is *absent*, so, since $(\Re\varphi(\beta))\cdot\Im\beta/|t-\beta|^2$ is in $L_1(\mathbb{R})$, *so is* $\psi(t)$. ψ being *entire and of exponential type* $\leqslant a$, we have, however,

$$\int_{-\infty}^{\infty} \psi(t)S(t)\mathrm{d}t = 0$$

for each of the sums S. Therefore, keeping in mind that $\Im\varphi(\beta) = 0$, we have for the latter

$$(\Re\varphi(\beta))\int_{-\infty}^{\infty} \frac{\Im\beta}{|t-\beta|^2}S(t)\mathrm{d}t = (\Im\beta)\int_{-\infty}^{\infty} \frac{\varphi(t)}{|t-\beta|^2}S(t)\,\mathrm{d}t.$$

The right side is in modulus

$$\leqslant \Im\beta\sqrt{\left(\int_{-\infty}^{\infty} \frac{|\varphi(t)|^2}{w(t)|t-\beta|^4}\mathrm{d}t\right)} \cdot \|S\|,$$

so we are *done* as long as $\Re\varphi(\beta) \neq 0$.

We need therefore only show that there are β, $\Im\beta > 0$, with $\Im\varphi(\beta) = 0$ but $\Re\varphi(\beta) \neq 0$. It is claimed in the first place that there are β, $\Im\beta > 0$, with $\Im\varphi(\beta) = 0$. *Otherwise*, the harmonic function $\Im\varphi(z)$ *would be of one sign*, say $\Im\varphi(z) \geqslant 0$, for $\Im z > 0$. In such case, the *second* theorem of §F.1, Chapter III, gives us a number $\alpha \geqslant 0$ and a positive measure μ on $\mathbb{R}$ with

$$\Im\varphi(z) = \alpha\Im z + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\mathrm{d}\mu(t), \qquad \Im z > 0.$$

Our function $\Im\varphi(z)$ is continuous right up to $\mathbb{R}$, φ being entire, so we readily see in the usual way that

$$\mathrm{d}\mu(t) = (\Im\varphi(t))\mathrm{d}t.$$

In our circumstances, however, $\Im\varphi(t) \equiv 0$, since we took $\varphi(t)$ to be *real on* $\mathbb{R}$. Hence $\Im\varphi(z) = \alpha\Im z$ for $\Im z > 0$ and finally

$$\varphi(z) = \alpha z + C.$$

Here, *we cannot have* $\alpha > 0$. For, if that were so, we would get

$$\int_{-\infty}^{\infty} \frac{|\alpha t + C|^2}{w(t)(t^2+1)^2}\mathrm{d}t = \int_{-\infty}^{\infty} \frac{|\varphi(t)|^2}{w(t)(t^2+1)^2}\mathrm{d}t < \infty,$$

whence, for A larger than $|C/\alpha|$,

$$\int_A^{\infty} \frac{\mathrm{d}t}{w(t)(t^2+1)} < \infty.$$

From this, however, it would follow that

$$\int_A^{\infty} \frac{dt}{\sqrt{(t^2+1)}} \quad \leqslant \quad \sqrt{\left(\int_A^{\infty} \frac{dt}{w(t)(t^2+1)} \int_A^{\infty} w(t)\,dt \right)} \quad < \quad \infty,$$

which is nonsense.

Thus, $\alpha = 0$ and $\varphi(z)$ reduces to a constant C. This possibility was, however, excluded at the very beginning of the present argument – *our* φ is *not constant*. The function $\Im\varphi(z)$, then, *cannot be of one sign* in $\Im z > 0$, and we *have* points β in that half plane for which $\Im\varphi(\beta) = 0$.

Take any one of those – call it β_0. Since $\Im\varphi(z)$ is harmonic (everywhere!), we have

$$\int_{-\pi}^{\pi} \Im\varphi(\beta_0 + \rho e^{i\vartheta})d\vartheta \quad = \quad 2\pi\Im\varphi(\beta_0) \quad = \quad 0$$

for $\rho > 0$, and there must be a point on *each circle about* β_0 where $\Im\varphi$ also vanishes. There is thus a sequence of points $\beta_n \neq \beta_0$ in the upper half plane with $\beta_n \xrightarrow[n]{} \beta_0$ and $\Im\varphi(\beta_n) = 0$ for each n. If now $\Re\varphi(\beta_n)$ also vanished for each n, we would have $\varphi(z) \equiv 0$. But $\varphi \not\equiv 0$. Hence,

$$\Im\varphi(\beta_n) \; = \; 0 \quad \text{but } \Re\varphi(\beta_n) \; \neq \; 0$$

for some n, and, taking that β_n as our β, we have what was needed. The *sufficiency* of our condition on w is thus established.

We are done.

When dealing with the harmonic functions

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} S(t)\,dt,$$

one usually needs estimates on them in the *whole* upper half plane, and not just for *certain values* of z therein. The availability of these for sums $S(t)$ like those figuring in the last theorem with

$$\int_{-\infty}^{\infty} |S(t)|^2 w(t)\,dt \quad \leqslant \quad 1$$

is governed by a different condition on w.

Theorem. *Let* $w(t) \geqslant 0$ *belong to* $L_1(\mathbb{R})$, *and let* $a > 0$. *In order that the finite sums*

$$S(t) \quad = \quad \sum_{|\lambda| \geqslant a} A_\lambda e^{i\lambda t}$$

satisfy a relation

$$\left|\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} S(t)\mathrm{d}t\right| \leqslant K_z \sqrt{\left(\int_{-\infty}^{\infty} |S(t)|^2 w(t)\mathrm{d}t\right)}$$

for every z, $\Im z > 0$, *it is necessary and sufficient that there exist a non-zero entire function* ψ *of exponential type* $\leqslant a$ *such that*

$$\int_{-\infty}^{\infty} \frac{|\psi(t)|^2}{w(t)(t^2+1)}\mathrm{d}t < \infty.$$

When this condition is met, the numbers K_z *can be taken to be bounded above on compact subsets of* $\{\Im z > 0\}$.

Proof: Sufficiency. If there *is* such a function ψ, we have, for any complex β,

$$\frac{\psi(t)}{t-\beta} = f_\beta(t) + \frac{\psi(\beta)}{t-\beta},$$

with

$$f_\beta(t) = \frac{\psi(t)-\psi(\beta)}{t-\beta}$$

entire and of exponential type $\leqslant a$. As long as $\beta \notin \mathbb{R}$, we may take $\psi(\beta)$ to be $\neq 0$. Indeed, we may assume that both $|\psi(\beta)|$ and $|\psi(\bar{\beta})|$ are bounded away from zero when β ranges over any compact subset E of $\{\Im z > 0\}$. To see this, observe that ψ has at most a finite number of zeros on $E \cup E^*$, where E^* is the reflection of E in $\mathbb{R}$. Calling those $z_1, z_2, \ldots, z_n$ (repetitions according to multiplicities, as usual), we may work with

$$\psi_E(t) = \frac{\psi(t)}{(t-z_1)(t-z_2)\cdots(t-z_n)}$$

instead of ψ. This function is entire, of exponential type $\leqslant a$, and bounded away from zero in modulus on $E \cup E^*$. And

$$\int_{-\infty}^{\infty} \frac{|\psi_E(t)|^2}{w(t)(t^2+1)}\mathrm{d}t < \infty$$

because none of the z_k, $1 \leqslant k \leqslant n$, are on the real axis.*

All this being granted, we fix a β with $\Im\beta \neq 0$ and look at the entire function f_β figuring in the above relation. It is claimed that $f_\beta(t)$ is in

* $\psi(z)$ may *need* to vanish at some points on the real axis in order to offset certain zeros that the given weight w might have there! See the scholium at the end of this §.

$L_2(\mathbb{R})$. We have

$$\int_{-\infty}^{\infty} \frac{|\psi(t)|}{|t-\beta|}\,\mathrm{d}t \quad \leqslant \quad \sqrt{\left(\int_{-\infty}^{\infty} \frac{|\psi(t)|^2}{w(t)|t-\beta|^2}\,\mathrm{d}t \int_{-\infty}^{\infty} w(t)\mathrm{d}t\right)} \quad < \quad \infty,$$

i.e., $|\psi(t)|/|t-\beta|$ is in $L_1(\mathbb{R})$. This ratio is also *bounded* on the real axis.

