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Riemann's Zeta Function

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Contents

Chapter 8 **Large-Scale Computations**

Chapter 9 **The Growth of Zeta as $t \to \infty$ and the Location of Its Zeros**

Chapter 10 **Fourier Analysis**

Chapter 11 **Zeros on the Line**

Chapter 12 **Miscellany**

Preface

My primary objective in this book is to make a point, not about analytic number theory, but about the way in which mathematics is and ought to be studied. Briefly put, I have tried to say to students of mathematics that they should *read the classics* and beware of secondary sources.

This is a point which Eric Temple Bell makes repeatedly in his biographies of great mathematicians in *Men of Mathematics*. In case after case, Bell points out that the men of whom he writes learned their mathematics not by studying in school or by reading textbooks, but by going straight to the sources and reading the best works of the masters who preceded them. It is a point which in most fields of scholarship at most times in history would have gone without saying.

No mathematical work is more clearly a classic than Riemann's memoir *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse*, published in 1859. Much of the work of many of the great mathematicians since Riemann—men like Hadamard, von Mangoldt, de la Vallée Poussin, Landau, Hardy, Littlewood, Siegel, Polya, Jensen, Lindelöf, Bohr, Selberg, Artin, Hecke, to name just a few of the most important—has stemmed directly from the ideas contained in this eight-page paper. According to legend, the person who acquired the copy of Riemann's collected works from the library of Adolph Hurwitz after Hurwitz's death found that the book would automatically fall open to the page on which the Riemann hypothesis was stated.

Yet it is safe to say both that the dictum "read the classics" is not much heard among contemporary mathematicians and that few students read *Ueber die Anzahl* . . . today. On the contrary, the mathematics of previous generations is generally considered to be unrigorous and naïve, stated in obscure terms which can be vastly simplified by modern terminology, and full of false starts and misstatements which a student would be best advised to avoid. Riemann in particular is avoided because of his reputation for lack of rigor (his "Dirichlet principle" is remembered more for the fact that Weierstrass pointed out that its proof was inadequate than it is for the fact that it was after all correct and that with it Riemann revolutionized the study of Abelian functions), because of his difficult style, and because of a general impression that the valuable parts of his work have all been gleaned and incorporated into subsequent more rigorous and more readable works.

These objections are all valid. When Riemann makes an assertion, it may be something which the reader can verify himself, it may be something which Riemann has proved or intends to prove, it may be something which was not proved rigorously until years later, it may be something which is still unproved, and, alas, it may be something which is not true unless the hypotheses are strengthened. This is especially distressing for a modern reader who is trained to digest each statement before going on to the next. Moreover, Riemann's style is extremely difficult. His tragically brief life was too occupied with mathematical creativity for him to devote himself to elegant exposition or to the polished presentation of all of his results. His writing is extremely condensed and *Ueber die anzahl . . .* in particular is simply a resumé of very extensive researches which he never found the time to expound upon at greater length; it is the only work he ever published on number theory, although Siegel found much valuable new material on number theory in Riemann's private papers. Finally, it is certainly true that most of Riemann's best ideas have been incorporated in later, more readable works.

Nonetheless, it is just as true that one should read the classics in this case as in any other. No secondary source can duplicate Riemann's insight. Riemann was so far ahead of his time that it was 30 years before anyone else began really to grasp his ideas—much less to have their own ideas of comparable value. In fact, Riemann was so far ahead of his time that the results which Siegel found in the private papers were a major contribution to the field when they were published in 1932, seventy years after Riemann discovered them. Any simplification, paraphrasing, or reworking of Riemann's ideas runs a grave risk of missing an important idea, of obscuring a point of view which was a source of Riemann's insight, or of introducing new technicalities or side issues which are not of real concern. There is no mathematician since Riemann whom I would trust to revise his work.

The perceptive reader will of course have noted the paradox here of a secondary source denouncing secondary sources. I might seem to be saying, "Do not read this book." But he will also have seen the answer to the paradox. What I am saying is: Read the classics, not just Riemann, but all the major contributions to analytic number theory that I discuss in this book. The purpose of a secondary source is to make the primary sources accessible to you. If you can read and understand the primary sources without reading this book, more power to you. If you read this book without reading the primary sources you are like a man who carries a sack lunch to a banquet.

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It is a pleasure to acknowledge my indebtedness to many individuals and institutions that have aided me in the writing of this book. The two institutions which have supported me during this time are the Washington Square College of New York University and the School of General Studies of the Australian National University; I have been proud to be a part of these particular schools because of their affirmation of the importance of undergraduate teaching in an age when the pressures toward narrow over-specialization are so great.

I am grateful to many libraries for the richness of the resources which they make available to us all. Of the many I have used during the course of the preparation of this book, I think particularly of the following institutions which provided access to relatively rare documents: The New York Public Library, the Courant Institute of Mathematical Sciences, the University of Illinois, Sydney University, the Royal Society of Adelaide, the Linnean Society of New South Wales, the Public Library of New South Wales, and the Australian National University. I am especially grateful to the University Library in Göttingen for giving me access to Riemann's *Nachlass* and for permitting me to photocopy the portions of it relevant to this book.

Among the individuals I would like to thank for their comments on the manuscript are Gabriel Stolzenberg of Northeastern University, David Lubell of Adelphi, Bruce Chandler of New York, Ian Richards of Minnesota, Robert Spira of Michigan State, and Andrew Coppel of the Australian National University. J. Barkley Rosser and D. H. Lehmer were very helpful in providing information on their researches. Carl Ludwig Siegel was very hospitable and generous with his time during my brief visit to Göttingen. And, finally, I am deeply grateful to Wilhelm Magnus for his understanding of my objectives and for his encouragement, which sustained me through many long days when it seemed that the work would never be done.

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Chapter 1

Riemann's Paper

1.1 THE HISTORICAL CONTEXT OF THE PAPER

This book is a study of Bernhard Riemann's epoch-making 8-page paper "On the Number of Primes Less Than a Given Magnitude,"† and of the subsequent developments in the theory which this paper inaugurated. This first chapter is an examination and an amplification of the paper itself, and the remaining 11 chapters are devoted to some of the work which has been done since 1859 on the questions which Riemann left unanswered.

The theory of which Riemann's paper is a part had its beginnings in Euler's theorem, proved in 1737, that the sum of the reciprocals of the prime numbers

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+\cdots \tag{1}$$

is a divergent series. This theorem goes beyond Euclid's ancient theorem that there are infinitely many primes [E2] and shows that the primes are rather *dense* in the set of all integers—denser than the squares, for example, in that the sum of the reciprocals of the square numbers converges.

Euler in fact goes beyond the mere statement of the divergence of (1) to say that since $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ diverges like the logarithm and since the series (1) diverges like‡ the logarithm of $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\cdots$, the series (1)

†The German title is *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.* An English translation of the paper is given in the Appendix.

