estimate. Hence, *restoring now the subscript k,*

$$\int_{\delta_k'}^{d_k}\int_{\delta_k'}^{d_k}\left(\frac{\dfrac{\nu(x)}{x}-\dfrac{\nu(y)}{y}}{x-y}\right)^2 \mathrm{d}x\,\mathrm{d}y \;\geqslant\; (3-2\log 2-15\eta)\left(\frac{d_k-\delta_k}{d_k}\right)^2,$$

as long as $\eta > 0$ is sufficiently small.

In the same way, one finds that

$$\int_{c_k}^{\gamma_k'}\int_{c_k}^{\gamma_k'}\left(\frac{\dfrac{\nu(x)}{x}-\dfrac{\nu(y)}{y}}{x-y}\right)^2 \mathrm{d}x\,\mathrm{d}y \;\geqslant\; (3-2\log 2-K\eta)\left(\frac{\gamma_k-c_k}{\gamma_k}\right)^2$$

for small enough $\eta > 0$, K being a certain numerical constant. Adding this to the previous relation gives us a lower estimate for

$$\int_{J_k}\int_{J_k}\left(\frac{\dfrac{\nu(x)}{x}-\dfrac{\nu(y)}{y}}{x-y}\right)^2 \mathrm{d}x\,\mathrm{d}y;$$

adding these estimates and referring again to the relation at the beginning of this proof, we obtain the theorem. Q.E.D.

From the initial discussion of this article, we see that the theorem has the following

Corollary. *Let $\mu(t)$ be the function constructed in article 3 and $\tilde{\Omega}$ be the complement, in $(0,\infty)$, of the set on which $\mu(t)$ is increasing. Then, if the parameter $\eta > 0$ used in constructing the J_k is sufficiently small,*

$$\int_{\tilde{\Omega}}\int_0^{\infty}\log\left|1-\frac{x^2}{t^2}\right|\mathrm{d}\mu(t)\frac{\mathrm{d}x}{x^2}$$
$$\geqslant\; \frac{p}{1-3p}\left(\tfrac{3}{2}-\log 2-K\eta\right)\sum_{k\geqslant 0}\left(\left(\frac{\gamma_k-c_k}{\gamma_k}\right)^2+\left(\frac{d_k-\delta_k}{d_k}\right)^2\right).$$

Here K is a numerical constant, independent of p or of the particular configuration of the J_k.

In the following work, our guiding idea will be to show that $\int_{\Omega}\int_0^{\infty}\log|1-x^2/t^2|\mathrm{d}\mu(t)(\mathrm{d}x/x^2)$ is *not too much less* than the left-hand integral in the above relation, in terms of the sum on the right.

7. Effect of taking x to be constant on each of the intervals J_k

We continue to write

$$\Omega \;=\; (0,\infty)\sim J,$$

where $J = \bigcup_{k\geqslant 0} J_k$ with $J_k = [c_k, d_k]$, and

$$\tilde{\Omega} = (0,\infty) \sim \tilde{J},$$

with

$$\tilde{J} = \bigcup_{k\geqslant 0} ((c_k, \gamma_k) \cup (\delta_k, d_k))$$

being the set on which $\mu(t)$ is increasing. The comparison of $\int_\Omega \int_0^\infty \log|1 - x^2/t^2| \mathrm{d}\mu(t)(\mathrm{d}x/x^2)$, object of our interest, with $\int_{\tilde{\Omega}} \int_0^\infty \log|1 - x^2/t^2| \mathrm{d}\mu(t)(\mathrm{d}x/x^2)$ is simplified by using two approximations to those quantities.

As in the previous article, we work in terms of

$$\nu(t) = \frac{1-3p}{p}\mu(t)$$

instead of $\mu(t)$. Put

$$\boxed{u(z) = \int_0^\infty \log\left|\frac{z+t}{z-t}\right| \mathrm{d}\left(\frac{\nu(t)}{t}\right).}$$

Then, by the corollary to the first lemma in article 4,

$$\int_\Omega \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\mu(t)\frac{\mathrm{d}x}{x^2} = \frac{p}{1-3p}\int_J u(x)\frac{\mathrm{d}x}{x}$$

and

$$\int_{\tilde{\Omega}} \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\mu(t)\frac{\mathrm{d}x}{x^2} = \frac{p}{1-3p}\int_{\tilde{J}} u(x)\frac{\mathrm{d}x}{x}.$$

Our approximation consists in the replacement of

$$\int_J u(x)\frac{\mathrm{d}x}{x} \quad \text{by} \quad \sum_{k\geqslant 0} \frac{1}{d_k}\int_{J_k} u(x)\mathrm{d}x$$

and of

$$\int_{\tilde{J}} u(x)\frac{\mathrm{d}x}{x} \quad \text{by} \quad \sum_{k\geqslant 0} \frac{1}{d_k}\left(\int_{c_k}^{\gamma_k} + \int_{\delta_k}^{d_k}\right)u(x)\mathrm{d}x.$$

To estimate the difference between the left-hand and right-hand quantities we use the positivity of the bilinear form $E(\ ,\ )$, proved in article 5.

Theorem. *If the parameter $\eta > 0$ used in the construction of the J_k is sufficiently small.*

$$\left| \int_J u(x) \frac{\mathrm{d}x}{x} - \sum_{k \geqslant 0} \frac{1}{d_k} \int_{J_k} u(x) \mathrm{d}x \right|$$

and

$$\left| \int_{\tilde{J}} u(x) \frac{\mathrm{d}x}{x} - \sum_{k \geqslant 0} \frac{1}{d_k} \left(\int_{c_k}^{\gamma_k} + \int_{\delta_k}^{d_k} \right) u(x) \mathrm{d}x \right|$$

are both

$$\leqslant C\eta^{\frac{1}{2}} E\left(\mathrm{d}\left(\frac{v(t)}{t} \right), \mathrm{d}\left(\frac{v(t)}{t} \right) \right),$$

where C is a purely numerical constant, independent of $p < \frac{1}{20}$ or the configuration of the J_k.

Remark. Here,

$$E\left(\mathrm{d}\left(\frac{v(t)}{t} \right), \mathrm{d}\left(\frac{v(t)}{t} \right) \right) = \int_{\tilde{J}} u(x) \frac{\mathrm{d}x}{x}$$

according to the corollary at the end of article 4.

Proof. Let us treat the *second* difference; the first is handled similarly. Take

$$\varphi(x) = \begin{cases} \dfrac{1}{x} - \dfrac{1}{d_k}, & c_k < x < \gamma_k, \quad k \geqslant 1; \\ \dfrac{1}{x} - \dfrac{1}{d_k}, & \delta_k < x < d_k, \quad k \geqslant 0; \\ 0 & \text{elsewhere.} \end{cases}$$

(Recall that $\gamma_0 = c_0$, so (c_0, γ_0) is empty.) The second of the expressions in question is then just the absolute value of

$$\int_0^\infty u(x)\varphi(x) \mathrm{d}x = \int_0^\infty \int_0^\infty \log\left| \frac{x+t}{x-t} \right| \mathrm{d}\left(\frac{v(t)}{t} \right) \varphi(x) \mathrm{d}x,$$

i.e., of $E(\mathrm{d}(v(t)/t), \varphi(t)\mathrm{d}t)$, in the notation of article 5. By the second corollary in that article and the remark at the end of it,

$$\left| E\left(\mathrm{d}\left(\frac{v(t)}{t} \right), \varphi(t) \mathrm{d}t \right) \right| \leqslant \sqrt{\left(E\left(\mathrm{d}\left(\frac{v(t)}{t} \right), \mathrm{d}\left(\frac{v(t)}{t} \right) \right) \right)}$$

$$\times \sqrt{(E(\varphi(t)\mathrm{d}t, \varphi(t)\mathrm{d}t))}.$$

The function $\varphi(x)$ is surely zero outside of the J_k, and, on J_k,

$$0 \leqslant \varphi(x) \leqslant \frac{d_k - x}{xd_k} \leqslant \frac{|J_k|}{xd_k}$$

with $|J_k|/d_k \leqslant 2\eta$ as in the proof of the theorem of article 6. Therefore,

$$0 < \int_0^\infty \log\left|\frac{x+t}{x-t}\right|\varphi(t)\mathrm{d}t \leqslant 2\eta\int_0^\infty \log\left|\frac{x+t}{x-t}\right|\frac{\mathrm{d}t}{t} = \pi^2\eta,$$

and

$$E(\varphi(t)\mathrm{d}t, \varphi(t)\mathrm{d}t) = \int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right|\varphi(t)\mathrm{d}t\,\varphi(x)\mathrm{d}x$$

$$\leqslant \pi^2\eta\sum_{k\geqslant 0}\int_{J_k}\frac{d_k - x}{xd_k}\mathrm{d}x \leqslant \tfrac{1}{2}\pi^2\eta\sum_{k\geqslant 0}\frac{|J_k|^2}{c_k d_k}.$$

We have $c_k = d_k - |J_k| \geqslant (1-2\eta)d_k$ (see above), and, according to property (iv) from the list near the end of article 3,

$$|J_k| = d_k - c_k = \frac{2}{1-3p}\{(\gamma_k - c_k) + (d_k - \delta_k)\}.$$

Since we are assuming (throughout this §) that $p < \frac{1}{20}$, this makes

$$|J_k|^2 < 2\left(\frac{40}{17}\right)^2\{(\gamma_k - c_k)^2 + (d_k - \delta_k)^2\},$$

yielding, by the preceding relation,

$$\frac{|J_k|^2}{c_k d_k} \leqslant \frac{12}{1-2\eta}\left\{\left(\frac{\gamma_k - c_k}{\gamma_k}\right)^2 + \left(\frac{d_k - \delta_k}{d_k}\right)^2\right\}.$$

Substitute this inequality into the previous estimate and then apply the theorem from the preceding article. One obtains

$$E(\varphi(t)\mathrm{d}t, \varphi(t)\mathrm{d}t)$$
$$\leqslant \frac{6\pi^2\eta}{(1-2\eta)(\frac{3}{2} - \log 2 - K\eta)}E\left(\mathrm{d}\left(\frac{\nu(t)}{t}\right), \mathrm{d}\left(\frac{\nu(t)}{t}\right)\right).$$

Using this in the above inequality for $|E(\mathrm{d}(\nu(t)/t), \varphi(t)\,\mathrm{d}t)|$, we immediately arrive at the desired bound on the difference in question. We are done.