Indeed, if ψ had no zeros at all its Hadamard factorization would reduce to $\psi(t) = C\mathrm{e}^{\gamma t}$ with constants C and γ. Then, however, $|\psi(t)|/|t-\beta|$ could not be integrable over $\mathbb{R}$. Consequently, ψ *has* a zero, say at z_0, and then $\psi(t)/(t-z_0)$ is *entire, of exponential type* $\leqslant a$, and in $L_1(\mathbb{R})$ since $\psi(t)/(t-\beta)$ is. A simple version of the Paley–Wiener theorem (Chapter III, §D) now shows that

$$\frac{\psi(t)}{t-z_0} \quad = \quad \int_{-a}^{a} \mathrm{e}^{it\lambda} p(\lambda)\,\mathrm{d}\lambda,$$

with (here) $p(\lambda)$ some *continuous* function on $[-a, a]$. By this formula, we see at once that $\psi(t)/(t-z_0)$ is bounded on $\mathbb{R}$ – so, then, is $\psi(t)/(t-\beta)$.

The ratio $|\psi(t)|/|t-\beta|$ is thus both in $L_1(\mathbb{R})$ and bounded on $\mathbb{R}$. Therefore it is in $L_2(\mathbb{R})$. So, however, is $\psi(\beta)/(t-\beta)$. The difference

$$f_\beta(t) \quad = \quad \frac{\psi(t)}{t-\beta} - \frac{\psi(\beta)}{t-\beta}$$

must hence also be square integrable.

Because f_β is entire and of exponential type $\leqslant a$, we now have

$$\underset{A\to\infty}{\mathrm{l.i.m.}} \int_{-A}^{A} \mathrm{e}^{\mathrm{i}\lambda t} f_\beta(t)\,\mathrm{d}t \quad = \quad 0$$

for almost all $\lambda \notin [-a, a]$ by the L_2 form of the Paley–Wiener theorem (Chapter III, §D). At the same time, when $A \longrightarrow \infty$, the integrals

$$\int_{-A}^{A} \mathrm{e}^{\mathrm{i}\lambda t} f_\beta(t)\,\mathrm{d}t$$

tend, for $\lambda \neq 0$, to a certain function of λ *continuous* on $\mathbb{R} \sim \{0\}$. This, indeed, is certainly true if, in those integrals, we replace $f_\beta(t)$ by $\psi(t)/(t-\beta) \in L_1(\mathbb{R})$. *Direct verification* shows that the same holds good when $f_\beta(t)$ is replaced by $\psi(\beta)/(t-\beta)$. The statement therefore holds for the difference $f_\beta(t)$ of these functions.

The (continuous) pointwise limit of the expressions

$$\int_{-A}^{A} e^{i\lambda t} f_\beta(t)\,dt, \qquad \lambda \neq 0,$$

for $A \longrightarrow \infty$ must, however, coincide a.e. with their limit in mean, known to be zero a.e. for $|\lambda| \geqslant a$, as we have just seen. Hence

$$\lim_{A\to\infty} \int_{-A}^{A} e^{i\lambda t} f_\beta(t)\,dt \quad = \quad 0, \qquad |\lambda| \geqslant a.$$

That is,

$$\lim_{A\to\infty} \int_{-A}^{A} \frac{\psi(\beta)}{t-\beta} e^{i\lambda t}\,dt \quad = \quad \int_{-\infty}^{\infty} \frac{\psi(t)e^{i\lambda t}}{(t-\beta)}\,dt$$

for $|\lambda| \geqslant a$, and finally,

$$\lim_{A\to\infty} \int_{-A}^{A} \frac{\psi(\beta)S(t)}{t-\beta}\,dt \quad = \quad \int_{-\infty}^{\infty} \frac{\psi(t)S(t)}{t-\beta}\,dt$$

for each of our sums $S(t)$.

Let us continue to write $\|\ \|$ for the norm appearing in the proof of the preceding theorem. In terms of this notation, we have for the modulus of the *right-hand* member of the last relation the upper bound

$$\sqrt{\left(\int_{-\infty}^{\infty} \frac{|\psi(t)|^2}{w(t)|t-\beta|^2}\,dt\right)} \cdot \|S\|.$$

The condition that

$$\int_{-\infty}^{\infty} \frac{|\psi(t)|^2}{w(t)(t^2+1)}\,dt \quad < \quad \infty$$

clearly makes the square root $\leqslant$ a quantity C_β, *bounded above* when β ranges over *compact subsets of* $\mathbb{C} \sim \mathbb{R}$. We thus have

$$\left|\lim_{A\to\infty} \int_{-A}^{A} \frac{S(t)}{t-\beta}\,dt\right| \quad \leqslant \quad \frac{C_\beta}{|\psi(\beta)|}\|S\|$$

for the sums S, when $\Im\beta > 0$. In like manner,

$$\left|\lim_{A\to\infty} \int_{-A}^{A} \frac{S(t)}{t-\bar\beta}\,dt\right| \quad \leqslant \quad \frac{C_{\bar\beta}}{|\psi(\bar\beta)|}\|S\|,$$

so finally, since

$$\frac{1}{t-\beta} - \frac{1}{t-\bar{\beta}} = \frac{2\mathrm{i}\Im\beta}{|t-\beta|^2},$$

$$\left|\int_{-\infty}^{\infty} \frac{\Im\beta}{|t-\beta|^2} S(t)\,\mathrm{d}t\right| \leqslant K_\beta \|S\|.$$

If $E \subseteq \{\Im z > 0\}$ is compact,

$$K_\beta = \frac{C_\beta}{2|\psi(\beta)|} + \frac{C_{\bar{\beta}}}{2|\psi(\bar{\beta})|}$$

may be taken to be bounded above on E, since, as explained at first, we can choose ψ with $|\psi(\beta)|$ and $|\psi(\bar{\beta})|$ bounded away from 0 on E. Sufficiency is proved.

Necessity. Suppose that for *every* β, $\Im\beta > 0$, the last inequality (at the end of the preceding discussion) *holds*, with some finite K_β. Then, by the previous theorem, we certainly have a non-zero entire function φ of exponential type $\leqslant a$, with

$$\int_{-\infty}^{\infty} \frac{|\varphi(t)|^2}{w(t)(t^2+1)^2}\,\mathrm{d}t < \infty.$$

As we saw at the beginning of the *sufficiency* part of that theorem's proof, we may take $\varphi(t)$ to be *real* on $\mathbb{R}$.

Let $\Im\beta \neq 0$. We have an identity

$$\frac{(\Im\beta)\varphi(t)}{|t-\beta|^2} = (\Im\beta)g_\beta(t) + (\Im\varphi(\beta))\frac{t-\Re\beta}{|t-\beta|^2} + (\Re\varphi(\beta))\frac{\Im\beta}{|t-\beta|^2}$$

like the one used in establishing the preceding theorem, where g_β is an entire function of exponential type $\leqslant a$.

The relation involving $\varphi(t)$ and w implies that $\varphi(t)/(t-\beta)^2 \in L_1(\mathbb{R})$ by the usual application of Schwarz' inequality. Then, if φ is *not* a pure exponential (when it *is*, it must be *bounded* on $\mathbb{R}$), it must have *at least two zeros*, for, if it had only *one*, $\varphi(t)/(t-\beta)^2$ would, by φ's resulting Hadamard factorization, be prevented from being in $L_1(\mathbb{R})$. This being the case, an argument like the one made during the preceding sufficiency proof shows that $\varphi(t)/(t-\beta)^2$ is *bounded* on $\mathbb{R}$. The conclusion is that $\varphi(t)/(t-\beta)^2$ is also in $L_2(\mathbb{R})$, and, referring to the above formula, we see that $g_\beta(t)$ is square integrable.

The function $g_\beta(t)$ is, however, entire and of exponential type $\leqslant a$, so we may essentially *repeat* the reasoning followed above, based on the

Paley–Wiener theorem, to conclude from the previous formula that

$$\Im\varphi(\beta)\cdot\lim_{A\to\infty}\int_{-A}^{A}\frac{t-\Re\beta}{|t-\beta|^2}S(t)\,\mathrm{d}t \;+\; \Re\varphi(\beta)\cdot\int_{-\infty}^{\infty}\frac{\Im\beta}{|t-\beta|^2}S(t)\,\mathrm{d}t$$

$$= \;\Im\beta\cdot\int_{-\infty}^{\infty}\frac{\varphi(t)}{|t-\beta|^2}S(t)\,\mathrm{d}t$$

for the sums $S(t)$.

