tells us that

$$E(\mathrm{d}\rho(t),\ \mathrm{d}\rho(t)) \quad = \quad \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{U_\rho(x)-U_\rho(y)}{x-y}\right)^2 \mathrm{d}x\,\mathrm{d}y.$$

What we have, then, to do is to *broaden the scope* of this relation, due to Jesse Douglas, so as to get it to apply to the whole class of measures ρ under consideration in this article. Instead to trying to extend the result directly, or to generalize the argument of problem 23(a) (based on the first lemma of §B.5 in Chapter VIII which was proved there under quite restrictive conditions), we will undertake a new derivation using different ideas.

The machinery employed for this purpose consists of the L_2 theory of Hilbert transforms, sketched in the scholium at the end of §C.1, Chapter VIII. The reader may have already noticed a connection between Hilbert transforms and logarithmic (and Green) potentials, appearing, for instance, in the first lemma of §C.3, Chapter VIII, and in problem 29(b) (Chapter IX, §B.1).

As usual, we write

$$\rho(x) \quad = \quad \int_0^x \mathrm{d}\rho(t) \qquad \text{for } x \geqslant 0$$

when working with real signed measures ρ on $[0, \infty)$. It will also be convenient to extend the definition of such functions ρ to *all of* $\mathbb{R}$ *by making them even* (*sic*!) there.

Lemma. *Let*

$$\int_0^{\infty}\int_0^{\infty} \log\left|\frac{x+t}{x-t}\right| |\mathrm{d}\rho(t)||\mathrm{d}\rho(x)| \quad < \quad \infty$$

for the real signed measure ρ on $[0, \infty)$ without point mass at the origin. Then $\rho(x)$ is $\mathrm{O}(\sqrt{x})$ for $x \longrightarrow \infty$.

Remark. This is a weak result.

Proof of lemma. Since

$$|\rho(x)| \quad \leqslant \quad \int_0^x |\mathrm{d}\rho(t)| \qquad \text{for } x > 0,$$

it is just as well to assume to begin with that $\mathrm{d}\rho(t)$ is *positive*, and thus $\rho(x)$ *increasing* on $[0, \infty)$.

Then, by the second lemma of §B.5, Chapter VIII,

$$\int_0^\infty \int_0^\infty \left(\frac{\rho(x)-\rho(y)}{x-y}\right)^2 \frac{x^2+y^2}{(x+y)^2}\,\mathrm{d}x\,\mathrm{d}y \quad = \quad E(\mathrm{d}\rho(t),\ \mathrm{d}\rho(t)) \quad < \quad \infty,$$

so

$$\int_0^\infty \left(\frac{\rho(x)-\rho(y)}{x-y}\right)^2 \mathrm{d}x \quad < \quad \infty$$

for almost all $y > 0$. Fix such a y; we get

$$\int_{2y}^\infty \left(\frac{\rho(x)}{x}\right)^2 \mathrm{d}x \quad \leqslant \quad 2\int_{2y}^\infty \left(\frac{\rho(x)-\rho(y)}{x-y}\right)^2 \mathrm{d}x \ + \ 2\frac{(\rho(y))^2}{y} \quad < \quad \infty,$$

and thence, for $x > 2y$,

$$(\rho(x))^2 \int_x^\infty \frac{\mathrm{d}t}{t^2} \quad \leqslant \quad \int_{2y}^\infty \left(\frac{\rho(t)}{t}\right)^2 \mathrm{d}t \quad < \quad \infty,$$

ρ being increasing. Thus,

$$(\rho(x))^2 \quad \leqslant \quad \text{const.}\, x \qquad \text{for } x > 2y.$$

Done.

Lemma. *Let ρ satisfy the hypothesis of the preceding lemma. Then*

$$U_\rho(x) \quad = \quad -\!\!\!\!\!\!\!\int_0^\infty \left(\frac{1}{x-t} + \frac{1}{x+t}\right)\rho(t)\,\mathrm{d}t \qquad \text{a.e., } x \in \mathbb{R}.$$

Proof. $U_\rho(x)$ is odd, so it is enough to establish the formula for almost all $x > 0$. Taking any such x for which the integral defining $U_\rho(x)$ converges absolutely, we have

$$U_\rho(x) \quad = \quad \lim_{\varepsilon \to 0}\left(\int_0^{x-\varepsilon} + \int_{x+\varepsilon}^\infty\right)\log\left|\frac{x+t}{x-t}\right|\mathrm{d}\rho(t).$$

Fixing for the moment a small $\varepsilon > 0$, we treat the two integrals on the right by partial integration, very much as in the proof of the lemma in §C.3, Chapter VIII (but going in the opposite direction). The integrated term

$$\rho(t)\,\log\left|\frac{x+t}{x-t}\right|$$

which thus arises vanishes at $t = 0$ and also when $t \longrightarrow \infty$, the latter thanks to the preceding lemma. Subtraction of its values for $t \ = \ x \pm \varepsilon$ also gives a small result when $\varepsilon > 0$ is small, *as long as $\rho'(x)$ exists and is finite*, and therefore for almost all x. We thus end with the desired formula on making $\varepsilon \longrightarrow 0$, Q.E.D.

Referring to our convention that $\rho(-t) = \rho(t)$, we immediately obtain the

Corollary

$$U_\rho(x) \quad = \quad -\!\!\!\!\!\!\int_{-\infty}^{\infty} \left(\frac{1}{x-t} + \frac{t}{t^2+1} \right) \rho(t)\,\mathrm{d}t \qquad \text{a.e.,} \quad x \in \mathbb{R}.$$

Remark. The dummy term $t/(t^2+1)$ is introduced in the integrand in order to guarantee absolute convergence of the integral near $\pm\infty$, and does so, $\rho(t)$ being $\mathrm{O}(\sqrt{|t|})$ there. We see that, aside from a missing factor of $-1/\pi$, $U_\rho(x)$ is just the harmonic conjugate (Hilbert transform) of $\rho(x)$, which should be very familiar to anyone who has read up to here in the present book.

Lemma. *Let the signed measure ρ satisfy the hypothesis of the first of the preceding two lemmas. Then, for almost every real y, the function of x equal to $(\rho(x)-\rho(y))/(x-y)$ belongs to $L_2(-\infty,\infty)$, and*

$$\frac{U_\rho(x) - U_\rho(y)}{x-y} \quad = \quad -\!\!\!\!\!\!\int_{-\infty}^{\infty} \frac{1}{x-t}\,\frac{\rho(t)-\rho(y)}{t-y}\,\mathrm{d}t \qquad \text{a.e.,} \quad x \in \mathbb{R}.$$

Proof. As at the beginning of the proof of the first of the above two lemmas,

$$\int_0^\infty \int_0^\infty \left(\frac{\rho(x)-\rho(y)}{x-y} \right)^2 \frac{x^2+y^2}{(x+y)^2}\,\mathrm{d}x\,\mathrm{d}y \quad < \quad \infty,$$

so, for almost every $y > 0$, $(\rho(x)-\rho(y))/(x-y)$ belongs to $L_2(0,\infty)$ as a function of x. But since ρ is even,

$$\left| \frac{\rho(x) \; - \; \rho(y)}{x \; - \; y} \right| \quad \leqslant \quad \left| \frac{\rho(|x|) \; - \; \rho(|y|)}{|x| \; - \; |y|} \right|;$$

we thus see by the statement just made that as a function of x, $(\rho(x)-\rho(y))/(x-y)$ belongs in fact to $L_2(-\infty,\infty)$ for almost all $y \in \mathbb{R}$.

For any real number C, we have (trick!):

$$-\!\!\!\!\!\!\int_{-\infty}^{\infty} \left(\frac{1}{x-t} + \frac{t}{t^2+1} \right) C\,\mathrm{d}t \quad = \quad 0.$$

Adding this relation to the formula given by the last corollary, we thus get

$$U_\rho(x) \quad = \quad -\!\!\!\!\!\!\int_{-\infty}^{\infty} \left(\frac{1}{x-t} + \frac{t}{t^2+1}\right)(\rho(t) - C)\,\mathrm{d}t$$

for almost all $x \in \mathbb{R}$, where the exceptional set *does not depend* on the number C. From this relation we subtract the similar one obtained on replacing x by any other value y for which it holds. That yields

$$U_\rho(x) - U_\rho(y) \quad = \quad -\!\!\!\!\!\!\int_{-\infty}^{\infty} \left(\frac{1}{x-t} - \frac{1}{y-t}\right)(\rho(t) - C)\,\mathrm{d}t \quad \text{a.e., } x, y \in \mathbb{R}.$$

In the Cauchy principal value standing on the right, the integrand involves *two* singularities, at $t = x$ and at $t = y$. Consider, however, what happens when y takes one of the values for which $\rho'(y)$ *exists and is finite. Then, we can put* $C = \rho(y)$ *in the preceding relation* (!), and, after dividing by $x - y$, it becomes

$$\frac{U_\rho(x) - U_\rho(y)}{x-y} \quad = \quad -\!\!\!\!\!\!\int_{-\infty}^{\infty} \frac{1}{x-t}\,\frac{\rho(t)-\rho(y)}{t-y}\,\mathrm{d}t,$$

in which the function $(\rho(t) - \rho(y))/(t-y)$ figuring on the right *remains bounded* for $t \longrightarrow y$. What we have on the right is thus just the *ordinary* Cauchy principal value involving an integrand with *one* singularity (at $t = x$), used in the study of Hilbert transforms.

We are, however, assured of the existence and finiteness of $\rho'(y)$ at almost every y. The last relation thus holds a.e. in both x and y, and we are done.

Theorem. *Let the real signed measure ρ on $[0, \infty)$, without point mass at the origin, be such that*

$$\int_0^\infty \log\left|\frac{x+t}{x-t}\right| |\mathrm{d}\rho(t)||\mathrm{d}\rho(x)| \quad < \quad \infty,$$

and put

$$U_\rho(x) \quad = \quad \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t),$$

thus specifying the value of U_ρ almost everywhere on $\mathbb{R}$. Then we have Jesse Douglas' formula:

$$\boxed{E(\mathrm{d}\rho(t), \mathrm{d}\rho(t)) \quad = \quad \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{U_\rho(x) - U_\rho(y)}{x-y}\right)^2 \mathrm{d}x\,\mathrm{d}y.}$$

Proof. The last lemma exhibits the function of x equal to $(U_\rho(x) - U_\rho(y))/(x-y)$ as $-\pi$ times the *Hilbert transform* of the one equal to $(\rho(x)-\rho(y))/(x-y)$ for almost every $y \in \mathbb{R}$, and also tells us that the latter function of x is in $L_2(-\infty,\infty)$ for almost every such y. We may therefore apply to these functions *the L_2 theory of Hilbert transforms* taken up in the scholium at the end of §C.1, Chapter VIII. By that theory,

$$\int_{-\infty}^{\infty}\left(\frac{U_\rho(x)-U_\rho(y)}{\pi(x-y)}\right)^2 \mathrm{d}x \quad = \quad \int_{-\infty}^{\infty}\left(\frac{\rho(x)-\rho(y)}{x-y}\right)^2 \mathrm{d}x$$

for the values of y in question, i.e., almost everywhere in y.

Integrating now with respect to y, this gives

$$\frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{U_\rho(x)-U_\rho(y)}{x-y}\right)^2 \mathrm{d}x\,\mathrm{d}y \quad = \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\rho(x)-\rho(y)}{x-y}\right)^2 \mathrm{d}x\,\mathrm{d}y.$$

Because ρ is even, the right side is just

$$2\int_0^{\infty}\int_0^{\infty}\left\{\left(\frac{\rho(x)-\rho(y)}{x-y}\right)^2 + \left(\frac{\rho(x)-\rho(y)}{x+y}\right)^2\right\}\mathrm{d}x\,\mathrm{d}y$$

$$= \quad 4\int_0^{\infty}\int_0^{\infty}\left(\frac{\rho(x)-\rho(y)}{x-y}\right)^2 \frac{x^2+y^2}{(x+y)^2}\,\mathrm{d}x\,\mathrm{d}y.$$

Dividing by 4 and referring to the second lemma of §B.5, Chapter VIII, we immediately obtain the desired result.

Corollary. *For any measure ρ satisfying the hypothesis of the theorem,*

$$E(\mathrm{d}\rho(t),\ \mathrm{d}\rho(t))$$

is determined when the Green potential $U_\rho(x)$ is specified almost everywhere on $\mathbb{R}$, and $\rho = 0$ if $U_\rho(x) = 0$ a.e. in $(0, \infty)$. (Here $U_\rho(x)$ is determined by its values on $(0, \infty)$ because it is odd.)

This corollary finally gives us the right to denote $\sqrt{(E(\mathrm{d}\rho(t),\ \mathrm{d}\rho(t)))}$ by $\|U_\rho\|_E$ for *any* measure ρ fulfilling the conditions of the theorem; indeed, we simply have

$$\|U_\rho\|_E \quad = \quad \frac{1}{2\pi}\sqrt{\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{U_\rho(x)-U_\rho(y)}{x-y}\right)^2 \mathrm{d}x\,\mathrm{d}y\right)}.$$

It will be convenient for us to use this formula for *arbitrary* real-valued Lebesgue measurable functions (*odd or not* !) defined on $\mathbb{R}$. Then, of course, it becomes a matter of

Notation. Given v, real-valued and Lebesgue measurable on $\mathbb{R}$, we write

$$\|v\|_E \quad = \quad \frac{1}{2\pi}\sqrt{\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{v(x)-v(y)}{x-y}\right)^2 \mathrm{d}x\,\mathrm{d}y\right)}.$$

This clearly defines $\|\ \ \|_E$ as a *norm* on the collection of such functions v (modulo the constants); if $\|v\|_E = 0$ we must have

$$v(x) \quad = \quad \text{const.} \qquad \text{a.e.,}\ x \in \mathbb{R}.$$

Near the beginning of this article, we said how the Hilbert space $\mathfrak{H}$ was to be formed: $\mathfrak{H}$ was specified as *the abstract completion in norm* $\|\ \ \|_E$ *of the collection of Green potentials* U_ρ *coming from the measures* ρ *satisfying the conditions of the last theorem.* An *element* of $\mathfrak{H}$ is, in other words, defined by a *Cauchy sequence*, $\{U_{\rho_n}\}$, of such potentials. According, however, to the corollary of the *first* theorem in this article, such a Cauchy sequence has in it a *subsequence*, which we may as well *also*, for the moment, denote by $\{U_{\rho_n}\}$, with $U_{\rho_n}(x)$ *pointwise convergent* at almost every $x \in \mathbb{R}$. Writing

$$\lim_{n\to\infty} U_{\rho_n}(x) \quad = \quad U(x)$$

wherever the limit exists, we see that $U(x)$ is *defined* a.e. and *Lebesgue measurable*; it is also *odd*, because the individual Green potentials $U_{\rho_n}(x)$ are odd.

Fixing any index m, we have, making the usual application of Fatou's lemma,

$$\|U \; - \; U_{\rho_m}\|_E^2 \quad = \quad \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{U(x)-U_{\rho_m}(x)-U(y)+U_{\rho_m}(y)}{x \; - \; y}\right)^2 \mathrm{d}x\,\mathrm{d}y$$

$$\leqslant \quad \liminf_{j\to\infty} \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{U_{\rho_j}(x)-U_{\rho_m}(x)-U_{\rho_j}(y)+U_{\rho_m}(y)}{x \; - \; y}\right)^2 \mathrm{d}x\,\mathrm{d}y$$

$$= \quad \liminf_{j\to\infty} \|U_{\rho_j} \; - \; U_{\rho_m}\|_E^2.$$

Since we started with a Cauchy sequence, the last quantity is *small* if m is large. This in fact holds for all the U_{ρ_m} from our *original* sequence, for the last chain of inequalities is valid for any of those potentials as long as the U_{ρ_j} appearing therein *run through the subsequence* just described. *Corresponding to the Cauchy sequence* $\{U_{\rho_n}\}$, *we have thus found an odd* *measurable function U with*

$$\|U - U_{\rho_n}\|_E \xrightarrow[n]{} 0.$$

In this fashion we can associate *an odd measurable function U, approximable in the norm $\| \quad \|_E$ by Green potentials U_ρ like the ones appearing in the last theorem, to each element of the space $\mathfrak{H}$*. It is, on the other hand, manifest that *each such function U does indeed correspond to some element of $\mathfrak{H}$* – the Green potentials U_ρ approximating U in norm $\| \quad \|_E$ *furnish us* with a Cauchy sequence of such potentials (in that norm)! There is thus a correspondence between *the collection of such functions U* and *the space $\mathfrak{H}$*.

It is necessary now to show that *this correspondence is one-one*. But that is easy. Suppose, in the first place that *two different odd functions*, say U and V, are associated to *the same* element of the space $\mathfrak{H}$ in the manner described. Then we have *two* Cauchy sequences of Green potentials, say $\{U_{\rho_n}\}$ and $\{U_{\sigma_n}\}$, with

$$\|U_{\rho_n} - U_{\sigma_n}\|_E \xrightarrow[n]{} 0,$$

and such that

$$\|U - U_{\rho_n}\|_E \xrightarrow[n]{} 0$$

while

$$\|V - U_{\sigma_n}\|_E \xrightarrow[n]{} 0.$$

It follows that

$$\|U - V\|_E = 0,$$

but then, as noted above,

$$U(x) - V(x) = \text{const.} \quad \text{a.e., } x \in \mathbb{R}.$$

Here, $U - V$ is *odd*, so the *constant* must be *zero*, and

$$U(x) = V(x) \quad \text{a.e., } x \in \mathbb{R}.$$

Given, on the other hand, *two* Cauchy sequences, $\{U_{\rho_n}\}$ and $\{U_{\sigma_n}\}$, of potentials associated to *the same odd function U*, we have

$$\|U - U_{\rho_n}\|_E \xrightarrow[n]{} 0$$

and

$$\|U - U_{\sigma_n}\|_E \xrightarrow[n]{} 0,$$

whence

$$\| U_{\rho_n} - U_{\sigma_n} \|_E \xrightarrow[n]{} 0.$$

Then, however, $\{U_{\rho_n}\}$ and $\{U_{\sigma_n}\}$ define *the same element* of the abstract completion $\mathfrak{H}$.

Our Hilbert space $\mathfrak{H}$ is thus in one-to-one correspondence with the collection of odd real measurable functions U approximable, in norm $\| \ \|_E$, by the potentials U_ρ under consideration here. There is hence nothing to keep us from *identifying the space $\mathfrak{H}$ with that collection of functions U, and we henceforth do so.*

We are now well enough equipped to give a strengthened version, promised earlier, of the corollary to the first theorem in this article.

Lemma. *Let the odd measurable function U be identified with an element of the space $\mathfrak{H}$ in the manner just described, and suppose that ρ is an* absolutely continuous *signed measure on* $[0, \infty)$ *with*

$$\int_0^\infty \int_0^\infty \log \left| \frac{x+t}{x-t} \right| d\rho(t)\, d\rho(x)$$

and

$$\int_0^\infty U(x)\, d\rho(x)$$

both absolutely convergent. Then

$$\int_0^\infty U(x) d\rho(x) = \langle U, U_\rho \rangle_E,$$

and especially

$$\boxed{\left| \int_0^\infty U(x)\, d\rho(x) \right| \leqslant \| U \|_E \sqrt{(E(d\rho(t), d\rho(t)))}.}$$

Remark. The second relation is very useful in certain applications.

Proof of lemma. We proceed to establish the first relation, using a somewhat repetitious crank-turning argument.

Starting with an absolutely continuous ρ fulfilling the conditions in the hypothesis, let us put, for $N \geqslant 1$,

$$\mathrm{d}\rho_N(t) \quad = \quad \begin{cases} \rho'(t)\,\mathrm{d}t & \text{if } |\rho'(t)| \leqslant N, \\ (N \operatorname{sgn} \rho'(t))\,\mathrm{d}t & \text{otherwise;} \end{cases}$$

it is claimed that

$$\| U_\rho \; - \; U_{\rho_N} \|_E \;\longrightarrow\; 0$$

for $N \longrightarrow \infty$.

By breaking $\rho'(t)$ up into positive and negative parts, we can reduce the general situation to one in which

$$\rho'(t) \;\geqslant\; 0,$$

so we may as well assume this property. Then, for each $x > 0$, the potentials

$$U_{\rho_N}(x) \quad = \quad \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \min(\rho'(t),\; N)\,\mathrm{d}t$$

increase and tend to $U_\rho(x)$ *as* $N \longrightarrow \infty$. Hence, by monotone convergence,

$$\int_0^\infty U_{\rho_N}(x)\,\mathrm{d}\rho(x) \;\xrightarrow[N]{}\; \int_0^\infty U_\rho(x)\,\mathrm{d}\rho(x).$$

From this, we see that

$$\| U_\rho \; - \; U_{\rho_N} \|_E^2 \quad = \quad E(\mathrm{d}\rho(t) - \mathrm{d}\rho_N(t),\; \mathrm{d}\rho(t) - \mathrm{d}\rho_N(t))$$

$$= \int_0^\infty (U_\rho(x) \; - \; U_{\rho_N}(x))(\rho'(x) \; - \; \min(\rho'(x), N))\,\mathrm{d}x$$

$$\leqslant \int_0^\infty (U_\rho(x) \; - \; U_{\rho_N}(x))\,\mathrm{d}\rho(x) \quad (!)$$

must *tend to zero* as $N \longrightarrow \infty$, verifying our assertion.

From what we have just shown, it follows that

$$\langle U,\; U_{\rho_N} \rangle_E \;\xrightarrow[N]{}\; \langle U,\; U_\rho \rangle_E.$$

But we clearly have

$$\int_0^\infty U(x)\,\mathrm{d}\rho_N(x) \;\xrightarrow[N]{}\; \int_0^\infty U(x)\,\mathrm{d}\rho(x)$$

by the given absolute convergence of the integral on the right. The desired first relation *will therefore follow* if we can prove that

$$\int_0^\infty U(x)\,\mathrm{d}\rho_N(x) \quad = \quad \langle U,\; U_{\rho_N} \rangle_E$$

for each N; that, however, simply amounts to *verifying the relation* in question for *measures* ρ satisfying the hypothesis and *having, in addition, bounded densities* $\rho'(x)$. We have thus brought down by one notch the generality of what is to be proven.

Suppose, then, that ρ satisfies the hypothesis and that $\rho'(x)$ *is also bounded*. For each a, $0 < a < 1$, put

$$\rho_a'(t) = \begin{cases} \rho'(t), & a \leqslant t \leqslant \dfrac{1}{a}, \\ 0 & \text{otherwise}, \end{cases}$$

and then define a measure ρ_a (*not to be confounded with the* ρ_N *just used!*) by taking $d\rho_a(t) = \rho_a'(t)\,dt$. An argument very similar to the one made above now shows that

$$\| U_\rho - U_{\rho_a} \|_E \longrightarrow 0 \quad \text{for } a \longrightarrow 0$$

(it suffices as before to consider the case where $\rho'(t) \geqslant 0$) and hence that

$$\langle U, U_{\rho_a} \rangle_E \longrightarrow \langle U, U_\rho \rangle_E \quad \text{as } a \longrightarrow 0.$$

At the same time,

$$\int_0^\infty U(x)\,d\rho_a(x) \longrightarrow \int_0^\infty U(x)\,d\rho(x),$$

so it is enough to check that

$$\int_0^\infty U(x)\,d\rho_a(x) = \langle U, U_{\rho_a} \rangle_E$$

for each a.

Here, however, $d\rho_a(x) = \rho_a'(x)\,dx$ *with* $\rho_a'(x)$ *bounded and of compact support in* $(0, \infty)$, so the *corollary of the first theorem* in this article is applicable. Since U is identified with an element of $\mathfrak{H}$ there is, by the discussion preceding this lemma, a sequence of Green potentials U_{σ_n} of the kind used to form that space such that

$$\| U - U_{\sigma_n} \|_E \xrightarrow[n]{} 0,$$

and also

$$U_{\sigma_n}(x) \xrightarrow[n]{} U(x) \quad \text{a.e., } x \in \mathbb{R}.$$

Then we must have

$$E(\mathrm{d}\sigma_n(t),\ \mathrm{d}\rho_a(t)) \quad = \quad \langle U_{\sigma_n},\ U_{\rho_a}\rangle_E \quad \xrightarrow[n]{} \quad \langle U,\ U_{\rho_a}\rangle_E,$$

and, by the corollary referred to,

$$E(\mathrm{d}\sigma_n(t),\ \mathrm{d}\rho_a(t)) \quad \xrightarrow[n]{} \quad \int_0^\infty U(x)\,\mathrm{d}\rho_a(x).$$

Thus,

$$\langle U,\ U_{\rho_a}\rangle_E \quad = \quad \int_0^\infty U(x)\,\mathrm{d}\rho_a(x),$$

what we needed to finish showing the first relation in the conclusion of the lemma.

From it, however, the *second* relation follows immediately by Schwarz' inequality and the preceding theorem, since

$$\|U_\rho\|_E \quad = \quad \sqrt{(E(\mathrm{d}\rho(t),\ \mathrm{d}\rho(t)))}.$$

Our lemma is proved.

What we have done so far still does not amount to a real *description* of the space $\mathfrak{H}$, for we do not yet know *which* odd measurable functions $U(x)$ with $\|U\|_E < \infty$ can be approximated in the norm $\|\ \ \|_E$ by potentials U_ρ of the kind appearing in the last theorem. The fact is that *all those functions U can be thus approximated.*

Theorem. *$\mathfrak{H}$ consists precisely of the real odd measurable functions $U(x)$ for which*

$$\|U\|_E^2 \quad = \quad \frac{1}{4\pi^2}\int_{-\infty}^\infty\int_{-\infty}^\infty\left(\frac{U(x)-U(y)}{x-y}\right)^2\mathrm{d}x\,\mathrm{d}y$$

is finite.

Proof. That $\mathfrak{H}$ *consists* of such functions U was shown in the course of the previous discussion; what we have to do here is prove the *converse*, to the effect that *any* odd function U with $\|U\|_E < \infty$ *is* in $\mathfrak{H}$. This involves an approximation argument.