8. An auxiliary harmonic function

We desire to use the lower bound furnished by the theorem of article 6 for

$$\int_{\tilde{J}} u(x)\frac{\mathrm{d}x}{x} = E\left(\mathrm{d}\left(\frac{\nu(t)}{t}\right), \mathrm{d}\left(\frac{\nu(t)}{t}\right)\right)$$

in order to obtain one for $\int_J u(x)(\mathrm{d}x/x)$, the quantity of interest to us. Our plan is to pass from

$$\int_{\tilde{J}} u(x)\frac{\mathrm{d}x}{x} \qquad \text{to} \qquad \sum_{k\geqslant 0}\frac{1}{d_k}\left(\int_{c_k}^{\gamma_k} + \int_{\delta_k}^{d_k}\right)u(x)\,\mathrm{d}x$$

and from

$$\sum_{k\geqslant 0}\frac{1}{d_k}\int_{c_k}^{d_k} u(x)\,\mathrm{d}x \qquad \text{to} \qquad \int_J u(x)\frac{\mathrm{d}x}{x};$$

according to the result of the preceding article (whose notation we maintain here), this will entail only small losses (relative to $\int_{\tilde{J}} u(x)(\mathrm{d}x/x)$), if $\eta > 0$ is small. This procedure still requires us, however, to get from the *first* sum to the *second*.

The simplest idea that comes to mind is to just compare corresponding terms of the two sums. That, however, would not be quite right, for in $\int_{c_k}^{d_k} u(x)\,\mathrm{d}x$, the integration takes place over a *set with larger Lebesgue measure* than in $(\int_{c_k}^{\gamma_k} + \int_{\delta_k}^{d_k})u(x)\,\mathrm{d}x$. In order to correct for this discrepancy, one should take an *appropriate multiple* of the second integral and then match the result against the first. The factor to be used here is obviously

$$\frac{2}{1-3p},$$

since (article 3),

$$\frac{\gamma_k - c_k + d_k - \delta_k}{d_k - c_k} = \frac{1-3p}{2}.$$

We are looking, then, at

$$\begin{aligned}\int_{c_k}^{d_k} u(x)\mathrm{d}x \;-\; \frac{2}{1-3p}\left(\int_{c_k}^{\gamma_k} + \int_{\delta_k}^{d_k}\right)u(x)\mathrm{d}x \\ = \int_{\gamma_k}^{\delta_k} u(x)\mathrm{d}x \;-\; \frac{1+3p}{1-3p}\left(\int_{c_k}^{\gamma_k} + \int_{\delta_k}^{d_k}\right)u(x)\mathrm{d}x.\end{aligned}$$

From now on, it will be convenient to write

$$\boxed{\lambda = \frac{1+3p}{1-3p};}$$

λ is > 1 and *very close to* 1 *if* $p > 0$ *is small.* It is also useful to split up each interval $(\gamma_k,\ \delta_k)$ into two pieces, associating the left-hand one with $(c_k,\ \gamma_k)$ and the other with $(\delta_k,\ d_k)$, and doing this in such a way that each piece has *λ times the length* of the interval to which it is associated. This is of course possible because

$$\frac{\delta_k - \gamma_k}{\gamma_k - c_k + d_k - \delta_k} \;=\; \frac{1+3p}{1-3p} \;=\; \lambda;$$

we thus take $g_k \in (\gamma_k,\ \delta_k)$ with

$$g_k \;=\; \gamma_k + \lambda(\gamma_k - c_k)$$

(and hence also $g_k = \delta_k - \lambda(d_k - \delta_k)$), and look at each of the two differences

$$\int_{\gamma_k}^{g_k} u(x)\mathrm{d}x \;-\; \lambda\int_{c_k}^{\gamma_k} u(x)\mathrm{d}x, \qquad \int_{g_k}^{\delta_k} u(x)\mathrm{d}x \;-\; \lambda\int_{\delta_k}^{d_k} u(x)\mathrm{d}x$$

separately; what we want to show is that *neither comes out too negative*, for we are trying to obtain a *positive lower bound* on $\int_J u(x)(\mathrm{d}x/x)$.

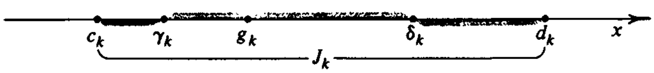

Figure 147

It is a fact that the two differences just written *can* be estimated in terms of $E(\mathrm{d}(\nu(t)/t),\ \mathrm{d}(\nu(t)/t))$.

Problem 23

(a) Show that for our function

$$u(z) \;=\; \int_0^\infty \log\left|\frac{z+t}{z-t}\right| \mathrm{d}\left(\frac{\nu(t)}{t}\right),$$

one has

$$E\left(\mathrm{d}\left(\frac{\nu(t)}{t}\right),\ \mathrm{d}\left(\frac{\nu(t)}{t}\right)\right) \;=\; \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{u(x)-u(y)}{x-y}\right)^2 \mathrm{d}x\,\mathrm{d}y.$$

This is Jesse Douglas' formula – I hope the coefficient on the right is correct. (Hint: Here, $u(x) = -(1/x)\int_0^\infty \log|1 - x^2/t^2|\mathrm{d}\nu(t)$ belongs to $L_2(-\infty,\infty)$ (it is *odd* on $\mathbb{R}$), so we can use Fourier–Plancherel transforms. In terms of

$$\hat{u}(\lambda) \;=\; \int_{-\infty}^{\infty} e^{i\lambda t} u(t)\mathrm{d}t$$

we have

$$u(x+iy) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|\lambda|y} e^{-i\lambda x}\hat{u}(\lambda)\mathrm{d}\lambda$$

for $y > 0$ (the left side being just the Poisson harmonic extension of the function $u(x)$ to $\Im z > 0$), and

$$\frac{u(x+h)-u(x)}{h} \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-i\lambda x}\frac{\mathrm{e}^{-i\lambda h}-1}{h}\hat{u}(\lambda)\,\mathrm{d}\lambda.$$

(All the right-hand integrals are to be understood in the l.i.m. sense.) Use Plancherel's theorem to express

$$\int_{-\infty}^{\infty}\left(\frac{u(x+h)-u(x)}{h}\right)^2 \mathrm{d}x \qquad \text{and} \qquad \int_{-\infty}^{\infty}[(u_x(z))^2+(u_y(z))^2]\,\mathrm{d}x$$

in terms of integrals involving $|\hat{u}(\lambda)|^2$, then integrate h from $-\infty$ to ∞ and y from 0 to ∞, and compare the results. Refer finally to the first lemma of article 5.)

(b) Show that

$$\left|\int_{\gamma_k}^{g_k} u(x)\mathrm{d}x \;-\; \lambda\int_{c_k}^{\gamma_k} u(x)\mathrm{d}x\right|$$
$$\leqslant \sqrt{\left(\frac{(1+\lambda)^4-1-\lambda^4}{12}\right)}\cdot(\gamma_k-c_k)\cdot\sqrt{\left(\int_{c_k}^{\gamma_k}\int_{\gamma_k}^{g_k}\left(\frac{u(x)-u(y)}{x-y}\right)^2\mathrm{d}y\,\mathrm{d}x\right)},$$

and obtain a similar estimate for

$$\int_{g_k}^{\delta_k} u(x)\mathrm{d}x \;-\; \lambda\int_{\delta_k}^{d_k} u(x)\mathrm{d}x.$$

(Hint: *Trick*:

$$\int_{\gamma_k}^{g_k} u(x)\mathrm{d}x \;-\; \lambda\int_{c_k}^{\gamma_k} u(x)\mathrm{d}x \;=\; \frac{1}{\gamma_k-c_k}\int_{c_k}^{\gamma_k}\int_{\gamma_k}^{g_k}[u(y)-u(x)]\,\mathrm{d}y\,\mathrm{d}x. \;)$$

(c) Use the result of article 6 with those of (a) and (b) to estimate

$$\left|\sum_{k\geqslant 0}\frac{1}{d_k}\left(\int_{\gamma_k}^{\delta_k} u(x)\mathrm{d}x \;-\; \lambda\left(\int_{c_k}^{\gamma_k}+\int_{\delta_k}^{d_k}\right)u(x)\mathrm{d}x\right)\right|$$

in terms of $E(\mathrm{d}(\nu(t)/t),\, \mathrm{d}(\nu(t)/t))$.

By working the problem, one finds that the difference considered in part (c) is in absolute value $\leqslant C\int_{\mathcal{J}} u(x)(\mathrm{d}x/x)$ for a certain numerical constant C. The trouble is, however, that the value of C obtained in this way comes out quite a bit *larger* than 1, so that the result cannot be used to yield a *positive* lower bound on $\int_J u(x)(\mathrm{d}x/x)$, λ being near 1. Too much is lost in following the simple reasoning of part (b); we need a more refined argument that will bring the value of C down below 1.