Here, we have

$$\int_{-\infty}^{\infty}\frac{|\varphi(t)|^2}{w(t)|t-\beta|^4}\mathrm{d}t \quad < \quad \infty,$$

so the *right side* of the preceding relation is in modulus

$$\leqslant \quad \text{const.}\,\|S\|,$$

where $\|\;\|$ has the same meaning as before. At the same time, our *assumption* is that

$$\left|\int_{-\infty}^{\infty}\frac{\Im\beta}{|t-\beta|^2}S(t)\,\mathrm{d}t\right| \quad \leqslant \quad \text{const.}\,\|S\|$$

for our sums S (with the constant depending, of course, on β). This may now be combined with the result just found to yield

$$\left|\Im\varphi(\beta)\cdot\lim_{A\to\infty}\int_{-A}^{A}\frac{t-\Re\beta+\mathrm{i}\Im\beta}{|t-\beta|^2}S(t)\,\mathrm{d}t\right| \quad \leqslant \quad \text{const.}\,\|S\|.$$

For each β, then, with $\Im\beta > 0$ there is a finite L_β such that

$$\left|\Im\varphi(\beta)\cdot\lim_{A\to\infty}\int_{-A}^{A}\frac{S(t)}{t-\beta}\mathrm{d}t\right| \quad \leqslant \quad L_\beta\|S\|$$

for the sums S.

Suppose that $\Im\varphi(\beta) \neq 0$ *for some* β *with* $\Im\beta > 0$. *Then we are done.* We can, indeed, argue as at the very *start* of the previous theorem's proof to obtain, thanks to the last relation, a $k(t)$ with $\|k\| < \infty$ (and hence $w(t)k(t) \in L_1(\mathbb{R})$ by Schwarz) such that

$$\lim_{A\to\infty}\int_{-A}^{A}\left(w(t)k(t) \;-\; \frac{1}{t-\beta}\right)\mathrm{e}^{\mathrm{i}\lambda t}\,\mathrm{d}t \quad = \quad 0$$

for $|\lambda| \geqslant a$.

Here, the integrable function $w(t)k(t)$ must also be in $L_2(\mathbb{R})$. Indeed, the (*bounded*!) Fourier transform

$$\int_{-\infty}^{\infty} e^{i\lambda t} w(t)k(t)\,dt$$

coincides with the L_2 Fourier transform of $1/(t-\beta)$ for large $|\lambda|$, and is thus itself in L_2. Then, however, $w(t)k(t) \in L_2(\mathbb{R})$ by Plancherel's theorem.

We may now apply the L_2 Paley–Wiener theorem (Chapter III, §D) to the function

$$w(t)k(t) \quad - \quad \frac{1}{t-\beta}$$

and conclude from the preceding relation that it coincides a.e. on $\mathbb{R}$ with an entire function $f(t)$ of exponential type $\leqslant a$. The function

$$\psi(t) \quad = \quad (t-\beta)f(t) \; + \; 1$$

is also entire and of exponential type a, and

$$\psi(t) \quad = \quad (t-\beta)w(t)k(t) \qquad \text{a.e., } t \in \mathbb{R}.$$

The above integral relation clearly implies that $w(t)k(t)$ cannot vanish a.e., so $\psi \not\equiv 0$. Finally,

$$\int_{-\infty}^{\infty} \frac{|\psi(t)|^2}{w(t)|t-\beta|^2}\,dt \quad = \quad \|k\|^2 \quad < \quad \infty,$$

so

$$\int_{-\infty}^{\infty} \frac{|\psi(t)|^2}{w(t)(t^2+1)}\,dt \quad < \quad \infty.$$

The *necessity is thus established* provided that for some β, $\Im\beta > 0$, the *original entire function* φ has *non-zero imaginary part at* β. If, however, there is *no* such β, we are *also* finished! *Then*, $\Im\varphi(\beta) \equiv 0$ for $\Im\beta > 0$, so $\varphi(z)$ must be *constant*, wlog, $\varphi(z) \equiv 1$. This means that

$$\int_{-\infty}^{\infty} \frac{1}{w(t)(t^2+1)^2}\,dt \quad < \quad \infty.$$

In that case,

$$\int_{-\infty}^{\infty} \frac{|\psi(t)|^2}{w(t)(t^2+1)}\,dt \quad < \quad \infty$$

with, e.g., $\psi(t) = \sin at/at$, and *this* function ψ is entire, of exponential type a, and $\not\equiv 0$.

The theorem is completely proved.

Scholium. The discrepancy between the conditions on w involved in the above two theorems is annoying. How can there be a $w \geqslant 0$ such that the sums

$$S(t) = \sum_{|\lambda| \geqslant a} A_\lambda e^{i\lambda t}$$

satisfying

$$\int_{-\infty}^{\infty} |S(t)|^2 w(t)\, dt \leqslant 1$$

yield harmonic functions

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} S(t)\, dt$$

with values bounded at *some* points z in the upper half plane, but not at *each* of those points? If there is a non-constant entire function $\varphi \not\equiv 0$ of exponential type $\leqslant a$ for which

$$\int_{-\infty}^{\infty} \frac{|\varphi(t)|^2}{w(t)(t^2+1)^2} dt < \infty,$$

can we not *divide out* one of the *zeros* of φ to get another such function ψ making

$$\int_{-\infty}^{\infty} \frac{|\psi(t)|^2}{w(t)(t^2+1)} dt < \infty \ ?$$

The present situation illustrates the care that must be taken in the investigation of such matters, straightforward though they may appear. The conditions involved in the two results are *not* equivalent, and there *really do* exist functions $w \geqslant 0$ *satisfying one, but not the other.* None of the zeros of φ can be divided out if they are all needed to cancel those of $w(t)$!

Here is a simple example. Let

$$w(t) = \frac{\sin^2 \pi t}{t^2+1}.$$

The condition

$$\int_{-\infty}^{\infty} \frac{|\varphi(t)|^2}{w(t)(t^2+1)^2} dt < \infty$$

is satisfied here with

$$\varphi(t) \quad = \quad \sin \pi t,$$

an entire function of exponential type π. *The kind of estimate furnished by the first theorem* is therefore *available* for the sums

$$S(t) \;=\; \sum_{|\lambda| \geqslant \pi} A_\lambda e^{i\lambda t}.$$

Here, however, the estimates provided by the second theorem are not all valid!

To see this, consider the functions

$$T_\eta(t) \quad = \quad \frac{i}{\sin \pi(t + i\eta)},$$

where η is a small parameter > 0. We have

$$\Re T_\eta(t) \quad = \quad \frac{\Im \sin \pi(t + i\eta)}{|\sin \pi(t + i\eta)|^2} \quad = \quad \frac{\sinh \pi\eta \cos \pi t}{\sin^2 \pi t + \sinh^2 \pi\eta}.$$

Clearly $\Re T_\eta(t + 2) = \Re T_\eta(t)$ and $\Re T_\eta$ is $\mathscr{C}_\infty$ on the real axis, so

$$\Re T_\eta(t) \quad = \quad \sum_{-\infty}^{\infty} a_n e^{\pi i n t}, \qquad t \in \mathbb{R},$$

the series being absolutely convergent. Since $\Re T_\eta(t - \frac{1}{2})$ and $\Re T_\eta(t + \frac{1}{2})$ are *odd* functions of t, we have

$$a_0 \quad = \quad \frac{1}{2} \int_{-1}^{1} (\Re T_\eta)(t)\, dt \quad = \quad 0,$$

and $\Re T_\eta(t)$ is a (uniform!) limit of sums

$$\sum_{1 \leqslant |n| \leqslant N} a_n e^{\pi i n t},$$

each of the form

$$\sum_{|\lambda| \geqslant \pi} A_\lambda e^{i\lambda t}.$$

If the estimates furnished by the second theorem held for the present w and for a $= \pi$, we would now have

$$\left| \int_{-\infty}^{\infty} \frac{(\Re T_\eta)(t)}{1 + t^2}\, dt \right| \quad \leqslant \quad C \sqrt{\left(\int_{-\infty}^{\infty} (\Re T_\eta(t))^2 w(t)\, dt \right)}$$

with a constant C *independent* of $\eta > 0$. That, however, is not the case.