Starting, then, with an odd function U such that $\|U\|_E < \infty$, we must *obtain* signed measures ρ on $[0, \infty)$ meeting the conditions of the last theorem, *for which*

$$\|U \ - \ U_\rho\|_E$$

is as small as we please. That will be done in essentially *three steps.*

The first step makes use of the notion of *contraction*, brought into potential theory by Beurling and Deny. For $M > 0$, put

$$U_M(x) = \begin{cases} U(x) & \text{if } |U(x)| < M, \\ M \operatorname{sgn} U(x) & \text{if } |U(x)| \geqslant M. \end{cases}$$

Then

$$|U_M(x) - U_M(y)| \leqslant |U(x) - U(y)|$$

(this is the contraction property), so

$$\left(\frac{U(x) - U_M(x) - U(y) + U_M(y)}{x - y}\right)^2 \leqslant 4\left(\frac{U(x) - U(y)}{x - y}\right)^2.$$

But $U_M(x) \longrightarrow U(x)$ as $M \longrightarrow \infty$, so the *left side* of this relation *tends to zero* a.e. in x and y (with respect to Lebesgue measure for $\mathbb{R}^2$) as $M \longrightarrow \infty$. (The *odd* function $U(x)$ *cannot be infinite* on a set of *positive measure*, for in that event U would be infinite on such a subset of $(-\infty, 0]$ or of $[0, \infty)$, and this would clearly make $\|U\|_E = \infty$.) Because $\|U\|_E < \infty$, the double integral of the *right side* of the relation over $\mathbb{R}^2$ is *finite*. Hence

$$\|U - U_M\|_E^2 \longrightarrow 0 \quad \text{for } M \longrightarrow \infty$$

by dominated convergence. *Any function U satisfying the hypothesis is thus $\| \ \|_E$ - approximable by bounded ones.*

It therefore suffices to show how to do the desired approximation of *bounded* functions U fulfilling the conditions of this theorem. Taking such a one, for which

$$|U(x)| \leqslant M, \quad \text{say,}$$

on the real axis, we look at the products

$$U_H(x) = \frac{H^2}{H^2 + x^2} U(x)$$

where H is *large*. (*These should not be confounded with the contractions U_M used in the previous step.*) Denoting by $v_H(x)$ the (even!) function

$$\frac{H^2}{H^2 + x^2},$$

we have

$$U_H(x) - U_H(y) = v_H(x)(U(x) - U(y)) + U(y)(v_H(x) - v_H(y)),$$

so that

$$\left|\frac{U_H(x)-U_H(y)}{x-y}\right| \quad \leqslant \quad \left|\frac{U(x)-U(y)}{x-y}\right| \quad + \quad M\left|\frac{v_H(x)-v_H(y)}{x-y}\right|.$$

From this last, it is easily seen that

$$\|U_H\|_E^2 \quad \leqslant \quad 2\|U\|_E^2 \;+\; 2M^2\|v_H\|_E^2,$$

and we must compute the norms $\|v_H\|_E$.

To do that, we note that $v_H(x)$ has a harmonic extension to the upper half plane given by

$$v_H(z) \quad = \quad \frac{H(H+y)}{(H+y)^2+x^2} \quad = \quad -\Im\left(\frac{H}{z+iH}\right),$$

and that this function is sufficiently well behaved for the identities employed in the solution of problem 23(a) (§B.8, Chapter VIII) to apply to it. By the help of those, one finds that

$$\|v_H\|_E^2 \quad = \quad \frac{1}{2\pi}\int_0^\infty\int_{-\infty}^\infty\left\{\left(\frac{\partial v_H(z)}{\partial x}\right)^2 \;+\; \left(\frac{\partial v_H(z)}{\partial y}\right)^2\right\}dx\,dy.$$

Using the Cauchy–Riemann equations, we convert the integral on the right to

$$\frac{H^2}{2\pi}\int_0^\infty\int_{-\infty}^\infty\left|\frac{d}{dz}\left(\frac{1}{z+iH}\right)\right|^2 dx\,dy \quad = \quad \frac{H^2}{2\pi}\int_0^\infty\int_{-\infty}^\infty\frac{1}{|z+iH|^4}\,dx\,dy,$$

which becomes

$$\frac{1}{2\pi}\int_0^\infty\int_{-\infty}^\infty\frac{d\xi\,d\eta}{(\xi^2+(\eta+1)^2)^2}$$

after putting $z = H(\xi + i\eta)$. The further substitution $\xi = (\eta+1)\tau$ converts the last double integral to

$$\frac{1}{2\pi}\int_0^\infty\frac{d\eta}{(\eta+1)^3}\int_{-\infty}^\infty\frac{d\tau}{(\tau^2+1)^4} \quad = \quad \frac{1}{8},$$

so we have

$$\|v_H\|_E \quad = \quad \frac{1}{2\sqrt{2}}$$

independently of the number H.

Plugging this into the previous inequality, we get

$$\|U_H\|_E^2 \quad \leqslant \quad 2\|U\|_E^2 \;+\; \tfrac{1}{4}M^2,$$

so the *norms on the left remain bounded as* $H \longrightarrow \infty$. From this it follows that *the functions* U_H *corresponding to some suitable sequence of values of* H *going out to infinity tend weakly* (in the Hilbert space of *all* real *odd* functions with finite $\| \ \|_E$ norm!) *to some odd function* W *with* $\|W\|_E < \infty$. That in turn implies that *some sequence of* (finite) *convex linear combinations* u_n *of those* U_H (formed by using *ever larger values* of H from the sequence just mentioned) *actually tends in norm* $\| \ \|_E$ *to* W. It is clear, however, that

$$U_H(x) \longrightarrow U(x)$$

(wherever the value on the right is defined) as $H \longrightarrow \infty$. Hence

$$u_n(x) \xrightarrow[n]{} U(x) \quad \text{a.e.}, \ x \in \mathbb{R},$$

so, since

$$\|W - u_n\|_E \xrightarrow[n]{} 0,$$

we see by the usual application of Fatou's lemma that

$$\|W - U\|_E = 0,$$

making (as in the previous discussion about the construction of $\mathfrak{H}$)

$$W(x) = U(x) \quad \text{a.e.}, \ x \in \mathbb{R},$$

since W and U are *both odd*. Therefore, we in fact have

$$\|U - u_n\|_E \xrightarrow[n]{} 0,$$

and *we can approximate the function* U *in norm* $\| \ \|_E$ *by finite linear combinations* u_n *of the functions*

$$\frac{H^2}{H^2 + x^2} U(x).$$

Since the function $U(x)$ is *bounded*, we see that each function u_n, *besides being odd* (like U), *satisfies a condition of the form*

$$|u_n(x)| \leqslant \frac{K_n}{x^2 + 1}, \qquad x \in \mathbb{R}$$

(where, of course, the constant K_n may be enormous, but we don't care about that).

For this reason it is enough if, using the potentials U_ρ, one can approximate in norm $\| \quad \|_E$ any odd function u with $\|u\|_E < \infty$ and, in addition,

$$|u(x)| \leqslant \frac{K}{x^2+1} \qquad \text{on } \mathbb{R}.$$

Showing how to do that is the last step in our proof.

Take any such u. For $\Im z > 0$, put

$$u(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\, u(t)\,\mathrm{d}t.$$

The *size* of $|u(t)|$ on $\mathbb{R}$ is here well enough controlled so that the Fourier integral argument used in working problem 23(a) may be applied to $u(z)$. In that way, one readily verifies that

$$\|u\|_E^2 = \frac{1}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty} \{(u_x(z))^2 + (u_y(z))^2\}\,\mathrm{d}x\,\mathrm{d}y;$$

this, in particular, makes the Dirichlet integral appearing on the right *finite*.

For $h > 0$ (which in a moment will be made to tend to zero) we now put

$$u_h(z) = u(z+\mathrm{i}h), \qquad \Im z \geqslant 0.$$

An evident adaptation of the argument just referred to then shows that

$$\|u - u_h\|_E^2 = \frac{1}{2\pi}\iint_{\Im z>0} \left\{\left(\frac{\partial}{\partial x}(u(z)-u_h(z))\right)^2 + \left(\frac{\partial}{\partial y}(u(z)-u_h(z))\right)^2\right\}\mathrm{d}x\,\mathrm{d}y.$$

The right-hand integral is just

$$\frac{1}{2\pi}\iint_{\Im z>0} \{(u_x(z)-u_x(z+\mathrm{i}h))^2 + (u_y(z)-u_y(z+\mathrm{i}h))^2\}\,\mathrm{d}x\,\mathrm{d}y,$$

and we see that *it must tend to zero when* $h \longrightarrow 0$ (by continuity of translation in $L_2(\mathbb{R}^2)$!), since

$$\iint_{\Im z>0} (u_x(z))^2\,\mathrm{d}x\,\mathrm{d}y \qquad \text{and} \qquad \iint_{\Im z>0} (u_y(z))^2\,\mathrm{d}x\,\mathrm{d}y$$

are *both finite*, according to the observation just made. Thus,

$$\|u - u_h\|_E \longrightarrow 0 \quad \text{as } h \longrightarrow 0$$

It is now claimed that *each function* $u_h(x)$ *is equal* (on $\mathbb{R}$) *to a potential* $U_\rho(x)$ of the required kind. Since $u(t)$ is odd, we have

$$u(x+iy) = u(-x+iy) \qquad \text{for } y>0,$$

so $u_h(x)$ is odd. Our condition on $u(x)$ implies a similar one,

$$|u_h(x)| \leqslant \frac{\text{const}}{x^2+1}, \qquad x \in \mathbb{R},$$

on u_h, so that function is (and by far!) in $L_2(-\infty, \infty)$, and we can apply to it the L_2 theory of Hilbert transforms from the scholium at the end of §C.1, Chapter VIII. In the present circumstances, $u_h(x) = u(x+ih)$ is $\mathscr{C}_\infty$ in x, so the Hilbert transform

$$\tilde{u}_h(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{u_h(t)}{x-t}\,dt = \frac{1}{\pi}\int_0^{\infty} \frac{u_h(x-\tau)-u_h(x+\tau)}{\tau}\,d\tau$$

is defined and continuous at each real x, the last integral on the right being absolutely convergent. From the Hilbert transform theory referred to (even a watered-down version of it will do here!) we thence get, by the inversion formula,

$$u_h(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\tilde{u}_h(t)}{x-t}\,dt \qquad \text{a.e., } x \in \mathbb{R}.$$

This relation, like the one preceding it, holds in fact at *each* $x \in \mathbb{R}$, for $\tilde{u}_h(x)$ is nothing but the value of a *harmonic conjugate to* $u(z)$ at $z = x+ih$ and is hence (like $u_h(x)$) $\mathscr{C}_\infty$ in x. Wishing to integrate the right-hand member by parts, we look at the behaviour of $\tilde{u}_h'(t)$.

By the Cauchy–Riemann equations,

$$\tilde{u}_h'(x) = \tilde{u}_x(x+ih) = -u_y(x+ih).$$

After differentiating the (Poisson) formula for $u(z)$ and then plugging in the given estimate on $|u(t)|$, we get (for small $h>0$)

$$|u_y(x+ih)| \leqslant \frac{\text{const.}}{h(x^2+1)},$$

so

$$|\tilde{u}_h'(t)| \leqslant \frac{\text{const.}}{t^2+1} \qquad \text{for } t \in \mathbb{R}.$$

As we have noted, $u_h(x)$ is *odd*. Its Hilbert transform $\tilde{u}_h(t)$ is therefore *even*, and the preceding formula for u_h can be written

$$u_h(x) \quad = \quad -\frac{1}{\pi}\int_0^\infty \left(\frac{1}{x-t} \; + \; \frac{1}{x+t}\right)\tilde{u}_h(t)\,\mathrm{d}t.$$

Here, we integrate by parts as in proving the lemma of §C.3, Chapter VIII and the third lemma of the present article. By the last inequality we actually have

$$\int_0^\infty |\tilde{u}_h'(t)|\,\mathrm{d}t \quad < \quad \infty,$$

so, $\tilde{u}_h(t)$ being $\mathscr{C}_\infty$, the partial integration readily yields the formula

$$u_h(x) \quad = \quad \frac{1}{\pi}\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \tilde{u}_h'(t)\,\mathrm{d}t,$$

valid for all real x.

This already exhibits $u_h(x)$ as a Green potential $U_\rho(x)$ with

$$\mathrm{d}\rho(t) \quad = \quad \frac{1}{\pi}\,\tilde{u}_h'(t)\,\mathrm{d}t,$$

and in order to complete this last step of the proof, it is only necessary to check that

$$\int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right| |\tilde{u}_h'(t)|\,|\tilde{u}_h'(x)|\,\mathrm{d}t\,\mathrm{d}x \quad < \quad \infty.$$

That, however, follows in straightforward fashion from the above estimate on $|\tilde{u}_h'(t)|$. Breaking up (for $x > 0$)

$$\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \frac{\mathrm{d}t}{1+t^2}$$

as $\int_0^{2x} + \int_{2x}^\infty$, we have

$$\int_0^{2x} \quad = \quad \int_0^2 \log\left|\frac{1+\tau}{1-\tau}\right| \frac{x\,\mathrm{d}\tau}{1+x^2\tau^2} \quad \leqslant \quad \frac{1}{2}\int_0^2 \log\left|\frac{1+\tau}{1-\tau}\right| \frac{\mathrm{d}\tau}{\tau},$$

a finite constant, whilst

$$\int_{2x}^\infty \quad \leqslant \quad \int_{2x}^\infty \mathrm{O}\left(\frac{x}{t}\right)\frac{\mathrm{d}t}{1+t^2} \quad = \quad \mathrm{O}\left(x\log\frac{1+x^2}{x^2}\right) \quad = \quad \mathrm{O}\left(\frac{1}{1+x}\right).$$

The double integral in question is thus

$$\leqslant \quad \text{const.}\int_0^\infty \left(1 \; + \; \frac{1}{1+x}\right)\frac{\mathrm{d}x}{1+x^2} \quad < \quad \infty,$$

showing that the measure ρ given by the above formula has the required property.

The three steps of our approximation have thus been carried out, and the theorem completely proved.

Remark. The Green potentials U_ρ furnished by this proof and approximating, in norm $\| \quad \|_E$, a given odd function $U(x)$ with $\|U\|_E < \infty$, are formed from signed measures ρ on $[0, \infty)$ having, in addition to the properties enumerated at the beginning of this article, the following special one:

each ρ is absolutely continuous, with $\mathscr{C}_\infty$ density satisfying a relation of the form

$$|\rho'(t)| \quad \leqslant \quad \frac{\text{const.}}{t^2+1}, \qquad t \geqslant 0.$$

The corresponding potentials $U_\rho(x)$ are also $\mathscr{C}_\infty$, and especially,

$$|U_\rho(x)| \quad \leqslant \quad \frac{\text{const.}}{x^2+1}, \qquad x \in \mathbb{R}.$$

This property will be used to advantage in the next article.

Scholium. Does the space $\mathfrak{H}$ actually *consist entirely* of Green potentials

$$U_\mu(x) \quad = \quad \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\mu(t)$$

formed from certain signed Borel measures μ on $(0, \infty)$, perhaps more general than the measures ρ considered in this article?

One easily proves that if the *positive* measures $\mathrm{d}\rho_n$ (without point mass at 0) form a Cauchy sequence in the norm $\sqrt{(E(\ , \ ))}$, *then* at least a *subsequence* of them (and in fact the original sequence) *does converge w* on every compact subinterval* $[a, 1/a]$ of $(0, \infty)$, thus yielding a *positive Borel measure* μ on $(0, \infty)$ (perhaps with $\mu((0,1)) = \infty$ as well as $\mu([1, \infty)) = \infty$). It is thence not hard to show that

$$U_\mu(x) \quad = \quad \lim_{n\to\infty} U_{\rho_n}(x) \qquad \text{a.e.,} \ x \in \mathbb{R},$$

and U_μ may hence be identified with the limit of the U_{ρ_n} in the space $\mathfrak{H}$.

Verification of the w^* convergence statement goes as follows: by the first lemma of this article, the integrals

$$\int_0^1 U_{\rho_n}(x)\,\mathrm{d}x$$

are surely bounded. However,

$$\int_0^1 U_{\rho_n}(x)\,\mathrm{d}x \quad = \quad \int_0^\infty \int_0^1 \log\left|\frac{x+t}{x-t}\right| \mathrm{d}x\,\mathrm{d}\rho_n(t)$$

with

$$\int_0^1 \log\left|\frac{x+t}{x-t}\right| \mathrm{d}x \quad \geqslant \quad 0 \qquad \text{for } t \geqslant 0$$

and clearly *bounded below by a number* > 0 *on any segment of the form* $\{a \leqslant t \leqslant 1/a\}$ with $a > 0$. Therefore, since $\mathrm{d}\rho_n(t) \geqslant 0$ for each n, the quantities

$$\rho_n([a,\ 1/a])$$

must stay bounded as $n \longrightarrow \infty$ for each $a > 0$. The existence of a subsequence of the ρ_n having the stipulated property now follows by the usual application of Helly's selection principle and the Cantor diagonal process.

As soon, however, as the measures $\mathrm{d}\rho_n$ *are allowed to be of variable sign*, the argument just made, and its conclusion as well, *cease to be valid*. There are thus plently of functions U in $\mathfrak{H}$ which are *not* of the form U_μ (unless one accepts to bring in certain *Schwartz distributions* μ). This fact is even familiar from physics: *lots* of functions $U(z)$, harmonic in the upper half plane and continuous up to the real axis, with $U(x)$ odd thereon, *cannot be obtained as logarithmic potentials of charge distributions on* $\mathbb{R}$, *even though they have finite Dirichlet integrals*

$$\iint_{\Im z > 0} \{(U_x(z))^2 \ + \ (U_y(z))^2\}\,\mathrm{d}x\,\mathrm{d}y.$$

Instead, physicists are obliged to resort to what they call a *double-layer distribution* on $\mathbb{R}$ (formed from 'dipoles'); mathematically, this simply amounts to using the Poisson representation

$$U(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\,U(t)\,\mathrm{d}t$$

in place of the formula

$$U(z) \quad = \quad -\frac{1}{\pi}\int_0^\infty \log\left|\frac{z+t}{z-t}\right| U_y(t+\mathrm{i}0)\,\mathrm{d}t,$$

which is not available unless $\partial U(z)/\partial y$ is sufficiently well behaved for $\Im z \longrightarrow 0$. ($U(x)$ may be *continuous* and the above Dirichlet integral *finite*, and yet the boundary value $U_y(x+\mathrm{i}0)$ *exist almost nowhere* on $\mathbb{R}$. This is most easily seen by first mapping the upper half plane conformally onto the unit disk and then working with lacunary Fourier series.)

Problem 61

Let $V(x)$ be *even and* > 0, with $\|V\|_E < \infty$. Given $U \in \mathfrak{H}$, define a function $U_V(x)$ by putting

$$U_V(x) \quad = \quad \begin{cases} U(x) & \text{if } |U(x)| < V(x), \\ V(x)\operatorname{sgn} U(x) & \text{if } |U(x)| \geqslant V(x); \end{cases}$$

the formation of U_V is illustrated in the following figure:

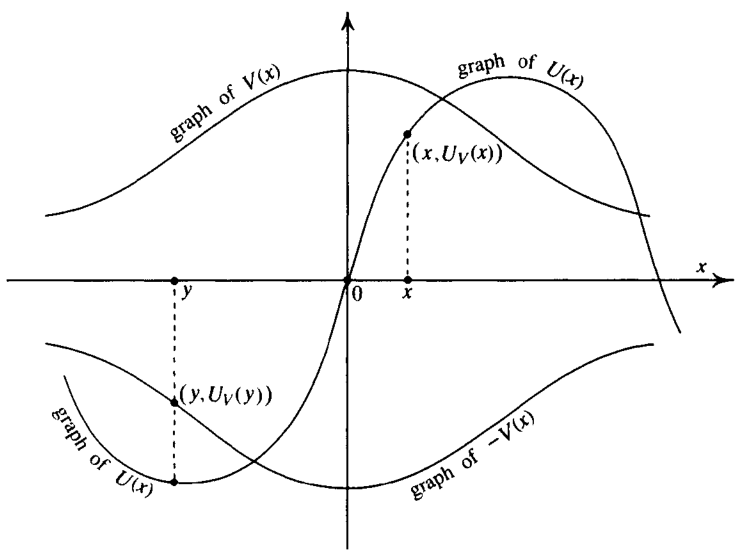

Figure 247

Show that $U_V \in \mathfrak{H}$.
(Hint: Show that

$$|U_V(x) \; - \; U_V(y)| \quad \leqslant \quad \max(|U(x) - U(y)|, \; |V(x) - V(y)|).$$

To check this, it is enough to look at six cases:

(i) $|U(x)| < V(x)$ and $|U(y)| < V(y)$

(ii) $U(x) \geqslant V(x)$ and $U(y) \geqslant V(y)$

(iii) $U(x) \geqslant V(x)$ and $U(y) \leqslant -V(y)$

(iv) $0 \leqslant U(x) < V(x)$ and $U(y) \leqslant -V(y)$

(v) $0 \leqslant U(x) < V(x)$, $V(y) \leqslant U(x)$ and $U(y) \geqslant V(y)$

(vi) $0 \leqslant U(x) < V(x)$, $V(y) > U(x)$ and $U(y) \geqslant V(y)$.)

Problem 62

(Beurling and Malliavin) Let $\omega(x)$ be *even*, $\geqslant 0$, and *uniformly* Lip 1, with

$$\int_0^\infty \frac{\omega(x)}{x^2}\,dx \quad < \quad \infty.$$

Show that then $\omega(x)/x$ belongs to $\mathfrak{H}$.
(Hint: The function $\omega(x)$ is certainly *continuous*, so, if the integral condition on it is to hold, we must have $\omega(0) = 0$. Thence, by the Lip 1 property, $\omega(x) \leqslant C|x|$, i.e.,

$$\left|\frac{\omega(x)}{x}\right| \leqslant C, \qquad x \in \mathbb{R}.$$

In the circumstances of this problem,

$$2\pi^2 \left\|\frac{\omega(x)}{x}\right\|_E^2 = \int_0^\infty \int_0^\infty \left(\frac{\dfrac{\omega(x)}{x} - \dfrac{\omega(y)}{y}}{x - y}\right)^2 dx\,dy$$

$$+ \int_0^\infty \int_0^\infty \left(\frac{\dfrac{\omega(x)}{x} + \dfrac{\omega(y)}{y}}{x + y}\right)^2 dx\,dy.$$

Using the inequality $(A+B)^2 \leqslant 2(A^2+B^2)$, the second double integral is immediately seen by symmetry to be

$$\leqslant 4\int_0^\infty \int_0^\infty \frac{1}{(x+y)^2}\left(\frac{\omega(x)}{x}\right)^2 dy\,dx \leqslant 4C\int_0^\infty \frac{\omega(x)}{x^2}\,dx.$$

The first double integral, by symmetry, is

$$2\int_0^\infty\int_y^\infty\left(\frac{\dfrac{\omega(x)}{x}-\dfrac{\omega(y)}{y}}{x-y}\right)^2 \mathrm{d}x\,\mathrm{d}y \;\leqslant\; 4\int_0^\infty\int_y^\infty \frac{1}{x^2}\left(\frac{\omega(x)-\omega(y)}{x-y}\right)^2 \mathrm{d}x\,\mathrm{d}y$$

$$+\quad 4\int_0^\infty\int_y^\infty\left(\frac{\dfrac{1}{x}-\dfrac{1}{y}}{x-y}\right)^2(\omega(y))^2\,\mathrm{d}x\,\mathrm{d}y.$$

The *second* of the expressions on the right boils down to

$$4\int_0^\infty\int_y^\infty\left(\frac{\omega(y)}{y}\right)^2\frac{\mathrm{d}x}{x^2}\mathrm{d}y$$

which is handled by reasoning already used; we are thus left with the *first* expression on the right.

That one we break up further as

$$4\int_0^\infty\int_y^{y+\omega(y)} \quad+\quad 4\int_0^\infty\int_{y+\omega(y)}^\infty\quad;$$

again, the first of these terms is readily estimated, and the second is

$$\leqslant\quad 8\int_0^\infty\int_{y+\omega(y)}^\infty\left\{\frac{(\omega(y))^2}{y^2}\,\frac{1}{(x-y)^2}\;+\;\frac{(\omega(x))^2}{x^2}\,\frac{1}{(x-y)^2}\right\}\mathrm{d}x\,\mathrm{d}y.$$

Integration of the first term in $\{\quad\}$ still does not give any problem, and we only need to deal with

$$8\int_0^\infty\int_{y+\omega(y)}^\infty \frac{(\omega(x))^2}{x^2}\,\frac{1}{(x-y)^2}\,\mathrm{d}x\,\mathrm{d}y.$$

By reversing the order of integration, show that this is

$$\leqslant\quad 8\int_0^\infty\frac{(\omega(x))^2}{x^2}\cdot\frac{1}{\omega(Y(x))}\,\mathrm{d}x,$$

where $Y(x)$ denotes the *largest value* of y for which $y + \omega(y) \leqslant x$. Then use the Lip 1 property of ω to get

$$|\omega(x) - \omega(Y(x))| \quad\leqslant\quad C\omega(Y(x)),$$

whence $1/\omega(Y(x)) \leqslant (C+1)/\omega(x)$. This relation is then substituted into the last integral.)

5. Even weights W with $\|\log W(x)/x\|_E < \infty$

Theorem (Beurling and Malliavin). *Let $W(x) \geqslant 1$ be continuous and even, with*

$$\int_0^\infty \frac{\log W(x)}{x^2}\,\mathrm{d}x \quad < \quad \infty,$$

and suppose that the odd function $\log W(x)/x$ belongs to the Hilbert space $\mathfrak{H}$ discussed in the preceding article. Then, given any $A > 0$, there is an increasing function $\nu(t)$, zero on a neighborhood of the origin, such that

$$\frac{\nu(t)}{t} \longrightarrow \frac{A}{\pi} \quad \text{as } t \longrightarrow \infty$$

and

$$\log W(x) \; + \; \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\nu(t) \quad \leqslant \quad \text{const.} \qquad \text{on } \mathbb{R}.$$

Proof. The argument, again based on the procedure explained in article 1, is much like those made in articles 2 and 3. For that reason, certain of its details may be omitted.

In order to have a weight going to ∞ for $x \longrightarrow \pm\infty$, we first take

$$W_\eta(x) \quad = \quad (1 + x^2)^\eta\, W(x)$$

with a small number $\eta > 0$. Then, given a value of $A > 0$, we form the function*

$$F(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^\infty \frac{|\Im z|}{|z - t|^2} \log W_\eta(t)\,\mathrm{d}t \quad - \quad A|\Im z|.$$

As before, the proof of our theorem boils down to showing that

$$(\mathfrak{M}F)(0) \quad < \quad \infty.$$

This, in turn, reduces to the *determination of an upper bound, independent of $\mathscr{D}$*, on

$$\int_{\partial\mathscr{D}} \log W_\eta(t)\,\mathrm{d}\omega_{\mathscr{D}}(t,\, 0) \quad - \quad A\,Y_{\mathscr{D}}(0),$$

where $\mathscr{D}$ is any domain of the kind studied in §C of Chapter VIII. We

* as usual, we put $F(x)$ equal to $\log W_\eta(x)$ on the real axis

see that what is needed is a comparison of

$$\int_{\partial\mathscr{D}} \log W_\eta(t)\,\mathrm{d}\omega_{\mathscr{D}}(t,\ 0)$$

with the quantity $Y_{\mathscr{D}}(0)$.