Any such refinement that works seems to involve bringing in (by use of Green's theorem, for instance) certain double integrals taken over portions of the first quadrant, in which the partial derivatives of u occur. Let us see how this comes about, considering the difference

$$\int_{\gamma_k}^{g_k} u(x)\,dx \;-\; \lambda \int_{c_k}^{\gamma_k} u(x)\,dx.$$

The latter can be rewritten as

$$\int_0^{(1+\lambda)\Delta_k} u(c_k + x)s_k(x)\,dx,$$

where $\Delta_k = \gamma_k - c_k$, and

$$s_k(x) \;=\; \begin{cases} -\lambda, & 0 < x < \Delta_k, \\ \;\;1, & \Delta_k < x < (1+\lambda)\Delta_k. \end{cases}$$

Suppose that we can find a function $V(z) = V_k(z)$, *harmonic* in the half-strip

$$S_k \;=\; \{z\colon\ 0 < \Re z < (1+\lambda)\Delta_k \text{ and } \Im z > 0\}$$

and having the following boundary behaviour:

$$V_y(x + i0) \;=\; -\,s_k(x), \qquad 0 < x < (1+\lambda)\Delta_k$$

$(V_y(x + i0)$ will be discontinuous at $x = \Delta_k)$,

$$V_x(iy) \;=\; 0, \qquad y > 0,$$

$$V_x(iy + (1+\lambda)\Delta_k) \;=\; 0, \qquad y > 0.$$

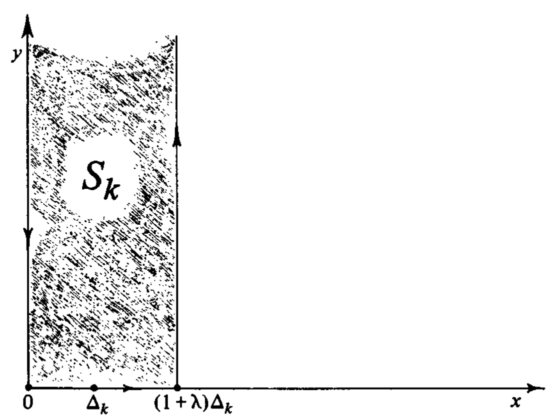

Figure 148

Then the previous integral becomes

$$\int_{\partial S_k} (-u(c_k + z)V_y(z)\,\mathrm{d}x \quad + \quad u(c_k + z)V_x(z)\,\mathrm{d}y),$$

∂S_k being oriented in the usual counterclockwise sense. Application of Green's theorem, if legitimate (which is easily shown to be the case here, as we shall see in due time), converts the line integral to

$$\iint_{S_k} (u_y(c_k + z)V_y(z) \quad + \quad u_x(c_k + z)V_x(z))\,\mathrm{d}x\,\mathrm{d}y$$

$$+ \quad \iint_{S_k} u(c_k + z)[V_{yy}(z) + V_{xx}(z)]\,\mathrm{d}x\,\mathrm{d}y.$$

The harmonicity of V in S_k will make the *second* integral vanish, and finally the *difference under consideration* will be equal to the *first* one. Referring to the first lemma of article 5, we see that the successful use of this procedure in order to get what we want necessitates our actually *obtaining* such a harmonic function $V = V_k$ and then *computing* (at least) its *Dirichlet integral*

$$\iint_{S_k} (V_x^2 + V_y^2)\,\mathrm{d}x\,\mathrm{d}y.$$

We will in fact need to know a little more than that. Let us proceed with the necessary calculations.

Our harmonic function $V_k(z)$ (assuming, of course, that there *is* one) will depend on *two* parameters, Δ_k and $\lambda = (1 + 3p)/(1 - 3p)$. The dependence on the *first* of these is nothing but a kind of homogeneity. Let $v(z, \lambda)$ be the function $V(z)$ corresponding to the special value $\pi/(1 + \lambda)$ of Δ_k, using the value of λ figuring in $V_k(z)$; $v(z, \lambda)$ is, in other words, to be harmonic in the half-strip

$$S \quad = \quad \{z\colon\ 0 < \Re z < \pi \ \text{ and } \ \Im z > 0\}$$

with $v_x(z, \lambda) = 0$ on the *vertical sides* of S and

$$v_y(x + \mathrm{i}0,\ \lambda) \quad = \quad \begin{cases} \lambda, & 0 < x < \dfrac{\pi}{1 + \lambda}, \\[2ex] -1, & \dfrac{\pi}{1 + \lambda} < x < \pi. \end{cases}$$

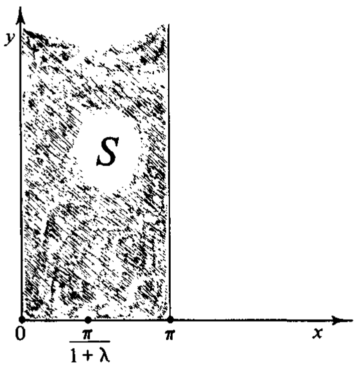

Figure 149

On the half-strip S_k of width $(1+\lambda)\Delta_k$ shown previously, the function

$$\frac{1}{\pi}(1+\lambda)\Delta_k v(\pi z/(1+\lambda)\Delta_k,\ \lambda)$$

is harmonic, and its partial derivatives clearly satisfy the boundary conditions on those of $V_k(z)$ stipulated above. We may therefore take

$$\boxed{V_k(z) \;=\; \frac{1}{\pi}(1+\lambda)\Delta_k v(\pi z/(1+\lambda)\Delta_k,\ \lambda)\,;}$$

this permits us to do all our calculations with the standard function v. Note that we will have, by simple change of variables,

$$\iint_{S_k}\left(\frac{\partial V_k}{\partial x}\right)^2 \mathrm{d}x\,\mathrm{d}y \;=\; \left(\frac{(1+\lambda)\Delta_k}{\pi}\right)^2 \iint_S (v_x(z,\lambda))^2\,\mathrm{d}x\,\mathrm{d}y$$

and

$$\iint_{S_k}\left[\left(\frac{\partial V_k}{\partial y}\right)^+\right]^2 \mathrm{d}x\,\mathrm{d}y \;=\; \left(\frac{(1+\lambda)\Delta_k}{\pi}\right)^2 \iint_S [(v_y(z,\lambda))^+]^2\,\mathrm{d}x\,\mathrm{d}y,$$

while

$$\iint_{S_k}\left|\frac{\partial V_k}{\partial y}\right| \mathrm{d}x\,\mathrm{d}y \;=\; \left(\frac{(1+\lambda)\Delta_k}{\pi}\right)^2 \iint_S |v_y(z,\lambda)|\,\mathrm{d}x\,\mathrm{d}y.$$

Lemma. *Given $\lambda \geqslant 1$, we can find a function $v(z,\lambda)$ harmonic in S whose partial derivatives satisfy the boundary conditions specified above. If $\varepsilon > 0$ is* *given, we have, for all $\lambda \geqslant 1$ sufficiently close to* 1,

$$\pi \iint_S (v_x(z,\lambda))^2 \mathrm{d}x\,\mathrm{d}y \;<\; 4\left(1+\frac{1}{3^3}+\frac{1}{5^3}+\cdots+\varepsilon\right),$$

$$\pi \iint_S [(v_y(z,\lambda))^+]^2 \mathrm{d}x\,\mathrm{d}y \;<\; 2\left(1+\frac{1}{3^3}+\frac{1}{5^3}+\cdots+\varepsilon\right),$$

and

$$\iint_S |v_y(z,\lambda)|\mathrm{d}x\,\mathrm{d}y \;\leqslant\; C,$$

C being a numerical constant, whose value we do not bother to calculate.

Remark. In the next article we will need the *numerical approximation*

$$\boxed{\frac{4}{\pi^2}\left(1+\frac{1}{3^3}+\frac{1}{5^3}+\cdots\right) \;<\; 0.4268.}$$

Proof of lemma. The method followed here (plain old 'separation of variables' from engineering mathematics) was suggested to me by Cedric Schubert. We look for a function v represented in the form

$$v(z,\lambda) \;=\; \sum_1^\infty A_n(\lambda)\mathrm{e}^{-ny}\cos nx$$

The series on the right, if convergent, *will* represent a function harmonic in S (each of its *terms* is harmonic!), and, for $y > 0$,

$$v_x(z,\lambda) \;=\; -\sum_1^\infty nA_n(\lambda)\mathrm{e}^{-ny}\sin nx$$

will vanish for $x = 0$ and $x = \pi$, for the exponentially decreasing factors e^{-ny} will make the series absolutely convergent.

For $y = 0$, by Abel's theorem,

$$v_y(x+\mathrm{i}0,\;\lambda) \;=\; -\sum_1^\infty nA_n(\lambda)\cos nx$$

at each x for which the series on the right is convergent. Let us choose the $A_n(\lambda)$ so as to make the right side the *Fourier cosine series* of the function

$$s(x,\lambda) \;=\; \begin{cases} \lambda, & 0 < x < \dfrac{\pi}{1+\lambda}, \\[2ex] -1, & \dfrac{\pi}{1+\lambda} < x < \pi. \end{cases}$$

We know from the very rudiments of Fourier series theory that this is accomplished by taking

$$-nA_n(\lambda) = \frac{2}{\pi}\int_0^\pi s(x,\lambda)\cos nx\,dx,$$

and that the resulting cosine series *does* converge to $s(x,\lambda)$ for $0 < x < \pi/(1+\lambda)$ and for $\pi/(1+\lambda) < x < \pi$. We can therefore *get* in this way a function $v(z\,\lambda)$ meeting all of our requirements.

