A. The Hyperbolic Space

We take first the case of negative curvature, that is $\epsilon = -1$. The transform $f \to \widehat{f}$ is now given by

$$\widehat{f}(\xi) = \int_{\xi} f(x)\, dm(x) \tag{2}$$

ξ being any k-dimensional totally geodesic submanifold of X $(1 \leq k \leq n-1)$ with the induced Riemannian structure and dm the corresponding measure. From our description of the geodesics in X it is clear that any two points in X can be joined by a unique geodesic. Let d be a distance function on X, and for simplicity we write o for the origin x^o in X. Consider now geodesic polar-coordinates for X at o; this is a mapping

$$\mathrm{Exp}_o Y \to (r, \theta_1, \ldots, \theta_{n-1})\,,$$

where Y runs through the tangent space $X_o, r = |Y|$ (the norm given by the Riemannian structure) and $(\theta_1, \ldots, \theta_{n-1})$ are coordinates of the unit vector $Y/|Y|$. Then the Riemannian structure of X is given by

$$ds^2 = dr^2 + (\sinh\, r)^2\, d\sigma^2\,, \tag{3}$$

where $d\sigma^2$ is the Riemannian structure

$$\sum_{i,j=1}^{n-1} g_{ij}(\theta_1, \cdots, \theta_{n-1})\, d\theta_i\, d\theta_j$$

on the unit sphere in X_o. The surface area $A(r)$ and volume $V(r) = \int_o^r A(t)\, dt$ of a sphere in X of radius r are thus given by

$$A(r) = \Omega_n (\sinh\, r)^{n-1}\,, \quad V(r) = \Omega_n \int_o^r \sinh^{n-1} t\, dt \tag{4}$$

so $V(r)$ increases like $e^{(n-1)r}$. This explains the growth condition in the next result where $d(o, \xi)$ denotes the distance of o to the manifold ξ.

Theorem 1.2. *(The support theorem.) Suppose $f \in C(X)$ satisfies*

(i) For each integer $m > 0, f(x)e^{md(o,x)}$ is bounded.

(ii) There exists a number $R > 0$ such that

$$\widehat{f}(\xi) = 0 \quad \text{for } d(o, \xi) > R\,.$$

Then

$$f(x) = 0 \quad \text{for } d(o, x) > R\,.$$

Taking $R \to 0$ we obtain the following consequence.

Corollary 1.3. *The Radon transform $f \to \widehat{f}$ is one-to-one on the space of continuous functions on X satisfying condition (i) of "exponential decrease".*

Proof of Theorem 1.2. Using smoothing of the form

$$\int_G \varphi(g) f(g^{-1} \cdot x)\, dg$$

($\varphi \in \mathcal{D}(G)$, dg Haar measure on G) we can (as in Theorem 2.6, Ch. I) assume that $f \in \mathcal{E}(X)$.

We first consider the case when f in (2) is a radial function. Let P denote the point in ξ at the minimum distance $p = d(o, \xi)$ from o, let $Q \in \xi$ be arbitrary and let

$$q = d(o, Q)\,, \quad r = d(P, Q)\,.$$

Since ξ is totally geodesic $d(P, Q)$ is also the distance between P and Q in ξ. Consider now the totally geodesic plane π through the geodesics oP and oQ as given by Lemma 1.1 (Fig. III.1). Since a totally geodesic submanifold contains the geodesic joining any two of its points, π contains the geodesic PQ. The angle oPQ being 90° (see e.g. [DS], p. 77) we conclude by hyperbolic trigonometry, (see e.g. Coxeter [1957])

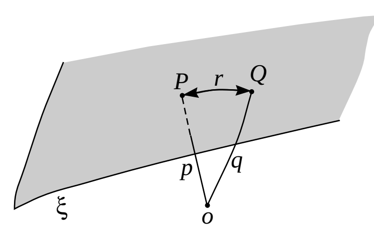

FIGURE III.1.

$$\cosh q = \cosh p \cosh r\,. \tag{5}$$

Since f is radial it follows from (5) that the restriction $f|\xi$ is constant on spheres in ξ with center P. Since these have area $\Omega_k (\sinh r)^{k-1}$ formula (2) takes the form

$$\widehat{f}(\xi) = \Omega_k \int_0^\infty f(Q)(\sinh r)^{k-1}\, dr\,. \tag{6}$$

Since f is a radial function it is invariant under the subgroup $K \subset G$ which fixes o. But K is not only transitive on each sphere $S_r(o)$ with center o, it is for each fixed k transitive on the set of k-dimensional totally geodesic submanifolds which are tangent to $S_r(o)$. Consequently, $\widehat{f}(\xi)$ depends only on the distance $d(o, \xi)$. Thus we can write

$$f(Q) = F(\cosh q)\,, \quad \widehat{f}(\xi) = \widehat{F}(\cosh p)$$

for certain 1-variable functions F and $\widehat{F}$, so by (5) we obtain

$$\widehat{F}(\cosh p) = \Omega_k \int_0^\infty F(\cosh p \, \cosh r)(\sinh r)^{k-1} \, dr \,. \tag{7}$$

Writing here $t = \cosh p$, $s = \cosh r$ this reduces to

$$\widehat{F}(t) = \Omega_k \int_1^\infty F(ts)(s^2-1)^{(k-2)/2} \, ds \,. \tag{8}$$

Here we substitute $u = (ts)^{-1}$ and then put $v = t^{-1}$. Then (8) becomes

$$v^{-1}\widehat{F}(v^{-1}) = \Omega_k \int_0^v \{F(u^{-1})u^{-k}\}(v^2-u^2)^{(k-2)/2} \, du \,.$$

This integral equation is of the form (19), Ch. I so we get the following analog of (20), Ch. I:

$$F(u^{-1})u^{-k} = cu \left(\frac{d}{d(u^2)}\right)^k \int_o^u (u^2-v^2)^{(k-2)/2} \widehat{F}(v^{-1}) \, dv \,, \tag{9}$$

where c is a constant. Now by assumption (ii) $\widehat{F}(\cosh p) = 0$ if $p > R$. Thus

$$\widehat{F}(v^{-1}) = 0 \quad \text{if } 0 < v < (\cosh R)^{-1} \,.$$

From (9) we can then conclude

$$F(u^{-1}) = 0 \quad \text{if } u < (\cosh R)^{-1}$$

which means $f(x) = 0$ for $d(o,x) > R$. This proves the theorem for f radial.

Next we consider an arbitrary $f \in \mathcal{E}(X)$ satisfying (i), (ii) . Fix $x \in X$ and if dk is the normalized Haar measure on K consider the integral

$$F_x(y) = \int_K f(gk \cdot y) \, dk \,, \quad y \in X \,,$$

where $g \in G$ is an element such that $g \cdot o = x$. Clearly, $F_x(y)$ is the average of f on the sphere with center x, passing through $g \cdot y$. The function F_x satisfies the decay condition (i) and it is radial. Moreover,

$$\widehat{F}_x(\xi) = \int_K \widehat{f}(gk \cdot \xi) \, dk \,. \tag{10}$$

We now need the following estimate

$$d(o, gk \cdot \xi) \geq d(o, \xi) - d(o, g \cdot o) \,. \tag{11}$$

For this let x_o be a point on ξ closest to $k^{-1}g^{-1} \cdot o$. Then by the triangle inequality

$$\begin{aligned} d(o, gk \cdot \xi) = d(k^{-1}g^{-1} \cdot o, \xi) &\geq d(o, x_o) - d(o, k^{-1}g^{-1} \cdot o) \\ &\geq d(o, \xi) - d(o, g \cdot o) \,. \end{aligned}$$

Thus it follows by (ii) that

$$\widehat{F}_x(\xi) = 0 \ \text{ if } \ d(o,\xi) > d(o,x) + R\,.$$

Since F_x is radial this implies by the first part of the proof that

$$\int_K f(gk \cdot y)\, dk = 0 \tag{12}$$

if

$$d(o,y) > d(o, g \cdot o) + R\,. \tag{13}$$

But the set $\{gk \cdot y : k \in K\}$ is the sphere $S_{d(o,y)}(g \cdot o)$ with center $g \cdot o$ and radius $d(o,y)$; furthermore, the inequality in (13) implies the inclusion relation

$$B_R(o) \subset B_{d(o,y)}(g \cdot o) \tag{14}$$

for the balls. But considering the part in $B_R(o)$ of the geodesic through o and $g \cdot o$ we see that conversely relation (14) implies (13). Theorem 1.2 will therefore be proved if we establish the following lemma.

Lemma 1.4. *Let $f \in C(X)$ satisfy the conditions:*

(i) *For each integer $m > 0$, $f(x)e^{m\, d(o,x)}$ is bounded.*

(ii) *There exists a number $R > 0$ such that the surface integral*

$$\int_S f(s)\, d\omega(s) = 0\,,$$

whenever the spheres S encloses the ball $B_R(o)$.

Then

$$f(x) = 0 \ \text{ for } \ d(o,x) > R\,.$$

Proof. This lemma is the exact analog of Lemma 2.7, Ch. I, whose proof, however, used the vector space structure of $\mathbf{R}^n$. By using a special model of the hyperbolic space we shall nevertheless adapt the proof to the present situation. As before we may assume f is smooth, i.e., $f \in \mathcal{E}(X)$.

Consider the unit ball $\{x \in \mathbf{R}^n : \sum_1^n x_i^2 < 1\}$ with the Riemannian structure

$$ds^2 = \rho(x_1, \ldots, x_n)^2 (dx_1^2 + \cdots + dx_n^2) \tag{15}$$

where

$$\rho(x_1, \ldots, x_n) = 2(1 - x_1^2 - \ldots - x_n^2)^{-1}\,.$$

This Riemannian manifold is well known to have constant curvature -1 so we can use it for a model of X. This model is useful here because the spheres in X are the ordinary Euclidean spheres inside the ball. This fact is obvious for the spheres Σ with center 0. For the general statement it suffices to prove that if T is the geodesic symmetry with respect to a point (which we can take on the x_1-axis) then $T(\Sigma)$ is a Euclidean sphere. The unit disk D in the x_1x_2-plane is totally geodesic in X, hence invariant under T. Now the isometries of the non-Euclidean disk D are generated by the complex conjugation $x_1 + ix_2 \to x_1 - ix_2$ and fractional linear transformations so they map Euclidean circles into Euclidean circles. In particular $T(\Sigma \cap D) = T(\Sigma) \cap D$ is a Euclidean circle. But T commutes with the rotations around the x_1-axis. Thus $T(\Sigma)$ is invariant under such rotations and intersects D in a circle; hence it is a Euclidean sphere.

After these preliminaries we pass to the proof of Lemma 1.4. Let $S = S_r(y)$ be a sphere in X enclosing $B_r(o)$ and let $B_r(y)$ denote the corresponding ball. Expressing the exterior $X - B_r(y)$ as a union of spheres in X with center y we deduce from assumption (ii)

$$\int_{B_r(y)} f(x)\,dx = \int_X f(x)\,dx\,, \tag{16}$$

which is a constant for small variations in r and y. The Riemannian measure dx is given by

$$dx = \rho^n\,dx_o\,, \tag{17}$$

where $dx_o = dx_1 \dots dx_n$ is the Euclidean volume element. Let r_o and y_o, respectively, denote the Euclidean radius and Euclidean center of $S_r(y)$. Then $S_{r_o}(y_o) = S_r(y), B_{r_o}(y_o) = B_r(y)$ set-theoretically and by (16) and (17)

$$\int_{B_{r_0}(y_0)} f(x_0)\rho(x_0)^n\,dx_o = \text{const.}\,, \tag{18}$$

for small variations in r_o and y_o; thus by differentiation with respect to r_o,

$$\int_{S_{r_0}(y_0)} f(s_0)\rho(s_0)^n\,d\omega_o(s_o) = 0\,, \tag{19}$$

where $d\omega_o$ is the Euclidean surface element. Putting $f^*(x) = f(x)\rho(x)^n$ we have by (18)

$$\int_{B_{r_o}(y_o)} f^*(x_o)\,dx_o = \text{const.}\,,$$

so by differentiating with respect to y_o, we get

$$\int_{B_{r_o}(o)} (\partial_i f^*)(y_o + x_o)\,dx_o = 0\,.$$

Using the divergence theorem (26), Chapter I, §2, on the vector field $F(x_o) = f^*(y_o + x_o)\partial_i$ defined in a neighborhood of $B_{r_o}(0)$ the last equation implies

$$\int_{S_{r_o}(0)} f^*(y_o + s)s_i \, d\omega_o(s) = 0$$

which in combination with (19) gives

$$\int_{S_{r_o}(y_o)} f^*(s)s_i \, d\omega_o(s) = 0 \,. \tag{20}$$

The Euclidean and the non-Euclidean Riemannian structures on $S_{r_o}(y_o)$ differ by the factor ρ^2. It follows that $d\omega = \rho(s)^{n-1} \, d\omega_o$ so (20) takes the form

$$\int_{S_r(y)} f(s)\rho(s)s_i \, d\omega(s) = 0 \,. \tag{21}$$

We have thus proved that the function $x \to f(x)\rho(x)x_i$ satisfies the assumptions of the theorem. By iteration we obtain

$$\int_{S_r(y)} f(s)\rho(s)^k s_{i_1} \ldots s_{i_k} \, d\omega(s) = 0 \,. \tag{22}$$

In particular, this holds with $y = 0$ and $r > R$. Then $\rho(s) =$ constant and (22) gives $f \equiv 0$ outside $B_R(o)$ by the Weierstrass approximation theorem. Now Theorem 1.2 is proved.

Now let L denote the Laplace-Beltrami operator on X. (See Ch. IV, §1 for the definition.) Because of formula (3) for the Riemannian structure of X, L is given by

$$L = \frac{\partial^2}{\partial r^2} + (n-1) \coth r \frac{\partial}{\partial r} + (\sinh r)^{-2} L_S \tag{23}$$

where L_S is the Laplace-Beltrami operator on the unit sphere in X_0. We consider also for each $r \geq 0$ the mean value operator M^r defined by

$$(M^r f)(x) = \frac{1}{A(r)} \int_{S_r(x)} f(s) \, d\omega(s) \,.$$

As we saw before this can also be written

$$(M^r f)(g \cdot o) = \int_K f(gk \cdot y) \, dk \tag{24}$$

if $g \in G$ is arbitrary and $y \in X$ is such that $r = d(o, y)$. If f is an analytic function one can, by expanding it in a Taylor series, prove from (24) that M^r is a certain power series in L (cf. Helgason [1959], pp. 270-272). In particular we have the commutativity

$$M^r L = L M^r \,. \tag{25}$$

This in turn implies the "Darboux equation"

$$L_x(F(x,y)) = L_y(F(x,y)) \tag{26}$$

for the function $F(x,y) = (M^{d(o,y)}f)(x)$. In fact, using (24) and (25) we have if $g \cdot o = x$, $r = d(o,y)$

$$\begin{aligned} L_x(F(x,y)) &= (LM^r f)(x) = (M^r Lf)(x) \\ &= \int_K (Lf)(gk \cdot y)\, dk = \int_K (L_y(f(gk \cdot y)))\, dk \end{aligned}$$

the last equation following from the invariance of the Laplacian under the isometry gk. But this last expression is $L_y(F(x,y))$.

We remark that the analog of Lemma 2.13 in Ch. IV which also holds here would give another proof of (25) and (26).

For a fixed integer $k(1 \leq k \leq n-1)$ let Ξ denote the manifold of all k-dimensional totally geodesic submanifolds of X. If φ is a continuous function on Ξ we denote by $\check{\varphi}$ the point function

$$\check{\varphi}(x) = \int_{x \in \xi} \varphi(\xi)\, d\mu(\xi)\,,$$

where μ is the unique measure on the (compact) space of ξ passing through x, invariant under all rotations around x and having total measure one.

Theorem 1.5. *(The inversion formula.) For k even let Q_k denote the polynomial*

$$Q_k(z) = [z + (k-1)(n-k)][z + (k-3)(n-k+2)] \ldots [z + 1 \cdot (n-2)]$$

of degree $k/2$. The k-dimensional Radon transform on X is then inverted by the formula

$$cf = Q_k(L)\big((\widehat{f})^\vee\big)\,, \quad f \in \mathcal{D}(X)\,.$$

Here c is the constant

$$c = (-4\pi)^{k/2}\Gamma(n/2)/\Gamma((n-k)/2)\,. \tag{27}$$

The formula holds also if f satisfies the decay condition (i) in Corollary 4.1.

Proof. Fix $\xi \in \Xi$ passing through the origin $o \in X$. If $x \in X$ fix $g \in G$ such that $g \cdot o = x$. As k runs through K, $gk \cdot \xi$ runs through the set of totally geodesic submanifolds of X passing through x and

$$\check{\varphi}(g \cdot o) = \int_K \varphi(gk \cdot \xi)\, dk\,.$$

Hence

$$(\widehat{f})^\vee(g \cdot o) = \int_K \left(\int_\xi f(gk \cdot y)\, dm(y) \right) dk = \int_\xi (M^r f)(g \cdot o)\, dm(y)\,,$$

where $r = d(o, y)$. But since ξ is totally geodesic in X, it has also constant curvature -1 and two points in ξ have the same distance in ξ as in X. Thus we have

$$(\widehat{f})^\vee(x) = \Omega_k \int_0^\infty (M^r f)(x)(\sinh r)^{k-1}\, dr\,. \tag{28}$$

We apply L to both sides and use (23). Then

$$(L(\widehat{f})^\vee)(x) = \Omega_k \int_0^\infty (\sinh r)^{k-1} L_r(M^r f)(x)\, dr\,, \tag{29}$$

where L_r is the "radial part" $\frac{\partial^2}{\partial r^2} + (n-1)\coth r \frac{\partial}{\partial r}$ of L. Putting now $F(r) = (M^r f)(x)$ we have the following result.

Lemma 1.6. *Let m be an integer $0 < m < n = \dim X$. Then*

$$\int_0^\infty \sinh^m r L_r F\, dr =$$
$$(m+1-n)\left[m \int_0^\infty \sinh^m r F(r)\, dr + (m-1)\int_0^\infty \sinh^{m-2} r F(r)\, dr\right].$$

If $m = 1$ the term $(m-1)\int_0^\infty \sinh^{m-2} r F(r)\, dr$ should be replaced by $F(0)$.

This follows by repeated integration by parts.

From this lemma combined with the Darboux equation (26) in the form

$$L_x(M^r f(x)) = L_r(M^r f(x)) \tag{30}$$

we deduce

$$[L_x + m(n-m-1)] \int_0^\infty \sinh^m r(M^r f)(x)\, dr$$
$$= -(n-m-1)(m-1)\int_0^\infty \sinh^{m-2} r(M^r f)(p)\, dr\,.$$

Applying this repeatedly to (29) we obtain Theorem 1.5.

B. The Spheres and the Elliptic Spaces

Now let X be the unit sphere $\mathbf{S}^n(0) \subset \mathbf{R}^{n+1}$ and Ξ the set of k-dimensional totally geodesic submanifolds of X. Each $\xi \in \Xi$ is a k-sphere. We shall now invert the Radon transform

$$\widehat{f}(\xi) = \int_\xi f(x)\, dm(x)\,, \quad f \in \mathcal{E}(X)$$

where dm is the measure on ξ given by the Riemannian structure induced by that of X. In contrast to the hyperbolic space, each geodesic X through a point x also passes through the antipodal point A_x. As a result, $\widehat{f} = (f \circ A)\widehat{}$ and our inversion formula will reflect this fact. Although we state our result for the sphere, it is really a result for the *elliptic space*, that is the sphere with antipodal points identified. The functions on this space are naturally identified with symmetric functions on the sphere.

Again let

$$\check{\varphi}(x) = \int_{x \in \xi} \varphi(\xi)\, d\mu(\xi)$$

denote the average of a continuous function on Ξ over the set of ξ passing through x.

Theorem 1.7. *Let k be an integer, $1 \leq k < n = \dim X$.*

(i) The mapping $f \to \widehat{f}$ $(f \in \mathcal{E}(X))$ has kernel consisting of the skew function (the functions f satisfying $f + f \circ A = 0$).

(ii) Assume k even and let P_k denote the polynomial

$$P_k(z) = [z-(k-1)(n-k)][z-(k-3)(n-k+2)]\dots[z-1(n-2)]$$

of degree $k/2$. The k-dimensional Radon transform on X is then inverted by the formula

$$c(f + f \circ A) = P_k(L)((\widehat{f})^\vee)\,, \quad f \in \mathcal{E}(X)$$

where c is the constant in (27).

Proof. We first prove (ii) in a similar way as in the noncompact case. The Riemannian structure in (3) is now replaced by

$$ds^2 = dr^2 + \sin^2 r\, d\sigma^2\,;$$

the Laplace-Beltrami operator is now given by

$$L = \frac{\partial^2}{\partial r^2} + (n-1)\cot r \frac{\partial}{\partial r} + (\sin r)^{-2} L_S \tag{31}$$

instead of (23) and

$$(\widehat{f})^\vee(x) = \Omega_k \int_0^\pi (M^r f)(x) \sin^{k-1} r\, dr\,.$$

For a fixed x we put $F(r) = (M^r f)(x)$. The analog of Lemma 1.6 now reads as follows.

Lemma 1.8. *Let m be an integer, $0 < m < n = \dim X$. Then*

$$\int_0^\pi \sin^m r L_r F\, dr =$$
$$(n-m-1)\left[m\int_0^\pi \sin^m rF(r)\,dr - (m-1)\int_0^\pi \sin^{m-2} rF(r)\,dr\right].$$

If $m = 1$, the term $(m-1)\int_0^\pi \sin^{m-2} rF(r)\,dr$ should be replaced by $F(o) + F(\pi)$.

Since (30) is still valid the lemma implies

$$\begin{aligned}\big[L_x - m(n-m-1)\big]&\int_0^\pi \sin^m r(M^r f)(x)\,dr \\ = \quad -(n-m-1)(m-1)&\int_0^\pi \sin^{m-2} r(M^r f)(x)\,dr\end{aligned}$$

and the desired inversion formula follows by iteration since

$$F(0) + F(\pi) = f(x) + f(Ax)\,.$$

In the case when k is even, Part (i) follows from (ii). Next suppose $k = n-1$, n even. For each ξ there are exactly two points x and Ax at maximum distance, namely $\frac{\pi}{2}$, from ξ and we write

$$\widehat{f}(x) = \widehat{f}(Ax) = \widehat{f}(\xi)\,.$$

We have then

$$\widehat{f}(x) = \Omega_n(M^{\frac{\pi}{2}} f)(x)\,. \tag{32}$$

Next we recall some well-known facts about spherical harmonics. We have

$$L^2(X) = \sum_0^\infty \mathcal{H}_s\,, \tag{33}$$

where the space $\mathcal{H}_s$ consist of the restrictions to X of the homogeneous harmonic polynomials on $\mathbf{R}^{n+1}$ of degree s.

(a) $Lh_s = -s(s+n-1)h_s \quad (h_s \in \mathcal{H}_s)$ for each $s \geq 0$. This is immediate from the decomposition

$$L_{n+1} = \frac{\partial^2}{\partial r^2} + \frac{n}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}L$$

of the Laplacian L_{n+1} of $\mathbf{R}^{n+1}$ (cf. (23)). Thus the spaces $\mathcal{H}_s$ are precisely the eigenspaces of L.

(b) Each $\mathcal{H}_s$ contains a function ($\not\equiv 0$) which is invariant under the group K of rotations around the vertical axis (the x_{n+1}-axis in $\mathbf{R}^{n+1}$). This function φ_s is nonzero at the North Pole o and is uniquely determined by the condition $\varphi_s(o) = 1$. This is easily seen since by (31) φ_s satisfies the ordinary differential equation

$$\frac{d^2\varphi_s}{dr^2} + (n-1)\cot r\frac{d\varphi_s}{dr} = -s(s+n-1)\varphi_s\,, \quad \varphi_s'(o) = 0\,.$$

It follows that $\mathcal{H}_s$ is irreducible under the orthogonal group $\mathbf{O}(n+1)$.

(c) Since the mean value operator $M^{\pi/2}$ commutes with the action of $\mathbf{O}(n+1)$ it acts as a scalar c_s on the irreducible space $\mathcal{H}_s$. Since we have

$$M^{\pi/2}\varphi_s = c_s\varphi_s\,, \quad \varphi_s(o) = 1\,,$$

we obtain

$$c_s = \varphi_s\left(\tfrac{\pi}{2}\right). \tag{34}$$

Lemma 1.9. *The scalar $\varphi_s(\pi/2)$ is zero if and only if s is odd.*

Proof. Let H_s be the K-invariant homogeneous harmonic polynomial whose restriction to X equals φ_s. Then H_s is a polynomial in $x_1^2 + \cdots + x_n^2$ and x_{n+1} so if the degree s is odd, x_{n+1} occurs in each term whence $\varphi_s(\pi/2) = H_s(1, 0, \ldots 0, 0) = 0$. If s is even, say $s = 2d$, we write

$$H_s = a_0(x_1^2 + \cdots + x_n^2)^d + a_1 x_{n+1}^2(x_1^2 + \cdots + x_n^2)^{d-1} + \cdots + a_d x_{n+1}^{2d}\,.$$

Using $L_{n+1} = L_n + \partial^2/\partial x_{n+1}^2$ and formula (31) in Ch. I the equation $L_{n+1}H_s \equiv 0$ gives the recursion formula

$$a_i(2d-2i)(2d-2i+n-2) + a_{i+1}(2i+2)(2i+1) = 0$$

($0 \leq i < d$). Hence $H_s(1, 0 \ldots 0)$, which equals a_0, is $\neq 0$; Q.e.d.

Now each $f \in \mathcal{E}(X)$ has a uniformly convergent expansion

$$f = \sum_0^\infty h_s \quad (h_s \in \mathcal{H}_s)$$

and by (32)

$$\widehat{f} = \Omega_n \sum_0^\infty c_s h_s\,.$$

If $\widehat{f} = 0$ then by Lemma 1.9, $h_s = 0$ for s even so f is skew. Conversely $\widehat{f} = 0$ if f is skew so Theorem 1.7 is proved for the case $k = n-1$, n even.

If k is odd, $0 < k < n-1$, the proof just carried out shows that $\widehat{f}(\xi)=0$ for all $\xi \in \Xi$ implies that f has integral 0 over every $(k+1)$-dimensional sphere with radius 1 and center o. Since $k+1$ is even and $< n$ we conclude by (ii) that $f + f \circ A = 0$ so the theorem is proved.

As a supplement to Theorems 1.5 and 1.7 we shall now prove an inversion formula for the Radon transform for general k (odd or even).

Let X be either the hyperbolic space $\mathbf{H}^n$ or the sphere $\mathbf{S}^n$ and Ξ the space of totally geodesic submanifolds of X of dimension k $(1 \leq k \leq n-1)$. We then generalize the transforms $f \to \widehat{f}, \varphi \to \check{\varphi}$ as follows. Let $p \geq 0$. We put

$$\widehat{f}_p(\xi) = \int_{d(x,\xi)=p} f(x)\, dm(x)\,, \quad \check{\varphi}_p(x) = \int_{d(x,\xi)=p} \varphi(\xi)\, d\mu(\xi)\,, \tag{35}$$

where dm is the Riemannian measure on the set in question and $d\mu$ is the average over the set of ξ at distance p from x. Let ξ_p be a fixed element of Ξ at distance p from 0 and H_p the subgroup of G leaving ξ_p stable. It is then easy to see that in the language of Ch. II, §1

$$x = gK\,, \quad \xi = \gamma H_p \text{ are incident } \Leftrightarrow d(x,\xi) = p\,. \tag{36}$$

This means that the transforms (35) are the Radon transform and its dual for the double fibration

$$\begin{tikzcd} & G/(K \cap H_p) \arrow[dl] \arrow[dr] & \\ G/K & & G/H_p \end{tikzcd}$$

For $X = \mathbf{S}^2$ the set $\{x : d(x,\xi) = p\}$ is two circles on $\mathbf{S}^2$ of length $2\pi \cos p$. For $X = \mathbf{H}^2$, the non-Euclidean disk, ξ a diameter, the set $\{x : d(x,\xi) = p\}$ is a pair of circular arcs with the same endpoints as ξ. Of course $\widehat{f}_0 = \widehat{f}\,, \quad \check{\varphi}_0 = \check{\varphi}$.

We shall now invert the transform $f \to \widehat{f}$ by invoking the more general transform $\varphi \to \check{\varphi}_p$. Consider $x \in X, \xi \in \Xi$ with $d(x,\xi) = p$. Select $g \in G$ such that $g \cdot o = x$. Then $d(o, g^{-1}\xi) = p$ so $\{kg^{-1} \cdot \xi : k \in K\}$ is the set of $\eta \in \Xi$ at distance p from o and $\{gkg^{-1} \cdot \xi : k \in K\}$ is the set of $\eta \in \Xi$ at distance p from x. Hence

$$\begin{aligned} (\widehat{f})^\vee_p(g \cdot o) &= \int_K \widehat{f}(gkg^{-1} \cdot \xi)\, dk = \int_K dk \int_\xi f(gkg^{-1} \cdot y)\, dm(y) \\ &= \int_\xi \left(\int_K f(gkg^{-1} \cdot y)\, dk \right) dm(y) \end{aligned}$$

so

$$(\widehat{f})^\vee_p(x) = \int_\xi (M^{d(x,y)} f)(x)\, dm(y)\,. \tag{37}$$

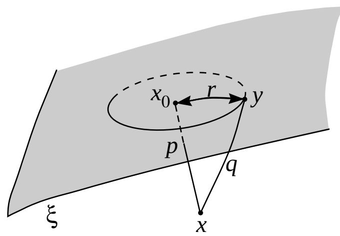

FIGURE III.2.

Let $x_0 \in \xi$ be a point at minimum distance (i.e., p) from x and let (Fig. III.2)

(38) $$r = d(x_0, y)\,, \quad q = d(x, y)\,, \quad y \in \xi\,.$$

Since $\xi \subset X$ is totally geodesic, $d(x_o, y)$ is also the distance between x_o and y in ξ. In (37) the integrand $(M^{d(x,y)}f)(x)$ is constant in y on each sphere in ξ with center x_o.

Theorem 1.10. *The k-dimensional totally geodesic Radon transform $f \to \widehat{f}$ on the hyperbolic space $\mathbf{H}^n$ is inverted by*

$$f(x) = c\left[\left(\frac{d}{d(u^2)}\right)^k \int_0^u (\widehat{f})^\vee_{\text{lm } v}(x)(u^2 - v^2)^{\frac{k}{2}-1}\, dv\right]_{u=1},$$

where $c^{-1} = (k-1)!\Omega_{k+1}/2^{k+1}$, $\text{lm } v = \cosh^{-1}(v^{-1})$.