‡This is true by dint of the Euler product formula which gives $\sum(1/n)=\prod(1-p^{-1})^{-1}$ (see Section 1.2); hence $\log\sum(1/n)=-\sum\log(1-p^{-1})=\sum(p^{-1}+\frac{1}{2}p^{-2}+\frac{1}{3}p^{-3}+\cdots)=\sum(1/p)+\text{convergent}$.

must diverge like the log of the log, which Euler writes [E4] as

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots = \log(\log \infty). \tag{2}$$

It is not clear exactly what Euler understood this equation to mean—if indeed he understood it as anything other than a mnemonic—but an obvious interpretation of it would be

$$\sum_{p<x} 1/p \sim \log(\log x) \qquad (x \longrightarrow \infty), \tag{2'}$$

where the left side denotes the sum of $1/p$ over all primes p less than x and where the sign $\sim$ means that the relative error is arbitrarily small for x sufficiently large or, what is the same, that the ratio of the two sides approaches one as $x \longrightarrow \infty$. Now

$$\log(\log x) = \int_1^{\log x} \frac{du}{u} = \int_e^x \frac{dv}{v \log v}$$

so (2′) says that the integral of $1/v$ relative to the measure $dv/\log v$ diverges in the same way as the integral of $1/v$ relative to the point measure which assigns weight 1 to primes and weight 0 to all other points. In this sense (2′) can be regarded as saying that the density of primes is roughly $1/\log v$. However, there is no evidence that Euler thought about the density of primes, and his methods were not adequate to prove the formulation (2′) of his statement (2).

Gauss states† in a letter [G2] written in 1849 that he had observed as early as 1792 or 1793 that the density of prime numbers appears on the average to be $1/\log x$ and he says that each new tabulation of primes which was published in the ensuing years had tended to confirm his belief in the accuracy of this approximation. However, he does not mention Euler's formula (2) and he gives no analytical basis for the approximation, which he presents solely as an empirical observation. He gives, in particular, Table I.

TABLE I[a]

x	Count of primes $< x$	$\int \frac{dn}{\log n}$	Difference
500,000	41,556	41,606.4	50.4
1,000,000	78,501	78,627.5	126.5
1,500,000	114,112	114,263.1	151.1
2,000,000	148,883	149,054.8	171.8
2,500,000	183,016	183,245.0	229.0
3,000,000	216,745	216,970.6	225.6

[a]From Gauss [G2].

†For some corroboration of Gauss's claim see his collected works [G3].

Gauss does not say exactly what he means by the symbol $\int (dn/\log n)$, but the data given in Table II, taken from D.N. Lehmer [L9], would indicate that he means n to be a continuous variable integrated from 2 to x, that is, $\int_2^x (dt/\log t)$. Note that Lehmer's count† of primes, which can safely be assumed to be accurate, differs from Gauss's information and that the difference is in *favor* of Gauss's estimate for the larger values of x.

TABLE II[a]

x	Count of primes $< x$	$\int_2^x \frac{dt}{\log t}$	Difference
500,000	41,538	41,606	68
1,000,000	78,498	78,628	130
1,500,000	114,155	114,263	108
2,000,000	148,933	149,055	122
2,500,000	183,072	183,245	173
3,000,000	216,816	216,971	155

[a]Data from Lehmer [L9].

Around 1800 Legendre published in his *Theorie des Nombres* [L11] an empirical formula for the number of primes less than a given value which amounted more or less to the same statement, namely, that the density of primes is $1/\log x$. Although Legendre made some slight attempt to prove his formula, his argument amounts to nothing more than the statement that if the density of primes is assumed to have the form

$$1/(A_1 x^{m_1} + A_2 x^{m_2} + \cdots)$$

where $m_1 > m_2 > \ldots$, then m_1 cannot be positive [because then the sum (1) would converge]; hence m_1 must be "infinitely small" and the density must be of the form

$$1/(A \log x + B).$$

He then determines A and B empirically. Legendre's formula was well known in the mathematical world and was mentioned prominently by Abel [A2], Dirichlet [D3], and Chebyshev [C2] during the period 1800–1850.

The first significant results beyond Euler's were obtained by Chebyshev around 1850. Chebyshev proved that the relative error in the approximation

$$\pi(x) \sim \int_2^x \frac{dt}{\log t}, \tag{3}$$

†Lehmer insists on counting 1 as a prime. To conform to common usage his counts have therefore been reduced by one in Table II.

where $\pi(x)$ denotes the number of primes less than x, is less than 11% for all sufficiently large x; that is, he proved† that

$$(0.89)\int_2^x \frac{dt}{\log t} < \pi(x) < (1.11)\int_2^x \frac{dt}{\log t}$$

for all sufficiently large x. He proved, moreover, that no approximation of Legendre's form

$$\pi(x) \sim x/(\mathrm{A}\log x + B)$$

can be better than the approximation (3) and that if the ratio of $\pi(x)$ to $\int_2^x (dt/\log t)$ approaches a limit as $x \longrightarrow \infty$, then this limit must be 1. It is clear that Chebyshev was attempting to prove that the relative error in the approximation (3) approaches zero as $x \longrightarrow \infty$, but it was not until almost 50 years later that this theorem, which is known as the "prime number theorem," was proved. Although Chebyshev's work was published in France well before Riemann's paper, Riemann does not refer to Chebyshev in his paper. He does refer to Dirichlet, however, and Dirichlet, who was acquainted with Chebyshev (see Chebyshev's report on his trip to Western Europe [C5, Vol. 5, p. 245 and pp. 254–255]) would probably have made Riemann aware of Chebyshev's work. Riemann's unpublished papers do contain several of Chebyshev's formulas, indicating that he had studied Chebyshev's work, and contain at least one direct reference to Chebyshev (see Fig. 1).

The real contribution of Riemann's 1859 paper lay not in its results but in its methods. The principal result was a formula‡ for $\pi(x)$ as the sum of an infinite series in which $\int_2^x (dt/\log t)$ is by far the largest term. However, Riemann's proof of this formula was inadequate; in particular, it is by no means clear from Riemann's arguments that the infinite series for $\pi(x)$ even *converges*, much less that its largest term $\int_2^x (dt/\log t)$ dominates it for large x. On the other hand, Riemann's methods, which included the study of the function $\zeta(s)$ as a function of a complex variable, the study of the complex zeros of $\zeta(s)$, Fourier inversion, Möbius inversion, and the representation of functions such as $\pi(x)$ by "explicit formulas" such as his infinite series, all have had important parts in the subsequent development of the theory.

For the first 30 years after Riemann's paper was published, there was

†Chebyshev did not state his result in this form. This form can be obtained from his estimate of the number of primes between l and L (see Chebyshev [C3, Section 6]) by fixing l, letting $L \longrightarrow \infty$, and using $\int_2^L (\log t)^{-1}\,dt \sim L/\log L$.

