and equal to f in Ω_1. Letting $\varepsilon \to 0+$ yields, by Theorem 5.4, $\sigma_u = \sigma_b \leq \frac{1}{2}$. Thus, by Theorem 5.24 and Claim 1, $\sigma_u = \frac{1}{2}$. □

LEMMA 5.29. *The number of monomials of degree m in n variables is* $\binom{n+m-1}{m}$.

PROOF: From a linear array of $n+m-1$ objects, choose $n-1$ and color them black. Let the power of z_i be the number of non-colored objects between the $(i-1)^{\text{st}}$ black one and the i^{th} one. □

EXERCISE 5.30. Fill in the details that the series in (5.28) converges.

EXERCISE 5.31. Show that for all $x \in [0, \frac{1}{2}]$, there is a Dirichlet series such that $\sigma_a - \sigma_u$ is exactly x.

(Hint: Although Bohnenblust and Hille did not spot it, this result is a one-line consequence of Theorem 5.26. If you find the right line!)

5.4. Notes

The proofs of the Bohnenblust-Hille theorem in Section 5.3 and Bohr's Theorem 5.4 are based on H. Boas's article [**Boa97**]. The original proofs are in [**BH31**] and [**Boh13b**], respectively. Theorem 5.24 was proved in [**Boh13a**].

Talk about recent advances, in particular [**DFOC$^+$11**].

CHAPTER 6

Hilbert Spaces of Dirichlet Series

6.1. Beurling's problem: The statement

We will motivate our discussion by considering a problem posed by A. Beurling in 1945. If we set $\beta(x) = \sqrt{2}\sin(\pi x)$, the set

$$\{\beta(nx) : \ n \in \mathbb{N}^+\}$$

forms an orthonormal basis of $L^2([0;1])$.

PROPOSITION 6.1. *If $\psi : \mathbb{R}^+ \to \mathbb{C}$ is 2-periodic, and $\{\psi(nx)\}_{n\in\mathbb{N}^+}$ is an orthonormal basis for $L^2([0;1])$, then $\psi = e^{i\theta}\beta$, for some $\theta \in \mathbb{R}$.*

Proof: Extend ψ to an odd function on $\mathbb{R}$. Then ψ is odd and 2-periodic, so we can expand it into a sine series $\psi(x) = \sum_{k=1}^{\infty} c_k\beta(kx)$. Since $\{\psi(nx)\}_{n\in\mathbb{N}^+}$ is an orthonormal basis, we have

$$\begin{aligned}
1 \ = \ \|\beta(mx)\|^2 \quad &= \quad \sum_{n=1}^{\infty} |\langle \beta(mx), \psi(nx)\rangle|^2 \\
&= \quad \sum_{n=1}^{\infty} |\langle \beta(mx), \sum_{k=1}^{\infty} c_k\beta(nkx)\rangle|^2 \\
&= \quad \sum_{n=1}^{\infty} \left| \sum_{k;\ kn=m} c_k \right|^2 \\
&= \quad \sum_{k|m} |c_k|^2.
\end{aligned}$$

Letting $m = 1$, we obtain $|c_1|^2 = 1$. Thus, for $m \geq 2$, we have $1 + \cdots + |c_m|^2 = 1$ (where the middle terms are non-negative) and so $|c_m| = 0$. $\square$

DEFINITION 6.2. Let $\{v_n\}$ be a set of vectors in a Hilbert space $\mathcal{H}$. We say that $\{v_n\}$ is a *Riesz basis*, if $\overline{\text{span}}\ \{v_n\} = \mathcal{H}$ and the *Gram matrix* G given by

$$G_{ij} := \langle v_j, v_i\rangle$$

is bounded and bounded below, that is, for all $\{a_n\}_{n=1}^{\infty} \in \ell^2$:

$$c_1 \sum_{j=1}^{\infty} |a_j|^2 \ \le \ \sum_{i,j=1}^{\infty} a_i \overline{a}_j G_{ij} \ \le \ c_2 \sum_{j=1}^{\infty} |a_j|^2. \tag{6.3}$$

PROPOSITION 6.4. *The set* $\{v_n\}_{n=1}^{\infty}$ *is a Riesz basis if and only if the map*

$$T : \ \sum_{n=1}^{\infty} a_n e_n \ \mapsto \ \sum_{n=1}^{\infty} a_n v_n$$

is bounded and invertible, where $\{e_n\}$ *is an orthonormal basis for* $\mathcal{H}$*.*

Proof: We have

$$\left\| T \sum_n a_n e_n \right\|^2 \ = \ \left\| \sum_n a_n v_n \right\|^2 \ = \ \sum_{m,n} a_n \overline{a}_m G_{mn}$$

and

$$\left\| \sum_n a_n e_n \right\|^2 = \sum_n |a_n|^2.$$

Thus condition (6.3) is equivalent to boundedness of T from below and above. Moreover, T is onto if and only if the span of $\{v_n\}$ is dense in $\mathcal{H}$. The claim follows by recalling that a map is invertible if and only if it is bounded, bounded from below, and onto. □

Here is Beurling's question.

QUESTION 6.5. **(Beurling)** For which odd 2-periodic functions $\psi : \mathbb{R} \to \mathbb{C}$ does the sequence $\{\psi(nx)\}_{n=1}^{\infty}$ form a Riesz basis for $L^2([0;1])$?

REMARK 6.6. A frame is a set of vectors $\{v_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ such that for some $c_1, c_2 > 0$

$$c_1 \|v\|^2 \ \le \ \sum_{n=1}^{\infty} |\langle v, v_n \rangle|^2 \ \le \ c_2 \|v\|^2$$

holds for every $v \in \mathcal{H}$. (Unlike a Riesz basis, they do not need to be linearly independent).

The following problem attracted a lot of attention; it has many equivalent reformulations.

CONJECTURE 6.7. **(Feichtinger)** Suppose that $\{v_n\}_{n=1}^{\infty}$ is a set of unit vectors in $\mathcal{H}$ that form a frame. Does it follow that $\{v_n\}_{n=1}^{\infty}$ is a finite union of Riesz bases?

The conjecture was proved, in the affirmative, by A. Marcus, D. Spielman and N. Srivastava [**MSS15**].

Beurling's idea was to consider the Hilbert space of Dirichlet series

$$\mathcal{H}^2 := \left\{ \sum_{n=1}^{\infty} a_n n^{-s} : \sum_n |a_n|^2 < \infty \right\}. \tag{6.8}$$

Let us first observe that for any $f \in \mathcal{H}^2$ we have $\sigma_a \le \frac{1}{2}$. Indeed, by the Cauchy-Schwarz inequality,

$$\left| \sum_n a_n n^{-s} \right| \le \left(\sum_n |a_n|^2 \right)^{\frac{1}{2}} \left(\sum_n n^{-2\sigma} \right)^{\frac{1}{2}} < \infty,$$

whenever $2\sigma > 1$. In fact, the above estimate shows that for any $s_0 \in \Omega_{1/2}$, the map $\mathcal{H}^2 \ni f \mapsto f(s_0)$ is a bounded linear functional. Therefore it is given by the inner product with a function $k_{s_0} \in \mathcal{H}^2$, the so-called *reproducing kernel* at s_0, i.e.,

$$f(s_0) = \langle f, k_{s_0} \rangle \qquad \text{for all } f \in \mathcal{H}^2.$$

For any Hilbert (or Banach) space of analytic functions $\mathcal{X}$, we define its *multiplier algebra* by

$$\mathrm{Mult}\,(\mathcal{X}) = \{\varphi;\ \varphi f \in \mathcal{X},\ \forall f \in \mathcal{X}\}.$$

It is easy to check that the following hold

- $1 \in \mathcal{X} \implies \mathrm{Mult}\,(\mathcal{X}) \subset \mathcal{X}$,
- $\mathrm{Mult}\,(\mathcal{X})$ is an algebra.

Clearly, multiplication by k^{-s} is isometric on $\mathcal{H}^2$, for all $k \in \mathbb{N}$. Consequently, every finite Dirichlet series lies in $\mathrm{Mult}\,(\mathcal{H}^2)$.

Also, note that $\sup_{s \in \Omega_{1/2}} |k^{-s}| = k^{-\frac{1}{2}} \to 0$ as k tends to ∞. Thus, $\|f\|_{\mathrm{Mult}\,(\mathcal{H}^2)} \not\lesssim \|f\|_{H^\infty(\Omega_{1/2})}$.

The following result — multiplication operators are bounded if they are everywhere defined — is true in great generality (see Section 11.4).

Proposition 6.9. *The multiplication operator M_φ is bounded on $\mathcal{H}^2$ for every $\varphi \in \mathrm{Mult}\,(\mathcal{H}^2)$.*

Proof: Multiplication operators on a Banach space of functions in which norm convergence implies pointwise convergence (or at least a.e. convergence) are easily seen to be closed. Indeed, suppose that $f_n \to f$ and $M_\varphi f_n \to g$. Then, for every $s \in \Omega_{1/2}$, $f_n(s) \to f(s)$ and so $(M_\varphi f_n)(s) = \varphi(s) f_n(s) \to \varphi(s) f(s) = (M_\varphi f)(s)$. On the other hand, $M_\varphi f_n(s) \to g(s)$, for all $s \in \Omega_{1/2}$. We conclude that $(M_\varphi f)(s) = g(s)$ for all $s \in \Omega_{1/2}$ and hence $M_\varphi f = g$. Thus, M_φ is closed. Hence, M_φ is an everywhere defined closed linear operator on a Banach space, and the closed graph theorem states that such operators are necessarily bounded. $\square$

Now let ψ be an odd 2-periodic function on $\mathbb{R}$. We can expand it into a Fourier series $\psi(x) = \sum_{n=1}^{\infty} c_n \beta(nx)$. The sequence $\{\psi(kx)\}_{k\in\mathbb{N}^+}$ is a Riesz basis, if and only if it spans L^2 and the operator $T : \sum_k a_k \beta(kx) \mapsto \sum_k a_k \psi(kx)$ is bounded and bounded below. Denote $\psi_k(x) := \psi(kx)$ and analyze the condition on T:

$$\begin{aligned}
\Big\| \sum_k a_k \psi_k \Big\|^2 &= \Big\| \sum_k a_k \sum_n c_n \beta(nkx) \Big\|^2 \\
&= \Big\langle \sum_{k,n} a_k c_n \beta(nkx), \sum_{j,m} a_j c_m \beta(mjx) \Big\rangle \\
&= \sum_{k,n,j,m;\ kn=jm} a_k c_n \overline{a}_j \overline{c}_m,
\end{aligned}$$

and thus we want

$$\sum_{k,n,j,m;\ kn=jm} a_k c_n \overline{a}_j \overline{c}_m \approx \sum_k |a_k|^2. \tag{6.9}$$

Let us define auxilliary functions in $\mathcal{H}^2$:

$$g(s) := \sum_n c_n n^{-s}, \qquad f(s) := \sum_k a_k k^{-s}.$$

We have

$$\|gf\|^2_{\mathcal{H}^2} = \Big\langle \sum_{n,k} c_n a_k (nk)^{-s}, \sum_{m,j} c_m a_j (mj)^{-s} \Big\rangle = \sum_{k,n,j,m;\ kn=jm} a_k c_n \overline{a}_j \overline{c}_m,$$

and so the condition (6.9) holds, if and only if $\|gf\|_{\mathcal{H}^2} \approx \|f\|_{\mathcal{H}^2}$, i.e., when M_g is bounded and bounded below.