Proof. Applying geodesic polar coordinates in ξ with center x_0 we obtain from (37)–(38),

(39) $$(\widehat{f})^\vee_p(x) = \Omega_k \int_0^\infty (M^q f)(x) \sinh^{k-1} r\, dr\,.$$

Using the cosine relation on the right-angled triangle (xx_0y) we have by (38) and $d(x_0, x) = p$,

(40) $$\cosh q = \cosh p \cosh r\,.$$

With x fixed we define F and $\widehat{F}$ by

(41) $$F(\cosh q) = (M^q f)(x)\,, \quad \widehat{F}(\cosh p) = (\widehat{f})^\vee_p(x)\,.$$

Then by (39),

(42) $$\widehat{F}(\cosh p) = \Omega_k \int_0^\infty F(\cosh p \cosh r) \sinh^{k-1} r\, dr\,.$$

Putting here $t = \cosh p$, $s = \cosh r$ this becomes

$$\widehat{F}(t) = \Omega_k \int_1^\infty F(ts)(s^2 - 1)^{\frac{k}{2}-1}\, ds\,,$$

which by substituting $u = (ts)^{-1}$, $v = t^{-1}$ becomes

$$v^{-1}\widehat{F}(v^{-1}) = \Omega_k \int_0^v F(u^{-1})u^{-k}(v^2 - u^2)^{\frac{k}{2}-1}\, du\,.$$

This is of the form (19), Ch. I, §2 and is inverted by

$$F(u^{-1})u^{-k} = cu\left(\frac{d}{d(u^2)}\right)^k \int_0^u (u^2 - v^2)^{\frac{k}{2}-1}\widehat{F}(v^{-1})\,dv\,, \tag{43}$$

where $c^{-1} = (k-1)!\,\Omega_{k+1}/2^{k+1}$. Defining $\operatorname{lm} v$ by $\cosh(\operatorname{lm} v) = v^{-1}$ and noting that $f(x) = F(\cosh 0)$ the theorem follows by putting $u = 1$ in (43).

For the sphere $X = \mathbf{S}^n$ we can proceed in a similar fashion. We assume f symmetric $(f(s) \equiv f(-s))$ because $\widehat{f} \equiv 0$ for f odd. Now formula (37) takes the form

$$(\widehat{f})_p^\vee(x) = 2\Omega_k \int_0^{\frac{\pi}{2}} (M^q f)(x) \sin^{k-1} r\, dr\,, \tag{44}$$

(the factor 2 and the limit $\pi/2$ coming from the symmetry assumption). This time we use spherical trigonometry on the triangle (xx_0y) to derive

$$\cos q = \cos p \cos r\,.$$

We fix x and put

$$F(\cos q) = (M^q f)(x)\,, \quad \widehat{F}(\cos p) = (\widehat{f})_p^\vee(x)\,. \tag{45}$$

and

$$v = \cos p\,, \quad u = v\cos r\,.$$

Then (44) becomes

$$v^{k-1}\widehat{F}(v) = 2\Omega_k \int_0^v F(u)(v^2 - u^2)^{\frac{k}{2}-1}\,du\,, \tag{46}$$

which is inverted by

$$F(u) = \frac{c}{2}u\left(\frac{d}{d(u^2)}\right)^k \int_0^u (u^2 - v^2)^{\frac{k}{2}-1} v^k \widehat{F}(v)\,dv\,,$$

c being as before. Since $F(1) = f(x)$ this proves the following analog of Theorem 1.10.

Theorem 1.11. *The k-dimensional totally geodesic Radon transform $f \to \widehat{f}$ on $\mathbf{S}^n$ is for f symmetric inverted by*

$$f(x) = \frac{c}{2}\left[\left(\frac{d}{d(u^2)}\right)^k \int_0^u (\widehat{f})^\vee_{\cos^{-1}(v)}(x) v^k (u^2 - v^2)^{\frac{k}{2}-1}\,dv\right]_{u=1}$$

where

$$c^{-1} = (k-1)!\Omega_{k+1}/2^{k+1}\,.$$

Geometric interpretation

In Theorems 1.10–1.11, $(\widehat{f})^\vee_p(x)$ is the average of the integrals of f over the k-dimensional totally geodesic submanifolds of X which have distance p from x.

We shall now look a bit closer at the geometrically interesting case $k = 1$. Here the transform $f \to \widehat{f}$ is called the *X-ray transform*.

We first recall a few facts about the *spherical transform* on the constant curvature space $X = G/K$, that is the hyperbolic space $\mathbf{H}^n = Q_-^+$ or the sphere $\mathbf{S}^n = Q_+$. A *spherical function* φ on G/K is by definition a K-invariant function which is an eigenfunction of the Laplacian L on X satisfying $\varphi(o) = 1$. Then the eigenspace of L containing φ consists of the functions f on X satisfying the functional equation

$$\int_K f(gk \cdot x)\, dk = f(g \cdot o)\varphi(x) \tag{47}$$

([GGA], p. 64). In particular, the spherical functions are characterized by

$$\int_K \varphi(gk \cdot x)\, dk = \varphi(g \cdot o)\varphi(x) \quad \varphi \not\equiv 0\,. \tag{48}$$

Consider now the case $\mathbf{H}^2$. Then the spherical functions are the solutions $\varphi_\lambda(r)$ of the differential equation

$$\frac{d^2\varphi_\lambda}{dr^2} + \coth r \frac{d\varphi_\lambda}{dr} = -(\lambda^2 + \tfrac{1}{4})\varphi_\lambda\,, \quad \varphi_\lambda(o) = 1\,. \tag{49}$$

Here $\lambda \in \mathbf{C}$ and $\varphi_{-\lambda} = \varphi_\lambda$. The function φ_λ has the integral representation

$$\varphi_\lambda(r) = \frac{1}{\pi}\int_0^\pi (\operatorname{ch} r - \operatorname{sh} r \cos\theta)^{-i\lambda + \frac{1}{2}}\, d\theta\,. \tag{50}$$

In fact, already the integrand is easily seen to be an eigenfunction of the operator L in (23) (for $n = 2$) with eigenvalue $-(\lambda^2 + 1/4)$.

If f is a radial function on X its *spherical transform* $\widetilde{f}$ is defined by

$$\widetilde{f}(\lambda) = \int_X f(x)\varphi_{-\lambda}(x)\, dx \tag{51}$$

for all $\lambda \in \mathbf{C}$ for which this integral exists. The continuous radial functions on X form a commutative algebra $C_c^\natural(X)$ under convolution

$$(f_1 \times f_2)(g \cdot o) = \int_G f_1(gh^{-1} \cdot o) f_2(h \cdot o)\, dh \tag{52}$$

and as a consequence of (48) we have

$$(f_1 \times f_2)^\sim(\lambda) = \widetilde{f_1}(\lambda)\widetilde{f_2}(\lambda)\,. \tag{53}$$

In fact,

$$\begin{aligned}(f_1 \times f_2)^{\sim}(\lambda) &= \int_G f_1(h\cdot o)\left(\int_G f_2(g\cdot o)\varphi_{-\lambda}(hg\cdot o)\,dg\right)dh\\ &= \int_G f_1(h\cdot o)\left(\int_G f_2(g\cdot o)\right)\left(\int_K \varphi_{-\lambda}(hkg\cdot o)\,dk\,dg\right)dh\\ &= \widetilde{f_1}(\lambda)\widetilde{f_2}(\lambda)\,.\end{aligned}$$

We know already from Corollary 1.3 that the Radon transform on $\mathbf{H}^n$ is injective and is inverted in Theorem 1.5 and Theorem 1.10. For the case $n = 2, k = 1$ we shall now obtain another inversion formula based on (53).

The spherical function $\varphi_\lambda(r)$ in (50) is the classical Legendre function $P_v(\cosh r)$ with $v = i\lambda - \frac{1}{2}$ for which we shall need the following result ([Prudnikov, Brychkov and Marichev], Vol. III, 2.17.8(2)).

Lemma 1.12.

$$2\pi \int_0^\infty e^{-pr} P_v(\cosh r)\,dr = \pi \frac{\Gamma(\frac{p-v}{2})\Gamma(\frac{p+v+1}{2})}{\Gamma(1+\frac{p+v}{2})\Gamma(\frac{1+p-v}{2})}\,, \tag{54}$$

for

$$Re(p-v) > 0\,, \quad Re(p+v) > -1\,. \tag{55}$$

We shall require this result for $p = 0, 1$ and λ real. In both cases, conditions (55) are satisfied.

Let τ and σ denote the functions

$$\tau(x) = \sinh\, d(o,x)^{-1}\,, \quad \sigma(x) = \coth(d(o,x)) - 1\,, \quad x \in X\,. \tag{56}$$

Lemma 1.13. *For $f \in \mathcal{D}(X)$ we have*

$$(\widehat{f})^\vee(x) = \pi^{-1}(f \times \tau)(x)\,. \tag{57}$$

Proof. In fact, the right hand side is

$$\int_X \sinh\, d(x,y)^{-1} f(y)\,dy = \int_0^\infty dr(\sinh\, r)^{-1} \int_{S_r(x)} f(y)\,dw(y)$$

so the lemma follows from (28).

Similarly we have

$$Sf = f \times \sigma\,, \tag{58}$$

where S is the operator

$$(Sf)(x) = \int_X (\coth(d(x,y)) - 1) f(y)\,dy\,. \tag{59}$$

Theorem 1.14. *The operator $f \to \widehat{f}$ is inverted by*

$$LS((\widehat{f})^{\vee}) = -4\pi f\,, \quad f \in \mathcal{D}(X)\,. \tag{60}$$

Proof. The operators $\widehat{}\,$, $^{\vee}$, S and L are all G-invariant so it suffices to verify (60) at o. Let $f^{\natural}(x) = \int_K f(k \cdot x)\, dk$. Then

$$(f \times \tau)^{\natural} = f^{\natural} \times \tau\,, \ \ (f \times \sigma)^{\natural} = f^{\natural} \times \sigma\,, \ \ (Lf)(o) = (Lf^{\natural})(o)\,.$$

Thus by (57)–(58)

$$\begin{aligned} LS((\widehat{f})^{\vee})(o) &= L(S((\widehat{f})^{\vee}))^{\natural}(o) = \pi^{-1} L(f \times \tau \times \sigma)^{\natural}(o) \\ &= LS(((f^{\natural})\widehat{}\,)^{\vee})\,)(o)\,. \end{aligned}$$

Now, if (60) is proved for a radial function this equals $cf^{\natural}(o) = cf(o)$. Thus (60) would hold in general. Consequently, it suffices to prove

$$L(f \times \tau \times \sigma) = -4\pi^2 f\,, \quad f \text{ radial in } \mathcal{D}(X)\,. \tag{61}$$

Since f, $\tau\varphi_\lambda$ (λ real) and σ are all integrable on X, we have by the proof of (53)

$$(f \times \tau \times \sigma)^{\sim}(\lambda) = \widetilde{f}(\lambda)\widetilde{\tau}(\lambda)\widetilde{\sigma}(\lambda)\,. \tag{62}$$

Since $\coth r - 1 = e^{-r}/\sinh r$, and since $dx = \sinh r\, dr\, d\theta$, $\widetilde{\tau}(\lambda)$ and $\widetilde{\sigma}(\lambda)$ are given by the left hand side of (54) for $p = 0$ and $p = 1$, respectively. Thus

$$\begin{aligned} \widetilde{\tau}(\lambda) &= \pi \frac{\Gamma(\frac{1}{4} - \frac{i\lambda}{2})\Gamma(\frac{i\lambda}{2} + \frac{1}{4})}{\Gamma(\frac{i\lambda}{2} + \frac{3}{4})\Gamma(\frac{3}{4} - \frac{i\lambda}{2})}\,, \\ \widetilde{\sigma}(\lambda) &= \pi \frac{\Gamma(\frac{3}{4} - \frac{i\lambda}{2})\Gamma(\frac{i\lambda}{2} + \frac{3}{4})}{\Gamma(\frac{i\lambda}{2} + \frac{5}{4})\Gamma(\frac{5}{4} - \frac{i\lambda}{2})}\,. \end{aligned}$$

Using the identity $\Gamma(x+1) = x\Gamma(x)$ on the denominator of $\widetilde{\sigma}(\lambda)$ we see that

$$\widetilde{\tau}(\lambda)\widetilde{\sigma}(\lambda) = 4\pi^2(\lambda^2 + \tfrac{1}{4})^{-1}\,. \tag{63}$$

Now

$$L(f \times \tau \times \sigma) = (Lf \times \tau \times \sigma)\,, \quad f \in \mathcal{D}^{\natural}(X)\,,$$

and by (49), $(Lf)^{\sim}(\lambda) = -(\lambda^2 + \frac{1}{4})\widetilde{f}(\lambda)$. Using the decomposition $\tau = \varphi\tau + (1-\varphi)\tau$ where φ is the characteristic function of a ball $B(0)$ we see that $f \times \tau \in L^2(X)$ for $f \in \mathcal{D}^{\sharp}(X)$. Since $\sigma \in L'(X)$ we have $f \times \tau \times \sigma \in L^2(X)$. By the Plancherel theorem, the spherical transform is injective on $L^2(X)$ so we deduce from (62)–(63) that (60) holds with the constant $-4\pi^2$.

It is easy to write down an analog of (60) for $\mathbf{S}^2$. Let o denote the North Pole and put

$$\tau(x) = \sin d(o,x)^{-1} \quad x \in \mathbf{S}^2\,.$$

Then in analogy with (57) we have

$$(\widehat{f})^\vee(x) = \pi^{-1}(f \times \tau)(x)\,, \tag{64}$$

where $\times$ denotes the convolution on $\mathbf{S}^2$ induced by the convolution on G. The spherical functions on G/K are the functions

$$\varphi_n(x) = P_n(\cos\, d(o,x)) \quad n \geq 0\,,$$

where P_n is the Legendre polynomial

$$P_n(\cos\theta) = \frac{1}{2\pi}\int_0^{2\pi}(\cos\theta + i\sin\theta\cos u)^n\, du\,.$$

Since $P_n(\cos(\pi-\theta)) = (-1)^n P_n(\cos\theta)$, the expansion of τ into spherical functions

$$\tau(x) \sim \sum_{n=0}^{\infty}(4n+1)\widetilde{\tau}(2n)P_{2n}(\cos\, d(o,x))$$

only involves even indices. The factor $(4n+1)$ is the dimension of the space of spherical harmonics containing φ_{2n}. Here the Fourier coefficient $\widetilde{\tau}(2n)$ is given by

$$\widetilde{\tau}(2n) = \frac{1}{4\pi}\int_{\mathbf{S}^2}\tau(x)\varphi_{2n}(x)\,dx\,,$$

which, since $dx = \sin\theta\, d\theta\, d\varphi$, equals

$$\frac{1}{4\pi}2\pi\int_0^{\pi}P_{2n}(\cos\theta)\,d\theta = \frac{\pi}{2^{4n-1}}\binom{2n}{n}^2\,, \tag{65}$$

by *loc. cit.*, Vol. 2, 2.17.6 (11). We now define the functional σ on $\mathbf{S}^2$ by the formula

$$\sigma(x) = \sum_{0}^{\infty}(4n+1)a_{2n}P_{2n}(\cos\, d(o,x))\,, \tag{66}$$

where

$$a_{2n} = \frac{2^{4n}\pi}{\binom{2n}{n}^2 n(2n+1)}\,. \tag{67}$$

To see that (66) is well-defined note that

$$\begin{aligned}\binom{2n}{n} &= 2^n 1\cdot 3\cdots(2n-1)/n! \geq 2^n 1\cdot 2\cdot 4\cdots(2n-2)/n!\\ &\geq 2^{2n-1}/n\end{aligned}$$

so a_{2n} is bounded in n. Thus σ is a distribution on $\mathbf{S}^2$. Let S be the operator

$$Sf = f \times \sigma \,. \tag{68}$$

Theorem 1.15. *The operator $f \to \widehat{f}$ is inverted by*

$$LS((\widehat{f})^\vee) = -4\pi f \,. \tag{69}$$

Proof. Just as is the case with Theorem 1.14 it suffices to prove this for f K-invariant and there it is a matter of checking that the spherical transforms on both sides agree. For this we use (64) and the relation

$$L\varphi_{2n} = -2n(2n+1)\varphi_{2n} \,.$$

Since

$$(\tau \times \sigma)^\sim(2n) = \widetilde{\tau}(2n)a_{2n} \,.$$

the identity (69) follows.

A drawback of (69) is of course that (66) is not given in closed form. We shall now invert $f \to \widehat{f}$ in a different fashion on $\mathbf{S}^2$. Consider the spherical coordinates of a point $(x_1, x_2, x_3) \in \mathbf{S}^2$.

$$x_1 = \cos\varphi \sin\theta \,, \quad x_2 = \sin\varphi \sin\theta \,, \quad x_3 = \cos\theta \tag{70}$$

and let $k_\varphi = K$ denote the rotation by the angle φ around the x_3-axis. Then f has a Fourier expansion

$$f(x) = \sum_{n \in \mathbb{Z}} f_n(x) \,, \quad f_n(x) = \frac{1}{2\pi} \int_0^{2\pi} f(k_\varphi \cdot x) e^{-in\varphi} \, d\varphi \,.$$

Then

$$f_n(k_\varphi \cdot x) = e^{in\varphi} f_n(x) \,, \quad \widehat{f}_n(k_\varphi \cdot \gamma) = e^{in\varphi} \widehat{f}_n(\gamma) \tag{71}$$

for each great circle γ. In particular, f_n is determined by its restriction $g = f_n|_{x_1=0}$, i.e.,

$$g(\cos\theta) = f_n(0, \sin\theta, \cos\theta) \,.$$

Since f_n is even, (70) implies $g(\cos(\pi - \theta)) = (-1)^n g(\cos\theta)$, so

$$g(-u) = (-1)^n g(u) \,.$$

Let Γ be the set of great circles whose normal lies in the plane $x_1 = 0$. If $\gamma \in \Gamma$ let x_γ be the intersection of γ with the half-plane $x_1 = 0, x_2 > 0$ and let α be the angle from o to x_γ, (Fig. III.3). Since f_n is symmetric,

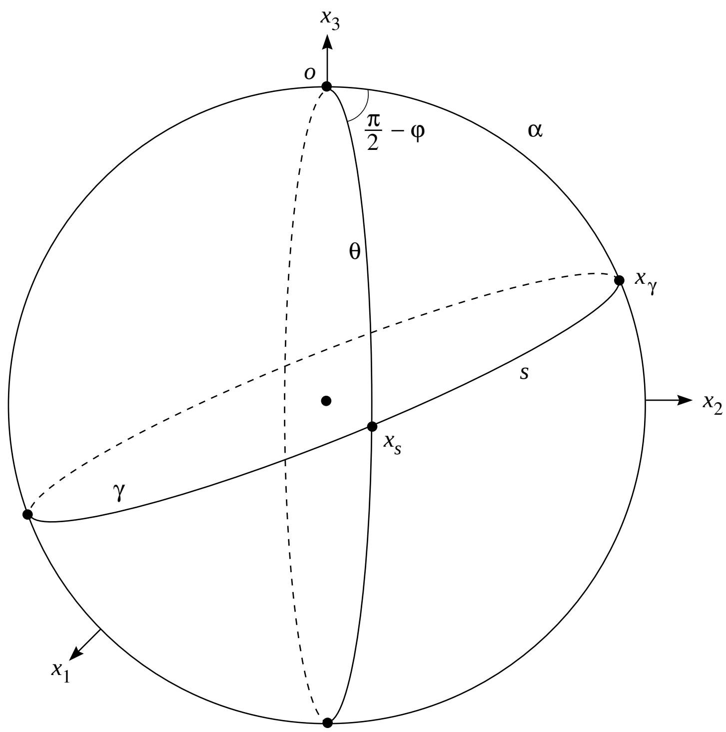

FIGURE III.3.

$$\widehat{f}_n(\gamma) = 2\int_0^{\pi} f_n(x_s)\, ds\,, \tag{72}$$

where x_s is the point on γ at distance s from x_γ (with $x_1(x_s) \geq 0$). Let φ and θ be the coordinates (70) of x_s. Considering the right angled triangle $x_s o x_\gamma$ we have

$$\cos\theta = \cos s\, \cos\alpha$$

and since the angle at o equals $\pi/2 - \varphi$, (71) implies

$$g(\cos\alpha) = f_n(x_\gamma) = e^{in(\pi/2-\varphi)} f_n(x_s)\,.$$

Writing

$$\widehat{g}(\cos\alpha) = \widehat{f}_n(\gamma) \tag{73}$$

equation (72) thus becomes

$$\widehat{g}(\cos\alpha) = 2(-i)^n \int_0^{\pi} e^{in\varphi} g(\cos\theta)\, ds\,.$$

Put $v = \cos\alpha\,, \quad u - v\cos s$, so

$$du = v(-\sin s)\, ds = -(v^2 - u^2)^{1/2}\, ds\,.$$

Then

$$\widehat{g}(v) = 2(-i)^n \int_{-v}^{v} e^{in\varphi(u,v)} g(u)(v^2 - u^2)^{-\frac{1}{2}}\, du\,, \tag{74}$$

where the dependence of φ on u and v is indicated (for $v \neq 0$).

Now $-u = v\cos(\pi - s)$ so by the geometry, $\varphi(-u, v) = -\varphi(u, v)$. Thus (74) splits into two Abel-type Volterra equations

$$\widehat{g}(v) \;=\; 4(-1)^{n/2} \int_0^v \cos(n\varphi(u,v)) g(u)(v^2 - u^2)^{-\frac{1}{2}}\, du\,, \quad n \text{ even} \tag{75}$$

$$\widehat{g}(v) \;=\; 4(-1)^{(n-1)/2} \int_0^v \sin(n\varphi(u,v)) g(u)(v^2 - u^2)^{-\frac{1}{2}}\, du\,, \quad n \text{ odd}\,. \tag{76}$$

For $n = 0$ we derive the following result from (43) and (75).

Proposition 1.16. *Let $f \in C^2(\mathbf{S}^2)$ be symmetric and K-invariant and $\widehat{f}$ its X-ray transform. Then the restriction $g(\cos d(o, x)) = f(x)$ and the function $\widehat{g}(\cos d(o, \gamma)) = \widehat{f}(\gamma)$ are related by*

$$\widehat{g}(v) = 4 \int_0^v g(u)(v^2 - u^2)^{-\frac{1}{2}}\, du \tag{77}$$

and its inversion

$$2\pi g(u) = \frac{d}{du} \int_0^u \widehat{g}(v)(u^2 - v^2)^{-\frac{1}{2}}\, v dv\,. \tag{78}$$

We shall now discuss the analog for $\mathbf{S}^n$ of the support theorem (Theorem 1.2) relative to the X-ray transform $f \to \widehat{f}$.

Theorem 1.17. *Let C be a closed spherical cap on $\mathbf{S}^n$, C' the cap on $\mathbf{S}^n$ symmetric to C with respect to the origin $0 \in \mathbf{R}^{n+1}$. Let $f \in C(\mathbf{S}^n)$ be symmetric and assume*

$$\widehat{f}(\gamma) = 0 \tag{79}$$

for every geodesic γ which does not enter the "arctic zones" C and C'. (See Fig. II.3.)

(i) If $n \geq 3$ then $f \equiv 0$ outside $C \cup C'$.

(ii) If $n = 2$ the same conclusion holds if all derivatives of f vanish on the equator.

Proof. (i) Given a point $x \in \mathbf{S}^n$ outside $C \cup C'$ we can find a 3-dimensional subspace ξ of $\mathbf{R}^{n+1}$ which contains x but does not intersect $C \cup C'$. Then $\xi \cap \mathbf{S}^n$ is a 2-sphere and f has integral 0 over each great circle on it. By Theorem 1.7, $f \equiv 0$ on $\xi \cap \mathbf{S}^n$ so $f(x) = 0$.

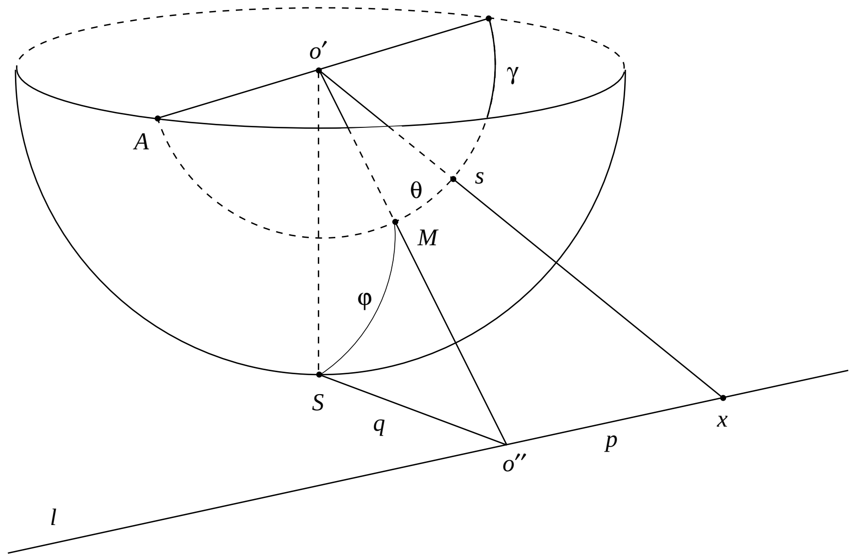

FIGURE III.4.

(ii) If f is K-invariant our statement follows quickly from Proposition 1.16. In fact, if C has spherical radius β, (79) implies $\widehat{g}(v) = 0$ for $0 < v < \cos\beta$ so by (78) $g(u) = 0$ for $0 < u < \cos\beta$ so $f \equiv 0$ outside $C \cup C'$.

Generalizing this method to f_n in (71) by use of (75)–(76) runs into difficulties because of the complexity of the kernel $e^{in\varphi(u,v)}$ in (74) near $v = 0$. However, if f is assumed $\equiv 0$ in a belt around the equator the theory of the Abel-type Volterra equations used on (75)–(76) does give the conclusion of (ii). The reduction to the K-invariant case which worked very well in the proof of Theorem 1.2 does not apply in the present compact case.

A better method, due to Kurusa, is to consider only the lower hemisphere $\mathbf{S}^2_-$ of the unit sphere and its tangent plane π at the South Pole S. The central projection μ from the origin is a bijection of $\mathbf{S}^2_-$ onto π which intertwines the two Radon transforms as follows: If γ is a (half) great circle on $\mathbf{S}^2_-$ and ℓ the line $\mu(\gamma)$ in π we have (Fig. III.4)

$$\cos d(S,\gamma)\widehat{f}(\gamma) = 2\int_\ell (f\circ\mu^{-1})(x)(1+|x|^2)^{-1}\,dm(x)\,. \tag{80}$$

The proof follows by elementary geometry: Let on Fig. III.4, $x = \mu(s)$, φ and θ the lengths of the arcs SM, Ms. The plane $o'So''$ is perpendicular to ℓ and intersects the semi-great circle γ in M. If $q = |So''|, p = |o''x|$ we have for $f \in C(\mathbf{S}^2)$ symmetric,

$$\widehat{f}(\gamma) = 2\int_\gamma f(s)d\theta = 2\int_\ell (f\circ\mu^{-1})(x)\frac{d\theta}{dp}\,dp\,.$$

Now

$$\tan\varphi = q\,, \quad \tan\theta = \frac{p}{(1+q^2)^{1/2}}\,, \quad |x|^2 = p^2 + q^2\,.$$

so

$$\frac{dp}{d\theta} = (1+q^2)^{1/2}(1+\tan^2\theta) = (1+|x|^2)/(1+q^2)^{1/2}\,.$$

Thus

$$\frac{dp}{d\theta} = (1+|x|^2)\cos\varphi$$

and since $\varphi = d(S,\gamma)$ this proves (80). Considering the triangle $o'xS$ we obtain

$$|x| = \tan d(S,s)\,. \tag{81}$$

Thus the vanishing of all derivatives of f on the equator implies rapid decrease of $f \circ \mu^{-1}$ at ∞.

Now if $\varphi > \beta$ we have by assumption, $\widehat{f}(\gamma) = 0$ so by (80) and Theorem 2.6 in Chapter I,

$$(f \circ \mu^{-1})(x) = 0 \quad \text{for } |x| > \tan\beta\,,$$

whence by (81),

$$f(s) = 0 \quad \text{for } d(S,s) > \beta\,.$$

Remark 1.18. Because of the example in Remark 2.9 in Chapter I the vanishing condition in (ii) cannot be dropped.

There is a generalization of (80) to d-dimensional totally geodesic submanifolds of $\mathbf{S}^n$ as well as of $\mathbf{H}^n$ (Kurusa [1992], [1994], Berenstein-Casadio Tarabusi [1993]). This makes it possible to transfer the range characterizations of the d-plane Radon transform in $\mathbf{R}^n$ (Chapter I, §6) to the d-dimensional totally geodesic Radon transform in $\mathbf{H}^n$. In addition to the above references see also Berenstein-Casadio Tarabusi-Kurusa [1997], Gindikin [1995] and Ishikawa [1997].

C. The Spherical Slice Transform

We shall now briefly consider a variation on the Funk transform and consider integrations over circles on $\mathbf{S}^2$ passing through the North Pole. This Radon transform is given by $f \to \widehat{f}$ where f is a function on $\mathbf{S}^2$,

$$\widehat{f}(\gamma) = \int_\gamma f(s)\,dm(s)\,, \tag{82}$$

γ being a circle on $\mathbf{S}^2$ passing through N and dm the arc-element on γ.

It is easy to study this transform by relating it to the X-ray transform on $\mathbf{R}^2$ by means of stereographic projection from N.

