(M(a, b))$. These conditions imply that

$$S(a, b, z)\, F_0(z) + A(a, z)\, F_1(z) + B(a, z)\, G_1(z)$$

vanishes identically. Since multiplication by $S(a, b, z)$ is an isometric transformation of the space $\mathcal{H}(E(a))$ into the space $\mathcal{H}(E(b))$, the functions $F_0(z)$, $F_1(z)$, and $G_1(z)$ vanish identically. The function $G_0(z)$ vanishes identically by the construction of the space $\mathcal{H}_{S(a)}(M(a))$. This completes the proof that $G(z)$ vanishes identically.

The proof of the factorization theorem for Hilbert spaces of analytic functions can now be applied. Since $T(z) = S(z)\, E(z)$ is an entire function without zeros in the upper half-plane such that $T(z)/(z - \bar{w})$ belongs to the space $\mathcal{F}(W)$ when w is in the upper half-plane, and since $T(z)/W(z)$ is of zero mean type in the upper half-plane, the space $\mathcal{F}(W)$ is the closed span of the functions of the form $T(z)/(z - \bar{w})$ with w in the upper half-plane. A space $\mathcal{L}(\psi)$ exists such that an isometric transformation of the space $\mathcal{F}(W)$ onto the space $\mathcal{L}(\psi)$ is obtained by taking $F(z)$ into the function $f(z)$ defined by the identity

$$\pi f(w) = \langle F(t), T(t)/(t - \bar{w}) \rangle_{\mathcal{F}(W)}$$

when w is in the upper half-plane.

A space $\mathcal{F}_T(W, \psi)$ exists such that an isometric transformation of the space onto the space $\mathcal{F}(W)$ is defined by taking

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

into

$$\sqrt{2}\, F(z)$$

and such that an isometric transformation of the space onto the space $\mathcal{L}(\psi)$ is defined by taking

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

into

$$\sqrt{2}\, [i\psi(z)\, F(z) + G(z)]/T(z).$$

The function $\psi(z)$ with these properties is unique within an added imaginary constant. The choice of constant can be made in a unique way such that

$$\begin{pmatrix} F(z)/S(z) \\ G(z)/S(z) \end{pmatrix}$$

belongs to the space $\mathcal{H}_{S(a)}(M(a))$ whenever

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

is an element of the space $\mathcal{F}_T(W, \psi)$ such that $F(z)/S(z)$ belongs to $\mathcal{H}(E)$. Multiplication by b^{iz} is then an isometric transformation of the space $\mathcal{H}_{S(b)}(M(b))$ into the space $\mathcal{F}_T(W, \psi)$ for every admissible parameter b. The union of the ranges of the transformations is dense in the space $\mathcal{F}_T(W, \psi)$.

An argument in the proof of the factorization theorem for Hilbert spaces of analytic functions can now be applied to show that multiplication by $M(a, z)$ is an isometric transformation of the space $\mathcal{F}_S(W_\varphi, \varphi)$ onto the orthogonal complement in the space $\mathcal{F}_T(W, \psi)$ of the image of the space $\mathcal{H}_{S(a)}(M(a))$. Multiplication by b^{iz} is an isometric transformation of the space $\mathcal{H}_{S(a,b)}(M(a, b))$ into the space $\mathcal{F}_S(W_\varphi, \varphi)$ for every admissible parameter b. The union of the ranges of the transformations is dense in the space $\mathcal{F}_S(W_\varphi, \varphi)$.

A computation of asymptotic behavior of reproducing kernel functions in the upper half-plane is a consequence of these relations. The identities

$$\lim b^{iz - i\bar w} M(b, z)\, I \bar M(b, w) = \frac{1}{2} i W(z)\, \bar W(w) \begin{pmatrix} 1 \\ -i\psi(z) \end{pmatrix} (1 \;\; i\bar\psi(w))$$

and

$$\lim b^{iz - i\bar w} M(a, b, z)\, I \bar M(a, b, w) = \frac{1}{2} i W_\varphi(z)\, \bar W_\varphi(w) \begin{pmatrix} 1 \\ -i\varphi(z) \end{pmatrix} (1 \;\; i\bar\varphi(w))$$

hold uniformly on compact subsets of the upper half-plane when w is in the upper half-plane. It follows that the identities

$$\psi(z) = \lim \frac{D(b, z) + iC(b, z)}{A(b, z) - iB(b, z)}$$

and

$$\varphi(z) = \lim \frac{D(a, b, z) + iC(a, b, z)}{A(a, b, z) - iB(a, b, z)}$$

hold uniformly on compact subsets of the upper half-plane.

A computation of asymptotic behavior will now be made in the half-plane $i\bar z - iz > -1$. Assume that an element

$$\begin{pmatrix} F(b, z) \\ G(b, z) \end{pmatrix}$$

of the space $\mathscr{H}_{S(b)}(M(b))$ is given for every admissible parameter b such that

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix} = \lim \begin{pmatrix} b^{iz} F(b, z) \\ b^{iz} G(b, z) \end{pmatrix}$$

exists in the metric topology of the space $\mathscr{F}_T(W, \psi)$. Then the functions

$$\frac{i\psi(z)\, F(b, z) + G(b, z)}{S(b, z)} = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{F(b, t)\, \bar S(b, t)\, dt}{W(t)\, \bar W(t)(t - z)}$$

converge to the limit

$$\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{F(t)\, \bar T(t)\, dt}{W(t)\, \bar W(t)(t - z)}$$

uniformly on compact subsets of the upper half-plane as b converges to infinity. If

$$\psi^*(z) = \bar\psi(\bar z),$$

the functions

$$\frac{-i\psi^*(z)\,F(b,z)+G(b,z)}{S(b,z)}=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{F(b,t)\,\bar{S}(b,t)\,dt}{W(t)\,\bar{W}(t)(t-z)}$$

converge to the limit

$$\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{F(t)\,\bar{T}(t)\,dt}{W(t)\,\bar{W}(t)(t-z)}$$

uniformly on compact subsets of the lower half-plane as b converges to infinity.

