Sharp density discrepancy for cut and project sets An approach via lattice point counting

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Abstract

Cut and project sets are obtained by taking an irrational slice of a lattice and projecting it to a lower dimensional subspace. We seek to quantify fluctuations from the asymptotic mean for point counts. We obtain uniform upper bounds on the discrepancy depending on the diophantine properties of the lattice. In an appendix, Michael Björklund and Tobias Hartnick obtain lower bounds on the L² norm of the discrepancy also depending on the diophantine class; these lower bounds match our uniform upper bounds and both are therefore sharp. This also allows us to find sharp bounds for the number of lattice points in thin slabs. Using a sufficient criteria of Burago–Kleiner and Aliste-Prieto–Coronel–Gambaudo we find an explicit full-measure class of cut and project sets that are biLipschitz equivalent to lattices; our lower bounds indicate that this is the largest class of cut and project sets for which those criteria can apply.

1 Introduction

1.1 Cut and project sets

A cut and project set Λ is a discrete subset of euclidean space which, despite being aperiodic, still exhibits repetitive structure and long-term order. Fix an orthogonal decomposition

$$\mathbb{R}^d \cong \mathbb{E} := \mathbb{E}_{\nabla} \oplus \mathbb{E}_{\triangleleft} \cong \mathbb{R}^{d_{\nabla}} \oplus \mathbb{R}^{d_{\triangleleft}}, \tag{1.1}$$

with associated projections π_{\triangledown} and π_{\triangleleft} . Fix a lattice $\Gamma \subseteq \mathbb{E}$, $\mathbf{z} \in \mathbb{E}$ and a bounded window $\Omega_{\triangleleft} \subseteq \mathbb{E}_{\triangleleft}$. The cut and project set associated with this data is

$$\Lambda := \Lambda(\mathbb{E}, \mathbb{E}_{\triangledown}, \mathbb{E}_{\lhd}; \Gamma + \mathbf{z}; \Omega_{\lhd}) := \pi_{\triangledown}((\Gamma + \mathbf{z}) \cap \pi_{\lhd}^{-1}(\Omega_{\lhd}));$$

it is said to be *regular* if Ω_{\triangleleft} is an open set whose boundary has zero measure, and *s-regular* if, in addition $\partial\Omega_{\triangleleft}$ has a finite $(d_{\triangleleft} - s)$ -Minkowski content for some $0 < s \le 1$ (for definitions, see Section 3.1). We also make the standard assumption that $\pi_{\triangleleft}(\Gamma)$ is dense in $\mathbb{E}_{\triangleleft}$ and $\pi_{\triangledown}|_{\Gamma}$ is injective; we give a characterisation of these via *complete irrationality* of Γ in Section 2.3. Since the decomposition (1.1) is fixed we generally drop them from the notation and write $\Lambda(\Gamma + \mathbf{z}, \Omega_{\triangleleft})$. Cut and project sets have received an increasing amount of interest of late as mathematical models for *quasicrystals*. Also referred to as *model sets*, they originated in mathematics as generalisations of lattices by Meyer in 1970 [Mey70].

Many of the standard questions about lattices extend naturally to cut and project sets; amongst which is the counting problem consisting of finding a good asymptotic description for the quantity $\#(\Lambda \cap \Omega_{\nabla})$

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where the search region $\Omega_{\nabla} \subset \mathbb{E}_{\nabla}$ is large, for example a large ball $\mathbb{B}_{\nabla}(0, t) \subset \mathbb{E}_{\nabla}$ or cube $[0, t]^{d_{\neg}} \subset \mathbb{E}_{\nabla}$. It is well-known (see, for example, [HKWS16], also see [BG13, Theorem 7.2]) that

$$\#(\Lambda \cap \mathbb{B}_{\nabla}(0,t)) = \frac{\operatorname{vol}(\mathbb{B}_{\nabla}(0,1))\operatorname{vol}(\Omega_{\triangleleft})}{\operatorname{covol}(\Gamma)}t^{d_{\nabla}} + \Delta(\Lambda; \mathbb{B}_{\nabla}(0,t)), \tag{1.2}$$

where the discrepancy is smaller than the main term, i.e. $\Delta(\Lambda; \mathbb{B}_{\nabla}(0, t)) = o(t^{d_{\nabla}})$ for every Λ . Here, the volumes are computed in the appropriate subspace and the covolume of a lattice Γ is defined as $\operatorname{covol}(\Gamma) := \operatorname{vol}(\mathbb{E}/\Gamma)$.

The purpose of the current work is to find estimates for the discrepancy Δ depending on the search region Ω_{∇} , the window Ω_{\triangleleft} and the diophantine properties of Γ . We establish a direct, explicit link between a quantifiable diophantine property of the lattice and a discrepancy bound for the corresponding cut and project set by reducing the problem to lattice point counting, see Theorem 1.2. We apply the discrepancy estimates to the problem of identifying new classes of cut and project sets that can be mapped by a biLipschitz map to a lattice, see Theorem 1.5. In particular, we note that for windows of finite perimeter the sharp classes we obtain depend only on the diophantine class of the lattice Γ and nothing else.

1.2 Uniform upper bounds for discrepancy

We will first state our main theorem on discrepancies of cut and project sets. We begin by defining what it means for a lattice to be ψ -repellent (for the origin of this term, see [BH23]).

Definition 1.1. Given an increasing $\psi : (0, \infty) \to (0, \infty)$ so that $\psi(t) \to \infty$ as $t \to \infty$, we call a lattice Γ ψ -repellent (with respect to the decomposition $\mathbb{E} = \mathbb{E}_{\nabla} \oplus \mathbb{E}_{\prec}$), if for every $\gamma \in \Gamma \setminus \{0\}$,

$$|\pi_{\triangleleft}(\gamma)| > \psi(|\pi_{\triangledown}(\gamma)|)^{-1}$$
.

If there is some $\mu > 0$ so that $\psi(r) = r^{\mu}$ at ∞ , we say that ψ grows at speed μ at infinity. If on the other hand $\psi(r) \lesssim_{\varepsilon} r^{\varepsilon}$ for every $\varepsilon > 0$, for instance $\psi(r) = \log(1+r)$, we say that ψ grows slowly at infinity.

Theorem 1.2. Suppose that $\Gamma \subset \mathbb{E}$ is completely irrational with respect to \mathbb{E}_{\prec} and let $0 < s \le 1$. Let the window $\Omega_{\prec} \subset \mathbb{E}_{\prec}$ have boundary with finite $(d_{\prec} - s)$ -Minkowski content, and assume that the dual lattice Γ^{\dagger} is ψ -repellent for some $\psi : (0,\infty) \to (0,\infty)$ that grows slowly. Let $\Omega_{\nabla} \subset \mathbb{E}_{\nabla}$ have finite perimeter.

Then, for any $\delta > 0$ there are $C_{\Lambda,\delta}$, $t_0 > 0$ so that for every $\mathbf{z} \in \mathbb{E}$ and every $t > t_0$, the cut and project set $\Lambda(\Gamma + \mathbf{z}; \Omega_{\lhd})$ satisfies,

$$\left| \#(\Lambda \cap t\Omega_{\triangledown}) - \frac{\operatorname{vol}(\Omega_{\triangledown}) \operatorname{vol}(\Omega_{\vartriangleleft})}{\operatorname{covol}(\Gamma)} t^{d_{\triangledown}} \right| \leq C_{\Lambda,\delta} t^{d_{\triangledown}} \psi(t)^{-s(1-\delta)}. \tag{1.3}$$

In Section 4 we show that proving Theorem 1.2 reduces to a lattice point count in anisotropically expanding domains — this is the content of Lemma 4.1. Our main theorems in the lattice point counting context are Theorems 4.3 and 4.5 of which Theorem 1.2 is then a special case. Note that by the Khintcine-Groshev Theorem (also see [BH23]) the property of being ψ -repellent for $\psi(t) \approx t^{\mu}$, and hence our results, hold in a conull set of lattices.

In Theorem 4.3 we assume that the search region is a strictly convex domain satisfying some boundary curvature assumption described later, whereas Theorem 4.5 merely assumes that the search region has finite perimeter, so that balls and cubes are special cases, respectively. A simple self-contained proof of the asymptotic (1.2) is included along the way to Theorem 1.2. We also formulate in Section 3.3 a corollary of Theorem 1.2 for discrepancies of frequencies of r-patterns in cut and project sets, see Theorem 3.13.

The main interest of the discrepancy bounds in Theorem 1.2 is that generically in a complementary set of lattices we obtain a matching lower bound along a subsequence, see Subsection 1.3 and the appendix of this text. This means that the class of lattices for which we obtain these upper bounds is sharp. Of course other upper bounds on the discrepancy have been proved in the past by Haynes, Julien, Koivusalo, and Walton [HJKW19] and Rühr, Smilansky and Weiss [RSW23]¹. Haynes *et al.* had a slightly different parametrisation of the cut and project scheme as compared to the one we have here, and one crucial difference is that they take both the window Ω_{\triangleleft} and the search region Ω_{∇} to be axes-aligned cubes. This, together with a diophantine condition on the lattice Γ allows them to deduce a very small discrepancy, in

¹To be more precise, these results are concerned with discrepancy to pattern frequencies. However, as we will explain in Section 3.3, there is no reason to make a distinction between the two problems.

a similar way to the anomalous small remainders proven for lattices in [KY15, Theorem 3.5]. Namely, they show that for any $\varepsilon > 0$, and almost every choice of Γ ,

$$\left| \#(\Lambda \cap t\Omega_{\triangledown}) - \frac{\operatorname{vol}(\Omega_{\lhd})\operatorname{vol}(\Omega_{\triangledown})}{\operatorname{covol}(\Gamma)} t^{d_{\triangledown}} \right| \leq C_{\varepsilon,\Gamma}(\log t)^{d_{\lhd} - \varepsilon}.$$

That these anomalously low bounds hold can also be deduced from an earlier work of Haynes, Kelly and Weiss [HKW14], but their parametrisation is yet different. The aim of both of these results was to obtain a conull set of lattices in which the discrepancy bounds hold. In particular, they did not work out the discrepancy bounds for classes broader than what was necessary for the statement on full measure, which amounts to a type of weak badly approximable condition on Γ . The weak bad approximation properties appearing in these works differ from the ψ -repellent condition of Theorem 1.2.

The result of Rühr *et al.* is also for Γ belonging to a full measure subset of the set of lattices, but they do allow a wider choice of Ω_{\triangleleft} and Ω_{\triangledown} . In particular, they assume that $\partial\Omega_{\triangleleft}$ has box dimension bounded from above by $d_{\triangleleft} - s$ and prove that for almost every lattice

$$\left|\#(\Lambda\cap\Omega_{\triangledown}) - \frac{\operatorname{vol}(\Omega_{\vartriangleleft})\operatorname{vol}(\Omega_{\triangledown})}{\operatorname{covol}(\Gamma)}\right| = O\left(\operatorname{vol}(\Omega_{\triangledown})^{1-\frac{s}{d_{\triangledown}-2s}}\right).$$

For this result also, an underlying generic diophantine condition is key, but the techniques obscure it and it seems difficult to track it down explicitly.

All of these results give a growth rate for the discrepancy which is smaller than that of Theorem 1.2, but there is no clarity on the behaviour of the discrepancy on the measure zero (but comeagre) set of lattices on which their result does not hold. Theorem 1.2 differs from these results in two ways. Foremost, it uses lattice point counting, a technique which is new to the study of discrepancies of cut and project sets. Further, in Theorem 1.2, the connection between the established discrepancy bound and the function ψ is explicit and, in comparison to [HJKW19] and [HKW14], we give a new class of cut and project sets with uniform upper bounds for discrepancy, as the diophantine condition of Theorem 1.2 is explicit and on the dual lattice and not the lattice Γ itself.

1.3 Averaging, and lower bounds for discrepancy

It is natural to ask what is the best possible bound for the discrepancy that one can expect. The reader may also have noticed that the translation parameter \mathbf{z} seems spurrious, as none of the previous results depended on it. This is the case because the methods we use are translation invariant. At the same time, translating Γ by \mathbf{z} is the same as translating the window by $\mathbf{z}_{\triangleleft}$ and translating the search region by $\mathbf{z}_{\triangledown}$; and up to a small set of parameters this should not affect the actual statistics for the distribution of the points in a cut and project set. With this heuristic in mind, we average the absolute value of the discrepancy over a fundamental domain of Γ to obtain lower bounds. The next theorem should be interpreted as saying that this is the best upper bound on the discrepancy that one can obtain on a full measure set of \mathbf{z} . We note that these lower bounds hold with even weaker assumptions on the lattice and the window. Indeed, no irrationality at all is assumed on Γ .

Theorem 1.3. Suppose $\Gamma \subset \mathbb{E}$ is any lattice, Ω_{\triangleleft} is any bounded window with nonzero volume. There is C, $t_0 > 0$ so that the cut and project sets $\Lambda(\Gamma + \mathbf{z}, \Omega_{\triangleleft})$ satisfy for all $t > t_0$

$$\int_{\mathbb{E}/\Gamma} |\Delta(\Lambda(\Gamma + \mathbf{z}, \Omega_{\triangleleft}); \mathbb{B}_{\triangledown}(0, t))| \, d\mathbf{z} \ge Cf(t) t^{\frac{d-1}{2}}. \tag{1.4}$$

where for some A > 0 the function f(t) is defined as

$$f(t) = \begin{cases} 1 & if d \not\equiv 1 \pmod{4} \\ \exp(-A\log\log(t)^4) & if d \equiv 1 \pmod{4}. \end{cases}$$

Furthermore,

$$\int_{\mathbb{F}/\Gamma} \Delta(\Lambda(\Gamma + \mathbf{z}, \Omega_{\triangleleft}); \mathbb{B}_{\triangledown}(0, t)) \, \mathrm{d}\mathbf{z} = 0.$$

Just as Theorem 1.2 was a special case of Theorem 4.3 through a characterisation as a lattice point counting problem as proved in Section 4, Theorem 1.3 is a consequence of a corresponding lattice result, Theorem 4.7. What Theorem 1.3 indicates is that for every cut and project set and every t large enough, there are positive measure subsets of \mathbb{E}/Γ so that the remainder terms are as big as indicated in (1.4), both in the positive and the negative direction. This is, in a way, as best as one can hope to do on full measure sets. In the Appendix, written by Michael Björklund and Tobias Hartnick, it is instead the L^2 norm in the parameter z which is considered, also known as the number variance. More precisely, define a lattice to be ψ -Liouvillean if it satisfies the complementary property to being ψ -repellant; in other words if there is a sequence $\gamma^{(n)}$ so that

$$|\pi_{\triangleleft}(\gamma^{(n)})| \leq \psi(|\pi_{\triangledown}(\gamma^{(n)})|^{-1}).$$

Björklund and Hartnick show that for almost every r > 0, every cut and project set Λ originating from the window $\mathbb{B}_{\triangleleft}(0, r)$ and lattice whose dual is ψ -Liouvillean satisfies the asymptotics

$$\limsup_{t\to\infty}\frac{\int_{\mathbb{E}/\Gamma}|\Delta(\Lambda(\Gamma+\mathbf{z},\mathbb{B}_{\lhd}(0,r));t\Omega_{\triangledown})|^2\,\mathrm{d}\mathbf{z}}{t^{2d}\psi(t)^{-d_{\lhd}-1-\delta}}=\infty.$$

for any $\delta > 0$. This means that, at least along subsequences, the discrepancy can be very large. In fact, they also show that there exist explicit ψ -Liouvillean lattices and Ω_{\triangleleft} so that cut and project sets Λ with this data satisfies

$$\limsup_{t \to \infty} \frac{\int_{\mathbb{E}/\Gamma} |\Delta(\Lambda(\Gamma + \mathbf{z}, \Omega_{\triangleleft}); t\Omega_{\triangledown})|^2 d\mathbf{z}}{t^{2d_{\triangledown}} \psi(t)^{-2-\delta}} = \infty$$
 (1.5)

for any $\delta > 0$; see Theorems A.5 and A.6. For those lattices satisfying (1.5) Hölder's inequality implies that

$$\limsup_{t\to\infty}\frac{|\Delta(\Lambda(\Gamma+\mathbf{z},\Omega_{\triangleleft});t\Omega_{\triangledown})|}{t^{d_{\triangledown}}\psi(t)^{-1+\delta}}=\infty.$$

so that Theorem 1.2 is sharp when the boundary of the window has Minkowski dimension d-1.

1.4 Alternative interpretation: lattice points in thin slabs

There is another geometric interpretation of our results in terms of lattice point counts which we describe now. Counting the number of points in the cut and project set $\Lambda(\Gamma + \mathbf{z}, \Omega_{\triangleleft})$ within dilates $t\Omega_{\nabla}$ of a search region is the same as counting the number of points in the lattice Γ in a thin slab $(t\Omega_{\nabla} \times \Omega_{\triangleleft}) - \mathbf{z}$. Our main result, Theorem 1.2 combined with Theorem A.6 can be rewritten in this language.

Theorem 1.4. Let $\mathbb{E} = \mathbb{E}_{\nabla} \oplus \mathbb{E}_{\triangleleft}$, and Γ be a lattice completely irrational with respect to this decomposition. Let $\Omega_{\triangleleft} \subset \mathbb{E}_{\triangleleft}$ and $\Omega_{\nabla} \subset \mathbb{E}_{\nabla}$ have finite perimeter, and assume furthermore that the dual lattice Γ^{\dagger} is ψ -repellent for some slowly growing $\psi : (0, \infty) \to (0, \infty)$. Then, for any $\delta > 0$ there is $C_{\Gamma, \delta}$ so that

$$\left| \#\Gamma \cap (t\Omega_{\nabla} \times \Omega_{\lhd}) - \frac{\operatorname{vol}(\Omega_{\nabla}) \operatorname{vol}(\Omega_{\lhd})}{\operatorname{covol}(\Gamma)} t^{d_{\nabla}} \right| \leq C_{\Gamma,\delta} t^{d_{\nabla}} \psi(t)^{-1+\delta}.$$

Furthermore, there are ψ -Liouvillean lattices and sets $\Omega_{\lhd} \subset \mathbb{E}_{\lhd}$ and a sequence $t_n \to \infty$ so that for every $\delta > 0$,

$$\left| \#\Gamma \cap (t_n\Omega_\nabla \times \Omega_{\lhd}) - \frac{\operatorname{vol}(\Omega_\nabla) \operatorname{vol}(\Omega_{\lhd})}{\operatorname{covol}(\Gamma)} t_n^{d_\nabla} \right| \ge t_n^{d_\nabla} \psi(t_n)^{-1-\delta}.$$

We note that this refines and corrects the lattice point counting results in anisotropically expanding domains found in any of [KY11, KY15, Lag20], where the contribution from Diophantine properties of the lattice was overlooked.

A classical problem in number theory is to prove existence of approximate integer solutions to polynomial equations, for instance Davenport and Heilbronn proved in [DH46] that there are infinitely many solutions to

$$\sum_{j=1}^{5} \lambda_j n_j^2 \le \varepsilon \tag{1.6}$$

provided the $\lambda_1, ..., \lambda_5$ are not zero, do not all share the same sign, and are pairwise rationally independent. Freeman obtained asymptotics for the number of solutions where every coordinate n_i is below some value T [Freo2]. This corresponds to counting the number of points of \mathbb{Z}^d in $\ker L \oplus [-\varepsilon, \varepsilon]$ where L is

the linear map given by the matrix $(\lambda_1 \quad \lambda_2 \quad \lambda_3 \quad \lambda_4 \quad \lambda_5)$ under the additional constraint that all coordinates are squares. After a linear change of variable, this can be interpreted in our setting, up to the additional constraint on the coordinates being squares.

Recently, in [Wal20], Walker has obtained results of this type for general matrices L in the case where the constraint is that all the coordinates belong to some set A of positive density, as opposed to the situation in (1.6) where the set of squares has density zero. In this situation, he obtains an asymptotic relation where the first term is given by the count of *all* lattice points in $\ker L \times [-\varepsilon, \varepsilon]$, plus an error term which is bounded in terms of the set A and Diophantine properties of the (dual) lattice [Wal20, Lemma 3.4]; see also [Wal21, Lemma 8.4] where the coefficients are assumed algebraic. As such, Theorem 1.4 makes more precise, and sharp, the contribution of the principal term in Walker's results.

1.5 BiLipschitz equivalence to lattices

We now turn our attention to the question of mapping cut and project sets onto lattices with a biLipschitz map $\varphi: \Lambda \to \mathbb{Z}^d$. If such a map exists, we say that Λ is biLipschitz (BL) equivalent to a lattice. The question of BL equivalence to lattices of relatively dense and uniformly discrete sets (separated nets) is an old problem originating in geometry: if two metric spaces have separated nets that are BL equivalent to each other, then they are quasi-isometric. Gromov [Gro93] asked whether there are choices of separated nets in the same metric space that are not BL equivalent to each other. The constructions of Burago and Kleiner [BK98] and McMullen [McM98] were the first to show that the answer is 'yes' in euclidean space: there are separated nets that are not BL equivalent to a lattice. Later it was found that in fact the collection of BL classes of separated nets in \mathbb{R}^d has the size of the continuum [MagII]. Burago and Kleiner [BK02] (in \mathbb{R}^2), and later, Aliste-Prieto, Coronel and Gambaudo [ACG13] (in \mathbb{R}^d) also showed that BL equivalence can be reduced to uniform point counting in large cubes.