‡See Section 1.17. Note that $Li(x) = \int_2^x (dt/\log t) + \text{const.}$

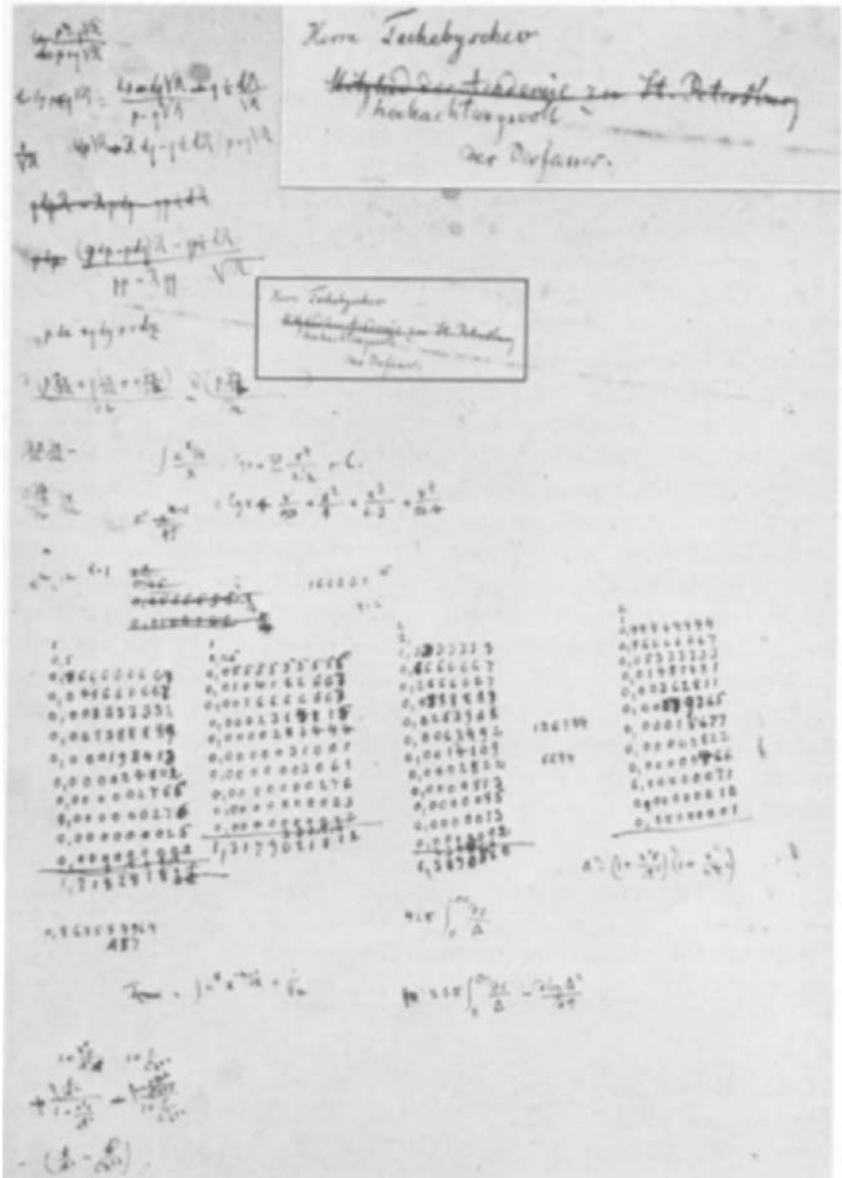

Fig. 1. A scrap sheet used to hold some other loose sheets in Riemann's papers. The note seems to prove that Riemann was aware of Chebyshev's work and intended to send him an offprint of his own paper. In all likelihood Riemann was practicing his penmanship in forming Roman, rather than German, letters to write a dedication to Chebyshev. (Reproduced with the permission of the Niedersächsische Staats- und Universitätsbibliothek, Handschriftenabteilung, Göttingen.)

virtually no progress† in the field. It was as if it took the mathematical world that much time to digest Riemann's ideas. Then, in a space of less than 10 years, Hadamard, von Mangoldt, and de la Vallée Poussin succeeded in proving both Riemann's main formula for $\pi(x)$ and the prime number theorem (3), as well as a number of other related theorems. In all these proofs Riemann's ideas were crucial. Since that time there has been no shortage of new problems and no shortage of progress in analytic number theory, and much of this progress has been inspired by Riemann's ideas.

Finally, no discussion of the historical context of Riemann's paper would be complete without a mention of the Riemann hypothesis. In the course of the paper, Riemann says that he considers it "very likely" that the complex zeros of $\zeta(s)$ all have real part equal to $\frac{1}{2}$, but that he has been unable to prove that this is true. This statement, that the zeros have real part $\frac{1}{2}$, is now known as the "Riemann hypothesis." The experience of Riemann's successors with the Riemann hypothesis has been the same as Riemann's—they also consider its truth "very likely" and they also have been unable to prove it. Hilbert included the problem of proving the Riemann hypothesis in his list [H9] of the most important unsolved problems which confronted mathematics in 1900, and the attempt to solve this problem has occupied the best efforts of many of the best mathematicians of the twentieth century. It is now unquestionably the most celebrated problem in mathematics and it continues to attract the attention of the best mathematicians, not only because it has gone unsolved for so long but also because it appears tantalizingly vulnerable and because its solution would probably bring to light new techniques of far-reaching importance.

1.2 THE EULER PRODUCT FORMULA

Riemann takes as his starting point the formula

$$\sum_n \frac{1}{n^s} = \prod_p \frac{1}{\left(1 - \frac{1}{p^s}\right)} \tag{1}$$

of Euler. Here n ranges over all positive integers ($n = 1, 2, 3, \ldots$) and p ranges over all primes ($p = 2, 3, 5, 7, 11, \ldots$). This formula, which is now known as the "Euler product formula," results from expanding each of the

†A major exception to this statement was Mertens's theorem [M5] of 1874 stating that (2′) is true in the strong sense that the difference of the two sides approaches a limit as $x \longrightarrow \infty$, namely, Euler's constant plus $\sum_p [\log(1 - p^{-1}) + p^{-1}]$. Another perhaps more natural statement of Mertens's theorem is

$$\lim_{x\to\infty} \log x \prod_{p<x} (1 - p^{-1}) = e^{-\gamma},$$

where γ is Euler's constant. See, for example, Hardy and Wright [H7].

factors on the right

$$\frac{1}{\left(1-\frac{1}{p^s}\right)}=1+\frac{1}{p^s}+\frac{1}{(p^2)^s}+\frac{1}{(p^3)^s}+\cdots$$

and observing that their product is therefore a sum of terms of the form

$$\frac{1}{(p_1^{n_1}p_2^{n_2}\cdots p_r^{n_r})^s},$$

where $p_1, \ldots, p_r$ are distinct primes and $n_1, n_2, \ldots, n_r$ are natural numbers, and then using the fundamental theorem of arithmetic (every integer can be written in essentially only one way as a product of primes) to conclude that this sum is simply $\sum(1/n)^s$. Euler used this formula principally as a formal identity and principally for integer values of s (see, for example, Euler [E5]).

Dirichlet also based his work† in this field on the Euler product formula. Since Dirichlet was one of Riemann's teachers and since Riemann refers to Dirichlet's work in the first paragraph of his paper, it seems certain that Riemann's use of the Euler product formula was influenced by Dirichlet. Dirichlet, unlike Euler, used the formula (1) with s as a real variable and, also unlike Euler, he proved‡ rigorously that (1) is true for all real $s > 1$.

Riemann, as one of the founders of the theory of functions of a complex variable, would naturally be expected to consider s as a *complex* variable. It is easy to show that both sides of the Euler product formula converge for complex s in the halfplane Re $s > 1$, but Riemann goes much further and shows that even though both sides of (1) diverge for other values of s, the function they define is meaningful for *all* values of s except for a pole at $s = 1$. This extension of the range of s requires a few facts about the factorial function which will be covered in the next section.

1.3 THE FACTORIAL FUNCTION

Euler extended the factorial function $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$ from the natural numbers n to all real numbers greater than -1 by observing that¶

$$n! = \int_0^\infty e^{-x}x^n\,dx \qquad (n = 1, 2, 3, \ldots) \tag{1}$$

†Dirichlet's major contribution to the theory was his proof that if m is relatively prime to n, then the congruence $p \equiv m \pmod n$ has infinitely many prime solutions p. He was also interested in questions concerning the density of the distribution of primes, but he did not have significant success with these questions.

‡Dirichlet [D3]. Since the terms p^{-s} are all positive, there is nothing subtle or difficult about this proof—it is essentially a reordering of absolutely convergent series—but it has the important effect of transforming (1) from a formal identity true for various values of s to an analytical formula true for all real $s > 1$.

¶However Euler wrote the integral in terms of $y = e^{-x}$ as $n! = \int_0^1 (\log 1/y)^n\,dy$ (see Euler [E3]).

