A CLASSIFICATION OF FOURIER SUMMATION FORMULAS AND CRYSTALLINE MEASURES

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ABSTRACT. We completely classify Fourier summation formulas, and in particular, all crystalline measures with quadratic decay. Our classification employs techniques from almost periodic functions, Hermite-Biehler functions, de Branges spaces and Poisson representation. We show how our classification generalizes recent results of Kurasov & Sarnak and Olevskii & Ulanovskii. As an application, we give a new classification result for nonnegative measures with uniformly discrete support that are bounded away from zero on their support. In the Appendix we attempt to give a systematic account of the several types of Fourier summation formulas, including a new construction using eta-products, generalizing an old example of Guinand.

1. Introduction

To understand the atomic structure of a crystal (of some material) one can fire a beam of electrons into the object and observe an electron diffraction pattern on a screen¹. In most materials one observes a periodic diffraction pattern, such as a two-dimensional lattice with 6-fold symmetry, as in Tantalum pentoxide Ta₂O₅. However, this is not the case with Holmium-magnesium-zinc Ho-Mg-Zn, where one observes some 5-fold and 10fold symmetry, hence a non-periodic diffraction pattern Fig. 1. The first to record such phenomena experimentally was materials scientist Dan Shechtman, in 1982, and today these materials are called quasicrystals. He was awarded the Nobel Prize in Chemistry in 2011 for his breakthrough. Mathematically, if μ represents the crystal (so $\mu \ge 0$ and $\operatorname{supp}(\mu)$ is uniformly discrete) then what we see in the diffraction pattern is effectively $|\hat{\mu}|^2$ (where $\hat{\mu}$ is the Fourier transform of μ). A famous inverse problem is that of reconstructing atomic structure from diffraction data, overcoming the issue that diffraction only determines intensity of spots but loses phase information. Real-world quasicrystals (such as Decagonite Al₇₁Ni₂₄Fe₅), although uniformly discrete in physical space, typically have everywhere dense diffraction spectrum². The diffraction pattern is a super-position of Dirac combs with several intensities (Bragg peaks) however, due to instrument precision and tuning, we only see peaks that have, say, an intensity above 10^{-4} of the brightest peak, hence it visually seems to be locally finite, such as in Fig. 1, nevertheless the real measure $\hat{\mu}$ has a dense support. On the other hand, stable 1D quasicrystals with locally

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¹This technique is called high resolution electron microscopy (HRTEM), but another common technique is X-ray microscopy.

²We thank Michael Baake for pointed this out, among several other remarks.

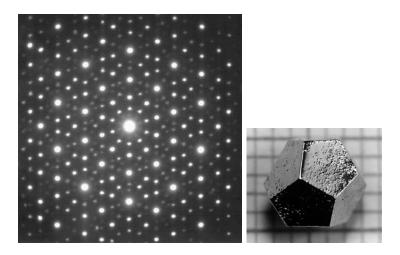


FIGURE 1. Diffraction pattern of Holmium-magnesium-zinc Ho-Mg-Zn.

finite spectra do exist in nature, and can be found as projections along pseudo axis's, see [43, Figure 1] and the references in [18, Ch. 7]. Nowadays there is unified framework to construct measures μ whose spectrum fits the data observed in the diffraction patterns of physical quasicrystals, these are called model sets (or Meyer sets) [8, 18, 27, 35, 36, 38].

From the mathematical standpoint, and in the last few decades, many mathematicians have wondered about what different types of measures μ with pure point support and spectrum are there, and what kind of geometrical properties of the spectrum are allowed under various conditions on μ ? The big classification question from the mathematical side is:

Can on one classify all measures μ with pure point support and spectrum?

Freeman Dyson addressed this question in 2009 in his famous paper [17], using the word quasicrystals for such measure, suggesting that such classification mighty be a very difficult mathematical problem, but made the "outrageous suggestion" it could potentially give a proof of the Riemann Hypothesis³. He said: "My suggestion is the following. Let us pretend that we do not know that the Riemann Hypothesis is true. Let us tackle the problem from the other end. Let us try to obtain a complete enumeration and classification of one-dimensional quasicrystals. That is to say, we enumerate and classify all point distributions that have a discrete point spectrum." Dyson stated this because of Guinand's prime summation formula (also known as Weyl's summation formula), which we explain in the Appendix. The notorious paper of Guinand [21], which inspired Dyson, grasps such classification, but it was forgotten for sometime. Meyer resurrected Guinand's paper in [36].

We say a measure μ is crystalline if it has locally finite support and its Fourier transform $\hat{\mu}$ also has locally finite support⁴. In this scenario, the first geometric classification

³We do not share the same belief.

⁴Such measures are sometimes also called quasicrystals, Fourier quasicrystals, or doubly sparse, depending on the author, area, context, decay conditions, etc. We will stick with Meyer's denomination of crystalline measure.

breakthrough occurred in the work of Lev & Olevskii [29, 30], where they classify classical Poisson summation as the only crystalline measures that have uniformly discrete support and spectrum. Guinand's example [21, p. 265] shows the theorem is tight (we generalize his construction in the Appendix). Recently, the field is very active, and the amazing works of Radchenko & Viazovska [40], Kurasov & Sarnak [25], Olevskii & Ulanovskii [39] and Alon et. al. [2, 1], but also, the not so recent works of Bohr [15], Besicovitch [10], Guinand [21], de Branges [12] and Meyer [34], deeply inspired this paper.

In this paper we completely classify (what we call) Fourier summation formulas, and in particular we classify crystalline measures that have quadratic decay, although our results are more general and can be applied to dense spectra. As far as we know, it is the first time classification results are proven in such generality. Our classification involves heavily the theory of almost periodic analytic functions [10] and de Branges spaces techniques [12]. Almost periodic functions have permeated this are for some time now, for example in the works of Guinand [21], Meyer [37] and Baake et. al. [4, 8, 9], among others. On the other hand, the connection with de Branges spaces that we make is totally new.

A Fourier summation formula should be one where the identity

$$\int_{\mathbb{R}} \widehat{\varphi}(t) d\mu(t) = \sum_{n \in \mathbb{N}} a(\lambda_n) \varphi(\lambda_n)$$

holds, at the bear minimum, for all functions $\varphi \in C_c^{\infty}(\mathbb{R})$, where $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ is some sequence of real nodes, $\{a(\lambda_n)\}_{n\in\mathbb{N}}\subset\mathbb{C}$ are some complex coefficients and μ is some complex-valued measure. There is a zoo of formulas of such type which we describe in the Appendix, from ancient and modern times. We choose the above form of because: (1) We want a summation formula, so there must be a sum somewhere in the definition and such summation must be singular with respect to Lebesgue measure; (2) Allowing the right hand side above to have a vanishing singular part is boring, as otherwise any L^1 -function would give rise to a Fourier summation formula.

1.1. Overview. Theorems 1 and 2 are our main general classification results and we later specialize them for measures with locally finite support (in particular, also crystalline measures) in Theorems 3 and 4. Finally, we derive a new characterization Theorem 5 for measures with uniformly discrete support such that $\mu|_{\text{supp}(\mu)} \geq \varepsilon$, but with no geometric assumption on $\hat{\mu}$, which perhaps makes this result most useful to physics.

2. Main Results

We will have to introduce several things before we can state our first main result. We say that a complex-valued measure μ on \mathbb{R} is **locally finite** if its total variation

$$|\mu|(K) = \sup_{\substack{\subseteq P_j \in \mathbb{N} \\ j \in \mathbb{N}}} \sum_{j \in \mathbb{N}} |\mu(P_j)| < \infty$$

is finite on every compact set $K \subset \mathbb{R}$, or equivalently, if there are nonnegative locally finite Borel measures μ_j such that $\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$. We say that μ is **tempered** if it is a complex-valued locally finite measure and there exists a tempered distribution $u \in \mathcal{S}'(\mathbb{R})$ such that $u(\varphi) = \int_{\mathbb{R}} \varphi d\mu$ for all $\varphi \in C_c^{\infty}(\mathbb{R})$. Some authors might not concord with this definition, but we are following the setup of [4]. We say μ is **strongly tempered** if $\varphi \in L^1(|\mu|)$ for all $\varphi \in \mathcal{S}(\mathbb{R})$. By [4, Prop. 2.5], this is equivalent to the **degree** of μ

$$\deg(\mu) := \min \left\{ n \in \mathbb{Z} : \int_{\mathbb{R}} \frac{\mathrm{d}|\mu|(t)}{(1+t^2)^{n/2}} < \infty \right\}$$

being $<\infty$. In particular, the functional $\varphi\mapsto \int_{\mathbb{R}}\varphi\mathrm{d}\mu$ defines a tempered distribution, hence μ is tempered. The implication [tempered] \Rightarrow [strongly tempered] is false. Indeed, the measure $\mu=\sum_{n\geqslant 0}3^{n^2}(\pmb{\delta}_n-\pmb{\delta}_{n+4^{-n^2}})$ is not strongly tempered but it is tempered, since $\int 2^{-t^2}\mathrm{d}|\mu|(t)\geqslant \sum_{n\geqslant 0}3^{n^2}2^{-n^2}=\infty$ but $\int 2^{-t^2}\mathrm{d}|\mu|(t)\geqslant \sum_{n\geqslant 0}3^{n^2}2^{-n^2}=\infty$

We say that a function $a : \mathbb{R} \to \mathbb{C}$ is **locally summable** if the support set $\sup(a) := \{\lambda \in \mathbb{R} : a(\lambda) \neq 0\}$ is at most countable and for some enumeration $\sup(a) = \{\lambda_n\}_{n \geq 1}$ we have

$$\sum_{\lambda_n \in [-T,T]} |a(\lambda_n)| < \infty,$$

for all T > 0. This is equivalent to say that $\nu = \sum_{n \ge 1} a(\lambda_n) \delta_{\lambda_n}$ is locally finite. In particular, local sums $\sum_{\lambda \in K} a(\lambda)$ are always well-defined for any bounded set $K \subset \mathbb{R}$.

Definition 1 (Fourier Summation Pairs). We say (μ, a) is a Fourier summation pair (FS-pair) if μ is a strongly tempered measure, $a : \mathbb{R} \to \mathbb{C}$ is locally summable and

$$\int_{\mathbb{R}} \widehat{\varphi}(t) d\mu(t) = \sum_{\lambda \in \mathbb{P}} a(\lambda) \varphi(\lambda)$$

for every $\varphi \in C_c^{\infty}(\mathbb{R})$. Equivalently, μ is a strongly tempered measure and for some locally summable function $a : \mathbb{R} \to \mathbb{C}$ we have $\widehat{\mu}|_{C_c^{\infty}} = \sum_{\lambda \in \mathbb{R}} a(\lambda) \delta_{\lambda}$.

We use the following definition for **Fourier transform**:

$$\widehat{\varphi}(\xi) = \int_{\mathbb{D}} \varphi(x) e^{-2\pi i x \xi} \mathrm{d}x.$$

Perhaps the two most simple examples of FS-pairs are

$$\int_{\mathbb{R}} \widehat{\varphi}(t) dt = \varphi(0) \quad \text{(Fourier Inversion)} \quad \text{and} \quad \sum_{n \in \mathbb{Z}} \widehat{\varphi}(n) = \sum_{n \in \mathbb{Z}} \varphi(n) \quad \text{(Poisson Summation)}.$$

Remark 1 (Real-Antipodal splitting). In the following we show that any FS-pair can be split into two real-antipodal FS-pairs. We say that a function $a: \mathbb{R} \to \mathbb{C}$ is antipodal if $a(-\lambda) = \overline{a(\lambda)}$ for every $\lambda \in \mathbb{R}$. We say that an FS-pair (μ, a) if real-antipodal is μ is real-valued and $a(\cdot)$ is (and must be) antipodal. Indeed, if we are given an arbitrary FS-pair (μ, a) then we can write $a_1(\lambda) = \frac{a(\lambda) + \overline{a(-\lambda)}}{2}$, $a_2(\lambda) = \overline{a(-\lambda) - a(\lambda)}$, and so $a = a_1 - ia_2$, with each a_j antipodal, and we can write $\mu = \mu_1 - i\mu_2$, where $\mu_1 = \text{Re } \mu$ and $\mu_2 = \text{Im } \mu$, and so each μ_j is real-valued. If $\varphi \in C_c^{\infty}(\mathbb{R})$ is antipodal then $\widehat{\varphi}$ is real, and if we further

⁵Whenever it is convenient, we write $A \ll B$ to mean that there is C > 0 such that $|A| \leqslant C|B|$.

assume that either φ is real-valued or imaginary-valued, then from the identity

$$\sum_{\lambda \in \mathbb{R}} (a_1(\lambda) - ia_2(\lambda))\varphi(\lambda) = \int_{\mathbb{R}} \widehat{\varphi}(t) d\mu_1(t) - i \int_{\mathbb{R}} \widehat{\varphi}(t) d\mu_2(t)$$

and the fact that $\sum_{\lambda \in \mathbb{R}} a_j(\lambda) \varphi(\lambda)$ is real, one deduces that

$$\sum_{\lambda \in \mathbb{R}} a_j(\lambda) \varphi(\lambda) = \int_{\mathbb{R}} \widehat{\varphi}(t) d\mu_j(t)$$

for j = 1, 2. Since for any $\varphi \in C_c^{\infty}(\mathbb{R})$ we have $\varphi = \varphi_1 - i\varphi_2$ where each φ_j is antipodal, and each $\varphi_j = \operatorname{Re} \varphi_j - i\operatorname{Im} \varphi_j$, where $\operatorname{Re} \varphi_j$ is real and even and $i\operatorname{Im} \varphi_j$ is imaginary and odd (hence both are antipodal as well), we deduce by linearity that (μ_j, a_j) is a real-antipodal FS-pair for j = 1, 2. As an example, note that $\sum_{n \in \mathbb{Z}} i^n \varphi(n) = \sum_{n \in \mathbb{Z}} \widehat{\varphi}(n - 1/4)$ has the real-antipodal splitting

$$\sum_{n \in \mathbb{Z}} (-1)^n \varphi(2n) = \frac{1}{2} \sum_{n \in \mathbb{Z}} (\widehat{\varphi}(n - 1/4) + \widehat{\varphi}(n + 1/4))$$
$$\sum_{n \in \mathbb{Z}} (-1)^n \varphi(2n - 1) = \frac{1}{2i} \sum_{n \in \mathbb{Z}} (\widehat{\varphi}(n + 1/4) - \widehat{\varphi}(n - 1/4)).$$

We say that a locally summable function $a: \mathbb{R} \to \mathbb{C}$ has **exponential growth** if

$$\sum_{\lambda \in \mathbb{R}} |a(\lambda)| e^{-c|\lambda|} < \infty$$

for some c > 0. We let $\mathbb{C}^+ = \{z = x + iy : y > 0\}$. Following Besicovitch⁶ [10, Ch. 3], we say a holomorphic function $f : \mathbb{C}^+ \to \mathbb{C}$ is **almost periodic** if for every $0 < \alpha < \beta < \infty$ and $\varepsilon > 0$ there is a relatively dense⁷ set of ε -translations $\tau \subset \mathbb{R}$ such that

$$\sup_{\alpha < \operatorname{Im} z < \beta} |f(z) - f(z+t)| \leqslant \varepsilon$$

for every $t \in \tau$. We write $f \in AP(\mathbb{C}^+)$. If $f \in AP(\mathbb{C}^+)$ it can be shown that

$$\mathbb{E}f(\lambda) := \lim_{T \to \infty} \frac{1}{2T} \int_{-T+iy}^{T+iy} f(z) e^{-2\pi i \lambda z} dz$$

exists for all $\lambda \in \mathbb{R}$ and is independent of y > 0 (see Section 4). It is straightforward to see that if instead $f(\cdot + ic) \in AP(\mathbb{C}^+)$ for some c > 0, then the above limit exists and is independent of y > c. If $f \in AP(\mathbb{C}^+)$ we define the **spectrum** of f by

$$\operatorname{spec}(f) := \{ \lambda \in \mathbb{R} : \mathbb{E}f(\lambda) \neq 0 \}.$$

We say a holomorphic function $g: \mathbb{C}^+ \to \mathbb{C}$ is of **bounded type**⁸ if there are two bounded holomorphic functions $P, Q: \mathbb{C}^+ \to \mathbb{C}$ such that g = P/Q.

⁶Besicovitch considers the half-plane Re z>0 instead of Im z>0, so one has to mentally perform a rotation in order to use his results.

⁷A set $\tau \subset \mathbb{R}$ is relatively dense if there exists l > 0 such that $(x, x + l) \cap \tau \neq \emptyset$ for every $x \in \mathbb{R}$.

⁸This is the same to say that $\log |g(z)|$ has a nonnegative harmonic majorant.

The following is the first main result of this paper.

Theorem 1. Let (μ, a) be a real-antipodal FS-pair. Assume that $deg(\mu) \leq 2$ and that $a(\cdot)$ has exponential growth. Then we have:

(I) The limit

$$f(z) = \frac{1}{2}a(0) + \lim_{T \to \infty} \sum_{0 < \lambda < T} a(\lambda)(1 - \lambda/T)e^{2\pi i\lambda z}$$
(1)

converges uniformly in compact sets of \mathbb{C}^+ ;

- (II) There is $c_1 > 0$ such that $f(\cdot + ic_1) \in AP(\mathbb{C}^+)$;
- (III) There are $c_2, B > 0$, with $c_2 \ge c_1$, such that for any trigonometric polynomial $p(x) = \sum_{n=1}^{N} p_n e^{2\pi i \theta_n x}$, with $\{\theta_n\} \subset [0, \infty)$, we have

$$\limsup_{T \to \infty} \left| \frac{1}{2T} \int_{-T}^{T} f(x + ic_2) \overline{p(x)} dx \right| \leqslant B \max |p_n|;$$

(IV) $\exp(f)$ is of bounded type and $\limsup_{y\to\infty} \frac{\operatorname{Re} f(iy)}{y} = 0$.

