Hypergeometric Motives

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1 Introduction

It must have been frustrating in the early days of calculus that an integral like

$$F(t) = \frac{1}{\pi} \int_0^1 \frac{1}{\sqrt{x(1-x)(1-tx)}}\, dx \qquad (1.1)$$

appeared not to be expressible in terms of known functions. This type of integral arises in computing the movement of the ideal pendulum or the length of an arc of an ellipse for example; they have remained relevant and are connected to a great deal of the mathematics of the last 200 years.

Indeed F is not an elementary function. Its Maclaurin expansion

$$F(t) = \sum_{k=0}^{\infty} \binom{2k}{k}^2 \left(\frac{t}{16}\right)^k \qquad (1.2)$$

is an example of a hypergeometric series. It satisfies a linear differential equation of order two of the type brilliantly analyzed by Riemann. As mentioned by Katz [Kat96, p.3], Riemann was lucky. His analysis only works because any rank two differential equation on $\mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\}$ is *rigid* in the sense that the local behavior of solutions around the missing points uniquely determines their global behavior.

Taking a more geometric perspective, (1.1) is presenting the function πF as a period of the family of elliptic curves defined by

$$E_t : \quad y^2 = x(1-x)(x-t). \qquad (1.3)$$

This fact implies as well that F satisfies an order two linear differential equation, ultimately because $H^1(E_t, \mathbb{Q})$ is two-dimensional.

Shifting now to more arithmetic topics, if we fix a rational number $t \neq 0, 1$ then for almost all primes p the number a_p defined by

$$|E_t(\mathbb{F}_p)| = p + 1 - a_p \qquad (1.4)$$

is of fundamental importance. With these a_p as the main

ingredients, one builds an *L*-function

$$L(E_t, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}. \qquad (1.5)$$

Much of the importance of the a_p is seen through this *L*-function. For example, the famous Birch-Swinnerton-Dyer conjecture says that the group $E_t(\mathbb{Q})$ modulo its torsion is isomorphic to $\mathbb{Z}^r$, where r is the order of vanishing of $L(E_t, s)$ at $s = 1$. A critical advance is the result of Wiles *et al.* that the function

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i z} \qquad (1.6)$$

on the upper half plane is a modular form. In particular, this result implies that $L(E_t, s)$ is at least well-defined at $s = 1$.

The equations displayed so far represent a standard general paradigm in arithmetic geometry. One can start with any variety X over $\mathbb{Q}$, not just the varieties (1.3). There are fully developed theories of periods and point counts, and in principle one can produce analogs of the period formulas (1.1)-(1.2) and the point count formula (1.4). Interacting now with deep but widely-believed conjectures, one can break the cohomology of X into irreducible motives, study *L*-functions like (1.5), and try to find corresponding automorphic forms like (1.6).

This survey is an informal invitation to *hypergeometric motives*, hereafter abbreviated HGMs; see Section 4 for their definition. We write them as $H(Q, t)$, with a rational function $Q \in \mathbb{Q}(T)$ satisfying certain conditions being the family parameter and $t \in \mathbb{Q} - \{0, 1\}$ the specialization parameter. The introductory family of examples is

$$H\left((T+1)^2/(T-1)^2, t\right) = H^1(E_t, \mathbb{Q}). \qquad (1.7)$$

Rigidity makes HGMs much more tractable than general motives: periods, point counts, and other invariants are given by explicit formulas in the parameters (Q, t). Our broader goal in this survey is to use HGMs to gain insight into the general theory of motives; we illustrate all topics with explicit examples throughout.

Sections 2-9 are geometric in nature. The main focus is on varieties generalizing (1.3) and the discrete aspect of periods like (1.1)-(1.2), as captured in vectors of Hodge numbers, $h = (h^{w,0}, \ldots, h^{0,w})$. A theme here is that HGMs form

quite a broad class of irreducible motives, as very general h arise. Sections 10-15 are arithmetic in nature, with the focus being on generalizations of (1.4), (1.6), and especially (1.5). Watkins has written a very useful hypergeometric motives package [Wat15] in *Magma* and throughout this article we indicate how to use it by including small snippets of *Magma* code. Together these snippets are enough to let *Magma* beginners numerically compute with L-functions $L(H(Q,t),s)$ using the free online *Magma* calculator.

2   Hypergeometric functions

We begin by generalizing (1.1)-(1.2) and explaining how this generalization leads to family parameters.

Integrals and series.   Let $\alpha = (\alpha_1,\ldots,\alpha_n)$, and $\beta = (\beta_1,\ldots,\beta_n)$ be vectors of complex numbers with $\mathrm{Re}(\beta_j) > \mathrm{Re}(\alpha_j) > 0$ and $\beta_n = 1$. For $|t| < 1$ define, making use of the standard Gamma function $\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}dx$,

$$F(\alpha,\beta,t) = \prod_{i=1}^{n} \frac{\Gamma(\beta_i)}{\Gamma(\alpha_i)\Gamma(\beta_i - \alpha_i)} \cdot \qquad (2.1)$$
$$\int_0^1 \cdots \int_0^1 \frac{\prod_{i=1}^{n-1}\left(x_i^{\alpha_i-1}(1-x_i)^{\beta_i-\alpha_i-1}dx_i\right)}{(1-tx_1\cdots x_{n-1})^{\alpha_n}}.$$

Via $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$, (1.1) is the special case $\alpha = (1/2, 1/2)$ and $\beta = (1, 1)$.

Expand the denominator of the integrand of (2.1) via the binomial theorem and use Euler's beta integral to evaluate the individual terms. Written in terms of Pochhammer symbols $(u)_k = u(+1)\cdots(u+k-1)$, the result is

$$F(\alpha,\beta,t) = {}_nF_{n-1}(\alpha,\beta,t) := \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_n)_k}{(\beta_1)_k \ldots (\beta_n)_k} t^k. \quad (2.2)$$

In other words, the integral (2.1) is an alternative definition of the standard hypergeometric power series (2.2). The case $\alpha = (1/2, 1/2)$ and $\beta = (1, 1)$ simplifies to (1.2).

Monodromy.   An excellent general reference for hypergeometric functions is [BH89], and we now give a summary sufficient for this survey. The function $F(\alpha,\beta,t)$ is in the kernel of an nth order differential operator $D(\alpha,\beta)$ with singularities only at 0, 1 and ∞. This means in particular that $F(\alpha,\beta,t)$, initially defined on the unit disk, extends to a "multivalued function" on the thrice-punctured projective line $\mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\} = \mathbb{C} - \{0, 1\}$. With respect to a given basis, this multivaluedness is codified by a representation ρ of the fundamental group $\pi_1(\mathbb{C} - \{0, 1\}, 1/2)$ into $GL_n(\mathbb{C})$. The fundamental group is free on g_0 and g_1, with these elements coming from counterclockwise circular paths of radius 1/2 about 0 and 1 respectively. To emphasize the equal status of ∞ and 0, it is better to present this group as generated by

g_∞, g_1, and g_0, subject to the relation $g_\infty g_1 g_0 = 1$. The assumption $\beta_n = 1$ was only imposed to present the classical viewpoint (2.1)-(2.2) cleanly; we henceforth drop it.