(integration by parts) and by observing that the integral on the right converges for noninteger values of n, provided only that $n > -1$. Gauss [G1] introduced the notation†

(2) $$\Pi(s) = \int_0^\infty e^{-x}x^s\,dx \qquad (s > -1)$$

for Euler's integral on the right side of (1). Thus $\Pi(s)$ is defined for all real numbers s greater than -1, in fact for all complex numbers s in the halfplane $\operatorname{Re} s > -1$, and $\Pi(s) = s!$ whenever s is a natural number. There is another representation of $\Pi(s)$ which was also known‡ to Euler, namely,

(3) $$\Pi(s) = \lim_{N\to\infty} \frac{1\cdot 2\cdots N}{(s+1)(s+2)\cdots(s+N)}(N+1)^s.$$

This formula is valid for all s for which (2) defines $\Pi(s)$, that is, for all s in the halfplane $\operatorname{Re} s > -1$. On the other hand, it is not difficult to show [use formula (4) below] that the limit (3) exists for *all* values of s, real or complex, provided only that the denominator is not zero, that is, provided only that s is not a negative integer. In short, formula (3) extends the definition of $\Pi(s)$ to all values of s other than $s = -1, -2, -3, \ldots$.

In addition to the fact that the two definitions (2) and (3) of $\Pi(s)$ coincide for real $s > -1$, the following facts will be used without proof:

(4) $$\Pi(s) = \prod_{n=1}^\infty \frac{n^{1-s}(n+1)^s}{s+n} = \prod_{n=1}^\infty \left(1+\frac{s}{n}\right)^{-1}\left(1+\frac{1}{n}\right)^s,$$

(5) $$\Pi(s) = s\Pi(s-1),$$

(6) $$\frac{\pi s}{\Pi(s)\Pi(-s)} = \sin \pi s,$$

(7) $$\Pi(s) = 2^s\Pi\left(\frac{s}{2}\right)\Pi\left(\frac{s-1}{2}\right)\pi^{-1/2}.$$

For the proofs of these facts the reader is referred to any book which deals with factorial function or the "Γ-function," for example, Edwards [E1, pp. 421–425]. Identity (4) is a simple reformulation of formula (3). Using it one can prove that $\Pi(s)$ *is an analytic function of the complex variable s which has simple poles at* $s = -1, -2, -3, \ldots$. *It has no zeros.* Identity (5) is

†Unfortunately, Legendre subsequently introduced the notation $\Gamma(s)$ for $\Pi(s-1)$. Legendre's reasons for considering $(n-1)!$ instead of $n!$ are obscure (perhaps he felt it was more natural to have the first pole occur at $s = 0$ rather than at $s = -1$) but, whatever the reason, this notation prevailed in France and, by the end of the nineteenth century, in the rest of the world as well. Gauss's original notation appears to me to be much more natural and Riemann's use of it gives me a welcome opportunity to reintroduce it.

‡See Euler [E3, E8].

called the "functional equation of the factorial function"; together with $\Pi(0) = 1$ [from (4)] it gives $\Pi(n) = n!$ immediately. Identity (6) is essentially the product formula for the sine; when $s = \frac{1}{2}$ it combines with (5) to give the important value $\Pi(-\frac{1}{2}) = \pi^{1/2}$. Identity (7) is known as the *Legendre relation*. It is the case $n = 2$ of a more general identity

$$\frac{\Pi(s)}{n^s \Pi\left(\frac{s}{n}\right)\Pi\left(\frac{s-1}{n}\right)\cdots\Pi\left(\frac{s-n+1}{n}\right)} = \left[\frac{2\pi n}{(2\pi)^n}\right]^{1/2}$$

which will not be needed.

1.4 THE FUNCTION $\zeta(s)$

It is interesting to note that Riemann does not speak of the "analytic continuation" of the function $\sum n^{-s}$ beyond the halfplane Re $s > 1$, but speaks rather of finding a formula for it which "remains valid for all s." This indicates that he viewed the problem in terms more analogous to the extension of the factorial function by formula (3) of the preceding section than to a piece-by-piece extension of the function in the manner that analytic continuation is customarily taught today. The view of analytic continuation in terms of chains of disks and power series convergent in each disk descends from Weierstrass and is quite antithetical to Riemann's basic philosophy that analytic functions should be dealt with *globally*, not locally in terms of power series.

Riemann derives his formula for $\sum n^{-s}$ which "remains valid for all s" as follows. Substitution of nx for x in Euler's integral for $\Pi(s-1)$ gives

$$\int_0^\infty e^{-nx} x^{s-1}\, dx = \frac{\Pi(s-1)}{n^s}$$

($s > 0$, $n = 1, 2, 3, \ldots$). Riemann sums this over n and uses $\sum_{n=1}^\infty r^{-n} = (r-1)^{-1}$ to obtain†

(1) $$\int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx = \Pi(s-1) \sum_{n=1}^\infty \frac{1}{n^s}$$

($s > 1$). Convergence of the improper integral on the left and the validity of

†This formula, with $s = 2n$, occurs in a paper [A1] of Abel which was included in the 1839 edition of Abel's collected works. It seems very likely that Riemann would have been aware of this. A very similar formula

$$\int_0^\infty (e^x - 1)^{-1} e^{-x} x^p\, dx = \Pi(p) \sum_{n=2}^\infty n^{-1-p}$$

is the point of departure of Chebyshev's 1848 paper [C2].

the interchange of summation and integration are not difficult to establish.

Next he considers the contour integral

$$\int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \frac{dx}{x}.$$

The limits of integration are intended to indicate a path of integration which begins at $+\infty$, moves to the left down the positive real axis, circles the origin once in the positive (counterclockwise) direction, and returns up the positive real axis to $+\infty$. The definition of $(-x)^s$ is $(-x)^s = \exp[s \log(-x)]$, where the definition of $\log(-x)$ conforms to the usual definition of $\log z$ for z not on the negative real axis as the branch which is real for positive real z; thus $(-x)^s$ is not defined on the positive real axis and, strictly speaking, the path of integration must be taken to be slightly above the real axis as it descends from $+\infty$ to 0 and slightly below the real axis as it goes from 0 back to $+\infty$. When this integral is written in the form

$$\int_{+\infty}^{\delta} \frac{(-x)^s\,dx}{(e^x - 1)x} + \int_{|x|=\delta} \frac{(-x)^s\,dx}{(e^x - 1)x} + \int_{\delta}^{+\infty} \frac{(-x)^s\,dx}{(e^x - 1)x},$$

the middle term is $2\pi i$ times the average value of $(-x)^s(e^x-1)^{-1}$ on the circle $|x| = \delta$ [because on this circle $i\,d\theta = (dx/x)$]. Thus the middle term approaches zero as $\delta \longrightarrow 0$ provided $s > 1$ [because $x(e^x - 1)^{-1}$ is nonsingular near $x = 0$]. The other two terms can then be combined to give

$$\begin{aligned}\int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1}\cdot\frac{dx}{x} &= \lim_{\delta\to 0}\left\{\int_{+\infty}^{\delta} \frac{\exp[s(\log x - i\pi)]\,dx}{(e^x - 1)x}\right. \\ &\qquad \left. + \int_{\delta}^{+\infty} \frac{\exp[s(\log x + i\pi)]\,dx}{(e^x - 1)x}\right\} \\ &= (e^{i\pi s} - e^{-i\pi s})\int_0^{\infty} \frac{x^{s-1}\,dx}{e^x - 1}\end{aligned}$$

which combines with the previous formula (1) to give

$$\int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1}\cdot\frac{dx}{x} = 2i\sin(\pi s)\Pi(s-1)\sum_{n=1}^{\infty}\frac{1}{n^s}$$

or, finally, when both sides are multiplied by $\Pi(-s)s/2\pi i s$ and identity (6) of the preceding section is used,

$$\frac{\Pi(-s)}{2\pi i}\int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1}\cdot\frac{dx}{x} = \sum_{n=1}^{\infty}\frac{1}{n^s}. \tag{2}$$