Conversely, suppose $f: \mathbb{C}^+ \to \mathbb{C}$ is a given holomorphic function such that items (II), (III) and (IV) hold true. Then there is $c \ge 0$ such that the limit

$$\mathbb{E}f(\lambda) := \lim_{T \to \infty} \frac{1}{2T} \int_{-T+iu}^{T+iy} f(z) e^{-2\pi i \lambda z} dz$$

exists for every $\lambda \in \mathbb{R}$ and y > c, is independent of y, vanishes identically for $\lambda < 0$ and defines a locally summable function with exponential growth such that

$$\sum_{\lambda \in \mathbb{R}} |\mathbb{E}f(\lambda)| e^{-2\pi y|\lambda|} < \infty$$

for every y > c. Moreover, there is a unique real-valued locally finite measure μ of degree at most 2 satisfying

$$\frac{f(z) + \overline{f(w)}}{z - \overline{w}} = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{(z - t)(\overline{w} - t)} \tag{2}$$

for all $w, z \in \mathbb{C}^+$. Furthermore, if we let

$$a(\lambda) = \begin{cases} \mathbb{E}f(\lambda) & \text{if } \lambda > 0\\ 2\operatorname{Re}\mathbb{E}f(0) & \text{if } \lambda = 0\\ \overline{\mathbb{E}f(-\lambda)} & \text{if } \lambda < 0 \end{cases}$$
 (3)

then (μ, a) is an real-antipodal FS-pair and identity (1) holds.

The proof we give for Theorem 1 in the converse part actually shows a stronger result.

Theorem 2. Let $f: \mathbb{C}^+ \to \mathbb{C}$ be a holomorphic function such that properties (II) and (IV) in Theorem 1 hold true, but also the following weaker version of property (III):

(III*) There is $c_2 \ge c_1$, and for every M > 0 there is $B_M > 0$, such that for any trigonometric polynomial $p(x) = \sum_{n=1}^{N} p_n e^{2\pi i \theta_n x}$, with $\{\theta_n\} \subset [0, M]$, we have

$$\lim_{T \to \infty} \left| \frac{1}{2T} \int_{-T}^{T} f(x + ic_2) \overline{p(x)} dx \right| \leq B_M \max |p_n|;$$

Then there is $c \ge 0$ such that the limit

$$\mathbb{E}f(\lambda) := \lim_{T \to \infty} \frac{1}{2T} \int_{-T+iy}^{T+iy} f(z) e^{-2\pi i \lambda z} dz$$

exists for every $\lambda \in \mathbb{R}$ and y > c, is independent of y, vanishes identically for $\lambda < 0$ and defines a locally summable function. Moreover, there is a unique real-valued locally finite measure μ of degree at most 2 satisfying (2), and if we define $a(\cdot)$ as in (3) then (μ, a) is an real-antipodal FS-pair and identity (1) holds.

Note that condition (III*) is equivalent to $\mathbb{E}f(\lambda)$ being locally summable.

Remark 2. We claim that if (μ, a) is an FS-pair such such that $\mu \geq 0$ and $a(\cdot)$ has exponential growth then $deg(\mu) \leq 2$. Indeed, let $\varphi \in C_c^{\infty}(\mathbb{R})$ be even, $0 \leq \varphi \leq 1$, $\int \varphi = \varphi(0) = 1 \text{ and } \operatorname{supp}(\varphi) \subset (-1,1). \text{ Let } \varphi_{\varepsilon}(x) = \varphi(x/\varepsilon)/\varepsilon \text{ and } \psi_{\varepsilon}(x) = \widehat{\varphi}(x/\varepsilon)/\varepsilon \text{ for } \varepsilon = 0$ $0 < \varepsilon < 1$. Let $g(x) = e^{-2\pi C|x|}$, with C > 0 sufficiently large, and note $\widehat{g}(\xi) = \frac{C}{\pi(C^2 + \xi^2)}$. We have

$$\int_{\mathbb{R}} (\widehat{g}\widehat{\varphi}_{\varepsilon}) * \psi_{\varepsilon}(t) d\mu(t) = \sum_{\lambda \in \mathbb{R}} a(\lambda) g * \varphi_{\varepsilon}(\lambda) \widehat{\psi}_{\varepsilon}(\lambda).$$

It is not hard to show that $|g * \varphi_{\varepsilon}(\lambda) \widehat{\psi}_{\varepsilon}(\lambda)| \leq e^{-2\pi C(|\lambda|-1)}$. Thus, we conclude the right hand side above is uniformly bounded for $0 < \varepsilon < 1$. Since $\mu \ge 0$ and $(\widehat{g}\widehat{\varphi}_{\varepsilon}) * \psi_{\varepsilon} \to \widehat{g}$ pointwise, as $\varepsilon \to 0$, we can apply Fatou's Lemma to deduce $\int \frac{d\mu(t)}{C^2+t^2} < \infty$.

Remark 3. Let (μ, a) be an FS-pair such that $\mu \ge 0$ and $a(\cdot)$ satisfies

$$\sum_{\lambda \in \mathbb{R}} |a(\lambda)| e^{-c|\lambda|/\sigma(\lambda)} < \infty, \tag{4}$$

for some c > 0 and a function $\sigma : \mathbb{R} \to [1, \infty)$ such that $\int_{|x| \ge 1} \frac{1}{x\sigma(x)} dx < \infty$. We claim there is C > 0 such that $\mu([x, x + 1]) \leq C$ for all $x \in \mathbb{R}$, hence $\deg(\mu) \leq 1$. For instance, one could take $\sigma(\lambda) = \log(10 + |\lambda|)(\log\log(10 + |\lambda|))^2$. Indeed, we only need to consider Ingham's construction [22]. This is an auxiliary even function $\varphi \in \mathcal{S}(\mathbb{R})$ such that $\varphi \geqslant 0$, $\operatorname{supp}(\widehat{\varphi}) \subset [-1,1], \ \widehat{\varphi} \geqslant 0, \ \widehat{\varphi}(0) = 1 \ and$

$$\varphi(x) \leqslant Ce^{-c|x|/\sigma(x)} \quad (for \ all \ x \in \mathbb{R})$$

for some C>0. Let c>0 be sufficiently small such that $\widehat{\varphi}(x)\geqslant 1/2$ for $x\in[-c,c]$. The decay condition on $a(\cdot)$ and a routine approximation argument shows the identity

$$\int_{\mathbb{R}} \widehat{\varphi}(t-s) d\mu(t) = \sum_{\lambda \in \mathbb{R}} a(\lambda) \varphi(\lambda) e^{2\pi i \lambda s}$$

holds for all $s \in \mathbb{R}$. In particular, there is C > 0 such that $\mu([s-c,s+c])$ for all $s \in \mathbb{R}$.

3. Applications

We now apply Theorems 1 and 2 to classify crystalline pairs. Some of the results and proofs presented here would be better understood after reading Sections 4, 5 and 6. We say an entire function E is of **Hermite-Biehler** class if

$$|E^*(z)| < |E(z)| \quad \text{for all} \quad z \in \mathbb{C}^+,$$
 (5)

where $E^*(z) = \overline{E(\overline{z})}$. In this case we always write E = A - iB, where A and B are real entire functions⁹ defined by $A = (E^* + E)/2$ and $B = (E^* - E)/(2i)$, and we always write $\Theta = E^*/E$. In particular we have the following identity

$$i\frac{A}{B} = \frac{1+\Theta}{1-\Theta}. (6)$$

Note that both A and B have only real zeros. We denote by $\varphi = \varphi_E$ a phase function associated to E. This is characterized by the condition $e^{i\varphi(x)}E(x) \in \mathbb{R}$ for all real x (φ is unique modulo an integer multiple of π). For instance, one could take $\varphi = \frac{1}{2i}\log\Theta$ after branch cutting all zeros and poles of Θ by vertical semi-lines. It is not hard to show that

$$\varphi'(x) = \operatorname{Re} i \frac{E'(x)}{E(x)} = \partial_y \log \Theta(x + iy)|_{y=0} > 0,$$

for all real x, and so $\varphi(x)$ is an increasing function of real x. We also have that

$$e^{2i\varphi(x)} = \frac{A(x)^2}{|E(x)|^2} - \frac{B(x)^2}{|E(x)|^2} + 2i\frac{A(x)B(x)}{|E(x)|^2},$$

for all real x. As a consequence, the points $\gamma \in \mathbb{R}$ such that $\varphi(\gamma) \equiv 0 \pmod{\pi}$ coincide with the real zeros of B/E and the points $s \in \mathbb{R}$ such that $\varphi(s) \equiv \pi/2 \pmod{\pi}$ coincide with the real zeros of A/E and, because $\varphi' > 0$, these zeros are simple. In particular, A/B has only simple real zeros and simple real poles, which interlace, and

$$\{\gamma \in \mathbb{R} : \varphi_j(\gamma) \equiv 0 \pmod{\pi}\} = \operatorname{Zeros}(B_j/A_j) \text{ and } \frac{1}{\varphi_j'(\gamma)} = \operatorname{Res}_{z=\gamma}(A_j/B_j) > 0.$$

Moreover, if E has no real zeros then A and B have only real simple zeros which interlace¹⁰. The next result characterizes FS-pairs (μ, a) for which $\mu \ge 0$ has locally finite support.

Theorem 3. Let E = A - iB be of Hermite-Biehler class such that $A/B \in AP(\mathbb{C}^+)$ and $\lambda \in \mathbb{R} \mapsto \mathbb{E}(A/B)(\lambda)$ is locally summable. Let

$$a(\lambda) = \overline{a(-\lambda)} = \mathbb{E}(iA/B)(\lambda) = \lim_{T \to \infty} \frac{1}{T} \int_{-T+i}^{T+i} i \frac{A(z)}{B(z)} e^{-2\pi i \lambda z} dz \quad (for \ \lambda > 0),$$

$$\frac{1}{2}a(0) = \operatorname{Re} \mathbb{E}(iA/B)(0) = \operatorname{Re} \lim_{T \to \infty} \frac{1}{T} \int_{-T+i}^{T+i} i \frac{A(z)}{B(z)} dz,$$

$$\mu = 2\pi \sum_{\varphi_E(\gamma) \equiv 0 \, (\text{mod } \pi)} \frac{1}{\varphi'_E(\gamma)} \delta_{\gamma}.$$

$$(7)$$

⁹Entire functions which attain only real values on the real line.

¹⁰For more on this we recommend [12] and the introduction of [20].

Then (μ, a) is a real-antipodal FS-pair such that $\mu \ge 0$ and μ has locally finite support¹¹. Conversely, if (μ, a) is a real-antipodal FS-pair such that $\mu \ge 0$, supp (μ) is locally finite and $a(\cdot)$ has exponential growth, then (μ, a) has to be built from the construction above.

Remark 4. Observe that by Lemma 12(iii), (iv), (v) and identity (6), $A/B \in AP(\mathbb{C}^+)$ if and only if $\Theta = E^*/E \in AP(\mathbb{C}^+)$ and A/B has locally finite spectrum if and only if Θ has. Note also that if $E, E^* \in AP(\mathbb{C}^+)$ then $\Theta \in AP(\mathbb{C}^+)$. In particular, if E is a trigonometric polynomial of Hermite-Biehler class, then iA/B belongs to $AP(\mathbb{C}^+)$ and has locally finite spectrum, hence Theorem 3 applies and the pair (μ, a) is crystalline $((\mu, a) \in CM, see Appendix)$.

Remark 5 (The Kurasov & Sarnak construction). Theorem 3 generalizes a construction of Kurasov & Sarnak [25]. They construct a crystalline FS-pair $(\mu, a) \in CM$ by letting

$$\mu = \sum_{Q(\gamma)=0} m(\gamma) \boldsymbol{\delta}_{\gamma},$$

where Q is any given trigonometric polynomial with only real zeros γ and multiplicities $m(\gamma)$. Theorem 3 majorates this construction. Indeed, since Q has finite exponential type, Hadamard's factorization implies that

$$Q(z) = z^n e^{c_1 z + c_2} \prod_{\gamma \neq 0} (1 - z/\gamma) e^{z/\gamma}$$

for some $c_1, c_2 \in \mathbb{C}$ and $n \geq 0$, where the $\gamma's$ are the real zeros of Q. In particular $e^{-c_2-iz\operatorname{Im} c_1}Q$ is a real entire trigonometric polynomial, so we can assume that Q is real on the real line. If that is the case then $c_1 \in \mathbb{R}$ and

Re
$$i\frac{Q'(z)}{Q(z)} = \sum_{\gamma} \frac{y}{(x-\gamma)^2 + y^2} > 0.$$

We obtain that E = Q' - iQ is a trigonometric polynomial of Hermite-Biehler class by a routine calculation. By the previous remark, Theorem 3 applies. It is also easy to see that the zeros of Q coincide with the points $\gamma \in \mathbb{R}$ such that $\varphi_E(\gamma) \equiv 0 \pmod{\pi}$ and $\varphi'_E(\gamma)^{-1} = \operatorname{Res}_{z=\gamma}(A/B) = m(\gamma)$. Thus, by Theorem 3 (with $a(\cdot)$ as in (7)), both μ and $a(\cdot)$ have locally finite support, and so (μ, a) is crystalline $((\mu, a) \in \operatorname{RRTP})$, see Appendix).

Remark 6 (The Ulanovskii & Oleveskii result). We now explain how one can prove the main result of Ulanovskii & Oleveskii [39] using Theorem 3, but under milder decay conditions. We claim that if (μ, a) is an FS-pair such $\mu \geq 0$ is N-valued, $a(\cdot)$ has locally finite support and exponential growth then the Hermite-Biehler function E = A - iBgiven by Theorem 3 is a trigonometric polynomial, and so the construction from Remark 5 applies. Indeed, since $\deg(\mu) \leq 2$ one can take \mathcal{B} to have the same zeros of B, but with multiplicities given by the weights of μ . We can take such \mathcal{B} with order(\mathcal{B}) ≤ 1 by [31, Thm. 6, p. 16]. Using (2) we conclude that $\operatorname{Re} i\mathcal{B}'/\mathcal{B} = \operatorname{Re} A/B$ in \mathbb{C}^+ , and

 $[\]overline{{}^{11}}$ A set $S \subset \mathbb{R}$ is locally finite if $\#(S \cap (a,b)) < \infty$ for any a < b.

so Poisson representation shows that $A/B = \mathcal{B}'/\mathcal{B} + h$ for some $h \in \mathbb{R}$, hence we can assume that A = B'. Since $a(\cdot)$ has locally finite support, then B'/B has locally finite spectrum contained in $[0, \infty)$. Lemma 13 shows that $B \in \operatorname{AP}(\mathbb{C}^+)$ and it has locally finite spectrum bounded from below. The class $\operatorname{AP}(\mathbb{C}^+)$ is closed under differentiation, and so $B' \in \operatorname{AP}(\mathbb{C}^+)$, and therefore, $E \in \operatorname{AP}(\mathbb{C}^+)$, both with locally finite spectrum bounded from below. Note that B' also has order ≤ 1 , and so E has order ≤ 1 . We can then apply Lemma 17 to conclude that E is a trigonometric polynomial.

Proof of Theorem 3. Let f = iA/B. Since E = A - iB is Hermite-Biehler, we obtain that $\operatorname{Re} f > 0$ in \mathbb{C}^+ , and so $|e^{-f}| < 1$ in \mathbb{C}^+ , hence e^f is of bounded type. Lemma 12(iv) shows that $\Theta = \frac{1-f}{1+f} \in \operatorname{AP}(\mathbb{C}^+)$ and that $|\Theta| < 1 - c_{\varepsilon}$ for $\operatorname{Im} z > \varepsilon$, so $f = \frac{1+\Theta}{1-\Theta}$ is bounded for $\operatorname{Im} z > \varepsilon$, for any $\varepsilon > 0$. In particular $\lim_{y \to \infty} \operatorname{Re} f(iy)/y = 0$. Moreover, $\mathbb{E} f(\lambda)$ is locally summable by hypothesis. We conclude that Theorem 2 applies. Poisson representation [12, Thm. 4 & Prob. 89] for f implies that μ has the form given by (7). Conversely, if (μ, a) is a real-antipodal FS-pair such that $\mu \geq 0$, $\operatorname{supp}(\mu)$ is locally finite and $a(\cdot)$ has exponential growth, then we can apply Theorem 4 to obtain that (7) holds for some Hermite-Biehler function E = A - iB such that A(z + ic)/B(z + ic) belongs to $\operatorname{AP}(\mathbb{C}^+)$ for some c > 0. However, Lemma 12 also shows that $\Theta(z + ic) = E^*(z + ic)/E(z + ic)$ belongs to $\operatorname{AP}(\mathbb{C}^+)$, and since $|\Theta| < 1$ in \mathbb{C}^+ , Lemma 11 shows that $\Theta \in \operatorname{AP}(\mathbb{C}^+)$ and, by Lemma 12 again we obtain that $A/B \in \operatorname{AP}(\mathbb{C}^+)$. Since $a(\lambda)$ is locally summable and $a(\lambda) = \mathbb{E} f(\lambda)$ for $\lambda > 0$, then $\mathbb{E} f(\lambda)$ is locally summable.

The next result classifies FS-pairs (μ, a) for which μ has locally finite support.

Theorem 4. Let (μ, a) be an FS-pair such that μ has locally finite support, $\deg(\mu) \leq 2$ and $a(\cdot)$ has exponential growth. Then there exists four Hermite-Biehler functions $E_j = A_j - iB_j$ and numbers $p_j \in \{0, 1\}$, for j = 1, 2, 3, 4, such that if we let $f = ip_1A_1/B_1 - ip_2A_2/B_2$ and $g = ip_3A_3/B_3 - ip_4A_4/B_4$ then $f(\cdot + ic)$ and $g(\cdot + ic)$ belong to $AP(\mathbb{C}^+)$ for some c > 0, $\mathbb{E}f(\lambda)$ and $\mathbb{E}g(\lambda)$ are locally summable, and:

$$\begin{split} \frac{1}{2\pi}\mu &= p_1 \sum_{\varphi_1(\gamma) \equiv 0 \, (\text{mod } \pi)} \frac{1}{\varphi_1'(\gamma)} \boldsymbol{\delta}_{\gamma} - p_2 \sum_{\varphi_2(\gamma) \equiv 0 \, (\text{mod } \pi)} \frac{1}{\varphi_2'(\gamma)} \boldsymbol{\delta}_{\gamma} \\ &- i p_3 \sum_{\varphi_3(\gamma) \equiv 0 \, (\text{mod } \pi)} \frac{1}{\varphi_3'(\gamma)} \boldsymbol{\delta}_{\gamma} + i p_4 \sum_{\varphi_4(\gamma) \equiv 0 \, (\text{mod } \pi)} \frac{1}{\varphi_4'(\gamma)} \boldsymbol{\delta}_{\gamma}; \\ a(\lambda) &= \mathbb{E} f(\lambda) - i \mathbb{E} g(\lambda) \quad and \quad a(-\lambda) = \overline{\mathbb{E} f(\lambda) + i \mathbb{E} g(\lambda)} \quad (for \, \lambda > 0); \\ \frac{1}{2} a(0) &= \text{Re} \, \mathbb{E} f(0) - i \text{Re} \, \mathbb{E} g(0), \end{split}$$

where φ_j is the phase function associated with E_j . Moreover: (i) $\mu \geqslant 0$ if and only if $p_1 = 1$ and $p_2 = p_3 = p_4 = 0$; (ii) $a(\cdot)$ has locally finite support if and only if f and g have locally finite spectrum. Furthermore, if μ is \mathbb{N} -valued then we can take $A_1 = B'_1$.