In the case of cut and project sets, Burago and Kleiner [BK02] were able to prove that for almost every choice of Γ , a regular cut and project set in dimension d=3 is BL equivalent to a lattice. This prompted them to ask whether every regular cut and project set has this property. Their result was extended to sregular cut and project sets in any dimension d by Haynes, Kelly and Weiss [HKW14]. Proofs of this kind rely on a result given in [ACG13, Theorem 3.1], see also [BK02], which establishes a link between discrepancy estimates of cut and project sets, and their BL equivalence class. In Theorem 1.2 we established an explicit connection between a diophantine property of Γ^{\dagger} and the discrepancy of the corresponding cut and project set. We can now apply it to find the following new class of cut and project sets which are BL equivalent to lattices.

Theorem 1.5. Suppose that $\Gamma \subset \mathbb{E}$ is completely irrational with respect to $\mathbb{E}_{\triangleleft}$, $\mathbf{z} \in \mathbb{E}$, and assume that $\Omega_{\triangleleft} \subset \mathbb{E}_{\triangleleft}$ is s-regular for some 0 < s < 1. Assume, furthermore, that there exists some $\eta > 0$ such that Γ^{\dagger} is $\log(1 + t)^{(1+\eta)/s}$ -repellant. Then, the cut and project set $\Lambda(\Gamma + \mathbf{z}, \Omega_{\triangleleft})$ is BL equivalent to a lattice.

As pointed out above, the property of being ψ -repellent for $\psi(t) \approx t^{\mu}$ holds in a conull set of lattices, and in particular the above result establishes that almost every lattice Γ corresponds to a cut and project set which is BL equivalent to a lattice. We note that Theorems A.5 and A.6 of the Appendix show that this is more or less the largest class of cut and project to which the sufficient conditions of Burago–Kleiner and Aliste-Prieto–Coronel–Gambaudo can be applied, up to a very small (both null and meagre) set. In other words, if we want to determine the biLipschitz class of more cut and project sets a weaker sufficient condition needs to be found, or an example which is not biLipschitz equivalent to a lattice.

Plan of the paper

In Section 2 and 3, we cover the background on lattices, the Fourier transform and cut and project sets. There, we also fix some notation and discuss the irrationality assumptions that are necessary for our results to hold. Section 4 connects the distribution of points in cut and project sets to lattice point counting, and states the main general theorems. Section 5 is a preparation section; we are aiming to use Poisson summation for the lattice point counts. As such, it is necessary to find smooth approximations to the functions being summed in order to ensure sufficient decay of the Fourier transforms. Section 6 is concerned with the proof of Theorems 4.3 which gives upper bounds to discrepancy. Section 7 is again a preparation section, this time providing uniform diophantine estimates for lattice projections. In Section 8 we prove the Theorem 4.7 on the average lower bound for the discrepancy. Finally, the appendix of Björklund and Hartnick gives the lower bounds on the number variance.

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2 Notation, definitions and assumptions

In this section we collect notations, definitions that we use in this paper, as well as recall some facts about them. The results stated in this section are standard or folklore, we state them for convenience of reference. As such, the reader who is short on time may wish to pass directly to Section 4.

2.1 Euclidean space

Total space is denoted by \mathbb{E} , and dim(\mathbb{E}) = d. The canonical decomposition with respect to which cut and project sets are defined is always denoted

$$\mathbb{E} = \mathbb{E}_{\nabla} \oplus \mathbb{E}_{\triangleleft}$$

with $\dim(\mathbb{E}_{\nabla}) = d_{\nabla}$ and $\dim(\mathbb{E}_{\triangleleft}) = d_{\triangleleft} = d - d_{\nabla}$. Sometimes, we require other subspaces or decompositions of \mathbb{E} , in which case we denote them $\mathbb{F} \subset \mathbb{E}$, or $\mathbb{E} = \mathbb{F}_{\nabla} \oplus \mathbb{F}_{\triangleleft}$. The projection on \mathbb{F} is denoted by $\pi_{\mathbb{F}}$.

Given a decomposition, whether the canonical one or an auxiliary one, we always indicate with subscripts ∇ or \triangleleft the association of a variable, a subset, an operator, etc. with one of the subspace, for instance we would write π_{∇} for $\pi_{\mathbb{F}_{\nabla}}$. We also indicate with $\bullet \in \{\nabla, \triangleleft\}$ an association with either of them.

When describing the volume of a set, we will not always specify it is the volume with respect to which subspace and it will be clear from context. For instance, for some $\Omega_{\nabla} \subset \mathbb{E}_{\nabla}$, $\operatorname{vol}(\Omega_{\nabla})$ is the d-dimensional volume of Ω_{∇} induced by the Lebesgue measure on \mathbb{E}_{∇} .

We give special names to some subsets of the fixed decomposition $\mathbb{E} = \mathbb{E}_{\nabla} \oplus \mathbb{E}_{\triangleleft}$

Definition 2.1. A window is a bounded subset $\Omega_{\triangleleft} \subset \mathbb{E}_{\triangleleft}$. If its interior is equal to the interior of its closure we say that it is a *regular window*.

Definition 2.2. A search region is a finite perimeter subset $\Omega_{\nabla} \subset \mathbb{E}_{\nabla}$.

We denote the indicator function of a set $\Omega \subset \mathbb{E}$ by $\chi_{\Omega} : \mathbb{E} \to \mathbb{R}$

$$\chi_{\Omega}(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \Omega \\ 0 & \text{otherwise.} \end{cases}$$

2.2 Asymptotic notation

We make frequent use of the Landau and Vinogradov asymptotic notation. To wit, given two real-valued functions *f* , *g* , we say that

- indiscriminately, f = O(g), $f \lesssim g$, or $g \gtrsim f$ to indicate that there exists C > 0 so that $|f| \le C|g|$;
- we use f = g to indicate that $f \leq g$ and $g \leq f$.

In either cases, an index on the asymptotic notation indicates dependence of the constants on this index. For instance, writing $f \lesssim_{\Gamma,s} g$ means that the constant C in the definition above may depend on Γ and s.

2.3 Lattices

Definition 2.3. Given $\mathbb{F} \subset \mathbb{E}$ with potentially $\mathbb{F} = \mathbb{E}$, a *lattice in* \mathbb{F} is a discrete cocompact subgroup Γ of the group of translations in \mathbb{F} . Through the identification of \mathbb{F} with its group of translations, we view a lattice as a subset $\Gamma \subset \mathbb{F}$, and \mathbb{F}/Γ is a torus which we can identify with any parallelotope (called a fundamental cell) generated by a basis of Γ . Given a lattice this identification is fixed once and for all, the fundamental cell is also denoted \mathbb{F}/Γ , and every $\mathbf{x} \in \mathbb{F}$ can therefore be written uniquely as $\mathbf{x} = \gamma + \mathbf{k}$, for some $\gamma \in \Gamma$ and $\mathbf{k} \in \mathbb{F}/\Gamma$. The covolume of a lattice $\Gamma \subset \mathbb{F}$ is defined as

$$\operatorname{covol}(\Gamma) := \operatorname{vol}(\mathbb{F}/\Gamma).$$

The space $\mathscr{X}_{\mathbb{E}}$ of all lattices in \mathbb{E} is identified with

$$\mathscr{X}_{\mathbb{E}} := \operatorname{GL}(d, \mathbb{Z}) \setminus \operatorname{GL}(d, \mathbb{R}) / S_d$$

where S_d is the symmetric group acting by permuting the columns of a $d \times d$ matrix. Note that this may differ from the standard description found in the litterature where S_d would be replaced by $SO(d,\mathbb{R})$; this is because for our purposes we do *not* want to identify lattices which are equivalent up to orthogonal transformations. From this identification we see that $\mathscr{X}_{\mathbb{E}}$ is an orbifold of dimension d^2 , and it inherits a (Haar) measure and a topology from $GL(d,\mathbb{R})$, on which we put the topology induced by the operator norm.

Definition 2.4. Let Γ be a lattice. Its dual lattice Γ^{\dagger} is defined as

$$\Gamma^{\dagger} := \left\{ \gamma^{\dagger} \in \mathbb{E} : \gamma^{\dagger} \cdot \gamma \in \mathbb{Z} \text{ for all } \gamma \in \Gamma \right\}.$$

The dual lattice is itself a lattice, and $(\Gamma^{\dagger})^{\dagger} = \Gamma$.

Definition 2.5. Let $\mathbb{F} \subset \mathbb{E}$ be a subspace. We denote $\Gamma(\mathbb{F}) := \Gamma \cap \mathbb{F}$ and we say that \mathbb{F} is a Γ -subspace if $\Gamma(\mathbb{F})$ spans \mathbb{F} .

We note the important distinction between $\Gamma^{\dagger}(\mathbb{F}):=\Gamma^{\dagger}\cap\mathbb{F}$ and

$$\Gamma(\mathbb{F})^{\dagger} := \left\{ \gamma^{\dagger} \in \operatorname{span}(\Gamma(\mathbb{F})) : \gamma^{\dagger} \cdot \gamma \in \mathbb{Z} \right\}.$$

There is a correspondence between Γ -subspaces of dimension m and Γ^{\dagger} -subspaces of codimension m. Indeed, if $\mathbb{F} \subset \mathbb{E}$ is a Γ -subspace, \mathbb{F}^{\perp} is a Γ^{\dagger} -subspace, see [KY15, Lemma 3.2]. This correspondence is showcased in the following lemma.

Lemma 2.6. Let $\Gamma \subset \mathbb{E}$ be a lattice. For every $x \in \mathbb{E}$, $\operatorname{span}(x)$ is a Γ^{\dagger} -subspace if and only if there exists $t \in \mathbb{R}$ and $\delta > 0$ so that

$$\{y \in \mathbb{E} : x \cdot y \in (t - \delta, t + \delta)\} \cap \Gamma = \emptyset. \tag{2.1}$$

Proof. We first suppose span(x) is a Γ^{\dagger} -subspace, so that there exists $a \in \mathbb{R} \setminus \{0\}$ such that $ax \in \Gamma^{\dagger}$. By definition of Γ^{\dagger} ,

$$\{y \in \mathbb{E} : ax \cdot y \in (1/4, 3/4)\} \cap \Gamma = \emptyset.$$

Therefore we can take $t = (2a)^{-1}$ and $\delta = (4a)^{-1}$.

Suppose on the other hand that (2.1) holds for some $x \in \mathbb{E}$. This implies we cannot make the quantity $|x \cdot \gamma|$ arbitrarily small. In particular, this implies that $|x \cdot \gamma|$ for $\gamma \in \Gamma$ is quantised: there exists a > 0 such that for every $\gamma \in \Gamma$ there exists $n \in \mathbb{Z}$ such that $x \cdot \gamma = an$. But then $a^{-1}x \in \Gamma^{\dagger}$, in other words span(x) is a Γ^{\dagger} -subspace.

Definition 2.7. Let $\Gamma \subset \mathbb{E}$ be a lattice and $\mathbb{F} \subset \mathbb{E}$ a subspace. We say that Γ is *irrational with respect to* \mathbb{F} if $\Gamma \subset \mathbb{F}$ is also irrational with respect to \mathbb{F} we say that Γ is *completely irrational with respect to* \mathbb{F} .

Given a subspace \mathbb{F} of codimension at least 1, the set of lattices which are not completely irrational with respect to \mathbb{F} is dense in the set of lattices, but is still small, as seen in the following proposition.

Proposition 2.8. Let $\mathbb{F} \subset \mathbb{E}$ be a subspace of dimension m < d. Then, the set of lattices completely irrational with respect to \mathbb{F} is a comeagre set whose complement has Hausdorff dimension dm.

Proof. The projection by the quotient map of a comeagre set is comeagre, so we can identify Γ with any of its (unordered) bases in $GL(d,\mathbb{R})/S_d$. In turn, this is viewed as an open subset of $\mathbb{R}^{d\times d}/S_d$. The first basis element θ_1 of Γ can be chosen arbitrarily outside \mathbb{F} , that is, outside a set of dimension m. Recursively, to ensure that $\mathbb{F} \cap \Gamma = \{0\}$, the jth basis vector needs to be chosen outside $\operatorname{span}_{\mathbb{Z}}(\theta_1,\ldots,\theta_{j-1}) + \mathbb{F}$. This is a d-fold product of a countable union of dimension m < d affine conditions, hence the set of irrational lattices with respect to \mathbb{F} is a comeagre set whose complement has Hausdorff dimension dm.

To see that the same holds for completely irrational lattices, we observe that the involution $\Gamma \mapsto \Gamma^{\dagger}$ corresponds in $GL(d,\mathbb{R})/S_d$ to the map $A \mapsto (A^{-1})^{\top}$. This is a locally Lipschitz homeomorphism, hence it preserves comeagreness and Hausdorff dimension. By the same construction as above, the set of lattices whose dual is irrational with respect to \mathbb{F} is also a comeagre set whose complement has Hausdorff dimension dm, taking their union yields our claim.

Remark 2.9. In our setting we will use (complete) irrationality with respect to $\mathbb{E}_{\triangleleft}$. Irrationality of Γ with respect to $\mathbb{E}_{\triangleleft}$ implies that the projection π_{∇} restricted to Γ is injective; this allows us to translate the problems at hand into lattice point counting problems (see Lemma 4.1). Irrationality of Γ^{\dagger} with respect to $\mathbb{E}_{\triangleleft}$ implies that Γ_{\triangleleft} is dense in $\mathbb{E}_{\triangleleft}$ (see Lemma 3.4). This latter fact is a standard condition in the definition of cut and project sets and it will appear technically in our use of the Poisson summation formula.

2.4 Fourier Transforms of indicator functions

Define $\mathbf{e}: \mathbb{R} \to \mathbb{C}$ as

$$\mathbf{e}(x) := e^{-2\pi i x}.$$

The Fourier transform of a function $f : \mathbb{E} \to \mathbb{C}$ is defined as

$$[\mathscr{F}f](\boldsymbol{\xi}) := \int_{\mathbb{F}} \mathbf{e}(\mathbf{x} \cdot \boldsymbol{\xi}) f(\mathbf{x}) \, \mathrm{d}\mathbf{x}$$

Given a decomposition $\mathbb{E} = \mathbb{F}_{\nabla} \oplus \mathbb{F}_{\triangleleft}$, we also define partial Fourier transforms relative to the decomposition via the formulae

$$[\mathscr{F}_{\triangledown}f](\mathbf{x}_{\triangleleft},\boldsymbol{\xi}_{\triangledown}) := \int_{\mathbb{E}_{\triangledown}} \mathbf{e}(\mathbf{x}_{\triangledown} \cdot \boldsymbol{\xi}_{\triangledown}) f(\mathbf{x}_{\triangledown},\mathbf{x}_{\triangleleft}) \, \mathrm{d}\mathbf{x}_{\triangledown}$$

and

$$[\mathscr{F}_{\lhd}f](\boldsymbol{\xi}_{\lhd},\mathbf{x}_{\triangledown}) := \int_{\mathbb{E}_{\lhd}} \mathbf{e}(\boldsymbol{\xi}_{\lhd} \cdot \mathbf{x}_{\lhd}) f(\mathbf{x}_{\triangledown},\mathbf{x}_{\lhd}) \, \mathrm{d}\mathbf{x}_{\lhd}.$$

A direct computation tells us that $\mathscr{F} = \mathscr{F}_{\triangledown}\mathscr{F}_{\dashv} = \mathscr{F}_{\dashv}\mathscr{F}_{\triangledown}$. We now provide three estimates on the decay of the Fourier transform of indicator functions, depending (in decreasing order of regularity) on whether the boundary of a set is strictly convex, has finite perimeter, or finite upper Minkowski content. We note that the second estimate could be proved by means of the third, but since the proof is simpler we keep it for clarity of exposition, in particular for readers who are only interested in the case where the boundary is regular.

Lemma 2.10. Let $\mathbb{E} = \mathbb{F}_{\nabla} \oplus \mathbb{F}_{\prec}$ be a decomposition of Euclidean space with $\dim(\mathbb{F}_{\nabla}) = m$. Let $\Omega \subset \mathbb{E}$ be compact and assume that for every $\mathbf{x}_{\prec} \in \mathbb{F}_{\prec}$, $\Omega \cap (\mathbb{F}_{\nabla} + \mathbf{x}_{\prec})$ either has zero m-dimensional measure or is a convex set with $C^{\frac{m-1}{2}}$ boundary and principal curvatures uniformly bounded away from 0. There exists C > 0 such that

$$\left| \left[\mathscr{F} \chi_{\Omega} \right] (\boldsymbol{\xi}) \right| \leq C (1 + \left| \boldsymbol{\xi}_{\nabla} \right|)^{\frac{-m-1}{2}}. \tag{2.2}$$

Remark 2.11. We note that the hypotheses of this statement differ slightly from those found in [KYII, Theorem 4], [KYI5, Theorem 2.3] (where this lemma is used in their proof without explicitly stating it separately) and [Lag20, Lemma 6.4]. We add here the hypotheses that the boundary of Ω is sufficiently regular and that the principal curvatures are uniformly bounded away from 0. These conditions are necessary to ensure the decay (2.2): if the principal curvatures vanish at some point then the optimal order of decay is worse, see e.g. [Ran66] for L^p balls in \mathbb{E} and [CdV77, Théorème 1] for general sets. As such, the hypotheses on regularity and geometry of the previous lemma should be added to the statements of the aforementioned theorems for their proof to go through in their current state.

Proof. We write the Fourier transform as

$$\begin{split} [\mathscr{F}\chi_{\Omega}](\xi) &= \int_{\mathbb{F}_{d}} \mathbf{e}(\mathbf{x}_{d} \cdot \boldsymbol{\xi}_{d}) \int_{\mathbb{F}_{\nabla}} \mathbf{e}(\mathbf{x}_{\nabla} \cdot \boldsymbol{\xi}_{\nabla}) \chi(\mathbf{x}_{\nabla}, \mathbf{x}_{d}) \, d\mathbf{x}_{\nabla} \mathbf{x}_{d} \\ &= [\mathscr{F}_{d}[\mathscr{F}_{\nabla} \chi_{\Omega \cap (\mathbb{F}_{\nabla} + \mathbf{x}_{d})}]](\xi). \end{split}$$

By the assumptions on $\Omega \cap (\mathbb{F}_{\nabla} + \mathbf{x}_{\triangleleft})$, it follows from [IL14, Theorem 2.29] with $\alpha = 0$ in their notation that under the hypotheses of the lemma

$$\left| \left[\mathscr{F}_{\triangledown} \chi_{\Omega \cap (\mathbb{F}_{\triangledown} + \mathbf{x}_{\lhd})} (\boldsymbol{\xi}_{\triangledown}) \right| \lesssim (1 + |\boldsymbol{\xi}_{\triangledown}|)^{\frac{-m-1}{2}},$$

uniformly with respect to $\mathbf{x}_{\triangleleft}$. The claim then follows by bounding the Fourier transform with respect to the $\mathbb{F}_{\triangleleft}$ direction by its value at 0.

Lemma 2.12. Let $\mathbb{E} = \mathbb{F}_{\nabla} \oplus \mathbb{F}_{\triangleleft}$ be a decomposition of Euclidean space with $\dim(\mathbb{F}_{\nabla}) = m$. Let $\Omega \subset \mathbb{E}$ be compact and assume that for every $\mathbf{x}_{\triangleleft} \in \mathbb{F}_{\triangleleft}$, $\Omega \cap (\mathbb{F}_{\nabla} + \mathbf{x}_{\triangleleft})$ either has zero measure or is a m-dimensional domain with finite perimeter, and assume that this perimeter is uniformly bounded with respect to $\mathbf{x}_{\triangleleft}$. There exists C > 0 such that

$$|[\mathcal{F}\chi_{\Omega}](\boldsymbol{\xi})| \leq C(1+|\boldsymbol{\xi}_{\nabla}|)^{-1}.$$

Proof. Assume without loss of generality that $|\xi_{\nabla}| \ge 1$. Write the Fourier transform again as

$$\begin{split} [\mathscr{F}\chi_{\Omega}](\xi) &= \int_{\mathbb{F}_{\lhd}} \mathbf{e}(\mathbf{x}_{\lhd} \cdot \boldsymbol{\xi}_{\lhd}) \int_{\mathbb{F}_{\bigtriangledown}} \mathbf{e}(\mathbf{x}_{\bigtriangledown} \cdot \boldsymbol{\xi}_{\bigtriangledown}) \chi(\mathbf{x}_{\bigtriangledown}, \mathbf{x}_{\lhd}) \, d\mathbf{x}_{\bigtriangledown} \mathbf{x}_{\lhd} \\ &= [\mathscr{F}_{\lhd}[\mathscr{F}_{\bigtriangledown}\chi_{\Omega \cap (\mathbb{F}_{\bigtriangledown} + \mathbf{x}_{\lhd})}]](\xi), \end{split}$$

where we interpret the indicator function of Ω as a family parametrised by $\mathbf{x}_{\triangleleft}$ of indicators in $\mathbb{F}_{\triangledown}$. Applying the Gauss–Green theorem we have that

$$\begin{split} \left| \int_{\Omega \cap (F_{\nabla} + \mathbf{x}_{\triangleleft})} \mathbf{e}(\mathbf{x}_{\nabla} \cdot \boldsymbol{\xi}_{\nabla}) \, d\mathbf{x}_{\nabla} \right| &= \left| \int_{\Omega \cap (F_{\nabla} + \mathbf{x}_{\triangleleft})} \frac{1}{|\boldsymbol{\xi}_{\nabla}|^{2}} \, div(\operatorname{grad}_{\mathbf{x}_{\nabla}} \mathbf{e}(\mathbf{x}_{\nabla} \cdot \boldsymbol{\xi}_{\nabla})) \, d\mathbf{x}_{\nabla} \right| \\ &= \left| \int_{\partial \Omega \cap (\mathbb{F}_{\nabla} + \mathbf{x}_{\triangleleft})} \frac{\boldsymbol{\xi}_{\nabla} \cdot \boldsymbol{\nu}}{\left|\boldsymbol{\xi}_{\nabla}\right|^{2}} \mathbf{e}(\mathbf{x}_{\nabla} \cdot \boldsymbol{\xi}_{\nabla}) \, d\mathbf{x}_{\nabla} \right| \\ &\lesssim \left| \boldsymbol{\xi}_{\nabla} \right|^{-1} \operatorname{vol}_{m-1} (\partial \Omega \cap (\mathbb{F}_{\nabla} + \mathbf{x}_{\triangleleft})), \end{split}$$

where v is the unit outward normal to the boundary. Uniform boundedness of the perimeter implies our claim.