Let us continue as long as we can without resorting to explicit computations. For fixed $y > 0$, Parseval's formula yields

$$\int_0^\pi (v_y(z,\lambda))^2\,dx = \frac{\pi}{2}\sum_1^\infty n^2(A_n(\lambda))^2 e^{-2ny},$$

and, in like manner,

$$\int_0^\pi [v_y(z,\lambda) - v_y(z,1)]^2\,dx = \frac{\pi}{2}\sum_1^\infty n^2(A_n(\lambda) - A_n(1))^2 e^{-2ny}.$$

Integrating both sides of this last relation with respect to y, we find that

$$\int_0^\infty\int_0^\pi (v_y(z,\lambda) - v_y(z,1))^2\,dx\,dy = \frac{\pi}{4}\sum_1^\infty n[A_n(\lambda) - A_n(1)]^2.$$

By Parseval's formula, we have, however,

$$\sum_1^\infty n^2[A_n(\lambda) - A_n(1)]^2 = \frac{2}{\pi}\int_0^\pi [s(x,\lambda) - s(x,1)]^2\,dx,$$

and it is evident that the right-hand integral tends to *zero* as $\lambda \to 1$. Hence, by the preceding relation,

$$\int_0^\infty\int_0^\pi [v_y(z,\lambda) - v_y(z,1)]^2\,dx\,dy \longrightarrow 0$$

for $\lambda \to 1$.

Now clearly

$$|(v_y(z,\lambda))^+ - (v_y(z,1))^+| \leqslant |v_y(z,\lambda) - v_y(z,1)|;$$

the result just obtained therefore implies that

$$\int_0^\infty\int_0^\pi [(v_y(z,\lambda))^+]^2\,dx\,dy \longrightarrow \int_0^\infty\int_0^\pi [(v_y(z,1))^+]^2\,dx\,dy$$

as $\lambda \to 1$. We see in the same fashion that

$$\int_0^\infty\int_0^\pi (v_x(z,\lambda))^2\,dx\,dy = \frac{\pi}{4}\sum_1^\infty n(A_n(\lambda))^2,$$

which $\longrightarrow$ $(\pi/4)\sum_1^\infty n(A_n(1))^2$ as $\lambda \to 1$.

For our purpose, it thus suffices to make the calculations for the limiting case $\lambda = 1$. Here,

$$-nA_n(1) = \frac{2}{\pi}\left(\int_0^{\pi/2} - \int_{\pi/2}^{\pi}\right)\cos nx\,\mathrm{d}x = \frac{4\sin\frac{\pi}{2}n}{\pi n},$$

so

$$\frac{\pi}{4}\sum_1^\infty n(A_n(1))^2 = \frac{4}{\pi}\left(1+\frac{1}{3^3}+\frac{1}{5^3}+\cdots\right),$$

whence, if $\lambda \geqslant 1$ is sufficiently close to 1,

$$\pi\int_0^\infty\int_0^\pi (v_x(z,\lambda))^2\,\mathrm{d}x\,\mathrm{d}y < 4\left(1+\frac{1}{3^3}+\frac{1}{5^3}+\cdots+\varepsilon\right).$$

Again

$$v_y(x+\mathrm{i}0,\ 1) = \begin{cases} 1, & 0<x<\dfrac{\pi}{2}, \\ -1, & \dfrac{\pi}{2}<x<\pi, \end{cases}$$

so by symmetry, for $y > 0$,

$$v_y\left(\frac{\pi}{2}-h+\mathrm{i}y,\ 1\right) = -v_y\left(\frac{\pi}{2}+h+\mathrm{i}y,\ 1\right) > 0, \qquad 0<h<\frac{\pi}{2}.$$

Hence,

$$\int_0^\infty\int_0^\pi [(v_y(z,1))^+]^2\,\mathrm{d}x\,\mathrm{d}y = \int_0^\infty\int_0^{\pi/2} (v_y(z,1))^2\,\mathrm{d}x\,\mathrm{d}y$$

$$= \frac{1}{2}\int_0^\infty\int_0^\pi (v_y(z,1))^2\,\mathrm{d}x\,\mathrm{d}y = \frac{\pi}{8}\sum_1^\infty n(A_n(1))^2$$

$$= \frac{2}{\pi}\left(1+\frac{1}{3^3}+\frac{1}{5^3}+\cdots\right).$$

Therefore, by the above observation,

$$\pi\int_0^\infty\int_0^\pi [(v_y(z,\lambda))^+]^2\,\mathrm{d}x\,\mathrm{d}y < 2\left(1+\frac{1}{3^3}+\frac{1}{5^3}+\cdots+\varepsilon\right)$$

for $\lambda \geqslant 1$ close enough to 1.

We are left with the integral $\int_0^\infty\int_0^\pi |v_y(z,\lambda)|\,\mathrm{d}x\,\mathrm{d}y$. This, by Schwarz' inequality, is

$$\leqslant \int_0^\infty \left(\pi \int_0^\pi (v_y(z,\lambda))^2\,\mathrm{d}x\right)^{\frac{1}{2}} \mathrm{d}y$$

$$= \int_0^\infty \left(\frac{\pi^2}{2}\sum_1^\infty n^2 (A_n(\lambda))^2 \mathrm{e}^{-2ny}\right)^{\frac{1}{2}} \mathrm{d}y$$

$$= \int_0^\infty \mathrm{e}^{-y/2}\left(\frac{\pi^2}{2}\sum_1^\infty n^2 (A_n(\lambda))^2 \mathrm{e}^{-(2n-1)y}\right)^{\frac{1}{2}} \mathrm{d}y$$

$$\leqslant \sqrt{\left(\int_0^\infty \mathrm{e}^{-y}\,\mathrm{d}y \cdot \int_0^\infty \frac{\pi^2}{2}\sum_1^\infty n^2 (A_n(\lambda))^2 \mathrm{e}^{-(2n-1)y}\,\mathrm{d}y\right)}$$

$$= \sqrt{\left(\frac{\pi^2}{2}\sum_1^\infty \frac{n^2}{2n-1}(A_n(\lambda))^2\right)} \leqslant \frac{\pi}{\sqrt{2}}\sqrt{\left(\sum_1^\infty n(A_n(\lambda))^2\right)}.$$

We have already seen that the sum inside the radical in the last of these terms tends to a definite (finite) limit as $\lambda \to 1$. So

$$\int_0^\infty \int_0^\pi |v_y(z,\lambda)|\,\mathrm{d}x\,\mathrm{d}y$$

is certainly *bounded* for $\lambda \geqslant 1$ near 1. The lemma is proved.

Referring to the remarks made just before the lemma and to the boxed numerical estimate immediately following its statement, we obtain, regarding our original functions V_k, the following

Corollary. *Given $\lambda \geqslant 1$ there is, for each k, a function $V_k(z)$ (depending on λ), harmonic in $S_k = \{z\colon\ 0 < \Re z < (1+\lambda)\Delta_k$ and $\Im z > 0\}$, with $\partial V_k(z)/\partial x = 0$ on the* vertical sides *of S_k and, on the latter's* base, *$\partial V_k/\partial y$ taking the boundary values λ and -1 along $(0,\ \Delta_k)$ and $(\Delta_k,\ (1+\lambda)\Delta_k)$ respectively.*

If $\lambda \geqslant 1$ is close enough to 1, *we have.*

$$\pi \iint_{S_k} \left(\frac{\partial V_k}{\partial x}\right)^2 \mathrm{d}x\,\mathrm{d}y \leqslant 0.44(1+\lambda)^2\Delta_k^2,$$

$$\pi \iint_{S_k} \left[\left(\frac{\partial V_k}{\partial y}\right)^+\right]^2 \mathrm{d}x\,\mathrm{d}y \leqslant 0.22(1+\lambda)^2\Delta_k^2,$$

and

$$\iint_{S_k} \left|\frac{\partial V_k}{\partial y}\right| \mathrm{d}x\,\mathrm{d}y \leqslant \alpha(1+\lambda)^2\Delta_k^2,$$

α being a certain numerical constant.

9. Lower estimate for $\int_\Omega \int_0^\infty \log|1 - x^2/t^2| \mathrm{d}\mu(t)(\mathrm{d}x/x^2)$

We return to the termwise comparison of

$$\sum_{k\geqslant 0} \frac{1}{d_k} \int_{c_k}^{d_k} u(x)\,\mathrm{d}x \quad \text{and} \quad \frac{2}{1-3p} \sum_{k\geqslant 0} \frac{1}{d_k}\left(\int_{c_k}^{\gamma_k} + \int_{\delta_k}^{d_k} \right) u(x)\,\mathrm{d}x,$$

which, as we saw in the first half of the preceding article, leads to the task of estimating

$$\int_{\gamma_k}^{g_k} u(x)\,\mathrm{d}x \; - \; \lambda \int_{c_k}^{\gamma_k} u(x)\,\mathrm{d}x$$

and

$$\int_{g_k}^{\delta_k} u(x)\,\mathrm{d}x \; - \; \lambda \int_{\delta_k}^{d_k} u(x)\,\mathrm{d}x$$

from below. The notation of the previous two articles is maintained here.