We have, in the first place, to take care of the factor $(1+x^2)^\eta$ used in forming $W_\eta(x)$. That is easy. Writing, for $t \geqslant 0$,

$$\Omega_{\mathscr{D}}(t) \quad = \quad \omega_{\mathscr{D}}(\partial\mathscr{D}\cap((-\infty, -t]\cup[t,\infty)),\ 0)$$

as in §C of Chapter VIII, we have

$$\int_{\partial\mathscr{D}} \eta\log(1+t^2)\,\mathrm{d}\omega_{\mathscr{D}}(t,\ 0) \quad = \quad -\eta\int_0^\infty \log(1+t^2)\,\mathrm{d}\Omega_{\mathscr{D}}(t)$$

$$= \quad \eta\int_0^\infty \frac{2t}{1+t^2}\Omega_{\mathscr{D}}(t)\,\mathrm{d}t.$$

By the fundamental result of §C.2, Chapter VIII,

$$\Omega_{\mathscr{D}}(t) \quad \leqslant \quad \frac{Y_{\mathscr{D}}(0)}{t},$$

so the last expression is

$$\leqslant \quad 2\eta Y_{\mathscr{D}}(0)\int_0^\infty \frac{\mathrm{d}t}{1+t^2} \quad = \quad \pi\eta Y_{\mathscr{D}}(0).$$

Thence,

$$\int_{\partial\mathscr{D}} \log W_\eta(t)\,\mathrm{d}\omega_{\mathscr{D}}(t,\ 0)$$

$$= \quad \int_{\partial\mathscr{D}} \eta\log(1+t^2)\,\mathrm{d}\omega_{\mathscr{D}}(t,\ 0) \quad + \quad \int_{\partial\mathscr{D}} \log W(t)\,\mathrm{d}\omega_{\mathscr{D}}(t,\ 0)$$

$$\leqslant \quad \pi\eta Y_{\mathscr{D}}(0) \quad + \quad \int_{\partial\mathscr{D}} \log W(t)\,\mathrm{d}\omega_{\mathscr{D}}(t,\ 0),$$

and our main work is the estimation of the integral in the last member.

For that purpose, we may as well make full use of the third theorem in the preceding article, having done the work to get it. The reader wishing to avoid use of that theorem will find a similar alternative procedure sketched in problems 63 and 64 below. According to our hypothesis, $\log W(x)/x \in \mathfrak{H}$ so, by the theorem referred to, there is, for any $\eta > 0$, a potential $U_\rho(x)$

of the sort considered in the last article, with

$$\left\| \frac{\log W(x)}{x} - U_\rho(x) \right\|_E < \eta$$

and also (by the remark to that theorem)

$$|U_\rho(x)| \leqslant \frac{K_\eta}{1+x^2} \quad \text{for } x \in \mathbb{R}.$$

Let us now proceed as in proving the theorem of §C.4, Chapter VIII, trying, however, to make use of the difference $(\log W(x)/x) - U_\rho(x)$.

We have

$$\int_{\partial\mathscr{D}} \log W(t)\,\mathrm{d}\omega_{\mathscr{D}}(t,\ 0) = \int_{\partial\mathscr{D}} tU_\rho(t)\,\mathrm{d}\omega_{\mathscr{D}}(t,\ 0)$$

$$+ \int_{\partial\mathscr{D}} (\log W(t) - tU_\rho(t))\,\mathrm{d}\omega_{\mathscr{D}}(t,\ 0).$$

Because $|tU_\rho(t)| \leqslant K_\eta/2$ by the last inequality, and $\omega_{\mathscr{D}}(\ \ ,\ 0)$ is a positive measure of total mass 1, the *first* integral on the right is

$$\leqslant K_\eta/2.$$

In terms of $\Omega_{\mathscr{D}}(t)$, the *second* right-hand integral is

$$-\int_0^\infty (\log W(t) - tU_\rho(t))\,\mathrm{d}\Omega_{\mathscr{D}}(t);$$

to this we now apply the trick used in the proof just mentioned, rewriting the last expression as

$$\int_0^\infty \left(\frac{\log W(t)}{t} - U_\rho(t)\right)\Omega_{\mathscr{D}}(t)\,\mathrm{d}t - \int_0^\infty \left(\frac{\log W(t)}{t} - U_\rho(t)\right)\mathrm{d}(t\Omega_{\mathscr{D}}(t)).$$

The *first* of these terms can be disposed of immediately. Taking a large number L, we break it up as

$$\int_0^L \frac{\log W(t)}{t}\Omega_{\mathscr{D}}(t)\,\mathrm{d}t - \int_0^\infty U_\rho(t)\,\Omega_{\mathscr{D}}(t)\,\mathrm{d}t + \int_L^\infty \frac{\log W(t)}{t}\Omega_{\mathscr{D}}(t)\,\mathrm{d}t.$$

We use the inequality $\Omega_{\mathscr{D}}(t) \leqslant 1$ in the first two integrals, plugging the above estimate on $U_\rho(t)$ into the second one. In the third integral, the relation $\Omega_{\mathscr{D}}(t) \leqslant Y_{\mathscr{D}}(0)/t$ is once again employed. In that way, the sum of these integrals is seen to be

$$\leqslant L\int_0^\infty \frac{\log W(t)}{t^2}\mathrm{d}t + \frac{\pi}{2}K_\eta + Y_{\mathscr{D}}(0)\int_L^\infty \frac{\log W(t)}{t^2}\mathrm{d}t.$$

We come to

$$\int_0^\infty \left(\frac{\log W(t)}{t} \;-\; U_\rho(t)\right) \mathrm{d}(t\Omega_{\mathscr{D}}(t)),$$

the *second* of the above terms. According to §C.3 of Chapter VIII, the double integral

$$\int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \,\mathrm{d}(t\Omega_{\mathscr{D}}(t))\, \mathrm{d}(x\Omega_{\mathscr{D}}(x))$$

is absolutely convergent, and its value,

$$E(\,\mathrm{d}(t\Omega_{\mathscr{D}}(t)),\; \mathrm{d}(t\Omega_{\mathscr{D}}(t))\,),$$

is

$$\leqslant \quad \pi(Y_{\mathscr{D}}(0))^2.$$

The measure $\mathrm{d}(t\Omega_{\mathscr{D}}(t))$ is *absolutely continuous* (albeit with *unbounded* density!), and acts like const.$(\mathrm{d}t/t^3)$ for large t. Near the origin, $\mathrm{d}(t\Omega_{\mathscr{D}}(t)) \;=\; \mathrm{d}t$, for $\Omega_{\mathscr{D}}(t) \;\equiv\; 1$ in a neighborhood of that point. These properties together with our given conditions on $W(t)$ and the above estimate for $U_\rho(t)$ ensure absolute convergence of

$$\int_0^\infty (\,(\log W(t)/t) \;-\; U_\rho(t)\,)\,\mathrm{d}(t\Omega_{\mathscr{D}}(t)),$$

which may hence be estimated by the *fifth* lemma of the last article. In that way this integral is found to be in absolute value

$$\leqslant \quad \left\|\frac{\log W(t)}{t} \;-\; U_\rho(t)\right\|_E \sqrt{(\,E(\mathrm{d}(t\Omega_{\mathscr{D}}(t)),\; \mathrm{d}(t\Omega_{\mathscr{D}}(t)))\,)}.$$

Referring to the previous relation, we see that for our choice of $U_\rho(t)$, the quantity just found is

$$\leqslant \quad \sqrt{\pi}\;\; \eta Y_{\mathscr{D}}(0),$$

and we have our upper bound for the second term in question.

Combining this with the estimate already obtained for the *first* term, we get

$$-\int_0^\infty (\log W(t) \;-\; tU_\rho(t)\,)\,\mathrm{d}\Omega_{\mathscr{D}}(t)$$

$$\leqslant \quad L\int_0^\infty \frac{\log W(t)}{t^2}\mathrm{d}t \;+\; \frac{\pi}{2}K_\eta \;+\; Y_{\mathscr{D}}(0)\int_L^\infty \frac{\log W(t)}{t^2}\mathrm{d}t \;+\; \sqrt{\pi}\;\;\eta Y_{\mathscr{D}}(0),$$

whence, by an earlier computation,

$$\int_{\partial\mathscr{D}} \log W(t)\,d\omega_{\mathscr{D}}(t,\ 0)$$

$$\leqslant \quad L\int_0^\infty \frac{\log W(t)}{t^2}dt \ + \ \frac{\pi+1}{2}K_\eta \ + \ \left(\sqrt{\pi}\,\eta \ + \ \int_L^\infty \frac{\log W(t)}{t^2}dt\right)Y_{\mathscr{D}}(0),$$

and thus finally

$$\int_{\partial\mathscr{D}} \log W_\eta(t)\,d\omega_{\mathscr{D}}(t,\ 0)$$

$$\leqslant \quad \left((\pi+\sqrt{\pi})\eta \ + \ \int_L^\infty \frac{\log W(t)}{t^2}dt\right)Y_{\mathscr{D}}(0)$$

$$+ \ L\int_0^\infty \frac{\log W(t)}{t^2}dt \ + \ \frac{\pi+1}{2}K_\eta.$$

Wishing now to have the initial term on the right outweighed by $-AY_{\mathscr{D}}(0)$ we first, for our given value of $A>0$, pick

$$\eta \quad \leqslant \quad \frac{A}{2(\pi+\sqrt{\pi})} \quad \text{(say)},$$

and then choose (and fix!) L large enough so as to have

$$\int_L^\infty \frac{\log W(t)}{t^2}dt \quad \leqslant \quad \frac{A}{2}.$$

For these particular values of η and L, it will follow that

$$\int_{\partial\mathscr{D}} \log W_\eta(t)\,d\omega_{\mathscr{D}}(t,\ 0) \ - \ AY_{\mathscr{D}}(0) \quad \leqslant \quad L\int_0^\infty \frac{\log W(t)}{t^2}dt \ + \ \frac{\pi+1}{2}K_\eta$$

for any of the domains $\mathscr{D}$, and hence that

$$(\mathfrak{M}F)(0) \quad \leqslant \quad L\int_0^\infty \frac{\log W(t)}{t^2}dt \quad + \quad \frac{\pi+1}{2}K_\eta,$$

by the method used in articles 2 and 3. This, however, proves the theorem.

We are done.

Referring now to the corollary of the next-to-the last theorem in §B.1, we immediately obtain the

Corollary. *Let the continuous weight $W(x) \geqslant 1$ satisfy the hypothesis of the theorem, and also fulfill the regularity requirement formulated in* §B.1. *Then W admits multipliers.*

This result and the one obtained in problem 62 (last article) give us once again a proposition due to Beurling and Malliavin, already deduced from their Theorem on the Multiplier (of article 2) in §C.1, Chapter X. That proposition may be stated in the following form:

Theorem. *Let $W(x) \geqslant 1$ be even, with* $\log W(x)$ *uniformly* Lip 1 *on* $\mathbb{R}$, *and*

$$\int_0^\infty \frac{\log W(x)}{x^2}\,\mathrm{d}x \quad < \quad \infty.$$

Then W admits multipliers.

It suffices to observe that the regularity requirement of §B.1 is certainly met by weights W with $\log W(x)$ uniformly Lip 1.

Originally, this theorem was essentially derived in such fashion from the preceding one by Beurling and Malliavin.

Problem 63

(a) Let ρ be a positive measure on $[0, \infty)$ without point mass at the origin, such that $E(\mathrm{d}\rho(t), \mathrm{d}\rho(t)) < \infty$. Show that there is a sequence of positive measures σ_n of *compact support* in $(0, \infty)$ with $\mathrm{d}\rho(t) - \mathrm{d}\sigma_n(t) \geqslant 0$, $U_{\sigma_n}(x)$ *bounded* on $\mathbb{R}$ for each n, and $\|U_\rho - U_{\sigma_n}\|_E \xrightarrow[n]{} 0$. (Hint: First argue as in the proof of the *fifth* lemma of the last article to verify that if ρ_n denotes the restriction of ρ to $[1/n,\ n]$, then $\|U_\rho - U_{\rho_n}\|_E \xrightarrow[n]{} 0$. Then, for each n, take σ_n as the restriction of ρ_n to the closed subset of $[1/n,\ n]$ on which $U_{\rho_n}(x) \leqslant$ some sufficiently large number M_n.)

(b) Let σ be a positive measure of *compact support* $\subseteq (0, \infty)$ with $\|U_\sigma\|_E < \infty$ and $U_\sigma(x)$ *bounded on* $\mathbb{R}$. Show that, corresponding to any $\varepsilon > 0$, there is a *signed* measure τ on $[0, \infty)$, without point mass at the origin, such that $U_\tau(x)$ is also bounded on $\mathbb{R}$, that $\|U_\sigma - U_\tau\|_E < \varepsilon$, and that $U_\tau(x) = 0$ for all sufficiently large x. (Hint: We have $U_\sigma(x) \longrightarrow 0$ for $x \longrightarrow \infty$. Taking a very large $R > 0$, far beyond the support K of σ, consider the domain $\mathscr{D}_R = \{\Re z > 0\} \sim [R, \infty)$, and the harmonic measure $\omega_R(\ \ , z)$ for $\mathscr{D}_R$. Define an absolutely continuous measure σ_R on $[R, \infty)$ by putting, for $t > R$,

$$\frac{\mathrm{d}\sigma_R(t)}{\mathrm{d}t} \quad = \quad \int_K \frac{\mathrm{d}\omega_R(t,\ \xi)}{\mathrm{d}t}\,\mathrm{d}\sigma(\xi).$$

Show that $U_{\sigma_R}(x) = U_\sigma(x)$ for $x > R$, that U_{σ_R} is bounded on $\mathbb{R}$, and that $\| U_{\sigma_R} \|_E < \varepsilon$ if R is taken large enough. Then put $\tau = \sigma - \sigma_R$. *Note*: Potential theorists say that σ_R has been obtained from σ by *balayage* (sweeping) onto the set $[R, \infty)$.)

(c) Hence show that if ρ is any *signed* measure on $[0, \infty)$ without point mass at 0 making $E(|\mathrm{d}\rho(t)|, |\mathrm{d}\rho(t)|) < \infty$, there is another such signed measure μ on $[0, \infty)$ with $\| U_\rho - U_\mu \|_E < \varepsilon$, $U_\mu(x)$ *bounded* on $\mathbb{R}$, and $U_\mu(x) = 0$ for all $x > R$, a number depending on ε. (Here, parts (a) and (b) are applied in turn to the *positive* part of ρ and to its *negative* part.)

Problem 64

Prove the first theorem of this article using the result of problem 63. (Hint: Given that $\log W(x)/x \in \mathfrak{H}$, take first a signed measure ρ on $[0, \infty)$ like the one in problem 63(c) such that

$$\|(\log W(x)/x) - U_\rho(x)\|_E < \eta/2,$$

and then a μ, furnished by that problem, with $\| U_\rho - U_\mu \|_E < \eta/2$. Argue as in the proof given above, working with the difference

$$\frac{\log W(x)}{x} - U_\mu(x),$$

and taking the number L figuring there to be *larger* than the R obtained in problem 63 (c).)

D. Search for the presumed essential condition

At the beginning of §B.1, it was proposed to limit a good part of the considerations of this chapter to weights $W(x) \geqslant 1$ satisfying a mild local regularity requirement:

There are three constants C, α and $L > 0$ (depending on W) such that, for each real x, one has an interval J_x of length L containing x with

$$W(t) \geqslant C(W(x))^\alpha \qquad \text{for } t \in J_x.$$

That restriction was accepted because, while leaving us with room enough to accommodate many of the weights arising in different circumstances, it serves, we believe, to rule out accidental and, so to say, *trivial* irregularities in a weight's behaviour that could spoil the existence of multipliers which might otherwise be forthcoming. Admittance of multipliers by a weight W was thought to be *really* governed *by some other condition* on its behaviour – an 'essential' one, probably not of strictly local character –

acting in conjunction with the growth requirement

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2} dx \quad < \quad \infty.$$

In adopting this belief, we of course made a tacit asumption that another condition regarding the weight (besides convergence of its logarithmic integral) *is in fact involved.* Up to now, however, we have not seen any reasons why that should be the case. *It is still quite conceivable that the integral condition and the local regularity requirement are, by themselves, sufficient to guarantee admittance of multipliers.*

Such a conclusion would be most satisfying, and indeed make a fitting end to this book. If its truth seemed likely, we would have to abandon our present viewpoint and think instead of looking for a proof. We have arrived at the place where one must decide which path to take.

It is for that purpose that the example given in the first article was constructed. This shows that an additional condition on our weights – what we are thinking of as the 'essential' one – *is really needed.* Our aim during the succeeding articles of this § will then be to find out *what that condition is* or at least arrive at some partial knowledge of it.

In working towards that goal, we will be led to the construction of a *second* example, actually quite similar to the one of the first article, but yielding a weight that *admits* multipliers although the weight furnished by the latter *does not.* Comparison of the two examples will enable us to form an idea of what the 'essential' condition on weights must look like, and, eventually, lead us to the *necessary and sufficient conditions for admittance of multipliers* (on weights meeting the local regularity requirement) formulated in the theorem of §E.

Before proceeding to the first example, it is worthwhile to see what the *absence* of an additional condition on our weights *would have* entailed. The local regularity requirement quoted at the beginning of this discussion is certainly satisfied by weights $W(x) \geqslant 1$ with

$$|\log^+ \log W(x) \; - \; \log^+ \log W(x')| \quad \leqslant \quad \text{const.}\,|x - x'|$$

on $\mathbb{R}$. Absence of an additional condition would therefore make

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2} dx \quad < \quad \infty$$

necessary and sufficient for the admittance of multipliers by such W. This would in turn have an obvious but quite interesting corollary: if, for a weight $W(x)$ with uniformly Lip 1 iterated logarithm, there is *even one* entire function $\Phi(z) \not\equiv 0$ of *some* (finite) *exponential type* making $W(x)\Phi(x)$ *bounded* on $\mathbb{R}$, there must be such functions $\varphi(z) \not\equiv 0$ of *arbitrarily small exponential type* having the *same property*. The example given in the first article will show that *even this corollary is false*.

The absence of an additional condition on *just* the weights with uniform Lip 1 iterated logarithms *would*, by the way, *imply* that absence for *all* weights meeting our (less stringent) local regularity requirement. Indeed, if $W(x) \geqslant 1$ fulfills the latter (with constants C, α and L), and

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2}\,dx \quad < \quad \infty,$$

we *know* from the proof of the first theorem in §B.1 that W *admits multipliers* (for which the last relation is at least *necessary*), *if and only if* the weight

$$W_1(x) \quad = \quad \exp\left\{\frac{4}{\pi\alpha}\int_{-\infty}^{\infty} \frac{L \log W(t)}{(x-t)^2+L^2}\,dt\right\}$$

also does. We see, however, that $|\,d\log W_1(x)/dx\,| \quad \leqslant \quad (1/L)\log W_1(x)$, i.e.,

$$\left|\frac{d\log\log W_1(x)}{dx}\right| \quad \leqslant \quad \frac{1}{L},$$

so W_1 *does* have a uniformly Lip 1 iterated logarithm.

Let us go on to the first example.

1. Example. Uniform Lip 1 condition on $\log\log W(x)$ not sufficient

Take the points

$$x_p \quad = \quad e^{p^{1/3}}, \quad p \quad = \quad 8, 9, 10, \ldots,$$

and put

$$\Delta_p \quad = \quad \begin{cases} x_8, & p = 8, \\ x_p - x_{p-1}, & p > 8. \end{cases}$$

Let then

$$F(z) \quad = \quad \prod_{p=8}^{\infty}\left(1 \; - \; \frac{z^2}{x_p^2}\right)^{[\Delta_p]},$$

where $[\Delta_p]$ denotes the largest integer $\leqslant \Delta_p$; it is not hard to see – is, indeed, a particular consequence of the following work – that the product is convergent, making $F(z)$ an entire function with a zero of order $[\Delta_p]$ at each of the points $\pm x_p$, $p \geqslant 8$, and no other zeros.

According to custom, we write $n(t)$ for the *number of zeros* of $F(z)$ in $[0, t]$ (counting multiplicities) when $t \geqslant 0$. Thus,

$$n(t) = 0 \quad \text{for } 0 \leqslant t < x_8$$

and

$$n(t) = [\Delta_8] + [\Delta_9] + \cdots + [\Delta_p] \qquad \text{for } x_p \leqslant t < x_{p+1}.$$

The *right side* of the last relation lies between

$$\Delta_8 + \Delta_9 + \cdots + \Delta_p - p = x_p - p$$

and

$$\Delta_8 + \Delta_9 + \cdots + \Delta_p = x_p,$$

so, since

$$p = (\log x_p)^3,$$

we have

$$t - \Delta_{p+1} - (\log x_p)^3 \leqslant n(t) \leqslant t \qquad \text{for } x_p \leqslant t < x_{p+1},$$

with the *second* inequality actually valid for *all* $t \geqslant 0$. Here,

$$\Delta_{p+1} = e^{(p+1)^{1/3}} - e^{p^{1/3}} = (\tfrac{1}{3}p^{-2/3} + O(p^{-5/3}))x_p$$

is

$$\sim \frac{x_p}{3(\log x_p)^2},$$

making

$$t\left(1 - \frac{1}{3(\log t)^2} - O\left(\frac{1}{\log t}\right)^5\right) \leqslant n(t) \leqslant t$$

for $x_p \leqslant t < x_{p+1}$. We thus certainly have

$$t - \frac{t}{(\log t)^2} \leqslant n(t) \leqslant t$$

for large values of t, with the *upper bound* in fact holding for all $t \geqslant 0$, as just noted.

We can write

$$\log|F(z)| \quad = \quad \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}n(t),$$

and the reader should now refer to problem 29 (§B.1, Chapter IX). Reasoning as in part (a) of that problem, one readily concludes that

$$\frac{\log|F(\mathrm{i}y)|}{y} \longrightarrow \pi \qquad \text{for } y \longrightarrow \infty,$$

since

$$\frac{n(t)}{t} \longrightarrow 1 \quad \text{as } t \longrightarrow \infty$$

by the previous relation. Clearly,

$$|F(z)| \leqslant F(\mathrm{i}|z|),$$

so our function $F(z)$ *is of exponential type* π.

To estimate $|F(x)|$ for real x, we refer to part (c) of the same problem, according to which

$$\log|F(x)| \quad \leqslant \quad 2n(x)\log\frac{1}{\lambda} \quad + \quad 2\int_0^\lambda \frac{\dfrac{n(xt)}{t} - tn\left(\dfrac{x}{t}\right)}{1 - t^2}\,\mathrm{d}t$$

for $x > 0$, where for λ we may take any number between 0 and 1. Assuming x *large*, we put

$$\lambda \quad = \quad 1 - \frac{1}{(\log x)^2}$$

and plug the above relation for $n(t)$ into the integral (using, of course, the upper bound with $n(xt)/t$ and the lower one with $tn(x/t)$). We thus find that

$$\log|F(x)| \quad \leqslant \quad 2n(x)\log\frac{1}{\lambda} \quad + \quad 2x\int_0^\lambda \frac{\mathrm{d}t}{(\log(x/t))^2(1-t^2)}$$

$$\leqslant \quad \text{const.}\frac{x}{(\log x)^2} \quad + \quad x\,\frac{\log 2 + 2\log\log x}{(\log x)^2} \quad \leqslant \quad C\,\frac{x\log\log x}{(\log x)^2}$$

for large values of x.

The quantity on the right is *increasing* when $x > 0$ is large enough, and satisfies

$$\int_{e}^{\infty} \frac{1}{x^2} C \frac{x \log\log x}{(\log x)^2}\,dx \quad = \quad C\int_1^{\infty} \frac{\log u}{u^2}\,du \quad = \quad C \quad < \quad \infty.$$

Therefore, since $\log|F(x)|$ is *even* and bounded above by that quantity when x is large, we can conclude by the elementary Paley–Wiener multiplier theorem of Chapter X, §A.1 (obtained by a different method far back in §D of Chapter IV!) that there is, corresponding to any $\eta > 0$, a non-zero entire function $\psi(z)$ of exponential type $\leqslant \eta$ with $F(x)\psi(x)$ *bounded* on the real axis. The function $\psi(z)$ obtained in Chapter X is in fact of the form $\varphi(z+\mathrm{i})$, where

$$\varphi(z) \quad = \quad \prod_k \left(1 - \frac{z^2}{\lambda_k^2}\right)$$

is *even* and has only the *real* zeros $\pm\lambda_k$; it is thus clear that for $x \in \mathbb{R}$,

$$|\psi(x)| \quad = \quad |\varphi(x+\mathrm{i})| \quad \geqslant \quad |\varphi(x)|.$$

(Observe that for $\zeta \;=\; \xi + \eta\mathrm{i}$,

$$|1-\zeta^2| \quad = \quad |1+\zeta||1-\zeta| \quad \geqslant \quad |1+\xi||1-\xi| \text{ !})$$

Hence,

$$|F(x)\varphi(x)| \quad \leqslant \quad \text{const.} \quad \text{on } \mathbb{R}$$

with an *even* function $\varphi(z)$ of exponential type $\leqslant$ η *having only real zeros.*

Fixing a constant $c > 0$ for which

$$c|F(x)\varphi(x)| \quad \leqslant \quad 1, \quad x \in \mathbb{R},$$

we put

$$\Psi(z) \quad = \quad cF(z)\varphi(z),$$

getting a certain *even* entire function Ψ, with *only real zeros*, having exponential type equal to a number B lying between π (the type of F) and $\pi + \eta$. For this function the Poisson representation

$$\log|\Psi(z)| \quad = \quad B\Im z \;+\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log|\Psi(t)|}{|z-t|^2}\,dt$$

from §G.1 of Chapter III is valid for $\Im z \;>\; 0$, the integral on the right being absolutely convergent. In particular,

$$\log\left|\frac{e^B}{\Psi(x+\mathrm{i})}\right| \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{(x-t)^2+1}\log\frac{1}{|\Psi(t)|}\,dt \qquad \text{for } x \in \mathbb{R},$$

where the integral is obviously $\geqslant$ 0.

We now take

$$W(x) \quad = \quad \frac{e^B}{|\Psi(x+i)|}, \qquad x \in \mathbb{R}.$$

Then $|W(x)| \geqslant 1$ and differentiation of the preceding formula immediately yields

$$\left|\frac{d \log W(x)}{dx}\right| \quad \leqslant \quad \log W(x),$$

making

$$|\log\log W(x) \; - \; \log\log W(x')| \quad \leqslant \quad |x - x'| \qquad \text{for } x,\ x' \in \mathbb{R}.$$

From the same formula we also see that

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2} dx \quad < \quad \infty.$$

It is claimed, however, that the weight W has no mutipliers of exponential type $<\ \pi$. Suppose, indeed, that there *is* a non-zero entire function $f(z)$ of exponential type $A' \ < \ \pi$ with $f(x)W(x)$ bounded on $\mathbb{R}$. According to the discussion following the *first* theorem of §B.1 we can, from f, obtain another non-zero entire function g of exponential type $A \leqslant A'$ (hence $A < \pi$ – A is in fact *equal* to A'), *having only real zeros*, with

$$|g(x)| \; \leqslant \; |f(x)| \quad \text{on } \mathbb{R},$$

so that $g(x)W(x)$ is also bounded for $x \in \mathbb{R}$. This function $g(z)$ (denoted by $C\psi(z)$ in the passage referred to) is a constant multiple of a product like

$$e^{bz} \prod_{\lambda'} \left(1 \; - \; \frac{z}{\lambda'}\right) e^{z/\lambda'}$$

formed from *real* numbers b and λ', and hence has the important property that

$$|g(\Re z)| \; \leqslant \; |g(z)|,$$

which we shall presently have occasion to use.

