Hence
$$\textstyle\sum_1 = (1-z)\sum_{n=0}^{N} a_n + O\Big(|1-z|^2 \sum_{n=1}^{N} n|a_n|\Big)$$
$$= (1-z)s_N + o(|1-z|^2 N)$$
$$= (1-z)s_N + o(|1-z|). \tag{4}$$
From (1), (3), and (4) it follows that $s_N = o(1)$, and this proves the theorem.

7.64. Littlewood's extension of Tauber's theorem. We now pass to an extension of quite a different kind. In all the forms of Tauber's theorem so far considered, the condition $a_n = o(1/n)$ has played an apparently essential part. It was, however, discovered by Littlewood that it can be replaced by the more general condition $a_n = O(1/n)$. Here we shall restrict ourselves for the sake of simplicity to the real axis, though it is possible to prove the theorem for complex paths.

7.65. We use the following lemma:

If $f(x)$ is a real function with differential coefficients of the first two orders for $0 \leqslant x < 1$, and, as $x \to 1$,
$$f(x) = o(1), \qquad f''(x) = O\left\{\frac{1}{(1-x)^2}\right\},$$
then
$$f'(x) = o\left(\frac{1}{1-x}\right).$$

Let $x' = x + \delta(1-x)$, where $0 < \delta < \frac{1}{2}$. Then
$$f(x') = f(x) + \delta(1-x)f'(x) + \tfrac{1}{2}\delta^2(1-x)^2 f''(\xi),$$
where $x < \xi < x'$. Hence
$$(1-x)f'(x) = \frac{f(x')-f(x)}{\delta} + \tfrac{1}{2}\delta(1-x)^2 f''(\xi)$$
$$= \frac{f(x')-f(x)}{\delta} + O(\delta), \tag{1}$$
since $\quad f''(\xi) = O\left\{\frac{1}{(1-\xi)^2}\right\} = O\left\{\frac{1}{(1-x')^2}\right\} = O\left\{\frac{1}{(1-x)^2}\right\}.$

By first choosing δ sufficiently small, and then x sufficiently near to 1, the right-hand side of (1) can be made as small as we please. This proves the lemma.

7.66. Littlewood's theorem. *If $f(x) = \sum a_n x^n \to s$ as $x \to 1$, and $a_n = O(1/n)$, then $\sum a_n$ converges to the sum s.*

The proof* depends on the theorem of § 7.51, and in proving

* The original proof, Littlewood (3), was different.

that theorem we have really overcome the most serious difficulties. We may obviously suppose, without loss of generality, that the limit s is zero. Then

$$f(x)=\sum_{n=0}^{\infty} a_n x^n = o(1)$$

as $x \to 1$. Also, since $a_n = O(1/n)$,

$$f''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = O\left\{\sum_{n=2}^{\infty}(n-1)x^{n-2}\right\} = O\left\{\frac{1}{(1-x)^2}\right\}.$$

Hence, by the lemma,

$$f'(x)=\sum_{n=1}^{\infty} na_n x^{n-1} = o\left(\frac{1}{1-x}\right).$$

Suppose that $|na_n| \leqslant c$. Then

$$\sum_{n=1}^{\infty}\left(1-\frac{na_n}{c}\right)x^{n-1} = \frac{1}{1-x}-\frac{f'(x)}{c} \sim \frac{1}{1-x}.$$

But the coefficients in this series are all positive, and so, by the theorem of § 7.51,

$$\sum_{\nu=1}^{n}\left(1-\frac{\nu a_\nu}{c}\right) \sim n,$$

or

$$\sum_{\nu=1}^{n} \nu a_\nu = o(n). \tag{1}$$

This is an asymptotic formula for a finite sum, and so is a considerable step in the right direction. To get exactly the required result, still another argument is required.

Let w_n denote the left-hand side of (1) if $n > 0$, and let $w_0 = 0$. Then

$$\begin{aligned} f(x)-a_0 &= \sum_{n=1}^{\infty}\frac{w_n-w_{n-1}}{n}x^n = \sum_{n=1}^{\infty} w_n\left(\frac{x^n}{n}-\frac{x^{n+1}}{n+1}\right) \\ &= \sum_{n=1}^{\infty} w_n\left\{\frac{x^n-x^{n+1}}{n+1}+\frac{x^n}{n(n+1)}\right\} \\ &= (1-x)\sum_{n=1}^{\infty}\frac{w_n}{n+1}x^n + \sum_{n=1}^{\infty}\frac{w_n}{n(n+1)}x^n. \end{aligned}$$

Since $w_n = o(n)$, the first term on the right is $o(1)$ as $x \to 1$. Hence, since $f(x) \to 0$,

$$\sum_{n=1}^{\infty}\frac{w_n}{n(n+1)}x^n \to -a_0.$$

But $w_n/\{n(n+1)\} = o(1/n)$, and so, by the ordinary form of Tauber's theorem,

$$\sum_{n=1}^{\infty} \frac{w_n}{n(n+1)} = -a_0.$$

The left-hand side is

$$\lim_{N\to\infty} \sum_{n=1}^{N} \frac{w_n}{n(n+1)} = \lim_{N\to\infty} \sum_{n=1}^{N} w_n\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$= \lim_{N\to\infty} \left\{\sum_{n=1}^{N} \frac{w_n-w_{n-1}}{n} - \frac{w_N}{N+1}\right\} = \lim_{N\to\infty} \sum_{n=1}^{N} a_n,$$

and the theorem is therefore proved.

7.7. Partial sums of a power series.* The study of the partial sums of a power series is facilitated by the use of the formulae of the theory of Fourier series. We shall use some of these formulae, and quote them from Chapter XIII; but in each case where they are used here the proof is an immediate consequence of uniform convergence.

Let
$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (|z|<1),$$
and
$$s_n(z) = a_0 + a_1 z + \dots + a_n z^n.$$
Let
$$k(r,\theta) = \frac{1-r^2}{2(1-2r\cos\theta+r^2)} = \tfrac{1}{2} + r\cos\theta + r^2\cos 2\theta + \dots,$$
and
$$k_n(r,\theta) = \frac{1-r^2-2r^{n+1}\{\cos(n+1)\theta - r\cos n\theta\}}{2(1-2r\cos\theta+r^2)}$$
$$= \tfrac{1}{2} + r\cos\theta + \dots + r^n\cos n\theta.$$
Then
$$s_n(re^{i\theta}) = \frac{1}{\pi}\int_0^{2\pi} f(\rho e^{i(\theta-\phi)}) k_n\left(\frac{r}{\rho},\phi\right) d\phi \qquad (0<r<\rho<1). \quad (1)$$

This may be proved directly by term-by-term integration. It is a case of Parseval's formula (§ 13.54).

Also, by Dirichlet's integral (§ 13.2),

$$k_n(r,\theta) = \frac{1}{\pi}\int_0^{2\pi} \frac{\sin(n+\frac{1}{2})(\theta-\phi)}{2\sin\frac{1}{2}(\theta-\phi)} k(r,\phi)\, d\phi. \quad (2)$$

* Landau (2), (3), (4), and *Ergebnisse*, Ch. I.

We can thus express s_n as a repeated integral involving f and k,

$$s_n(re^{i\theta}) = \frac{1}{2\pi^2}\int_0^{2\pi} f(\rho e^{i(\theta-\phi)})\,d\phi \int_0^{2\pi} \frac{\sin(n+\frac{1}{2})(\phi-\psi)}{\sin\frac{1}{2}(\phi-\psi)}\,k\left(\frac{r}{\rho},\psi\right)d\psi. \quad (3)$$

We consider also the arithmetic means of the partial sums,

$$\sigma_n(z) = \{s_0(z)+s_1(z)+\ldots+s_{n-1}(z)\}/n.$$

By (1),
$$\sigma_n(re^{i\theta}) = \frac{1}{\pi}\int_0^{2\pi} f(\rho e^{i(\theta-\phi)})K_n\left(\frac{r}{\rho},\phi\right)d\phi, \quad (4)$$

where
$$K_n(r,\theta) = \frac{1}{n}\sum_{\nu=0}^{n-1} k_\nu(r,\theta);$$

and, by Fejér's integral (§ 13.31),

$$K_n(r,\theta) = \frac{1}{2n\pi}\int_0^{2\pi} \frac{\sin^2\frac{1}{2}n(\theta-\phi)}{\sin^2\frac{1}{2}(\theta-\phi)}\,k(r,\phi)\,d\phi. \quad (5)$$

7.71. Bounded power series. Suppose now that $f(z)$ is bounded in the unit circle.

If $|f(z)| \leqslant M$ for $|z|<1$, then $|\sigma_n(z)| \leqslant M$ for all values of n and $|z|<1$; and, conversely, if $|\sigma_n(z)| \leqslant M$ for all n and $|z|<1$, then $|f(z)| \leqslant M$.

It is clear from the above formulae that $k(r,\theta)$ and $K_n(r,\theta)$ are positive for $r<1$. Hence, if $|f(z)| \leqslant M$, it follows from (4) that

$$|\sigma_n(re^{i\theta})| \leqslant \frac{1}{\pi}\int_0^{2\pi} MK_n\left(\frac{r}{\rho},\phi\right)d\phi.$$

But the right-hand side is what σ_n reduces to in the case $f(z)=M$, viz. M. This proves the first part.