Let us also look the density of the span of $\{\psi_n\}_n$. It is equivalent to

$$\begin{aligned}
\overline{\operatorname{span}} \Big\{ \sum_n c_n \beta(nkx) \Big\}_{k\in\mathbb{N}^+} = L^2([0;1]) \quad &\iff \quad \overline{\operatorname{span}} \Big\{ \sum_n c_n e_{nk} \Big\}_{k\in\mathbb{N}^+} = \ell^2(\mathbb{N}) \\
&\iff \quad \overline{\operatorname{span}} \Big\{ \sum_n c_n (nk)^{-s} \Big\}_{k\in\mathbb{N}^+} = \mathcal{H}^2 \\
&\iff \quad \overline{\operatorname{span}} \Big\{ k^{-s} g(s) \Big\}_{k\in\mathbb{N}^+} = \mathcal{H}^2.
\end{aligned}$$

The last condition implies that range of M_g is dense. But since M_g is bounded below, it has a closed range and thus is onto. Therefore M_g is invertible, or $M_{1/g}$ is bounded. Conversely, if M_g is invertible, the image of the dense set span $\{k^{-s}\}_{k\in\mathbb{N}^+}$ is dense, and so the density of span $\{\psi_k\}_k$ follows by the above equivalences. We have proved:

Proposition 6.10. *Let* $\psi(x) = \sum_{n=1}^{\infty} c_n \beta(nx)$ *be a odd 2-periodic function on* $\mathbb{R}$*. Then* $\{\psi(kx)\}_{k\in\mathbb{N}^+}$ *is a Riesz basis, if and only if both* g *and* $1/g$ *are multipliers of* $\mathcal{H}^2$*, where* $g(s) = \sum_{n=1}^{\infty} c_n n^{-s}$*.*

In view of Proposition 6.10, Beurling's question 6.5 would be answered if we could answer the following question:

Question 6.11. What are the multipliers of $\mathcal{H}^2$?

6.2. Reciprocals of Dirichlet Series

Proposition 6.12. *If* $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ *is a Dirichlet series that converges somewhere and satisfies* $a_1 \neq 0$*, then* $g(s) = \frac{1}{f(s)}$ *is also given by the sum of a somewhere-convergent Dirichlet series. Moreover,* $\sigma_b(g) = \inf\{\rho : \inf |f|\big|_{\Omega_\rho} > 0\}$*.*

Proof: By rescaling, we may assume that $a_1 = 1$, and by shifting the series so that $\sigma_a < 0$, we have $\sup |a_n| \leq M$. We will construct the coefficients b_k of g inductively. Clearly, $b_1 = 1$. For $n \geq 2$, we have

$$0 \;=\; \widehat{fg}(n) \;=\; \sum_{k|n} a_{n/k} b_k. \tag{6.13}$$

Equations (6.13) can be solved for b_k, first when k is a prime, then a power of a prime, then when k has two distinct prime factors, and so on.
Claim: If $n = p_1^{i_1} \dots p_r^{i_r}$, then $|b_n| \leq n^2 M^{|i|}$.

Proof: For $n = 1$, $b_n = 1$ and so the claim holds. Assume inductively that the claim holds for all $m < n$. By (6.13), we have

$$\begin{aligned}
|b_n| &\leq \sum_{k|n,\ k\geq 2} |a_k b_{n/k}| \\
&\leq M \sum_{k\geq 2} b_{n/k} \\
&\leq M \sum_{k\geq 2} \left(\frac{n}{k}\right)^2 M^{|i|-1} \\
&\leq M^{|i|} n^2 \sum_{k\geq 2} \frac{1}{k^2} \\
&= M^{|i|} n^2 \left(\frac{\pi^2}{6} - 1\right),
\end{aligned}$$

and the claim follows, since $\frac{\pi^2}{6} < 2$.

Since $|i| \le \log_2 n$, we obtain

$$\begin{aligned} |b_n| &\le M^{|i|} n^2 \\ &\le M^{\log_2 n} n^2 \\ &= n^{\log_2 M} n^2 \\ &= n^{2+\log_2 M}. \end{aligned}$$

Hence, for Re $s > 3 + \log_2 M$, the Dirichlet series $\sum_n b_n n^{-s}$ converges absolutely.

Now, g is bounded in Ω_ρ, if and only if $\inf |f|\big|_{\Omega_\rho} > 0$. As g is given by a convergent Dirichlet series in $\Omega_{3+\log_2 M}$, by Theorem 5.4,

$$\sigma_b(g) \le \inf\{\rho : \ \inf |f|\big|_{\Omega_\rho} > 0\}.$$

The reverse inequality is obvious. $\square$

Note that the condition $a_1 \neq 0$ is necessary, since $a_1 = \lim_{\sigma \to \infty} f(\sigma)$.

6.3. Kronecker's Theorem

THEOREM 6.14. **(Kronecker)**

(1) *Let $\theta_1, \dots, \theta_k \in \mathbb{R}$ be linearly independent over $\mathbb{Q}$, and let $\alpha_1, \dots, \alpha_k \in \mathbb{R}$, $T, \varepsilon > 0$ be given. Then there exist $t > T$ and $q_1, \dots, q_k \in \mathbb{Z}$ such that*

$$|t\theta_j - \alpha_j - q_j| < \varepsilon, \ 1 \le j \le k.$$

(2) *Let $1, \theta_1, \dots, \theta_k \in \mathbb{R}$ be linearly independent over $\mathbb{Q}$, and let $\alpha_1, \dots, \alpha_k \in \mathbb{R}$, $T, \varepsilon > 0$ be given. Then there exist $\mathbb{N} \ni n > T$ and $q_1, \dots, q_k \in \mathbb{Z}$ such that*

$$|n\theta_j - \alpha_j - q_j| < \varepsilon, \ 1 \le j \le k.$$

Proof: (1) $\implies$ (2): Assume that all θ_j's lie in $(-M, M)$. Fix $0 < \varepsilon < 1$, and apply (1) to the $(k+1)$-tuples $\theta_1, \dots, \theta_k, 1$ and $\alpha_1, \dots, \alpha_k, 0$, $T = N + 1$ and $\varepsilon/(M+1)$. Let $n = q_{k+1}$, then $|t - n| < \varepsilon/(M+1)$. Thus, for $1 \le j \le k$, we have

$$\begin{aligned} |n\theta_j - \alpha_j - q_j| &\le |n - t|\theta_j + |t\theta_j - \alpha_j - q_j| \\ &< \frac{M\varepsilon}{M+1} + \frac{\varepsilon}{M+1}. \end{aligned}$$

To prove (1), define $F(t) := 1 + \sum_{j=1}^k e^{2\pi i[\theta_j t - \alpha_j]}$. We need to show that $\limsup_{t\to\infty} |F(t)| = k + 1$. Fix $m \in \mathbb{N}$, and define $\alpha = (0, \alpha_1, \dots, \alpha_k)$, $\theta = (0, \theta_1, \dots, \theta_k)$ and $j = (j_0, \dots, j_k)$. Then

$$[F(t)]^m = \sum_{|j| = j_0 + \cdots + j_k = m} a_j e^{2\pi i t \gamma_j},$$

where $a_j = \frac{m!}{j!} e^{-2\pi i j\cdot\alpha}$ and $\gamma_j = j \cdot \theta$. Indeed, there are $\frac{m!}{j!}$ ways to get $\prod_l e^{2\pi i t j_l \theta_l}$ in the product, and, by independence of θ_j's over $\mathbb{Q}$, distinct j's yield distinct γ_j's. Also, $\sum_{|j|=m} |a_j| = (k+1)^m$, since there are $(k+1)$ terms, each with a coefficient of modulus 1.

Suppose that $\limsup_{t\to\infty} F(t) < k+1$. Then there exist $M > 0$ and $\lambda < k+1$ such that $|F(t)| \leq \lambda$ for all $t > M$. Consequently,

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T |F(t)|^m \, dt \;\leq\; \lambda^m.$$

Since $[F(t)]^m$ is a finite combination of exponentials,

$$\begin{aligned} |a_j| &= \left| \lim_{T\to\infty} \frac{1}{T} \int_0^T [F(t)]^m e^{-2\pi i t \gamma_j} \, dt \right| \\ &\leq \limsup_{T\to\infty} \frac{1}{T} \int_0^T |F(t)|^m \, dt \\ &\leq \lambda^m. \end{aligned} \tag{6.15}$$

Note that there are $\binom{m+k}{k} \leq (m+1)^k$ possible j's. Thus, summing the inequality (6.15) over all j's yields

$$\begin{aligned} (k+1)^m &= \sum_{|j|=m} |a_j| \\ &\leq (m+1)^k \lambda^m, \end{aligned}$$

a contraction for large m. $\square$

Remark 6.16. Let $q_1, \ldots, q_k$ be distinct primes. Then $\log q_1, \ldots, \log q_k$ are linearly independent over $\mathbb{Q}$.