For our final estimate we start with a definition of (upper) Minkowski content.

Definition 2.13. Let $Y \subset \mathbb{E}$ be a bounded set and let $0 \le s \le d$. The *s-codimensional (upper) Minkowski content of* Y *with respect to* \mathbb{E} is defined as

$$\mathfrak{M}_s(\Upsilon; \mathbb{E}) := \limsup_{r \searrow 0} r^{-s} \operatorname{vol}(\Upsilon_r),$$

where Υ_r is the *r*-neigbourhood of Υ in \mathbb{E} .

Lemma 2.14. Let $\mathbb{E} = \mathbb{F}_{\nabla} \oplus \mathbb{F}_{\prec}$ be a decomposition of Euclidean space with $\dim(\mathbb{F}_{\nabla}) = m$. Let 0 < s < 1 and let $\Omega \subset \mathbb{E}$ be compact and assume that for every $\mathbf{x}_{\prec} \in \mathbb{F}_{\prec}$, $\Omega \cap (\mathbb{F}_{\nabla} + \mathbf{x}_{\prec})$ either has zero measure or is a m-dimensional domain whose boundary has finite s-codimensional Minkowski content with respect to \mathbb{F}_{∇} , and assume furthermore that this Minkowski content is uniformly bounded with respect to \mathbb{X}_{\prec} . There exists C > 0 such that

$$|[\mathscr{F}\chi_{\Omega}](\boldsymbol{\xi})| \leq C(1+|\boldsymbol{\xi}_{\nabla}|)^{-s}.$$

Proof. Let $\Omega_{\nabla} \subset \mathbb{F}_{\nabla}$ be a set whose boundary has finite *s*-codimensional Minkowski content with respect to \mathbb{F}_{∇} . For r>0, cover the boundary of Ω_{∇} with finitely many disjoint cubes of sidelength r>0, of which we need at most $\lesssim r^{s-m}$. Define $\Omega^{(r)}$ to be Ω with those cubes removed; this is a set with Lipschitz boundary, hence to which we can apply the Gauss–Green theorem. Furthermore, $\operatorname{vol}_{m-1}(\partial\Omega^{(r)})\lesssim r^{s-1}$. Consequently,

$$\left| \int_{\Omega_{\nabla}} \mathbf{e}(\mathbf{x}_{\nabla} \cdot \boldsymbol{\xi}_{\nabla}) \, d\mathbf{x}_{\nabla} \right| \leq \left| \int_{\Omega \setminus \Omega^{(r)}} \mathbf{e}(\mathbf{x}_{\nabla} \cdot \boldsymbol{\xi}_{\nabla}) \, d\mathbf{x}_{\nabla} \right| + \left| \int_{\partial \Omega^{(r)}} \frac{\boldsymbol{\xi}_{\nabla} \cdot \boldsymbol{\nu}}{\left| \boldsymbol{\xi}_{\nabla} \right|^{2}} \mathbf{e}(\mathbf{x}_{\nabla} \cdot \boldsymbol{\xi}_{\nabla}) \, d\mathbf{x}_{\nabla} \right|$$

$$\leq r^{s} + \left| \boldsymbol{\xi}_{\nabla} \right|^{-1} r^{s-1}.$$

We now assume without loss of generality that $|\xi_{\nabla}| \ge 1$ is large enough and we take $r = |\xi_{\nabla}|^{-1}$ to obtain

$$r^{s} + |\boldsymbol{\xi}_{\nabla}|^{-1} r^{s-1} \lesssim |\boldsymbol{\xi}_{\nabla}|^{-s}$$
.

Uniformity of the Minkowski content allows us to decompose the integral in the \triangleleft and \triangledown direction as in the proof of Lemma 2.12, whence we obtain our desired result.

3 Cut and project sets

In this section we introduce the definitions and preliminary results for cut and project sets, and prove Theorem 1.5 as a consequence of Theorem 1.2. We reserve the letter Λ for cut and project sets, and M for subsets of Λ which are themselves a cut and project set.

Definition 3.1. Let $\mathbb{E} = \mathbb{E}_{\nabla} \oplus \mathbb{E}_{\neg}$ be a decomposition of d-dimensional Euclidean space. Let $\Gamma \subset \mathbb{E}$ be a lattice, $\Omega_{\neg} \subset \mathbb{E}_{\neg}$ be a window and $\mathbf{z} \in \mathbb{E}$. The cut and project set associated with this data is

$$\Lambda(\Gamma; \mathbf{z}; \Omega_{\triangleleft}) := \pi_{\triangledown}((\Gamma + \mathbf{z}) \cap \pi_{\triangleleft}^{-1}(\Omega_{\triangleleft}))$$

We fix the decomposition $\mathbb{E} = \mathbb{E}_{\nabla} \oplus \mathbb{E}_{\triangleleft}$ throughout and this is why it has been dropped from the notation in this definition of cut and project sets.

Remark 3.2. If $\gamma \in \Gamma$, then $\Lambda(\Gamma; \mathbf{z}; \Omega_{\triangleleft}) = \Lambda(\Gamma; \mathbf{z} + \gamma; \Omega_{\triangleleft})$, in other words the map $\mathbf{z} \mapsto \Lambda(\Gamma; \mathbf{z}; \Omega_{\triangleleft})$ is Γ -periodic, so that we may consider $\mathbf{z} \in \mathbb{E}/\Gamma$ is an element of the torus rather than of \mathbb{E} . In addition, we can observe that for any $\mathbf{z} \in \mathbb{E}$,

$$\Lambda(\Gamma; \mathbf{z}; \Omega_{\triangleleft}) = \Lambda(\Gamma; 0; \Omega_{\triangleleft} + \mathbf{z}_{\triangleleft}) + \mathbf{z}_{\triangledown},$$

in other words every cut and project set with a parameter $\mathbf{z} \in \mathbb{E}$ can be realised as the translation of a cut and project set with a parameter $\mathbf{z} = 0$ by translating the window appropriately.

3.1 Assumptions on the cut and project set

We do not need to give any conditions on the parameter \mathbf{z} for our results to hold. We will require some conditions on the other pieces of data which we list now.

- **(O)** The decomposition $\mathbb{E}_{\nabla} \oplus \mathbb{E}_{\triangleleft}$ is orthogonal.
- **(D)** The image Γ_{\triangleleft} is dense in $\mathbb{E}_{\triangleleft}$.
- **(I)** The projection $\pi_{\nabla}|_{\Gamma}$ is injective,
- (s-Reg) The window Ω_{\triangleleft} is regular, meaning that it is relatively compact and has non-empty interior, and its boundary is of measure 0. We assume further that the s-codimensional Minkowski content of $\partial\Omega_{\triangleleft}$ in $\mathbb{E}_{\triangleleft}$ is finite.

The condition **(O)** is simply stated for convenience; every cut and project set can be realised as a cut and project set satisfying **(O)**, see for example the discussion in [RSW23, p. 9]. The conditions **(I)** and **(D)** below are standard assumptions in the area (see, for example, [RSW23, Section 2]), but we will prove as Lemma 3.4 below that they are equivalent to the assumption of Γ being completely irrational with respect to $\mathbb{E}_{\triangleleft}$, a formulation which is more convenient for our purposes. Finally, some condition on the regularity on the boundary of the window is needed; otherwise, not even the patch counting function behaves as expected [JLO19]. Condition (s-Reg) is the weakest condition that allows for our techniques. Notice that, given these assumptions the inverse $(\pi_{\nabla})^{-1}: \Lambda \to \Gamma$ is well defined. For any subset $A \subset \Lambda$, we set the notation $A^{\Delta} = (\pi_{\nabla})^{-1}(A)$, and the notation $A^* = \pi_{\prec}((\pi_{\nabla})^{-1}(A))$.

The following lemmas tell us that complete irrationality of the lattice with respect to $\mathbb{E}_{\triangleleft}$ is equivalent to conditions **(D)** and **(I)**. We note that even outside our current purposes, complete irrationality is *a priori* a much easier condition to check directly on a given lattice than density of the projection on $\mathbb{E}_{\triangleleft}$, say.

Lemma 3.3. The lattice projection Γ_{\triangleleft} is dense in $\mathbb{E}_{\triangleleft}$ if and only if for every r > 0, the intersection $\mathbb{B}(0, r) \cap \Gamma_{\triangleleft}$ spans $\mathbb{E}_{\triangleleft}$.

Proof. We see immediately that density of Γ_{\lhd} in \mathbb{E}_{\lhd} implies the existence of arbitrarily small subsets of Γ_{\lhd} spanning \mathbb{E}_{\lhd} ; we now prove the other direction. We observe that $\pi_{\lhd}: \Gamma \hookrightarrow \mathbb{E}_{\lhd}$ is a group homomorphism. In particular, for every linearly independent $\Sigma \subset \Gamma_{\lhd}$ of cardinality d_{\lhd} , $\operatorname{span}_{\mathbb{Z}}(\Sigma)$ is a lattice in \mathbb{E}_{\lhd} . In order to prove our claim, it is sufficient to show that for every $x_{\lhd} \in \mathbb{E}_{\lhd}$ and $\varepsilon > 0$, there exists $\Sigma \subset \Gamma_{\lhd}$ such that $\operatorname{dist}(x_{\lhd}, \operatorname{span}_{\mathbb{Z}}(\Sigma)) < \varepsilon$. We note that for any full rank lattice $\Theta \subset \mathbb{E}_{\lhd}$,

$$\sup_{x_{\triangleleft} \in \mathbb{E}_{\dashv}} \operatorname{dist}(x_{\triangleleft}, \Theta) < \operatorname{diam}(\mathbb{E}_{\triangleleft}/\Theta). \tag{3.1}$$

Now, if Σ is a basis for a lattice Θ , then identifying $\mathbb{E}_{\triangleleft}/\Theta$ with the parallelotope generated by Σ we see that

$$\operatorname{diam}(\mathbb{E}_{\triangleleft}/\Theta) < d_{\triangleleft} \max\{|\theta| : \theta \in \Sigma\}.$$

In particular, by hypothesis for every $\varepsilon > 0$ there exists $\Sigma_{\varepsilon} \in \mathbb{B}_{\prec}(0, d_{\prec}^{-1}\varepsilon) \cap \Gamma_{\prec}$ of cardinality d_{\prec} spanning \mathbb{E}_{\prec} , so that we deduce from (3.1) that $\operatorname{dist}(x, \operatorname{span}_{\mathbb{Z}}(\Sigma_{\varepsilon})) < \varepsilon$, and indeed existence of arbitrarily small subsets of Γ_{\prec} spanning \mathbb{E}_{\prec} implies density of Γ_{\prec} in \mathbb{E}_{\prec} .

Lemma 3.4. The lattice Γ is completely irrational with respect to $\mathbb{E}_{\triangleleft}$, if and only if conditions **(D)** and **(I)** hold.

Proof. The proof is split in two: we show that irrationality of Γ with respect to $\mathbb{E}_{\triangleleft}$ is equivalent to **(I)**, whereas irrationality of Γ^{\dagger} with respect to $\mathbb{E}_{\triangleleft}$ is equivalent to **(D)**. The first assertion is direct: $\mathbb{E}_{\triangleleft} = \ker(\pi_{\nabla})$, so that we see directly that Γ being irrational with respect to $\mathbb{E}_{\triangleleft}$ is equivalent to π_{∇} being injective when restricted to Γ .

Now, suppose **(D)**, that Γ_{\triangleleft} is dense in $\mathbb{E}_{\triangleleft}$. Then, for every $x_{\triangleleft} \in \mathbb{E}_{\triangleleft} \setminus \{0\}$, there exists $\gamma \in \Gamma$ such that

$$0 < |x_{\triangleleft} \cdot \gamma| = |x_{\triangleleft} \cdot \gamma_{\triangleleft}| < 1/2,$$

in particular $x_{\triangleleft} \not\in \Gamma^{\dagger}$ and Γ^{\dagger} is irrational with respect to $\mathbb{E}_{\triangleleft}$. On the other hand, if Γ_{\triangleleft} is not dense in $\mathbb{E}_{\triangleleft}$, use Lemma 3.3 to find r > 0 such that

$$r = \inf\{t : \operatorname{span}(\mathbb{B}(0, t) \cap \Gamma_{\triangleleft}) = \mathbb{E}_{\triangleleft}\}.$$

Choose a codimension 1 subspace $\mathbb{F} \subset \mathbb{E}_{\triangleleft}$ spanned by elements of $\overline{\mathbb{B}(0,r)} \cap \Gamma_{\triangleleft}$ and containing every $\gamma \in \mathbb{B}(0,r) \cap \Gamma_{\triangleleft}$. (In particular for $d_{\triangleleft} = 1$ take \mathbb{F}^{\perp} to be the whole $\mathbb{E}_{\triangleleft}$.) Let $x \in \mathbb{F}^{\perp} \cap \mathbb{E}_{\triangleleft}$ have norm 1. By construction,

$$\{y \in \mathbb{E} : x \cdot y \in (r/4, 3r/4)\} \cap \Gamma = \emptyset$$

so that Lemma 2.6 implies that span(x) is a Γ^{\dagger} -subspace, in other words $\Gamma^{\dagger} \cap \mathbb{E}_{\triangleleft} \neq \{0\}$.

Remark 3.5. We observe that complete irrationality of Γ with respect to $\mathbb{E}_{\triangleleft}$ does not preclude Λ from containing periodic subsets of rank strictly smaller than d, so that these are also included in our analysis.

We call a window satisfying Condition (s-Reg) a s-regular window. Boundedness of the window ensures that Λ is uniformly discrete, and non-empty interior of the window guarantees that Λ is relatively dense. The Minkowski content of $\partial\Omega_{\triangleleft}$ will determine how good of an upper bound we are able to get on the discrepancy.

3.2 BL equivalence classes

Recall that we call a subset Y of euclidean space a *separated net* if it is uniformly discrete and relatively dense. Two separated nets Y, Y' are said to be *biLipschitz* (BL) equivalent if there is a biLipschitz map $\varphi: Y \to Y'$. We now describe sufficient conditions under which we can obtain BL equivalence of a separated net Y, a fortiori of a cut and project set, to a lattice. We note that since lattices are all in the same BL equivalence class, these characterisations are often given in terms of equivalence to $\mathbb{Z}^{d_{\nabla}}$.

Given a separated net $Y \subset \mathbb{E}_{\nabla}$, and a search region Ω_{∇} and $\alpha > 0$, put

$$\zeta_{\alpha}(\Omega_{\nabla}) = \max \left\{ \frac{\alpha \operatorname{vol}(\Omega_{\nabla})}{\#(Y \cap \Omega_{\nabla})}, \frac{\#(Y \cap \Omega_{\nabla})}{\alpha \operatorname{vol}(\Omega_{\nabla})} \right\}.$$

Set $Q_{\nabla} \subset \mathbb{E}_{\nabla}$ to be the unit cube and for t > 0 put

$$Z_{\alpha}(t) = \sup_{\mathbf{n} \in \mathbb{Z}^{d_{\nabla}}} \zeta_{\alpha}(\mathbf{n} + tQ_{\nabla})$$

The following lemma appears first as [BKo2] for $d_{\nabla} = 2$ and [ACG13, Theorem 3.1] for any $d_{\nabla} \ge 2$.

Lemma 3.6. Let $Y \subset \mathbb{E}_{\nabla}$ be a separated net, and suppose that there is $\alpha > 0$ such that

$$\sum_{n=1}^{\infty} \log Z_{\alpha}(2^n) < \infty.$$

Then, Y is BL-equivalent to $\mathbb{Z}^{d_{\nabla}}$.

We use this lemma as well as our Theorem 1.2 in order to prove Theorem 1.5.

Proof of Theorem 1.5. Recall from Remark 3.2 that it is equivalent to translate Ω_{∇} or Γ by \mathbf{z}_{∇} . Therefore, since Theorem 1.2 is uniform in the parameter \mathbf{z} , we can read from (1.3) that for any s-regular window Ω_{\lhd} , any lattice Γ completely irrational with respect to \mathbb{E}_{\lhd} and so that Γ^{\dagger} is ψ -repelled by \mathbb{E}_{\lhd} , any $\mathbf{z}, \mathbf{z}' \in \mathbb{E}$ and $\delta > 0$:

$$\frac{\#(\Lambda(\Gamma + \mathbf{z}; \Omega_{\lhd}) \cap (\mathbf{z}_{\nabla}' + [0, t]^d))}{\alpha t^d} = 1 + O_{\Gamma, \Omega_{\lhd}} \left(\psi(t)^{-s + \delta s} \right),$$

where we have set $\alpha = \frac{\text{vol}(\Omega_d)}{\text{covol}(\Gamma)}$. In particular, if there is $\eta > 0$ so that $\psi(t) \gtrsim \log(t)^{\frac{1+\eta}{s}}$ then

$$\sum_{n=1}^{\infty} \log Z_{\alpha}(2^{n}) \lesssim_{\delta} \sum_{n=1}^{\infty} \log \left(1 + \log(2^{-n})^{(-1+\delta)(1+\eta)} \right)$$

$$\lesssim_{\delta} \sum_{n=1}^{\infty} n^{-1-\eta+\delta+\eta\delta}.$$
(3.2)

In particular, taking δ sufficiently small ensures that this is a convergent sum. Therefore, by Lemma 3.6 the cut and project set $\Lambda(\Gamma + \mathbf{z}; \Omega_{\triangleleft})$ is BL equivalent to \mathbb{Z}^d .

Remark 3.7. Suppose that $\psi(r)$ has a slower growth at infinity, that is that for every $\eta > 0$, as $r \setminus 0$, $\psi(r) \lesssim \psi(r)^{\frac{1+\eta}{s}}$, and that Γ^{\dagger} is not ψ' -repelled by \mathbb{E}_{\lhd} for any ψ' with a faster growth at infinity than this. Then, our methods cannot prove that the associated cut and project set is BL equivalent to a lattice from Lemma 3.6. Indeed, in that situation the sum at the end of (3.2) diverges. This means that from the point of view of our results on discrepancy this is the largest class of lattices which can be shown to be BL equivalent to a lattice; in order to improve it one needs to either find weaker sufficient conditions or improved results on the discrepancy. It is indicated in [RSW23] that the bound $t^{-d}\Delta(\Lambda(\Gamma+\mathbf{z};\Omega_{\lhd});\mathbf{n}+[0,t]^d)=o(1)$ cannot be improved to a different decay rate on the right-hand side that would hold for every cut and project set. In the appendix, Michael Björklund and Tobias Hartnick give variance estimates proving that our diophantine conditions is the weakest under which we can use Lemma 3.6 to prove BL equivalence of a cut and project set to a lattice.

3.3 Acceptance domains

The goal of this section is to describe r-patterns in cut and project sets as cut and project sets themselves. The upshot of this procedure is that any theorem we prove for the statistics of cut and project sets is also valid for statistics of r-patterns.

Definition 3.8. For r > 0, define

$$\Gamma(r) := \left\{ \gamma \in \Gamma : \gamma_{\nabla} \in \mathbb{B}_{\nabla}(0, r) \right\}.$$

For every $P \subset \Gamma(r)$, the *acceptance domain* A_P is defined as

$$A_{P} := \bigcap_{\gamma \in P} (\Omega_{\triangleleft} - \gamma_{\triangleleft}) \cap \bigcap_{\gamma \in \Gamma(r) \setminus P} (\operatorname{int}((\Omega_{\triangleleft})^{c}) - \gamma_{\triangleleft}), \tag{3.3}$$

where the interior and the complement of Ω_{\triangleleft} are taken relative to $\mathbb{E}_{\triangleleft}$. An *r-admissible pattern* is a subset $P \subset \Gamma(r)$ containing the origin and such that $A_P \neq \emptyset$. The set of all acceptance domains of size r is denoted

$$\mathcal{A}(r) := \{A_P : P \text{ is a } r\text{-admissible pattern}\}.$$

Remark 3.9. • We note a difference in our definition with the one from [KW21, Definition 3.3]: There patches P are thought to be subsets of Λ , but here they are subsets of $\Gamma(r)$.

- We take the interior of the complements rather than of the window as our windows are open whereas in [KW21, Definition 3.3] they are closed; the advantage of our choice is that it makes acceptance domains windows in their own right, see Lemma 3.10.
- Another difference to [KW21, Definition 3.3] is that we consider in the intersections all $\gamma \in \Gamma(r)$ rather than considering only the 'slab' of them so that $\gamma_{\lhd} \in \Omega_{\lhd} \Omega_{\lhd}$. This has no bearing on the definition: since $0 \in P$ for all r-admissible patterns, $\gamma_{\lhd} \not\in \Omega_{\lhd} \Omega_{\lhd}$ implies that $\Omega_{\lhd} \cap (\Omega_{\lhd} \gamma_{\lhd}) = \varnothing$. In other words, if such a $\gamma \in P$, then $A_P = \varnothing$. If such a $\gamma \not\in P$, $\Omega_{\lhd} \subset \operatorname{int}((\Omega_{\lhd})^c) \gamma_{\lhd}$ so that intersecting with that set changes nothing. In the end, the intersections in (3.3) can always be supposed to be intersections over a finite set.

