Suppose indeed that $0 < L < A$. The space $\mathscr{C}(-L, L)$ is contained in each of the $L_p(-L, L)$ and *dense* in the latter. Therefore, since uniform convergence on $[-L, L]$ implies L_p convergence thereon, the linear combinations of the $e^{i\lambda_n t}$, being uniformly dense in $\mathscr{C}(-L, L)$, will be $\| \quad \|_p$ dense in $L_p(-L, L)$.

Let, on the other hand, $L > A$. *Then the finite linear combinations of the* $e^{i\lambda_n t}$ *are not* $\| \quad \|_p$ *dense in any of the spaces* $L_p(-L, L)$. To see this, we can take an L', $A < L' < L$, and apply duality as above to get a non-zero complex measure μ on $[-L', L']$ (*sic!*) with $\hat{\mu}(\lambda_n) = 0$ for each n. If $h > 0$ is sufficiently small, the function

$$\varphi(t) = \frac{1}{2h}\int_{t-h}^{t+h} d\mu(\tau)$$

is supported on $[-L, L]$; φ is clearly *bounded*, hence in each of the *duals* $L_q(-L, L)$, $1 < q \leqslant \infty$, to our L_p spaces. We have

$$\hat{\varphi}(z) = \frac{\sin hz}{hz}\hat{\mu}(z),$$

so in particular $\hat{\varphi}(z) \not\equiv 0$, hence $\varphi(t)$ *cannot vanish a.e. on* $[-L, L]$. By the same token, however,

$$\int_{-L}^{L} e^{i\lambda_n t}\varphi(t)\,dt = \hat{\varphi}(\lambda_n) = 0$$

for each n, so finite linear combinations of the $e^{i\lambda_n t}$ cannot be dense in $L_p(-L, L)$.

A considerable refinement of the preceding observation is due to L. Schwartz – Levinson also certainly knew of it:

Theorem. *If, for* $L > 0$, *the finite linear combinations of the* $e^{i\lambda_n t}$ *are not uniformly dense in* $\mathscr{C}(-L, L)$, *removal of any one of the exponentials* $e^{i\lambda_n t}$ *leaves us with a collection whose finite linear combinations are not dense in* $L_1(-L, L)$ *(hence not dense in any of the* $L_p(-L, L)$, $1 \leqslant p < \infty$*).*

Taking the exponentials e^{int}, $n \in \mathbb{Z}$, on $[-\pi, \pi]$, we see that this result is sharp. Finite linear combinations of the former *are* dense in each of the $L_p(-\pi, \pi)$, $1 \leqslant p < \infty$, but can *uniformly* approximate *only* those $f \in \mathscr{C}(-\pi, \pi)$ for which $f(-\pi) = f(\pi)$. In order to have (uniform) completeness on $[-\pi, \pi]$ we must use *one additional* imaginary exponential $e^{i\lambda z}$, $\lambda \notin \mathbb{Z}$ (*any* such one will do!).

Problem 31

Prove the above theorem. (Hint: Assuming a non-zero measure μ on

$[-L, L]$ with $\hat{\mu}(\lambda_n) = 0$, take, for instance, λ_1, and look at the function φ supported on $[-L, L]$ given by

$$\varphi(t) = e^{-i\lambda_1 t}\int_{-L}^{t} e^{i\lambda_1\tau}\,d\mu(\tau)$$

for $-L \leqslant t \leqslant L$. Compute $\hat{\varphi}(z)$.)

1. Application of the formula from §C

Let us, without further, ado, proceed to this chapter's basic result about completeness.

Theorem. *Given the sequence of distinct frequencies* $\lambda_n > 0$, *suppose that for some* $D > 0$ *there are* disjoint *half-open intervals* $(a_k, b_k]$ *in* $(0, \infty)$ *such that, for each* k,

$$\frac{\text{number of } \lambda_n \text{ in } (a_k, b_k]}{b_k - a_k} \geqslant D,$$

and that

$$\sum_k \left(\frac{b_k - a_k}{a_k}\right)^2 = \infty.$$

Then, if $0 < L < \pi D$, *the exponentials* $e^{i\lambda_n t}$ *are complete on* $[-L, L]$.

Remark. The *second* condition on the intervals $(a_k, b_k]$ has already figured in Beurling's gap theorem (§A.2, Chapter VII).

Proof of Theorem. Assume that the $e^{i\lambda_n t}$ are *not* complete on $[-L, L]$, where $0 < L < \pi D$. Then, as in the discussion immediately preceding the present article, there is a *non-zero complex measure* μ on $[-L, L]$ with $\hat{\mu}(\lambda_n) = 0$. The function $\hat{\mu}(z)$ is entire, of exponential type $\leqslant L$, and bounded on the real axis, indeed, wlog,

$$(*) \qquad |\hat{\mu}(z)| \leqslant e^{L|\Im z|},$$

as one sees by direct inspection of the Fourier–Stieltjes integral used to define $\hat{\mu}(z)$. Our aim is to show that

$$\int_{-\infty}^{\infty} \frac{\log^- |\hat{\mu}(x)|}{1 + x^2}\,dx = \infty,$$

which, by §G.2 of Chapter III, implies that $\hat{\mu}(z) \equiv 0$, contrary to μ's being non-zero.

If $\hat{\mu}(z)$ is not to vanish identically, we must have $b_k \xrightarrow[k]{} \infty$, for each interval $(a_k, b_k]$ contains at least *one* zero λ_n of $\hat{\mu}$. We may therefore re-enumerate the $(a_k, b_k]$ so as to ensure that
$0 < a_1 < b_1 \leqslant a_2 < b_2 \leqslant a_3 < \cdots$ (with $a_k \xrightarrow[k]{} \infty$); this we henceforth suppose done.

Following the idea mentioned at the beginning of this chapter, we proceed to apply the Jensen formula from the preceding § to certain ellipses whose centres have been *moved* from the origin to the midpoints of the $(a_k, b_k]$. Let us fix our attention on *any one* of the latter which, for the moment, we designate as $(c - R,\ c + R]$. We take a fixed small number $\gamma > 0$ (whose value will be assigned presently) and, with

$$\frac{R}{\cosh \gamma} < r < R,$$

apply the corollary at the end of §C to $\hat{\mu}(c + z)$ (*sic!*) in the ellipse

$$z = r\cosh(\gamma + i\vartheta), \quad 0 \leqslant \vartheta \leqslant 2\pi;$$

we are, in other words, looking at $\hat{\mu}(z)$ in an ellipse whose major axis is $[c - r\cosh\gamma,\ c + r\cosh\gamma]$:

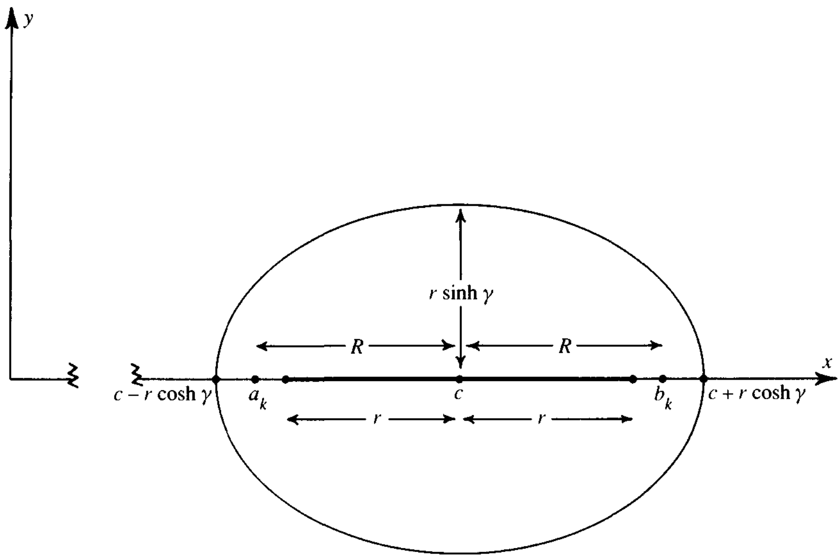


Figure 169

If η, $0 < \eta < \gamma$, is a number such that $r\cosh\eta \geqslant R$ and N denotes the number of λ_n in $(c - R,\ c + R]$, we find that

$$N(\gamma-\eta) \leqslant \frac{1}{2\pi}\int_0^{2\pi} \log|\hat{\mu}(c+r\cosh(\gamma+\mathrm{i}\vartheta))|\,\mathrm{d}\vartheta$$
$$- \frac{1}{\pi}\int_{-r}^{r} \frac{\log|\hat{\mu}(c+t)|}{\sqrt{(r^2-t^2)}}\,\mathrm{d}t.$$

By $(*)$, $\log|\hat{\mu}(c+r\cosh(\gamma+\mathrm{i}\vartheta))| \leqslant Lr\sinh\gamma\,|\sin\vartheta|$, and this, substituted in the first integral on the right, yields

$$\frac{1}{\pi}\int_{-r}^{r} \frac{\log|\hat{\mu}(c+t)|}{\sqrt{(r^2-t^2)}}\,\mathrm{d}t \leqslant \frac{2L}{\pi}r\sinh\gamma - N(\gamma-\eta).$$

However, the number N of λ_n in $(a_k, b_k]$ is by hypothesis $\geqslant 2RD \geqslant 2Dr$. Hence

$$(\dagger) \qquad \frac{1}{\pi}\int_{-r}^{r} \frac{\log|\hat{\mu}(c+t)|}{\sqrt{(r^2-t^2)}}\,\mathrm{d}t \leqslant \frac{2}{\pi}\Big(L\sinh\gamma - \pi D(\gamma-\eta)\Big)r$$

for $\dfrac{R}{\cosh\gamma} < \dfrac{R}{\cosh\eta} \leqslant r \leqslant R.$

Now we are assuming that $\pi D > L$. We can therefore *fix* $\gamma > 0$ small enough (independently of k !) so that

$$\pi D\frac{\gamma}{\sinh\gamma} > L$$

and *then fix* $\eta > 0$ *much smaller than the small number* γ *so as to still have*

$$\pi D\frac{\gamma-\eta}{\sinh\gamma} > L.$$

With such values of γ and η *the right side of* $(\dagger)$ *is negative.*

Multiply both sides of $(\dagger)$ by $r\mathrm{d}r$ (the purpose of the factor r being to make our computation a little easier) and then integrate r from $R/\cosh\eta$ to R. After changing the order of integration on the left, we get

$$\int_{-R}^{R} \varphi_R(t)\log|\hat{\mu}(c+t)|\,\mathrm{d}t \leqslant \frac{2}{3\pi}\Big(L\sinh\gamma - \pi D(\gamma-\eta)\Big)\frac{\cosh^3\eta-1}{\cosh^3\eta}R^3,$$

where

$$\pi\varphi_R(t) = \int_{\max(|t|,R/\cosh\eta)}^{R} \frac{r\mathrm{d}r}{\sqrt{(r^2-t^2)}}$$
$$= \begin{cases} \sqrt{(R^2-t^2)} - \sqrt{(R^2/\cosh^2\eta - t^2)} & \text{for } |t| < R/\cosh\eta, \\ \sqrt{(R^2-t^2)} & \text{for } R/\cosh\eta \leqslant |t| \leqslant R. \end{cases}$$

The function $\pi\varphi_R(t)$ assumes its maximum on $[-R, R]$ for $t = R/\cosh\eta$, where it equals $R\tanh\eta$. Hence, since $\log|\hat{\mu}(x)| \leqslant 0$ by $(*)$,

$$R\tanh\eta\int_{-R}^{R}\log|\hat{\mu}(c+t)|\,dt \leqslant \frac{2}{3}\left(L\sinh\gamma - \pi D(\gamma-\eta)\right)\frac{\cosh^3\eta-1}{\cosh^3\eta}R^3,$$

i.e.

$$(\begin{smallmatrix}*\\*\end{smallmatrix})\qquad \int_{a_k}^{b_k}\log|\hat{\mu}(x)|\,dx \leqslant -\frac{\pi D(\gamma-\eta)-L\sinh\gamma}{2}\cdot\frac{\tanh(\eta/2)}{\cosh^2\eta}(b_k-a_k)^2,$$

a relation holding for all the intervals $(a_k, b_k]$, with our fixed γ and η for which

$$\pi D(\gamma-\eta) - L\sinh\gamma > 0.$$

As we saw at the beginning of this discussion, $a_k \xrightarrow[k]{} \infty$. If $\hat{\mu}(z) \not\equiv 0$, we must also have

$$\frac{b_k}{a_k} \longrightarrow 1 \quad \text{as } k \longrightarrow \infty.$$

Indeed, such a function $\hat{\mu}(z)$ satisfies the hypothesis of *Levinson's theorem* (Chapter III, §H.3), after being multiplied by a suitable exponential $e^{i\alpha z}$ (see the observation at the beginning of §H.2, Chapter III). According to that result, if we denote the number of zeros of $\hat{\mu}(z)$ with modulus $\leqslant r$ *in the right half plane* by $n_+(r)$, we have

$$\frac{n_+(r)}{r} \longrightarrow \frac{L'}{\pi} \quad \text{for } r \longrightarrow \infty,$$

where (here) $0 \leqslant L' \leqslant L$. Therefore, if $\varepsilon > 0$, we will certainly have

$$n_+(a_k) \geqslant \left(\frac{L'}{\pi} - \varepsilon^2\right)a_k$$

and

$$n_+(b_k) \leqslant \left(\frac{L'}{\pi} + \varepsilon^2\right)b_k$$

for all sufficiently large k, since a_k and b_k tend to ∞ with k. If now $a_k < (1-\varepsilon)b_k$ for some large enough k, the previous inequalities yield

$$n_+(b_k) - n_+(a_k) \leqslant \frac{L'}{\pi}(b_k-a_k) + \varepsilon^2(a_k+b_k) \leqslant \left(\frac{L}{\pi} + 2\varepsilon\right)(b_k-a_k).$$

In particular, the interval $(a_k, b_k]$ can contain at most

$$\left(\frac{L}{\pi} + 2\varepsilon\right)(b_k-a_k)$$

of the points λ_n, since $\hat{\mu}(\lambda_n) = 0$. By hypothesis, however, that interval contains at least $D(b_k - a_k)$ of those points. Hence, since $L < \pi D$, we have a contradiction if $\varepsilon > 0$ is small enough.

Once we know that $a_k \longrightarrow \infty$ and $b_k/a_k \longrightarrow 1$ for $k \longrightarrow \infty$, we can be sure that the quantity

$$M = \sup_k \frac{b_k^2 + 1}{a_k^2}$$

is finite. Then, since $\log|\hat{\mu}(x)| \leqslant 0$, $(\ast_\ast)$ yields

$$\sum_{k=1}^{\infty} \int_{a_k}^{b_k} \frac{\log|\hat{\mu}(x)|}{1+x^2}\,dx \quad \leqslant \quad -\frac{\pi D(\gamma-\eta) - L\sinh\gamma}{2M} \cdot \frac{\tanh(\eta/2)}{\cosh^2\eta} \sum_{k=1}^{\infty} \left(\frac{b_k - a_k}{a_k}\right)^2,$$

and the right side equals $-\infty$ *by hypothesis.* That is,

$$\int_{-\infty}^{\infty} \frac{\log^-|\hat{\mu}(x)|}{1+x^2}\,dx = \infty,$$

the relation sought. The proof is complete.

Remark 1. Beurling and Malliavin originally proved this result using the ordinary Jensen formula (for circles). For such a proof, a covering lemma for intervals on $\mathbb{R}$ is required.

Remark 2. In case the λ_n are *not* distinct but $\hat{\mu}(z)$, having the other properties assumed in the proof has, at each of the former, a zero of order equal to that point's number of occurences in the sequence, we still conclude that $\hat{\mu}(z) \equiv 0$ by reasoning the same as above.

Remark 3. If we assume the apparently stronger condition

$$\sum_{k=1}^{\infty} \left(\frac{b_k - a_k}{b_k}\right)^2 = \infty$$

on the intervals $(a_k, b_k]$, the appeal to Levinson's theorem in the above argument can be avoided. In that way one arrives at what looks like a weaker criterion for completeness.

That criterion is in fact *not* weaker, for the condition just written is *implied* by the divergence of $\sum_k ((b_k - a_k)/a_k)^2$. Convergence of *either* series is actually equivalent to that of the other; see the top of p. 81.

Be that as it may, we *are* in possession of Levinson's theorem. It was therefore just as well to use it.

2. Beurling and Malliavin's effective density $\tilde{D}_\Lambda$

A certain notion of *density* for positive real sequences, different from the one used in the first two §§ of the present chapter, is suggested by the result proved in the preceding article.

Starting with a sequence Λ of numbers $\lambda_n > 0$ tending to ∞, we denote by $n_\Lambda(t)$ the number* of λ_n in $[0, t]$ when $t \geqslant 0$ (as in §A), and take $n_\Lambda(t)$ as *zero* for $t < 0$. Fixing a $D > 0$, we then consider the set $\mathcal{O}_D$ of $t > 0$ such that

$$\frac{n_\Lambda(\tau) - n_\Lambda(t)}{\tau - t} > D$$

for at least one $\tau > t$. Since $n_\Lambda(t) = n_\Lambda(t+)$, $\mathcal{O}_D$ is *open*, and hence the union of a sequence of disjoint open intervals $(a_k, b_k) \subseteq (0, \infty)$, perhaps only finite in number. It is convenient in the present article to have the *index k start* from the value *zero.*

The (a_k, b_k) are yielded by a geometric construction reminiscent of that of the Bernstein intervals made at the beginning (first stage) of §B.2, Chapter VIII, but different from the latter. Imagine *light shining downwards and from the right, in a direction of slope D, onto the graph* of $n_\Lambda(t)$ vs. t for $t \geqslant 0$. The intervals (a_k, b_k) will then lie *under those portions of that graph which are left in shadow*:

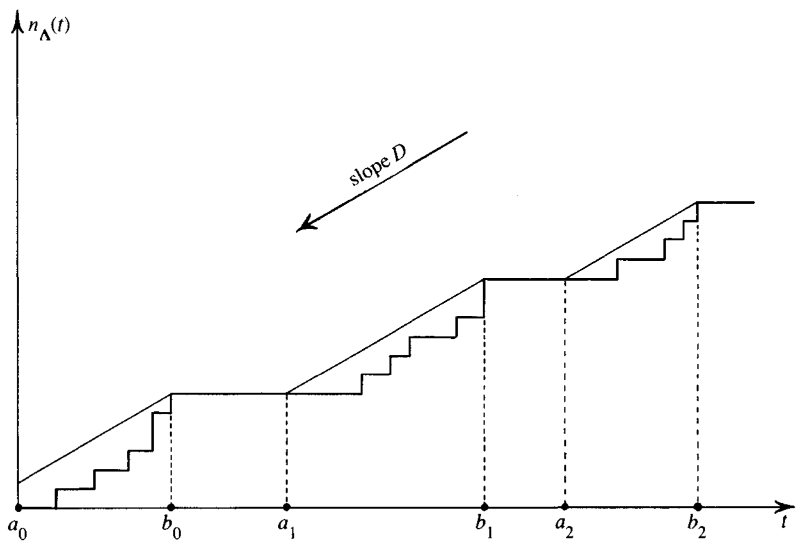


Figure 170

* We allow *repetitions* in the sequence Λ; $n_\Lambda(t)$ thus counts the points in Λ *with their appropriate multiplicities.*

It is clear that *each of the points* λ_n (where $n_\Lambda(t)$ has a jump) *must lie in one of the half-open intervals* $(a_k,\ b_k]$. Therefore since $\lambda_n \xrightarrow[n]{} \infty$ we may enumerate those intervals so as to have

$$0 \leqslant a_0 < b_0 \leqslant a_1 < b_1 \leqslant a_2 < b_2 \leqslant \cdots,$$

with $a_k \xrightarrow[k]{} \infty$ (see beginning of proof of the theorem in the preceding article). This having been done, we see from the diagram that when $b_k < \infty$,

$$\frac{n_\Lambda(b_k) - n_\Lambda(a_k)}{b_k - a_k} = D,$$

except perhaps for $k = 0$, where the left side may be $> D$.

Let us, for the work of this article, agree to *call the set* $\mathcal{O}_D$ *substantial if either* $b_0 = \infty$ *or*

$$\sum_{k \geqslant 1} \left(\frac{b_k - a_k}{a_k}\right)^2 = \infty.$$

Lemma. *If, for some* $D > 0$, $\mathcal{O}_D$ *is substantial, then* $\mathcal{O}_{D'}$ *is substantial for* $0 < D' < D$.

Proof. It is clear from our definition that $\mathcal{O}_D \subseteq \mathcal{O}_{D'}$ for $0 < D' < D$. If, then, we write $\mathcal{O}_D$ and $\mathcal{O}_{D'}$ as disjoint unions

$$\mathcal{O}_D = \bigcup_{k \geqslant 0} (a_k,\ b_k),$$

$$\mathcal{O}_{D'} = \bigcup_{l \geqslant 0} (a'_l,\ b'_l)$$

of intervals enumerated in the way just described, *each of the* $(a_k,\ b_k)$ *is contained in some* $(a'_l,\ b'_l)$, and in particular $(a_0,\ b_0) \subseteq (a'_0,\ b'_0)$. Hence, if $b_0 = \infty$, surely $b'_0 = \infty$, and $\mathcal{O}_{D'}$ is substantial.

When $b_0 < \infty$ and $\mathcal{O}_D$ is substantial, we have

$$(*) \qquad \sum_{k \geqslant 1} \left(\frac{b_k - a_k}{a_k}\right)^2 = \infty.$$

If there are only *finitely many* $(a_k,\ b_k)$, b_k must be infinite for the *last one* of those by $(*)$, so, if that one is contained in $(a'_l,\ b'_l)$, say, surely $b'_l = \infty$, making $\mathcal{O}_{D'}$ substantial. Otherwise, $(*)$ consists of *infinitely many terms*, each of which is *finite. Then*, in case $b'_0 = \infty$, we are done. When $b'_0 < \infty$, however, there must be intervals $(a'_l,\ b'_l)$ with $l \geqslant 1$ since $\lambda_n \xrightarrow[n]{} \infty$ and each λ_n is contained in some interval of $\mathcal{O}_{D'}$. Taking, then, any $l \geqslant 1$ and denoting by

$$N_l, \; N_l+1, \; . \; . \; . \; , \; M_l$$

the indices k – there *are* some – for which

$$(a_k, \; b_k) \; \subseteq \; (a'_l, \; b'_l),$$

we get

$$b'_l - a'_l \; \geqslant \; \sum_{k=N_l}^{M_l} (b_k - a_k),$$

so that

$$(b'_l - a'_l)^2 \; \geqslant \; \sum_{k=N_l}^{M_l} (b_k - a_k)^2$$

(the possibility that $M_l = \infty$ is not excluded here). Because $0 \; < \; a'_l \; \leqslant \; a_k$ for $k \; \geqslant \; N_l$, we see that

$$\left(\frac{b'_l - a'_l}{a'_l}\right)^2 \; \geqslant \; \sum_{k=N_l}^{M_l} \left(\frac{b_k - a_k}{a_k}\right)^2.$$

Adding both sides for the values of $l \; \geqslant \; 1$, we obtain *on the right* a sum which differs from the one in $(*)$ *by at most a finite number of terms* (those, if any, for which $1 \leqslant k < N_1$). The former sum must thus *diverge*, making

$$\sum_{l \geqslant 1} \left(\frac{b'_l - a'_l}{a'_l}\right)^2 \quad = \quad \infty,$$

and $\mathcal{O}_{D'}$ is substantial.

We are done.

Definition. If a sequence Λ of (perhaps repeated) strictly positive numbers has no finite limit point, its *effective density* $\tilde{D}_\Lambda$ is the *supremum* of the $D > 0$ for which the sets $\mathcal{O}_D$ corresponding to Λ in the way described above *are substantial*. If *none* of the $\mathcal{O}_D$ with $D > 0$ *are substantial*, we put $\tilde{D}_\Lambda \; = \; 0$. Finally, if Λ *has* a finite limit point, we put $\tilde{D}_\Lambda \; = \; \infty$.

The density $\tilde{D}_\Lambda$ was brought into the investigation of completeness for sets of exponentials $e^{i\lambda_n t}$ by Beurling and Malliavin; its rôle there turns out to be analogous to the one played by the Pólya maximum density D^*_Λ in studying singularities of Taylor series on their circles of convergence. We will see at the end of this article that $\tilde{D}_\Lambda$ is *a kind of upper density*, being the *infimum of the* (ordinary) *densities of those measurable sequences containing* Λ *that enjoy a certain definite property*, to be described presently.

It is convenient to extend our definition of $\tilde{D}_\Lambda$ to *arbitrary real sequences* Λ.

Definition. If the real sequence Λ includes 0 infinitely often, $\tilde{D}_\Lambda = \infty$. Otherwise, $\tilde{D}_\Lambda$ is the *greater* of $\tilde{D}_{\Lambda+}$ and $\tilde{D}_{\Lambda-}$ for the positive sequences

$$\begin{aligned} \Lambda_+ &= \Lambda \cap (0,\ \infty), \\ \Lambda_- &= (-\Lambda) \cap (0,\ \infty) \quad (sic!). \end{aligned}$$

($-\Lambda$ denotes the sequence $\{-\lambda_n\}$ when $\Lambda = \{\lambda_n\}$.)

The result from the preceding article can then be reformulated as follows:

Theorem. *Let Λ be a sequence of distinct real numbers λ_n with $\tilde{D}_\Lambda > 0$. Then, if $0 < L < \pi D_\Lambda$, the exponentials $e^{i\lambda_n t}$ are complete on $[-L,\ L]$.*

Proof. Consists mainly of reductions to the result referred to.

Suppose in the first place that the λ_n have a finite limit point, making $\tilde{D}_\Lambda = \infty$. We see then as at the beginning of the proof of the theorem from the preceding article that the $e^{i\lambda_n t}$ are complete on *any* finite interval. Having disposed of this trivial case, we look at $\Lambda_+ = \Lambda \cap (0,\ \infty)$ and $\Lambda_- = (-\Lambda) \cap (0,\ \infty)$. Assume, wlog, that $\tilde{D}_\Lambda = \tilde{D}_{\Lambda_+}$; in that case we re-enumerate Λ so as to make Λ_+ consist of the λ_n with $n \geqslant 1$, and then claim that the $e^{i\lambda_n t}$ with $n \geqslant 1$ are already complete on $[-L,\ L]$ for

$$0 < L < \pi\tilde{D}_{\Lambda_+} = \pi\tilde{D}_\Lambda.$$

Fix a number D with

$$\frac{L}{\pi} < D < \tilde{D}_{\Lambda_+} = \tilde{D}_\Lambda,$$

and form the open set

$$\mathcal{O}_D = \bigcup_{k \geqslant 0} (a_k,\ b_k)$$

in the manner described above. By definition of $\tilde{D}_{\Lambda_+}$ there must be a D',

$$D < D' < \tilde{D}_{\Lambda_+}$$

such that the set $\mathcal{O}_{D'}$ corresponding to it is substantial; *$\mathcal{O}_D$ is therefore substantial by the above lemma.* We thus either have $b_k = \infty$ for some k, or else there are infinitely many finite intervals $(a_k,\ b_k)$ with

$$\sum_{k \geqslant 1} \left(\frac{b_k - a_k}{a_k}\right)^2 = \infty$$

Let us consider the first possibility. If, say, $b_{k_0} = \infty$, there must be arbitrarily large $\tau_j > a_{k_0}$ for which

$$\frac{n_\Lambda(\tau_j) - n_\Lambda(a_{k_0}+)}{\tau_j - a_{k_0}} \geqslant D.$$

There is, indeed, a $\tau_1 > a_{k_0}$ such that

$$\frac{n_\Lambda(\tau_1) - n_\Lambda(a_{k_0}+)}{\tau_1 - a_{k_0}} \geqslant D$$

(see the above diagram, and keep in mind that $n_\Lambda(t) = n_\Lambda(t+)$). Then, however, $\tau_1 \in (a_{k_0}, \infty) \subseteq \mathcal{O}_D$, so there is a $\tau_2 > \tau_1$ with

$$\frac{n_\Lambda(\tau_2) - n_\Lambda(\tau_1)}{\tau_2 - \tau_1} > D,$$

and similarly a $\tau_3 > \tau_2$ with

$$\frac{n_\Lambda(\tau_3) - n_\Lambda(\tau_2)}{\tau_3 - \tau_2} > D,$$

and so forth. Since $n_\Lambda(t)$ *increases by at least unity at each of its discontinuities* λ_n, we must have $n_\Lambda(\tau_j) \xrightarrow[j]{} \infty$. But then $\tau_j \xrightarrow[j]{} \infty$ since we are in the case where $\lambda_n \xrightarrow[n]{} \infty$.