If the function $F(z)$ has an analytic extension to the half-plane $i\bar{z}-iz>-1$, and if the functions $b^{iz}F(b,z)$ converge to $F(z)$ uniformly on compact subsets of the half-plane, then the functions $b^{i}G(b,z)$ converge uniformly on compact subsets of the half-plane. Convergence is obtained across the real axis because the functions

$$(z-\bar{z})\,b^{iz}G(b,z)$$

remain uniformly bounded on compact subsets of the half-plane. The function $G(z)$ is analytic in the half-plane. The identity

$$\frac{i\psi(z)\,F(z)+G(z)}{T(z)}=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{F(t)\,\bar{T}(t)\,dt}{W(t)\,\bar{W}(t)(t-z)}$$

is satisfied at the points of the half-plane which lie above the real axis. The identity

$$\frac{-i\psi^*(z)\,F(z)+G(z)}{T(z)}=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{F(t)\,\bar{W}(t)\,dt}{W(t)\,\bar{W}(t)(t-z)}$$

is satisfied at the points of the half-plane which lie below the real axis.

These conditions are satisfied when

$$b^{\;iw}F(b,z)=\frac{[A(b,z)-iB(b,z)]\,S(b,w)-S(b,z)[A(b,w)-iB(b,w)]}{z-w}$$

with w in the upper half-plane, in which case

$$b^{-iw}G(b,z)=\frac{[C(b,z)-iD(b,z)]\,S(b,w)-S(b,z)[C(b,w)-iD(b,w)]}{z-w}.$$

The functions $b^{iz}F(b,z)$ converge in the metric topology of the space $\mathcal{F}(W)$ to the function

$$F(z)=\frac{W(z)\,T(w)-T(z)\,W(w)}{z-w}.$$

Convergence also takes place uniformly on compact subsets of the half-plane $i\bar{z} - iz > -1$.

The functions $b^{iz} G(b, z)$ converge uniformly on compact subsets of the half-plane to the function $G(z)$ defined by the identity

$$\frac{i\psi(z)\, F(z) + G(z)}{T(z)} = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{[W(t)\, T(w) - T(t)\, W(w)]\, \bar{T}(t)\, dt}{W(t)\, \bar{W}(t)(t - z)(t - w)}$$

when z lies above the real axis and by the identity

$$\frac{-i\psi^*(z)\, F(z) + G(z)}{T(z)} = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{[W(t)\, T(w) - T(t)\, W(w)]\, \bar{T}(t)\, dt}{W(t)\, \bar{W}(t)(t - z)(t - w)}$$

when z lies below the real axis. Since the identity

$$\int_{-\infty}^{+\infty} \frac{\bar{T}(t)\, dt}{\bar{W}(t)(t - z)(t - w)} = 0$$

holds when z lies above the real axis, and since the identity

$$-\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\bar{T}(t)\, dt}{\bar{W}(t)(t - z)(t - w)} = \frac{T^*(z)}{W^*(z)(z - w)}$$

holds when z lies below the real axis, and since the identity

$$G(z) = -i\, \frac{\psi(z)\, W(z)\, T(w) - T(z)\, \psi(w)\, W(w)}{z - w}$$

holds when z lies above the real axis, and the identity

$$G(z) = -2i\, \frac{T(z)\, T^*(z)\, T(w)}{W^*(z)(z - w)} + i\, \frac{\psi^*(z)\, W(z)\, T(w) + T(z)\, \psi(w)\, W(w)}{z - w}$$

holds when z lies below the real axis. The functions

$$\frac{D(b, z) + iC(b, z)}{A(b, z) - iB(b, z)}$$

therefore converge uniformly on compact subsets of the half-plane to a function which is equal to $\psi(z)$ above the real axis and which is equal to

$$-\psi^*(z) + 2\, \frac{T(z)\, T^*(z)}{W(z)\, W^*(z)}$$

below the real axis.

These results show in particular that the function $\psi(z)$ has an analytic extension to the half-plane $i\bar{z} - iz > -1$. If the extension is also denoted $\psi(z)$, then the identity

$$\psi(z) + \psi^*(z) = 2\,\frac{T(z)\,T^*(z)}{W(z)\,W^*(z)}$$

holds when z lies below the real axis.

Similar results will now be obtained for the function $\varphi(z)$. The calculations are simplified by assuming that $S(a, z) = E(a, z)$. Since the identities

$$\begin{aligned} &A(b, z) - iB(b, z) - D(b, z) - iC(b, z) \\ &\quad = [A(a, z) + iB(a, z)] \\ &\qquad \times [A(a, b, z) - iB(a, b, z) - D(a, b, z) - iC(a, b, z)] \end{aligned}$$

and

$$\begin{aligned} &A(b, z) - iB(b, z) + D(b, z) + iC(b, z) \\ &\quad = [A(a, z) - iB(a, z)] \\ &\qquad \times [A(a, b, z) - iB(a, b, z) + D(a, b, z) + iC(a, b, z)] \end{aligned}$$

then hold for every admissible parameter b, the functions

$$b^{iz}[A(a, b, z) - iB(a, b, z) - D(a, b, z) - iC(a, b, z)]$$

converge uniformly on compact subset of the half-plane $i\bar{z} - iz > -1$ to a function which is equal to

$$[1 - \psi(z)]\ W(z)/E^*(z) = [1 - \varphi(z)]\ W(z)/[A(z) - i\varphi(z)\,B(z)]$$

above the real axis and which is equal to

$$[1 + \psi^*(z)]\ W(z)/E(z)$$

below the real axis. And the functions

$$b^{iz}[A(a, b, z) - iB(a, b, z) + D(a, b, z) + iC(a, b, z)]$$

converge uniformly on compact subsets of the half-plane $i\bar{z} - iz > -1$ to a function which is equal to

$$[1 + \psi(z)]\ W(z)/E(z) = [1 + \varphi(z)]\ W(z)/[A(z) + i\varphi(z)\,B(z)]$$

above the real axis and which is equal to

$$[1 - \psi^*(z)]\ W(z)/E(z)$$

below the real axis. These conditions imply that $[1 - \psi(z)]/E^*(z)$ is analytic in the upper half-plane.