There is no loss of generality in our assuming that

$$|g(x)W(x)| \quad \leqslant \quad e^{-B} \qquad \text{for } x \in \mathbb{R}.$$

Referring to our definition of W, we see that this is the same as the relation

$$|g(x)| \quad \leqslant \quad e^{-2B}|\Psi(x+i)|, \qquad x \in \mathbb{R}.$$

To the function g, having all its zeros on the real axis (and surely *bounded* there!) we may apply the Poisson representation from §G.1 of Chapter III to get

$$\log|g(z)| \quad = \quad A\Im z \;+\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log|g(t)|}{|z-t|^2}\,dt, \qquad \Im z \;>\; 0.$$

In like manner,

$$\log|\Psi(z+\mathrm{i})| \quad = \quad B\Im z \;+\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log|\Psi(t+\mathrm{i})|}{|z-t|^2}\,dt, \qquad \Im z \;>\; 0,$$

so that, for $\Im z \;>\; 0$,

$$\log\left|\frac{g(z)}{\Psi(z+\mathrm{i})}\right| \quad = \quad (A-B)\Im z \quad + \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log\left|\frac{g(t)}{\Psi(t+\mathrm{i})}\right|\,dt.$$

Since $A < \pi \leqslant B$, the right side in the last relation is $\leqslant\; -2B$ by the preceding inequality, so we have in particular

$$|g(x+\mathrm{i})| \quad \leqslant \quad \mathrm{e}^{-2B}|\Psi(x+2\mathrm{i})|, \qquad x\in\mathbb{R}.$$

Let us note moreover that $\mathrm{e}^{-2B}|\Psi(x+2\mathrm{i})| \;\leqslant\; 1$ on the real axis by the third Phragmén–Lindelöf theorem of Chapter III, §C, $\Psi(z)$ being of exponential type B and of modulus $\leqslant\; 1$ for real z. Another application of the same Phragmén–Lindelöf theorem thence shows that

$$\mathrm{e}^{-2B}|\Psi(z+2\mathrm{i})| \quad \leqslant \quad \mathrm{e}^{B|\Im z|}$$

This estimate will also be of use to us.

Our idea now is to show that $|g(x)|$ must get so small near the zeros $\pm x_p$ of our original function $F(z)$ as to make

$$\int_{-\infty}^{\infty} \frac{\log^-|g(x)|}{1+x^2}\,dx \quad = \quad \infty$$

and thus imply that $g(z) \;\equiv\; 0$ (*a contradiction!*) by §G.2 of Chapter III. We start by looking at $|g(x_p+\mathrm{i})|$, which a previous relation shows to be $\leqslant\; \mathrm{e}^{-2B}|\Psi(x_p+2\mathrm{i})|$. The latter quantity we estimate by Jensen's formula.

For the moment, let us denote by $N(r,\; z_0)$ the *number of zeros of* $\Psi(z)$ inside any closed disk of the form $\{|z-z_0| \;\leqslant\; r\}$. Then we have, for any $R > 0$,

$$\log|\mathrm{e}^{-2B}\Psi(x_p+2\mathrm{i})| \quad = \quad \frac{1}{2\pi}\int_{-\pi}^{\pi}\log|\mathrm{e}^{-2B}\Psi(x_p+2\mathrm{i}+R\mathrm{e}^{\mathrm{i}\vartheta})|\,d\vartheta$$

$$- \quad \int_0^R \frac{N(r,\; x_p+2\mathrm{i})}{r}\,dr.$$

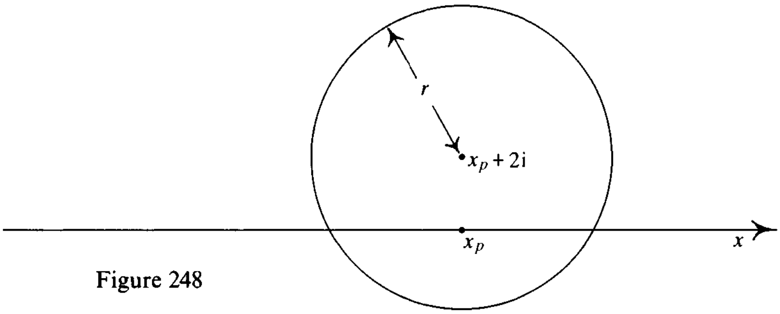

Figure 248

Substituting the *last* of the above relations involving Ψ into the first integral on the right and noting that $\Psi(z)$ has *at least a* $[\Delta_p]$ *-fold zero at* x_p, we see that for $R > 2$,

$$\log|\mathrm{e}^{-2B}\Psi(x_p+2\mathrm{i})| \;\leqslant\; \frac{2}{\pi}BR \;-\; [\Delta_p]\log\frac{R}{2}.$$

Write now $\nu(r,\ z_0)$ for the *number of zeros of* $g(z)$ in the closed disk $\{|z-z_0| \leqslant r\}$. Application of Jensen's formula to g then yields

$$\log|g(x_p+\mathrm{i})| \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi}\log|g(x_p+\mathrm{i}+R\mathrm{e}^{\mathrm{i}\vartheta})|\,\mathrm{d}\vartheta \;-\; \int_0^R \frac{\nu(r,\ x_p+\mathrm{i})}{r}\mathrm{d}r.$$

Using the inequality $|g(x_p+\mathrm{i})| \leqslant \mathrm{e}^{-2B}|\Psi(x_p+2\mathrm{i})|$ and referring to the preceding relation we find, after transposing, that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|g(x_p+\mathrm{i}+R\mathrm{e}^{\mathrm{i}\vartheta})|\,\mathrm{d}\vartheta$$

$$\leqslant\; \frac{2}{\pi}BR \;+\; \int_0^R\frac{\nu(r,\ x_p+\mathrm{i})}{r}\mathrm{d}r \;-\; [\Delta_p]\log\frac{R}{2} \qquad \text{for } R > 2.$$

Since all the zeros of $g(z)$ *are real,* $\nu(r,\ x_p+\mathrm{i})$ *is certainly zero for* $r < 1$, whence

$$\int_0^R\frac{\nu(r,\ x_p+\mathrm{i})}{r}\mathrm{d}r \;\leqslant\; \nu(R,\ x_p+\mathrm{i})\log R, \qquad R > 1,$$

which, together with the last, gives

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|g(x_p+\mathrm{i}+R\mathrm{e}^{\mathrm{i}\vartheta})|\,\mathrm{d}\vartheta$$

$$\leqslant\; \frac{2}{\pi}BR \;+\; \big(\nu(R,\ x_p+\mathrm{i}) \;-\; [\Delta_p]\big)\log R \;+\; [\Delta_p]\log 2, \qquad R > 2.$$

We want to use this to show that for a certain R_p, $\int_{-R_p}^{R_p} \log|g(x_p+t)|\,\mathrm{d}t$ comes out *very negative.*

To do that, we simply (trick!) plug the inequality $|g(\Re z)| \leqslant |g(z)|$ noted above into the left side of the last relation. We are, in other words, *flattening* the circle involved in Jensen's formula to its horizontal diameter which is then moved down to the real axis:

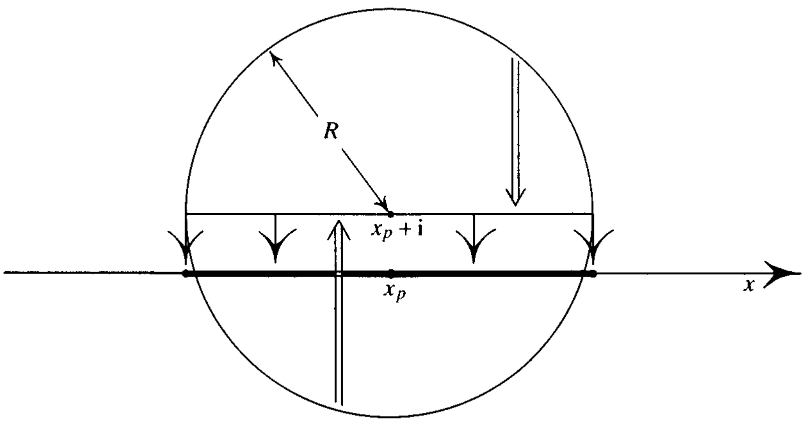

Figure 249

That causes $\log|g(x_p + \mathrm{i} + R\mathrm{e}^{\mathrm{i}\vartheta})|$ to be replaced by $\log|g(x_p + R\cos\vartheta)|$ in the integral appearing in the relation in question; the resulting integral then becomes

$$\frac{1}{\pi}\int_{-R}^{R} \frac{\log|g(x_p+s)|}{\sqrt{(R^2-s^2)}}\,\mathrm{d}s$$

on making the substitution $R\cos\vartheta = s$. What we have just written is hence

$$\leqslant \quad \frac{2}{\pi}BR \;+\; \big(\nu(R,\; x_p+\mathrm{i}) \;-\; [\Delta_p]\big)\log R \;+\; [\Delta_p]\log 2 \qquad \text{for } R > 2.$$

Our reasoning at this point is much like that in §D.1 of Chapter IX. Taking

$$R_p \quad = \quad \frac{\Delta_p}{2},$$

we multiply the preceding integral and the expression immediately following it by $R\,\mathrm{d}R$ and integrate from $R_p/2$ to R_p. That yields

$$\frac{1}{\pi}\int_{R_p/2}^{R_p}\int_{-R}^{R}\frac{R\log|g(x_p+s)|}{\sqrt{(R^2-s^2)}}\,\mathrm{d}s\,\mathrm{d}R$$
$$\leqslant\quad \frac{7B}{12\pi}R_p^3 \;+\; \frac{3}{8}R_p^2\,\nu(R_p,\ x_p+\mathrm{i})\log R_p$$
$$-\;\frac{3}{8}R_p^2[\Delta_p]\log\frac{R_p}{2} \;+\; \frac{3\log 2}{8}[\Delta_p]R_p^2.$$

An integral like the one on the left (involving $\log|\hat{\mu}(c+t)|$ instead of $\log|g(x_p+s)|$) has already figured in the proof of the theorem from the passage just referred to. Here, we may argue as in that proof (reversing the order of integration), for $\log|g(x)| \leqslant 0$ on the real axis, as follows from the inequalities $|g(x)|W(x) \leqslant \mathrm{e}^{-B}$ and $W(x) \geqslant 1$, valid thereon. In that way, one finds the left-hand integral to be

$$\geqslant\quad \frac{\sqrt{3}}{2\pi}R_p\int_{-R_p}^{R_p}\log|g(x_p+s)|\,\mathrm{d}s.$$

After dividing by $R_p x_p^2$ and clearing out some coefficients, we see that

$$\frac{1}{x_p^2}\int_{-R_p}^{R_p}\log|g(x_p+s)|\,\mathrm{d}s$$
$$\leqslant\quad \frac{7B}{6\sqrt{3}}\frac{R_p^2}{x_p^2} \;+\; \frac{\sqrt{3}\,\pi}{4}\left\{\nu(R_p,\ x_p+\mathrm{i}) \;-\; [\Delta_p]\right\}\frac{R_p\log R_p}{x_p^2}$$
$$+\;\frac{\sqrt{3}\,\pi\log 2}{2}[\Delta_p]\frac{R_p}{x_p^2}$$

(the actual values of the numerical coefficients on the right are not so important).

The quantities $R_p = \Delta_p/2 = (x_p - x_{p-1})/2$ are *increasing* when $p > 8$ because $x_p = \exp(p^{1/3})$, and the latter function has a *positive second derivative* for $p > 8$. (*Now* we see why we use the sequence of points x_p beginning with x_8 !) The intervals $[x_p - R_p,\ x_p + R_p]$ *therefore do not overlap when* $p > 8$, so our desired conclusion, namely, that

$$\int_{-\infty}^{\infty}\frac{\log^-|g(x)|}{1+x^2}\mathrm{d}x \quad=\quad \infty,$$

will surely follow if we can establish that

$$\sum_{p>8}\frac{1}{x_p^2}\int_{-R_p}^{R_p}\log|g(x_p+s)|\,\mathrm{d}s \quad=\quad -\infty$$

with the help of the preceding relation.

Here, we are guided by a simple idea. Everything turns on the *middle term* figuring on the right side of our relation, for the sums of the *first* and *third* terms are readily seen to be *convergent*. To see how the middle term behaves, we observe that by *Levinson's theorem* (!), the function $g(z)$, bounded on the real axis and of exponential type A, should, *on the average*, have about

$$\frac{2}{\pi}AR_p \quad = \quad \frac{A}{\pi}\Delta_p$$

zeros on the interval $(x_p - R_p,\ x_p + R_p]$, for all of g's zeros are real. The quantity $\nu(R_p,\ x_p + \mathrm{i})$ is clearly *not more* than that number of zeros, so the factor in $\{\quad\}$ from our middle term should, *on the average*, be

$$\leqslant \quad -\frac{\pi - A}{\pi}\Delta_p$$

(approximately). Straightforward computation easily shows, however, that

$$\frac{\Delta_p R_p \log R_p}{x_p^2} \quad \sim \quad \frac{1}{18p}$$

for large values of p. It is thus quite plausible that the series

$$\sum_p \left\{\nu(R_p,\ x_p + \mathrm{i}) \ - \ [\Delta_p]\right\} \frac{R_p \log R_p}{x_p^2}$$

should diverge to $-\infty$. This inference is in fact *correct*, but for its justification we must resort to a technical device.

Picking a number $\gamma > 1$ *close* to 1 (the exact manner of choosing it will be described presently), we form the sequence

$$X_m \quad = \quad \gamma^m, \qquad m \ = \ 1, 2, 3, \ldots .$$

We think of $\{X_m\}$ as a *coarse* sequence of points, amongst which those of $\{x_p\}$ – regarded as a *fine* sequence – are interspersed:

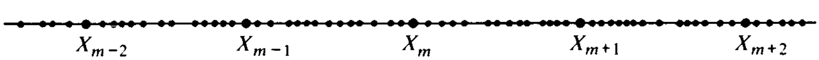

Figure 250

It is convenient to denote by $\nu(t)$ the *number of zeros* of $g(t)$ in $[0,\ t]$ for $t \geqslant 0$; then, as remarked above,

$$\nu(R_p,\ x_p + \mathrm{i}) \quad \leqslant \quad \nu(x_p + R_p) \ - \ \nu(x_p - R_p).$$

For any large value of m we thus have, recalling that $R_p = \Delta_p/2$,

$$\sum_{X_m < x_p \leqslant X_{m+1}} \nu(R_p, x_p + \mathrm{i}) \frac{R_p \log R_p}{x_p^2}$$

$$\leqslant \sum_{X_m < x_p \leqslant X_{m+1}} \left\{ \nu\left(x_p + \frac{\Delta_p}{2}\right) - \nu\left(x_p - \frac{\Delta_p}{2}\right) \right\} \cdot \sup_{X_m < x_p \leqslant X_{m+1}} \frac{\Delta_p \log \Delta_p}{2x_p^2}.$$

Denote by h_m the value of $\Delta_p/2$ corresponding to the *smallest* $x_p > X_m$, and by h'_{m+1} the value of that quantity corresponding to the *largest* $x_p \leqslant X_{m+1}$. Since, for $p > 8$, the intervals $[x_p - \frac{1}{2}\Delta_p,\ x_p + \frac{1}{2}\Delta_p]$ don't overlap, we have

$$\sum_{X_m < x_p \leqslant X_{m+1}} \left\{ \nu\left(x_p + \frac{\Delta_p}{2}\right) - \nu\left(x_p - \frac{\Delta_p}{2}\right) \right\} \leqslant \nu(X_{m+1} + h'_{m+1}) - \nu(X_m - h_m).$$

According to Levinson's theorem (the simpler version from §H.2 of Chapter III is adequate here), we have

$$\frac{A}{\pi}t - \mathrm{o}(t) \leqslant \nu(t) \leqslant \frac{A}{\pi}t + \mathrm{o}(t)$$

for t tending to ∞. Since, in our construction, $\Delta_p = \mathrm{o}(x_p)$ for large p, it follows that $h_m = \mathrm{o}(X_m)$ and $h'_{m+1} = \mathrm{o}(X_{m+1})$ for $m \longrightarrow \infty$; the preceding relation thus implies that

$$\nu(X_{m+1} + h'_{m+1}) - \nu(X_m - h_m) \leqslant \frac{A}{\pi}(X_{m+1} - X_m) + \mathrm{o}(X_{m+1})$$

when $m \longrightarrow \infty$. Hence, since $X_{m+1} - X_m = (1 - 1/\gamma)X_{m+1}$, we have

$$\sum_{X_m < x_p \leqslant X_{m+1}} \left\{ \nu\left(x_p + \frac{\Delta_p}{2}\right) - \nu\left(x_p - \frac{\Delta_p}{2}\right) \right\} \leqslant \left(\frac{A}{\pi} + \varepsilon\right)(X_{m+1} - X_m)$$

for any given $\varepsilon > 0$, as long as m is sufficiently large.

We require an estimate of $(\Delta_p \log \Delta_p)/x_p^2$ for $X_m < x_p \leqslant X_{m+1}$. As p tends to ∞,

$$\Delta_p \log \Delta_p \sim \tfrac{1}{3}p^{-2/3} e^{p^{1/3}} \log\left(\tfrac{1}{3}p^{-2/3} e^{p^{1/3}}\right) \sim \frac{x_p}{3p^{1/3}},$$

so

$$\frac{\Delta_p \log \Delta_p}{2x_p^2} \sim \frac{1}{6x_p \log x_p},$$

a *decreasing* function of p. Therefore, since $X_m = \gamma^m$,

$$\sup_{X_m < x_p \leqslant X_{m+1}} \frac{\Delta_p \log \Delta_p}{2x_p^2} \leqslant \left(\frac{1}{6} + o(1)\right)\frac{1}{mX_m \log\gamma}$$

for large values of m.

Use this estimate together with the preceding one in the above relation. It is found that

$$\sum_{X_m < x_p \leqslant X_{m+1}} v(R_p,\ x_p + i)\frac{R_p \log R_p}{x_p^2} \leqslant \frac{1}{6}\left(\frac{A}{\pi} + 2\varepsilon\right)\frac{\gamma - 1}{\log\gamma}\cdot\frac{1}{m}$$

for sufficiently large values of m, where $\varepsilon > 0$ is arbitrary.

We turn to the sum

$$\sum_{X_m < x_p \leqslant X_{m+1}} \frac{[\Delta_p]\, R_p \log R_p}{x_p^2},$$

for which a *lower bound* is needed. We have

$$\frac{[\Delta_p]\, R_p \log R_p}{x_p^2} \sim \frac{1}{2x_p^2}\left(\tfrac{1}{3}p^{-2/3}x_p\right)^2 \log\left(\tfrac{1}{6}p^{-2/3}x_p\right) \sim \frac{1}{18p^{4/3}}\log x_p$$

$$= \frac{1}{18p} \quad \text{for } p \longrightarrow \infty.$$

Thence, calling p_m the *smallest* value of p for which $x_p > X_m$ and p'_{m+1} the *largest* such value with $x_p \leqslant X_{m+1}$, the preceding sum works out to

$$\left(\frac{1}{18} + o(1)\right)\log\frac{p'_{m+1}}{p_m}$$

when m is large. In that case, $x_{p_m} = \exp(p_m^{1/3})$ is nearly $X_m = \gamma^m$, and $\exp((p'_{m+1})^{1/3})$ nearly γ^{m+1}. So then, p'_{m+1}/p_m is practically equal to $((m+1)/m)^3 \sim 1 + 3/m$, and

$$\log\frac{p'_{m+1}}{p_m} \sim \frac{3}{m}.$$

Thus,

$$\sum_{X_m < x_p \leqslant X_{m+1}} \frac{[\Delta_p]\, R_p \log R_p}{x_p^2} \geqslant \left(\frac{1}{6} - o(1)\right)\cdot\frac{1}{m}$$

for large values of m.

Now combine this result with the one obtained previously. One gets

$$\sum_{X_m < x_p \leqslant X_{m+1}} \{v(R_p,\ x_p + \mathrm{i}) \ - \ [\Delta_p]\} \frac{R_p \log R_p}{x_p^2}$$

$$\leqslant \ \frac{1}{6}\left\{\left(\frac{A}{\pi} \ + \ 2\varepsilon\right)\frac{\gamma - 1}{\log\gamma} \ - \ 1 \ + \ \varepsilon\right\} \cdot \frac{1}{m}$$

(with $\varepsilon > 0$ arbitrary) for large m. *We had, however*, $A/\pi < 1$. It is thus possible to *choose* $\gamma > 1$ *so as to make*

$$\frac{A}{\pi}\frac{\gamma - 1}{\log\gamma} \ < \ 1 \ - \ \delta,$$

say, where δ is a certain number > 0. *Fixing* such a γ, we can *then* take an $\varepsilon > 0$ small enough to ensure that

$$\frac{1}{6}\left\{\left(\frac{A}{\pi} \ + \ 2\varepsilon\right)\frac{\gamma - 1}{\log\gamma} \ - \ 1 \ + \ \varepsilon\right\} \ < \ -\frac{\delta}{7}.$$

The left-hand sum in the previous relation is then

$$\leqslant \ -\frac{\delta}{7m}$$

for sufficiently large m, so

$$\frac{\sqrt{3}\pi}{4}\sum_p \{v(R_p,\ x_p + \mathrm{i}) \ - \ [\Delta_p]\} \ \frac{R_p \log R_p}{x_p^2}$$

does diverge to $-\infty$.

Aside from the general term of this series, our upper bound on

$$\frac{1}{x_p^2}\int_{-R_p}^{R_p} \log|g(x_p + s)|\,\mathrm{d}s$$

involved *two other terms*, each of which is $\sim$ const.Δ_p^2/x_p^2 when p is large. But

$$\frac{\Delta_p^2}{x_p^2} \ \sim \ \frac{1}{9}p^{-4/3} \qquad \text{for } p \longrightarrow \infty,$$

so *the sum of those remaining terms is certainly convergent*. The divergence just established therefore does imply that

$$\sum_{p>8} \frac{1}{x_p^2}\int_{-R_p}^{R_p} \log|g(x_p + s)|\,\mathrm{d}s \ = \ -\infty,$$

and hence that

$$\int_{-\infty}^{\infty} \frac{\log^- |g(x)|}{1+x^2} \mathrm{d}x \quad = \quad \infty$$

as claimed, yielding finally our desired contradiction.

The weight

$$W(x) \quad = \quad \frac{\mathrm{e}^B}{|\Psi(x+\mathrm{i})|} \quad \geqslant \quad 1$$

constructed above thus *admits no multipliers f of exponential type $< \pi$, even though* it enjoys the regularity property

$$|\log\log W(x) \; - \; \log\log W(x')| \quad \leqslant \quad |x-x'|, \qquad x, \; x' \in \mathbb{R},$$

and satisfies the condition

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2} \mathrm{d}x \quad < \quad \infty.$$

2. Discussion

We see that our local regularity requirement and the convergence of the logarithmic integral do not, by themselves, ensure admittance of multipliers. Some other property of the weight is thus really involved.

For the weight W constructed in the example just given we actually had

$$|\log\log W(x) \; - \; \log\log W(x')| \quad \leqslant \quad |x-x'|$$

on $\mathbb{R}$. By this we are reminded that another regularity condition of similar appearance *has* previously been shown to *be* sufficient when combined with the requirement that $\int_{-\infty}^{\infty} (\log W(x)/(1+x^2))\mathrm{d}x \; < \; \infty$. The theorem proved in §C.1 of Chapter X (and reestablished by a different method at the end of §C.5 in this chapter) states that a weight W *does* admit multipliers if its logarithmic integral converges and

$$|\log W(x) \; - \; \log W(x')| \quad \leqslant \quad \text{const.}\,|x-x'|$$

on $\mathbb{R}$. A uniform Lipschitz condition on $\log W(x)$ thus *gives us* enough regularity, although such a requirement on $\log\log W(x)$ *does not.*

An *intermediate property* is in fact *already sufficient.* Consider a continuous weight W with $\log W(x) \; = \; O(x^2)$ near the origin (not a real restriction), and suppose, besides, that $W(x)$ is *even.* As remarked near the end of §B.1, that involves no loss of generality *either*, because $W(x)$ admits multipliers if and only if $W(x)W(-x)$ does. In these circumstances, convergence of the logarithmic integral is equivalent to the condition that

$$\int_0^\infty \frac{\log W(x)}{x^2}\,dx \quad < \quad \infty.$$

When this holds, we know, however, by the corollary at the end of §C.5 that $W(x)$, if it meets the local regularity requirement, *admits multipliers* as long as $\log W(x)/x$ belongs to the Hilbert space $\mathfrak{H}$ studied in §C.4, i.e., that

$$\|\log W(x)/x\|_E \quad < \quad \infty.$$

Problem 62 tells us on the other hand that an even weight $W(x)$ will *have* that property when the above integral is convergent and $\log W(x)$ uniformly Lip 1. These last conditions are thus *more stringent* than the *sufficient* ones furnished by the corollary of §C.5.

This fact leads us to believe, or at least to hope, that the intermediate property just spoken of could serve as basis for the formulation of *necessary and sufficient conditions for admittance of multipliers* by weights satisfying the local regularity requirement. But how one could set out to accomplish that is not immediately apparent, because pointwise behaviour *of the weight itself* seems at the same time *to be involved* and *not to be involved* in the matter.

Behaviour of the weight *itself* seems to *not* be directly involved (beyond the local regularity requirement), because, if $W(x)$ *admits multipliers, so does any weight* $W_1(x)$ *with* $1 \leqslant W_1(x) \leqslant W(x)$. Even when such a $W_1(x)$ meets the local regularity requirement, its behaviour may be very wild in comparison to whatever we may otherwise stipulate for $W(x)$. Were one, for instance to prescribe that $\|\log W(x)/x\|_E < \infty$, there would be weights W_1 *failing* to meet that criterion even though W *answered* to it – see the formula provided by the last theorem of §C.4.

Nevertheless, *some* condition on the behaviour of our weights *does* appear to be involved! Support for this point of view is obtained by putting the theorem on the multiplier (from §C.2) together with those of Szegő (from Chapter II!) and de Branges (Chapter VI, §F).