- In [KW21, Definition 3.3] the windows are assumed to be in a *generic* position (see [BG13, Definition 7.2]), so that $\partial\Omega_{\triangleleft}\cap\Gamma_{\triangleleft}=\emptyset$. We will not make such an assumption.
- For every r-admissible pattern P the acceptance domain A_P has non-empty interior since it is open
 and non-empty. As such, by the assumption (D), there exists λ ∈ Λ such that P_r(λ) = P_∇.

Lemma 3.10. Suppose that Ω_{\triangleleft} is a s-regular window and let r > 0. Then for every r-admissible pattern $P \in \Gamma(r)$, A_P is a s-regular window.

Proof. The domain A_P is a non-empty intersection of translates of Ω_{\triangleleft} and of the interior of its complement. As such it is bounded, open, and its boundary is a union of boundary pieces for $\partial\Omega_{\triangleleft}$. Finiteness of the upper Minkowski content is stable under finite unions, so the boundary of A_P satisfies the same condition as that of Ω_{\triangleleft} and we conclude that A_P is a s-regular window.

The following lemma completely characterises pattern equivalence in terms of maximal r-admissible patterns and their acceptance domains. It has appeared in the literature multiple times in various forms. For a proof we refer to [KW21, Section 2].

Lemma 3.11. For every $\lambda \in \Lambda$,

- 1. $P_r(\lambda)^{\triangle} \lambda^{\triangle}$ is the unique r-admissible pattern $P \subset \Gamma(r)$ such that $P_r(\lambda) = P_{\nabla}$.
- 2. If $\zeta \in \Lambda$ is such that $\zeta^* \in A_{P_r(\lambda)^{\triangle}}$, then ζ is r-pattern equivalent to λ . Conversely, if ζ is r-pattern equivalent to λ , then ζ^* is in the closure of $A_{P_r(\lambda)^{\triangle}}$.

Remark 3.12. Let us make a few remarks about the statement, as well as other definitions of acceptance domains in the literature.

- For a window Ω_{\triangleleft} in a generic position, the statement 2 in the previous lemma does not need to refer to the closure, and becomes an equivalence. In either case, our results are the same whether or not some γ_{\triangleleft} hit $\partial\Omega_{\triangleleft}$.
- The previous lemma implies that our definition of acceptance domains makes them a tiling of Ω_{\triangleleft} . If, rather, we are interested in the frequency of *r-clusters* of points in Λ , that is subsets of Λ of diameter at most 2r, the acceptance domains take the form of (3.3) without the intersection over translates of int($(\Omega_{\triangleleft})^c$). In particular, this turns the relation "belongs to an acceptance domain" into a partial order (through reverse inclusion) rather than an equivalence, with that partial order expressing whether a cluster is a subset of another up to translation. This corresponds to the definition of acceptance domains found in, say [BG13, Section 7.2] and since acceptance domains are still *s*-regular windows (with the same proof as in Lemma 3.10), our results on asymptotic frequency also apply to them.

We finish this section with the statement of a theorem on discrepancies for the frequences of r-patterns. From the above discussion, Theorem 3.13 follows directly from Theorem 1.2.

Theorem 3.13. Under the hypotheses of Theorem 1.2, let $M \subset \Lambda$ be the set of representatives for a r-pattern equivalence class. Then, there is a regular $\Omega_{\lhd,M} \subset \Omega_{\lhd}$ so that $\partial(\Omega_{\lhd,M})$ has finite $(d_{\lhd} - s)$ -dimensional Minkowski content so that $M = \Lambda(\Gamma + \mathbf{z}; \Omega_{\lhd,M})$. In particular, for every $\delta > 0$ there is $C_{M,s,\delta}$ so that

$$\left|\#(\mathbf{M}\cap\mathbb{B}_{\triangledown}(0,t))-\frac{\operatorname{vol}(\mathbb{B}_{\triangledown}(0,1))\operatorname{vol}(\Omega_{\vartriangleleft,\mathbf{M}})}{\operatorname{covol}(\Gamma)}\right|\leq C_{\mathbf{M},s,\delta}\,t^{d_{\triangledown}}\psi(t)^{-s+\delta s}$$

Of course, any improvement to Theorem 1.2 obtained when ψ grows at speed μ also applies here.

4 Lattice point counts and the statement of the general theorems

In this section, we reduce Theorem 1.2 to a lattice point count for Γ in an anisotropically expanding set. Counting lattice points in ever expanding sets is a standard problem in the geometry of numbers, we refer to the methods and results in [KY11, KY15, Lag20], where lattice points are counted in anisotropically expanding domains. The main difference in our situation compared to these works is that the relevant domains are products of domains in $\mathbb{E}_{\triangleleft}$ and $\mathbb{E}_{\triangledown}$; any expansion occurs only along $\mathbb{E}_{\triangledown}$. In particular, this allows us to consider sets whose $\mathbb{E}_{\triangleleft}$ slices have very bad boundary regularity.

In that spirit, we start with the following general lemma which tells us that we can reduce the count of elements in a cut and project set to a sum over the lattice Γ in the total space.

Lemma 4.1. Let $\Gamma \subset \mathbb{E}$ be a lattice which is irrational with respect to $\mathbb{E}_{\triangleleft}$, $\mathbf{z} \in \mathbb{E}$ and $\Omega_{\triangleleft} \subset \mathbb{E}_{\triangleleft}$, $\Omega_{\nabla} \subset \mathbb{E}_{\nabla}$. Then

$$\#(\Lambda(\Gamma+\mathbf{z};\Omega_{\lhd})\cap\Omega_{\triangledown})=\sum_{\gamma\in\Gamma}\chi_{\Omega_{\triangledown}\times\Omega_{\lhd}}(\gamma-\mathbf{z}).$$

Proof. Irrationality of Γ with respect to \mathbb{E}_{\neg} implies that $\pi^{\triangle}: \Lambda \to (\Gamma + \mathbf{z}) \cap (\mathbb{E}_{\triangledown} \times \Omega_{\neg})$ is a bijection. By definition, $\lambda \in \Lambda \cap \Omega_{\triangledown}$ if and only if $(\lambda^{\triangle})_{\triangledown} \in \Omega_{\triangledown}$ and $(\lambda^{\triangle})_{\neg} \in \Omega_{\neg}$; in other words $\lambda^{\triangle} \in \Omega_{\triangledown} \times \Omega_{\neg}$. We deduce that

$$\#\{\Lambda\cap\Omega_{\triangledown}\} = \#\left\{\Lambda^{\triangle}\cap(\Omega_{\triangledown}\times\Omega_{\dashv})\right\} = \#\{(\Gamma+\mathbf{z})\cap(\Omega_{\triangledown}\times\Omega_{\dashv})\} = \sum_{\gamma\in\Gamma}\chi_{\Omega_{\triangledown}\times\Omega_{\dashv}}(\gamma-\mathbf{z}).$$

This is our claim.

Remark 4.2. We note that irrationality of Γ with respect to $\mathbb{E}_{\triangleleft}$ is only used to ensure that π^{\triangle} is well defined (and in particular that π_{\triangledown} is injective). This can also be ensured from taking a window whose diameter is smaller than the element of $\Gamma \cap \mathbb{E}_{\triangleleft}$ with smallest norm.

In order to simplify the statement of theorems, we now set the notation, for $\Omega = \Omega_{\nabla} \times \Omega_{\triangleleft}$, t > 0, $\Gamma \subset \mathbb{E}$ a lattice, and $\mathbf{z} \in \mathbb{E}$,

$$N(\Omega; \Gamma + \mathbf{z}; t) := \#((\Gamma + \mathbf{z}) \cap (t\Omega_{\nabla} \times \Omega_{\lhd}) = \sum_{\gamma \in \Gamma} \chi_{t\Omega_{\nabla} \times \Omega_{\lhd}} (\gamma - \mathbf{z}).$$

We also slightly abuse notation and use the same symbol for the discrepancy in the cut and project counting function and for the discrepancy in the lattice point counting function, that is

$$\Delta(\Omega; \Gamma + \mathbf{z}; t) = N(\Omega; \Gamma + \mathbf{z}; t) - \frac{\operatorname{vol}(\Omega) t^{d_{\nabla}}}{\operatorname{covol}(\Gamma)}.$$
(4.1)

It follows from Lemma 4.1 that any estimates on the discrepancy as defined in (4.1) transfers to estimates on the discrepancy for cut and project sets as defined in (1.2). The next theorems in this sections are equivalent reinterprations of Theorems 1.2 and 1.3 in the language of lattice point counting. They are a refinement of [KY15, Theorems 2.1–2.3] and [Lag20, Theorem 1.11], where looking at product sets allows us to be more precise in the dependence of the remainder estimates on the regularity of the window, and with precise control on diophantine properties of Γ , see Definition 1.1.

Theorem 4.3. Let $\Gamma \subset \mathbb{E}$ be a lattice completely irrational with respect to \mathbb{E}_{\neg} so that Γ^{\dagger} is ψ -repellent with respect to \mathbb{E}_{\neg} $\oplus \mathbb{E}_{\neg}$. Let $\Omega_{\neg} \subset \mathbb{E}_{\neg}$ be a s-regular window and $\Omega_{\nabla} \subset \mathbb{E}_{\nabla}$ be a convex set with $C^{\frac{d_{\nabla}-1}{2}}$ boundary and principal curvatures uniformly bounded away from 0. If ψ grows slowly at infinity, for every $\delta > 0$ there are $C_{\Lambda,\delta}$, $t_0 > 0$, such that for every $\mathbf{z} \in \mathbb{E}$ and every $t > t_0$,

$$|\Delta(\Omega;\Gamma+\mathbf{z};t)| \leq C_{\Lambda,\delta} t^{d_{\triangledown}} \psi(t)^{-s(1-\delta)}.$$

If ψ grows at speed μ at infinity, then for every $\delta > 0$ there are C, $t_0 > 0$ such that for every $\mathbf{z} \in \mathbb{E}$ and $t > t_0$,

$$|\Delta(\Omega;\Gamma+\mathbf{z};t)| \leq Ct^{d_\nabla - \frac{2sd_\nabla}{(d_\nabla+1)(s+\frac{1}{\mu})+2d_\lhd} + \delta}.$$

Remark 4.4. Theorem 1.2 is an immediate corollary of the first part of Theorem 4.3 in the case where Ω_{∇} is the unit ball – note that by the above considerations

$$|\#(\Lambda(\Gamma+\mathbf{z};\Omega_{\lhd})\cap t\Omega_{\triangledown}) - \frac{\operatorname{vol}(\Omega_{\triangledown})\operatorname{vol}(\Omega_{\lhd})}{\operatorname{covol}(\Gamma)}t^{d_{\triangledown}}| = |\Delta(\Omega;\Gamma+\mathbf{z};t)|.$$

Further, Theorem 3.13 is also an immediate consequence, after observing that by Lemma 3.10, the representatives of an *r*-pattern form cut and project set with a *s*-regular window.

Even when the boundary of the search region merely has finite perimeter we can still obtain estimates on the discrepancy.

Theorem 4.5. Let $\Gamma \subset \mathbb{E}$ be a lattice completely irrational with respect to \mathbb{E}_{\neg} so that Γ^{\dagger} is ψ -repellent with respect to $\mathbb{E}_{\neg} \oplus \mathbb{E}_{\neg}$. Let $\Omega_{\neg} \subset \mathbb{E}_{\neg}$ be a s-regular window and $\Omega_{\nabla} \subset \mathbb{E}_{\nabla}$ be a set with finite perimeter. If ψ grows slowly at infinity, for every $\delta > 0$ there are C, $t_0 > 0$, such that for every $\mathbf{z} \in \mathbb{E}$ and every $t > t_0$,

$$|\Delta(\Omega; \Gamma + \mathbf{z}; t)| \leq C t^{d_{\nabla}} \psi(t)^{-s(1-\delta)}$$
.

If ψ grows at speed μ at infinity, then for every $\delta > 0$ there are C, $t_0 > 0$ such that for every $\mathbf{z} \in \mathbb{E}$ and $t > t_0$,

$$|\Delta(\Omega; \Gamma + \mathbf{z}; t)| \le C t^{d_{\nabla} - \frac{d_{\nabla} s}{d_{\nabla} s + d_{\triangleleft} + \mu^{-1}} + \delta}$$

Remark 4.6. We first observe that when ψ grows slowly at infinity, the estimate on the discrepancy does not depend on convexity properties of the search region. In fact, it depends only on the Diophantine properties of Γ and the regularity of the boundary of the window.

Theorems 4.3, 4.5 (and hence Theorems 1.2 and 3.13) are proved in Section 6. The two previous theorems are uniform bounds on the discrepancy Δ with respect to the parameter \mathbf{z} . When we average over this parameter \mathbf{z} we obtain lower bounds without conditions on the lattice Γ . We note that when the dimension $d_{\nabla} \equiv 1 \pmod{4}$ there are arithmetic obstructions which make our lower bounds slightly worse. This is similar to what is observed in [PSo1, KSSo3, LP16], and is not an artifact of the proof methods. We note that for these lower bounds it is only required that the window be bounded with finite volume, no other regularity assumptions is made. Even the boundedness assumption can be weakened, see Remark 8.2

Theorem 4.7. Let Ω_{\triangleleft} be any bounded window with non-zero volume, $\Omega_{\nabla} = \mathbb{B}_{\nabla}(0,1)$, $\Omega = \Omega_{\nabla} \times \Omega_{\triangleleft}$ and $\Gamma \subset \mathbb{E}$ be any lattice. Then, there exists $A, C, t_0 > 0$ such that

$$\int_{\mathbb{F}/\Gamma} \Delta(\Omega, \Gamma + \mathbf{z}, t) \, \mathrm{d}\mathbf{z} = 0$$

and for every $t > t_0$

$$\int_{\mathbb{F}/\Gamma} |\Delta(\Omega, \Gamma + \mathbf{z}, t)| \, d\mathbf{z} \ge C f(t) t^{\frac{d_{\nabla} - 1}{2}},$$

where $f:(0,\infty)\to(0,\infty)$ is defined as

$$f(t) := \begin{cases} 1 & \text{if } d_{\nabla} \not\equiv 1 \pmod{4} \\ \exp(-A \log \log(t)^4) & \text{if } d_{\nabla} \equiv 1 \pmod{4}. \end{cases}$$

For the proof, see Section 8. From this theorem we immediately obtain Theorem 1.3 and the following lower bounds on the best possible upper bounds.

Corollary 4.8. Under the hypotheses of Theorem 4.7, there are C, $t_0 > 0$ such that for every $t > t_0$ there are two subsets $\mathcal{S}_+(t)$ and $\mathcal{S}_-(t)$ of \mathbb{E}/Γ of positive measure so that

$$\forall \mathbf{z} \in \mathcal{S}_{\pm} \qquad \qquad \pm \Delta(\Omega, \Gamma + \mathbf{z}, t) \geq C f(t) t^{\frac{d-1}{2}}.$$

5 Interlude I: approximation and mollification

5.1 Approximation of sets and volume estimates

In this section we will provide a smoothing of sets well-adapted to the products of sets appearing in the study of pattern statistics. We use $\bullet \in \{ \nabla, \triangleleft \}$ to represent either direction since the construction itself is symmetric with respect to the decomposition, but different choices end up being made in either direction.

For $\Omega_{\bullet} \subset \mathbb{E}_{\bullet}$ and $a_{\bullet} > 0$ we define

$$\Omega_{a_{\bullet}}^{+} := \Omega_{\bullet} + \mathbb{B}_{\bullet}(0, a_{\bullet})$$
 and $\Omega_{a_{\bullet}}^{-} := \mathbb{E}_{\bullet} \setminus (\mathbb{E}_{\bullet} \setminus \Omega_{\bullet})_{a_{\bullet}}^{+}$.

These are exterior and interior approximations to Ω_{\bullet} , respectively. We note that since $\mathbb{B}_{\bullet}(0, a_{\bullet})$ is an open convex set containing 0,

$$\Omega_{a_{\bullet}}^{+} = \Omega_{\bullet} \cup (\partial \Omega_{\bullet})_{a_{\bullet}}^{+} \quad \text{and} \quad \Omega_{a_{\bullet}}^{-} = \Omega_{\bullet} \setminus (\partial \Omega_{\bullet})_{a_{\bullet}}^{+}.$$
 (5.1)

In particular we see that if Ω_{\bullet} has empty interior then $\Omega_{a_{\bullet}}^{-}$ is empty.

Lemma 5.1. For any $\bullet \in \{ \nabla, \triangleleft \}$, let $a_{\bullet}, t_{\bullet} > 0$ and $\Omega \subset \mathbb{E}_{\bullet}$. We have the identity

$$(a_{\bullet}\Omega_{\bullet})_{a_{\bullet}t_{\bullet}}^{\pm} = a_{\bullet}\Omega_{t_{\bullet}}^{\pm}.$$

Proof. The equality for + can be proven by observing that by linearity

$$a_{\bullet}(\Omega_{t_{\bullet}}^{+}) = a_{\bullet}(\Omega_{\bullet} + \mathbb{B}_{\bullet}(0, t_{\bullet})) = a_{\bullet}\Omega_{\bullet} + \mathbb{B}_{\bullet}(0, a_{\bullet}t_{\bullet}) = (a_{\bullet}\Omega_{\bullet})_{a_{\bullet}t_{\bullet}}^{+}.$$

Similarly and using this first result we have that

$$a_{\bullet}\Omega_{t_{\bullet}}^{-} = \mathbb{E}_{\bullet} \setminus a_{\bullet}((\mathbb{E}_{\bullet} \setminus \Omega_{\bullet})_{t_{\bullet}}^{+}) = \mathbb{E}_{\bullet} \setminus (\mathbb{E}_{\bullet} \setminus a_{\bullet}\Omega_{\bullet})_{a_{\bullet}t_{\bullet}}^{+} = (a_{\bullet}\Omega_{\bullet})_{a_{\bullet}t_{\bullet}}^{-}$$

We need the following volume estimate.

Lemma 5.2. Let $\Omega_{\bullet} \subset \mathbb{E}_{\bullet}$ be a bounded set whose boundary has finite s-codimensional Minkowski content with respect to \mathbb{E}_{\bullet} , for some $0 \le s \le d_{\bullet}$. Then, there exists $C, \varepsilon_0 > 0$ so that for every $0 < a_{\bullet} < \varepsilon_0$ we have

$$\operatorname{vol}(\Omega_{a_{\bullet}}^{+} \setminus \Omega_{a_{\bullet}}^{-}) \leq Ca_{\bullet}^{s}$$

Proof. It follows from identity (5.1) that $\Omega_{a_{\bullet}}^+ \setminus \Omega_{a_{\bullet}}^- = (\partial \Omega_{\bullet})_{a_{\bullet}}^+$, i.e. is the tubular neighbourhood of $\partial \Omega_{\bullet}$ of radius a_{\bullet} . The claim then follows from the definition of Minkowski content.

5.2 Anisotropic mollifiers of indicators

Now that we have found approximations of sets, it is time to find smooth approximations to the indicator functions of those sets.