The functions

$$b^{iz}[A(a, b, z) - iB(a, b, z)]$$

converge uniformly on compact subsets of the half-plane $i\bar{z} - iz > -1$ to a function which is equal to

$$W(z)/[A(z) - i\varphi(z)\,B(z)]$$

above the real axis. The functions

$$b^{iz}[D(a, b, z) + iC(a, b, z)]$$

converge uniformly on compact subsets of the half-plane $i\bar{z} - iz > -1$ to a function which is equal to

$$\varphi(z)\,W(z)/[A(z) - i\varphi(z)\,B(z)]$$

above the real axis. The functions

$$\frac{D(a, b, z) + iC(a, b, z)}{A(a, b, z) - iB(a, b, z)}$$

converge uniformly on compact subsets of the half-plane $i\bar{z} - iz > -1$ to a function which is equal to $\varphi(z)$ in the upper half-plane.

These results imply that $\varphi(z)$ has an analytic extension to the half-plane $i\bar{z} - iz > -1$. If the extended function is also denoted $\varphi(z)$, then the identity

$$\varphi(z) + \varphi^*(z) = 2\,\frac{S(z)\,S^*(z)}{W_\varphi(z)\,W_\varphi^*(z)}$$

holds when z lies below the real axis, where

$$W_\varphi(z) = W(z)/[A(z) - i\varphi(z)\,B(z)].$$

Since $W_\varphi(z)$ is analytic in the half-plane $i\bar{z} - iz > -1$ and since $W(z)$ has no zeros in the half-plane, the function

$$A(z) - i\varphi(z)\,B(z)$$

has no zeros in the half-plane.

Since the expression

$$\frac{M(a, b, z)\,I\bar{M}(a, b, w) - S^*(a, b, z)\,IS(a, b, \bar{w})}{\pi(z - \bar{w})}$$

is the reproducing kernel function of the space $\mathcal{H}_{S^*(a,b)}(M(a,b))$, it is a positive-definite function of z and w in the complex plane. Since the identity

$$M^*(a, b, z)\, IM(a, b, z) = S^*(a, b, z)\, IS(a, b, z)$$

is satisfied, the expression

$$\frac{S(a, b, z)\, I\bar{S}(a, b, w) - M^*(a, b, z)\, IM(a, b, \bar{w})}{\pi(z - \bar{w})}$$

is a positive-definite function of z and w in the complex plane. Multiplication on the left by the row vector $(1, -i)$ and on the right by the conjugate transpose column vector produces a positive-definite function of z and w in the complex plane which can be written as a fraction with numerator

$$\begin{aligned}&[A(a, b, z) - iB(a, b, z)][\bar{D}(a, b, w) + i\bar{C}(a, b, w)]\\ &\quad + [D(a, b, z) + iC(a, b, z)][\bar{A}(a, b, w) + i\bar{B}(a, b, w)]\\ &\quad - 2S(a, b, z)\, \bar{S}(a, b, w)\end{aligned}$$

and denominator

$$\pi i(\bar{w} - z).$$

Since the identities

$$\lim \frac{D(a, b, z) + iC(a, b, z)}{A(a, b, z) - iB(a, b, z)} = \varphi(z)$$

and

$$\lim \frac{S(a, b, z)}{A(a, b, z) - iB(a, b, z)} = \frac{S(z)[A(z) - i\varphi(z)\, B(z)]}{W(z)}$$

hold uniformly on compact subsets of the half-plane $i\bar{z} - iz > -1$, the expression

$$\frac{\varphi(z) + \bar{\varphi}(w)}{\pi i(\bar{w} - z)} - 2\frac{S(z)/W_\varphi(z)\, \bar{S}(w)/\bar{W}_\varphi(w)}{\pi i(\bar{w} - z)}$$

is a positive-definite function of z and w in the half-plane.

This completes the proof of the theorem.

If λ is a given point in the lower half-plane, it may be convenient to normalize the defining function $E(z)$ of a space $\mathcal{H}(E)$ so that $E(\lambda) = 0$. Assume that $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ are given spaces such that the identity

$$(A(b, z), B(b, z)) = (A(a, z), B(a, z))\, M(a, b, z)$$

holds for a space $\mathcal{H}_S(M(a,b))$. If $E(a,\lambda)$ and $E(b,\lambda)$ both vanish, the identity

$$A(a,b,\lambda)-iB(a,b,\lambda)=D(a,b,\lambda)+iC(a,b,\lambda)$$

is satisfied. Estimates of matrix-valued entire functions result from such normalizations.

THEOREM 15. *If $\mathcal{H}_S(M)$ is a given space, the matrix*

$$S^*(z)\begin{pmatrix} p & i \\ -i & p\end{pmatrix} S(\bar z) - M(z)\begin{pmatrix} q & i \\ -i & q\end{pmatrix}\bar M(z)$$

is nonnegative when z is in the lower half-plane and when p and q are numbers, $-1 \leqslant q \leqslant p \leqslant 1$, such that

$$\begin{aligned}
&(p^2-q^2)\,S^*(z)\,S(\bar z)\\
&\quad+\tfrac{1}{2}(p-q)(1-q)[A(z)+iB(z)][\bar A(z)-i\bar B(z)]\\
&\quad+\tfrac{1}{2}(p-q)(1-q)[D(z)-iC(z)][\bar D(z)+i\bar C(z)]\\
&\quad-\tfrac{1}{2}(p-q)(1+q)[A(z)-iB(z)][\bar A(z)+i\bar B(z)]\\
&\quad-\tfrac{1}{2}(p-q)(1+q)[D(z)+iC(z)][\bar D(z)-i\bar C(z)]\\
&\quad+(1-q^2)[A(z)\,\bar D(z)-B(z)\,\bar C(z)+D(z)\,\bar A(z)\\
&\quad-C(z)\,\bar B(z)-S(z)\,\bar S(z)-S^*(z)\,S(\bar z)]\\
&\quad+\tfrac{1}{2}q(1-q)\,|A(z)+iB(z)-D(z)+iC(z)|^2\\
&\quad-\tfrac{1}{2}q(1+q)\,|A(z)-iB(z)-D(z)-iC(z)|^2
\end{aligned}$$

is nonnegative.