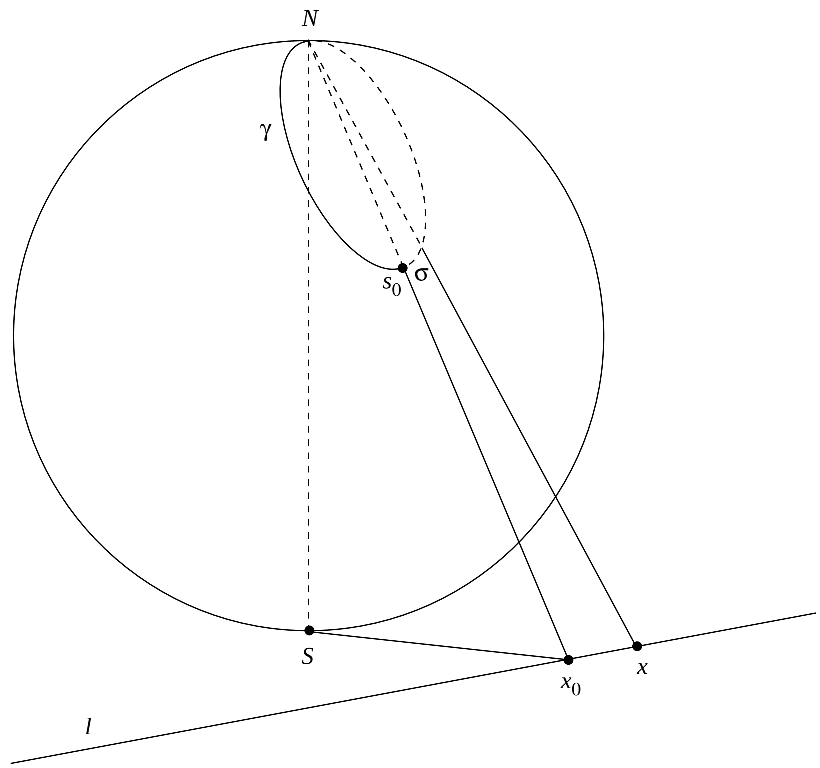

FIGURE III.5.

We consider a two-sphere $\mathbf{S}^2$ of *diameter* 1, lying on top of its tangent plane $\mathbf{R}^2$ at the South Pole. Let $\nu : \mathbf{S}^2 - N \to \mathbf{R}^2$ be the stereographic projection. The image $\nu(\gamma)$ is a line $\ell \subset \mathbf{R}^2$. (See Fig. III.5.) The plane through the diameter NS perpendicular to ℓ intersects γ in s_0 and ℓ in x_0. Then Ns_0 is a diameter in γ, and in the right angle triangle NSx_0, the line Ss_0 is perpendicular to Nx_0. Thus, d denoting the Euclidean distance in $\mathbf{R}^3$, and $q = d(S, x_0)$, we have

$$d(N, s_0) = (1 + q^2)^{-1/2}\,, \quad d(s_0, x_0) = q^2(1 + q^2)^{-1/2}\,. \tag{83}$$

Let σ denote the circular arc on γ for which $\nu(\sigma)$ is the segment (x_0, x) on ℓ. If θ is the angle between the lines Nx_0, Nx then

$$\sigma = (2\theta) \cdot \tfrac{1}{2}(1 + q^2)^{-1/2}\,, \quad d(x_0, x) = \tan\theta(1 + q^2)^{1/2}\,. \tag{84}$$

Thus, $dm(x)$ being the arc-element on ℓ,

$$\begin{aligned}\frac{dm(x)}{d\sigma} &= \frac{dm(x)}{d\theta}\cdot\frac{d\theta}{d\sigma} = (1+q^2)^{1/2}\cdot(1+\tan^2\theta)(1+q^2)^{1/2}\\ &= (1+q^2)\left(1+\frac{d(x_0,x)^2}{1+q^2}\right) = 1+|x|^2\,.\end{aligned}$$

Hence we have

$$\widehat{f}(\gamma) = \int_\ell (f\circ\nu^{-1})(x)(1+|x|^2)^{-1}\,dm(x)\,, \tag{85}$$

a formula quite similar to (80).

If f lies on $C^1(\mathbf{S}^2)$ and vanishes at N then $f\circ\nu^{-1} = 0(x^{-1})$ at ∞. Also of $f\in\mathcal{E}(\mathbf{S}^2)$ and all its derivatives vanish at N then $f\circ\nu^{-1}\in\mathcal{S}(\mathbf{R}^2)$. As in the case of Theorem 1.17 (ii) we can thus conclude the following corollaries of Theorem 3.1, Chapter I and Theorem 2.6, Chapter I.

Corollary 1.19. *The transform $f\to\widehat{f}$ is one-to-one on the space $C_0^1(\mathbf{S}^2)$ of C^1-functions vanishing at N.*

In fact, $(f\circ\nu^{-1})(x)/(1+|x|^2) = 0(|x|^{-3})$ so Theorem 3.1, Chapter I applies.

Corollary 1.20. *Let B be a spherical cap on $\mathbf{S}^2$ centered at N. Let $f\in\mathcal{C}^\infty(\mathbf{S}^2)$ have all its derivatives vanish at N. If*

$$\widehat{f}(\gamma) = 0 \textit{ for all } \gamma \textit{ through } N\,, \quad \gamma\subset B$$

then $f\equiv 0$ on B.

In fact $(f\circ\nu^{-1})(x) = 0(|x|^{-k})$ for each $k\geq 0$. The assumption on $\widehat{f}$ implies that $(f\circ\nu^{-1})(x)(1+|x|^2)^{-1}$ has line integral 0 for all lines outside $\nu(B)$ so by Theorem 2.6, Ch. I, $f\circ\nu^{-1}\equiv 0$ outside $\nu(B)$.

Remark 1.21. In Cor. 1.20 the condition of the vanishing of all derivatives at N cannot be dropped. This is clear from Remark 2.9 in Chapter I where the rapid decrease at ∞ was essential for the conclusion of Theorem 2.6.

If according to Remark 3.3, Ch. I $g\in\mathcal{E}(\mathbf{R}^2)$ is chosen such that $g(x) = 0(|x|^{-2})$ and all its line integrals are 0, the function f on $\mathbf{S}^2 - N$ defined by

$$(f\circ\nu^{-1})(x) = (1+|x|^2)g(x)$$

is bounded and by (85), $\widehat{f}(\gamma) = 0$ for all γ. This suggests, but does not prove, that the vanishing condition at N in Cor. 1.19 cannot be dropped.

§2 Compact Two-point Homogeneous Spaces. Applications

We shall now extend the inversion formula in Theorem 1.7 to compact two-point homogeneous spaces X of dimension $n > 1$. By virtue of Wang's classification [1952] these are also the compact symmetric spaces of rank one (see Matsumoto [1971] and Szabo [1991] for more direct proofs), so their geometry can be described very explicitly. Here we shall use some geometric and group theoretic properties of these spaces ((i)–(vii) below) and refer to Helgason ([1959], p. 278, [1965a], §5–6 or [DS], Ch. VII, §10) for their proofs.

Let U denote the group $I(X)$ of isometries X. Fix an origin $o \in X$ and let K denote the isotropy subgroup U_o. Let $\mathfrak{k}$ and $\mathfrak{u}$ be the Lie algebras of K and U, respectively. Then $\mathfrak{u}$ is semisimple. Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ and $\mathfrak{u}$ with respect to the Killing form B of $\mathfrak{u}$. Changing the distance function on X by a constant factor we may assume that the differential of the mapping $u \to u \cdot o$ of U onto X gives an isometry of $\mathfrak{p}$ (with the metric of $-B$) onto the tangent space X_o. This is the canonical metric X which we shall use.

Let L denote the diameter of X, that is the maximal distance between any two points. If $x \in X$ let A_x denote the set of points in X of distance L from x. By the two-point homogeneity the isotropy subgroup U_x acts transitively on A_x; thus $A_x \subset X$ is a submanifold, *the antipodal manifold* associated to x.

(i) *Each A_x is a totally geodesic submanifold of X; with the Riemannian structure induced by that of X it is another two-point homogeneous space.*

(ii) *Let Ξ denote the set of all antipodal manifolds in X; since U acts transitively on Ξ, the set Ξ has a natural manifold structure. Then the mapping $j : x \to A_x$ is a one-to-one diffeomorphism; also $x \in A_y$ if and only if $y \in A_x$.*

(iii) *Each geodesic in X has period $2L$. If $x \in X$ the mapping* $\mathrm{Exp}_x : X_x \to X$ *gives a diffeomorphism of the ball $B_L(0)$ onto the open set $X - A_x$.*

Fix a vector $H \in \mathfrak{p}$ of length L (i.e., $L^2 = -B(H, H)$). For $Z \in \mathfrak{p}$ let T_Z denote the linear transformation $Y \to [Z, [Z, Y]]$ of $\mathfrak{p}$, $[\,,\,]$ denoting the Lie bracket in $\mathfrak{u}$. For simplicity, we now write Exp instead of Exp_o. A point $Y \in \mathfrak{p}$ is said to be *conjugate* to o if the differential $d\mathrm{Exp}$ is singular at Y.

The line $\mathfrak{a} = \mathbf{R}H$ is a maximal abelian subspace of $\mathfrak{p}$. The eigenvalues of T_H are 0, $\alpha(H)^2$ and possibly $(\alpha(H)/2)^2$ where $\pm\alpha$ (and possibly $\pm\alpha/2$) are the roots of $\mathfrak{u}$ with respect to $\mathfrak{a}$. Let

$$\mathfrak{p} = \mathfrak{a} + \mathfrak{p}_\alpha + \mathfrak{p}_{\alpha/2} \tag{86}$$

be the corresponding decomposition of $\mathfrak{p}$ into eigenspaces; the dimensions $q = \dim(\mathfrak{p}_\alpha)$, $p = \dim(\mathfrak{p}_{\alpha/2})$ are called the *multiplicities* of α and $\alpha/2$, respectively.

(iv) *Suppose H is conjugate to o. Then* $\mathrm{Exp}(\mathfrak{a}+\mathfrak{p}_\alpha)$*, with the Riemannian structure induced by that of X, is a sphere, totally geodesic in X, having o and* $\mathrm{Exp}H$ *as antipodal points and having curvature $\pi^2 L^2$. Moreover*

$$A_{\mathrm{Exp}H} = \mathrm{Exp}(\mathfrak{p}_{\alpha/2})\,.$$

(v) *If H is not conjugate to o then $\mathfrak{p}_{\alpha/2} = 0$ and*

$$A_{\mathrm{Exp}H} = \mathrm{Exp}\;\mathfrak{p}_\alpha\,.$$

(vi) *The differential at Y of* Exp *is given by*

$$d\mathrm{Exp}_Y = d\tau(\exp Y) \circ \sum_0^\infty \frac{T_Y^k}{(2k+1)!}\,,$$

where for $u \in U$, $\tau(u)$ is the isometry $x \to u \cdot x$.

(vii) *In analogy with (23) the Laplace-Beltrami operator L on X has the expression*

$$L = \frac{\partial^2}{\partial r^2} + \frac{1}{A(r)} A'(r) \frac{\partial}{\partial r} + L_{S_r}\,,$$

where L_{S_r} is the Laplace-Beltrami operator on $S_r(o)$ and $A(r)$ its area.

(viii) *The spherical mean-value operator M^r commutes with the Laplace-Beltrami operator.*

Lemma 2.1. *The surface area $A(r)$ $(0 < r < L)$ is given by*

$$A(r) = \Omega_n \lambda^{-p} (2\lambda)^{-q} \sin^p(\lambda r) \sin^q(2\lambda r)$$

where p and q are the multiplicities above and $\lambda = |\alpha(H)|/2L$.

Proof. Because of (iii) and (vi) the surface area of $S_r(o)$ is given by

$$A(r) = \int_{|Y|=r} \det\Big(\sum_0^\infty \frac{T_Y^k}{(2k+1)!}\Big)\, d\omega_r(Y)\,,$$

where $d\omega_r$ is the surface on the sphere $|Y| = r$ in $\mathfrak{p}$. Because of the two-point homogeneity the integrand depends on r only so it suffices to evaluate it for $Y = H_r = \frac{r}{L} H$. Since the nonzero eigenvalues of T_{H_r} are $\alpha(H_r)^2$ with multiplicity q and $(\alpha(H_r)/2)^2$ with multiplicity p, a trivial computation gives the lemma.

We consider now Problems A, B and C from Chapter II, §2 for the homogeneous spaces X and Ξ, which are acted on transitively by the same group U. Fix an element $\xi_o \in \Xi$ passing through the origin $o \in X$. If $\xi_o = A_0$, then an element $u \in U$ leaves ξ_o invariant if and only if it lies in the isotropy subgroup $K' = U_o$; we have the identifications

$$X = U/K\,, \quad \Xi = U/K'$$

and $x \in X$ and $\xi \in \Xi$ are incident if and only if $x \in \xi$.

On Ξ we now choose a Riemannian structure such that the diffeomorphism $j : x \to A_x$ from (ii) is an isometry. Let L and Λ denote the Laplacians on X and Ξ, respectively. With $\check{x}$ and $\widehat{\xi}$ defined as in Ch. II, §1, we have

$$\widehat{\xi} = \xi\,, \quad \check{x} = \{j(y) : y \in j(x)\}\,;$$

the first relation amounts to the incidence description above and the second is a consequence of the property $x \in A_y \Leftrightarrow y \in A_x$ listed under (ii).

The sets $\check{x}$ and $\widehat{\xi}$ will be given the measures $d\mu$ and dm, respectively, induced by the Riemannian structures of Ξ and X. The Radon transform and its dual are then given by

$$\widehat{f}(\xi) = \int_\xi f(x)\,dm(x)\,, \quad \check{\varphi}(x) = \int_{\check{x}} \varphi(\xi)\,d\mu(\xi)\,.$$

However

$$\check{\varphi}(x) = \int_{\check{x}} \varphi(\xi)\,d\mu(\xi) = \int_{y\in j(x)} \varphi(j(y))\,d\mu(j(y)) = \int_{j(x)} (\varphi\circ j)(y)\,dm(y)$$

so

$$\check{\varphi} = (\varphi \circ j)\widehat{\ } \circ j\,. \tag{87}$$

Because of this correspondence between the transforms $f \to \widehat{f}$, $\varphi \to \check{\varphi}$ it suffices to consider the first one. Let $\mathbf{D}(X)$ denote the algebra of differential operators on X, invariant under U. It can be shown that $\mathbf{D}(X)$ is generated by L. Similarly $\mathbf{D}(\Xi)$ is generated by Λ.

Theorem 2.2. *(i) The mapping $f \to \widehat{f}$ is a linear one-to-one mapping of $\mathcal{E}(X)$ onto $\mathcal{E}(\Xi)$ and*

$$(Lf)\widehat{\ } = \Lambda \widehat{f}\,.$$

(ii) Except for the case when X is an even-dimensional elliptic space

$$f = P(L)((\widehat{f})^\vee)\,, \quad f \in \mathcal{E}(X)\,,$$

where P is a polynomial, independent of f, explicitly given below, (90)–(93). In all cases

$$\text{degree } P = \tfrac{1}{2} \text{ dimension of the antipodal manifold}\,.$$

Proof. (Indication.) We first prove (ii). Let dk be the Haar measure on K such that $\int dk = 1$ and let Ω_X denote the total measure of an antipodal manifold in X. Then $\mu(\check{o}) = m(A_o) = \Omega_X$ and if $u \in U$,

$$\check{\varphi}(u \cdot o) = \Omega_X \int_K \varphi(uk \cdot \xi_o)\, dk\,.$$

Hence

$$(\widehat{f})^\vee(u \cdot o) = \Omega_X \int_K \left(\int_{\xi_o} f(uk \cdot y)\, dm(y) \right) dk = \Omega_X \int_{\xi_o} (M^r f)(u \cdot o)\, dm(y)\,,$$

where r is the distance $d(o, y)$ in the space X between o and y. If $d(o, y) < L$ there is a unique geodesic in X of length $d(o, y)$ joining o to y and since ξ_0 is totally geodesic, $d(o, y)$ is also the distance in ξ_0 between o and y. Thus using geodesic polar coordinates in ξ_0 in the last integral we obtain

$$(88) \qquad (\widehat{f})^\vee(x) = \Omega_X \int_0^L (M^r f)(x) A_1(r)\, dr\,,$$

where $A_1(r)$ is the area of a sphere of radius r in ξ_0. By Lemma 2.1 we have

$$(89) \qquad A_1(r) = C_1 \sin^{p_1}(\lambda_1 r) \sin^{q_1}(2\lambda_1 r)\,,$$

where C_1 and λ_1 are constants and p_1, q_1 are the multiplicities for the antipodal manifold. In order to prove (ii) on the basis of (88) we need the following complete list of the compact symmetric spaces of rank one and their corresponding antipodal manifolds:

X		A_0
Spheres	$\mathbf{S}^n (n = 1, 2, \ldots)$	point
Real projective spaces	$\mathbf{P}^n(\mathbf{R}) (n = 2, 3, \ldots)$	$\mathbf{P}^{n-1}(\mathbf{R})$
Complex projective spaces	$\mathbf{P}^n(\mathbf{C}) (n = 4, 6, \ldots)$	$\mathbf{P}^{n-2}(\mathbf{C})$
Quaternian projective spaces	$\mathbf{P}^n(\mathbf{H}) (n = 8, 12, \ldots)$	$\mathbf{P}^{n-4}(\mathbf{H})$
Cayley plane	$\mathbf{P}^{16}(\mathbf{Cay})$	$\mathbf{S}^8$

Here the superscripts denote the real dimension. For the lowest dimensions, note that

$$\mathbf{P}^1(\mathbf{R}) = \mathbf{S}^1\,, \quad \mathbf{P}^2(\mathbf{C}) = \mathbf{S}^2\,, \quad \mathbf{P}^4(\mathbf{H}) = \mathbf{S}^4\,.$$

For the case $\mathbf{S}^n$, (ii) is trivial and the case $X = \mathbf{P}^n(\mathbf{R})$ was already done in Theorem 1.7. The remaining cases are done by classification starting with (88). The mean value operator M^r still commutes with the Laplacian L

$$M^r L = L M^r$$

and this implies

$$L_x((M^r f)(x)) = L_r((M^r f)(x))\,,$$

where L_r is the radial part of L. Because of (vii) above and Lemma 2.1 it is given by

$$L_r = \frac{\partial^2}{\partial r^2} + \lambda\{p\cot(\lambda r) + 2q\cot(2\lambda r)\}\frac{\partial}{\partial r}\,.$$

For each of the two-point homogeneous spaces we prove (by extensive computations) the analog of Lemma 1.8. Then by the pattern of the proof of Theorem 1.5, part (ii) of Theorem 2.2 can be proved. The full details are carried out in Helgason ([1965a] or [GGA], Ch. I, §4).

The polynomial P is explicitly given in the list below. Note that for $\mathbf{P}^n(\mathbf{R})$ the metric is normalized by means of the Killing form so it differs from that of Theorem 1.7 by a nontrivial constant.

The polynomial P is now given as follows:

For $X = \mathbf{P}^n(\mathbf{R})$, n odd

$$\begin{aligned} P(L) &= c\left(L - \tfrac{(n-2)1}{2n}\right)\left(L - \tfrac{(n-4)3}{2n}\right)\cdots\left(L - \tfrac{1(n-2)}{2n}\right) \\ c &= \tfrac{1}{4}(-4\pi^2 n)^{\frac{1}{2}(n-1)}\,. \end{aligned} \tag{90}$$

For $X = \mathbf{P}^n(\mathbf{C})$, $n = 4, 6, 8, \ldots$

$$\begin{aligned} P(L) &= c\left(L - \tfrac{(n-2)2}{2(n+2)}\right)\left(L - \tfrac{(n-4)4}{2(n+2)}\right)\cdots\left(L - \tfrac{2(n-2)}{2(n+2)}\right) \\ c &= (-8\pi^2(n+2))^{1-\frac{n}{2}}\,. \end{aligned} \tag{91}$$

For $X = \mathbf{P}^n(\mathbf{H})$, $n = 8, 12, \ldots$

$$\begin{aligned} P(L) &= c\left(L - \tfrac{(n-2)4}{2(n+8)}\right)\left(L - \tfrac{(n-4)6}{2(n+8)}\right)\cdots\left(L - \tfrac{4(n-2)}{2(n+8)}\right) \\ c &= \tfrac{1}{2}[-4\pi^2(n+8)]^{2-n/2}\,. \end{aligned} \tag{92}$$

For $X = \mathbf{P}^{16}(\mathbf{Cay})$

$$P(L) = c\left(L - \tfrac{14}{9}\right)^2\left(L - \tfrac{15}{9}\right)^2\,, \quad c = 3^6\pi^{-8}2^{-13}\,. \tag{93}$$

That $f \to \widehat{f}$ is injective follows from (ii) except for the case $X = \mathbf{P}^n(\mathbf{R})$, n even. But in this exceptional case the injectivity follows from Theorem 1.7.

For the surjectivity we use once more the fact that the mean-value operator M^r commutes with the Laplacian (property (viii)). We have

$$\widehat{f}(j(x)) = c(M^L f)(x)\,, \tag{94}$$

where c is a constant. Thus by (87)

$$(\widehat{f})^\vee(x) = (\widehat{f}\circ j)\widehat{\ }(j(x)) = cM^L(\widehat{f}\circ j)(x)$$

so

$$(\widehat{f})^{\vee} = c^2 M^L M^L f\,. \tag{95}$$

Thus if X is not an even-dimensional projective space f is a constant multiple of $M^L P(L) M^L f$ which by (94) shows $f \to \widehat{f}$ surjective. For the remaining case $\mathbf{P}^n(\mathbf{R})$, n even, we use the expansion of $f \in \mathcal{E}(\mathbf{P}^n(\mathbf{R}))$ in spherical harmonics

$$f = \sum_{k,m} a_{km} S_{km} \quad (k \text{ even})\,.$$

Here $k \in \mathbb{Z}^+$, and $S_{km}(1 \leq m \leq d(k))$ is an orthonormal basis of the space of spherical harmonics of degree k. Here the coefficients a_{km} are rapidly decreasing in k. On the other hand, by (32) and (34),

$$\widehat{f} = \Omega_n M^{\frac{\pi}{2}} f = \Omega_n \sum_{k,m} a_{km} \varphi_k\left(\tfrac{\pi}{2}\right) S_{km} \quad (k \text{ even})\,. \tag{96}$$

The spherical function φ_k is given by

$$\varphi_k(s) = \frac{\Omega_{n-1}}{\Omega_n} \int_0^{\pi} (\cos\theta + i \sin\theta \cos\varphi)^k \sin^{n-2}\varphi \, d\varphi$$

so $\varphi_{2k}(\frac{\pi}{2}) \sim k^{-\frac{n-1}{2}}$. Thus every series $\sum_{k,m} b_{k,m} S_{2k,m}$ with $b_{2k,m}$ rapidly decreasing in k can be put in the form (96). This verifies the surjectivity of the map $f \to \widehat{f}$.

It remains to prove $(Lf)\widehat{\ } = \Lambda \widehat{f}$. For this we use (87), (vii), (41) and (94). By the definition of Λ we have

$$(\Lambda\varphi)(j(x)) = L(\varphi \circ j)(x)\,, \qquad x \in X, \varphi \in \mathcal{E}(X)\,.$$

Thus

$$(\Lambda\widehat{f})(j(x)) = (L(\widehat{f} \circ j))(x) = cL(M^L f)(x) = cM^L(Lf)(x) = (Lf)\widehat{\ }(j(x))\,.$$

This finishes our indication of the proof of Theorem 2.2.

Corollary 2.3. *Let X be a compact two-point homogeneous space and suppose f satisfies*

$$\int_\gamma f(x)\, ds(x) = 0$$

for each (closed) geodesic γ in X, ds being the element of arc-length. Then

(i) If X is a sphere, f is skew.

(ii) If X is not a sphere, $f \equiv 0$.

Taking a convolution with f we may assume f smooth. Part (i) is already contained in Theorem 1.7. For Part (ii) we use the classification; for $X = \mathbf{P}^{16}(\mathbf{Cay})$ the antipodal manifolds are totally geodesic spheres so using Part (i) we conclude that $\widehat{f} \equiv 0$ so by Theorem 2.2, $f \equiv 0$. For the remaining cases $\mathbf{P}^n(\mathbf{C})$ $(n = 4, 6, \ldots)$ and $\mathbf{P}^n(\mathbf{H}), (n = 8, 12, \ldots)$ (ii) follows similarly by induction as the initial antipodal manifolds, $\mathbf{P}^2(\mathbf{C})$ and $\mathbf{P}^4(\mathbf{H})$, are totally geodesic spheres.

Corollary 2.4. *Let B be a bounded open set in $\mathbf{R}^{n+1}$, symmetric and star-shaped with respect to 0, bounded by a hypersurface. Assume for a fixed k $(1 \leq k < n)$*

$$\text{Area } (B \cap P) = \text{ constant} \tag{97}$$

for all $(k+1)$-planes P through 0. Then B is an open ball.

In fact, we know from Theorem 1.7 that if f is a symmetric function on $X = \mathbf{S}^n$ with $\widehat{f}(\mathbf{S}^n \cap P)$ constant (for all P) then f is a constant. We apply this to the function

$$f(\theta) = \rho(\theta)^{k+1} \quad \theta \in \mathbf{S}^n$$

if $\rho(\theta)$ is the distance from the origin to each of the two points of intersection of the boundary of B with the line through 0 and θ; f is well defined since B is symmetric. If $\theta = (\theta_1, \ldots, \theta_k)$ runs through the k-sphere $\mathbf{S}^n \cap P$ then the point

$$x = \theta r \quad (0 \leq r < \rho(\theta))$$

runs through the set $B \cap P$ and

$$\text{Area } (B \cap P) = \int_{\mathbf{S}^n \cap P} d\omega(\theta) \int_0^{\rho(\theta)} r^k \, dr \, .$$

It follows that Area $(B \cap P)$ is a constant multiple of $\widehat{f}(\mathbf{S}^n \cap P)$ so (97) implies that f is constant. This proves the corollary.

§3 Noncompact Two-point Homogeneous Spaces

Theorem 2.2 has an analog for noncompact two-point homogeneous spaces which we shall now describe. By Tits' classification [1955], p. 183, of homogeneous manifolds L/H for which L acts transitively on the tangents to L/H it is known, in principle, what the noncompact two-point homogeneous spaces are. As in the compact case they turn out to be symmetric. A direct proof of this fact was given by Nagano [1959] and Helgason [1959]. The theory of symmetric spaces then implies that the noncompact two-point homogeneous spaces are the Euclidean spaces and the noncompact spaces $X = G/K$ where G is a connected semisimple Lie group with finite center and real rank one and K a maximal compact subgroup.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the direct decomposition of the Lie algebra of G into the Lie algebra $\mathfrak{k}$ of K and its orthogonal complement $\mathfrak{p}$ (with respect to the Killing form of $\mathfrak{g}$). Fix a 1-dimensional subspace $\mathfrak{a} \subset \mathfrak{p}$ and let

$$\mathfrak{p} = \mathfrak{a} + \mathfrak{p}_\alpha + \mathfrak{p}_{\alpha/2} \tag{98}$$

be the decomposition of $\mathfrak{p}$ into eigenspaces of T_H (in analogy with (86)). Let ξ_o denote the totally geodesic submanifold $\operatorname{Exp}(\mathfrak{p}_{\alpha/2})$; in the case $\mathfrak{p}_{\alpha/2} = 0$ we put $\xi_o = \operatorname{Exp}(\mathfrak{p}_\alpha)$. By the classification and duality for symmetric spaces we have the following complete list of the spaces G/K. In the list the superscript denotes the real dimension; for the lowest dimensions note that

$$\mathbf{H}^1(\mathbf{R}) = \mathbf{R}\,, \quad \mathbf{H}^2(\mathbf{C}) = \mathbf{H}^2(\mathbf{R})\,, \quad \mathbf{H}^4(\mathbf{H}) = \mathbf{H}^4(\mathbf{R})\,.$$

	X	ξ_0
Real hyperbolic spaces	$\mathbf{H}^n(\mathbf{R})(n = 2, 3, \ldots)$,	$\mathbf{H}^{n-1}(\mathbf{R})$
Complex hyperbolic spaces	$\mathbf{H}^n(\mathbf{C})(n = 4, 6, \ldots)$,	$\mathbf{H}^{n-2}(\mathbf{C})$
Quaternian hyperbolic spaces	$\mathbf{H}^n(\mathbf{H})(n = 8, 12, \ldots)$,	$\mathbf{H}^{n-4}(\mathbf{H})$
Cayley hyperbolic spaces	$\mathbf{H}^{16}(\mathbf{Cay})$,	$\mathbf{H}^8(\mathbf{R})$.

Let Ξ denote the set of submanifolds $g \cdot \xi_0$ of X as g runs through G; Ξ is given the canonical differentiable structure of a homogeneous space. Each $\xi \in \Xi$ has a measure m induced by the Riemannian structure of X and the Radon transform on X is defined by

$$\widehat{f}(\xi) = \int_\xi f(x)\, dm(x)\,, \quad f \in C_c(X)\,.$$

The dual transform $\varphi \to \check{\varphi}$ is defined by

$$\check{\varphi}(x) = \int_{\xi \ni x} \varphi(\xi)\, d\mu(\xi)\,, \quad \varphi \in C(\Xi)\,,$$

where μ is the invariant average on the set of ξ passing through x. Let L denote the Laplace-Beltrami operator on X, Riemannian structure being that given by the Killing form of $\mathfrak{g}$.

Theorem 3.1. *The Radon transform $f \to \widehat{f}$ is a one-to-one mapping of $\mathcal{D}(X)$ into $\mathcal{D}(\Xi)$ and, except for the case $X = \mathbf{H}^n(\mathbf{R})$, n even, is inverted by the formula $f = Q(L)((\widehat{f})^\vee)$. Here Q is given by*

$X = \mathbf{H}^n(\mathbf{R}), n$ *odd:*

$$Q(L) = \gamma\left(L + \tfrac{(n-2)1}{2n}\right)\left(L + \tfrac{(n-4)3}{2n}\right)\cdots\left(L + \tfrac{1(n-2)}{2n}\right).$$

$X = \mathbf{H}^n(\mathbf{C})$:

$$Q(L) = \gamma\left(L + \tfrac{(n-2)2}{2(n+2)}\right)\left(L + \tfrac{(n-4)4}{2(n+2)}\right)\cdots\left(L + \tfrac{2(n-2)}{2(n+2)}\right).$$

$X = \mathbf{H}^n(\mathbf{H})$:
$$Q(L) = \gamma \left(L + \tfrac{(n-2)4}{2(n+8)}\right)\left(L + \tfrac{(n-4)6}{2(n+8)}\right)\cdots\left(L + \tfrac{4(n-2)}{2(n+8)}\right).$$
$X = \mathbf{H}^{16}(\boldsymbol{Cay})$:
$$Q(L) = \gamma \left(L + \tfrac{14}{9}\right)^2 \left(L + \tfrac{15}{9}\right)^2.$$

The constants γ are obtained from the constants c in (90)–(93) by multiplication by the factor Ω_X which is the volume of the antipodal manifold in the compact space corresponding to X. This factor is explicitly determined for each X in [GGA], Chapter I, §4.