Consider *any* continuous function $W(x) \geqslant 1$ tending to ∞ for

$x \longrightarrow \pm\infty$, and such that

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2} dx \quad < \quad \infty.$$

Then the weight $\Omega(x) = (1+x^2)W(x)$ also has convergent logarithmic integral, so, by the version of Szegő's theorem set as problem 2 (in Chapter II), there is *no finite sum*

$$s(x) \quad = \quad \sum_{\lambda \geqslant 1} a_\lambda e^{i\lambda x}$$

which can make

$$\int_{-\infty}^{\infty} \frac{|1 \; - \; s(x)|}{\Omega(x)} dx$$

smaller than a certain $\delta > 0$. Hence, for any such $s(x)$, we must have

$$\sup_{x \in \mathbb{R}} \frac{|1 \; - \; s(x)|}{W(x)} \quad \geqslant \quad \frac{\delta}{\pi}.$$

Given any $L > 0$, this holds a fortiori for sums $s(x)$ of the form

$$s(x) \quad = \sum_{1 \leqslant \lambda \leqslant 2L+1} a_\lambda e^{i\lambda x}.$$

Problem 65

(a) Show that in this circumstance there is, corresponding to any L, an entire function $\Phi(z)$ of exponential type L such that

$$\int_{-\infty}^{\infty} \frac{\log^+ |\Phi(x)|}{1+x^2} dx \quad < \quad \infty$$

and

$$W(x_n) \quad \leqslant \quad |\Phi(x_n)|$$

at the points x_n of a *two-way* real sequence Λ with $x_n \neq x_m$ for $n \neq m$ and

$$\frac{n_\Lambda(t)}{t} \quad \longrightarrow \quad \frac{L}{\pi} \qquad \text{for } t \longrightarrow \pm\infty.$$

Here, $n_\Lambda(t)$ denotes (as usual) the number of points of Λ in $[0, \; t]$ when $t \geqslant 0$ and *minus* the number of such points in $[t, \; 0)$ when $t < 0$. (Hint: See §F.3, Chapter VI.)

(b) Show that for the sequence $\{x_n\} = \Lambda$ obtained in (a) we also have

$$\tilde{D}_{\Lambda_+} = \tilde{D}_{\Lambda_-} = \frac{L}{\pi}$$

for $\Lambda_+ = \Lambda \cap [0, \infty)$ and $\Lambda_- = (-\Lambda) \cap (0, \infty)$, with $\tilde{D}$ the Beurling–Malliavin effective density defined in §D.2 of chapter IX. (Hint: See the very end of §E.2, Chapter IX.)

(c) Show that for any given $A > 0$ there is a non-zero entire function $f(z)$ of exponential type $\leqslant A$, bounded on $\mathbb{R}$, with

$$W(x_n)|f(x_n)| \leqslant \text{const.}$$

at the points x_n of the sequence from part (a).

The result obtained in part (c) of this problem holds on the *mere assumptions that* $W(x) \geqslant 1$ is *continuous* and tends to ∞ for $x \longrightarrow \pm\infty$, and that

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2} \mathrm{d}x < \infty.$$

The points x_n on which any of the products $W(x)f(x)$ is *bounded* behave, however, *rather closely* like the ones of the *arithmetic progression*

$$\frac{\pi}{L}n, \qquad n = 0, \pm 1, \pm 2, \ldots.$$

which, for large enough L, seem to 'fill out' the real axis. From this standpoint it appears to be plausible that *some regularity property* of the weight $W(x)$ would be both *necessary and sufficient* to ensure *boundedness of the products* $W(x)f(x)$ *on* $\mathbb{R}$.

These considerations illustrate our present difficulty, but also suggest a way out of it, which is to look for an additional condition *pertaining to a majorant of* $W(x)$ rather than *directly* to the latter. That such an approach is *reasonable* is shown by the first theorem of §B.1, according to which a weight $W(x) \geqslant 1$ meeting the local regularity requirement (with constants C, α and L) and satisfying $\int_{-\infty}^{\infty} (\log W(x)/(1+x^2))\mathrm{d}x < \infty$ *admits multipliers if and only if a certain* $\mathscr{C}_\infty$ *majorant of it also does so.* For that majorant one may take the weight

$$\Omega(x) = M \exp\left\{\frac{4}{\pi\alpha}\int_{-\infty}^{\infty} \frac{L \log W(t)}{(x-t)^2 + L^2}\mathrm{d}t\right\}$$

where M is a large constant, and then $|\mathrm{d}(\log\log\Omega(x))/\mathrm{d}x| \leqslant 1/L$ on $\mathbb{R}$.

This idea actually underlies much of what is done in §B.1. One may of course use the *even* $\mathscr{C}_\infty$ *majorant* $\Omega(x)\Omega(-x)$ *instead* – see the remark just preceding problem 52.

Let us try then to characterize a weight's admittance of multipliers by the *existence for it of some even majorant also admitting multipliers and having, in addition, some specific kind of regularity.* What we have in mind at present is essentially the regularity embodied in the *intermediate property* described earlier in this article. We think the criterion should be that $W(x)$ have an even $\mathscr{C}_\infty$ majorant $\Omega(x)$ with $\log^+\log\Omega(x)$ uniformly Lip 1, $\int_0^\infty (\log\Omega(x)/x^2)\,dx < \infty$, and $\|\log\Omega(x)/x\|_E < \infty$.

A minor hitch encountered at this point is easily taken care of. The trouble is that neither of the last two of the conditions on Ω is compatible with Ω's being a *majorant* of W when $W(x) > 1$ on a neighborhood of the origin. That, however, should not present a real problem because admittance of multipliers by a finite weight W meeting the local regularity requirement *does not depend* on the behaviour of $W(x)$ near 0 – according to the first lemma of §B.1, $W(x)$, if *not bounded* on finite intervals, would have to be *identically infinite* on one of length > 0. We can thus allow majorants $\Omega(x)$ which are merely $\geq W(x)$ *for* $|x|$ *sufficiently large*, instead of for *all* real x. In that way we arrive at a statement having (we hope) some chances of being true:

A finite weight $W(x) \geq 1$ *meeting the local regularity requirement admits multipliers if and only if there exists an even* $\mathscr{C}_\infty$ *function* $\Omega(x) \geq 1$ *with* $\Omega(0) = 1$ (*making* $\log\Omega(x) = O(x^2)$ *near* 0),

$\log^+\log\Omega(x)$ *uniformly* Lip 1 *on* $\mathbb{R}$,

$\Omega(x) \geq W(x)$ *whenever* $|x|$ *is sufficiently large,*

$$\int_0^\infty \frac{\log\Omega(x)}{x^2}\,dx < \infty,$$

and

$$\|\log\Omega(x)/x\|_E < \infty.$$

According to what we already know, the 'if' part of this proposition *is* valid, because a weight Ω with the stipulated properties *does* admit multipliers (it enjoys the *intermediate property*), and hence W must *also* do so. But the 'only if' part is still just a *conjecture.*

Support for believing 'only if' to be *correct* comes from a review of how the energy norm $\|\log W(x)/x\|_E$ entered into the argument of §C.5. There, as in §C.4 of Chapter VIII, that was through the use of Schwarz' inequality for the inner product $\langle\ ,\ \rangle_E$. This encourages us to look for a proof of the 'only if' part based on the Schwarz inequality's being best possible.

There is, on the other hand, nothing to prevent anyone's *doubting* the truth of 'only if'. We have again to choose between two approaches – to look for a proof or try constructing a counterexample. The *second* approach proves fruitful here.

In article 4 we give an example showing that the *existence* of an Ω having the properties enumerated above is *not necessary* for the admittance of multipliers by a weight W. The 'essential' condition we are seeking turns out to be more elusive than at first thought.

The reader who is still following the present discussion is urged not to lose patience with this §'s chain of seesaw arguments and interspersed seemingly artificial examples. By going on in such fashion we will arrive at a clear vision of the object of our search. See the first paragraph of article 5.

Our example's construction depends on an auxiliary result relating the norm $\|\ \ \|_E$ of a certain kind of Green potential to the same norm of a majorant for it. This we attend to in the next article.

3. Comparison of energies

The weight W to be presently constructed is similar to the one considered in article 1, being of the form

$$W(x) \quad = \quad \frac{\text{const.}}{\exp F(x+\mathrm{i})},$$

where $F(z)$, bounded above on the real axis, is given by the formula

$$F(z) \quad = \quad \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}\mu(t)$$

with $\mu(t)$ *increasing* and $O(t)$ (for both large and small values of t) on $[0, \infty)$. $W(x)$ is thus much like the *reciprocal* of the modulus of an entire function of exponential type.

From $\mu(t)$ one can, as in §C.5 of Chapter VIII, form another increasing function $\nu(t)$, this one defined* and infinitely differentiable on $\mathbb{R}$, $O(|t|)$

* by the formula $\nu'(t) = (1/\pi)\int_0^\infty \{((t+s)^2+1)^{-1} + ((t-s)^2+1)^{-1}\}\,\mathrm{d}\mu(t)$; see next article, about 3/4 of the way through.

there and *odd*, such that

$$F(x+\mathrm{i}) - F(\mathrm{i}) = \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\nu(t)$$

for $x \in \mathbb{R}$. The right-hand integral can in turn be converted to

$$-x\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\left(\frac{\nu(t)}{t}\right),$$

and our weight $W(x)$ thereby expressed in the form

$$\text{const.} \quad + \quad x\int_{-\infty}^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\left(\frac{\nu(t)}{t}\right).$$

The reader should take care to distinguish between this representation and the one which has frequently been used in this book for certain entire functions $G(z)$ of exponential type. The latter also involves a function $\nu(t)$, increasing and $\mathrm{O}(t)$ on $[0, \infty)$, but reads

$$\log|G(x)| = -x\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\left(\frac{\nu(t)}{t}\right)$$

with a *minus sign* in front of the integral. It will eventually become clear that this *difference in sign* is very important for the matter under discussion.

The weight W we will be working with in the next article is closely related to the *Green potentials* studied in §C.4, since

$$\frac{1}{x}\log\left(\frac{W(x)}{W(0)}\right) = \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\left(\frac{\nu(t)}{t}\right).$$

We will want to be able to affirm that this expression belongs to the Hilbert space $\mathfrak{H}$ considered in §C.4 provided that there is some even $\Omega(x) \geqslant 1$ with $\log\Omega(x)/x$ in $\mathfrak{H}$ and $\int_0^\infty (\log\Omega(x)/x^2)\mathrm{d}x$ finite, such that $W(x) \leqslant \Omega(x)$ for all x of sufficiently large modulus.

This kind of comparison is well known for the simpler circumstance involving *pure potentials*. Those are the potentials

$$U_\rho(x) = \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t)$$

corresponding to *positive* measures ρ. *Cartan's lemma* says that if for two of them, U_ρ and U_σ, we have $U_\rho(x) \leqslant U_\sigma(x)$ for $x \geqslant 0$, then

$\|U_\rho\|_E \leqslant \|U_\sigma\|_E$. Proof:

$$\|U_\rho\|_E^2 = \int_0^\infty U_\rho(x)\mathrm{d}\rho(x) \leqslant \int_0^\infty U_\sigma(x)\mathrm{d}\rho(x) = \int_0^\infty U_\rho(x)\mathrm{d}\sigma(x)$$
$$\leqslant \int_0^\infty U_\sigma(x)\mathrm{d}\sigma(x) = \|U_\sigma\|_E^2 \quad !$$

The result obviously depends greatly on the positivity of ρ and σ.

For our weight W, $(1/x)\log(W(x)/W(0))$ is of the form $U_\rho(x)$, but the measure ρ is *not positive*. Instead,

$$\mathrm{d}\rho(t) = \mathrm{d}\left(\frac{v(t)}{t}\right) = \frac{\mathrm{d}v(t)}{t} - \frac{v(t)}{t}\frac{\mathrm{d}t}{t},$$

and all that the properties of our v give us is the relation

$$\mathrm{d}\rho(t) \geqslant -\text{const.}\frac{\mathrm{d}t}{t}.$$

As we know,

$$\int_0^\infty \log\left|\frac{x+t}{x-t}\right|\frac{\mathrm{d}t}{t} = \frac{\pi^2}{2} \quad \text{for } x > 0;$$

the measure σ on $(0, \infty)$ with $\mathrm{d}\sigma(t) = \mathrm{d}t/t$ thus *just misses* having finite energy. *Finiteness* of $\|U_\rho\|_E$, *if realized*, must hence be due to *interference* between $\mathrm{d}v(t)/t$ and $v(t)\,\mathrm{d}t/t^2$. A version of Cartan's result is nevertheless still available in this situation.

In order to deal with the measure $\mathrm{d}t/t$ we will use the following two elementary lemmas.

Lemma. *For* $A > 0$, *we have*:

$$\int_0^A\int_A^\infty \log\left|\frac{x+t}{x-t}\right|\frac{\mathrm{d}t}{t}\frac{\mathrm{d}x}{x} = 2 + \frac{2}{3^3} + \frac{2}{5^3} + \cdots;$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_A^\infty \log\left|\frac{x+t}{x-t}\right|\frac{\mathrm{d}t}{t} = \frac{1}{x}\log\left|\frac{x+A}{x-A}\right| \quad \text{for } x > 0,\ x \neq A;$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^A \log\left|\frac{x+t}{x-t}\right|\frac{\mathrm{d}t}{t} = -\frac{1}{x}\log\left|\frac{x+A}{x-A}\right| \quad \text{for } x > 0,\ x \neq A.$$

Proof. To establish the first relation, make the changes of variable

$$\xi = \frac{x}{A}, \quad \tau = \frac{t}{A}$$

and expand the logarithm in powers of ξ/τ, then integrate term by term.

For the last two relations, we use a different change of variable, putting $s = t/x$. Then the left side of the *second* relation becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{A/x}^{\infty}\log\left|\frac{1+s}{1-s}\right|\frac{\mathrm{d}s}{s},$$

and this may be worked out for $x \neq A$ by the fundamental theorem of calculus. The third relation follows in like manner.

Lemma. *Let $\rho(t) = \int_0^t \mathrm{d}\rho(\tau)$ be bounded for $0 \leqslant t < \infty$. Then, for $A > 0$, the two expressions*

$$\int_0^A \int_A^{\infty} \log\left|\frac{x+t}{x-t}\right| \frac{\mathrm{d}t}{t}\,\mathrm{d}\rho(x),$$

$$\int_0^A \int_A^{\infty} \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t)\,\frac{\mathrm{d}x}{x},$$

are bounded in absolute value by quantities independent of A.

Proof. Considering the *second* expression, we have, for large $M > A$ and any $M' > M$,

$$\int_M^{M'} \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t)$$

$$= \quad \rho(M')\log\left|\frac{1+(x/M')}{1-(x/M')}\right| \; - \; \rho(M)\log\left|\frac{1+(x/M)}{1-(x/M)}\right| \quad + \quad 2x\int_M^{M'} \frac{\rho(t)}{t^2 - x^2}\,\mathrm{d}t$$

whenever $0 < x < A$. Because $|\rho(t)|$ is bounded, the right side is equal to *x times a quantity uniformly small for* $0 < x < A$ when M and M' are both large. The second expression is therefore equal to the *limit*, for $M \longrightarrow \infty$, of the double integrals

$$\int_0^A \int_A^M \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t)\,\frac{\mathrm{d}x}{x}.$$

Any one of these is equal to

$$\int_A^M \left(\int_0^A \log\left|\frac{x+t}{x-t}\right| \frac{\mathrm{d}t}{t} \right) \mathrm{d}\rho(x);$$

here we use partial integration on the *outer* integral and refer to the third formula provided by the preceding lemma. In that way we get

$$\rho(M)\int_0^A \log\left|\frac{M+t}{M-t}\right|\frac{dt}{t} \quad - \quad \rho(A)\int_0^A \log\left|\frac{A+t}{A-t}\right|\frac{dt}{t}$$
$$+ \quad \int_A^M \log\left|\frac{x+A}{x-A}\right|\frac{\rho(x)}{x}dx.$$

Remembering that

$$\int_0^\infty \log\left|\frac{x+t}{x-t}\right|\frac{dt}{t} \quad = \quad \frac{\pi^2}{2} \qquad \text{for } x > 0,$$

we see that the last expression is $\leqslant 3\pi^2 K/2$ in absolute value if $|\rho(t)| \leqslant K$ on $[0, \infty)$. Thence,

$$\left|\int_0^A \int_A^\infty \log\left|\frac{x+t}{x-t}\right| d\rho(t)\frac{dx}{x}\right| \quad \leqslant \quad \frac{3\pi^2}{2}K$$

independently of $A > 0$.

Treatment of the *first* expression figuring in the lemma's statement is similar (and easier). We are done.

Now we are ready to give our version of Cartan's lemma. So as not to obscure its main idea with fussy details, we avoid insisting on more generality than is needed for the next article. An alternative formulation is furnished by problem 68 below.

Theorem. *Let $\omega(x)$, even and tending to ∞ for $x \longrightarrow \pm\infty$, be given by a formula*

$$\omega(x) \quad = \quad -\int_0^\infty \log\left|1 \; - \; \frac{x^2}{t^2}\right| d\nu(t),$$

where $\nu(t)$, odd and increasing, is $\mathscr{C}_\infty$ on $\mathbb{R}$, with $\nu(t)/t$ bounded there. Suppose there is an even function $\Omega(x) \geqslant 1$, with

$$\int_0^\infty \frac{\log\Omega(x)}{x^2}dx \quad < \quad \infty$$

and $\log\Omega(x)/x$ in the Hilbert space $\mathfrak{H}$ of §C.4, such that

$$\omega(x) \;\leqslant\; \log\Omega(x)$$

for all x of sufficiently large absolute value. Then $\omega(x)/x$ also belongs to

$\mathfrak{H}$, *and*

$$\int_0^\infty \frac{\omega(x)}{x^2}\,\mathrm{d}\nu(x) \quad < \quad \infty.$$

Proof. If there is an Ω meeting the stipulated conditions, there is an L such that

$$\omega(x) \;\leqslant\; \log\Omega(x) \qquad \text{for } x \geqslant L.$$

Because $\omega(x) \longrightarrow \infty$ for $x \longrightarrow \infty$, we can take (and *fix*) L large enough to also make

$$\omega(x) \;\geqslant\; 0 \qquad \text{for } x \geqslant L.$$

The given properties of $\nu(t)$ make $\omega(0) = 0$ and $\omega(x)$ infinitely differentiable* on $\mathbb{R}$. Therefore, since $\omega(x)$ is even, we have $\omega(x) = O(x^2)$ near 0, and, having chosen L, we can find an M such that

$$-x^2M \;\leqslant\; \omega(x) \;\leqslant\; x^2M \qquad \text{for } 0 \leqslant x \leqslant L.$$

According to the first lemma of §B.4, Chapter VIII, the given formula for $\omega(x)$ can be rewritten

$$\omega(x) \quad = \quad x\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\left(\frac{\nu(t)}{t}\right), \qquad x > 0.$$

We put

$$\rho(t) \quad = \quad \frac{\nu(t)}{t},$$

making $\rho(t) \;\geqslant\; 0$ and bounded by hypothesis, with

$$\mathrm{d}\rho(t) \;\geqslant\; -\frac{\nu(t)}{t}\frac{\mathrm{d}t}{t} \;\geqslant\; -C\,\frac{\mathrm{d}t}{t}.$$

* To check infinite differentiability of $\omega(x)$ in $(-A,\ A)$, say, take any *even* $\mathscr{C}_\infty$ function $\varphi(t)$ equal to 1 for $|t| \leqslant A$ and to 0 for $|t| \geqslant 2A$. Then, since $\nu'(t)$ is also even, we have

$$\omega(x) \;=\; \int_A^\infty \log|1 - x^2/t^2|(1-\varphi(t))\nu'(t)\,\mathrm{d}t \;+\; \int_{-2A}^{2A} \log|x-t|\,\varphi(t)\nu'(t)\,\mathrm{d}t$$
$$-\;\int_{-2A}^{2A} \log|t|\,\varphi(t)\nu'(t)\,\mathrm{d}t.$$

The *first* integral on the right is clearly $\mathscr{C}_\infty$ in x for $|x| < A$. When $|x| < A$, the *second* one can be rewritten as $\int_{-3A}^{3A}\varphi(x-s)\nu'(x-s)\log|s|\,\mathrm{d}s$, and this, like φ and ν', is $\mathscr{C}_\infty$ (in x), since $\log|s| \in L_1(-3A, 3A)$.

In order to keep our notation simple, let us, without real loss of generality, assume that $C = 1$, i.e., that

$$0 \leqslant \rho(t) \leqslant 1$$

and

$$\mathrm{d}\rho(t) \geqslant -\frac{\mathrm{d}t}{t}.$$

We now consider the Green potentials

$$U_A(x) = \int_0^A \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t),$$

where $A > L$. Since $v(t)$ is $\mathscr{C}_\infty$ and $\rho(t) = v(t)/t$ bounded, it is readily verified with the help of l'Hôpital's rule that $\rho(t)$ (taken as $v'(0)$ for $t = 0$) is *differentiable right down to the origin*, and that

$$\rho'(t) = \frac{tv'(t) - v(t)}{t^2}$$

stays bounded as $t \longrightarrow 0$. The quantity $|\rho'(t)|$ is thus bounded on each of the finite segments $[0, A]$, and the double integrals

$$\int_0^A \int_0^A \log\left|\frac{x+t}{x-t}\right| |\mathrm{d}\rho(t)|\,|\mathrm{d}\rho(x)|$$

hence finite. Each of the potentials U_A therefore belongs to $\mathfrak{H}$. We proceed to obtain an upper bound on $\|U_A\|_E$ which, for $A > L$, is *independent* of A.

By the absolute convergence just noted we have, according to §C.4,

$$\|U_A\|_E^2 = \int_0^A U_A(x)\,\mathrm{d}\rho(x).$$

In view of the above formula for $\omega(x)$, we can write

$$U_A(x) = \frac{\omega(x)}{x} - \int_A^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t).$$

When $A \longrightarrow \infty$, the integral on the right tends to zero *uniformly* for $0 \leqslant x \leqslant L$ (see beginning of the proof of the *second* of the above lemmas). Therefore, $|\rho'(x)|$ being bounded for $0 \leqslant x \leqslant L$,

$$\int_0^L U_A(x)\,\mathrm{d}\rho(x) = \int_0^L \frac{\omega(x)}{x}\rho'(x)\,\mathrm{d}x + o(1)$$

for $A \longrightarrow \infty$. Referring to one of the initial inequalities for $\omega(x)$, we see that

$$\int_0^L U_A(x)\,\mathrm{d}\rho(x) \quad \leqslant \quad \int_0^L xM|\rho'(x)|\,\mathrm{d}x \; + \; \mathrm{o}(1)$$

for large A.

Our main work is with the integral $\int_L^A U_A(x)\,\mathrm{d}\rho(x)$. Since $\mathrm{d}\rho(t) \geqslant -\mathrm{d}t/t$, it is convenient to put

$$\mathrm{d}\rho(t) + \frac{\mathrm{d}t}{t} \quad = \quad \mathrm{d}\sigma(t),$$

getting a *positive* measure σ on $[0,\infty)$. The above relation connecting $U_A(x)$ and $\omega(x)/x$ then gives us

$$U_A(x) \quad \leqslant \quad \frac{\omega(x)}{x} \quad + \quad \int_A^\infty \log\left|\frac{x+t}{x-t}\right| \frac{\mathrm{d}t}{t}, \qquad x > 0;$$

$$U_A(x) \quad \geqslant \quad \frac{\omega(x)}{x} \quad - \quad \int_A^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\sigma(t), \qquad x > 0.$$

Thence, by our initial relations for $\omega(x)$,

$$U_A(x) \quad \leqslant \quad \frac{\log\Omega(x)}{x} \quad + \quad \int_A^\infty \log\left|\frac{x+t}{x-t}\right| \frac{\mathrm{d}t}{t}, \qquad x > L;$$

$$U_A(x) \quad \geqslant \quad -\int_A^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\sigma(t), \qquad x \geqslant L.$$

Since $\log\Omega(x) \geqslant 0$ by hypothesis and $\log|(x+t)/(x-t)| \geqslant 0$ for x and $t \geqslant 0$, the first of these inequalities yields

$$\int_L^A U_A(x)\,\mathrm{d}\sigma(x) \quad \leqslant \quad \int_0^A \frac{\log\Omega(x)}{x}\,\mathrm{d}\sigma(x) \quad + \quad \int_0^A \int_A^\infty \log\left|\frac{x+t}{x-t}\right| \frac{\mathrm{d}t}{t}\,\mathrm{d}\sigma(x)$$

$$\leqslant \quad \int_0^\infty \frac{\log\Omega(x)}{x^2}\,\mathrm{d}x \quad + \quad \int_0^A \frac{\log\Omega(x)}{x}\,\mathrm{d}\rho(x) \quad + \quad \int_0^A \int_A^\infty \log\left|\frac{x+t}{x-t}\right| \frac{\mathrm{d}t}{t}\,\mathrm{d}\rho(x)$$

$$+ \quad \int_0^A \int_A^\infty \log\left|\frac{x+t}{x-t}\right| \frac{\mathrm{d}t}{t}\,\frac{\mathrm{d}x}{x}.$$

Similarly, from the second inequality,

$$-\int_L^A U_A(x)\,\frac{\mathrm{d}x}{x} \quad \leqslant \quad \int_0^A \int_A^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\sigma(t)\,\frac{\mathrm{d}x}{x}$$

$$= \quad \int_0^A \int_A^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t)\,\frac{\mathrm{d}x}{x} \quad + \quad \int_0^A \int_A^\infty \log\left|\frac{x+t}{x-t}\right| \frac{\mathrm{d}t}{t}\,\frac{\mathrm{d}x}{x}.$$

Putting these results together and then adding on the one obtained previously, we find that for large $A > L$,

$$\|U_A\|_E^2 = \int_0^A U_A(x)\,d\rho(x)$$

$$\leqslant o(1) + \int_0^L xM|\rho'(x)|\,dx + \int_0^\infty \frac{\log\Omega(x)}{x^2}\,dx + \int_0^A \frac{\log\Omega(x)}{x}\,d\rho(x)$$

$$+ \int_0^A\int_A^\infty \log\left|\frac{x+t}{x-t}\right| \frac{dt}{t}\,d\rho(x) + \int_0^A\int_A^\infty \log\left|\frac{x+t}{x-t}\right| d\rho(t)\,\frac{dx}{x}$$

$$+ 2\int_0^A\int_A^\infty \log\left|\frac{x+t}{x-t}\right| \frac{dt}{t}\,\frac{dx}{x}.$$

It is part of our hypothesis that

$$\int_0^\infty \frac{\log\Omega(x)}{x^2}\,dx < \infty.$$

Because $0 \leqslant \rho(t) \leqslant 1$, the *fourth* and *fifth* of the right-hand integrals are *bounded* (by quantities independent of A) according to the second of the above lemmas. By the first of those lemmas, the *sixth* integral is equal to a finite constant independent of A. We thus have a constant c independent of A such that

$$\|U_A\|_E^2 \leqslant c + \int_0^A \frac{\log\Omega(x)}{x}\,d\rho(x)$$

for large $A > L$.