Proof of Theorem 15. Note that a space $\mathcal{H}_{S^*}(M)$ exists whenever a space $\mathcal{H}_S(M)$ exists. The transformation which takes

$$\begin{pmatrix} F(z) \\ G(z)\end{pmatrix}$$

into

$$\begin{pmatrix} F^*(z) \\ G^*(z)\end{pmatrix}$$

is an isometry of each space onto the other. The function $S(z)$ can therefore be replaced by the function $S^*(z)$ in the statement of the theorem. By continuity it is sufficient to give a proof of the nonnegativity of the matrix at points where $S^*(z)$ is nonzero. The matrix is nonnegative when $p=0$ by the positivity properties of reproducing kernel functions since z is in the lower half-plane.

A necessary and sufficient condition for the matrix to be nonnegative is for its determinant and trace to be nonnegative. For any given p the determinant is a quadratic function of q whose values are easily computed at the points 0, 1, and -1. The determinant is obtained by interpolation for other values of q as $S^*(z)\,S(\bar{z})$ times

$$\begin{aligned}
&(p^2-q^2)\,S^*(z)\,S(\bar{z})\\
&\quad+\tfrac{1}{2}(p-q)(1-q)[A(z)+iB(z)][\bar{A}(z)-i\bar{B}(z)]\\
&\quad+\tfrac{1}{2}(p-q)(1-q)[D(z)-iC(z)][\bar{D}(z)+i\bar{C}(z)]\\
&\quad-\tfrac{1}{2}(p-q)(1+q)[A(z)-iB(z)][\bar{A}(z)+i\bar{B}(z)]\\
&\quad-\tfrac{1}{2}(p-q)(1+q)[D(z)+iC(z)][\bar{D}(z)-i\bar{C}(z)]\\
&\quad+(1-q^2)[A(z)\,\bar{D}(z)-B(z)\,\bar{C}(z)+D(z)\,\bar{A}(z)\\
&\quad-C(z)\,\bar{B}(z)-S(z)\,\bar{S}(z)-S^*(z)\,S(\bar{z})]\\
&\quad+\tfrac{1}{2}q(1-q)|A(z)+iB(z)-D(z)+iC(z)|^2\\
&\quad-\tfrac{1}{2}q(1+q)|A(z)-iB(z)-D(z)-iC(z)|^2.
\end{aligned}$$

The determinant is a quadratic function of p for every q. The function has at least one zero. The matrix is nonnegative and invertible when p is greater than the largest zero. The matrix is nonpositive and invertible when p is less than the smallest zero. The matrix is nonnegative at the largest zero and nonpositive at the smallest zero. The matrix is indefinite when p lies between two zeros.

A necessary and sufficient condition for the matrix to be nonnegative is for the determinant and its partial derivative with respect to p to be nonnegative. The theorem is proved by showing that the partial derivative with respect to p is automatically nonnegative when the determinant is nonnegative and when $-1 \leqslant q \leqslant p \leqslant 1$. Argue by contradiction assuming that the derivative is nonpositive at such a point. Then the determinant remains nonnegative when p is replaced by q. It is therefore sufficient to give a proof of nonnegativity of the matrix when $p=q$.

When $p=q$, the determinant of the matrix reads

$$\begin{aligned}
&(1-p^2)[A(z)\,\bar{D}(z)-B(z)\,\bar{C}(z)+D(z)\,\bar{A}(z)\\
&\quad-C(z)\,\bar{D}(z)-S(z)\,\bar{S}(z)-S^*(z)\,S(\bar{z})]\\
&\quad+\tfrac{1}{2}p(1-p)|A(z)+iB(z)-D(z)+iC(z)|^2\\
&\quad-\tfrac{1}{2}p(1+p)|A(z)-iB(z)-D(z)-iC(z)|^2.
\end{aligned}$$

The determinant is a quadratic function of p which is nonpositive outside of the interval $(-1, 1)$. The points where the determinant is nonnegative are points where the matrix is nonnegative.

This completes the proof of the theorem.

An example of a space $\mathscr{H}_S(M)$ is obtained when $S(z) = z - w$ for a complex number w and when the elements of the space are constants. If the space contains a nonzero element, it is spanned by an element

$$\begin{pmatrix} u \\ v \end{pmatrix}$$

the square of whose norm is equal to 2π. The identity for difference-quotients implies the identity

$$w - \bar{w} = u\bar{v} - v\bar{u}.$$

The identity

$$M(z) = \begin{pmatrix} z - w + u\bar{v} & -u\bar{u} \\ v\bar{v} & z - w - v\bar{u} \end{pmatrix} \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

holds for real numbers P, Q, R, and S such that

$$PS - QR = 1.$$

These conditions imply that the function

$$A(z) - iB(z) - D(z) - iC(z)$$

is linear. If it has a zero λ, then the identities

$$\begin{aligned} P - iQ &= [\lambda - w - i(u - iv)\,\bar{u}]/k \\ &= [\lambda - w - i(\bar{u} - i\bar{v})\,u]/k \end{aligned}$$

and

$$\begin{aligned} S + iR &= [\lambda - w + (u - iv)\,\bar{v}]/k \\ &= [\lambda - w + (\bar{u} - i\bar{v})\,v]/k \end{aligned}$$

are satisfied for a nonzero number k such that

$$\begin{aligned} 2k\bar{k} &= (\lambda - w)(\bar{\lambda} - \bar{w}) + (i\lambda - i\bar{\lambda})(u - iv)(\bar{u} + i\bar{v}) \\ &= (\lambda - \bar{w})(\bar{\lambda} - w) + (i\lambda - i\bar{\lambda})(u + iv)(\bar{u} - i\bar{v}). \end{aligned}$$

These conditions imply that the right side is positive. It follows that

$$M(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix},$$

where $A(z)$, $B(z)$, $C(z)$, and $D(z)$ are the unique linear functions, which are real for real z, such that

$$\begin{aligned} A(z) - iB(z) &= (z - \bar{w})(\lambda - w)/k - i(z - \lambda)(u - iv)\,\bar{u}/k \\ &= (z - w)(\lambda - \bar{w})/k - i(z - \lambda)(\bar{u} - i\bar{v})\,u/k \end{aligned}$$

and

$$\begin{aligned} D(z) + iC(z) &= (z - \bar{w})(\lambda - w)/k + (z - \lambda)(u - iv)\,\bar{v}/k \\ &= (z - w)(\lambda - \bar{w})/k + (z - \lambda)(\bar{u} - i\bar{v})\,v/k. \end{aligned}$$

Note that the identity

$$\begin{aligned} &A(z)\,\bar{D}(w) - B(z)\,\bar{C}(w) + D(z)\,\bar{A}(w) - C(z)\,\bar{B}(w) \\ &\quad = S(z)\,\bar{S}(z) + S^*(z)\,S(\bar{w}) \end{aligned}$$

holds for all complex numbers z and w.