Following the idea of the last article, we use the harmonic function $V_k(z)$ described there to express the *first* of the above differences as a *line integral*

$$\int_{\partial S_k} \left(-u(c_k+z)\frac{\partial V_k(z)}{\partial y}\mathrm{d}x \; + \; u(c_k+z)\frac{\partial V_k(z)}{\partial x}\mathrm{d}y \right)$$

around the vertical half-strip S_k whose base is the segment $[0,\ (1+\lambda)\Delta_k] = [0,\ (1+\lambda)(\gamma_k - c_k)]$ of the real axis. By use of Green's theorem, this line integral is converted to

$$\iint_{S_k} \left(u_x(c_k+z)\frac{\partial V_k(z)}{\partial x} \; + \; u_y(c_k+z)\frac{\partial V_k(z)}{\partial y} \right) \mathrm{d}x\,\mathrm{d}y,$$

thanks to the harmonicity of $V_k(z)$ in S_k. The justification of the present application of Green's theorem proceeds as follows.

We have

$$\int_0^{(1+\lambda)(\gamma_k - c_k)} u(c_k+x)(V_k)_y(x+\mathrm{i}0)\,\mathrm{d}x$$

$$= \lim_{h\to 0} \int_0^{(1+\lambda)(\gamma_k - c_k)} u(c_k+x+\mathrm{i}h)(V_k)_y(x+\mathrm{i}h)\,\mathrm{d}x,$$

because $u(z)$ is continuous up to the real axis, and, as one verifies by referring to the computations with v and v_y near the end of the previous article,

$$\int_0^{(1+\lambda)(\gamma_k - c_k)} [(V_k)_y(x+\mathrm{i}h) \; - \; (V_k)_y(x+\mathrm{i}0)]^2\,\mathrm{d}x \longrightarrow 0$$

for $h \to 0$.

However, for $h>0$ and $0<x<(1+\lambda)(\gamma_k-c_k)$,

$$\int_h^\infty \frac{\partial}{\partial y}\left(u(c_k+z)\frac{\partial V_k(z)}{\partial y}\right)\mathrm{d}y \;=\; -u(c_k+x+\mathrm{i}h)(V_k)_y(x+\mathrm{i}h),$$

since

$$|u(c_k+z)| \;\leqslant\; \text{const.}\frac{\log|z|}{|z|}, \qquad z\in S_k,$$

by an estimate used in proving the first lemma of article 5, and $V_k(z)$, together with its partial derivatives, tends (exponentially) to zero as $z\to\infty$ in S_k (see the calculations at end of the previous article).

Again, since $\partial V_k(z)/\partial x = 0$ on the vertical sides of S_k,

$$\int_0^{(1+\lambda)(\gamma_k-c_k)} \frac{\partial}{\partial x}\left(u(c_k+z)\frac{\partial V_k(z)}{\partial x}\right)\mathrm{d}x \;=\; 0, \qquad y>0.$$

By integrating y in this formula from h to ∞ and x in the previous one from 0 to $(1+\lambda)(\gamma_k-c_k)$, and then adding the results, we express

$$-\int_0^{(1+\lambda)(\gamma_k-c_k)} u(c_k+x+\mathrm{i}h)(V_k)_y(x+\mathrm{i}h)\,\mathrm{d}x$$

as the sum of two iterated integrals. For $h>0$, both of the latter are *absolutely convergent*, and the order of integration in *one of them* may be *reversed*. Doing this and remembering that $\nabla^2 V_k = 0$ in S_k, we see that the sum in question boils down to

$$\int_h^\infty\int_0^{(1+\lambda)(\gamma_k-c_k)}\left(u_x(c_k+z)\frac{\partial V_k(z)}{\partial x} \;+\; u_y(c_k+z)\frac{\partial V_k(z)}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y.$$

Making $h\to 0$ in this expression finally gives us the corresponding double integral over S_k (whose absolute convergence readily follows from the first lemma in article 5 and the work at the end of the previous one by Schwarz' inequality).

This, together with our initial observation, shows that the double integral over S_k is equal to

$$-\int_0^{(1+\lambda)(\gamma_k-c_k)} u(c_k+x)(V_k)_y(x+\mathrm{i}0)\,\mathrm{d}x,$$

a quantity clearly identical with the above line integral around ∂S_k.* In this way, we see that our use of Green's theorem is legitimate.

The line integral is, as we recall (and as we see by glancing at the preceding expression), the same as

$$\int_{\gamma_k}^{g_k} u(x)\,\mathrm{d}x \;-\; \lambda\int_{c_k}^{\gamma_k} u(x)\,\mathrm{d}x.$$

* and actually coinciding with the original expression on p. 499 (the second one displayed there) from which the line integral was elaborated

That difference is therefore equal to

$$\iint_{S_k}\left(u_x(c_k+z)\frac{\partial V_k(z)}{\partial x} \;+\; u_y(c_k+z)\frac{\partial V_k(z)}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y.$$

What we want is a *lower bound* for the difference, and that means we have to *find one for this double integral.*

Our intention is to express such a lower bound as a certain portion of $E(\mathrm{d}(v(t)/t), \mathrm{d}(v(t)/t))$, the hope being that when all these portions are added (and also all the ones corresponding to the differences

$$\int_{g_k}^{\delta_k} u(x)\,\mathrm{d}x \;-\; \lambda\int_{\delta_k}^{d_k} u(x)\,\mathrm{d}x\ ),$$

we will end with a multiple of $E(\mathrm{d}(v(t)/t), \mathrm{d}(v(t)/t))$ that is not too large. In view, then, of the first lemma of article 5, we are interested in getting a lower bound in terms of

$$\frac{1}{\pi}\iint_{S_k}[(u_x(c_k+z))^2 \;+\; (u_y(c_k+z))^2]\,\mathrm{d}x\,\mathrm{d}y.$$

The present situation allows for very little leeway, and we have to be quite careful.

We start by writing

$$\iint_{S_k} u_x(c_k+z)\frac{\partial V_k(z)}{\partial x}\mathrm{d}x\,\mathrm{d}y$$
$$\geqslant -\sqrt{\left(\pi\iint_{S_k}\left(\frac{\partial V_k(x)}{\partial x}\right)^2\mathrm{d}x\,\mathrm{d}y\right)}$$
$$\times\sqrt{\left(\frac{1}{\pi}\iint_{S_k}(u_x(c_k+z))^2\,\mathrm{d}x\,\mathrm{d}y\right)}.$$

According to the corollary at the end of the last article, the right side is in turn

$$\geqslant\; -(0.44)^{\frac{1}{2}}(1+\lambda)(\gamma_k - c_k)\sqrt{\left(\frac{1}{\pi}\iint_{S_k}(u_x(c_k+z))^2\,\mathrm{d}x\,\mathrm{d}y\right)},$$

provided that $\lambda = (1+3p)/(1-3p)$ is close enough to 1 (recall that the Δ_k of the previous article equals $\gamma_k - c_k$).

For the estimation of

$$\iint_{S_k} u_y(c_k+z)\frac{\partial V_k(z)}{\partial y}\mathrm{d}x\,\mathrm{d}y,$$

we split up S_k into *two pieces*,

$$S_k^+ \;=\; \left\{z\in S_k:\;\; \frac{\partial V_k(z)}{\partial y} > 0\right\}$$

and

$$S_k^- \;=\; S_k \sim S_k^+.$$

We have

$$\iint_{S_k^+} u_y(c_k+z)\frac{\partial V_k(z)}{\partial y}\,\mathrm{d}x\,\mathrm{d}y$$
$$\geqslant\; -\sqrt{\left(\pi\iint_{S_k^+}\left(\frac{\partial V_k(z)}{\partial y}\right)^2\mathrm{d}x\,\mathrm{d}y\right)}$$
$$\times\sqrt{\left(\frac{1}{\pi}\iint_{S_k^+}(u_y(c_k+z))^2\,\mathrm{d}x\,\mathrm{d}y\right)},$$

which, by the corollary of the preceding article, is

$$\geqslant\; -(0.22)^{\frac{1}{2}}(1+\lambda)(\gamma_k-c_k)\sqrt{\left(\frac{1}{\pi}\iint_{S_k^+}(u_y(c_k+z))^2\,\mathrm{d}x\,\mathrm{d}y\right)}$$

for λ close enough to 1. In this last expression, the integral involving u_y may, if we wish, be replaced by one over S_k, yielding a worse result.

We are left with

$$\iint_{S_k^-} u_y(c_k+z)\frac{\partial V_k(z)}{\partial y}\,\mathrm{d}x\,\mathrm{d}y,$$

in which $\partial V_k(z)/\partial y\;\leqslant\;0$. To handle this integral, we recall that

$$u(z) \;=\; \int_0^\infty \log\left|\frac{z+t}{z-t}\right|\mathrm{d}\left(\frac{\nu(t)}{t}\right),$$

which makes

$$u_y(z) \;=\; \int_0^\infty\left[\frac{y}{(x+t)^2+y^2}-\frac{y}{(x-t)^2+y^2}\right]\mathrm{d}\left(\frac{\nu(t)}{t}\right),$$

with the quantity in brackets obviously *negative* for x, y and $t>0$. Since $\nu(t)/t \leqslant 2\eta$ by our construction of the intervals J_k, we have

$$\mathrm{d}\left(\frac{\nu(t)}{t}\right) \;=\; \frac{\mathrm{d}\nu(t)}{t} \;-\; \frac{\nu(t)\,\mathrm{d}t}{t^2} \;\geqslant\; -2\eta\frac{\mathrm{d}t}{t},$$

and therefore, for x and $y > 0$,

$$\begin{aligned} u_y(z) &\leqslant 2\eta \int_0^\infty \left(\frac{y}{(x-t)^2 + y^2} - \frac{y}{(x+t)^2 + y^2} \right) \frac{dt}{t} \\ &= 2\eta \lim_{\delta \to 0} \int_{-\infty}^\infty \frac{t}{t^2 + \delta^2} \, \frac{y}{(x-t)^2 + y^2} dt \\ &= 2\eta\pi \lim_{\delta \to 0} \frac{x}{x^2 + (y+\delta)^2} = 2\pi\eta \frac{x}{x^2 + y^2}. \end{aligned}$$

(We have simply used the Poisson representation for the function $\Re(1/(z + i\delta))$, harmonic in $\Im z > 0$.) Thus,

$$u_y(c_k + z) \leqslant \frac{2\pi\eta}{c_k}, \qquad z \in S_k,$$

whence

$$\iint_{S_k^-} u_y(c_k + z) \frac{\partial V_k(z)}{\partial y} dx\, dy \geqslant -\frac{2\pi\eta}{c_k} \iint_{S_k^-} \left| \frac{\partial V_k(z)}{\partial y} \right| dx\, dy.$$

For λ close to 1, the right side is

$$\geqslant -2\pi\alpha\eta \frac{(1+\lambda)^2(\gamma_k - c_k)^2}{c_k}$$

by the corollary from the previous article, α being a numerical constant.