A useful fact due to Levelt is the explicit description of the matrices $h_\tau = \rho(g_\tau) \in GL_n(\mathbb{C})$ with respect to a certain well-chosen basis. Define polynomials

$$q_\infty := (T - e^{2\pi i \alpha_1})\cdots(T - e^{2\pi i \alpha_n}),$$
$$q_0 := (T - e^{-2\pi i \beta_1})\cdots(T - e^{-2\pi i \beta_n}).$$

Then h_∞ and h_0 are companion matrices of q_∞ and q_0, while h_1 is determined by $h_\infty h_1 h_0 = I$. The matrix h_1 differs minimally from the identity in that $h_1 - I$ has rank 1. We will henceforth consider only cases where no $\alpha_i - \beta_j$ is an integer. This ensures that the h_τ generate an irreducible subgroup Γ of $GL_n(\mathbb{C})$. Moreover the representation is rigid, in the following sense: suppose h'_∞, h'_1, and h'_0 are conjugate to h_∞, h_1, and h_0 respectively. Then there is a single matrix c such that $h_\tau = ch'_\tau c^{-1}$ for all three τ.

Family parameters.   The parameters (α, β) contain information which is irrelevant for the sequel. First, the individual α_j and β_j are important only modulo integers. Second, the orderings of the α_j and β_j do not matter. To remove these irrelevancies, we will regard the degree n rational function $Q = q_\infty/q_0$ as the primary index in the sequel, calling it the *family parameter*. A bonus of this shift in emphasis is that an important field $E \subset \mathbb{C}$ is made evident, the field generated by the coefficients of q_∞ and q_0. By construction, all three h_τ lie in $GL_n(E)$.

The cases which naturally have underlying motives are exactly the ones with all α_j and β_j rational, so that E is some cyclotomic field. In this survey we will substantially simplify by restricting to cases with $E = \mathbb{Q}$. With this simplification, there are two natural ways to present Q as follows. Write $\Psi_m = T^m - 1$ and consider its factorization into irreducible polynomials, $\Psi_m = \prod_{d|m} \Phi_d$. So the factors are cyclotomic polynomials $\Phi_d = \prod_{j \in (\mathbb{Z}/d)^\times}(T - e^{2\pi i j/d})$ and thus have degree the Euler totient $\phi(d) = |(\mathbb{Z}/d)^\times|$.

In our introductory example, the ways are

$$Q = \frac{(T^2-1)^2}{(T-1)^4} = \frac{\Psi_2^2}{\Psi_1^4} = \frac{(T+1)^2}{(T-1)^2} = \frac{\Phi_2^2}{\Phi_1^2}. \quad (2.3)$$

In general, the second way is just the canonical factorization into irreducibles, while the first way is the unique "unreduction" to products of Ψ_m in which no factor appears in both a numerator and denominator.

To enter a family parameter Q into *Magma*, one can use either of these two ways, as in the equivalent commands

```
Q:=HypergeometricData([*-2,-2,1,1,1,1*]);
Q:=HypergeometricData([1,1],[2,2]);    (2.4)
```

In the first method, one inputs just the *gamma vector* $\gamma = [\gamma_1,\ldots,\gamma_l]$ formed by subscripts on the Ψ's, using signs to

distinguish between numerator and denominator. In the second method, one inputs just the subscripts of the denominator and then numerator Φ's, these being called the *cyclotomic parameters*. When working with underlying varieties, the gamma vectors are so useful that we often simply write $H(\gamma, t)$ rather than $H(Q, t)$. After the transition to motives, the cyclotomic presentation is generally more convenient. To simplify slightly, we henceforth require that $\gcd(\gamma_1, \ldots, \gamma_l) = 1$.

Note that initialization commands like (2.4) do nothing by themselves; in this survey *Magma* will first start returning useful information in Sections 5 and 11. Note also that *Magma* requires a semicolon at the end of all commands, as in (2.4). We often omit these semicolons in the sequel.

Orthogonal vs. symplectic. The number $Q(0) = \det(h_1)$ is either -1 or 1 under our restriction $E = \mathbb{Q}$. This dichotomy is strongly felt throughout this survey. It can also be expressed in terms of the fundamental bilinear form $(\cdot, \cdot)$ on $\mathbb{Q}^n$ preserved by the monodromy group $\Gamma = \langle h_\infty, h_0 \rangle$; see [BH89, §4], [JRV14, §3.5]. In the *orthogonal case*, h_1 is conjugate to $\mathrm{diag}(-1, 1, \ldots, 1)$ and $(\cdot, \cdot)$ is symmetric. In the *symplectic case*, h_1 is conjugate to $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \oplus \mathrm{diag}(1, \ldots, 1)$, and $(\cdot, \cdot)$ is antisymmetric.

3 Source varieties

We now describe varieties which give rise to hypergeometric motives.

Euler varieties. We have already generalized (1.1) to (2.1) and (1.2) to (2.2). Assuming briefly $\beta_n = 1$ again, a natural generalization of (1.3) is to

$$y^m = \prod_{j=1}^{n-1} x_j^{a_j} (1 - x_j)^{b_j} (1 - tx_1 \cdots x_{n-1})^{a_n}. \qquad (3.1)$$

Here m is the least common denominator of the α_j and β_j, and the exponents are integers $0 \le a_j, b_j < m$ such that

$$a_j \equiv -m\alpha_j \bmod m, \qquad b_j \equiv m(\alpha_j - \beta_j) \bmod m,$$

for $j = 1, \ldots, n-1$ and $a_n \equiv m\alpha_n \bmod m$. The equations (2.1)-(2.2) show that a specified scalar multiple of $_nF_{n-1}(\alpha, \beta, t)$ arises as a period of this variety. However the varieties (3.1) depend on how the parameters are paired: $(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)$. This dependence complicates the arithmetic of these varieties, so we will use an alternative collection of varieties to define hypergeometric motives.

Canonical varieties. The alternative varieties appear under the term "circuits" in [GKZ94] and are studied at greater length in [BCM15]. For a gamma vector γ and a complex number t, define $X_{\gamma,t}^{\mathrm{bcm}} \subset \mathbb{P}^{l-1}$ by two homogeneous equations,

$$\sum_{j=1}^{l} y_j = 0, \qquad \prod_{\gamma_j > 0} y_j^{\gamma_j} = u \prod_{\gamma_i < 0} y_j^{-\gamma_j}. \qquad (3.2)$$

Here and in the sequel, we systematically use the abbreviation $u = t \prod_j \gamma_j^{\gamma_j}$. The canonical variety is by definition the open subvariety $X_{\gamma,t}$ on which all the homogeneous coordinates y_j are nonzero. The point $(\gamma_1 : \cdots : \gamma_l)$ is an ordinary double point on $X_{\gamma,1}$ and otherwise all the $X_{\gamma,t}$ are smooth. Because of this double point, we exclude the case $t = 1$ from consideration until Section 9.

Toric models. From a dimension-count viewpoint, the BCM equations (3.2) for canonical varieties are inefficient. They start with the $l = \kappa + 3$ variables y_i and use two equations and projectivization to get the desired κ-dimensional variety $X_{\gamma,t}$. The toric models from [GKZ94] start instead with $d = \kappa + 1$ variables x_i and present $X_{\gamma,t}$ by just one equation.