Recalling, however, that

$$U_A(x) = \int_0^A \log\left|\frac{x+t}{x-t}\right| d\rho(t)$$

and that $\log\Omega(x)/x$ is in $\mathfrak{H}$ by hypothesis, we see from the fifth lemma of §C.4 that

$$\int_0^A \frac{\log\Omega(x)}{x}\,d\rho(x) = \left\langle \frac{\log\Omega(x)}{x},\ U_A(x)\right\rangle_E \leqslant \left\|\frac{\log\Omega(x)}{x}\right\|_E \|U_A\|_E.$$

The preceding relation thus becomes

$$\|U_A\|_E^2 \leqslant c + \|\log\Omega(x)/x\|_E\,\|U_A\|_E.$$

Knowing, then, that $\|U_A\|_E < \infty$, we get by 11th grade algebra (!) that

$$\|U_A\|_E \leqslant \frac{1}{2}\left(\|\log\Omega(x)/x\|_E + \sqrt{(\|\log\Omega(x)/x\|_E^2 + 4c)}\right)$$

for large $A > L$, *with the bound on the right independent of* A.

Now it is easy to show that $\omega(x)/x$ belongs to $\mathfrak{H}$. Since $\omega(x)/x$ is *odd*, we need, according to the last theorem of §C.4, merely check that $\|\omega(x)/x\|_E < \infty$ where, for $\|\ \ \|_E$, the general definition adopted towards the middle of §C.4 is taken. As observed earlier,

$$U_A(x) \longrightarrow \frac{\omega(x)}{x} \qquad \text{u.c.c. in } [0, \infty)$$

for $A \longrightarrow \infty$. Thence, by the second theorem of §C.4 and Fatou's lemma,

$$\|\omega(x)/x\|_E^2 \leqslant \liminf_{A\to\infty} \|U_A\|_E^2.$$

(Cf. the discussion of how $\mathfrak{H}$ is formed, about half way into §C.4.) The result just found therefore implies that

$$\|\omega(x)/x\|_E \leqslant \frac{1}{2}\big(\|\log\Omega(x)/x\|_E + \sqrt{(\|\log\Omega(x)/x\|_E^2 + 4c)}\big),$$

making $\omega(x)/x \in \mathfrak{H}$. (Appeal to the *last* theorem of §C.4 can be avoided here. A sequence of the U_A with $A \longrightarrow \infty$ certainly *converges weakly* to *some* element, say U, of $\mathfrak{H}$. Some convex linear combinations of those U_A then *converge in norm* $\|\ \ \|_E$ to U, which then can be easily identified with $\omega(x)/x$, reasoning as in the discussion towards the middle of §C.4.)

Once it is known that $\omega(x)/x \in \mathfrak{H}$, the rest of the theorem is almost immediate. The relations for $\omega(x)$ given near the beginning of this proof make

$$\int_0^\infty \frac{|\omega(x)|}{x^2}\,dx \leqslant \int_0^L M\,dx + \int_L^\infty \frac{\log\Omega(x)}{x^2}\,dx$$

$$\leqslant ML + \int_0^\infty \frac{\log\Omega(x)}{x^2}\,dx < \infty,$$

so, since (here) $0 \leqslant \nu(x)/x \leqslant 1$, we have

$$\int_0^\infty \frac{|\omega(x)|}{x}\,\frac{\nu(x)}{x^2}\,dx < \infty.$$

It is thus enough to verify that

$$\int_0^\infty \frac{\omega(x)}{x}\,d\left(\frac{\nu(x)}{x}\right) < \infty$$

in order to show that

$$\int_0^\infty \frac{\omega(x)}{x^2}\,d\nu(x)$$

is finite. Since $|\mathrm{d}(v(x)/x)/\mathrm{d}x|$ is bounded for $0 \leqslant x \leqslant L$ with $|\omega(x)/x^2| \leqslant M$ there, while $\omega(x) \geqslant 0$ for $x > L$, the *first* of these integrals is *perfectly unambiguous* according to the observation just made, and equal to the limit, for $A \longrightarrow \infty$, of

$$\int_0^A \frac{\omega(x)}{x}\,\mathrm{d}\left(\frac{v(x)}{x}\right).$$

Here we may again resort to the fifth lemma of §C.4, according to which any one of the last integrals, identical with $\int_0^A (\omega(x)/x)\,\mathrm{d}\rho(x)$, is just the inner product

$$\left\langle \frac{\omega(x)}{x},\ U_A(x)\right\rangle_E \quad \leqslant \quad \left\|\frac{\omega(x)}{x}\right\|_E \|U_A\|_E.$$

Plugging in the bounds found above, we see that for large $A > L$,

$$\int_0^A \frac{\omega(x)}{x}\,\mathrm{d}\left(\frac{v(x)}{x}\right) \quad \leqslant \quad \frac{1}{4}\left(\left\|\frac{\log \Omega(x)}{x}\right\|_E + \sqrt{\left(\left\|\frac{\log\Omega(x)}{x}\right\|_E^2 + 4c\right)}\right)^2,$$

and, making $A \longrightarrow \infty$, we arrive at the desired conclusion.

The theorem is proved.

The variant of this result referred to earlier, to be given in problem 68, applies to functions $\omega(x)$ of the form

$$\omega(x) \quad = \quad x\int_0^\infty \log\left|\frac{x+t}{x-t}\right|\mathrm{d}\rho(t),$$

where the *only* assumptions on the measure ρ are that it is absolutely continuous, with $\rho'(t)$ bounded on each finite interval, and that

$$\mathrm{d}\rho(t) \quad \geqslant \quad -C\frac{\mathrm{d}t}{t} \qquad \text{for } t \geqslant 1.$$

That generalization is related to some material of independent interest taken up in problems 66 and 67.

Let us, as usual, write

$$U_\rho(x) \quad = \quad \int_0^\infty \log\left|\frac{x+t}{x-t}\right|\mathrm{d}\rho(t).$$

Under our assumption on ρ, the integral on the right is certainly unambiguously defined because

$$\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \min(1,\ 1/t)\,\mathrm{d}t$$

is finite for $x > 0$ and, if K is large enough,

$$d\sigma(t) \quad = \quad d\rho(t) \ + \ K\min(1,\ 1/t)\,dt$$

is $\geqslant 0$ for $t \geqslant 0$.* The preceding integral is indeed $O(x\log(1/x))$ for small values of $x > 0$, so, by applying Fubini's theorem separately to

$$\int_0^A \int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\sigma(t)\frac{dx}{x}$$

and to the similar expression with $d\sigma(t) - d\rho(t)$ standing in place of $d\sigma(t)$, we see that for each finite $A > 0$, $\int_0^A (U_A(x)/x)\,dx$ is well defined and equal to

$$\int_0^\infty \int_0^A \log\left|\frac{x+t}{x-t}\right| \frac{dx}{x}\, d\rho(t).$$

By writing $d\rho(t)$ one more time as the difference of the two positive measures $d\sigma(t)$ and $K\min(1,\ 1/t)\,dt$, one verifies that the last expression is in turn equal to

$$\lim_{M\to\infty} \int_0^M \int_0^A \log\left|\frac{x+t}{x-t}\right| \frac{dx}{x}\, d\rho(t).$$

Problem 66

In this problem, we suppose that the above assumptions on the measure ρ hold, and that in addition the integrals

$$\int_0^A \frac{U_\rho(x)}{x}\,dx$$

are *bounded* as $A \longrightarrow \infty$. The object is to then obtain a preliminary grip on the magnitude of $|\rho(t)|$.

(a) Show that for each M and A.

$$\int_0^M \int_0^A \log\left|\frac{x+t}{x-t}\right| \frac{dx}{x}\, d\rho(t) \quad = \quad \rho(M)\int_0^A \log\left|\frac{x+M}{x-M}\right| \frac{dx}{x}$$
$$+ \quad \int_0^M \log\left|\frac{t+A}{t-A}\right| \frac{\rho(t)}{t}\,dt.$$

(Hint: cf. proof of second lemma, beginning of this article.)

* Only *this* property of ρ is used in problems 66 and 67; absolute continuity of that measure plays no rôle in them (save that $\rho(t)$ should be replaced by $\rho(t) - \rho(0+)$ throughout if ρ has point mass at the origin).

(b) Hence show that

$$\frac{\rho(M)}{M} \longrightarrow 0 \quad \text{as } M \longrightarrow \infty.$$

(Hint: $\rho(t) \geqslant -K(1+\log^+ t)$ for $t > 0$, making

$$-\int_0^M \log\left|\frac{t+A}{t-A}\right| \frac{\rho(t)}{t}\,\mathrm{d}t \quad \leqslant \quad \text{const.}\log A$$

with a constant independent of M, for $A > \mathrm{e}$, say. Deduce that for fixed large A and $M \longrightarrow \infty$,

$$2\frac{\rho(M)}{M}A \quad \leqslant \quad \mathrm{O}(1) \;+\; \text{const.}\log A. \qquad)$$

(c) Then show that

$$\int_0^A \frac{U_\rho(x)}{x}\,\mathrm{d}x \quad = \quad \int_0^\infty \log\left|\frac{t+A}{t-A}\right| \frac{\rho(t)}{t}\,\mathrm{d}t.$$

(d) Show that for large $t > 1$ we not only have $\rho(t) \geqslant -\text{const.}\log t$ but also $\rho(t) \leqslant \text{const.}\log t$. (Hint: Wlog, $\mathrm{d}\rho(t) \geqslant -\mathrm{d}t/t$ for $t > 1$. Assuming that for some large A we have $\rho(A) \geqslant k\log A$ with a number $k > 0$, it follows that

$$\rho(t) \quad \geqslant \quad k\log A \;-\; \log\frac{t}{A} \qquad \text{for } t > A.$$

At the same time, $\rho(t) \geqslant -\mathrm{O}(1) - \log^+ t$ for $0 < t < A$. Use result of (c) with these relations to get a *lower bound* on $\int_0^A (U_\rho(x)/x)\mathrm{d}x$ involving k and $\log A$, thus arriving at an *upper bound* for k.)

Problem 67

Continuing with the material of the preceding problem, we now assume that

$$\lim_{A\to\infty} \int_0^A \frac{U_\rho(x)}{x}\,\mathrm{d}x$$

exists (and is finite). It is proposed to show by means of an elementary Tauberian argument that $\rho(t)$ then *also has a limit* (equal to $2/\pi^2$ times the preceding one) for $t \longrightarrow \infty$. Essentially this result was used by Beurling and Malliavin* in their original proof of the Theorem on the

* under the milder condition on ρ pointed out in the preceding footnote – they in fact assumed only that the measure ρ on $[0, \infty)$ satisfies $\mathrm{d}\rho(t) \geqslant -\text{const.}\,\mathrm{d}t/t$ there, but then the conclusion of problem 67 holds just as well because the existence

Multiplier.

(a) Show that for a and $b > 0$,

$$\int_0^\infty \log\left|\frac{x+a}{x-a}\right| \log\left|\frac{x+b}{x-b}\right| \mathrm{d}x = \pi^2 \min(a,\ b).$$

(Hint: We have

$$\frac{1}{\pi}\log\left|\frac{x+a}{x-a}\right| = \frac{1}{\pi}\int_{-a}^{a} \frac{\mathrm{d}t}{x-t}.$$

Apply the L_2 theory of Hilbert transforms sketched at the end of §C.1, Chapter VIII.)

(b) Hence derive the formula

$$\int_0^\infty \log\left|\frac{t+x}{t-x}\right| \left\{2\log\left|\frac{x+A}{x-A}\right| - \log\left|\frac{x+(1+\delta)A}{x-(1+\delta)A}\right| - \log\left|\frac{x+(1-\delta)A}{x-(1-\delta)A}\right|\right\}\mathrm{d}x$$

$$= \pi^2(\delta A - |t-A|)^+,$$

valid for $t > 0$, $A > 0$ and $0 < \delta < 1$.

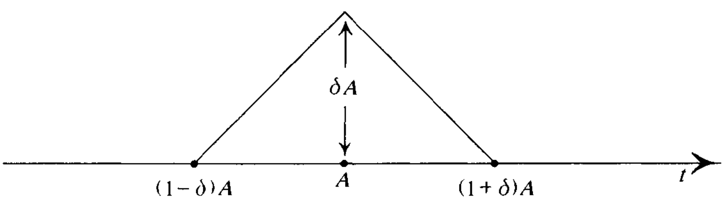

Figure 251

(c) Then, referring to part (c) of the previous problem, prove that

$$\frac{1}{A}\int_0^\infty \left(\int_0^x \frac{U_\rho(\xi)}{\xi}\mathrm{d}\xi\right)\left\{2\log\left|\frac{x+A}{x-A}\right| - \log\left|\frac{x+(1+\delta)A}{x-(1+\delta)A}\right|\right.$$

$$\left. - \log\left|\frac{x+(1-\delta)A}{x-(1-\delta)A}\right|\right\}\mathrm{d}x = \pi^2\delta\int_{(1-\delta)A}^{(1+\delta)A}\left(1 - \frac{|t-A|}{\delta A}\right)\frac{\rho(t)}{t}\mathrm{d}t.$$

(Hint: Use the formula obtained in (b) together with Fubini's theorem. In justifying application of the latter, the bound on $|\rho(t)|$ found at the end of the preceding problem comes in handy. One should also observe that the expression in $\{\quad\}$ involved in the left-hand integrand belongs to $L_1(0, \infty)$ and is in fact $O(1/x^3)$ for large x.)

of $\lim_{A\to\infty}\int_0^A (U_\rho(x)/x)\,\mathrm{d}x$ is not affected when ρ is replaced by its restriction to $[1, \infty)$

(d) Assume now that

$$\int_0^x \frac{U_\rho(\xi)}{\xi}\,d\xi \quad\longrightarrow\quad l$$

for $x \longrightarrow \infty$. Show that then the *right side* of the relation establish in (c) *tends to a limit*, equal to $(2\delta^2 + O(\delta^3))l$, as $A \longrightarrow \infty$. (Hint: The *left side* of the relation referred to can be rewritten as

$$\int_0^\infty \left(\int_0^{uA} \frac{U_\rho(\xi)}{\xi}\,d\xi \right) \varphi_\delta(u)\,du,$$

where $\varphi_\delta(u)$ is a certain L_1 function not involving A. To compute $\int_0^\infty \varphi_\delta(u)\,du$, look at $\int_0^M \varphi_\delta(u)\,du$. By making appropriate changes of variable, the last integral is thrown into the form

$$(1+\delta)\int_{M/(1+\delta)}^{M} \log\left|\frac{v+1}{v-1}\right| dv \quad - \quad (1-\delta)\int_M^{M/(1-\delta)} \log\left|\frac{v+1}{v-1}\right| dv,$$

and this is readily evaluated for large M by expanding the integrands in powers of $1/v$.)

(e) Hence show that under the assumption in (d),

$$\rho(t) \quad\longrightarrow\quad \frac{2}{\pi^2}l \qquad \text{for } t \longrightarrow \infty.$$

(Hint: Picking a small $\delta > 0$, assume that for some large A we have

$$\rho((1-\delta)A) \;>\; m,$$

a number $> 2l/\pi^2$. Recalling that $d\rho(t) \geqslant -dt/t$ for $t > 1$, we then get

$$\rho(t) \quad > \quad m \;-\; \log\left(\frac{1+\delta}{1-\delta}\right) \qquad \text{for } (1-\delta)A \;\leqslant\; t \;\leqslant\; (1+\delta)A,$$

and this will contradict the result in (d) if A is large, and m much bigger than $2l/\pi^2$.

In case we have

$$\rho((1+\delta)A) \;<\; m',$$

a number $< 2l/\pi^2$ for some large A, we get

$$\rho(t) \quad < \quad m' \;+\; \log\left(\frac{1+\delta}{1-\delta}\right) \qquad \text{for } (1-\delta)A \;\leqslant\; t \;\leqslant\; (1+\delta)A,$$

and then the same kind of argument can be made.)

Problem 68

Formulate and prove a theorem analogous to the one of this article for functions $\omega(x)$ of the form

$$\omega(x) \quad = \quad x\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \,\mathrm{d}\rho(t),$$

where ρ is a measure subject to the assumption* stated above. (Hint: Work directly in terms of $\rho(t)$ and $\mathrm{d}\rho(t)$. *Boundedness* of $\rho(t)$ – needed to apply the second lemma of this article – is guaranteed by the last problem.)

4. Example. The finite energy condition not necessary

The construction starts out as in article 1; again we take

$$x_p \quad = \quad \exp(p^{1/3}) \qquad \text{for } p \;=\; 8, 9, 10, \ldots .$$

and put

$$\begin{aligned} \Delta_8 \quad &= \quad x_8 \\ \Delta_p \quad &= \quad x_p - x_{p-1} \qquad \text{for } p \geqslant 9. \end{aligned}$$

We also use a sequence of strictly positive numbers $\lambda_p < 1$ tending monotonically to 1, but at so slow a rate that

$$\sum_8^\infty \frac{(1-\lambda_p)^2}{p} \quad = \quad \infty .$$

It will turn out to be convenient to specify the λ_p explicitly near the end of this article.

Based on these sequences, we form an increasing function $\nu(t)$, defined for $t \geqslant 0$ according to the rule

$$\nu(t) \quad = \quad \begin{cases} \lambda_8 t, & 0 \leqslant t < x_8, \\ x_{p-1} \;+\; \lambda_p(t - x_{p-1}) & \text{for } x_{p-1} \;\leqslant\; t \;<\; x_p \text{ with } p \geqslant 9. \end{cases}$$

This function has a jump of magnitude $(1-\lambda_p)\Delta_p$ at each of the points x_p; its behaviour is shown by the following figure:

* in its original form, including absolute continuity of ρ and boundedness of $\rho'(t)$ on finite intervals

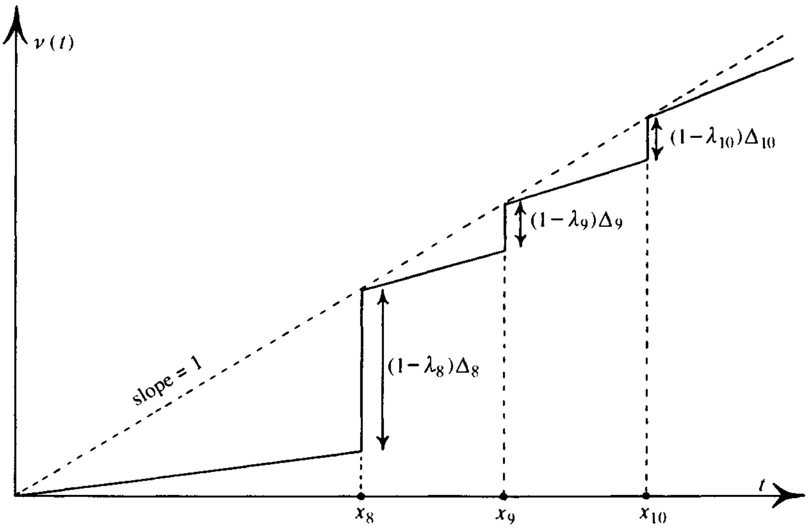

Figure 252

Obviously,

$$\nu(t) \leqslant t.$$

Also, since

$$\Delta_p \sim \tfrac{1}{3}p^{-2/3}x_p = \frac{x_p}{3(\log x_p)^2}$$

and

$$\frac{x_p}{x_{p-1}} \longrightarrow 1$$

as $p \longrightarrow \infty$, it is evident that

$$\nu(t) \geqslant t - \frac{t}{(\log t)^2} \qquad \text{for large } t.$$

Thence, putting

$$F_1(z) = \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| d\nu(t),$$

we see by computations just like those at the beginning of article 1 that

$$F_1(z) \leqslant F_1(i|z|) = \pi|z| + o(|z|)$$

for $|z|$ large and that moreover, for real values of x with sufficiently large absolute value,

$$F_1(x) \quad \leqslant \quad C\frac{|x|\log\log|x|}{(\log|x|)^2}$$

where C is a certain constant.

The right side of the last relation is an increasing function of $|x|$ when that quantity is large. Choosing, then, a large number l in a manner to be described presently, we take

$$T(x) \quad = \quad \begin{cases} 0, \quad 0 \leqslant x < l, \\ C\dfrac{x\log\log x}{(\log x)^2}, \quad x \geqslant l, \end{cases}$$

thus getting an increasing function T such that

$$\int_0^\infty \frac{T(x)}{x^2}\mathrm{d}x \quad < \quad \infty.$$

By the above two inequalities for F_1, we then have

$$F_1(x) \quad \leqslant \quad T(|x|) \; + \; \text{const.} \qquad \text{for } x \in \mathbb{R}.$$

We now follow the procedure used in Chapter X, §A.1, to prove the elementary multiplier theorem of Paley and Wiener. Using a constant B to be determined shortly, write

$$\mu(t) \quad = \quad Bt\int_t^\infty \frac{T(\tau)}{\tau^2}\mathrm{d}\tau$$

and then let

$$F_2(z) \quad = \quad \int_0^\infty \log\left|1 \; - \; \frac{z^2}{t^2}\right|\mathrm{d}\mu(t).$$

Since $T(t)$ is increasing, we have

$$\mu'(t) \quad = \quad B\int_t^\infty \frac{T(\tau)}{\tau^2}\mathrm{d}\tau \; - \; B\frac{T(t)}{t} \quad \geqslant \quad 0$$

for $t > 0$, and at the same time,

$$\frac{\mu(t)}{t} \longrightarrow 0 \quad \text{as } t \longrightarrow \infty.$$

The first lemma of Chapter VIII, §B.4, may thus be applied to the right side of our formula for F_2, yielding

$$\begin{aligned} F_2(x) &= -x\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\left(\frac{\mu(t)}{t}\right) \\ &= Bx\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \frac{T(t)}{t^2}\mathrm{d}t \qquad \text{for } x \geqslant 0. \end{aligned}$$

Therefore, because $T(t)$ is increasing, we have

$$F_2(x) \geqslant BT(x)\int_1^\infty \log\left|\frac{1+\tau}{1-\tau}\right| \frac{\mathrm{d}\tau}{\tau^2}, \qquad x \geqslant 0.$$

The integral on the right is just a certain strictly positive numerical quantity. We can thus pick B large enough (independently of the value of the large number l used in the specification of T) so as to ensure that

$$F_2(x) \geqslant 2T(x) \qquad \text{for } x \geqslant 0.$$

Fix such a value of B – it will be clear later on why we want the coefficient 2 on the right. Then, taking

$$F(z) = F_1(z) - F_2(z),$$

we will have

$$F(x) \leqslant -T(|x|) + \text{const.} \leqslant \text{const.}$$

for real values of x.

The function F is given by the formula

$$F(z) = \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}(\nu(t) - \mu(t)),$$

in which $\nu(t) - \mu(t)$ is *increasing*, provided that the parameter l entering into the definition of T is chosen properly. Because $\nu(t)$ and $\mu(t)$ are each increasing, with the second function absolutely continuous, this may be verified by looking at $\nu'(t) - \mu'(t)$. For $x_{p-1} < t < x_p$ with $p > 8$, we have

$$\nu'(t) - \mu'(t) = \lambda_p + B\frac{T(t)}{t} - B\int_t^\infty \frac{T(\tau)}{\tau^2}\mathrm{d}\tau,$$

and an analogous relation holds in the interval $(0, x_8)$. *Choose*, therefore, *l large enough to make*

$$B\int_0^\infty \frac{T(\tau)}{\tau^2}\mathrm{d}\tau = BC\int_l^\infty \frac{\log\log\tau}{(\log\tau)^2\,\tau}\,\mathrm{d}\tau < \lambda_8.$$

Then, the sequence $\{\lambda_p\}$ being increasing, we will have $\nu'(t) - \mu'(t) > 0$ for $t > 0$ different from any of the points x_p, and $\nu(t) - \mu(t)$ will be increasing.

It is also clear that

$$\frac{\nu(t) - \mu(t)}{t} \longrightarrow 1 \quad \text{as } t \longrightarrow \infty.$$

Hence

$$F(z) \leqslant F(\mathrm{i}|z|) = \pi|z| + \mathrm{o}(|z|)$$

for large $|z|$. $F(x)$ is, on the other hand, *bounded above* for real x. From these two properties and the formula for $F(z)$ we can now deduce the representation of §G.1, Chapter III,

$$F(z) = \pi|\Im z| + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|F(t)}{|z-t|^2}\,\mathrm{d}t,$$

by an argument like one used in the proof of the second theorem of §B.1.

Let K be any upper bound for $F(x)$ on $\mathbb{R}$, and then, proceeding much as in article 1, put

$$W(x) = \frac{\mathrm{e}^{\pi+K}}{\exp F(x+\mathrm{i})}, \qquad x \in \mathbb{R}.$$

From the preceding relation, we get

$$\log W(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(K - F(t))}{(x-t)^2 + 1}\,\mathrm{d}t,$$

and from this we see that

$$W(x) \geqslant 1,$$

besides which

$$\left|\frac{\mathrm{d}\log W(x)}{\mathrm{d}x}\right| \leqslant \log W(x),$$

making $\log\log W(x)$ uniformly Lip 1 on $\mathbb{R}$. *The present weight W thus meets the local regularity requirement from* §B.1, quoted at the beginning of this §. Since $F(t)$ is even, so is $W(x)$, and the relation $F(t) \leqslant -T(|t|) + \text{const.}$, together with $T(t)$'s tending to ∞ for $t \longrightarrow \infty$, implies that

$$W(x) \longrightarrow \infty \qquad \text{for } x \longrightarrow \pm\infty.$$

(*That's* why we chose B so as to have $F_2(x) \geqslant 2T(|x|)$ with a factor of 2.)

It will now be shown that $W(x)$ *admits multipliers,* but that *there can be no even function* $\Omega(x) \geqslant 1$ *with*

$$\int_0^\infty \frac{\log \Omega(x)}{x^2}\,dx \quad < \quad \infty$$

and $\log \Omega(x)/x$ *in* $\mathfrak{H}$ *such that*

$$W(x) \leqslant \Omega(x)$$

for large values of $|x|$.

To show that W *admits multipliers,* we start from the relation

$$\log W(x) \quad = \quad \pi + K - F_1(x+\mathrm{i}) + F_2(x+\mathrm{i})$$

and deal separately with the terms $F_1(x+\mathrm{i})$ and $F_2(x+\mathrm{i})$ standing on the right. One handles each of those by first moving down to the real axis and working with $F_1(x)$ and $F_2(x)$; afterwards, one goes back up to the line $z = x+\mathrm{i}$.