Assume that $E(z)$ is an entire function which has no zeros in the upper half-plane and which satisfies the inequality

$$|E(x - iy)| \leqslant |E(x + iy)|$$

for $y > 0$. A phase function $\phi(x)$ associated with $E(z)$ is a continuous function of real x such that

$$E(x)\exp[i\phi(x)]$$

is nonnegative for all real x. Such a function $\phi(x)$ is nonincreasing and differentiable. It may be convenient to choose the function $\phi(x)$ to vanish at the origin when $E(z)$ has a positive value at the origin.

If the strict inequality

$$|E(x - iy)| < |E(x + iy)|$$

holds for $y > 0$, the phase function $\phi(x)$ is an increasing function of x which gives information about the space $\mathscr{H}(E)$. If α is a given real number, an orthogonal set in the space is formed by the functions of the form

$$\frac{E(z)\exp(i\alpha) - E^*(z)\exp(-i\alpha)}{z - t}$$

for a real number t such that

$$\phi(t) \equiv 0 \text{ modulo } \pi.$$

The orthogonal set is complete if the function

$$E(z)\exp(i\alpha) - E^*(z)\exp(-i\alpha)$$

does not belong to the space. If the function belongs to the space, it spans the orthogonal complement of the domain of multiplication by z in the space. A complete orthogonal set is then obtained by including this function. At most one real number α modulo π exists such that the function belongs to the space.

A comparison theorem holds for phase functions in factorizations.

THEOREM 16. *Assume that $S(a, c, z)$ is an entire function such that multiplication by $S(a, c, z)$ is an isometric transformation of a space $\mathcal{H}(E(a))$ into a space $\mathcal{H}(E(c))$ and that $E(a, z)$ has no real zeros. Let $M(a, c, z)$ be the unique matrix-valued entire function such that a space $\mathcal{H}_{S(a,c)}(M(a, c))$ exists, such that the identity*

$$(A(c, z), B(c, z)) = (A(a, z), B(a, z))\, M(a, c, z)$$

is satisfied, and such that an isometric transformation of the space $\mathcal{H}_{S(a,c)}(M(a, c))$ onto the orthogonal complement in $\mathcal{H}(E(c))$ of the image of $\mathcal{H}(E(a))$ is defined by taking

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

into

$$\sqrt{2}\,[A(a, z)\,F(z) + B(a, z)\,G(z)].$$

Assume that $E(a, z)$ and $E(c, z)$ have positive values at the origin, that $S(a, c, z)$ has value one at the origin, and that the value of $M(a, c, z)$ at the origin is the identity matrix. Let $\phi(a, x)$ and $\phi(c, x)$ be the phase functions associated with $E(a, z)$ and $E(c, z)$ which vanish at the origin. Then the inequality

$$\phi(a, x)/x \leqslant \phi(c, x)/x$$

holds for all real numbers x.

Proof of Theorem 16. A proof of the theorem will first be given in the case that

$$\lambda S(a, c, z) = \lambda - z$$

for a nonreal number λ and that

$$\lambda\bar{\lambda}M(a, c, z) = \begin{pmatrix} z - \lambda + u\bar{v} & -u\bar{u} \\ v\bar{v} & z - \lambda - v\bar{u} \end{pmatrix} \begin{pmatrix} -\lambda - v\bar{u} & u\bar{u} \\ -v\bar{v} & -\lambda + u\bar{v} \end{pmatrix}$$

for complex numbers u and v such that

$$\lambda - \bar{\lambda} = u\bar{v} - v\bar{u}.$$

The identities

$$\bar{\lambda}A(c, z) = (\bar{\lambda} - z)\, A(a, z) - (z/\lambda)[A(a, z)\,\bar{u} + B(a, z)\,\bar{v}]\, v$$

and

$$\bar{\lambda}B(c, z) = (\bar{\lambda} - z)\, B(a, z) + (z/\lambda)[A(a, z)\,\bar{u} + B(a, z)\,\bar{v}]\, u$$

are satisfied. The identity

$$\tan \phi(c, x) = \frac{(\bar{\lambda} - x)\tan \phi(a, x) + (x/\lambda)[\bar{u} + \tan \phi(a, x)\,\bar{v}]\, u}{(\bar{\lambda} - x) - (x/\lambda)[\bar{u} + \tan \phi(a, x)\,\bar{v}]\, v}$$

holds for all real numbers x. Since $\tan \phi(a, x)$ and $\tan \phi(c, x)$ are real, the equation

$$\tan \phi(a, x) = \tan \phi(c, x)$$

has no solution when x is real and nonzero. Since $\phi(c, x) - \phi(a, x)$ is a continuous real-valued function of real x, it makes no change of sign on the positive or the negative half-line. Since the function has the positive derivative

$$\frac{u\bar{u}}{\lambda\bar{\lambda}}$$

at the origin, it is positive for positive x and negative for negative x. It follows that the inequality

$$\phi(a, x)/x \leqslant \phi(c, x)/x$$

holds for all real numbers x. It was assumed in the construction that λ is not real. But the same inequality holds by continuity when λ is real and nonzero. The inequality is strict when λ is not real.

An inductive argument is used to prove the general case of the theorem. A partial ordering of equivalence classes of spaces $\mathcal{H}_S(M)$ is considered. Spaces $\mathcal{H}_{S_0}(M_0)$ and $\mathcal{H}_{S_1}(M_1)$ are considered equivalent if the function $S_1(z)/S_0(z)$ is analytic and without zeros in the upper and lower half-planes and if multiplication by the function is an isometry of the space $\mathcal{H}_{S_0}(M_0)$ onto the space $\mathcal{H}_{S_1}(M_1)$.