In other words, if $\zeta(s)$ is defined by the formula†

$$\zeta(s) = \frac{\Pi(-s)}{2\pi i}\int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1}\cdot\frac{dx}{x}, \tag{3}$$

†This formula is misstated by the editors of Riemann's works in the notes; they put the factor π on the wrong side of their equation.

then, for real values of s greater than one, $\zeta(s)$ is equal to Dirichlet's function

(4) $$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

However, formula (3) for $\zeta(s)$ "remains valid for all s." In fact, since the integral in (3) clearly converges for all values of s, real or complex (because e^x grows much faster than x^s as $x \longrightarrow \infty$), and since the function it defines is complex analytic (because convergence is uniform on compact domains), the function $\zeta(s)$ of (3) is defined and analytic at all points with the possible exception of the points $s = 1, 2, 3, \ldots$, where $\Pi(-s)$ has poles. Now at $s = 2, 3, 4, \ldots$, formula (4) shows that $\zeta(s)$ has no pole [hence the integral in (3) must have a zero which cancels the pole of $\Pi(-s)$ at these points, a fact which also follows immediately from Cauchy's theorem], and at $s = 1$ formula (4) shows that $\lim \zeta(s) = \infty$ as $s \downarrow 1$, hence that $\zeta(s)$ has a simple [because the pole of $\Pi(-s)$ is simple] pole at $s = 1$. Thus *formula (3) defines a function $\zeta(s)$ which is analytic at all points of the complex s-plane except for a simple pole at $s = 1$.* This function coincides with $\sum n^{-s}$ for real values of $s > 1$ and in fact, by analytic continuation, throughout the halfplane $\operatorname{Re} s > 1$.

The function $\zeta(s)$ is known as the Riemann zeta function.

1.5 VALUES OF $\zeta(s)$

The function $x(e^x - 1)^{-1}$ is analytic near $x = 0$; therefore it can be expanded as a power series

(1) $$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}$$

valid near zero [in fact valid in the disk $|x| < 2\pi$ which extends to the nearest singularities $x = \pm 2\pi i$ of $x(e^x - 1)^{-1}$]. The coefficients B_n of this expansion are by definition the *Bernoulli numbers;* the first few are easily determined to be

$$\begin{aligned}
B_0 &= 1, & B_1 &= -\tfrac{1}{2}, \\
B_2 &= \tfrac{1}{6}, & B_3 &= 0, \\
B_4 &= -\tfrac{1}{30}, & B_5 &= 0, \\
B_6 &= \tfrac{1}{42}, & B_7 &= 0, \\
B_8 &= -\tfrac{1}{30}, & B_9 &= 0.
\end{aligned}$$

The odd Bernoulli numbers B_{2n+1} are all zero† after the first, and the even Bernoulli numbers B_{2n} can be determined successively, but there is no simple

†This can be proved directly by noting that $(-t)(e^{-t} - 1)^{-1} + (-t/2) = (-te^t + t - t)(1 - e^t)^{-1} - (t/2) = t(e^t - 1)^{-1} + (t/2)$, that is, $t(e^t - 1)^{-1} + (t/2)$ is an even function. For alternative proofs see the note of Section 1.6 and formula (10) of Section 6.2.

computational formula for them. (See Euler [E6] for a list of the values of $(-1)^{n-1} B_{2n}$ up to B_{30}.)

When $s = -n$ $(n = 0, 1, 2, \ldots)$, this expansion (1) can be used in the defining equation of $\zeta(s)$ to obtain

$$\begin{aligned}\zeta(-n) &= \frac{\Pi(n)}{2\pi i}\int_{+\infty}^{\infty}\frac{(-x)^{-n}}{e^x - 1}\cdot\frac{dx}{x} \\ &= \frac{\Pi(n)}{2\pi i}\int_{|x|=\delta}\left(\sum_m \frac{B_m x^m}{m!}\right)\frac{(-x)^{-n}}{x}\cdot\frac{dx}{x} \\ &= \sum_m \Pi(n)\frac{B_m}{m!}(-1)^n \cdot \frac{1}{2\pi}\int_0^{2\pi} x^{m-n-1}\, d\theta \\ &= n!\frac{B_{n+1}}{(n+1)!}(-1)^n = (-1)^n\frac{B_{n+1}}{n+1}.\end{aligned}$$

Riemann does not give this formula for $\zeta(-n)$, but he does state the particular consequence $\zeta(-2) = \zeta(-4) = \zeta(-6) = \cdots = 0$. He was surely aware, however, not only of the values†

$$\zeta(0) = -1/2, \qquad \zeta(-1) = -1/12, \qquad \zeta(-3) = 1/120,$$

etc., which it implies, but also of the values

$$\zeta(2) = \pi^2/6, \zeta(4) = \pi^4/90, \ldots,$$

and, in general,

$$\zeta(2n) = \frac{(2\pi)^{2n}(-1)^{n+1}B_{2n}}{2\cdot(2n)!} \tag{2}$$

which had been found by Euler [E6]. There is no easy way to deduce this famous formula of Euler's from Riemann's integral formula for $\zeta(s)$ [(3) of Section 1.4] and it may well have been this problem of deriving (2) anew which led Riemann to the discovery‡ of the functional equation of the zeta function which is the subject of the next section.

1.6 FIRST PROOF OF THE FUNCTIONAL EQUATION

For negative real values of s, Riemann evaluates the integral

$$\zeta(s) = \frac{\Pi(-s)}{2\pi i}\int_{+\infty}^{+\infty}\frac{(-x)^s}{e^x - 1}\cdot\frac{dx}{x} \tag{1}$$

†The editors of Riemann's collected works give the erroneous value $\zeta(0) = \frac{1}{2}$.

‡Actually the functional equation occurs in Euler's works [E7] in a slightly different form, and it is entirely possible that Riemann found it there. (See also Hardy [H5, pp. 23–26].) In any case, Euler had nothing but an empirical (!) proof of the functional equation and Riemann, in a reversal of his usual role, gave the first rigorous proof of a statement which had been made, but not adequately proved, by someone else.

as follows. Let D denote the domain in the s-plane which consists of all points other than those which lie within ϵ of the positive real axis or within ϵ of one of the singularities $x = \pm 2\pi i n$ of the integrand of (1). Let ∂D be the boundary of D oriented in the usual way. Then, ignoring for the moment the fact that D is not compact, Cauchy's theorem gives

(2) $$\frac{\Pi(-s)}{2\pi i}\int_{\partial D}\frac{(-x)^s}{e^x - 1}\cdot\frac{dx}{x} = 0.$$

Now one component of this integral is the integral (1) with the orientation reversed, whereas the others are integrals over the circles $|x \pm 2\pi i n| = \epsilon$ oriented clockwise. Thus when the circles are oriented in the usual counterclockwise sense, (2) becomes

(3) $$-\zeta(s) - \sum \frac{\Pi(-s)}{2\pi i}\int_{|x\pm 2\pi i n|=\epsilon}\frac{(-x)^s}{e^x - 1}\cdot\frac{dx}{x} = 0.$$