Proof: If not, then for some rational numbers $r_1, \ldots, r_k$ we have $\sum_j r_j \log q_k = 0$ and by clearing the denominators, there exists integers $n_1, \ldots, n_k$ so that

$$\sum_k n_k \log q_k = 0 \quad\Longrightarrow\quad \prod_k q_k^{n_k} = 1.$$

Thus all n_k's must be zero by the uniqueness of prime factorization. $\square$

6.4. Power series in infinitely many variables

Recall from (5.8) that given $f \in \mathcal{H}^2$, $f = \sum_{n=1}^\infty a_n n^{-s}$, we have a formal power series in infinitely many variables

$$(\mathcal{Q}f)(z) = \sum_{n=1}^\infty a_n z^{r(n)}.$$

Let $\mathbb{D}^\infty$ denote $\{(z_i)_{i=1}^\infty;\ |z_i| < 1\}$ — the *infinite polydisk*.

PROPOSITION 6.17. *If $f \in \mathcal{H}^2$ and $z \in \mathbb{D}^\infty \cap \ell^2$, then $(\mathcal{Q}f)(z)$ is well-defined.*

Proof: Using the Cauchy-Schwarz inequality, we obtain

$$|\mathcal{Q}f(z)|^2 \ \leq \ \left(\sum_{n=1}^\infty |a_n|^2\right)\left(\sum_{n=1}^\infty |z|^{2r(n)}\right).$$

For $z \in \mathbb{D}^\infty$, observe that the map $n \mapsto \psi_z(n) := z^{[n]}$ is multiplicative and satisfies $|\psi_z(n)| \leq 1$, for all $n \in \mathbb{N}^+$. It follows that

$$\begin{aligned}\sum_{n=1}^\infty \left|z^{r(n)}\right|^2 \quad &= \quad \prod_{i=1}^\infty \frac{1}{1-|z_i|^2} \\ &= \quad \prod_{p\in\mathbb{P}} \frac{1}{1-|\phi(p)|^2}.\end{aligned}$$

Therefore

$$|\mathcal{Q}f(z)| \ \leq \ \|f\|_{\mathcal{H}^2}\left[\prod_{i=1}^\infty \frac{1}{1-|z_i|^2}\right]^{1/2}.$$

This is finite if $z \in \mathbb{D}^\infty \cap \ell^2$. ☐

REMARK 6.18. A character on $(\mathbb{N}^+, \cdot)$ is a multiplicative map from $\mathbb{N}^+$ to $\mathbb{T}$. A *quasi-character* on $(\mathbb{N}^+, \cdot)$ is a multiplicative map from $\mathbb{N}^+$ to $\overline{\mathbb{D}}$. So ψ_z is a quasi-character.

Hilbert, in 1909, asked:

QUESTION 6.19. Does $\mathcal{Q}f(z)$ make sense on a larger set than $\mathbb{D}^\infty \cap \ell^2$?

This was his answer. Let $z = (z_1, z_2, \dots)$, and let $z_{(m)}$ denote $(z_1, \dots, z_m, 0, 0, \dots)$. Consider the sequence $F_m(z) := F(z_{(m)})$; this is called the m^{te}-Abschnitt (or cut-off). If $f \in \mathcal{H}^2$ and $F = \mathcal{Q}f$, then the functions F_m are well-defined on $\mathbb{D}^\infty$ by Proposition 6.17.

PROPOSITION 6.20. **(Hilbert)** *Suppose that there exists $C > 0$ such that*

$$|F_m(z)| \leq C \quad \forall\ z \ \in \ \mathbb{D}^\infty,\ \forall\ m \in \mathbb{N}^+.$$

Then, for every $z \in \mathbb{D}^\infty \cap c_0$, the limit

$$\lim_{m\to\infty} F_m(z) =: F(z)$$

exists.

Proof: Fix $z \in \mathbb{D}^\infty \cap c_0$ and an $\varepsilon > 0$. Then, there exists $K \in \mathbb{N}$ such that $|z_k| < \frac{\varepsilon}{2C}$ holds for all $k > K$. Fix $n > m > K$, and consider the function $f \in H^\infty(\mathbb{D}^{n-m})$ given by

$$f(w_{m+1}, \dots, w_n) \; := \; F(z_1, \dots, z_m, w_{m+1}, \dots, w_n, 0, 0, \dots).$$

Now, we apply the polydisk version of Schwarz's lemma, Lemma 11.2, to $g(w) := \frac{f(w)-f(0)}{2C}$. Since $g : \mathbb{D}^{n-m} \to \mathbb{D}$, we conclude that

$$|g(z_{m+1}, \dots, z_n)| \; \leq \; \max_{i=m+1,\dots,n} |z_i| \; < \; \frac{\varepsilon}{2C},$$

so that

$$|f(z) - f(0)| \; \leq \; \frac{\varepsilon}{2C} \cdot 2C = \varepsilon.$$

Thus, the sequence $\{F_m(z)\}_m$ is Cauchy. □

Definition 6.21. We define $H^\infty(\mathbb{D}^\infty)$ by

$$H^\infty(\mathbb{D}^\infty) \; := \left\{ F(z) = \sum_{n=1}^{\infty} a_n z^{r(n)} \; : \; |F_m(z)| \leq C, \; \forall \, m \in \mathbb{N}, z \in \mathbb{D}^\infty \right\}. \tag{6.22}$$

The norm of $F \in H^\infty(\mathbb{D}^\infty)$ is the smallest C that satisfies the inequality in (6.22).

6.5. Besicovitch's Theorem

Definition 6.23.

(1) Let $f \in \mathrm{Hol}\,(\Omega_\rho)$, let $\varepsilon > 0$. We say that $\tau \in \mathbb{R}$ is an *ε-translation number* of f, if

$$\sup_{s \in \Omega_\rho} |f(s + i\tau) - f(s)| < \varepsilon.$$

We shall let $E(\varepsilon, f)$ denotes the set of ε-translation numbers of f.

(2) A set $S \subset \mathbb{R}$ is called *relatively dense*, if there exists $L < \infty$ such that each interval of length L contains at least one element of S.

(3) A function $f \in \mathrm{Hol}\,(\Omega_\rho)$ is *uniformly almost periodic* in Ω_ρ, if for all $\varepsilon > 0$, the set of ε-translation numbers of f is relatively dense.

Example 6.24. The function $f(s) = 2^{-s} + 3^{-s}$ is uniformly almost periodic in the half-plane Ω_ρ for every $\rho \in \mathbb{R}$.

It follows from Kronecker's theorem that for every $\varepsilon > 0$ there exists an arbitrarily large ε-translation number. Indeed, let $\theta_1 = \frac{\log 2}{2\pi}$,

$\theta_2 = \frac{\log 3}{2\pi}$ and $\alpha_1 = \alpha_2 = 0$. Then there exists an arbitrarily large $\tau \in \mathbb{R}$ so that

$$\text{dist}\left\{\frac{\tau \log 2}{2\pi}, \mathbb{Z}\right\} < \varepsilon \quad \text{and} \quad \text{dist}\left\{\frac{\tau \log 3}{2\pi}, \mathbb{Z}\right\} < \varepsilon.$$

Thus,

$$\begin{aligned} |2^{-(s+i\tau)} - 2^{-s}| &= |2^{-s}(e^{-i\tau \log 2} - 1)| \\ &\leq 2^{-\rho} 2\pi (\log 2) \text{ dist } \left\{\frac{\tau \log 2}{2\pi}, \mathbb{Z}\right\} \\ &< C\varepsilon. \end{aligned}$$

Similarly, one obtains

$$|3^{-(s+i\tau)} - 3^{-s}| \leq 3^{-\rho} 2\pi (\log 3) \text{ dist } \{\frac{\tau \log 3}{2\pi}, \mathbb{Z}\} < C\varepsilon.$$

There exists a refined version of Kronecker's theorem that implies that the ε-translation numbers of f are relatively dense, so f is uniformly almost periodic. However, the claim also follows from Corollary 6.28 below.

THEOREM 6.25. **(Besicovitch)** *Suppose $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ and the series converges uniformly in Ω_ρ. Then f is uniformly almost periodic.*

LEMMA 6.26. *Suppose f is uniformly almost periodic and uniformly continuous in Ω_ρ, and let $0 < \varepsilon_1 < \varepsilon_2$ be arbitrary. Then there exists a $\delta > 0$ such that for each $\tau \in E(\varepsilon_1, f)$, the inclusion $(\tau - \delta, \tau + \delta) \subset E(\varepsilon_2, f)$ holds.*

Proof: Let $\delta > 0$ be such that for every $0 < \delta' < \delta$ and $z \in \Omega_\rho$,

$$|f(z + i\delta') - f(z)| < \varepsilon_2 - \varepsilon_1.$$

For any $\tau' \in (\tau - \delta, \tau + \delta)$, write $\tau' = \tau + \delta'$ with $0 < |\delta'| < \delta$. Then the inequality

$$\begin{aligned} |f(z + i\tau') - f(z)| &\leq |f(z + i(\tau + \delta') - f(z + i\tau)| + |f(z + i\tau) - f(z)| \\ &< (\varepsilon_2 - \varepsilon_1) + \varepsilon_1 = \varepsilon_2 \end{aligned}$$

holds. $\square$

LEMMA 6.27. *Let $\varepsilon, \delta > 0$ and let f_1, f_2 be uniformly almost periodic and uniformly continuous functions. Then the set*

$$P = \{\tau \in E(\varepsilon, f_1) : \text{ dist } (\tau, E(\varepsilon, f_2)) < \delta\}$$

is relatively dense.