Let $\widetilde{\rho} \in C_c^{\infty}(\mathbb{R})$ be a smooth nonnegative bump function supported in [-1,1]. Define $\rho_{\bullet} \in C_c^{\infty}(\mathbb{E}_{\bullet})$ as

$$\rho_{\bullet}(\mathbf{x}_{\bullet}) = \frac{\widetilde{\rho}(|\mathbf{x}_{\bullet}|)}{\int_{\mathbb{R}} \rho(|\mathbf{x}_{\bullet}|) \, \mathrm{d}\mathbf{x}_{\bullet}},$$

and $\rho \in C_c^{\infty}(\mathbb{E})$ as

$$\rho(\mathbf{x}) = \rho_{\nabla}(\mathbf{x}_{\nabla})\rho_{\triangleleft}(\mathbf{x}_{\triangleleft}).$$

For $a_{\bullet} > 0$, $\bullet \in \{ \nabla, \triangleleft \}$ define the bidisk

$$\mathbb{D}(a_{\triangledown}, a_{\triangleleft}) := \mathbb{B}_{\triangledown}(0, a_{\triangledown}) \times \mathbb{B}_{\triangleleft}(0, a_{\triangleleft}) \subset \mathbb{E},$$

and the mollifier $\rho^{(a_{\triangledown},a_{\dashv})}$ to be

$$\rho^{(a_{\triangledown},a_{\dashv})}(\mathbf{x}) := \frac{1}{a_{\triangledown}^{d_{\triangledown}} a_{\dashv}^{d_{\dashv}}} \rho(a_{\triangledown}^{-1} \mathbf{x}_{\triangledown}, a_{\dashv}^{-1} \mathbf{x}_{\dashv})$$

We observe that $\rho^{(a_{\nabla},a_{\lhd})}$ is supported in $\mathbb{D}(a_{\nabla},a_{\lhd})$, is smooth and has unit mass. By separation of variables, its Fourier transform satisfies

$$[\mathcal{F}\rho^{(a_{\triangledown},a_{\vartriangleleft})}](\boldsymbol{\xi}) = \left[\mathcal{F}_{\triangledown}\rho_{\triangledown}\right](a_{\triangledown}\boldsymbol{\xi}_{\triangledown})\left[\mathcal{F}_{\vartriangleleft}\rho_{\vartriangleleft}\right](a_{\triangledown}\boldsymbol{\xi}_{\triangledown})$$

For any measurable $f : \mathbb{E} \to \mathbb{R}$ we define

$$f^{(a_{\triangledown},a_{\vartriangleleft})} := [f * \rho^{(a_{\triangledown},a_{\vartriangleleft})}](\mathbf{x}) := \int_{\mathbb{E}} f(\mathbf{x} - \mathbf{y}) \rho^{(a_{\triangledown},a_{\vartriangleleft})}(\mathbf{y}) \, \mathrm{d}\mathbf{y}.$$

Proposition 5.3. *For* $\bullet \in \{ \nabla, \triangleleft \}$, *let* $\Omega_{\bullet} \in \mathbb{E}_{\bullet}$ *and* $a_{\bullet} > 0$, *and* put $\Omega = \Omega_{\nabla} \times \Omega_{\triangleleft}$. *For all* $\mathbf{x} \in \mathbb{E}_{\bullet}$

$$\chi^{(a_{\triangledown},a_{\lhd})}_{\Omega^-_{a_{\triangledown},a_{\lhd}}}(\mathbf{x}) \leq \chi_{\Omega}(\mathbf{x}) \leq \chi^{(a_{\triangledown},a_{\lhd})}_{\Omega^+_{a_{\triangledown},a_{\lhd}}}(\mathbf{x}).$$

Proof. We start with the rightmost inequality. Since $0 \le \chi_{\Upsilon}^{(a_{\nabla}, a_{\neg})} \le 1$ for any $\Upsilon \subset \mathbb{E}$, it suffices to show that for every $\mathbf{x} \in \Omega$, $\chi_{\Omega_{a_{\nabla}, a_{\neg}}^+}^{(a_{\nabla}, a_{\neg})}(\mathbf{x}) = 1$. But by definition of $\Omega_{a_{\nabla}, a_{\neg}}^+$, $\mathbf{x} - \mathbb{D}(a_{\nabla}, a_{\neg}) \subset \Omega_{a_{\nabla}, a_{\neg}}^+$. Since $\operatorname{supp}(\rho_{a_{\nabla}, a_{\neg}}) \subset \mathbb{D}(a_{\nabla}, a_{\neg})$ we deduce that

$$\begin{split} \chi_{\Omega_{a_{\nabla},a_{\dashv}}^{-}}^{(a_{\nabla},a_{\dashv})}(\mathbf{x}) &= \int_{\mathbb{D}(a_{\nabla},a_{\dashv})} \chi_{\Omega_{a_{\nabla},a_{\dashv}}^{+}}(\mathbf{x} - \mathbf{y}) \rho_{a_{\nabla},a_{\dashv}}(\mathbf{y}) \, \mathrm{d}\mathbf{y} \\ &= \int_{\mathbb{D}(a_{\nabla},a_{\dashv})} \rho_{a_{\nabla},a_{\dashv}}(\mathbf{y}) \, \mathrm{d}\mathbf{y} \\ &= 1. \end{split}$$

Similarly, to prove the leftmost inequality it suffices to show that for $\mathbf{x} \in \mathbb{E} \setminus \Omega$, we have $\chi_{\Omega_{\overline{a}_{\nabla}, a_{\triangleleft}}}^{(a_{\nabla}, a_{\triangleleft})}(\mathbf{x}) = 0$. This time, observe that for $\bullet \in \{\nabla, \triangleleft\}$

$$\mathbf{x} - \mathbb{B}_{\bullet}(0, a_{\bullet}) \subset (\mathbb{E}_{\bullet} \setminus \Omega_{\bullet})_{a_{\bullet}}^{+} = \mathbb{E}_{\bullet} \setminus \Omega_{a_{\bullet}}^{-}$$

so that $\mathbf{x} - \mathbb{D}(A_{\nabla}, A_{\triangleleft}) \cap \Omega_{a_{\nabla}, a_{\triangleleft}}^- = \emptyset$. This implies that

$$\chi_{\Omega_{a_{\nabla},a_{\lhd}}^{-}}^{(a_{\nabla},a_{\lhd})}(\mathbf{x}) = \int_{\mathbb{D}(a_{\nabla},a_{\lhd})} \chi_{\Omega_{a_{\nabla},a_{\lhd}}}^{-}(\mathbf{x} - \mathbf{y}) \rho_{a_{\nabla},a_{\lhd}}(\mathbf{y}) \, \mathrm{d}\mathbf{y} = 0,$$

proving our claim.

6 Uniform upper bounds for the discrepancy

We now aim at proving Theorem 4.3. Let $\Omega_{\neg} \subset \mathbb{E}_{\neg}$ be an open set whose boundary has finite s-codimensional Minkowski content, and $\Omega_{\nabla} \subset \mathbb{E}_{\neg}$ be an open convex set satisfying the hypothesis of either Lemma 2.10 or 2.12, and put $\Omega = \Omega_{\nabla} \times \Omega_{\neg}$. We suppose without loss of generality that the parameter $\mathbf{z} = 0$; as mentioned in Remark 3.2 this can be achieved by translating Ω instead of Γ . For t > 0, we put

$$N(\Omega;\Gamma;t) = \sum_{\gamma \in \Gamma} \chi_{t\Omega_{\triangledown} \times \Omega_{\lhd}}(\gamma) = \sum_{\gamma \in \Gamma} \chi_{\Omega}(t^{-1}\gamma_{\triangledown},\gamma_{\lhd}).$$

Since our goal is to use the Poisson summation formula, we need to smooth out this sum. For $a_{\bullet} > 0$, define

$$\begin{split} N^{\pm}_{a_{\triangledown},a_{\dashv}}(\Omega;\Gamma;t) &:= \sum_{\gamma \in \Gamma} \chi^{(a_{\triangledown},a_{\dashv})}_{(t\Omega_{\triangledown} \times \Omega_{\dashv})^{\pm}_{a_{\triangledown},a_{\dashv}}}(\gamma) \\ &= \sum_{\gamma \in \Gamma} [\chi_{t\Omega^{\pm}_{t^{-1}a_{\triangledown}} \times \Omega^{\pm}_{a_{\dashv}}} * \rho^{(a_{\triangledown},a_{\dashv})}](\gamma) \\ &= \sum_{\gamma \in \Gamma} [\chi_{\Omega^{\pm}_{t^{-1}a_{\triangledown}} \times \Omega^{\pm}_{a_{\dashv}}} * \rho^{(a_{\triangledown}t^{-1},a_{\dashv})}](t^{-1}\gamma_{\triangledown},\gamma_{\dashv}), \end{split}$$

where we went from the first to the second line using Lemma 5.1, and the second to third changing variables in the integral defining convolution. It follows from Proposition 5.3 that

$$N^-_{a_{\triangledown},a_{\vartriangleleft}}(\Omega;\Gamma;t) \leq N(\Omega;\Gamma;t) \leq N^+_{a_{\triangledown},a_{\vartriangleleft}}(\Omega;\Gamma;t)$$

as such we now aim to find an asymptotic expression for $N^{\pm}_{a_{\nabla},a_{\lhd}}(\Omega;\Gamma;t)$. We will choose a_{∇} and a_{\lhd} at the end of the process, as functions of t, the dimensions d_{∇} and d_{\lhd} , as well as the parameter s. Recall from Section 2.3 that Γ^{\dagger} is the dual lattice of Γ , that is the lattice

$$\Gamma^{\dagger} := \left\{ \gamma^{\dagger} \in \mathbb{E} : \gamma^{\dagger} \cdot \gamma \in \mathbb{Z} \text{ for all } \gamma \in \Gamma \right\}.$$

From the Poisson summation formula, we have that

$$\begin{split} N^{\pm}_{a_{\triangledown},a_{\dashv}}(\Omega;\Gamma,t) &= \frac{1}{\operatorname{covol}(\Gamma)} \sum_{\gamma^{\dagger} \in \Gamma^{\dagger}} [\mathscr{F}\chi^{(a_{\triangledown},a_{\dashv})}_{(t\Omega_{\triangledown} \times \Omega_{\dashv})^{\pm}_{a_{\triangledown},a_{\dashv}}}](\gamma^{\dagger}) \\ &= \frac{t^{d_{\triangledown}}}{\operatorname{covol}(\Gamma)} \sum_{\gamma^{\dagger} \in \Gamma^{\dagger}} \left[\mathscr{F}_{\triangledown}\chi_{\Omega^{\pm}_{t^{-1}A_{\triangledown}}}\right] (t\gamma^{\dagger}_{\triangledown}) \left[\mathscr{F}_{\dashv}\chi_{\Omega^{\pm}_{a_{\dashv}}}\right] (\gamma^{\dagger}_{\dashv}) \left[\mathscr{F}_{\triangledown}\rho_{\triangledown}\right] (a_{\triangledown}\gamma^{\dagger}_{\triangledown}) \left[\mathscr{F}_{\dashv}\rho_{\dashv}\right] (a_{\dashv}\gamma^{\dagger}_{\dashv}). \end{split}$$

This last sum can be further split into two terms, the term at 0 and the remainder:

$$N_{a_{\nabla},a_{\lhd}}^{\pm}(\Omega;\Gamma;t) = V(\Omega;\Gamma;t;a_{\nabla},a_{\lhd}) + R(\Omega;\Gamma;t;a_{\nabla},a_{\lhd})$$

Here *V*, the volume term, is the term at 0 in the sum:

$$V(\Omega; \Gamma; t; a_{\nabla}, a_{\neg}) = \frac{t^{d_{\nabla}}}{\operatorname{covol}(\Gamma)} \left[\mathscr{F}_{\nabla} \chi_{\Omega_{t^{-1}a_{\nabla}}^{\pm}} \right] (0) \left[\mathscr{F}_{\neg} \chi_{\Omega_{a_{\neg}}^{\pm}} \right] (0) \left[\mathscr{F}_{\nabla} \rho_{\nabla} \right] (0) \left[\mathscr{F}_{\neg} \rho_{\neg} \right] (0). \tag{6.1}$$

On the other hand *R*, the remainder term, is the sum over all other elements of the dual lattice:

$$R(\Omega;\Gamma;t;a_{\triangledown};a_{\lhd}) = \frac{t^{d_{\triangledown}}}{\operatorname{covol}(\Gamma)} \sum_{\gamma^{\dagger} \in \Gamma^{\dagger} \backslash \{0\}} \left[\mathscr{F}_{\triangledown} \chi_{\Omega_{t^{-1}a_{\triangledown}}^{\pm}} \right] (t\gamma^{\dagger}_{\triangledown}) \left[\mathscr{F}_{\lhd} \chi_{\Omega_{a_{\lhd}}^{\pm}} \right] (\gamma^{\dagger}_{\lhd}) \left[\mathscr{F}_{\triangledown} \rho_{\triangledown} \right] (a_{\triangledown} \gamma^{\dagger}_{\triangledown}) \left[\mathscr{F}_{\lhd} \rho_{\lhd} \right] (a_{\lhd} \gamma^{\dagger}_{\lhd}).$$

6.1 The volume term

We first analyse V. The mollifiers ρ_{\bullet} were constructed specifically to have unit mass so that $[\mathscr{F}_{\bullet}\rho_{\bullet}](0) = 1$. On to the Fourier transforms of indicators, we have that for any $s_{\bullet} > 0$

$$[\mathscr{F}_{\bullet}\chi_{\Omega_{s}^{\pm}}](0) = \operatorname{Vol}(\Omega_{s_{\bullet}}^{\pm}).$$

Since

$$\Omega_{s}^{-} \subset \Omega_{\bullet} \subset \Omega_{s}^{+}$$

we deduce that

$$\left|\operatorname{vol}(\Omega_{s_{\bullet}}^{\pm}) - \operatorname{vol}(\Omega_{\bullet})\right| \leq \operatorname{vol}(\Omega_{s_{\bullet}}^{+} \setminus \Omega_{s_{\bullet}}^{-}),$$

and inserting into (6.1) with $t^{-1}a_{\nabla}$ and a_{\triangleleft} in place of s_{∇} and s_{\triangleleft} gives us

$$\left| V(\Omega; \Gamma; t; a_{\nabla}, a_{\triangleleft}) - \frac{t^{d_{\nabla}} \operatorname{vol}(\Omega)}{\operatorname{covol}(\Gamma)} \right| \leq \left(\operatorname{vol}(\Omega_{\nabla}) \operatorname{vol}(\Omega_{a_{\triangleleft}}^{+} \setminus \Omega_{a_{\triangleleft}}^{-}) + \operatorname{vol}(\Omega_{\triangleleft}) \operatorname{vol}(\Omega_{t^{-1}a_{\nabla}}^{+} \setminus \Omega_{t^{-1}a_{\nabla}}^{-}) + \operatorname{vol}(\Omega_{t^{-1}a_{\nabla}}^{+} \setminus \Omega_{t^{-1}a_{\nabla}}^{-}) \operatorname{vol}(\Omega_{a_{\triangleleft}}^{+} \setminus \Omega_{a_{\triangleleft}}^{-}) \right) \frac{t^{d_{\nabla}} \operatorname{vol}(\Omega)}{\operatorname{covol}(\Gamma)}.$$

$$(6.2)$$

Since we end up choosing $a_{\triangledown} = o(t)$ and $a_{\triangleleft} = o(1)$ in such a way that

$$\operatorname{vol}(\Omega^+_{a_{\lhd}} \setminus \Omega^-_{a_{\lhd}}) \to 0 \qquad \text{and} \qquad \operatorname{vol}(\Omega^+_{t^{-1}a_{\triangledown}} \setminus \Omega^-_{t^{-1}a_{\triangledown}}) \to 0$$

as $t \to \infty$, it is sufficient to bound only the first two terms in (6.2). Lemma 5.2 tells us that as $t^{-1}a_{\nabla} \to 0$ and $a_{\lhd} \to 0$,

$$\operatorname{vol}(\Omega_{t^{-1}a_{\triangledown}}^+ \setminus \Omega_{t^{-1}a_{\triangledown}}^-) \lesssim t^{-1}a_{\triangledown} \qquad \text{and} \qquad \operatorname{vol}(\Omega_{a_{\dashv}}^+ \setminus \Omega_{a_{\dashv}}^-) \lesssim a_{\dashv}^s$$

Putting together the last two displays we deduce that

$$\left| V(\Omega; \Gamma; t; a_{\nabla}; a_{\triangleleft}) - \frac{t^{d_{\nabla}} \operatorname{Vol}(\Omega)}{\operatorname{covol}(\Gamma)} \right| \lesssim_{\Omega, \Gamma} t^{d_{\nabla}} \left(a_{\triangleleft}^{s} \operatorname{vol}(\Omega_{\nabla}) + t^{-1} a_{\nabla} \operatorname{vol}(\Omega_{\triangleleft}) \right)$$

6.2 The remainder term

Since ρ_{\bullet} is smooth $\mathscr{F}_{\bullet}\rho_{\bullet}$ is a Schwartz function. That is, for every $K_{\bullet} > 0$

$$|[\mathscr{F}_{\bullet}\rho_{\bullet}](a_{\bullet}\xi_{\bullet})| \lesssim_{K_{\bullet}} (1+|a_{\bullet}\xi_{\bullet}|)^{-K_{\bullet}}.$$

For $\mathscr{F}_{\bullet}\chi_{\Omega^{\pm}_{a}}$ we use the estimate, uniform in ξ_{\bullet}

$$\left| \left[\mathscr{F}_{\bullet} \chi_{\Omega_{a_{\bullet}}^{\pm}} \right] (\boldsymbol{\xi}_{\triangleleft}) \right| \lesssim_{\Omega_{\triangleleft}} (1 + \left| \boldsymbol{\xi}_{\bullet} \right|)^{-L_{\bullet}}$$
(6.3)

where

$$L_{\bullet} = \begin{cases} \frac{d_{\bullet}+1}{2} & \text{if } \Omega_{\bullet} \text{ is strictly convex with principal curvatures bounded away from zero;} \\ 1 & \text{if } \Omega_{\bullet} \text{ has finite perimeter;} \\ s & \text{if } \Omega_{\bullet} \text{ is } s\text{-regular, } 0 < s < 1, \end{cases}$$
(6.4)

Combining (6.3) and (6.4), we have for every $K_{\bullet} > d_{\bullet} - L_{\bullet}$ (so that the sum converges)

$$\begin{split} |R(\Omega;\Gamma;t;a_{\triangledown},a_{\dashv})| \lesssim & \frac{t^{d_{\triangledown}}}{\operatorname{covol}(\Gamma)} \sum_{\gamma^{\dagger} \in \Gamma^{\dagger} \backslash \{0\}} \frac{(1+|t\gamma^{\dagger}_{\triangledown}|)^{-L_{\triangledown}} (1+|\gamma^{\dagger}_{\dashv}|)^{-L_{\dashv}}}{\left(1+|a_{\triangledown}\gamma^{\dagger}_{\triangledown}|\right)^{K_{\triangledown}} \left(1+|a_{\dashv}\gamma^{\dagger}_{\dashv}|\right)^{K_{\dashv}}} \\ \lesssim_{\Gamma} t^{d_{\triangledown}} \left(\underbrace{\sum_{0 < |\gamma^{\dagger}_{\triangledown}| < t^{-\sigma}} \frac{(1+|\gamma^{\dagger}_{\dashv}|)^{-L_{\dashv}}}{(1+|a_{\dashv}\gamma^{\dagger}_{\dashv}|)^{K_{\dashv}}}}_{\Sigma_{1}:=} + \underbrace{\sum_{|\gamma^{\dagger}_{\triangledown}| > t^{-\sigma}} \frac{(1+|t\gamma^{\dagger}_{\triangledown}|)^{-L_{\triangledown}} (1+|\gamma^{\dagger}_{\dashv}|)^{-L_{\dashv}}}{\left(1+|a_{\triangledown}\gamma^{\dagger}_{\dashv}|\right)^{K_{\dashv}}}_{\Sigma_{2}:=} \right)}_{\Sigma_{2}:=} \end{split},$$

where $\sigma > 0$ is a parameter to determine later.

First region: small γ_{∇} . To estimate Σ_1 , we first use Peetre's inequality to see that for every $\xi \in \mathbb{E}/\Gamma^{\dagger}$,

$$\begin{split} \Sigma_{1} &\leq \sum_{0 < |\gamma_{\triangledown}^{\dagger}| < t^{-\sigma}} \frac{(1 + |\gamma_{\lhd}^{\dagger} + \xi_{\lhd}|)^{-L_{\lhd}}}{(1 + |a_{\lhd}(\gamma_{\lhd}^{\dagger} + \xi_{\lhd}))^{K_{\lhd}}} \frac{(1 + |\xi_{\lhd}|)^{L_{\lhd}}}{(1 + |\xi_{\lhd}|)^{-K_{\lhd}}} \\ &\leq (1 + \operatorname{diam}(\mathbb{E}/\Gamma^{\dagger}))^{L_{\lhd} + K_{\lhd}} \sum_{0 < |\gamma_{\triangledown}^{\dagger}| < t^{-\sigma}} \frac{(1 + |\gamma_{\lhd}^{\dagger} + \xi_{\lhd}|)^{-L_{\lhd}}}{(1 + |a_{\lhd}(\gamma_{\lhd}^{\dagger} + \xi_{\lhd})|)^{K_{\lhd}}}. \end{split}$$

Integrating both sides over $\mathbb{E}/\Gamma^{\dagger}$ therefore gives us

$$\Sigma_1 \lesssim_{\Gamma} \sum_{0 < |\gamma_{\neg}^{\dagger}| < t^{-\sigma}} \int_{\mathbb{E}/\Gamma^{\dagger}} \frac{(1 + |\gamma_{\neg}^{\dagger} + \xi_{\neg}|)^{-L_{\neg}}}{(1 + |a_{\neg}(\gamma_{\neg}^{\dagger} + \xi_{\neg}|)^{K_{\neg}}} d\xi.$$

Since Γ^{\dagger} is ψ -repulsive, in this region $|\gamma_{\lhd}^{\dagger}| > \psi(t^{\sigma})$, and in particular since $\mathbb{E}/\Gamma^{\dagger}$ is bounded we see that $|\gamma_{\lhd}^{\dagger}| + \xi_{\lhd}| > \psi(t^{\sigma})/2$ as soon as t is large enough, whereas in this region $\gamma_{\nabla} + \xi_{\nabla}$ is merely uniformly bounded. Therefore, we can unfold the previous sum in that region to obtain

$$\begin{split} & \Sigma_{1} \lesssim_{\Gamma} \int_{|\xi_{\triangledown}| < 2\operatorname{diam}(E/\Gamma^{\dagger})} \int_{|\xi_{\dashv}| > \psi(t^{\sigma})/2} (1 + |\xi_{\dashv}|)^{-L_{\dashv}} (1 + a_{\dashv}|\xi_{\dashv}|)^{-K_{\dashv}} \, \mathrm{d}\xi_{\dashv} \, \mathrm{d}\xi_{\triangledown} \\ & \lesssim_{\Gamma} a_{\dashv}^{-K_{\dashv}} \int_{|\xi_{\dashv}| > \psi(t^{\sigma})/2} \frac{\mathrm{d}\xi_{\dashv}}{|\xi_{\dashv}|^{L_{\dashv} + K_{\dashv}}} \\ & \lesssim_{\Gamma} a_{\dashv}^{-K_{\dashv}} \psi(t^{\sigma})^{d_{\dashv} - L_{\dashv} - K_{\dashv}}. \end{split}$$