Because

$$(\Re T_\eta(t))^2 w(t) \;\leqslant\; |T_\eta(t)|^2 w(t) \;=\; |T_\eta(t)|^2 \frac{\sin^2 \pi t}{t^2+1} \;\leqslant\; \frac{1}{t^2+1}$$

and

$$\Re T_\eta(t) \longrightarrow 0 \qquad \text{as } \eta \longrightarrow 0$$

for $t \neq 0, \pm 1, \pm 2, \ldots$, we have

$$\int_{-\infty}^{\infty} (\Re T_\eta(t))^2 w(t)\,\mathrm{d}t \longrightarrow 0$$

when $\eta \longrightarrow 0$, by dominated convergence.

At the same time, since each of the functions

$$T_\eta(z) \;=\; \frac{\mathrm{i}}{\sin \pi(z + \mathrm{i}\eta)}$$

is *analytic* and *bounded* in $\Im z > 0$,

$$\int_{-\infty}^{\infty} \frac{(\Re T_\eta)(t)}{1+t^2}\,\mathrm{d}t \;=\; \pi \Re T_\eta(\mathrm{i}) \;=\; \frac{\pi}{\sinh \pi(1+\eta)} \;\longrightarrow\; \frac{\pi}{\sinh \pi} \;>\; 0$$

as $\eta \longrightarrow 0$. This does it.

It is not hard to see that here, for the sums

$$S(t) \;=\; \sum_{|\lambda| \geqslant \pi} A_\lambda \mathrm{e}^{\mathrm{i}\lambda t},$$

the condition

$$\int_{-\infty}^{\infty} |S(t)|^2 w(t)\,\mathrm{d}t \;\leqslant\; 1$$

gives us control on the integrals

$$\int_{-\infty}^{\infty} \frac{\Im \beta}{|t-\beta|^2} S(t)\,\mathrm{d}t$$

when $\Re \beta = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots,$ $\Im \sin \pi\beta$ vanishing precisely for such values of β.

One may pose a problem similar to the one discussed in this §, but with the sums

$$S(t) \;=\; \sum_{|\lambda| \geqslant a} A_\lambda \mathrm{e}^{\mathrm{i}\lambda t}$$

replaced by others of the form

$$\sum_{|\lambda|\leqslant a} A_\lambda e^{i\lambda t}$$

(i.e., by entire functions of exponential type $\leqslant a$ bounded on $\mathbb{R}$!). That seems *harder*. Some of the material in the first part of de Branges' book is relevant to it.

E. Hilbert transforms of certain functions having given weighted quadratic norms.

We continue along the lines of the preceding §'s discussion. Taking, as we did there, some fixed $w \geqslant 0$ belonging to $L_1(\mathbb{R})$, let us suppose that we are given a certain class of functions $U(t)$, bounded on the real axis, whose harmonic extensions

$$U(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\, U(t)\,dt$$

to the upper half plane are controlled by the weighted norm

$$\sqrt{\left(\int_{-\infty}^{\infty} |U(t)|^2 w(t)\,dt\right)}.$$

A suitably defined *harmonic conjugate* $\tilde{U}(z)$ of each of our functions $U(z)$ will then also be controlled by that norm. As we have seen in Chapter III, §F.2 and in the scholium to §H.1 of that chapter, the $\tilde{U}(z)$ have well defined non-tangential boundary values a.e. on $\mathbb{R}$ and thereby give rise to Lebesgue measurable functions $\tilde{U}(t)$ of the real variable t. Each of the latter is a *Hilbert transform* of the corresponding original bounded function $U(t)$; we say *a* Hilbert transform because that object, like the harmonic conjugate, is really only defined to within an additive constant. The reader can arrive at a fairly clear idea of these transforms by referring first to the §§ mentioned above and then to the middle of §C.1, Chapter VIII, and the scholium at the end of it.

Whatever specification is adopted for the Hilbert transforms $\tilde{U}(t)$ of our functions U, one may ask whether their *size* is governed by the weighted norm in question when that is the case for the harmonic extensions $U(z)$. To be more definite, let us ask whether there is some integrable function $\omega(t) \geqslant 0$, not a.e. zero on $\mathbb{R}$, such that

$$\int_{-\infty}^{\infty} |\tilde{U}(t)|^2 \omega(t)\,dt \quad\leqslant\quad \int_{-\infty}^{\infty} |U(t)|^2 w(t)\,dt$$

for the particular class of functions U under consideration. In the present

§, we study this question for the exponential sums

$$U(t) = \sum_{|\lambda| \geqslant a} A_\lambda e^{i\lambda t}$$

worked with in §D. Although the problem, as formulated, no longer refers directly to the harmonic extensions $U(z)$, it will turn out to have a positive solution (for given w) precisely when the latter are controlled by $\int_{-\infty}^{\infty} |U(t)|^2 w(t)\,dt$ in $\{\Im z > 0\}$ (and only then). For this reason, multipliers will again be involved in our discussion.

The work will require some material from the theory of H_p spaces. In order to save the reader the trouble of digging up that material elsewhere, we give it (and no more) in the next article, starting from scratch. This is *not* a book about H_p spaces, and anyone wishing to really learn about them should refer to such a book. Several are now available, including (and *why* not!) my own.*

1. H_p spaces for people who don't want to really learn about them

We will need to know some things about H_1, H_∞ and H_2, and proceed to take up those spaces in that order. Most of the real work involved here has actually been done already in various parts of the present book.

For our purposes, it is most convenient to use the

Definition. $H_1(\mathbb{R})$, or, as we usually write, H_1, is the set of f in $L_1(\mathbb{R})$ for which the Fourier transform

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda t} f(t)\,dt$$

vanishes for all $\lambda \geqslant 0$.

* As much as I want that book to sell, I should warn the reader that there are a fair number of misprints and also some actual mistakes in it. The *statement* of the lemma on p. 104 is inaccurate; boundedness only holds for r away from 0 when $F(0) = 0$. *Statement* of the lemma on p. 339 is wrong; ν may also contain a point mass at 0. That, however, makes no difference for the subsequent application of the lemma. The *argument* at the bottom of p. 116 is nonsense. *Instead*, one should say that if $\text{Б} | B_\alpha$ and $d\sigma' \leqslant d\sigma_\alpha$ for each α, then every f_α is in ΩH_2, where Ω is given by the formula displayed there. Hence $\omega H_2 = E$ is $\subseteq \Omega H_2$, so $\text{Б} | B$ and $d\sigma' \leqslant d\sigma$ by reasoning like that at the top of p. 116. There are confusing *misprints* in the proof of the first theorem on p 13; near the end of that proof, F should be replaced by G.

Lemma. *If $f \in H_1$, $e^{i\lambda t}f(t) \in H_1$ for each $\lambda \geqslant 0$.*

Proof. Clear.

Lemma. *If $f \in H_1$ and $\Im z > 0$, $f(t)/(t-\bar{z}) \in H_1$.*

Proof. For $\Im z > 0$ (i.e., $\Re(-i\bar{z}) < 0$), we have

$$\frac{i}{t-\bar{z}} = \int_0^\infty e^{-i\bar{z}\lambda}e^{i\lambda t}\,d\lambda, \qquad t \in \mathbb{R}.$$

Therefore, if $f \in H_1$,

$$i\int_{-\infty}^\infty \frac{f(t)}{t-\bar{z}}\,dt = \int_{-\infty}^\infty \int_0^\infty e^{-i\bar{z}\lambda}e^{i\lambda t}f(t)\,d\lambda\,dt.$$

The double integral on the right is absolutely convergent, and hence can be rewritten as

$$\int_0^\infty \int_{-\infty}^\infty e^{-i\bar{z}\lambda}e^{i\lambda t}f(t)\,dt\,d\lambda = \int_0^\infty e^{-i\bar{z}\lambda}\hat{f}(\lambda)\,d\lambda = 0.$$

If $\alpha \geqslant 0$ and $f \in H_1$, $e^{i\alpha t}f(t)$ is also in H_1 by the preceding lemma, so, using it in place of $f(t)$ in the computation just made, we get

$$\int_{-\infty}^\infty e^{i\alpha t}\frac{f(t)}{t-\bar{z}}\,dt = 0.$$

$f(t)/(t-\bar{z})$ is thus in H_1 by definition.