§4 The X-ray Transform on a Symmetric Space

Let X be a complete Riemannian manifold of dimension > 1 in which any two points can be joined by a unique geodesic. The *X-ray transform* on X assigns to each continuous function f on X the integrals

$$\widehat{f}(\gamma) = \int_\gamma f(x)\, ds(x)\,, \tag{99}$$

γ being any complete geodesic in X and ds the element of arc-length. In analogy with the X-ray reconstruction problem on $\mathbf{R}^n$ (Ch.I, §7) one can consider the problem of inverting the X-ray transform $f \to \widehat{f}$. With d denoting the distance in X and $o \in X$ some fixed point we now define two subspaces of $C(X)$. Let

$$\begin{aligned} F(X) &= \{f \in C(X) : \sup_x d(o,x)^k |f(x)| < \infty \text{ for each } k \geq 0\} \\ \mathfrak{F}(X) &= \{f \in C(X) : \sup_x e^{kd(o,x)} |f(x)| < \infty \text{ for each } k \geq 0\}\,. \end{aligned}$$

Because of the triangle inequality these spaces do not depend on the choice of o. We can informally refer to $F(X)$ as the space of continuous *rapidly decreasing functions* and to $\mathfrak{F}(X)$ as the space of continuous *exponentially decreasing functions*. We shall now prove the analog of the support theorem (Theorem 2.6, Ch. I, Theorem 1.2, Ch. III) for the X-ray transform on a symmetric space of the noncompact type. This general analog turns out to be a direct corollary of the Euclidean case and the hyperbolic case, already done.

Corollary 4.1. *Let X be a symmetric space of the noncompact type, B any ball in M.*

(i) If a function $f \in \mathfrak{F}(X)$ satisfies

$$\widehat{f}(\xi) = 0 \quad \textit{whenever } \xi \cap B = \emptyset\,, \quad \xi \textit{ a geodesic,} \tag{100}$$

then

$$f(x) = 0 \quad \textit{for } x \notin B\,. \tag{101}$$

In particular, the X-ray transform is one-to-one on $\mathfrak{F}(X)$.

(ii) If X has rank greater than one statement (i) holds with $\mathfrak{F}(X)$ *replaced by* $F(X)$.

Proof. Let o be the center of B, r its radius, and let γ be an arbitrary geodesic in X through o.

Assume first X has rank greater than one. By a standard conjugacy theorem for symmetric spaces γ lies in a 2-dimensional, flat, totally geodesic submanifold of X. Using Theorem 2.6, Ch. I on this Euclidean plane we deduce $f(x) = 0$ if $x \in \gamma, d(o,x) > r$. Since γ is arbitrary (101) follows.

Next suppose X has rank one. Identifying $\mathfrak{p}$ with the tangent space X_o let $\mathfrak{a}$ be the tangent line to γ. We can then consider the eigenspace decomposition (98). If $\mathfrak{b} \subset \mathfrak{p}_\alpha$ is a line through the origin then $S = \operatorname{Exp}(\mathfrak{a} + \mathfrak{b})$ is a totally geodesic submanifold of X (cf. (iv) in the beginning of §2). Being 2-dimensional and not flat, S is necessarily a hyperbolic space. From Theorem 1.2 we therefore conclude $f(x) = 0$ for $x \in \gamma$, $d(o,x) > r$. Again (101) follows since γ is arbitrary.

§5 Maximal Tori and Minimal Spheres in Compact Symmetric Spaces

Let $\mathfrak{u}$ be a compact semisimple Lie algebra, θ an involutive automorphism of $\mathfrak{u}$ with fixed point algebra $\mathfrak{k}$. Let U be the simply connected Lie group with Lie algebra $\mathfrak{u}$ and $\operatorname{Int}(\mathfrak{u})$ the adjoint group of $\mathfrak{u}$. Then θ extends to an involutive automorphism of U and $\operatorname{Int}(\mathfrak{u})$. We denote these extensions also by θ and let K and K_θ denote the respective fixed point groups under θ. The symmetric space $X_\theta = \operatorname{Int}(\mathfrak{u})/K_\theta$ is called the *adjoint space* of $(\mathfrak{u}, \theta)$ (Helgason [1978], p. 327), and is covered by $X = U/K$, this latter space being simply connected since K is automatically connected.

The flat totally geodesic submanifolds of X_θ of maximal dimension are permuted transitively by $\operatorname{Int}(\mathfrak{u})$ according to a classical theorem of Cartan. Let E_θ be one such manifold passing through the origin eK_θ in X_θ and let H_θ be the subgroup of $\operatorname{Int}(\mathfrak{u})$ preserving E_θ. We then have the pairs of homogeneous spaces

$$X_\theta = \operatorname{Int}(\mathfrak{u})/K_\theta\,, \quad \Xi_\theta = \operatorname{Int}(\mathfrak{u})/H_\theta\,. \tag{102}$$

The corresponding Radon transform $f \to \widehat{f}$ from $C(X_\theta)$ to $C(\Xi_\theta)$ amounts to

$$\widehat{f}(E) = \int_E f(x)\,dm(x)\,, \quad E \in \Xi_\theta\,, \tag{103}$$

E being any flat totally geodesic submanifold of X_θ of maximal dimension and dm the volume element. If X_θ has rank one, E is a geodesic and we are in the situation of Corollary 2.3. The transform (103) is often called the *flat Radon transform.*

Theorem 5.1. *Assume X_θ is irreducible. Then the flat Radon transform is injective.*

For a proof see Grinberg [1992].

The sectional curvatures of the space X lie in an interval $[0, \kappa]$. The space X contains totally geodesic spheres of curvature κ and all such spheres S of maximal dimension are conjugate under U (Helgason [1966b]). Fix one such sphere S_0 through the origin eK and let H be the subgroup of U preserving S_0. Then we have another double fibration

$$X = U/K\,, \quad \Xi = U/H$$

and the accompanying Radon transform

$$\widehat{f}(S) = \int_S f(x)\, d\sigma(x)\,.$$

$S \in \Xi$ being arbitrary and $d\sigma$ being the volume element on S.

It is proved by Grinberg [1994] that injectivity holds in many cases although the general question is not fully settled.

Bibliographical Notes

As mentioned earlier, it was shown by Funk [1916] that a function f on the two-sphere, symmetric with respect to the center, can be determined by the integrals of f over the great circles. When f is rotation-invariant (relative to a vertical axis) he gave an explicit inversion formula, essentially (78) in Proposition 1.16.

The Radon transform on hyperbolic and on elliptic spaces corresponding to k-dimensional totally geodesic submanifolds was defined in the author's paper [1959]. Here and in [1990] are proved the inversion formulas in Theorems 1.5, 1.7, 1.10 and 1.11. See also Semyanisty [1961] and Rubin [1998b]. The alternative version in (60) was obtained by Berenstein and Casadio Tarabusi [1991] which also deals with the case of $\mathbf{H}^k$ in $\mathbf{H}^n$ (where the regularization is more complex). Still another interesting variation of Theorem 1.10 (for $k = 1, n = 2$) is given by Lissianoi and Ponomarev [1997]. By calculating the dual transform $\check{\varphi}_p(z)$ they derive from (30) in Chapter II an inversion formula which has a formal analogy to (38) in Chapter II. The underlying reason may be that to each geodesic γ in $\mathbf{H}^2$ one can associate a pair of horocycles tangential to $|z| = 1$ at the endpoints of γ having the same distance from o as γ.

The support theorem (Theorem 1.2) was proved by the author ([1964], [1980b]) and its consequence, Cor. 4.1, pointed out in [1980d]. Interesting generalizations are contained in Boman [1991], Boman and Quinto [1987], [1993]. For the case of $\mathbf{S}^{n-1}$ see Quinto [1983] and in the stronger form of Theorem 1.17, Kurusa [1994]. The variation (82) of the Funk transform has also been considered by Abouelaz and Daher [1993] at least for K-invariant functions. The theory of the Radon transform for antipodal manifolds in compact two-point homogeneous spaces (Theorem 2.2) is from Helgason [1965a]. R. Michel has in [1972] and [1973] used Theorem 2.2 in establishing certain infinitesimal rigidity properties of the canonical metrics on the real and complex projective spaces. See also Guillemin [1976] and A. Besse [1978], Goldschmidt [1990], Estezet [1988].

CHAPTER IV

ORBITAL INTEGRALS AND THE WAVE OPERATOR FOR ISOTROPIC LORENTZ SPACES

In Chapter II, §3 we discussed the problem of determining a function on a homogeneous space by means of its integrals over generalized spheres. We shall now solve this problem for the *isotropic Lorentz spaces* (Theorem 4.1 below). As we shall presently explain these spaces are the Lorentzian analogs of the two-point homogeneous spaces considered in Chapter III.

§1 Isotropic Spaces

Let X be a manifold. A *pseudo-Riemannian structure* of signature (p,q) is a smooth assignment $y \to g_y$ where $y \in X$ and g_y is a symmetric nondegenerate bilinear form on $X_y \times X_y$ of signature (p,q). This means that for a suitable basis $Y_1, \ldots, Y_{p+q}$ of X_y we have

$$g_y(Y) = y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_{p+q}^2$$

if $Y = \sum_1^{p+q} y_i Y_i$. If $q = 0$ we speak of a *Riemannian* structure and if $p = 1$ we speak of a *Lorentzian* structure. Connected manifolds X with such structures g are called pseudo-Riemannian (respectively Riemannian, Lorentzian) manifolds.

A manifold X with a pseudo-Riemannian structure g has a differential operator of particular interest, the so-called Laplace-Beltrami operator. Let $(x_1, \ldots, x_{p+q})$ be a coordinate system on an open subset U of X. We define the functions g_{ij}, g^{ij}, and $\overline{g}$ on U by

$$g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right), \quad \sum_j g_{ij} g^{jk} = \delta_{ik}, \quad \overline{g} = |\det(g_{ij})|.$$

The *Laplace-Beltrami operator* L is defined on U by

$$Lf = \frac{1}{\sqrt{\overline{g}}}\left(\sum_k \frac{\partial}{\partial x_k}\left(\sum_i g^{ik}\sqrt{\overline{g}}\frac{\partial f}{\partial x_i}\right)\right)$$

for $f \in \mathcal{C}^\infty(U)$. It is well known that this expression is invariant under coordinate changes so L is a differential operator on X.

An *isometry* of a pseudo-Riemannian manifold X is a diffeomorphism preserving g. It is easy to prove that L is *invariant* under each isometry φ, that is $L(f \circ \varphi) = (Lf) \circ \varphi$ for each $f \in \mathcal{E}(X)$. Let $I(X)$ denote the group of all isometries of X. For $y \in X$ let $I(X)_y$ denote the subgroup of $I(X)$

fixing y (the isotropy subgroup at y) and let H_y denote the group of linear transformations of the tangent space X_y induced by the action of $I(X)_y$. For each $a \in \mathbf{R}$ let $\sum_a(y)$ denote the "sphere"

$$\Sigma_a(y) = \{Z \in X_y : g_y(Z, Z) = a\,, \quad Z \neq 0\}\,. \tag{1}$$

Definition. The pseudo-Riemannian manifold X is called *isotropic* if for each $a \in \mathbf{R}$ and each $y \in X$ the group H_y acts transitively on $\sum_a(y)$.

Proposition 1.1. *An isotropic pseudo-Riemannian manifold X is homogeneous; that is, $I(X)$ acts transitively on X.*

Proof. The pseudo-Riemannian structure on X gives an affine connection preserved by each isometry $g \in I(X)$. Any two points $y, z \in X$ can be joined by a curve consisting of finitely many geodesic segments $\gamma_i (1 \leq i \leq p)$. Let g_i be an isometry fixing the midpoint of γ_i and reversing the tangents to γ_i at this point. The product $g_p \cdots g_1$ maps y to z, whence the homogeneity of X.

A. The Riemannian Case

The following simple result shows that the isotropic spaces are natural generalizations of the spaces considered in the last chapter.

Proposition 1.2. *A Riemannian manifold X is isotropic if and only if it is two-point homogeneous.*

Proof. If X is two-point homogeneous and $y \in X$ the isotropy subgroup $I(X)_y$ at y is transitive on each sphere $S_r(y)$ in X with center y so X is clearly isotropic. On the other hand if X is isotropic it is homogeneous (Prop. 1.1) hence complete; thus by standard Riemannian geometry any two points in X can be joined by means of a geodesic. Now the isotropy of X implies that for each $y \in X, r > 0$, the group $I(X)_y$ is transitive on the sphere $S_r(y)$, whence the two-point homogeneity.

B. The General Pseudo-Riemannian Case

Let X be a manifold with pseudo-Riemannian structure g and curvature tensor R. Let $y \in X$ and $S \subset X_y$ a 2-dimensional subspace on which g_y is nondegenerate. The curvature of X along the section S spanned by Z and Y is defined by

$$K(S) = -\frac{g_p(R_p(Z, Y)Z, Y)}{g_p(Z, Z)g_p(Y, Y) - g_p(Z, Y)^2}$$

The denominator is in fact $\neq 0$ and the expression is independent of the choice of Z and Y.

We shall now construct isotropic pseudo-Riemannian manifolds of signature (p,q) and constant curvature. Consider the space $\mathbf{R}^{p+q+1}$ with the flat pseudo-Riemannian structure

$$B_e(Y) = y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_{p+q}^2 + e\, y_{p+q+1}^2\,, \quad (e = \pm 1)\,.$$

Let Q_e denote the quadric in $\mathbf{R}^{p+q+1}$ given by

$$B_e(Y) = e\,.$$

The orthogonal group $\mathbf{O}(B_e)$ $(= \mathbf{O}(p,q+1)$ or $\mathbf{O}(p+1,q))$ acts transitively on Q_e; the isotropy subgroup at $o = (0,\ldots,0,1)$ is identified with $\mathbf{O}(p,q)$.

Theorem 1.3. *(i) The restriction of B_e to the tangent spaces to Q_e gives a pseudo-Riemannian structure g_e on Q_e of signature (p,q).*

(ii) We have

(2) $$Q_{-1} \cong \mathbf{O}(p,q+1)/\mathbf{O}(p,q) \qquad \textit{(diffeomorphism)}$$

and the pseudo-Riemannian structure g_{-1} on Q_{-1} has constant curvature -1.

(iii) We have

(3) $$Q_{+1} = \mathbf{O}(p+1,q)/\mathbf{O}(p,q) \qquad \textit{(diffeomorphism)}$$

and the pseudo-Riemannian structure g_{+1} on Q_{+1} has constant curvature $+1$.

(iv) The flat space $\mathbf{R}^{p+q}$ with the quadratic form $g_o(Y) = \sum_1^p y_i^2 - \sum_{p+1}^{p+q} y_j^2$ and the spaces

$$\mathbf{O}(p,q+1)/\mathbf{O}(p,q)\,, \quad \mathbf{O}(p+1,q)/\mathbf{O}(p,q)$$

are all isotropic and (up to a constant factor on the pseudo-Riemannian structure) exhaust the class of pseudo-Riemannian manifolds of constant curvature and signature (p,q) except for local isometry.

Proof. If s_o denotes the linear transformation

$$(y_1,\ldots,y_{p+q},y_{p+q+1}) \to (-y_1,\ldots,-y_{p+q},y_{p+q+1})$$

then the mapping $\sigma : g \to s_o g s_o$ is an involutive automorphism of $\mathbf{O}(p,q+1)$ whose differential $d\sigma$ has fixed point set $\mathfrak{o}(p,q)$ (the Lie algebra of $\mathbf{O}(p,q)$). The (-1)-eigenspace of $d\sigma$, say $\mathfrak{m}$, is spanned by the vectors

(4) $$Y_i = E_{i,p+q+1} + E_{p+q+1,i} \qquad (1 \le i \le p)\,,$$

(5) $$Y_j = E_{j,p+q+1} - E_{p+q+1,j} \qquad (p+1 \le j \le p+q)\,.$$

Here E_{ij} denotes a square matrix with entry 1 where the i^{th} row and the j^{th} column meet, all other entries being 0.

The mapping $\psi : g\mathbf{O}(p,q) \to g \cdot o$ has a differential $d\psi$ which maps $\mathfrak{m}$ bijectively onto the tangent plane $y_{p+q+1} = 1$ to Q_{-1} at o and $d\psi(X) = X \cdot o$ $(X \in \mathfrak{m})$. Thus

$$d\psi(Y_k) = (\delta_{1k}, \ldots, \delta_{p+q+1,k}) \,, \quad (1 \leq k \leq p+q) \,.$$

Thus

$$B_{-1}(d\psi(Y_k)) = 1 \quad \text{if} \quad 1 \leq k \leq p \text{ and } -1 \text{ if } p+1 \leq k \leq p+q \,,$$

proving (i). Next, since the space (2) is symmetric its curvature tensor satisfies

$$R_o(X,Y)(Z) = [[X,Y],Z] \,,$$

where $[\, ,\,]$ is the Lie bracket. A simple computation then shows for $k \neq \ell$

$$K(\mathbf{R}Y_k + \mathbf{R}Y_\ell) = -1 \quad (1 \leq k, \ell \leq p+q)$$

and this implies (ii). Part (iii) is proved in the same way. For (iv) we first verify that the spaces listed are isotropic. Since the isotropy action of $\mathbf{O}(p,q+1)_o = \mathbf{O}(p,q)$ on $\mathfrak{m}$ is the ordinary action of $\mathbf{O}(p,q)$ on $\mathbf{R}^{p+q}$ it suffices to verify that $\mathbf{R}^{p+q}$ with the quadratic form g_o is isotropic. But we know $\mathbf{O}(p,q)$ is transitive on $g_e = +1$ and on $g_e = -1$ so it remains to show $\mathbf{O}(p,q)$ transitive on the cone $\{Y \neq 0 : g_e(Y) = 0\}$. By rotation in $\mathbf{R}^p$ and in $\mathbf{R}^q$ it suffices to verify the statement for $p = q = 1$. But for this case it is obvious. The uniqueness in (iv) follows from the general fact that a symmetric space is determined locally by its pseudo-Riemannian structure and curvature tensor at a point (see e.g. [DS], pp. 200–201). This finishes the proof.

The spaces (2) and (3) are the pseudo-Riemannian analogs of the spaces $\mathbf{O}(p,1)/\mathbf{O}(p)$, $\mathbf{O}(p+1)/\mathbf{O}(p)$ from Ch. III, §1. But the other two-point homogeneous spaces listed in Ch. III, §2–§3 have similar pseudo-Riemannian analogs (indefinite elliptic and hyperbolic spaces over **C**, **H** and **Cay**). As proved by Wolf [1967], p. 384, each non-flat isotropic pseudo-Riemannian manifold is locally isometric to one of these models.

We shall later need a lemma about the connectivity of the groups $\mathbf{O}(p,q)$. Let $I_{p,q}$ denote the diagonal matrix (d_{ij}) with

$$d_{ii} = 1 \quad (1 \leq i \leq p) \,, \, d_{jj} = -1 \quad (p+1 \leq j \leq p+q)$$

so a matrix g with transpose ${}^t g$ belongs to $\mathbf{O}(p,q)$ if and only if

$$(6) \qquad {}^t g I_{p,q} g = I_{p,q} \,.$$

If $y \in \mathbf{R}^{p+q}$ let

$$y^T = (y_1, \ldots, y_p, 0 \ldots 0), y^S = (0, \ldots, 0, y_{p+1}, \ldots, y_{p+q})$$

and for $g \in \mathbf{O}(p,q)$ let g_T and g_S denote the matrices

$$\begin{array}{ll} (g_T)_{ij} = g_{ij} & (1 \le i,j \le p)\,, \\ (g_S)_{k\ell} = g_{k\ell} & (p+1 \le k,\ell \le p+q) \end{array} .$$

If $g_1, \dots, g_{p+q}$ denote the column vectors of the matrix g then (6) means for the scalar products

$$\begin{array}{ll} g_i^T \cdot g_i^T - g_i^S \cdot g_i^S = 1\,, & 1 \le i \le p\,, \\ g_j^T \cdot g_j^T - g_j^S \cdot g_j^S = -1\,, & p+1 \le j \le p+q\,, \\ g_j^T \cdot g_k^T = g_j^S \cdot g_k^S\,, & j \ne k\,. \end{array}$$

Lemma 1.4. *We have for each $g \in \mathbf{O}(p,q)$*

$$|\det(g_T)| \ge 1\,, \quad |\det(g_S)| \ge 1\,.$$

The components of $\mathbf{O}(p,q)$ are obtained by

$$\det g_T \ge 1 \quad , \quad \det g_S \ge 1; \quad \textit{(identity component)} \tag{7}$$

$$\det g_T \le -1 \quad , \quad \det g_S \ge 1; \tag{8}$$

$$\det g_T \ge -1 \quad , \quad \det g_S \le -1, \tag{9}$$

$$\det g_T \le -1 \quad , \quad \det g_S \le -1\,. \tag{10}$$

Thus $\mathbf{O}(p,q)$ has four components if $p \ge 1, q \ge 1$, two components if p or $q = 0$.

Proof. Consider the Gram determinant

$$\det \begin{pmatrix} g_1^T \cdot g_1^T & g_1^T \cdot g_2^T & \cdots & g_1^T \cdot g_p^T \\ g_2^T \cdot g_1^T & \cdot & & \\ \vdots & & & \\ g_p^T \cdot g_1^T & \cdots & & g_p^T \cdot g_p^T \end{pmatrix},$$

which equals $(\det g_T)^2$. Using the relations above it can also be written

$$\det \begin{pmatrix} 1 + g_1^S \cdot g_1^S & g_1^S \cdot g_2^S & \cdots & g_1^S \cdot g_p^S \\ g_2^S \cdot g_1^S & \cdot & \cdots & \\ \vdots & & & \\ g_p^S \cdot g_1^S & & & 1 + g_p^S \cdot g_p^S \end{pmatrix},$$

which equals 1 plus a sum of lower order Gram determinants each of which is still positive. Thus $(\det g_T)^2 \ge 1$ and similarly $(\det g_S)^2 \ge 1$. Assuming now $p \ge 1, q \ge 1$ consider the decomposition of $\mathbf{O}(p,q)$ into the four pieces (7), (8), (9), (10). Each of these is $\ne \emptyset$ because (8) is obtained from (7) by multiplication by $I_{1,p+q-1}$ etc. On the other hand, since the functions $g \to$

$\det(g_T)$, $g \to \det(g_S)$ are continuous on $\mathbf{O}(p,q)$ the four pieces above belong to different components of $\mathbf{O}(p,q)$. But by Chevalley [1946], p. 201, $\mathbf{O}(p,q)$ is homeomorphic to the product of $\mathbf{O}(p,q) \cap \mathbf{U}(p+q)$ with a Euclidean space. Since $\mathbf{O}(p,q) \cap \mathbf{U}(p+q) = \mathbf{O}(p,q) \cap \mathbf{O}(p+q)$ is homeomorphic to $\mathbf{O}(p) \times \mathbf{O}(q)$ it just remains to remark that $\mathbf{O}(n)$ has two components.

C. The Lorentzian Case

The isotropic Lorentzian manifolds are more restricted than one might at first think on the basis of the Riemannian case. In fact there is a theorem of Lichnerowicz and Walker [1945] (see Wolf [1967], Ch. 12) which implies that an isotropic Lorentzian manifold has constant curvature. Thus we can deduce the following result from Theorem 1.3.

Theorem 1.5. *Let X be an isotropic Lorentzian manifold (signature $(1,q)$, $q \geq 1$). Then X has constant curvature so (after a multiplication of the Lorentzian structure by a positive constant) X is locally isometric to one of the following:*

$$\mathbf{R}^{1+q}(\textit{flat, signature } (1,q))\,,$$
$$Q_{-1} = \mathbf{O}(1,q+1)/\mathbf{O}(1,q):\ y_1^2 - y_2^2 - \cdots - y_{q+2}^2 = -1\,,$$
$$Q_{+1} = \mathbf{O}(2,q)/\mathbf{O}(1,q):\ y_1^2 - y_2^2 - \cdots - y_{q+1}^2 + y_{q+2}^2 = 1\,,$$

the Lorentzian structure being induced by $y_1^2 - y_2^2 - \cdots \mp y_{q+2}^2$.

§2 Orbital Integrals

The orbital integrals for isotropic Lorentzian manifolds are analogs to the spherical averaging operator M^r considered in Ch. I, §1, and Ch. III, §1. We start with some geometric preparation.

For manifolds X with a Lorentzian structure g we adopt the following customary terminology: If $y \in X$ the cone

$$C_y = \{Y \in X_y : g_y(Y,Y) = 0\}$$

is called the *null cone* (or the *light cone*) in X_y with vertex y. A nonzero vector $Y \in X_y$ is said to be *timelike, isotropic* or *spacelike* if $g_y(Y,Y)$ is positive, 0, or negative, respectively. Similar designations apply to geodesics according to the type of their tangent vectors.

While the geodesics in $\mathbf{R}^{1+q}$ are just the straight lines, the geodesics in Q_{-1} and Q_{+1} can be found by the method of Ch. III, §1.

Proposition 2.1. *The geodesics in the Lorentzian quadrics Q_{-1} and Q_{+1} have the following properties:*

(i) The geodesics are the nonempty intersections of the quadrics with two-planes in $\mathbf{R}^{2+q}$ *through the origin.*

(ii) For Q_{-1} *the spacelike geodesics are closed, for* Q_{+1} *the timelike geodesics are closed.*

(iii) The isotropic geodesics are certain straight lines in $\mathbf{R}^{2+q}$*.*

Proof. Part (i) follows by the symmetry considerations in Ch. III, §1. For Part (ii) consider the intersection of Q_{-1} with the two-plane

$$y_1 = y_4 = \cdots = y_{q+2} = 0\,.$$

The intersection is the circle $y_2 = \cos t$, $y_3 = \sin t$ whose tangent vector $(0, -\sin t, \cos t, 0, \ldots, 0)$ is clearly spacelike. Since $\mathbf{O}(1, q+1)$ permutes the spacelike geodesics transitively the first statement in (ii) follows. For Q_{+1} we intersect similarly with the two-plane

$$y_2 = \cdots = y_{q+1} = 0\,.$$

For (iii) we note that the two-plane $\mathbf{R}(1, 0, \ldots, 0, 1) + \mathbf{R}(0, 1, \ldots, 0)$ intersects Q_{-1} in a pair of straight lines

$$y_1 = t, y_2 \pm 1, y_3 = \cdots = y_{q+1} = 0, y_{q+2} = t$$

which clearly are isotropic. The transitivity of $\mathbf{O}(1, q+1)$ on the set of isotropic geodesics then implies that each of these is a straight line. The argument for Q_{+1} is similar.

Lemma 2.2. *The quadrics* Q_{-1} *and* Q_{+1} $(q \geq 1)$ *are connected.*

Proof. The q-sphere being connected, the point $(y_1, \ldots, y_{q+2})$ on $Q_{\mp 1}$ can be moved continuously on $Q_{\mp 1}$ to the point

$$(y_1, (y_2^2 + \cdots + y_{q+1}^2)^{1/2}, 0, \ldots, 0, y_{q+2})$$

so the statement follows from the fact that the hyperboloids $y_1^2 - y_1^2 \mp y_3^2 = \mp 1$ are connected.

Lemma 2.3. *The identity components of* $\mathbf{O}(1, q+1)$ *and* $\mathbf{O}(2, q)$ *act transitively on* Q_{-1} *and* Q_{+1}*, respectively, and the isotropy subgroups are connected.*

Proof. The first statement comes from the general fact (see e.g [DS], pp. 121–124) that when a separable Lie group acts transitively on a connected manifold then so does its identity component. For the isotropy groups we use the description (7) of the identity component. This shows quickly that

$$\begin{aligned} \mathbf{O}_o(1, q+1) \cap \mathbf{O}(1, q) &= \mathbf{O}_o(1, q)\,, \\ \mathbf{O}_o(2, q) \cap \mathbf{O}(1, q) &= \mathbf{O}_o(1, q) \end{aligned}$$

the subscript o denoting identity component. Thus we have

$$\begin{aligned} Q_{-1} &= \mathbf{O}_o(1, q+1)/\mathbf{O}_o(1, q)\,, \\ Q_{+1} &= \mathbf{O}_o(2, q)/\mathbf{O}_o(1, q)\,, \end{aligned}$$

proving the lemma.

We now write the spaces in Theorem 1.5 in the form $X = G/H$ where $H = \mathbf{O}_o(1, q)$ and G is either $G^0 = \mathbf{R}^{1+q} \cdot \mathbf{O}_o(1, q)$ (semi-direct product) $G^- = \mathbf{O}_o(1, q+1)$ or $G^+ = \mathbf{O}_o(2, q)$. Let o denote the origin $\{H\}$ in X, that is

$$\begin{aligned} &o = (0, \dots, 0) && \text{if } X = \mathbf{R}^{1+q} \\ &o = (0, \dots, 0, 1) && \text{if } X = Q_{-1} \text{ or } Q_{+1}\,. \end{aligned}$$

In the cases $X = Q_{-1}, X = Q_{+1}$ the tangent space X_o is the hyperplane $\{y_1, \dots, y_{q+1}, 1\} \subset \mathbf{R}^{2+q}$.

The timelike vectors at o fill up the "interior" C_o^o of the cone C_o. The set C_o^o consists of two components. The component which contains the timelike vector

$$v_o = (-1, 0, \dots, 0)$$

will be called the *solid retrograde cone* in X_o. It will be denoted by D_o. The component of the hyperboloid $g_o(Y, Y) = r^2$ which lies in D_o will be denoted $S_r(o)$. If y is any other point of X we define $C_y, D_y, S_r(y) \subset X_y$ by

$$C_y = g \cdot C_o\,,\ D_y = g \cdot D_o\,,\ S_r(y) = g \cdot S_r(o)$$

if $g \in G$ is chosen such that $g \cdot o = y$. This is a valid definition because the connectedness of H implies that $h \cdot D_o \subset D_o$. We also define

$$B_r(y) = \{Y \in D_y : 0 < g_y(Y, Y) < r^2\}\,.$$

If Exp denotes the exponential mapping of X_y into X, mapping rays through 0 onto geodesics through y we put

$$\begin{aligned} \mathbf{D}_y = \operatorname{Exp} D_y\,, &\quad \mathbf{C}_y = \operatorname{Exp} C_y \\ \mathbf{S}_r(y) = \operatorname{Exp} S_r(y)\,, &\quad \mathbf{B}_r(y) = \operatorname{Exp} B_r(y)\,. \end{aligned}$$

Again $\mathbf{C}_y$ and $\mathbf{D}_y$ are respectively called the *light cone* and *solid retrograde cone* in X with vertex y. For the spaces $X = Q_+$ we always assume $r < \pi$ in order that Exp will be one-to-one on $B_r(y)$ in view of Prop. 2.1(ii).