The integrals over the circles can be evaluated by setting $x = 2\pi i n + y$ for $|y| = \epsilon$ to find

$$\begin{aligned}
&\frac{\Pi(-s)}{2\pi i}\int_{|y|=\epsilon}\frac{(-2\pi i n - y)^s}{e^{2\pi i n + y} - 1}\frac{dy}{2\pi i n + y}\\
&\quad = -\frac{\Pi(-s)}{2\pi i}\int_{|y|=\epsilon}(-2\pi i n - y)^{s-1}\cdot\frac{y}{e^y - 1}\cdot\frac{dy}{y}\\
&\quad = -\Pi(-s)(-2\pi i n)^{s-1}
\end{aligned}$$

by the Cauchy integral formula. Summing over all integers n other than $n = 0$ and using (3) then gives

$$\begin{aligned}
\zeta(s) &= \sum_{n=1}^{\infty}\Pi(-s)[(-2\pi i n)^{s-1} + (2\pi i n)^{s-1}]\\
&= \Pi(-s)(2\pi)^{s-1}[i^{s-1} + (-i)^{s-1}]\sum_{n=1}^{\infty} n^{s-1}.
\end{aligned}$$

Finally, using the simplification

$$\begin{aligned}
i^{s-1} + (-i)^{s-1} &= \frac{1}{i}[e^{s\log i} - e^{s\log(-i)}]\\
&= \frac{1}{i}[e^{s\pi i/2} - e^{-s\pi i/2}] = 2\sin\frac{s\pi}{2},
\end{aligned}$$

one obtains the desired formula

(4) $$\zeta(s) = \Pi(-s)(2\pi)^{s-1}\,2\sin(s\pi/2)\zeta(1 - s).$$

This relationship between $\zeta(s)$ and $\zeta(1 - s)$ is known as the *functional equation of the zeta function.*

In order to prove rigorously that (4) holds for $s < 0$, it suffices to modify the above argument by letting D_n be the intersection of D with the disk $|s| \le (2n + 1)\pi$ and letting $n \to \infty$; then the integral (2) splits into two parts, one being an integral over the circle $|s| = (2n + 1)\pi$ with the points within ϵ of the positive real axis deleted, and the other being an integral whose limit as $n \longrightarrow \infty$ is the left side of (3). The first of these two parts approaches zero because the length of the path of integration is less than $2\pi(2n + 1)\pi$, because the factor $(e^x - 1)^{-1}$ is bounded on the circle $|s| = (2n + 1)\pi$, and because the modulus of $(-x)^s/x$ on this circle is $|x|^{s-1} \leq [(2n + 1)\pi]^{-\delta-1}$ for $s \leq -\delta < 0$. Thus the second part, which by Cauchy's theorem is the negative of the first part, also approaches zero, which implies (3) and hence (4).

This completes the proof of the functional equation (4) in the case $s < 0$. However, both sides of (4) are analytic functions of s, so this suffices to prove (4) for all values of s [except for $s = 0, 1, 2, \ldots,$ where† one or more of the terms of (4) have poles].

For $s = 1 - 2n$ the functional equation plus the identity

$$\zeta(-(2n - 1)) = (-1)^{2n-1}\frac{B_{2n}}{2n}$$

of the previous section gives

$$(-1)^{2n-1}\frac{B_{2n}}{2n} = \Pi(2n - 1)(2\pi)^{-2n}2(-1)^n\zeta(2n)$$

and hence Euler's famous formula for $\zeta(2n)$ [(2) of Section 1.5].

Riemann uses two of the basic identities of the factorial function [(6) and (7) of Section 1.3] to rewrite the functional equation (4) in the form

$$\zeta(s) = \pi^{-1/2}2^{-s}\Pi\left(-\frac{s}{2}\right)\Pi\left(-\frac{s+1}{2}\right)2^s\pi^{s-1}\frac{\pi s/2}{\Pi\left(\frac{s}{2}\right)\Pi\left(-\frac{s}{2}\right)}\zeta(1 - s)$$

and hence in the form

$$\Pi\left(\frac{s}{2} - 1\right)\pi^{-s/2}\zeta(s) = \Pi\left(\frac{1-s}{2} - 1\right)\pi^{-(1-s)/2}\zeta(1 - s). \tag{5}$$

In words, then, *the function on the left side of (5) is unchanged by the substitution $s = 1 - s$.*

Riemann appears to consider this symmetrical statement (5) as the natural statement of the functional equation, because he gives‡ an alternative proof

†When $s = 2n + 1$, the fact that $\zeta(s)$ has no pole at $2n + 1$ implies, since Π has a pole at $-2n - 1$ and $\sin(s\pi/2)$ has no zero at $2n + 1$, that $\zeta(-2n) = 0$ and hence, by the formula for $\zeta(-2n)$ of the preceding section, that the odd Bernoulli numbers $B_3, B_5, B_7, \ldots$ are all zero.

‡Since the second proof renders the first proof wholly unnecessary, one may ask why Riemann included the first proof at all. Perhaps the first proof shows the argument by which he originally discovered the functional equation or perhaps it exhibits some properties which were important in his understanding of ζ.

which exhibits this symmetry in a more satisfactory way. This second proof is given in the next section.

1.7 SECOND PROOF OF THE FUNCTIONAL EQUATION

Riemann first observes that the change of variable $x = n^2\pi x$ in Euler's integral for $\Pi(s/2 - 1)$ gives

$$\frac{1}{n^s}\pi^{-s/2}\Pi\left(\frac{s}{2} - 1\right) = \int_0^\infty e^{-n^2\pi x}x^{s/2}\frac{dx}{x} \qquad (\operatorname{Re} s > 1).$$

Thus summation over n gives

$$\Pi\left(\frac{s}{2} - 1\right)\pi^{-s/2}\zeta(s) = \int_0^\infty \psi(x)x^{s/2}\cdot\frac{dx}{x} \qquad (\operatorname{Re} s > 1), \tag{1}$$

where† $\psi(x) = \sum_{n=1}^\infty \exp(-n^2\pi x)$. The symmetrical form of the functional equation is the statement that the function (1) is unchanged by the substitution $s = 1 - s$. To prove directly that the integral on the right side of (1) is unchanged by this substitution Riemann uses the *functional equation of the theta function* in a form taken from Jacobi,‡ namely, in the form

$$\frac{1 + 2\psi(x)}{1 + 2\psi(1/x)} = \frac{1}{\sqrt{x}}. \tag{2}$$

[Since $\psi(x)$ approaches zero very rapidly as $x \longrightarrow \infty$, this shows in particular that $\psi(x)$ is like $\frac{1}{2}(x^{-1/2} - 1)$ for x near zero and hence that the integral on the right side of (1) is convergent for $s > 1$. Once this has been established, the validity of (1) for $s > 1$ can be proved by an elementary argument using absolute convergence to justify the interchange of summation and integration.] Using (2), Riemann reformulates the integral on the right side of (1) as

$$\begin{aligned}\int_0^\infty \psi(x)x^{s/2}\frac{dx}{x} &= \int_1^\infty \psi(x)x^{s/2}\cdot\frac{dx}{x} - \int_\infty^1 \psi\left(\frac{1}{x}\right)x^{-s/2}\cdot\frac{dx}{x}\\ &= \int_1^\infty \psi(x)x^{s/2}\cdot\frac{dx}{x} + \int_1^\infty\left[x^{1/2}\psi(x) + \frac{x^{1/2}}{2} - \frac{1}{2}\right]x^{-s/2}\frac{dx}{x}\\ &= \int_1^\infty \psi(x)[x^{s/2} + x^{(1-s)/2}]\frac{dx}{x}\\ &\quad + \frac{1}{2}\int_1^\infty [x^{-(s-1)/2} - x^{-s/2}]\frac{dx}{x}.\end{aligned}$$

†This function $\psi(x)$ has nothing whatsoever to do with the function $\psi(x)$ which appears in Chapter 3.