Proof: For a uniformly almost periodic function f and $\varepsilon > 0$, let $L(\varepsilon, f)$ denote the infimum of those $L > 0$ such that any interval of length L contains an ε-translation number of f. Choose $K \in \mathbb{N}$ so that $L = \delta K$ is greater than $\max\{L(\frac{\varepsilon}{2}, f_1), L(\frac{\varepsilon}{2}, f_2)\}$. Write

$$\mathbb{R} \;=\; \bigcup_{n\in\mathbb{Z}} \big[(n-1)L, nL\big) \;=\; \bigcup_{n\in\mathbb{Z}} I_n.$$

In each I_n there exist $\tau_1^{(n)} \in E(\frac{\varepsilon}{2}, f_1)$ and $\tau_2^{(n)} \in E(\frac{\varepsilon}{2}, f_2)$ and clearly $-L < \tau_1^{(n)} - \tau_2^{(n)} \leq L$. Decompose $[-L, L)$ into $2K$ disjoint intervals J_l of length δ. Since this is a finite number, there exists $n_0 \in \mathbb{N}$ such that if any interval J_l contains some point in the set $\{\tau_1^{(n)} - \tau_2^{(n)}\}_{n\in\mathbb{Z}}$, then it contains a point in the set $\{\tau_1^{(n)} - \tau_2^{(n)}\}_{n=-n_0}^{n_0}$. Thus, for any $n \in \mathbb{Z}$, there exists $n' \in \{-n_0, \ldots, n_0\}$ such that

$$\left|(\tau_1^{(n)} - \tau_2^{(n)}) - (\tau_1^{(n')} - \tau_2^{(n')})\right| \;<\; \delta.$$

Equivalently,

$$\tau \;:=\; \left(\tau_1^{(n)} - \tau_1^{(n')}\right) \;=\; \left(\tau_2^{(n)} - \tau_2^{(n')}\right) + \theta\delta,$$

with $|\theta| < 1$. By the triangle inequality, this implies that τ lies in $E(\varepsilon, f_1)$, and is closer than δ to an element of $E(\varepsilon, f_2)$, namely $(\tau_2^{(n)} - \tau_2^{(n')})$. In other words, $\tau \in P$.

We will now show that P is relatively dense. Consider an arbitrary interval I of length $(2n_0 + 3)L$ and find the integer n for which $\tau_1^{(n)}$ is closest to the center of I. Then the distance of $\tau_1^{(n)}$ from the center of I is at most L. Find the corresponding n' and τ, and conclude that

$$|\tau - \tau_1^{(n)}| \;=\; |\tau_1^{(n')}| \;\leq\; n_0 L.$$

This means that τ lies in I, and so the set P intersects every interval of length $(2n_0 + 3)L$. $\square$

COROLLARY 6.28. *Let f_1 and f_2 be both uniformly almost periodic and uniformly continuous. Then $f_1 + f_2$ is also uniformly almost periodic.*

Proof: Fix $\varepsilon > 0$, and apply Lemma 6.26 to $f = f_2$, $\varepsilon_1 = \frac{\varepsilon}{3}$ and $\varepsilon_2 = \frac{2\varepsilon}{3}$. We obtain $\delta > 0$ such that $\{\tau :\ \text{dist}\ (\tau, E(\frac{\varepsilon}{3}, f_2) < \delta\} \subseteq E(\frac{2\varepsilon}{3}, f_2)$. Now apply Lemma 6.27 to conclude that

$$\{\tau \in E(\tfrac{\varepsilon}{3}, f_1) :\ \text{dist}\ (\tau, E(\tfrac{\varepsilon}{3}, f_2)) < \delta\}$$

is relatively dense. But, by the triangle inequality, any τ in the above set is an ε-translation number for $f_1 + f_2$. $\square$

Proof: (of Theorem 6.25) Since a finite Dirichlet series is uniformly continuous, it follows inductively from Corollary 6.28 that it is also uniformly almost periodic. Therefore it is sufficient to prove that the uniform limit of uniformly almost periodic functions is also uniformly almost periodic.

Fix $\varepsilon > 0$. Find N so that $\|f_n - f\|_\infty < \varepsilon/3$ holds for all $n \geq N$. Then any $\varepsilon/3$-translation number τ of f_N is an ε-translation number of f, since

$$\begin{aligned} |f(z+\tau) - f(z)| &\leq |f(z+\tau) - f_N(z+\tau)| + |f_N(z+\tau) - f_N(z)| + |f_N(z) - f(z)| \\ &< \varepsilon \qquad \forall\, z. \end{aligned}$$

□

6.6. The spaces $\mathcal{H}^2_w$

Definition 6.29. Let $w = \{w_n\}_{n=1}^\infty$ be a sequence of positive real numbers which are in this context called a *weight*. Define the Hilbert space $\mathcal{H}^2_w$ of Dirichlet series by

$$\mathcal{H}^2_w \ := \left\{ \sum_n a_n n^{-s} : \ \sum_n |a_n|^2 w_n < \infty \right\}.$$

Remark 6.30. Note that if $f \in \mathcal{H}^2_w$, then f' is in the space with weights $w_n (\log n)^2$.

One way to obtain interesting weights is from measures on the positive real axis. Let μ be a positive Radon measure on $[0, \infty)$ such that

$$0 \in \operatorname{supp} \mu \tag{6.31}$$

$$\textstyle\int_0^\infty 4^{-\sigma}\ d\mu(\sigma) < \infty. \tag{6.32}$$

We define the weight sequence by

$$w_n \ := \int_0^\infty n^{-2\sigma}\ d\mu(\sigma). \tag{6.33}$$

One example of course is when μ is the Dirac measure at 0 denoted by δ_0, and all the weights are 1, giving $\mathcal{H}^2$. Here is another class.

Example 6.33. For each $\alpha < 0$, define μ_α on $[0, \infty)$ by

$$d\mu_\alpha(\sigma) \ = \ \frac{2^{-\alpha}}{\Gamma(-\alpha)} \sigma^{-1-\alpha} d\sigma.$$

Then for each $n \geq 2$, we have from (6.33)

$$w_n \ = \ (\log n)^\alpha. \tag{6.34}$$

Since w_1 is infinite, it is convenient to assume that sums $\sum_n a_n n^{-s}$ start at $n = 2$ when dealing with these spaces.

REMARK 6.35. On the unit disk, one can define spaces H^2_w by

$$H^2_w \ := \left\{ \sum_n a_n z^n : \ \sum_n |a_n|^2 w_n < \infty \right\}. \tag{6.36}$$

A special case is when

$$w_n \ = \ (n+1)^\alpha.$$

Then $\alpha = 0$ corresponds to the Hardy space, $\alpha = -1$ to the Bergman space, and $\alpha = 1$ to the Dirichlet space, the space of functions whose derivatives are in the Bergman space. The theory of the Hardy space on the disk is fairly well-developed – see *e.g.* [**Koo80, Dur70**] for a first course, or [**Nik85**] for a second. The Bergman space (and the other spaces with $\alpha < 0$ in this scale, that all come from L^2-norms of radial measures) is more complicated — see *e.g.* [**DS04, HKZ00**]. The Dirichlet space on the disk is even more complicated analytically, though it does have the complete Pick property. See *e.g.* [**EFKMR14**].

This section should be seen as an attempt to continue the analogy of Remark 6.35. The case $\alpha = 0$ in (6.34) we think of as a Hardy-type space, and the case $\alpha = -1$ in (6.34) we think of as a Bergman-type space. When $\alpha > 0$, we can still define weights by (6.34), though they do not come from a measure as in (6.33). By Remark 6.30, we can think of $\alpha = 1$, for example, as the space of functions whose first derivatives lie in the space with $\alpha = -1$. This would render this space a "Dirichlet space" of Dirichlet series, which is perhaps a surfeit of Dirichlet.

If the weights are defined by (6.33), then, for every $\varepsilon > 0$,

$$\begin{aligned} w_n \ &\geq \ \int_0^\varepsilon n^{-2\sigma} \, d\mu \\ &\geq \ \mu([0,\varepsilon]) n^{-2\varepsilon}, \end{aligned} \tag{6.37}$$

and, consequently, the weight sequence cannot decrease to 0 very fast.

PROPOSITION 6.38. *Suppose w_n is a weight sequence that is bounded below by $n^{-2\varepsilon}$ for every $\varepsilon > 0$. Then for any $f \in \mathcal{H}^2_w$, we have $\sigma_a(f) \leq \frac{1}{2}$.*

Proof: Take $\sigma > \frac{1}{2}$, and choose $\varepsilon > 0$ such that $\sigma - \varepsilon > \frac{1}{2}$. Then, by the Cauchy-Schwarz inequality,

$$\begin{aligned} \sum_n |a_n| n^{-\sigma} \ &= \ \sum_n |a_n n^{-\varepsilon}| n^{-(\sigma-\varepsilon)} \\ &\leq \ \left(\sum_n |a_n|^2 n^{-2\varepsilon} \right)^{\frac{1}{2}} \left(\sum n^{-2(\sigma-\varepsilon)} \right)^{\frac{1}{2}}. \end{aligned}$$

The first term is finite by (6.37), and the second since $2(\sigma - \varepsilon) > 1$. $\square$

The following theorem, in the case that $\mu = \delta_0$, is due to F. Carlson [**Car22**]. If $w_1 < \infty$, we assume that the Dirichlet series for f starts at $n = 1$; if w_1 is infinite, we start the series at $n = 2$ (Condition (6.32) says that $w_2 < \infty$).