Second region: larger γ_{∇} . In this region, we estimate $(1+|\gamma_{\triangleleft}^{\dagger}|)^{-L_{\triangleleft}} \lesssim 1$, and use Peetre's inequality, assuming that we choose $a_{\bullet} \leq 1$ small, to get

$$\begin{split} \left(1 + |a_{\bullet}\gamma_{\bullet}^{\dagger}|\right)^{-K_{\bullet}} & \leq \left(1 + |a_{\bullet}(\gamma_{\bullet}^{\dagger} + \xi_{\bullet})|\right)^{-K_{\bullet}} \left(1 + |a_{\bullet}\xi_{\bullet}|\right)^{K_{\bullet}} \\ & \lesssim_{K,\Gamma} \left(1 + |a_{\bullet}(\gamma_{\bullet}^{\dagger} + \xi_{\bullet})|\right)^{-K_{\bullet}}. \end{split}$$

We refine Peetre's inequality on the region $|\gamma^{\dagger}_{\nabla}| > t^{-\sigma}$. Observe that

$$\frac{1+|t(\gamma_{\nabla}+\xi_{\nabla})|}{1+t|\gamma_{\nabla}|} \leq \frac{1+t(|\gamma_{\nabla}|+|\xi_{\nabla}|)}{1+t|\gamma_{\nabla}|} \\
\leq \frac{1+t(|\gamma_{\nabla}|+\operatorname{diam}(\mathbb{E}/\Gamma^{\dagger}))}{1+t|\gamma_{\nabla}|}.$$
(6.5)

It is a straightforward calculus exercise to see that for every t, c > 0, the function

$$x \mapsto \frac{1 + t(x + c)}{1 + tx}$$

is strictly decreasing on $(0,\infty)$, so that the right-hand side in (6.5) is bounded by the value at $|\gamma_{\nabla}| = t^{-\sigma}$. Evaluating gives us

$$(1+t|\gamma_{\triangledown}|)^{-L_{\triangledown}} \lesssim_{L_{\triangledown},\Gamma} t^{L_{\triangledown}\sigma} (1+t|\gamma_{\triangledown}+\xi_{\triangledown}|)^{-L_{\triangledown}}.$$

Putting all these estimates back into Σ_2 , integrating over $\mathbb{E}/\Gamma^{\dagger}$ then unfolding the sum gives us

$$\begin{split} \Sigma_{2} \lesssim_{K_{\bullet},\Gamma,L_{\bullet}} t^{L_{\nabla}\sigma} \sum_{|\gamma_{\nabla}^{\dagger}| > t^{-\sigma}} \int_{\mathbb{E}/\Gamma^{\dagger}} \frac{(1+t|\gamma_{\nabla}^{\dagger} + \xi_{\nabla}|)^{-L_{\nabla}}}{(1+a_{\nabla}|\gamma_{\nabla}^{\dagger} + \xi_{\nabla}|)^{K_{\nabla}}(1+a_{\neg}|\gamma_{\neg}^{\dagger} + \xi_{\neg}|)^{K_{\neg}}} \, \mathrm{d}\boldsymbol{\xi} \\ \lesssim_{K_{\bullet},\Gamma,L_{\bullet}} t^{L_{\nabla}\sigma} \int_{\mathbb{E}} \frac{(1+t|\xi_{\nabla}|)^{-L_{\nabla}}}{(1+a_{\nabla}|\xi_{\nabla}|)^{K_{\nabla}}(1+a_{\neg}|\xi_{\neg}|)^{K_{\neg}}} \, \mathrm{d}\boldsymbol{\xi}. \end{split}$$

Changing variables as $(\xi_{\triangledown}, \xi_{\lhd}) \mapsto (a_{\triangledown}^{-1} \xi_{\triangledown}, a_{\lhd}^{-1} \xi_{\lhd})$ gives us

$$\begin{split} \Sigma_2 \lesssim_{K_\bullet,\Gamma,L_\bullet} \frac{t^{-(1-\sigma)L_\nabla} a_\nabla^{L_\nabla}}{a_\nabla^{d_{\triangledown}} a_{\vartriangleleft}^{d_{\vartriangleleft}}} \int_{\mathbb{E}} \frac{|\xi_\nabla|^{-L_\nabla}}{(1+|\xi_\nabla|)^{K_\nabla} (1+|\xi_{\vartriangleleft}|)^{K_{\vartriangleleft}}} \, \mathrm{d}\xi \\ \lesssim_{K_\bullet,\Gamma,L_\bullet} t^{-(1-\sigma)L_\nabla} a_\nabla^{L_\nabla-d_\nabla} a_{\vartriangleleft}^{-d_{\vartriangleleft}}, \end{split}$$

as long as $K_{\triangleleft} > k - d$, and we take K_{\triangledown} arbitrary large (since it plays no role in the asymptotics appart from the constant in front).

Combining the remainder terms. Summing up Σ_1 and Σ_2 we obtain in the end that

$$|R(\Omega,\Gamma,t,a_{\triangledown},a_{\lhd})| \lesssim_{\Omega,\Gamma,K_{\bullet},L_{\bullet}} t^{d_{\triangledown}} a_{\lhd}^{-K_{\lhd}} \psi(t^{\sigma})^{d_{\lhd}-L_{\lhd}-K_{\lhd}} + t^{d_{\triangledown}-(1-\sigma)L_{\triangledown}} a_{\triangledown}^{L_{\triangledown}-d_{\triangledown}} a_{\lhd}^{-d_{\lhd}}.$$

6.3 A balancing act

Along the proof, we have introduced arbitrary parameters a_{∇} , a_{\lhd} , K_{\lhd} and σ ; it is now time to choose them carefully in order to obtain the required bounds on the discrepancy. Note that a_{\bullet} will be functions of t, which is our asymptotic parameter; as such any constants in asymptotic estimates is not allowed to depend on them. On the other hand, σ and K_{\lhd} will depend only on ψ which is part of the geometry of the cut and project set, so our asymptotic estimates may depend without issue from them. Technically, K_{∇} is also arbitrary but it plays no role in the asymptotics appart from needing to be large enough for some integral to converge. Summing up the contributions from Σ_1, Σ_2 and the remainder term and factoring $t^{d_{\nabla}}$ out we arrive to

$$t^{-d_{\triangledown}}\Delta(\Gamma;t;\Omega) \lesssim_{\Omega,\Gamma,K_{\triangleleft}} a_{\triangleleft}^{s} + t^{-1}a_{\triangledown} + a_{\triangleleft}^{-K_{\triangleleft}}\psi(t^{\sigma})^{d_{\triangleleft}-L_{\triangleleft}-K_{\triangleleft}} + t^{-(1-\sigma)L_{\triangledown}}a_{\triangledown}^{L_{\triangledown}-d_{\triangledown}}a_{\triangleleft}^{-d_{\triangleleft}}. \tag{6.6}$$

Since $d_{\lhd} - L_{\lhd} > 0$ and $\psi(t^{\sigma}) \to \infty$ as $t \to \infty$, we see immediately that if we choose $a_{\lhd}(t) \lesssim \psi(t^{\sigma})^{-1}$ the third term in (6.6) is unbounded. As such, we choose $a_{\lhd}(t) = \psi(t^{\sigma})^{-1+\delta}$ for some fixed $0 < \delta < 1$ (recalling that we assumed in our earlier analysis that a_{\lhd} should be sufficiently small). Once δ is fixed, taking K_{\lhd} arbitrarily large will make the third term smaller than the first. From these choices we are now left with

$$t^{-d_{\nabla}}\Delta(\Gamma;t;\Omega) \lesssim_{\Omega,\Gamma,K_{\triangleleft},\delta} \psi(t^{\sigma})^{-s+\delta s} + t^{-1}a_{\nabla} + t^{-(1-\sigma)L_{\nabla}}\psi(t^{\sigma})^{-d_{\triangleleft}(-1+\delta)}a_{\nabla}^{L_{\nabla}-d_{\nabla}}. \tag{6.7}$$

There are now two situations, depending on the growth of ψ at infinity.

Case 1: ψ grows slowly at infinity

In this situation $\psi(t^{\sigma}) \lesssim_{\varepsilon} t^{\varepsilon}$ for all $\varepsilon > 0$. Here, the first term on the right-hand side of (6.7) would preclude any polynomial bound on $t^{-d}\Delta$, but choosing $a_{\nabla} = t^{1/2}$ and $\sigma = 1$, we see that the second and third term are bounded above by a negative power of t. Therefore, in this situation we obtain

$$t^{-d_{\nabla}}\Delta(\Gamma;t;\Omega) \lesssim_{\Omega,\Gamma,K_{\sigma},\delta} \psi(t)^{-s+\delta s}$$

for any $\delta > 0$.

Case 2: ψ grows at speed μ at infinity.

In this case $\psi(t^{\sigma}) = t^{\sigma \mu}$ and we can rewrite (6.7) as

$$t^{-d_{\nabla}}\Delta(\Gamma;t;\Omega) \lesssim_{\Omega,\Gamma,K_{\sigma},\delta} t^{-\sigma\mu s + \delta\sigma\mu s} + t^{-1}a_{\nabla} + t^{-(1-\sigma)L_{\nabla}-d_{\neg}\sigma\mu(-1+\delta)}a_{\nabla}^{L_{\nabla}-d_{\nabla}}.$$

From this we see that the discrepancy will be made as small as possible if we choose the parameters $0 < \sigma < 1, 0 < a_{\nabla} \ll t$, and $0 < \delta < 1$ in such a way that

$$t^{\sigma\mu(-s+\delta s)} = t^{-1}a_{\triangledown} = t^{-(1-\sigma)L_{\triangledown}-d_{\lhd}\sigma\mu(-1+\delta)}a_{\triangledown}^{L_{\triangledown}-d_{\triangledown}}.$$

Indeed, the first two terms are independent of each other and have opposite monotonicity with the third term in every parameter. This leads directly to taking $a_{\nabla} = t^{1+\sigma\mu(-s+\delta s)}$. Replacing in the third term and equating with the first we choose

$$\sigma = \frac{d_{\nabla}}{s\mu(1-\delta)(d_{\nabla} - L_{\nabla} + 1) + L_{\nabla} + (1-\delta)d_{\triangleleft}\mu}$$

so that

$$t^{-d_{\nabla}}\Delta(\Gamma;t;\Omega) \lesssim_{\Omega,\Gamma,\delta} t^{\frac{-d_{\nabla}(1-\delta)}{(1-\delta)(d_{\nabla}-L_{\nabla}+1+\frac{d_{\Delta}}{s})+\frac{L_{\nabla}}{\mu s}}}$$

We see that this exponent is smaller when δ approaches 0, which means that in the end, for any $\delta > 0$ we get

$$t^{-d_{\nabla}} \Delta(\Gamma; t; \Omega) \lesssim_{\Omega, \Gamma, \delta} t^{\frac{-d_{\nabla}}{d_{\nabla} - L_{\nabla} + 1 + \frac{d_{\Delta}}{S} + \frac{L_{\nabla}}{\mu_{S}}}} + \delta.$$

7 Interlude II: Diophantine properties of lattices

In order to obtain averaged bounds on the discrepancy in Theorem 4.7, we need to get quantitative lower bounds on the sizes of lattice vector projections. Notice that we do not make any irrationality assumptions on Γ in this section.

Lemma 7.1. Let $\Gamma \subset \mathbb{E}$ be a lattice and suppose that $d_{\nabla} \ge 2$. There exist C, $t_0 > 0$ such that for every $t > t_0$ there are linearly independent $\gamma_1, \gamma_2 \in \Gamma$ such that

- *for* $j \in \{1, 2\}$, $t \le |\gamma_{j, \nabla}| \le 5t$;
- the angle between $\gamma_{1,\nabla}$ and $\gamma_{2,\nabla}$ is at least $\pi/3$.
- $for j \in \{1,2\}, |\gamma_{j,\triangleleft}| \le Ct^{\frac{-d_{\nabla}}{d_{\triangleleft}}}$.

Proof. Suppose first that $\mathbb{E}_{\triangleleft}$ is a Γ^{\dagger} -subspace. Then, \mathbb{E}_{∇} is a Γ -subspace and we can use standard lattice point counting results in \mathbb{E}_{∇} to see that there exists at least two linearly independent vectors in $\Gamma \cap (\mathbb{B}_{\nabla}(3t) \setminus \mathbb{B}_{\nabla}(t))$ for every t large enough.

Suppose now that $\mathbb{E}_{\triangleleft}$ is not a Γ^{\dagger} -subspace, so that the Γ -subspace $\mathbb{F} := \Gamma^{\dagger}(\mathbb{E}_{\triangleleft})^{\perp}$ strictly contains $\mathbb{E}_{\triangledown}$. Then $\Theta := \Gamma \cap \mathbb{F}$ is a sublattice of Γ spanning \mathbb{F} . Note that it is possible that Θ^{\dagger} (when viewed as a lattice in \mathbb{F}) intersects $\mathbb{F}_{\triangleleft}$, but repeating the previous argument reduces the dimension of the left subspace everytime, so either we are left with $\mathbb{E}_{\triangledown}$ eventually a Γ -subspace, in which case the original lattice point counting argument apply, or eventually Θ^{\dagger} is irrational with respect to $\mathbb{F}_{\triangleleft}$, which we now assume,

Put s=1 if $\Gamma \cap \mathbb{F} = \emptyset$, or, following Remark 4.2 put $s=\min\{|\gamma|: 0 \neq \gamma \in \Gamma \cap \mathbb{E}_{\triangleleft}\}$, so that π_{\triangledown} is injective when restricted to $\Theta \cap (\mathbb{E}_{\triangledown} \times \mathbb{B}_{\mathbb{F}}(0,s))$. Then, $\Lambda(\mathbb{F},\mathbb{E}_{\triangledown};\mathbb{F}_{\triangleleft};\Theta;\mathbb{B}_{\mathbb{F}_{\triangleleft}}(0,s))$ is a cut and project set. Let $\mathbf{u}_1,\mathbf{u}_2 \in \mathbb{E}_{\triangledown}$ be orthogonal unit vectors. For $j \in \{1,2\}$, let

$$\Omega_{j,\nabla} := \left\{ \mathbf{x}_{\nabla} \in \mathbb{E}_{\nabla} : \frac{|\mathbf{x}_{\nabla} \cdot \mathbf{u}_{j}|}{|\mathbf{x}_{\nabla}|} \leq \frac{\pi}{12}, 0 < |\mathbf{x}_{\nabla}| \leq 1 \right\}.$$

Then, for any t > 0, if $\mathbf{x}_1 \in t\Omega_{1,\nabla}$ and $\mathbf{x}_2 \in t\Omega_{2,\nabla}$, then \mathbf{x}_1 and \mathbf{x}_2 have angle at least $\pi/3$. By Theorem 1.2, we have that for $j \in \{1,2\}$, t > 0 and $n \in \mathbb{N}$,

$$\#(\Lambda \cap (n+1)t\Omega_{j,\nabla}) - \#\Lambda \cap nt\Omega_{j,\nabla} \gtrsim t^{d_{\nabla}}$$

By the pigeonhole principle, this means that there are at least two points $\lambda_j \in \Lambda \cap (2t\Omega_{j, \nabla} \setminus t\Omega_{j, \nabla})$ and $\zeta_j \in \Lambda \cap (4t\Omega_{j, \nabla} \setminus 3t\Omega_{j, \nabla})$ so that $|\lambda_j^{\star} - \zeta_j^{\star}| < t^{-d_{\nabla}/d_{\neg}}$. Put $\gamma_j = \lambda_j^{\triangle} - \zeta_j^{\triangle}$, this satisfies all the requirements. \square

The following porism is obtained by inspecting the construction used to prove [KSSo3, Theorem 1.1] (rather than the statement of the theorem itself).

Porism 7.2. Let \mathbb{F} be some euclidean space, let $\theta_1, \theta_2 \in \mathbb{F}$ be linearly independent, and put $\Theta = \operatorname{span}_{\mathbb{Z}}(\theta_1, \theta_2) \subset \mathbb{F}$. There exists $t_0, C > 0$ depending on $\frac{|\theta|_1}{|\theta|_2}$ so that for every $t > t_0$ there exists $n, m \leq \log(t)^7$ such that $\theta := n\theta_1 + m\theta_2$ satisfies

$$\operatorname{dist}(2t|\theta|,\mathbb{Z}) \ge \exp(-C\operatorname{loglog}(t)^4).$$

We are now ready to prove our main lemma of this section.

Lemma 7.3. Let $\Gamma \subset \mathbb{E}$ be a lattice and suppose that $d_{\nabla} \ge 2$. For every $R_0 > 0$, there exists $t_0, c, C > 0$ such that for every $t > t_0$ there is $\theta \in \Gamma$ such that

- $|\theta_{\triangleleft}| \leq R_0$;
- $|\theta_{\nabla}| \leq c \log(t)^{8 \frac{d_{\triangleleft}}{d_{\nabla}} + 7}$;
- $\operatorname{dist}(2t|\theta_{\triangledown}|,\mathbb{Z}) \ge \exp(-C\operatorname{loglog}(t)^4)$.

Proof. Applying Lemma 7.1, we find t_0 such that for every $t > t_0$ there are θ_1, θ_2 such that

- I. The projection on $\mathbb{E}_{\triangleleft}$ is small: $|\theta_{j,\triangleleft}| \leq R_0 \log(t)^{-8}$;
- 2. the projection on \mathbb{E}_{∇} is not too large: $|\theta_{j,\nabla}| \lesssim_{\Gamma,R_0} \log(t)^{\frac{8d_q}{d\nabla}}$;
- 3. the projections on \mathbb{E}_{∇} have comparable norms: $\theta_{1,\nabla} \asymp_{\Gamma} \theta_{2,\nabla}$;
- 4. the projections on \mathbb{E}_{∇} don't have a small angle between them:

$$\frac{\theta_{1,\nabla} \cdot \theta_{2,\nabla}}{\left|\theta_{1,\nabla}\right| \left|\theta_{2,\nabla}\right|} \leq \frac{1}{2}.$$

Put $\Theta = \operatorname{span}_{\mathbb{Z}}(\theta_{1,\nabla}, \theta_{2,\nabla}) \subset \mathbb{E}_{\nabla}$. By the Porism 7.2, there are $n, m \leq \log(t)^7$ such that $\lambda := n\theta_{1,\nabla} + m\theta_{2,\nabla}$ satisfies $\operatorname{dist}(2t|\lambda|,\mathbb{Z}) \geq \exp(-C\log\log(t)^4)$. We put $\theta = n\theta_1 + m\theta_2$, whence $\theta_{\nabla} = \lambda$. We now observe by the triangle inequality that θ is the element whose existence we asserted:

- I. the projection on $\mathbb{E}_{\triangleleft}$ remains uniformly bounded: $|\theta_{\triangleleft}| \leq 2R_0 \log(t)^{-1} \leq R_0$ as long as $t > e^2$;
- 2. the projection on \mathbb{E}_{∇} remains not too large: $|\theta_{\nabla}| \lesssim \log(t)^{8\frac{d_{\Delta}}{d_{\nabla}}+7}$.

These properties form our claim.

8 Averaged lower bounds for the discrepancy

In this section, we prove the averaged bounds on the discrepancy. Our first lemma proves that for every cut and project set, no matter how bad the window or the search region is, the average of the discrepancy over translates of the lattice is zero.