‡Riemann refers to Section 65 of Jacobi's treatise "Fundmenta Nova Theoriae Functionum Ellipticarum." Although the needed formula is not given explicitly there, Jacobi in another place [J1] shows how the needed formula follows from formula (6) of Section 65. Jacobi attributes the formula to Poisson. For a proof of the formula see Section 10.4.

Now $\int_1^\infty x^{-a}(dx/x) = 1/a$ for $a > 0$ so the second integral is

$$\frac{1}{2}\left[\frac{1}{(s-1)/2} - \frac{1}{s/2}\right] = \frac{1}{s(s-1)}$$

for $s > 1$. Thus for $s > 1$ the formula

$$\text{(3)} \qquad \Pi\left(\frac{s}{2} - 1\right)\pi^{-s/2}\zeta(s) = \int_1^\infty \psi(x)[x^{s/2} + x^{(1-s)/2}]\frac{dx}{x} - \frac{1}{s(1-s)}$$

holds. But, because $\psi(x)$ decreases more rapidly than any power of x as $x \longrightarrow \infty$, the integral in this formula converges for all† s. Since both sides are analytic, the same equation holds for all s. Because the right side is obviously unchanged by the substitution $s = 1 - s$, this proves the functional equation of the zeta function.

1.8 THE FUNCTION $\xi(s)$

The function $\Pi((s/2) - 1)\pi^{-s/2}\zeta(s)$, which occurs in the symmetrical form of the functional equation, has poles at $s = 0$ and $s = 1$. [This follows immediately from (3) of the preceding section.] Riemann multiplies it by $s(s-1)/2$ and defines‡

$$\text{(1)} \qquad \xi(s) = \Pi(s/2)(s-1)\pi^{-s/2}\zeta(s).$$

Then $\xi(s)$ is an entire function—that is, an analytic function of s which is defined for all values of s—and the functional equation of the zeta function is equivalent to $\xi(s) = \xi(1-s)$.

Riemann next derives the following representation of $\xi(s)$. Equation (3) of the preceding section gives

$$\begin{aligned}
\xi(s) &= \frac{1}{2} - \frac{s(1-s)}{2}\int_1^\infty \psi(s)(x^{s/2} + x^{(1-s)/2})\frac{dx}{x} \\
&= \frac{1}{2} - \frac{s(1-s)}{2}\int_1^\infty \frac{d}{dx}\left\{\psi(x)\left[\frac{x^{s/2}}{s/2} + \frac{x^{(1-s)/2}}{(1-s)/2}\right]\right\} dx \\
&\quad + \frac{s(1-s)}{2}\int_1^\infty \psi'(x)\left[\frac{x^{s/2}}{s/2} + \frac{x^{(1-s)/2}}{(1-s)/2}\right] dx
\end{aligned}$$

†Note that this gives, therefore, another formula for $\zeta(s)$ which is "valid for all s" other than $s = 0, 1$; that is, it gives an alternative proof of the fact that $\zeta(s)$ can be analytically continued.

‡Actually Riemann uses the letter ξ to denote the function which it is now customary to denote by Ξ, namely, the function $\Xi(t) = \xi(\frac{1}{2} + it)$, where ξ is defined as above. I follow Landau, and almost all subsequent writers, in rejecting Riemann's change of variable $s = \frac{1}{2} + it$ in formula (1) as being confusing. In fact, there is reason to believe that Riemann himself was confused by it [see remarks concerning $\xi(0)$ in Section 1.16].

$$= \frac{1}{2} + \frac{s(1-s)}{2}\psi(1)\left[\frac{2}{s} + \frac{2}{1-s}\right]$$

$$+ \int_1^\infty \psi'(x)[(1-s)x^{s/2} + sx^{(1-s)/2}]\,dx$$

$$= \frac{1}{2} + \psi(1) + \int_1^\infty x^{3/2}\psi'(x)[(1-s)x^{[(s-1)/2]-1} + sx^{-(s/2)-1}]\,dx$$

$$= \frac{1}{2} + \psi(1) + \int_1^\infty \frac{d}{dx}[x^{3/2}\psi'(x)(-2x^{(s-1)/2} - 2x^{-s/2})]\,dx$$

$$- \int_1^\infty \frac{d}{dx}[x^{3/2}\psi'(x)]\,[-2x^{(s-1)/2} - 2x^{-s/2}]\,dx$$

$$= \frac{1}{2} + \psi(1) - \psi'(1)[-2-2]$$

$$+ \int_1^\infty \frac{d}{dx}[x^{3/2}\psi'(x)]\,(2x^{(s-1)/2} + 2x^{-s/2})\,dx.$$

Now differentiation of

$$2\psi(x) + 1 = x^{-1/2}[2\psi(1/x) + 1]$$

easily gives

$$\tfrac{1}{2} + \psi(1) + 4\psi'(1) = 0,$$

and using this puts the formula in the final form

$$\xi(s) = 4\int_1^\infty \frac{d[x^{3/2}\psi'(x)]}{dx}\,x^{-1/4}\cosh\left[\frac{1}{2}\left(s-\frac{1}{2}\right)\log x\right]dx \tag{2}$$

or, as Riemann writes it,

$$\xi\left(\frac{1}{2}+it\right) = 4\int_1^\infty \frac{d[x^{3/2}\psi'(x)]}{dx}\,x^{-1/4}\cos\left(\frac{t}{2}\log x\right)dx.$$

If $\cosh[\frac{1}{2}(s-\frac{1}{2})\log x]$ is expanded in the usual power series $\cosh y = \frac{1}{2}(e^y + e^{-y}) = \sum y^{2n}/(2n)!$, formula (2) shows that

$$\xi(s) = \sum_{n=0}^\infty a_{2n}(s-\tfrac{1}{2})^{2n} \tag{3}$$

where

$$a_{2n} = 4\int_1^\infty \frac{d[x^{3/2}\psi'(x)]}{dx}\,x^{-1/4}\frac{(\frac{1}{2}\log x)^{2n}}{(2n)!}\,dx.$$

Riemann states that this series representation of $\xi(s)$ as an even function of $s-\frac{1}{2}$ "converges very rapidly," but he gives no explicit estimates and he does not say what role this series plays in the assertions which he makes next.

The two paragraphs which follow the formula (2) for $\xi(s)$ are the most difficult portion of Riemann's paper. Their goal is essentially to prove that

$\xi(s)$ can be expanded as an infinite product

$$\xi(s) = \xi(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right), \tag{4}$$

where ρ ranges† over the roots of the equation $\xi(\rho) = 0$. Now any *polynomial* $p(s)$ can be expanded as a finite product $p(s) = p(0) \prod_{\rho} [1 - (s/\rho)]$, where ρ ranges over the roots of the equation $p(\rho) = 0$ [except that the product formula for $p(s)$ is slightly different if $p(0) = 0$]; hence the product formula (4) states that $\xi(s)$ is *like a polynomial of infinite degree*. (Similarly, Euler thought of $\sin x$ as a "polynomial of infinite degree" when he conjectured, and finally proved, the formula $\sin \pi x = \pi x \prod_{n=1}^{\infty} [1 - (x/n)^2]$.) On the other hand, the statement that the series (3) converges "very rapidly" is also a statement that $\xi(s)$ is like a polynomial of infinite degree—a finite number of terms gives a very good approximation in any finite part of the plane. Thus there is some relationship between the series (3) and the product formula (4)—in fact it is *precisely* the rapid decrease of the coefficients a_n which Hadamard (in 1893) proved was necessary and sufficient for the validity of the product formula—but the steps of the argument by which Riemann went from the one to the other are obscure, to say the very least.