The function F_1 is easier to take care of on account of $\nu(t)$'s special form. Knowing that

$$-F_1(x) \quad = \quad -\int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\nu(t),$$

we proceed, for given arbitrary $\eta > 0$, to build an increasing $\sigma_1(t)$ with $\sigma_1(t)/t \leqslant \eta/2$ having *jumps* that will *cancel out* most of ν's, making, indeed, $\sigma_1(t) - \nu(t)$ a *constant multiple* of t for large values of that variable. The property that $\lambda_p \longrightarrow 1$ as $p \longrightarrow \infty$ enables us to do this.

Given the quantity $\eta > 0$, there is a number $p(\eta)$ such that

$$\lambda_p \quad > \quad 1 - \frac{\eta}{2} \qquad \text{for } p > p(\eta).$$

We put

$$\sigma_1(t) \quad = \quad \begin{cases} 0, \quad t < x_{p(\eta)}, \\ \dfrac{\eta}{2}x_{p-1} + \left\{\lambda_p - \left(1 - \dfrac{\eta}{2}\right)\right\}(t - x_{p-1}) \quad \text{for} \\ \qquad\qquad\qquad x_{p-1} \leqslant t < x_p \text{ with } p > p(\eta). \end{cases}$$

This increasing function $\sigma_1(t)$ is related to $\nu(t)$ in the way shown by the following diagram:

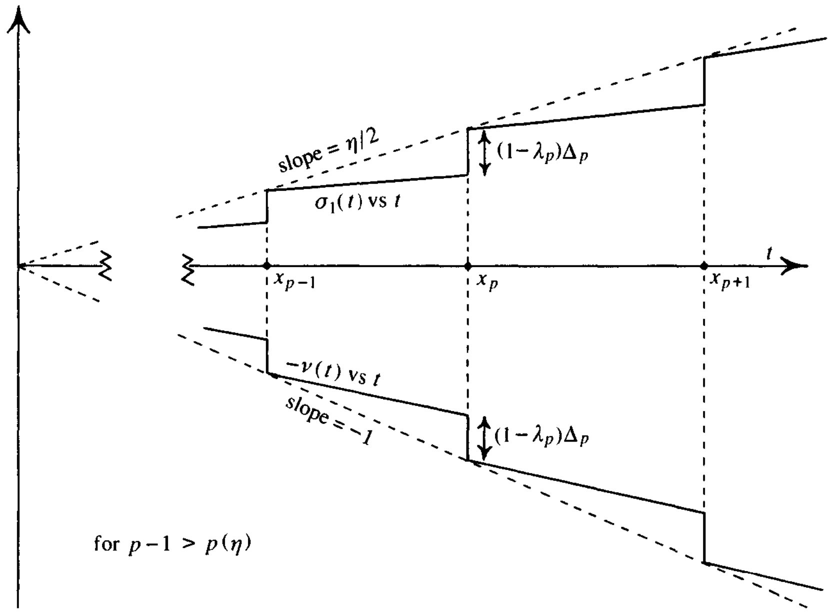

Figure 253

It is clear that

$$\sigma_1(t) \; - \; \nu(t) \quad = \quad -\left(1 \; - \; \frac{\eta}{2}\right)t \qquad \text{for } t \; \geqslant \; x_{p(\eta)}.$$

Take now

$$G_1(z) \quad = \quad \int_0^\infty \log\left|1 \; - \; \frac{z^2}{t^2}\right| \mathrm{d}\sigma_1(t).$$

We have

$$\frac{\sigma_1(t)}{t} \longrightarrow \frac{\eta}{2} \quad \text{as } t \longrightarrow \infty,$$

so for large values of $|z|$,

$$G_1(z) \quad \leqslant \quad G_1(\mathrm{i}|z|) \quad = \quad \frac{\pi\eta}{2}|z| \; + \; \mathrm{o}(|z|).$$

The first lemma of §B.4, Chapter VIII, tells us that

$$G_1(x) \; - \; F_1(x) \quad = \quad \int_0^\infty \log\left|1 \; - \; \frac{x^2}{t^2}\right| \mathrm{d}(\sigma_1(t) - \nu(t))$$

$$= \quad x\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\left(\frac{\nu(t) - \sigma_1(t)}{t}\right), \qquad x \in \mathbb{R}.$$

As we have just seen, $(\nu(t) - \sigma_1(t))/t$ is *constant* for $t \geqslant x_{p(\eta)}$; the last expression on the right thus reduces to

$$x\int_0^{x_{p(\eta)}} \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\left(\frac{\nu(t)-\sigma_1(t)}{t}\right).$$

This, however, is clearly bounded (above *and* below!) for $|x| \geqslant 2x_{p(\eta)}$, say. Therefore

$$G_1(x) \;-\; F_1(x) \;\leqslant\; \text{const.}, \qquad |x| \;\geqslant\; 2x_{p(\eta)}.$$

This relation does not hold *everywhere* on $\mathbb{R}$; $G_1(x) - F_1(x)$ is indeed *infinite* at each of the points $\pm x_p$ with $8 \leqslant p < p(\eta)$. But at those places (corresponding to the points where $\sigma_1(t) - \nu(t)$ jumps *downwards*) the infinities of $G_1(z) - F_1(z)$ are *logarithmic*, and hence *harmless* as far as we are concerned. Besides becoming $-\infty$ (logarithmically again) at $\pm x_{p(\eta)}$, the function $G_1(x) - F_1(x)$ is otherwise well behaved on $\mathbb{R}$, and belongs to $L_1(-2x_{p(\eta)}, 2x_{p(\eta)})$. We can now reason once again as in the proof of the second theorem, §B.1, and deduce from the properties of $G_1(x) - F_1(x)$ just noted, and from those of $G_1(z)$ and $F_1(z)$ in the complex plane, pointed out previously, that

$$G_1(z) \;-\; F_1(z) \;=\; -\pi\left(1 \;-\; \frac{\eta}{2}\right)|\Im z| \;+\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|(G_1(t)-F_1(t))}{|z-t|^2}\,\mathrm{d}t.$$

Keeping in mind the behaviour of $G_1(t) - F_1(t)$ on the real axis, we see by this relation that

$$G_1(x+\mathrm{i}) \;-\; F_1(x+\mathrm{i}) \;\leqslant\; \text{const.}, \qquad x\in\mathbb{R}.$$

We turn to the function $F_2(x)$, equal, as we have seen, to

$$Bx\int_0^{\infty} \log\left|\frac{x+t}{x-t}\right| \frac{T(t)}{t^2}\,\mathrm{d}t$$

for $x\in\mathbb{R}$ (both $F_2(x)$ and this expression being even). Here we proceed just as in the passage from the function $Cx(\log\log x)/(\log x)^2$ to $F_2(z)$. A change of variable shows that

$$F_2(x) \;=\; B\int_0^{\infty} \log\left|\frac{1+\tau}{1-\tau}\right| \frac{T(x\tau)}{\tau^2}\,\mathrm{d}\tau \qquad \text{for } x \geqslant 0,$$

from which it is manifest that $F_2(x)$, like $T(x)$, is *increasing* on $[0, \infty)$. Again, by Fubini's theorem,

$$\int_0^{\infty} \frac{F_2(x)}{x^2}\,\mathrm{d}x \;=\; B\int_0^{\infty}\int_0^{\infty} \log\left|\frac{x+t}{x-t}\right| \frac{\mathrm{d}x}{x}\,\frac{T(t)}{t^2}\,\mathrm{d}t \;=$$

$$= \quad \frac{\pi^2 B}{2}\int_0^\infty \frac{T(t)}{t^2}\,\mathrm{d}t \quad < \quad \infty.$$

Pick, then, a large number m (in a way to be described in a moment), and put

$$\Theta(t) \quad = \quad \begin{cases} 0, & 0 \leqslant t < m, \\ F_2(t), & t \geqslant m. \end{cases}$$

Bringing in once more the given quantity $\eta > 0$, we form the function

$$\sigma_2(t) \quad = \quad \frac{\eta}{2}t \quad - \quad Bt\int_t^\infty \frac{\Theta(\tau)}{\tau^2}\,\mathrm{d}\tau$$

and observe that it is *increasing provided that m is chosen large enough* – verification of this statement is just like that of the corresponding one about $\nu(t) - \mu(t)$. *Fixing* once and for all such a value of m, we take

$$G_2(z) \quad = \quad \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}\sigma_2(t).$$

The function $\sigma_2(t)$, besides being increasing, has the property that

$$\frac{\sigma_2(t)}{t} \longrightarrow \frac{\eta}{2} \qquad \text{as } t \longrightarrow \infty.$$

Thence,

$$G_2(z) \quad \leqslant \quad G_2(\mathrm{i}|z|) \quad = \quad \frac{\pi\eta}{2}|z| \; + \; \mathrm{o}(|z|)$$

for large values of $|z|$.

For $x > 0$, by the first lemma of §B.4, Chapter VIII,

$$G_2(x) \quad = \quad -x\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\left(\frac{\sigma_2(t)}{t}\right)$$

$$= \quad -Bx\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \frac{\Theta(t)}{t^2}\,\mathrm{d}t.$$

Thanks to our choice of B, the last quantity is $\leqslant -2\Theta(x)$ (cf. the previous examination of $F_2(x)$'s relation to $T(x)$). Thus,

$$G_2(x) \quad \leqslant \quad -2F_2(x) \qquad \text{for } |x| \geqslant m,$$

and we certainly have

$$G_2(x) \; + \; F_2(x) \quad \leqslant \quad 0$$

for such real x, $F_2(x)$ being clearly positive.

Since $\mu(t)/t \longrightarrow 0$ for $t \longrightarrow \infty$, with $\mu(t)$ increasing, we must have

$$F_2(z) \leqslant F_2(\mathrm{i}|z|) = \mathrm{o}(|z|)$$

for large $|z|$. Using this estimate and the corresponding one on $G_2(z)$ given above we deduce from the behaviour of $G_2(x) + F_2(x)$ on $\mathbb{R}$ just found that

$$G_2(z) + F_2(z) = \frac{\pi\eta}{2}|\Im z| + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|(G_2(t)+F_2(t))}{|z-t|^2}\,\mathrm{d}t;$$

this is done by the procedure followed twice already. It then follows from this relation and from the fact that $G_2(t) + F_2(t) \leqslant 0$ for $|t| \geqslant m$ that

$$G_2(x+\mathrm{i}) + F_2(x+\mathrm{i}) \leqslant \text{const.}, \qquad x \in \mathbb{R}.$$

Going back to $\log W(x)$ we find, recalling the above formula for it and using the two results now obtained, that

$$\log W(x) + G(x+\mathrm{i}) \leqslant \text{const.}, \qquad x \in \mathbb{R},$$

where

$$G(z) = G_1(z) + G_2(z) = \int_0^{\infty} \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}(\sigma_1(t) + \sigma_2(t)).$$

Consider now the entire function $\varphi(z)$ given by the formula

$$\log|\varphi(z)| = \int_0^{\infty} \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}[\sigma_1(t) + \sigma_2(t)].$$

Because

$$\frac{[\sigma_1(t)+\sigma_2(t)]}{t} = \frac{\sigma_1(t)}{t} + \frac{\sigma_2(t)}{t} + \mathrm{o}(1) \longrightarrow \eta$$

for $t \longrightarrow \infty$, we have

$$\log|\varphi(z)| \leqslant \log|\varphi(\mathrm{i}|z|)| = \pi\eta|z| + \mathrm{o}(|z|)$$

for z of large modulus, making $\varphi(z)$ *of exponential type* $\pi\eta$. The lemma from Chapter X, §A.1, now yields

$$\log|\varphi(x+\mathrm{i})| \leqslant G(x+\mathrm{i}) + \log^+|x|, \qquad x \in \mathbb{R},$$

which, with the previous relation, gives

$$W(x)|\varphi(x+\mathrm{i})| \leqslant \text{const.}\sqrt{(x^2+1)}, \qquad x \in \mathbb{R}.$$

However, $\sigma_1(t)+\sigma_2(t) \longrightarrow \infty$ for $t \longrightarrow \infty$, so $\varphi(z)$ certainly *has* zeros.

Dividing out any one of them then yields a new entire function, $\psi(z)$, *also of exponential type* $\pi\eta$, with

$$W(x)|\psi(x+\mathrm{i})| \leqslant \text{const.}, \qquad x \in \mathbb{R}.$$

The number $\eta > 0$ was, however, *arbitrary*. Our weight $W(x)$ therefore *admits multipliers*, as claimed.

To see that there is *no* even function $\Omega(x) \geqslant 1$ with $\log\Omega(x)/x$ in $\mathfrak{H}$ and

$$\int_0^\infty \frac{\log\Omega(x)}{x^2}\,\mathrm{d}x \quad < \quad \infty$$

such that

$$W(x) \leqslant \Omega(x)$$

for large $|x|$, we use the theorem from the last article. Because $W(0) \geqslant 1$, the relation just written would make

$$\frac{W(x)}{W(0)} \leqslant \Omega(x) \qquad \text{for } |x| \text{ large},$$

so, since $W(x) \longrightarrow \infty$ for $x \longrightarrow \pm\infty$ as we have noted, the theorem referred to is applicable *provided that*

$$\log\left(\frac{W(x)}{W(0)}\right) = -\int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\rho(t),$$

where $\rho(t)$ is an *increasing*, *infinitely differentiable* odd function *defined on* $\mathbb{R}$, with $\rho(t)/t$ *bounded* for $t > 0$.

In our present circumstances,

$$\log\left(\frac{W(x)}{W(0)}\right) = F(\mathrm{i}) - F(x+\mathrm{i})$$

where, as already pointed out,

$$F(z) = \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}(\nu(t) - \mu(t))$$

with $\nu(t) - \mu(t)$ increasing and $O(t)$ on $[0, \infty)$. Taking note of the identity

$$\left|1 - \frac{z+\mathrm{i}}{t}\right| = \left|1 - \frac{z}{t-\mathrm{i}}\right|\left|1 - \frac{\mathrm{i}}{t}\right|, \qquad t \in \mathbb{R},$$

we see that

$$F(z+\mathrm{i}) \;-\; F(\mathrm{i}) \;=\; \int_0^\infty \log\left|\left(1 - \frac{z}{t-\mathrm{i}}\right)\left(1 + \frac{z}{t+\mathrm{i}}\right)\right| \mathrm{d}(\nu(t)-\mu(t)).$$

Now for any particular z, $\Im z \geqslant 0$, the function of w equal to $\log|1 + (z/w)|$ is *harmonic* for $\Im w > 0$. We can thence conclude, just as in proving the first lemma of §C.5, Chapter VIII, that the right-hand integral in the preceding relation is equal to

$$\int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}\rho(t) \qquad \text{for } \Im z \geqslant 0,$$

with an absolutely continuous increasing function $\rho(t)$ defined on $\mathbb{R}$, having there the derivative

$$\frac{\mathrm{d}\rho(t)}{\mathrm{d}t} \;=\; \frac{1}{\pi}\int_0^\infty \left(\frac{1}{(t-\tau)^2+1} + \frac{1}{(t+\tau)^2+1}\right) \mathrm{d}(\nu(\tau)-\mu(\tau)).$$

Infinite differentiability of $\rho(t)$ is manifest from the last formula. Taking

$$\rho(0) \;=\; 0$$

(which makes $\rho(t)$ *odd*), we can also verify boundedness of $\rho(t)/t$ in $(0, \infty)$ without much difficulty. One way is to simply refer to the *second* lemma of §C.5, Chapter VIII, using the formula

$$F(z+\mathrm{i}) \;-\; F(\mathrm{i}) \;=\; \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}\rho(t), \qquad \Im z \geqslant 0,$$

just established together with the fact noted above that $F(z) \leqslant \pi|z| + \mathrm{o}(|z|)$ for large $|z|$. We see in this way that the *hypothesis of the theorem from the preceding article is fulfilled* for the function

$$\omega(x) \;=\; \log\left(\frac{W(x)}{W(0)}\right) \;=\; F(\mathrm{i}) \;-\; F(x+\mathrm{i}).$$

According to that theorem, *if an* Ω *having the properties described above did exist, we would have*

$$\int_0^\infty \frac{\omega(x)}{x^2}\,\mathrm{d}\rho(x) \;<\; \infty,$$

or, what comes to the same thing,

$$\int_1^\infty \frac{F(x+\mathrm{i})}{x^2}\,\mathrm{d}\rho(x) \;>\; -\infty,$$

$\rho(x)$ being increasing and $O(x)$. It thus *suffices to prove that*

$$\int_1^\infty \frac{F(x+\mathrm{i})}{x^2}\,\mathrm{d}\rho(x) \quad = \quad -\infty$$

in order to show that no such function Ω *can exist.*

For this purpose, we first obtain an *upper* bound on $F(x+\mathrm{i})$ for x near one of the points x_p, arguing somewhat as in article 1. Given $x > 0$ and $0 < r \leqslant x$, denote by $N(r,\ x+\mathrm{i})$ the quantity

$$\int_J \mathrm{d}(\nu(t) - \mu(t)),$$

where J is the *intersection* of the *disk* of radius r about $x+\mathrm{i}$ with the *real axis*:

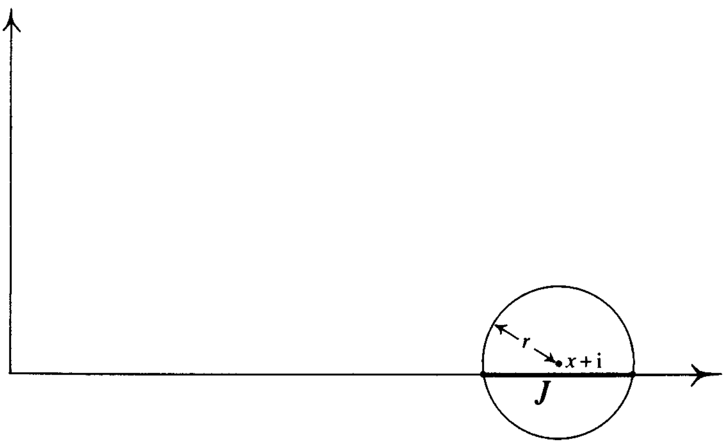

Figure 254

Keeping in mind the relation

$$F(z) \quad = \quad \int_0^\infty \log\left|1 \; - \; \frac{z^2}{t^2}\right| \mathrm{d}(\nu(t) - \mu(t)),$$

we see then, by an evident adaptation of Jensen's formula (cf. near the beginning of the proof of the *first* theorem, §B.3), that

$$F(x+\mathrm{i}) \quad \leqslant \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x+\mathrm{i}+R\mathrm{e}^{\mathrm{i}\vartheta})\,\mathrm{d}\vartheta \quad - \quad \int_0^R \frac{N(r,\ x+\mathrm{i})}{r}\,\mathrm{d}r$$

as long as $R \leqslant x$. (Some such restriction on R is *necessary* in order to ensure that the disk of radius R about $x+\mathrm{i}$ *not intersect* with the *negative real axis.*) We use this formula for

$$x_p - 1 \leqslant x \leqslant x_p + 1$$

with p large, remembering that the increasing function $\nu(t) - \mu(t)$ *jumps upwards by* $(1-\lambda_p)\Delta_p$ *units* at $t = x_p$. That makes

$$N(r,\ x+\mathrm{i}) \geqslant (1-\lambda_p)\Delta_p$$

for such x as soon as r exceeds $\sqrt{2}$. Since $\Delta_p/x_p \longrightarrow 0$ for $p \longrightarrow \infty$ we may, for large p, take $R = \sqrt{2}\,\Delta_p$ in the formula, which, in view of the relation just written, then yields

$$F(x+\mathrm{i}) \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x+\mathrm{i}+\sqrt{2}\,\Delta_p e^{\mathrm{i}\vartheta})\,d\vartheta - (1-\lambda_p)\Delta_p \log \Delta_p$$

for $x_p - 1 \leqslant x \leqslant x_p + 1$.

From the relations $F(x) \leqslant$ const., $x \in \mathbb{R}$, and $F(z) \leqslant \pi|z| + o(|z|)$ it now follows by the third Phragmén–Lindelöf theorem of §C, Chapter III, that

$$F(x+\mathrm{i}+\sqrt{2}\,\Delta_p e^{\mathrm{i}\vartheta}) \leqslant \text{const.} + \sqrt{2}\,\pi\Delta_p|\sin\vartheta|.$$

Plugging this into the preceding inequality, we find that

$$F(x+\mathrm{i}) \leqslant \text{const.} + 2\sqrt{2}\,\Delta_p - (1-\lambda_p)\Delta_p \log \Delta_p$$

for $x_p - 1 \leqslant x \leqslant x_p + 1$, p being large.

We need also to see how much $\rho(t)$ increases on the interval $[x_p - 1,\ x_p + 1]$. Because $\nu(t) - \mu(t)$ has a jump of magnitude $(1-\lambda_p)\Delta_p$ at x_p, we see by the above formula for $\rho'(t)$ that

$$\frac{d\rho(t)}{dt} \geqslant \frac{(1-\lambda_p)\Delta_p}{\pi((t-x_p)^2+1)}.$$

Integrating, we get

$$\int_{x_p-1}^{x_p+1} d\rho(x) \geqslant \frac{1}{2}(1-\lambda_p)\Delta_p.$$

We can simplify our work at this point by specifying the sequence $\{\lambda_p\}$ *in precise fashion. Take, namely,*

$$\lambda_p = 1 - \frac{1}{\sqrt{(\log p)}}, \qquad p \geqslant 8.$$

Then, for large p,

$$(1-\lambda_p)\Delta_p \log \Delta_p \quad \sim \quad \frac{\exp(p^{1/3})}{3p^{1/3}(\log p)^{1/2}}$$

is *much bigger* than

$$2\sqrt{2}\,\Delta_p \quad \sim \quad \frac{2\sqrt{2}\,\exp(p^{1/3})}{3p^{2/3}},$$

and thus *the third term in our last estimate for $F(x+\mathrm{i})$ will greatly outweigh both of the first two.* When $p \geqslant$ a certain p_0, the estimate therefore reduces to

$$F(x+\mathrm{i}) \quad \leqslant \quad -\frac{1}{2}(1-\lambda_p)\Delta_p \log \Delta_p, \qquad x_p - 1 \leqslant x \leqslant x_p + 1.$$

Use this relation together with the one just found involving ρ. That gives

$$\int_{x_p-1}^{x_p+1} F(x+\mathrm{i})\,\mathrm{d}\rho(x) \quad \leqslant \quad -\frac{1}{4}(1-\lambda_p)^2\,\Delta_p^2 \log \Delta_p, \qquad p \geqslant p_0.$$

Here, $F(x+\mathrm{i})$ is, as we know, bounded above for real x, and the increasing function $\rho(x)$ is $\mathrm{O}(x)$. The *divergence* of

$$\int_1^\infty \frac{F(x+\mathrm{i})}{x^2}\,\mathrm{d}\rho(x)$$

to $-\infty$ is hence *implied* by that of the *sum*

$$\sum_{p \geqslant p_0} \frac{1}{x_p^2} \int_{x_p-1}^{x_p+1} F(x+\mathrm{i})\,\mathrm{d}\rho(x).$$

By the preceding inequality,

$$\frac{1}{x_p^2} \int_{x_p-1}^{x_p+1} F(x+\mathrm{i})\,\mathrm{d}\rho(x) \quad \leqslant \quad -\frac{1}{4}(1-\lambda_p)^2 \left(\frac{\Delta_p}{x_p}\right)^2 \log \Delta_p$$

for $p \geqslant p_0$, and the right side is

$$\sim \quad -\frac{1}{4}(1-\lambda_p)^2 \cdot \frac{1}{9} p^{-4/3} \cdot p^{1/3} \quad = \quad -\frac{1}{36 p \log p}$$

for $p \longrightarrow \infty$. So, since $\sum_p (1/p \log p)$ is divergent, we do indeed have

$$\int_1^\infty \frac{F(x+\mathrm{i})}{x^2}\,\mathrm{d}\rho(x) \quad = \quad -\infty.$$

Therefore, *no* even function $\Omega(x) \geqslant 1$ having the properties stated above and with

$$W(x) \leqslant \Omega(x)$$

for large $|x|$ *can exist. Nevertheless,* $W(x)$ *admits multipliers.*

5. Further discussion and a conjecture

At this point, the reader may have the impression that we have been merely raising up straw men in order to knock them down again, but that is not so. Considerable insight about the nature of the 'essential' condition we are seeking may be gained by studying the examples constructed above.

Suppose that we have an even weight $W(x) \geqslant 1$ meeting our local regularity requirement, with

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2}\,dx \quad < \quad \infty.$$

Let us, for purposes of discussion, also assume $W(x)$ to be infinitely differentiable. In these circumstances, the odd function

$$u(x) \quad = \quad \frac{1}{x}\log\left(\frac{W(x)}{W(0)}\right)$$

has a $\mathscr{C}_\infty$ Hilbert transform*

$$\tilde{u}(x) \quad = \quad \frac{1}{\pi}\fint_{-\infty}^{\infty} \frac{u(t)}{x-t}\,dt,$$

and it is frequently possible to justify the formula

$$u(x) \quad = \quad \frac{1}{\pi}\int_0^{\infty} \log\left|\frac{x+t}{x-t}\right| \tilde{u}'(t)\,dt$$

by an argument like the one made near the end of the proof of the *last* theorem in §C.4. Provided that $|\tilde{u}(t)|$ does not get very big for $t \longrightarrow \infty$, further manipulation will yield

$$\log\left(\frac{W(x)}{W(0)}\right) \quad = \quad xu(x) \quad = \quad -\frac{1}{\pi}\int_0^{\infty} \log\left|1 \; - \; \frac{x^2}{t^2}\right| d(t\tilde{u}(t)).$$

* regarding the infinite differentiability of $\tilde{u}(x)$, cf. initial footnote to the *third* lemma of §E.1 below

In this article, let us not worry further about the restrictions on W needed in order to justify these transformations; what we *have* is a representation of the form

$$\log W(x) \quad = \quad \log W(0) \quad - \quad \int_0^\infty \log\left|1 \ - \ \frac{x^2}{t^2}\right| \mathrm{d}\lambda(t)$$

for a fairly general collection of weights W, involving *signed* (and very smooth) measures λ on $[0, \infty)$. How is it for the admittance of multipliers by such weights?

We can see already from the work of §C.2 that *negative measures λ are 'good'* insofar as this question is concerned. In the case of a weight with convergent logarithmic integral given by such a measure λ, one readily shows with help of the argument in §H.2, Chapter III, that the *increasing* function

$$-\lambda(t) \quad = \quad -\int_0^t \mathrm{d}\lambda(\tau)$$

must be $\mathrm{O}(t)$ on $[0, \infty)$. The proof of the Theorem on the Multiplier in §C.2 may then be taken over, essentially without change, to conclude that $W(x)$ admits multipliers.*

From this point of view, *positive measures λ are 'bad'*; the example in article 1 shows that weights with convergent logarithmic integrals given by *positive* λ's *need not* admit multipliers.