Proof of Theorem 4. By real-antipodal splitting we can assume that μ is real-valued (so $p_3 = p_4 = 0$). We can then apply Theorem 1 and obtain that

$$f(z) = \frac{a(0)}{2} + \lim_{T \to \infty} \sum_{0 < \lambda < T} (1 - \lambda/T) a(\lambda) e^{2\pi i \lambda z} = ih + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{1 + tz}{t - z} \frac{\mathrm{d}\mu(t)}{1 + t^2},$$

for some $h \in \mathbb{R}$, where f is holomorphic in \mathbb{C}^+ and $f(\cdot + ic) \in AP(\mathbb{C}^+)$ for some c > 0. Let $\mu = \sum_{\gamma \in \Lambda} r(\gamma) \delta_{\gamma}$ where $\Lambda \subset \mathbb{R}$ is locally finite and $r(\gamma) = r_1(\gamma) - r_2(\gamma)$, with $r_j \ge 0$. Let

$$f_j(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{1+tz}{t-z} \frac{\mathrm{d}\mu_j(t)}{1+t^2}$$

for j=1,2, where $\mu_j=\sum_{\gamma\in\Lambda}r_j(\gamma)\boldsymbol{\delta}_{\gamma}$. Observe that each f_j is meromorphic in $\mathbb C$ with only simple poles (possibly) at Λ . Since $\mu_j\geqslant 0$ we have $if_j(\mathbb C^+)\subset\mathbb C^+\cup\{0\}$, and we can then apply a classical result [31, p. 308, Thm. 1] to deduce this happens exactly when there are two real entire functions A_j and B_j with only real zeros that interlace and such that $if_j=-p_jA_j/B_j$, with $p_j\in\{0,1\}$ to account for the case when $\mu_j=0$ for some j=0,1. Since $\operatorname{Re} iA_j/B_j>0$, we deduce that $E_j=A_j-iB_j$ is of Hermite-Biehler class, and so, Poisson representation shows that

$$\mu = 2\pi p_1 \sum_{\varphi_1(\gamma) \equiv 0 \, (\text{mod } \pi)} \frac{1}{\varphi_1'(\gamma)} \delta_{\gamma} - 2\pi p_2 \sum_{\varphi_2(\gamma) \equiv 0 \, (\text{mod } \pi)} \frac{1}{\varphi_2'(\gamma)} \delta_{\gamma}.$$

Noting that $f = f_1 - f_2 + ih$, we obtain $f_1(\cdot + ic) - f_2(\cdot + ic) \in AP(\mathbb{C}^+)$, and since $a(\cdot)$ has exponential growth we conclude that $a(\lambda) = \mathbb{E}(f_1 - f_2)(\lambda)$ for $\lambda > 0$ and $\frac{1}{2}a(0) = \text{Re}\,\mathbb{E}(f_1 - f_2)(0)$. If $\mu \ge 0$ then $(p_1, p_2) = (1, 0)$. Also, $a(\cdot)$ has locally finite support if and only if $f_1 - f_2$ has locally finite spectrum. Finally, if μ is \mathbb{N} valued then, since $\deg(\mu) \le 2$, one might take \mathcal{B} to have zeros exactly at support of μ with multiplicity determined by the weights of μ and order at most 1. In particular, Hadamard factorization and log-differentiation show that $\operatorname{Re} i\mathcal{B}'/\mathcal{B} = \operatorname{Re} iA_1/B_1$ and so $f_1 = ih + i\mathcal{B}'/\mathcal{B}$, and $\mathcal{E} = \mathcal{B}' - i\mathcal{B}$ is now Hermite-Biehler.

In what follows we say that an entire function E is of **Pólya class** if E has no zeros in \mathbb{C}^+ , $|E^*/E| \leq 1$ in \mathbb{C}^+ and $y \mapsto |E(x+iy)|$ is nondecreasing for y > 0. A classical theorem [12, Thm. 7] shows that a function E is of Pólya class if and only if

$$E(z) = E^{(r)}(0)(z^r/r!)e^{-dz^2-ibz}\prod_n (1-z/\overline{z_n})e^{z\operatorname{Re} 1/z_n},$$

where $d \ge 0$, Re $b \ge 0$, $\{\overline{z}_n = x_n - iy_n\}$ are the nonzero zeros of E (with $y_n \ge 0$) and

$$\sum_{n} \frac{1 + y_n}{x_n^2 + y_n^2} < \infty.$$

The Pólya class can be also defined as functions that are uniform limits in compact sets of polynomials with no zeros in \mathbb{C}^+ (see [12, Problem 12]). Moreover, any function of Pólya class has order at most 2 ([12, Problem 10]).

The next result is a brand new classification theorem¹² for nonnegative crystalline measures. It generalizes a result of Olevekii & Ulanovskii [39] in the uniformly discrete case, dropping the N-valuedness assumption.

Theorem 5. Let (μ, a) be an FS-pair. Assume that:

- (1) μ has uniformly discrete support and there is $\delta > 0$ such that $\mu(\{x\}) \geqslant \delta$ for all $x \in \text{supp}(\mu)$;
- (2) There is b > 0 such that $supp(a) \cap (0, b) = \emptyset$ and $a(\cdot)$ has the decay (4).

Then there exists an entire function E = A - iB of Hermite-Biehler class and of finite exponential type such that $E, E^* \in AP(\mathbb{C}^+)$ (and so $A/B \in AP(\mathbb{C}^+)$), spec(E) is bounded and (7) holds. If in addition supp(a) is locally finite, then E is a trigonometric polynomial and Remark 5 applies, and so $(\mu, a) \in RRTP$ (see Appendix).

Proof. Step 1. We first apply Theorem 3 to conclude there is an Hermite-Biehler function E = A - iB such that $A/B \in \operatorname{AP}(\mathbb{C}^+)$ and (7) holds. We can assume that E has no real zeros as these can be removed (with a Weierstrass product) without altering A/B or the fact that E is Hermite-Biehler. In particular, $\operatorname{supp}(\mu) = \{\gamma \in \mathbb{R} : \varphi(\gamma) \equiv 0 \pmod{\pi}\} = \operatorname{Zeros}(B)$, where φ is the phase function associated with E, and these zeros are all simple. The zeros of E are also simple and interlace those of E. Since E is bounded from below on its support and E has the decay (4), we can apply Remark 3 to obtain

$$c < \varphi'(\gamma) < 1/c,$$

for some 0 < c < 1, whenever $B(\gamma) = 0$. In particular, we have $\sum_{B(\gamma)=0} \frac{1}{1+\gamma^2} < \infty$ and, since the zeros of A interlace those of B, we also have $\sum_{A(s)=0} \frac{1}{1+s^2} < \infty$. We claim that we can assume E is has order at most 1. Indeed, define the canonical products

$$\mathcal{B}(z) = bz^p \prod_{B(\gamma)=0, \ \gamma \neq 0} (1 - z/\gamma)e^{z/\gamma} \text{ and } \mathcal{A}(z) = z^q \prod_{A(s)=0, \ s \neq 0} (1 - z/s)e^{z/s},$$

for some $p, q \in \{0, 1\}$, where $b \in \mathbb{R}$ is chosen so that (A(z)/B(z))/(A(z)/B(z)) = 1 + O(z) for $z \to 0$. Hence both \mathcal{A} and \mathcal{B} have order at most 1 by a classical result of Borel [31, Thm. 6, p. 16]. Since $\operatorname{Re} iA/B \geqslant 0$ in \mathbb{C}^+ and, a straitfoward computation, shows that $\operatorname{Re} iA/B \geqslant 0$ in \mathbb{C}^+ , we conclude that both A/B and A/B are of bounded type and so F = (A/B)/(A/B) also is and has no zeros. Since $F = F^*$, a classical result of Krein [24] (see also [32, Thm. 1, p. 115]) shows that F is of exponential type, and so $F(z) = e^{hz}$, for some $h \in \mathbb{R}$. Since F is of bounded type we have h = 0. We can then define $\widetilde{E} = A - i\mathcal{B}$ to finish the claim.

Step 2. Let $\{\overline{z}_n = x_n - iy_n\}$ be the zeros of E with $y_n > 0$. We claim $\sup_n y_n < \infty$ and that E is of Póyla class. Indeed, since $f = iA/B \in AP(\mathbb{C}^+)$, by Remark 4 we have $\Theta = E^*/E = (1-f)/(1+f) \in AP(\mathbb{C}^+)$. Since $\operatorname{spec}(f) \subset [0,\infty)$, a simple computation using Lemma 12(i) shows $\mathbb{E}\Theta(0) = \frac{1-\mathbb{E}f(0)}{1+\mathbb{E}f(0)}$. However, $2\operatorname{Re}\mathbb{E}f(0) = a(0) > 0$ (because $\mu \geq 0$),

¹²In a way, this is the most technically hard result of the paper.

thus $|\mathbb{E}\Theta(0)| < 1$. Assume now $\mathbb{E}\Theta(0) \neq 0$. Then Lemma 12(i) shows straightforwardly that $\Theta(x+iy)$ has no zeros for large y>0. Assume otherwise that $\mathbb{E}\Theta(0)=0$, that is, $\mathbb{E}f(0)=1$. Since $a(\lambda)=\mathbb{E}f(\lambda)$ for $\lambda>0$ we deduce that $\mathrm{spec}(f)\subset\{0\}\cup[b,\infty)$ where $b=\inf\{\lambda>0: a(\lambda)\neq 0\}>0$ by assumption. Since f has no zeros in \mathbb{C}^+ , we can then apply Lemma 12(ii) to conclude that $b\in\mathrm{spec}(f)$. Writing f=1-2g, with $\mathrm{spec}(g)\subset[b,\infty)$ and $\mathbb{E}g(b)\neq 0$, we can apply Lemma 12(i) to conclude that |g(x+iy)|<1 for large y>0, and so

$$\Theta(z) = \frac{g(z)}{1 - g(z)} = \sum_{n \ge 1} g(z)^n,$$

hence $\operatorname{spec}(\Theta) \subset [b, \infty)$ and $\mathbb{E}\Theta(b) \neq 0$. We can then apply again Lemma 12(i) to conclude that $\lim_{y\to\infty} \sup_x |e^{-2\pi i b(x+iy)}\Theta(x+iy) - \mathbb{E}\Theta(b)| = 0$, hence $\Theta(z)$ has no zeros for large Im z. Finally, since E has order at most 1 the following sum is now finite

$$\sum_{n} \frac{1 + y_n}{x_n^2 + y_n^2} < \infty.$$

and E has the following factorization

$$E(z) = E(0)e^{-ihz} \prod_{n} (1 - z/\overline{z_n})e^{z\operatorname{Re} 1/z_n}$$

for some $h \ge 0$. Hence E is of Póyla class.

Step 3. We claim now that $\sup_{x\in\mathbb{R}} \varphi'(x) < \infty$. Since (2) holds true for f = iA/B, we obtain that

$$K(w,z) := \frac{B(z)\overline{A(w)} - \overline{B(w)}A(z)}{\pi(z - \overline{w})} = \sum_{B(\gamma)=0} \frac{1}{\pi\varphi'(\gamma)} \frac{B(z)\overline{B(w)}}{(\gamma - z)(\gamma - \overline{w})}$$

for all $z, w \in \mathbb{C}$. A routine computation shows that

$$\varphi'(x) = \operatorname{Re} i \frac{E'(x)}{E(x)} = \frac{\pi K(x, x)}{|E(x)|^2} = \sum_{B(\gamma)=0} \frac{\sin^2 \varphi(x)}{\varphi'(\gamma)(\gamma - x)^2}.$$

Enumerate the Zeros(B) = $\{\gamma_n\}$ and let c > 0 be so small that $\gamma_{n+1} - \gamma_n \ge c$ for all n. Let $0 < \varepsilon < c/10$ to be chosen later. Observe first that if $\operatorname{dist}(x, \{\gamma_n\}) = |x - \gamma| \ge \varepsilon$ (with $\gamma = \gamma_{n_0}$) then $|x - \gamma_{n_0+k}| \ge \varepsilon + |k|c/2$, and so

$$\varphi'(x) \leqslant 2 \sum_{k \geqslant 0} \frac{1}{c(\varepsilon + kc/2)^2} = O(\varepsilon^{-2}).$$

On the other hand, if $\operatorname{dist}(x,\{\gamma_n\}) = |x - \gamma| \leq \varepsilon$, then $|\gamma_{n_0+k} - x| \geq \frac{c}{2}|k|$, for $k \neq 0$, and

$$\varphi'(\gamma)\varphi'(x) \leqslant \frac{\sin^2 \varphi(x)}{(\gamma - x)^2} + C\sin^2 \varphi(x)$$

for some C > 0, independent of ε and γ . Using the inequality $x^2 \ge \sin^2 x$ for all real x, and factoring $\sin^2 \varphi(x)$, we obtain

$$\frac{\varphi'(\gamma)\varphi'(x)}{\sin^2\varphi(x)} \le \frac{(\pi/c)^2}{\sin^2[(\gamma - x)\pi/c]} + C$$

We conclude that the function

$$\xi(x) = \varphi'(\gamma)\cot\varphi(x) - \frac{\pi}{c}\cot[\pi(x-\gamma)/c] + C(x-\gamma),$$

which is analytic in a complex neighbourhood of the segment $I = [\gamma - \varepsilon, \gamma + \varepsilon]$, is nondecreasing for $x \in I$ and $\xi(\gamma) = 0$. We conclude that

$$(x-\gamma)\varphi'(\gamma)\cot\varphi(x) \geqslant \frac{\pi(x-\gamma)}{c}\cot[\pi(x-\gamma)/c] - C(x-\gamma)^2$$

for $x \in I$. Since the function $\frac{\pi(x-\gamma)}{c} \cot[\pi(x-\gamma)/c]$ has value 1 for $x = \gamma$, there is $\varepsilon_0 = \varepsilon_0(c,C) > 0$, with $0 < \varepsilon_0 < c/10$, such that the right hand side above is bounded from below by $\frac{\pi(x-\gamma)}{2c} \cot[\pi(x-\gamma)/c]$ for $x \in I_0 := (\gamma - \varepsilon_0, \gamma + \varepsilon_0)$, and so

$$(x-\gamma)\varphi'(\gamma)\cot\varphi(x) \geqslant \frac{\pi(x-\gamma)}{2c}\cot[\pi(x-\gamma)/c]$$

for $x \in I_0$. We now select $\varepsilon = \varepsilon_0$. Observing the right hand side above is positive for $x \in I_0$, we can square both sides above, factor out $(x - \gamma)^2$, lower bound $\varphi'(\gamma) \ge c$ and isolate $\sin^2 \varphi$ to obtain

$$\sin^2 \varphi(x) \leqslant \frac{1}{1 + C \cot^2 [\pi(x - \gamma)/c]}$$

for $C = \pi^2/(4c^4)$. Note also that for $x \in I_0$ we have $|x - \gamma_{n_0 + k}| \ge |x - \gamma_{n_0} - kc|$ for any $k \in \mathbb{Z}$, hence

$$\varphi'(x) \leqslant \frac{1}{c} \sum_{k \in \mathbb{Z}} \frac{\sin^2 \varphi(x)}{(x - \gamma + kc)^2} = \frac{\pi^2}{c^3} \frac{\sin^2 \varphi(x)}{\sin^2 [\pi (x - \gamma)/c]}$$

$$\leqslant \frac{\pi^2}{c^3} \frac{1}{\sin^2 [\pi (x - \gamma)/c] + C \cos^2 [\pi (x - \gamma)/c]} \leqslant \frac{\pi^2}{c^3}.$$

for $x \in I_0$. This proves the claim.

Step 4. We claim the zeros of E are separated from the real axis, that is, there is c > 0 such that $y_n \ge c$ for all n. Indeed, observe that since $E(z)e^{i\varphi(z)} = E^*(z)e^{-i\varphi(z)}$ whenever $\varphi(z)$ is defined (any neighborhood of \mathbb{R} not containing $\{z_n\}_{n\in\mathbb{Z}} \cup \{\overline{z_n}\}_{n\in\mathbb{Z}}$) we obtain that

$$\varphi'(z) = \partial_z \frac{1}{2i} \log \Theta(z) = \frac{1}{2i} \frac{\Theta'(z)}{\Theta(z)},$$

with $\Theta = E^*/E$. Using the factorization of E we obtain

$$\Theta(z) = e^{2ihz} \prod_{n \in \mathbb{Z}} \frac{1 - z/z_n}{1 - z/\overline{z_n}},$$

and so

$$\varphi'(x) = h + \sum_{n \in \mathbb{Z}} \frac{y_n}{|x - z_n|^2}$$

for real x. Hence $\varphi'(x_n) \ge 1/y_n$, and since $\varphi'(x)$ is bounded, the claim follows.

Step 5. We claim that $\varphi' \in AP(\mathbb{R})$ and $\operatorname{spec}(\varphi') \subset [b, \infty)$. Let c > 0 be small enough such that Θ has no zeros or poles in the strip $S = \{-c < \operatorname{Im} z < c\}$. First we show that

 Θ is bounded in every horizontal strip contained in S. We only need to show this for $z \in S$ with $\text{Im } z \leq 0$. Indeed, since

$$\frac{1}{2}\partial_y \log |\Theta^*(x+iy)| = h + \sum_{n \in \mathbb{Z}} \frac{y_n[(x-x_n)^2 + y_n^2 - y^2]}{|z-z_n|^2 |z-\overline{z_n}|^2}$$

and $y_n^2 - y^2 \le 3(y_n - y)^2$ for $0 \le y \le c$, we deduce that

$$\frac{1}{2}\partial_y \log |\Theta^*(x+iy)| \le h + 3\sum_{n \in \mathbb{Z}} \frac{y_n}{|x-z_n|^2} \le 3\varphi'(x) \le C$$

for some C>0. Since $\log |\Theta^*(x)|=0$, integration shows that $\Theta^*(x+iy)\leqslant e^{2Cy}$ for $0\leqslant y\leqslant c$. Since $\Theta\in \operatorname{AP}(\mathbb{C}^+)$, we can now apply Lemma 11 to deduce that $\Theta(\cdot-ic)\in\operatorname{AP}(\mathbb{C}^+)$ and $\Theta'(\cdot-ic)\in\operatorname{AP}(\mathbb{C}^+)$. By Lemma 12(iv), we get that $|\Theta|$ is bounded away from zero in every horizontal strip contained in S. We deduce that $\Theta'/\Theta\in\operatorname{AP}(\mathbb{R})$, and $\operatorname{so}^{13}\varphi'\in\operatorname{AP}(\mathbb{R})$. Step 1 shows that $\operatorname{spec}(\Theta)\subset\{0\}\cup[b,\infty)$ and $\mathbb{E}\Theta(b)\neq 0$, hence $\operatorname{spec}(\Theta'/\Theta)\subset[b,\infty)$ and $\operatorname{spec}(\varphi')\subset[b,\infty)$.