Combining the three estimates just obtained, we find with the help of Schwarz' inequality that

$$\begin{aligned} &\iint_{S_k} \left(u_x(c_k + z) \frac{\partial V_k(z)}{\partial x} + u_y(c_k + z) \frac{\partial V_k(z)}{\partial y} \right) dx\, dy \\ &\geqslant -(0.44)^{\frac{1}{2}}(1+\lambda)(\gamma_k - c_k) \sqrt{\left(\frac{1}{\pi} \iint_{S_k} (u_x(c_k + z))^2 \, dx\, dy \right)} \\ &\quad - (0.22)^{\frac{1}{2}}(1+\lambda)(\gamma_k - c_k) \sqrt{\left(\frac{1}{\pi} \iint_{S_k} (u_y(c_k + z))^2 \, dx\, dy \right)} \\ &\quad - 2\pi\alpha\eta \frac{(1+\lambda)^2(\gamma_k - c_k)^2}{c_k} \\ &\geqslant -(0.66)^{\frac{1}{2}}(1+\lambda)(\gamma_k - c_k) \\ &\qquad \times \sqrt{\left(\frac{1}{\pi} \iint_{S_k} ((u_x(c_k + z))^2 + (u_y(c_k + z))^2) dx\, dy \right)} \\ &\qquad\qquad - 2\pi\alpha\eta \frac{(1+\lambda)^2(\gamma_k - c_k)^2}{c_k}, \end{aligned}$$

provided that λ is close enough to 1. The double integral on the left is nothing but a complicated expression for the first of the two differences with which we are concerned – that was, indeed, our reason for bringing the function $V_k(z)$ into this work. Hence the relation just proved can be rewritten

$$\int_{\gamma_k}^{g_k} u(x)\,\mathrm{d}x \;-\; \lambda\int_{c_k}^{\gamma_k} u(x)\,\mathrm{d}x$$
$$\geqslant\; -\;(0.66)^{\frac{1}{2}}(1+\lambda)(\gamma_k-c_k)\sqrt{\left(\frac{1}{\pi}\int_0^\infty\int_{c_k}^{g_k}((u_x(z))^2+(u_y(z))^2)\,\mathrm{d}x\,\mathrm{d}y\right)}$$
$$-\;2\pi\alpha\eta\,\frac{(1+\lambda)^2(\gamma_k-c_k)^2}{c_k}.$$

The difference $\int_{g_k}^{\delta_k} u(x)\,\mathrm{d}x - \lambda\int_{\delta_k}^{d_k} u(x)\,\mathrm{d}x$ can also be estimated by the method of this and the preceding articles. One finds in exactly the same way as above that

$$\int_{g_k}^{\delta_k} u(x)\,\mathrm{d}x \;-\; \lambda\int_{\delta_k}^{d_k} u(x)\,\mathrm{d}x$$
$$\geqslant\; -\;(0.66)^{\frac{1}{2}}(1+\lambda)(d_k-\delta_k)\sqrt{\left(\frac{1}{\pi}\int_0^\infty\int_{g_k}^{d_k}((u_x(z))^2+(u_y(z))^2)\mathrm{d}x\,\mathrm{d}y\right)}$$
$$-\;2\pi\alpha\eta\,\frac{(1+\lambda)^2(d_k-\delta_k)^2}{g_k}$$

for λ close enough to 1. The following diagram shows the regions over which the double integrals involved in this and the previous inequalities are taken:

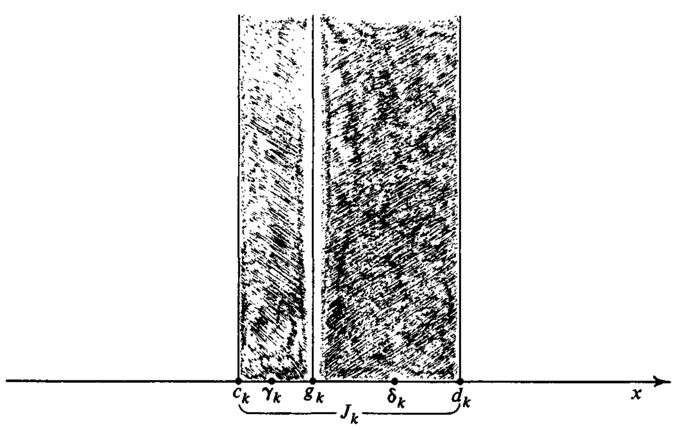

Figure 150

We now add the two relations just obtained. After dividing by d_k and using Schwarz' inequality again together with the fact that

$$c_k \leqslant \gamma_k \leqslant g_k < \delta_k < d_k < (1+2\eta)c_k,$$

we get, recalling that $\lambda = (1+3p)/(1-3p)$,

$$\begin{aligned}
&\frac{1}{d_k}\int_{c_k}^{d_k} u(x)\,\mathrm{d}x - \frac{2}{1-3p}\cdot\frac{1}{d_k}\left(\int_{c_k}^{\gamma_k} + \int_{\delta_k}^{d_k}\right)u(x)\,\mathrm{d}x \\
&\geqslant -(0.66)^{\frac{1}{2}}\frac{2}{1-3p}(1+2\eta)\sqrt{\left(\left(\frac{\gamma_k - c_k}{\gamma_k}\right)^2 + \left(\frac{d_k-\delta_k}{d_k}\right)^2\right)} \\
&\qquad\times\sqrt{\left(\frac{1}{\pi}\int_0^\infty\int_{c_k}^{d_k}(u_x^2+u_y^2)\,\mathrm{d}x\,\mathrm{d}y\right)} \\
&\quad - \frac{8\pi\alpha(1+2\eta)}{(1-3p)^2}\eta\left[\left(\frac{\gamma_k-c_k}{\gamma_k}\right)^2 + \left(\frac{d_k-\delta_k}{d_k}\right)^2\right]
\end{aligned}$$

for λ close enough to 1, *in other words, for $p > 0$ close enough to zero.*

We have now carried out the program explained in the first half of article 8 and at the beginning of the present one. Summing the preceding relation over k and using Schwarz' inequality once more, we obtain, for small $p > 0$,

$$\begin{aligned}
&\sum_{k\geqslant 0}\frac{1}{d_k}\int_{c_k}^{d_k} u(x)\,\mathrm{d}x - \frac{2}{1-3p}\sum_{k\geqslant 0}\frac{1}{d_k}\left(\int_{c_k}^{\gamma_k} + \int_{\delta_k}^{d_k}\right)u(x)\,\mathrm{d}x \\
&\geqslant -(0.66)^{\frac{1}{2}}(1+2\eta)\frac{2}{1-3p}\sqrt{\sum_{k\geqslant 0}\left(\left(\frac{\gamma_k - c_k}{\gamma_k}\right)^2 + \left(\frac{d_k-\delta_k}{d_k}\right)^2\right)} \\
&\qquad\times\sqrt{\left(\frac{1}{\pi}\sum_{k\geqslant 0}\int_0^\infty\int_{J_k}(u_x^2+u_y^2)\,\mathrm{d}x\,\mathrm{d}y\right)} \\
&\quad - \frac{8\pi\alpha(1+2\eta)}{(1-3p)^2}\eta\sum_{k\geqslant 0}\left(\left(\frac{\gamma_k-c_k}{\gamma_k}\right)^2 + \left(\frac{d_k-\delta_k}{d_k}\right)^2\right).
\end{aligned}$$

To the right-hand expression we apply the theorem of article 6 together with its remark and the first lemma of article 5. In this way, we find that the right side is

$$\begin{aligned}
&\geqslant -\frac{2}{1-3p}\sqrt{\left(\frac{0.66}{0.80-K\eta}\right)}\cdot(1+2\eta)E\left(\mathrm{d}\left(\frac{\nu(t)}{t}\right),\mathrm{d}\left(\frac{\nu(t)}{t}\right)\right) \\
&\qquad - K'\eta E\left(\mathrm{d}\left(\frac{\nu(t)}{t}\right),\mathrm{d}\left(\frac{\nu(t)}{t}\right)\right)
\end{aligned}$$

for small enough positive values of η and p, K and K' being certain numerical constants independent of p and of the configuration of the J_k.