Putting together the inequalities from the chain just obtained, we see that

$$\frac{\text{number of } \lambda_n \text{ in } (a_{k_0}, \tau_j]}{\tau_j - a_{k_0}} \geqslant D$$

with $\tau_j \xrightarrow[j]{} \infty$. If now the $e^{i\lambda_n t}$ were *incomplete* on $[-L, L]$ for $0 < L < \pi D$, we would as in the previous article get a non-zero complex measure μ on $[-L, L]$ with $\hat{\mu}(\lambda_n) = 0$ for $n \geqslant 1$, and the zeros of $\hat{\mu}(z)$ in the right half plane would have density $\leqslant L/\pi < D$ by Levinson's theorem (Chapter III, §H.3). This, however, is incompatible with the previous relation. The $e^{i\lambda_n t}$ with $n \geqslant 1$ must hence be complete on $[-L, L]$ in the event that one of the b_k is infinite.

There remains the case where $\mathcal{O}_D$ consists of infinitely many finite intervals (a_k, b_k) with

$$\sum_{k \geq 1} \left(\frac{b_k - a_k}{a_k} \right)^2 = \infty.$$

Here, however,

$$\frac{\text{number of } \lambda_n \text{ in } (a_k, b_k]}{b_k - a_k} = D$$

for $k \geqslant 1$, and the completeness of the $e^{i\lambda_n t}$ on $[-L, L]$ for $0 < L < \pi D$ is an immediate consequence of the preceding article's result.

We are done.

Corollary. *The completeness radius associated with Λ is $\geqslant \pi\tilde{D}_\Lambda$.*

Remark. Work in the next chapter will show that in the corollary we actually have *equality*. That is the real reason for $\tilde{D}_\Lambda$'s having been defined as it was. This extension, due also to Beurling and Malliavin, lies much deeper than the results of the present §.

We proceed to look at how $\tilde{D}_\Lambda$ can be regarded as an upper density. The following lemma and corollary will be used in Chapter X.

Lemma. *If $\tilde{D}_\Lambda < \infty$ for a sequence of (perhaps repeated) strictly positive numbers Λ there is, corresponding to any $D > \tilde{D}_\Lambda$, a sequence $\Sigma \supseteq \Lambda$ of strictly positive numbers for which*

$$\int_0^\infty \frac{|n_\Sigma(t) - Dt|}{1+t^2}\,dt \quad < \quad \infty.$$

Remark. Such a sequence Σ does not differ by much from the straight *arithmetic progression* $1/D,\ 2/D,\ 3/D\ ,\ldots.$ About this more later on.

Proof of lemma. Is based on a very simple geometric construction.

Starting with a fixed $D > \tilde{D}_\Lambda$, we form the set

$$\mathcal{O}_D \quad = \quad \bigcup_{k \geq 0} (a_k,\ b_k)$$

corresponding to Λ in the manner described above, with the intervals $(a_k,\ b_k)$ enumerated from left to right. By choice of D, $\mathcal{O}_D$ *cannot be substantial*, hence $b_0 < \infty$ and

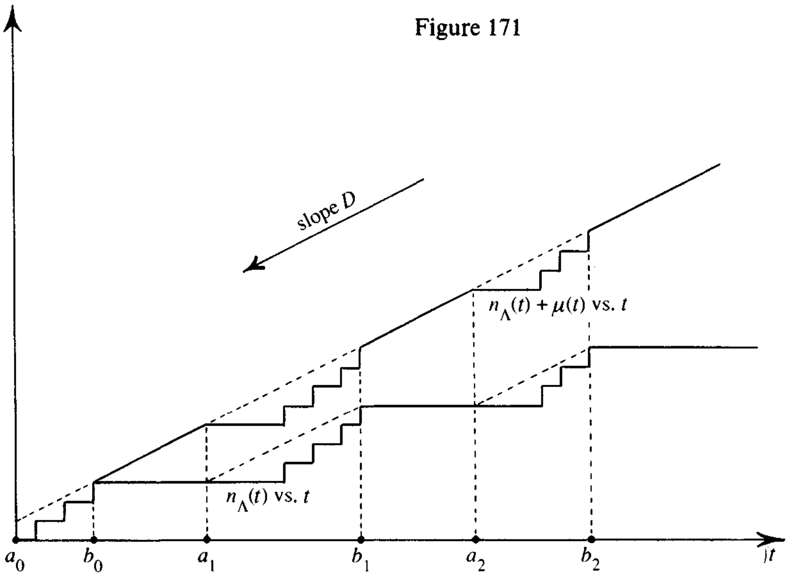
$$\sum_{k \geq 1} \left(\frac{b_k - a_k}{a_k}\right)^2 \quad < \quad \infty.$$

Let us first find a continuous increasing function $\mu(t)$ such that

$$(\dagger) \qquad \int_0^\infty \frac{|n_\Lambda(t) + \mu(t) - Dt|}{1+t^2}\,dt \quad < \quad \infty.$$

This is not difficult: take $\mu(t)$ to be the piecewise linear continuous function having *slope D* on *each interval* of $[0,\ \infty)$ *complementary* to $\mathcal{O}_D$ and *slope zero* on each of the *components* $(a_k,\ b_k)$ of $\mathcal{O}_D$, with $\mu(0) = 0$:

Figure 171



Outside of $\mathcal{O}_D$ on the positive real axis, $n_\Lambda(t)+\mu(t)$ is simply *equal* to $n_\Lambda(b_0) - D(b_0-a_0) + Dt$, so

$$\int_{[0,\infty)\sim\mathcal{O}_D} \frac{|n_\Lambda(t)+\mu(t)-Dt|}{1+t^2}\,\mathrm{d}t \quad < \quad \infty.$$

Writing $n_\Lambda(b_0)-D(b_0-a_0) = c$, we see from the figure that *on each of the intervals* $(a_k,\ b_k)$,

$$Da_k+c \;=\; n_\Lambda(a_k)+\mu(a_k) \;\leqslant\; n_\Lambda(t)+\mu(t) \;\leqslant\; n_\Lambda(b_k)+\mu(b_k) \;=\; Db_k+c,$$

so

$$|n_\Lambda(t)+\mu(t)-Dt| \quad\leqslant\quad D(b_k-a_k) + c$$

for $a_k < t < b_k$, and

$$\int_{a_k}^{b_k} \frac{|n_\Lambda(t)+\mu(t)-Dt|}{1+t^2}\,\mathrm{d}t \quad\leqslant\quad D\frac{(b_k-a_k)^2}{a_k^2+1} + c\int_{a_k}^{b_k}\frac{\mathrm{d}t}{t^2+1}.$$

The convergence of

$$\int_{\mathcal{O}_D} \frac{|n_\Lambda(t)+\mu(t)-Dt|}{1+t^2}\,\mathrm{d}t$$

thus follows from that of the sum $\sum_{k\geq 1}(b_k-a_k)^2/a_k^2$, and, referring to the previous relation, we get (†).

In virtue of (†) we have also

$$\int_0^\infty \frac{|n_\Lambda(t)+[\mu(t)]-Dt|}{1+t^2}\,\mathrm{d}t \quad < \quad \infty,$$

where, as usual, $[\mu(t)]$ denotes the greatest integer $\leqslant \mu(t)$. The increasing function $n_\Lambda(t) + [\mu(t)]$ takes, however, only *integral values*, and it vanishes at 0. It is therefore equal to $n_\Sigma(t)$ for some strictly positive sequence Σ. Σ clearly consists of *the points where $n_\Lambda(t)$ jumps together with those where $[\mu(t)]$ jumps*, so $\Sigma \supseteq \Lambda$. And

$$\int_0^\infty \frac{|n_\Sigma(t) - Dt|}{1+t^2}\,dt \quad < \quad \infty,$$

as required.

Remark. For the sequence Σ actually furnished by the construction we have

$$n_\Sigma(t) \;\leqslant\; Dt + c.$$

Corollary. *If Λ is a real sequence for which $\tilde{D}_\Lambda < \infty$ and $D > \tilde{D}_\Lambda$, there is a real sequence Σ including Λ such that*

$$\int_{-\infty}^\infty \frac{|n_\Sigma(t) - Dt|}{1+t^2}\,dt \quad < \quad \infty.$$

N.B. Here, $n_\Sigma(t)$ has its usual meaning for $t \geqslant 0$, *but denotes the negative of the number of members of Σ in $[t, 0)$ when $t < 0$* (convention of Chapter III, §H.2).

Proof of corollary. Write $\Lambda_+ = \Lambda \cap (0, \infty)$ and $\Lambda_- = (-\Lambda) \cap (0, \infty)$. Given $D >$ both $\tilde{D}_{\Lambda_+}$ and $\tilde{D}_{\Lambda_-}$ we apply the lemma to Λ_+ and Λ_- separately, and then put the two results together to get Σ, adjoining thereto, if needed, the point 0* so as to ensure that $\Lambda \subseteq \Sigma$.

The preceding lemma has a *converse* whose proof requires somewhat more work.

Lemma. *If $A \geqslant 0$ and Σ is a strictly positive sequence such that*

$$\int_0^\infty \frac{|n_\Sigma(t) - At|}{1+t^2}\,dt \quad < \quad \infty,$$

we have $\tilde{D}_\Sigma \leqslant A$.

Proof. Let us take any $D > A$ and form a set

$$\mathcal{O}_D \;=\; \bigcup_{k \geq 0} (a_k,\ b_k)$$

corresponding to the sequence Σ, following the procedure used up to now with positive sequences Λ. According to the definition of $\tilde{D}_\Sigma$, it is enough

* with appropriate multiplicity

to show that $\mathcal{O}_D$ is *not substantial*. In the following discussion, we assume that $A > 0$. When $A = 0$, the treatment is similar (and easier).

Order the interval components $(a_k,\ b_k)$ of $\mathcal{O}_D$ in the now familiar fashion:

$$0 \leqslant a_0 < b_0 \leqslant a_1 < b_1 \leqslant a_2 < \cdots.$$

It is claimed first of all that $b_0 < \infty$. Suppose indeed that $b_0 = \infty$. Then, as in the proof of the previous lemma, we obtain a sequence $\tau_j \xrightarrow[j]{} \infty$ such that

$$\frac{n_\Sigma(\tau_j) - n_\Sigma(a_0)}{\tau_j - a_0} \geqslant D,$$

i.e.,

$$n_\Sigma(\tau_j) \geqslant D(\tau_j - a_0),$$

since of course $n_\Sigma(a_0) = 0$ (see figure near the beginning of this article). This means that $n_\Sigma(t) \geqslant D(\tau_j - a_0)$ for $t \geqslant \tau_j$, $n_\Sigma(t)$ being increasing. Therefore, if τ_j is large enough to make

$$\frac{D}{A}\tau_j - \frac{D}{A}a_0 > \tau_j$$

(we *are taking* $D > A$!), we have

$$n_\Sigma(t) - At \geqslant D\tau_j - At - Da_0 > 0$$

for $\tau_j \leqslant t < (D/A)\tau_j - (D/A)a_0$:

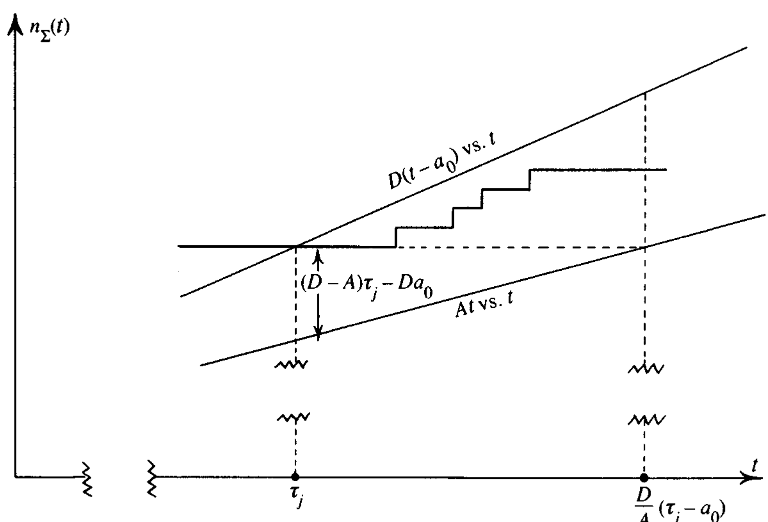


Figure 172

Looking at a τ_j larger than $(2D/(D-A))a_0$, we see from the figure that

$$\int_{\tau_j}^{(D/A)\tau_j-(D/A)a_0} \frac{n_\Sigma(t) - At}{1+t^2}\,dt$$

$$\geqslant \frac{\frac{1}{2}\left\{(D-A)\tau_j - Da_0\right\}\left\{\frac{D}{A}\tau_j - \frac{D}{A}a_0 - \tau_j\right\}}{1+\left(\frac{D}{A}\tau_j - \frac{D}{A}a_0\right)^2} \geqslant \frac{(D-A)^2}{8A}\,\frac{\tau_j^2}{1+\frac{D^2}{A^2}\tau_j^2},$$

and *this* is $\geqslant A(D-A)^2/16D^2$ (say) for large enough τ_j. Since the τ_j tend to ∞, selection of a suitable subsequence of them shows that

$$\int_0^\infty \frac{|n_\Sigma(t)-At|}{1+t^2}\,dt = \infty,$$

a contradiction.

Having thus proved that $b_0 < \infty$, we are assured of the existence of intervals (a_k, b_k) with $k \geqslant 1$, and need to show that

$$\sum_{k\geq 1}\left(\frac{b_k - a_k}{a_k}\right)^2 < \infty.$$

Considering any one of the intervals $(a_k,\ b_k)$ in question,* we denote by $\mathscr{L}_k$ the straight line of slope D through $(a_k,\ n_\Sigma(a_k))$ and $(b_k,\ n_\Sigma(b_k))$, and look at the *abscissa* c_k of the *point where* $\mathscr{L}_k$ *and the line of slope* A *through the origin intersect*:

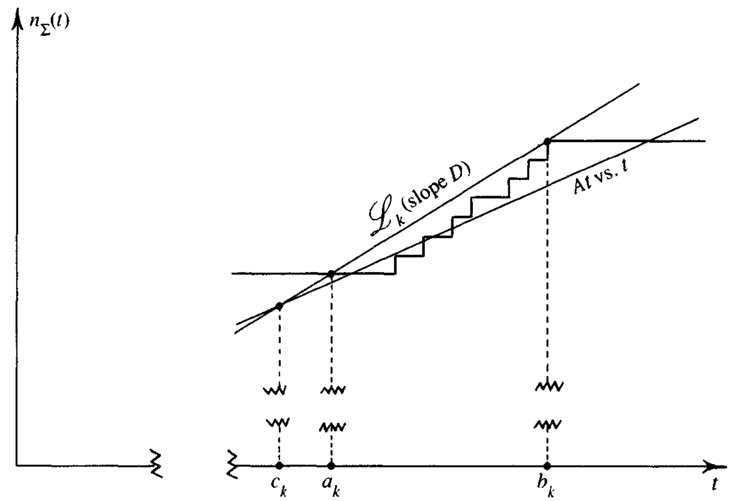


Figure 173

* $b_k < \infty$ by the argument just made for b_0

If c_k *lies to the right of the midpoint*, $(a_k + b_k)/2$, of (a_k, b_k), we say that the index $k \geqslant 1$ *belongs to the set* R. Otherwise, when $c_k < (a_k + b_k)/2$ (as in the last picture), we say that $k \geqslant 1$ *belongs to the set* S.

Let us first show that

$$\sum_{k \in R} \left(\frac{b_k - a_k}{a_k} \right)^2 < \infty.$$

When $k \in R$, the situation is as follows:

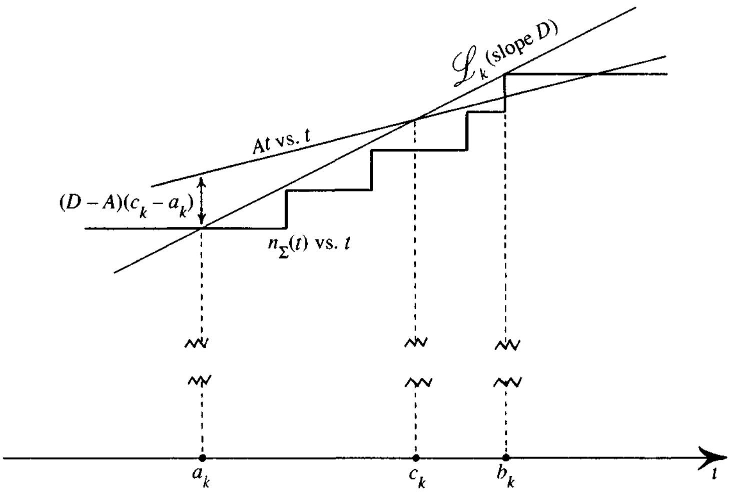


Figure 174

It may, of course, happen that $c_k > b_k$. In order to allow for that possibility, we work with

$$c'_k = \min(c_k, b_k).$$

The preceding figure shows that here

$$\int_{a_k}^{c'_k} \frac{At - n_\Sigma(t)}{1+t^2} dt \geqslant \frac{\frac{1}{2}(D-A)(c_k - a_k)(c'_k - a_k)}{1 + (c'_k)^2} \geqslant \frac{D-A}{8} \cdot \frac{(b_k - a_k)^2}{1 + b_k^2},$$

since $c'_k - a_k \geqslant \frac{1}{2}(b_k - a_k)$ for $k \in R$. However, $(a_k, c'_k) \subseteq (a_k, b_k)$ with the latter intervals *disjoint*. On adding the previous inequalities for $k \in R$ it thus follows by the hypothesis that

$$\sum_{k \in R} \frac{(b_k - a_k)^2}{1 + b_k^2} < \infty,$$

whence

$$\sum_{k\in R}\left(\frac{b_k-a_k}{b_k}\right)^2 < \infty,$$

since we are dealing with numbers $b_k \geqslant a_1 > 0$. The last relation certainly implies that there cannot be infinitely many $k\in R$ with

$$\frac{b_k}{a_k} > 4, \quad \text{say,}$$

so we must also have

$$\sum_{k\in R}\left(\frac{b_k-a_k}{a_k}\right)^2 < \infty.$$

We now show that

$$\sum_{k\in S}\left(\frac{b_k-a_k}{a_k}\right)^2 < \infty;$$

this involves a covering argument. Given $k\in S$, we put

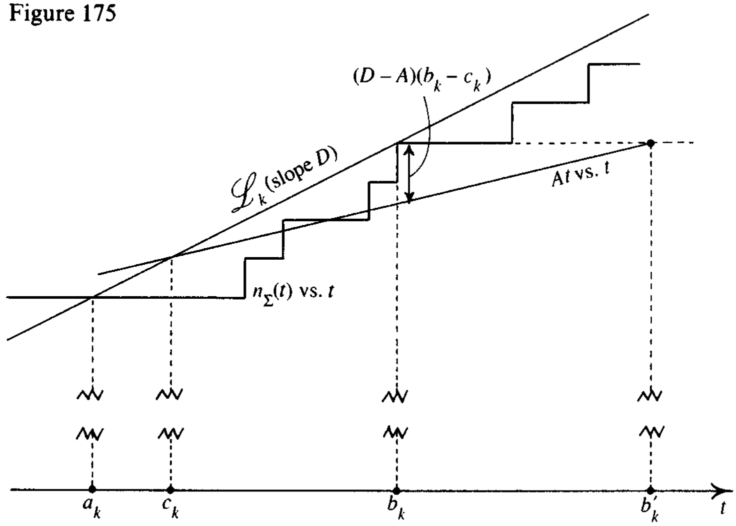
$$b_k' = \min\left(b_k+\frac{D-A}{A}(b_k-c_k),\quad b_k+\frac{D-A}{A}(b_k-a_k)\right),$$

so that

$$\frac{D-A}{2A}(b_k-a_k) < b_k'-b_k \leqslant \frac{D-A}{A}(b_k-a_k),$$

and observe that $n_\Sigma(t) > At$ for $b_k \leqslant t < b_k'$:

Figure 175



We see that in the present case

$$\int_{b_k}^{b'_k} \frac{n_\Sigma(t) - At}{1+t^2}\,dt \;\geqslant\; \frac{\frac{1}{2}(D-A)(b_k - c_k)(b'_k - b_k)}{1+(b'_k)^2}$$

$$\geqslant\; \frac{(D-A)^2}{8A}\cdot\frac{(b_k - a_k)^2}{1 + \dfrac{D^2}{A^2}b_k^2}.$$

What prevents us now from reasoning as we did when examining the sum $\sum_{k\in R}(b_k - a_k)^2/b_k^2$ is that the intervals $(b_k,\ b'_k)$, $k \in S$, *may overlap*, although of course the $(a_k,\ b_k)$ do not.

To deal with this complication, we fix for the moment *any finite subset* S' of S, and set out to obtain a bound *independent of* S' on the sum

$$\sum_{k\in S'} \frac{(b_k - a_k)^2}{1 + (Db_k/A)^2}.$$

For this purpose, we *select certain of the intervals* $(a_k,\ b_k)$, $k \in S'$, in the following manner.

First of all, we take the *leftmost of the* $(a_k,\ b_k)$, $k \in S'$, and *denote it by* $(\alpha_1,\ \beta_1)$. If $(\alpha_1,\ \beta_1)$ is $(a_{k_1},\ b_{k_1})$, say, *we denote* b'_{k_1} by β'_1; thus,

$$\frac{D-A}{2A}(\beta_1 - \alpha_1) \;<\; \beta'_1 - \beta_1 \;\leqslant\; \frac{D-A}{A}(\beta_1 - \alpha_1).$$

Having picked $(\alpha_1,\ \beta_1)$, we *skip over* any of the *remaining* $(a_k,\ b_k)$, $k \in S'$, which happen to be *entirely contained* in $(\alpha_1,\ \beta'_1)$ (*sic*!), and then, if there are *still any* $(a_k,\ b_k)$ left over for $k \in S'$, choose $(\alpha_2,\ \beta_2)$ as the *leftmost of those*. If $(\alpha_2,\ \beta_2) = (a_{k_2},\ b_{k_2})$, say, we write β'_2 for b'_{k_2}, which makes

$$\frac{D-A}{2A}(\beta_2 - \alpha_2) \;<\; \beta'_2 - \beta_2 \;\leqslant\; \frac{D-A}{A}(\beta_2 - \alpha_2).$$

It is important that $(\beta_1,\ \beta'_1)$ *cannot overlap with* $(\beta_2,\ \beta'_2)$, *even though* $(\alpha_1,\ \beta'_1)$ *may well overlap with* $(\alpha_2,\ \beta_2)$:

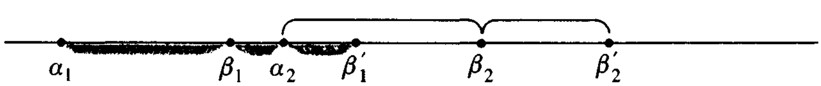


Figure 176

Otherwise, $(\alpha_2,\ \beta_2)$ would certainly be *included* in $(\alpha_1,\ \beta'_1)$, contrary to the way it was chosen. It is *also true* that *any of the intervals* $(a_k,\ b_k)$, $k \in S'$, *skipped over* in going from $(\alpha_1,\ \beta_1)$ to $(\alpha_2,\ \beta_2)$ *must lie* in $(\beta_1,\ \beta'_1)$. The former are indeed included in (α_1, β_1'), but certainly have no intersection with (α_1, β_1), which, as a *particular* (a_k, b_k) is *disjoint from all the others.*

If, after choosing (α_2, β_2), there are still some (a_k, b_k) left over with $k \in S'$, we go on in the same fashion, *first skipping over any that may be entirely included in* (α_2, β_2') – by the argument just made, *those must in fact lie in* (β_2, β_2') – and *then* taking (α_3, β_3) as the *leftmost of the remaining* (a_k, b_k), $k \in S'$, if such there be. Defining β_3' as we did β_1' and β_2' above, we see that the three intervals

$$(\beta_1, \beta_1'), \qquad (\beta_2, \beta_2'), \qquad (\beta_3, \beta_3')$$

must be *disjoint.*

This process can be continued as long as there are any (a_k, b_k) left with $k \in S'$. After a finite number of steps we finish, ending with certain intervals

$$(\alpha_1, \beta_1), \qquad (\alpha_2\ \beta_2), \ \ldots, (\alpha_p, \beta_p),$$

selected from among the (a_k, b_k) with $k \in S'$, and having the following two properties:

(i) the intervals (β_l, β_l') are *disjoint* for $1 \leqslant l \leqslant p$;

(ii) *each* of the *remaining* (a_k, b_k), $k \in S'$, is entirely contained in one of the (β_l, β_l'), $1 \leqslant l \leqslant p$.

For each l, $1 \leqslant l \leqslant p$, the inequality proved above can be rewritten

$$\int_{\beta_l}^{\beta_l'} \frac{n_\Sigma(t) - At}{1+t^2}\,dt \quad \geqslant \quad \frac{(D-A)^2}{8A} \cdot \frac{(\beta_l - \alpha_l)^2}{1 + (D\beta_l/A)^2}.$$

Denoting by S_l the set of $k \in S'$ for which $(a_k, b_k) \subseteq (\beta_l, \beta_l')$, we have

$$\sum_{k \in S_l} (b_k - a_k) \quad \leqslant \quad \beta_l' - \beta_l \quad \leqslant \quad \frac{D-A}{A}(\beta_l - \alpha_l),$$

the (a_k, b_k) being disjoint, and b_k is of course $> \beta_l$ for $k \in S_l$. Hence,

$$\sum_{k \in S_l} \frac{(b_k - a_k)^2}{1 + (Db_k/A)^2} \quad \leqslant \quad \left(\frac{D-A}{A}\right)^2 \cdot \frac{(\beta_l - \alpha_l)^2}{1 + (D\beta_l/A)^2}$$

which is

$$\leqslant \quad \frac{8}{A}\int_{\beta_l}^{\beta_l'} \frac{n_\Sigma(t) - At}{1+t^2}\,dt$$

by the previous relation, and finally

$$\frac{(\beta_l - \alpha_l)^2}{1 + (D\beta_l/A)^2} + \sum_{k \in S_l} \frac{(b_k - a_k)^2}{1 + (Db_k/A)^2} \quad \leqslant \quad \left(\frac{8A}{(D-A)^2} + \frac{8}{A}\right)\int_{\beta_l}^{\beta_l'} \frac{n_\Sigma(t) - At}{1+t^2}\,dt.$$

By property (i) we find, summing over l, that

$$\sum_{l=1}^{p} \frac{(\beta_l - \alpha_l)^2}{1 + (D\beta_l/A)^2} \quad + \quad \sum_{l=1}^{p} \sum_{k \in S_l} \frac{(b_k - a_k)^2}{1 + (Db_k/A)^2}$$
$$\leqslant \quad \left(\frac{8A}{(D-A)^2} + \frac{8}{A}\right) \int_0^\infty \frac{|n_\Sigma(t) - At|}{1 + t^2} \mathrm{d}t$$

According to property (ii), the sum on the left is just

$$\sum_{k \in S'} \frac{(b_k - a_k)^2}{1 + (Db_k/A)^2};$$

that quantity is therefore *bounded by the right hand member, obviously independent* of S', of the relation just written.

Since S' was *any* finite subset of S, we thus have

$$\sum_{k \in S} \frac{(b_k - a_k)^2}{1 + (Db_k/A)^2} \quad \leqslant \quad \left(\frac{8A}{(D-A)^2} + \frac{8}{A}\right) \int_0^\infty \frac{|n_\Sigma(t) - At|}{1 + t^2} \mathrm{d}t,$$

and from this point we may argue just as during the consideration of $\sum_{k \in R} (b_k - a_k)^2 / a_k^2$ to show that

$$\sum_{k \in S} \left(\frac{b_k - a_k}{a_k}\right)^2 \quad < \quad \infty.$$

Knowing that the corresponding sum over R is finite, we conclude that

$$\sum_{k \geq 1} \left(\frac{b_k - a_k}{a_k}\right)^2 \quad < \quad \infty.$$

That, however, was what we needed to establish in order to finish showing the *non-substantiability* of $\mathcal{O}_D$, from which it follows that $\tilde{D}_\Sigma \leqslant D$. Thus $\tilde{D}_\Sigma \leqslant A$, since $D > A$ was arbitrary. Q.E.D.

Putting together this and the preceding lemma, we immediately obtain the

Theorem. *Let Λ be a strictly positive sequence.* Then $\tilde{D}_\Lambda$ is the infimum of the positive numbers A such that there exist positive sequences $\Sigma \supseteq \Lambda$ with*

$$\int_0^\infty \frac{|n_\Sigma(t) - At|}{1 + t^2} \mathrm{d}t \quad < \quad \infty.$$

* perhaps with repetitions.

Corollary. *If Λ is a real sequence, $\tilde{D}_\Lambda$ is the* infimum *of the positive numbers A for which there exist real sequences $\Sigma \supseteq \Lambda$ with*

$$\int_{-\infty}^{\infty} \frac{|n_\Sigma(t) - At|}{1+t^2}\,dt \quad < \quad \infty.$$

This corollary follows from the theorem in the same way that the one to the *first* of the preceding two lemmas does from that lemma.

A positive sequence Σ such that

$$\int_0^{\infty} \frac{|n_\Sigma(t) - At|}{1+t^2}\,dt \quad < \quad \infty$$

is *measurable* according to the definition in §E.3, Chapter VI.