Step 6. Since E is of Pólya class we have $\operatorname{Re} iE'/E \geqslant 0$ in \mathbb{C}^+ , and Poisson representation guarantees that

$$\frac{iE'(z)}{E(z)} = id - ipz + \frac{1}{\pi i} \int_{\mathbb{R}} \frac{1 + tz}{t - z} \frac{\varphi'(t) dt}{1 + t^2},$$

for some $d \in \mathbb{R}$ and $p \ge 0$, where $\deg(\varphi'(t)dt) \le 2$. This fact, in conjunction with the factorization of E shows that

Re
$$\frac{iE'(z)}{E(z)} = py + \frac{y}{\pi} \int_{\mathbb{R}} \frac{\varphi'(t)dt}{(x-t)^2 + y^2} = h + \sum_{n} \frac{y + y_n}{(x-x_n)^2 + (y+y_n)^2}.$$

Since $\sum_n \frac{1+y_n}{x_n^2+y_n^2} < \infty$, we conclude that $p = \lim_{y\to\infty} \operatorname{Re} \frac{iE'(iy)}{yE(iy)} = 0$. We can apply Lemma 16 so deduce that $iE'/E \in \operatorname{AP}(\mathbb{C}^+)$ and $\operatorname{spec}(iE'/E) \subset \{0\} \cup [b,\infty)$. We can apply Lemma 13 to conclude that $E \in \operatorname{AP}(\mathbb{C}^+)$ and that $\operatorname{spec}(E)$ is bounded from below. Since $E^* = \Theta E$, we conclude that $E^* \in \operatorname{AP}(\mathbb{C}^+)$ also and that $\operatorname{spec}(E^*)$ is bounded from below. Finally, we can use Lemma 17 to deduce the spectrum of E (and of E^*) is bounded and E is of exponential type. To finish, if in addition $\operatorname{supp}(a)$ is locally finite, then so is $\operatorname{spec}(A/B)$. Since $a(\cdot)$ has exponential growth (due to (4)), we obtain that Θ has locally finite spectrum, thus so φ' has locally finite spectrum. The previous applications of Lemmas 16 and 13 (and their content) show that E has locally finite spectrum as well, and so Lemma 17 shows that E is a trigonometric polynomial.

We believe that Theorem 5 should still hold only assuming that $\operatorname{supp}(\mu)$ is locally finite. Note that by the decay of $a(\cdot)$ and Remark 3, $\operatorname{supp}(\mu)$ is contained in a finite union of uniformly discrete sets. Also, the uniformly discreteness of $\operatorname{supp}(\mu)$ plays a role only in Step 3 above, where we show φ' is bounded as a stepping stone to show the zeros of E cannot get close to the real axis. Note that if φ' is bounded then $\operatorname{supp}(\mu)$

¹³If we let AP(S) be defined in the same way as in $AP(\mathbb{C}^+)$, but considering only horizontal strips strictly contained in S, we have showed that $\varphi' \in AP(S)$.

is uniformly discrete. However, we still think there must be a way to circumvent this issue, and nevertheless conclude that the zeros of E are separated from the real axis. Unfortunately we were not able to come up with such maneuver, despite many efforts.

- 3.1. de Branges spaces a Hilbert space interpretation of FS-pairs. A de Branges space [12] (see also the introduction of [20]) is a Hilbert space $(\mathfrak{H}, \|\cdot\|)$ of entire functions $F: \mathbb{C} \to \mathbb{C}$ satisfying:
 - (H1) If $F \in \mathfrak{H}$ and F(w) = 0 for some $w \in \mathbb{C}$, then $G(z) = F(z) \frac{z \overline{w}}{z w}$ belongs to \mathfrak{H} and $\|G\| = \|F\|$;
 - (H2) The functional $F \in \mathfrak{H} \mapsto F(w)$ is continuous for every $w \in \mathbb{C}$;
 - (H3) If $F \in \mathfrak{H}$ then $F^* \in \mathfrak{H}$ and $||F^*|| = ||F||$.

Because of (H2), the space \mathfrak{H} comes equipped with an unique reproducing kernel K(w, z), that is, $F(w) = \langle F, K(w, \cdot) \rangle$ for any $w \in \mathbb{C}$ and $F \in \mathfrak{H}$. De Branges proved [12, Thm. 23] that for any such Hilbert space \mathfrak{H} there exists an Hermite-Biehler function E such that $\mathfrak{H} = \mathfrak{H}(E)$ isometrically, where $\mathfrak{H}(E)$ is the space of entire functions F such that $\mathfrak{H}(E)$

$$||F||^2 := \sup_{y \in \mathbb{R}} \int_{\mathbb{R}} \left| \frac{F(x+iy)}{E(x+i|y|)} \right|^2 dx < \infty,$$

in which case the sup above is attained at y = 0. De Branges also shows [12, Thm. 19] that any such space $\mathfrak{H}(E)$ is a Hilbert space that satisfies the above axioms. When we identify $\mathfrak{H} = \mathfrak{H}(E)$ for some Hermite-Biehler function E = A - iB we have

$$K(w,z) = \frac{B(z)\overline{A(w)} - \overline{B(w)}A(z)}{\pi(z - \overline{w})} = \frac{E(z)\overline{E(w)} - E^*(z)E(\overline{w})}{2\pi i(\overline{w} - z)}$$
(8)

and

$$F(w) = \int_{\mathbb{R}} \frac{F(t)\overline{K(w,t)}}{|E(t)|^2} dt$$

for every $w \in \mathbb{C}$ and $F \in \mathfrak{H}$. The function realizing $\mathfrak{H} = \mathfrak{H}(E)$ is not unique. Indeed, if $\mathfrak{H}(E) = \mathfrak{H}(E_1)$ isometrically, one can then apply [12, Problem 69] (for $S = A_1$ and $S = B_1$) to obtain

$$E_1(z) = \frac{e^{i\beta}}{\sqrt{1 - |p|^2}} (E(z) - \overline{p}E^*(z))$$

for some $\beta \in \mathbb{R}$ and $p \in \mathbb{C}$, with |p| < 1. Conversely, a routine computation shows that any E_1 defined in the above way satisfies (8), and since K(z,z) > 0 for $z \in \mathbb{C}^+$, we conclude that E_1 is Hermite-Biehler, and so $\mathfrak{H}(E) = \mathfrak{H}(E_1)$ isometrically. Moreover, their theta functions are related by the Möbius transformation

$$\Theta_1(z) = e^{2i\beta} \frac{\Theta(z) - p}{1 - \Theta(z)\overline{p}}.$$

A major result [12, Thm. 22] is that, for any $\alpha \in [0, \pi)$, the set $\{K(\gamma, z)\}_{\varphi(\gamma) \equiv \alpha \pmod{\pi}}$ is orthogonal in $\mathfrak{H}(E)$, and its orthogonal complement is one-dimensional and spanned

¹⁴The original definition involves bounded type theory, but this is an equivalent definition, see [20].

by $e^{i\alpha}E - e^{-i\alpha}E^*$. In particular, when $\alpha = 0$, E has no real zeros and $B \notin \mathfrak{H}(E)$, then representation of functions in the basis $\{K(\gamma,z)\}_{B(\gamma)=0}$ implies the identities

$$\pi K(w,z) := \frac{B(z)\overline{A(w)} - \overline{B(w)}A(z)}{z - \overline{w}} = \sum_{B(\gamma)=0} \frac{1}{\varphi'(\gamma)} \frac{B(z)\overline{B(w)}}{(\gamma - z)(\gamma - \overline{w})}$$

and

$$F(z) = \sum_{B(\gamma)=0} \frac{F(\gamma)B(z)}{B'(\gamma)(z-\gamma)}, \text{ for any } F \in \mathfrak{H}(E),$$

with convergence in $\mathfrak{H}(E)$ and uniformly (and absolutely) in compact sets of \mathbb{C} . Letting f = iA/B, if one divides the expansion of K above by iB(z)B(w) we get

$$\frac{f(z) + \overline{f(w)}}{z - \overline{w}} = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{(t - z)(t - \overline{w})}$$

with $\mu = \sum_{B(\gamma)=0} \frac{2\pi}{\varphi'(\gamma)} \delta_{\gamma}$. Assume now that $f = iA/B \in AP(\mathbb{C}^+)$ and that $\mathbb{E}f(\cdot)$ is locally summable. Thus, Theorem 3 applies and (μ, a) is a FS-pair (with $a(\cdot)$ as in (7)). There is no particular reason (other than convenience) to take $\alpha = 0$ here. One could select as well any $\alpha \in [0, \pi)$, but now the FS-pair would be¹⁵

$$\mu = \sum_{\varphi(\gamma) \equiv \alpha \pmod{\pi}} \frac{2\pi}{\varphi'(\gamma)} \delta_{\gamma},$$

$$a(\lambda) = \overline{a(-\lambda)} = \mathbb{E}f(\lambda) \ (\lambda > 0)$$

$$a(0) = 2\operatorname{Re} \mathbb{E}f(0),$$

assuming $f = (1 + e^{-2i\alpha}\Theta)/(1 - e^{-2i\alpha}\Theta) \in AP(\mathbb{C}^+)$ (note if $E_{\alpha} = e^{i\alpha}E = A_{\alpha} - iB_{\alpha}$ then $f = iA_{\alpha}/B_{\alpha}$).

One could ask the question whether there is another natural axiom, as the ones above, that would force the space \mathfrak{H} to "produce" an FS-pair. Such axiom indeed exists.

Proposition 6. Let $(\mathfrak{H}, \|\cdot\|)$ be a Hilbert space of entire functions satisfying (H1), (H2) and (H3), with reproducing kernel K(w,z). Assume also that:

(H4) There is $\alpha \in \mathbb{C}^+$ such that the function

$$f(z) = \frac{(\overline{\alpha} - z)K(\alpha, z) - (\alpha - z)K(\overline{\alpha}, z)}{(\overline{\alpha} - z)K(\alpha, z) + (\alpha - z)K(\overline{\alpha}, z)}$$

belongs to $AP(\mathbb{C}^+)$ and $\mathbb{E}f(\cdot)$ is locally summable.

Then there is an Hermite-Biehler function E = A - iB such that $\mathfrak{H} = \mathfrak{H}(E)$ isometrically, and, if we define μ and $a(\cdot)$ as in (7), we have that (μ, a) is a FS-pair.

Proof. We can then apply [12, Thm. 23, p. 58] to obtain that the identity

$$\frac{(K(w,z)-K(\beta,z)K(\beta,\beta)^{-1}K(w,\beta))(z-\overline{\beta})}{z-\beta} = \frac{(K(w,z)-K(\overline{\beta},z)K(\overline{\beta},\overline{\beta})^{-1}K(w,\overline{\beta}))(\overline{w}-\overline{\beta})}{\overline{w}-\beta}$$

The proof of Theorem 1 implies that $(e^{i\alpha}E - e^{-i\alpha}E^*) \notin \mathfrak{H}(E)$

holds for all $w, z \in \mathbb{C}$ and $\beta \in \mathbb{C}^+$. Let $L(w, z) = 2\pi i(\overline{w} - z)K(w, z)$ and

$$E_1(z) = L(\alpha, \alpha)^{-1/2} L(\alpha, z).$$

A routine computation, using the above identity, now shows that

$$K(w,z) = \frac{E_1(z)\overline{E_1(w)} - E_1^*(z)E_1(\overline{w})}{2\pi i(\overline{w} - z)}.$$

Since 0 < K(z, z) for $z \in \mathbb{C}^+$, we conclude that E_1 is of Hermite-Biehler class, thus $\mathfrak{H} = \mathfrak{H}(E_1)$ isometrically. Letting $E_1 = A_1 - iB_1$ and noting that $L(\alpha, z)^* = -L(\overline{\alpha}, z)$, we deduce that

$$f(z) = \frac{1 + \frac{L(\alpha, z)^*}{L(\alpha, z)}}{1 - \frac{L(\alpha, z)^*}{L(\alpha, z)}} = \frac{1 + \Theta_1(z)}{1 - \Theta_1(z)} = i \frac{A_1(z)}{B_1(z)}.$$

We can then apply Theorem 3 to finish the proof.

4. Almost Periodic Functions in \mathbb{R}

We say that a continuous function $f: \mathbb{R} \to \mathbb{C}$ in almost periodic (in the sense of Bohr [15]) if for every $\varepsilon > 0$ the following set is relatively dense

$$\tau_{\varepsilon}(f) := \{ t \in \mathbb{R} : \sup_{x \in \mathbb{R}} |f(x) - f(x+t)| \leqslant \varepsilon \}.$$

This set is called the set of ε -translations for f. We denote by $\operatorname{AP}(\mathbb{R})$ the set of continuous almost period functions $f:\mathbb{R}\to\mathbb{C}$. It is not hard to show that any $f\in\operatorname{AP}(\mathbb{R})$ is bounded and uniformly continuous, and that $\tau_{\varepsilon}(f)$ is also closed with non-empty interior. The following is a very useful criteria for almost periodicity (see [3, p. 7]). In what follows $C(\mathbb{R})$ denotes the usual Banach algebra of bounded continuous functions with the topology induced by the sup-norm $\|f\|_{\infty} = \sup_{x\in\mathbb{R}} |f(x)|$.

In the remaining part of this section we compile necessary facts about almost periodic functions and we provide some proofs for completeness.

Theorem 7 (Bochner's criterion). Let $f : \mathbb{R} \to \mathbb{C}$ be continuous. Then $f \in AP(\mathbb{R})$ if and only if the set of functions $\{f(\cdot + h)\}_{h \in \mathbb{R}}$ is pre-compact in $C(\mathbb{R})$.

Remark 7. Bochner's criterion works also for functions $f : \mathbb{R} \to \mathbb{C}^n$. In particular, if $f_j \in AP(\mathbb{R})$ for j = 1, ..., N, then $\bigcap_{j=1}^N \tau_{\varepsilon}(f_j)$ is nonempty and relatively dense.

It is now straightforward to show that $AP(\mathbb{R})$ closed under multiplication, addition and uniform convergence, hence $AP(\mathbb{R})$ is a closed subalgebra of $C(\mathbb{R})$. Moreover, it also follows that for every continuous $g: \mathbb{C} \to \mathbb{C}$ we have $g \circ f \in AP(\mathbb{R})$ whenever $f \in AP(\mathbb{R})$. Since exponentials $e^{2\pi ix\lambda}$, with $\lambda \in \mathbb{R}$, are periodic, we conclude by Bochner's criterion that any **trigonometric polynomial**

$$p(x) = \sum_{n=1}^{N} a_n e^{2\pi i \lambda_n x} \qquad (\lambda_n \in \mathbb{R})$$

belongs to $AP(\mathbb{R})$.

Lemma 8. Given any $f \in AP(\mathbb{R})$ and $\varepsilon > 0$, there is a trigonometric polynomial p such that $||f - p||_{\infty} < \varepsilon$. In particular, $AP(\mathbb{R})$ is the closure in $C(\mathbb{R})$ of the algebra of trigonometric polynomials.

Proof. We can assume $||f||_{\infty} \leq 1$. For a given $\varepsilon > 0$ let τ_{ε} be the set of ε -translations for f. First we will need some auxiliary functions. We claim that for every sufficiently large $M \geq 1$ there exists $\varphi_M \in C_c^{\infty}(\tau_{\varepsilon} \cap (-M, M))$ such that $0 \leq \varphi_M \leq 1$, $\int_{\mathbb{R}} \varphi_M = 1$, $\int_{\mathbb{R}} |\varphi_M'| = O_{\varepsilon}(1)$ and

$$\sum_{n\in\mathbb{Z}} |\widehat{\varphi}(n/(4M))|^{1/2} = O_{\varepsilon}(1).$$

Assuming this claim is true, we now finish the proof. Let

$$f_M(x) = \int_{\mathbb{R}} f(x+t)\varphi_M(t)dt$$

and note that $||f - f_M||_{\infty} \leq \varepsilon$ for any M. Also note that if $|x| \leq M$ then

$$f_M(x) = \int_{-2M}^{2M} f(t)\varphi_M(t-x)dt.$$

The function $\varphi_M(x)$ can be identified with a 4M-periodic C^{∞} -function and so we have

$$\varphi_m(x) = \sum_{n \in \mathbb{Z}} \theta_n e^{2\pi i n x/(4M)}$$

where $\sum_{n\in\mathbb{Z}} |\theta_n|^2 = (4M)^{-1} \int_{-2M}^{2M} |\varphi_M(x)|^2 dx$ and

$$\theta_n = (4M)^{-1} \int_{-2M}^{2M} \varphi_M(x) e^{-2\pi i n x/(4M)} dx = \widehat{\varphi}_M(n/(4M))/(4M).$$

We obtain

$$f_M(x) = \sum_{n \in \mathbb{Z}} \widetilde{\theta}_n e^{-2\pi i n x/(4M)}$$
 for $|x| \leqslant M$,

where $\widetilde{\theta}_n = \theta_n \int_{-2M}^{2M} f(t) e^{2\pi i n t/(4M)} dt$. Note this representation converges absolutely since $|\widetilde{\theta}_n| \leq 4M |\theta_n| = |\widehat{\varphi}_M(n/(4M))| \leq 1$, $\sum_{n \in \mathbb{Z}} |\widetilde{\theta}_n|^{1/2} = O_{\varepsilon}(1)$, and so, $\sum_{n \in \mathbb{Z}} |\widetilde{\theta}_n| = O_{\varepsilon}(1)$. Now enumerate $(\widetilde{\theta}_n)_{n \in \mathbb{Z}}$ in decreasing order of magnitude and call this new sequence $(\alpha_{M,n})_{n \geq 1}$. We obtain

$$f_M(x) = \sum_{n \ge 1} \alpha_{M,n} e^{2\pi i \lambda_{M,n} x}$$
 for $|x| \le M$,

for some $\lambda_{M,n} \in \frac{1}{4M}\mathbb{Z}$. Noticing that

$$|\alpha_{M,n}|^{1/2}n \leq \sum_{j=1}^{n} |\alpha_{M,j}|^{1/2} = O_{\varepsilon}(1),$$

we obtain $|\alpha_{M,n}| \leq n^{-2}$. A standard Cantor's diagonal procedure guarantees the existence of $(\alpha_n)_{n\geqslant 1}$ such that $\sum_{n\in\mathbb{Z}} |\alpha_n| = O_{\varepsilon}(1)$ and a subsequence $M_k \to \infty$ such that $\lim_k \alpha_{M_k,n} = \alpha_n$ for all $n\geqslant 1$. Let now $I\subset\mathbb{N}$ be the n's such that $\sup_k |\lambda_{M_k,n}| < \infty$. By

further taking a subsequence of the M_k 's we can assume that $\lim_k \lambda_{M_k,n} \to \lambda_n$ for $n \in I$. Note now that

$$|\alpha_{M_k,n}| \leq |\widehat{\varphi}_M(-\lambda_{M_k,n})| \leq \frac{1}{2\pi |\lambda_{M_k,n}|} \int_{\mathbb{R}} |\varphi'_M(x)| dx = O_{\delta}(|\lambda_{M_k,n}|^{-1}).$$

In particular, $\lim_k \alpha_{M_k,n} = \alpha_n = 0$ if $n \notin I$. Finally, the uniform bound $|\alpha_{M,n}| \leq n^{-2}$ forces f_{M_k} to converge uniformly in compact sets to

$$g(x) = \sum_{n \ge 1} \alpha_n e^{2\pi i \lambda_n x},$$

which in particular implies that

$$||f - g||_{\infty} \le \varepsilon.$$

However, we can now simply truncated g to find a trigonometric polynomial p such that $||f - p||_{\infty} \leq 2\varepsilon$.