Theorem. *If, for $f \in H_1$, we write*

$$f(z) = \frac{1}{\pi}\int_{-\infty}^\infty \frac{\Im z}{|z-t|^2}f(t)\,dt$$

for $\Im z > 0$, the function $f(z)$ is analytic *in the upper half plane.*

Proof. We have

$$f(z) = \frac{1}{2\pi i}\int_{-\infty}^\infty \left(\frac{1}{t-z} - \frac{1}{t-\bar{z}}\right)f(t)\,dt.$$

By the last lemma, the right side equals

$$\frac{1}{2\pi i}\int_{-\infty}^\infty \frac{f(t)}{t-z}\,dt$$

for $\Im z > 0$, and this expression is clearly analytic in the upper half plane. We are done.

Theorem. *The function $f(z)$ defined in the statement of the preceding result has the following properties:*

(i) *$f(z)$ is continuous and bounded in each half plane $\{\Im z \geqslant h\}$, $h > 0$, and tends to 0 as $z \longrightarrow \infty$ in any one of those;*

(ii) $$\int_{-\infty}^{\infty} |f(x+iy)|\,dx \leqslant \|f\|_1 \quad \text{for } y > 0;$$

(iii) $$\int_{-\infty}^{\infty} |f(t+iy) - f(t)|\,dt \longrightarrow 0 \quad \text{as } y \longrightarrow 0;$$

(iv) $f(t+iy) \longrightarrow f(t)$ a.e. as $y \longrightarrow 0$.

Remark. Properties (iii) and (iv) justify our denoting

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} f(t)\,dt$$

by $f(z)$.

Proof of theorem. Property (i) is verified by inspection; (ii) and (iii) hold because the Poisson kernel is a (positive) approximate identity. Property (iv) comes out of the discussion beginning in Chapter II, §B and then continuing in §F.2 of Chapter III and in the scholium to §H.1 of that chapter. These ideas have already appeared frequently in the present book.

Theorem. *If $f(t) \in H_1$ is not zero a.e. on $\mathbb{R}$, we have*

$$\boxed{\int_{-\infty}^{\infty} \frac{\log^- |f(t)|}{1+t^2}\,dt < \infty,}$$

and, for each z, $\Im z > 0$,

$$\log|f(z)| \leqslant \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} \log|f(t)|\,dt,$$

the integral on the right being absolutely convergent. Here, $f(z)$ has the same meaning as in the preceding two results.

Proof. For each $h > 0$ we can apply the results from Chapter III, §G.2 to $f(z+ih)$ in the half plane $\Im z > 0$, thanks to property (i), guaranteed by the last theorem. In this way we get

$$\log|f(z+ih)| \leqslant \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} \log|f(t+ih)|\,dt$$

for $\Im z > 0$, with the integral on the right absolutely convergent.

Fix for the moment any z, $\Im z > 0$, for which $f(z) \neq 0$. The *left* side of the relation just written then tends to a limit $> -\infty$ as $h \longrightarrow 0$. At the same time, the *right* side is equal to

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log^+|f(t+ih)|\,dt \quad - \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log^-|f(t+ih)|\,dt,$$

where

$$\int_{-\infty}^{\infty} \left|\, \log^+|f(t+ih)| - \log^+|f(t)|\,\right| dt \quad \leqslant \quad \int_{-\infty}^{\infty} \left|\, |f(t+ih)| - |f(t)|\,\right| dt,$$

which tends to zero as h does, according to property (iii) in the preceding result. Therefore

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log^+|f(t+ih)|\,dt \quad \longrightarrow \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log^+|f(t)|\,dt,$$

a finite quantity (by the inequality between arithmetic and geometric means), as $h \longrightarrow 0$.

From property (iv) in the preceding theorem and Fatou's lemma, we have, however,

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log^-|f(t)|\,dt \quad \leqslant \quad \liminf_{h\to 0}\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log^-|f(t+ih)|\,dt.$$

Using this and the preceding relation we see, by making $h \longrightarrow 0$ in our initial one, that

$$-\infty \quad < \quad \log|f(z)| \quad \leqslant \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log^+|f(t)|\,dt$$
$$- \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log^-|f(t)|\,dt.$$

Since the *first* integral on the right is finite, the *second* must also be so. That, however, is equivalent to the relation

$$\int_{-\infty}^{\infty} \frac{\log^-|f(t)|}{1+t^2}\,dt \quad < \quad \infty.$$

Putting the two right-hand integrals together, we see that

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log|f(t)|\,dt$$

is absolutely convergent for our particular z, and hence for any z with

$\Im z > 0$. That quantity is $\geqslant \log|f(z)|$ as we have just seen, provided that $|f(z)| > 0$. It is of course $> \log|f(z)|$ in case $f(z) = 0$. We are done.

Corollary. *If $f(t) \in H_1$ is not a.e. zero, $|f(t)|$ is necessarily > 0 a.e.*.

Proof. The theorem's boxed inequality makes $\log^-|f(t)| > -\infty$ a.e..

Definition. $H_\infty(\mathbb{R})$, or, as we frequently write, H_∞, is the collection of g in $L_\infty(\mathbb{R})$ satisfying

$$\int_{-\infty}^{\infty} g(t)f(t)\,\mathrm{d}t = 0$$

for all $f \in H_1$.

H_∞ is thus the subspace of L_∞, dual of L_1, consisting of functions *orthogonal* to the closed subspace H_1 of L_1. As such, it is closed, and even w^* closed, in L_∞.

By definition of H_1 we have the

Lemma. *Each of the functions $\mathrm{e}^{\mathrm{i}\lambda t}$, $\lambda \geqslant 0$, belongs to H_∞.*

Corollary. *A function $f \in L_1(\mathbb{R})$ belongs to H_1 iff*

$$\int_{-\infty}^{\infty} g(t)f(t)\,\mathrm{d}t = 0$$

for all $g \in H_\infty$.

Lemma. *If $f \in H_1$ and $g \in H_\infty$, $g(t)f(t) \in H_1$.*

Proof. First of all, $gf \in L_1$. Also, when $\lambda \geqslant 0$, $\mathrm{e}^{\mathrm{i}\lambda t}f(t) \in H_1$ by a previous lemma, so by definition of H_∞,

$$\int_{-\infty}^{\infty} g(t)\mathrm{e}^{\mathrm{i}\lambda t}f(t)\,\mathrm{d}t = 0,$$

i.e.,

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\lambda t}g(t)f(t)\,\mathrm{d}t = 0$$

for *each* $\lambda \geqslant 0$. Therefore $gf \in H_1$.

Lemma. *If g and h belong to H_∞, $g(t)h(t)$ does also.*

Proof. If f is any member of H_1, gf is also in H_1 by the previous lemma. Therefore

$$\int_{-\infty}^{\infty} h(t)\cdot g(t)f(t)\,\mathrm{d}t = 0.$$

This, holding for all $f \in H_1$, makes $hg \in H_\infty$ by definition.

Theorem. *Let $g \in H_\infty$. Then the function*

$$g(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\, g(t)\,\mathrm{d}t$$

is analytic for $\Im z > 0$.

Proof. Fix z, $\Im z > 0$, and, for the moment, a large $A > 0$. The function

$$f(t) = \frac{1}{t-\bar{z}}\,\frac{\mathrm{i}A}{t+\mathrm{i}A}$$

belongs to H_1. This is easily verified directly by showing that

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\lambda t} f(t)\,\mathrm{d}t = 0$$

for $\lambda \geqslant 0$ using contour integration. One takes large semi-circular contours in the upper half plane with base on the real axis; the details are left to the reader.