Again $s_n(z)$, and so also $\sigma_n(z)$, tends to $f(z)$ as $n\to\infty$. The second part follows at once from this.

7.72. The corresponding results for $s_n(z)$ are not so simple. This is due to the fact that k_n, unlike K_n, is not always positive. It is not necessarily true that $|s_n(z)| \leqslant M$ for all values of n and z. In fact it is known* that the upper bound of $|s_n(z)|$, for all functions $f(z)$ such that $|f(z)| \leqslant M$, tends to infinity with n. We have, however, the following result:

* Landau, *Ergebnisse*, § 2.

There is an absolute constant A such that

$$|s_n(z)| < AM \log n$$

for all functions $f(z)$ such that $|f(z)| \leqslant M$.

If $|f(z)| \leqslant M$, by § 7.7 (3),

$$|s_n(re^{i\theta})| \leqslant \frac{M}{2\pi^2} \int_0^{2\pi} k\left(\frac{r}{\rho}, \psi\right) d\psi \int_0^{2\pi} \left|\frac{\sin(n+\frac{1}{2})(\phi-\psi)}{\sin\frac{1}{2}(\phi-\psi)}\right| d\phi.$$

The inner integral is equal to

$$2\int_0^{\pi} \left|\frac{\sin(n+\frac{1}{2})\alpha}{\sin\frac{1}{2}\alpha}\right| d\alpha < 2\int_0^{1/(n+\frac{1}{2})} \frac{(n+\frac{1}{2})\alpha}{\sin\frac{1}{2}\alpha}\, d\alpha + 2\int_{1/(n+\frac{1}{2})}^{\pi} \frac{d\alpha}{\sin\frac{1}{2}\alpha}$$

$$= O(1) + O(\log n);$$

and, putting $n = 0$ in 7.7 (2),

$$\frac{1}{2\pi}\int_0^{2\pi} k\left(\frac{r}{\rho}, \psi\right) d\psi = \tfrac{1}{2}.$$

This proves the theorem.

7.73. It is easily seen that $s_n(z)$ is bounded in a circle of radius r' less than 1; for $k_n(r, \theta)$ is obviously bounded in such a circle. The upper bound for $s_n(z)$ depends on M and on r'. What is not so obvious is that we can choose r', independent of M, so that the upper bound is exactly M.

*If $|f(z)| \leqslant M$, then $|s_n(z)| \leqslant M$ for $|z| \leqslant \frac{1}{2}$.**

It is clear that

$$k_n(r, \phi) \geqslant \frac{1-r^2-2r^{n+1}(1+r)}{2(1-r)^2},$$

and if $r \leqslant \frac{1}{2}$, $n \geqslant 1$, the numerator is not less than

$$1-\tfrac{1}{4}-2\,.\,\tfrac{1}{4}(1+\tfrac{1}{2}) = 0.$$

Hence $k_n(r, \phi) \geqslant 0$ for $r \leqslant \frac{1}{2}$, and we can now proceed as in § 7.71. We have

$$|s_n(re^{i\theta})| \leqslant \frac{1}{\pi}\int_0^{2\pi} Mk_n\left(\frac{r}{\rho}, \phi\right) d\phi \qquad (r \leqslant \tfrac{1}{2}\rho),$$

and the right-hand side is what $s_n(z)$ reduces to when $f(z) = M$, viz. M. This proves the theorem.

* Fejér (5).

The number $\frac{1}{2}$ is the greatest number with this property. For consider the function

$$f(z) = \frac{z-a}{az-1} \quad (0 < a < 1).$$

Then $|f(e^{i\theta})| = 1$, so that $|f(z)| \leqslant 1$ for $|z| \leqslant 1$. Also

$$s_1(z) = a+(a^2-1)z,$$

$$s_1\left(-\frac{1}{2a}\right) = \frac{a^2+1}{2a} > 1,$$

and the point $-\frac{1}{2}/a$ where $s_1(z) > 1$ is arbitrarily near to $|z| = \frac{1}{2}$, since a is arbitrarily near to 1.

7.8. The zeros of partial sums.* Let

$$f(z) = a_0+a_1z+\ldots \qquad (a_0 \neq 0),$$

be a power series with radius of convergence 1, and let

$$s_n(z) = a_0+a_1z+\ldots+a_nz^n.$$

Then $s_n(z)$, being a polynomial of degree n, has n zeros.

If $f(z)$ has zeros inside the circle of convergence, then by Hurwitz's theorem (§ 3.45) every such zero is a limit-point of zeros of the polynomials $s_n(z)$.

Now consider the simplest function of the above type,

$$f(z) = \frac{1}{1-z} = 1+z+z^2+\ldots.$$

Here $$s_n(z) = 1+z+\ldots+z^n = \frac{1-z^{n+1}}{1-z}.$$

Hence $s_n(z)$ has zeros distributed evenly round the circle, and it is plain that every point of the circle is a limit-point of such zeros.

It is somewhat remarkable that the general case is so nearly like this simple case. This was discovered by Jentzsch, who proved that, *for every power series, every point of the circle of convergence is a limit-point of zeros of partial sums.*

We shall deduce this from some quite simple ideas depending on the theory of equations.

Let δ be a given positive number, n a number such that

$$|a_n| > \frac{|a_0|}{(1+\delta)^n}. \tag{1}$$

* Jentzsch (1).

This is true for arbitrarily large values of n, or the radius of convergence would be greater than 1.

Let $z_1, z_2, \dots, z_n$ be the zeros of the corresponding $s_n(z)$. Then

$$z_1 z_2 \dots z_n = (-1)^n a_0/a_n,$$

and so

$$|z_1 z_2 \dots z_n| < (1+\delta)^n.$$

Let $z_1, \dots, z_k$ be the zeros of $s_n(z)$ in the circle $|z| \leqslant 1-\delta$. By Hurwitz's theorem (§ 3.45), k is constant for sufficiently large values of n, and

$$|z_1 z_2 \dots z_k| > K,$$

where K depends on δ only.

Let $z_{n-p+1}, \dots, z_n$ be the zeros for which $|z| > 1+\epsilon$. Then

$$(1+\epsilon)^p < |z_{n-p+1} \dots z_n| = \left|\frac{z_1 z_2 \dots z_n}{z_1 \dots z_k . z_{k+1} \dots z_{n-p}}\right| < \frac{(1+\delta)^n}{K(1-\delta)^{n-p}}.$$

Hence

$$p < \frac{n\{\log(1+\delta)-\log(1-\delta)\}-\log K}{\log(1+\epsilon)-\log(1-\delta)} < \frac{An\delta-\log K}{A\epsilon}.$$

By choosing first ϵ, then δ, and then n, we can make p/n arbitrarily small.

Hence, for given δ, ϵ, and η, the number of zeros in the circle $|z| \leqslant 1+\epsilon$ is greater than $n(1-\eta)$, if n is a sufficiently large integer for which (1) *is true.*

7.81. It is clear from the above result that *the zeros of partial sums have at least one limit-point on the circle of convergence.*

We can obtain a little more information by considering the sum

$$\sum_{\nu=1}^{n} \frac{1}{z_\nu} = -\frac{a_1}{a_0}.$$

Putting $z_\nu = r_\nu e^{i\theta_\nu}$, we have

$$\sum_{\nu=1}^{n} \frac{\cos\theta_\nu}{r_\nu} = -\mathbf{R}\left(\frac{a_1}{a_0}\right). \tag{1}$$

If $\theta_\nu > \frac{1}{2}\pi+\alpha$, or $\theta_\nu < -\frac{1}{2}\pi-\alpha$, where $\alpha > 0$, for every ν, the left-hand side is less than

$$\frac{-n(1-\eta)\sin\alpha}{1+\epsilon}$$

by the above theorem. This is inconsistent with (1), if n is large enough. Hence there must be zeros in any angle including

$(-\frac{1}{2}\pi, \frac{1}{2}\pi)$. Similarly there must be zeros in any angle greater than π.

To prove Jentzsch's theorem we have to replace such an angle by an arbitrarily small one. This is done by using a conformal transformation which magnifies the effect of the zeros in the immediate neighbourhood of the point considered.

7.82. Let

$$w = \frac{\cos\lambda - z}{z\cos\lambda - 1}, \qquad z = \frac{w+\cos\lambda}{1+w\cos\lambda}, \tag{1}$$

where $0<\lambda<\frac{1}{2}\pi$, and where $f(\cos\lambda) \neq 0$. This transforms the unit circle in the z-plane into the unit circle in the w-plane. The point $z=1$ becomes $w=1$. The point $z=e^{i\lambda}$ becomes

$$w = \frac{\frac{1}{2}(e^{-i\lambda}-e^{i\lambda})}{\frac{1}{2}(e^{2i\lambda}-1)} = -e^{-i\lambda} = e^{i(\pi-\lambda)},$$

and similarly $z=e^{-i\lambda}$ becomes $w=e^{-i(\pi-\lambda)}$. Thus, if $z=re^{i\theta}$, $w=\rho e^{i\phi}$, the arc $-\lambda\leqslant\theta\leqslant\lambda$ of the unit circle is transformed into the arc $-\pi+\lambda\leqslant\phi\leqslant\pi-\lambda$.