Lemma 8.1. Let $\Omega_{\triangleleft} \subset \mathbb{E}_{\triangleleft}$, $\Omega_{\nabla} \subset \mathbb{E}_{\nabla}$ and put $\Omega = \Omega_{\nabla} \times \Omega_{\triangleleft}$. For every lattice Γ and t > 0

$$\int_{\mathbb{R}/\Gamma} \Delta(\Omega; \Gamma + \mathbf{z}; t) \, d\mathbf{z} = 0.$$

Proof. By definition,

$$\Delta(\Omega, \Gamma + \mathbf{z}, t) = \sum_{\gamma \in \Gamma} \chi_{\Omega}(\gamma - \mathbf{z}) - \frac{\operatorname{vol}(\Omega)}{\operatorname{covol}(\Gamma)}.$$
(8.1)

Integrating over \mathbb{E}/Γ a sum over elements of Γ is the same as integrating over \mathbb{E} without taking the sum, in other words:

$$\int_{\mathbb{E}/\Gamma} \Delta(\Omega, \Gamma + \mathbf{z}, t) \, d\mathbf{z} = -\operatorname{vol}(\Omega) + \int_{\mathbb{E}} \chi_{\Omega}(-\mathbf{z}) \, d\mathbf{z} = 0$$

Lower bounds on the averaged discrepancy, Proof of Theorem 4.7

When the dimension $d_{\nabla}=1$, the claim is simply that there is a constant bound below for the average of the discrepancy, which is readily seen to hold. We now assume that $d_{\nabla} \ge 2$. This proof is in the same spirit as [DT82, Lemma 1] and [LP16, Section 4]. Recall that for this theorem Ω_{\triangleleft} is any bounded window with nonzero volume, and $\Omega_{\nabla}=\mathbb{B}_{\nabla}(0,1)$. We first observe that $\Delta(\Omega,\Gamma+\mathbf{z},t)$ is a Γ -periodic function in the parameter \mathbf{z} , it therefore makes sense to compute the coefficients at $\gamma^{\dagger} \in \Gamma^{\dagger}$ of its Fourier series:

$$\widetilde{\Delta}(\Omega, \Gamma, t)_{\gamma^{\dagger}} := \int_{\mathbb{F}/\Gamma} \Delta(\Lambda(\Gamma; \mathbf{z}; \Omega_{\lhd})) \mathbf{e}(\gamma^{\dagger} \cdot \mathbf{z}) \, \mathrm{d}\mathbf{z}.$$

Lemma 8.1 tells us that the zero'th Fourier coefficient vanishes, for other $\gamma^{\dagger} \in \Gamma^{\dagger} \setminus \{0\}$ we obtain in the same way as in (8.1)

$$\begin{split} \widetilde{\Delta}(\Omega, \Gamma, t)_{\gamma^{\dagger}} &= \int_{\mathbb{E}/\Gamma} \left(-\frac{\operatorname{vol}(\Omega)}{\operatorname{covol}(\Gamma)} + \sum_{\gamma \in \Gamma} \chi_{t\Omega_{\nabla} \times \Omega_{\neg}}(\gamma - \mathbf{z}) \right) \mathbf{e}(\gamma^{\dagger} \cdot \mathbf{z}) \, d\mathbf{z} \\ &= \int_{\mathbb{E}} \chi_{t\Omega_{\nabla} \times \Omega_{\neg}}(\mathbf{z}) \mathbf{e}(-\gamma^{\dagger} \cdot \mathbf{z}) \, d\mathbf{z} \\ &= [\mathscr{F}\chi_{t\Omega_{\nabla} \times \Omega_{\neg}}](-\gamma^{\dagger}) \\ &= [\mathscr{F}\chi_{t\Omega_{\nabla}}\chi_{t\Omega_{\nabla}}](-\gamma^{\dagger}) [\mathscr{F}_{\neg}\chi_{\Omega_{\neg}}](-\gamma^{\dagger}_{\neg}) \\ &= t^{d} [\mathscr{F}_{\nabla}\chi_{\Omega_{\nabla}}](-t\gamma^{\dagger}_{\nabla}) [\mathscr{F}_{\neg}\chi_{\Omega_{\neg}}](-\gamma^{\dagger}_{\neg}). \end{split}$$

It follows from the triangle inequality that,

$$\|\Delta\|_{L^{1}(\mathbb{F}/\Gamma)} \ge \|\widetilde{\Delta}\|_{\ell^{\infty}(\Gamma^{\dagger})},\tag{8.2}$$

so that finding large Fourier coefficients implies lower bounds on the L¹ norm of the discrepancy. Let us first examine the term $\mathscr{F}_{\lhd}\chi_{\Omega_{\lhd}}$. We assumed Ω_{\lhd} is bounded, for definiteness suppose that Ω_{\lhd} is contained in a ball of radius R in \mathbb{E}_{\lhd} . Then,

$$\left| \nabla_{\xi_{\triangleleft}} \mathscr{F}_{\triangleleft} \chi_{\Omega_{\triangleleft}}(\xi_{\triangleleft}) \right| \leq \int_{\mathbb{E}_{\triangleleft}} |x_{\triangleleft}| \, \chi_{\Omega_{\triangleleft}}(x_{\triangleleft}) \, \mathrm{d}x_{\triangleleft} \leq \mathrm{Vol}(\Omega_{\triangleleft}) R \tag{8.3}$$

so that since $[\mathscr{F}_{\lhd}\chi_{\Omega_{\lhd}}](0) = \text{Vol}(\Omega_{\lhd})$, Taylor's theorem implies that as long as $|\gamma_{\lhd}^{\dagger}| \leq (2R)^{-1}$,

$$\left| \left[\mathscr{F}_{\triangleleft} \chi_{\Omega_{\triangleleft}} \right] (-\gamma_{\triangleleft}^{\dagger}) \right| \geqslant \frac{\text{Vol}(\Omega_{\triangleleft})}{2}. \tag{8.4}$$

On the other hand, the Fourier transform of the indicator of a unit ball can be explicitly computed in terms of Bessel functions, for which there are precise asymptotic descriptions. Following [GR07, Formula 8.451.1], there is z_0 such that as long as $t|\gamma_{\nabla}^{\dagger}| \ge z_0$

$$t^{d_{\nabla}} [\mathscr{F}_{\nabla} \chi_{\Omega_{\nabla}}] (-t \gamma_{\nabla}^{\dagger}) = \frac{t^{d_{\nabla}/2}}{|\gamma_{\nabla}^{\dagger}|^{d_{\nabla}/2}} J_{d_{\nabla}/2} \left(2\pi t |\gamma_{\nabla}^{\dagger}| \right)$$

$$= \frac{t^{\frac{d_{\nabla}-1}{2}}}{2\pi |\gamma_{\nabla}^{\dagger}|^{\frac{d_{\nabla}+1}{2}}} \sin \left(2\pi t |\gamma_{\nabla}^{\dagger}| + \frac{1 - d_{\nabla}}{4} \pi \right) + O\left(t^{\frac{d_{\nabla}-3}{2}} |\gamma_{\nabla}^{\dagger}|^{\frac{-d_{\nabla}-3}{2}} \right). \tag{8.5}$$

Case 1, $d_{\triangledown} \not\equiv 1 \pmod{4}$: In this situation, we use that

$$0 < \inf_{x \in \mathbb{R}} \max \left\{ \left| \sin \left(x + \frac{1 - d_{\nabla}}{4} \pi \right) \right|, \left| \sin \left(2x + \frac{1 - d_{\nabla}}{4} \pi \right) \right| \right\}$$
 (8.6)

so that for any fixed $\gamma^{\dagger} \in \Gamma^{\dagger}$ with $|\gamma_{\triangleleft}^{\dagger}| \leq (4R)^{-1}$,

$$\max\left\{\left|t^{d_{\triangledown}}\mathscr{F}_{\triangledown}\chi_{\Omega_{\triangledown}}(-t\gamma_{\triangledown}^{\dagger})\right|,\left|t^{d_{\triangledown}}\mathscr{F}_{\triangledown}\chi_{\Omega_{\triangledown}}(-2t\gamma_{\triangledown}^{\dagger})\right|\right\}\gtrsim t^{\frac{d_{\triangledown}-1}{2}},$$

and inserting this along with the estimate (8.4) in (8.2) is our claim.

Case 2, $d_{\nabla} \equiv 1 \pmod{4}$: Here the situation is more delicate as (8.6) doesn't hold anymore. In this situation we read from (8.5) that as long as $\operatorname{dist}(2t|\gamma_{\nabla}^{\dagger}|,\mathbb{Z}) \geq t^{-1}|\gamma_{\nabla}^{\dagger}|^{-1}$,

$$\left| t^{d_{\nabla}} [\mathscr{F}_{\nabla} \chi_{\Omega_{\nabla}}] (-t \gamma_{\nabla}^{\dagger}) \right| = \left| \frac{t^{\frac{d_{\nabla} - 1}{2}}}{2\pi |\gamma_{\nabla}^{\dagger}|^{\frac{d_{\nabla} + 1}{2}}} \sin\left(2\pi t |\gamma_{\nabla}^{\dagger}|\right) \right| + O\left(t^{\frac{d_{\nabla} - 3}{2}} |\gamma_{\nabla}^{\dagger}|^{\frac{-d_{\nabla} - 3}{2}}\right)
\gtrsim \frac{t^{\frac{d_{\nabla} - 1}{2}}}{|\gamma_{\nabla}^{\dagger}|^{\frac{d_{\nabla} + 1}{2}}} \operatorname{dist}(2t |\gamma_{\nabla}^{\dagger}|, \mathbb{Z}).$$
(8.7)

Following Lemma 7.3 there exists $\gamma^{\dagger} \in \Gamma^{\dagger}$ such that

$$|\gamma_{\lhd}^{\dagger}| < (2R)^{-1}, \qquad |\gamma_{\bigtriangledown}^{\dagger}| \lesssim \log(t)^{8\frac{d_{\lhd}}{d\bigtriangledown} + 7}, \quad \text{and} \quad \operatorname{dist}(2t|\gamma_{\bigtriangledown}^{\dagger}|, \mathbb{Z}) \geqslant \exp(-C \log\log(t)^{4}).$$

Inserting into (8.7), along with the observation that for every $s \ge 0$

$$\log(t)^{-s} \exp(-C \log \log(t)^4) \gtrsim_{\varepsilon, s} \exp(-(C + \varepsilon) \log \log(t)^4)$$

yields the existence of a constant A > 0 such that

$$\left|t^{d_\nabla}[\mathcal{F}_\nabla\chi_{\Omega_\nabla}](-t\gamma_\nabla^\dagger)\right|\gtrsim t^{\frac{d_\nabla-1}{2}}\exp(-A\mathrm{loglog}(t)^4).$$

Remark 8.2. It follows from (8.3) that we would be able to weaken the assumptions on the window Ω_{\triangleleft} to

$$\int_{\Omega_{\triangleleft}} |\mathbf{x}_{\triangleleft}| \; d\mathbf{x}_{\triangleleft} < \infty$$

and obtain the same lower bounds on the average of the discrepancy. While this allows for some unbounded windows, we decided to keep the statement for bounded windows to keep in line with the standard assumptions of the field.

A L^2 -bounds for the discrepancy of Liouville cut and project sets by Michael Björklund and Tobias Hartnick

A.1 Statement of the main result

As in the body of the text we fix an orthogonal decomposition

$$\mathbb{R}^d \cong \mathbb{E} := \mathbb{E}_{\triangledown} \oplus \mathbb{E}_{\triangleleft} \cong \mathbb{R}^{d_{\triangledown}} \oplus \mathbb{R}^{d_{\triangleleft}}$$

with associated projections π_{\triangledown} and π_{\vartriangleleft} and a lattice $\Gamma \subset \mathbb{E}$ such that $\pi_{\vartriangleleft}(\Gamma)$ is dense in $\mathbb{E}_{\vartriangleleft}$ and $\pi_{\triangledown}|_{\Gamma}$ is injective. To simplify notation we will assume without loss of generality that $\mathbb{E} = \mathbb{R}^d$ and that $\mathbb{E}^{\triangledown} = \mathbb{R}^{d_{\triangledown}} \times \{0\}$ and $\mathbb{E}^{\vartriangleleft} = \{0\} \times \mathbb{R}^{d_{\vartriangleleft}}$. We write elements of \mathbb{E} as $\mathbf{z} = (\mathbf{z}_{\triangledown}, \mathbf{z}_{\vartriangleleft})$, where $\mathbf{z}_{\triangledown} \in \mathbb{R}^{d_{\triangledown}}$ and $\mathbf{z}_{\vartriangleleft} \in \mathbb{R}^{d_{\vartriangleleft}}$. Given $\mathbf{z} \in \mathbb{E}$ and a bounded $window \Omega_{\vartriangleleft} \subset \mathbb{E}_{\vartriangleleft}$ we denote by

$$\Lambda(\Gamma+\mathbf{z};\Omega_{\lhd}):=\pi_{\triangledown}((\Gamma+\mathbf{z})\cap\pi_{\lhd}^{-1}(\Omega_{\lhd}))$$

the associated cut and project set. If Λ is such a cut and project set and $\Omega_{\nabla} \subset \mathbb{E}_{\nabla}$ is a bounded Borel subset, then we define the *discrepancy* of Λ with respect to Ω_{∇} as

$$\Delta(\Lambda; \Omega_{\nabla}) = \#(\Lambda \cap \Omega_{\nabla}) - \frac{\operatorname{vol}_{\nabla}(\Omega_{\nabla}) \operatorname{vol}_{\triangleleft}(\Omega_{\triangleleft})}{\operatorname{covol}(\Gamma)}.$$
(A.1)

In the study of the discrepancy of cut and project sets one can vary the underlying lattice Γ , underlying window Ω_{\triangleleft} and the bounded set Ω_{∇} . We will always assume that Ω_{∇} is a fixed set of bounded perimeter and study its dilates $t\Omega_{\nabla}$. In this specific case we have $\operatorname{vol}_{\nabla}(t\Omega_{\nabla}) = t^{d_{\nabla}} \operatorname{vol}_{\nabla}(\Omega_{\nabla})$, and hence formula (A.1) reduces to the one from Theorem 1.2 in the introduction. Moreover we have $\Delta(\Lambda; \Omega_{\nabla}) = o(t^{d_{\nabla}})$ which means that the number of points in dilates of Ω_{∇} is asymptotically proportional to the volume of Ω_{∇} with an error term given by the discrepancy. If we fix the window Ω_{\triangleleft} and only vary the lattice by translations, then we can consider the discrepancy as a function

$$\Delta_{\Omega_{\lhd}}: \Gamma \backslash \mathbb{E} \times [0, \infty) \to \mathbb{R}, \quad (\Gamma + \mathbf{z}, t) \mapsto \Delta_{\Omega_{\lhd}}(\Gamma + \mathbf{z}, t) := \Delta(\Lambda(\Gamma + \mathbf{z}; \Omega_{\lhd}), t\Omega_{\nabla}).$$

In the context of the bi-Lipschitz problem for model sets one is mostly interested in estimates of the form

$$\varphi_{\Omega_{\lhd}}^{-}(t) \le \sup_{\Gamma + \mathbf{z} \in \Gamma \setminus \mathbb{F}} |\Delta_{\Omega_{\lhd}}(\Gamma + \mathbf{z}, t)| \le \varphi_{\Omega_{\lhd}}^{+}(t),$$

which are uniform in the torus variable. However, there is also some interest in L^p -estimates of the form

$$\varphi_{\Omega_{\triangleleft}}^{p,-}(t) \leq \int_{\Gamma \setminus \mathbb{R}} |\Delta_{\Omega_{\triangleleft}}(\Gamma + \mathbf{z}, t)|^p \, \mathrm{d}\mathbf{z} \leq \varphi_{\Omega_{\triangleleft}}^{p,+}(t).$$

This appendix is specifically concerned with lower L²-bounds for the discrepancy for specific choices of Γ and t (and Ω_{\triangleleft} given by Euclidean balls of some "generic" radius or cylinders of some "generic" sidelength), i.e. we will provide lower bounds for the *number variance*

$$NV_t(\Gamma, \Omega_{\triangleleft}) := \int_{\Gamma^{\backslash \mathbb{F}}} |\Delta_{\Omega_{\triangleleft}}(\Gamma + \mathbf{z}, t)|^2 d\mathbf{z}.$$

As the term "number variance" suggests, this L²-norm can be interpreted as the variance of a certain random variable and hence can be estimated using probabilistic tools. Before we discuss this probabilistic interpretation, we state our main result. For this we need the following notion. We denote by ||x|| the nearest distance from a real number x to an integer and say that an increasing function $\psi: (0,\infty) \to (0,\infty)$ is increasing slowly at infinity if $\psi(r) \lesssim_{\varepsilon} r^{\varepsilon}$ for every $\varepsilon > 0$ as $r \to \infty$. Typical examples of such functions are given by $\psi(r) = \log(1+r)^{\beta}$ for some fixed $\beta > 0$ or $\psi(r) = \log\log(e+r)$.

Definition A.1. Let $a \in \mathbb{R}$ and let $\psi : (0, \infty) \to (0, \infty)$ be a function which is increasing slowly at infinity.

(i) a is a Liouville number if there is a sequence of integers (q_n) so that

$$q_n \to \infty$$
 and $||q_n a||^{-1/n} \ge q_n$.

(ii) a is a ψ -Liouville number if there is a sequence of integers $\{q_n\}$ so that

$$q_n \to \infty$$
 and $\psi(\|q_n a\|^{-1}) \ge q_n$.

Remark A.2. Every ψ -Liouville number is a Liouville number, and conversely every Liouville number is ψ -Liouville for *some* ψ . Indeed, we can just set $\psi(\|q_n a\|^{-1}) := \|q_n a\|^{-1/n}$, and interpolate linearly. There exist ψ -Liouville numbers for any ψ which increases slowly at infinity.

There is a similar notion for lattices:

Definition A.3. Let $\psi: (0,\infty) \to (0,\infty)$ be slowly increasing at infinity. A lattice $\Gamma \subset \mathbb{E}$ is said to be ψ -Liouvillean (with respect to the splitting $\mathbb{E} = \mathbb{E}_{\nabla} \oplus \mathbb{E}_{\lhd}$) if it is irrational with respect to $\mathbb{E}_{\nabla} \oplus \mathbb{E}_{\lhd}$ and there is a sequence $\{\gamma^{(n)}\} = \{(\gamma^{(n)}_{\nabla}, \gamma^{(n)}_{\lhd})\} \subset \Gamma \setminus \{(0,0)\}$ such that

$$\lim_{n\to\infty} \gamma_{\nabla}^{(n)} = 0 \quad \text{and} \quad \lim_{n\to\infty} |\gamma_{\lhd}^{(n)}| = \infty \quad \text{and} \quad |\gamma_{\lhd}^{(n)}| \le \psi(|\gamma_{\nabla}^{(n)}|^{-1}) \quad \text{for all } n. \tag{A.2}$$

It is called a *Liouvillean* if it is ψ -Liouvillean for some ψ .

Example A.4. We can construct ψ -Liouvillean lattices for any slowly increasing ψ : Let $e_{\nabla} \in \mathbb{E}_{\nabla}$, $e_{\neg} \in \mathbb{E}_{\neg}$ be unit vectors and further decompose the splitting $\mathbb{E}_{\nabla} \oplus \mathbb{E}_{\neg}$ as

$$\mathbb{E} = \operatorname{span}(e_{\triangledown}) \oplus \mathbb{F}_{\triangledown} \oplus \operatorname{span}(e_{\vartriangleleft}) \oplus \mathbb{F}_{\vartriangleleft},$$

where $\mathbb{F}_{\bullet} = \mathbb{E}_{\bullet} \ominus \operatorname{span}(e_{\bullet})$. Putting $\mathbb{F} = \mathbb{F}_{\triangledown} \oplus \mathbb{F}_{\triangleleft}$, let $\widetilde{\Gamma} \subset \mathbb{F}$ be a lattice which is irrational with respect to $\mathbb{F}_{\triangledown} \oplus \mathbb{F}_{\triangleleft}$. If $\frac{1}{2} < a < \frac{3}{2}$ is a ψ -Liouville number and $\frac{3}{4} < b < \frac{5}{4}$ is irrational, then the lattice defined by

$$\Gamma = \operatorname{span}_{\mathbb{Z}}(ae_{\nabla} + e_{\triangleleft}, e_{\nabla} + be_{\triangleleft}) \oplus \widetilde{\Gamma} \subset \mathbb{E}$$

is a ψ -Liouvillean lattice with respect to the decomposition $\mathbb{E}_{\nabla} \oplus \mathbb{E}_{\triangleleft}$. Indeed, it is easy to see that it is a lattice irrational with respect to $\mathbb{E}_{\nabla} \oplus \mathbb{E}_{\triangleleft}$. To see that it it is ψ -Liouvillean, consider a sequence $q_n \to \infty$ so that $\psi(\|aq_n\|^{-1}) \ge q_n$, and put m_n to be the nearest integer to aq_n . Then,

$$\gamma^{(n)} = q_n(ae_{\nabla} + e_{\triangleleft}) - m_n(e_{\nabla} + e_{\triangleleft}) \in \Gamma$$

is so that

$$|\gamma_{\nabla}^{(n)}| = ||q_n a||$$
 and $|\gamma_{\Delta}^{(n)}| = |q_n - bm_n| < |q_n|$,

and this sequence makes Γ a ψ -Liouvillean lattice.

Theorem A.5. Let $f:(0,\infty)\to (0,\infty)$ be a function diverging at infinity, let $\psi:(0,\infty)\to (0,\infty)$ be a slowly increasing function, and let $\Omega_{\nabla}\subset \mathbb{E}_{\nabla}$ be of bounded perimeter. Then there exist a lattice Γ , a window Ω_{\prec} , an increasing sequence (t_n) of positive real numbers such that

$$\overline{\lim_{n \to \infty}} \frac{\text{NV}_{t_n}(\Gamma, \Omega_{\triangleleft})}{t_n^{2d_{\triangledown}} \psi(t_n)^{-d_{\triangleleft} - 1} f(t_n)^{-1}} = \infty. \tag{A.3}$$

In fact, (A.3) holds whenever the dual lattice Γ^{\dagger} of Γ is ψ -Liouvillean and $\Omega_{\triangleleft} = \mathbb{B}_{\triangleleft}(0,r)$ for r in some Lebesgue conull subset $A \subset (0,\infty)$ depending on Γ^{\dagger} .