By definition of H_∞, we thus have

$$\int_{-\infty}^{\infty} g(t)\frac{1}{t-\bar{z}}\,\frac{\mathrm{i}A}{\mathrm{i}A+t}\,\mathrm{d}t = 0.$$

Subtracting the left side from

$$\int_{-\infty}^{\infty} g(t)\cdot\frac{1}{t-z}\,\frac{\mathrm{i}A}{\mathrm{i}A+t}\,\mathrm{d}t$$

and then dividing by 2i, we see that

$$\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\,\frac{\mathrm{i}A}{\mathrm{i}A+t}\,g(t)\,\mathrm{d}t = \frac{1}{2\mathrm{i}}\int_{-\infty}^{\infty} \frac{1}{t-z}\,\frac{\mathrm{i}A}{\mathrm{i}A+t}\,g(t)\,\mathrm{d}t.$$

For each $A > 0$, then,

$$g_A(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\,\frac{\mathrm{i}A}{\mathrm{i}A+t}\,g(t)\,\mathrm{d}t$$

is analytic for $\Im z > 0$ (by inspection).

As $A \longrightarrow \infty$, the functions $g_A(z)$ tend u.c.c. in $\{\Im z > 0\}$ to

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\, g(t)\,\mathrm{d}t \quad = \quad g(z).$$

The latter is therefore also analytic there.

Remark. For the function $g(z)$ figuring in the above theorem we have, for each z, $\Im z > 0$,

$$|g(z)| \leqslant \|g\|_\infty,$$

where the L_∞ norm on the right is taken for $g(t)$ on $\mathbb{R}$. This is evident by inspection. The same reasoning which shows that

$$f(t+\mathrm{i}y) \longrightarrow f(t) \quad \text{a.e. as } y \longrightarrow 0$$

for functions f in H_1 also applies here, yielding the result that

$$g(t+\mathrm{i}y) \longrightarrow g(t) \quad \text{a.e. as } y \longrightarrow 0$$

when $g \in H_\infty$. Unless $g(t)$ is uniformly continuous, however, we do *not* have

$$\|g(t+\mathrm{i}y) - g(t)\|_\infty \longrightarrow 0$$

for $y \longrightarrow 0$. Instead, we are only able to affirm that $g(t+\mathrm{i}y)$ tends w^* to $g(t)$ (in $L_\infty(\mathbb{R})$) as $y \longrightarrow 0$.

The theorem just proved has an important *converse*:

Theorem. *Let $G(z)$ be analytic and bounded for $\Im z > 0$. Then there is a $g \in H_\infty$ such that*

$$G(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\, g(t)\,\mathrm{d}t$$

for $\Im z > 0$, and

$$\|g\|_\infty \quad = \quad \sup_{\Im z > 0} |G(z)|.$$

Proof. It is claimed first of all that each of the functions $G(t+\mathrm{i}h)$, $h > 0$, belongs to H_∞ (as a function of t). Take any $f \in H_1$, and put

$$f(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\, f(t)\,\mathrm{d}t, \qquad \Im z > 0.$$

Our definition of H_∞ requires us to verify that

$$\int_{-\infty}^{\infty} G(t+\mathrm{i}h) f(t)\,\mathrm{d}t \quad = \quad 0.$$

Since

$$\|f(t+ib) - f(t)\|_1 \longrightarrow 0$$

as $b \longrightarrow 0$, it is enough to show that

$$\int_{-\infty}^{\infty} G(t+ih)f(t+ib)\,dt = 0$$

for each $b > 0$.

Fix any such b. According to a previous result, $f(z+ib)$ is then analytic and *bounded* for $\Im z > 0$, and continuous up to the real axis. The same is true for $G(z+ih)$. These properties make it easy for us to see by contour integration that

$$\int_{-\infty}^{\infty} \left(\frac{iA}{iA+t}\right)^2 G(t+ih)f(t+ib)\,dt = 0$$

for $A > 0$; one just integrates

$$\left(\frac{iA}{iA+z}\right)^2 G(z+ih)f(z+ib)$$

around large semi-circles in $\Im z \geqslant 0$ having their diameters on the real axis. Since $f(t+ib) \in L_1(\mathbb{R})$, we may now make $A \longrightarrow \infty$ in the relation just found to get

$$\int_{-\infty}^{\infty} G(t+ih)f(t+ib)\,dt = 0$$

and thus ensure that $G(t+ih) \in H_\infty(\mathbb{R})$.

For each $h > 0$ the first lemma of §H.1, Chapter III, makes

$$G(z+ih) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} G(t+ih)\,dt$$

when $\Im z > 0$. Here,

$$|G(t+ih)| \leqslant \sup_{\Im z > 0} |G(z)| < \infty.$$

Hence, since L_∞ is the *dual* of L_1, a procedure just like the one used in establishing the first theorem of §F.1, Chapter III, gives us a sequence of numbers $h_n > 0$ tending to zero and a g in L_∞ with

$$G(t+ih_n) \longrightarrow g(t) \quad w^*$$

as $n \longrightarrow \infty$. From this we see, referring to the preceding formula, that

$$G(z) = \lim_{n\to\infty} G(z+ih_n) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} g(t)\,dt$$

for $\Im z > 0$. By the w^* convergence we also have

$$\|g\|_\infty \leqslant \liminf_{n\to\infty} \|G(t+ih_n)\|_\infty \leqslant \sup_{\Im z>0} |G(z)|.$$

However, the representation just found for $G(z)$ implies the reverse inquality, so

$$\|g\|_\infty = \sup_{\Im z>0} |G(z)|.$$

As we have seen, each of the functions $G(t+ih_n)$ is in H_∞. Their w^* limit $g(t)$ must then also be in H_∞.

The theorem is proved.

Remark. An analogous theorem is true about H_1. Namely, if $F(z)$, analytic for $\Im z > 0$, is such that the integrals

$$\int_{-\infty}^{\infty} |F(x+iy)|\,dx$$

are *bounded* for $y > 0$, there *is* an $f \in H_1$ for which

$$F(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} f(t)\,dt, \qquad \Im z > 0.$$

This result will not be needed in the present §; it is deeper than the one just found because $L_1(\mathbb{R})$ is *not the dual* of any Banach space. The F. and M. Riesz theorem is required for its proof; see §B.4 of Chapter VII.

Problem 41

Let $g \in H_\infty$, and write

$$g(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} g(t)\,dt$$

for $\Im z > 0$.

(a) If $\Im c > 0$, both functions

$$\frac{g(t)-g(c)}{t-\bar{c}} \quad\text{and}\quad \frac{g(t)-g(c)}{t-c}$$

belong to H_∞. (Hint: In considering the first function, begin by noting that $1/(t-\bar{c}) \in H_\infty$ according to the second lemma about H_1. To investigate the second function, look at $(g(z)-g(c))/(z-c)$ in the upper half plane.)

(b) Hence show that if $f \in H_1$ and

$$f(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} f(t)\,dt$$

for $\Im z > 0$, one has

$$f(c)g(c) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im c}{|c-t|^2} f(t)g(t)\,\mathrm{d}t$$

for each c with $\Im c > 0$.

(c) If, for the $f(z)$ of part (b) one has $f(c) = 0$ for some c, $\Im c > 0$, show that $f(t)/(t-c)$ belongs to H_1. (Hint: Follow the argument of (b) using the function $g(t) = \mathrm{e}^{i\lambda t}$, where $\lambda \geqslant 0$ is arbitrary.)

Theorem. *If $g(t) \in H_\infty$ is* not *a.e. zero on $\mathbb{R}$, we have*

$$\boxed{\int_{-\infty}^{\infty} \frac{\log^-|g(t)|}{1+t^2}\,\mathrm{d}t \quad < \quad \infty,}$$

and, for $\Im z > 0$,

$$\log|g(z)| \quad \leqslant \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} \log|g(t)|\,\mathrm{d}t,$$

the integral on the right being absolutely convergent. Here, $g(z)$ has its usual meaning:

$$g(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} g(t)\,\mathrm{d}t.$$

Proof. By the first of the preceding two theorems, $g(z)$ is analytic (and of course bounded) for $\Im z > 0$. Therefore, by the results of §G.2 in Chapter III, for each $h > 0$,

$$\log|g(z+ih)| \quad \leqslant \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} \log|g(t+ih)|\,\mathrm{d}t$$

when $\Im z > 0$.