The zeros z_ν of $s_n(z)$ are transformed into the zeros $w_\nu=\rho_\nu e^{i\phi_\nu}$ of the function

$$(1+w\cos\lambda)^n s_n\left(\frac{w+\cos\lambda}{1+w\cos\lambda}\right)$$
$$= s_n(\cos\lambda)+w\{n\cos\lambda\, s_n(\cos\lambda)+\sin^2\lambda\, s_n'(\cos\lambda)\}+\dots$$
$$= b_0+b_1w+\dots,$$

say; and corresponding to § 7.81 (1) we have

$$\sum_{\nu=1}^{n}\frac{\cos\phi_\nu}{\rho_\nu} = -\mathbf{R}\left(\frac{b_1}{b_0}\right) = -n\cos\lambda - \sin^2\lambda\,\mathbf{R}\left\{\frac{s_n'(\cos\lambda)}{s_n(\cos\lambda)}\right\}. \tag{2}$$

The last term tends to a limit as $n\to\infty$, since $s_n(\cos\lambda)\to f(\cos\lambda)$, which we have supposed is not 0, and $s_n'(\cos\lambda)\to f'(\cos\lambda)$. Hence as $n\to\infty$

$$\sum_{\nu=1}^{n}\frac{\cos\phi_\nu}{\rho_\nu} \sim -n\cos\lambda. \tag{3}$$

Suppose now that the region of the w-plane

$$1-\epsilon<\rho<1+\epsilon, \qquad -(\pi-\lambda+\alpha)<\phi<\pi-\lambda+\alpha, \tag{4}$$

where $0<\epsilon<1$, $0<\alpha<\lambda$, is free from zeros. Put

$$\sum_{\nu=1}^{n}\frac{\cos\phi_\nu}{\rho_\nu} = \sum_{\rho_\nu\leqslant 1-\epsilon} + \sum_{1-\epsilon<\rho_\nu<1+\epsilon} + \sum_{\rho_\nu\geqslant 1+\epsilon} = \Sigma_1+\Sigma_2+\Sigma_3. \tag{5}$$

Since $\rho=1$ corresponds to $r=1$, it follows from considerations of continuity that the circles $\rho = 1-\epsilon$, $\rho = 1+\epsilon$ correspond to curves (in fact circles) inside and outside $r = 1$ respectively, which can be made as near to it as we please by taking ϵ small enough.

The number of terms in $\sum_1$ is less than $K = K(\delta, \epsilon, \lambda)$; and ρ_ν has a positive lower bound, since the zeros of $s_n(\cos\lambda)$ in question tend to zeros of $f(\cos\lambda)$. Hence

$$\sum\nolimits_1 < K. \tag{6}$$

The number of terms in $\sum_3$ is, by § 7.8, less than ηn, where $\eta = \eta(n, \delta, \epsilon, \lambda)$ tends to 0 as $n \to \infty$ through a certain sequence of values. Hence

$$\sum\nolimits_3 < \frac{\eta n}{1+\epsilon}. \tag{7}$$

In $\sum_2$ the number of terms exceeds $n(1-\eta)-K$, and by hypothesis $\cos\phi_\nu < -\cos(\lambda-\alpha)$ for each term of this sum. Hence

$$\sum\nolimits_2 < -\frac{n(1-\eta)-K}{1+\epsilon}\cos(\lambda-\alpha). \tag{8}$$

From (5), (6), (7), (8) it follows that

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{\nu=1}^{n} \frac{\cos\phi_\nu}{\rho_\nu} \leqslant -\frac{\cos(\lambda-\alpha)}{1+\epsilon}.$$

This contradicts (3) if $\alpha > 0$ and ϵ is small enough. There are therefore zeros in the region (4), and hence, since ϵ and α may be as small as we please, in any region containing the arc $\rho = 1$, $-\pi+\lambda < \phi < \pi-\lambda$. Hence, in the z-plane, there are zeros in any region containing the arc $r = 1$, $-\lambda < \theta < \lambda$. Finally, since λ may be as small as we please, it follows that $z = 1$ is a limit-point of zeros. Similarly every point on the unit circle is a limit-point of zeros.

MISCELLANEOUS EXAMPLES

1. If $|a_n/a_{n+1}| \to R$, then the radius of convergence of $\sum a_n z^n$ is R.

2. If
$$\left|\frac{a_n}{a_{n+1}}\right| = \left\{1+\frac{c}{n}+o\left(\frac{1}{n}\right)\right\}R$$
where $c > 1$, then $\sum a_n z^n$ converges absolutely everywhere on its circle of convergence.

3. If $a_n/a_{n+1} \to 1$, then
$$\lim_{n\to\infty} \frac{s_n(z)}{a_n z^n} = \frac{z}{z-1}$$
uniformly for $|z| \geqslant 1+\delta > 1$. Hence show that all the limit-points of the zeros of partial sums are inside or on the unit circle. [S. Izumi (1).]

4. Show that as $x \to 1$

$$\sum_{n=0}^{\infty} x^{n^2} \sim \frac{1}{2}\sqrt{\left(\frac{\pi}{1-x}\right)},$$

and that, as $z \to e^{2i\pi p/q}$ along the radius vector,

$$\sum_{n=0}^{\infty} z^{n^2} \sim \frac{1}{2q}\sqrt{\left(\frac{\pi}{1-|z|}\right)} \sum_{r=0}^{q-1} e^{2i\pi pr^2/q}.$$

5. If $a_n \sim \log n$, then, as $x \to 1$,

$$\sum_{n=0}^{\infty} a_n x^n \sim \frac{1}{1-x} \log \frac{1}{1-x}.$$

$\left[\text{The right-hand side is } \sum \left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)x^n\right]$.

6. If $a_n \sim 1/\log n$, then, as $x \to 1$,

$$\sum_{n=0}^{\infty} a_n x^n \sim \frac{1}{(1-x)\log\{1/(1-x)\}}.$$

[If $\sum_p$ denotes a sum over the range $\epsilon p/\log(1/x) < n \leqslant \epsilon(p+1)/\log(1/x)$, then

$$\sum\nolimits_p \frac{x^n}{\log n} < \frac{\epsilon e^{-\epsilon p}}{\log(1/x)\log\{1/\log(1/x)\}}, \text{ etc.]}$$

7. Show that if $a_n \geqslant 0$, and

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \sim \frac{1}{(1-x)^2},$$

then

$$s_n \sim \tfrac{1}{2}n^2.$$

[We have

$$f_1(x) = \int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1} \sim \frac{1}{1-x}.$$

Hence

$$\sum_{\nu=0}^{n} \frac{a_\nu}{\nu+1} \sim n,$$

and the result then follows by partial summation.]

8. Generally, if $a_n \geqslant 0$, and $f(x) \sim (1-x)^{-\alpha}$, where $\alpha > 1$, then

$$s_n \sim \frac{n^\alpha}{\Gamma(\alpha+1)}.$$

[We have

$$f_{\alpha-1}(x) = \frac{1}{\Gamma(\alpha-1)} \int_0^x (x-t)^{\alpha-2} f(t)\,dt = \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+1)}{\Gamma(n+\alpha)} x^{\alpha+n-1} \sim \sum_{n=1}^{\infty} \frac{a_n x^n}{n^{\alpha-1}},$$

and, on the other hand,

$$f_{\alpha-1}(x) \sim \frac{1}{\Gamma(\alpha-1)} \int_0^x (x-t)^{\alpha-2}(1-t)^{-\alpha}\,dt$$

$$= \frac{x^{\alpha-1}}{\Gamma(\alpha)(1-x)} \sim \frac{1}{\Gamma(\alpha)(1-x)}.$$

Hence
$$\sum_{\nu=1}^{n} \frac{a_\nu}{\nu^{\alpha-1}} \sim \frac{n}{\Gamma(\alpha)},$$
and the result now follows without difficulty.]

9. If $f(z)$ is regular in a region including the origin, and $f(0) = 1$, then $f(z)$ can be expanded in the form
$$f(z) = (1+a_1 z)(1+a_2 z^2)(1+a_3 z^3)...$$
for sufficiently small values of z.

[Ritt (1): Assuming an expansion of the above form, we write $f'(z)/f(z) = c_1+c_2 z+...$, and determine the numbers a_n in succession by equating coefficients in the equation
$$c_1+c_2 z+... = \sum_{n=1}^{\infty} \frac{n a_n z^{n-1}}{1+a_n z^n}.$$
If $\mu_n = \max_{\nu \leqslant n} |a_\nu|^{1/\nu}$, we deduce from the recurrence relation that $\mu_n^n \leqslant n\mu_{\frac{1}{2}n}^n + |c_n|$. Hence μ_n is bounded, and the process can be justified.]

10. Show that the circle of convergence of the above product is the same as that of the series $\sum a_n z^n$, but that the power series for $f(z)$ may have a larger circle of convergence.

11. If each of the series
$$\sum_{n=0}^{\infty} a_n z^n, \qquad \sum_{n=0}^{\infty} b_n z^n, \qquad \sum_{n=0}^{\infty} a_n b_n z^n$$
has a radius of convergence equal to 1, then so have the series
$$\sum_{n=0}^{\infty} a_n b_n^2 z^n, \qquad \sum_{n=0}^{\infty} a_n^2 b_n z^n.$$

12. If each of the series
$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=0}^{\infty} b_n z^n, \qquad F(z) = \sum_{n=0}^{\infty} a_n b_n z^n$$
has a radius of convergence equal to 1, if $f(z)$ is regular on its circle of convergence except at $z = 1$, and $b_n \geqslant 0$ for all values of n, then $F(z)$ has a singularity at $z = 1$.