THEOREM 6.39. *Let μ satisfy* (6.31) *and* (6.32)*, and define w_n by* (6.33)*. Assume that $f = \sum_n a_n n^{-s}$ has $\sigma_b(f) \leq 0$. Then*

$$\sum_n |a_n|^2 w_n \;=\; \lim_{c\to 0+} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_0^\infty |f(s+c)|^2 \, d\mu(\sigma) \, dt. \tag{6.40}$$

Moreover, if $\mu(\{0\}) = 0$, then the right-hand side becomes

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_0^\infty |f(s)|^2 \, d\mu(\sigma) \, dt.$$

Proof: Fix $0 < c < 1$, and let $0 < \varepsilon < 1$. Define δ by

$$\delta \;=\; \frac{\varepsilon}{(1 + \mu[0, \frac{1}{c}])(1 + 2\|f\|_{\Omega_c})}.$$

Since the Dirichlet series of f converges uniformly in $\overline{\Omega_c}$, there exists N such that

$$|\sum_{n\leq N'} a_n n^{-s} - f(s)| \;<\; \delta, \quad \forall\, s \in \overline{\Omega_c},\ \forall\, N' > N.$$

Then

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_0^{1/c} |f(s+c)|^2 \, d\mu(\sigma) \, dt \;&=\; \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_0^{1/c} |\sum_{n\leq N'} a_n n^{-s-c}|^2 \, d\mu(\sigma) \, dt + O(\varepsilon) \\ &=\; \sum_{n\leq N'} |a_n|^2 \int_0^{1/c} n^{-2\sigma-2c} \, d\mu(\sigma) + O(\varepsilon)\end{aligned}$$

Let N' tend to infinity, and c tend to 0, to get that the difference between the left and right sides of (6.40) are at most ε; since this is arbitrary, the two sides must be equal.

As $\lim_{T\to\infty} \fint_{-T}^{T} |f(s+c)|^2 dt$ is monotonically increasing as $c \to 0^+$, the monotone convergence theorem proves the second part of the theorem. $\square$

In particular, if $d\mu = d\mu_{-1} = 2dm$, we obtain

$$\sum_n |a_n|^2 \frac{1}{\log n} \;= 2 \lim_{T\to\infty} \fint_{-T}^{T} \int_0^\infty |f(s)|^2 \, dm(\sigma) \, dt,$$

and for $\mu = \delta_0$, we get

$$\sum_n |a_n|^2 \;=\; \lim_{c\to 0+} \lim_{T\to\infty} \fint_{-T}^{T} |f(c+it)|^2 \, dt.$$

6.7. Multiplier algebras of $\mathcal{H}^2$ and $\mathcal{H}^2_w$

NOTATION 6.41. Let us denote by $\mathcal{D}$ the set of functions expressible as Dirichlet series which converge somewhere, that is,

$$\mathcal{D} \;:=\; \big\{ f :\; \exists \rho \text{ such that } f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \text{ in } \Omega_\rho \big\}.$$

Since $\sigma_a \le \sigma_c + 1$, $\mathcal{D}$ is also the set of Dirichlet series that converge absolutely in some half-plane.

The following theorem is due to H. Hedenmalm, P. Lindqvist and K. Seip, in their ground-breaking paper [**HLS97**].

THEOREM 6.42. *Let μ and $\{w_n\}$ satisfy* (6.31) – (6.33). *Then* Mult $(\mathcal{H}^2_w)$ *is isometrically isomorphic to $H^\infty(\Omega_0) \cap \mathcal{D}$.*

REMARK 6.43. Before we prove the theorem, note that it implies that the multiplier algebra is independent of the weight w. The situation is analogous to a similar phenomenon on the disk. For any sequence $w = \{w_n\}_{n=0}^\infty$, one can define a Hilbert space of holomorphic functions H^2_w by (6.36). If the sequence w comes from a radial positive Radon measure μ on $\overline{\mathbb{D}}$ such that $\mathbb{T} \subset \operatorname{supp} \mu$ as

$$w_n \;=\; \int_{\overline{\mathbb{D}}} |z|^{2n} \, d\mu(z),$$

then $\{w_n\}_n$ is non-increasing and, since the measure is radial, the sequence $\{z^n\}_{n\in\mathbb{N}}$ is an orthogonal basis of H^2_w. (Saying the measure is radial means $d\mu = d\theta d\nu(r)$ for some measure ν on $[0,1]$). Thus, the norm on H^2_w is given by integration:

$$\|f\|^2 \;=\; \int_{\overline{\mathbb{D}}} |f(z)|^2 \, d\mu(z).$$

For all these spaces,

$$\operatorname{Mult}(H^2_w) \;=\; H^\infty(\mathbb{D}), \tag{6.44}$$

the bounded analytic functions on the disk. Indeed, if μ is carried by the open disk, this follows from Proposition 11.9. If μ puts weight on the circle, the theorem is still true, and can most easily be seen by writing

$$\int_{\overline{\mathbb{D}}} |f(z)|^2 \, d\mu(z) \;=\; \lim_{r \nearrow 1} \int_{\overline{\mathbb{D}}} |f(rz)|^2 \, d\mu(z).$$

In particular, (6.44) holds for all the spaces with $w_n = (n+1)^\alpha$ for $\alpha \leq 0$.

REMARK 6.45. There exist many functions in $H^\infty(\Omega_0) \setminus \mathcal{D}$, for example $f(s) = \left(\frac{3}{2}\right)^{-s}$ and $g(s) = \frac{s}{(s+1)^2}$.

Before embarking on the proof of the theorem, recall the following fact. It is a version of the Phragmén-Lindelöf principle — a maximum modulus principle for unbounded domains. This particular version is known as the three line lemma.

LEMMA 6.46. *Let f be a bounded holomorphic function in $\{z \in \mathbb{C};\ a < Re\ z < b\}$, let $N(\sigma) := \sup_{t \in \mathbb{R}} |f(\sigma + it)|$. Then the function N is logarithmically convex, that is,*

$$N(\sigma) \;\leq\; N(a)^{\frac{b-\sigma}{b-a}} N(b)^{\frac{\sigma-a}{b-a}}.$$

Proof: See Theorem 12.8, p. 274 in [**Rud86**]. □

REMARK 6.47. The lemma does not hold without the assumption that f is bounded in the strip. Indeed, consider the function $f(z) = e^{e^{iz}}$. It is holomorphic in the strip $\{-\frac{\pi}{2} < \text{Re } z < \frac{\pi}{2}\}$, bounded on its boundary $\{|\text{Re } z| = \frac{\pi}{2}\}$, but $\lim_{t \to -\infty} f(it) = \infty$. However, one can weaken the assumption of boundedness of f to an appropriate restriction on the growth of f.

The following lemma is trivial if $1 \in \mathcal{H}^2_w$.

LEMMA 6.48. *Any multiplier of $\mathcal{H}^2_w$ lies in $\mathcal{D}$.*

PROOF: If φ belongs to Mult$\,(\mathcal{H}^2_w)$, then both $\varphi(s)2^{-s}$ and $\varphi(s)3^{-s}$ are in $\mathcal{D}$. So

$$\begin{aligned} \varphi(s)2^{-s} &= \sum a_n n^{-s} \\ \varphi(s)3^{-s} &= \sum b_n n^{-s}. \end{aligned}$$

Multiplying the first equation by 3^{-s} and the second by 2^{-s}, we conclude that a_n is zero when n is odd (and b_n is zero when n is not divisible by 3), so φ itself can be represented by an ordinary Dirichlet series. □

PROPOSITION 6.49. *Let $\varphi(s) = \sum_{n=1}^\infty b_n n^{-s}$ with $\sigma_b \leq 0$. Then $\|M_\varphi\| = \|\varphi\|_{\Omega_0}$.*

Proof: Let $f(s) = \sum_{n \le N} a_n n^{-s}$, then $\sigma_b(\varphi f) \le 0$. By Theorem 6.39,

$$\begin{aligned} \|\varphi f\|^2_{\mathcal{H}^2_w} &= \lim_{c \to 0+} \lim_{T \to \infty} ⨍_{-T}^{T} \int_0^\infty |\varphi(s+c)|^2 |f(s+c)|^2 \, d\mu(\sigma) \, dt \\ &\le \|\varphi\|^2_{\Omega_0} \cdot \|f\|^2_{\mathcal{H}^2_w}. \end{aligned}$$

Hence M_φ is bounded on a dense subset of $\mathcal{H}^2_w$, and therefore extends to a bounded operator on all of $\mathcal{H}^2_w$, which must be multiplication by ϕ. (Why?) Also, the estimate above shows that $\|M_\varphi\| \le \|\varphi\|_{\Omega_0}$.

Conversely, assume that $\|M_\varphi\| = 1$ and $1 < \|\varphi\|_{\Omega_0}$ (possibly infinite). Let

$$N(\sigma) := \sup_{t \in \mathbb{R}} |\varphi(\sigma + it)|.$$

Clearly, $N(\sigma) \to |b_1|$ as $\sigma \to \infty$, and for any $\sigma > 0$, we have

$$N^2(\sigma) \ \ge \ \lim_{T \to \infty} ⨍_{-T}^{T} |\varphi(\sigma + it)|^2 \, dt \ = \ \sum_{n=1}^{\infty} |b_n|^2 n^{-2\sigma} \ > \ |b_1|^2,$$

unless φ is a constant (in which case the Proposition is obvious). For any $0 < a < b$ one can apply the three line lemma, 6.46, to conclude that $\log N$ is convex, so it must be convex on the half-line $(0, \infty)$. Since

$$\lim_{\sigma \to \infty} \log N(\sigma) \ = \ \log |b_1| \ < \infty,$$

we must have that $\log N$, and hence N, is a decreasing function on $(0, \infty)$.

For each $c > 0$, $\sum_n b_n n^{-s}$ converges uniformly in $\overline{\Omega_c}$, and hence by Theorem 6.25, φ is uniformly continuous and uniformly almost periodic in this half-plane. Thus, there exist $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and ε_4 positive such that

$$\Big| \{ t : \ |\varphi(\sigma + it)| \ \ge \ 1 + \varepsilon_1, \ -T < t < T \} \Big| \ \ge \ \varepsilon_2 (2T) \qquad (6.50)$$

holds for every sufficiently large $T > 0$, and $\sigma \in (\varepsilon_3, \varepsilon_3 + \varepsilon_4)$. Indeed, choose ε_3 so that $N(\varepsilon_3) > 1$. Then there is some $\varepsilon_1 > 0$ and some rectangle R with non-empty interior,

$$R \ = \ \{ \sigma + it : \varepsilon_3 \le \sigma \le \varepsilon_3 + \varepsilon_4, \ t_1 \le t \le t_1 + h \},$$

such that $|\varphi| > 1 + 2\varepsilon_1$ on R. By the definition of uniform almost periodicity, there exists some L such that every interval of length L contains an ε_1 translation number of φ. For $T > L$, every interval of length $2T$ contains at least $\frac{T}{L}$ disjoint sub-intervals of length L, so for any $\sigma \in [\varepsilon_3, \varepsilon_3 + \varepsilon_4]$ the left-hand side of (6.50) is at least $\frac{T}{L} h$. Setting $\varepsilon_2 = \frac{h}{2L}$ yields the inequality (6.50).