We may, wlog, take $\|g\|_\infty$ to be $\leqslant 1$, so that $|g(z)| \leqslant 1$ and $\log|g(z)| \leqslant 0$ for $\Im z > 0$. As $h \longrightarrow 0$, $g(t+ih) \longrightarrow g(t)$ a.e. according to a previous remark, so, by Fatou's lemma,

$$\limsup_{h\to 0} \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} \log|g(t+ih)|\,\mathrm{d}t \quad \leqslant \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} \log|g(t)|\,\mathrm{d}t.$$

The right-hand quantity must thus be $\geqslant \log|g(z)|$ by the previous relation, proving the second inequality of our theorem.

In case $g(t)$ is not a.e. zero, there must be some z, $\Im z > 0$, with $g(z) \neq 0$, again because $g(t+ih) \longrightarrow g(t)$ a.e. for $h \longrightarrow 0$. Using this z in the inequality just proved, we see that

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log|g(t)|\,dt \quad > \quad -\infty,$$

whence

$$\int_{-\infty}^{\infty} \frac{\log^-|g(t)|}{1+t^2}\,dt \quad < \quad \infty,$$

and the former integral is actually absolutely convergent for all z with $\Im z > 0$, whether $g(z) \neq 0$ or not.

We are done.

Come we now to the space H_2.

Definition. A function $f \in L_2(\mathbb{R})$ belongs to $H_2(\mathbb{R})$, usually designated as H_2, iff

$$\int_{-\infty}^{\infty} \frac{f(t)}{t-\bar{z}}\,dt \quad = \quad 0$$

for all z with $\Im z > 0$.

H_2 is clearly a closed subspace of $L_2(\mathbb{R})$.

Theorem. *If $f \in H_2$, the function*

$$f(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} f(t)\,dt$$

is analytic for $\Im z > 0$.

Proof. Is like that of the corresponding result for H_1.

Theorem. *If $f \in H_2$, the function $f(z)$ in the preceding theorem has the following properties:*

(i) $|f(z)| \quad \leqslant \quad \|f\|_2/\sqrt{(\pi\Im z)}, \qquad \Im z > 0;$

(ii) $\displaystyle\int_{-\infty}^{\infty} |f(x+iy)|^2 dx \quad \leqslant \quad \|f\|_2^2 \qquad$ for $y > 0$;

(iii) $\displaystyle\int_{-\infty}^{\infty} |f(t+iy)-f(t)|^2\,dt \longrightarrow 0$ as $y \longrightarrow 0$;

(iv) $f(t+iy) \longrightarrow f(t)$ a.e. as $y \longrightarrow 0$.

Proof. Property (i) follows by applying Schwarz' inequality to the formula for $f(z)$. The remaining properties are verified by arguments like those used in proving the corresponding theorem about H_1, given above.

As is the case for H_∞ (and for H_1), these results have a converse:

Theorem. *Let $F(z)$ be analytic for $\Im z > 0$, and suppose that*

$$\int_{-\infty}^{\infty} |F(x+iy)|^2\,dx$$

is bounded for $y > 0$. Then there is an $f \in H_2$ with

$$F(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} f(t)\,dt, \qquad \Im z > 0,$$

and

$$\|f\|_2^2 = \sup_{y>0} \int_{-\infty}^{\infty} |F(x+iy)|^2\,dx.$$

Proof. For each $h > 0$, put

$$F_h(z) = \frac{1}{2h}\int_{-h}^{h} F(z+s)\,ds, \qquad \Im z > 0.$$

By Schwarz' inequality,

$$|F_h(z)| \leqslant (2h)^{-1/2}\sqrt{\left(\int_{-\infty}^{\infty} |F(z+s)|^2\,ds\right)} \leqslant \frac{C}{\sqrt{(2h)}}$$

where C is independent of z or h; each function $F_h(z)$ is therefore *bounded* in $\Im z > 0$, besides being analytic there.

A previous theorem therefore gives us functions $f_h \in H_\infty$ such that

$$F_h(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} f_h(t)\,dt, \qquad \Im z > 0,$$

and, as already remarked,

$$F_h(t+iy) \longrightarrow f_h(t) \quad \text{a.e. for } y \longrightarrow 0.$$

We have, for each h and $y > 0$,

$$\int_{-\infty}^{\infty} |F_h(x+\mathrm{i}y)|^2\,\mathrm{d}x \;\leqslant\; \frac{1}{2h}\int_{-\infty}^{\infty}\int_{-h}^{h} |F(x+s+\mathrm{i}y)|^2\,\mathrm{d}s\,\mathrm{d}x$$
$$= \int_{-\infty}^{\infty} |F(x+\mathrm{i}y)|^2\,\mathrm{d}x$$

by Schwarz' inequality and Fubini. Since the right side is bounded by a quantity $M < \infty$ independent of y (and h), the limit relation just written guarantees that

$$\|f_h\|_2^2 \;\leqslant\; M$$

for $h > 0$, according to Fatou's lemma.

Once it is known that the norms $\|f_h\|_2$ are bounded we can, as in the proof of the corresponding theorem about H_∞, get a sequence of $h_n > 0$ tending to zero for which the f_{h_n} converge *weakly*, *this time* in L_2, to some $f \in L_2(\mathbb{R})$. Then, for each z, $\Im z > 0$,

$$F_{h_n}(z) \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} f_{h_n}(t)\,\mathrm{d}t \;\longrightarrow\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} f(t)\,\mathrm{d}t$$

as $n \longrightarrow \infty$. At the same time,

$$F_{h_n}(z) \underset{n}{\longrightarrow} F(z),$$

so we *have* our desired representation of $F(z)$ if we can show that $f \in H_2$.

For this purpose, it is enough to verify that when $\Im z > 0$,

$$\int_{-\infty}^{\infty} \frac{f_h(t)}{t-\bar{z}}\,\mathrm{d}t \;=\; 0,$$

since the f_{h_n} tend to f weakly in L_2. However, the f_h belong to H_∞, and, when $\Im z > 0$ and $A > 0$, the function

$$\frac{1}{t-\bar{z}}\,\frac{\mathrm{i}A}{\mathrm{i}A+t}$$

belongs to H_1, as we have noted during the proof of a previous result. Hence

$$\int_{-\infty}^{\infty} \frac{\mathrm{i}A}{\mathrm{i}A+t}\,\frac{f_h(t)}{t-\bar{z}}\,\mathrm{d}t \;=\; 0.$$

Here, $f_h(t)/(t-\bar{z})$ belongs to L_1, so we may make $A \longrightarrow \infty$ in this relation, which yields the desired one.

We still need to show that $\|f\|_2^2 = \sup_{y>0}\int_{-\infty}^{\infty}|F(x+iy)|^2\,dx$. Here, we now know that the function $F(z)$ is nothing but the $f(z)$ figuring in the preceding theorem. The statement in question thus follows from properties (ii) and (iii) of that result.

We are done.

Remark. Using the theorems just proved, one readily verifies that H_2 consists precisely of the functions $u(t) + i\tilde{u}(t)$, with u an arbitrary real-valued member of $L_2(\mathbb{R})$ and $\tilde{u}$ its L_2 Hilbert transform – the one studied in the scholium to §C.1 of Chapter VIII. The reader should carry out this verification.

Our use of the space H_2 in the following articles of this § is based on a relation between H_2 and H_1, established by the following two results.

Theorem. *If f and g belong to H_2, $f\cdot g$ is in H_1.*

Proof. Certainly $fg \in L_1$, so the quantity

$$\int_{-\infty}^{\infty} e^{i\lambda t} f(t)\,g(t)\,dt$$

varies continuously with λ. It is therefore enough to show that it vanishes for $\lambda > 0$ (*sic*) in order to prove that $fg \in H_1$.

Let, as usual,

$$f(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} f(t)\,dt$$

for $\Im z > 0$, and

$$g(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} g(t)\,dt$$

there.

Using the facts that $\|f(t+ih) - f(t)\|_2 \longrightarrow 0$ and $\|g(t+ih) - g(t)\|_2 \longrightarrow 0$ for $h \longrightarrow 0$ (property (iii) in the first of the preceding two theorems) and applying Schwarz' inequality to the identity

$$\begin{aligned} &f(t+ih)g(t+ih) \;-\; f(t)g(t) \\ &\quad = \; [f(t+ih) - f(t)]\,g(t) \;+\; f(t+ih)\,[g(t+ih) - g(t)], \end{aligned}$$

one readily sees that

$$\|f(t+ih)g(t+ih) \;-\; f(t)g(t)\|_1 \;\longrightarrow\; 0$$

as $h \longrightarrow 0$. It is therefore sufficient to check that

$$\int_{-\infty}^{\infty} e^{i\lambda t} f(t+ih)\,g(t+ih)\,dt \quad = \quad 0$$

for each $h > 0$ when $\lambda > 0$.