To obtain a toric model from a gamma vector γ, one proceeds as illustrated by Table 3.1. First, for each new variable x_i choose a row vector m_{i*} in $\mathbb{Z}^l$ which is orthogonal to the given l-vector γ. These row vectors are required to be such that $\mathbb{Z}^l / \langle m_{i*} \rangle$ is torsion-free. Second, choose a row vector $k \in \mathbb{Z}^l$ which satisfies $\gamma \cdot k = 1$. The toric model is then

$$\sum_{i=1}^{l} u^{k_i} \prod_{j=1}^{d} x_i^{m_{ij}} = 0. \qquad (3.3)$$

So in the example of Table 3.1, the resulting equation is

$$x_1^2 + ux_1 x_2^2 + u + x_1^3 x_2 = 0, \qquad (3.4)$$

with $u = -2^6 3^3 t / 5^5$. In general, the variety $X_{\gamma,t}$ is the subvariety of the torus $\mathbb{G}_m^d$ given by the equation (3.3).

	γ_1	γ_2	γ_3	γ_4		-5	-2	3	4
$x_1:$	m_{11}	m_{12}	m_{13}	m_{14}	$=$	2	1	0	3
$x_2:$	m_{21}	m_{22}	m_{23}	m_{24}		0	2	0	1
$u:$	k_1	k_2	k_3	k_4		0	1	1	0

Table 3.1: Derivation of the equation (3.4) for $X_{[-5,-2,3,4],t}$

The relation between the BCM equation for $X_{\gamma,t}$ and a toric model for $X_{\gamma,t}$ is very simple:

$$y_j = u^{k_j} \prod_{i=1}^{d} x_i^{m_{ij}}. \qquad (3.5)$$

When one uses (3.5) to write (3.2) in terms of the x_i, the second equation is identically satisfied while the first becomes (3.3). Conversely, any point $(y_1 : \cdots : y_l)$ comes from a unique $(x_1, \ldots, x_d)$ because of the torsion-free condition.

Polytopes. A toric model gives a polytope $\Delta \subset \mathbb{R}^d$ which is an aid to understanding the $X_{\gamma,t}$. The case $d = 2$ is readily visualizable and Figure 3.1 continues our example. In general, one interprets the column vectors m_{*j} of the chosen matrix as points in $\mathbb{Z}^d$ and Δ is their convex hull. Let Δ_j be the convex hull of all the points except the jth one. Normalize volume so that the standard d-dimensional simplex has volume 1, and thus $[0,1]^d$ has volume $d!$. Then the volume of Δ_j is $|\gamma_j|$. The Δ_j with $\gamma_j > 0$ form one triangulation of Δ, while the Δ_j with $\gamma_j < 0$ form another. The total volume of Δ is the important number $\text{vol}(\gamma) = \frac{1}{2} \sum_{j=1}^{l} |\gamma_j|$
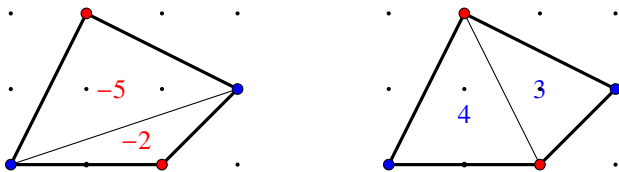

Figure 3.1: The triangulations $\Delta = \Delta_1 \cup \Delta_2$ and $\Delta = \Delta_3 \cup \Delta_4$ of the polytope Δ for the family with $\gamma = [-5, -2, 3, 4]$. The points are at the column vectors $\binom{m_{1j}}{m_{2j}}$ of Table 3.1, and γ_j is printed in the opposite triangle.

The common topology of the $X_{\gamma,t}$ with $t \neq 1$ is reflected in the combinatorics of Δ. In the case of $d = 2$, the genus g of $X_{\gamma,t}$ is the number of lattice points on the interior, while the number of punctures k is the number of lattice points on the boundary. Pick's theorem then says that the Euler characteristic $\chi = 2 - 2g - k$ of $X_{\gamma,t}$ is $-\text{vol}(\gamma)$. In the example of Figure 3.1, $(g, k, \chi) = (2, 5, -7)$. For larger ambient dimension d, the situation is of course much more complicated, but always $\chi = (-1)^{d-1}\text{vol}(\gamma)$.

Compactifications. In algebraic geometry, one normally wants to compactify a given open variety such as $X_{\gamma,t}$ and there are typically many natural ways of doing it.

We already saw the compactification $X_{\gamma,t}^{\text{bcm}}$. It is a hypersurface of degree $\text{vol}(\gamma)$ in the projective space $\mathbb{P}^d$ defined by the first equation of (3.2). On the other hand, for any choice of matrix m with all entries nonnegative, homogenization of (3.3) gives a alternative compactification $\overline{X}_{\gamma,t} \subset \mathbb{P}^d$.

In our continuing example $\gamma = [-5, -2, 3, 4]$, the plane curve $X_{\gamma,t}^{\text{bcm}}$ has degree seven. In contrast, the plane curve $\overline{X}_{\gamma,t}$ has degree just four, this number arising as the maximum column sum of the matrix m in Table 3.1. Smooth curves in these degrees have genera 15 and 3 respectively. For $t \neq 1$, $X_{\gamma,t}$ has genus 2 so $X_{\gamma,t}^{\text{bcm}}$ must have bad singularities while $\overline{X}_{\gamma,t}$ has just a single node.

Another compactification $X_{\gamma,t}^{\text{BCM}}$ is a major focus of [BCM15]. It is typically not smooth, but only has quotient singularities. These singularities are mild in the sense that $X_{\gamma,t}^{\text{BCM}}$ looks smooth from the viewpoint of rational cohomology, and may be ignored when discussing motives as in the

next section.

4 HGMs from cohomology

Here we define HGMs and explain how their behavior is simpler than other similar motives.

Motivic formalism. Let K and E be subfields of $\mathbb{C}$; the case of principal interest to us is $K = E = \mathbb{Q}$. Minimally modifying Grothendieck's original conditional definitions, André unconditionally defined a category $\mathcal{M}(K, E)$ of pure motives over K with coefficients in E [And04]. The formal structures of this category can best be understood in terms of a huge proreductive algebraic group $\mathbb{G}_K$ over $\mathbb{Q}$, the absolute motivic Galois group of K. Then $\mathcal{M}(K, E)$ is exactly the category of representations of $\mathbb{G}_K$ on finite-dimensional E vector spaces.

When taking cohomology, we are always implicitly working with the complex points of a variety. For a smooth projective variety X over K and an integer w, the singular cohomology space $M = H^w(X, E)$ is an object of $\mathcal{M}(K, E)$. The image G_M of $\mathbb{G}_K$ in the general linear group of M is by definition the motivic Galois group of M. The purpose of G_M, as the rest of this survey will make clear, is to group-theoretically coordinate very concrete structures on the vector space $H^w(X, E)$.