Now, on one hand, we have

$$\|M_\varphi^j 2^{-s}\|_{\mathcal{H}^2_w} \ \leq \ \|M_\varphi\|^j \cdot \|2^{-s}\|_{\mathcal{H}^2_w} \ = \ \|2^{-s}\|_{\mathcal{H}^2_w},$$

so that this sequence of norms is bounded by $\sqrt{w_2}$. On the other hand,

$$\begin{aligned} \|M_\varphi^j 2^{-s}\|^2_{\mathcal{H}^2_w} \quad &\geq \quad \lim_{T\to\infty} \int_0^{\varepsilon_4} \fint_{-T}^{T} |2^{-(s+\varepsilon_3)}\varphi^j(s+\varepsilon_3)|^2 \ dt \ d\mu(\sigma) \\ &\geq \quad \mu([0,\varepsilon_4])2^{-2(\varepsilon_3+\varepsilon_4)}\varepsilon_2(1+\varepsilon_1)^{2j}, \end{aligned}$$

and this tends to infinity as j tends to ∞, a contradiction. □

For later use, note that the proof of Proposition 6.49 shows:

Lemma 6.51. *If $\varphi = \sum_{n=1}^\infty b_n n^{-s}$ satisfies $\sigma_b(\varphi) \leq 0$, and $\|\varphi\|_{\Omega_0} > 1$, then*

$$\sup_{j\in\mathbb{N}^+} \|M_\varphi^j 2^{-s}\| \ = \ \infty.$$

For $K \in \mathbb{N}^+$, define

$$\mathbb{N}_K \ = \ \{n = p_1^{r_1} \cdot \ldots \cdot p_K^{r_K}; \ r_j \in \mathbb{N}\},$$

where, as usual, p_l is the l-th prime. Clearly, $n \in \mathbb{N}_K$, if and only if $p_l \not| n$ for all $l > K$. Let $Q_K : \mathcal{D} \to \mathcal{D}$ be the map defined by

$$Q_K\left(\sum_{n=1}^\infty a_n n^{-s}\right) \ = \ \sum_{n\in\mathbb{N}_K} a_n n^{-s}.$$

The map Q_K is well-defined, since if $\sum_{n=1}^\infty a_n n^{-s}$ converges absolutely in Ω_ρ, then so does $\sum_{n\in\mathbb{N}_K} a_n n^{-s}$.

We need the following observations.

Lemma 6.52. *For any $K \in \mathbb{N}^+$, the map Q_K has the following properties:*

1. *The restriction of Q_K to $\mathcal{H}^2_w$ is the orthogonal projection onto $\overline{\text{span}}\ \{n^{-s} : \ n \in \mathbb{N}_K\}$.*
2. *For any $\varphi, f \in \mathcal{D}$, $Q_K(\varphi f) = (Q_K\varphi)(Q_K f)$.*
3. *If $\varphi \in \text{Mult}\,(\mathcal{H}^2_w)$, then $Q_K M_\varphi Q_K = M_{Q_K\varphi} Q_K = Q_K M_\varphi$.*

Proof: (1) follows immediately from the orthogonality of the functions $\{n^{-s}\}_{n\in\mathbb{N}^+}$.

(2) By linearity, we only need to check that $Q_K(n^{-s}m^{-s}) = Q_K(n^{-s})Q_K(m^{-s})$, for all $m, n \in \mathbb{N}^+$. This follows from the facts that if p is prime, then $p \not| nm$ if and only if $p \not| n$ and $p \not| m$, and

$$Q_K n^{-s} = \begin{cases} n^{-s}, \ p_l \not| n, \ \text{for all } l > K, \\ 0, \ \text{otherwise.} \end{cases}$$

(3) Let $f \in \mathcal{H}^2_w$, then, using (2), we get

$$\begin{aligned} Q_K M_\varphi Q_K f &= Q_K(\varphi Q_K f) \\ &= (Q_K\varphi)(Q_K^2 f) \\ &= Q_K(\varphi f). \end{aligned}$$

Also,

$$\begin{aligned} M_{Q_K\varphi} Q_K f &= (Q_K\varphi)(Q_K f) \\ &= Q_K(\varphi f). \end{aligned}$$

□

PROPOSITION 6.53. Mult $(\mathcal{H}^2_w) \subset H^\infty(\Omega_0) \cap \mathcal{D}$.

Proof: Let $f = \sum a_n n^{-s} \in \mathcal{H}^2_w$, and fix $K \in \mathbb{N}^+$, $s \in \Omega_0$. Then

$$\begin{aligned} |Q_K f(s)| &= \left| \sum_{n\in\mathbb{N}_K} a_n n^{-s} \right| \\ &\leq \left[\sum_{n\in\mathbb{N}_K} n^{-\sigma} \right] \sup_{n\in\mathbb{N}_K} |a_n| \\ &= \left[\prod_{j=1}^{K} \frac{1}{1-p_j^{-\sigma}} \right] \sup_{n\in\mathbb{N}_K} |a_n|. \end{aligned}$$

So, if $\sup_{n\in\mathbb{N}_K} |a_n|$ is finite, then $Q_K(f)$ is bounded in Ω_ρ for all $\rho > 0$. Since $\sum_n |a_n|^2\omega_n$ converges, $\{|a_n|^2\omega_n\}$ is bounded, and hence by (6.37), $|a_n| = O(n^\varepsilon)$ for all $\varepsilon > 0$. Thus, for any $\varepsilon > 0$, the Dirichlet series of $f_\varepsilon(s) := f(s+\varepsilon)$ has bounded coefficients. Consequently, $Q_K f_\varepsilon \in H^\infty(\Omega_\rho)$, which is the same as saying $Q_K f_{\varepsilon+\rho} \in H^\infty(\Omega_0)$. Since $\varepsilon > 0$ and $\rho > 0$ were arbitrary, we conclude that

$$Q_K f_\varepsilon \in H^\infty(\Omega_0), \quad \forall\ K \in \mathbb{N}^+,\ \varepsilon > 0,\ f \in \mathcal{H}^2_w.$$

Let φ be in Mult $(\mathcal{H}^2_w)$. Then $\varphi 2^{-s} \in \mathcal{H}^2_w$, and so $2^{-s}Q_K(\varphi) = Q_K(\varphi 2^{-s}) \in H^\infty(\Omega_\varepsilon)$, for all $\varepsilon > 0$. Since we know $\varphi \in \mathcal{D}$ by Lemma 6.48 it follows that $\sigma_b(Q_K\varphi) \leq 0$.

By Lemma 6.51, applied to $Q_K\varphi$, we get

$$\|Q_K\varphi\|_{\Omega_0} \ \leq\ \|M_{Q_K\varphi}|_{Q_K\mathcal{H}^2_w}\|. \tag{6.54}$$

By Lemma 6.52,

$$\begin{aligned} \|M_{Q_K\varphi}|_{Q_K\mathcal{H}^2_w}\| &= \|Q_K M_\varphi Q_K\| \\ &\leq \|M_\varphi\|. \end{aligned}$$

So by (6.54),

$$\|Q_K\varphi\|_{\Omega_0} \ \leq\ \|M_\varphi\| \quad \forall\ K \in \mathbb{N}^+.$$

Using normal families, we conclude that some subsequence $Q_{K_l}\varphi$ converges to some function $\psi \in H^\infty(\Omega_0)$ uniformly on compact subsets of Ω_0. But $Q_K\varphi \to \varphi$ uniformly on compact subsets of $\Omega_{\sigma_u(\varphi)}$ and hence, $\varphi = \psi$ in $\Omega_{\sigma_u(\varphi)}$. By uniqueness of analytic functions, we conclude that $\varphi = \psi$ in Ω_0. $\square$

Combining Propositions 6.49 and 6.53, we complete the proof of Theorem 6.42. This also concludes the solution to Beurling's problem [**HLS97**].

COROLLARY 6.55. **(Hedenmalm, Lindqvist, Seip)** *Let* $\psi(x) = \sqrt{2}\sum_{n=1}^\infty c_n \sin(n\pi x)$ *be an odd, 2-periodic function on* $\mathbb{R}$. *Then* $\{\psi(nx)\}_{n\in\mathbb{N}^+}$ *forms a Riesz basis for* $L^2([0,1])$ *if and only if the function* $\varphi(s) = \sum_{n=1}^\infty c_n n^{-s}$ *is bounded and bounded from below in* Ω_0.

6.8. Cyclic Vectors

Consider the following variant of Beurling's question. Let $\psi : [0;1] \to \mathbb{C}$ be in L^2. When is the set $\{\psi(nx) : n \in \mathbb{N}^+\}$ complete, i.e., when do we have

$$\overline{\text{span}}\ \{\psi(nx) :\ n \in \mathbb{N}^+\} = L^2([0;1])?$$

As before, we can write $\psi(x) = \sum_{n=1}^\infty c_n\beta(x)$, and translate this problem to $\mathcal{H}^2$. Let $f(s) = \sum_{n=1}^\infty c_n n^{-s}$. When is

$$\overline{\text{span}}\ \{f(ns) :\ n \in \mathbb{N}^+\}\ =\ \mathcal{H}^2?$$

Since $f(ns) = (M_{n^{-s}}f)(s)$, it is equivalent to requiring that

$$\overline{\text{span}}\ \{f \cdot \mathcal{D}\} = \mathcal{H}^2,$$

i.e. that f is a *cyclic vector* for the collection of multipliers $\{M_{p^{-s}} : p \in \mathbb{P}\}$. An obvious necessary condition is that f does not vanish in $\Omega_{1/2}$. We record this open question.