[Bohnenblust (1): the series
$$\phi(z) = \sum_{n=0}^{\infty} |a_n|^2 b_n z^n$$
has the radius of convergence 1, and so by § 7.21 has a singularity at $z = 1$. By Hadamard's multiplication theorem (§ 4.6) the singularities of $\phi(z)$ are products of those of $F(z)$ and of
$$\bar{f}(z) = \sum_{n=0}^{\infty} \bar{a}_n z^n.$$
Thus $1 = \alpha\beta$, where α is a singularity of $F(z)$, β of $\bar{f}(z)$; and β must be 1. Hence $\alpha = 1$.]

13. If $f(z) = \sum a_n z^n$ is regular on its circle of convergence except at z_0, then every series which consists of a selection of terms from $\sum a_n z^n$, and which has the same radius of convergence, has a singularity at z_0.

14. Show that the theorem of § 7.21 is still true if the coefficients a_n are complex, provided that $|\arg a_n| < \alpha < \frac{1}{2}\pi$ for all values of n.

[We have $|a_n| \leq \sec\alpha \, \mathbf{R}a_n$.]

15. The function
$$\sum_{n=1}^{\infty} \frac{z^{2^n}}{n^2}$$
is continuous in and on the unit circle; but every point of the circle is a singularity.

16. If $f(z)$ is bounded in the unit circle, then $\sum |a_n|^2$ is convergent. [See § 2.5.]

The following examples are on the border-line between theory of power series and theory of real functions. It seems most convenient to insert them here, but some of them assume the theory of mean convergence given in § 12.5.

17. If $\sum |a_n|^2$ is convergent, then
$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})-f(r'e^{i\theta})|^2\, d\theta = \sum_{n=0}^{\infty} |a_n|^2 (r^n - r'^n)^2.$$
Hence, show that, as $r \to 1$, $f(re^{i\theta})$ converges in mean to a limit-function $F(\theta)$ of the class $L^2(0, 2\pi)$.

18. If $f(z) = u+iv$, $F(\theta) = U+iV$ in the previous example, show that Poisson's formulae
$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2}\, U(\phi)\, d\phi,$$
$$v(r,\theta) - v(0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{2r\sin(\theta-\phi)}{1-2r\cos(\theta-\phi)+r^2}\, U(\phi)\, d\phi$$
hold for $r < 1$.

19. Show that, in the above examples, $u(r,\theta) \to U(\theta)$ as $r \to 1$ for every value of θ in the Lebesgue set of $U(\theta)$. Deduce that $f(re^{i\theta}) \to F(\theta)$ as $r \to 1$ for almost all values of θ.

[The analysis is similar to that of § 13.34.]

20. Show that a bounded analytic function tends to a limit radially at almost all points of its circle of convergence.

21. If $U(\theta) \geq 0$ for all values of θ, then $u(r,\theta) \geq 0$ for all values of r and θ.

22. If $f(z)$ is regular and bounded for $|z| < 1$, and $f(z) \to 0$ as $r \to 1$ throughout an interval of values of θ, then $f(z)$ is identically zero.

[If $0 \leqslant \theta \leqslant 2\pi/p$ is part of the interval, consider the function

$$g(z) = f(z)f(ze^{2i\pi/p})...f(ze^{2(p-1)i\pi/p}).]$$

23. More generally, if $f(z)$ is bounded and tends to zero radially for values of θ in a set of positive measure, then $f(z)$ is identically zero.

[See Bieberbach, ii, p. 156. Let E be the set where $f(z) \to 0$, and let $m(E) = \mu > 0$. Let $u_1(\theta) = \lambda/\mu$ in E, and $= -\lambda/(2\pi-\mu)$ in CE. Let $g(z)$ be the corresponding analytic function defined by the formulae of ex. 18. Then $g(0) = 0$. Let $h(z) = e^{g(z)}$, so that $h(0) = 1$. Then

$$f(0) = f(0)h(0) = \frac{1}{2\pi i}\int\limits_{|z|=1} f(z)h(z)\,\frac{dz}{z} = \frac{1}{2\pi i}\int_{CE} f(z)h(z)\,\frac{dz}{z},$$

$$|f(0)| < Ae^{-\lambda/(2\pi-\mu)}.$$

Since λ may be as large as we please, $f(0) = 0$. Applying the same argument to $f(z)/z$, $f'(0) = 0$, etc.]

24. If $U(\theta)$ is any function integrable in the Lebesgue sense, and

$$f(z) = \frac{1}{2\pi}\int\limits_0^{2\pi} \frac{1+ze^{-i\phi}}{1-ze^{i\phi}}\,U(\phi)\,d\phi \qquad (|z| < 1),$$

then $f(z)$ tends to a limit as $r \to 1$ for almost all values of θ.

[Plessner (1): We may suppose without loss of generality that $U(\phi) \geqslant 0$. Then $\mathbf{R}\{f(z)\} \geqslant 0$. Hence the function $1/\{1+f(z)\}$ is bounded in the unit circle, and so tends to a limit for almost all values of θ. This limit is different from zero almost everywhere.]

CHAPTER VIII

INTEGRAL FUNCTIONS

8.1. Factorization of integral functions. An integral function is an analytic function which has no singularities except at infinity. The simplest such functions are polynomials. A polynomial $f(z)$ which has zeros at the points $z_1, z_2,\ldots, z_n$ can be factorized in the form

$$f(z) = f(0)\left(1-\frac{z}{z_1}\right)\left(1-\frac{z}{z_2}\right)\ldots\left(1-\frac{z}{z_n}\right).$$

The zeros of integral functions in general are equally important. An integral function which is not a polynomial may have an infinity of zeros z_n; and the product

$$\prod\left(1-\frac{z}{z_n}\right)$$

taken over these zeros may be divergent. So we cannot always factorize an integral function in the same way as a polynomial, and we have to consider less simple factors than $1-z/z_n$.

The expressions

$$E(u,0) = 1-u, \qquad E(u,p) = (1-u)e^{u+\frac{u^2}{2}+\ldots+\frac{u^p}{p}} \quad (p = 1, 2,\ldots),$$

are called *primary factors*. Each primary factor vanishes when $u = 1$; but the behaviour of $E(u,p)$ as $u \to 0$ depends on p. For $|u| < 1$,

$$\log E(u,p) = -\frac{u^{p+1}}{p+1}-\frac{u^{p+2}}{p+2}-\ldots.$$

Hence, if $k > 1$, and $|u| \leqslant 1/k$,

$$|\log E(u,p)| \leqslant |u|^{p+1}+|u|^{p+2}+\ldots$$
$$\leqslant |u|^{p+1}\left\{1+\frac{1}{k}+\frac{1}{k^2}+\ldots\right\} = \frac{k}{k-1}|u|^{p+1}.$$

It is this inequality which determines the convergence of a product of primary factors.

8.11. The theorem of Weierstrass. If $f(z)$ is an integral function, what can we say about its zeros?

Since $f(z)$ is analytic except at infinity, the zeros can have no limit-point except at infinity. In general, this is all that we can say. This follows from the following theorem of Weierstrass.

Given any sequence of numbers $z_1, z_2,\ldots$ whose sole limiting-point *is at infinity, there is an integral function with zeros at these points, and these points only.*

We may suppose the zeros arranged so that $|z_1| \leqslant |z_2| \leqslant \ldots$. Let $|z_n| = r_n$, and let $p_1, p_2, \ldots$ be a sequence of positive integers such that the series

$$\sum_{n=1}^{\infty} \left(\frac{r}{r_n}\right)^{p_n}$$

is convergent for all values of r. It is always possible to find such a sequence; for $r_n \to \infty$, since otherwise the zeros would have a limiting-point other than infinity; and we may take $p_n = n$, since

$$\left(\frac{r}{r_n}\right)^n < \frac{1}{2^n}$$

for $r_n > 2r$, and the series is therefore convergent.

Let
$$f(z) = \prod_{n=1}^{\infty} E\left(\frac{z}{z_n}, p_n - 1\right). \tag{1}$$

This function has the required property; for, if $|z_n| > 2|z|$,

$$\left|\log E\left(\frac{z}{z_n}, p_n - 1\right)\right| \leqslant 2\left(\frac{r}{r_n}\right)^{p_n}; \tag{2}$$

hence the series
$$\sum_{|z_n| > 2R} \log E\left(\frac{z}{z_n}, p_n - 1\right)$$

is uniformly convergent for $|z| \leqslant R$, and hence* so is the product

$$\prod_{|z_n| > 2R} E\left(\frac{z}{z_n}, p_n - 1\right).$$

Hence $f(z)$ is regular for $|z| \leqslant R$, and its only zeros in this region are those of

$$\prod_{|z_n| \leqslant 2R} E\left(\frac{z}{z_n}, p_n - 1\right),$$

i.e. the points $z_1, z_2, \ldots$. Since R may be as large as we please, this proves the theorem.

The function $f(z)$ is, of course, not uniquely determined by the zeros, since we have a wide choice of the numbers p_n.