Two copies of the multiplicative group $\mathbb{G}_m = GL_1$ play important roles in the formalism of motives. One is a normal subgroup and the other a quotient: $\mathbb{G}_m \subset \mathbb{G}_K \twoheadrightarrow \mathbb{G}_m$. A rank n motive M is said to be of weight w if the representation restricted to the subgroup $\mathbb{G}_m$ consists of n copies the representation $r \mapsto r^w$. The motives $H^w(X, E)$ all have weight w. The representation of $\mathbb{G}_K$ on the rank one motive $E(-1) := H^2(\mathbb{P}^1, E)$ corresponds to the representation $t \mapsto t$ of the quotient group $\mathbb{G}_m$. The motive corresponding to the representation $t \mapsto t^j$ is denoted $E(-j)$. The motives $M(j) := M \otimes E(j)$ are called the Tate twists of M.

Each category of pure motives $\mathcal{M}(K, E)$ is contained in a larger category $\mathcal{MM}(K, E)$ of mixed motives, where an irreducible motive has a canonical weight filtration with subquotients in $\mathcal{M}(K, E)$. We will mention mixed motives at several junctures, but our focus is sharply on pure motives.

Definition of HGMs. Let γ be a gamma vector of length $\kappa + 3$ with r negative entries and let $t \in \mathbb{Q}^{\times} - \{1\}$. The hypergeometric motive $H(\gamma, t)$ is defined from the cohomology of the affine variety $X_{\gamma,t}$ [RV19]. We start with the compactly supported middle cohomology space $H_c^{\kappa}(X_{\gamma,t}, \mathbb{Q})$ and begin by cutting out a subquotient $H'(\gamma, t)$ in two steps.

First, we eliminate the contribution of the ambient d-dimensional torus to obtain the primitive subspace $PH_c^{\kappa}(X_{\gamma,t}, \mathbb{Q})$. Second, we take any smooth compactification $\overline{X}$ of $X_{\gamma,t}$, or one with at worst mild singularities as mentioned above, and consider the image $H'(\gamma, t)$ of

$PH_c^\kappa(X_{\gamma,t}, \mathbb{Q})$ under the natural map to $H^\kappa(\bar{X}, \mathbb{Q})$. As a quotient of $PH_c^\kappa(X_{\gamma,t}, \mathbb{Q})$, the space $H'(\gamma, t)$ is independent of the choice of compactification.

For example, for the compactification $X_{\gamma,t}^{BCM}$ there is a decomposition of its middle cohomology, $H^\kappa(X_{\gamma,t}^{BCM}, \mathbb{Q}) = H'(\gamma, t) \oplus T$. It is described at the level of point counts in [BCM15, Thm 1.5]. Here T is zero if κ is odd and the sum of $\binom{\kappa+1}{r-1}$ copies of $\mathbb{Q}(-\kappa/2)$ if κ is even.

Finally, we define the hypergeometric motive $H(\gamma, t) \in \mathcal{M}(\mathbb{Q}, \mathbb{Q})$ as the Hodge-normalized Tate twist $H'(\gamma, t)(j)$, as discussed in the next section. So $H(\gamma, t)$ has weight $w = \kappa - 2j$ with $j \in \mathbb{Z}_{\geq 0}$ specified there.

More conceptually, $PH_c^\kappa(X_{\gamma,t}, \mathbb{Q})$ is a mixed motive of rank $\mathrm{vol}(\gamma) - 1$ and the pure motive $H'(\gamma, t)$ is its weight κ quotient. The passage from $PH_c^\kappa(X_{\gamma,t}, \mathbb{Q})$ to $H'(\gamma, t)$ is closely related to the reduction of fractions as in (2.3). In particular, $H'(\gamma, t)$ has rank $n = \deg(Q)$.

It is worth stressing that the full mixed motive $PH_c^\kappa(X_{\gamma,t}, \mathbb{Q})$ is itself of great interest, with its lower weight parts playing an important role in deeper studies of hypergeometric motives.

Motivic Galois groups of HGMs. For $t \in \mathbb{C}^\times - \{1\}$, one likewise gets a motive $M = H(\gamma, t) \in \mathcal{M}(\mathbb{Q}(t), \mathbb{Q})$. If t is transcendental then the motivic Galois group of M can be cleanly expressed in terms of the monodromy group Γ of Section 2, as follows. If $w = 0$, then Γ is finite and $G_M = \Gamma$. If $w > 0$, then Γ is infinite and G_M is the smallest algebraic group containing both Γ and scalars; more explicitly, G_M is the conformal symplectic group CSp_n in the symplectic case of odd w, and a conformal orthogonal group CO_n in the orthogonal case of even w. In the case that t is algebraic, including our main case that t is rational, the same identification of G_M holds almost always.

Related motives. The toric model viewpoint is part of the program in [GKZ94] to approach algebraic geometry by emphasizing the number $\kappa + \epsilon$ of terms in polynomials defining κ-dimensional varieties. By scaling to normalize coefficients, such varieties come in $(\epsilon - 2)$-dimensional families.

For $\epsilon = 2$, the Newton polytope Δ is a simplex. An abelian group A of order $\mathrm{vol}(\Delta)$ and some exponent m acts on the single associated complex variety X. The essential cases here are the Fermat varieties in $\mathbb{P}^{\kappa+1}$, defined by

$$x_1^m + \cdots + x_{\kappa+2}^m = 0. \tag{4.1}$$

The group A comes from scaling the variables by m^{th} roots of unity and has order $m^{\kappa+1}$ and exponent m. Writing $K = \mathbb{Q}(e^{2\pi i/m})$, the action decomposes $H^\kappa(X, K)$ into one-dimensional motives in $\mathcal{M}(K, K)$. This setting of $\epsilon = 2$ was the focus of several influential papers of Weil from around 1950, and the rank one motives appearing are *Jacobi motives*.

The case $\epsilon = 3$ corresponds to *general hypergeometric motives* where the α_j and β_j can be arbitrary rational num-

bers. The group A now has order a divisor of $\mathrm{vol}(\Delta)$ and some exponent m. For example, for $m = \kappa + 2 \geq 3$ one can add the term $ux_1 \cdots x_m$ to (4.1); then A is reduced to having order m^κ but still has exponent m. The action of A again decomposes $H^\kappa(X, K)$ in $\mathcal{M}(K, K)$ and the summands include general hypergeometric motives. Our torsion-free requirement for exponent matrices is equivalent to requiring that the column vectors affine span $\mathbb{Z}^d$; in turn, this means that our HGMs constitute exactly the case $|A| = 1$.

Much of what we are describing in this article both has simpler analogs for Jacobi motives and extends to general hypergeometric motives. Indeed [BH89] and [Kat90] are in the latter setting. However the associated L-functions correspond to motives that have been descended to $\mathcal{M}(\mathbb{Q}, \mathbb{Q})$ and have rank $\phi(m)n$. Because of the factor $\phi(m)$, inclusion of these other settings would only modestly increase the collection of computationally accessible L-functions. Also the resulting motives in $\mathcal{M}(\mathbb{Q}, \mathbb{Q})$ have motivic Galois groups which are more complicated than the CSp_n and CO_n arising ubiquitously in our setting of $H(\gamma, t)$.