QUESTION 6.56. Which Dirichlet series f satisfy $\overline{\text{span}}\ \{f \cdot \mathcal{D}\} = \mathcal{H}^2$?

6.9. Exercises

EXERCISE 6.57. Show that $\mathcal{H}^2$ contains a function f with $\sigma_a(f) = \frac{1}{2}$.

EXERCISE 6.58. Prove that the reproducing kernel for $\mathcal{H}^2_w$ is given by

$$k(s,u)\ =\ \sum_n \frac{1}{w_n} n^{-s-\bar{u}}. \tag{6.59}$$

EXERCISE 6.60. Prove (6.34).

EXERCISE 6.61. Show that if $\alpha \in \mathbb{Z}$, and $w_n = (\log n)^\alpha$, the reproducing kernel for $\mathcal{H}^2_w$ can be written in terms of the ζ function (if $\alpha = 0$), its derivatives (if $\alpha < 0$) or integrals (if $\alpha > 0$), after adjusting if necessary for the constant term.

EXERCISE 6.62. Prove that $\sum a_n n^{-s}$ is in $\mathcal{D}$ if and only if a_n is bounded by a polynomial in n.

6.10. Notes

The proof we give of Besicovitch's theorem 6.25 is from his book [**Bes32**, p. 144]. In the book he also develops the theory of functions that are almost periodic in the L^p-sense (where the L^p-norm of the difference between f and a vertical translate of it is less than ϵ).

The solution to Beurling's problem, and the proof of Theorem 6.42 (in the most important case, $\mathcal{H}^2_w = \mathcal{H}^2$) is due to Hedenmalm, Lindqvist and Seip [**HLS97**]. The spaces $\mathcal{H}^2_w$ were first studied in [**McCa04**].

In Carlson's theorem 6.39, if μ has a point mass at 0, then one cannot take the limit with respect to c inside the integral in (6.40). Indeed, E. Saksman and K. Seip prove the following theorem in [**SS09**]:

THEOREM 6.63. *(1) There exists a function f in $H^\infty(\Omega_0) \cap \mathcal{D}$ such that $\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(it)|^2\, dt$ does not exist.*

(2) For all $\varepsilon > 0$, there exists $g = \sum_{n=1}^\infty a_n n^{-s} \in H^\infty(\Omega_0) \cap \mathcal{D}$ that is a singular inner function and such that $\sum |a_n|^2 < \varepsilon$.

For a more refined version of Carlson's theorem, see [**QQ13**, Section 7.4].

CHAPTER 7

Characters

7.1. Vertical Limits

Let us return to the map $\mathcal{Q} : \mathcal{D} \to \mathrm{Hol}\,(\mathbb{D}^\infty)$. Consider the group $(\mathbb{Q}^+, \cdot)$ equipped with discrete topology. Its *dual group* K — the group of all characters,

$$K = \{\chi : \mathbb{Q}^+ \to \mathbb{T};\ \chi(mn) = \chi(m)\chi(n), \text{for all } m, n \in \mathbb{Q}^+\}$$

is isomorphic (as a topological group) to $\mathbb{T}^\infty$ via the map $\chi \mapsto \{\chi(p_k)\}_{k\in\mathbb{N}^+} = (\chi(2), \chi(3), \chi(5), \dots)$. The topology on K is the topology of pointwise convergence. It corresponds to the product topology on $\mathbb{T}^\infty$. The group $\mathbb{T}^\infty$ is also equipped with a Haar measure, which is the infinite product of the Haar measures on $\mathbb{T}$. We shall use ρ to denote Haar measure on $\mathbb{T}^\infty$.

Given any set X, a *flow* on X is family of maps $T_t : X \to X$, where t is a real parameter, that satisfy T_0 is the identity, and $T_s \circ T_t = T_{s+t}$. If X is equipped with some structure (measure space, topological space, smooth manifold, ...), we usually assume that T_t is compatible with this structure (*i.e.* each T_t is measurable, continuous, smooth, ...).

Given a sequence of real number $\{\alpha_n\}_{n\in\mathbb{N}}$, we define a flow on $\mathbb{T}^\infty$ by

$$T_t(z_1, z_2, \dots) := (e^{-it\alpha_1} z_1, e^{-it\alpha_2} z_2, \dots),$$

the so-called *Kronecker flow*. Note that the Kronecker flow is continuous and measurable.

Definition 7.1. A measurable flow on a probability space is *ergodic*, if all invariant sets have measure 0 or 1.

Theorem 7.2. *The Kronecker flow is ergodic if and only if* $\{\alpha_n\}$ *are linearly independent over* $\mathbb{Q}$.

Proof: See [**CFS82**]. □

In particular, if $\alpha_n = \log p_n$, the Kronecker flow is ergodic. (See Theorem 6.14.) The ergodic theorem (of which there are many variants) says that for an ergodic flow, the time average (the left-hand side of (7.4)) equals the space average (the right-hand side).

THEOREM 7.3. **(Birkhoff-Khinchin)** *Let T_t be an ergodic flow on K. Then*

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} g(T_t\chi_0)\ dt \ = \ \int_K g(\chi)d\rho(\chi), \tag{7.4}$$

for all χ_0, if $g \in \mathcal{C}(K)$, and for a.e. χ_0, if $g \in L^1$.

Proof: See [**CFS82**]. □

LEMMA 7.5. *Let $f \sim \sum_{n=1}^{\infty} a_n n^{-s}$ safisfies $\sigma_u(f) < 0$. Then $\mathcal{Q}f \in \mathcal{C}(\mathbb{T}^\infty)$.*

Proof: It suffices to show that the series for $\mathcal{Q}f$ is uniformly Cauchy, since the partial sums are clearly continuous.

Let $L = \sup_n |a_n| < \infty$. Fix $0 < \varepsilon < 1$, and find $N \in \mathbb{N}$ such that for all $M_2 > M_1 > N$

$$\Big|\sum_{n=M_1}^{M_2} a_n n^{it}\Big| \ < \ \varepsilon.$$

Thus, for all $t \in \mathbb{R}$,

$$\Big|\sum_{n=M_1}^{M_2} a_n \big[e^{it\log p_1}\big]^{r_1(n)} \dots \big[e^{it\log p_k}\big]^{r_k(n)}\Big| \ < \ \varepsilon.$$

Note that, if $w_1, \dots, w_k, \zeta_1, \dots, \zeta_k \in \mathbb{T}$, then

$$|w_1 \dots w_k - \zeta_1 \dots \zeta_k| \ \le \ |w_1 - \zeta_1| + \dots + |w_k - \zeta_k|. \tag{7.5}$$

This can be proven by induction on k using the inequality $|w_1 w_2 - \zeta_1\zeta_2| \le |w_1 - \zeta_1| + |w_2 - \zeta_2|$, which follows easily from the triangle inequality.

Fix $z \in \mathbb{T}^\infty$ and $M_2 > M_1 > N$ as above. By Kronecker's theorem 6.14, we can find $t \in \mathbb{R}$ such that $|e^{it\log p_j} - z_j| < \frac{\varepsilon}{M_2 L}$ holds for all j's such that $p_j \le M_2$. Thus we have,

$$\begin{aligned}
\Big|\sum_{n=M_1}^{M_2} a_n z^{r(n)}\Big| \quad &\le \quad \Big|\sum_{n=M_1}^{M_2} a_n \Big[z^{r(n)} - \big[e^{it\log p_1}\big]^{r_1(n)} \dots \big[e^{it\log p_k}\big]^{r_k(n)}\Big]\Big| \\
&+ \quad \Big|\sum_{n=M_1}^{M_2} a_n \big[e^{it\log p_1}\big]^{r_1(n)} \dots \big[e^{it\log p_k}\big]^{r_k(n)}\Big| \\
&< \quad \varepsilon + \varepsilon = 2\varepsilon,
\end{aligned}$$

where we used the inequality (7.5) to estimate the first term. □

This gives another proof of Carlson's theorem, 6.39.

THEOREM 7.6. *Let $f \sim \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{H}^2$, and let $x > \frac{1}{2}$. Then*

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(x+it)|^2 \, ds \;=\; \sum_{n=1}^{\infty} |a_n|^2 n^{-2x}. \tag{7.7}$$

Proof: Since $\sigma_u(f) \leq \sigma_a(f) \leq \frac{1}{2}$, we obtain $\sigma_u(f_x) < 0$, for $x > \frac{1}{2}$.
Since $\mathcal{Q}f_x$ is continuous on $\mathbb{T}^\infty$ by Lemma 7.5, we can apply the Birkhoff-Khinchin ergodic theorem 7.3 for any character $\chi_0 \in K$ to get

$$\begin{aligned}
\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |\mathcal{Q}f_x(T_t\chi_0)|^2 \, dt \quad &= \quad \int_K |\mathcal{Q}f_x(\chi)|^2 \, d\rho(\chi) \\
&= \quad \sum_{q\in\mathbb{Q}_+} |\widehat{\mathcal{Q}f_x}(q)|^2 \qquad (7.8) \\
&= \quad \sum_{n=1}^{\infty} |a_n|^2 n^{-2x}. \qquad (7.9)
\end{aligned}$$

We used Plancherel's theorem to obtain (7.8), and the fact that $\mathcal{Q}f$ is a sum only over positive powers of z means the only non-zero terms in (7.8) are when $q \in \mathbb{N}^+$, giving (7.8). Choosing the trivial character $\chi_0(n) \equiv 1$ yields

$$\begin{aligned}
(\mathcal{Q}f_x)(T_t\chi_0) \quad &= \quad \sum_n a_n n^{-x} n^{-it} \chi_0(n) \\
&= \quad f(x+it),
\end{aligned}$$

giving (7.7). ☐

For every $\tau \in \mathbb{R}$, the map $f \mapsto f_{i\tau}$ is unitary on $\mathcal{H}^2$. Thus, by Corollary 11.8, for every sequence $\{\tau_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$, there is a subsequence τ_{k_l} such that $\{f_{i\tau_{k_l}}\}$ converges uniformly on compact subsets of $\Omega_{1/2}$.