The (equivalence class of a) space $\mathcal{H}_{S(a)}(M(a))$ is considered less than or equal to the (equivalence class of a) space $\mathcal{H}_{S(b)}(M(b))$ if the identities

$$S(b, z) = S(a, z)\, S(a, b, z)$$

and

$$M(b, z) = M(a, z)\, M(a, b, z)$$

hold for a space $\mathcal{H}_{S(a,b)}(M(a, b))$.

A space $\mathcal{H}_{S(a,b)}(M(a, b))$ is considered admissible if it is less than or equal to the space $\mathcal{H}_{S(a,c)}(M(a, c))$, if $S(a, b, z)$ has value one at the origin, if the value of $M(a, b, z)$ at the origin is the identity matrix, and if the space $\mathcal{H}(E(b))$ defined by the identity

$$(A(b, z), B(b, z)) = (A(a, z), B(a, z))\, M(a, b, z)$$

has phase function $\phi(b, x)$, zero at the origin, which satisfies the inequality

$$\phi(a, x)/x \leqslant \phi(b, x)/x$$

for all real numbers x.

A maximal equivalence class of admissible spaces exists by Zorn's lemma. Let $\mathcal{H}_{S(a,b)}(M(a, b))$ be the choice of a representative in a maximal equivalence class. Consider the space $\mathcal{H}_{S(b,c)}(M(b, c))$ which is defined by the identities

$$S(a, c, z) = S(a, b, z)\, S(b, c, z)$$

and

$$M(a, c, z) = M(a, b, z)\, M(b, c, z).$$

It will be shown that the transformation which takes

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

into

$$\begin{pmatrix} [F(z)\, S(b, c, w) - S(b, c, z)\, F(w)]/(z - w) \\ [G(z)\, S(b, c, w) - S(b, c, z)\, G(w)]/(z - w) \end{pmatrix}$$

for a complex number w which is not a zero of $S(b, c, z)$ has no nonzero eigenvalues.

Argue by contradiction assuming that a nonzero eigenvalue exists. Then the eigenvalue is of the form $S(b, c, w)/(\lambda - w)$ for a zero λ of $S(b, c, z)$. The entries of the eigenvector are constant multiples of $S(b, c, z)/(z - \lambda)$. If $S(b, t, z)$ is defined by

$$\lambda S(b, t, z) = \lambda - z,$$

then

$$S(b, c, z) = S(b, t, z)\, S(t, c, z)$$

for an entire function $S(t, c, z)$. The identity

$$M(b, c, z) = M(b, t, z)\, M(t, c, z)$$

holds for spaces $\mathscr{H}_{S(b,t)}(M(b, t))$ and $\mathscr{H}_{S(t,c)}(M(t, c))$, where

$$\lambda\bar{\lambda} M(b, t, z) = \begin{pmatrix} z - \lambda + u\bar{v} & -u\bar{u} \\ v\bar{v} & z - \lambda - v\bar{u} \end{pmatrix} \begin{pmatrix} -\bar{\lambda} - v\bar{u} & u\bar{u} \\ -v\bar{v} & -\bar{\lambda} + u\bar{v} \end{pmatrix}$$

for complex numbers u and v such that

$$\lambda - \bar{\lambda} = u\bar{v} - v\bar{u}.$$

If a space $\mathscr{H}(E(t))$ is defined by the factorization

$$(A(t, z), B(t, z)) = (A(b, z), B(b, z))\, M(b, t, z)$$

and if the phase function $\phi(t, x)$ of $E(t, z)$ is normalized so as to have value zero at the origin, then it has been seen that the inequality

$$\phi(b, x)/x \leqslant \phi(t, x)/x$$

holds for all real numbers x. It follows that an admissible space $\mathscr{H}_{S(a,t)}(M(a, t))$ is defined by

$$S(a, t, z) = S(a, b, z)\, S(b, t, z)$$

and

$$M(a, t, z) = M(a, b, z)\, M(b, t, z).$$

A contradiction of the maximal choice of the (equivalence class of the) space $\mathscr{H}_{S(a,b)}(M(a, b))$ is obtained since it is less than the (equivalence class of the) space $\mathscr{H}_{S(a,t)}(M(a, t))$.

Since the transformation which takes

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

into

$$\begin{pmatrix} [F(z)\,S(b,c,w) - S(b,c,z)\,F(w)]/(z-w) \\ [G(z)\,S(b,c,w) - S(b,c,z)\,G(w)]/(z-w) \end{pmatrix}$$

in the space $\mathscr{H}_{S(b,c)}(M(b,c))$ has no nonzero eigenvalues, the zeros of $S(b,c,z)$ are real. A zero of $S(b,c,z)$ is a zero of $F(z)$ and $G(z)$ of at least equal multiplicity for every element

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

of the space $\mathscr{H}_{S(b,c)}(M(b,c))$. Since a zero of $S(b,c,z)$ is a zero of $A(b,c,z)$, $B(b,c,z)$, $C(b,c,z)$, and $D(b,c,z)$ of at least equal multiplicity, it is a zero of $A(c,z)$ and $B(c,z)$ of at least equal multiplicity.

Note that a space $\mathscr{H}_{S^*(b,c)}(M(b,c))$ exists and that an isometric transformation of the space onto the space $\mathscr{H}_{S(b,c)}(M(b,c))$ is defined by taking

$$\begin{pmatrix} F(z) \\ G(z) \end{pmatrix}$$

into

$$\begin{pmatrix} F^*(z) \\ G^*(z) \end{pmatrix}.$$

Multiplication by $S(a,b,z)\,S^*(b,c,z)$ is an isometric transformation of the space $\mathscr{H}(E(a))$ into the space $\mathscr{H}(E(c))$. Since the ratio

$$S^*(b,c,z)/S(b,c,z)$$

is an entire function which is of bounded type in the upper half-plane, has absolute value one on the real axis, and has value one at the origin, it is of the form

$$\exp(2i\tau z)$$

for a real number τ. The entire function

$$S(b,c,z)\exp(i\tau z)$$

is real for real z, has only real zeros, and has value one at the origin. The choice of representatives can be made in equivalence classes so that the function is identically one.