8.12. It is possible to factorize any given integral function in the following way.

If $f(z)$ is an integral function, and $f(0) \neq 0$, then

$$f(z) = f(0)\, P(z)\, e^{g(z)}$$

* See § 1.43, end.

where $P(z)$ is a product of primary factors, and $g(z)$ is an integral function.

We form $P(z)$ as in the above theorem from the zeros of $f(z)$. Let

$$\phi(z) = \frac{f'(z)}{f(z)} - \frac{P'(z)}{P(z)}.$$

Then $\phi(z)$ is an integral function, since the poles of one term are cancelled by those of the other. Hence also

$$g(z) = \int_0^z \phi(t)\,dt = \log f(z) - \log f(0) - \log P(z)$$

is an integral function, and the result stated follows on taking exponentials.

If $f(z)$ has a zero of order p at $z = 0$, a factor z^p has to be inserted.

This factorization is not unique.

8.2. Functions of finite order. The general factorization theorem is not precise enough to be of much use; in general the numbers p_n increase indefinitely with n, and we can say little about the function $g(z)$. There is, however, one case in which we can put the theorem into a perfectly definite form, that of functions of finite order.

An integral function $f(z)$ is said to be of finite order if there is a positive number A such that, as $|z| = r \to \infty$,

$$f(z) = O(e^{r^A}).$$

The lower bound ρ of numbers A for which this is true is called the *order* of the function. Thus, if $f(z)$ is of order ρ,

$$f(z) = O(e^{r^{\rho+\epsilon}})$$

for every positive value of ϵ, but not for any negative value. In this, and similar statements throughout the chapter, ϵ is thought of as taking arbitrarily small values, and the constant implied in the O depends in general on ϵ. If it were independent of ϵ, we could replace ϵ by 0 in the formula.

Functions of finite order are, after polynomials, the simplest integral functions. A polynomial is of order zero; some of the properties of functions of small order are similar to those of polynomials.

Many familiar functions are easily seen to be of finite order; e^z is of order 1; so are $\sin z$ and $\cos z$; $\cos \sqrt{z}$ is an integral function of order $\frac{1}{2}$; e^{z^k} is an integral function of order k, if k is a positive integer (if k is not an integer, it is not an integral function). The function e^{e^z} is of infinite order.

In what follows we shall suppose generally that $f(0)$ is not 0. This simplifies the analysis a little, and division by a factor z^k does not affect the order.

8.21. The function $n(r)$. Let $n(r)$ denote the number of zeros z_1, z_2,... of an integral function $f(z)$ for which $|z_n| \leqslant r$. Then $n(r)$ is a non-decreasing function of r which is constant in intervals; it is zero for $r < |z_1|$, if $f(0)$ is not zero.

This function is, as we have seen in § 3.61, connected with $f(z)$ by means of Jensen's formula. In fact

$$\int_0^r \frac{n(x)}{x}\,dx = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta - \log|f(0)|. \tag{1}$$

If $f(z)$ is an integral function, this holds for all values of r.

If $f(z)$ is of order ρ, then $n(r) = O(r^{\rho+\epsilon})$. For

$$\log|f(re^{i\theta})| < Kr^{\rho+\epsilon},$$

K depending on ϵ only. Hence, by (1),

$$\int_0^{2r} \frac{n(x)}{x}\,dx < Kr^{\rho+\epsilon}. \tag{2}$$

But, since $n(r)$ is non-decreasing,

$$\int_r^{2r} \frac{n(x)}{x}\,dx \geqslant n(r)\int_r^{2r} \frac{dx}{x} = n(r)\log 2.$$

Hence
$$n(r) \leqslant \frac{1}{\log 2}\int_0^{2r} \frac{n(x)}{x}\,dx < Kr^{\rho+\epsilon}$$

by (2).

We may thus say, roughly, that the higher the order of a function is, the more zeros it may have in a given region.

8.22. *If r_1, r_2,... are the moduli of the zeros of $f(z)$, then the series $\sum r_n^{-\alpha}$ is convergent if $\alpha > \rho$.*

Let β be a number between α and ρ. Then $n(r) < Ar^{\beta}$. Putting $r = r_n$, this gives

$$n < Ar_n^{\beta}.$$

Hence $$r_n^{-\alpha} < An^{-\alpha/\beta},$$
and the result follows.

The lower bound of positive numbers α for which $\sum r_n^{-\alpha}$ is convergent is called the *exponent of convergence of the zeros*, and is denoted by ρ_1. What we have just proved is that $\rho_1 \leqslant \rho$. We may have $\rho_1 < \rho$; for example, if $f(z) = e^z$, $\rho = 1$; but there are no zeros, so that $\rho_1 = 0$.

Notice that $\rho_1 = 0$ for any function with a finite number of zeros; thus $\rho_1 > 0$ implies that there are an infinity of zeros.

8.23. Canonical products. An important consequence of the above theorem is that, if $f(z)$ is of finite order, then there is an integer p, independent of n, such that the product

$$\prod_{n=1}^{\infty} E\left(\frac{z}{z_n}, p\right) \tag{1}$$

is convergent for all values of z; for by 8.11 (1), with $p_n = p+1$, this product is convergent if

$$\sum \left(\frac{r}{r_n}\right)^{p+1} \tag{2}$$

is convergent;* and this is true for all values of r if $p+1 > \rho_1$, and so it is certainly true if $p+1 > \rho$.

If p is the smallest integer for which (2) is convergent, the product (1) is called the *canonical product* formed with the zeros of $f(z)$; and p is called its *genus*.

If ρ_1 is not an integer, then $p = [\rho_1]$; if ρ_1 is an integer, $p = \rho_1$ if $\sum r_n^{-\rho_1}$ is divergent, while $p = \rho_1 - 1$ if it is convergent. In any case
$$p \leqslant \rho_1 \leqslant \rho.$$

8.24. Hadamard's factorization theorem. *If $f(z)$ is an integral function of order ρ, with zeros $z_1, z_2, \ldots$ ($f(0) \neq 0$), then*
$$f(z) = e^{Q(z)} P(z),$$
where $P(z)$ is the canonical product formed with the zeros of $f(z)$, and $Q(z)$ is a polynomial of degree not greater than ρ.

We can now take the $P(z)$ of § 8.12 to be the canonical product. It follows from the factorization theorem of § 8.12 that there is an expression for $f(z)$ in the above form, in which $Q(z)$ is an integral function. What we have to prove is that in this case $Q(z)$ is a polynomial.

* Compare § 1.43, ex. (vii).

Let $\nu = [\rho]$, so that $p \leqslant \nu$. Taking logarithms and differentiating $\nu+1$ times, we obtain

$$\left(\frac{d}{dz}\right)^{\nu}\left\{\frac{f'(z)}{f(z)}\right\} = Q^{(\nu+1)}(z) - \nu! \sum_{n=1}^{\infty} \frac{1}{(z_n - z)^{\nu+1}}.$$

To prove that $Q(z)$ is a polynomial of degree ν at most, we have to prove that $Q^{(\nu+1)}(z) = 0$.

Let
$$g_R(z) = \frac{f(z)}{f(0)} \prod_{|z_n| \leqslant R} \left(1 - \frac{z}{z_n}\right)^{-1}.$$

Since $|1 - z/z_n| \geqslant 1$ for $|z| = 2R$, $|z_n| \leqslant R$, we have

$$|g_R(z)| \leqslant |f(z)/f(0)| = O(e^{(2R)^{\rho+\epsilon}}) \tag{1}$$

for $|z| = 2R$. Since $g_R(z)$ is an integral function, this holds for $|z| < 2R$ also.

Let $h_R(z) = \log g_R(z)$, the logarithm being determined so that $h_R(0) = 0$. Then $h_R(z)$ is regular for $|z| \leqslant R$, and, by (1),

$$\mathbf{R}\{h_R(z)\} < KR^{\rho+\epsilon}. \tag{2}$$

Hence, by § 5.51,

$$|h_R^{(\nu+1)}(z)| \leqslant \frac{2^{\nu+2}(\nu+1)!\,R}{(R-r)^{\nu+2}} KR^{\rho+\epsilon}$$

for $|z| = r < R$; and for $|z| = \frac{1}{2}R$ this gives

$$h_R^{(\nu+1)}(z) = O(R^{\rho+\epsilon-\nu-1}). \tag{3}$$

Hence
$$\begin{aligned} Q^{(\nu+1)}(z) &= h_R^{(\nu+1)}(z) + \nu! \sum_{|z_n| > R} \frac{1}{(z_n - z)^{\nu+1}} \\ &= O(R^{\rho+\epsilon-\nu-1}) + O\Big(\sum_{|z_n| > R} |z_n|^{-\nu-1}\Big) \end{aligned}$$

for $|z| = \frac{1}{2}R$, and so also for $|z| < \frac{1}{2}R$. The first term on the right tends to 0 as $R \to \infty$ if ϵ is small enough, since $\nu+1 > \rho$; and the second term tends to 0 since $\sum |z_n|^{-\nu-1}$ is convergent. Since the left-hand side is independent of R it must be zero, and the theorem follows.*

8.25. *The order of a canonical product is equal to the exponent of convergence of its zeros.*

We know that, for any function, $\rho_1 \leqslant \rho$. Hence we have to prove that, for a canonical product $P(z)$, $\rho \leqslant \rho_1$. Let $r_1, r_2, \ldots$