DEFINITION 7.10. Let $f \in \mathcal{H}^2$, and let $\{\tau_k\}_{k\in\mathbb{N}}$ be a sequence of real numbers. If the sequence $f_{i\tau_k}$ converges uniformly on compact subsets of $\Omega_{1/2}$ to a function g, then g is called a *vertical limit function* of f.

PROPOSITION 7.11. *Let $f \in \mathcal{H}^2$, and let χ be a character. Then $f_\chi(s) := \sum_{n=1}^{\infty} a_n \chi(n) n^{-s}$ is a vertical limit function of f. Conversely, all vertical limit functions have this form for some character χ.*

Proof: Fix a character χ and let $k \in \mathbb{N}^+$. By Kronecker's theorem, we can find $\tau_k \in \mathbb{R}$ such that $|e^{i\tau_k \log p_j} - \chi(p_j)| \leq 1/k$ holds for $j = 1, \ldots, k$. Define $f_k := f_{i\tau_k}$. Then using inequality (7.5), we conclude that for any $n \in \mathbb{N}^+$, $n = p_1^{r_1(n)} \ldots p_l^{r_l(n)}$,

$$\left|\widehat{f_k}(n) - \widehat{f_\chi}(n)\right| \quad = \quad \left|\hat{f}(n) n^{i\tau_k} - \hat{f}(n)\chi(n)\right|$$

$$
\begin{aligned}
&= \left|\hat{f}(n)\right|\cdot\left|\prod_{j=1}^{l}\left[e^{i\log p_j\tau_k}\right]^{r_j}-\prod_{j=1}^{l}\chi(p_j)^{r_j}\right| \\
&\leq \|f\|_{\mathcal{H}^2}\sum_{j=1}^{l} r_j\left|e^{i\log p_j\tau_k}-\chi(p_j)\right| \\
&\leq \frac{1}{k}\,\|f\|_{\mathcal{H}^2}\sum_{j=1}^{l} r_j,
\end{aligned}
$$

and this last expression tends to 0 as $k \to \infty$. Proposition 11.7 now implies that f_χ is a vertical limit function of f.

Conversely, let g be a vertical limit function of f. Using Proposition 11.7 again, we conclude that there exists a sequence $\{\tau_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ such that $\widehat{f_{i\tau_k}}(n) \to \hat{g}(n)$ for all $n \in \mathbb{N}$. Equivalently,

$$
n^{i\tau_k} \to \frac{\hat{g}(n)}{\hat{f}(n)}, \text{ as } k \to \infty.
$$

Since $n \mapsto n^{i\tau_k}$ is a character for all $k \in \mathbb{N}$, so is the limit: $n \mapsto \hat{g}(n)/\hat{f}(n)$. □

Let us now turn to the Lindelöf hypothesis, a conjecture weaker than the Riemann hypothesis, but one that could be possibly approached by the tools of functional analysis.

Recall that the alternating zeta function is given by $\tilde{\zeta}(s) = \sum_{n=1}^{\infty}(-1)^n n^{-s}$. We have seen that $\tilde{\zeta}(s) = (2^{1-s}-1)\zeta(s)$. This implies that $\tilde{\zeta}(s)$ and $\zeta(s)$ are of comparable size in $\{s \in \mathbb{C} : \ \mathrm{Re}\ s > 0,\ |1-\mathrm{Re}\ s| > \varepsilon\}$, for any $\varepsilon > 0$.

Conjecture 7.12. **(Lindelöf hypothesis)** For every $\sigma > \frac{1}{2}$ and $k \in \mathbb{N}^+$

$$
\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\tilde{\zeta}^k(\sigma+it)|^2\,dt < \infty
$$

holds.

Recall that $d_k(n)$, defined in Corollary 1.17, is the number of ways n can be factored into exactly k factors, allowing 1 and where the order matters.

Lemma 7.13. *Let k be a natural number and let $\varepsilon > 0$. Then*

$$
d_k(n) = O(n^\varepsilon) \text{ as } n \to \infty.
$$

Proof: Note that $d_2(n)$ is the number of divisors of n. Also, $d_3(n) \leq d_2(n)^2$, since

$$d_3(n) \;=\; \sum_{l|n} d_2\left(\frac{n}{l}\right) \;\leq\; \sum_{l|n} d_2(n) \;=\; d_2(n)^2.$$

Applying this argument inductively, we obtain $d_k(n) \leq d_2(n)^{k-1}$ and thus it is enough to show that $d_2(n) = O(n^\varepsilon)$ for all $\varepsilon > 0$.

Fix $\varepsilon > 0$. We need to show that there exist $C = C(\varepsilon)$ such that $d_2(n) \leq Cn^\varepsilon$ holds for all $n \in \mathbb{N}^+$, or equivalently, that

$$\log d_2(n) \;\leq\; \varepsilon \log n + \log C.$$

Write $n = \prod_{j=1}^{l} p_j^{t_j}$ with $t_j \geq 0$ and $t_l > 0$, then $d_2(n) = \prod_{j=1}^{l}(1 + t_j)$. We want to show that

$$\sum_{j=1}^{l} [\log(1 + t_j) - \varepsilon t_j \log p_j] \;\leq\; \log C$$

for all $n \in \mathbb{N}$. Clearly, if $\log p_j \geq 1/\varepsilon$, then the j^{th} summand is non-positive, because $\log(1 + t_j) < t_j$. As $t_j \to \infty$, the j^{th} summand tends to $-\infty$. Hence each of the finitely many summands with $\log p_j < 1/\varepsilon$ is bounded. □

Suppose that the Carlson theorem applied to $\tilde{\zeta}^k(s)$. Then

$$\begin{aligned}
\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\tilde{\zeta}^k(\sigma + it)|^2 \, dt \quad &= \quad \sum_{n=1}^{\infty} n^{-2\sigma} |\widehat{\tilde{\zeta}^k}(n)|^2 \\
&\leq \quad \sum_{n=1}^{\infty} n^{-2\sigma} |\widehat{\zeta^k}(n)|^2 \\
&< \quad \infty,
\end{aligned}$$

since $\widehat{\zeta^k}(n) = d_k(n) = O(n^\varepsilon)$ for all $\varepsilon > 0$ by Lemma 7.13. Thus we would have proved the Lindelöf hypothesis. Conversely, the following is known.

Theorem 7.14. **(Titchmarsh)** *If*

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\tilde{\zeta}^k(\sigma + it)|^2 \, dt < \infty,$$

then it equals to $\sum_{n=1}^{\infty} n^{-2\sigma} |\widehat{\tilde{\zeta}^k}(n)|^2$.

7.2. Helson's Theorem

We will need some properties of Hardy spaces of the right half-plane Ω_0. There is more than one natural definition. We will consider two of them. Let $\psi : \Omega_0 \to \mathbb{D}$ be the standard conformal mapping of the right half-plane onto the disk, that is, $\psi(z) = \frac{1-z}{1+z}$. For $1 \leq p \leq \infty$, we define the conformally invariant Hardy space as

$$H^p_i(\Omega_0) = \{g \circ \psi;\ g \in H^p(\mathbb{D})\}.$$

For $1 \leq p < \infty$, writing $e^{i\theta} = \psi(-it) = \frac{1+it}{1-it}$, and changing variables yields

$$\begin{aligned} \|g\|^p_{H^p(\mathbb{D})} &= \int_{\mathbb{T}} |g(e^{i\theta})|^p \frac{d\theta}{2\pi} \\ &= \int_{\mathbb{R}} |(g \circ \psi)(-it)|^p \left|\frac{d\theta}{dt}\right| \frac{dt}{2\pi} \\ &= \int_{\mathbb{R}} |(g \circ \psi)(-it)|^p \frac{dt}{\pi(1+t^2)}. \end{aligned}$$

Any function $g \in H^p(\mathbb{D})$ extends to an L^p function on $\mathbb{T}$ satisfying

$$\int_{\mathbb{T}} g(e^{i\theta}) e^{in\theta} \frac{d\theta}{2\pi} \ = \ 0, \text{ for all } n \in \mathbb{N}^+. \tag{7.10}$$

Conversely, any L^p function on $\mathbb{T}$ satisfying (7.10) is the boundary value of function in $H^p(\mathbb{D})$.

Let μ be the measure on the real axis give by $d\mu(t) = \frac{dt}{\pi(1+t^2)}$.

We deduce that a Lebesgue measurable function $f : i\mathbb{R} \to \mathbb{C}$ belongs to $H^p_i(\Omega_0)$, if and only if

$$\|f\|^p_{H^p_i(\Omega_0)} \ := \ \int_{\mathbb{R}} |f(it)|^p \ d\mu(t) \ < \ \infty,$$

and

$$\int_{\mathbb{R}} f(it) \left(\frac{1-it}{1+it}\right)^n \ d\mu(t) \ = \ 0, \text{ for all } n \in \mathbb{N}^+. \tag{7.11}$$

Here is the second definition for the Hardy spaces of the half-plane. For $1 \leq p < \infty$, set

$$H^p(\Omega_0) := \Big\{ f \in \mathrm{Hol}\ (\Omega_0) \ \Big|\ \|f\|^p_{H^p(\Omega_0)} := \sup_{\sigma > 0} \int_{-\infty}^{\infty} |f(\sigma + it)|^p \ dt < \infty \Big\}.$$

For any function $f \in H^p(\Omega_0)$ and almost every $t \in \mathbb{R}$, the limit $\tilde{f}(it) := \lim_{\sigma \to 0+} f(\sigma + it)$ exists and satisfies $\tilde{f} \in L^p(i\mathbb{R})$. One can recover f from $\tilde{f}$ by convolution with the Poisson kernel. For both $H^p_i(\Omega_0)$ and $H^p(\Omega_0)$ we identify the functions with their boundary values.