According to the theorem of article 7, the left-hand difference in the above relation is within

$$\frac{3-3p}{1-3p}C\eta^{\frac{1}{2}}E\left(\mathrm{d}\left(\frac{\nu(t)}{t}\right),\ \mathrm{d}\left(\frac{\nu(t)}{t}\right)\right)$$

of

$$\int_J u(x)\frac{\mathrm{d}x}{x} \ -\ \frac{2}{1-3p}\int_{\tilde{J}} u(x)\frac{\mathrm{d}x}{x}$$

for small enough $\eta > 0$, where C is a numerical constant independent of p or the configuration of the J_k. So, since

$$\int_{\tilde{J}} u(x)\frac{\mathrm{d}x}{x} \ = \ E\left(\mathrm{d}\left(\frac{\nu(t)}{t}\right),\mathrm{d}\left(\frac{\nu(t)}{t}\right)\right)$$

(see remark to the theorem of article 7), what we have boils down, for small enough p and $\eta > 0$, to

$$\int_J u(x)\frac{\mathrm{d}x}{x} \ \geqslant\ \frac{2}{1-3p}\left(1-\sqrt{\left(\frac{0.66}{0.80}\right)}-A\eta-B\sqrt{\eta}\right) \times E\left(\mathrm{d}\left(\frac{\nu(t)}{t}\right),\mathrm{d}\left(\frac{\nu(t)}{t}\right)\right)$$

with numerical constants A and B independent of p and the configuration of the J_k. Here,

$$\sqrt{\left(\frac{0.66}{0.80}\right)} \ = \ 0.9083^-,$$

so, the coefficient on the right is

$$\geqslant\ \frac{2}{1-3p}(0.0917-A\eta-B\sqrt{\eta}).$$

Not much at all, but still enough!

We have finally arrived at the point where a *value* for the parameter η must be chosen. This quantity, independent of p, was introduced during the *third stage* of the long construction in article 2, where it was necessary to take $0<\eta<\frac{2}{3}$. Aside from that requirement, we were free to assign ▶ any value we liked to it. *Let us now choose, once and for all, a numerical value* >0 *for* η, *small enough to ensure that all the estimates of articles* 6, 7 *and the present one hold good, and that besides*

$$0.0917-A\eta-B\sqrt{\eta} \ > \ 1/20.$$

That value is henceforth fixed. This matter having been settled, the relation finally obtained above reduces to

$$\int_J u(x)\frac{dx}{x} \geqslant \frac{1}{10(1-3p)}E\left(d\left(\frac{\nu(t)}{t}\right), d\left(\frac{\nu(t)}{t}\right)\right).$$

To get a lower bound on the right-hand member, we use again the inequality

$$E\left(d\left(\frac{\nu(t)}{t}\right), d\left(\frac{\nu(t)}{t}\right)\right)$$
$$\geqslant (0.80 - K\eta)\sum_{k\geqslant 0}\left(\left(\frac{\gamma_k - c_k}{\gamma_k}\right)^2 + \left(\frac{d_k - \delta_k}{d_k}\right)^2\right)$$

(valid for *our* fixed value of η!), furnished by the theorem of article 6. In article 2, the intervals J_k were constructed so as to make $d_0 - c_0 = |J_0| \geqslant \eta d_0$ (see property (v) in the description near the end of that article), and in the construction of the function $\mu(t)$ we had

$$\frac{d_0 - \delta_0}{d_0 - c_0} = \frac{1-3p}{2}$$

(property (iii) of the specification near the end of article 3). Therefore

$$\frac{d_0 - \delta_0}{d_0} \geqslant \frac{1-3p}{2}\eta,$$

which, substituted into the previous inequality, yields

$$E\left(d\left(\frac{\nu(t)}{t}\right), d\left(\frac{\nu(t)}{t}\right)\right) \geqslant (0.80 - K\eta)\left(\frac{1-3p}{2}\right)^2\eta^2.$$

We substitute this into the relation written above, and get

$$\boxed{\int_J u(x)\frac{dx}{x} \geqslant (1-3p)c}$$

with a certain purely numerical constant c. (We see that it is finally just the ratio $|J_0|/d_0$ associated with the *first* of the intervals J_k that enters into these last calculations. If only we had been able to avoid consideration of the other J_k in the above work!) In terms of the function $\mu(t) = (p/(1-3p))\nu(t)$ constructed in article 3, we have, as at the beginning of article 7,

$$\int_\Omega \int_0^\infty \log\left|1-\frac{x^2}{t^2}\right|\mathrm{d}\mu(t)\frac{\mathrm{d}x}{x^2} = \frac{p}{1-3p}\int_J u(x)\frac{\mathrm{d}x}{x}.$$

By the preceding boxed formula and the work of article 3 we therefore have the

Theorem. *If $p \geqslant 0$ is small enough and if, for our original polynomial $P(x)$, the zero counting function $n(t)$ satisfies*

$$\sup_{t>0}\frac{n(t)}{t} > \frac{p}{1-3p},$$

then, for the function $\mu(t)$ constructed in article 3, we have

$$\int_\Omega \int_0^\infty \log\left|1-\frac{x^2}{t^2}\right|\mathrm{d}\mu(t)\frac{\mathrm{d}x}{x^2} \geqslant pc,$$

c being a numerical constant independent of $P(x)$. Here,

$$\Omega = (0,\infty) \sim \bigcup_{k\geqslant 0} J_k,$$

where the J_k are the intervals constructed in article 2.

In this way the task described at the very end of article 3 has been carried out, and the main work of the present § completed.

Remark. One reason why the present article's estimations have had to be so delicate is the *smallness* of the lower bound on

$$E\left(\mathrm{d}\left(\frac{\nu(t)}{t}\right),\mathrm{d}\left(\frac{\nu(t)}{t}\right)\right)$$

obtained in article 6. If we could be sure that this quantity was considerably *larger*, a much simpler procedure could be used to get from $\int_{\tilde{J}} u(x)(\mathrm{d}x/x)$ to $\int_J u(x)(\mathrm{d}x/x)$; the one of problem 23 (article 8) for instance.

It is possible that $E(\mathrm{d}(\nu(t)/t), \mathrm{d}(\nu(t)/t))$ *is* quite a bit larger than the lower bound we have found for it. One can write

$$E\left(\mathrm{d}\left(\frac{\nu(t)}{t}\right),\mathrm{d}\left(\frac{\nu(t)}{t}\right)\right) = \int_{\tilde{J}}\int_{\tilde{J}}\frac{1}{x^2}\log\left|\frac{1}{1-x^2/t^2}\right|\mathrm{d}t\,\mathrm{d}x.$$

If the intervals J_k are *very far apart from each other* (so that the cross terms

$$\int_{J_k\cap\tilde{J}}\int_{J_l\cap\tilde{J}}\frac{1}{x^2}\log\left|\frac{1}{1-x^2/t^2}\right|\mathrm{d}t\,\mathrm{d}x, \qquad k\neq l,$$

are all *very small*), the right-hand integral behaves like a constant multiple of

$$\sum_{k\geqslant 0}\left(\frac{|J_k|}{d_k}\right)^2\log\left(\frac{d_k}{|J_k|}\right).$$

When $\eta>0$ is taken to be *small*, this, on account of the inequality $|J_k|/d_k\leqslant 2\eta$, is much *larger* than the bound furnished by the theorem of article 6, which is essentially a *fixed* constant multiple of

$$\sum_{k\geqslant 0}\left(\frac{|J_k|}{d_k}\right)^2.$$

I have not been able to verify that the first of the above sums can be used to give a lower bound for $E(\mathrm{d}(\nu(t)/t),\ \mathrm{d}(\nu(t)/t))$ when the J_k are *not* far apart. That, however, is perhaps still worth trying.

10. Return to polynomials

Let us now combine the theorem from the end of article 3 with the one finally arrived at above. We obtain, without further ado, the

Theorem. *If $p>0$ is sufficiently small and $P(x)$ is any polynomial of the form*

$$\prod_k\left(1-\frac{x^2}{x_k^2}\right)$$

with the $x_k>0$, the condition

$$\sup_{t>0}\frac{n(t)}{t} > \frac{p}{1-3p}$$

for $n(t)=$ number of x_k (counting multiplicities) in $[0,t]$ implies that

$$\sum_1^\infty\frac{\log^+|P(m)|}{m^2} \geqslant \frac{cp}{5}.$$

Here, $c>0$ is a numerical constant independent *of p and of $P(x)$.*

Corollary. *Let $Q(z)$ be any even polynomial (with, in general, complex zeros) such that $Q(0)=1$. There is an absolute constant k, independent of Q, such that, for all z,*

$$\frac{\log|Q(z)|}{|z|} \leqslant k\sum_1^\infty\frac{\log^+|Q(m)|}{m^2},$$

provided that the sum on the right is less than some number $\gamma>0$, also independent of Q.

Proof. We can write

$$Q(z) \;=\; \prod_k \left(1 - \frac{z^2}{\zeta_k^2}\right).$$

Put $x_k = |\zeta_k|$ and then let

$$P(z) \;=\; \prod_k \left(1 - \frac{z^2}{x_k^2}\right);$$

we have $|P(x)| \leqslant |Q(x)|$ on $\mathbb{R}$, so

$$\sum_1^\infty \frac{\log^+ |P(m)|}{m^2} \;\leqslant\; \sum_1^\infty \frac{\log^+ |Q(m)|}{m^2}.$$

To $P(x)$ we apply the theorem, which clearly implies that

$$\sup_{t>0} \frac{n(t)}{t} \;\leqslant\; \frac{10}{c} \sum_1^\infty \frac{\log^+ |P(m)|}{m^2}$$

for $n(t)$, the number of x_k in $[0, t]$, whenever the sum on the right is small enough. For $z \in \mathbb{C}$,

$$\log |Q(z)| \;\leqslant\; \sum_k \log\left(1 + \frac{|z|^2}{|\zeta_k|^2}\right) \;=\; \int_0^\infty \log\left(1 + \frac{|z|^2}{t^2}\right) \mathrm{d}n(t),$$

and partial integration converts the last expression to

$$\int_0^\infty \frac{n(t)}{t} \frac{2|z|^2}{|z|^2 + t^2} \mathrm{d}t \;\leqslant\; \pi |z| \sup_{t>0} \frac{n(t)}{t}.$$

In view of our initial relation, we therefore have

$$\frac{\log |Q(z)|}{|z|} \;\leqslant\; \frac{10\pi}{c} \sum_1^\infty \frac{\log^+ |Q(m)|}{m^2}$$

whenever the right-hand sum is small enough. Done.