It remains to construct the auxiliary functions φ_M . Since f is also uniformly continuous, it is easy to show that $\{t_n\}_{n\in\mathbb{Z}}+(-\delta,\delta)\subset\tau_{\varepsilon/4}$ for some small $\delta=\delta_\varepsilon>0$ and some sequence $\{t_n\}_{n\in\mathbb{Z}}$ satisfying $1/\delta\leqslant t_{n+1}-t_n\leqslant 3/\delta$ for all n. Take $h\in C_c^\infty(-1,1)$ even with $0\leqslant h\leqslant 1$ and $\int_{\mathbb{R}}h=1$. Let $h_\delta(x)=h(x/\delta)/\delta$, $\psi(x)=(2N+1)^{-1}\sum_{n=-N}^Nh_\delta(x-t_n)$ and $\varphi_M(x)=\psi*\psi*\psi*\psi$, where $N\geqslant 1$ is selected to be the largest such that $\{t_n\}_{|n|\leqslant N}+(-\delta,\delta)\subset\tau_{\varepsilon/4}\cap(-M/4,M/4)$. Note we must have $N\geqslant\frac{\delta M}{10}$ for sufficiently large M. Since $\tau_{\varepsilon/4}+\tau_{\varepsilon/4}+\tau_{\varepsilon/4}=\tau_\varepsilon$ we conclude that $\varphi_M\in C_c^\infty(\tau_\varepsilon\cap(-M,M))$. Also note that $\int_{\mathbb{R}}\varphi_M=(\int_{\mathbb{R}}\psi)^4=1$ and, since $\varphi_M'=\psi'*\psi*\psi$, that

$$\int_{\mathbb{R}} |\varphi'_M| \leqslant \int_{\mathbb{R}} |\psi'| = O(1/\delta).$$

Finally note that

$$\sum_{n \in \mathbb{Z}} |\widehat{\varphi}_M(n/(4M))|^{1/2} = \sum_{n \in \mathbb{Z}} |\widehat{\psi}(n/(4M))|^2 = 4M \int_{-2M}^{2M} |\psi(x)|^2 dx$$

$$= \frac{4M}{\delta(2N+1)} \int_{-1}^{1} |h(x)|^2 dx \leqslant \frac{4M}{\delta(2N+1)} \ll \delta^{-2},$$

for M large enough. This finishes the proof.

We define

$$\mathbb{E}f := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(x) dx \quad \text{and} \quad \mathbb{E}f(\lambda) := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(x) e^{-2\pi i \lambda x} dx,$$

whenever these limits exist (so $\mathbb{E}f = \mathbb{E}f(0)$).

Lemma 9. For any $f \in AP(\mathbb{R})$ we have:

- (1) The average $\mathbb{E}f(\lambda)$ exists;
- (2) The set spec $(f) := \{ \lambda \in \mathbb{R} : \mathbb{E}f(\lambda) \neq 0 \}$ it at most countable;
- (3) If $\operatorname{spec}(f) = \emptyset$ then f = 0;

(4) If $\sum_{\lambda \in \mathbb{R}} |\mathbb{E}f(\lambda)| < \infty$, then the following series converges absolutely and uniformly

$$f(x) = \sum_{\lambda \in \mathbb{R}} \mathbb{E} f(\lambda) e^{2\pi i \lambda x};$$

(5) We have $\mathbb{E}|f|^2 = \sum_{\lambda} |\mathbb{E}f(\lambda)|^2$ and for every $\varepsilon > 0$ there is $S \subset \operatorname{spec}(f)$ finite such that $\mathbb{E}_x|f(x) - \sum_{\lambda \in S} \mathbb{E}f(\lambda)e^{2\pi i\lambda x}|^2 < \varepsilon$.

Proof. For item (1), we can assume $\lambda = 0$. Let $\varepsilon > 0$ be given and take a trigonometric polynomial p such that $||f - p||_{\infty} \le \varepsilon$. Let $A = \mathbb{E}p$, which exists by direct computation. We have

$$\frac{1}{2T} \int_{-T}^{T} f(x) dx - \frac{1}{2T'} \int_{-T'}^{T'} f(x) dx = O(2\varepsilon) + \frac{1}{2T} \int_{-T}^{T} p(x) dx - \frac{1}{2T'} \int_{-T'}^{T'} p(x) dx
= O(2\varepsilon) + A - A + o_{T,T'}(1).$$

For item (2), note first that $\mathbb{E}f(\lambda)$ exists since $f(x)e^{-2\pi i\lambda x}$ is also almost periodic (by Bochner's criterion). Let p_n be a trigonometric polynomial such that $\|f - p_n\|_{\infty} < \frac{1}{n}$. Assume that $|\mathbb{E}f(\lambda)| > 0$ and let $1/n < |\mathbb{E}f(\lambda)|$. We obtain that $|\mathbb{E}p_n(\lambda)| \geqslant |\mathbb{E}f(\lambda)| - \frac{1}{n} > 0$, hence $\mathbb{E}p_n(\lambda) \neq 0$. We conclude that $\operatorname{spec}(f) \subset \bigcup_{n\geqslant 1} \operatorname{spec}(p_n)$. For item (3), let $\varepsilon > 0$ be given and take a trigonometric polynomial p such that $\|\bar{f} - p\|_{\infty} \leqslant \varepsilon$. Now, observing that

$$\frac{1}{2T} \int_{-T}^{T} |f(x)|^2 dx = \frac{1}{2T} \int_{-T}^{T} f(x) p(x) dx + \frac{1}{2T} \int_{-T}^{T} f(x) (\bar{f}(x) - p(x)) dx = o_T(1) + O(\varepsilon ||f||_{\infty}),$$

we conclude that $\lim_T \frac{1}{2T} \int_{-T}^T |f(x)|^2 dx = 0$. This is impossible if f is nonzero, because $|f|^2$ is also almost periodic and uniformly continuous, and so if $|f(x)|^2 \ge c$ for x in some interval I then $|f(x)|^2 \ge c/2$ for $x \in \tau_{c/2}(|f|^2) + I$. Since $\tau_{c/2}(|f|^2)$ contains some increasing sequence $\{t_n\}_{n\in\mathbb{Z}}$ satisfying $t_{n+1} - t_n = O(1)$, it implies that $\mathbb{E}|f|^2 \ge \mathbb{E}(\mathbf{1}_{\tau_{c/2}(|f|^2)+I}) > 0$, which is absurd. We conclude that f = 0. For item (4), note that

$$g(x) = \sum_{\lambda \in \mathbb{R}} \mathbb{E}f(\lambda)e^{2\pi i\lambda x},$$

is well-defined and converges absolutely and uniformly on \mathbb{R} . Also note that dominated convergence implies that $\mathbb{E}g(\lambda) = \mathbb{E}f(\lambda)$ for all $\lambda \in \mathbb{R}$, and so by item (3) f = g. For item (5), let $\Lambda' := \bigcup_{m \geqslant 1} \operatorname{spec}(p_m)$ for some trigonometric polynomials p_m such that $||f - p_m||_{\infty} < 1/m$ and enumerate $\Lambda' = \{\lambda_n\}_{n \geqslant 1}$. It is enough to show that $\mathbb{E}|f|^2 = \sum_{n \geqslant 1} |\mathbb{E}f(\lambda_n)|^2$. Let $f_N(x) = \sum_{n=1}^N \mathbb{E}f(\lambda_n)e^{2\pi i\lambda_n x}$. A straightforward computation shows that $\mathbb{E}|f|^2 - \mathbb{E}|f_N|^2 = \mathbb{E}|f - f_N|^2$ and so $\sum_{n=1}^N |\mathbb{E}f(\lambda_n)|^2 \leqslant \mathbb{E}|f|^2$ for all N. Another routine computation shows

$$\mathbb{E}|f - p|^2 = \mathbb{E}|f|^2 - \mathbb{E}|f_N|^2 + \mathbb{E}|f_N - p|^2 \geqslant \mathbb{E}|f|^2 - \mathbb{E}|f_N|^2,$$

for any trigonometric polynomial of the form $p(x) = \sum_{n=1}^{N} b_n e^{2\pi i \lambda_n x}$, for some $b_n \in \mathbb{C}$. Since p_m is of such form for some N_m , we conclude that $\mathbb{E}|f - f_{N_m}|^2 < 1/m$ and that

$$\mathbb{E}|f_{N_m}|^2 \leqslant \mathbb{E}|f|^2 \leqslant \mathbb{E}|f_{N_m}|^2 + 2/m.$$

The following proposition (see [10, p. 46]) justifies the notation

$$f(x) \sim \sum_{\lambda \in \mathbb{R}} \mathbb{E} f(\lambda) e^{2\pi i \lambda x}.$$

Proposition 10 (Bochner's approximation). For any $f \in AP(\mathbb{R})$ there is an sequence of functions $c_n : \mathbb{R} \to [0,1]$, $c_n \leq c_{n+1}$, each c_n with finite support, satisfying

$$\lim_{n \to \infty} c_n(\lambda) = \begin{cases} 1 & \text{if } \mathbb{E}f(\lambda) \neq 0 \\ 0 & \text{if } \mathbb{E}f(\lambda) = 0, \end{cases}$$

and such that

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}} |f(x) - \sum_{\lambda \in \mathbb{R}} c_n(\lambda) \mathbb{E}f(\lambda) e^{2\pi i \lambda x}| = 0.$$

5. Almost Periodic Functions in \mathbb{C}^+

Recall that we have defined in the introduction the space $AP(\mathbb{C}^+)$ of holomorphic almost periodic functions $f:\mathbb{C}^+\to\mathbb{C}$ such that for every $\varepsilon>0$ there is a relatively dense set of ε -translations $\tau_{\varepsilon}(f)\subset\mathbb{R}$ satisfying

$$\sup_{\varepsilon < \operatorname{Im} z < 1/\varepsilon} |f(z) - f(z+t)| < \varepsilon$$

for every $t \in \tau_{\varepsilon}(f)$. We say a function $f: \mathbb{C}^+ \to \mathbb{C}$ is **bounded on strips** if

$$\sup_{\varepsilon < \operatorname{Im} z < 1/\varepsilon} |f(z)| < \infty$$

for every $\varepsilon > 0$. We now give an alternative characterization of almost periodicity.

Lemma 11. Let $f: \mathbb{C}^+ \to \mathbb{C}$ be holomorphic. Then following are equivalent:

- (1) f is bounded on strips and for every h > 0 we have $f(\cdot + ih) \in AP(\mathbb{R})$;
- (2) f is bounded on strips and there is h > 0 such that $f(\cdot + ih) \in AP(\mathbb{R})$;
- (3) $f \in AP(\mathbb{C}^+)$;

In this case the quantity

$$\mathbb{E}f(\lambda) := \lim_{T \to \infty} \frac{1}{2T} \int_{-T+iy}^{T+iy} f(z) e^{-2\pi i \lambda z} dz$$

exists for every $\lambda \in \mathbb{R}$, is independent of y > 0, it is nonzero in at most countably many λ 's and

$$\sum_{\lambda \in \mathbb{R}} |\mathbb{E}f(\lambda)|^2 e^{-4\pi y\lambda} = \mathbb{E}[|f(\cdot + iy)|^2]$$

for every y > 0.

Proof. The implications $(1) \Rightarrow (2)$ and $(3) \Rightarrow (1)$ are obvious. We now show $(2) \Rightarrow (3)$. Let τ_{ε} be the set of ε -translations for $f(\cdot + ih)$. We claim that τ_{ε} also works in any horizontal strip containing the line Im z = h. Indeed, let $t \in \tau_{\varepsilon}$ and $0 < y_1 = y_2/2 < y_2 < h$. For all y with $y_2 < y < h$, we can then apply Hadamard's there lines lemma (for Im $z \in \{y_1, y, h\}$) to conclude that

$$\sup_{x} \log |f(x+iy) - f(x+iy+t)|$$

$$\leq \frac{y - y_1}{h - y_1} \sup_{x} \log |f(x+ih) - f(x+ih+t)| + \frac{h - y}{h - y_1} \sup_{x} \log |f(x+iy_1) - f(x+iy_1+t)|$$

$$\leq \frac{y_2 - y_1}{h - y_1} \log \varepsilon + \frac{h - y_2}{h - y_1} B_1$$

for some $B_1 > 0$. Hence, there is $\alpha = \alpha(y_2, h) > 0$ such that

$$\sup_{y_2 < \operatorname{Im} z < h} |f(z) - f(z+t)| \le \frac{1}{\alpha} \varepsilon^{\alpha}.$$

We can apply the same procedure for $h < \text{Im } z < y_3$ for any $y_3 > h$ to prove the claim. This shows that $f \in AP(\mathbb{C}^+)$. Now notice that by Lemma 9 the limit $\mathbb{E}f(\lambda)$ exists for every y > 0. To show is independent of y, we can use Cauchy's formula to deduce that if $0 < y_1 < y_2$ then

$$\left[\frac{1}{2T} \int_{-T+iy_1}^{T+iy_1} f(z) e^{-2\pi i \lambda z} dz - \frac{1}{2T} \int_{-T+iy_2}^{T+iy_2} f(z) e^{-2\pi i \lambda z} dz \right]
= \left[\frac{1}{2T} \int_{-T+iy_1}^{-T+iy_2} f(z) e^{-2\pi i \lambda z} dz - \frac{1}{2T} \int_{T+iy_1}^{T+iy_2} f(z) e^{-2\pi i \lambda z} dz \right] = O((y_2 - y_1) e^{2\pi |\lambda| y_2} / T).$$

Taking $T \to \infty$, we conclude that $\mathbb{E}f(\lambda)$ is independent of y. Since $e^{-2\pi\lambda y}\mathbb{E}f(\lambda) = \mathbb{E}[f(\cdot + iy)](\lambda)$ we deduce that $\lambda \in \mathbb{R} \mapsto \mathbb{E}f(\lambda)$ has countable support and $\sum_{\lambda \in \mathbb{R}} |e^{-2\pi\lambda y}\mathbb{E}f(\lambda)|^2 = \mathbb{E}[f(\cdot + iy)]^2$. This finishes the proof.

For a function $f \in AP(\mathbb{C}^+)$ (or $f \in AP(\mathbb{R})$) we define the **spectrum** of f to be

$$\operatorname{spec}(f) := \{ \lambda \in \mathbb{R} : \mathbb{E}f(\lambda) \neq 0 \}.$$

Lemma 12. The following hold:

- (i) Let $f \in AP(\mathbb{C}^+)$. Then f is bounded in $\mathbb{C}^+ + ic$, for some c > 0, if and only if $\operatorname{spec}(f) \subset [0, \infty)$. In this case f is bounded in $\mathbb{C}^+ + ic$ for any c > 0 and $\lim_{y \to \infty} \sup_x |f(x+iy) \mathbb{E}f(0)| = 0$;
- (ii) If $f \in AP(\mathbb{C}^+)$, spec(f) is bounded from below and $f(z) \neq 0$ for all Im z > c, for some c > 0, then $\inf \text{spec}(f) \in \text{spec}(f)$;
- (iii) If $f, g \in AP(\mathbb{C}^+)$ have spectrum bounded from below then so has $fg \in AP(\mathbb{C}^+)$. If in addition f, g have locally finite spectrum, then so has fg.
- (iv) If $f \in AP(\mathbb{C}^+)$ and $f \neq 0$ in \mathbb{C}^+ , then $\inf_{\varepsilon < \operatorname{Im} z < 1/\varepsilon} |f(z)| > 0$ for all $\varepsilon > 0$ and $1/f \in AP(\mathbb{C}^+)$. Moreover, if f has spectrum bounded from below then so has 1/f, and if in addition f has locally finite spectrum then so has 1/f.

(v) If $f \in AP(\mathbb{C}^+)$ and |f(z)| < 1 for all $z \in \mathbb{C}^+$ then for every $\varepsilon > 0$ there is c > 0 such that |f(z)| < 1 - c if $\text{Im } z > \varepsilon$.