A space $\mathscr{H}_S(M)$ is denoted $\mathscr{H}(M)$ in the case $S(z) = 1$. The proof of the theorem now reduces to a case in which it is already known [2]. The existence of a space $\mathcal{H}_{S(b,c)}(M(b,c))$ with

$$S(b, c, z) = \exp(-i\tau z)$$

for a real number τ implies the existence of a space $\mathcal{H}(M(b,c))$. Since the inequality

$$\phi(b, x)/x \leqslant \phi(c, x)/x$$

holds for all real numbers x, the desired inequality

$$\phi(a, x)/x \leqslant \phi(c, x)/x$$

holds for all real numbers x.

This completes the proof of the theorem.

Similar results will now be obtained for the classical zeta-function, which is associated with the (principal) character modulo one. The results obtained will be stated more generally for the principal character modulo r where r is any positive integer.

Related Hilbert spaces of entire functions exist as in the case of nonprincipal characters. The definition of the spaces now requires additional linear factors because of the singularities of the zeta-functions. If χ is the principal character modulo r, a space $\mathcal{H}(E)$ exists,

$$E(z) = (r/\pi)^{-(1/2)iz}(-\tfrac{1}{2}iz)\,\Gamma(\tfrac{3}{2} - \tfrac{1}{2}iz)\,\zeta_\chi(1 - iz).$$

The function $E(z)$ is of Pólya class.

A variant of these Hilbert spaces of entire functions is now used. If χ is the principal character modulo r and if ν is a given number, $0 < \nu < \frac{1}{2}$, then a space $\mathcal{H}(E)$ exists,

$$E(z) = (r/\pi)^{-iz}\,W(z)\,\zeta_\chi(1 + \nu - 2iz),$$

where

$$W(z) = (\tfrac{1}{2}\nu - iz)\,\Gamma(\tfrac{1}{2}\nu + \tfrac{3}{2} - iz).$$

The function $E(z)$ is of Pólya class. The functional identity states that

$$E(z - ik) = E^*(z)$$

when χ is the principal character modulo one and $k = \nu + \frac{1}{2}$. In this notation the Riemann hypothesis is the conjecture that the zeros of $E(z)$ lie on the line $z + ik = \bar{z}$.

A new weight function $W(z)$ appears. The situation is different from the previous one in that the function

$$W(z)/W(z+ik)$$

does not have positive real part in the half-plane $i\bar{z}-iz>-k$. In fact the function vanishes at the point $-\frac{1}{2}i\nu$, which lies within the half-plane. The space $\mathscr{F}(W)$ does not have the previous positivity property.

A generalization of the Sonine spaces can however be constructed. Define

$$W(z)=(\mu-iz)\,\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}-iz)$$

for given numbers μ and ν, $\mu>0$ and $\nu>-1$. Let c be a given positive number. The set of entire functions $F(z)$ such that $c^{iz}F(z)$ and $c^{iz}F^*(z)$ belong to the space $\mathscr{F}(W)$ is a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) when considered with the unique scalar product such that the multiplication by c^{iz} is an isometric transformation of the space into the space $\mathscr{F}(W)$. The space is called the μ-augmented Sonine space of order ν and parameter c.

The space is a space $\mathscr{H}(E)$ for an entire function $E(z)$ which satisfies the identity

$$E^*(z)=E(-z).$$

It is convenient to choose $E(z)$ to vanish at the point $-i\mu$ and to normalize it so as to have positive values on the upper half of the imaginary axis. A space $\mathscr{H}(E_0)$ exists

$$E_0(z)=E(z)/(\mu-iz),$$

and it is isometrically equal to the Sonine space of order ν and parameter c. Multiplication by $\mu-iz$ is an isometric transformation of the Sonine space of order ν and parameter c onto the set of elements of the space $\mathscr{H}(E)$ which vanish at $-i\mu$.

A theory of asymptotic behavior for the augmented Sonine spaces is derived from the known asymptotic behavior of the Sonine spaces.

THEOREM 17. *Define*

$$W(z)=(\mu-iz)\,\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}-iz)$$

for given numbers μ and ν, $\mu>0$, and $\nu\geqslant 0$. For each positive number c let $E(c,z)$ be the defining function of the μ-augmented Sonine space of order ν

and parameter c, chosen so as to vanish at $-i\mu$ and to have positive values on the upper half of the imaginary axis. Then the identity

$$W(z) = \lim t^{iz} E(t, z)$$

holds uniformly on compact subsets of the half-plane $i\bar{z} - iz > -1$.

Proof of Theorem 17. By definition the space $\mathscr{H}(E(c))$ is the set of entire functions $F(z)$ such that $c^{iz}F(z)$ and $c^{iz}F^*(z)$ belong to the space $\mathscr{F}(W)$. Since the defining function $E(c, z)$ of the μ-augmented Sonine space of order ν and parameter c has been chosen to vanish at $-i\mu$, a defining function of the $(\frac{1}{2}\nu + \frac{1}{2})$-augmented Sonine space of order ν is

$$E(c, z)(\tfrac{1}{2}\nu + \tfrac{1}{2} - iz)/(\mu - iz).$$

It is therefore sufficient to give a proof of the theorem in the case $\mu = \frac{1}{2}\nu + \frac{1}{2}$. Since the identity

$$W(z) = \Gamma(\tfrac{1}{2}\nu + \tfrac{3}{2} - iz)$$

is then satisfied, the space $\mathscr{H}(E(c))$ is then the Sonine space of order $\nu + 2$ and parameter c.

It has been shown that a defining function $E_0(c, z)$ of the Sonine space of order $\nu + 2$ and parameter c can be chosen so that $E_0(c, z - \frac{1}{2}i)$ is of Pólya class and so that the identity

$$W(z) = \lim t^{iz} E_0(t, z)$$

holds uniformly on compact subsets of the half-plane $i\bar{z} - iz > -1$.