Lemma. *If the relation just written holds for the positive sequence Σ and some $A \geqslant 0$, we have*

$$\frac{n_\Sigma(t)}{t} \longrightarrow A \quad \textit{for } t \longrightarrow \infty.$$

Proof. If, for some $\eta > 0$, we have $n_\Sigma(t_0) > (A+\eta)t_0$ with $t_0 > 1$, we see from the following diagram that

$$\int_{t_0}^{(1+\eta/A)t_0} \frac{n_\Sigma(t) - At}{1+t^2}\,dt \quad \geqslant \quad \frac{\frac{1}{2}\eta t_0 \cdot \frac{\eta}{A} t_0}{1 + \left(1+\frac{\eta}{A}\right)^2 t_0^2} \quad \geqslant \quad \frac{\eta^2}{4A\left(1+\frac{\eta}{A}\right)^2}:$$

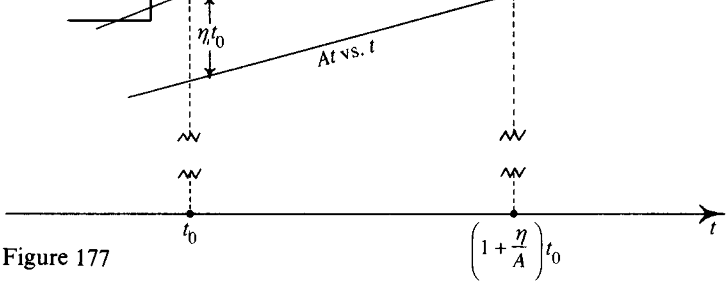


Figure 177

If, then, this happens for *arbitrarily large* t_0, we can get a sequence $\{t_k\}$,

$$t_{k+1} > \left(1+\frac{\eta}{A}\right)t_k,$$

such that

$$\int_{t_k}^{(1+\eta/A)t_k} \frac{n_\Sigma(t) - At}{1+t^2}\,dt \geqslant \frac{\eta^2 A}{4(A+\eta)^2}$$

for each k, making

$$\int_0^\infty \frac{|n_\Sigma(t) - At|}{1+t^2}\,dt = \infty,$$

contrary to hypothesis. Therefore $n_\Sigma(t)/t$ must be $\leqslant A+\eta$ for all sufficiently large t.

By working with an integral over $((1-\eta/A)t_0,\ t_0)$, one shows in like manner that $n_\Sigma(t_0)$ cannot be $< (A-\eta)t_0$ for arbitrarily large values of t_0.

The lemma is proved.

This result shows that the (ordinary) *density* of any positive sequence Σ for which

$$\int_0^\infty \frac{|n_\Sigma(t) - At|}{1+t^2}\,dt < \infty$$

is *defined* (in the sense of §E.3, Chapter VI) *and equal to* A. The previous theorem thus furnishes our *characterization of* $\tilde{D}_\Lambda$ *as an upper density*:

> *For a positive sequence* Λ, $\tilde{D}_\Lambda$ *is the infimum of the densities* D_Σ *of the positive measurable sequences* $\Sigma \supseteq \Lambda$ *such that*
>
> $$\int_0^\infty \frac{|n_\Sigma(t) - D_\Sigma t|}{1+t^2}\,dt < \infty.$$

Referring to the definition of *Pólya's maximum density* D^*_Λ given in §E.3 of Chapter VI, we also see that

$$D^*_\Lambda \leqslant \tilde{D}_\Lambda$$

for positive sequences Λ. Simple examples show that $\tilde{D}_\Lambda$ can be *really bigger* than D^*_Λ; it is recommended, for instance, that the reader construct a *measurable* sequence Λ of *positive integers* for which the *ordinary density* D_Λ is *zero*, while $\tilde{D}_\Lambda = 1$.

E. Extension of the results in §D to the zero distribution of entire functions $f(x)$ of exponential type, with $\int_{-\infty}^{\infty}(\log^+|f(x)|/(1+x^2))\,dx$ convergent

In the proof of the theorem from §D.1 we may, thanks to the third Phragmén–Lindelöf theorem of §C, Chapter III, replace the Fourier transform $\hat{\mu}(z)$ by any entire function $f(z)$ of exponential type $\leqslant L$, *bounded on the real axis*, and vanishing* at the points $\lambda_n > 0$. This yields a result about the real zeros of such functions which is best formulated in terms of the effective density $\tilde{D}_\Lambda$ introduced in §D.2:

> *If $f(z)$, a non-zero entire function of exponential type $\leqslant L$, bounded on the real axis, vanishes at the points of the real sequence Λ, then $\tilde{D}_\Lambda \leqslant L/\pi$.*

In cases where $\Lambda \subseteq \mathbb{R}$ consists of *all* the zeros of $f(z)$ (each counted according to its multiplicity) and where the two quantities

$$\limsup_{y\to\infty}\frac{\log|f(iy)|}{y}, \qquad \limsup_{y\to-\infty}\frac{\log|f(iy)|}{|y|}$$

are both *equal* to L, we in fact have $\tilde{D}_{\Lambda_+} = \tilde{D}_{\Lambda_-} = L/\pi$ for the two 'halves' $\Lambda_+ = \Lambda\cap(0,\infty)$, $\Lambda_- = (-\Lambda)\cap(0,\infty)$ of Λ, which is a considerable amelioration of Levinson's theorem. This is so because under the stated conditions Λ_+ and Λ_- will both be *measurable* and of (ordinary) density L/π by the Levinson theorem from §H.2, Chapter III. We know on the other hand that $\tilde{D}_{\Lambda_+}$ and $\tilde{D}_{\Lambda_-}$ must be $\geqslant$ the respective ordinary densities of Λ_+ and Λ_-, according to the work at the end of §D.2.

The requirement that $f(x)$ be *bounded* on the real axis can be relaxed. From the *Beurling–Malliavin multiplier theorem*, to be proved in Chapter XI, it readily follows that the boundedness can be replaced by the milder condition that

$$\int_{-\infty}^{\infty}\frac{\log^+|f(x)|}{1+x^2}\,dx \quad < \quad \infty,$$

and the above result, together with its consequences, *will still hold*. The multiplier theorem is, however, a deep result about *existence*, so an argument which depends on it for merely refining the simple estimation procedure of §D.1 hardly seems satisfactory. That is why Beurling and Malliavin gave a *direct* proof of the more general result in their 1967 *Acta*

* With the appropriate multiplicity at any *repeated* value λ_n; see *Remark 2* at the end of §D.1

paper. Their work is presented in article 2 below. In it, an estimate for harmonic measure going back to Ahlfors and Carleman is used in somewhat unusual fashion. That estimate is derived in article 1.

1. Introduction to extremal length and to its use in estimating harmonic measure

The notion of extremal length, due to Beurling, is a natural development of a more special idea already appearing in his thesis, and is closely related to material in Grötzsch's work. Ahlfors also is closely associated with the early study of it.

The use of extremal length (or rather of its *reciprocal*) is very helpful and convenient in the investigation of various problems involving analytic functions, due partly to the strong appeal such use makes to our geometric intuition. This technique, based on a simple and beautiful idea, has been valuable in the study of quasiconformal mappings and even of problems in $\mathbb{R}^n$, as well as in ordinary function theory. It is really a pity that familiarity with extremal length is not more widespread among analysts, and that most textbooks on analytic functions do not discuss it. The most accessible introduction is in W. Fuchs' little book; material is also contained in the one by Ahlfors on conformal invariants. Hersch's *Commentarii Helvetici* paper from the 1950s has a longer (and somewhat pedantic) development, as does the book of Ohtsuka. This last is *not* the place for beginning one's study of the subject.*

Extremal length is defined for a *given family G of curves in a domain $\mathscr{D}$.* One usually requires the curves belonging to G to be at least locally rectifiable. Once G and $\mathscr{D}$ are prescribed, we look at certain *positive Borel functions ('weights') $p(z)$ defined on $\mathscr{D}$.* We say that one of these is *admissible for G* if

$$\int_\gamma p(z)|dz| \;\geqslant\; 1$$

for every curve γ in the family G. The *reciprocal extremal length* $\Lambda(\mathscr{D}, G)$ (often called the *modulus*) associated with G and $\mathscr{D}$ is simply the *infimum of*

$$\iint_{\mathscr{D}} (p(z))^2\,dx\,dy$$

for all the p admissible for G.

* The beautiful outline in Beurling's *Collected Works* (vol. I, pp. 361–85) has now appeared (*Collected Works of Arne Beurling*; 2 vols, edited by L. Carleson, P. Malliavin, J. Neuberger and J. Wermer; Birkhäuser, Boston, 1989).

The idea here is natural and straightforward. We think of $p(z)$ as some kind of varying *gauge* or *conversion factor* which must be used *to the first power* to get (infinitesimal) *lengths* and *to the second power* to get *areas.* Saying that $\int_\gamma p(z)|dz| \geqslant 1$ for all the curves γ of the family G means that *we require our p to make each of those have* (gauged) *length* $\geqslant 1$. We then look to see *how small* the (gauged) *areas* $\iint_{\mathscr{D}}(p(z))^2\,dx\,dy$ can come out *using the different conversion factors p fulfilling that requirement. The infimum of those gauged areas is our quantity* $\Lambda(\mathscr{D}, G)$.

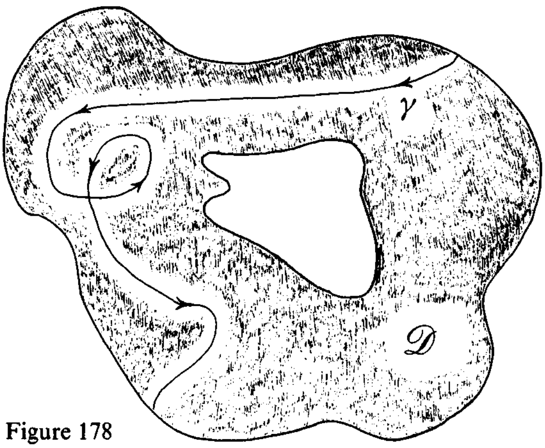


Figure 178

> *WARNING* Most authors work with the *actual extremal length* $\lambda(\mathscr{D}, G)$ equal to $1/\Lambda(\mathscr{D}, G)$, although at least *one* uses $\lambda(\mathscr{D}, G)$ to denote *our* $\Lambda(\mathscr{D}, G)$ and calls *it* extremal length. Some write $\Lambda(\mathscr{D}, G)$ where we have $1/\Lambda(\mathscr{D}, G)$. *Care must therefore be taken* when consulting the formulas in other publications *not to confound what we call* $\Lambda(\mathscr{D}, G)$ *with its reciprocal.*

Here are some practically obvious properties of reciprocal extremal length:

1. *If* $G' \subseteq G$, $\Lambda(\mathscr{D}, G') \leqslant \Lambda(\mathscr{D}, G)$.

Indeed, there are certainly *at least as many* weights $p \geqslant 0$ admissible for G' as there are for G.

2. $\Lambda(\mathscr{D}, G)$ *is a conformal invariant*; in other words, *if* φ *is a conformal mapping of* $\mathscr{D}$ *onto* $\tilde{\mathscr{D}}$, *say, and* $\tilde{G}$ *consists of the images under* φ *of the*

curves belonging to G, we have

$$\Lambda(\tilde{\mathscr{D}}, \tilde{G}) = \Lambda(\mathscr{D}, G).$$

To verify this, observe that if $\gamma \in G$ and $\varphi(\gamma) = \tilde{\gamma}$, then

$$\int_\gamma p(z)|dz| = \int_{\tilde{\gamma}} \tilde{p}(\zeta)|d\zeta|$$

where, for $\zeta = \varphi(z) \in \tilde{\mathscr{D}}$, $\tilde{p}(\zeta) = p(z)/|\varphi'(z)|$. Thus, to each $p \geqslant 0$ defined on $\mathscr{D}$ and admissible for G corresponds a $\tilde{p} \geqslant 0$ defined on $\tilde{\mathscr{D}}$, admissible for $\tilde{G}$; we obviously can get all weights on $\tilde{\mathscr{D}}$ admissible for $\tilde{G}$ in this fashion. Since φ is conformal, we also have (with $\zeta = \xi + i\eta$):

$$\iint_{\tilde{\mathscr{D}}} |\tilde{p}(\zeta)|^2 d\xi\, d\eta = \iint_{\mathscr{D}} \left|\frac{p(z)}{\varphi'(z)}\right|^2 |\varphi'(z)|^2 dx\, dy = \iint_{\mathscr{D}} (p(z))^2 dx\, dy$$

for each such p.

It is *property 2* that makes extremal length so useful in the study of analytic functions. For those applications, we can go a long way using just the two properties and the result of the following simple

Calculation. To find $\Lambda(\mathscr{D}, G)$ when $\mathscr{D}$ is a *rectangle* of *height h* and *length l*, and G consists of the *curves* in $\mathscr{D}$ *joining the two vertical sides* of $\mathscr{D}$.

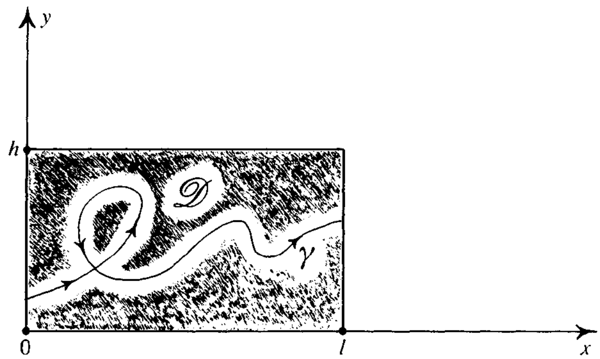


Figure 179

Choosing coordinates in the manner shown, we look at any function $p(z) \geqslant 0$ defined on $\mathscr{D}$ for which $\int_\gamma p(z)|dz| \geqslant 1$ when γ is *any* curve like the one in the diagram. That relation must in particular hold when γ is a *line parallel to* the x-axis, so we must have

$$\int_0^l p(x+iy)\,dx \geqslant 1 \quad \text{for } 0 < y < h.$$

From this, by Schwarz' inequality,

$$\int_0^l (p(x+iy))^2\,\mathrm{d}x \cdot l \quad \geqslant \quad \left(\int_0^l p(x+iy)\,\mathrm{d}x\right)^2 \quad \geqslant \quad 1,$$

so

$$\int_0^h \int_0^l (p(x+iy)^2\,\mathrm{d}x\,\mathrm{d}y \quad \geqslant \quad \frac{h}{l},$$

and $\Lambda(\mathscr{D}, G) \;\geqslant\; h/l$.

However, the function $p(z) \;\equiv\; 1/l$ *is* admissible for G, because if γ is *any* curve like the one shown,

$$\int_\gamma \frac{1}{l}|\mathrm{d}z| \quad = \quad \frac{\text{length}\,\gamma}{l} \quad \geqslant \quad \frac{l}{l} \quad = \quad 1 \;(!)$$

This p gives us *exactly* the value h/l for $\iint_{\mathscr{D}} p^2\,\mathrm{d}x\,\mathrm{d}y$. Therefore

$$\boxed{\Lambda(\mathscr{D}, G) \quad = \quad \frac{h}{l}}$$

for this particular situation.

It is important to note that the computation just made *goes through in the same way, and yields the same result, when we except a finite number of values of* y *from the requirement* (on $p(z)$) *that*

$$\int_0^l p(x+iy)\,\mathrm{d}x \quad \geqslant \quad 1$$

for $0 < y < h$. *This means that we obtain the same value,* h/l, *for* $\Lambda(\mathscr{D}, G)$ *when* $\mathscr{D}$ *is a rectangle of height* h *and length* l *with a finite number of horizontal slits in it, and* G *consists of the curves in* $\mathscr{D}$ *joining* $\mathscr{D}$*'s vertical sides* (*and avoiding those horizontal slits*):

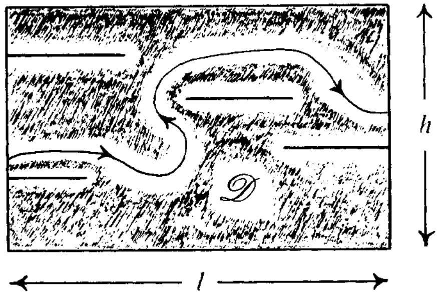


Figure 180

We can now show how extremal length can be used to *express* the *harmonic measure of a single arc* on the boundary of a *simply connected domain* $\mathscr{D}$. Given such a domain $\mathscr{D}$ with a Jordan curve boundary $\partial\mathscr{D}$ and an arc σ on $\partial\mathscr{D}$, we take *any fixed* $z_0 \in \mathscr{D}$ and consider the *family* G *of curves in* $\mathscr{D}$ *which start out from* σ, *loop around* z_0, *and then* (eventually) *go back to* σ:

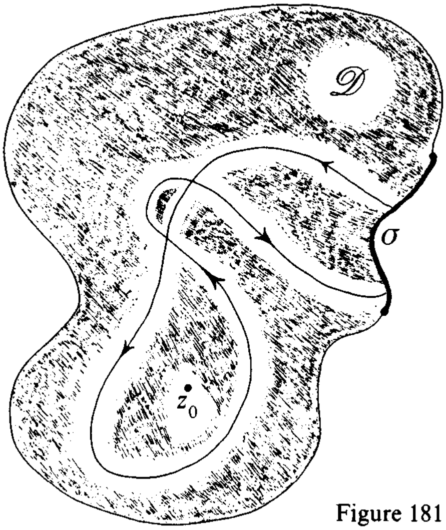


Figure 181

(The curves belonging to G are not *required* to *intersect themselves,* although they are *allowed* to do so.) There is a *precise relation* between $\Lambda(\mathscr{D}, G)$ and $\omega_{\mathscr{D}}(\sigma,\ z_0)$, the harmonic measure of σ in $\mathscr{D}$, as seen from z_0.

This is due to the *conformal invariance* enjoyed by both $\Lambda(\mathscr{D}, G)$ and $\omega_{\mathscr{D}}(\sigma,\ z_0)$. Let us first *map* $\mathscr{D}$ *conformally* onto the *unit disk* Δ in such a way that z_0 *goes to* 0 and σ *to an arc* $\tilde{\sigma}$ of the unit circle *with midpoint at* -1:

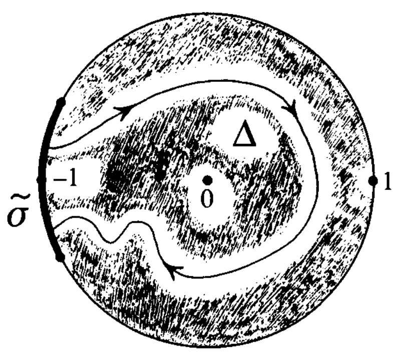


Figure 182

Then

$$\omega_{\mathscr{D}}(\sigma,\ z_0) \;=\; \omega_{\Delta}(\tilde{\sigma},\ 0) \;=\; |\tilde{\sigma}|/2\pi,$$

and, if $\tilde{G}$ denotes the *family of curves* in Δ which *leave* $\tilde{\sigma}$, *loop around* 0, and *then come back to* $\tilde{\sigma}$ (see the figure),

$$\Lambda(\mathscr{D}, G) \;=\; \Lambda(\Delta, \tilde{G})$$

according to property 2. From this, we already see that $\Lambda(\mathscr{D}, G)$ is a *function* of $|\tilde{\sigma}| = 2\pi\omega_{\mathscr{D}}(\sigma, z_0)$, because the *whole configuration* used to *define* $\Lambda(\Delta, \tilde{G})$ is *completely determined* by the *size* of the arc $\tilde{\sigma}$. Calling that function ψ, we have

$$\boxed{\Lambda(\mathscr{D}, G) \;=\; \psi(2\pi\omega_{\mathscr{D}}(\sigma,\; z_0)),}$$

the relation referred to above.

If the boxed formula is to be of any use, we need some information about ψ. With that in mind, we look first at the way $\Lambda(\Delta, \tilde{G})$, equal to $\Lambda(\mathscr{D}, G)$, is obtained. The *reflection*, $\tilde{\gamma}^*$, *of any curve* $\tilde{\gamma} \in \tilde{G}$ *in the real axis also belongs to* $\tilde{G}$, because the arc $\tilde{\sigma}$ is *symmetric with respect to the real axis*, due to our having *chosen* it that way:

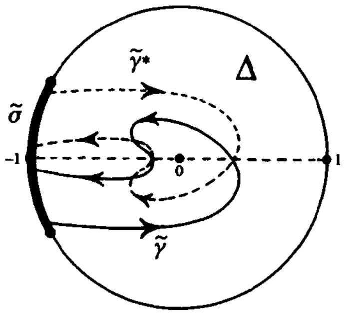


Figure 183

Therefore, if $p \geqslant 0$ (defined on Δ) is *admissible for* $\tilde{G}$, we *not only have* $\int_{\tilde{\gamma}} p(z)|dz \geqslant 1$ for any $\tilde{\gamma} \in \tilde{G}$, but *also*

$$\int_{\tilde{\gamma}^*} p(z)|dz| \;\geqslant\; 1$$

for such curves $\tilde{\gamma}$, i.e.,

$$\int_{\tilde{\gamma}} p(\bar{z})|dz| \;\geqslant\; 1.$$

This means that $p(\bar{z})$ is *also admissible* for $\tilde{G}$, from which it follows that

$$\tfrac{1}{2}(p(z) + p(\bar{z}))$$

is admissible for $\tilde{G}$, whenever p is. However,

$$\iint_\Delta \left(\frac{p(z)+p(\bar{z})}{2}\right)^2 dx\,dy \;\leqslant\; \frac{1}{2}\iint_\Delta (p(z))^2\,dx\,dy \;+\; \frac{1}{2}\iint_\Delta (p(\bar{z}))^2\,dx\,dy$$

$$= \iint_\Delta (p(z))^2\,dx\,dy,$$

so it follows that *for the computation of* $\Lambda(\Delta, \tilde{G})$, *we need only look at the functions p admissible for $\tilde{G}$ such that*

$$p(z) = p(\bar{z}).$$

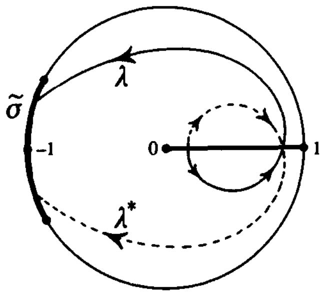


Figure 184

For *such* admissible weights p,

$$\int_\lambda p(z)|dz| \;\geqslant\; \frac{1}{2}$$

when λ is *any curve*, going *from the segment* $[0, 1)$ *to the arc* $\tilde{\sigma}$ and *lying in* Δ. Indeed, *given* such a curve λ, *it* and its *reflection* λ^* in the real axis *together make up a curve $\tilde{\gamma}$ belonging to $\tilde{G}$* (when λ^* is traversed in the reverse direction), so that

$$\int_\lambda p(z)|dz| + \int_{\lambda^*} p(z)|dz| \;\geqslant\; 1,$$

i.e.,

$$\int_\lambda (p(z)+p(\bar{z}))|dz| \;\geqslant\; 1,$$

which implies the relation in question. *The weight $2p(z)$ is thus admissible for the family H consisting of the curves λ just described.* It is, moreover, clear that *all the $q \geqslant 0$ admissible for H with $q(z) = q(\bar{z})$ are of the form $2p$ where $p(z) = p(\bar{z})$ is admissible for $\tilde{G}$.*

To *obtain* $\Lambda(\Delta, H)$ we may, however, *limit our attention* to the q admissible for H with $q(z) = q(\bar{z})$. This is seen by arguing as we did for

$\Lambda(\Delta, \tilde{G})$. In view of the preceding observation, we therefore have

$$\Lambda(\Delta, H) = 4\Lambda(\Delta, \tilde{G}).$$

If we now *restrict* the family H so as to only have in it curves λ *lying entirely in the slit disk* $\Omega = \Delta \sim [0, 1)$ *except for their endpoints*, $\Lambda(\Delta, H)$, according to property 1, *will not be augmented. It will not be diminished either*, for such restriction of H does not give us any *new* admissible functions q.* *We may therefore take* H *to consist only of curves* λ *of the kind just mentioned* without affecting the last relation. Once this is done, $\Lambda(\Delta, H)$ becomes identical with $\Lambda(\Omega, H)$,† so we have finally

$$\Lambda(\Omega, H) = 4\Lambda(\Delta, \tilde{G}),$$

and if ψ is the function introduced above,

$$\psi(|\tilde{\sigma}|) = \frac{1}{4}\Lambda(\Omega, H).$$

Another conformal mapping will enable us to identify $\Lambda(\Omega, H)$ with the reciprocal extremal length already worked out in the above special calculation. *Let it be granted* for the moment that $\Omega = \Delta \sim [0, 1)$ can be *mapped conformally* onto a certain *rectangle* in such a way as to make $\tilde{\sigma}$ *go onto one side of that rectangle*, while the *slit* $[0, 1]$ *goes onto the opposite side*. We shall see presently *why* there always *is* such a mapping.

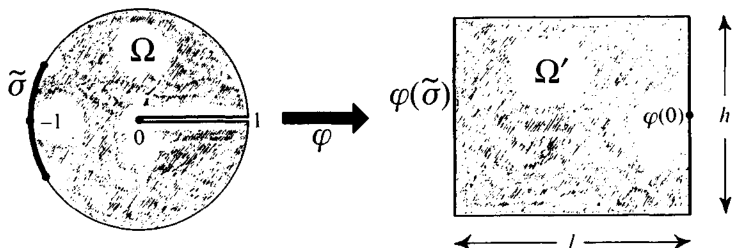


Figure 185

The rectangle may evidently be oriented so that the images of $\tilde{\sigma}$ and the slit are *vertical*; the *mapping* – call it φ – will then take the curves of our (restricted) family H to the *ones in the rectangle joining its two vertical sides*. Denoting the latter family of curves by H' and the rectangle itself by Ω', we see by property 2 that

$$\Lambda(\Omega, H) = \Lambda(\Omega', H').$$

* That's because any curve λ in Δ running from $[0, 1)$ to $\tilde{\sigma}$ has on it *an arc lying in the smaller domain* Ω and joining $[0, 1)$ to $\tilde{\sigma}$.

† The slit $[0, 1]$ has *zero area*!

If Ω' has height h and length l, we know by the special computation* that $\Lambda(\Omega', H') = h/l$. Hence

$$\Lambda(\Omega, H) = h/l.$$

As we have already seen, however,

$$\Lambda(\Omega, H) = 4\psi(|\tilde{\sigma}|),$$

ψ being the function presently under investigation. We have therefore arrived at two conclusions:

(i) that the *side ratio* h/l of the *rectangle* Ω' (assuming always that a mapping φ exists!) is *determined by* $|\tilde{\sigma}|$ even though, for the *same* arc $\tilde{\sigma}$, *different* mappings φ of the kind described onto *different* rectangles Ω' may be (and in fact *are*) possible;

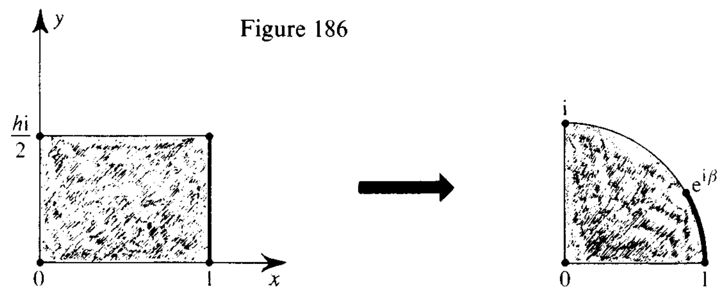
(ii) that the *function value* $\psi(|\tilde{\sigma}|)$ can be *evaluated*, once a mapping φ is available, by the formula

$$\boxed{\psi(|\tilde{\sigma}|) = h/4l}$$

With a little more work we can show that a mapping φ *really does exist* and, at the same time, obtain a simpler description of the function ψ. The idea here is to get at φ by *going backwards.*

We start by taking an arbitrary $h > 0$ and *mapping* the *rectangle* $\{z\colon\ 0 < \Re z < 1,\ 0 < \Im z < h/2\}$ *conformally* onto the *quarter circle* $\{z\colon\ |z| < 1,\ \Re z > 0,\ \Im z > 0\}$ in such a way as to take 1 to 1, $hi/2$ to i, and 0 to 0:

Figure 186



Under this mapping, the *upper right-hand corner* of the rectangle goes to a certain point $e^{i\beta}$, $0 < \beta < \pi/2$, where β evidently depends on h. Successive Schwarz reflections in the x and y axes will now yield a conformal mapping of the *enlarged* rectangle $\{z\colon\ -1 < \Re z < 1,\ -h/2 < \Im z < h/2\}$

* The curves in H' join the left vertical side of Ω' to its right one *with endpoints omitted.* That does not affect the computation; see the observation following it.

onto the unit disk Δ, which takes 0 to 0, the *right vertical side of the new rectangle* to the *arc* $(\widehat{e^{-i\beta}, e^{i\beta}})$, and its *left vertical side* to the *opposite arc* $(\widehat{-e^{i\beta}, -e^{-i\beta}})$:

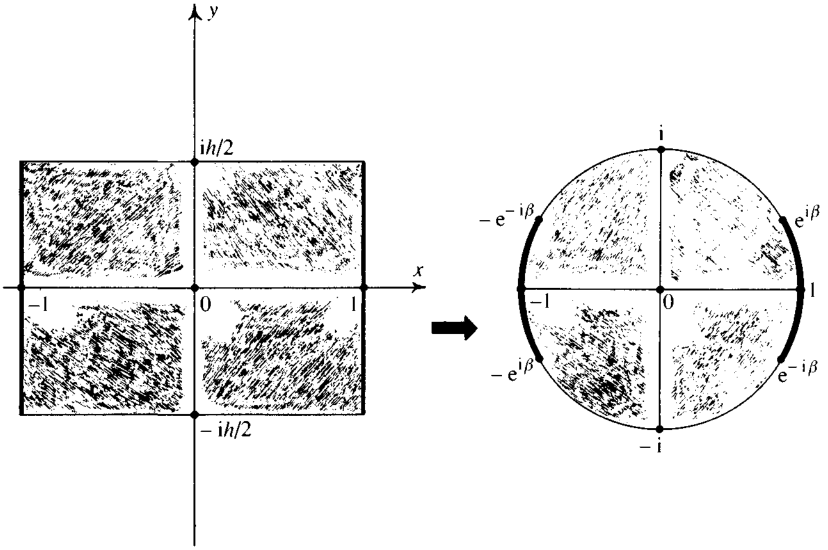


Figure 187

From this we see that $2\beta/\pi$ is equal to the *harmonic measure of the rectangle's two vertical sides* relative to that rectangle, as seen from 0. It is, however, obvious by the principle of extension of domain that *this harmonic measure increases when h does*, in fact, *grows steadily from* 0 *towards* 1 as h increases from 0 to ∞:

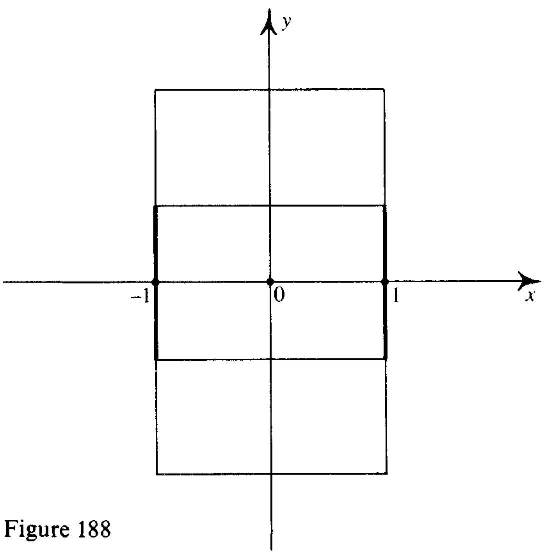


Figure 188

Given α, $0<\alpha<2\pi$, we may therefore *adjust h so as to make $\beta = \alpha/4$*, and there will be *only one value* of h for which this happens.