The next section contains a discussion of the distribution of the roots ρ of $\xi(\rho) = 0$, and the following section returns to the discussion of the product formula for $\xi(s)$.

1.9 THE ROOTS ρ OF ξ

In order to prove the convergence of the product $\xi(s) = \xi(0) \prod_{\rho} [1 - (s/\rho)]$, Riemann needed, of course, to investigate the distribution of the roots ρ of $\xi(\rho) = 0$. He begins by observing that the Euler product formula

$$\zeta(s) = \prod_{p} (1 - p^{-s})^{-1} \qquad (\operatorname{Re} s > 1)$$

shows immediately that $\zeta(s)$ has no zeros in the halfplane $\operatorname{Re} s > 1$ (because a convergent infinite product can be zero only if one of its factors is zero). Since $\xi(s) = \Pi(s/2)(s - 1)\pi^{-s/2}\zeta(s)$ and since the factors other than $\zeta(s)$ have only the simple zero at $s = 1$, it follows that none of the roots ρ of $\xi(\rho) = 0$ lie in the halfplane $\operatorname{Re} s > 1$. Since $1 - \rho$ is a root if and only if ρ is, this implies that none of the roots lie in the halfplane $\operatorname{Re} s < 0$ either, and hence that *all the roots ρ of $\xi(\rho) = 0$ lie in the strip* $0 \leq \operatorname{Re} \rho \leq 1$.

He then goes on to say that the number of roots ρ whose imaginary parts

†Here, and in the many formulas in the remainder of the book which involve sums or products over the roots ρ, it is understood that multiple roots—if there are any—are to be counted with multiplicities.

lie between 0 and T is approximately

$$\frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi} \tag{1}$$

and that the relative† error in this approximation is of the order of magnitude $1/T$. His "proof" of this is simply to say that the number of roots in this region is equal to the integral of $\xi'(s)\,ds/2\pi i\xi(s)$ around the boundary of the rectangle $\{0 \le \operatorname{Re} s \le 1,\ 0 \le \operatorname{Im} s \le T\}$ and that this integral is equal to (1) with a relative error T^{-1}. Unfortunately he gives no hint whatsoever of the method he used to estimate the integral. He himself was a master at evaluating and estimating definite integrals (see, for example, Section 1.14 or 7.4) and it is quite possible that he assumed that his readers would be able to carry out their own estimation of this integral, but if so he was wrong; it was not until 1905 that von Mangoldt succeeded in proving that Riemann's estimate was correct (see Section 6.7).

Riemann's next statement is even more baffling. He states that the number of roots *on the line* $\operatorname{Re} s = \frac{1}{2}$ is also "about" (1). He does not make precise the sense in which this approximation is true, but it is generally assumed that he meant that the relative error in the approximation of the number of zeros of $\xi(\frac{1}{2}+it)$ for $0 \le t \le T$ by (1) approaches zero as $T \to \infty$. He gives no indication of a proof at all, and no one since Riemann has been able to prove (or disprove) this statement. It was proved in 1914 that $\xi(\frac{1}{2}+it)$ has infinitely many real roots (Hardy [H3]), in 1921 that the number of real roots between 0 and T is at least KT for some positive constant K and all sufficiently large T (Hardy and Littlewood [H6]), in 1942 that this number is in fact at least $KT\log T$ for some positive K and all large T (Selberg, [S1]), and in 1914 that the number of complex roots t of $\xi(\frac{1}{2}+it)=0$ in the range $\{0 \le \operatorname{Re} t \le T, -\epsilon \le \operatorname{Im} t \le \epsilon\}$ is equal, for any $\epsilon > 0$, to (1) with a relative error which approaches zero as $T \to \infty$ (Bohr and Landau, [B8]). However, these partial results are still far from Riemann's statement. We can only guess what lay behind this statement (see Siegel [S4 p. 67], Titchmarsh [T8, pp. 213–214], or Section 7.8 of this book), but we do know that it led Riemann to conjecture an even stronger statement, namely, that *all* the roots lie on $\operatorname{Re} s = \frac{1}{2}$.

This is of course the famous "Riemann hypothesis." He says he considers it "very likely" that the roots all do lie on $\operatorname{Re} s = \frac{1}{2}$, but says that he was not able to prove it (which would seem to imply, incidentally, that he did feel he had rigorous proofs of the preceding two statements). Since it is not necessary for his main goal, which is the proof of his formula for the number of primes less than a given magnitude, he simply leaves the matter there—where it has remained ever since—and goes on to the product formula for $\xi(s)$.

†Titchmarsh, in an unfortunate lapse which he did not catch in the 21 years between the publication of his two books on the zeta function, failed to realize that Riemann meant the *relative* error and believed that Riemann had made a mistake at this point. See Titchmarsh [T8, p. 213].

1.10 THE PRODUCT REPRESENTATION OF $\xi(s)$

A recurrent theme in Riemann's work is the *global characterization of analytic functions by their singularities.*† Since the function $\log \xi(s)$ has logarithmic singularities at the roots ρ of $\xi(s)$ and no other singularities, it has the same singularities as the formal sum

$$\sum_{\rho} \log\left(1 - \frac{s}{\rho}\right). \tag{1}$$

Thus if this sum converges and if the function it defines is in some sense as well behaved near ∞ as $\log \xi(s)$ is, then it should follow that the sum (1) differs from $\log \xi(s)$ by at most an additive constant; setting $s = 0$ gives the value $\log \xi(0)$ for this constant, and hence exponentiation gives

$$\xi(s) = \xi(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right) \tag{2}$$

as desired. This is essentially the proof of the product formula (2) which Riemann sketches.

There are two problems associated with the sum (1). The first is the determination of the imaginary parts of the logarithms it contains. Riemann passes over this point without comment and, indeed, it is not a very serious problem. For any fixed s the ambiguity in the imaginary part of $\log[1 - (s/\rho)]$ disappears for large ρ; hence the sum (1) is defined except for a (finite) multiple of $2\pi i$ which drops out when one exponentiates (2). Furthermore, one can ignore the imaginary parts altogether; the real parts of the terms of (1) are unambiguously defined and their sum is a harmonic function which differs from Re $\log \xi(s)$ by a harmonic function without singularities, and if this difference function can be shown to be constant, it will follow that its harmonic conjugate is constant also.

The second problem associated with the sum (1) is its convergence. It is in fact a conditionally convergent sum, and the *order* of the series must be specified in order for the sum to be well determined. Roughly speaking the natural order for the terms would be the order of increasing $|\rho|$, or perhaps of increasing $|\rho - \frac{1}{2}|$, but specifically it suffices merely to stipulate that each

†See, for example, the Inauguraldissertation, especially article 20 (*Werke*, pp. 37–39) or part 3 of the introduction to the article "Theorie der Abel'schen Functionen," which is entitled "Determination of a function of a complex variable by boundary values and singularities [R1]." See also Riemann's introduction to Paper XI of the collected works, where he writes ". . . our method, which is based on the determination of functions by means of their singularities (*Unstetigkeiten und Unendlichwerden*) . . . [R1]." Finally, see Ahlfors [A3], the section at the end entitled "Riemann's point of view."