* Hadamard (2). This proof is due to Landau (5). For an alternative proof see § 8.72.

be the moduli of the zeros, and k a constant greater than unity. Let

$$\log|P(z)| = \sum_{r_n \leqslant kr} \log\left|E\left(\frac{z}{z_n}, p\right)\right| + \sum_{r_n > kr} \log\left|E\left(\frac{z}{z_n}, p\right)\right| = \sum\nolimits_1 + \sum\nolimits_2.$$

In $\sum_2$ we use the inequality 8.11 (2), and obtain

$$\sum\nolimits_2 = O\left\{\sum_{r_n > kr} \left(\frac{r}{r_n}\right)^{p+1}\right\} = O\left\{r^{p+1} \sum_{r_n > kr} \frac{1}{r_n^{p+1}}\right\}.$$

If $p = \rho_1 - 1$, this is $O(r^{p+1}) = O(r^{\rho_1})$. Otherwise $\rho_1 + \epsilon < p+1$ if ϵ is small enough, and then

$$r^{p+1} \sum_{r_n > kr} r_n^{-p-1} = r^{p+1} \sum_{r_n > kr} r_n^{\rho_1+\epsilon-p-1} r_n^{-\rho_1-\epsilon}$$
$$< r^{p+1}(kr)^{\rho_1+\epsilon-p-1} \sum r_n^{-\rho_1-\epsilon} = O(r^{\rho_1+\epsilon}).$$

Again in $\sum_1$ we have terms involving $E(u,p)$, where $|u| \geqslant 1/k$, so that

$$\log|E(u,p)| \leqslant \log(1+|u|) + |u| + \ldots + \frac{|u|^p}{p} < K|u|^p,$$

where K depends on k only. Hence

$$\sum\nolimits_1 < O\left(r^p \sum_{r_n \leqslant kr} r_n^{-p}\right) = O\left(r^p \sum_{r_n \leqslant kr} r_n^{\rho_1+\epsilon-p} r_n^{-\rho_1-\epsilon}\right)$$
$$= O\{r^p (kr)^{\rho_1+\epsilon-p} \sum r_n^{-\rho_1-\epsilon}\} = O(r^{\rho_1+\epsilon}).$$

Hence $$\log|P(z)| < O(r^{\rho_1+\epsilon}),$$

and the result follows.

8.26. *If ρ is not an integer, $\rho_1 = \rho$.*

We have in any case $\rho_1 \leqslant \rho$. Suppose that $\rho_1 < \rho$. Then $P(z)$ is of order ρ_1, i.e. of order less than ρ. Also, if $Q(z)$ is of degree q, $e^{Q(z)}$ is of order q; and $q \leqslant \rho$, and in this case $q < \rho$, since q is an integer and ρ is not. Hence $f(z)$ is the product of two functions, each of order less than ρ. Hence $f(z)$ is of order less than ρ, which gives a contradiction. Hence $\rho_1 = \rho$.

In particular, a function of non-integral order must have an infinity of zeros. In fact, if the order is not an integer, the function is dominated by the canonical product $P(z)$; whereas, if the order is an integer, $P(z)$ may reduce to a polynomial or a constant, and the order then depends entirely on the factor $e^{Q(z)}$.

In any case, since $P(z)$ is of order ρ_1, and $e^{Q(z)}$ of order q, we have

$$\rho = \max(q, \rho_1).$$

8.27. Genus. The genus of the integral function $f(z)$ is the greater of the two integers p and q, and is therefore an integer.

Since $p \leqslant \rho$ and $q \leqslant \rho$, the genus does not exceed the order. The actual determination of the genus of a given function is sometimes not easy.

Example. Prove that the genus is not less than $\rho-1$.

8.3. The coefficients in the expansion of a function of finite order. *A necessary and sufficient condition that*

$$f(z)=\sum_{n=0}^{\infty} a_n z^n \tag{1}$$

should be an integral function of finite order ρ is that

$$\varliminf_{n\to\infty} \frac{\log(1/|a_n|)}{n\log n}=\frac{1}{\rho}.$$

The argument depends on the fact that $\sum |a_n z^n|$ does not differ very much from its greatest term, and that $|f(z)|$ lies between the two. This is further illustrated by the example which follows.

(i) Let
$$\varliminf \frac{\log(1/|a_n|)}{n\log n}=\mu,$$

where μ is 0, positive, or infinite. Then, for every positive ϵ,

$$\log(1/|a_n|)>(\mu-\epsilon)n\log n \qquad (n>n_0),$$

i.e.
$$|a_n|<n^{-n(\mu-\epsilon)}.$$

If $\mu>0$, it follows that (1) converges for all values of z, so that $f(z)$ is an integral function. Also, if μ is finite,

$$|f(z)|<Ar^{n_0}+\sum_{n=n_0+1}^{\infty} r^n n^{-n(\mu-\epsilon)} \qquad (r>1).$$

Let $\sum_1$ denote the part of the last series for which $n\leqslant(2r)^{\frac{1}{\mu-\epsilon}}$, $\sum_2$ the remainder. Then in $\sum_1$

$$r^n\leqslant \exp\{(2r)^{\frac{1}{\mu-\epsilon}}\log r\},$$

so that

$$\sum\nolimits_1<\exp\{(2r)^{\frac{1}{\mu-\epsilon}}\log r\}\sum n^{-n(\mu-\epsilon)}<K\exp\{(2r)^{\frac{1}{\mu-\epsilon}}\log r\}.$$

In $\sum_2$, $rn^{-(\mu-\epsilon)}<\frac{1}{2}$, so that

$$\sum\nolimits_2<\sum(\tfrac{1}{2})^n<1.$$

Hence
$$|f(z)|<K\exp\{(2r)^{\frac{1}{\mu-\epsilon}}\log r\},$$

i.e. $\rho\leqslant 1/(\mu-\epsilon)$. Making $\epsilon\to 0$, $\rho\leqslant 1/\mu$. In the case $\mu=\infty$, the argument, with an arbitrarily large number instead of μ, shows that $\rho = 0$.

On the other hand, given ϵ, there is a sequence of values of n for which

$$\log(1/|a_n|) < (\mu+\epsilon)n\log n,$$

i.e.

$$|a_n| > n^{-n(\mu+\epsilon)},$$

i.e.

$$|a_n|r^n > \{rn^{-(\mu+\epsilon)}\}^n.$$

Taking $r = (2n)^{\mu+\epsilon}$, this gives

$$|a_n|r^n > 2^{(\mu+\epsilon)n} = \exp\left\{\tfrac{1}{2}(\mu+\epsilon)\log 2 \, . \, r^{\frac{1}{\mu+\epsilon}}\right\}.$$

Since by Cauchy's inequality $M(r) \geqslant |a_n|r^n$, it follows that, for a sequence of values of r tending to infinity,

$$M(r) > \exp\{Ar^{1/(\mu+\epsilon)}\}.$$

Hence $\rho \geqslant 1/(\mu+\epsilon)$, and, making $\epsilon \to 0$, $\rho \geqslant 1/\mu$. If $\mu = 0$, the argument shows that $f(z)$ is of infinite order.

(ii) Let $f(z)$ be a function of finite order ρ. Then $a_n \to 0$, so that μ, defined as before, is not negative. The argument then shows that $\mu = 1/\rho$.

8.4. Examples. (i) Prove that the order of the function

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^\alpha}$$

is $1/\alpha$.

[We may use the above theorem, or proceed more directly as follows. Suppose z real and positive. The terms of the series increase until n is approximately $z^{1/\alpha}$, and then decrease. Hence, if $z = n^\alpha$, we get a maximum term

$$\frac{n^{n\alpha}}{(n!)^\alpha} \sim \frac{n^{n\alpha}}{(n^{n+\frac{1}{2}}e^{-n}2^{\frac{1}{2}}\pi^{\frac{1}{2}})^\alpha} = \frac{e^{n\alpha}}{n^{\frac{1}{2}\alpha}(2^{\frac{1}{2}}\pi^{\frac{1}{2}})^\alpha} = \frac{e^{\alpha z^{1/\alpha}}}{z^{\frac{1}{2}}(2^{\frac{1}{2}}\pi)^\alpha}.$$

Since $|f(z)|$ is greater than this term, its order is at least $1/\alpha$.

On the other hand, $|f(z)| \leqslant f(|z|)$, and if z is real

$$\begin{aligned} f(z) &= \sum_{n=0}^{N} \frac{z^n}{(n!)^\alpha} + \sum_{N+1}^{\infty} \frac{z^n}{(n!)^\alpha} \\ &< \sum_{n=0}^{N} \frac{z^N}{(n!)^\alpha} + \sum_{N+1}^{\infty} \frac{z^n}{\{(N+1)!\,N^{n-N-1}\}^\alpha} \\ &< Az^N + \frac{z^{N+1}}{\{(N+1)!\}^\alpha(1-z/N^\alpha)}, \end{aligned}$$

provided that $N^\alpha > z$. Taking $N = [(2z)^{1/\alpha}]$, we obtain

$$f(z) = O(z^N) = O\{z^{(2z)^{1/\alpha}}\} = O(e^{z^{1/\alpha+\epsilon}}),$$

so that the order does not exceed $1/\alpha$. Hence $\rho = 1/\alpha$. (See Hardy's *Orders of Infinity*, ed. 1, p. 55.)]