Remark 1. These results hold for objects more general than polynomials. Instead of $|Q(z)|$, we can consider any *finite product of the form*

$$\prod_k \left|1 - \frac{z^2}{\zeta_k^2}\right|^{\lambda_k}$$

where the exponents λ_k are all $\geqslant$ some fixed $\alpha > 0$. Taking $|P(x)|$ as

$$\prod_k \left|1 - \frac{x^2}{x_k^2}\right|^{\lambda_k}$$

with $x_k = |\zeta_k|$, and writing

$$n(t) \;=\; \sum_{x_k \in [0,t]} \lambda_k$$

(so that each 'zero' x_k is counted with 'multiplicity' λ_k), we easily convince ourselves that the arguments and constructions of articles 1 and 2 go through for *these* functions $|P(x)|$ and $n(t)$ without essential change. What was important there is the property, valid here, that $n(t)$ *increase by at least some fixed amount* $\alpha > 0$ *at each of its jumps*, crucial use having been made of this during the second and third stages of the construction in article 2. The work of articles 3–8 can thereafter be taken over *as is*, and we end with analogues of the above results for our present functions $|P(x)|$ and $|Q(z)|$.

Thus, in the case of *polynomials* $P(z)$, *it is not so much the single-valuedness* of the *analytic function* with *modulus* $|P(z)|$ as the *quantization of the point masses* associated with the *subharmonic function* $\log|P(z)|$ that is essential in the preceding development.

Remark 2. The specific arithmetic character of $\mathbb{Z}$ plays no rôle in the above work. Analogous results hold if we replace the sums

$$\sum_1^\infty \frac{\log^+|P(m)|}{m^2}, \qquad \sum_1^\infty \frac{\log^+|Q(m)|}{m^2},$$

by others of the form

$$\sum_{\lambda\in\Lambda} \frac{\log^+|P(\lambda)|}{\lambda^2}, \qquad \sum_{\lambda\in\Lambda} \frac{\log^+|Q(\lambda)|}{\lambda^2},$$

Λ being any fixed set of points in $(0,\infty)$ having at least one element in each interval of length $\geqslant h$ with $h > 0$ and fixed. This generalization requires some rather self-evident modification of the work in article 1. The reasoning in articles 2–8 then applies with hardly any change.

Problem 24

Consider entire functions $F(z)$ *of very small exponential type* α having the special form

$$F(z) \;=\; \prod_k \left(1 - \frac{z^2}{x_k^2}\right)$$

where the x_k are > 0, and such that

$$\int_{-\infty}^{\infty} \frac{\log^+|F(x)|}{1+x^2}\,dx \;<\; \infty.$$

Investigate the possibility of adapting the development of this § to such functions $F(z)$ (instead of *polynomials* $P(z)$).

Here, if the small numbers 2η and p are both several times larger than α, the constructions of article 2 can be made to work (by problem 1(a), Chapter I!), yielding an infinite number of intervals J_k. The statement of the *second lemma* from article 4 has to be modified.

I have not worked through this problem.

We now come to the *principal result* of this whole §, an extension of the above corollary to *general polynomials*. To establish it, we need a simple

Lemma. *Let* $\alpha > 0$ *be given. There is a number* M_α *depending on* α *such that, for* any *real valued function* f *on* $\mathbb{Z}$ *satisfying*

$$\sum_{-\infty}^{\infty} \frac{\log^+ |f(n)|}{1+n^2} \leqslant \alpha,$$

we have

$$\sum_1^\infty \frac{1}{n^2} \log\left(1 + \frac{n^2(f(n)+f(-n))^2}{M_\alpha^2}\right) \leqslant 6\alpha$$

and

$$\sum_1^\infty \frac{1}{n^2} \log\left(1 + \frac{(f(n)-f(-n))^2}{M_\alpha^2}\right) \leqslant 6\alpha$$

Proof. When $q \geqslant 0$, the function $\log(1+q) - \log^+ q$ assumes its maximum for $q = 1$. Hence

$$\log(1+q) \leqslant \log 2 + \log^+ q, \qquad q \geqslant 0.$$

Also,

$$\log^+(qq') \leqslant \log^+ q + \log^+ q', \qquad q,\ q' \geqslant 0.$$

Therefore, if $M \geqslant 1$,

$$\begin{aligned} &\log\left(1 + \frac{n^2(f(n)+f(-n))^2}{M^2}\right) \\ &\quad \leqslant \log 2 + 2\log^+ n + 2\log^+(|f(n)| + |f(-n)|) \\ &\quad \leqslant 3\log 2 + 2\log n + 2\max(\log^+|f(n)|,\ \log^+|f(-n)|) \end{aligned}$$

for $n \geqslant 1$.

Given $\alpha > 0$, choose (and then *fix*) an N sufficiently large to make

$$\sum_{n>N} \frac{3\log 2 + 2\log n}{n^2} < \alpha$$

Then, if f is any real valued function with

$$\sum_{-\infty}^{\infty} \frac{\log^+|f(n)|}{1+n^2} \leqslant \alpha,$$

we will surely have

$$\sum_{n>N} \frac{1}{n^2} \log\left(1+\frac{n^2(f(n)+f(-n))^2}{M^2}\right) < 5\alpha$$

by the previous relation, as long as $M \geqslant 1$. Similarly,

$$\sum_{n>N} \frac{1}{n^2} \log\left(1+\frac{(f(n)-f(-n))^2}{M^2}\right) < 5\alpha$$

for such f, if $M \geqslant 1$.

Our condition on f certainly implies that

$$\log^+|f(n)| \leqslant \alpha(1+n^2),$$

so

$$|f(n)|+|f(-n)| \leqslant 2e^{(1+N^2)\alpha}$$

for $1 \leqslant n \leqslant N$. Choosing $M_\alpha \geqslant 1$ sufficiently large so as to have

$$\sum_{1}^{N} \frac{1}{n^2} \log\left(1+\frac{4n^2 e^{2\alpha(1+N^2)}}{M_\alpha^2}\right) < \alpha$$

will thus ensure that

$$\sum_{1}^{N} \frac{1}{n^2} \log\left(1+\frac{n^2(f(n)+f(-n))^2}{M_\alpha^2}\right) < \alpha$$

and

$$\sum_{1}^{N} \frac{1}{n^2} \log\left(1+\frac{(f(n)-f(-n))^2}{M_\alpha^2}\right) < \alpha.$$

Adding each of these relations to the corresponding one obtained above, we have the lemma.

Theorem. *There are numerical constants $\alpha_0 > 0$ and k such that, for any polynomial $p(z)$ with*

$$\sum_{-\infty}^{\infty} \frac{\log^+|p(n)|}{1+n^2} = \alpha \leqslant \alpha_0,$$

we have, for all z,

$$|p(z)| \leqslant K_\alpha e^{3k\alpha|z|},$$

where K_α is a constant depending only on α (and not on p).

Proof. Given a polynomial p, we may as well assume to begin with that $p(x)$ is *real for real* x – otherwise we just work separately with the polynomials $(p(z)+\overline{p(\bar{z})})/2$ and $(p(z)-\overline{p(\bar{z})})/2i$ which both have that property.

Considering, then, p to be real on $\mathbb{R}$ and assuming that it satisfies the condition in the hypothesis, we take the number M_α furnished by the lemma and form each of the *polynomials*

$$Q_1(z) \;=\; 1+\frac{z^2(p(z)+p(-z))^2}{M_\alpha^2},$$

$$Q_2(z) \;=\; 1+\frac{(p(z)-p(-z))^2}{M_\alpha^2}.$$

The polynomials Q_1 and Q_2 are both *even*, and

$$Q_1(0) \;=\; Q_2(0) \;=\; 1.$$

By the lemma,

$$\sum_1^\infty \frac{1}{n^2}\log^+|Q_1(n)| \;\leqslant\; 6\alpha$$

and

$$\sum_1^\infty \frac{1}{n^2}\log^+|Q_2(n)| \;\leqslant\; 6\alpha,$$

since (here) $Q_1(x)\geqslant 1$ and $Q_2(x)\geqslant 1$ on $\mathbb{R}$.

If $\alpha>0$ is *small enough*, these inequalities imply *by the above corollary* that $(\log|Q_1(z)|)/|z|$ and $(\log|Q_2(z)|)/|z|$ are both $\leqslant 6k\alpha$, k being a certain numerical constant. Therefore

$$\left|1+\frac{z^2(p(z)+p(-z))^2}{M_\alpha^2}\right| \;\leqslant\; e^{6k\alpha|z|},$$

and

$$\left|1+\frac{(p(z)-p(-z))^2}{M_\alpha^2}\right| \;\leqslant\; e^{6k\alpha|z|}.$$

From these relations we get

$$|z^2(p(z)+p(-z))^2| \;\leqslant\; M_\alpha^2(1+e^{6k\alpha|z|}) \;\leqslant\; 2M_\alpha^2 e^{6k\alpha|z|},$$

whence

$$|p(z)+p(-z)| \;\leqslant\; \sqrt{2}M_\alpha e^{3k\alpha|z|} \qquad \text{for } |z|\geqslant 1,$$