Figure IV.1 illustrates the situation for Q_{-1} in the case $q = 1$. Then Q_{-1} is the hyperboloid

$$y_1^2 - y_2^2 - y_3^2 = -1$$

and the y_1-axis is vertical. The origin o is

$$o = (0, 0, 1)$$

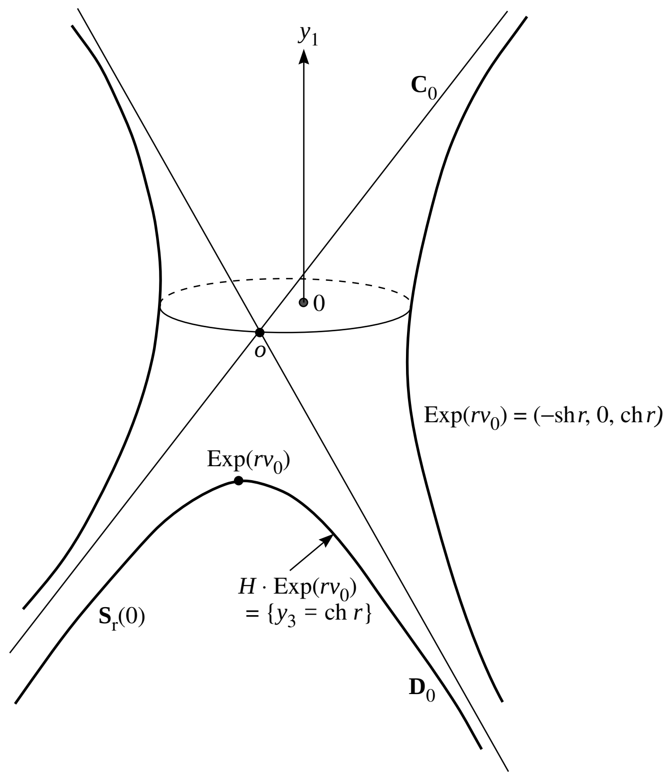

FIGURE IV.1.

and the vector $v_o = (-1, 0, 0)$ lies in the tangent space

$$(Q_{-1})_o = \{y : y_3 = 1\}$$

pointing downward. The mapping $\psi : gH \to g \cdot o$ has differential $d\psi : \mathfrak{m} \to (Q_{-1})_o$ and

$$d\psi(E_{1\,3} + E_{3\,1}) = -v_o$$

in the notation of (4). The geodesic tangent to v_o at o is

$$t \to \mathrm{Exp}(tv_o) = \exp(-t(E_{1\,3} + E_{3\,1})) \cdot o = (-\sinh t, 0, \cosh t)$$

and this is the section of Q_{-1} with the plane $y_2 = 0$. Note that since H preserves each plane $y_3 =$ const., the “sphere” $\mathbf{S}_r(o)$ is the plane section $y_3 = \cosh r, y_1 < 0$ with Q_{-1}.

Lemma 2.4. *The negative of the Lorentzian structure on $X = G/H$ induces on each $\mathbf{S}_r(y)$ a Riemannian structure of constant negative curvature $(q > 1)$.*

Proof. The manifold X being isotropic the group $H = \mathbf{O}_o(1,q)$ acts transitively on $\mathbf{S}_r(o)$. The subgroup leaving fixed the geodesic from o with tangent vector v_o is $\mathbf{O}_o(q)$. This implies the lemma.

Lemma 2.5. *The timelike geodesics from y intersect $\mathbf{S}_r(y)$ under a right angle.*

Proof. By the group invariance it suffices to prove this for $y = o$ and the geodesic with tangent vector v_o. For this case the statement is obvious.

Let $\tau(g)$ denote the translation $xH \to gxH$ on G/H and for $Y \in \mathfrak{m}$ let T_Y denote the linear transformation $Z \to [Y,[Y,Z]]$ of $\mathfrak{m}$ into itself. As usual, we identify $\mathfrak{m}$ with $(G/H)_o$.

Lemma 2.6. *The exponential mapping* $\operatorname{Exp} : \mathfrak{m} \to G/H$ *has differential*

$$d\mathrm{Exp}_Y = d\tau(\exp Y) \circ \sum_0^\infty \frac{T_Y^n}{(2n+1)!} \qquad (Y \in \mathfrak{m}).$$

For the proof see [DS], p. 215.

Lemma 2.7. *The linear transformation*

$$A_Y = \sum_0^\infty \frac{T_Y^n}{(2n+1)!}$$

has determinant given by

$$\det A_Y = \left\{ \frac{\sinh(g(Y,Y))^{1/2}}{(g(Y,Y))^{1/2}} \right\}^q \quad \textit{for } Q_{-1}$$

$$\det A_Y = \left\{ \frac{\sin(g(Y,Y))^{1/2}}{(g(Y,Y))^{1/2}} \right\}^q \quad \textit{for } Q_{+1}$$

for Y timelike.

Proof. Consider the case of Q_{-1}. Since $\det(A_Y)$ is invariant under H it suffices to verify this for $Y = cY_1$ in (4), where $c \in \mathbf{R}$. We have $c^2 = g(Y,Y)$ and $T_{Y_1}(Y_j) = Y_j \quad (2 \le j \le q+1)$. Thus T_Y has the eigenvalue 0 and $g(Y,Y)$; the latter is a q-tuple eigenvalue. This implies the formula for the determinant. The case Q_{+1} is treated in the same way.

From this lemma and the description of the geodesics in Prop. 2.1 we can now conclude the following result.

Proposition 2.8. *(i) The mapping* $\operatorname{Exp} : \mathfrak{m} \to Q_{-1}$ *is a diffeomorphism of D_o onto $\mathbf{D}_o$.*

(ii) The mapping $\operatorname{Exp} : \mathfrak{m} \to Q_{+1}$ *gives a diffeomorphism of $B_\pi(o)$ onto $\mathbf{B}_\pi(o)$.*

Let dh denote a bi-invariant measure on the unimodular group H. Let $u \in \mathcal{D}(X)$, $y \in X$ and $r > 0$. Select $g \in G$ such that $g \cdot o = y$ and select $x \in \mathbf{S}_r(o)$. Consider the integral

$$\int_H u(gh \cdot x)\, dh\,.$$

Since the subgroup $K \subset H$ leaving x fixed is compact it is easy to see that the set

$$C_{g,x} = \{h \in H : gh \cdot x \in \text{ support } (u)\}$$

is compact; thus the integral above converges. By the bi-invariance of dh it is independent of the choice of g (satisfying $g \cdot o = y$) and of the choice of $x \in \mathbf{S}_r(o)$. In analogy with the Riemannian case (Ch. III, §1) we thus define the operator M^r (*the orbital integral*) by

$$(11) \qquad (M^r u)(y) = \int_H u(gh \cdot x)\, dh\,.$$

If g and x run through suitable compact neighborhoods, the sets $C_{g,x}$ are enclosed in a fixed compact subset of H so $(M^r u)(y)$ depends smoothly on both r and y. It is also clear from (11) that the operator M^r is invariant under the action of G: if $m \in G$ and $\tau(m)$ denotes the transformation $nH \to mnH$ of G/H onto itself then

$$M^r(u \circ \tau(m)) = (M^r u) \circ \tau(m)\,.$$

If dk denotes the normalized Haar measure on K we have by standard invariant integration

$$\int_H u(h \cdot x)\, dh = \int_{H/K} d\dot{h} \int_K u(hk \cdot x)\, dk = \int_{H/K} u(h \cdot x)\, d\dot{h}\,,$$

where $d\dot{h}$ is an H-invariant measure on H/K. But if $d\mathbf{w}^r$ is the volume element on $\mathbf{S}_r(o)$ (cf. Lemma 2.4) we have by the uniqueness of H-invariant measures on the space $H/K \approx \mathbf{S}_r(o)$ that

$$(12) \qquad \int_H u(h \cdot x)\, dh = \frac{1}{A(r)} \int_{\mathbf{S}_r(o)} u(z)\, d\mathbf{w}^r(z)\,,$$

where $A(r)$ is a positive scalar. But since g is an isometry we deduce from (12) that

$$(M^r u)(y) = \frac{1}{A(r)} \int_{\mathbf{S}_r(y)} u(z)\, d\mathbf{w}^r(z)\,.$$

Now we have to determine $A(r)$.

Lemma 2.9. *For a suitable fixed normalization of the Haar measure dh on H we have*

$$A(r) = r^q\,, \quad (\sinh r)^q\,, \quad (\sin r)^q$$

for the cases

$$\mathbf{R}^{1+q}\,, \quad \mathbf{O}(1, q+1)/\mathbf{O}(1, q)\,, \quad \mathbf{O}(2, q)/\mathbf{O}(1, q)\,,$$

respectively.

Proof. The relations above show that $dh = A(r)^{-1}\, d\mathbf{w}^r\, dk$. The mapping $\mathrm{Exp} : D_o \to \mathbf{D}_o$ preserves length on the geodesics through o and maps $S_r(o)$ onto $\mathbf{S}_r(o)$. Thus if $z \in S_r(o)$ and Z denotes the vector from 0 to z in X_o the ratio of the volume of elements of $\mathbf{S}_r(o)$ and $S_r(o)$ at z is given by $\det(d\mathrm{Exp}_Z)$. Because of Lemmas 2.6–2.7 this equals

$$1, \left(\frac{\sinh r}{r}\right)^q, \left(\frac{\sin r}{r}\right)^q$$

for the three respective cases. But the volume element $d\omega^r$ on $S_r(o)$ equals $r^q d\omega^1$. Thus we can write in the three respective cases

$$dh = \frac{r^q}{A(r)}\, d\omega^1\, dk\,, \quad \frac{\sinh^q r}{A(r)}\, d\omega^1\, dk\,, \quad \frac{\sin^q r}{A(r)}\, d\omega^1\, dk\,.$$

But we can once and for all normalize dh by $dh = d\omega^1\, dk$ and for this choice our formulas for $A(r)$ hold.

Let $\Box$ denote the *wave operator* on $X = G/H$, that is the Laplace-Beltrami operator for the Lorentzian structure g.

Lemma 2.10. *Let $y \in X$. On the solid retrograde cone $\mathbf{D}_y$, the wave operator $\Box$ can be written*

$$\Box = \frac{\partial^2}{\partial r^2} + \frac{1}{A(r)}\, \frac{dA}{dr}\, \frac{\partial}{\partial r} - L_{\mathbf{S}_r(y)}\,,$$

where $L_{\mathbf{S}_r(y)}$ is the Laplace-Beltrami operator on $\mathbf{S}_r(y)$.

Proof. We can take $y = o$. If $(\theta_1, \ldots, \theta_q)$ are coordinates on the "sphere" $S_1(o)$ in the flat space X_o then $(r\theta_1, \ldots, r\theta_q)$ are coordinates on $S_r(o)$. The Lorentzian structure on D_o is therefore given by

$$dr^2 - r^2\, d\theta^2\,,$$

where $d\theta^2$ is the Riemannian structure of $S_1(o)$. Since A_Y in Lemma 2.7 is a diagonal matrix with eigenvalues 1 and $r^{-1}A(r)^{1/q}$ (q-times) it follows from Lemma 2.6 that the image $\mathbf{S}_r(o) = \mathrm{Exp}(S_r(o))$ has Riemannian structure

$r^2\,d\theta^2$, $\sinh^2 r\,d\theta^2$ and $\sin^2 r\,d\theta^2$ in the cases $\mathbf{R}^{1+q}$, Q_{-1} and Q_{+1}, respectively. By the perpendicularity in Lemma 2.5 it follows that the Lorentzian structure on $\mathbf{D}_o$ is given by

$$dr^2 - r^2\,d\theta^2\,, \quad dr^2 - \sinh^2 r\,d\theta^2\,, \quad dr^2 - \sin^2 r\,d\theta^2$$

in the three respective cases. Now the lemma follows immediately.

The operator M^r is of course the Lorentzian analog to the spherical mean value operator for isotropic Riemannian manifolds. We shall now prove that in analogy to the Riemannian case (cf. (41), Ch. III) the operator M^r commutes with the wave operator $\square$.

Theorem 2.11. *For each of the isotropic Lorentz spaces $X = G^-/H$, G^+/H or G^0/H the wave operator $\square$ and the orbital integral M^r commute:*

$$\square M^r u = M^r \square u \qquad \text{for } u \in \mathcal{D}(X)\,.$$

(For G^+/H we assume $r < \pi$.)

Given a function u on G/H we define the function $\widetilde{u}$ on G by $\widetilde{u}(g) = u(g\cdot o)$.

Lemma 2.12. *There exists a differential operator $\widetilde{\square}$ on G invariant under all left and all right translations such that*

$$\widetilde{\square}\widetilde{u} = (\square u)^{\sim} \quad \text{for } u \in \mathcal{D}(X)\,.$$

Proof. We consider first the case $X = G^-/H$. The bilinear form

$$K(Y,Z) = \tfrac{1}{2}\,\mathrm{Tr}(YZ)$$

on the Lie algebra $\mathfrak{o}(1,q+1)$ of G^- is nondegenerate; in fact K is nondegenerate on the complexification $\mathfrak{o}(q+2,\mathbf{C})$ consisting of all complex skew symmetric matrices of order $q+2$. A simple computation shows that in the notation of (4) and (5)

$$K(Y_1,Y_1) = 1\,, \quad K(Y_j,Y_j) = -1 \quad (2 \le j \le q+1)\,.$$

Since K is symmetric and nondegenerate there exists a unique left invariant pseudo-Riemannian structure $\widetilde{K}$ on G^- such that $\widetilde{K}_e = K$. Moreover, since K is invariant under the conjugation $Y \to gYg^{-1}$ of $\mathfrak{o}(1,q+1)$, $\widetilde{K}$ is also right invariant. Let $\widetilde{\square}$ denote the corresponding Laplace-Beltrami operator on G^-. Then $\widetilde{\square}$ is invariant under all left and right translations on G^-. Let $u = \mathcal{D}(X)$. Since $\widetilde{\square}\widetilde{u}$ is invariant under all right translations from H there is a unique function $v \in \mathcal{E}(X)$ such that $\widetilde{\square}\widetilde{u} = \widetilde{v}$. The mapping $u \to v$ is a differential operator which at the origin must coincide with $\square$, that is $\widetilde{\square}\widetilde{u}(e) = \square u(o)$. Since, in addition, both $\square$ and the operator $u \to v$ are invariant under the action of G^- on X it follows that they coincide. This proves $\widetilde{\square}\widetilde{u} = (\square u)^{\sim}$.

The case $X = G^+/H$ is handled in the same manner. For the flat case $X = G^0/H$ let

$$Y_j = (0, \ldots, 1, \ldots, 0)\,,$$

the j^{th} coordinate vector on $\mathbf{R}^{1+q}$. Then $\square = Y_1^2 - Y_2^2 - \cdots - Y_{q+1}^2$. Since $\mathbf{R}^{1+q}$ is naturally embedded in the Lie algebra of G^0 we can extend Y_j to a left invariant vector field $\widetilde{Y}_j$ on G^0. The operator

$$\widetilde{\square} = \widetilde{Y}_1^2 - \widetilde{Y}_2^2 - \cdots - \widetilde{Y}_{q+1}^2$$

is then a left and right invariant differential operator on G^0 and again we have $\widetilde{\square}\widetilde{u} = (\square u)^{\sim}$. This proves the lemma.

We can now prove Theorem 2.11. If $g \in G$ let $L(g)$ and $R(g)$, respectively, denote the left and right translations $\ell \to g\ell$, and $\ell \to \ell g$ on G. If $\ell \cdot o = x, x \in \mathbf{S}_r(o)$ $(r > 0)$ and $g \cdot o = y$ then

$$(M^r u)(y) = \int_H \widetilde{u}(gh\ell)\, dh$$

because of (11). As g and ℓ run through sufficiently small compact neighborhoods the integration takes place within a fixed compact subset of H as remarked earlier. Denoting by subscript the argument on which a differential operator is to act we shall prove the following result.

Lemma 2.13.

$$\widetilde{\square}_\ell \left(\int_H \widetilde{u}(gh\ell)\, dh \right) = \int_H (\widetilde{\square}\widetilde{u})(gh\ell)\, dh = \widetilde{\square}_g \left(\int_H \widetilde{u}(gh\ell)\, dh \right).$$

Proof. The first equality sign follows from the left invariance of $\widetilde{\square}$. In fact, the integral on the left is

$$\int_H (\widetilde{u} \circ L(gh))(\ell)\, dh$$

so

$$\begin{aligned} \widetilde{\square}_\ell \left(\int_H \widetilde{u}(gh\ell)\, dh \right) &= \int_H \left[\widetilde{\square}(\widetilde{u} \circ L(gh)) \right] (\ell)\, dh \\ &= \int_H \left[(\widetilde{\square}\widetilde{u}) \circ L(gh) \right] (\ell)\, dh = \int_H (\widetilde{\square}\widetilde{u})(gh\ell)\, dh\,. \end{aligned}$$

The second equality in the lemma follows similarly from the right invariance of $\widetilde{\square}$. But this second equality is just the commutativity statement in Theorem 2.11.

Lemma 2.13 also implies the following analog of the Darboux equation in Lemma 3.2, Ch. I.

Corollary 2.14. *Let $u \in \mathcal{D}(X)$ and put*

$$U(y,z) \quad = \quad (M^r u)(y) \quad \textit{if } z \in \mathbf{S}_r(o)\,.$$

Then

$$\Box_y(U(y,z)) \quad = \quad \Box_z(U(y,z))\,.$$

Remark 2.15. In $\mathbf{R}^n$ the solutions to the Laplace equation $Lu = 0$ are characterized by the spherical mean-value theorem $M^r u = u$ (all r). This can be stated equivalently: $M^r u$ is a constant in r. In this latter form the mean value theorem holds for the solutions of the wave equation $\Box u = 0$ in an isotropic Lorentzian manifold: *If u satisfies $\Box u = 0$ and if u is suitably small at ∞ then $(M^r u)(o)$ is constant in r.* For a precise statement and proof see Helgason [1959], p. 289. For $\mathbf{R}^2$such a result had also been noted by Ásgeirsson.

§3 Generalized Riesz Potentials

In this section we generalize part of the theory of Riesz potentials (Ch. V, §5) to isotropic Lorentz spaces.

Consider first the case

$$X = Q_{-1} = G^-/H = \mathbf{O}_o(1,n)/\mathbf{O}_o(1,n-1)$$

of dimension n and let $f \in \mathcal{D}(X)$ and $y \in X$. If $z = \mathrm{Exp}_y Y$ $(Y \in D_y)$ we put $r_{yz} = g(Y,Y)^{1/2}$ and consider the integral

$$(I_-^\lambda f)(y) = \frac{1}{H_n(\lambda)} \int_{\mathbf{D}_y} f(z) \sinh^{\lambda-n}(r_{yz})\, dz\,, \tag{13}$$

where dz is the volume element on X, and

$$H_n(\lambda) = \pi^{(n-2)/2} 2^{\lambda-1} \Gamma\left(\lambda/2\right) \Gamma\left((\lambda+2-n)/2\right)\,. \tag{14}$$

The integral converges for $\mathrm{Re}\,\lambda \geq n$. We transfer the integral in (13) over to D_y via the diffeomorphism $\mathrm{Exp}(= \mathrm{Exp}_y)$. Since

$$dz = dr d\mathbf{w}^r = dr \left(\frac{\sinh r}{r}\right)^{n-1} d\omega^r$$

and since $dr d\omega^r$ equals the volume element dZ on D_y we obtain

$$(I^\lambda f)(y) = \frac{1}{H_n(\lambda)} \int_{D_y} (f \circ \mathrm{Exp})(Z) \Big(\frac{\sinh r}{r}\Big)^{\lambda-1} r^{\lambda-n}\, dZ\,,$$

where $r = g(Z,Z)^{1/2}$. This has the form

$$\frac{1}{H_n(\lambda)} \int_{D_y} h(Z,\lambda) r^{\lambda-n}\, dZ\,, \tag{15}$$

where $h(Z,\lambda)$, as well as each of its partial derivatives with respect to the first argument, is holomorphic in λ and h has compact support in the first variable. The methods of Riesz [1949], Ch. III, can be applied to such integrals (15). In particular we find that the function $\lambda \to (I_-^\lambda f)(y)$ which by its definition is holomorphic for $\operatorname{Re}\lambda > n$ admits a holomorphic continuation to the entire λ-plane and that its value at $\lambda = 0$ is $h(0,0) = f(y)$. (In Riesz' treatment $h(Z,\lambda)$ is independent of λ, but his method still applies.) Denoting the holomorphic continuation of (13) by $(I_-^\lambda)f(y)$ we have thus obtained

$$I_-^0 f = f\,. \tag{16}$$

We would now like to differentiate (13) with respect to y. For this we write the integral in the form $\int_F f(z)K(y,z)\,dz$ over a bounded region F which properly contains the intersection of the support of f with the closure of $\mathbf{D}_y$. The kernel $K(y,z)$ is defined as $\sinh^{\lambda-n} r_{yz}$ if $z \in \mathbf{D}_y$, otherwise 0. For $\operatorname{Re}\lambda$ sufficiently large, $K(y,z)$ is twice continuously differentiable in y so we can deduce for such λ that $I_-^\lambda f$ is of class C^2 and that

$$(\Box I_-^\lambda f)(y) = \frac{1}{H_n(\lambda)} \int_{\mathbf{D}_y} f(z)\Box_y(\sinh^{\lambda-n} r_{yz})\,dz\,. \tag{17}$$

Moreover, given $m \in \mathbb{Z}^+$ we can find k such that $I_-^\lambda f \in C^m$ for $\operatorname{Re}\lambda > k$ (and all f). Using Lemma 2.10 and the relation

$$\frac{1}{A(r)}\frac{dA}{dr} = (n-1)\coth r$$

we find

$$\begin{aligned}\Box_y(\sinh^{\lambda-n} r_{yz}) &= \Box_z(\sinh^{\lambda-n} r_{yz}) \\ &= (\lambda-n)(\lambda-1)\sinh^{\lambda-n} r_{yz} + (\lambda-n)(\lambda-2)\sinh^{\lambda-n-2} r_{yz}\,.\end{aligned}$$

We also have

$$H_n(\lambda) = (\lambda-2)(\lambda-n)H_n(\lambda-2)$$

so substituting into (17) we get

$$\Box I_-^\lambda f = (\lambda-n)(\lambda-1)I_-^\lambda f + I_-^{\lambda-2} f\,.$$

Still assuming $\operatorname{Re}\lambda$ large we can use Green's formula to express the integral

$$\int_{\mathbf{D}_y} [f(z)\Box_z(\sinh^{\lambda-n} r_{yz}) - \sinh^{\lambda-n} r_{yz}(\Box f)(z)]\,dz \tag{18}$$

as a surface integral over a part of $\mathbf{C}_y$ (on which $\sinh^{\lambda-n} r_{yz}$ and its first order derivatives vanish) together with an integral over a surface inside $\mathbf{D}_y$

(on which f and its derivatives vanish). Hence the expression (18) vanishes so we have proved the relations

$$\Box(I_-^\lambda f) = I_-^\lambda(\Box f) \tag{19}$$

$$I_-^\lambda(\Box f) = (\lambda - n)(\lambda - 1)I_-^\lambda f + I_-^{\lambda-2} f \tag{20}$$

for $\operatorname{Re}\lambda > k$, k being some number (independent of f).

Since both sides of (20) are holomorphic in λ this relation holds for all $\lambda \in \mathbf{C}$. We shall now deduce that for each $\lambda \in \mathbf{C}$, we have $I_-^\lambda f \in \mathcal{E}(X)$ and (19) holds. For this we observe by iterating (20) that for each $p \in \mathbb{Z}^+$

$$I_-^\lambda f = I_-^{\lambda+2p}(Q_p(\Box)f)\,, \tag{21}$$

Q_p being a certain p^{th}-degree polynomial. Choosing p arbitrarily large we deduce from the remark following (17) that $I_-^\lambda f \in \mathcal{E}(X)$; secondly (19) implies for $\operatorname{Re}\lambda + 2p > k$ that

$$\Box I_-^{\lambda+2p}(Q_p(\Box)f) = I_-^{\lambda+2p}(Q_p(\Box)\Box f)\,.$$

Using (21) again this means that (19) holds for all λ.

Putting $\lambda = 0$ in (20) we get

$$I_-^{-2} = \Box f - nf\,. \tag{22}$$

Remark 3.1. In Riesz' paper [1949], p. 190, an analog I^α of the potentials in Ch. V, §5, is defined for any analytic Lorentzian manifold. These potentials I^α are however different from our I_-^λ and satisfy the equation $I^{-2}f = \Box f$ in contrast to (22).

We consider next the case

$$X = Q_{+1} = G^+/H = \mathbf{O}_o(2, n-1)/\mathbf{O}_o(1, n-1)$$

and we define for $f \in \mathcal{D}(X)$

$$(I_+^\lambda f)(y) = \frac{1}{H_n(\lambda)} \int_{\mathbf{D}_y} f(z) \sin^{\lambda-n}(r_{yz})\, dz\,. \tag{23}$$

Again $H_n(\lambda)$ is given by (14) and dz is the volume element. In order to bypass the difficulties caused by the fact that the function $z \to \sin r_{yz}$ vanishes on $\mathbf{S}_\pi$ we assume that f has support disjoint from $\mathbf{S}_\pi(o)$. Then the support of f is disjoint from $\mathbf{S}_\pi(y)$ for all y in some neighborhood of o in X. We can then prove just as before that

$$(I_+^0 f)(y) = f(y) \tag{24}$$

$$(\Box I_+^\lambda f)(y) = (I_+^\lambda \Box f)(y) \tag{25}$$

$$(I_+^\lambda \Box f)(y) = -(\lambda - n)(\lambda - 1)(I_+^\lambda f)(y) + (I_+^{\lambda-2} f)(y) \tag{26}$$

for all $\lambda \in \mathbf{C}$. In particular

$$I_+^{-2} f = \Box f + n f\,. \tag{27}$$

Finally we consider the flat case

$$X = \mathbf{R}^n = G^0/H = \mathbf{R}^n \cdot \mathbf{O}_o(1, n-1)/\mathbf{O}_o(1, n-1)$$

and define

$$(I_o^\lambda f)(y) = \frac{1}{H_n(\lambda)} \int_{\mathbf{D}_y} f(z) r_{yz}^{\lambda - n}\, dz\,.$$

These are the potentials defined by Riesz in [1949], p. 31, who proved

$$I_o^0 f = f,\ \Box I_o^\lambda f = I_o^\lambda \Box f = I_o^{\lambda-2} f\,. \tag{28}$$

§4 Determination of a Function from its Integral over Lorentzian Spheres

In a Riemannian manifold a function is determined in terms of its spherical mean values by the simple relation $f = \lim_{r\to 0} M^r f$. We shall now solve the analogous problem for an even-dimensional isotropic Lorentzian manifold and express a function f in terms of its orbital integrals $M^r f$. Since the spheres $\mathbf{S}_r(y)$ do not shrink to a point as $r \to 0$ the formula (cf. Theorem 4.1) below is quite different.

For the solution of the problem we use the geometric description of the wave operator $\Box$ developed in §2, particularly its commutation with the orbital integral M^r, and combine this with the results about the generalized Riesz potentials established in §3.

We consider first the negatively curved space $X = G^-/H$. Let $n = \dim X$ and assume n even. Let $f \in \mathcal{D}(X)$, $y \in X$ and put $F(r) = (M^r f)(y)$. Since the volume element dz on $\mathbf{D}_y$ is given by $dz = dr\, d\mathbf{w}^r$ we obtain from (12) and Lemma 2.9 ,

$$(I_-^\lambda f)(y) = \frac{1}{H_n(\lambda)} \int_0^\infty \sinh^{\lambda-1} r F(r)\, dr\,. \tag{29}$$

Let $Y_1, \ldots, Y_n$ be a basis of X_y such that the Lorentzian structure is given by

$$g_y(Y) = y_1^2 - y_2^2 - \cdots - y_n^2\,, \quad Y = \sum_1^n y_i Y_i\,.$$

If $\theta_1, \ldots, \theta_{n-2}$ are geodesic polar coordinates on the unit sphere in $\mathbf{R}^{n-1}$ we put

$$\begin{aligned} y_1 &= -r\cosh\zeta \qquad (0 \le \zeta < \infty, 0 < r < \infty) \\ y_2 &= r\sinh\zeta\cos\theta_1 \\ &\vdots \\ y_n &= r\sinh\zeta\sin\theta_1 \ldots \sin\theta_{n-2}\,. \end{aligned}$$

Then $(r, \zeta, \theta_1, \ldots, \theta_{n-2})$ are coordinates on the retrograde cone D_y and the volume element on $S_r(y)$ is given by

$$d\omega^r = r^{n-1} \sinh^{n-2} \zeta \, d\zeta \, d\omega^{n-2}$$

where $d\omega^{n-2}$ is the volume element on the unit sphere in $\mathbf{R}^{n-1}$. It follows that

$$d\mathbf{w}^r = \sinh^{n-1} r \sinh^{n-2} \zeta \, d\zeta \, d\omega^{n-2}$$

and therefore

$$F(r) = \iint (f \circ \mathrm{Exp})(r, \zeta, \theta_1, \ldots, \theta_{n-2}) \sinh^{n-2} \zeta d\zeta d\omega^{n-2}, \tag{30}$$

where for simplicity

$$(r, \zeta, \theta_1, \ldots, \theta_{n-2})$$

stands for

$$(-r \cosh \zeta, r \sinh \zeta \cos \theta_1, \ldots, r \sinh \zeta \sin \theta_1 \ldots \sin \theta_{n-2}).$$

Now select A such that $f \circ \mathrm{Exp}$ vanishes outside the sphere $y_1^2 + \cdots + y_n^2 = A^2$ in X_y. Then, in the integral (30), the range of ζ is contained in the interval $(0, \zeta_o)$ where

$$r^2 \cosh^2 \zeta_o + r^2 \sinh^2 \zeta_o = A^2.$$

Then

$$r^{n-2} F(r) = \int_{\mathbf{S}^{n-2}} \int_0^{\zeta_o} (f \circ \mathrm{Exp})(r, \zeta, (\theta))(r \sinh \zeta)^{n-2} \, d\zeta \, d\omega^{n-2}.$$

Since

$$|r \sinh \zeta| \leq r e^{\zeta} \leq 2A \text{ for } \zeta \leq \zeta_o$$

this implies

$$|r^{n-2} (M^r f)(y)| \leq C A^{n-2} \sup |f|, \tag{31}$$

where C is a constant independent of r. Also substituting $t = r \sinh \zeta$ in the integral above, the ζ-integral becomes

$$\int_0^k \varphi(t) t^{n-2} (r^2 + t^2)^{-1/2} \, dt,$$

where $k = [(A^2 - r^2)/2]^{1/2}$ and φ is bounded. Thus if $n > 2$ the limit

$$a = \lim_{r \to 0} \sinh^{n-2} r F(r) \quad n > 2 \tag{32}$$

exist and is $\not\equiv 0$. Similarly, we find for $n = 2$ that the limit

$$b = \lim_{r \to 0} (\sinh r) F'(r) \qquad (n = 2) \tag{33}$$

exists.