Taking that value of h (which depends on α), we denote by F_α the *inverse* of the *last* of the above conformal mappings (the one from the enlarged rectangle to Δ). If α is the *length* of our arc $\tilde{\sigma}$ on the boundary of the slit disk Ω, the transformation

$$z \longrightarrow F_\alpha(\sqrt{(-z)})$$

maps Ω conformally onto the *rectangle*

$$\left\{z\colon\ 0<\Re z<1,\ -\frac{h}{2} < \Im z < \frac{h}{2}\right\}$$

and sends $\tilde{\sigma}$ to the *right vertical side* of that *rectangle*, taking, at the same time, the slit $[0, 1]$ to its *left vertical side*. Indeed, the simple conformal mapping $z \longrightarrow \sqrt{(-z)}$ does take Ω onto the *right half* of Δ, and sends the slit $[0, 1]$ to the *vertical diameter* of that semi-circle, while the arc $\tilde{\sigma}$ of length α (having midpoint at -1) goes to the *arc* $(\widehat{e^{-i\alpha/4}, e^{i\alpha/4}})$:

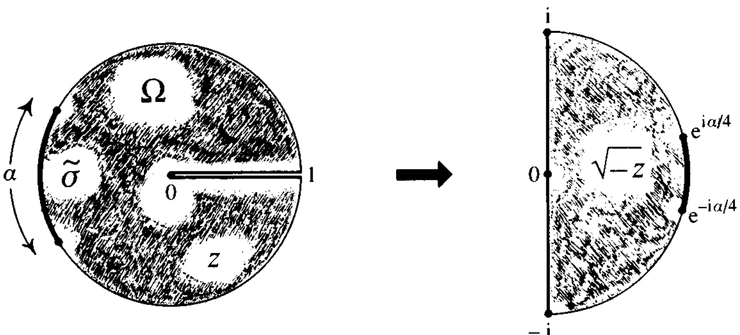


Figure 189

And, since $\beta = \alpha/4$ for our value of h, F_α does, according to one of the preceding diagrams, take the *right half* of Δ onto the above mentioned *rectangle*, making the *diameter* and *arc* go onto that rectangle's *vertical sides*. We *have*, in other words, *obtained* a mapping φ of the required sort:

$$\varphi(z) = F_\alpha(\sqrt{(-z)}).$$

The rectangle onto which this φ takes Ω has *height h* and *length unity*. Therefore, with h corresponding to $\alpha = |\tilde{\sigma}|$ in the manner described,

$$\psi(\alpha) = h/4.$$

We see by the preceding discussion that $\psi(\alpha)$ is *strictly increasing* (from 0 to ∞) when α increases from 0 to 2π; this means, of course, that the *inverse function* ψ^{-1} *exists.* In fact, if β is related to h through the conformal mapping of a *rectangle onto a quarter disk* used above, we simply have

$$\psi^{-1}(h/4) = 4\beta:$$

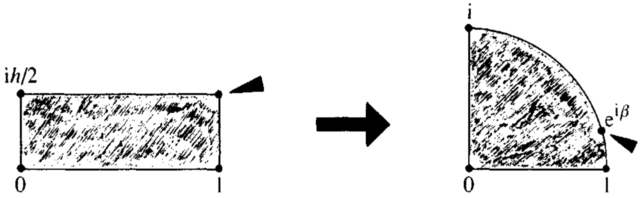


Figure 190

Let us now return to our original boxed relation between $\Lambda(\mathscr{D}, G)$ and $\omega_{\mathscr{D}}(\sigma, z_0)$. From it and the above reasoning we obtain without further ado the following

Theorem. *Given a simply connected domain* $\mathscr{D}$ *with Jordan curve boundary* $\partial\mathscr{D}$, *we have, for any* arc σ *on* $\partial\mathscr{D}$ *and any* $z_0 \in \mathscr{D}$,

$$\omega_{\mathscr{D}}(\sigma, z_0) = \frac{1}{2\pi}\psi^{-1}(\Lambda(\mathscr{D}, G)),$$

where G is the family of curves in $\mathscr{D}$ *beginning and ending on* σ *and looping around* z_0. *Here,* ψ^{-1} *is the strictly increasing function just described.*

Remark. The relation between $\omega_{\mathscr{D}}(\sigma, z_0)$ and $\Lambda(\mathscr{D}, G)$ is thus *one–one*; either of these quantities determines the other.

As we have seen, the description of ψ^{-1} is based on a certain conformal mapping of a *rectangle onto a disk.* Since elliptic functions are needed for the precise expression of such mappings, those must be required for the *explicit formula* for ψ^{-1}, which in fact involves *elliptic modular functions.* Fortunately, the *exact value* of $\psi^{-1}(\Lambda(\mathscr{D}, G))$ is *hardly ever needed* in applications, and an *approximation, asymptotically correct for small values of* $\Lambda(\mathscr{D}, G)$, *suffices.* That can be obtained in completely elementary fashion.

Lemma. *For small values of* $\alpha > 0$,

$$\psi(\alpha) = \frac{\pi}{2\log(1/Q\alpha^2)},$$

with a quantity Q tending to a limit $\neq 0$ *as* $\alpha \to 0$

Proof. The mapping φ used above to express ψ is obtained by putting together a chain of simpler conformal mappings. The whole construction can be most easily presented using the following diagram. In it, $C.\alpha$ *denotes a quantity asymptotic to some non-zero constant multiple of* α for $\alpha \longrightarrow 0$. That constant need not be the same in the different steps.

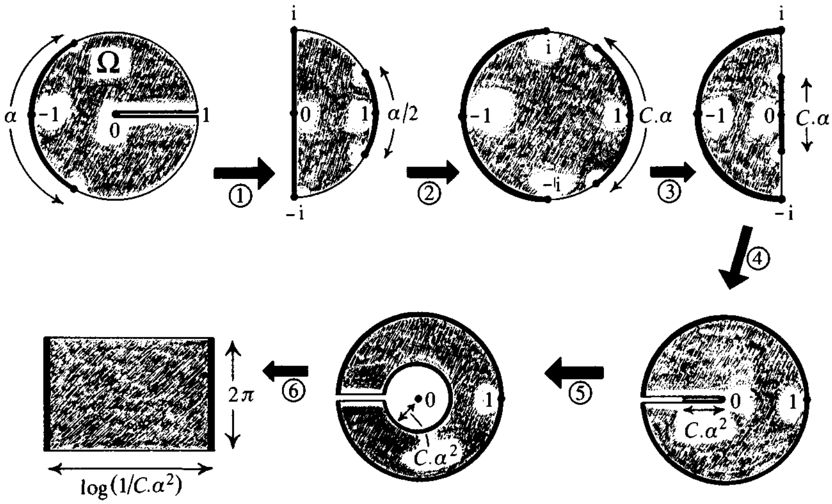


Figure 191

$$\therefore\ \psi(\alpha) \;=\; \frac{1}{4}\times\frac{2\pi}{\log(1/C.\alpha^2)} \;=\; \frac{\pi}{2\log(1/C.\alpha^2)}, \qquad \text{Q.E.D.}$$

Theorem. *In the preceding theorem,* $\omega_{\mathscr{D}}(\sigma,\ z_0)$ *lies between two constant multiples of*

$$e^{-\pi/4\Lambda(\mathscr{D},G)}$$

(*with* absolute *constants*).

Proof. For *small* values of $\alpha = 2\pi\omega_{\mathscr{D}}(\sigma,\ z_0)$, the statement follows immediately from the lemma in view of the relation $\Lambda(\mathscr{D}, G) = \psi(\alpha)$.

Because ψ is *strictly* increasing, when *either* of the quantities $\omega_{\mathscr{D}}(\sigma,\ z_0)$, $\Lambda(\mathscr{D}, G)$, is *not* small, the *other* is *not small either*. Hence, since $0 \leqslant \omega_{\mathscr{D}}(\sigma,\ z_0) \leqslant 1$, the statement holds generally.

Remark. Usually what is used is the inequality

$$\boxed{\omega_{\mathscr{D}}(\sigma,\ z_0) \;\leqslant\; Ce^{-\pi/4\Lambda(\mathscr{D},G)}\ ;}$$

for most applications the precise value of the numerical constant C does not matter.

Problem 32

Show that one may replace $\leqslant$ in the above boxed formula by $=$ and C by $(8/\pi) + o(1)$ for values of $\Lambda(\mathscr{D}, G)$ tending to zero (which covers just about all the situations where the formula is ever used).

(Hint: In the proof of the lemma, all the mappings of the chain shown are elementary except ⑤. Approximate the latter by a Joukowski transformation which takes the *small inner slit* onto a *circle* about 0 and the *outer circle* onto a curve that is *nearly* a circle about 0 when α is small.)

According to the preceding boxed inequality, *any upper bound for* $\Lambda(\mathscr{D}, G)$ *yields one for* $\omega_{\mathscr{D}}(\sigma,\ z_0)$. An upper bound for $\Lambda(\mathscr{D}, G)$ is obtained, however, *as soon as we are able to specify a weight* $p(z) \geqslant 0$ *on* $\mathscr{D}$ *admissible for the family* G. To this feature is due the great practical value of the inequality.

Suppose we have a *simply connected unbounded domain* $\mathscr{D}$, with reasonably nice boundary. We *fix* some $z_0 \in \mathscr{D}$ and take any $R > \operatorname{dist}(z_0, \partial\mathscr{D})$. The *intersection* of $\mathscr{D}$ with the disk $|z - z_0| < R$ will then have a *connected component containing* z_0, which we denote by $\mathscr{O}_R$:

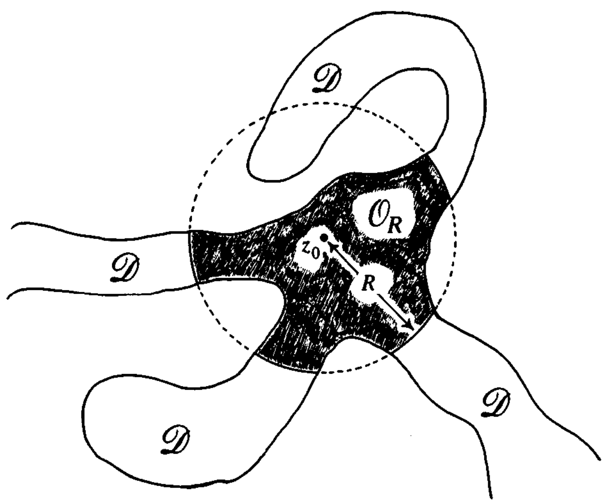


Figure 192

The boundary of $\mathscr{O}_R$ consists of part of $\partial\mathscr{D}$ and a certain number of *arcs* on the circle $|z - z_0| = R$. Some of those separate $\mathscr{O}_R$ from *unbounded* components of $\mathscr{D} \sim \bar{\mathcal{O}}_R$ (which must be present, $\mathscr{D}$ being *assumed* unbounded); we call the former *distinguished arcs*. Let us denote by $\mathscr{D}_R$ the set of points in $\mathscr{D}$ which can be joined to z_0 by paths *lying entirely in* $\mathscr{D}$ *and not crossing any distinguished arc*. If $\partial\mathscr{D}$ has *sufficient regularity, which we are assuming*, $\mathscr{D}_R$ will be a *bounded domain*. It *may*, however, contain points z with $|z - z_0| > R$, and hence *include* $\mathcal{O}_R$ *properly*:

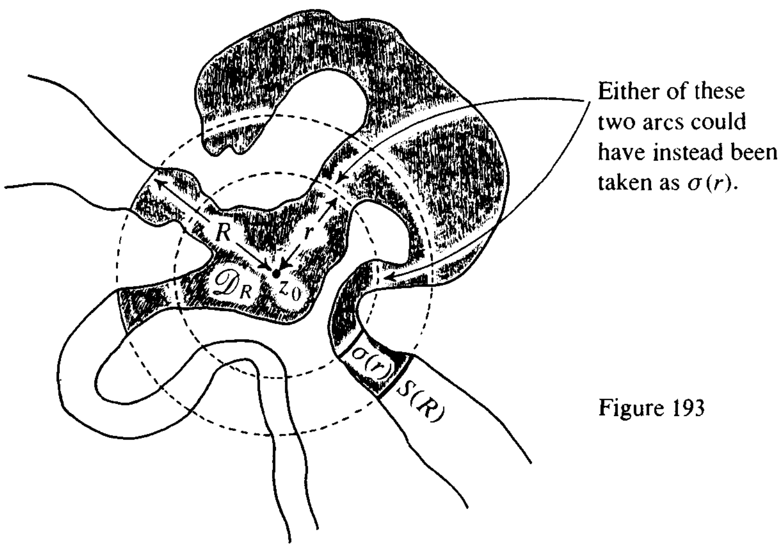


Figure 193

We *now single out at pleasure one* of the *distinguished arcs* on $|z - z_0| = R$ and call it $S(R)$. $S(R)$ is part of $\partial\mathscr{D}_R$. *We are interested in estimating the harmonic measure*

$$\omega_{\mathscr{D}_R}(S(R),\ z_0)$$

from above.

For each r, $\operatorname{dist}(z_0, \partial\mathscr{D}) < r < R$, the circle of radius r about z_0 intersects $\mathscr{D}_R$ (*sic*!) in a number of *open arcs. One or more* of these must *separate* z_0 from $S(R)$ in $\mathscr{D}_R$; in other words, any *path* in $\mathscr{D}_R$ from z_0 to $S(R)$ must *pass through* it (or them). *We choose such an arc and call it* $\sigma(r)$ (see the preceding figure). When several choices are possible, this may be done in *fairly arbitrary fashion*; *we do require*, however, that the selection be done in such a way as to make the *union of the* $\sigma(r)$ *at least a Borel set in* $\mathbb{C}$. As long as $\partial\mathscr{D}$ is *decent* (which we are assuming), this is certainly possible.

Put $\vartheta(r) = |\sigma(r)|/r$ for $\operatorname{dist}(z_0, \partial\mathscr{D}) < r < R$; $\vartheta(r)$ is simply the *angle subtended by the arc* $\sigma(r)$ *at* z_0. *For* $0 < r < \operatorname{dist}(z_0, \partial\mathscr{D})$ *we take* $\vartheta(r) = \infty$ (*sic*!). In our present set-up, we then have the

Theorem. (due essentially to Ahlfors and Carleman)

$$\omega_{\mathscr{D}_R}(S(R),\ z_0) \ \leqslant\ C\exp\left(-\pi\int_0^R \frac{\mathrm{d}r}{r\vartheta(r)}\right),$$

C being an absolute constant.

Proof. We use the preceding theorem, specifying a suitable weight p on $\mathscr{D}_R$ admissible for the family G of curves in $\mathscr{D}_R$ that loop around z_0 and have both ends on $S(R)$. In this we are guided by the special computation for a rectangle made earlier.

Denote by S the *union* of the $\sigma(r)$, $\operatorname{dist}(z_0, \partial\mathscr{D}) < r < R$. Then $S \subseteq \mathscr{D}_R$; note that S need not be connected! We are *assuming* that it is a Borel set. For $z \in S$, we put

$$p(z) \quad = \quad \frac{k}{|z-z_0|\,\vartheta(|z-z_0|)}$$

with a constant k to be presently determined. When $z \in \mathscr{D}_R$ *lies outside the set* S, we put $p(z) = 0$; this is consistent with our having taken $\vartheta(r) = \infty$ for $0 < r < \operatorname{dist}(z_0, \partial\mathscr{D})$. Our weight p will be admissible for G provided that

$$\int_\gamma p(z)|\mathrm{d}z| \quad \geqslant \quad 1$$

for each curve γ in $\mathscr{D}_R$ starting from $S(R)$, looping around z_0, and then returning to $S(R)$.

In terms of the polar coordinates

$$re^{i\varphi} \ = \ z - z_0$$

with origin at z_0, we have

$$|\mathrm{d}z| \quad = \quad \sqrt{((\mathrm{d}r)^2 + r^2(\mathrm{d}\varphi)^2)} \quad \geqslant \quad |\mathrm{d}r|$$

(with possible equality) along the curves γ; we thus require that

$$\int_\gamma p(z)|\mathrm{d}r| \quad \geqslant \quad 1$$

for $\gamma \in G$. Any of these γ must, however, *pass at least twice through each of the arcs* $\sigma(r)$, $\operatorname{dist}(z_0, \partial\mathscr{D}) < r < R$; *going and coming back*:

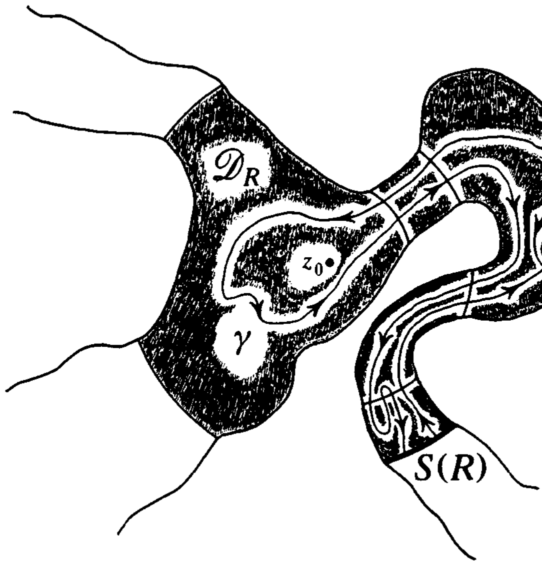


Figure 194

Our condition on $p(z) = k/r\vartheta(r)$ will therefore be *met* if k is adjusted so as to make

$$2\int_{\mathrm{dist}(z_0,\partial\mathscr{D})}^{R}\frac{k\,\mathrm{d}r}{r\vartheta(r)} = 1,$$

i.e., if

$$k\int_0^R\frac{\mathrm{d}r}{r\vartheta(r)} = \frac{1}{2}.$$

Choosing the value of k which satisfies this last relation, we then have

$$\iint_{\mathscr{D}_R}(p(z))^2\,\mathrm{d}x\,\mathrm{d}y = \iint_S\frac{k^2}{r^2(\vartheta(r))^2}r\,\mathrm{d}\varphi\,\mathrm{d}r$$

$$= \int_{\mathrm{dist}(z_0,\partial\mathscr{D})}^{R}\int_{\sigma(r)}\frac{k^2}{r^2(\vartheta(r))^2}r\,\mathrm{d}\varphi\,\mathrm{d}r$$

$$= \int_{\mathrm{dist}(z_0,\partial\mathscr{D})}^{R}\frac{k^2}{r\vartheta(r)}\mathrm{d}r = k\int_0^R\frac{k}{r\vartheta(r)}\mathrm{d}r = \frac{k}{2}.$$

Here, the *very first* of the above integrals is by definition $\geqslant \Lambda(\mathscr{D}, G)$, so

$$\Lambda(\mathscr{D}, G) \leqslant \frac{k}{2} = 1\Big/4\int_0^R\frac{\mathrm{d}r}{r\vartheta(r)}.$$

Our result now follows by the previous theorem.

Remark 1. There is a version of this result for *horizontal curvilinear strips* in which *vertical crosscuts* of those strips play the rôle of the arcs $\sigma(r)$. That version, obtainable formally from ours by a change of variable, is best derived *ab initio* by again reasoning as above. From it and the principle of extension of domain one immediately gets the harmonic measure estimate used in the scholium at the end of §D.5, Chapter VII. The reader should go through the verification of this because that estimate has many practical applications.

Remark 2. If there are several candidates for $\sigma(r)$ as in the preceding two diagrams whenever $r < R$ belongs to a *set of positive measure*, an *improvement* of the estimate provided by the theorem is available. This is pointed out in an *Arkiv* article by K. Haliste.

Let the candidates in question be denoted by $\sigma_l(r)$ with $1 \leqslant l \leqslant n(r)$, and write $r\vartheta_l(r)$ for the length of each $\sigma_l(r)$, taking $\vartheta_1(r) = \infty$ when $r < \operatorname{dist}(z_0, \partial\mathscr{D})$. *Then the integral* $\int_0^R (1/r\vartheta(r))\,\mathrm{d}r$ *figuring in the theorem's statement can be replaced by*

$$\int_0^R \sum_{l=1}^{n(r)} (1/r\vartheta_l(r))\,\mathrm{d}r.$$

The proof of this better result is just like that of the theorem. One takes for S the union of *all* the $\sigma_l(r)$ for $\operatorname{dist}(z_0, \partial\mathscr{D}) < r < R$ and then works with a weight $p(z)$, equal to *zero* outside S, and to

$$\frac{k}{|z - z_0|\,\vartheta_l(|z - z_0|)}$$

for $z \in S$ if $\sigma_l(|z - z_0|)$ is the arc on which it lies.

Remark 3. If $\Sigma(R)$ denotes the *union of the arcs* on $|z - z_0| = R$ bounding $\mathcal{O}_R$ (the component of $\mathscr{D} \cap \{|z - z_0| < R\}$ containing z_0), it is possible to get an estimate for

$$\omega_{\mathcal{O}_R}(\Sigma(R),\ z_0)$$

similar to the one for $\omega_{\mathscr{D}_R}(S(R),\ z_0)$ furnished by the last theorem. One form of that estimate usually goes under the name of *Tsuji's inequality*, although a better version of it can already be found in Beurling's thesis. This matter will be taken up in §F.3 below.

A celebrated application of the preceding result is in the proof, also due to Ahlfors and Carleman, of the *Denjoy conjecture*, which should be part of every analyst's general background.

The conjecture deals with the number of limiting values that an entire function of finite order may tend to when its argument moves out to ∞ along various continuous paths. We say that an entire function $f(z)$ is of *order* p if, for *each* $\varepsilon > 0$, there is a constant K_ε such that

$$|f(z)| \leqslant K_\varepsilon e^{|z|^{p+\varepsilon}}.$$

(The entire functions of exponential type considered so often in this book are thus *of order* 1.) A *finite* number a is called an *asymptotic value* for $f(z)$ if there is a *curve* γ going out to ∞ with

$$f(\zeta) \longrightarrow a \quad \text{as} \quad \zeta \longrightarrow \infty \quad \text{along } \gamma.$$

For example, 0 is an asymptotic value for the function e^z.

An entire function may have *more than one* asymptotic value. Let us for instance take *any integer* $p > 1$. Then the functions $(\sin z^p)/z^p$ and

$$f(z) = \int_0^z \frac{\sin \zeta^p}{\zeta^p}\, d\zeta$$

are both entire and of order p. When z goes out to ∞ along any one of the $2p$ rays

$$\arg z = \frac{\pi}{p}k, \quad k = 1, 2, 3, \ldots, 2p,$$

$f(z)$ tends to the finite value

$$e^{\pi i k/p} \int_0^\infty \frac{\sin t^p}{t^p}\, dt.$$

Since the integral itself is $\neq 0$,* these values are all *different*. The function $f(z)$ thus has $2p$ asymptotic values.

Denjoy made the conjecture that the example just given represents an *extreme case*, and that indeed an *entire function of order p (integral or not) cannot have more than* $[2p]$ *asymptotic values. To prove this, Ahlfors and Carleman argued as follows.*

Let us assume that $f(z)$ has n different asymptotic values, where $n > 1$. The case where $n = 1$ needs also to be considered when $f(z)$ has order $< \frac{1}{2}$; its treatment is like that of the one for $n > 1$, and somewhat easier

* It is equal to $\cos(\pi/2p)\Gamma(1/p)/(p-1)$, as is readily seen on putting $t^p = x$, integrating by parts and then taking the integral of $z^{1/p-1}e^{iz}$ around a contour consisting of the positive real and imaginary axes. Here, $\Gamma(1/p)$ is clearly > 0 – look at the integral representation for $\Gamma(x+1)$ in §B.3!

than the latter. Taking, then, $n > 1$, we have certain curves

$$\gamma_1, \gamma_2, \ldots, \gamma_n$$

going out to ∞ with $f(\zeta)$ tending to some limit as $\zeta \longrightarrow \infty$ along any one of them, *these limits being all different*. We may obviously take each of the γ_k to be *polygonal*, and *without self-intersections* (just cut off any closed loops that γ_k may have:

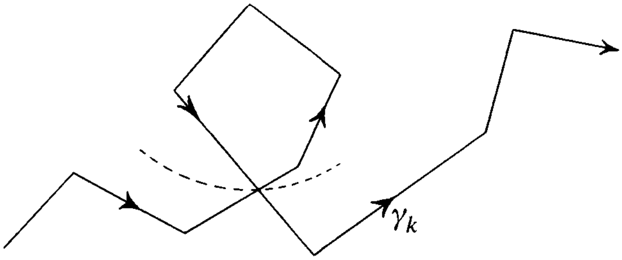


Figure 195

Since $f(\zeta)$ tends to different limits as $\zeta \longrightarrow \infty$ along the different curves γ_k, *two* of those *cannot intersect* at points *arbitrarily far out from* 0. There is thus no loss of generality in taking the γ_k *disjoint*, and in assuming that *the origin does not lie on any one or them*. The γ_k then bound n separate *channels*, or *tracts*, starting from a common central neighbourhood of 0 and going out to ∞:

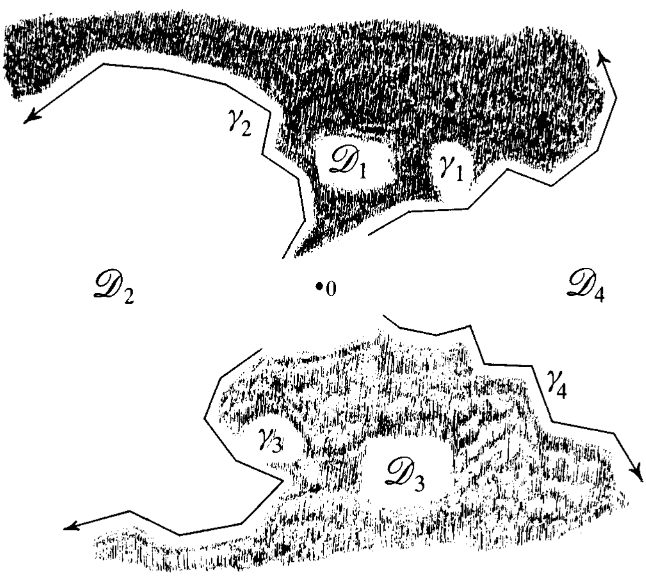


Figure 196

We can index the γ_k in such fashion that for $k = 1,2,\ldots,n-1$, γ_k and γ_{k+1} together bound one of these tracts, denoted by $\mathscr{D}_k$, and that γ_n and γ_1 bound one, called $\mathscr{D}_n$. The preceding figure shows how things could look when $n = 4$; in it, the tracts $\mathscr{D}_1$ and $\mathscr{D}_3$ are shaded.