5 Hodge numbers

One of the very first things one wants to know about a motive is its Hodge numbers. Fortunately, this desire is easily satisfied for HGMs by an appealing procedure.

Background. For a smooth projective variety X over $K \subseteq \mathbb{C}$, there is a decomposition of complex vector spaces $H^w(X, \mathbb{C}) = \bigoplus_{p=0}^{w} H^{p,w-p}$, with complex conjugation on coefficients switching $H^{p,q}$ and $H^{q,p}$. The Hodge numbers $h^{p,q} := \dim(H^{p,q})$ therefore satisfy Hodge symmetry $h^{p,q} = h^{q,p}$ and sum to the Betti number $b_w := \dim(H^w(X, \mathbb{C}))$. Classical examples are given in (6.1)-(6.2) below.

Likewise, the rank of a weight w motive $M \in \mathcal{M}(K, E)$ is decomposed into Hodge numbers $h^{p,w-p}$. The decomposition has a simple group-theoretic reformulation: $\mathbb{G}_K(\mathbb{R})$ contains a subgroup $\mathbb{C}^\times$ which acts on $H^{p,q}$ by $z^p \bar{z}^q$. If either K or E is in $\mathbb{R}$, as will generally be the case for us, then Hodge symmetry continues to hold.

If a motive M has Hodge numbers $\underline{h}^{p,q}$ then the Hodge numbers of its Tate twist $M(j)$ are $h^{p-j,q-j} = \underline{h}^{p,q}$. The Hodge-normalization of a pure weight motive is the Tate twist for which all the nonzero Hodge numbers are in the vector $h = (h^{w,0}, \ldots, h^{0,w})$ and at least one of the outer ones is nonzero.

Zigzag procedure. The procedure we are about to describe is equivalent to a formula conjectured by Corti and Golyshev [CG11] and proved by different methods in Fedorov [Fed18] and [RV19]. The procedure is completely combinatorial and only depends on the interlacing pattern of the roots of q_∞ and q_0 in the unit circle.

To pass from a family parameter $Q = q_\infty/q_0$ to its Hodge vector h one proceeds as illustrated by Figure 5.1. One or-

α_j's: $\frac{1}{4}$ $\frac{1}{3}$ $\frac{1}{2}$ $\frac{2}{3}$ $\frac{3}{4}$

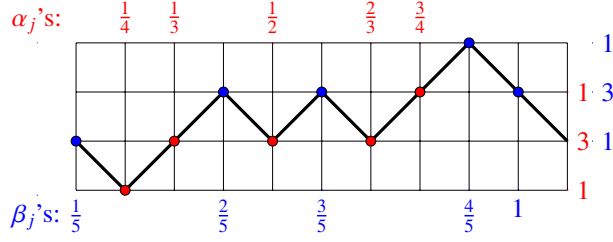β_j's: $\frac{1}{5}$ $\frac{2}{5}$ $\frac{3}{5}$ $\frac{4}{5}$ 1

Figure 5.1: The zigzag procedure with input the family parameter $Q = \Phi_2\Phi_3\Phi_4/\Phi_1\Phi_5$ and output the Hodge vector $h = (1, 3, 1)$

ders the parameters α_j and β_j, viewed as elements in say $(0, 1]$; for more immediate readability, we associate the colors red and blue to ∞ and 0 respectively. One draws a point at $(0, 0)$ corresponding to the smallest parameter in a Cartesian plane. One then proceeds in uniform steps from left to right, drawing a point for each parameter and then moving diagonally upwards after red points and diagonally downwards after blue points. One focuses on one color or the other, counting the number of points on horizontal lines. The numbers obtained form the Hodge vector h. The red and blue dots yield the same Hodge vector but contain more information. They may be used to describe the limiting mixed Hodge structure at $t = \infty$ and 0 respectively.

The completely intertwined case. Complete intertwining of the α_j and β_j gives the extreme where the resulting Hodge vector is just (n). Beukers and Heckman [BH89] proved that complete intertwining is exactly the condition one needs for the monodromy group $\langle h_\infty, h_0\rangle$ to be finite. They also established the complete list of such pairs (α, β). Actually they, like Schwarz who famously treated the $n = 2$ case more than a century earlier, worked without our standing assumption $E = \mathbb{Q}$. Then one needs to require complete intertwining of all the natural conjugates of (α, β) and the list obtained is longer.

In our setting of $E = \mathbb{Q}$, the corresponding γ-vectors are of odd lengths 3 to 9. There are infinite collections of length 3 and 5 given by coprime positive integers a, b:

$$
\begin{array}{ll}
(i) & [-(a+b), a, b], \quad\quad\quad\quad\quad\quad\quad (5.1)\\
(ii) & [-2(a+b), -a, 2a, b, a+b],\\
(iii) & [-2a, -2b, a, b, a+b].
\end{array}
$$

Here and always when discussing classification, we omit consideration of $-\gamma$ whenever γ is listed. In case (i), the canonical variety consists of just $a + b$ points. Removing a variable, the BCM presentation takes the form

$$X_{a,b,t} : y^a(1-y)^b - \frac{a^a b^b}{(a+b)^{a+b}} t = 0.$$

For $b = 1$ this presentation is already trinomial; in general, one has to make a non-trivial change of variables to pass to the trinomial presentation of $X_{a,b,t}$ given by a toric model.

Beyond the closely related collections (i) - (iii), there are only finitely many further γ, all related to Weyl groups. [BH89, Table 8.3] says that, modulo the quadratic twisting operation $Q(T) \mapsto Q(-T)$, there are just one, five, five, and fifteen respectively for the groups $W(F_4)$, $W(E_6)$, $W(E_7)$, and $W(E_8)$. One of the $W(E_6)$ cases is discussed in Section 7 and the remaining $W(E_n)$ cases are similarly treated in [Rob18].

The completely separated case. Complete separation of the α_j and β_j gives the extreme where the resulting Hodge vector is $(1, 1, \dots, 1, 1)$. The subcase where $q_0 = (T-1)^n$ has the simplifying feature that h_0 consists of a single Jordan block. Families in this subcase have received particular attention in the literature; the condition is sometimes verbalized as MUM, for *maximal unipotent monodromy*.

Classification of families in the completely separated case is easier than in the completely intertwined case. It becomes trivial in the MUM subcase because q_∞ is arbitrary except for the fact that it contains no factors of $(T-1)$. Accordingly, the number c_n of rank n families in the MUM subcase is given by a generating function

$$
\begin{aligned}
\sum_{n=0}^{\infty} c_n x^n &= \prod_{k=2}^{\infty} \frac{1}{1 - x^{\phi(k)}} \quad\quad\quad (5.2)\\
&= 1 + x + 4x^2 + 4x^3 + 14x^4 + 14x^5 + \cdots
\end{aligned}
$$

Always $c_{2j} = c_{2j+1}$ as under the MUM restriction multiplying by $(T+1)/(T-1)$ gives a bijection on parameters. Arithmetic information about the list underlying $c_4 = 14$ is in [RV03]. For general n, the case $q_\infty = 1+T+\cdots+T^n$ is the "mirror dual" of the Dwork case discussed after (4.1), and so motives of this family have been given special attention in the physics literature.