Fix any such h. By property (i) from the result just referred to,

$$|f(z+ih)| \quad \leqslant \quad \frac{\text{const.}}{\sqrt{h}} \qquad \text{for } \Im z \ \geqslant \ 0.$$

Also, since $f(t) \in L_2(\mathbb{R})$, the function

$$f(z+ih) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z + h}{|z+ih-t|^2} f(t)\,dt$$

tends uniformly to zero for z tending to ∞ in any *fixed strip* $0 \ \leqslant \ \Im z \ \leqslant \ L$. The function $g(z+ih)$ has the same behaviour.

These properties make it possible for us to now virtually *copy* the contour integral argument made in proving the Paley–Wiener theorem, Chapter III, §D, replacing the function $f_h(z)$ figuring there* by $f(z+ih)g(z+ih)$. In that way we find that

$$\int_{-\infty}^{\infty} e^{i\lambda t} f(t+ih)\,g(t+ih)\,dt \quad = \quad 0$$

for $\lambda > 0$, the relation we needed.

The theorem is proved.

The last result has an important converse:

Theorem. *Given $\varphi \in H_1$, there are functions f and g in H_2 with $\varphi = fg$ and* $\|f\|_2 \ = \ \|g\|_2 \ = \ \sqrt{(\|\varphi\|_1)}$.

Proof. There is no loss of generality in assuming that $\varphi(t)$ is not a.e. zero on $\mathbb{R}$, for otherwise our theorem is trivial. Putting, then,

$$\varphi(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\varphi(t)\,dt$$

for $\Im z \ > \ 0$, we know by previous results that $\varphi(z)$ is analytic in the upper

* Here, the condition $\lambda \ > \ 0$ plays the rôle that the relation $\lambda \ > \ A$ did in the discussion referred to.

half plane and that

$$\log|\varphi(z)| \quad \leqslant \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log|\varphi(t)|\,\mathrm{d}t$$

there, the integral on the right being absolutely convergent.

Thanks to the absolute convergence, we can define a function $F(z)$ analytic for $\Im z > 0$ by writing

$$F(z) \quad = \quad \exp\left\{\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\left(\frac{1}{t-z} - \frac{t}{t^2+1}\right)\log|\varphi(t)|\,\mathrm{d}t\right\};$$

the idea here is that $F(z) \neq 0$ for $\Im z > 0$, with

$$\log|F(z)| \quad = \quad \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2}\log|\varphi(t)|\,\mathrm{d}t,$$

one half the right side of the preceding inequality. The ratio

$$G(z) \quad = \quad \frac{\varphi(z)}{F(z)}$$

is then *analytic* for $\Im z > 0$, and we have

$$\log|G(z)| \quad \leqslant \quad \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2}\log|\varphi(t)|\,\mathrm{d}t \quad = \quad \log|F(z)|,$$

i.e.,

$$|G(z)| \leqslant |F(z)|, \qquad \Im z > 0.$$

By the inequality between arithmetic and geometric means,

$$|F(z)|^2 \quad \leqslant \quad \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2}|\varphi(t)|\,\mathrm{d}t,$$

so, for each $y > 0$,

$$\int_{-\infty}^{\infty}|G(x+\mathrm{i}y)|^2\,\mathrm{d}x \quad \leqslant \quad \int_{-\infty}^{\infty}|F(x+\mathrm{i}y)|^2\,\mathrm{d}x$$

$$\leqslant \quad \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{y|\varphi(t)|}{(x-t)^2+y^2}\,\mathrm{d}t\,\mathrm{d}x \quad = \quad \|\varphi\|_1.$$

According to a previous theorem, there are thus functions f and g in H_2 with

$$F(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2}f(t)\,\mathrm{d}t, \qquad \Im z > 0,$$

$$G(z) \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2}g(t)\,dt, \qquad \Im z \;>\; 0,$$

and

$$\|g\|_2^2 \;\leqslant\; \|f\|_2^2 \;\leqslant\; \|\varphi\|_1.$$

For $\Im z > 0$, we have

$$\varphi(z) \;=\; F(z)G(z),$$

However, when $y \longrightarrow 0$,

$$\varphi(t+iy) \;\longrightarrow\; \varphi(t) \quad \text{a.e.}$$

while at the same time

$$F(t+iy) \;\longrightarrow\; f(t) \quad \text{a.e.}$$

and

$$G(t+iy) \;\longrightarrow\; g(t) \quad \text{a.e..}$$

Therefore,

$$\varphi(t) \;=\; f(t)g(t) \qquad \text{a.e., } t \in \mathbb{R},$$

our desired factorization.

Schwarz' inequality now yields

$$\|\varphi\|_1 \;\leqslant\; \|f\|_2\|g\|_2.$$

We already know, however, that

$$\|g\|_2 \;\leqslant\; \|f\|_2 \;\leqslant\; \sqrt{(\|\varphi\|_1)}.$$

Hence $\|g\|_2 \;=\; \|f\|_2 \;=\; \sqrt{(\|\varphi\|_1)}$.

We are done.

Remark. For the function $F(z)$ used in the above proof, we have

$$\log|F(z)| \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2}\log|\varphi(t)|\,dt,$$

so

$$\log|F(t+iy)| \;\longrightarrow \left|\tfrac{1}{2}\log|\varphi(t)|\right. \quad \text{a.e.}$$

as $y \longrightarrow 0$ by the property of the Poisson kernel already used frequently in this article. This means, however, that

$$|F(t+iy)| \;\longrightarrow\; \sqrt{(|\varphi(t)|)} \quad \text{a.e.}$$

for $y \longrightarrow 0$. At the same time,

$$F(t+\mathrm{i}y) \longrightarrow f(t) \quad \text{a.e.},$$

so we have

$$|f(t)| = \sqrt{(|\varphi(t)|)} \qquad \text{a.e., } t \in \mathbb{R},$$

for the H_2 function f furnished by the last theorem.

Since $\varphi \in H_1$, we must have $|\varphi(t)| > 0$ a.e. by a previous corollary (unless $\varphi(t) \equiv 0$ a.e., a trivial special case which we are excluding). The H_2 function g with $fg = \varphi$ must then also satisfy

$$|g(t)| = \sqrt{(|\varphi(t)|)} \qquad \text{a.e., } t \in \mathbb{R}.$$

In spite of the fact that the H_2 functions f and g involved here have a.e. the same moduli on $\mathbb{R}$, they are in general essentially different. It is usually true that their extensions F and G to the upper half plane satisfy

$$|G(z)| < |F(z)|$$

there.

Later on in this §, our work will involve the products

$$\mathrm{e}^{\mathrm{i}\lambda t} f(t)$$

with $\lambda \geqslant 0$, where f is a given function in H_2. Our first observation about these is the

Lemma. *If $f \in H_2$ and $\lambda \geqslant 0$, $\mathrm{e}^{\mathrm{i}\lambda t} f(t) \in H_2$.*

Proof. If $\Im z > 0$, the function $1/(t-\bar{z})$ belongs to H_2. This is most easily checked by referring to the definition of H_2 and doing a contour integral; such verification is left to the reader. According to a previous theorem, then, $f(t)/(t-\bar{z})$ belongs to H_1. Hence

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\lambda t} f(t)}{t-\bar{z}}\,\mathrm{d}t = 0$$

for each $\lambda \geqslant 0$. Here, z with $\Im z > 0$ is arbitrary, so the functions $\mathrm{e}^{\mathrm{i}\lambda t} f(t)$ with $\lambda \geqslant 0$ belong to H_2 by definition. Done.

When $f \in H_2$, finite linear combinations of the products $\mathrm{e}^{\mathrm{i}\lambda t} f(t)$ with $\lambda \geqslant 0$ form, by the lemma just proved, a certain vector subspace of H_2. We want to know when the L_2 closure of that subspace is *all* of H_2. This question was answered by Beurling. His argument uses material from the