Consider now the case $n > 2$. We can rewrite (29) in the form

$$(I_-^\lambda f)(y) = \frac{1}{H_n(\lambda)} \int_0^A \sinh^{n-2} r F(r) \sinh^{\lambda-n+1} r \, dr \,,$$

where $F(A) = 0$. We now evaluate both sides for $\lambda = n-2$. Since $H_n(\lambda)$ has a simple pole for $\lambda = n-2$ the integral has at most a simple pole there and the residue is

$$\lim_{\lambda \to n-2} (\lambda - n+2) \int_0^A \sinh^{n-2} r F(r) \sinh^{\lambda-n+1} r \, dr \,.$$

Here we can take λ real and greater than $n-2$. This is convenient since by (32) the integral is then absolutely convergent and we do not have to think of it as an implicitly given holomorphic extension. We split the integral in two parts

$$(\lambda - n+2) \int_0^A (\sinh^{n-2} r F(r) - a) \sinh^{\lambda-n+1} r \, dr$$
$$+ a(\lambda - n + 2) \int_0^A \sinh^{\lambda-n+1} r \, dr \,.$$

For the last term we use the relation

$$\lim_{\mu \to 0+} \mu \int_0^A \sinh^{\mu-1} r \, dr = \lim_{\mu \to 0+} \mu \int_0^{\sinh A} t^{\mu-1}(1+t^2)^{-1/2} \, dt = 1$$

by (38) in Chapter V. For the first term we can for each $\epsilon > 0$ find a $\delta > 0$ such that

$$|\sinh^{n-2} r F(r) - a| < \epsilon \quad \text{for } 0 < r < \delta \,.$$

If $N = \max |\sinh^{n-2} r F(r)|$ we have for $n-2 < \lambda < n-1$ the estimate

$$\big|(\lambda - n+2) \int_\delta^A (\sinh^{n-2} r F(r) - a) \sinh^{\lambda-n+1} r \, dr\big|$$
$$\leq (\lambda - n+2)(N + |a|)(A - \delta)(\sinh \delta)^{\lambda-n+1} \,;$$
$$\big|(\lambda + n-2) \int_0^\delta (\sinh^{n-2} r F(r) - a) \sinh^{\lambda-n+1} r \, dr\big|$$
$$\leq \epsilon(\lambda - n + 2) \int_0^\delta r^{\lambda-n+1} \, dr = \epsilon \delta^{\lambda-n+2} \,.$$

Taking $\lambda - (n-2)$ small enough the right hand side of each of these inequalities is $< 2\epsilon$. We have therefore proved

$$\lim_{\lambda \to n-2} (\lambda - n + 2) \int_0^\infty \sinh^{\lambda-1} r F(r) \, dr = \lim_{r \to 0} \sinh^{n-2} r \, F(r) \,.$$

Taking into account the formula for $H_n(\lambda)$ we have proved for the integral (29):

$$I_-^{n-2} f = (4\pi)^{(2-n)/2} \frac{1}{\Gamma((n-2)/2)} \lim_{r\to o} \sinh^{n-2} r \; M^r f \,. \tag{34}$$

On the other hand, using formula (20) recursively we obtain for $u \in \mathcal{D}(X)$

$$I_-^{n-2}(Q(\Box)u) = u$$

where

$$Q(\Box) = (\Box + (n-3)2)(\Box + (n-5)4)\cdots(\Box + 1(n-2))\,.$$

We combine this with (34) and use the commutativity $\Box M^r = M^r \Box$. This gives

$$u = (4\pi)^{(2-n)/2} \frac{1}{\Gamma((n-2)/2)} \lim_{r\to 0} \sinh^{n-2} r \, Q(\Box) M^r u \,. \tag{35}$$

Here we can for simplicity replace $\sinh r$ by r.

For the case $n = 2$ we have by (29)

$$(I_-^2 f)(y) = \frac{1}{H_2(2)} \int_0^\infty \sinh r F(r)\, dr \,. \tag{36}$$

This integral which in effect only goes from 0 to A is absolutely convergent because our estimate (31) shows (for $n = 2$) that $rF(r)$ is bounded near $r = 0$. But using (20), Lemma 2.10, Theorem 2.11 and Cor. 2.14, we obtain for $u \in \mathcal{D}(X)$

$$\begin{aligned} u &= I_-^2 \Box u = \tfrac{1}{2} \int_0^\infty \sinh r M^r \Box u \, dr \\ &= \tfrac{1}{2} \int_0^\infty \sinh r \Box M^r u \, dr = \tfrac{1}{2} \int_0^\infty \sinh r \left(\frac{d^2}{dr^2} + \coth r \frac{d}{dr} \right) M^r u \, dr \\ &= \tfrac{1}{2} \int_0^\infty \frac{d}{dr} \left(\sinh r \frac{d}{dr} M^r u \right) dr = -\tfrac{1}{2} \lim_{r \to 0} \sinh r \frac{d(M^r u)}{dr} \,. \end{aligned}$$

This is the substitute for (35) in the case $n = 2$.

The spaces G^+/H and G^o/H can be treated in the same manner. We have thus proved the following principal result of this chapter.

Theorem 4.1. *Let X be one of the isotropic Lorentzian manifolds G^-/H, G^o/H, G^+/H. Let κ denote the curvature of X ($\kappa = -1, 0, +1$) and assume $n = \dim X$ to be even, $n = 2m$. Put*

$$Q(\Box) = (\Box - \kappa(n-3)2)(\Box - \kappa(n-5)4)\cdots(\Box - \kappa 1(n-2))\,.$$

Then if $u \in \mathcal{D}(X)$

$$\begin{aligned} u &= c \lim_{r \to 0} r^{n-2} Q(\Box)(M^r u)\,, \qquad (n \neq 2) \\ u &= \tfrac{1}{2} \lim_{r \to 0} r \frac{d}{dr}(M^r u) \qquad (n = 2)\,. \end{aligned}$$

Here $c^{-1} = (4\pi)^{m-1}(m-2)!$ *and* $\Box$ *is the Laplace-Beltrami operator on* X*.*

§5 Orbital Integrals and Huygens' Principle

We shall now write out the limit in (35) and thereby derive a statement concerning Huygens' principle for $\Box$. As $r \to 0$, $\mathbf{S}_r(o)$ has as limit the boundary $C_R = \partial \mathbf{D}_o - \{o\}$ which is still an H-orbit. The limit

(37) $$\lim_{r \to 0} r^{n-2}(M_r v)(o) \qquad v \in C_c(X - o)$$

is by (31)–(32) a positive H-invariant functional with support in the H-orbit C_R, which is closed in $X - o$. Thus the limit (37) only depends on the restriction $v|C_R$. Hence it is "the" H-invariant measure on C_R and we denote it by μ. Thus

(38) $$\lim_{r \to 0} r^{n-2}\,(M_r v)(o) = \int_{C_R} v(z)\, d\mu(z)\,.$$

To extend this to $u \in \mathcal{D}(X)$, let $A > 0$ be arbitrary and let φ be a "smoothed out" characteristic function of $\operatorname{Exp} B_A$. Then if

$$u_1 = u\varphi\,,\; u_2 = u(1-\varphi)$$

we have

$$\begin{aligned} &\left| r^{n-2}(M^r u)(o) - \int_{C_R} u(z)\, d\mu(z) \right| \\ &\quad \leq \left| r^{n-2}(M^r u_1)(o) - \int_{C_R} u_1(z)\, d\mu(z) \right| + \left| r^{n-2}(M^r u_2)(o) - \int_{C_R} u_2(z)\, d\mu(z) \right| . \end{aligned}$$

By (31) the first term on the right is $O(A)$ uniformly in r and by (38) the second tends to 0 as $r \to 0$. Since A is arbitrary (38) holds for $u \in \mathcal{D}(X)$.

Corollary 5.1. *Let* $n = 2m\,(m > 1)$ *and* δ *the delta distribution at* o*. Then*

(39) $$\delta = c\, Q(\Box)\mu\,,$$

where $c^{-1} = (4\pi)^{m-1}(m-2)!$.

In fact, by (35), (38) and Theorem 2.11

$$u = c \lim_{r \to 0} r^{n-2} (M^r Q(\Box)u)(o) = c \int_{C_R} (Q(\Box)u)(z)\, d\mu(z)$$

and this is (39).

Remark 5.2. Formula (39) shows that each factor

$$\Box_k = \Box - \kappa(n-k)(k-1) \quad k = 3, 5, \ldots, n-1 \tag{40}$$

in $Q(\Box)$ has fundamental solution supported on the *retrograde conical surface* $\overline{C}_R$. This is known to be the equivalent to the validity of Huygens' principle for the Cauchy problem for the equation $\Box_k u = 0$ (see Gűnther [1991] and [1988], Ch. IV, Cor. 1.13). For a recent survey on Huygens' principle see Berest [1998].

Bibliographical Notes

§1. The construction of the constant curvature spaces (Theorems 1.3 and 1.5) was given by the author ([1959], [1961]). The proof of Lemma 1.4 on the connectivity is adapted from Boerner [1955]. For more information on isotropic manifolds (there is more than one definition) see Tits [1955], p. 183 and Wolf [1967].

§§2-4. This material is based on Ch. IV in Helgason [1959]. Corollary 5.1 with a different proof and the subsequent remark were shown to me by Schlichtkrull. See Schimming and Schlichtkrull [1994] (in particular Lemma 6.2) where it is also shown that the constants $c_k = -\kappa(n-k)(k-1)$ in (40) are *the only ones* for which $\Box + c_k$ satisfies Huygens' principle. Here it is of interest to recall that in the flat Lorentzian case $\mathbf{R}^{2m}$, $\Box + c$ satisfies Huygens' principle only for $c = 0$. Theorem 4.1 was extended to pseudo-Riemannian manifolds of constant curvature by Orloff [1985], [1987]. For recent representative work on orbital integrals see e.g. Bouaziz [1995], Flicker [1996], Harinck [1998], Renard [1997].

CHAPTER V

FOURIER TRANSFORMS AND DISTRIBUTIONS. A RAPID COURSE

§1 The Topology of the Spaces $\mathcal{D}(\mathbf{R}^n)$, $\mathcal{E}(\mathbf{R}^n)$ and $\mathcal{S}(\mathbf{R}^n)$

.

Let $\mathbf{R}^n = \{x = (x_1, \ldots, x_n) : x_i \in \mathbf{R}\}$ and let ∂_i denote $\partial/\partial x_i$. If $(\alpha_1, \ldots, \alpha_n)$ is an n-tuple of integers $\alpha_i \geq 0$ we put $\alpha! = \alpha_1! \cdots \alpha_n!$,

$$D^\alpha = \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}\,,\ x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}\,,\ |\alpha| = \alpha_1 + \cdots + \alpha_n\,.$$

For a complex number c, $\operatorname{Re} c$ and $\operatorname{Im} c$ denote respectively, the real part and the imaginary part of c. For a given compact set $K \subset \mathbf{R}^n$ let

$$\mathcal{D}_K = \mathcal{D}_K(\mathbf{R}^n) = \{f \in \mathcal{D}(\mathbf{R}^n) : \operatorname{supp}(f) \subset K\}\,,$$

where supp stands for support. The space $\mathcal{D}_K$ is topologized by the seminorms

$$\|f\|_{K,m} = \sum_{|\alpha| \leq m} \sup_{x \in K} |(D^\alpha f)(x))|, \quad m \in \mathbb{Z}^+\,. \tag{1}$$

The topology of $\mathcal{D} = \mathcal{D}(\mathbf{R}^n)$ is defined as the largest locally convex topology for which all the embedding maps $\mathcal{D}_K \to \mathcal{D}$ are continuous. This is the so-called *inductive limit* topology. More explicitly, this topology is characterized as follows:

A *convex* set $C \subset \mathcal{D}$ is a neighborhood of 0 in $\mathcal{D}$ if and only if for each compact set $K \subset \mathbf{R}^n$, $C \cap \mathcal{D}_K$ is a neighborhood of 0 in $\mathcal{D}_K$.

A fundamental system of neighborhoods in $\mathcal{D}$ can be characterized by the following theorem. If B_R denotes the ball $|x| < R$ in $\mathbf{R}^n$ then

$$\mathcal{D} = \cup_{j=0}^{\infty} \mathcal{D}_{\overline{B}_j}\,. \tag{2}$$

Theorem 1.1. *Given two monotone sequences*

$$\begin{aligned} \{\epsilon\} &= \epsilon_0, \epsilon_1, \epsilon_2, \ldots, \qquad && \epsilon_i \to 0 \\ \{N\} &= N_0, N_1, N_2, \ldots, && N_i \to \infty \quad N_i \in \mathbb{Z}^+ \end{aligned}$$

let $V(\{\epsilon\}, \{N\})$ *denote the set of functions* $\varphi \in \mathcal{D}$ *satisfying for each* j *the conditions*

$$|(D^\alpha \varphi)(x)| \leq \epsilon_j \quad \textit{for } |\alpha| \leq N_j, \quad x \notin B_j\,. \tag{3}$$

Then the sets $V(\{\epsilon\}, \{N\})$ *form a fundamental system of neighborhoods of* 0 *in* $\mathcal{D}$.

Proof. It is obvious that each $V(\{\epsilon\},\{N\})$ intersects each $\mathcal{D}_K$ in a neighborhood of 0 in $\mathcal{D}_K$. Conversely, let W be a *convex* subset of $\mathcal{D}$ intersecting each $\mathcal{D}_K$ in a neighborhood of 0. For each $j \in \mathbb{Z}^+$, $\exists N_j \in \mathbb{Z}^+$ and $\eta_j > 0$ such that each $\varphi \in \mathcal{D}$ satisfying

$$|D^\alpha \varphi(x)| \leq \eta_j \text{ for } |\alpha| \leq N_j \quad \operatorname{supp}(\varphi) \subset \overline{B}_{j+2}$$

belongs to W. Fix a sequence (β_j) with

$$\beta_j \in \mathcal{D}, \beta_j \geq 0\,,\ \Sigma\beta_j = 1\,,\ \operatorname{supp}(\beta_j) \subset \overline{B}_{j+2} - B_j$$

and write for $\varphi \in \mathcal{D}$,

$$\varphi = \sum_j \frac{1}{2^{j+1}} (2^{j+1}\beta_j\varphi)\,.$$

Then by the convexity of W, $\varphi \in W$ if each function $2^{j+1}\beta_j\varphi$ belongs to W. However, $D^\alpha(\beta_j\varphi)$ is a finite linear combination of derivatives $D^\beta\beta_j$ and $D^\gamma\varphi$, $(|\beta|, |\gamma| \leq |\alpha|)$. Since (β_j) is fixed and only values of φ in $\overline{B}_{j+2} - B_j$ enter, $\exists$ constant k_j such that the condition

$$|(D^\alpha\varphi)(x)| \leq \epsilon_j \text{ for } |x| \geq j \text{ and } |\alpha| \leq N_j$$

implies

$$|2^{j+1} D^\alpha(\beta_j\varphi)(x)| \leq k_j\epsilon_j \quad \text{for } |\alpha| \leq N_j, \quad \text{all } x\,.$$

Choosing the sequence $\{\epsilon\}$ such that $k_j\epsilon_j \leq \eta_j$ for all j we deduce for each j

$$\varphi \in V(\{\epsilon\},\{N\}) \Rightarrow 2^{j+1}\beta_j\varphi \in W\,,$$

whence $\varphi \in W$.

The space $\mathcal{E} = \mathcal{E}(\mathbf{R}^n)$ is topologized by the seminorms (1) for the varying K. Thus the sets

$$V_{j,k,\ell} = \{\varphi \in \mathcal{E}(\mathbf{R}^n) : \|\varphi\|_{\overline{B}_j,k} < 1/\ell \quad j,k,\ell \in \mathbb{Z}^+$$

form a fundamental system of neighborhoods of 0 in $\mathcal{E}(\mathbf{R}^n)$. This system being countable the topology of $\mathcal{E}(\mathbf{R}^n)$ is defined by sequences: A point $\varphi \in \mathcal{E}(\mathbf{R}^n)$ belongs to the closure of a subset $A \subset \mathcal{E}(\mathbf{R}^n)$ if and only if φ is the limit of a sequence in A. It is important to realize that this fails for the topology of $\mathcal{D}(\mathbf{R}^n)$ since the family of sets $V(\{\epsilon\},\{N\})$ is uncountable.

The space $\mathcal{S} = \mathcal{S}(\mathbf{R}^n)$ of rapidly decreasing functions on $\mathbf{R}^n$ is topologized by the seminorms (6), Ch. I. We can restrict the P in (6), Ch. I to polynomials with rational coefficients.

In contrast to the space $\mathcal{D}$ the spaces $\mathcal{D}_K$, $\mathcal{E}$ and $\mathcal{S}$ are Fréchet spaces, that is their topologies are given by a countable family of seminorms.

The spaces $\mathcal{D}_K(M)$, $\mathcal{D}(M)$ and $\mathcal{E}(M)$ can be topologized similarly if M is a manifold.

§2 Distributions

A *distribution* by definition is a member of the dual space $\mathcal{D}'(\mathbf{R}^n)$ of $\mathcal{D}(\mathbf{R}^n)$. By the definition of the topology of $\mathcal{D}$, $T \in \mathcal{D}'$ if and only if the restriction $T|\mathcal{D}_K$ is continuous for each compact set $K \subset \mathbf{R}^n$. Each locally integrable function F on $\mathbf{R}^n$ gives rise to a distribution $\varphi \to \int \varphi(x)F(x)\,dx$. A measure on $\mathbf{R}^n$ is also a distribution.

The *derivative* $\partial_i T$ of a distribution T is by definition the distribution $\varphi \to -T(\partial_i\varphi)$. If $F \in C^1(\mathbf{R}^n)$ then the distributions $T_{\partial_i F}$ and $\partial_i(T_F)$ coincide (integration by parts).

A *tempered distribution* by definition is a member of the dual space $\mathcal{S}'(\mathbf{R}^n)$. Since the imbedding $\mathcal{D} \to \mathcal{S}$ is continuous the restriction of a $T \in \mathcal{S}'$ to $\mathcal{D}$ is a distribution; since $\mathcal{D}$ is dense in $\mathcal{S}$ two tempered distributions coincide if they coincide on $\mathcal{D}$. In this sense we have $\mathcal{S}' \subset \mathcal{D}'$.

Since distributions generalize measures it is sometimes convenient to write

$$T(\varphi) = \int \varphi(x)\,dT(x)$$

for the value of a distribution on the function φ. A distribution T is said to be 0 on an open set $U \subset \mathbf{R}^n$ if $T(\varphi) = 0$ for each $\varphi \in \mathcal{D}$ with support contained in U. Let U be the union of all open sets $U_\alpha \subset \mathbf{R}^n$ on which T is 0. Then $T = 0$ on U. In fact, if $f \in \mathcal{D}(U)$, $\operatorname{supp}(f)$ can be covered by finitely many U_α, say $U_1, \ldots, U_r$. Then $U_1, \ldots, U_r, \mathbf{R}^n - \operatorname{supp}(f)$ is a covering of $\mathbf{R}^n$. If $1 = \sum_1^{r+1} \varphi_i$ is a corresponding partition of unity we have $f = \sum_1^r \varphi_i f$ so $T(f) = 0$. The complement $\mathbf{R}^n - U$ is called the *support of* T, denoted $\operatorname{supp}(T)$.

A distribution T of compact support extends to a unique element of $\mathcal{E}'(\mathbf{R}^n)$ by putting

$$T(\varphi) = T(\varphi\varphi_0)\,, \quad \varphi \in \mathcal{E}(\mathbf{R}^n)$$

if φ_0 is any function in $\mathcal{D}$ which is identically 1 on a neighborhood of $\operatorname{supp}(T)$. Since $\mathcal{D}$ is dense in $\mathcal{E}$, this extension is unique. On the other hand let $\tau \in \mathcal{E}'(\mathbf{R}^n)$, T its restriction to $\mathcal{D}$. Then $\operatorname{supp}(T)$ is compact. Otherwise we could for each j find $\varphi_j \in \mathcal{E}$ such that $\varphi_j \equiv 0$ on $\overline{B}_j$ but $T(\varphi_j) = 1$. Then $\varphi_j \to 0$ in $\mathcal{E}$, yet $\tau(\varphi_j) = 1$ which is a contradiction.

This identifies $\mathcal{E}'(\mathbf{R}^n)$ with the space of distributions of compact support and we have the following canonical inclusions:

$$\begin{array}{ccc} \mathcal{D}(\mathbf{R}^n) & \subset \mathcal{S}(\mathbf{R}^n) & \subset \mathcal{E}(\mathbf{R}^n) \\ \cap & \cap & \cap \\ \mathcal{E}'(\mathbf{R}^n) & \subset \mathcal{S}'(\mathbf{R}^n) & \subset \mathcal{D}'(\mathbf{R}^n)\,. \end{array}$$

If S and T are two distributions, at least one of compact support, their *convolution* is the distribution $S * T$ defined by

$$\varphi \to \int \varphi(x+y)\, dS(x)\, dT(y), \quad \varphi \in \mathcal{D}(\mathbf{R}^n)\,. \tag{4}$$

If $f \in \mathcal{D}$ the distribution $T_f * T$ has the form T_g where

$$g(x) = \int f(x-y)\, dT(y)$$

so we write for simplicity $g = f * T$. Note that $g(x) = 0$ unless $x - y \in \operatorname{supp}(f)$ for some $y \in \operatorname{supp}(T)$. Thus $\operatorname{supp}(g) \subset \operatorname{supp}(f) + \operatorname{supp} T$. More generally,

$$\operatorname{supp}(S * T) \subset \operatorname{supp}(S) + \operatorname{supp} T$$

as one sees from the special case $S = T_g$ by approximating S by functions $S * \varphi_\epsilon$ with $\operatorname{supp}(\varphi_\epsilon) \subset B_\epsilon(0)$.

The convolution can be defined for more general S and T, for example if $S \in \mathcal{S}$, $T \in \mathcal{S}'$ then $S * T \in \mathcal{S}'$. Also $S \in \mathcal{E}'$, $T \in \mathcal{S}'$ implies $S * T \in \mathcal{S}'$.

§3 The Fourier Transform

For $f \in L^1(\mathbf{R}^n)$ the *Fourier transform* is defined by

$$\widetilde{f}(\xi) = \int_{\mathbf{R}^n} f(x) e^{-i\langle x, \xi\rangle}\, dx, \quad \xi \in \mathbf{R}^n\,. \tag{5}$$

If f has compact support we can take $\xi \in \mathbf{C}^n$. For $f \in \mathcal{S}(\mathbf{R}^n)$ one proves quickly

$$i^{|\alpha|+|\beta|} \xi^\beta (D^\alpha \widetilde{f})(\xi) = \int_{\mathbf{R}^n} D^\beta (x^\alpha f(x)) e^{-i\langle x, \xi\rangle}\, dx \tag{6}$$

and this implies easily the following result.

Theorem 3.1. *The Fourier transform is a linear homeomorphism of $\mathcal{S}$ onto $\mathcal{S}$.*

The function $\psi(x) = e^{-x^2/2}$ on $\mathbf{R}$ satisfies $\psi'(x) + x\psi = 0$. It follows from (6) that $\widetilde{\psi}$ satisfies the same differential equation and thus is a constant multiple of $e^{-\xi^2/2}$. Since $\widetilde{\psi}(0) = \int e^{-\frac{x^2}{2}}\, dx = (2\pi)^{1/2}$ we deduce $\widetilde{\psi}(\xi) = (2\pi)^{1/2} e^{-\xi^2/2}$. More generally, if $\psi(x) = e^{-|x|^2/2}$, $(x \in \mathbf{R}^n)$ then by product integration

$$\widetilde{\psi}(\xi) = (2\pi)^{n/2} e^{-|\xi|^2/2}\,. \tag{7}$$

Theorem 3.2. *The Fourier transform has the following properties.*

(i) $f(x) = (2\pi)^{-n} \int \widetilde{f}(\xi) e^{i\langle x,\xi\rangle}\, d\xi$ *for* $f \in \mathcal{S}$.

(ii) $f \to \widetilde{f}$ *extends to a bijection of* $L^2(\mathbf{R}^n)$ *onto itself and*

$$\int_{\mathbf{R}^n} |f(x)|^2 = (2\pi)^{-n} \int_{\mathbf{R}^n} |\widetilde{f}(\xi)|^2\, d\xi\,.$$

(iii) $(f_1 * f_2)^\sim = \widetilde{f}_1 \widetilde{f}_2$ *for* $f_1, f_2 \in \mathcal{S}$.

(iv) $(f_1 f_2)^\sim = (2\pi)^{-n} \widetilde{f}_1 * \widetilde{f}_2$ *for* $f_1, f_2 \in \mathcal{S}$.

Proof. (i) The integral on the right equals

$$\int e^{i\langle x,\xi\rangle} \left(\int f(y) e^{-i\langle y,\xi\rangle}\, dy \right) d\xi$$

but here we cannot exchange the integrations. Instead we consider for $g \in \mathcal{S}$ the integral

$$\int e^{i\langle x,\xi\rangle} g(\xi) \left(\int f(y) e^{-i\langle y,\xi\rangle}\, dy \right) d\xi\,,$$

which equals the expressions

$$\int \widetilde{f}(\xi) g(\xi) e^{i\langle x,\xi\rangle}\, d\xi = \int f(y) \widetilde{g}(y - x)\, dy = \int f(x + y) \widetilde{g}(y)\, dy\,. \tag{8}$$

Replace $g(\xi)$ by $g(\epsilon\xi)$ whose Fourier transform is $\epsilon^{-n}\widetilde{g}(y/\epsilon)$. Then we obtain

$$\int \widetilde{f}(\xi) g(\epsilon\xi) e^{i\langle x,\xi\rangle}\, d\xi = \int \widetilde{g}(y) f(x + \epsilon y)\, dy\,,$$

which upon letting $\epsilon \to 0$ gives

$$g(0) \int \widetilde{f}(\xi) e^{i\langle x,\xi\rangle}\, d\xi = f(x) \int \widetilde{g}(y)\, dy\,.$$

Taking $g(\xi)$ as $e^{-|\xi|^2/2}$ and using (7) Part (i) follows. The identity in (ii) follows from (8) (for $x = 0$) and (i). It implies that the image $L^2(\mathbf{R}^n)^\sim$ is closed in $L^2(\mathbf{R}^n)$. Since it contains the dense subspace $\mathcal{S}(\mathbf{R}^n)$ (ii) follows. Formula (iii) is an elementary computation and now (iv) follows taking (i) into account.

If $T \in \mathcal{S}'(\mathbf{R}^n)$ its Fourier transform is the linear form $\widetilde{T}$ on $\mathcal{S}(\mathbf{R}^n)$ defined by

$$\widetilde{T}(\varphi) = T(\widetilde{\varphi})\,. \tag{9}$$

Then by Theorem 3.1, $\widetilde{T} \in \mathcal{S}'$. Note that

$$\int \varphi(\xi) \widetilde{f}(\xi)\, d\xi = \int \widetilde{\varphi}(x) f(x)\, dx \tag{10}$$

for all $f \in L^1(\mathbf{R}^n)$, $\varphi \in \mathcal{S}(\mathbf{R}^n)$. Consequently

$$(T_f)^\sim = T_{\widetilde{f}} \quad \text{for } f \in L^1(\mathbf{R}^n) \tag{11}$$

so the definition (9) extends the old one (5). If $S_1, S_2 \in \mathcal{E}'(\mathbf{R}^n)$ then $\widetilde{S}_1$ and $\widetilde{S}_2$ have the form T_{s_1} and T_{s_2} where $s_1, s_2 \in \mathcal{E}(\mathbf{R}^n)$ and in addition $(S_1 * S_2)^\sim = T_{s_1 s_2}$. We express this in the form

$$(S_1 * S_2)^\sim = \widetilde{S}_1 \widetilde{S}_2 \,. \tag{12}$$

This formula holds also in the cases

$$S_1 \in \mathcal{S}(\mathbf{R}^n), \quad S_2 \in \mathcal{S}'(\mathbf{R}^n)\,,$$
$$S_1 \in \mathcal{E}'(\mathbf{R}^n), \quad S_2 \in \mathcal{S}'(\mathbf{R}^n)$$

and $S_1 * S_2 \in \mathcal{S}'(\mathbf{R}^n)$ (cf. Schwartz [1966], p. 268).

The classical Paley-Wiener theorem gave an intrinsic description of $L^2(0, 2\pi)^\sim$. We now prove an extension to a characterization of $\mathcal{D}(\mathbf{R}^n)^\sim$ and $\mathcal{E}'(\mathbf{R}^n)^\sim$.