The function $f(z)$ cannot be bounded in any of the tracts $\mathscr{D}_k$. Suppose, for instance, that $f(z)$ is bounded in $\mathscr{D}_1$. Closing up the 'base' of $\mathscr{D}_1$ in any convenient fashion then gives us a simply connected region, part of whose boundary consists of the curves γ_1 and γ_2, both going out to ∞:

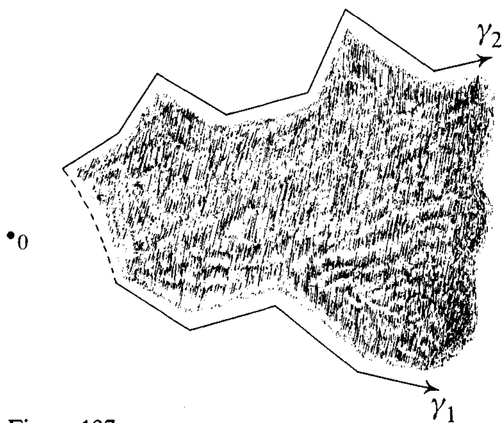


Figure 197

$f(z)$, *bounded* in that region and continuous up to γ_1 and γ_2, then tends to *two limits*, say a_1 and a_2, according as $z \longrightarrow \infty$ along γ_1 or along γ_2. In this circumstance, a well known theorem of Lindelöf implies that $a_1 = a_2$. Since, however, a_1 and a_2 are *two different asymptotic values* of f, we have a contradiction.

Problem 33

Prove Lindelöf's theorem. (Hint: By means of a conformal mapping, one may convert the region in question to the upper half plane and the function f to a new one, $F(z)$, *analytic* and *bounded* for $\Im z > 0$, *continuous* up to $\mathbb{R}$, and having the property that $F(x) \longrightarrow a_2$ for $x \longrightarrow -\infty$ while $F(x) \longrightarrow a_1$ for $x \longrightarrow \infty$. Apply the Poisson representation to $G(z) = (F(z) - a_1)(F(z) - a_2)$ (*sic*!), thus showing that $G(z) \longrightarrow 0$ *uniformly* for $z \longrightarrow \infty$ in $\{\Im z \geqslant 0\}$.)

Having established that $f(z)$ *cannot be bounded* in any of the $\mathscr{D}_k$, it suffices, in order to prove the Denjoy conjecture, to *assume that* $n > [2p]$ (with $f(z)$ of order p) and *deduce* that *then $f(z)$ must be bounded in some* $\mathscr{D}_k$. For this purpose, we take large values of r and look at the intersections

$\Sigma_k(r)$ of each of the tracts $\mathscr{D}_k$ with the circle $|z| = r$. Each $\Sigma_k(r)$ is the union of one or more arcs; we single out one of them, called $\sigma_k(r)$, in such a way as to ensure that *any path from* 0 *to* $\sigma_k(r)$ which touches *neither* γ_k nor γ_{k+1} (and hence *stays in* $\mathscr{D}_k$ after once entering that channel) must *necessarily cut every* $\sigma_k(r')$ *with* $r' < r$ (the latter being *defined for all sufficiently large* r').

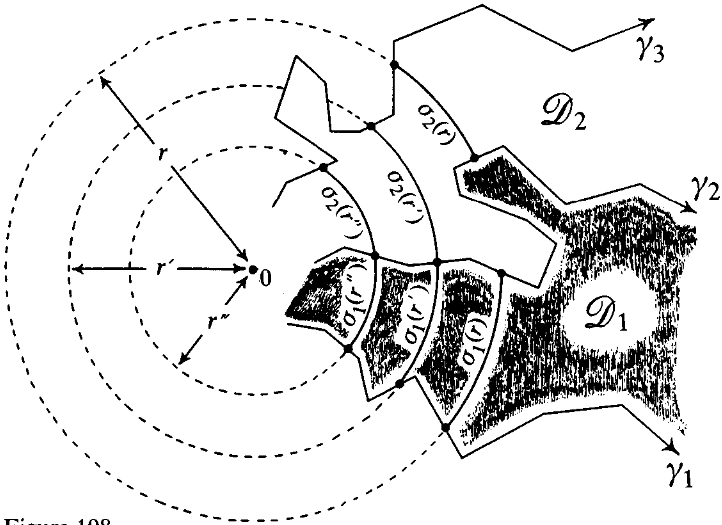


Figure 198

This we do for each k, taking care to select the $\sigma_k(r)$ for different values of r in such fashion as to *make their union a Borel set*, which is clearly *possible* since the γ_k are *polygonal curves*.

Calling $\vartheta_k(r) = |\sigma_k(r)|/r$, it is then evident that

$$\vartheta_1(r) + \vartheta_2(r) + \cdots + \vartheta_n(r) \leqslant 2\pi.$$

The above picture shows that the sum on the left may actually be $< 2\pi$.

Problem 34

(a) Show that if r_0 is *fixed and large enough* and $R > r_0$, we have

$$\sum_{k=1}^{n} \int_{r_0}^{R} \frac{dr}{r\vartheta_k(r)} \geqslant \frac{n^2}{2\pi}\log\frac{R}{r_0}.$$

(Hint: $\sum_{k=1}^{n} (1/\sqrt{(\vartheta_k(r))})\cdot\sqrt{(\vartheta_k(r))} = n$ (!).)

(b) Hence show that for some (fixed) k there must be arbitrarily large values of R for which

$$\int_{r_0}^{R} \frac{dr}{r\vartheta_k(r)} \geqslant \frac{n}{2\pi}\log\frac{R}{r_0}.$$

(c) Wlog, let the index k in (b) be unity. Take, then, the tract $\mathscr{D}_1$ and attach to it a bounded region containing 0 so as to obtain a *simply connected unbounded domain* $\mathscr{D}$:

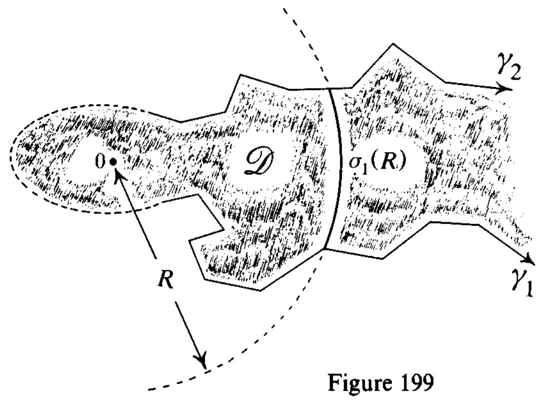


Figure 199

For large $R > 0$, denote by $\mathscr{D}_R$ the *set of points in* $\mathscr{D}$ *which can be reached by paths in* $\mathscr{D}$ *starting at* 0 *and not crossing* $\sigma_1(R)$. Show that for each $z \in \mathscr{D}$ there is a number C_z such that, for large enough R,

$$\omega_{\mathscr{D}_R}(\sigma_1(R),\ z) \quad \leqslant \quad C_z \exp\left(-\pi \int_{r_0}^{R} \frac{dr}{r\vartheta_1(r)}\right).$$

(Hint: First do this for $z = 0$. Then use Harnack.)

(d) Assuming that $n \geqslant [2p]+1$, show that $f(z)$ is *bounded in* $\mathscr{D}$, and thus *bounded in* $\mathscr{D}_1$, *yielding a contradiction that proves the Denjoy conjecture.* (Hint: $f(z)$ is bounded on $\partial\mathscr{D}$ since the part of that boundary lying outside some large circle consists of points either on γ_1 or on γ_2. Fix any $z \in \mathscr{D}$, take large values of R for which the conclusion of (b) holds (with, as we are assuming, the index $k = 1$), and use the *theorem on harmonic estimation* (Chapter VII, §B.1) to estimate $\log|f(z)|$ in the domains $\mathscr{D}_R$. Note that on $\partial\mathscr{D}_R \cap \mathscr{D} \;=\; \sigma_1(R)$,

$$\log|f(\zeta)| \quad \leqslant \quad O(1) + R^{p+\varepsilon}$$

with $\varepsilon > 0$ arbitrary. Apply the conclusion of (c).)

2. Real zeros of functions $f(z)$ of exponential type with $\int_{-\infty}^{\infty}(\log^+|f(x)|/(1+x^2))\,dx < \infty$

Now that the Ahlfors–Carleman estimate for harmonic measure is at our disposal, we are ready to carry out the extension of the results from the preceding § described at the beginning of the present one. With that in mind, we turn again to the *proof* of the theorem in §D.1, considering, instead of the Fourier–Stieltjes transform $\hat{\mu}(z)$, an entire function $f(z)$ of exponential type $\leqslant L$ with

$$(*) \qquad \int_{-\infty}^{\infty} \frac{\log^+|f(x)|}{1+x^2}\,dx \;<\; \infty.$$

Taking $f(z)$ to *vanish** at each point of a certain positive sequence $\{\lambda_n\}$, we *assume* as in §D.1 that for some number $D > L/\pi$ there is a sequence of *disjoint half-open intervals* $(a_k, b_k]$, $a_k > 0$, such that

$$\frac{\text{number of } \lambda_n \text{ in } (a_k, b_k]}{b_k - a_k} \;\geqslant\; D$$

and

$$\sum_k \left(\frac{b_k - a_k}{a_k}\right)^2 \;=\; \infty.$$

Our *object* is to prove that *then*

$$\int_{-\infty}^{\infty} \frac{\log^-|f(x)|}{1+x^2}\,dx \;=\; \infty,$$

from which it will follow by §G.2 of Chapter III that $f(z) \equiv 0$.

The argument starts out exactly as in §D.1, and proceeds as it did there until we arrive at the examination of $f(z)$ in the ellipses

$$z \;=\; c + r\cosh(\gamma + i\vartheta)$$

about the midpoint

$$c \;=\; (a_k + b_k)/2$$

of one of the intervals $(a_k, b_k]$, where

$$\frac{b_k - a_k}{2\cosh\gamma} \;<\; r \;<\; \frac{b_k - a_k}{2}.$$

Here, γ is a *small fixed number* > 0, *the same for all of the intervals* $(a_k, b_k]$. We continue to write

$$R \;=\; (b_k - a_k)/2$$

as we did in §D.1, so that $(a_k, b_k] = (c - R, c + R]$.

* with the appropriate multiplicity at any repeated point of the sequence – in the next displayed formula, such points are counted according to the multiplicity of their repetition.

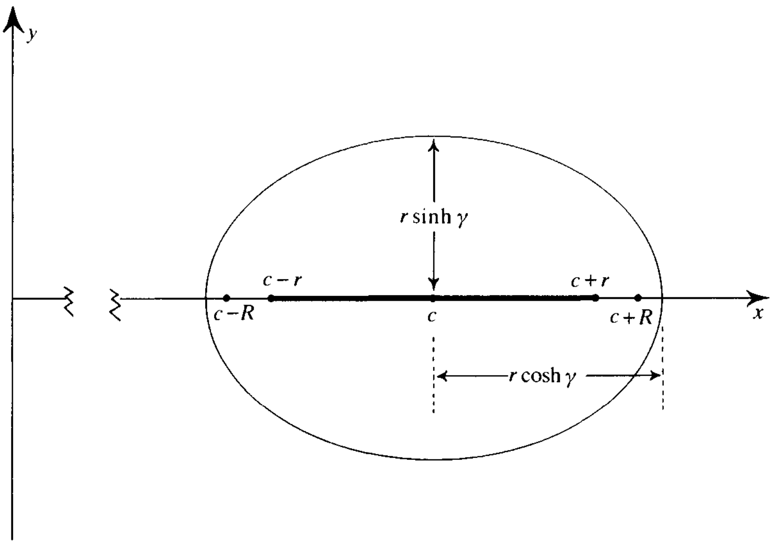


Figure 200

Picking a *fixed* $\eta > 0$ *much smaller than* γ, and denoting the number of λ_n in $(c-R,\ c+R]$ by N (a quantity $\geqslant\ 2RD$), we find as before that

$$N(\gamma-\eta) \quad \leqslant \quad \frac{1}{2\pi}\int_0^{2\pi}\log|f(c+r\cosh(\gamma+\mathrm{i}\vartheta))|\,\mathrm{d}\vartheta$$

$$-\quad \frac{1}{\pi}\int_{-r}^{r}\frac{\log|f(c+t)|}{\sqrt{(r^2-t^2)}}\mathrm{d}t$$

for $r\ \geqslant\ R/\cosh\eta$. *Here*, however, the simple inequality

$$\log|f(c+r\cosh(\gamma+\mathrm{i}\vartheta))| \ \leqslant\ Lr\sinh\gamma|\sin\vartheta|$$

is *no longer available* for the estimation of the first integral on the right, because *f is no longer assumed to be bounded on the real axis. Instead of boundedness,* $(*)$ *is all we have to work with.*

Our adaptation of the earlier reasoning to the present circumstances is nevertheless not altogether thwarted. In the passage from the previous relation to what corresponds to (†) of §D.1, there is a certain amount of leeway. *Provided that the constant* $\delta > 0$ *is small enough, it is sufficient to have*

$$(\substack{*\\*}) \qquad \log|f(c+r\cosh(\gamma+\mathrm{i}\vartheta))| \ \leqslant\ Lr\sinh\gamma|\sin\vartheta| \ + \ \delta R$$

$$\text{for } 0\ \leqslant\ r\ \leqslant\ R$$

in place of the stronger inequality written above. Indeed, substitution of this into

$$\frac{1}{2\pi}\int_0^{2\pi}\log|f(c+r\cosh(\gamma+\mathrm{i}\vartheta))|\,\mathrm{d}\vartheta$$

yields, for $R/\cosh\eta \;\leqslant\; r \;\leqslant\; R$,

$$\frac{1}{\pi}\int_{-r}^{r}\frac{\log|f(c+t)|}{\sqrt{(r^2-t^2)}}\mathrm{d}t \quad\leqslant\quad \frac{2(L\sinh\gamma-\pi D(\gamma-\eta)+\frac{1}{2}\pi\delta\cosh\eta)}{\pi}r$$

instead of (†), §D.1, so we certainly have*

$$-\frac{1}{\pi}\int_{-r}^{r}\frac{\log^-|f(c+t)|}{\sqrt{(r^2-t^2)}}\mathrm{d}t \quad\leqslant\quad \frac{2(L\sinh\gamma-\pi D(\gamma-\eta)+\frac{1}{2}\pi\delta\cosh\gamma)}{\pi}r$$

for the values of r just indicated. (*Remember*: our *convention in this book* is that $\log^-|f| \;\geqslant\; 0$!)

Our *assumption* here is that $\pi D \;>\; L$, from which it follows that the *coefficient of r in the preceding relation is surely* (strictly) *negative* when the constants γ, η and δ (all $>$ 0) are *chosen properly, as we henceforth suppose they are*. This being the case we may now deduce from that relation the inequality

$$(\S)\qquad \int_{a_k}^{b_k}\log^-|f(x)|\,\mathrm{d}x \;\geqslant\; \frac{(\pi D(\gamma-\eta)-L\sinh\gamma-\frac{1}{2}\pi\delta\cosh\gamma)\tanh(\eta/2)}{2\cosh^2\eta}\times(b_k-a_k)^2$$

by arguing as in §D.1. *This is valid*, then, for any of the intervals $(a_k,\ b_k]$ for which ($\ast$) holds with $c \;=\; (a_k+b_k)/2$ and $R \;=\; (b_k-a_k)/2$.

Let us denote by S the set of indices k such that

$$\log|f(z)| \;\leqslant\; L|\Im z| \;+\; \frac{\delta}{2}(b_k-a_k)$$

on the whole rectangle

$$a_k-(b_k-a_k) \;\leqslant\; \Re z \;\leqslant\; b_k+(b_k-a_k)$$
$$|\Im z| \;\leqslant\; (b_k-a_k)/2.$$

* remember, $0 \;<\; \eta \;<\; \gamma$!

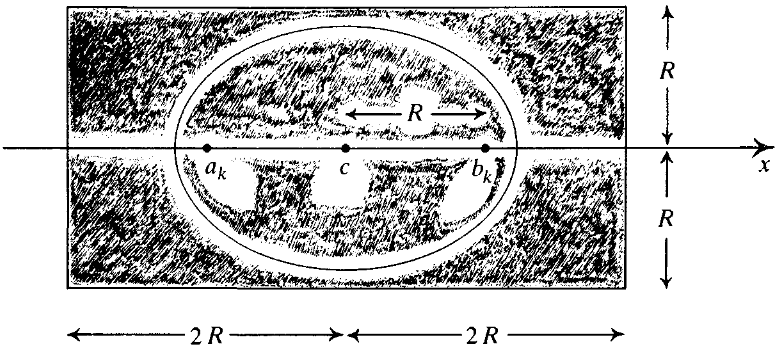


Figure 201

(Do not be misled into looking at the term $\delta(b_k - a_k)/2$ on the right side of the inequality just written as a *majorant* for the other right-hand term in the rectangle. The constant δ is usually *very much smaller* than L.) Since we are working with a *small* constant $\gamma > 0$, there is no loss of generality in *assuming – which we do from now on –* that $\cosh\gamma < 2$ and $\sinh\gamma < 1$. Then the ellipses $z = c + r\cosh(\gamma + i\vartheta)$, $0 < r \leqslant R$, *are all contained in the rectangle* (we are writing $c = (a_k + b_k)/2$ and $R = (b_k - a_k)/2$), so, if $k \in S$, $(\ast_\ast)$ *holds on those ellipses, and* (§) *is therefore true.* From this we see, reasoning as near the end of §D.1, that

$$\sum_{k\in S}\int_{a_k}^{b_k}\frac{\log^-|f(x)|}{1+x^2}\,dx$$

$$\geqslant \frac{(\pi D(\gamma-\eta) - L\sinh\gamma - \frac{1}{2}\pi\delta\cosh\gamma)\tanh(\eta/2)}{2M\cosh^2\eta}\sum_{k\in S}\left(\frac{b_k-a_k}{a_k}\right)^2,$$

where M is a certain finite constant.

The last relation shows that *if*

$$\sum_{k\in S}((b_k - a_k)/a_k)^2 = \infty,$$

we will have

$$\int_{-\infty}^{\infty}\frac{\log^-|f(x)|}{1+x^2}\,dx = \infty,$$

whence $f(z) \equiv 0$, the conclusion sought. It was, however, *given* that

$$\sum_{k=1}^{\infty}((b_k - a_k)/a_k)^2 = \infty.$$

Our extension will thus be *fully established* if it can be proved that

$$\sum_{k\notin S}^{\infty} ((b_k - a_k)/a_k)^2 \quad < \quad \infty.$$

We set out to verify this convergence.

Our starting point here is just the simple inequality from §E of Chapter III, which can be applied to any of the functions $f(z - \mathrm{i}h)$, $h \in \mathbb{R}$, in either the *upper* or *lower* half plane, f being *entire*. After making a change of variable, this gives

$$(\ddagger) \qquad \log|f(z)| \;\leqslant\; L|\Im z + h| \;+\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z + h|\log^+|f(t - \mathrm{i}h)|}{|z - t + \mathrm{i}h|^2}\,\mathrm{d}t,$$

because f is of exponential type $\leqslant L$. Picking any k, we write $c = (a_k + b_k)/2$, $R = (b_k - a_k)/2$ as before, and fix our attention on the rectangle

$$|\Re z - c| \;\leqslant\; 2R \qquad |\Im z| \;\leqslant\; R,$$

wishing to see whether or not

$$\log|f(z)| \;\leqslant\; L|\Im z| \;+\; \delta R$$

therein. For this purpose we use $(\ddagger)$ twice, making a *hall of mirrors* argument like the one resorted to several times in Chapter VI (cf. §§A.3, B.1 and E.4 there).

Let us look in the *top half*, $\mathcal{O}$, of the rectangle in question:

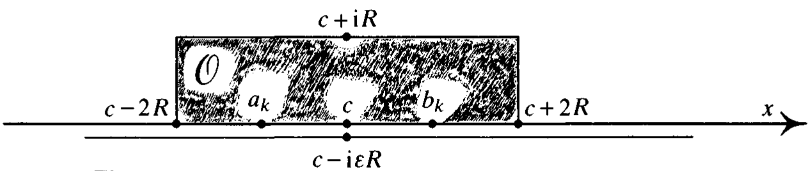


Figure 202

Fix a small $\varepsilon > 0$, *the same for all the indices* k – in a moment we will see *how small* ε must be taken. Putting first $h = \varepsilon R$ in $(\ddagger)$ we get, for $z \in \mathcal{O}$:

$$\log|f(z)| \;\leqslant\; L\Im z + \varepsilon LR + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(\Im z + \varepsilon R)\log^+|f(s - \mathrm{i}\varepsilon R)|}{|z - s + \mathrm{i}\varepsilon R|^2}\,\mathrm{d}s.$$

Using then $(\ddagger)$ with $h = 0$, we find that

$$\log^+|f(s - \mathrm{i}\varepsilon R)| \quad \leqslant \quad \varepsilon LR + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\varepsilon R\log^+|f(t)|}{(s - t)^2 + (\varepsilon R)^2}\,\mathrm{d}t,$$

which, substituted into the previous, yields

$$\log|f(z)| \leqslant L\Im z + 2\varepsilon LR + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(\Im z + 2\varepsilon R)\log^+|f(t)|}{|z-t+2i\varepsilon R|^2}\,dt, \quad z \in \mathcal{O},$$

by Fubini's theorem and the reproducing property of the Poisson kernel. The desired estimate for $\log|f(z)|$ will therefore *hold in $\mathcal{O}$, provided that*

$$2\varepsilon L = \frac{\delta}{4}, \quad \text{say},$$

and also

$$(\S\S) \qquad \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(\Im z + 2\varepsilon R)\log^+|f(t)|}{|z-t+2i\varepsilon R|^2}\,dt < \frac{3\delta}{4}R \quad \text{for } z \in \mathcal{O}.$$

Now, however, the simplest form of Harnack's inequality shows that for $z \in \mathcal{O}$,

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(\Im z + 2\varepsilon R)\log^+|f(t)|}{|z-t+2i\varepsilon R|^2}\,dt \leqslant C_\varepsilon \cdot \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{R\log^+|f(t)|}{(c-t)^2+R^2}\,dt$$

with a constant C_ε depending only on ε – the exact form of the dependence is not important here. (Actually, C_ε acts like $1/\varepsilon$.) Therefore, picking ε equal to $\delta/8L$, we may then determine a small constant $\alpha > 0$ such that (§§) *is ensured as long as*

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{R\log^+|f(t)|}{(c-t)^2+R^2}\,dt < \alpha R.$$

If this relation holds, we will *have*

$$\log|f(z)| \leqslant L|\Im z| + \delta R$$

for $|c - \Re z| \leqslant 2R$ and $0 \leqslant \Im z \leqslant R$ and then, by obvious symmetry, for such $\Re z$ and $-R \leqslant \Im z \leqslant 0$.

We now bring the index k back into our notation, writing

$$(a_k + b_k)/2 = c_k \quad (\text{instead of } c)^*$$

and

$$(b_k - a_k)/2 = R_k \quad (\text{instead of } R).$$

What we have just shown is that k *belongs to the set S* introduced above, *provided that*

* Note that wlog, $c_k \xrightarrow[k]{} \infty$; see top of p. 66

$$\boxed{\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{R_k\log^+|f(t)|}{(c_k-t)^2+R_k^2}\mathrm{d}t \quad < \quad \alpha R_k.}$$

Regarding this condition there is, however, the important

Lemma (Beurling and Malliavin, 1967). *Let $\alpha > 0$, and suppose that $c_k \xrightarrow[k]{} \infty$ with the intervals $(c_k - R_k,\ c_k + R_k)$ lying on $(0, \infty)$ and* disjoint. *If f satisfies $(*)$ and S' is the set of indices k for which* the preceding boxed relation fails to hold,

$$\sum_{k\in S'}\left(\frac{R_k}{c_k}\right)^2 \quad < \quad \infty.$$

Proof. Is based on the Ahlfors–Carleman inequality for harmonic measure derived in the preceding article.

The function

$$U(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z\log^+|f(t)|}{|z-t|^2}\mathrm{d}t,$$

harmonic and *positive* in $\Im z > 0$, is available thanks to $(*)$. For each fixed real x,

$$\frac{U(x+\mathrm{i}y)}{y} \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\log^+|f(t)|}{(x-t)^2+y^2}\mathrm{d}t$$

tends to zero as $y \longrightarrow \infty$ and is a *decreasing* function of y for $y > 0$, indeed, *strictly decreasing* unless $\log^+|f(t)| = 0$ a.e., i.e., $|f(t)| \leqslant 1$ on $\mathbb{R}$, in which case the lemma is obviously true anyway. Wlog, then, there is a certain value $Y(x)$ of y (perhaps equal to zero), such that

$$\frac{\alpha y}{2} - U(x+\mathrm{i}y) \quad \begin{cases} = 0 & \text{for } y = Y(x), \\ > 0 & \text{for } y > Y(x). \end{cases}$$

Continuity of $Y(x)$ as a function of x follows easily from that of $U(z)$ for $\Im z > 0$; the set

$$\mathscr{D} \quad = \quad \{(x,y):\quad y \;>\; Y(x)\}$$

is therefore *open*, besides being obviously *simply connected.*

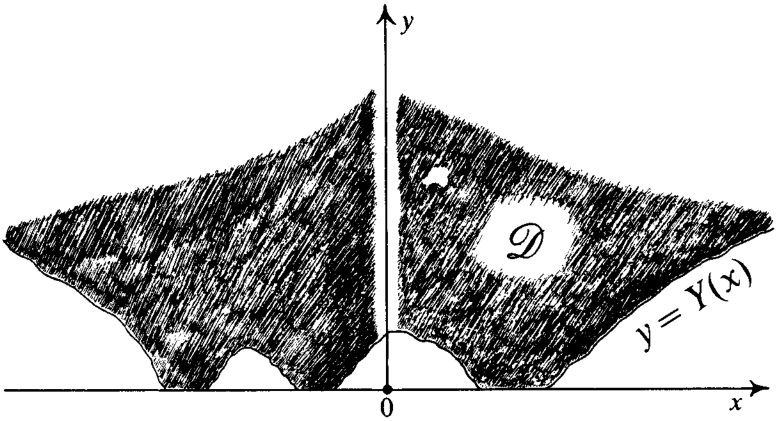


Figure 203

We work with the function

$$V(z) = \frac{\alpha}{2}\Im z - U(z)$$

in the domain $\mathscr{D}$; $V(z)$ is *harmonic* and (by construction) *strictly positive* there, and *zero* on $\partial\mathscr{D}$. Wlog, $\mathrm{i} \in \mathscr{D}$, so that

$$V(\mathrm{i}) > 0.$$

The idea now is to *assume* that

$$\sum_{k \in S'} (R_k/c_k)^2 = \infty$$

and then *derive* therefrom the *contradictory* conclusion that $V(\mathrm{i}) = 0$. That will prove the lemma.

To say that $k \in S'$ means that

$$U(c_k + \mathrm{i}R_k) \geqslant \alpha R_k,$$

whence, $U(z)$ being $\geqslant 0$ in $\{\Im z > 0\}$,

$$U(x + \mathrm{i}R_k) \geqslant \frac{\alpha}{2} R_k \quad \text{for } c_k - \tfrac{1}{3}R_k \leqslant x \leqslant c_k + \tfrac{1}{3}R_k$$

by Harnack, in other words,

$$Y(x) \geqslant R_k \quad \text{for } c_k - \tfrac{1}{3}R_k \leqslant x \leqslant c_k + \tfrac{1}{3}R_k \text{ when } k \in S'.$$

This observation we will use presently.

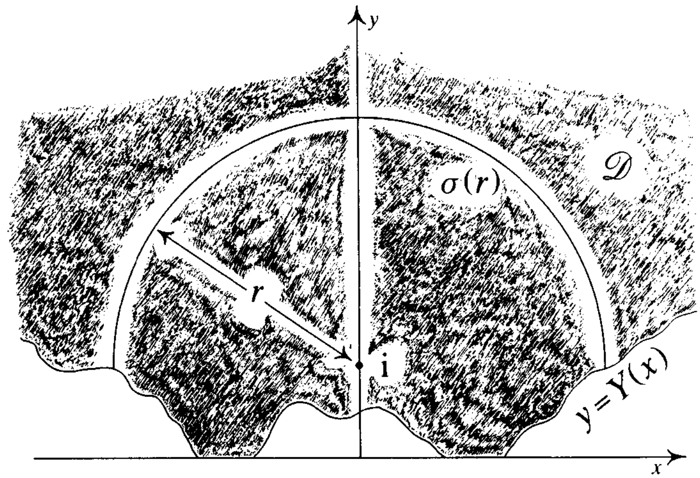


Figure 204

For large r, let $\sigma(r)$ be the circular arc of radius r about i (*sic!*), *lying entirely in $\mathscr{D}$ save for its two endpoints* which are on the curve $y = Y(x)$. Given R, the arc $\sigma(R)$ divides $\mathscr{D}$ into at least two simply connected regions, one of which contains i and is denoted by $\mathscr{D}_R$. We apply the *theorem on harmonic estimation* (Chapter VII, §B.1) to $V(z)$ in $\mathscr{D}_R$. Since

$$V(z) \leqslant \frac{\alpha}{2}\Im z \leqslant \frac{\alpha}{2}(R+1)$$

for z on $\sigma(R)$ and $V(z) = 0$ on $\partial\mathscr{D}$, we find that

$$V(\mathrm{i}) \leqslant \frac{\alpha}{2}(R+1)\omega_{\mathscr{D}_R}(\sigma(R), \mathrm{i}).$$

To estimate the harmonic measure appearing on the right, we now apply the *last theorem* of the preceding article (Ahlfors–Carleman). *Fixing* any large r_0 (more or less at pleasure) and writing $\vartheta(r) = |\sigma(r)|/r$ as usual, we have by that result

$$\omega_{\mathscr{D}_R}(\sigma(R), \mathrm{i}) \leqslant C_0 \exp\left(-\pi\int_{r_0}^{R}\frac{\mathrm{d}r}{r\vartheta(r)}\right)$$

for $R > r_0$; here C_0 is a constant, depending, of course, on the choice of r_0, but independent of R.