(ii) Discuss in a similar way the function

$$\sum_{n=1}^{\infty} \frac{z^n}{n^{\alpha n}}.$$

(iii) If $\lambda \neq 0$, and $p(z)$ is a polynomial, $e^{\lambda z} - p(z)$ has an infinity of zeros.

[If not, $e^{\lambda z} - p(z) = e^{az+b}P(z)$, where $P(z)$ is a polynomial. By comparing rates of increase in various directions we find that $a = \lambda$, then $e^{\lambda z} =$ rational function.]

(iv) If $f(z)$ is of order ρ, and $g(z)$ of order $\rho' \leqslant \rho$, and the zeros of $g(z)$ are all zeros of $f(z)$, then $f(z)/g(z)$ is of order ρ at most.

[For $f(z) = P_1(z)e^{Q_1(z)}, g(z) = P_2(z)e^{Q_2(z)}$, and P_1/P_2 is either the canonical product formed with the zeros of f_1/f_2, or this product multiplied by an exponential factor of order not exceeding ρ. Hence the order of P_1/P_2 does not exceed ρ.]

(v) $\cos z$ and $\sin z$ are of order 1; the product formulae (§ 3.23) are cases of Hadamard's theorem.

(vi) $1/\Gamma(z)$ is of order 1; deduce the product formula (§ 4.41) from Hadamard's theorem.

[With the notation of § 4.41, $f(1-z) = -2i\pi/\Gamma(z)$, and

$$f(z) = O\left\{e^{\pi|z|}\left(1+\int_1^\infty t^{|z|}e^{-t}\,dt\right)\right\} = O\{e^{\pi|z|}(|z|+1)^{|z|+1}\}.\Big]$$

(vii) $\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s)\zeta(s)$ is an integral function with $\rho = 1$, $\rho_1 = 1$.

[To prove that $\rho \leqslant 1$, use § 4.43 (3) and ex. (iv); and $\rho \geqslant 1$ since $\log\zeta(s) \sim 2^{-s}$, $\log\xi(s) \sim \frac{1}{2}s\log s$ as $s \to \infty$ by real values. Next the functional equation gives $\xi(s) = \xi(1-s)$. Hence $\Xi(z) = \xi(\frac{1}{2}+iz)$ is even, and $\Xi(\sqrt{z})$ is an integral function of order $\frac{1}{2}$, and so has convergence-exponent $\frac{1}{2}$.]

(viii) $z^{-\nu}J_\nu(z)$ is an integral function with $\rho = 1$, $\rho_1 = 1$. Verify the result of § 8.3 in this case. [See p. 60, ex. 5.]

(ix) $F_\alpha(z) = \int_0^\infty e^{-t^\alpha}\cos zt\,dt$ $(\alpha > 1)$ is of order $\alpha/(\alpha-1)$.

[Either directly from the integral, or from the power-series.]

(x) $\vartheta_1(z) = -i\sum_{n=-\infty}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}e^{(2n+1)iz}$, where $|q| < 1$, is an integral function with $\rho = 2$, $\rho_1 = 2$.

[If $\lambda = (2|z|+\log 2)/\log|1/q|-\frac{1}{2}$,

$$\vartheta_1(z) \leqslant 2\sum_{n\leqslant\lambda}|q|^{(n+\frac{1}{2})^2}e^{(2n+1)|z|}+2\sum_{n>\lambda}(\tfrac{1}{2})^{n+\frac{1}{2}} = O(e^{(2\lambda+1)|z|}) = O(e^{K|z|^2}).$$

$\vartheta_1(z)$ has simple zeros at $z = m\pi+n\pi\tau$, where m and n run through all integers (see e.g. Whittaker and Watson, *Modern Analysis*, § 21.12). Hence $\rho_1 = 2$.]

(xi) $\vartheta_1(z)$ is an integral function of $\sin z$ of order 0.

[If $2\sin z = w$, $\vartheta_1(z) = g(w)$, then

$$g(w) = \sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2}\{w^{2n+1}-(2n+1)w^{2n-1}+\ldots\}$$

$$= O\left\{\sum_{n=0}^{\infty}|q|^{(n+\frac{1}{2})^2}(|w|+1)^{2n+1}\right\} = O\{e^{K\log^2(|w|+1)}\}.$$

It was proved by Pólya (2) that if g and h are integral functions, and $g\{h(z)\}$ of finite order, then either h is a polynomial and g of finite order, or h is not a polynomial but of finite order, and g of zero order.]

(xii) If $$f(z) = \prod_1^\infty\left(1+\frac{z}{r_n}\right) \qquad (r_n > 0)$$

is of order ρ, $0 < \rho < 1$, then for $\rho < \sigma < 1$

$$\int_0^\infty \frac{\log f(x)}{x^{\sigma+1}}\,dx = \frac{\pi}{\sigma \sin \pi\sigma} \sum_1^\infty \frac{1}{r_n^\sigma}, \qquad \int_0^\infty \frac{\log|f(-x)|}{x^{\sigma+1}}\,dx = \frac{\pi}{\sigma \tan \pi\sigma} \sum_1^\infty \frac{1}{r_n^\sigma}.$$

[We have

$$\int_0^\infty \frac{\log(1+x)}{x^{\sigma+1}}\,dx = \frac{\pi}{\sigma \sin \pi\sigma}, \qquad \int_0^\infty \frac{\log|1-x|}{x^{\sigma+1}}\,dx = \frac{\pi}{\sigma \tan \pi\sigma}.]$$

8.5. The derived function. Many of the properties of the derived function of an integral function are the same as those of the primitive function. The following theorems are examples of this.

8.51. *The derived function $f'(z)$ is of the same order as $f(z)$.*

Let $M'(r) = \max\limits_{|z|=r} |f'(z)|$. Then

$$\frac{M(r) - |f(0)|}{r} \leqslant M'(r) \leqslant \frac{M(R)}{R-r}. \tag{1}$$

For
$$f(z) = \int_0^z f'(t)\,dt + f(0),$$
the integral being taken along the straight line. Hence

$$M(r) \leqslant rM'(r) + |f(0)|.$$

On the other hand,

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z)^2}\,dw,$$

where C is the circle $|w-z| = R-r$ $(|z| = r < R)$. Hence, choosing z so that $|f'(z)| = M'(r)$, we have

$$M'(r) \leqslant \frac{M(R)}{R-r}.$$

The result stated now follows on taking, say, $R = 2r$ in (1).

8.52. The well-known theorem, that if $f(z)$ is a polynomial with all its roots real, then $f'(z)$ has the same property, can be

* Pólya (2).

extended to a certain class of integral functions. The result is expressed by the following theorem of Laguerre:

If $f(z)$ is an integral function, real for real z, of order less than 2, with real zeros, then the zeros of $f'(z)$ are also all real, and are separated from each other by the zeros of $f(z)$.

We have
$$f(z) = cz^k e^{az} \prod_{n=1}^{\infty} \left(1-\frac{z}{z_n}\right) e^{\frac{z}{z_n}},$$
where k is zero or a positive integer, and c, a, and z_1, z_2,... are all real. Taking logarithms and differentiating,
$$\frac{f'(z)}{f(z)} = \frac{k}{z} + a + \sum_{n=1}^{\infty} \left(\frac{1}{z-z_n} + \frac{1}{z_n}\right).$$
Hence, if $z = x+iy$,
$$\mathbf{I}\left\{\frac{f'(z)}{f(z)}\right\} = -iy\left\{\frac{k}{x^2+y^2} + \sum_{n=1}^{\infty} \frac{1}{(x-z_n)^2+y^2}\right\},$$
which is zero if $y = 0$ only. Hence $f'(z)$ cannot be zero except on the real axis.

Again,
$$\frac{d}{dz}\left\{\frac{f'(z)}{f(z)}\right\} = -\frac{k}{z^2} - \sum_{n=1}^{\infty} \frac{1}{(z-z_n)^2},$$
which is real and negative if z is real. Hence $f'(z)/f(z)$ decreases steadily as z increases through real values from z_n to z_{n+1}, and so it cannot vanish more than once between z_n and z_{n+1}. Clearly it changes sign, and so vanishes just once in this interval. This proves the theorem.

It is clear from the above result that, if the zeros of $f'(z)$ are z_1', z_2',..., then the series
$$\sum \frac{1}{|z_n|^\alpha}, \quad \sum \frac{1}{|z_n'|^\alpha},$$
converge or diverge together. Hence the zeros of $f'(z)$ have the same exponent of convergence as those of $f(z)$. It may be shown further that $f(z)$ and $f'(z)$ have the same genus, but this is not quite so easy to prove (see ex. 16 at the end of the chapter).

Since $f'(z)$ is of the same order as $f(z)$, and has real zeros only, the theorem may now be applied to it, and we see that $f''(z)$ has real zeros only; and so for $f'''(z)$, etc.

The proof also applies to a function $f(z)$ of order 2, but of genus 1. In this case, however, we cannot extend the result to $f''(z)$,... without considering the problem of the genus of $f'(z)$.