Theorem 3.3. *(i) A holomorphic function $F(\zeta)$ on $\mathbf{C}^n$ is the Fourier transform of a distribution with support in $\overline{B}_R$ if and only if for some constants C and $N \geq 0$ we have*

$$|F(\zeta)| \leq C(1 + |\zeta|^N) e^{R|\mathrm{Im}\,\zeta|}\,. \tag{13}$$

(ii) $F(\zeta)$ is the Fourier transform of a function in $\mathcal{D}_{\bar{B}_R}(\mathbf{R}^n)$ if and only if for each $N \in \mathbb{Z}^+$ there exists a constant C_N such that

$$|F(\zeta)| \leq C_N (1 + |\zeta|)^{-N} e^{R|\mathrm{Im}\,\zeta|}\,. \tag{14}$$

Proof. First we prove that (13) is necessary. Let $T \in \mathcal{E}'$ have support in $\overline{B}_R$ and let $\chi \in \mathcal{D}$ have support in $\overline{B_{R+1}}$ and be identically 1 in a neighborhood of $\overline{B}_R$. Since $\mathcal{E}(\mathbf{R}^n)$ is topologized by the semi-norms (1) for varying K and m we have for some $C_0 \geq 0$ and $N \in \mathbb{Z}^+$

$$|T(\varphi)| = |T(\chi\varphi)| \leq C_0 \sum_{|\alpha| \leq N} \sup_{x \in \overline{B_{R+1}}} |(D^\alpha(\chi\varphi))(x)|\,.$$

Computing $D^\alpha(\chi\varphi)$ we see that for another constant C_1

$$|T(\varphi)| \leq C_1 \sum_{|\alpha| \leq N} \sup_{x \in \mathbf{R}^n} |D^\alpha \varphi(x)|, \quad \varphi \in \mathcal{E}(\mathbf{R}^n). \tag{15}$$

Let $\psi \in \mathcal{E}(\mathbf{R})$ such that $\psi \equiv 1$ on $(-\infty, \frac{1}{2})$, and $\equiv 0$ on $(1, \infty)$. Then if $\zeta \neq 0$ the function

$$\varphi_\zeta(x) = e^{-i\langle x, \zeta\rangle} \psi(|\zeta|(|x| - R))$$

belongs to $\mathcal{D}$ and equals $e^{-i\langle x,\zeta\rangle}$ in a neighborhood of $\overline{B}_R$. Hence

$$|\widetilde{T}(\zeta)| = |T(\varphi_\zeta)| \leq C_1 \sum_{|\alpha|\leq N} \sup |D^\alpha \varphi_\zeta| \,. \tag{16}$$

Now $\mathrm{supp}(\varphi_\zeta) \subset \overline{B}_{R+|\zeta|^{-1}}$ and on this ball

$$|e^{-i\langle x,\zeta\rangle}| \leq e^{|x|\,|\mathrm{Im}\,\zeta|} \leq e^{(R+|\zeta|^{-1})|\mathrm{Im}\,\zeta|} \leq e^{R|\mathrm{Im}\,\zeta|+1} \,.$$

Estimating $D^\alpha\varphi_\zeta$ similarly we see that by (16), $\widetilde{T}(\zeta)$ satisfies (13).

The necessity of (14) is an easy consequence of (6).

Next we prove the sufficiency of (14). Let

$$f(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} F(\xi) e^{i\langle x,\xi\rangle}\, d\xi \,. \tag{17}$$

Because of (14) we can shift the integration in (17) to the complex domain so that for any fixed $\eta \in \mathbf{R}^n$,

$$f(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} F(\xi + i\eta) e^{i\langle x,\xi+i\eta\rangle}\, d\xi \,.$$

We use (14) for $N = n+1$ to estimate this integral and this gives

$$|f(x)| \leq C_N e^{R|\eta| - \langle x,\eta\rangle} (2\pi)^{-n} \int_{\mathbf{R}^n} (1+|\xi|)^{-(n+1)}\, d\xi \,.$$

Taking now $\eta = tx$ and letting $t \to +\infty$ we deduce $f(x) = 0$ for $|x| > R$.

For the sufficiency of (13) we note first that F as a distribution on $\mathbf{R}^n$ is tempered. Thus $F = \widetilde{f}$ for some $f \in \mathcal{S}'(\mathbf{R}^n)$. Convolving f with a $\varphi \in \mathcal{D}_{\overline{B}_\epsilon}$ we see that $f * \varphi$ satisfies estimates (14) with R replaced by $R+\epsilon$. Thus $\mathrm{supp}(f * \varphi_\epsilon) \subset \overline{B_{R+\epsilon}}$. Letting $\epsilon \to 0$ we deduce $\mathrm{supp}(f) \subset \overline{B_R}$, concluding the proof.

We shall now prove a refinement of Theorem 3.3 in that the topology of $\mathcal{D}$ is described in terms of $\widetilde{\mathcal{D}}$. This has important applications to differential equations as we shall see in the next section.

Theorem 3.4. *A convex set $V \subset \mathcal{D}$ is a neighborhood of* 0 *in $\mathcal{D}$ if and only if there exist positive sequences*

$$M_0, M_1, \ldots,\ \delta_0, \delta_1, \ldots$$

such that V contains all $u \in \mathcal{D}$ satisfying

$$|\widetilde{u}(\zeta)| \leq \sum_{k=0}^{\infty} \delta_k \frac{1}{(1+|\zeta|)^{M_k}} e^{k|\mathrm{Im}\,\zeta|}, \quad \zeta \in \mathbf{C}^n \,. \tag{18}$$

The proof is an elaboration of that of Theorem 3.3. Instead of the contour shift $\mathbf{R}^n \to \mathbf{R}^n + i\eta$ used there one now shifts $\mathbf{R}^n$ to a contour on which the two factors on the right in (14) are comparable.

Let $W(\{\delta\},\{M\})$ denote the set of $u \in \mathcal{D}$ satisfying (18). Given k the set

$$W_k = \{u \in \mathcal{D}_{\overline{B}_k} : |\widetilde{u}(\zeta)| \le \delta_k (1+|\zeta|)^{-M_k} e^{k|\operatorname{Im}\zeta|}\}$$

is contained in $W(\{\delta\},\{M\})$. Thus if V is a convex set containing $W(\{\delta\},\{M\})$ then $V \cap \mathcal{D}_{\overline{B}_k}$ contains W_k which is a neighborhood of 0 in $\mathcal{D}_{\overline{B}_k}$ (because the bounds on $\widetilde{u}$ correspond to the bounds on the $\|u\|_{\overline{B}_k, M_k}$). Thus V is a neighborhood of 0 in $\mathcal{D}$.

Proving the converse amounts to proving that given $V(\{\epsilon\},\{N\})$ in Theorem 1.1 there exist $\{\delta\},\{M\}$ such that

$$W(\{\delta\},\{M\}) \subset V(\{\epsilon\},\{N\})\,.$$

For this we shift the contour in (17) to others where the two factors in (14) are comparable. Let

$$\begin{aligned} x &= (x_1,\dots,x_n), \qquad & x' &= (x_1,\dots,x_{n-1}) \\ \zeta &= (\zeta_1,\dots,\zeta_n) & \zeta' &= (\zeta_1,\dots,\zeta_{n-1}) \\ \zeta &= \xi + i\eta, & \xi,\eta &\in \mathbf{R}^n\,. \end{aligned}$$

Then

$$\text{(19)} \qquad \int_{\mathbf{R}^n} \widetilde{u}(\xi) e^{i\langle x,\xi\rangle}\, d\xi = \int_{\mathbf{R}^{n-1}} e^{i\langle x',\xi'\rangle}\, d\xi' \int_{\mathbf{R}} e^{ix_n\xi_n} \widetilde{u}(\xi',\xi_n)\, d\xi_n\,.$$

In the last integral we shift from $\mathbf{R}$ to the contour in $\mathbf{C}$ given by

$$\text{(20)} \qquad \gamma_m : \zeta_n = \xi_n + im\log(1 + (|\xi'|^2 + \xi_n^2)^{1/2})$$

$m \in \mathbb{Z}^+$ being fixed.

We claim that (cf. Fig. V.1)

$$\text{(21)} \qquad \int_{\mathbf{R}} e^{ix_n\xi_n} \widetilde{u}(\xi',\xi_n)\, d\xi_n = \int_{\gamma_m} e^{ix_n\zeta_n} \widetilde{u}(\xi',\zeta_n)\, d\zeta_n\,.$$

Since (14) holds for each N, $\widetilde{u}$ decays between ξ_n-axis and γ_m faster than any $|\zeta_n|^{-M}$. Also

$$\left|\frac{d\zeta_n}{d\xi_n}\right| = \left|1 + i\,m\,\frac{1}{1+|\xi|}\cdot\frac{\partial(|\xi|)}{\partial\xi_n}\right| \le 1+m\,.$$

Thus (21) follows from Cauchy's theorem in *one* variable. Putting

$$\Gamma_m = \{\zeta \in \mathbf{C}^n : \zeta' \in \mathbf{R}^{n-1}, \zeta_n \in \gamma_m\}$$

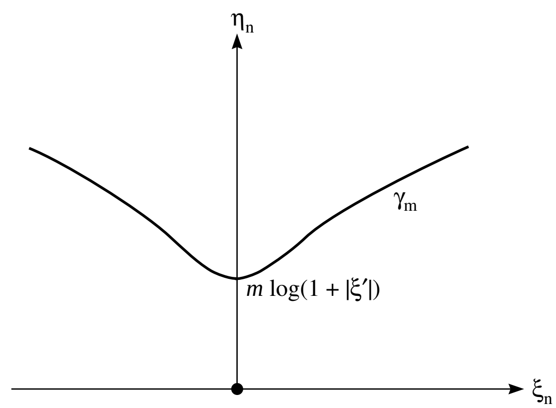

FIGURE V.1.

we thus have with $d\zeta = d\xi_1 \dots d\xi_{n-1}\, d\zeta_n$,

$$u(x) = (2\pi)^{-n} \int_{\Gamma_m} \widetilde{u}(\zeta) e^{i\langle x,\zeta\rangle}\, d\zeta\,. \tag{22}$$

Now suppose the sequences $\{\epsilon\}$, $\{N\}$ and $V(\{\epsilon\},\{N\})$ are given as in Theorem 1.1. We have to construct sequences $\{\delta\}$ $\{M\}$ such that (18) implies (3). By rotational invariance we may assume $x = (0,\dots,0,x_n)$ with $x_n > 0$. For each n-tuple α we have

$$(D^\alpha u)(x) = (2\pi)^{-n} \int_{\Gamma_m} \widetilde{u}(\zeta)(i\zeta)^\alpha e^{i\langle x,\zeta\rangle}\, d\zeta\,.$$

Starting with positive sequences $\{\delta\}$, $\{M\}$ we shall modify them successively such that (18) $\Rightarrow$ (3). Note that for $\zeta \in \Gamma_m$

$$e^{k|\operatorname{Im}\zeta|} \leq (1+|\xi|)^{km}$$

$$|\zeta^\alpha| \leq |\zeta|^{|\alpha|} \leq ([|\xi|^2 + m^2(\log(1+|\xi|))^2]^{1/2})^{|\alpha|}\,.$$

For (3) with $j = 0$ we take $x_n = |x| \geq 0$, $|\alpha| \leq N_0$ so

$$|e^{i\langle x,\zeta\rangle}| = e^{-\langle x,\operatorname{Im}\zeta|} \leq 1 \quad \text{for } \zeta \in \Gamma_m\,.$$

Thus if u satisfies (18) we have by the above estimates

$$\begin{aligned} &|(D^\alpha u)(x)| \\ &\leq \sum_0^\infty \delta_k \int_{\mathbf{R}^n} (1 + [|\xi|^2 + m^2(\log(1+|\xi|))^2]^{1/2})^{N_0 - M_k} (1+|\xi|)^{km}(1+m)\, d\xi\,. \end{aligned} \tag{23}$$

We can choose sequences $\{\delta\}$, $\{M\}$ (all δ_k, $M_k > 0$) such that this expression is $\leq \epsilon_0$. This then verifies (3) for $j = 0$. We now fix δ_0 and M_0. Next we want to prove (3) for $j = 1$ by shrinking the terms in $\delta_1, \delta_2, \dots$ and increasing the terms in $M_1, M_2, \dots$ (δ_0, M_0 having been fixed).

Now we have $x_n = |x| \geq 1$ so

$$|e^{i\langle x,\zeta\rangle}| = e^{-\langle x, \operatorname{Im}\zeta\rangle} \leq (1+|\xi|)^{-m} \text{ for } \zeta \in \Gamma_m \tag{24}$$

so in the integrals in (23) the factor $(1+|\xi|)^{km}$ is replaced by $(1+|\xi|)^{(k-1)m}$.

In the sum we separate out the term with $k = 0$. Here M_0 has been fixed but now we have the factor $(1+|\xi|)^{-m}$ which assures that this $k = 0$ term is $< \frac{\epsilon_1}{2}$ for a sufficiently large m which we now fix. In the remaining terms in (23) (for $k > 0$) we can now increase $1/\delta_k$ and M_k such that the sum is $< \epsilon_1/2$. Thus (3) holds for $j = 1$ and it will remain valid for $j = 0$. We now fix this choice of δ_1 and M_1.

Now the inductive process is clear. We assume $\delta_0, \delta_1, \dots, \delta_{j-1}$ and $M_0, M_1, \dots, M_{j-1}$ having been fixed by this shrinking of the δ_i and enlarging of the M_i.

We wish to prove (3) for this j by increasing $1/\delta_k$, M_k for $k \geq j$. Now we have $x_n = |x| \geq j$ and (24) is replaced by

$$|e^{i\langle x,\zeta\rangle}| = e^{-\langle x, \operatorname{Im}\zeta\rangle} \leq 1 + |\xi|^{-jm}$$

and since $|\alpha| \leq N_j$, (23) is replaced by

$$\begin{aligned} &|(D^\alpha f)(x)| \\ &\leq \sum_{k=0}^{j-1} \delta_k \int_{\mathbf{R}^n} (1 + [|\xi|^2 + m^2(\log(1+|\xi|))^2]^{1/2})^{N_j - M_k} (1+|\xi|)^{(k-j)m} (1+m)\, d\xi \\ &+ \sum_{k \geq j} \delta_k \int_{\mathbf{R}^n} (1 + [|\xi|^2 + m^2(\log(1+|\xi|))^2]^{1/2})^{N_j - M_k} (1+|\xi|)^{(k-j)m} (1+m)\, d\xi\,. \end{aligned}$$

In the first sum the M_k have been fixed but the factor $(1+|\xi|)^{(k-j)m}$ decays exponentially. Thus we can fix m such that the first sum is $< \frac{\epsilon_j}{2}$.

In the latter sum the $1/\delta_k$ and the M_k can be increased so that the total sum is $< \frac{\epsilon_j}{2}$. This implies the validity of (3) for this particular j and it remains valid for $0, 1, \dots j-1$. Now we fix δ_j and M_j.

This completes the induction. With this construction of $\{\delta\}$, $\{M\}$ we have proved that $W(\{\delta\}, \{M\}) \subset V(\{\epsilon\}, \{N\})$. This proves Theorem 3.4.

§4 Differential Operators with Constant Coefficients

The description of the topology of $\mathcal{D}$ in terms of the range $\widetilde{\mathcal{D}}$ given in Theorem 3.4 has important consequences for solvability of differential equations on $\mathbf{R}^n$ with constant coefficients.

Theorem 4.1. *Let $D \neq 0$ be a differential operator on $\mathbf{R}^n$ with constant coefficients. Then the mapping $f \to \mathcal{D}f$ is a homeomorphism of $\mathcal{D}$ onto $D\mathcal{D}$.*

Proof. It is clear from Theorem 3.3 that the mapping $f \to Df$ is injective on $\mathcal{D}$. The continuity is also obvious.

For the continuity of the inverse we need the following simple lemma.

Lemma 4.2. *Let $P \neq 0$ be a polynomial of degree m, F an entire function on $\mathbf{C}^n$ and $G = PF$. Then*

$$|F(\zeta)| \leq C \sup_{|z| \leq 1} |G(z+\zeta)|, \quad \zeta \in \mathbf{C}^n\,,$$

where C is a constant.

Proof. Suppose first $n = 1$ and that $P(z) = \sum_0^m a_k z^k (a_m \neq 0)$. Let $Q(z) = z^m \sum_0^m \overline{a}_k z^{-k}$. Then, by the maximum principle,

$$(25) \qquad |a_m F(0)| = |Q(0)F(0) \leq \max_{|z|=1} |Q(z)F(z)| = \max_{|z|=1} |P(z)F(z)|\,.$$

For general n let A be an $n \times n$ complex matrix, mapping the ball $|\zeta| < 1$ in $\mathbf{C}^n$ into itself and such that

$$P(A\zeta) = a\zeta_1^m + \sum_0^{m-1} P_k(\zeta_2, \ldots, \zeta_n)\zeta_1^k, \quad a \neq 0\,.$$

Let

$$F_1(\zeta) = F(A\zeta), \quad G_1(\zeta) = G(A\zeta), \quad P_1(\zeta) = P(A\zeta)\,.$$

Then

$$G_1(\zeta_1 + z, \zeta_2, \ldots, \zeta_n) = F_1(\zeta_1 + z, \zeta_2, \ldots, \zeta_n) P_1(\zeta + z, \zeta_2, \ldots, \zeta_n)$$

and the polynomial

$$z \to P_1(\zeta_1 + z, \ldots, \zeta_n)$$

has leading coefficient a. Thus by (25)

$$|aF_1(\zeta)| \leq \max_{|z|=1} |G_1(\zeta_1 + z, \zeta_2, \ldots, \zeta_n)| \leq \max_{\substack{z \in \mathbf{C}^n \\ |z| \leq 1}} |G_1(\zeta + z)|\,.$$

Hence by the choice of A

$$|aF(\zeta)| \leq \sup_{\substack{z \in \mathbf{C}^n \\ |z| \leq 1}} |G(\zeta + z)|$$

proving the lemma.

For Theorem 4.1 it remains to prove that if V is a convex neighborhood of 0 in $\mathcal{D}$ then there exists a convex neighborhood W of 0 in $\mathcal{D}$ such that

$$f \in \mathcal{D},\, Df \in W \Rightarrow f \in V\,. \tag{26}$$

We take V as the neighborhood $W(\{\delta\},\{M\})$. We shall show that if $W = W(\{\epsilon\},\{M\})$ (same $\{M\}$) then (26) holds provided the ϵ_j in $\{\epsilon\}$ are small enough. We write $u = Df$ so $\widetilde{u}(\zeta) = P(\zeta)\widetilde{f}(\zeta)$ where P is a polynomial. By Lemma 4.2

$$|\widetilde{f}(\zeta)| \leq C \sup_{|z|\leq 1} |\widetilde{u}(\zeta + z)|\,. \tag{27}$$

But $|z| \leq 1$ implies

$$(1 + |z+\zeta|)^{-M_j} \leq 2^{M_j}(1+|\zeta|)^{-M_j}\,, \quad |\mathrm{Im}\,(z+\zeta)| \leq |\mathrm{Im}\,\zeta| + 1\,,$$

so if $C2^{M_j}e^j\epsilon_j \leq \delta_j$ then (26) holds.

Q.e.d.

Corollary 4.3. *Let $D \neq 0$ be a differential operator on $\mathbf{R}^n$ with constant (complex) coefficients. Then*

$$D\mathcal{D}' = \mathcal{D}'\,. \tag{28}$$

In particular, there exists a distribution T on $\mathbf{R}^n$ such that

$$DT = \delta\,. \tag{29}$$

Definition. A distribution T satisfying (29) is called a *fundamental solution* for D.

To verify (28) let $L \in \mathcal{D}'$ and consider the functional $D^*u \to L(u)$ on $D^*\mathcal{D}$ ($*$ denoting adjoint). Because of Theorem 3.3 this functional is well defined and by Theorem 4.1 it is continuous. By the Hahn-Banach theorem it extends to a distribution $S \in \mathcal{D}'$. Thus $S(D^*u) = Lu$ so $DS = L$, as claimed.

Corollary 4.4. *Given $f \in \mathcal{D}$ there exists a smooth function u on $\mathbf{R}^n$ such that*

$$Du = f\,.$$

In fact, if T is a fundamental solution one can put $u = f * T$.

We conclude this section with the mean value theorem of Ásgeirsson which entered into the range characterization of the X-ray transform in Chapter I. For another application see Theorem 5.9 below

Theorem 4.5. *Let u be a C^2 function on $B_R \times B_R \subset \mathbf{R}^n \times \mathbf{R}^n$ satisfying*

$$L_x u = L_y u\,. \tag{30}$$

Then

$$\int_{|y|=r} u(0,y)\,dw(y) = \int_{|x|=r} u(x,0)\,dw(x) \quad r < R\,. \tag{31}$$

Conversely, if u is of class C^2 near $(0,0) \subset \mathbf{R}^n \times \mathbf{R}^n$ and if (31) holds for all r sufficiently small then

$$(L_x u)(0,0) = (L_y u)(0,0)\,. \tag{32}$$

Remark 4.6. Integrating Taylor's formula it is easy to see that on the space of analytic functions the mean value operator M^r (Ch. I, §2) is a power series in the Laplacian L. (See (44) below for the explicit expansion.) Thus (30) implies (31) for analytic functions.

For u of class C^2 we give another proof.

We consider the mean value operator on each factor in the product $\mathbf{R}^n \times \mathbf{R}^n$ and put

$$U(r,s) = (M_1^r M_2^s u)(x,y)$$

where the subscript indicates first and second variable, respectively. If u satisfies (30) then we see from the Darboux equation (Ch. I, Lemma 3.2) that

$$\frac{\partial^2 U}{\partial r^2} + \frac{n-1}{r}\frac{\partial U}{\partial r} = \frac{\partial^2 U}{\partial s^2} + \frac{n-1}{s}\frac{\partial U}{\partial s}\,.$$

Putting $F(r,s) = U(r,s) - U(s,r)$ we have

$$\frac{\partial^2 F}{\partial r^2} \quad + \quad \frac{n-1}{r}\frac{\partial F}{\partial r} - \frac{\partial^2 F}{\partial s^2} - \frac{n-1}{s}\frac{\partial F}{\partial s} = 0\,, \tag{33}$$

$$F(r,s) \quad = \quad -F(s,r)\,. \tag{34}$$

After multiplication of (33) by $r^{n-1}\frac{\partial F}{\partial s}$ and some manipulation we get

$$-r^{n-1}\tfrac{\partial}{\partial s}\left[\left(\tfrac{\partial F}{\partial r}\right)^2 + \left(\tfrac{\partial F}{\partial s}\right)^2\right] + 2\tfrac{\partial}{\partial r}\left(r^{n-1}\tfrac{\partial F}{\partial r}\tfrac{\partial F}{\partial s}\right) - 2r^{n-1}\tfrac{n-1}{s}\left(\tfrac{\partial F}{\partial s}\right)^2 = 0\,.$$

Consider the line MN with equation $r + s =$ const. in the (r, s)-plane and integrate the last expression over the triangle OMN (see Fig. V.2).

Using the divergence theorem (Ch. I, (26)) we then obtain, if $\mathbf{n}$ denotes the outgoing unit normal, $d\ell$ the element of arc length, and $\cdot$ the inner product,

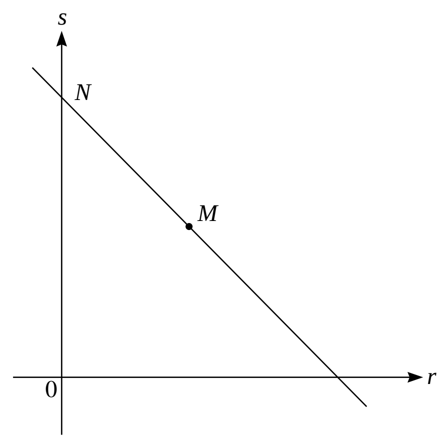

FIGURE V.2.

$$(35)\quad \int_{OMN} \left(2r^{n-1}\frac{\partial F}{\partial r}\frac{\partial F}{\partial s}\,, -r^{n-1}\left[\left(\frac{\partial F}{\partial r}\right)^2 + \left(\frac{\partial F}{\partial s}\right)^2\right]\right) \cdot \mathbf{n}\, d\ell$$

$$= 2\iint_{OMN} r^{n-1}\frac{n-1}{s}\left(\frac{\partial F}{\partial s}\right)^2 dr\, ds\,.$$

On $OM : \overline{n} = (2^{-1/2}, -2^{-1/2})$, $\quad F(r, r) = 0$ so $\frac{\partial F}{\partial r} + \frac{\partial F}{\partial s} = 0$.
On $MN : \overline{n} = (2^{-1/2}, 2^{-1/2})$.

Taking this into account, (35) becomes

$$2^{-\frac{1}{2}}\int_{MN} r^{n-1}\left(\frac{\partial F}{\partial r} - \frac{\partial F}{\partial s}\right)^2 d\ell + 2\iint_{OMN} r^{n-1}\frac{n-1}{s}\left(\frac{\partial F}{\partial s}\right)^2 dr\, ds = 0\,.$$

This implies F constant so by (34) $F \equiv 0$. In particular, $U(r, 0) = U(0, r)$ which is the desired relation (31).

For the converse we observe that the mean value $(M^r f)(0)$ satisfies (by Taylor's formula)

$$(M^r f)(0) = f(0) + c_n r^2 (Lf)(0) + o(r^2)$$

where $c_n \neq 0$ is a constant. Thus

$$r^{-1}\frac{dM^r f(0)}{dr} \to 2c_n (Lf)(0) \ \text{ as } \ r \to 0\,.$$

Thus (31) implies (32) as claimed.

§5 Riesz Potentials

We shall now study some examples of distributions in detail. If $\alpha \in \mathbf{C}$ satisfies $\operatorname{Re}\alpha > -1$ the functional

$$x_+^\alpha : \varphi \to \int_0^\infty x^\alpha \varphi(x)\, dx, \quad \varphi \in \mathcal{S}(\mathbf{R})\,, \tag{36}$$

is a well-defined tempered distribution. The mapping $\alpha \to x_+^\alpha$ from the half-plane $\operatorname{Re}\alpha > -1$ to the space $\mathcal{S}'(\mathbf{R})$ of tempered distributions is holomorphic (that is $\alpha \to x_+^\alpha(\varphi)$ is holomorphic for each $\varphi \in \mathcal{S}(\mathbf{R})$). Writing

$$x_+^\alpha(\varphi) = \int_0^1 x^\alpha(\varphi(x) - \varphi(0))\, dx + \frac{\varphi(0)}{\alpha+1} + \int_1^\infty x^\alpha \varphi(x)\, dx$$

the function $\alpha \to x_+^\alpha$ is continued to a holomorphic function in the region $\operatorname{Re}\alpha > -2, \alpha \neq -1$. In fact

$$\varphi(x) - \varphi(0) = x \int_0^\infty \varphi'(tx)\, dt\,,$$

so the first integral above converges for $\operatorname{Re}\alpha > -2$. More generally, $\alpha \to x_+^\alpha$ can be extended to a holomorphic $\mathcal{S}'(\mathbf{R})$-valued mapping in the region

$$\operatorname{Re}\alpha > -n-1, \quad \alpha \neq -1, -2, \ldots, -n\,,$$

by means of the formula

$$\begin{aligned}(37)\ x_+^\alpha(\varphi) = &\int_0^1 x^\alpha \left[\varphi(x) - \varphi(0) - x\varphi'(0) - \cdots - \frac{x^{n-1}}{(n-1)!}\varphi^{(n-1)}(0)\right] dx \\ &+ \int_1^\infty x^\alpha \varphi(x)\, dx + \sum_{k=1}^n \frac{\varphi^{(k-1)}(0)}{(k-1)!(\alpha+k)}\,.\end{aligned}$$

In this manner $\alpha \to x_+^\alpha$ is a meromorphic distribution-valued function on $\mathbf{C}$, with simple poles at $\alpha = -1, -2, \ldots$. We note that the residue at $\alpha = -k$ is given by

$$\operatorname*{Res}_{\alpha=-k} x_+^\alpha = \lim_{\alpha \to -k} (\alpha+k) x_+^\alpha = \frac{(-1)^{k-1}}{(k-1)!}\delta^{(k-1)}\,. \tag{38}$$

Here $\delta^{(h)}$ is the h^{th} derivative of the delta distribution δ. We note that x_+^α is always a tempered distribution.

Next we consider for $\operatorname{Re}\alpha > -n$ the distribution r^α on $\mathbf{R}^n$ given by

$$r^\alpha : \varphi \to \int_{\mathbf{R}^n} \varphi(x)|x|^\alpha\, dx, \quad \varphi \in \mathcal{D}(\mathbf{R}^n)\,.$$

Lemma 5.1. *The mapping $\alpha \to r^\alpha$ extends uniquely to a meromorphic mapping from $\mathbf{C}$ to the space $\mathcal{S}'(\mathbf{R}^n)$ of tempered distributions. The poles are the points*

$$\alpha = -n - 2h \quad (h \in \mathbb{Z}^+)$$

and they are all simple.

Proof. We have for $\operatorname{Re}\alpha > -n$

$$r^\alpha(\varphi) = \Omega_n \int_0^\infty (M^t\varphi)(0) t^{\alpha+n-1}\,dt\,. \tag{39}$$

Next we note (say from (15) in §2) that the mean value function $t \to (M^t\varphi)(0)$ extends to an even $\mathcal{C}^\infty$ function on $\mathbf{R}$, and its odd order derivatives at the origin vanish. Each even order derivative is nonzero if φ is suitably chosen. Since by (39)

$$r^\alpha(\varphi) = \Omega_n t_+^{\alpha+n-1}(M^t\varphi)(0) \tag{40}$$

the first statement of the lemma follows. The possible (simple) poles of r^α are by the remarks about x_+^α given by $\alpha + n - 1 = -1, -2, \ldots$. However if $\alpha + n - 1 = -2, -4, \ldots$, formula (38) shows, since $(M^t\varphi(0))^{(h)} = 0$, ($h$ odd) that $r^\alpha(\varphi)$ is holomorphic at the points $a = -n-1, -n-3, \ldots$.

The remark about the even derivatives of $M^t\varphi$ shows on the other hand, that the points $\alpha = -n-2h$ ($h \in \mathbb{Z}^+$) are genuine poles. We note also from (38) and (40) that

$$\operatorname*{Res}_{\alpha=-n} r^\alpha = \lim_{\alpha\to -n} (\alpha+n) r^\alpha = \Omega_n \delta\,. \tag{41}$$

We recall now that the Fourier transform $T \to \widetilde{T}$ of a tempered distribution T on $\mathbf{R}^n$ is defined by

$$\widetilde{T}(\varphi) = T(\widetilde{\varphi}) \qquad \varphi = \mathcal{S}(\mathbf{R}^n)\,.$$

We shall now calculate the Fourier transforms of these tempered distributions r^α.