Proof. Items (i) and (ii): These are direct applications of [10, Thm. p. 162 & p. 152]. Item (iii): Note that $f_1(z) = e^{-2\pi i M z} f(z)$ and $g_1(z) = e^{-2\pi i M z} g(z)$ have spectrum contained in $[0,\infty)$ for some M>0, thus both are bounded for $\operatorname{Im} z>1$ and so is $h=f_1g_1$. Thus, $\operatorname{spec}(fg)=\operatorname{spec}(h)+2M\subset [0,\infty)$, hence fg have spectrum bounded from below. We also have $\operatorname{spec}(fg)\subset\operatorname{spec}(f)+\operatorname{spec}(g)=\operatorname{spec}(f_1)+\operatorname{spec}(g_1)-2M$, and since sums of locally finite sets contained in $[0,\infty)$ is also locally finite and contained in $[0,\infty)$, we conclude that the spectrum of fg is locally finite if both f and g have locally finite spectrum. Item (iv): By a clever application of Hadamard's Three-Lines theorem [10, Thm. 11, p. 139 & Thm. 9, p. 144], one can show that if w is an accumulation point of f in an horizontal line, then f(z)=w has a solution in any horizontal strip containing this line. In particular, if f never vanishes in \mathbb{C}^+ then |f| is bounded away from zero in any horizontal strip, thus $1/f \in \operatorname{AP}(\mathbb{C}^+)$. If $f \sim \sum_{\lambda \geqslant M} \mathbb{E} f(\lambda) e^{2\pi i \lambda z}$ where $M=\inf \operatorname{spec}(f)$, since f has no zeros, $M \in \operatorname{spec}(f)$, $p=\mathbb{E} f(M) \neq 0$ and $\lim_{y\to\infty} \sup_x |f(x+iy)e^{-2\pi i M(x+it)}-p|=0$. In particular f(z)=1 and f(z)=1 is bounded in absolute value by 1/2 for f(z)=1 for f(z)=1 for some large f(z)=1 and f(z)=1. We obtain

$$\frac{1}{f(z)} = \frac{e^{-2\pi i M z}}{p} \frac{1}{1 - g(z)} = \frac{e^{-2\pi i M z}}{p} \sum_{n \ge 0} g(z)^n,$$

where the sum converges absolutely and uniformly for $\operatorname{Im} z > c$. We conclude that $\mathbb{E}(1/f)(\lambda) = \sum_{n \geq 0} \mathbb{E}(g^n)(\lambda + M)$ (with absolute convergence), and so $\mathbb{E}(1/f)(\lambda) = 0$ if $\lambda < -M$. If in addition $\operatorname{spec}(f)$ is locally finite, then $\operatorname{spec}(g)$ is locally finite and contained in $[\delta, \infty)$, for some $\delta > 0$. Since $\operatorname{spec}(1/f) + \{M\} \subset \cup_{n \geq 0} (\bigoplus^n \operatorname{spec}(g))$, we conclude that $(\operatorname{spec}(1/f) + \{M\}) \cap [0, T] \subset \cup_{0 \leq n \leq T/\delta} (\bigoplus^n \operatorname{spec}(g) \cap [0, T])$, and so $\operatorname{spec}(1/f)$ is locally finite. Item (v): Since f is bounded we deduce that $\mathbb{E}f(\lambda) = 0$ for $\lambda < 0$ and, by [10, Thm. $\overline{9}$, \overline{p} , $\overline{144}$], that $\sup_{\varepsilon < \operatorname{Im} z < 1/\varepsilon} |f(z)| < 1$ for every $\varepsilon > 0$. Hence $|\mathbb{E}f(0)| < 1$. On the other hand $\lim_{y \to \infty} \sup_x |f(x + iy) - \mathbb{E}f(0)| = 0$.

Lemma 13. Let $f \in AP(\mathbb{C}^+)$ and assume $\operatorname{spec}(f) \subset \{0\} \cup [b, \infty)$ for some b > 0. If $F : \mathbb{C}^+ \to \mathbb{C}$ is holomorphic and iF'/F = f, then $F \in AP(\mathbb{C}^+)$ and $\operatorname{spec}(F) \subset \{-\mathbb{E}f(0)/(2\pi)\} \cup [-\mathbb{E}f(0)/(2\pi) + b, \infty)$. Moreover, if in addition $\operatorname{spec}(f)$ is locally finite, then so it is $\operatorname{spec}(F)$.

Proof. Let $p = \mathbb{E}f(0)$. Since F has no zeros in \mathbb{C}^+ we can write $F(z) = e^{-ig(z)-ipz}$ for some $g: \mathbb{C}^+ \to \mathbb{C}$ holomorphic. Since g'(z) + p = iF'(z)/F(z) = f(z) we conclude that $g' \in \operatorname{AP}(\mathbb{C}^+)$, $\operatorname{spec}(g) \subset [b,\infty)$ and $\mathbb{E}(g')(\lambda) = \mathbf{1}_{\lambda \geqslant b}\mathbb{E}f(\lambda)$. We can now apply [10, Thm. 9, p. 152] to conclude that $g \in \operatorname{AP}(\mathbb{C}^+)$ and that $\mathbb{E}g(\lambda) = \mathbf{1}_{\lambda \geqslant b}\mathbb{E}f(\lambda)/(2\pi i\lambda)$ for $\lambda \neq 0$. If $\Phi: \mathbb{C} \to \mathbb{C}$ is entire and $h \in \operatorname{AP}(\mathbb{C}^+)$ then it is easy to conclude that $\Phi \circ h \in \operatorname{AP}(\mathbb{C}^+)$. Indeed, since h is bounded in strips and since Φ' is bounded in bounded sets, we conclude that for any horizontal strip contained in \mathbb{C}^+ there is C > 0 such that

 $|\Phi(h(z)) - \Phi(h(w))| \leq C|h(z) - h(w)|$ for all z and w in that strip. This shows that $\Phi \circ h \in AP(\mathbb{C}^+)$. This implies that the power series expansion

$$\Phi(h(z)) = \sum_{n \ge 0} \frac{\Phi^{(n)}(0)}{n!} h(z)^n.$$

converges absolutely and uniformly on any horizontal strip. Now assume that $\operatorname{spec}(h) \subset [b,\infty)$ for some b>0. Since $\operatorname{spec}(h^n) \subset [nb,\infty)$ and

$$\mathbb{E}[\Phi \circ h](\lambda) = \sum_{n \geqslant 0} \frac{\Phi^{(n)}(0)}{n!} \mathbb{E}[h^n](\lambda),$$

we conclude that $\operatorname{spec}(\Phi \circ h) \subset \{0\} \cup [b, \infty)$. If in addition $\operatorname{spec}(h)$ is locally finite, since

$$\operatorname{spec}(\Phi \circ h) \cap [0,T] \subset \cup_{0 \leqslant n \leqslant T/b} (\oplus^n \operatorname{spec}(h) \cap [0,T]),$$

we conclude that $\operatorname{spec}(\Phi \circ h)$ is locally finite. Considering $\Phi(z) = e^{-iz}$ and $h(z) = g(z) - \mathbb{E}g(0)$, we deduce that $F(z) = \Phi(h(z))e^{-i\mathbb{E}g(0)-ipz}$ belongs to $\operatorname{AP}(\mathbb{C}^+)$ and that $\operatorname{spec}(F) \subset \{-p/(2\pi)\} \cup [-p/(2\pi) + b, \infty)$. Moreover, $\operatorname{spec}(F)$ is locally finite if $\operatorname{spec}(f)$ also is.

We also have approximation by trigonometric polynomials (see [10, p. 148])

Proposition 14 (Bochner's approximation). Let $f \in AP(\mathbb{C}^+)$. Then for every b > 1 there is a sequence of functions $c_n : \mathbb{R} \to [0,1]$ as in Proposition 10 such that

$$\lim_{n \to \infty} \sup_{1/b < \operatorname{Im} z < b} |f(z) - \sum_{\lambda \in \mathbb{R}} c_n(\lambda) \mathbb{E} f(\lambda) e^{2\pi i \lambda z}| = 0.$$

6. Representations of Analytic Functions

We say a holomorphic function $g: \mathbb{C}^+ \to \mathbb{C}$ if of **bounded type** if there are two bounded holomorphic functions $P, Q: \mathbb{C}^+ \to \mathbb{C}$ such that g = P/Q. We write $g \in \mathrm{BT}(\mathbb{C}^+)$. If g is of bounded type then the **mean type**

$$\vartheta(g) := \limsup_{y \to \infty} \frac{\log |g(iy)|}{y}$$

exists and is finite. For more information about these definitions we recommend [12, 19].

Lemma 15. Let $f: \mathbb{C}^+ \to \mathbb{C}$ be holomorphic and write $g = \exp(f)$. The following are equivalent:

(1) There is a real-valued locally finite measure μ with $\deg(\mu) \leq 2$ such that

$$\frac{f(z) + \overline{f(w)}}{z - \overline{w}} = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{(t - z)(t - \overline{w})} \tag{9}$$

holds for every $z, w \in \mathbb{C}^+$;

(2) $g \in BT(\mathbb{C}^+)$ and $\vartheta(g) = 0$.

Proof. First we show that $(1) \Rightarrow (2)$. Indeed, note first that

$$\operatorname{Re} f(x+iy) = \frac{y}{\pi} \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{(t-x)^2 + y^2}.$$

Writing $\mu = \mu_1 - \mu_2$ where each μ_j is nonnegative and defining

$$f_j(z) = \frac{1}{\pi i} \int_{\mathbb{R}} \frac{1 + tz}{t - z} \frac{\mathrm{d}\mu_j(t)}{1 + t^2}$$

we conclude that Re $f_j \ge 0$ in \mathbb{C}^+ and that Re $f = \text{Re } f_1 - \text{Re } f_2$, hence $f = f_1 - f_2 + ih$ for some constant $h \in \mathbb{R}$. However, since each $g_j = \exp(-f_j)$ is bounded in absolute value by 1 in \mathbb{C}^+ and $g = e^{ih}g_2/g_1$, we deduce that $g \in \text{BT}(\mathbb{C}^+)$. Note also that

$$\vartheta(g) = \limsup_{y \to \infty} \frac{\operatorname{Re} f(iy)}{y} = \limsup_{y \to \infty} \frac{1}{\pi} \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{t^2 + y^2} = 0.$$

Now we show $(2) \Rightarrow (3)$. Since g has no zeros, Nevalinnas's factorization for functions of bounded type [12, Theprem 9] implies that there exists a unique real-valued locally finite measure μ , with $\deg(\mu) \leq 2$, and some $c, h \in \mathbb{R}$ such that

$$e^{f(z)} = g(z) = e^{-ihz} \exp\left(ic + \frac{1}{\pi i} \int_{\mathbb{R}} \frac{1+tz}{t-z} \frac{d\mu(t)}{1+t^2}\right).$$

Since $h = \vartheta(g) = 0$, a simple computation shows that (9) holds.

Lemma 16. Let $f \in AP(\mathbb{R})$ and assume that $spec(f) \cap (0,b) = \emptyset$ for some b > 0. Then

$$F(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{1+tz}{t-z} \frac{f(t)dt}{1+t^2}$$

belongs to $AP(\mathbb{C}^+)$ and $\mathbb{E}F(\lambda) = \mathbf{1}_{\lambda \geqslant b}\mathbb{E}f(\lambda)$ for $\lambda \neq 0$.

Proof. Since any $f \in AP(\mathbb{R})$ is bounded, the integral defining F converges absolutely and defines an holomorphic function F. Noticing that $\frac{1+tz}{(t-z)(1+t^2)} = \frac{1}{t-z} - \frac{t}{1+t^2}$ we obtain that

$$F'(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(t)dt}{(t-z)^2},$$

and so $|F'(x+iy)| \leq \frac{1}{2y} \max_{t \in \mathbb{R}} |f(t)|$ for y > 0, hence F' is bounded on strips. Since

$$F'(x+i) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(t+x)dt}{(t-i)^2}$$

it is clear that any set of ε -translations for f is also one for $F'(\cdot + i)$, and so $F'(\cdot + i) \in AP(\mathbb{R})$, and Lemma 11 shows that $F' \in AP(\mathbb{C}^+)$. Dominated Convergence implies that

$$\mathbb{E}F'(\lambda) = e^{2\pi\lambda} \mathbb{E}f(\lambda) \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{e^{2\pi i \lambda t}}{(t-i)^2} = e^{2\pi\lambda} \mathbb{E}f(\lambda) 2\pi i |\lambda| e^{-2\pi |\lambda|} \mathbf{1}_{\lambda>0} = 2\pi i \max\{\lambda, 0\} \mathbb{E}f(\lambda),$$

and, since $\operatorname{spec}(f) \cap (0, b) = \emptyset$ for some b > 0, we obtain $\operatorname{spec}(F') \subset [b, \infty)$. We can now apply [10, Thm. 9, p. 152] to conclude that $F \in \operatorname{AP}(\mathbb{C}^+)$.

Lemma 17. Let $f: \mathbb{C} \to \mathbb{C}$ be entire such that $f, f^* \in AP(\mathbb{C}^+)$ and both have spectrum bounded from below. If f has finite order, that is,

$$\operatorname{order}(f) := \inf\{\rho > 0 : \limsup_{|z| \to \infty} \frac{\log^+ |f(z)|}{|z|^\rho} < \infty\} < \infty,$$

then f has finite exponential type, that is,

$$\tau = \limsup_{|z| \to \infty} \frac{\log |f(z)|}{|z|} < \infty.$$

In this case f and f^* are of bounded type, f is bounded on the real line and $|f(z)| \leq Me^{\tau|y|}$ for all z = x + iy, where $M = \sup_{x \in \mathbb{R}} |f(x)|$ and $\tau = \max\{\vartheta(f^*), \vartheta(f)\}$. Moreover, $\mathbb{E}f(\lambda) = \overline{\mathbb{E}(f^*)(-\lambda)}$ for all $\lambda \in \mathbb{R}$ and

$$\operatorname{spec}(f) = -\operatorname{spec}(f^*) \subset [-\tau/(2\pi), \tau/(2\pi)].$$

If in addition $\operatorname{spec}(f)$ is locally finite, then f is trigonometric polynomial.

Proof. Observe that since f has spectrum bounded from below, then Lemma 12(i) shows that for $m = \min\{\inf \operatorname{spec}(f), \inf \operatorname{spec}(f^*)\}$ we have that $|f(z)e^{-2\pi imz}| + |f^*(z)e^{-2\pi imz}|$ is bounded for $\operatorname{Im} z > c$, for any c > 0. This in particular shows that $f(z)e^{-2\pi |m|z}$ is bounded on the sides of any sector $S_c = \{z = x + iy : x > |y|/c\}$. Since f has finite order, one can apply Phranmën-Lindelöf for sectors (with small c > 0) [32, Thm. 1, p. 37] to conclude that $f(z)e^{-2\pi |m|z}$ is bounded in S_c . We then apply the same argument replacing f(z) by $f^*(-z)$ to conclude that $f^*(-z)e^{-2\pi |m|z}$ is bounded in S_c . All these imply that f has finite exponential type, that is,

$$|f(z)| \ll e^{C|z|}$$

for some C>0. However, since $f(z+i)e^{-2\pi i m(z+i)}$ is bounded on the real line, another application of Phranmën-Lindelöf [32, Thm. 3, p. 38] shows that $|f(z+i)e^{-2\pi i m(z+i)}| \ll e^{C'|y|}$ for some C'>0, and so $|f(z)| \ll e^{C''|y|}$ for some C''>0. In particular f is bounded on every horizontal strip and, by the same result, we can take $C''=\tau$. A straightforward calculation using Cauchy's integral formula shows that

$$\mathbb{E}f(\lambda) - \overline{\mathbb{E}(f^*)(-\lambda)} = \lim_{T \to \infty} \frac{1}{2T} \left(\int_{T-i}^{T+i} f(z) e^{-2\pi i \lambda z} dz - \int_{-T-i}^{-T+i} f(z) e^{-2\pi i \lambda z} dz \right) = 0,$$

since f is now bounded on horizontal strips. In particular, $\operatorname{spec}(f) = -\operatorname{spec}(f^*)$. A well-known result of Krein [24] (see also [32, Thm. 1, p. 115]) shows that $f, f^* \in \operatorname{BT}(\mathbb{C}^+)$ and that $\tau = \max\{\vartheta(f^*), \vartheta(f)\}$. Since $g(z) = f(z)e^{i\tau z}$ is bounded in \mathbb{C}^+ , Lemma 12(i) shows that $\operatorname{spec}(f) + \tau/(2\pi) = \operatorname{spec}(g) \subset [0, \infty)$, hence $\operatorname{spec}(f) \subset [-\tau/(2\pi), \infty)$. The same argument shows that $\operatorname{spec}(f^*) \subset [-\tau/(2\pi), \infty)$, but since $\mathbb{E}f(\lambda) = \overline{\mathbb{E}(f^*)(-\lambda)}$, we obtain that $\operatorname{spec}(f) \subset [-\tau/(2\pi), \tau/(2\pi)]$. Finally, if $\operatorname{spec}(f)$ is locally finite, Proposition 14 proves straightforwardly that f is a trigonometric polynomial.

7. Proof of Theorem 1

We start we some lemmas.

Lemma 18. For $w, z \in \mathbb{C}^+$ let

$$g(w, z, x) = \frac{e^{-2\pi i \overline{w}|x|} \mathbf{1}_{x < 0} + e^{2\pi i z|x|} \mathbf{1}_{x \ge 0}}{z - \overline{w}}$$

Then

$$\widehat{g}(w,z,\xi) = \frac{1}{2\pi i (\xi - z)(\xi - \overline{w})}.$$

Proof. Letting $f_z(x) = e^{2\pi i z|x|} \mathbf{1}_{x>0} + \frac{1}{2} \mathbf{1}_{x=0}$ we have

$$\widehat{f}_z(\xi) = \int_0^\infty e^{2\pi i x(z-\xi)} \mathrm{d}x = \frac{1}{2\pi i (\xi-z)}.$$

The lemma follows since $g(w,z,x) = (z-\overline{w})^{-1}(f_{-\overline{w}}(-x)+f_z(x)).$

Lemma 19. Let (μ, a) be a FS-pair with $deg(\mu) \leq 2$. Then

$$\lim_{T \to \infty} \sum_{|\lambda| < T} a(\lambda) g(w, z, \lambda) (1 - |\lambda|/T) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{(t - z)(t - \overline{w})}$$

uniformly for w and z in the region

$$R_c := \{ z \in \mathbb{C}^+ : |\operatorname{Re} z| \le 1/c \text{ and } \operatorname{Im} z \ge c \}, \quad (for \ any \ c > 0).$$

Proof. Let $\varphi \in C_c^{\infty}(\mathbb{R})$, $\int \varphi = 1$, $\varphi \geqslant 0$, supp $\varphi \subset (-1,1)$ and define $\varphi_{\varepsilon}(x) = \varphi(x/\varepsilon)/\varepsilon$ for $0 < \varepsilon < 1$. Let $w, z \in R_c$, T > 2/c and

$$g_{\varepsilon,T}(x) := (g(w,z,\cdot)(1-\frac{1}{T}|\cdot|)_+) * \varphi_{\varepsilon}(x),$$

where $s_+ = \max\{0, s\}$. Note that $g_{\varepsilon,T} \in C_c^{\infty}(-\varepsilon - T, T + \varepsilon)$ and

$$\widehat{g}_{\varepsilon,T}(\xi) = (\widehat{g}(w,z,\cdot) * S_T(\xi))\widehat{\varphi}(\varepsilon\xi),$$

where $S_T(x) = \frac{\sin^2(T\pi x)}{T(\pi x)^2}$. Note that $|\widehat{\varphi}(\varepsilon\xi)| \leq 1$. We claim that 16

$$|\widehat{g}(w,z,\cdot) * S_T(\xi)| \ll_c \frac{1}{\xi^2 + c^2}.$$
 (10)

We shall prove this claim in the end. Since (μ, a) is a FS-pair, in particular we obtain

$$\sum_{|\lambda| < T + \varepsilon} a(\lambda) g_{\varepsilon,T}(\lambda) = \int_{\mathbb{R}} \widehat{g}_{\varepsilon,T}(t) d\mu(t).$$

Since $a(\cdot)$ is locally summable, and as $\varepsilon \to 0$, $g_{\varepsilon,T}(x) \to g(w,z,x)(1-|x|/T)_+$ uniformly in $x \in \mathbb{R}$ and $\hat{g}_{\varepsilon,T}(\xi) \to \hat{g}(w,z,\cdot) * S_T(\xi)$ pointwise, Dominated Convergence implies that

$$\sum_{|\lambda| < T} a(\lambda)g(w, z, \lambda)(1 - |\lambda|/T) = \int_{\mathbb{R}} \widehat{g}(w, z, \cdot) * S_T(t) d\mu(t).$$

¹⁶We write $A \ll_p B$ if there is a numerical constant K > 0, depending only in the parameter p, such that $A \leqslant KB$.