It is easily seen by means of examples that the theorem is not true for functions of genus 2. For example, in the case

$$f(z) = ze^{z^2}, \qquad f'(z) = (2z^2+1)e^{z^2},$$

the zeros of $f'(z)$ are complex; and in the case

$$f(z) = (z^2-4)e^{\frac{1}{3}z^2}, \qquad f'(z) = \tfrac{2}{3}z(z^2-1)e^{\frac{1}{3}z^2},$$

the zeros of $f'(z)$ are real, but are not separated by those of $f(z)$.

Example. The differential equation

$$y\frac{d^2y}{dt^2} = -\sin^2 t$$

has no real solution, other than $y = \pm\sin t$, which is an integral function of finite order.

[Suppose that y is a function of finite order ρ. Then

$$y = e^{Q(t)}P(t),$$

where $P(t)$ is a canonical product, and $Q(t)$ a polynomial of degree not greater than ρ. Since the zeros of $P(t)$ are zeros of $\sin^2 t$, $P(t)$ is of order 1 at most.

Now $$\frac{dy}{dt} = e^{Q(t)}\{P'(t)+P(t)Q'(t)\},$$

$$\frac{d^2y}{dt^2} = e^{Q(t)}\{P''(t)+2P'(t)Q'(t)+P(t)Q'^2(t)+P(t)Q''(t)\} = e^{Q(t)}f(t),$$

where $f(t)$ is of order 1 at most. Hence

$$f(t) = e^{at+b}P_1(t),$$

where $P_1(t)$ is a canonical product. Hence

$$e^{2Q(t)+at+b}P(t)P_1(t) = -\sin^2 t,$$

i.e. $$e^{2Q(t)+at+b}P(t) = -\sin^2 t/P_1(t)$$

is of order 1 at most (§ 8.4, ex. (iv)). Hence $P(t)$ is of order 1 and $Q(t)$ is linear.

Hence y is a function of order 1.

We can now use Laguerre's theorem. y is a function of order 1 with real zeros. The zeros of $\frac{dy}{dt}$ are separated by those of y, so that, as y has no triple zeros, all the zeros of $\frac{dy}{dt}$ are simple. So all the zeros of $\frac{d^2y}{dt^2}$ are simple. Hence y has zeros at *all* the zeros of $\sin t$. Suppose y has a double zero at $t = k\pi$. Then $\frac{dy}{dt}$ has a zero between $(k-1)\pi$ and $k\pi$, a zero at $k\pi$, and a zero between $k\pi$ and $(k+1)\pi$. Hence $\frac{d^2y}{dt^2}$ has two zeros between

$(k-1)\pi$ and $(k+1)\pi$, which is impossible. Hence y has all the zeros of $\sin t$ just once. Hence

$$y = e^{\alpha t+\beta}\sin t.$$

Inserting this in the differential equation, we obtain

$$(\alpha^2-1)\sin t+2\alpha\cos t = -e^{-2\alpha t-2\beta}\sin t.$$

Since the left-hand side is bounded for real t, so is the right-hand side, and hence $\alpha = 0$. Then $\beta = 0$ or πi.]

8.6. Functions with real zeros only. A number of important functions have no complex zeros; for example, all the zeros of $1/\Gamma(z)$ are real. On the other hand it is sometimes very difficult to decide whether the zeros are real or not; for example, it was conjectured by Riemann, in 1859, that all the zeros of the function $\Xi(z)$ of § 8.45 are real, but this has never been proved.

8.61. The theorems of Laguerre. In some cases the question can be decided by the following theorems of Laguerre.*

Let $f(z)$ be a polynomial,

$$f(z)=a_0+a_1z+...+a_pz^p,$$

all of whose zeros are real; and let $\phi(w)$ be an integral function of genus 0 or 1, which is real for real w, and all the zeros of which are real and negative. Then the polynomial

$$g(z)=a_0\phi(0)+a_1\phi(1)z+...+a_p\phi(p)z^p$$

has all its zeros real, and as many positive, zero and negative zeros as $f(z)$.

Let
$$\phi(w)=ae^{kw}\prod_{n=1}^{\infty}\left(1+\frac{w}{\alpha_n}\right)e^{-\frac{w}{\alpha_n}},$$

where $\alpha_n>0$ for all values of n. Consider the function

$$\begin{aligned} g_1(z) &= f(z)+\frac{z}{\alpha_1}f'(z) \\ &= \frac{z^{1-\alpha_1}}{\alpha_1}\frac{d}{dz}\{z^{\alpha_1}f(z)\} \qquad (z>0) \\ &= a_0+a_1\left(1+\frac{1}{\alpha_1}\right)z+...+a_p\left(1+\frac{p}{\alpha_1}\right)z^p. \end{aligned}$$

Obviously $g_1(z)$ has as many zeros at $z=0$ as $f(z)$; and the second expression for it shows, by Rolle's theorem, that it has

* *Œuvres*, t. 1, p. 200.

the same number of positive zeros as $f(z)$. Similarly it has the same number of negative zeros.

By repeating the argument, we can obtain the same result for the function

$$g_n(z) = a_0 + a_1\phi_n(1)z + \ldots + a_p\phi_n(p)z^p,$$

where $$\phi_n(w) = \left(1+\frac{w}{\alpha_1}\right)\ldots\left(1+\frac{w}{\alpha_n}\right).$$

Next, the transformation $z = e^{k_n}z'$, where $k_n = k - \sum_1^n 1/\alpha_\nu$, shows that the same result holds for

$$G_n(z) = a_0\Phi_n(0) + a_1\Phi_n(1)z + \ldots + a_p\Phi_n(p)z^p,$$

where $$\Phi_n(w) = ae^{kw}\prod_{\nu=1}^{n}\left(1+\frac{w}{\alpha_\nu}\right)e^{-\frac{w}{\alpha_\nu}} = ae^{k_n w}\phi_n(w).$$

Finally, $\Phi_n(w) \to \phi(w)$ uniformly in any finite region. Hence $G_n(z) \to g(z)$ uniformly in any finite region; by Hurwitz's theorem (§ 3.45) the zeros of $g(z)$ are the limits of the zeros of $G_n(z)$; it is clear that $g(z)$ has the same number of zeros at $z = 0$ as $f(z)$; and this completes the proof.

8.62. *Suppose that $\phi(w)$ satisfies the conditions of the previous theorem, and that $f(z)$ is an integral function of the form*

$$f(z) = e^{az+b}\prod_{n=1}^{\infty}\left(1+\frac{z}{z_n}\right),$$

the numbers a and z_n being all positive. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Then $$g(z) = \sum_{n=0}^{\infty} a_n\phi(n)z^n$$

is an integral function, all of whose zeros are real and negative.

In the first place, $g(z)$ is an integral function; for, since $(1+x)e^{-x} \leqslant 1$ for $x \geqslant 0$,

$$|\phi(n)| \leqslant |a|e^{kn},$$

and so the series for $g(z)$ is everywhere convergent.

Let $$f_p(z) = e^b\left(1+\frac{az}{p}\right)^p\prod_{n=1}^{p}\left(1+\frac{z}{z_n}\right)$$
$$= \sum_{n=0}^{2p} a_{n,p}z^n.$$

All the zeros of this are real and negative, and hence, by the previous theorem, so are the zeros of

$$g_p(z) = \sum_{n=0}^{2p} a_{n,p}\,\phi(n)z^n.$$

Finally, $g_p(z) \to g(z)$ uniformly in any finite region. In fact it is clear from the expression

$$\left(1+\frac{az}{p}\right)^p = 1+az+\left(1-\frac{1}{p}\right)\frac{a^2z^2}{2!}+\ldots+\frac{a^pz^p}{p^p}$$

that $a_{n,p} \to a_n$ as $p \to \infty$ for every fixed n, while $|a_{n,p}| \leqslant a_n$ for all values of n and p. Hence, if $N < 2p$,

$$|g(z)-g_p(z)| \leqslant \left|\sum_{n=1}^{N}(a_n-a_{n,p})z^n\right| + \left|\sum_{N+1}^{\infty} a_n z^n\right| + \left|\sum_{N+1}^{2p} a_{n,p}z^n\right|$$

$$\leqslant \left|\sum_{n=1}^{N}(a_n-a_{n,p})z^n\right| + 2\sum_{N+1}^{\infty}|a_n z^n|.$$

We can now choose N so large that the second term is less than any given ϵ, and then, having fixed N, the first term tends to zero. Hence $g_p(z) \to g(z)$.

As in the previous proof, the result now follows from Hurwitz's theorem.

8.63. The simplest case is that of the function $f(z) = e^z$. From this we deduce that *if $\phi(w)$ satisfies the conditions of the previous theorems, then*

$$F(z) = \sum_{n=0}^{\infty} \frac{\phi(n)}{n!} z^n$$

is an integral function, and all of its zeros are real and negative.

Examples. (i) Let

$$\phi(w) = 1/\Gamma(w+\nu+1) \qquad (\nu > -1).$$

This is an integral function of genus 1, with zeros at $w = -\nu-1$, $-\nu-2, \ldots$. These are all real and negative, and hence the zeros of

$$\sum_{n=0}^{\infty} \frac{z^n}{n!\,\Gamma(n+\nu+1)} = \frac{J_\nu(2i\sqrt{z})}{(i\sqrt{z})^\nu}$$

are all real and negative. Hence the zeros of $J_\nu(z)$ are all real.

(ii) The function*

$$F_\alpha(z) = \int_0^\infty e^{-t^\alpha}\cos zt\,dt$$

* Pólya (1).