Signature and the *Magma* implementation. A motive defined over a subfield of $\mathbb{R}$ has a signature σ, which is the trace of complex conjugation. For odd weight motives, it is always zero. For even weight HGMs $H(Q, t)$, it depends only on Q and the interval $(-\infty, 0)$, $(0, 1)$, or $(1, \infty)$ in which t lies. *Magma*'s command `HodgeStructure` returns both the Hodge vector and the signature in coded form. To see just the Hodge vector clearly, one can implement Q as in (2.4) and get the Hodge vector from say

    HodgeVector(HodgeStructure(Q,2));

For example, from the gamma vector $[-21, 1, 2, 3, 4, 5, 6]$ one gets the Hodge vector $(1, 2, 12, 2, 1)$.

6 Projective Hypersurfaces

Here we realize some HGMs in the cohomology of the most classical varieties of all, smooth hypersurfaces in projective

space.

Hodge numbers. Let $X \subset \mathbb{P}^{\kappa+1}$ be a smooth hypersurface of degree δ. Let $PH^\kappa(X, \mathbb{Q})$ be the primitive part of its middle cohomology, meaning the part that does not come from the ambient projective space. If κ is odd, then this primitive part is all of $H^\kappa(X, \mathbb{Q})$. If κ is even, then the complementary piece that we are discarding is $\mathbb{Q}(-\kappa/2)$.

Hirzebruch gave a formula for the Hodge numbers of $PH^\kappa(X, \mathbb{Q})$ as a function of κ and δ. For example, the sum of the Hodge numbers and first Hodge number are respectively

$$b_\kappa = \frac{(\delta - 1)^{\kappa+2} + (-1)^\kappa(\delta - 1)}{\delta}, \quad h^{\kappa,0} = \binom{\delta - 1}{\kappa + 1}. \quad (6.1)$$

These special cases and Hodge symmetry are sufficient to give Hodge vectors when $\kappa \leq 3$:

δ	Curves in $\mathbb{P}^2$	Surfaces in $\mathbb{P}^3$	Threefolds in $\mathbb{P}^4$	
3	$(1, 1)$	$(0, 6, 0)$	$(0, 5, 5, 0)$	
4	$(3, 3)$	$(1, 19, 1)$	$(0, 30, 30, 0)$	(6.2)
5	$(6, 6)$	$(4, 44, 4)$	$(1, 101, 101, 1)$	
6	$(10, 10)$	$(10, 85, 10)$	$(5, 255, 255, 5)$.	

For $\kappa = 1$, either part of (6.1) reduces to the genus formula for smooth plane curves, $g = (\delta - 1)(\delta - 2)/2$.

One example for every (δ, κ). Let $\delta = e + 1 \geq 3$ be a desired degree and let κ be a desired dimension. Define

$$\gamma = [1, -e, e^2, \ldots, (-e)^\kappa, (-e)^{\kappa+1} - 1, \frac{(-e)^{\kappa+2} + e}{e + 1}]. \quad (6.3)$$

The toric procedure illustrated by Table 3.1 yields the completed canonical variety:

$$X_t : \quad ux_{\kappa+1}x_1^e + \sum_{i=2}^{\kappa+2} x_{i-1}x_i^e + x_{\kappa+2}^{e+1} = 0. \quad (6.4)$$

The necessary orthogonality relations on each variable's exponents are illustrated by the case of cubic fourfolds where $\gamma = [1, -2, 4, -8, 16, -33, 22]$. Then (6.4) becomes

$$X_t : ux_5x_1^2 + x_1x_2^2 + x_2x_3^2 + x_3x_4^2 + x_4x_5^2 + x_5x_6^2 + x_6^3 = 0.$$

For x_5, the relation is that $m_{5*} = (1, 0, 0, 0, 2, 1, 0)$ is orthogonal to γ. In general, partial derivatives of (6.4) are very simple since the row vectors m_{i*} have just two nonzero entries, except for the case $i = \kappa + 1$ and its three nonzero entries. It is then a pleasant exercise to check via the Jacobian criterion that X_t is smooth for $t \in \mathbb{C}^\times - \{1\}$.

The degree of the rational function Q determined by γ can be computed uniformly in (δ, κ) as the cancellations to be analyzed are very structured. This degree agrees with the Betti number b_κ from (6.1). Thus $H(\gamma, t)$ is the full primitive middle cohomology of X_t, while *a priori* it might have been a proper subspace. The zigzag procedure for computing Hodge numbers must agree in the end with the Hirzebruch formula. The reader might want to check the above case of cubic fourfolds, where Hirzebruch's full formula gives $(0, 1, 20, 1, 0)$.

All examples for a given (δ, κ). An interesting problem is to find all γ which give projective smooth κ-folds of degree δ. For small parameters, this problem can be solved by direct computation. For example, consider $(\delta, \kappa) = (3, 4)$, thus cubic fourfolds. In this case, one has the standardization $[-33, -8, -2, 1, 4, 16, 22]$ of the above example, and then exactly ten more:

$[-48, -15, -12, 5, 16, 24, 30]$, $[-36, -9, -4, 3, 8, 18, 20]$,
$[-48, -12, -3, 1, 6, 24, 32]$, $[-33, -16, -4, 2, 8, 11, 32]$,
$[-48, -12, -3, 6, 16, 17, 24]$, $[-33, -10, -7, 5, 11, 14, 20]$,
$[-36, -16, -9, 3, 8, 18, 32]$, $[-33, -4, -1, 2, 8, 11, 17]$,
$[-36, -9, -8, 4, 15, 16, 18]$, $[-21, -20, -16, 7, 8, 10, 32]$.

7 Dimension reduction

An HGM $H(\gamma, t)$ is defined in terms of a κ-dimensional variety but its Hodge vector $(h^{w,0}, \ldots, h^{0,w})$ raises the question of whether it also comes from a variety of dimension $w = \kappa - 2j$. The exterior zeros for low degree projective hypersurfaces as illustrated in (6.2) raise the same question. The generalized Hodge conjecture says that this dimension reduction is always possible. We illustrate here some of the appealing geometry that arises from reducing dimension.

Reduction to points. When $w = 0$ the reduction to dimension zero is possible in all cases. For example, $\gamma = [-12, -3, 1, 6, 8]$ corresponds to entry 45 on the Beukers-Heckman list [BH89, Table 8.3]. Formula (3.3) then gives a family X_t of cubic surfaces. An equation whose roots correspond to the famous twenty-seven lines on X_t is

$$2^4tx^3(x^2 - 3)^{12} - 3^9(x^3 - 3x^2 + x + 1)^8(x - 2) = 0. \quad (7.1)$$

The Galois group of this polynomial $g(t, x)$ for generic $t \in \mathbb{Q}^\times - \{1\}$ is $W(E_6)$. It has $51840 = 2^73^45$ elements and is also the monodromy group $\Gamma = \langle h_\infty, h_0 \rangle$.