In the case where $Y(x)$ is *zero at both ends* of an arc $\sigma(r)$, we have $\vartheta(r) = \pi + 2\arcsin(1/r)$. In the general situation, one may bring in the angles $\varphi(r)$ and $\psi(r)$ shown in the following figure, and in terms of them,

$$\vartheta(r) \quad = \quad \pi \; + \; 2\arcsin\frac{1}{r} \; - \; \varphi(r) \; - \; \psi(r):$$

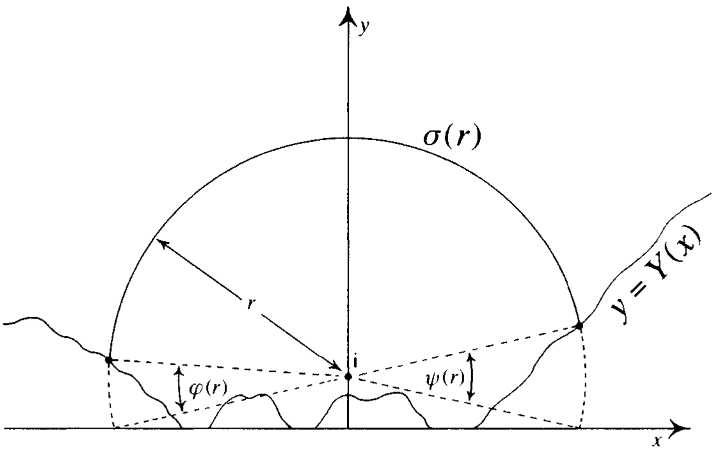


Figure 205

To investigate the integral occurring on the right side of the previous inequality, it is better to first work directly with $\varphi(r)$ and $\psi(r)$, bringing in the quantities c_k, R_k (with $k \in S'$) only towards the end.

We have:

$$\frac{\pi}{\pi + 2\arcsin\frac{1}{r} - \varphi(r) - \psi(r)}$$

$$\geqslant \quad \frac{\pi}{\pi + 2\arcsin\frac{1}{r}} \; + \; \frac{\pi}{\left(\pi + 2\arcsin\frac{1}{r}\right)^2}\Big(\varphi(r) + \psi(r)\Big)$$

$$= \quad 1 \; - \; O\left(\frac{1}{r}\right) \; + \; \frac{\varphi(r) + \psi(r)}{\pi}.$$

Hence

$$\pi\int_{r_0}^{R}\frac{dr}{r\vartheta(r)} \quad \geqslant \quad \log\frac{R}{r_0} \; - \; O(1) \; + \; \frac{1}{\pi}\int_{r_0}^{R}\frac{\varphi(r) + \psi(r)}{r}\,dr.$$

In terms of the *polar coordinates about* i

$$re^{i\theta} \;=\; z-\mathrm{i},$$

the *integral on the right* has a simple interpretation. It is none other than

$$\frac{1}{\pi}\iint_{\Omega\cap\{r_0<|z-\mathrm{i}|<R\}} \frac{r\,\mathrm{d}\theta\,\mathrm{d}r}{r^2},$$

where Ω is the *complement*, in

$$\{z\colon \Im z \;>\; 0 \text{ and } |z-\mathrm{i}| \;>\; r_0\},$$

of the union of the arcs $\sigma(r)$ *for* $r > r_0$. Ω *certainly includes the complement of* $\mathscr{D}$ in the region just mentioned, and may, *indeed*, include the latter *properly*:

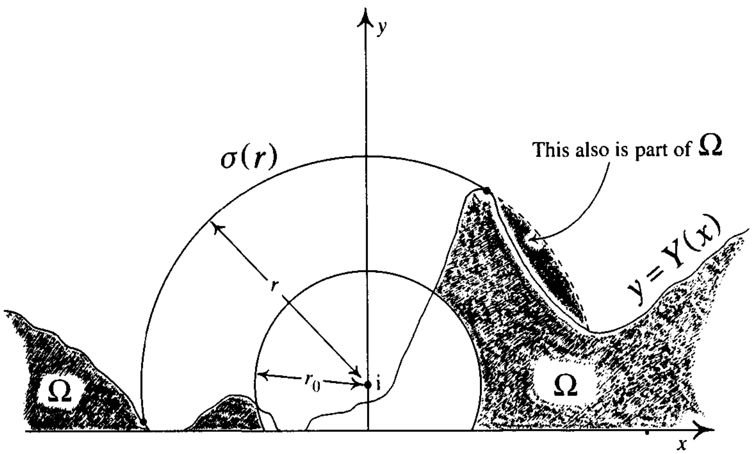


Figure 206

Writing

$$\Omega_R \;=\; \Omega \cap \{z\colon r_0<|z-i|<R\},$$

we see, going over to rectangular coordinates, that

$$\frac{1}{\pi}\int_{r_0}^{R}\frac{\varphi(r)+\psi(r)}{r}\mathrm{d}r \;=\; \frac{1}{\pi}\iint_{\Omega_R}\frac{\mathrm{d}x\,\mathrm{d}y}{x^2+(y-1)^2},$$

whence finally

$$\pi\int_{r_0}^{R}\frac{\mathrm{d}r}{r\vartheta(r)} \;\geqslant\; \log\frac{R}{r_0} \;-\; \mathrm{O}(1) \;+\; \frac{1}{\pi}\iint_{\Omega_R}\frac{\mathrm{d}x\,\mathrm{d}y}{1+x^2+y^2}.$$

Substitution of this into the above estimate for harmonic measure yields

$$\omega_{\mathscr{D}_R}(\sigma(R),\ i) \;\leqslant\; \frac{\text{const.}}{R}\exp\left(-\frac{1}{\pi}\iint_{\Omega_R}\frac{dx\,dy}{1+x^2+y^2}\right),$$

from which

$$V(i) \;\leqslant\; \text{const.}\frac{R+1}{R}\exp\left(-\frac{1}{\pi}\iint_{\Omega_R}\frac{dx\,dy}{1+x^2+y^2}\right).$$

The constant here depends on r_0, but is independent of R.

Let us now make $R \longrightarrow \infty$. By the last relation, *divergence* of

$$\iint_{\Omega}\frac{dx\,dy}{1+x^2+y^2}$$

will then force $V(i) = 0$. As we have seen, however,

$$Y(x) \;\geqslant\; R_k \quad \text{for } c_k - \tfrac{1}{3}R_k \;\leqslant\; x \;\leqslant\; c_k + \tfrac{1}{3}R_k$$

when $k \in S'$, so, when $k \in S'$ is large, the *rectangle* of *height* R_k with *base* on $[c_k - \frac{1}{3}R_k,\ c_k + \frac{1}{3}R_k]$ *must lie in* Ω (*recall* that the intervals $(c_k - R_k,\ c_k + R_k) \subseteq (0,\infty)$ are *disjoint* and that $c_k \xrightarrow[k]{} \infty$!):

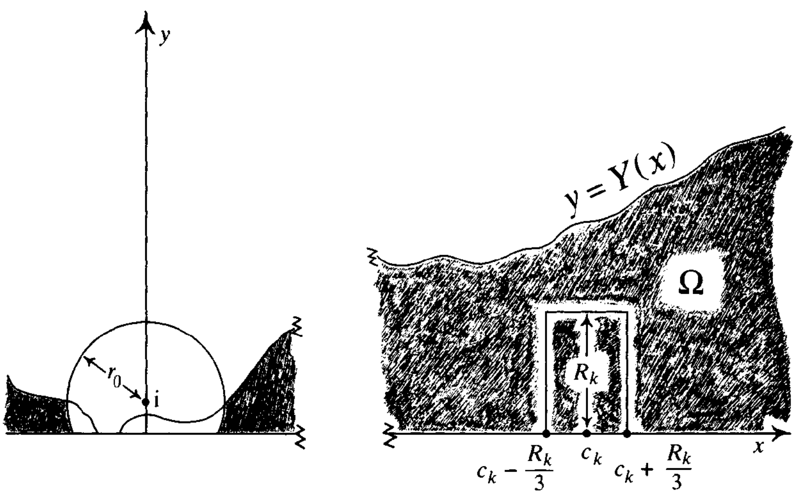


Figure 207

Therefore, fixing a suitable k_0,

$$\iint_{\Omega}\frac{dx\,dy}{1+x^2+y^2} \;\geqslant\; \sum_{\substack{k\in S'\\ k\geqslant k_0}}\frac{R_k^2}{3(1+c_k^2+R_k^2)}.$$

Again, since $(c_k - R_k,\ c_k + R_k) \subseteq (0, \infty)$, $c_k \geqslant R_k$, and, since $c_k \xrightarrow[k]{} \infty$, $c_k \geqslant$ some number $A > 0$ for $k \geqslant k_0$. From the preceding relation we thus get

$$\iint_{\Omega} \frac{\mathrm{d}x\,\mathrm{d}y}{1+x^2+y^2} \quad \geqslant \quad \frac{A^2}{6A^2+3} \sum_{\substack{k \in S' \\ k \geqslant k_0}} \left(\frac{R_k}{c_k}\right)^2.$$

Divergence of the sum on the right therefore makes the integral on the left infinite, which, as we have just shown, implies that $V(\mathrm{i}) = 0$.

But $V(\mathrm{i}) > 0$. The right-hand sum must therefore *converge*, so that

$$\sum_{k \in S'} (R_k/c_k)^2 \quad < \quad \infty.$$

This is what we had to prove, however. We are done.

Let us return to the discussion preceding the lemma, where we saw that *the index k belongs to the set S provided that*

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{R_k \log^+|f(t)|}{(t-c_k)^2 + R_k^2}\mathrm{d}t \quad < \quad \alpha R_k.$$

Knowing that

$$\sum_{k=1}^{\infty} \left(\frac{b_k - a_k}{a_k}\right)^2 \quad = \quad \infty,$$

we wished to conclude that

$$\sum_{k \in S} \left(\frac{b_k - a_k}{a_k}\right)^2 \quad = \quad \infty,$$

for, as had already been deduced then, this would imply that

$$\int_{-\infty}^{\infty} \frac{\log^-|f(x)|}{1+x^2}\mathrm{d}x \quad = \quad \infty$$

and hence that $f(z) \equiv 0$ (our desired result) by §G.2 of Chapter III.

The lemma shows, however, that the above condition on the numbers $c_k = (a_k + b_k)/2$ and $R_k = (b_k - a_k)/2$ *certainly holds* except for those k belonging to a set S' for which*

$$\sum_{k \in S'} \left(\frac{R_k}{c_k}\right)^2 \quad < \quad \infty.$$

* see footnote, p. 116

As in §D.1, we may, wlog, assume that

$$\frac{b_k}{a_k} \xrightarrow[k]{} 1,$$

so, since the $a_k > 0$, the preceding relation becomes

$$\sum_{k\in S'}\left(\frac{b_k-a_k}{a_k}\right)^2 < \infty.$$

Thus, we surely have

$$\sum_{k\notin S}\left(\frac{b_k-a_k}{a_k}\right)^2 < \infty,$$

from which it follows that

$$\sum_{k\in S}\left(\frac{b_k-a_k}{a_k}\right)^2 = \infty$$

in view of the divergence of the corresponding sum over all k.

In this way, we have *proved* our generalization of the result in §D.1:

Theorem. *Let $f(z)$, entire and of exponential type $\leqslant L$, satisfy*

$$\int_{-\infty}^{\infty}\frac{\log^+|f(x)|}{1+x^2}\,dx < \infty$$

and vanish at each of the numbers $\lambda_n > 0$. If, for some $D > L/\pi$, there is a sequence of disjoint intervals $(a_k, b_k]$, $a_k > 0$, with*

$$\sum_k\left(\frac{b_k-a_k}{a_k}\right)^2 = \infty$$

and

$$\frac{\text{number of }\lambda_n\text{ in }(a_k,b_k]}{b_k-a_k} \geqslant D$$

for each k, then $f(z) \equiv 0$.

Remark. The number L enters into this theorem *solely on account* of (‡). We may therefore replace the condition, figuring in the hypothesis, that f be of exponential type $\leqslant L$ by the simpler requirement that f be of

* with appropriate multiplicity at any repeated value λ_n

(some) exponential type, with

$$\limsup_{y\to\infty}\frac{\log|f(\mathrm{i}y)|}{y} \quad \text{and} \quad \limsup_{y\to-\infty}\frac{\log|f(\mathrm{i}y)|}{|y|}$$

both $\leqslant L$ (see Chapter III, §E). Of course, the latter in fact *implies* that $f(z)$ is of exponential type $\leqslant L$ (see the last theorem in §E.2, Chapter VI and especially the discussion in §B.2 there), so that, *logically*, we have gained *nothing*. It is nevertheless often *easier in practice* to estimate the two limsups than to obtain a good upper bound on f's exponential type by direct examination of that function.

Once the result just stated is available, it is useful to bring in the effective density $\tilde{D}_\Lambda$ discussed in §D.2. Arguing as at the very beginning of the present §, we then obtain the following propositions:

Theorem. *Let* Λ *be a sequence of real numbers on which a non-zero entire function* f *of exponential type vanishes,* with*

$$\limsup_{y\to\infty}\frac{\log|f(\mathrm{i}y)|}{y} \quad \text{and} \quad \limsup_{y\to-\infty}\frac{\log|f(\mathrm{i}y)|}{|y|}$$

both $\leqslant L$ *and*

$$\int_{-\infty}^{\infty}\frac{\log^+|f(x)|}{1+x^2}\mathrm{d}x \quad < \quad \infty.$$

Then $\tilde{D}_\Lambda \leqslant L/\pi$.

Theorem. *Let the real sequence* Λ *consist of* all the zeros (*with repetitions according to multiplicities*) *of the entire function* f *figuring in the previous result, and suppose that the two limsups occurring there are each* exactly equal *to* L. *Then for* $\Lambda_+ = \Lambda\cap(0,\infty)$ *and* $\Lambda_- = (-\Lambda)\cap(0,\infty)$, *we have*

$$\tilde{D}_{\Lambda_+} \;=\; \tilde{D}_{\Lambda_-} \;=\; \frac{L}{\pi}.$$

As remarked at the beginning of this §, the second of these two theorems is a considerable improvement of the one of Levinson for functions with real zeros (§H.2, Chapter III). It can be used to *replace Pólya maximum density by effective density* in some of the earlier results in this book, thus

* with appropriate multiplicity at any repeated member of Λ

strengthening them. For instance, the *first* theorem from §E.3, Chapter VI, can be extended as follows (keeping the notation of the place where it appears originally):

Theorem. *Given a weight $W(x) \geqslant 1$ tending to ∞ as $x \longrightarrow \pm\infty$ and a number $A > 0$, suppose that*

$$\int_{-\infty}^{\infty} \frac{\log W_A(x)}{1+x^2}\,dx \quad < \quad \infty$$

and that $W(x_k) < \infty$ for each of the positive numbers x_k. If the effective density $\tilde{D}$ *of the sequence $\{x_k\}$ is $> A/\pi$,* $\mathscr{E}_A$ *is* not $\|\ \|_W$*-dense in* $\mathscr{C}_W(\mathbb{R})$.

To compare this improved result with the original one, it suffices to note that there exist positive sequences *having ordinary density equal to zero* for which the *effective density is infinite.*

Again, we can now see that the entire function $\Phi(z)$ of exponential type A with

$$\int_{-\infty}^{\infty} \frac{\log^+|\Phi(x)|}{1+x^2}\,dx \quad < \quad \infty$$

whose existence was treated during the discussion of de Branges' theorem (near the beginning of §F.3, Chapter VI) in fact satisfies $|\Phi(x_n)| \geqslant W(x_n)$ on a sequence Λ of real x_n for which

$$\tilde{D}_{\Lambda_+} \quad = \quad \tilde{D}_{\Lambda_-} \quad = \quad \frac{A}{\pi}.$$

This represents a certain improvement over the asymptotic relation

$$x_n \quad \sim \quad \frac{\pi}{A}n, \qquad n \longrightarrow \pm\infty,$$

obtained for that sequence by use of Levinson's theorem.

F. Scholium. Extension of results in §E.1. Pfluger's theorem and Tsuji's inequality

Extremal length is not only useful for finding the harmonic measure of *single arcs* on the boundaries of simply connected domains (as in §E.1); it can also be applied when examining the harmonic measure of *fairly general sets* lying on those boundaries. In the latter situation one obtains *an estimate* instead of the *exact relation* holding for *arcs*. The main result there is based on work of Pfluger which was published in the same volume of *Commentarii Helvetici* as the one where Hersch's article appeared. Another form of the result, involving, however, a more special notion than extremal length (or rather one *formulated* in more particular fashion), can already be found in Beurling's thesis.

Logarithmic capacity and the logarithmic conductor potential play an important rôle in Pfluger's work; these are explained in the first of the following three articles. In order to apply his result so as to obtain an analogue of the Ahlfors–Carleman estimate given near the end of §E.1, some elementary theorems about univalent functions are needed. Those may be found in many standard texts on complex variable theory.

1 Logarithmic capacity and the conductor potential

If $\mathscr{D}$ is a *bounded* domain having reasonably decent boundary $\partial\mathscr{D}$, we have, for the Green's function $G_{\mathscr{D}}(z, w)$ associated thereto,

$$G_{\mathscr{D}}(z,w) \quad = \quad \log\frac{1}{|z-w|} + \int_{\partial\mathscr{D}} \log|\zeta - w|\, d\omega_{\mathscr{D}}(\zeta,\ z); \quad z, w \in \mathscr{D}.$$

We saw in §C.1 of Chapter VIII that this formula remains valid for certain *unbounded* domains $\mathscr{D}$ whose boundary *includes* the point at ∞ and is *not too sparse* there. It *cannot*, however, be true for *unbounded* domains with *compact* boundary. Indeed, when $\partial\mathscr{D}$ is compact but $\mathscr{D}$ unbounded, the *right side* of the relation tends to $-\infty$ as $z \longrightarrow \infty$ if $w \in \mathscr{D}$ is fixed and finite. At the same time, $G_{\mathscr{D}}(z, w)$ is supposed *by definition* to stay > 0.

In complex variable theory, there is a standard procedure for adapting the various notions of local behaviour originally defined for the points of $\mathbb{C}$ to the point at ∞ on the Riemann sphere. One first uses a linear fractional transformation to map ∞ to a finite point a, and then says that a function defined near and at ∞ has such and such behaviour there if the one related to it through the transformation behaves thus (in the usual accepted sense) at a. The Green's function for a domain $\mathscr{D}$ on the Riemann sphere having reasonable boundary, with $\infty \in \mathscr{D}$, is defined in accordance with that convention: taking a linear fractional transformation φ which maps $\mathscr{D}$ onto some domain $\mathscr{E} \subseteq \mathbb{C}$, we put

$$G_{\mathscr{D}}(z,w) \quad = \quad G_{\mathscr{E}}(\varphi(z), \varphi(w)), \quad z, w \in \mathscr{D}.$$

In this extension of the definition of $G_{\mathscr{D}}(z, w)$ to domains $\mathscr{D}$ containing ∞, all of the usual general properties of that function holding for domains $\mathscr{D} \subseteq \mathbb{C}$ are preserved. That is, in particular, true of the important *symmetry* relation

$$G_{\mathscr{D}}(z,w) \quad = \quad G_{\mathscr{D}}(w,z),$$

established at the end of §A.2, Chapter VIII. We see also that when $w \in \mathscr{D}$ is *not* equal to ∞, $G_{\mathscr{D}}(z, w)$ is described by just the ordinary specification:

> As a function of z, $G_{\mathscr{D}}(z, w)$ is continuous and $\geqslant 0$ on $\bar{\mathscr{D}} \sim \{w\}$, harmonic in $\mathscr{D} \sim \{w\}$ (including at ∞), and zero on $\partial\mathscr{D}$. Near w, it equals $\log(1/|z-w|)$ plus a harmonic function of z.

For $w = \infty$, however, our definition leads to the following description:

> $G_{\mathscr{D}}(z, \infty)$ is continuous and $\geqslant 0$ on $\bar{\mathscr{D}} \sim \{\infty\}$, harmonic in $\mathscr{D} \sim \{\infty\}$, and zero on $\partial\mathscr{D}$. Near ∞, it equals $\log|z|$ (*sic*!) plus a function harmonic in z (including at ∞).

Keeping these characterizations in mind, we easily obtain an integral representation for $G_{\mathscr{D}}(z, \infty)$ akin to the one given above for $G_{\mathscr{D}}(z, w)$ and *bounded* domains $\mathscr{D}$. Fix, for the moment, a *finite* $w \in \mathscr{D}$, and consider the function

$$h_w(z) = \log|z-w| + G_{\mathscr{D}}(z, w) - G_{\mathscr{D}}(z, \infty).$$

According to the descriptions just made $h_w(z)$ is *bounded* and *harmonic* in $\mathscr{D}$ (including at $z = w$ and at $z = \infty$), and *continuous* up to $\partial\mathscr{D}$. Therefore, as long as $\partial\mathscr{D}$ is *decent* (which we are *assuming throughout this* § !),

$$h_w(z) = \int_{\partial\mathscr{D}} h_w(\zeta)\, d\omega_{\mathscr{D}}(\zeta, z) = \int_{\partial\mathscr{D}} \log|\zeta - w|\, d\omega_{\mathscr{D}}(\zeta, z).$$

As $z \longrightarrow \infty$, $\log|z-w| - G_{\mathscr{D}}(z, \infty) = o(1) + \log|z| - G_{\mathscr{D}}(z, \infty)$ tends to *a certain finite limit*, which we denote by $-\gamma_{\mathscr{D}}$, so then

$$h_w(z) \longrightarrow G_{\mathscr{D}}(\infty, w) - \gamma_{\mathscr{D}}.$$

Making $z \longrightarrow \infty$ in the previous relation, we thus obtain

$$-\gamma_{\mathscr{D}} + G_{\mathscr{D}}(\infty, w) = \int_{\partial\mathscr{D}} \log|\zeta - w|\, d\omega_{\mathscr{D}}(\zeta, \infty)$$

After using the above mentioned symmetry property and then replacing w by z, this becomes

$$\boxed{G_{\mathscr{D}}(z, \infty) = \gamma_{\mathscr{D}} + \int_{\partial\mathscr{D}} \log|z - \zeta|\, d\omega_{\mathscr{D}}(\zeta, \infty),}$$

the integral representation sought.

From the last formula, we see in particular that

$$\boxed{\int_{\partial\mathscr{D}} \log \frac{1}{|z-\zeta|}\, d\omega_{\mathscr{D}}(\zeta, \infty) \quad = \quad \gamma_{\mathscr{D}} \quad \text{for } z \in \partial\mathscr{D}.}$$

The positive measure $\omega_{\mathscr{D}}(\ \ , \infty)$ *of total mass* 1 *supported on* $\partial\mathscr{D}$ *has constant logarithmic potential equal to* $\gamma_{\mathscr{D}}$ *thereon.* $\gamma_{\mathscr{D}}$ is called the *Robin constant* for $\mathscr{D}$.

If we imagine a very long metallic cylinder perpendicular to the $x-y$ plane, cutting the latter, near its own middle, precisely in $\partial\mathscr{D}$, and having its different pieces joined to each other by thin perfectly conducting wires, an electric charge placed on it will distribute itself so as to make the electrostatic potential constant thereon. Near the $x-y$ plane, that equilibrium distribution will depend mainly on $z = x + iy$ and hardly at all on distance measured perpendicularly to the cylinder's cross section; the same is true of the corresponding electrostatic potential. To within a constant factor, the latter*, near the $x-y$ plane, is practically equal to a *logarithmic potential* in $z = x + iy$ corresponding to a *measure* giving the *amount of electric charge per unit of cylinder generator length.* The *second* of the aboved boxed formulas shows that if the whole cylinder carries *one unit of electric charge per unit of length* (measured along a generator), the electrostatic potential (see footnote) at equilibrium is

$$\int_{\partial\mathscr{D}} \log \frac{1}{|z-\zeta|}\, d\omega_{\mathscr{D}}(\zeta, \infty)$$

(near the $x-y$ plane); this is called the *logarithmic conductor potential* (or *equilibrium potential*) for $\partial\mathscr{D}$. The *constant value* that this potential assumes on $\partial\mathscr{D}$ is equal to the Robin constant $\gamma_{\mathscr{D}}$.

In physics, the *capacity* of a conductor is the *quantity of electricity* which must be placed on it *in order to raise its (equilibrium) electrostatic potential to unity.* That potential is there taken as $\log L + \gamma_{\mathscr{D}}$ instead of $\gamma_{\mathscr{D}}$ when dealing with a long cylinder of length L bearing one unit of electric charge per unit of length (see footnote); in this way one arrives at a value $L/(\log L + \gamma_{\mathscr{D}})$ for the capacity of the cylinder. Even after division by L

* measured from a certain reference value depending on the cylinder's length and net electric charge, but *not* on its cross-sectional form

(in order to obtain a *capacity per unit length*), this quantity shows practically *no dependence* on $\gamma_{\mathscr{D}}$, i.e., *on the cylinder's cross section*, because the length L is assumed to be very large. That is why mathematicians have agreed on a different specification of logarithmic capacity.

Definition. The *logarithmic capacity*, $\operatorname{Cap}\partial\mathscr{D}$, of the compact boundary $\partial\mathscr{D}$ is equal to

$$e^{-\gamma_{\mathscr{D}}},$$

where $\gamma_{\mathscr{D}}$ is the Robin constant for $\mathscr{D}$.

The logarithmic conductor potential and measure $\omega_{\mathscr{D}}(\ \ ,\infty)$ corresponding to it are characterized by an important extremal property. From physics, we expect that the *equilibrium charge distribution* $\omega_{\mathscr{D}}(\ \ ,\infty)$ on $\partial\mathscr{D}$ should be the positive measure μ of total mass 1 carried thereon for which the *energy* (cf. Chapter VIII, §B.5)

$$\int_{\partial\mathscr{D}}\int_{\partial\mathscr{D}} \log\frac{1}{|z-\zeta|}\,d\mu(\zeta)\,d\mu(z)$$

is as small as possible. That is true.

Lemma (goes back to Gauss). *If μ is a positive measure on $\partial\mathscr{D}$ with $\mu(\partial\mathscr{D}) = 1$,*

$$\int_{\partial\mathscr{D}}\int_{\partial\mathscr{D}} \log\frac{1}{|z-\zeta|}\,d\mu(\zeta)\,d\mu(z) \quad\geqslant\quad \gamma_{\mathscr{D}},$$

and equality is realized for $\mu = \omega_{\mathscr{D}}(\ \ ,\infty)$.

Remark. It is not too hard to show that equality holds *only* for $\mu = \omega_{\mathscr{D}}(\ \ ,\infty)$. We will not require that fact.