The identity

$$\begin{aligned} &E(c, z)\, \bar{E}(c, \tfrac{1}{2}i\nu + \tfrac{1}{2}i) \\ &\quad = E_0(c, z)\, \bar{E}_0(c, \tfrac{1}{2}i\nu + \tfrac{1}{2}i) - E_0^*(c, z)\, E_0(c, -\tfrac{1}{2}i\nu - \tfrac{1}{2}i) \end{aligned}$$

is satisfied because $E(c, z)$ vanishes at $-\frac{1}{2}i\nu - \frac{1}{2}i$. It will be shown that the identity

$$W(z)\, \bar{W}(\tfrac{1}{2}i\nu + \tfrac{1}{2}i) = \lim t^{iz - (1/2)\nu - (1/2)} E(c, z)\, \bar{E}(t, \tfrac{1}{2}i\nu + \tfrac{1}{2}i)$$

holds uniformly on compact subsets of the half-plane $i\bar{z} - iz > -1$. Since the identity

$$W(z)\, \bar{W}(\tfrac{1}{2}i\nu + \tfrac{1}{2}i) = \lim t^{iz - (1/2)\nu - (1/2)} E_0(t, z)\, \bar{E}_0(t, \tfrac{1}{2}i\nu + \tfrac{1}{2}i)$$

holds uniformly on compact subsets of the half-plane, it is sufficient to show that the identity

$$0 = \lim t^{iz - (1/2)\nu - (1/2)} E_0^*(t, z)\, E_0(t, -\tfrac{1}{2}i\nu - \tfrac{1}{2}i)$$

holds uniformly on compact subsets of the half-plane.

Since the inequality

$$|E_0(t, -\tfrac{1}{2}iv - \tfrac{1}{2}i)| \leqslant |E_0(t, \tfrac{1}{2}iv - \tfrac{1}{2}i)|$$

holds because the entire function $E_0(t, z)$ is of Pólya class, it is sufficient to show that the identity

$$0 = \lim t^{iz-(1/2)v-(1/2)} E_0^*(t, z)\, E_0(t, \tfrac{1}{2}iv - \tfrac{1}{2}i)$$

holds uniformly on compact subsets of the half-plane. Since the identity

$$W(\tfrac{1}{2}iv - \tfrac{1}{2}i) = \lim t^{-(1/2)v+(1/2)} E_0(t, \tfrac{1}{2}iv - \tfrac{1}{2}i)$$

is satisfied, it is sufficient to show that the identity

$$0 = \lim t^{iz-1} E_0^*(t, z)$$

holds uniformly on compact subsets of the half-plane.

The identity holds when z is in the upper half-plane because the inequality

$$|E_0^*(t, z)| < |E_0(t, z)|$$

holds in the half-plane and because the identity

$$W(z) = \lim t^{iz} E_0(t, z)$$

is satisfied. The desired identity also holds when $-1 < i\bar{z} - iz \leqslant 0$ because the identity

$$W^*(z) = \lim t^{-iz} E_0^*(t, z)$$

is then satisfied.

The identity

$$W(\tfrac{1}{2}iv + \tfrac{1}{2}i) = \lim t^{-(1/2)v-(1/2)} E(t, \tfrac{1}{2}iv + \tfrac{1}{2}i)$$

now follows because $E(t, z)$ has positive values on the upper half of the imaginary axis. The desired identity

$$W(z) = \lim t^{iz} E(t, z)$$

therefore holds uniformly on compact subsets of the half-plane $i\bar{z} - iz > -1$.

This completes the proof of the theorem.

A construction of Hilbert spaces of entire functions associated with Dirichlet zeta-functions is made from the augmented Sonine spaces of entire functions.

THEOREM 18. *Assume that χ is the principal character modulo r. Let ν be a given positive number. Consider the space $\mathscr{H}(E)$ corresponding to the entire function*

$$E(z) = (r/\pi)^{-iz}\, W(z)\, \zeta_\chi(1 + \nu - 2iz),$$

where

$$W(z) = (\tfrac{1}{2}\nu - iz)\, \Gamma(\tfrac{1}{2}\nu + \tfrac{3}{2} - iz).$$

For each positive integer n, which is divisible by r, such that n/r is relatively prime to r and is not divisible by the square of a prime, define

$$S_n(a, b, z) = \prod (p^{-iz} - \chi(p)\, p^{-1-\nu+iz}),$$

where the product is taken over the prime divisors p of n which are not divisors of r. For each such index n define $\mathscr{H}(E_n(b))$ to be the $(\frac{1}{2}\nu)$-augmented Sonine space of order $\nu + 2$ and parameter n/π. Then the set $\mathscr{H}(E)$ becomes a space $\mathscr{H}(E_n(a))$ when considered with the unique scalar product such that multiplication by $S_n(a, b, z)$ is an isometric transformation of the space $\mathscr{H}(E_n(a))$ into the space $\mathscr{H}(E_n(b))$. If λ is a given zero of $E(z)$, then $E_n(a, z)$ can be chosen to have a zero at λ. The identity

$$E(z)\, \bar{E}(w) = \lim E_n(a, z)\, \bar{E}_n(a, w)$$

then holds uniformly for z and w in any compact subset of the half-plane. Multiplication by

$$(n/\pi)^{iz}\, S_n(a, b, z)$$

is an isometric transformation of the space $\mathscr{H}(E_n(a))$ into the space $\mathscr{F}(W)$ for every index n. The expression

$$\frac{W(z)\, \bar{W}(\lambda)}{2\pi i(\lambda - z)} - (n/\pi)^{iz - i\lambda}\, \frac{S_n(a, b, z)\, E_n(a, z)\, \bar{E}_n(a, \bar{\lambda})\, \bar{S}_n(a, b, \bar{\lambda})}{2\pi i(\lambda - z)}$$

is an element of the space $\mathscr{F}(W)$ which is orthogonal to the image of the space $\mathscr{H}(E_n(a))$ and the square of whose norm is less than or equal to

$$\frac{|W(\bar{\lambda})|^2}{2\pi i(\lambda - \bar{\lambda})} \times \left[1 - \prod (1 - p^{-1-\nu})^2 \prod (1 - p^{-1-\nu+i\bar{\lambda}-i\lambda})^2\right]$$

where the products are taken over the primes p which are not divisors of n.

Proof of Theorem 18. The construction of Hilbert spaces of entire functions from the augmented Sonine spaces is similar to the construction from the Sonine spaces in view of the results on asymptotic behavior for the augmented Sonine spaces obtained in the previous theorem.

This completes the proof of the theorem.

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