Reduction via splicing. Suppose γ can be written as the concatenation of two lists each summing to zero. Then one can use a general splicing technique from [BCM15, §6] to reduce the dimension by two. This technique is behind the scenes even of our introduction: in the family of examples there, the canonical varieties for $[-2, -2, 1, 1, 1, 1]$ are three-dimensional, although the more familiar source varieties are just the Legendre curves (1.3).

For an example complicated enough to be representative of the general case, take $\gamma = [-12, -3, -2, 1, 1, 1, 6, 8]$ so that the canonical variety has dimension $\kappa = 5$. The Hodge vector is just $(3, 3)$, so one would like to realize $H(\gamma, t)$ in the cohomology of a curve.

Splicing is possible because both $[-12, -3, 1, 6, 8]$ and $[-2, 1, 1]$ sum to zero. No further splicing is possible, but fortunately we have just treated the first sublist by other means. Splicing corresponds to taking a fiber product over

the t-line which in turn corresponds to just multiplying rational functions. In our case, solving (7.1) for t to get the first factor, the dimension-reduced variety is given by

$$\frac{3^9(x-2)\left(x^3 - 3x^2 + x + 1\right)^8}{2^4 x^3 \left(x^2 - 3\right)^{12}} \cdot \frac{2y-1}{y^2} = t. \qquad (7.2)$$

The variable y from $[-2, 1, 1]$ enters only quadratically and so (7.2) defines a double cover of the x-line. Taking the discriminant with respect to y and removing unneeded square factors presents this hyperelliptic curve in standard form:

$$z^2 = -3(x-2)g(t, x).$$

As the right side has degree 28, this curve has genus 13.

In both the new examples of this section, the middle cohomology of the dimension-reduced varieties contains not only the desired motives, with Hodge vectors (6) and $(3, 3)$ respectively, but also parasitical motives, with Hodge vectors (21) and $(10, 10)$. In this regard, they are less attractive than the original canonical varieties. HGMs provide many illustrations like these two of the motivic principle that a motive M comes from many varieties X, and often no single X should be viewed as the best source.

8 Distribution of Hodge vectors

In this section, we explain one of the great features of HGMs: they represent many Hodge vectors.

Completeness in ranks ≤ 19. By direct computation starting from all family parameters Q in degrees ≤ 19, we have verified the following fact. *Let $h = (h^{w,0}, \cdots, h^{0,w})$ be a vector of positive integers satisfying $h^{q,p} = h^{p,q}$ for all $p + q = w$ and let $n = \sum_{i=0}^{w} h^{p,w-p}$. Then if $n \leq 19$ there exists an HGM with Hodge vector h.*

Many families per Hodge vector in ranks ≤ 100. In ranks 20 to 23, the only vectors not realized by a family of HGMs are

$$
\begin{aligned}
20 : &\quad (6, 1, 1, 1, 2, 1, 1, 1, 6), \\
22 : &\quad (6, 1, 1, 1, 1, 2, 1, 1, 1, 1, 6), \\
22 : &\quad (4, 1, 2, 1, 1, 1, 2, 1, 1, 1, 2, 1, 4), \\
23 : &\quad (1, 21, 1).
\end{aligned}
$$

Table 8.1 gives a fuller sense of the situation for $n = 24$, where there are about $460,000,000$ family parameters. It gives the extremes of the list of 4096 possible Hodge vectors h, sorted by how many families realize h.

The ratio of the numbers just reported say that the number of family parameters per Hodge vector in degree 24 is about $113,000$. This ratio increases to a maximum at $n = 58$ where it is about four million. It then decreases to zero, with some approximate sample values being two million for $n = 100$ but only 0.00001 for $n = 300$. These numbers are computed via generating functions, similar to (5.2) but more complicated.

h	#
$(9, 1, 1, 2, 1, 1, 9)$	0
$(7, 1, 1, 1, 1, 2, 1, 1, 1, 1, 7)$	0
$(1, 6, 1, 1, 1, 1, 2, 1, 1, 1, 1, 6, 1)$	0
$(4, 1, 3, 1, 1, 1, 2, 1, 1, 1, 3, 1, 4)$	0
$(5, 1, 2, 1, 1, 1, 2, 1, 1, 1, 2, 1, 5)$	0
$(6, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 6)$	0
$(4, 1, 1, 2, 1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 4)$	0
$(4, 1, 2, 1, 1, 1, 1, 2, 1, 1, 1, 1, 2, 1, 4)$	0
$(6, 2, 1, 1, 1, 2, 1, 1, 1, 2, 6)$	2
$(8, 1, 1, 1, 2, 1, 1, 1, 8)$	4
$(1, 22, 1)$	4
$(8, 1, 1, 4, 1, 1, 8)$	6
$\vdots$	$\vdots$
$(1, 5, 6, 6, 5, 1)$	7637828
$(1, 2, 4, 5, 5, 4, 2, 1)$	7982874
$(2, 4, 6, 6, 4, 2)$	9504072
$(1, 4, 7, 7, 4, 1)$	9905208

Table 8.1: Hodge vectors with total 24 and their number of hypergeometric realizations

Perspective. Section 6 offers some perspective on the general inverse problem of finding an irreducible motive $M \in \mathcal{M}(\mathbb{Q}, \mathbb{Q})$ with a given Hodge vector. From (6.1)-(6.2), one sees that the Hodge vectors coming from hypersurfaces are very sparse. When one looks at broader standard classes of varieties, such as complete intersections in projective spaces, more Hodge vectors arise, but they all have the same rough form: bunched in the middle. *Ad hoc* techniques, such as reducing Hodge numbers by imposing singularities, give many more Hodge vectors. But for many h, it does not seem easy to find a corresponding motive and then prove irreducibility in this geometric way. For example, imposing k ordinary double points on a sextic surface reduces the Hodge vector to $(10, 85 - k, 10)$. However the family of sextic surfaces is only 68-dimensional, and so it would it seem to be difficult to get down to e.g. $(10, 1, 10)$. There does not seem to be even a conjectural expectation of which Hodge vectors arise from irreducible motives in $\mathcal{M}(\mathbb{Q}, \mathbb{Q})$.

The cases $(1, b, 1)$. One could go into much more detail about the families behind any given Hodge vector. Here we say a little more about the cases $(1, b, 1)$, which are particularly interesting for several reasons. The γ giving Hodge vectors of the form $(1, b, 1)$ typically have canonical dimension $\kappa = \dim(X_{\gamma,t})$ greater than two, posing instances of the dimension reduction problem. If $b \leq 19$, then the moduli theory of $K3$ surfaces says that there is at least one family Y_t of $K3$ surfaces also realizing $H(\gamma, t)$. Finding such a family is a challenge.

Cases with $b \geq 20$ present a greater challenge, as they

cannot be realized by K3 surfaces. There are seventy-two parameters giving $(1, 20, 1)$. None of the eleven listed in Section 6 can be spliced, underscoring the difficulty of dimension reduction. One of the four gamma vectors giving $(1, 22, 1)$ has canonical dimension eight, namely $[-60, -5, -4, -3, -2, 8, 9, 10, 12, 15, 20]$. The other three have canonical dimension ten:

$$[-66, -11, -6, -5, -4, -4, 1, 2, 8, 12, 18, 22, 33],$$
$$[-60, -15, -9, -6, -4, -2, 3, 5, 8, 12, 18, 20, 30],$$
$$[-33, -10, -6, -4, -4, -1, 2, 2, 5, \ 8, 11, 12, 18].$$

In all four cases, there are many ways to splice, but no path to a surface.