How bad is bad? The example in article 4 *does*, after all, furnish a weight admitting multipliers and given by a positive measure λ. The first thing to be observed is that absolutely continuous λ's with $\lambda'(t)$ *bounded above* on $[0, \infty)$ are *just as good* as the *negative* ones. For, since

$$\int_0^\infty \log\left|1 \ - \ \frac{x^2}{t^2}\right| \mathrm{d}t \quad = \quad 0, \qquad x \in \mathbb{R},$$

we have, for any weight $W(x)$ given by such a λ with $\lambda'(t) \leqslant K$, say,

$$\log W(x) \quad = \quad \log W(0) \quad - \quad \int_0^\infty \log\left|1 \ - \ \frac{x^2}{t^2}\right| \mathrm{d}(\lambda(t) - Kt),$$

showing that W is also given by the *negative measure* ρ with $\mathrm{d}\rho(t) \ = \ \mathrm{d}\lambda(t) \ - \ K\,\mathrm{d}t$. Things can hence go wrong only for measures λ with $\lambda'(t)$ *very large* in certain places. It is therefore reasonable, when trying to find out *how* the *positive part* of a *signed measure* λ can bring about *failure of the weight given by it to admit multipliers*, to *slough off*

* See also the footnote on p. 556.

from λ its *portions having densities bounded above by ever larger constants,* and then *look each time at what is left.* That amounts to examining the behaviour of

$$\max(\lambda'(t),\ K) \ - \ K$$

on $[0, \infty)$ for larger and larger values of K.

The weights constructed in articles 1 and 4 (one *admitting* multipliers and the other *not*) are given by positive measures λ so similar in behaviour that something should be learned by treating those measures in the way described. It is better to first look at the measure giving the weight *of article 4.*

For that weight W we had

$$\log W(x) \quad = \quad \log W(0) \quad - \quad \int_0^\infty \log\left|1 \ - \ \frac{x^2}{t^2}\right| \mathrm{d}\rho(t)$$

with the absolutely continuous (indeed, $\mathscr{C}_\infty$) positive measure ρ furnished by the formula

$$\frac{\mathrm{d}\rho(t)}{\mathrm{d}t} \quad = \quad \frac{1}{\pi}\int_0^\infty \left(\frac{1}{(t-\tau)^2 \ + \ 1} \ + \ \frac{1}{(t+\tau)^2+1}\right)\mathrm{d}(\nu(\tau)-\mu(\tau)).$$

Here, $\nu(\tau)$ and $\mu(\tau)$, as well as the difference $\nu(\tau) \ - \ \mu(\tau)$ figuring in the integral, are increasing functions. The function $\mu(t)$, equal, in the notation of the last article, to

$$Bt\int_t^\infty \frac{T(\tau)}{\tau^2}\,\mathrm{d}\tau,$$

is *absolutely continuous,* with *bounded derivative,* and the behaviour of $\nu(t)$ is shown by the figure at the beginning of article 4. The latter consists of an *absolutely continuous part,* again with *bounded derivative,* together with a *singular* part having *jumps of magnitude* $(1-\lambda_p)\Delta_p$ *at the points* x_p, $p \geqslant 8$. The difference $\nu(t) \ - \ \mu(t)$ has therefore the *same description,* and, since $(1-\lambda_p)\Delta_p$ and $x_p - x_{p-1}$ both tend to ∞ with p in our example, the function

$$\rho(t) \quad = \quad \int_0^t \rho'(\tau)\,\mathrm{d}\tau,$$

really nothing but a regularized version of that difference, shows almost the same behaviour as the latter for large t, except for being somewhat smoother.

Thus, when K is big, a good representation of the graph of the residual function

$$\rho_K(t) \quad = \quad \int_0^t \bigl(\max(\rho'(\tau),\ K) \ - \ K\bigr)\,\mathrm{d}\tau$$

will, for large values of t, be provided by one simply showing the *jumps* of $\nu(t)$ that go to make up the singular part of $\nu(t) \ - \ \mu(t)$.

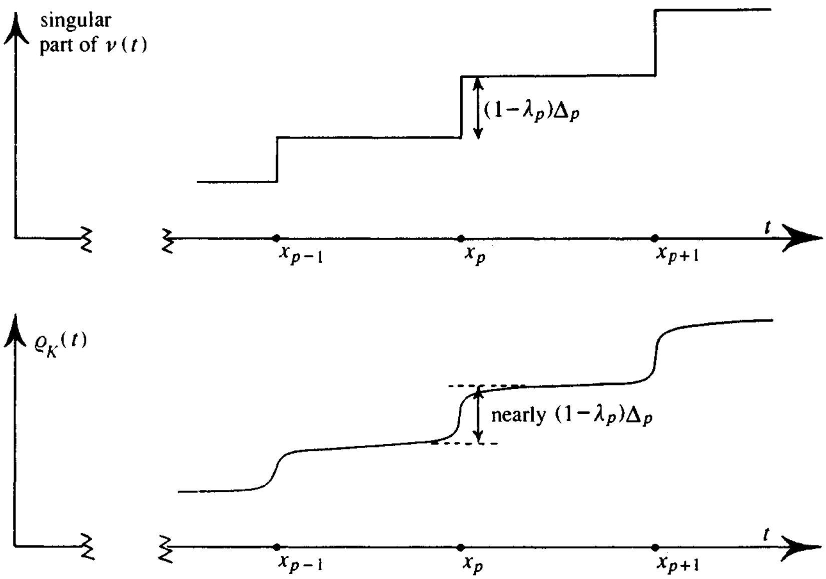

Figure 255

We turn to the weight W considered in *article 1*. In the notation of that article, it is given by the formula

$$W(x) \quad = \quad \frac{\text{const.}}{|F(x+\mathrm{i})\varphi(x+\mathrm{i})|},$$

where $F(z)$ and $\varphi(z)$ are certain even entire functions, of exponential type π and η respectively, having only real zeros. For the first of these, we had

$$\log|F(z)| \quad = \quad \int_0^\infty \log\left|1 \ - \ \frac{z^2}{t^2}\right| \mathrm{d}n(t)$$

with a function $n(t)$, increasing by a *jump* of magnitude $[\Delta_p]$ at each of the points x_p, $\quad p \geqslant 8$, and constant on the intervals separating those points (as well as on $[0,\ x_8)$). The function $\varphi(z)$, obtained from §A.1 of

Chapter X, has the representation

$$\log|\varphi(z)| \quad = \quad \int_0^\infty \log\left|1 \; - \; \frac{z^2}{t^2}\right| \mathrm{d}[s(t)],$$

with

$$s(t) \quad = \quad \left(\frac{\eta}{\pi}t \; - \; \mu_1(t) \; - \; 1\right)^+$$

an increasing function formed from a certain $\mu_1(t)$ very much like the $\mu(t)$ appearing in the example of article 4. Thus, although $[s(t)]$ is composed exclusively of *jumps*, it is *based* on the function $(\eta/\pi)t \; - \; \mu_1(t)$ which increases *quite uniformly*, having *derivative* between 0 and η/π in value at each $t \geqslant 0$.

Referring to the first lemma of §C.5, Chapter VIII, we see that for the weight W of article 1,

$$\log W(x) \quad = \quad \log W(0) \quad - \quad \int_0^\infty \log\left|1 \; - \; \frac{x^2}{t^2}\right| \mathrm{d}\sigma(t),$$

where $\sigma(t)$ is an absolutely continuous increasing function determined by the relation

$$\frac{\mathrm{d}\sigma(t)}{\mathrm{d}t} \quad = \quad \frac{1}{\pi}\int_0^\infty \left(\frac{1}{(t-\tau)^2+1} \; + \; \frac{1}{(t+\tau)^2+1}\right)\mathrm{d}(n(\tau)+[s(\tau)]).$$

By feeding just the increasing function $[s(\tau)]$ into the integral on the right (which has the effect of smoothing out the former's jumps), one obtains an *increasing function* having a *bounded derivative* (given by the integral in question), thanks to the moderate behaviour of $(\eta/\pi)t \; - \; \mu_1(t)$ just noted. Therefore, when K is *big*, the residual function

$$\sigma_K(t) \quad = \quad \int_0^t (\max(\sigma'(\tau, \; K) \; - \; K)\mathrm{d}\tau$$

acts, for large t, essentially like $n(t)$, which has the quite substantial jumps of height $[\Delta_p]$ at the points x_p. In this respect, the present situation is much like the one described above corresponding to the *weight from article 4*, involving the functions $\rho_K(t)$ and $\nu(t)$.

If now we compare the graph of $\rho_K(t)$, corresponding to the weight *admitting* multipliers, with the one for $\sigma_K(t)$, corresponding to the *weight that does not*, only *one difference* is apparent, and that is in the *relative heights of the steps*. Wishing to arrive at a quantitative notion of this difference, one soon thinks of performing the F. Riesz construction on

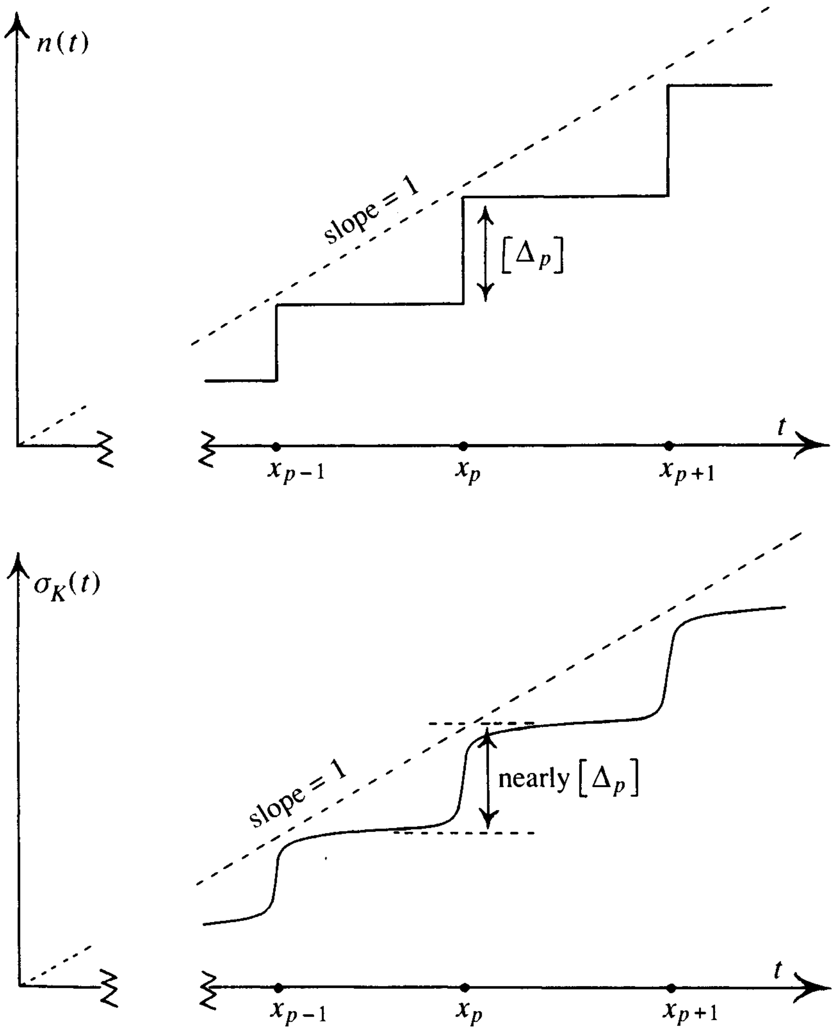

Figure 256

both graphs, letting light shine downwards on each of them from the right along a direction of small positive slope. On account of the great similarity just described between the graphs of $\sigma_K(t)$ and $n(t)$ for large t, and between those of $\rho_K(t)$ and the singular part of $\nu(t)$, it seems quite certain that we will (for large t) arrive at the same results by instead carrying out the Riesz construction for $n(t)$ and for the singular part of $\nu(t)$. This we do in order to save time, simply *assuming*, without bothering to verify the fact, that the results thus obtained really are the same as those that would be gotten (for large t), were the constructions to be made for $\sigma_K(t)$ and for $\rho_K(t)$. We are, after all, trying to *find* a theorem and not to *prove* one!

Taking, then, any small $\delta > 0$, we look at the set of *large* t with the property that

$$\frac{\text{sing. part of } \nu(t') \; - \; \text{sing. part of } \nu(t)}{t' - t} \; > \; \delta$$

for some $t' > t$ (depending, of course, on t). Here a crucial rôle is played by the fact that

$$\lambda_p \longrightarrow 1 \quad \text{as } p \longrightarrow \infty.$$

That makes $1 - \lambda_p < \delta$ for large enough p, and then the *jump* which $\nu(t)$ has at x_p, equal to $(1-\lambda_p)\Delta_p$, will be $< \delta\Delta_p$, with $\Delta_p = x_p - x_{p-1}$, the distance from x_p to the *preceding* point of discontinuity for $\nu(t)$. Therefore the t fulfilling the last condition will, beyond a certain point, all lie in a collection of *disjoint intervals* (x'_p, x_p) with

$$x_p - x'_p \;=\; \frac{1-\lambda_p}{\delta}\Delta_p \;<\; \Delta_p$$

and

$$\frac{\text{sing. part of } \nu(x_p) \; - \; \text{sing. part of } \nu(x'_p)}{x_p - x'_p} \;=\; \delta.$$

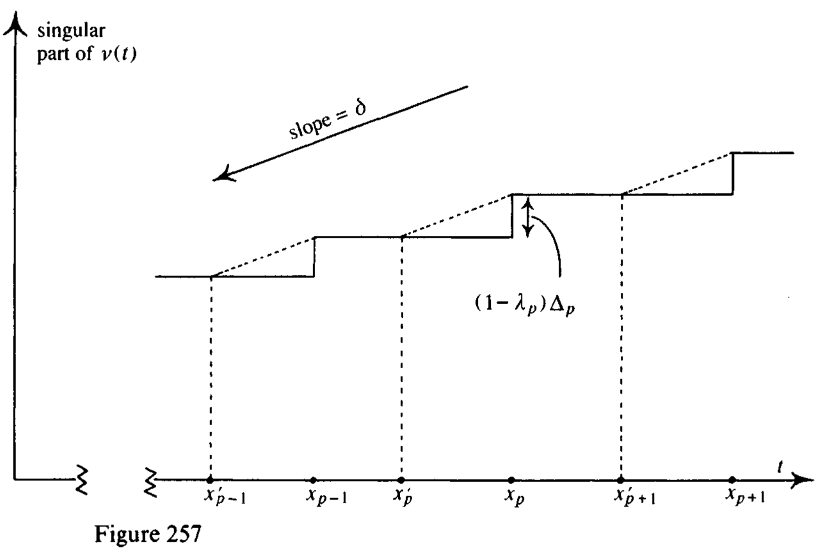

Figure 257

How *big* are the intervals (x'_p, x_p) ? In the present circumstances,

$$\Delta_p \sim \frac{1}{3}p^{-2/3}x_p \qquad \text{for } p \longrightarrow \infty,$$

so then

$$\frac{x_p - x'_p}{x'_p} < \frac{\Delta_p}{x_p - \Delta_p} \sim \frac{1}{3}p^{-2/3},$$

and we have

$$\sum_p \left(\frac{x_p - x'_p}{x'_p}\right)^2 < \infty;$$

the intervals (x'_p, x_p) *satisfy the Beurling condition that has played such an important rôle in this book!*

What can we (with almost certain confidence) conclude from this about the residual functions $\rho_K(t)$? The function $\rho(t)$ is, after all, $\mathscr{C}_\infty$, so a large enough K will *swamp out the derivative* $\rho'(t)$ for *all save the very large values of* t. The residual $\rho_K(t)$ will, in other words, *stay equal to zero* until t gets so large that the singular part of $\nu(t)$ shows the behaviour just described; *thereafter*, however, $\rho_K(t)$ and the latter function have *almost the same behaviour*, as we have seen. This means that ***for given*** $\boldsymbol{\delta > 0}$**, *we can, by making K sufficiently large, ensure that*** $\boldsymbol{\rho'_K(t) \leqslant \delta}$ ***for all*** $\boldsymbol{t \geqslant 0}$ ***save those belonging to a certain collection of disjoint intervals*** $\boldsymbol{(a_n, b_n)}$ ***(like the*** $\boldsymbol{(x'_p, x_p)}$ ***), with***

$$\frac{\rho_K(b_n) - \rho_K(a_n)}{b_n - a_n} = \delta$$

and

$$\sum_n \left(\frac{b_n - a_n}{a_n}\right)^2 < \infty.$$

Now what distinguishes the functions $\sigma_K(t)$ from the $\rho_K(t)$ is that *the analogous statement does not hold for the former when* $\delta < 1$. This is evident if we look at the graph of $n(t)$ which, for large enough t, is almost the same as that of any of the $\sigma_K(t)$. When $\delta < 1$, the Riesz construction, applied to $n(t)$, *will not even yield an infinite sequence of disjoint intervals like the* (x'_p, x_p); instead, one simply obtains a single big interval of infinite length. That's because at each x_p, $n(t)$, instead of jumping by a *small multiple* of Δ_p, *jumps by* $[\Delta_p]$, which is, for all intents and purposes, *the same as* $\Delta_p = x_p - x_{p-1}$ when p is large.

The *size* of these jumps of $n(t)$ was, by the way, the *key property ensuring* that the weight constructed in *article 1 did not admit multipliers of* *exponential type* $< \pi$. Cutting the jumps down to $(1-\lambda_p)\Delta_p$ for the construction in *article 4* was also what *made* the weight obtained there *admit multipliers*; it did so because $\lambda_p \longrightarrow 1$ as $p \longrightarrow \infty$. That, however, is just what guarantees the truth of the above statement about the $\rho_K(t)$! It thus seems likely that the distinction we have observed between behaviour of the $\rho_K(t)$ and that of the $\sigma_K(t)$ is the *source* of the corresponding two weights' difference in behaviour regarding the admittance of multipliers. *The 'essential' condition we have been seeking may well involve a requirement that the above statement hold for the* ρ_K *corresponding to a certain function* ρ, *associated with whatever weight one may have under consideration.*

Having been carried thus far by inductive reasoning, let us continue on grounds of pure speculation. We have been looking at $\mathscr{C}_\infty$ weights $W(x) \geqslant 1$ corresponding to monotone $\mathscr{C}_\infty$ functions $\lambda(t)$ according to the formula

$$\log W(x) \quad = \quad \log W(0) \quad - \quad \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\lambda(t).$$

Mostly, we have been considering *increasing* functions λ, and we have come around to the view that a weight W corresponding to one of these admits multipliers if (and, in *some* sense, *only if*) the above statement holds, with the functions

$$\lambda_K(t) \quad = \quad \int_0^t (\max(\lambda'(\tau), K) \; - \; K)\mathrm{d}\tau$$

standing in place of the $\rho_K(t)$. Insofar as *decreasing* functions $\lambda(t)$ were concerned, we simply observed near the beginning of this article that they were *good*, for a weight W given by any of *those admits* multipliers as long as

$$\int_{-\infty}^\infty \frac{\log W(x)}{1+x^2}\mathrm{d}x \quad < \quad \infty.$$

Let us now *drop* any requirement that the function $\lambda(t)$ be monotone, but *keep* the criterion that *the above statement hold for the* $\lambda_K(t)$.

The *increase* of $\lambda(t)$ is thereby *limited*, but *not* its *decrease*! Observe that for any $\mathscr{C}_\infty$ function $\omega(x)$ of the form

$$\omega(x) \quad = \quad \omega(0) \quad - \quad \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\lambda(t)$$

with

$$\int_{-\infty}^{\infty} \frac{|\omega(x)|}{1+x^2}\,dx \quad < \quad \infty,$$

the *Hilbert transform* $\tilde{\omega}(x)$ is defined (everywhere) and infinitely differentiable*, and *it differs from* $\pi\lambda(x)$ *by a constant multiple of* x. The statement involving the $\lambda_K(t)$ may thus be rephrased in terms of the Hilbert transform of $\log W(x)$, *eliminating* any direct reference to a *particular representation* for W.

Let us go one step further and guess at a criterion applicable to *any* weight $W(x)$ meeting the local regularity requirement of §B.1. Here, we *give up* trying to have the Hilbert transform of $\log W(x)$ fit the above statement. Instead, we let the latter apply to $\tilde{\omega}(x)$, where $\exp \omega(x)$ is *some even* $\mathscr{C}_\infty$ *majorant* of $W(x)$, as is in keeping with the guiding idea of this §. In that way, we arrive at the following

Conjecture. *A weight* $W(x) \geqslant 1$ *meeting the local regularity requirement admits multipliers iff it has an even* $\mathscr{C}_\infty$ *majorant* $\Omega(x)$ *with the following properties*:

(i) $$\int_{-\infty}^{\infty} \frac{\log \Omega(x)}{1+x^2}\,dx \quad < \quad \infty,$$

(ii) *To any* $\delta > 0$ *corresponds a* K *such that the* ($\mathscr{C}_\infty$) *Hilbert transform* $\tilde{\omega}(x)$ *of* $\omega(x) = \log \Omega(x)$ *has* derivative $\leqslant K + \delta$ *at all positive* x *save those contained in a set of* disjoint *intervals* (a_n, b_n), *with*

$$\sum_n \left(\frac{b_n - a_n}{a_n}\right)^2 \quad < \quad \infty$$

and

$$\int_{a_n}^{b_n} (\max(\tilde{\omega}'(x), K) - K)\,dx \quad \leqslant \quad \delta(b_n - a_n)$$

for each n.

E. A necessary and sufficient condition for weights meeting the local regularity requirement

The conjecture advanced at the end of the last § is true. One may look on its statement as an expression of the 'essential' condition for the admittance of multipliers that we had set out in that § to bring to light. A proof, which turns out to be not all that difficult, involves techniques

* cf. initial footnote to the *third* lemma of §E.1 below

like those employed in the determination of the completeness radius for a set of imaginary exponentials, carried out in Chapters IX and X. That proof is given below in article 2. Some auxiliary results are needed for it; we attend to those first.

1. Five lemmas

Lemma. *Let $\nu(t)$ be increasing on $[0, \infty)$ with $\nu(t)/t$ bounded for $t > 0$, and put*

$$F(x) \quad = \quad \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\nu(t) \qquad \text{for } x \in \mathbb{R}.$$

Suppose that $\rho(\xi)$, positive and infinitely differentiable, has compact support in $(0, \infty)$. Then the function

$$F_\rho(x) \quad = \quad \int_0^\infty F(\xi x) \frac{\rho(\xi)}{\xi} \mathrm{d}\xi$$

is equal to

$$\int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\nu_\rho(t),$$

where

$$\nu_\rho(t) \quad = \quad \int_0^\infty \nu(\xi t) \frac{\rho(\xi)}{\xi} \mathrm{d}\xi$$

is increasing and $\mathscr{C}_\infty$ in $(0, \infty)$, with

$$\nu_\rho'(t) \;\leqslant\; \text{const.} \quad \text{for } t > 0.$$

Proof. Is essentially an exercise about multiplicative convolution. Because the function $\log|1 - (x^2/t^2)|$ is neither bounded above nor below the justification of the transformations involved is a bit tricky.

Let us, as usual, write

$$F(z) \quad = \quad \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}\nu(t)$$

for complex z, noting that $F(x + \mathrm{i}y)$, for fixed $x \in \mathbb{R}$, is an *increasing* function of y when $y \geqslant 0$ (because $\nu(t)$ increases), and that

$$F(z) \;\leqslant\; \text{const.}\,|z|$$

(because $\nu(t)$ is also $\mathrm{O}(t)$ for $t \geqslant 0$). Supposing, then, that $\rho(\xi)$ has its support in $[a, b]$, $0 < a < b < \infty$, we have, for each fixed $x \in \mathbb{R}$,

$$F_\rho(x) \;=\; \int_a^b F(\xi x)\,\frac{\rho(\xi)}{\xi}\,d\xi \;=\; \lim_{y\to 0}\int_a^b F(\xi(x+iy))\,\frac{\rho(\xi)}{\xi}\,d\xi$$

by monotone convergence*.

For $z = x+iy$ with $y \neq 0$ it is easy, thanks to the properties of ν, to show by partial integration that

$$F(z) \;=\; 2\Re\int_0^\infty \frac{z^2}{z^2-t^2}\,\frac{\nu(t)}{t}\,dt.$$

Hence

$$\int_a^b F(\xi z)\,\frac{\rho(\xi)}{\xi}\,d\xi \;=\; 2\Re\int_a^b\int_0^\infty \frac{\xi^2 z^2}{\xi^2 z^2-t^2}\,\frac{\nu(t)}{t}\,\frac{\rho(\xi)}{\xi}\,dt\,d\xi.$$

On making the change of variable $t/\xi = \tau$, this becomes

$$2\Re\int_a^b\int_0^\infty \frac{z^2}{z^2-\tau^2}\,\frac{\nu(\xi\tau)}{\tau}\,\frac{\rho(\xi)}{\xi}\,d\tau\,d\xi.$$

Here, it is legitimate to change the order of integration, for the double integral is absolutely convergent. The last expression is thus equal to

$$2\Re\int_0^\infty \frac{z^2}{z^2-\tau^2}\,\frac{\nu_\rho(\tau)}{\tau}\,d\tau,$$

where

$$\nu_\rho(\tau) \;=\; \int_a^b \frac{\nu(\xi\tau)}{\xi}\,\rho(\xi)\,d\xi.$$

Since $\rho(\xi) \geqslant 0$ and $\nu(t)$ is increasing, $\nu_\rho(\tau)$ is also increasing. For $\tau > 0$, we can make the change of variable $\xi\tau = s$ in the preceding integral, getting

$$\nu_\rho(\tau) \;=\; \int_{a\tau}^{b\tau} \rho\!\left(\frac{s}{\tau}\right)\frac{\nu(s)}{s}\,ds.$$

Noting that $\rho(\xi)$ is $\mathscr{C}_\infty$ and vanishes (together with all its derivatives) for $\xi = a$ and $\xi = b$, we see that the expression on the right can be differentiated with respect to τ as many times as we want, making $\nu_\rho(\tau)$ $\mathscr{C}_\infty$ for $\tau > 0$. For the first derivative, we find

$$\nu_\rho'(\tau) \;=\; -\int_{a\tau}^{b\tau} \frac{s}{\tau^2}\,\rho'\!\left(\frac{s}{\tau}\right)\frac{\nu(s)}{s}\,ds, \qquad \tau > 0.$$

We have $\nu(s)/s \leqslant C$, say, for $s > 0$, so, denoting the maximum value of $|\rho'(\xi)|$ for $a \leqslant \xi \leqslant b$ by K, we get for the right side of the last relation a

* the integrand on the right being *bounded above* by the preceding inequality