Proof of lemma. Since

$$\int_{\partial\mathscr{D}} \log\frac{1}{|z-\zeta|}\,d\omega_{\mathscr{D}}(\zeta,\infty) \quad=\quad \gamma_{\mathscr{D}}, \qquad z\in\partial\mathscr{D},$$

we have

$$\int_{\partial\mathscr{D}}\int_{\partial\mathscr{D}} \log\frac{1}{|z-\zeta|}\,d\omega_{\mathscr{D}}(\zeta,\infty)\{d\mu(z)-d\omega_{\mathscr{D}}(z,\infty)\} \quad=\quad 0$$

for any measure μ on $\partial\mathscr{D}$ with $\mu(\partial\mathscr{D}) = \omega_{\mathscr{D}}(\partial\mathscr{D}, \infty) = 1$. Now*

$$\int_{\partial\mathscr{D}}\int_{\partial\mathscr{D}} \log\frac{1}{|z-\zeta|}\,\mathrm{d}\mu(\zeta)\,\mathrm{d}\mu(z) - \gamma_{\mathscr{D}}$$

$$= \int_{\partial\mathscr{D}}\int_{\partial\mathscr{D}} \log\frac{1}{|z-\zeta|}\left\{\mathrm{d}\mu(\zeta)\,\mathrm{d}\mu(z) - \mathrm{d}\omega_{\mathscr{D}}(\zeta,\infty)\,\mathrm{d}\omega_{\mathscr{D}}(z,\infty)\right\}$$

$$= \int_{\partial\mathscr{D}}\int_{\partial\mathscr{D}} \log\frac{1}{|z-\zeta|}(\mathrm{d}\mu(\zeta) + \mathrm{d}\omega_{\mathscr{D}}(\zeta,\infty))(\mathrm{d}\mu(z) - \mathrm{d}\omega_{\mathscr{D}}(z,\infty))$$

$$= \int_{\partial\mathscr{D}}\int_{\partial\mathscr{D}} \log\frac{1}{|z-\zeta|}(\mathrm{d}\mu(\zeta) - \mathrm{d}\omega_{\mathscr{D}}(\zeta,\infty))(\mathrm{d}\mu(z) - \mathrm{d}\omega_{\mathscr{D}}(z,\infty))$$

$$+ \; 2\int_{\partial\mathscr{D}}\int_{\partial\mathscr{D}} \log\frac{1}{|z-\zeta|}\,\mathrm{d}\omega_{\mathscr{D}}(\zeta,\infty)\left\{\mathrm{d}\mu(z) - \mathrm{d}\omega_{\mathscr{D}}(z,\infty)\right\}.$$

In the last expression, the *second* double integral is *zero*, as we have just seen. However, $\int_{\partial\mathscr{D}}(\mathrm{d}\mu(\zeta) - \mathrm{d}\omega_{\mathscr{D}}(\zeta,\infty)) = 0$, so the *first* double integral in the last expression is *positive*, according to the *scholium and warning* just past the middle of §B.5, Chapter VIII. (The argument alluded to there works at least for sufficiently *smooth* signed measures of total mass zero supported on compact sets. The positivity thus established can be extended to our present signed measures $\mu - \omega_{\mathscr{D}}(\;\;,\infty)$, with μ *not* necessarily *smooth*, by an appropriate limiting argument (regularization).) We see in this way that the *first* member in the above chain of equalities is *positive*. That's what we had to prove.

The following exercise gives us an alternative procedure for verifying that

$$\int_{\partial\mathscr{D}}\int_{\partial\mathscr{D}} \log\frac{1}{|z-\zeta|}(\mathrm{d}\mu(\zeta) - \mathrm{d}\omega_{\mathscr{D}}(\zeta,\infty))(\mathrm{d}\mu(z) - \mathrm{d}\omega_{\mathscr{D}}(z,\infty)) \;\geqslant\; 0$$

when $\mu(\partial\mathscr{D}) = 1$.

Problem 35. (Ahlfors)

(a) Show that for large R,

$$\iint_{|z|<R} \frac{\mathrm{d}x\,\mathrm{d}y}{|z-\zeta|\,|z-w|} = 2\pi\log\frac{1}{|\zeta-w|} + 2\pi\log R + C + \delta(\zeta,w,R),$$

where C is a certain numerical constant (its value will not be needed) and $\delta(\zeta,w,R)\longrightarrow 0$ *uniformly* for ζ and w ranging over any compact set in $\mathbb{C}$, as $R\longrightarrow\infty$. (Hint: Wlog, $w-\zeta = a > 0$. Use the polar coordinates $re^{i\vartheta} = z-\zeta$ and, in the double integral thus obtained, *integrate r first*.)

* note that $\int_{\partial\mathscr{D}}\int_{\partial\mathscr{D}}\log(1/|z-\zeta|)\mathrm{d}\omega_{\mathscr{D}}(\zeta,\infty)\mathrm{d}\mu(z) = \gamma_{\mathscr{D}}$, a *finite* quantity!

(b) Hence show that

$$\int_{\partial\mathscr{D}}\int_{\partial\mathscr{D}} \log\frac{1}{|\zeta - w|}(\mathrm{d}\mu(\zeta) - \mathrm{d}\omega_{\mathscr{D}}(\zeta,\infty))(\mathrm{d}\mu(w) - \mathrm{d}\omega_{\mathscr{D}}(w,\infty)) \quad \geqslant \quad 0$$

for any positive measure μ on $\partial\mathscr{D}$ having total mass 1 (the double integral may be infinite). Pay attention to the problems of convergence and of possibly getting $\infty - \infty$. (Note that under our assumptions on $\partial\mathscr{D}$, $\gamma_{\mathscr{D}}$ is certainly *finite*.)

2 A conformal mapping. Pfluger's theorem

Let us fix any finite union E of closed arcs on $\{|z| = 1\}$ and a simple closed curve Γ about 0 lying in the unit disk Δ. The unit circumference and Γ bound a certain ring domain Δ_Γ lying in Δ, and we denote by G the *family of curves λ lying in Δ_Γ and going from the set E to Γ*:

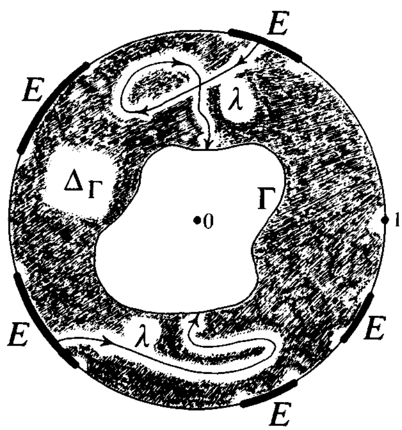


Figure 208

In this article we will be mainly concerned with the *reciprocal extremal length*

$$\Lambda(\Delta_\Gamma, G)$$

(see §E.1 for the definition and elementary properties of reciprocal extremal length).

Pfluger found a *relation* between $\Lambda(\Delta_\Gamma, G)$ and the *logarithmic capacity* of E defined in the preceding article. That relation also involves the curve Γ of course, but in fairly straightforward fashion. To arrive at Pfluger's result, we construct a *conformal mapping* of Δ onto a certain *disk with radial slits.*

Let $\mathscr{D} = (\mathbb{C} \cup \{\infty\}) \sim E$, so that $\partial\mathscr{D} = E$. With the harmonic measure $\omega_{\mathscr{D}}(\ \ , z)$ for this domain, we form the *conductor potential* for E,

$$U_E(z) \quad = \quad \int_E \log \frac{1}{|z-\zeta|} \mathrm{d}\omega_{\mathscr{D}}(\zeta, \ \infty),$$

described in the last article. Our conformal mapping will be constructed from the function

$$V_E(z) \quad = \quad U_E(z) \ + \ U_E\left(\frac{1}{\bar{z}}\right).$$

We have, first of all,

$$V_E(z) \quad = \quad V_E(1/\bar{z}).$$

Explicitly, since $E \subseteq \{|\zeta| = 1\}$,

$$\begin{aligned} V_E(z) \quad &= \quad \int_E \log \left| \frac{z}{(z-\zeta)(1-\bar{\zeta}z)} \right| \mathrm{d}\omega_{\mathscr{D}}(\zeta, \ \infty) \\ &= \quad \log|z| \quad + \quad 2\int_E \log \frac{1}{|z-\zeta|} \mathrm{d}\omega_{\mathscr{D}}(\zeta, \ \infty), \end{aligned}$$

from which we see in particular that

$$V_E(z) \longrightarrow -\infty \quad \text{for } z \longrightarrow 0$$

and hence also, that

$$V_E(z) \longrightarrow -\infty, \quad z \longrightarrow \infty.$$

Since $V_E(z)$ is *harmonic* in $\mathbb{C} \sim E \sim \{0\}$ and $V_E(z) = 2\gamma_{\mathscr{D}}$ for $z \in E$, the previous relations and the maximum principle imply that

$$V_E(z) \ < \ 2\gamma_{\mathscr{D}} \quad \text{for } z \notin E.$$

The function $V_E(z)$ has a *harmonic conjugate* $\tilde{V}_E(z)$ in $\{0 < |z| < 1\}$. $\tilde{V}_E(z)$ is of course *multiple valued* there; we proceed to investigate its behaviour. For $0 < r < 1$, we have

$$V_E(re^{i\vartheta}) \quad = \quad \int_E \log \left(\frac{r}{1 + r^2 - 2r\cos(\vartheta - \tau)} \right) \mathrm{d}\omega_{\mathscr{D}}(e^{i\tau}, \ \infty),$$

from which

$$\frac{\partial V_E(re^{i\vartheta})}{\partial r} \quad = \quad \frac{1}{r} \int_E \frac{1 - r^2}{1 + r^2 - 2r\cos(\vartheta - \tau)} \mathrm{d}\omega_{\mathscr{D}}(e^{i\tau}, \ \infty),$$

whence, by the Cauchy–Riemann equations,

$$\frac{\partial \tilde{V}_E(re^{i\vartheta})}{\partial \vartheta} \;=\; \int_E \frac{1-r^2}{1+r^2-2r\cos(\vartheta-\tau)}\, d\omega_{\mathscr{D}}(e^{i\tau}, \infty).$$

From this we see, *firstly*, that $\tilde{V}_E(re^{i\vartheta})$ is a *strictly increasing* function of ϑ for each fixed r, $0<r<1$, and, *secondly*, that $V_E(re^{i\vartheta})$ *increases by*

$$\int_0^{2\pi}\int_E \frac{1-r^2}{1+r^2-2r\cos(\vartheta-\tau)}\, d\omega_{\mathscr{D}}(e^{i\tau}, \infty)\, d\vartheta$$

$$= \;2\pi\int_E d\omega_{\mathscr{D}}(e^{i\tau}, \infty) \;=\; 2\pi$$

when ϑ goes from 0 to 2π. The determinations of $\tilde{V}_E(z)$ are, however, well defined and *single valued* in the simply connected region

$$\{z:\;\; 0<\arg z<2\pi,\;\; 0<|z|<1\}.$$

It therefore follows from the calculation just made that

$$\tilde{V}_E(z) \;-\; \arg z$$

is single valued (and harmonic) in $\{0<|z|<1\}$, and hence that

$$f_E(z) \;=\; \exp(V_E(z)+i\tilde{V}_E(z)) \;=\; z\exp(V_E(z)+i\tilde{V}_E(z)-\log z)$$

is *single valued and analytic* there. Since $V_E(z) \;=\; \log|z|+O(1)$ near 0, $f_E(z)$ is *bounded* in a punctured neighbourhood of that point and thus analytic at 0 also (with $f_E(0) \;=\; 0$). The function $f_E(z)$ is therefore *analytic* in Δ. We are going to show that it maps Δ conformally onto a disk with radial slits.

As we have seen,

$$V_E(e^{i\vartheta}) \;=\; 2\gamma_{\mathscr{D}} \quad \text{for } e^{i\vartheta}\in E$$

while

$$V_E(e^{i\vartheta}) \;<\; 2\gamma_{\mathscr{D}} \quad \text{when } e^{i\vartheta}\notin E.$$

The boundary value

$$\tilde{V}_E(e^{i\vartheta}) \;=\; \lim_{r\to 1-} \tilde{V}_E(re^{i\vartheta})$$

must, on the other hand, be an *increasing* function of ϑ like each of the $\tilde{V}_E(re^{i\vartheta})$, augmenting by 2π when ϑ does. In fact, $\tilde{V}_E(e^{i\vartheta})$ increases *only* on the arcs making up E; on *each* of the *complementary* arcs of $\{|z|=1\}$ it is *constant*. To see this, it suffices to observe that

$$V_E(z) = U_E(z) + U_E(1/\bar{z})$$

is *harmonic*, hence *infinitely differentiable*, along each of those complementary arcs, on which, by *symmetry*,

$$\frac{\partial V_E(z)}{\partial r} = 0.$$

Thence, by the Cauchy–Riemann equations,

$$\frac{\partial \tilde{V}_E(e^{i\vartheta})}{\partial \vartheta} = 0 \quad \text{for } e^{i\vartheta} \notin E.$$

We see that when ϑ increases from 0 to 2π, $f_E(e^{i\vartheta})$ describes a closed path of the following form, going through it *exactly once*:

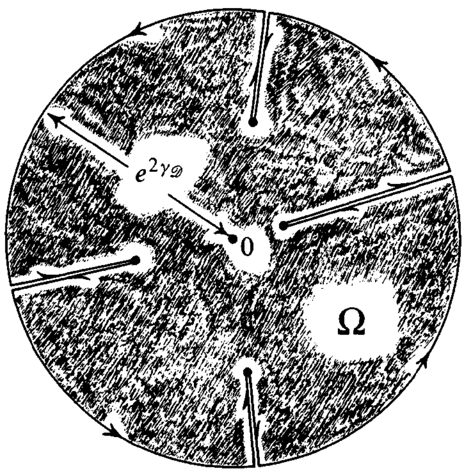


Figure 209

The arcs of E are taken onto those on the circle $\{|w| = e^{2\gamma_{\mathscr{D}}}\}$, and the ones *complementary* to E go onto the *radial slits*. Each of the *latter*, by the way, *really is* traversed *exactly once in each direction* (inward and then outwards) as $e^{i\vartheta}$ runs through the complementary arc corresponding to it. This is due to the *convexity* of $V_E(e^{i\vartheta})$ in ϑ on each of those complementary arcs, which may be verified by noting that

$$V_E(e^{i\vartheta}) = \int_E \log\left(\frac{1}{4\sin^2\frac{(\vartheta-\tau)}{2}}\right) d\omega_{\mathscr{D}}(e^{i\tau}, \infty)$$

for $e^{i\vartheta} \notin E$, whence

$$\frac{\partial^2 V_E(e^{i\vartheta})}{\partial\vartheta^2} = \int_E \frac{1}{2\sin^2\left(\frac{\vartheta-\tau}{2}\right)} d\omega_{\mathscr{D}}(e^{i\tau}, \infty) > 0$$

for such ϑ.

Let Ω denote the (bounded) region enclosed by the path traced out by $f_E(e^{i\vartheta})$ as ϑ goes from 0 to 2π. Then, if $w \in \Omega$, $\arg(f_E(e^{i\vartheta}) - w)$ *increases by exactly* 2π when ϑ does, and $f_E(z)$ must *assume the value* w *exactly once* in Δ, by the *principle of argument*. We see in the same way that *none* of the values w *outside* of $\bar{\Omega}$ are assumed by $f_E(z)$ in Δ; f_E *thus maps* Δ *conformally onto* Ω.

It is good to have a more compact formula for $f_E(z)$ at our disposal. Since

$$V_E(z) = \log|z| + 2\int_E \log\frac{1}{|1-\bar{\zeta}z|} d\omega_{\mathscr{D}}(\zeta, \infty),$$

we may take the harmonic conjugate $\tilde{V}_E(z)$ equal to

$$\arg z + 2\int_E \arg\left(\frac{1}{1-\bar{\zeta}z}\right) d\omega_{\mathscr{D}}(\zeta, \infty)$$

for $0 < |z| < 1$, and hence

$$f_E(z) = z\exp\left\{2\int_E \log\left(\frac{1}{1-\bar{\zeta}z}\right) d\omega_{\mathscr{D}}(\zeta, \infty)\right\}$$

for $z \in \Delta$.

These results are important enough to be summarized in the following

Theorem. *Let E be a finite union of arcs on $\{|z| = 1\}$, and denote by $\omega_{\mathscr{D}}(\ \ , z)$ the harmonic measure for*

$$\mathscr{D} = (\mathbb{C} \cup \{\infty\}) \sim E.$$

The function

$$\boxed{f_E(z) = z\exp\left\{2\int_E \log\left(\frac{1}{1-\bar{\zeta}z}\right) d\omega_{\mathscr{D}}(\zeta, \infty)\right\}}$$

maps the unit disk Δ conformally onto a domain Ω obtained by removing certain radial sits from the disk $\{|w| < e^{2\gamma_{\mathscr{D}}}\}$. Under the mapping, the arcs *of E go onto* others, *lying on the* circumference $\{|w| = e^{2\gamma_{\mathscr{D}}}\}$ *and* precisely covering it, *and the* complementary arcs *of the unit circumference go onto the* radial slits.

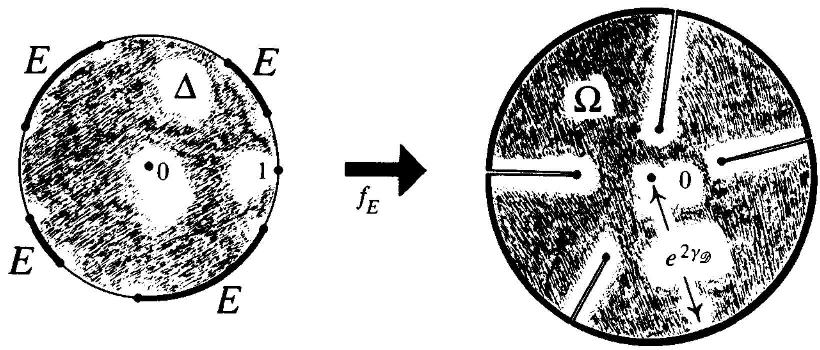


Figure 210

Once the conformal mapping f_E is available, it is easy to obtain estimates for the reciprocal extremal length $\Lambda(\Delta_\Gamma, G)$ specified at the beginning of the present article. To do that, we need two lemmas.

Lemma. $e^{\gamma_{\mathscr{D}}} \geqslant 1$.

Proof. Uses the minimum property of the conductor potential. Denoting the *unit circumference* by K, let us write

$$\mathscr{E} = \{|z|>1\}\cup\{\infty\},$$

so that $\partial\mathscr{E} = K$. If $E \subseteq K$, any positive measure μ of total mass 1 supported on E is *certainly* supported on K, so, by the extremal property in question (lemma at the end of the preceding article),

$$\gamma_{\mathscr{E}} \leqslant \int_E\int_E \log\frac{1}{|z-\zeta|}\,d\mu(\zeta)\,d\mu(z)$$

Choosing $\mu = \omega_{\mathscr{D}}(\ \ ,\infty)$, we get

$$\gamma_{\mathscr{E}} \leqslant \gamma_{\mathscr{D}}.$$

Referring to the formula

$$\frac{1}{2\pi}\int_0^{2\pi}\log\frac{1}{|z-e^{i\vartheta}|}\,d\vartheta = \log\frac{1}{|z|}, \qquad |z| \geqslant 1,$$

we see that $\gamma_{\mathscr{E}} = 0$. Therefore $\gamma_{\mathscr{D}} \geqslant 0$, as required.

Lemma. *For* $|z| < 1$,

$$\frac{|z|}{(1+|z|)^2} \leqslant |f_E(z)| \leqslant \frac{|z|}{(1-|z|)^2}.$$

Proof. By the boxed formula from the theorem,

$$\log\left|\frac{f_E(z)}{z}\right| = 2\int_E \log\frac{1}{|1-\bar{\zeta}z|}\, d\omega_{\mathscr{D}}(\zeta, \infty).$$

The integral on the right evidently lies between $2\log(1/(1+|z|))$ and $2\log(1/(1-|z|))$. The lemma follows.

Consider now any simple closed curve Γ about 0 lying in Δ, and write

$$\boxed{\begin{aligned} M_\Gamma &= \sup_{z\in\Gamma} \frac{|z|}{(1-|z|)^2}, \\ m_\Gamma &= \inf_{z\in\Gamma} \frac{|z|}{(1+|z|)^2}. \end{aligned}}$$

For a finite union E of arcs on the unit circumference, $\operatorname{Cap} E = e^{-\gamma_{\mathscr{D}}}$ and the quantity $\Lambda(\Delta_\Gamma, G)$ are then related through

Pfluger's Theorem. *If* $|z| < \frac{1}{2}(3-\sqrt{5})$ *for* $z \in \Gamma$ *(so as to make* $M_\Gamma < 1$*), the logarithmic capacity of* E *satisfies the double inequality*

$$M_\Gamma^{-1/2}\, e^{-\pi/\Lambda(\Delta_\Gamma, G)} \leqslant \operatorname{Cap} E \leqslant m_\Gamma^{-1/2}\, e^{-\pi/\Lambda(\Delta_\Gamma, G)}$$

Proof. The conformal mapping f_E described in the preceding theorem takes the ring domain Δ_Γ bounded by the unit circumference and Γ onto another, $\Omega_{\tilde{\Gamma}}$, bounded by $\partial\Omega$ and the curve $\tilde{\Gamma} = f_E(\Gamma)$:

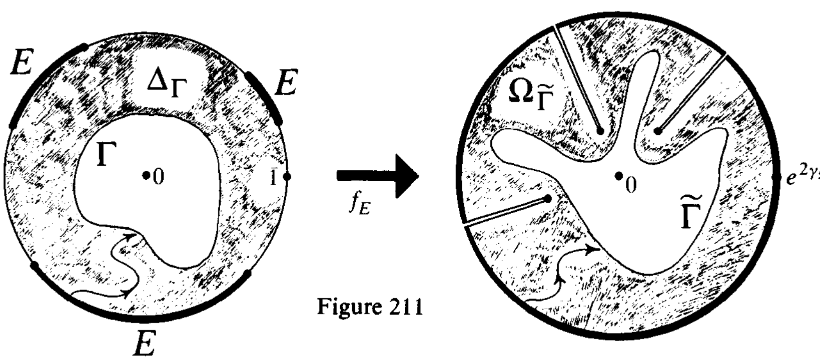


Figure 211

Under this mapping the curves of the family G – those lying in Δ_Γ and joining E to Γ – are taken to the ones lying in $\Omega_{\tilde{\Gamma}}$ which join *the circumference* $\{|w| = e^{2\gamma_{\mathscr{D}}}\}$ to $\tilde{\Gamma}$. Denoting the family of the latter curves by $\tilde{G}$, we have

$$\Lambda(\Delta_\Gamma, G) = \Lambda(\Omega_{\tilde{\Gamma}}, \tilde{G})$$

on account of property 2 of extremal length (§E.1).

By the *second* of the preceding two lemmas, $\tilde{\Gamma}$ lies inside the circle of radius M_Γ about 0, and in the present circumstances, $M_\Gamma < 1 \leqslant e^{2\gamma_{\mathscr{D}}}$ thanks to the *first* of those lemmas. The picture is thus as follows:

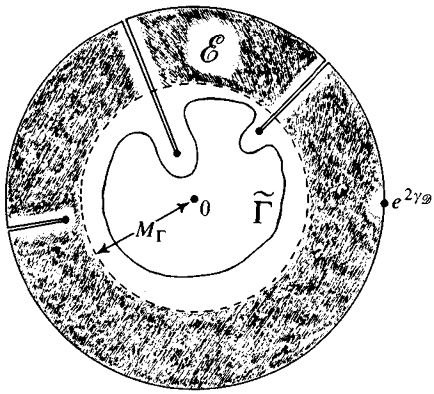


Figure 212

For the moment, let us denote the ring

$$\{M_\Gamma < |w| < e^{2\gamma_{\mathscr{D}}}\}$$

by $\mathscr{E}$, and the *family of curves* in $\mathscr{E}$ joining its *inner* to its *outer* boundary by H. Then, if $p(z) \geqslant 0$ is any weight on $\mathscr{E}$ *admissible for the family H* (in the parlance of §E.1), the weight $p^*(z)$ on $\Omega_{\tilde{\Gamma}}$ equal to $p(z)$ on $\Omega_{\tilde{\Gamma}} \cap \mathscr{E}$ and to *zero* on $\Omega_{\tilde{\Gamma}} \cap \sim \mathscr{E}$ *is certainly admissible* for $\tilde{G}$, so, by definition (§E.1),

$$\Lambda(\Omega_{\tilde{\Gamma}}, \tilde{G}) \leqslant \iint_{\Omega_{\tilde{\Gamma}}} (p^*(z))^2 \mathrm{d}x\,\mathrm{d}y = \iint_{\mathscr{E}} (p(z))^2 \mathrm{d}x\mathrm{d}y$$

(the presence of radial slits in Ω makes no difference here). Since the infimum of the right hand integral for the weights p in question is just $\Lambda(\mathscr{E}, H)$, we have $\Lambda(\Omega_{\tilde{\Gamma}}, \tilde{G}) \leqslant \Lambda(\mathscr{E}, H)$; that is, in view of the previous relation,

$$\Lambda(\Delta_\Gamma, G) \leqslant \Lambda(\mathscr{E}, H).$$

The right side of this inequality can easily be calculated explicitly by a procedure much like that of the special computation for a rectangle in §E.1,

reducible, in fact, to the latter by logarithmic substitution. In that way one finds without difficulty that

$$\Lambda(\mathscr{E}, H) \quad = \quad \frac{2\pi}{\log(e^{2\gamma_{\mathscr{D}}}/M_\Gamma)}.$$

Plugging this into the preceding relation we get

$$\Lambda(\Delta_\Gamma, G) \quad \leqslant \quad \frac{\pi}{\gamma_{\mathscr{D}} - (\log M_\Gamma/2)}.$$

The second of the above two lemmas also implies that the circle of radius m_Γ about 0 lies entirely *inside* the curve $\tilde{\Gamma}$. From this we see by an argument like the one just made that

$$\frac{\pi}{\gamma_{\mathscr{D}} - (\log m_\Gamma/2)} \quad \leqslant \quad \Lambda(\Delta_\Gamma, G)$$

(again the radial slits of Ω cause no trouble).

Combining the last inequalities, we find that

$$\gamma_{\mathscr{D}} \; - \; \tfrac{1}{2}\log M_\Gamma \quad \leqslant \quad \frac{\pi}{\Lambda(\Delta_\Gamma, G)} \quad \leqslant \quad \gamma_{\mathscr{D}} \; - \; \tfrac{1}{2}\log m_\Gamma,$$

or, since $\operatorname{Cap} E \; = \; e^{-\gamma_{\mathscr{D}}}$,

$$M_\Gamma^{-1/2} e^{-\pi/\Lambda(\Delta_\Gamma, G)} \quad \leqslant \quad \operatorname{Cap} E \quad \leqslant \quad m_\Gamma^{-1/2} e^{-\pi/\Lambda(\Delta_\Gamma, G)}. \qquad \text{Q.E.D.}$$

3. Application to the estimation of harmonic measure. Tsuji's inequality

The use of Pfluger's theorem in estimating harmonic measure is made possible by the fact that $\operatorname{Cap} E$ is a majorant for $|E|$ when E is a closed subset of the unit circumference. We restrict our attention to *finite unions of closed arcs*; results valid for such sets are general enough for most purposes.*

A sharp (i.e., *best possible*) relation between Cap E and $|E|$ (for E on the unit circumference) is known; its derivation is set as problem 36, given further on. For us a less precise result, having, however, a more straightforward proof, suffices:

Lemma. *If E is a finite union of closed arcs on the unit circumference,*

$$|E| \; \leqslant \; 4\pi \operatorname{Cap} E.$$

* See below, at end of the proof of Tsuji's inequality.

Proof. Is based on the extremal property of the conductor potential.

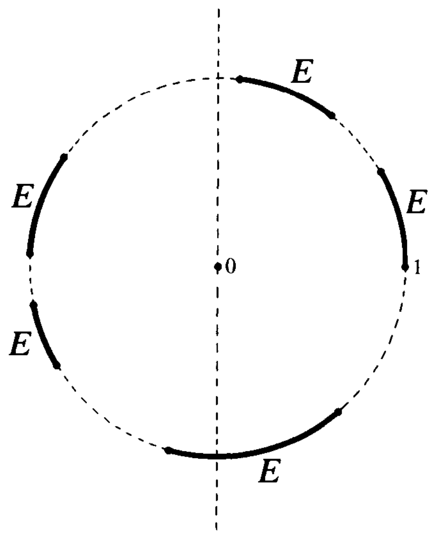


Figure 213

The part of E lying to *one of the two sides* of the imaginary axis has measure $\geqslant |E|/2$; suppose, wlog, that for E_+, the part lying to the *right* of that axis, we have

$$|E_+| \quad \geqslant \quad \tfrac{1}{2}|E|.$$

Arguing as in the proof of the *first* of the two lemmas in article 2, we see that

$$\operatorname{Cap} E_+ \quad \leqslant \quad \operatorname{Cap} E.$$

We have, however, $\operatorname{Cap} E_+ = e^{-\gamma}$, where, by the extremal property referred to (lemma, end of article 1), γ is the *minimum value* of the expressions

$$\int_{E_+}\int_{E_+} \log\frac{1}{|z-\zeta|}\,d\mu(z)\,d\mu(\zeta)$$

formed using positive measures μ on E_+ of total mass 1.

Suppose that E_+ consists of the arcs $I_1, I_2, \ldots, I_n$. By means of different rotations about the origin, we may *move* the I_j to *new arcs* I'_j on the *right half of the unit circumference*, arranged in the same order as the I_j but just touching each other, with $|I'_j| = |I_j|$ for $j = 1, 2, \ldots, n$.