For the dual lattices of the specific lattices constructed in Example A.4 and windows adapted to these examples one can also get a lower bound, which matches the upper bounds obtained in the main body of this article:

Theorem A.6. Let $f:(0,\infty) \to (0,\infty)$ be a function diverging at infinity, let ψ be a slowly growing function and let Γ be the dual of the ψ -Liouvillean lattice from Example A.4. Moreover, let $\Omega_{\nabla} \subset \mathbb{E}_{\nabla}$ be of bounded perimeter and let $\Omega_{\neg} = [-r, r] \times Y_{\neg} \subset \mathbb{E}_{\neg} = \operatorname{span}(e_{\neg}) \oplus \mathbb{F}_{\neg}$ for some Y_{\neg} of bounded perimeter. Then there is a sequence $\{t_n\}$ of positive real numbers such that $t_n \to \infty$ and a Lebesgue conull subset $A \subset (0,\infty)$ such that for all $r \in A$ and all functions $f:(0,\infty) \to (0,\infty)$ diverging at infinity

$$\overline{\lim_{n\to\infty}}\,\frac{\mathrm{NV}_{t_n}(\Gamma,\Omega_{\vartriangleleft})}{t_n^{2d_{\triangledown}}\psi(t_n)^{-2}f(t_n)^{-1}}=\infty.$$

A.2 The diffraction formula for the number variance

It will be convenient for us to use the language of point processes; see e.g. [Kal17] for background. Given $n \in \mathbb{N}$, we denote by $M(\mathbb{R}^n)$ the space of Radon measures on \mathbb{R}^n and by $\mathscr{LF}(\mathbb{R}^n)$ the space of locally finite (i.e. closed and discrete) subsets of \mathbb{R}^n . We consider $\mathscr{LF}(\mathbb{R}^n)$ as a subset of $M(\mathbb{R}^n)$ by identifying each set Λ with its Dirac comb $\delta_{\Lambda} := \sum_{x \in \Lambda} \delta_x$. Every bounded Borel function $f : \mathbb{R}^n \to \mathbb{C}$ with bounded support defines a *linear statistic*

$$\mathscr{P}f: M(G) \to \mathbb{C}, \quad p \mapsto p(f),$$

and we equip $\mathscr{LF}(\mathbb{R}^n) \subset M(\mathbb{R}^n)$ with the smallest σ -algebra \mathscr{B} for which all of these linear statistics are measurable. Now assume that $(\Omega, \mathscr{F}, \mathbb{P})$ is some auxiliary probability space on which \mathbb{R}^n acts measurably, preserving \mathbb{P} ; then an equivariant measurable map

$$\Lambda: \Omega \to \mathscr{LF}(\mathbb{R}^n), \quad \omega \mapsto \Lambda_{\omega}.$$

is called a *stationary simple point process* with *distribution* $\mu = \Lambda_* \mathbb{P}$. We will only consider processes which are *locally square-integrable* in the sense that the real-value random variable

$$\Lambda \cap B : \Omega \to \mathbb{R}, \quad z \mapsto \#(\Lambda_z \cap B)$$

is square-integrable for every bounded Borel set $B \subset \mathbb{R}^n$; note that this is automatically the case if Λ_{ω} is almost surely r-uniformly discrete for some fixed r > 0.

Definition A.7. If $\Lambda: \Omega \to \mathscr{LF}(\mathbb{R}^n)$ is locally square-integrable, then the random variable

$$\operatorname{disc}_B(\Lambda): \Omega \to \mathbb{R}, \quad \operatorname{disc}_B(\Lambda)(z) := \#(\Lambda_z \cap B) - E[\Lambda \cap B].$$

is called the *discrepancy* of Λ on B, and the variance

$$N_B(\Lambda) := \operatorname{Var}(\Lambda \cap B) = \int_{\Omega} |\operatorname{disc}_B(\Lambda)(\omega)|^2 d\mathbb{P}(\omega)$$

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is called the *discrepancy* of Λ on B, and the variance

$$N_B(\Lambda) := \operatorname{Var}(\Lambda \cap B) = \int_{\Omega} |\operatorname{disc}_B(\Lambda)(\omega)|^2 d\mathbb{P}(\omega)$$

is called the *number variance* of Λ at B.

Note that the number variance is finite, since $\Lambda \cap B$ is square-integrable.

Remark A.8 (Diffraction formula). According to [BH23, Prop. 2.3] there exists a positive-definite signed Radon measure η_{Λ} , the *(reduced) autocorrelation measure* of Λ , such that

$$N_B(\Lambda) = \eta_{\Lambda}(\chi_B * \chi_B^*), \tag{A.4}$$

where χ_B denotes the characteristic function of B. The Fourier transform $\mathscr{F}\eta_{\Lambda}$ is a positive Radon measure on \mathbb{R}^n , called the *(reduced) diffraction* of Λ , and it follows from (A.4) that for sufficiently regular B

$$N_B(\Lambda) = [\mathcal{F}\eta_{\Lambda}](|\mathcal{F}\chi_B|^2). \tag{A.5}$$

This is called the *diffraction formula* for the number variance.

Example A.9. We return to the general setting of Subsection A.1. If we fix the lattice $\Gamma \subset \mathbb{E}$ and the window Ω_{\triangleleft} then the map

$$\Lambda(-;\Omega_{\triangleleft}):\Gamma\backslash\mathbb{E}\to\mathscr{LF}(\mathbb{E}_{\triangledown}),\quad\Gamma+\mathbf{z}\mapsto\Lambda(\Gamma+\mathbf{z};\Omega_{\triangleleft})$$

is a stationary simple point process with respect to the unique \mathbb{E} -invariant probability measure on $\Gamma \setminus \mathbb{E}$, called the *cut and project process* associated with Γ and Ω_{\lhd} . For a bounded Borel set $\Omega_{\nabla} \subset \mathbb{E}_{\nabla}$ we have

$$E[\Lambda(-;\Omega_{\triangleleft}) \cap B] = \frac{\operatorname{vol}_{\triangledown}(\Omega_{\triangledown}) \operatorname{vol}_{\triangleleft}(\Omega_{\triangleleft})}{\operatorname{covol}(\Gamma)},$$

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and hence the discrepancy of $\Lambda(-;\Omega_{\triangleleft})$ is given by

$$\operatorname{disc}_{\Omega_{\nabla}}(\Lambda(-;\Omega_{\triangleleft}))(\Gamma+\mathbf{z}) = \Delta_{\Omega_{\triangleleft}}(\Gamma+\mathbf{z};\Omega_{\nabla}).$$

On the other hand, it follows from classical results of Meyer [Mey70] that the reduced diffraction of $\Lambda(-;\Omega_{\triangleleft})$ is given by [BH23, Thm. 2.9]

$$\mathscr{F}\eta = \frac{1}{\operatorname{covol}(\Gamma)^2} \sum_{\xi = (\xi_{\triangledown}, \xi_{\blacktriangleleft}) \in \Gamma^{\dagger} \setminus \{(0,0)\}} \left| \left[\mathscr{F} \chi_{\Omega_{\triangleleft}} \right] (\xi_{\triangleleft}) \right|^2 \cdot \delta_{\xi_{\triangledown}}.$$

The diffraction formula (A.5) for the number variance of $\Lambda(-;\Omega_{\lhd})$ may thus be stated as follows; here we use that since $\chi_{t\Omega_{\nabla}}(x_{\nabla}) = \chi_{\Omega_{\nabla}}(x_{\nabla}/t)$ we have $[\mathscr{F}\chi_{t\Omega_{\nabla}}](\xi_{\nabla}) = t^{d_{\nabla}}[\mathscr{F}\chi_{\Omega_{\nabla}}](t\xi_{\nabla})$.

Proposition A.10. If $\Omega \triangleleft$ has sufficient Fourier decay, then for all t > 0 we have

$$\mathrm{NV}_t(\Gamma,\Omega_{\lhd}) = \frac{t^{2d_{\triangledown}}}{\mathrm{covol}(\Gamma)^2} \sum_{\xi = (\xi_{\triangledown},\xi_{\lhd}) \in \Gamma^{\dagger} \setminus \{(0,0)\}} |[\mathscr{F}\chi_{\Omega_{\triangledown}}](t\xi_{\triangledown})|^2 |[\mathscr{F}\chi_{\Omega_{\lhd}}](\xi_{\lhd})|^2. \quad \Box$$

A.3 Proof of Theorem A.5

We now return to the setting of Theorem A.5; thus Γ^{\dagger} is assumed to be ψ -Liouvillean with parameters $\psi, \xi_{\nabla}^{(n)}, \xi_{\prec}^{(n)}$ as in (A.2) and $\Omega_{\prec} = \mathbb{B}_{\prec}(0, r)$ for some r > 0. Let $f : (0, \infty) \to (0, \infty)$ be a function such that $\lim_{t \to \infty} f(t) = \infty$. If we abbreviate $\mathbb{B}_{\prec} := \mathbb{B}_{\prec}(0, 1)$, then by Proposition A.10 we have

$$\Psi_{f}(t,r) := \frac{\operatorname{NV}_{t}(\Gamma, \mathbb{B}_{\lhd}(0,r))}{t^{2d_{\triangledown}} \psi(t)^{-d_{\dashv}-1} f(t)^{-1}} = \frac{\psi(t)^{d_{\dashv}+1} f(t) r^{2d_{\dashv}}}{\operatorname{covol}(\Gamma)^{2}} \sum_{\xi = \epsilon \Gamma^{\dagger} \setminus \{(0,0)\}} |[\mathscr{F}\chi_{\Omega_{\triangledown}}](t\xi_{\triangledown})|^{2} |[\mathscr{F}\chi_{\mathbb{B}_{\dashv}}](r\xi_{\dashv})|^{2}. \tag{A.6}$$

We have to prove that there exists a sequence t_n such that for all r in a Lebesgue conull subset $E \subset (0, \infty)$ we have

$$\overline{\lim}_{n \to \infty} \Psi_f(t_n, r) = \infty. \tag{A.7}$$

Following (8.3),

$$\inf\{|\widehat{\chi}_{\mathbb{B}_{\nabla}}(\xi_{\nabla})|:|\xi_{\nabla}|\leq (2\operatorname{diam}(\Omega_{\nabla}))^{-1}\}\geq \frac{1}{2}\operatorname{vol}_{\nabla}(\mathbb{B}_{\nabla}). \tag{A.8}$$

If we set

$$\mathscr{C}_t := \{ \xi = (\xi_{\nabla}, \xi_{\triangleleft}) \in \Gamma^{\dagger} \setminus \{(0,0)\} : |\xi_{\nabla}| \leq 1/t \},$$

we thus obtain for any $t \ge (2 \operatorname{diam}(\Omega_{\triangledown}))^{-1}$

$$\begin{split} \Psi_f(t,r) & \geq & \frac{\psi(t)^{d_{\dashv}+1}f(t)r^{2d_{\dashv}}}{\operatorname{covol}(\Gamma)^2} \sum_{\xi \in \mathscr{C}_t} |[\mathscr{F}\chi_{\Omega_{\triangledown}}](t\xi_{\triangledown})|^2 |[\mathscr{F}\chi_{\mathbb{B}_{\dashv}}](r\xi_{\dashv})|^2 \\ & \geq & \frac{\psi(t)^{d_{\dashv}+1}f(t)r^{2d_{\dashv}}\operatorname{vol}_{\triangledown}(\Omega_{\triangledown})^2}{4\operatorname{covol}(\Gamma)^2} \sum_{\xi \in \mathscr{C}_t} |[\mathscr{F}\chi_{\mathbb{B}_{\dashv}}](r\xi_{\dashv})|^2 \\ & \geq & \frac{\psi(t)^{d_{\dashv}+1}f(t)r^{2d_{\dashv}}\operatorname{vol}_{\triangledown}(\Omega_{\triangledown})^2}{4\operatorname{covol}(\Gamma)^2} \sup_{\xi \in \mathscr{C}_t} |[\mathscr{F}\chi_{\mathbb{B}_{\dashv}}](r\xi_{\dashv})|^2. \end{split}$$

Lemma A.11. If we define $t_n := \|\xi_{\nabla}^{(n)}\|^{-1}$, then there exists a constant C > 0 depending only on d_{∇} and the covolume of Γ such that

$$\Psi_f(t_n, r) \ge C \cdot |\xi_{\le 1}^{(n)}| f(\psi^{-1}(|\xi_{\le 1}^{(n)}|)) \cdot |J_{d_{\le 1}/2}(2\pi r |\xi_{\le 1}^{(n)}|)|^2. \tag{A.9}$$

Proof. Since $\xi_{\nabla}^{(n)} \to 0$ we have $t_n \to \infty$, and for all $n \in \mathbb{N}$ sufficiently large we have $|\xi_{\nabla}^{(n)}| \le 1/t_n$, and hence $\xi_n \in \mathcal{C}_{t_n}$. We deduce that, for all $n \in \mathbb{N}$ large enough we have

$$\sup_{\xi \in \mathcal{C}_{t_n}} |[\mathcal{F}\chi_{\mathbb{B}_{\triangleleft}}](r\xi_{\triangleleft})|^2 \ge |[\mathcal{F}\chi_{\mathbb{B}_{\triangleleft}}](r\xi_{\triangleleft}^{(n)})|^2.$$

Moreover, by (A.2) we have $t_n \ge \psi^{-1}(|\xi_{\triangleleft}^{(n)}|)$. Plugging both into the inequality above, we deduce that

$$\begin{split} \Psi_f(t_n,r) &\geq \frac{\psi(t_n)^{d_{\lhd}+1} f(t_n) r^{2d_{\lhd}} \operatorname{vol}_{\triangledown}(\Omega_{\triangledown})^2}{4 \operatorname{covol}(\Gamma)^2} |[\mathscr{F}\chi_{\mathbb{B}_{\lhd}}](r\xi_{\lhd}^{(n)})|^2 \\ &\geq \frac{\|\xi_{\lhd}^{(n)}\|^{d_{\lhd}+1} f(\psi^{-1}(|\xi_{\lhd}^{(n)}|)) r^{2d_{\lhd}} \operatorname{vol}_{\triangledown}(\mathbb{B}_{\triangledown})^2}{4 \operatorname{covol}(\Gamma)^2} |[\mathscr{F}\chi_{\mathbb{B}_{\lhd}}](r\xi_{\lhd}^{(n)})|^2. \end{split}$$

We recall, following [GR07, Formula 8.145.1], that the Fourier transform of the indicator of a unit ball can be expressed in terms of Bessel functions as

$$[\mathcal{F}\chi_{B_{\lhd}}](\xi_{\lhd}) = \frac{J_{d_{\lhd}/2}(2\pi|\xi_{\lhd}|)}{|\xi_{\lhd}|^{d_{\lhd}/2}},$$

hence we obtain the desired inequality with $C = \frac{\text{vol}_{\nabla}(\Omega_{\nabla})^2}{4\text{covol}(\Gamma)^2}$

Let us put $\tau_n := 2\pi |\xi_{\leq 1}^{(n)}|$ and $\tilde{f} = f \circ \psi^{-1}$. By (A.9) we thus have

$$\Psi_f(t_n,r) \gtrsim [\tau_n^{1/2} \widetilde{J}_{d_{\lhd}/2}(r\tau_n)]^2 \widetilde{f}(\tau_n).$$

To deduce (A.7) and thereby prove Theorem A.5 it thus suffices to establish the following lemma.

Lemma A.12. Let (τ_n) be a sequence of positive real numbers such that $\tau_n \to \infty$. Then there is a Lebesgue conull subset $E \subset (0,\infty)$ such that for all $r \in E$, and $d \subseteq \mathbb{N}$

$$\overline{\lim_{n\to\infty}} \tau_n^{1/2} \cdot |J_{d_{\triangleleft}/2}(r\tau_n)| \cdot \widetilde{f}(\tau_n) = \infty.$$

Proof. Since $\tau_n \to \infty$ we may without loss of generality assume that $\tau_n > 1$ for all n. By [Wat95, Subsection 7.21], there exist two real constants φ and κ depending on d_{\triangleleft} such that

$$\left| J_{d_{\triangleleft}/2}(\tau) - \left(\frac{2}{\pi \tau} \right)^{1/2} \cos(\tau - \varphi) \right| \le \frac{\kappa}{\tau^{3/2}}, \quad \text{for all } \tau \ge 1.$$

In particular, for all r > 0 and n,

$$\begin{split} |J_{d_{\lhd}/2}(r\tau_n)| &= \left|J_{d_{\lhd}/2}(r\tau_n) - \left(\frac{2}{\pi r \tau_n}\right)^{1/2} \cos(r\tau_n - \varphi) + \left(\frac{2}{\pi r \tau_n}\right)^{1/2} \cos(r\tau_n - \varphi)\right| \\ &\geq \left(\frac{2}{\pi r \tau_n}\right)^{1/2} |\cos(r\tau_n - \varphi)| - \frac{\kappa}{(r\tau_n)^{3/2}}, \end{split}$$

and thus,

$$\tau_n^{\frac{1}{2}}\widetilde{f}(\tau_n)\cdot |J_{d_{\triangleleft}/2}(r\tau_n)| \geq \widetilde{f}(\tau_n)\cdot \left(\frac{2}{\pi r}\right)^{1/2} |\cos(r\tau_n-\varphi)| - \widetilde{f}(\tau_n)\tau_n^{-1}\frac{\kappa}{r^{3/2}}.$$

Since $\tau_n \to \infty$, [KN74, Chapter 1, Theorem 4.1] tells us that we can find a Lebesgue conull subset $A \subset (0,\infty)$ such that the sequence $(r\tau_n)$ is equidistributed modulo 2π for all $r \in A$. Hence, for all $r \in A$, we can find a further sub-sequence (τ_{n_k}) such that $r\tau_{n_k} - \varphi \to 0$ (modulo 2π) as $k \to \infty$. In particular,

$$|\cos(r\tau_{n_k} - \varphi)| \ge \frac{1}{2}$$
 for all large enough k ,

and thus

$$\tau_{n_k}^{\frac{1}{2}}\widetilde{f}(\tau_{n_k})\cdot |J_{d_{\triangleleft}/2}(r\tau_{n_k})| \geq \frac{1}{2}\cdot \widetilde{f}(\tau_{n_k})\cdot \left(\frac{2}{\pi r}\right)^{1/2} - \widetilde{f}(\tau_{n_k})\tau_{n_k}^{-1}\frac{\kappa}{r^{3/2}}.$$

The right-hand side clearly tends to infinity as $k \to \infty$, which finishes the proof.

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A.4 Proof of Theorem A.6

We now turn our eye to Theorem A.6; thus Γ^{\dagger} is the lattice from Example A.4. In particular there is a ψ -Liouville number 0 < a < 2 and unit length vectors $e_{\lhd} \in \mathbb{E}_{\lhd}$ and $e_{\nabla} \in \mathbb{E}_{\nabla}$ such that $\operatorname{span}(ae_{\nabla} + e_{\lhd}, e_{\nabla} + e_{\lhd}) \subset \Gamma^{\dagger}$, and the rest of the lattice is orthogonal to this span. We fix once and for all the sequence $\xi^{(n)} = q_n(ae_{\nabla} + e_{\lhd}) - m_n(e_{\nabla} + e_{\lhd})$ where $q_n \in \mathbb{N}$ satisfies $\psi(\|aq_n\|^{-1}) \ge q_n$ and m_n is the nearest integer to aq_n . Note that $|\xi^{(n)}_{\lhd}| \sim |a-1|q_n \to \infty$. Our goal is to use the specific knowledge of the sequence realising the ψ -Liouvilleanness of Γ^{\dagger} in order to choose a window making the discrepancy particularly large; the proof is almost identical to the proof of Theorem A.5 but in fact more direct because we can put our hands on the explicit sequence.

We put Ω_{∇} again to be any set of bounded perimeter, and $\Omega_{\neg} = [-r, r] \times \Upsilon_{\neg}$, where the interval is taken to be in the direction of e_{\neg} . This time, we have that for every $n \in \mathbb{N}$,

$$[\mathcal{F}\chi_{\Omega_{\lhd}}](\xi^{(n)}) = 2r \operatorname{vol}(\Upsilon_{\lhd}) \frac{\sin(2\pi r (q_n - m_n))}{2\pi r (q_n - m_n)}$$

Let $f:(0,\infty)\to (0,\infty)$ be any function such that $\lim_{t\to\infty} f(t) = \infty$. Set $t_n = |q_n a_n - m_n|^{-1} = |\xi_{\nabla}^{(n)}|^{-1}$, and assume n is large enough so that the bound in (A.8) holds. By Proposition A.10 we have just as in (A.6) that

$$\begin{split} \Psi_{f}(t_{n},r) &:= \frac{\text{NV}_{t_{n}}(\Gamma,r\Omega_{\triangleleft})}{t_{n}^{2d_{\triangledown}}\psi(t_{n})^{-2}f(t_{n})^{-1}} = \frac{\psi(t_{n})^{2}f(t_{n})}{\text{covol}(\Gamma)^{2}} \sum_{\xi \in \Gamma^{\dagger} \setminus \{(0,0)\}} |[\mathscr{F}\chi_{\Omega_{\triangledown}}](t_{n}\xi_{\triangledown})|^{2} |[\mathscr{F}\chi_{\Omega_{\triangleleft}}](\xi_{\triangleleft})|^{2} \\ &\geqslant \frac{\psi(t_{n})^{2}f(t_{n})r^{2}\text{vol}(\Omega_{\triangledown})^{2}\text{vol}(\Upsilon_{\triangleleft})^{2}}{\text{covol}(\Gamma)^{2}} \frac{\sin^{2}(2\pi r(q_{n}-m_{n}))}{(2\pi r(q_{n}-m_{n}))^{2}} \\ &\geqslant \frac{f(t_{n})\text{vol}(\Omega_{\triangledown})^{2}\text{vol}(\Upsilon_{\triangleleft})^{2}}{4\pi^{2}\text{covol}(\Gamma)^{2}} \sin^{2}(2\pi r(q_{n}-m_{n})) \end{split} \tag{A.10}$$

where we have used that $|q_n - m_n| \le \psi(t_n)$. But again, [KN74, Chapter 1, Theorem 4.1] tells us that we can find a Lebesgue conull set $A \subset (0,\infty)$ for which the sequence $\{r(q_n - m_n)\}$ is equidistributed modulo 1 for all $r \in A$. Hence, for all $r \in A$ we can find a further subsequence such that $r(q_{n_k} - m_{n_k}) - 1/4 \to 0$ (modulo 1) as $k \to \infty$. In particular, the last line in (A.10) diverges to infinity as $k \to \infty$ since f is divergent.

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