Lemma 5.2. *We have the following identity*

$$(r^\alpha)^\sim = 2^{n+\alpha}\pi^{\frac{n}{2}} \frac{\Gamma((n+\alpha)/2)}{\Gamma(-\alpha/2)} r^{-\alpha-n}, \quad -\alpha - n \notin 2\mathbb{Z}^+\,. \tag{42}$$

For $\alpha = 2h$ ($h \in \mathbb{Z}^+$) the singularity on the right is removable and (42) takes the form

$$(r^{2h})^\sim = (2\pi)^n (-L)^h \delta, \quad h \in \mathbb{Z}^+\,. \tag{43}$$

Proof. We use the fact that if $\psi(x) = e^{-|x|^2/2}$ then $\widetilde{\psi}(u) = (2\pi)^{\frac{n}{2}} e^{-|u|^2/2}$ so by the formula $\int f\widetilde{g} = \int \widetilde{f}g$ we obtain for $\varphi \in \mathcal{S}(\mathbf{R}^n), t > 0$,

$$\int \widetilde{\varphi}(x) e^{-t|x|^2/2}\, dx = (2\pi)^{n/2} t^{-n/2} \int \varphi(u) e^{-|u|^2/2t}\, du\,.$$

We multiply this equation by $t^{-1-\alpha/2}$ and integrate with respect to t. On the left we obtain the expression

$$\Gamma(-\alpha/2) 2^{-\frac{\alpha}{2}} \int \widetilde{\varphi}(x) |x|^{\alpha}\, dx\,,$$

using the formula

$$\int_0^{\infty} e^{-t|x|^2/2} t^{-1-\alpha/2}\, dt = \Gamma(-\tfrac{\alpha}{2}) 2^{-\frac{\alpha}{2}} |x|^{\alpha}\,,$$

which follows from the definition

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1}\, dt\,.$$

On the right we similarly obtain

$$(2\pi)^{\frac{n}{2}} \Gamma((n+\alpha)/2)\, 2^{\frac{n+\alpha}{2}} \int \varphi(u) |u|^{-\alpha-n}\, du\,.$$

The interchange of the integrations is valid for α in the strip $-n < \operatorname{Re}\alpha < 0$ so (42) is proved for these α. For the remaining ones it follows by analytic continuation. Finally, (43) is immediate from the definitions and (6).

By the analytic continuation, the right hand sides of (42) and (43) agree for $\alpha = 2h$. Since

$$\operatorname*{Res}_{\alpha=2h} \Gamma(-\alpha/2) = -2(-1)^h/h!$$

and since by (40) and (38),

$$\operatorname*{Res}_{\alpha=2h} r^{-\alpha-n}(\varphi) = -\Omega_n \frac{1}{(2h)!} \left[\left(\frac{d}{dt}\right)^{2h} (M^t \varphi) \right]_{t=0}$$

we deduce the relation

$$\left[\left(\frac{d}{dt}\right)^{2h} (M^t \varphi) \right]_{t=0} = \frac{\Gamma(n/2)}{\Gamma(h+n/2)} \frac{(2h)!}{2^{2h} h!} (L^h \varphi)(0)\,.$$

This gives the expansion

$$M^t = \sum_{h=0}^{\infty} \frac{\Gamma(n/2)}{\Gamma(h+n/2)} \frac{(t/2)^{2h}}{h!} L^h \tag{44}$$

on the space of analytic functions so M^t is a modified Bessel function of $tL^{1/2}$. This formula can also be proved by integration of Taylor's formula (cf. end of §4).

Lemma 5.3. *The action of the Laplacian is given by*

$$Lr^{\alpha} = \alpha(\alpha+n-2)r^{\alpha-2}, \quad (-\alpha-n+2\notin 2\mathbb{Z}^+) \tag{45}$$

$$Lr^{2-n} = (2-n)\Omega_n\delta \quad (n\neq 2)\,. \tag{46}$$

For $n=2$ this 'Poisson equation' is replaced by

$$L(\log r) = 2\pi\delta\,. \tag{47}$$

Proof. For $\operatorname{Re}\alpha$ sufficiently large (45) is obvious by computation. For the remaining ones it follows by analytic continuation. For (46) we use the Fourier transform and the fact that for a tempered distribution S,

$$(-LS)^{\sim} = r^2\widetilde{S}\,.$$

Hence, by (42),

$$(-Lr^{2-n})^{\sim} = 4\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}-1)} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}(n-2)\widetilde{\delta}\,.$$

Finally, we prove (47). If $\varphi\in\mathcal{D}(\mathbf{R}^2)$ we have, putting $F(r)=(M^r\varphi)(0)$,

$$(L(\log r))(\varphi) = \int_{\mathbf{R}^2}\log r(L\varphi)(x)\,dx = \int_0^{\infty}(\log r)2\pi r(M^rL\varphi)(0)\,dr\,.$$

Using Lemma 3.2 in Chapter I this becomes

$$\int_0^{\infty}\log r\; 2\pi r(F''(r)+r^{-1}F'(r))\,dr\,,$$

which by integration by parts reduces to

$$\Big[\log r(2\pi r)F'(r)\Big]_0^{\infty} - 2\pi\int_0^{\infty}F'(r)\,dr = 2\pi F(0)\,.$$

This proves (47).

Another method is to write (45) in the form $L(\alpha^{-1}(r^{\alpha}-1)) = \alpha r^{\alpha-2}$. Then (47) follows from (41) by letting $\alpha\to 0$.

We shall now define fractional powers of L, motivated by the formula

$$(-Lf)^{\sim}(u) = |u|^2\widetilde{f}(u)\,,$$

so that formally we should like to have a relation

$$((-L)^pf)^{\sim}(u) = |u|^{2p}\widetilde{f}(u)\,. \tag{48}$$

Since the Fourier transform of a convolution is the product of the Fourier transforms, formula (42) (for $2p=-\alpha-n$) suggests defining

$$(-L)^pf = I^{-2p}(f)\,, \tag{49}$$

where I^γ is the *Riesz potential*

$$(I^\gamma f)(x) = \frac{1}{H_n(\gamma)} \int_{\mathbf{R}^n} f(y)|x-y|^{\gamma-n}\,dy \tag{50}$$

with

$$H_n(\gamma) = 2^\gamma \pi^{\frac{n}{2}} \frac{\Gamma(\frac{\gamma}{2})}{\Gamma(\frac{n-\gamma}{2})}\,. \tag{51}$$

Note that if $-\gamma \in 2\mathbb{Z}^+$ the poles of $\Gamma(\gamma/2)$ cancel against the poles of $r^{\gamma-n}$ because of Lemma 5.1. Thus if $\gamma - n \notin 2\mathbb{Z}^+$ we can write

$$(I^\gamma f)(x) = (f * (H_n(\gamma)^{-1} r^{\gamma-n}))(x), \quad f \in \mathcal{S}(\mathbf{R}^n)\,. \tag{52}$$

By (12) and Lemma 5.2 we then have

$$(I^\gamma f)^\sim(u) = |u|^{-\gamma}\widetilde{f}(u)\,, \qquad \gamma - n \notin 2\mathbb{Z}^+ \tag{53}$$

as tempered distributions. Thus we have the following result.

Lemma 5.4. *If $f \in \mathcal{S}(\mathbf{R}^n)$ then $\gamma \to (I^\gamma f)(x)$ extends to a holomorphic function in the set $\mathbf{C}_n = \{\gamma \in \mathbf{C} : \gamma - n \notin 2\mathbb{Z}^+\}$. Also*

$$I^0 f = \lim_{\gamma \to 0} I^\gamma f = f\,, \tag{54}$$

$$I^\gamma L f = L I^\gamma f = -I^{\gamma-2} f\,. \tag{55}$$

We now prove an important property of the Riesz' potentials. Here it should be observed that $I^\gamma f$ is defined for all f for which (50) is absolutely convergent and $\gamma \in \mathbf{C}_n$.

Proposition 5.5. *The following identity holds:*

$$I^\alpha(I^\beta f) = I^{\alpha+\beta} f \text{ for } f \in \mathcal{S}(\mathbf{R}^n), \quad \operatorname{Re}\alpha, \operatorname{Re}\beta > 0, \quad \operatorname{Re}(\alpha+\beta) < n\,,$$

$I^\alpha(I^\beta f)$ being well defined. The relation is also valid if

$$f(x) = 0(|x|^{-p}) \text{ for some } p > \operatorname{Re}\alpha + \operatorname{Re}\beta\,.$$

Proof. We have

$$\begin{aligned} I^\alpha(I^\beta f)(x) &= \frac{1}{H_n(\alpha)} \int |x-z|^{\alpha-n} \left(\frac{1}{H_n(\beta)} \int f(y)|z-y|^{\beta-n}\,dy\right) dz \\ &= \frac{1}{H_n(\alpha)H_n(\beta)} \int f(y) \left(\int |x-z|^{\alpha-n}|z-y|^{\beta-n}\,dz\right) dy\,. \end{aligned}$$

The substitution $v = (x-z)/|x-y|$ reduces the inner integral to the form

$$|x-y|^{\alpha+\beta-n} \int_{\mathbf{R}^n} |v|^{\alpha-n}|w-v|^{\beta-n}\,dv\,, \tag{56}$$

where w is the unit vector $(x-y)/|x-y|$. Using a rotation around the origin we see that the integral in (56) equals the number

$$c_n(\alpha,\beta) = \int_{\mathbf{R}^n} |v|^{\alpha-n}|e_1-v|^{\beta-n}\,dv\,, \tag{57}$$

where $e_1 = (1,0,\dots,0)$. The assumptions made on α and β insure that this integral converges. By the Fubini theorem the exchange order of integrations above is permissible and

$$I^\alpha(I^\beta f) = \frac{H_n(\alpha+\beta)}{H_n(\alpha)H_n(\beta)}c_n(\alpha,\beta)I^{\alpha+\beta}f\,. \tag{58}$$

It remains to calculate $c_n(\alpha,\beta)$. For this we use the following lemma which was already used in Chapter I, §2. As there, let $\mathcal{S}^*(\mathbf{R}^n)$ denote the set of functions in $\mathcal{S}(\mathbf{R}^n)$ which are orthogonal to all polynomials.

Lemma 5.6. *Each $I^\alpha(\alpha\in\mathbf{C}_n)$ leaves the space $\mathcal{S}^*(\mathbf{R}^n)$ invariant.*

Proof. We recall that (53) holds in the sense of tempered distributions. Suppose now $f\in\mathcal{S}^*(\mathbf{R}^n)$. We consider the sum in the Taylor formula for $\widetilde{f}$ in $|u|\le 1$ up to order m with $m>|\alpha|$. Since each derivative of $\widetilde{f}$ vanishes at $u=0$ this sum consists of terms

$$(\beta!)^{-1}u^\beta(D^\beta\widetilde{f})(u^*)\,, \qquad |\beta| = m$$

where $|u^*|\le 1$. Since $|u^\beta|\le|u|^m$ this shows that

$$\lim_{u\to 0}|u|^{-\alpha}\widetilde{f}(u) = 0\,. \tag{59}$$

Iterating this argument with $\partial_i(|u|^{-\alpha}\widetilde{f}(u))$ etc. we conclude that the limit relation (59) holds for each derivative $D^\beta(|u|^{-\alpha}\widetilde{f}(u))$. Because of (59), relation (53) can be written

$$\int_{\mathbf{R}^n}(I^\alpha f)^\sim(u)g(u)\,du = \int_{\mathbf{R}^n}|u|^{-\alpha}\widetilde{f}(u)g(u)\,du\,, \quad g\in\mathcal{S}\,, \tag{60}$$

so (53) holds as an identity for functions $f\in\mathcal{S}^*(\mathbf{R}^n)$. The remark about $D^\beta(|u|^{-\alpha}\widetilde{f}(u))$ thus implies $(I^\alpha f)^\sim\in\mathcal{S}_0$ so $I^\alpha f\in\mathcal{S}^*$ as claimed.

We can now finish the proof of Prop. 5.5. Taking $f_o\in\mathcal{S}^*$ we can put $f=I^\beta f_o$ in (53) and then

$$\begin{aligned}(I^\alpha(I^\beta f_0))^\sim(u) &= (I^\beta f_0)^\sim(u)|u|^{-\alpha} = \widetilde{f}_0(u)|u|^{-\alpha-\beta}\\ &= (I^{\alpha+\beta}f_0)^\sim(u)\,.\end{aligned}$$

This shows that the scalar factor in (58) equals 1 so Prop. 5.5 is proved. In the process we have obtained the evaluation

$$\int_{\mathbf{R}^n}|v|^{\alpha-n}|e_1-v|^{\beta-n}\,dv = \frac{H_n(\alpha)H_n(\beta)}{H_n(\alpha+\beta)}\,.$$

We now prove a variation of Prop. 5.5 needed in the theory of the Radon transform.

Proposition 5.7. *Let $0 < k < n$. Then*

$$I^{-k}(I^k f) = f \qquad f \in \mathcal{E}(\mathbf{R}^n)$$

if $f(x) = 0(|x|^{-N})$ for some $N > n$.

Proof. By Prop. 5.5 we have if $f(y) = 0(|y|^{-N})$

$$I^{\alpha}(I^k f) = I^{\alpha+k} f \quad \text{for } 0 < \operatorname{Re}\alpha < n-k\,. \tag{61}$$

We shall prove that the function $\varphi = I^k f$ satisfies

$$\sup_x |\varphi(x)|\,|x|^{n-k} < \infty\,. \tag{62}$$

For an $N > n$ we have an estimate $|f(y)| \leq C_N(1+|y|)^{-N}$ where C_N is a constant. We then have

$$\begin{aligned}\left(\int_{\mathbf{R}^n} f(y)|x-y|^{k-n}\,dy\right) \leq C_N &\int_{|x-y|\leq\frac{1}{2}|x|} (1+|y|)^{-N}|x-y|^{k-n}\,dy \\ +C_N &\int_{|x-y|\geq\frac{1}{2}|x|} (1+|y|)^{-N}|x-y|^{k-n}\,dy\,.\end{aligned}$$

In the second integral, $|x-y|^{k-n} \leq (\frac{|x|}{2})^{k-n}$ so since $N > n$ this second integral satisfies (62). In the first integral we have $|y| \geq \frac{|x|}{2}$ so the integral is bounded by

$$\left(1+\frac{|x|}{2}\right)^{-N} \int_{|x-y|\leq\frac{|x|}{2}} |x-y|^{k-n}\,dy = \left(1+\frac{|x|}{2}\right)^{-N} \int_{|z|\leq\frac{|x|}{2}} |z|^{k-n}\,dz$$

which is $0(|x|^{-N}|x|^k)$. Thus (62) holds also for this first integral. This proves (62) provided

$$f(x) = 0(|x|^{-N}) \text{ for some } N > n\,.$$

Next we observe that $I^{\alpha}(\varphi) = I^{\alpha+k}(f)$ is holomorphic for $0 < \operatorname{Re}\alpha < n-k$. For this note that by (39)

$$\begin{aligned}(I^{\alpha+k}f)(0) &= \frac{1}{H_n(\alpha+k)}\int_{\mathbf{R}^n} f(y)|y|^{\alpha+k-n}\,dy \\ &= \frac{1}{H_n(\alpha+k)}\Omega_n \int_0^{\infty} (M^t f)(0)t^{\alpha+k-1}\,dt\,.\end{aligned}$$

Since the integrand is bounded by a constant multiple of $t^{-N}t^{\alpha+k-1}$, and since the factor in front of the integral is harmless for $0 < k + \operatorname{Re}\alpha < n$, the holomorphy statement follows.

We claim now that $I^\alpha(\varphi)(x)$, which as we saw is holomorphic for $0 < \operatorname{Re}\alpha < n-k$, extends to a holomorphic function in the half-plane $\operatorname{Re}\alpha < n-k$. It suffices to prove this for $x=0$. We decompose $\varphi = \varphi_1 + \varphi_2$ where φ_1 is a smooth function identically 0 in a neighborhood $|x| < \epsilon$ of 0, and $\varphi_2 \in \mathcal{S}(\mathbf{R}^n)$. Since φ_1 satisfies (62) we have for $\operatorname{Re}\alpha < n-k$,

$$\begin{aligned}\left|\int \varphi_1(x)|x|^{\operatorname{Re}\alpha-n}\,dx\right| &\leq C\int_\epsilon^\infty |x|^{k-n}|x|^{\operatorname{Re}\alpha-n}|x|^{n-1}d|x| \\ &= C\int_\epsilon^\infty |x|^{\operatorname{Re}\alpha+k-n-1}d|x| < \infty\end{aligned}$$

so $I^\alpha\varphi_1$ is holomorphic in this half-plane. On the other hand $I^\alpha\varphi_2$ is holomorphic for $\alpha \in \mathbf{C}_n$ which contains this half-plane. Now we can put $\alpha = -k$ in (61). As a result of (39), $f(x) = 0(|x|^{-N})$ implies that $(I^\lambda f)(x)$ is holomorphic near $\lambda = 0$ and $I^0 f = f$. Thus the proposition is proved.

Denoting by C_N the class of continuous functions f on $\mathbf{R}^n$ satisfying $f(x) = 0(|x|^{-N})$ we proved in (62) that if $N > n$, $0 < k < n$, then

$$I^k C_N \subset C_{n-k}\,. \tag{63}$$

More generally, we have the following result.

Proposition 5.8. *If $N > 0$ and $0 < \operatorname{Re}\gamma < N$, then*

$$I^\gamma C_N \subset C_s$$

where $s = \min(n,N) - \operatorname{Re}\gamma \quad (n \neq N)$.

Proof. Modifying the proof of Prop. 5.7 we divide the integral

$$I = \int (1+|y|)^{-N}|x-y|^{\operatorname{Re}\gamma-n}\,dy$$

into integrals I_1, I_2 and I_3 over the disjoint sets

$$A_1 = \{y : |y-x| \leq \tfrac{1}{2}|x|\}, \qquad A_2 = \{y : |y| < \tfrac{1}{2}|x|\,,$$

and the complement $A_3 = \mathbf{R}^n - A_1 - A_2$. On A_1 we have $|y| \geq \frac{1}{2}|x|$ so

$$I_1 \leq \left(1+\frac{|x|}{2}\right)^{-N}\int_{A_1}|x-y|^{\operatorname{Re}\gamma-n}\,dy = \left(1+\frac{|x|}{2}\right)^{-N}\int_{|z|\leq|x|/2}|z|^{\operatorname{Re}\gamma-n}\,dz$$

so

$$I_1 = 0(|x|^{-N+\operatorname{Re}\gamma})\,. \tag{64}$$

On A_2 we have $|x| + \frac{1}{2}|x| \geq |x-y| \geq \frac{1}{2}|x|$ so

$$|x-y|^{\operatorname{Re}\gamma-n} \leq C|x|^{\operatorname{Re}\gamma-n}, \quad C = \text{const.}.$$

Thus

$$I_2 \le C|x|^{\mathrm{Re}\,\gamma - n} \int_{A_2} (1+|y|)^{-N} \,.$$

If $N > n$ then

$$\int_{A_2} (1+|y|)^{-N}\,dy \le \int_{\mathbf{R}^n} (1+|y|)^{-N}\,dy < \infty \,.$$

If $N < n$ then

$$\int_{A_2} (1+|y|)^{-N}\,dy \le C|x|^{n-N} \,.$$

In either case

$$I_2 = 0(|x|^{\mathrm{Re}\,\gamma - \min(n,N)}) \,. \tag{65}$$

On A_3 we have $(1+|y|)^{-N} \le |y|^{-N}$. The substitution $y = |x|u$ gives (with $e = x/|x|$)

$$I_3 \le |x|^{\mathrm{Re}\,\gamma - N} \int_{|u|\ge\frac{1}{2},|e-u|\ge\frac{1}{2}} |u|^{-N}|e-u|^{\mathrm{Re}\,\gamma - n}\,du = 0(|x|^{\mathrm{Re}\,\gamma - N}) \,. \tag{66}$$

Combining (64)–(66) we get the result.

We conclude with a consequence of Theorem 4.5 observed in John [1935]. Here the Radon transform maps functions $\mathbf{R}^n$ into functions on a space of $(n+1)$ dimensions and the range is the kernel of a single differential operator. This may have served as a motivation for the range characterization of the X-ray transform in John [1938]. As before we denote by $(M^r f)(x)$ the average of f on $S_r(x)$.

Theorem 5.9. *For f on $\mathbf{R}^n$ put*

$$\widehat{f}(x,r) = (M^r f)(x) \,.$$

Then

$$\mathcal{E}(\mathbf{R}^n)\widehat{\ } = \{\varphi \in \mathcal{E}(\mathbf{R}^n \times \mathbf{R}^+) : L_x\varphi = \partial_r^2\varphi + \frac{n-1}{r}\partial_r\varphi\} \,.$$

The inclusion $\subset$ follows from Lemma 3.2, Ch. I. Conversely suppose φ satisfies the Darboux equation. The extension $\Phi(x,y) = \varphi(x,|y|)$ then satisfies $L_x\Phi = L_y\Phi$. Using Theorem 4.5 on the function $(x,y) \to \Phi(x + x_0, y)$ we obtain $\varphi(x_0,r) = (M^r f)(x_0)$ so $\widehat{f} = \varphi$ as claimed.

Bibliographical Notes

§1-2 contain an exposition of the basics of distribution theory following Schwartz [1966]. The range theorems (3.1–3.3) are also from there but we have used the proofs from Hörmander [1963]. Theorem 3.4 describing the topology of $\mathcal{D}$ in terms of $\widetilde{\mathcal{D}}$ is from Hörmander [1983], Vol. II, Ch. XV. The idea of a proof of this nature involving a contour like Γ_m appears already in Ehrenpreis [1956] although not correctly carried out in details. In the proof we specialize Hörmander's convex set K to a ball; it simplifies the proof a bit and requires Cauchy's theorem only in a single variable. The consequence, Theorem 4.1, and its proof were shown to me by Hörmander in 1972. The theorem appears in Ehrenpreis [1956].

Theorem 4.5, with the proof in the text, is from Ásgeirsson [1937]. Another proof, with a refinement in odd dimension, is given in Hörmander [1983], Vol. I. A generalization to Riemannian homogeneous spaces is given by the author in [1959]. The theorem is used in the theory of the X-ray transform in Chapter I.

§5 contains an elementary treatment of the results about Riesz potentials used in the book. The examples x_+^λ are discussed in detail in Gelfand-Shilov [1959]. The potentials I^λ appear there and in Riesz [1949] and Schwartz [1966]. In the proof of Proposition 5.7 we have used a suggestion by R. Seeley and the refinement in Proposition 5.8 was shown to me by Schlichtkrull. A thorough study of the composition formula (Prop. 5.5) was carried out by Ortner [1980] and a treatment of Riesz potentials on L^p-spaces (Hardy-Littlewood-Sobolev inequality) is given in Hörmander [1983], Vol. I, §4.

Bibliography

ABOUELAZ, A. AND DAHER, R.
1993 Sur la transformation de Radon de la sphere $\mathbf{S}^d$, *Bull. Soc. Math. France* **121** (1993), 353–382.

AGRANOVSKI, M.L. AND QUINTO, E.T.
1996 Injectivity sets for the Radon transform over circles and complete systems of radial functions, *J. Funct. Anal.* **139** (1996), 383–414.

AGUILAR, V., EHRENPREIS, L., AND KUCHMENT, P.
1996 Range conditions for the exponential Radon transform, *J. d'Analyse Math.* **68** (1996), 1–13.

AMEMIYA, I. AND ANDO, T.
1965 Convergence of random products of contractions in Hilbert space, *Acta Sci. Math. (Szeged)* **26** (1965), 239–244.

ARMITAGE, D.H.
1994 A non-constant function on the plane whose integral on every line is 0, *Amer. Math. Monthly* **101** (1994), 892–894.

ARMITAGE, D.H. AND GOLDSTEIN, M.
1993 Nonuniqueness for the Radon transform, *Proc. Amer. Math. Soc.* **117** (1993), 175–178.

ÁSGEIRSSON, L.
1937 Über eine Mittelwertseigenschaft von Lösungen homogener linearer partieller Differentialgleichungen 2 Ordnung mit konstanten Koefficienten, *Math. Ann.* **113** (1937), 321–346.

BERENSTEIN, C.A., KURUSA, A., AND CASADIO TARABUSI, E.
1997 Radon transform on spaces of constant curvature, *Proc. Amer. Math. Soc.* **125** (1997), 455–461.

Berenstein, C.A. and Shahshahani, M.
1983 Harmonic analysis and the Pompeiu problem, *Amer. J. Math.* **105** (1983), 1217–1229.
Berenstein, C.A. and Casadio Tarabusi, E.
1991 Inversion formulas for the k-dimensional Radon transform in real hyperbolic spaces, *Duke Math. J.* **62** (1991), 613–631.
1992 Radon- and Riesz transform in real hyperbolic spaces, *Contemp. Math.* **140** (1992), 1–18.
1993 Range of the k-dimensional Radon transform in real hyperbolic spaces, *Forum Math.* **5** (1993), 603–616.
1994 An inversion formula for the horocyclic Radon transform on the real hyperbolic space, *Lectures in Appl. Math.* **30** (1994), 1–6.
Berenstein, C.A. and Walnut, D.E.
1994 "Local inversion of the Radon transform in even dimensions using wavelets," in: *Proc. Conf.* 75 *Years of Radon Transform,* Vienna, 1992, International Press, Hong Kong, 1994.
Berenstein, C.A. and Zalcman, L.
1976 Pompeiu's problem on spaces of constant curvature, *J. Anal. Math.* **30** (1976), 113–130.
1980 Pompeiu's problem on symmetric spaces, *Comment. Math. Helv.* **55** (1980), 593–621.
Berest, Y.
1998 Hierarchies of Huygens' operators and Hadamard's conjecture, *Acta Appl. Math.* **53** (1998), 125–185.
Besse, A.
1978 *Manifolds all of whose geodesics are closed*, Ergeb. Math. Grenzgeb. **93**, Springer, New York, 1978.
Bockwinkel, H.B.A.
1906 On the propagation of light in a biaxial crystal about a midpoint of oscillation, *Verh. Konink. Acad. V. Wet. Wissen. Natur.* **14** (1906), 636.
Boerner, H.
1955 *Darstellungen der Gruppen,* Springer-Verlag, Heidelberg, 1955.
Boman, J.
1990 On generalized Radon transforms with unknown measures, *Contemp. Math.* **113** (1990), 5–15.
1991 "Helgason's support theorem for Radon transforms: A new proof and a generalization," in: *Mathematical Methods in Tomography,* Lecture Notes in Math. No. 1497, Springer-Verlag, Berlin and New York, 1991, 1–5.
1992 Holmgren's uniqueness theorem and support theorems for real analytic Radon transforms, *Contemp. Math.* **140** (1992), 23–30.
1993 An example of non-uniqueness for a generalized Radon transform, *J. Analyse Math.* **61** (1993), 395–401.

Boman, J. and Quinto, E.T.

1987 Support theorems for real-analytic Radon transforms, *Duke Math. J.* **55** (1987), 943–948.

1993 Support theorems for Radon transforms on real-analytic line complexes in three space, *Trans. Amer. Math. Soc.* **335** (1993), 877–890.

Borovikov, W.A.

1959 Fundamental solutions of linear partial differential equations with constant coefficients, *Trudy Moscov. Mat. Obshch.* **8** (1959), 199–257.

Bouaziz, A.

1995 Formule d'inversion des intégrales orbitales sur les groupes de Lie réductifs, *J. Funct. Anal.* **134** (1995), 100–182.

Bracewell, R.N. and Riddle, A.C.

1967 Inversion of fan beam scan in radio astronomy, *Astrophys. J.* **150** (1967), 427–434.

Branson, T.P., Ólafsson, G., and Schlichtkrull, H.

1994 A bundle-valued Radon transform with applications to invariant wave equations, *Quart. J. Math. Oxford 45* (1994), 429–461.

Chern, S.S.

1942 On integral geometry in Klein spaces, *Ann. of Math.* **43** (1942), 178–189.

Chevalley, C.

1946 *The Theory of Lie Groups, Vol.* I, Princeton University Press, Princeton, NJ, 1946.

Cormack, A.M.

1963–64 Representation of a function by its line integrals, with some radiological applications I, II, *J. Appl. Phys.* **34** (1963), 2722–2727; **35** (1964), 2908–2912.

Cormack, A.M. and Quinto, E.T.

1980 A Radon transform on spheres through the origin in $\boldsymbol{R}^n$ applications to the Darboux equation, *Trans. Amer. Math. Soc.* **260** (1980), 575–581.

Courant, R. and Lax, A.

1955 Remarks on Cauchy's problem for hyperbolic partial differential equations with constant coefficints in several independent varaiables, *Comm. Pure Appl. Math.* **8** (1955), 497–502.

Coxeter, H.S.M.

1957 *Non-Euclidean Geometry,* University of Toronto Press, Toronto, 1957.

Deans, S.R.

1983 *The Radon Transform and Some of Its Applications,* Wiley, New York, 1983.

Debiard, A. and Gaveau, B.

1983 Formule d'inversion en geométrie intégrale Lagrangienne, *C. R. Acad. Sci. Paris Sér.* I *Math.* **296** (1983), 423–425.

Droste, B.
1983 A new proof of the support theorem and the range characterization of the Radon transform, *Manuscripta Math.* **42** (1983), 289–296.

Ehrenpreis, L.
1956 Solutions of some problems of division, part III, *Amer. J. Math.* **78** (1956), 685–715.

Ehrenpreis, L., Kuchment, P., and Panchenko, A.
1997 The exponential X-ray transform, F. John equation, and all that I: Range description, preprint, 1997.

Estezet, P.
1988 Tenseurs symétriques a énergie nulle sur les variétés a courbure constante, thesis, Université de Grenoble, Grenoble, France, 1988.

Farah, S.B. and Kamoun, L.
1990 Distributions coniques sur le cone des matrices de rang un et de trace nulle, *Bull. Soc. Math. France* **118** (1990), 251–272.

Faraut, J.
1982 Un théorème de Paley-Wiener pour la transformation de Fourier sur un espace Riemannian symmétrique de rang un, *J. Funct. Anal.* **49** (1982), 230–268.

Faraut, J. and Harzallah, K.
1984 Distributions coniques associées au groupe orthogonal $O(p, q)$, *J. Math. Pures Appl.* **63** (1984), 81–119.

Felix, R.
1992 Radon Transformation auf nilpotenten Lie Gruppen, *Invent. Math.* **112** (1992), 413–443.

Finch, D.V. and Hertle, A.
1987 The exponential Radon transform, *Contemp. Math.* **63** (1987), 67–73.

Flensted-Jensen, M.
1977 Spherical functions on a simply connected semisimple Lie group II, *Math. Ann.* **228** (1977), 65–92.

Flicker, Y.Z.
1996 Orbital integrals on symmetric spaces and spherical characters, *J. Algebra* **184** (1996), 705–754.

Friedlander, F.C.
1975 *The Wave Equation in Curved Space,* Cambridge University Press, London and New York, 1975.

Fuglede, B.
1958 An integral formula, *Math. Scand.* **6** (1958), 207–212.

Funk, P.
1916 Über eine geometrische Anwendung der Abelschen Integralgleichnung, *Math. Ann.* **77** (1916), 129–135.

Gårding, L.
1961 Transformation de Fourier des distributions homogènes, *Bull. Soc. Math. France* **89** (1961), 381–428.

Gardner, R.J.
1995 *Geometric Tomography,* Cambridge Univ. Press, New York, 1995.