It is now enough to show the right hand side above converges uniformly in the region $(w, z) \in \mathbb{R}^2_c$. This can be done by noticing that

$$|\widehat{g}(w,z,\xi_1) - \widehat{g}(w,z,\xi_2)| = \frac{1}{2\pi|z - \overline{w}|} \left| \frac{1}{\xi_1 - z} - \frac{1}{\xi_1 - \overline{w}} - \frac{1}{\xi_2 - z} + \frac{1}{\xi_2 - \overline{w}} \right| \leqslant \frac{|\xi_1 - \xi_2|}{4\pi c^3},$$

and so $\widehat{g}(w,z,\cdot) * S_T(\xi) \to \widehat{g}(w,z,\xi)$ uniformly in $x \in \mathbb{R}$ and $(w,z) \in R_c$, as $T \to \infty$. Finally, the uniform bound (10) guarantees the desired uniform convergence. It remains to prove the claim (10). First note that

$$|\hat{g}(w,z,\xi)| \ll \frac{1}{(\xi - \operatorname{Re} z)^2 + c^2} + \frac{1}{(\xi - \operatorname{Re} w)^2 + c^2} \ll_c \frac{1}{c^2 + \xi^2},$$

and so we can use a contour change (and that T > 2/c) to obtain

$$|g(w,z,\cdot)*S_T(\xi)| \ll_c \frac{1}{T} \int_{\mathbb{R}} \frac{\sin^2(\pi T t)}{((\xi-t)^2+c^2)t^2} dt = \frac{1}{T} \int_{\mathbb{R}} \frac{\sin^2(\pi T (t+i/T))}{((\xi-t-i/T)^2+c^2)(t+i/T)^2} dt$$
$$\ll \frac{1}{T} \int_{\mathbb{R}} \frac{1}{((\xi-t)^2+c^2)(t^2+1/T^2)} dt = \frac{\pi(c+1/T)/c}{\xi^2+(c+1/T)^2} \ll \frac{1}{c^2+\xi^2}.$$

Proof of Theorem 1 (and Theorem 2).

<u>Necessity</u>. Assume that (μ, a) is an real-antipoal FS-pair such that $\deg(\mu) \leq 2$ and $a(\cdot)$ has exponential growth. Since $a(-\lambda) = \overline{a(\lambda)}$, we can apply Lemma 19 (multiplying both sides by $(z - \overline{w})$) to obtain

$$a(0) + \lim_{T \to \infty} \left(\sum_{0 < \lambda < T} a(\lambda) e^{2\pi i \lambda z} (1 - \lambda/T) + \sum_{0 < \lambda < T} \overline{a(\lambda)} e^{-2\pi i \lambda \overline{w}} (1 - \lambda/T) \right)$$

$$= \frac{1}{2\pi i} \int_{\mathbb{R}} \left(\frac{1}{t - z} - \frac{1}{t - \overline{w}} \right) d\mu(t) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{1 + tz}{t - z} \frac{d\mu(t)}{1 + t^2} - \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{1 + t\overline{w}}{t - \overline{w}} \frac{d\mu(t)}{1 + t^2},$$

where the limit exists and is uniform for w and z in compact sets of \mathbb{C}^+ . In particular, we conclude there exists $h \in \mathbb{R}$ such that

$$f(z) = \frac{1}{2}a(0) + \lim_{T \to \infty} \sum_{0 \le \lambda \le T} a(\lambda)e^{2\pi i\lambda z} (1 - \lambda/T) = ih + \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{1 + tz}{t - z} \frac{\mathrm{d}\mu(t)}{1 + t^2},$$

where the limit above exists and is uniform in compact sets. A simple computation shows

$$\frac{f(z) + \overline{f(w)}}{z - \overline{w}} = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{(t - z)(t - \overline{w})}.$$
 (11)

This proves assertion (I). Since $a(\cdot)$ has exponential growth, we let $c=\inf\{b>0:$ $\sum_{\lambda\in\mathbb{R}}|a(\lambda)|e^{-2\pi b|\lambda|}<\infty\}$. Then we must have

$$f(z) = \frac{1}{2}a(0) + \sum_{\lambda > 0} a(\lambda)e^{2\pi i\lambda z}$$

for Im z > c, as the above series converges absolutely and uniformly for Im $z > c + \varepsilon$, for any $\varepsilon > 0$. Thus, after choosing an enumeration for $\mathrm{supp}(a)$, we conclude that $f(\cdot + ih)$ is the uniform limit of trigonometric polynomials and we obtain that $f(\cdot + ih) \in \mathrm{AP}(\mathbb{R})$ for every h > c. By Lemma 11 we obtain $f(\cdot + ic) \in \mathrm{AP}(\mathbb{C}^+)$. This proves assertion (II).

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Assertion (III) is direct since if $p(x) = \sum_{\theta \in S} p_{\theta} e^{2\pi i \theta x}$, for some finite set $S \subset [0, \infty)$, then

$$\limsup_{T \to \infty} \left| \frac{1}{2T} \int_{-T}^{T} f(x+2ic) \overline{p(x)} dx \right| = \left| \frac{1}{2} a(0) \delta_{S}(0) + \sum_{\lambda > 0} a(\lambda) e^{-4\pi c \lambda} \delta_{S}(\lambda) p_{\lambda} \right|$$

$$\ll \max |p_{\theta}| \sum_{\lambda \in \mathbb{R}} |a(\lambda)| e^{-4\pi c |\lambda|} \ll \max |p_{\theta}|.$$

Finally, by Lemma 15, assertion (IV) is implied by representation (11).

<u>Sufficiency</u>. We prove sufficiency replacing property (III) by (III*) as in Theorem 2, and so also proving Theorem 2. Let $c_1 = \inf\{b > 0 : f(\cdot + ib) \in AP(\mathbb{C}^+)\}$ and $f_1(z) = f(z+c_1)$. By condition (II), we can apply Lemma 11 to deduce the limit

$$\mathbb{E}f_1(\lambda) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T+iy}^{T+iy} f_1(z) e^{-2\pi i \lambda z} dz = e^{-2\pi \lambda c_1} \lim_{T \to \infty} \frac{1}{2T} \int_{-T+iy+ic_1}^{T+iy+ic_1} f(z) e^{-2\pi i \lambda z} dz$$

exists, does not depend on y > 0 and is nonzero only for countable many real λ 's. We also have that $\sum_{\lambda} |\mathbb{E} f_1(\lambda)|^2 e^{-4\pi h\lambda} < \infty$ for all h > 0. This shows that

$$\mathbb{E}f(\lambda) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T+iy}^{T+iy} f(z) e^{-2\pi i \lambda z} dz$$

is independent of $y > c_1$ and is nonzero only for countable many real λ 's. By condition (IV), we can apply Lemma 15 to deduce there is a real-valued locally finite measure μ of degree at most 2 such that (11) is true. In particular,

$$2\operatorname{Re} f(x+iy) = \frac{y}{\pi} \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{(t-x)^2 + y^2}.$$

As in Lemma 15 one can then write $f = f_1 - f_2$ such that $\operatorname{Re} f_j \ge 0$ in \mathbb{C}^+ and $f_j(i) = \int_{\mathbb{R}} \frac{d\mu_j(t)}{2\pi(1+t^2)}$. We can then apply Harnack's inequality [10, p. 136], that translates to

$$\left| \frac{f_j(z) - f_j(i)}{f_j(z) + f_j(i)} \right| \le \left| \frac{z - i}{z + i} \right|.$$

Since

$$\frac{|z+i|+|z-i|}{|z+i|-|z-i|} \le \frac{(1+|z|)^2}{4y} \quad (z=x+iy),$$

we conclude that $|f_j(z)| \leq \frac{(1+|z|)^2}{4y} f_j(i)$ and so $|f(z)| \leq \frac{(1+|z|)^2}{4y} \int_{\mathbb{R}} \frac{\mathrm{d}|\mu|(t)}{2\pi(1+t^2)}$. We deduce that

$$\liminf_{r \to \infty} \frac{1}{r} \int_0^{\pi} \log^+ |f(re^{i\theta})| \sin \theta d\theta = 0,$$

and so one can now apply the Phranmën-Lindelöf principle (as in [12, Thm. 1]) to obtain that f(z+ic) is bounded in \mathbb{C}^+ for every $c>c_1$. Lemma 12(i) shows that $\mathbb{E}f(\lambda)=0$ for $\lambda<0$. By condition (IV), there is $B_M>0$ and $c_2\geqslant c_1$ (with equality only if $f(\cdot+ic_1)\in AP(\mathbb{R})$) such that

$$B_{M} \max |p_{\theta}| \geqslant \limsup_{T \to \infty} \left| \frac{1}{2T} \int_{-T}^{T} f(x + ic_{2}) \overline{p(x)} dx \right| = \left| \sum_{0 \leqslant \lambda \leqslant M} \mathbb{E}f(\lambda) e^{-2\pi c_{2}\lambda} \delta_{S}(\lambda) p_{\lambda} \right|$$

whenever $p(x) = \sum_{\theta \in S} p_{\theta} e^{2\pi i \theta x}$ for some finite set $S \subset [0, M]$. We conclude that

$$\sum_{0 \le \lambda \le M} |\mathbb{E}f(\lambda)| \le B_M e^{2\pi c_2 M}$$

for every M > 0, so the function $a(\lambda)$ defined in (3) is locally summable. Finally, let us prove that (μ, a) is a FS-pair. For $z \in \mathbb{C}^+$ and $t \in \mathbb{R}$ let $P_z(t) = \frac{z}{\pi i (t^2 - z^2)}$. Since identity (11) holds true, a simple computation shows

$$f(z+s) + \overline{f(-\overline{z}+s)} = P_z * \mu(s)$$

for all $s \in \mathbb{R}$. By Bochner's approximation on $AP(\mathbb{R})$, for every h > 0, one can find a sequence of functions $d_n : \mathbb{R} \to [0, 1)$, each with finite support and such that $\lim_n d_n(\lambda) = \delta_{\operatorname{spec}(f)}$, and that

$$\lim_{n} \sup_{\mathrm{Im} z = c_1 + h} |f(z) - \sum_{\lambda \ge 0} \mathbb{E}f(\lambda) d_n(\lambda) e^{2\pi i \lambda z}| = 0.$$

Let $\varphi \in C_c^{\infty}((-M, M))$ be antipodal. It is now straightforward (using Bochner's approximation, Dominated Convergence and local summability) to conclude that

$$\int_{\mathbb{R}} f(z+s)\widehat{\varphi}(s) ds = \sum_{0 \le \lambda \le M} \mathbb{E}f(\lambda)\varphi(\lambda)e^{2\pi i\lambda z}.$$

if Im $z > c_1$. Another routine computation (using that $a(0) = 2 \operatorname{Re} \mathbb{E} f(0)$) shows

$$\sum_{|\lambda| < M} a(\lambda) \varphi(\lambda) e^{2\pi i |\lambda| z} = \int_{\mathbb{R}} [f(z+s) + \overline{f(-\overline{z}+s)}] \widehat{\varphi}(s) ds = \int_{\mathbb{R}} P_z * \mu(s) \widehat{\varphi}(s) ds$$
$$= \int_{\mathbb{R}} P_z * \widehat{\varphi}(t) d\mu(t)$$

if $\operatorname{Im} z > c_1$. By real-antipodal splitting and linearity, the above formula holds for all $\varphi \in C_c^{\infty}(\mathbb{R})$. Now observe that both sides extend analytically to $\operatorname{Im} z > 0$. Taking z = iy and noting that P_{iy} is Poisson's kernel, a routine application of Dominated Convergence and approximate identity arguments (using that $a(\cdot)$ is locally summable and $\deg(\mu) \leq 2$) allows us to take $y \to 0$ and conclude that

$$\sum_{|\lambda| < M} a(\lambda)\varphi(\lambda) = \int_{\mathbb{R}} \widehat{\varphi}(t) d\mu(t).$$

Therefore (μ, a) is an FS-pair. This finishes the proof.

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APPENDIX: FS-PAIRS, AN ACCOUNT OF VARIOUS EXAMPLES.

Throughout this section we define certain families of FS-pairs which will be closed under scaling, translation, modulation and multiplication by scalar.

Finite Support. Let (μ, a) be a FS-pair such that $a(\cdot)$ has finite support, this is, there is $\{p_n\}_{n=1}^N \subset \mathbb{C}$ and $\{\lambda_n\}_{n=1}^N \subset \mathbb{R}$ such that

$$\int_{\mathbb{R}} \widehat{\varphi}(t) d\mu(t) = \sum_{n=1}^{N} p_n \varphi(\lambda_n)$$

for all $\varphi \in C_c^{\infty}(\mathbb{R})$. It is easy to see that we must have $d\mu(t) = \sum_{n=1}^N p_n e^{2\pi i \lambda_n t} dt$, since the difference of these two measures would vanish identically over $C_c^{\infty}(\mathbb{R})$.

Uniformly Discrete Poisson Summation (UDPS). We denote UDPS the set of FS-pairs (μ, a) such that there is $\alpha > 0$, $\{\theta_j\}_{j=1}^N \subset [0, \alpha)$ and trig-polynomials $\{Q_j\}_{j=1}^N$ where

$$\mu = \sum_{j=1}^{N} \sum_{\lambda \in \alpha \mathbb{Z} + \theta_j} Q_j(\lambda) \delta_{\lambda}.$$

It is easy to see that if $(\mu, a) \in \text{UDPS}$ then $a(\cdot)$ is also of the above form. The break-through of Lev & Olveskii [30] shows that $(\mu, a) \in \text{UDPS}$ if and only if μ and $a(\cdot)$ have uniformly discrete support (assuming both are strongly tempered). We observe that we can always assume that all Q'_js are equal since by classical interpolation results (for instance, Lagrange interpolation via Chebyshev polynomials) one can always find trigonometric polynomials P_j , which are α -periodic, and such that $P_j(\theta_k) = \delta_{j,k}$. Thus, if we let $Q = \sum_{j=1}^N P_j Q_j$ then $Q(\alpha n + \theta_j) = Q_j(\alpha n + \theta_j)$ for all $n \in \mathbb{Z}$, and so

$$\mu = \sum_{\lambda \in \cup_{i=1}^{N} (\alpha \mathbb{Z} + \theta_{i})} Q(\lambda) \delta_{\lambda}.$$

Real Rooted Trigonometric Polynomial (RRTP). We let the class RRTP to be the family of FS-pairs (μ, a) such that there are four trigonometric polynomials E_j (j = 1, 2, 3, 4) of Hermite-Biehler class, that is, $|E_j(z)| > |E_j(\overline{z})|$ for Im z > 0, and such that

$$\mu = \sum_{\varphi_1(\gamma) \equiv 0 \pmod{\pi}} \frac{Q_1(\gamma)}{\varphi_1'(\gamma)} \delta_{\gamma} - \sum_{\varphi_2(\gamma) \equiv 0 \pmod{\pi}} \frac{Q_2(\gamma)}{\varphi_2'(\gamma)} \delta_{\gamma}$$
$$-i \sum_{\varphi_3(\gamma) \equiv 0 \pmod{\pi}} \frac{Q_3(\gamma)}{\varphi_3'(\gamma)} \delta_{\gamma} + i \sum_{\varphi_4(\gamma) \equiv 0 \pmod{\pi}} \frac{Q_4(\gamma)}{\varphi_4'(\gamma)} \delta_{\gamma},$$

where Q_j are trigonometric polynomials, φ_j is the phase function associated with E_j , defined by the condition that $E_j(x)e^{i\varphi_j(x)}$ is real for all $x \in \mathbb{R}$ (φ_j is uniquely defined modulo π , see (5)). We observe that if we write $E_j = A_j - iB_j$, were A_j, B_j are trigonometric polynomials which are real on the real line, then both A, B have only real zeros, A/B have only simple real zeros and simple poles which interlace,

$$\{\gamma \in \mathbb{R} : \varphi_j(\gamma) \equiv 0 \pmod{\pi}\} = \operatorname{Zeros}(B_j/A_j) \text{ and } \frac{1}{\varphi_j'(\gamma)} = \operatorname{Res}_{z=\gamma}(A_j/B_j) > 0,$$