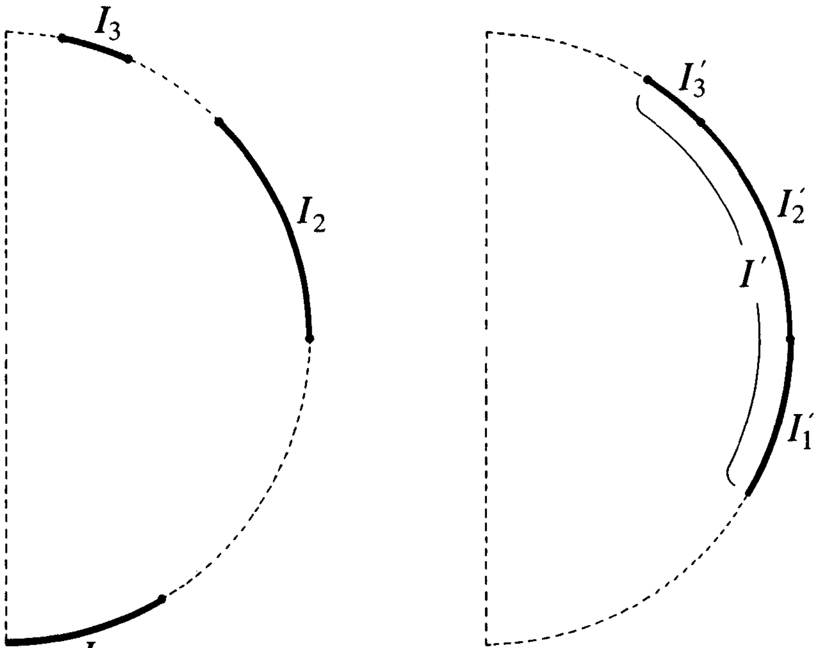


Figure 214

We denote $\bigcup_{j=1}^{n} I'_j$ by I'; I' is a *single arc* and $|I'| = |E_+|$.

A mapping ψ from E_+ to I' can now be defined in the following way: if $\zeta \in I_j$, we let $\psi(\zeta)$ be the point on I'_j *having the same position, relative to the endpoints of that arc, that* ζ *has, relative to the endpoints of* I_j. We then have

$$|\psi(z) - \psi(\zeta)| \leqslant |z - \zeta| \quad \text{for } z \text{ and } \zeta \in E_+.$$

Indeed, when z and ζ belong to the *same* arc I_j, there is *equality*, and, if z and ζ belong to *different arcs*, *strict inequality*, the effect of ψ being to move *each* arc of E_+ *closer to all the others* (since E_+ lies on a *semi-circumference*):

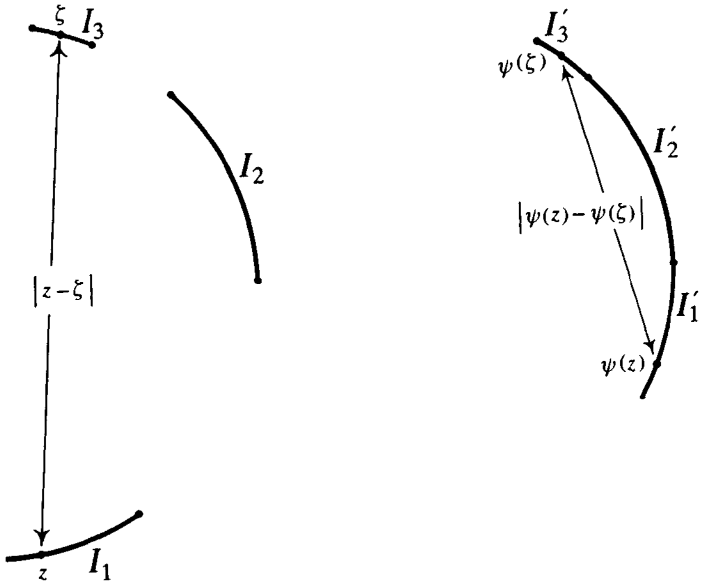


Figure 215

From the last inequality, we get

$$\int_{E_+}\int_{E_+}\log\frac{1}{|z-\zeta|}\,\mathrm{d}\mu(\zeta)\mathrm{d}\mu(z) \quad\leqslant\quad \int_{E_+}\int_{E_+}\log\frac{1}{|\psi(z)-\psi(\zeta)|}\,\mathrm{d}\mu(\zeta)\,\mathrm{d}\mu(z).$$

If μ_ψ is the measure on I' given by the formula

$$\mu_\psi(A) \quad=\quad \mu(\psi^{-1}(A)), \quad A \subseteq I',$$

the integral on the right can be rewritten

$$\int_{I'}\int_{I'}\log\frac{1}{|z'-\zeta'|}\,\mathrm{d}\mu_\psi(z')\mathrm{d}\mu_\psi(\zeta');$$

it is, moreover clear that *any* positive measure of total mass 1 on I' can be obtained as a μ_ψ for proper choice of the measure μ on E_+ with $\mu(E_+) = 1$. Choose, then, μ so as to make μ_ψ the *equilibrium charge distribution* for the arc I' (article 1). The integral just written is then equal to γ', the *conductor potential* for I', so, by the previous relation,

$$\gamma' \quad\geqslant\quad \int_{E_+}\int_{E_+}\log\frac{1}{|z-\zeta|}\,\mathrm{d}\mu(\zeta)\,\mathrm{d}\mu(z).$$

The expression on the right is, however, $\geqslant\ \gamma$ according to the observation made above. Thus.

$$\gamma' \geqslant \gamma,$$

or, in terms of $\operatorname{Cap} I' = e^{-\gamma'}$ and $\operatorname{Cap} E_+$,

$$\operatorname{Cap} I' \leqslant \operatorname{Cap} E_+.$$

The logarithmic capacity of the *single arc* I' may be expressed directly in terms of $|I'| = |E_+|$ after going through an elementary but somewhat tedious computation. We, however, do not require any great precision, and content ourselves with a simple lower bound on $\operatorname{Cap} I'$. Suppose, wlog, that I' is the (counterclockwise) arc from $e^{-i\alpha}$ to $e^{i\alpha}$, where $0 < \alpha \leqslant \pi/2$, so that $|I'| = 2\alpha$, and denote by φ the horizontal projection from I' onto the vertical line through the point 1:

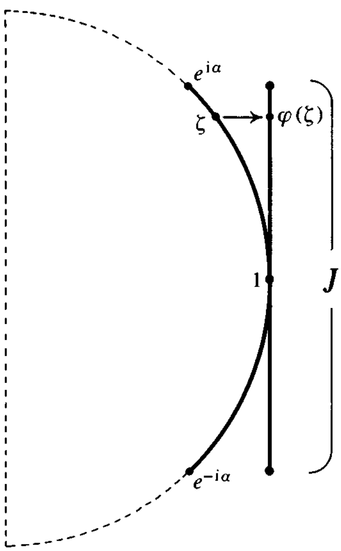


Figure 216

$\varphi(I')$ is thus the straight segment

$$J = [1 - i\sin\alpha,\ 1 + i\sin\alpha].$$

For z and $\zeta \in I'$, we clearly have

$$|\varphi(z) - \varphi(\zeta)| \leqslant |z - \zeta|,$$

whence, by an argument like the one made above,

$$\operatorname{Cap} J \leqslant \operatorname{Cap} I'.$$

The *left side* of this inequality is easily calculated. The Joukowski transformation

$$z \longrightarrow \frac{z-1}{\mathrm{i}\sin\alpha} + \sqrt{\left(\left(\frac{z-1}{\mathrm{i}\sin\alpha}\right)^2 - 1\right)} = w$$

takes the *exterior* of J conformally onto the domain $\{|w|>1\}$, with ∞ going to ∞; so, if $\mathscr{D}$ denotes the domain

$$(\mathbb{C}\cup\{\infty\})\sim J,$$

we have

$$G_{\mathscr{D}}(z,\infty) = \log|w| = \log\left|\frac{z-1}{\sin\alpha} + \sqrt{\left(\left(\frac{z-1}{\sin\alpha}\right)^2 + 1\right)}\right|.$$

For large values of $|z|$, the expression on the right reduces to

$$\log|z| + \log\left(\frac{2}{\sin\alpha}\right) + O\left(\frac{1}{|z|}\right),$$

whence, from article 1,

$$\gamma_{\mathscr{D}} = \lim_{z\to\infty}(G_{\mathscr{D}}(z,\infty) - \log|z|) = \log\left(\frac{2}{\sin\alpha}\right),$$

and $\operatorname{Cap} J = e^{-\gamma_{\mathscr{D}}} = \frac{1}{2}\sin\alpha$.

Since $0 < \alpha \leqslant \pi/2$, $\alpha \leqslant \frac{1}{2}\pi\sin\alpha$, so

$$|I'| = 2\alpha \leqslant 2\pi\operatorname{Cap} J$$

by the calculation just made and hence, in view of the relations established above,

$$|I'| \leqslant 2\pi\operatorname{Cap} I' \leqslant 2\pi\operatorname{Cap} E_+ \leqslant 2\pi\operatorname{Cap} E.$$

On the other hand,

$$|E| \leqslant 2|E_+| = 2|I'|,$$

so finally

$$|E| \leqslant 4\pi\operatorname{Cap} E.$$

We are done.

Remark. For sets E on the unit circumference, $4\pi\operatorname{Cap} E$ is in general a *poor upper bound* for $|E|$; it has the same order of magnitude as $|E|$ only when E is an arc or at least a set of fairly simple structure. As a rule, the comparison between $|E|$ and $\operatorname{Cap} E$ becomes *worse* as the set E under consideration becomes *more disconnected.* It is not hard to construct totally disconnected closed sets E on the unit circle for which $|E| = 0$ and yet $\operatorname{Cap} E > 0$. Descriptions of such constructions are found in many books, including the ones by Nevanlinna, Kahane & Salem, and Tsuji.

Problem 36

(a) Let E be compact and composed of a finite number of smooth arcs, and put $\mathscr{D} = (\mathbb{C} - E) \cup \{\infty\}$. If $f(z)$, analytic in $\mathscr{D}$, has modulus < 1 there and is zero at ∞, show that

$$\lim_{z \to \infty} |zf(z)| \leqslant \operatorname{Cap} E$$

(Hint: Look at $\log|f(z)| + G_{\mathscr{D}}(z, \infty)$.)

(b) Let E be a finite union of closed arcs on the unit circumference. Show that

$$\operatorname{Cap} E \geqslant \sin(|E|/4).$$

(Hint: With

$$\varphi(z) = \frac{1}{2\pi} \int_E \frac{e^{it} + z}{e^{it} - z} dt, \quad z \in \mathscr{D},$$

take

$$f(z) = \frac{e^{\pi i \varphi(z)/2} - e^{\pi i \varphi(0)/2}}{z(e^{\pi i \varphi(z)/2} + e^{-\pi i \varphi(0)/2})}$$

and use the result of part (a).)

Remarks. This proof of the inequality in (b) is from Lebedev's book on the area principle. The inequality becomes an equality for single arcs E. The supremum of the left-hand limits in (a) for the kind of functions f considered there is called the *analytic capacity* of E, and the f given in (b) actually realizes that supremum. For more about analytic capacity and its rôle in approximation theory the reader should consult the monographs by Garnett and by Zalcman.

Combination of the lemma just proved with Pfluger's theorem shows that if E is a finite union of arcs on the unit circumference and Γ a simple closed curve about 0 on which $|z| < \frac{1}{2}(3 - \sqrt{5})$, we have

$$\frac{|E|}{2\pi} \leqslant 2m_\Gamma^{-1/2} e^{-\pi/\Lambda(\Delta_\Gamma, G)}$$

where $m_\Gamma = \inf_{z \in \Gamma}(|z|/(1 + |z|)^2)$, and $\Lambda(\Delta_\Gamma, G)$ is related to E and Γ in the way described at the beginning of article 2. The use of this relation to estimate harmonic measure for simply connected domains comes immediately to mind on account of the conformal invariance of both harmonic measure and extremal length. Some control on the quantity m_Γ is of course needed if the results obtained are to have any practical value. We use a couple of the elementary properties of univalent functions for that; those are covered in many texts, for instance, the ones by Nehari and by Markushevich.

Lemma. *Let $\mathcal{O}$ be a simply connected domain with $z_0 \in \mathcal{O}$, and put*

$$R_0 = \operatorname{dist}(z_0, \partial\mathcal{O}).$$

If σ is the circle $|z - z_0| = R_0/16$ and φ is a conformal mapping of $\mathcal{O}$ onto the unit disk Δ with $\varphi(z_0) = 0$, φ takes σ onto a closed curve Γ about 0, such that

$$|w| < \tfrac{1}{2}(3 - \sqrt{5}) \quad \textit{for } w \in \Gamma$$

and that

$$|w|/(1 + |w|)^2 > 1/75, \quad w \in \Gamma.$$

Proof.

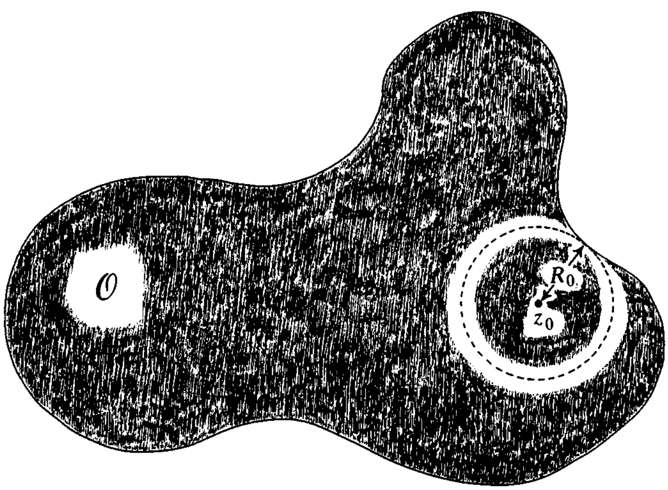


Figure 217

The function

$$f(\zeta) = \frac{\varphi(z_0 + R_0\zeta)}{R_0\varphi'(z_0)}$$

is certainly univalent for $|\zeta| < 1$, and has a Taylor expansion of the form

$$\zeta + A_2\zeta^2 + A_3\zeta^3 + \cdots$$

there. f also maps $\{|\zeta| < 1\}$ conformally onto a region *included* in the disk

$$\{|w| < 1/R_0|\varphi'(z_0)|\}.$$

Thence, by the Koebe 1/4-theorem, $1/R_0|\varphi'(z_0)| \geqslant 1/4$, or

$$R_0|\varphi'(z_0)| \leqslant 4.$$

Denoting by Φ the *inverse* to φ, we have, on the other hand,

$$\frac{\Phi(w) - z_0}{\Phi'(0)} = w + B_2 w^2 + B_3 w^3 + \cdots$$

for $|w| < 1$, with the left side *univalent* there. Here, $\Phi(0) = 0$ is distant by at least $R_0/|\Phi'(0)|$ units from the boundary of $\Phi(\Delta)$, so, by Koebe's 1/4-theorem, $R_0/|\Phi'(0)| \geqslant 1/4$, i.e.,

$$R_0|\varphi'(z_0)| \geqslant \frac{1}{4}.$$

According to the distortion theorem,

$$\frac{|\zeta|}{(1+|\zeta|)^2} \leqslant |f(\zeta)| \leqslant \frac{|\zeta|}{(1-|\zeta|)^2}$$

for $|\zeta| < 1$, so, in terms of $z = z_0 + R_0\zeta$ and φ,

$$\frac{R_0|\varphi'(z_0)||\zeta|}{(1+|\zeta|)^2} \leqslant |\varphi(z)| \leqslant \frac{R_0|\varphi'(z_0)||\zeta|}{(1-|\zeta|)^2}.$$

Hence, if $|z - z_0| = R_0/16$,

$$|\varphi(z)| \leqslant \frac{4/16}{(1-1/16)^2} < \frac{1}{3} < \frac{3-\sqrt{5}}{2}$$

by the *first* of the above inequalities, in other words,

$$|w| < \frac{3-\sqrt{5}}{2} \quad \text{for } w \in \Gamma = \varphi(\sigma).$$

When $|z - z_0| = R_0/16$, we also see by the second of the above inequalities and the relation involving $|\varphi(z)|$ and ζ, that

$$|\varphi(z)| \geqslant \frac{1/4 \cdot 1/16}{(1+1/16)^2} = \frac{4}{(17)^2} > \frac{1}{72}.$$

Thence, since $r/(1+r)^2$ is increasing for $0 \leqslant r \leqslant 1$,

$$\frac{|w|}{(1+|w|)^2} > \frac{1/72}{(73/72)^2} > \frac{1}{75} \quad \text{for } w \in \Gamma.$$

The lemma is proved

Now we can give the

Theorem. *Let $\mathcal{O}$ be a simply connected domain bounded by a Jordan curve. Taking a $z_0 \in \mathcal{O}$, we denote by σ the circle of radius $\frac{1}{16}\operatorname{dist}(z_0, \partial\mathcal{O})$ about z_0, and by $\mathcal{O}_\sigma$ the ring domain bounded by $\partial\mathcal{O}$ and σ. If F is a finite union of closed arcs on $\partial\mathcal{O}$, let S be the family of curves in $\mathcal{O}_\sigma$ joining F to σ. Then*

$$\boxed{\omega_{\mathcal{O}}(F, z_0) \leqslant 18e^{-\pi/\Lambda(\mathcal{O}_\sigma, S)}.}$$

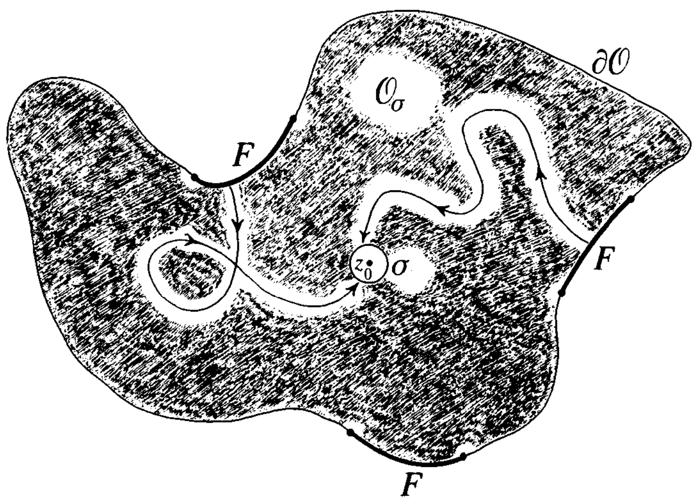


Figure 218

Proof. Let φ be a conformal mapping of $\mathcal{O}$ onto the unit disk Δ with $\varphi(z_0) = 0$; φ takes $F \subseteq \partial\mathcal{O}$ onto a finite union E of arcs on the unit circumference, and the circle σ to a simple closed curve Γ about 0, lying in Δ. The function φ thus maps the ring domain $\mathcal{O}_\sigma$ conformally onto Δ_Γ, the region bounded by Γ and the unit circumference, and takes the curves of the family S to those lying in Δ_Γ, joining E to Γ. Denoting, as usual, the collection of the latter curves by G, we have, by property 2 from article 1 (conformal invariance of extremal length),

$$\Lambda(\mathcal{O}_\sigma, S) \;=\; \Lambda(\Delta_\Gamma, G).$$

On the other hand,

$$\omega_{\mathcal{O}}(F,\ z_0) \;=\; \frac{|E|}{2\pi}.$$

According to the *second* of the preceding two lemmas, we can apply Pfluger's theorem with our curve Γ, so

$$\operatorname{Cap} E \;\leqslant\; m_\Gamma^{-1/2} e^{-\pi/\Lambda(\Delta_\Gamma, G)}$$

where

$$m_\Gamma \;=\; \inf_{w\in\Gamma} \frac{|w|}{(1+|w|)^2}.$$

By the same lemma, we have here,

$$m_\Gamma^{-1/2} \;\leqslant\; 9.$$

From the *first* of the above lemmas we now get

$$\frac{|E|}{2\pi} \;\leqslant\; 2\operatorname{Cap} E \;\leqslant\; 18e^{-\pi/\Lambda(\Delta_\Gamma, G)},$$

so, by the preceding two relations,

$$\omega_{\mathcal{O}}(F,\ z_0) \;\leqslant\; 18e^{-\pi/\Lambda(\mathcal{O}_\sigma, S)}. \qquad \text{Q.E.D.}$$

Remark 1. In the theorem, one may replace $\Lambda(\mathcal{O}_\sigma, S)$ by $\Lambda(\mathcal{O}, S_\sigma)$ where S_σ is the family of *curves in* $\mathcal{O}$ joining σ to F, for the two reciprocal extremal lengths are *equal.* Indeed, every curve in S_σ has on it an *arc* which, *by itself*, belongs to S. Therefore, if a weight $p(z)$ on $\mathcal{O}_\sigma$ is admissible for S, the weight $p_1(z)$ on $\mathcal{O}$ equal to $p(z)$ in $\mathcal{O}_\sigma$ and to *zero* inside the circle σ is admissible for S_σ, and hence $\Lambda(\mathcal{O}, S_\sigma) \leqslant \Lambda(\mathcal{O}_\sigma, S)$. On the other hand, $S \subseteq S_\sigma$, so if a weight $p_2(z)$ on $\mathcal{O}$ is admissible for S_σ, its *restriction*, $p(z)$, to $\mathcal{O}_\sigma$ must be admissible for S. This makes $\Lambda(\mathcal{O}_\sigma, S) \leqslant \Lambda(\mathcal{O}, S_\sigma)$.

Remark 2. As stated at the beginning of this §, Beurling's thesis already contained practically the same result.

Remark 3. Note that *here* we have the exponential

$$e^{-\pi/\Lambda(\mathcal{O}_\sigma, S)}$$

where the corresponding theorem of §E.1 (for *single* arcs) involves the expression

$$e^{-\pi/4\Lambda(\mathscr{D},G)}.$$

There is, however, *no real discrepancy* between the two results. The *extra factor of* 4 figuring in the second exponential is due to the fact that the curves of the family G involved there *start* from an arc on $\partial\mathscr{D}$ (the one whose harmonic measure at z_0 is in question), *loop around* z_0, and then *go back out* to that arc. The curves of our *present* family S *just go in* from the set F on $\partial\mathcal{O}$ to the circle σ about z_0; *they don't go back out* to F again.

Remark 4. The present result, unlike the corresponding one of §E.1, furnishes an inequality whose two sides are generally *not* of the same order of magnitude. According to the remark following the first of the above lemmas, the estimate given will usually tend to get *less and less precise* when applied to sets $F \subseteq \partial\mathcal{O}$ having *more and more components.* The *right* side of the inequality is *really* a measure (roughly speaking) of *logarithmic capacity* rather than of *harmonic measure.* Our result *does* nevertheless *have its uses.*

One application of the last theorem is to the derivation of a generalization, usually known as *Tsuji's inequality*, of the Ahlfors–Carleman estimate given at the end of §E.1. Suppose we have a simply connected domain $\mathcal{O}$ (bounded or not), with $\partial\mathcal{O}$ consisting of a Jordan curve, or of Jordan arcs going out to ∞. Let $z_0 \in \mathcal{O}$.

If $r > \operatorname{dist}(z_0, \partial\mathcal{O})$, we denote by $\mathcal{O}_r$ the *component* of

$$\mathcal{O} \cap \{|z-z_0| < r\}$$

containing the point z_0. $\mathcal{O}_r$ is bounded by all or part of $\partial\mathcal{O}$, and, in the second case, *by certain arcs on the circle* $|z-z_0| = r$ as well. We call the union of these $\Sigma(r)$ (understanding that $\Sigma(r)$ may be *empty*), and write

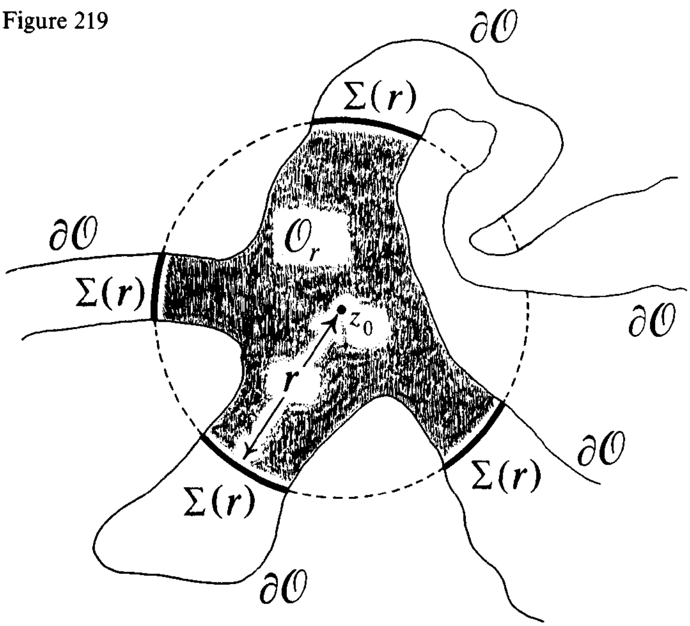
$$\theta(r) = |\Sigma(r)|/r.$$

For $0 < r < \operatorname{dist}(z_0, \partial\mathcal{O})$, $\Sigma(r)$ consists of the *entire circumference* $|z-z_0| = r$, and *then we put*

$$\theta(r) = \infty \ (\textit{sic!})$$

as in §E.1.

Figure 219



Fixing any $R > \operatorname{dist}(z_0, \partial\mathcal{O})$, we look at harmonic measure $\omega_{\mathcal{O}_R}(\ \ , z)$ for the domain $\mathcal{O}_R$. Concerning the the latter, we have the

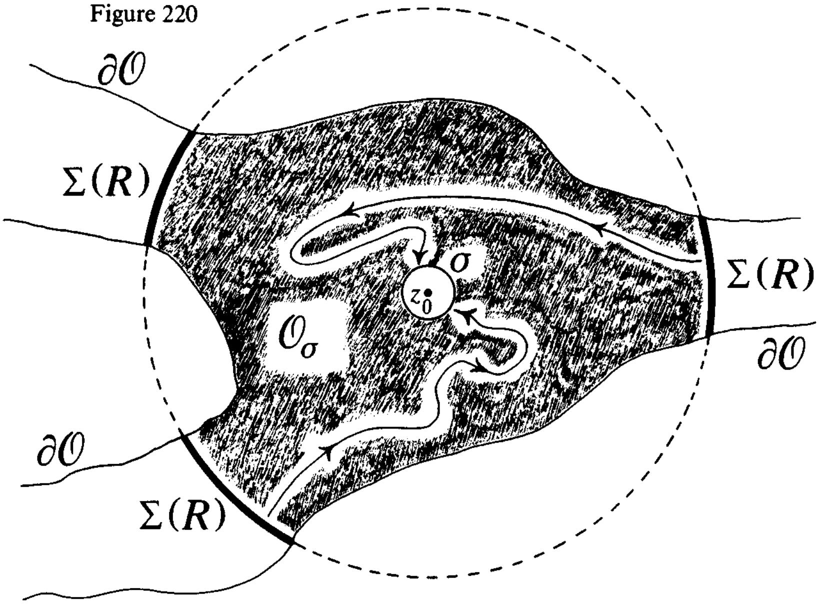
Theorem. (Tsuji's inequality)

$$\boxed{\omega_{\mathcal{O}_R}(\Sigma(R),\ z_0) \quad \leqslant \quad 18\mathrm{e}^{-\pi\int_0^R \frac{\mathrm{d}r}{r\theta(r)}}.}$$

Proof. Consider first the case where $\Sigma(R)$ consists of a *finite* number of arcs which we may just as well look on as *closed*. Taking the circle σ of radius $\frac{1}{16}\operatorname{dist}(z_0, \partial\mathcal{O})$ about z_0, we proceed to obtain an *upper bound* for the reciprocal extremal length $\Lambda(\mathcal{O}_\sigma, S)$, where S is the family of curves in $\mathcal{O}_\sigma$, the ring domain bounded by $\partial\mathcal{O}_R$ (*sic*!) and σ, joining $\Sigma(R)$ to σ. For this purpose, it is enough to exhibit a *single weight* $p(z) \geqslant 0$ on $\mathcal{O}_\sigma$, *admissible for the family* S. Then, by definition (§E.1), we'll have

$$\Lambda(\mathcal{O}_\sigma, S) \quad \leqslant \quad \iint_{\mathcal{O}_\sigma} (p(z))^2\,\mathrm{d}x\,\mathrm{d}y.$$

Figure 220



Write

$$\Sigma \quad = \bigcup_{\mathrm{dist}(z_0,\partial\mathcal{O})\leqslant r<R} \Sigma(r) \quad ,$$

and, for $z\in\Sigma$, put

$$p(z) \quad = \quad \frac{k}{|z-z_0|\theta(|z-z_0|)},$$

where k is a constant yet to be determined. For $z\notin\Sigma$, we take $p(z) \ = \ 0$.

If λ is *any curve in* $\mathcal{O}_\sigma$ *joining* $\Sigma(R)$ *to* σ, we have (cf. proof of Ahlfors–Carleman inequality, end of §E.1),

$$\int_\lambda p(z)|\mathrm{d}z| \quad \geqslant \quad \int_\lambda p(z)|\mathrm{d}|z-z_0||.$$

Putting $\mathrm{dist}(z_0,\partial\mathcal{O}) \ = \ R_0$ and writing

$$z-z_0 \ = \ re^{i\vartheta},$$

we see that the integral on the right is

$$\geqslant \quad \int_{R_0}^{R} p(z)\,\mathrm{d}r \quad = \quad k\int_{R_0}^{R}\frac{\mathrm{d}r}{r\theta(r)}.$$

For $k \ = \ 1/\int_{R_0}^{R}\mathrm{d}r/r\theta(r)$, the last expression is equal to *unity, making* $p(z)$ *admissible for the family* S of curves λ.