By Theorem 3 we conclude that $(\mu, \hat{\mu}) \in CM$ (see below). By Remark 5, RRTP contains the pairs $(\mu, \hat{\mu})$ where

$$\mu = \sum_{P(\gamma)=0} m(\gamma) Q(\gamma) \boldsymbol{\delta}_{\gamma}$$

and Q, P are trigonometric polynomials, P with only real roots and $m(\gamma)$ is the multiplicity of the zero γ . These measures were first introduced and studied by Kurasov and Sarnak [25]. Observe that UDPS \subset RRTP since $\cup_j \alpha \mathbb{Z} + \theta_j = \operatorname{Zeros}(\prod_j \sin(\pi(z - \theta_j)/\alpha))$. Indeed RRTP is a much richer class since

$$P(z) = \det(U + \operatorname{diag}(e^{2\pi i l_1 z}, ..., e^{2\pi i l_n z}))$$

has only real roots if U is a $n \times n$ unitary matrix and $l_1,, l_n \in \mathbb{R}$. The zeros of such polynomials are generically not contained in a finite union of arithmetic progressions [2]. A simple example is $P(x) = \sin(x) + \frac{1}{10}\sin(\sqrt{2}x)$. Trigonometric polynomials with only real zeros have been classified recently in [1] as restrictions of Lee-Yang polynomials. A polynomial $P: \mathbb{C}^n \to \mathbb{C}$ is Lee-Yang if $P(z_1, ..., z_n) \neq 0$ whenever $\min\{\max\{|z_1|, ..., |z_n|\}, \max\{|z_1|^{-1}, ..., |z_n|^{-1}\}\} < 1$. Indeed, the main result in [1] states that if Q is a trigonometric polynomial with only real zeros then there exists a Lee-Yang polynomial $P(z_1, ..., z_n)$ and positive reals $l_1, ..., l_n$, linear independent over \mathbb{Q} , such that

$$Q(x) = P(e^{il_1x}, ..., e^{il_nx}).$$

Crystalline Measures (CM). We let CM be the class of FS-pairs (μ, a) such that both μ and $a(\cdot)$ have locally finite support. It is not hard to see that RRTP \subset CM. One main theme investigated in [25, 2, 1] is to find pairs $(\mu, a) \in$ CM such that supp(a) and/or supp (μ) intersects any infinite arithmetic progression at most finitely¹⁷. Guinand [21, p. 265], in 1959, was "almost the first" to produce such pair, as we shall explain below. He constructs a (self-dual) FS-pair (μ, μ) with

$$\mu = \sum_{n\geqslant 0} c_n (\boldsymbol{\delta}_{\sqrt{n+1/9}} + \boldsymbol{\delta}_{-\sqrt{n+1/9}}),$$

where c_n are the cofficients of the modular form¹⁸

$$\frac{\eta(z)^{2/3}\eta(4z)^{2/3}}{\eta(2z)^{1/3}} = q^{1/9} \left(1 - \frac{2}{3}q - \frac{4}{9}q^2 - \frac{40}{81}q^3 - \frac{160}{243}q^4 + \frac{268}{729}q^5 + \frac{1808}{6561}q^6 + \cdots \right)
= \sum_{n\geqslant 0} c_n q^{1/9+n}.$$
(12)

Above

$$\eta(z) = q^{1/24} \prod_{n \ge 1} (1 - q^n) = \sum_{n \ge 1} \chi_{12}(n) q^{n^2/24}$$

is the Dedekind eta-function, $q=e^{2\pi iz}$ and χ_{12} is the Dirichlet character of modulus 12 with $\chi(1)=\chi(11)=-\chi(5)=-\chi(7)=1$ and zero otherwise. Hence, if $n=9m^2+2m$

¹⁷So very non-periodic

¹⁸Guinand did not realized this nor constructed his example in the way we do here.

then $\sqrt{n+1/9}=3m+1/3$, so the support of μ contains an arithmetic progression. We will show below that, with the right point of view, one can embed Guinand's example and also classical Poisson summation in a real one parameter family, for which almost all members have the property that their support intersect any infinite arithmetic progression at most finitely. For that we need first the following lemma.

Lemma 20. Let $F \in AP(\mathbb{C}^+)$, with locally finite spectrum contained in $[0, \infty)$ and assume that

$$\sum_{n>0} |\mathbb{E}F(\gamma_n)| (1+\gamma_n)^{-k} < \infty$$

for some k > 0, where $\operatorname{spec}(F) = \{\gamma_n\}_{n \geq 0}$, Assume that $G(z) = F(-1/z)\sqrt{i/z}$ also belongs to $\operatorname{AP}(\mathbb{C}^+)$, has locally finite spectrum contained in $[0, \infty)$ and $\mathbb{E}G$ decays similarly as $\mathbb{E}F$. Let

$$\mu = \sum_{n \ge 0} \mathbb{E} F(\gamma_n) (\boldsymbol{\delta}_{-\sqrt{2\gamma_n}} + \boldsymbol{\delta}_{-\sqrt{2\gamma_n}}).$$

Then μ is a strongly tempered measure and

$$\hat{\mu} = \sum_{n>0} \mathbb{E}G(\lambda_n) (\boldsymbol{\delta}_{\sqrt{2\lambda_n}} + \boldsymbol{\delta}_{-\sqrt{2\lambda_n}})$$

where spec $(G) = \{\lambda_n\}_{n \geq 0}$. Hence $(\mu, \widehat{\mu}) \in CM$.

Proof. The proof is inspired by [16, Lemma 2.1]. Consider the gaussian $g_z(t) = e^{\pi i z t^2}$ for $z \in \mathbb{C}^+$. Observe that $\hat{g}_z(t) = \sqrt{i/z} g_{-1/z}(t)$ and that $g_z(\sqrt{2s}) = e^{2\pi i z s}$ for $s \ge 0$. Assume first that $\mathbb{E}G$ also has polynomial growth. Then the identity

$$G(z) = \sum_{n \ge 0} \mathbb{E}G(\lambda_n) g_z(\sqrt{2\lambda_n}) = F(-1/z) \sqrt{i/z} = \sum_{n \ge 0} \mathbb{E}F(\gamma_n) \widehat{g}_z(\sqrt{2\lambda_n})$$

shows that

$$\sum_{n\geq 0} \mathbb{E}F(\gamma_n)(\widehat{\varphi}(\sqrt{2\gamma_n}) + \widehat{\varphi}(\sqrt{-2\gamma_n})) = \sum_{n\geq 0} \mathbb{E}G(\lambda_n)(\varphi(\sqrt{2\lambda_n}) + \varphi(-\sqrt{2\lambda_n})),$$

for any $\varphi \in \operatorname{span}_{\mathbb{C}} \{g_z : z \in \mathbb{C}^+\}$. An approximation argument similar to the one employed in [16, Lemma 2.1] shows that the above identity actually holds for any $\varphi \in \mathcal{S}(\mathbb{R})$.

The following construction was inspired by a similar one partially communicated in 2018 by Danylo Radchenko in a seminar at the University of Bonn. Let $N \ge 1$ be an integer and $\mathbf{r} = \{r_d\}_{d|N}$ be a sequence of reals indexed by the divisors of N such that

$$r_d = r_{N/d}$$
, $\sum_{d|N} r_d = 1$ and $\sum_{d|N} dr_d = \frac{24k}{b}$

where $b \ge 1$ and $k \ge 0$ are coprime integers with b = 1 if k = 0. Consider the eta-product

$$\eta(\boldsymbol{r},z) = \prod_{d|N} \eta(dz)^{r_d} = \sum_{n\geqslant k} \alpha_n q^{n/b}.$$

The eta-function $\eta: \mathbb{C}^+ \to \mathbb{C}$ is an holomorphic function that satisfies the functional identities $\eta(z+\ell) = e^{\pi i \ell/12} \eta(z)$, for $\ell \in \mathbb{Z}$, and $\eta(-1/z) = \sqrt{z/i} \, \eta(z)$. We obtain that

$$\eta(\boldsymbol{r}, z + b) = e^{\pi i b(\sum_{d|N} dr_d)/12} \eta(\boldsymbol{r}, z) = \eta(\boldsymbol{r}, z)$$

$$\eta(\mathbf{r}, -1/z) = \prod_{d|N} \eta(-d/z)^{r_d} = \prod_{d|N} (z/(di))^{r_d/2} \prod_{d|N} \eta(z/d)^{r_d} = \sqrt{z/(i\sqrt{N})} \prod_{d|N} \eta(dz/N)^{r_{N/d}} \\
= \sqrt{z/(i\sqrt{N})} \eta(\mathbf{r}, z/N).$$

Above we use that $\prod_{d|N} d^{-r_d/2} = N^{-1/4}$, which follows from the conditions on r_d . In particular we conclude that the function

$$F_{+}(\boldsymbol{r},z) = \eta(\boldsymbol{r},z/\sqrt{N}) = \sum_{n \geqslant k} \alpha_n q^{n/(b\sqrt{N})}.$$

satisfies the functional equations

$$F_{+}(\mathbf{r}, z + b\sqrt{N}) = F_{+}(\mathbf{r}, z)$$
 and $\sqrt{i/z} F_{+}(\mathbf{r}, -1/z) = F_{+}(\mathbf{r}, z)$

for all $z \in \mathbb{C}^+$. It can be shown that the function $\eta(\boldsymbol{r}, z)$ is a holomorphic modular form of level N, and so the Hecke bound produces $|\alpha_n| \ll n^{1/4}$ (see also [11]). In particular, μ_+ is strongly tempered. Hence F_+ satisfies the hypotheses of Lemma 20. Thus, the measure

$$\mu_{+} = \sum_{n \geq k} \alpha_{n} (\boldsymbol{\delta}_{\sqrt{2n/(b\sqrt{N})}} + \boldsymbol{\delta}_{-\sqrt{2n/(b\sqrt{N})}}),$$

satisfies

$$\widehat{\mu}_+ = \mu_+.$$

Poisson Summation can be recovered this way by

$$\frac{\eta(2z)^5}{\eta(z)^2\eta(4z)^2} = 1 + 2q + 2q^4 + 2q^9 + 2q^{25} + \cdots,$$

with parameters N = 4, $(r_1, r_2, r_4) = (-2, 5, -2)$, b = 1 and k = 0. Guinand's example (12) has parameters N = 4, $(r_1, r_2, r_4) = (2/3, -1/3, 2/3)$, with the others $r_d = 0$ for d|36, b = 3 and k = 1.

Assume now that N is a square. Using the modular λ -invariant (Hauptmodul for $\Gamma(2)$)

$$\lambda(z) = \frac{16\eta(2z)^{16}\eta(z/2)^8}{\eta(z)^{24}} = 16q^{1/2} - 128q + 704q^{3/2} + \cdots,$$

which can be shown to satisfy the transformation laws

$$\lambda(-1/z) = 1 - \lambda(z)$$
 and $\lambda(z+1) = \lambda(z)/(\lambda(z)-1)$,

we can define instead

$$F_{-}(\boldsymbol{r},z) = (1 - 2\lambda(z))\eta(\boldsymbol{r},z/\sqrt{N}) = \sum_{n \ge 2k} \beta_n q^{n/(2b\sqrt{N})}$$

and deduce that

$$F_{-}(\mathbf{r}, z + 2b\sqrt{N}) = F_{-}(\mathbf{r}, z)$$
 and $\sqrt{i/z} F_{-}(\mathbf{r}, -1/z) = -F_{-}(\mathbf{r}, z)$.

Again we conclude that the measure

$$\mu_{-} = \sum_{n>2k} \beta_n (\boldsymbol{\delta}_{\sqrt{n/(b\sqrt{N})}} + \boldsymbol{\delta}_{-\sqrt{n/(b\sqrt{N})}}),$$

satisfies $\hat{\mu}_{-} = -\mu_{-}$. It is now obvious that one can build infinitely many examples of such measures $\mu_{\pm} = \pm \hat{\mu}_{\pm}$ with locally finite support.

Consider now the following family

$$\frac{\eta(z)^{l}\eta(4z)^{l}}{\eta(2z)^{2l-1}} = q^{(l+2)/24} \left(1 - lq + \frac{(l-1)(2+l)}{2} q^{2} + \sum_{n \geq 3} \alpha_{n,l} q^{n} \right)$$

$$(1 - 2\lambda(2z)) \frac{\eta(z)^{l}\eta(4z)^{l}}{\eta(2z)^{2l-1}} = q^{(l+2)/24} \left(1 - (32+l)q + \frac{1}{2}(l^{2} + 65l + 510)q^{2} + \sum_{n \geq 3} \beta_{n,l} q^{n} \right)$$

for N=4 and $l\geqslant -2$. The examples above produce measures $\mu_{\pm}=\pm \hat{\mu}_{\pm}$ with support contained in

$$\{\sqrt{c+n}\}_{n\geqslant 0}\cup\{-\sqrt{c+n}\}_{n\geqslant 0},$$

for any real $c=(l+2)/24\geqslant 0$. Note that for l=-2 we recover classical Poisson summation and for l=2/3 we recover Guinand's example. Perhaps other interesting constructions may be derived from [28] and the references therein. It is not hard to show that $n!\alpha_{n,l}$ and $n!\beta_{n,l}$ are polynomials in l with integer coefficients and degree at most n. Claim: We claim that if c is irrational then $\{\sqrt{c+n}\}_{n\geqslant 0}$ intersects any infinite arithmetic progression at most 2 times. Indeed, having three hits implies that

$$\frac{\sqrt{n_2 + c} - \sqrt{n_1 + c}}{k_1} = \frac{\sqrt{n_3 + c} - \sqrt{n_2 + c}}{k_2}$$

is satisfied for some integers $n_3 > n_2 > n_1 \ge 0$ and $k_1, k_2 > 0$. A boring computation to isolate $n_2 + c$ shows that c is rational, which is absurd.

Such measures are also connected with new Fourier interpolation formulae produced in [40], which as a byproduct produces crystalline pairs with space and spectral supports contained in $\{\pm\sqrt{n}\}_{n\geqslant 0} \cup \{x_0\}$, for any x_0 not an integer square root. These nodes can also be perturbed to be $\{\pm\sqrt{n+\varepsilon_n}\}_{n\geqslant 0}$, as long as $|\varepsilon_n| \leq \delta n^{-5/4}$ for some small $\delta > 0$, see [42]. The recent papers [41, 26] also suggest that there should exist a crystalline pair with space and spectral supports contained in $\{\pm n^{\alpha}\}_{n\geqslant 1}$ and $\{\pm n^{(1-\alpha)}\}_{n\geqslant 1}$ respectively, whenever $0 < \alpha < 1$. Moreover, [26, Cor. 3] proves the striking result that for every (supercritical) pair of increasing sequences $\mathcal{T} = \{t_j\}_{j\in\mathbb{Z}}$ and $\mathcal{S} = \{s_j\}_{j\in\mathbb{Z}}$ such that

$$\max \left(\limsup_{j \to \infty} |t_j|^{p-1} (t_{j+1} - t_j), \limsup_{j \to \infty} |s_j|^{q-1} (s_{j+1} - s_j) \right) < \frac{1}{2},$$

for some $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, there is a FS-pair (μ, a) with μ supported in \mathcal{T} and $a(\cdot)$ supported in \mathcal{S} . An important simple example is $\mathcal{T} = \{\pm n^{\alpha}\}_{n \geq 0}$ and $\mathcal{S} = \{\pm n^{\beta}\}_{n \geq 0}$ for positive α, β with $\alpha + \beta < 1$. In fact, $(\mathcal{T}, \mathcal{S})$ is both a Fourier uniqueness pair and an interpolating pair.

Almost Crystalline Measures. We say (μ, a) is an almost Fourier summation pair (AFS-pair) if μ is strongly tempered, $a(\cdot)$ is locally summable function and

$$(a-\widehat{\mu})|_{C_c^{\infty}(\mathbb{R})} = (\nu_1 - \widehat{\nu}_2)|_{C_c^{\infty}(\mathbb{R})},$$

where ν_1 and ν_2 are absolutely continuous, $\nu_1 \neq 0$ is tempered and ν_2 is strongly tempered. In particular, if $\varphi \in C_c^{\infty}(\mathbb{R})$ we have

$$\sum_{\lambda \in \mathbb{R}} a(\lambda) \varphi(\lambda) - \int_{\mathbb{R}} \widehat{\varphi}(t) d\mu(t) = \int_{\mathbb{R}} \varphi(t) d\nu_1(t) - \int_{\mathbb{R}} \widehat{\varphi}(t) d\nu_2(t).$$

We then let the class ACM (Almost Crystalline Measures) denote the AFS-pairs (μ, a) such that both μ and $a(\cdot)$ have locally finite support. A typical example is Guinand's prime summation formula (also known as Riemann–Weil explicit formula), which states that for every $\varphi \in C_c^{\infty}(\mathbb{R})$ we have

$$\frac{\log(\pi/N)}{2\pi}\varphi(0) + \frac{1}{2\pi} \sum_{n\geqslant 1} \frac{\Lambda(n)\chi(n)}{\sqrt{n}} \left(\varphi\left(\frac{\log n}{2\pi}\right) + \varphi\left(\frac{-\log n}{2\pi}\right)\right)$$
$$= -\sum_{\gamma} \widehat{\varphi}(\gamma) + 2\int_{\mathbb{R}} \varphi(t) \cosh(\pi t) dt + \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{\varphi}(t) \operatorname{Re} \psi(\frac{1}{4} + i\frac{t}{2}) dt$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function, $\Lambda(n)$ is the von Mangoldt function, $\rho = 1/2 + i\gamma$ are the zeros of the Dirichlet *L*-function

$$L(\chi, s) = \sum_{n \geqslant 1} \chi(n) n^{-s}$$

on the critical strip $0 < \text{Re } \rho < 1$ and χ is a primitive, real and even Dirichlet character of modulus N. Since $|\psi(1/4+it/2)| \ll \log(2+|t|)$ and $\sum_{\gamma} \frac{1}{1+\gamma^2} < \infty$, we conclude, assuming the Riemann Hypothesis for $L(\chi, s)$, that if we let

$$a = \frac{\log(\pi/N)}{2\pi} \mathbf{1}_0 + \frac{1}{2\pi} \sum_{n \ge 1} \frac{\Lambda(n)\chi(n)}{\sqrt{n}} (\mathbf{1}_{\frac{\log n}{2\pi}} + \mathbf{1}_{-\frac{\log n}{2\pi}})$$

$$\mu = -\sum_{\gamma} \boldsymbol{\delta}_{\gamma}$$

$$\nu_1 = 2\cosh(\pi t) dt$$

$$\nu_2 = -\frac{1}{2\pi} \operatorname{Re} \psi(\frac{1}{4} + i\frac{t}{2}) dt,$$

then $(\mu, a) \in ACM$. The difference of any two of such pairs above for two characters χ_1 and χ_2 real, even and primitive, but different moduli N, produces a pair in CM (this was pointed out first by Guinand [21]). Such measures are also connected with new Fourier

interpolation formulae produced in [14], and these produce, under GRH, CM-pairs (μ, a) with $\operatorname{supp}(\mu) = \{ \gamma \in \mathbb{R} : L(\chi, 1/2 + i\gamma) = 0 \} \cup \{ \gamma_0 \}$ and $\operatorname{supp}(a) = \{ \operatorname{sgn}(n) \frac{\log |n|}{4\pi} \}_{n \geq 1}$, where γ_0 is any real such that $L(\chi, 1/2 + i\gamma_0) \neq 0$.

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