9   Special and semi HGMs

We have so far been excluding the singular specialization point $t = 1$ from consideration. Now we explain how it yields a particularly interesting motive $H(Q, 1) \in \mathcal{M}(\mathbb{Q}, \mathbb{Q})$. We also explain how other interesting motives arise when the family parameter Q is reflexive, in the sense of satisfying $Q(-T) = Q(T)^{-1}$.

Interior zeros. A Hodge-normalized motive $M \in \mathcal{M}(\mathbb{Q}, \mathbb{Q})$ of weight w has Hodge vector $h = (h^{w,0}, \dots, h^{0,w})$ with $h^{w,0} = h^{0,w} > 0$. But for the Hodge vectors explicitly considered so far, the remaining numbers $h^{p,w-p}$ are also positive. There is a reason for this restriction: Griffiths transversality says that any collection of motives moving in a family with irreducible monodromy group has Hodge vector with no interior zeros. Special and semi HGMs do not move in families, and they include cases with interior zeros.

Special HGMs. The way to account for the double point on the canonical variety $X_{\gamma,1}$ is to first of all take inertial invariants with respect to the monodromy operator h_1. In the orthogonal case, this already give the right motive $H(Q, 1)$. Its Hodge vector differs from the generic Hodge vector only in that $h^{w/2,w/2}$ is decreased by 1. In the symplectic case, the motive of inertial invariants is mixed, and quotienting out by its submotive of weight $w - 1$ and rank 1 gives $H(Q, 1)$. Its Hodge vector now comes from the generic one by decreasing the two central Hodge numbers by 1. These drops obviously can cause interior zeros, as in $(10, 1, 10) \rightarrow (10, 0, 10)$ or $(1, 1, 1, 1, 1, 1) \rightarrow (1, 1, 0, 0, 1, 1)$.

Semi HGMs. For a reflexive parameter Q and any $t \in \mathbb{Q}^{\times}$, the motives $H(Q, t)$ and $H(Q, t^{-1})$ are quadratic twists of one another. The interest in reflexive parameters is that non-generic behavior is thereby forced at $t = \pm 1$. The motive $H(Q, (-1)^n)$ is a direct sum of two motives in $\mathcal{M}(\mathbb{Q}, \mathbb{Q})$ of roughly equal rank. We call the summands semi HGMs and their Hodge vectors can have many interior zeros. For example, the summands of $H(\Phi_2^{16}/\Phi_1^{16}, 1)$ are studied in [Rob19]

and the two Hodge vectors are

$$(1, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 1),$$
$$(1, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 1). \tag{9.1}$$

There is a similar decomposition of $H(Q, -(-1)^n)$, but only after viewing it in $\mathcal{M}(\mathbb{Q}(i), \mathbb{Q}(i))$.

10   Point counts

We now turn to arithmetic. The point counts of this section form the principal raw material from which the L-functions studied in the remaining sections are built.

Background. Let X be a smooth projective variety over $\mathbb{Q}$. Then for all primes p outside a finite set S, the equations defining X have good reduction and so define a smooth projective variety over $\mathbb{F}_p$. For any power $q = p^e$, one has the finite set of solutions $X(\mathbb{F}_q)$ to the defining equations. The key invariants that need to be input into the motivic formalism are the cardinalities $|X(\mathbb{F}_q)|$, and famous results of Grothendieck, Deligne, and others provide the tools.

The vector spaces $H^k(X, \mathbb{Q})$ do not see that X is defined over $\mathbb{Q}$. The arithmetic origin of X yields extra structure as follows. For any prime ℓ, one can extend coefficients to obtain vector spaces $H^k(X, \mathbb{Q}_\ell)$ over the field $\mathbb{Q}_\ell$ of ℓ-adic numbers. Then the group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $H^k(X, \mathbb{Q}_\ell)$.

For every prime p the group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ contains Frobenius elements Fr_p, well-defined up to ambiguities that will disappear from our considerations. For any power $q = p^e$ of a prime $p \notin S$, and any $\ell \neq p$, one has the trace of the operator $\mathrm{Fr}_q = \mathrm{Fr}_p^e$ acting on $H^k(X, \mathbb{Q}_\ell)$. These ℓ-adic numbers are in fact rational and independent of ℓ. We emphasize the independence of ℓ by denoting them $\mathrm{Tr}(\mathrm{Fr}_q|H^k(X, \mathbb{Q}))$. The connection with point counts is the Lefschetz trace formula: $|X(\mathbb{F}_q)| = \sum_k (-1)^k \mathrm{Tr}(\mathrm{Fr}_q|H^k(X, \mathbb{Q}))$. The left side for fixed p and varying e determines the summands on the right side in principle because the complex eigenvalues of Fr_p on weight k cohomology have absolute value $p^{k/2}$.

Much of this transfers formally to the motivic setting. Thus for a motive M and a prime ℓ, there is an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the corresponding ℓ-adic vector space M_ℓ. This action has image in $G_M(\mathbb{Q}_\ell)$. Indeed the Tate conjecture predicts that the $\mathbb{Q}_\ell$-Zariski closure of the image of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is all of $G_M(\mathbb{Q}_\ell)$.

One technical problem with André's category $\mathcal{M}(\mathbb{Q}, \mathbb{Q})$ is that the projectors used to define motives are not known to come from algebraic cycles. As a consequence, for a general $M \in \mathcal{M}(\mathbb{Q}, \mathbb{Q})$ the above compatibility of Frobenius traces is not known. However this problem does not arise for hypergeometric motives, because they are essentially the entire middle cohomology of varieties. Accordingly one has well-defined rational numbers $\mathrm{Tr}(\mathrm{Fr}_q|H(\gamma, t))$. There are similar technical problems at the primes $p \in S$, but they do not affect our computations and we will ignore them.

Wild, tame, and good primes. Returning now to very concrete considerations, we sort primes for a parameter (γ, t) as follows. A prime p is *wild* if it divides a γ_j. For $t \neq 1$, a prime p is *tame* if it is not wild but it divides either the numerator of t, the denominator of t, or the numerator of $t - 1$; these last three conditions say that t is p-adically close to the special points 0, 1, and ∞ respectively. For $t = 1$, no primes are tame. We say that a prime is *bad* if it is either wild or tame, and all other primes are *good*.

Split powers of a good prime p. A power q of a good prime p is *split* for γ if $q \equiv 1 \bmod m$, where m is the least common multiple of the γ_j. One then has a collection of Jacobi sums indexed by characters χ of $\mathbb{F}_q^\times$:

$$J(\gamma, \chi) := \prod_{j=1}^{n} g(\omega^{\alpha_j(q-1)}\chi, \psi)\overline{g(\omega^{\beta_j(q-1)}\chi, \psi)}.$$

Here $\psi : \mathbb{F}_q \to \mathbb{C}^\times$ is any nonzero additive character, $\omega : \mathbb{F}_q^\times \to \mathbb{C}^\times$ is any generator of the group of multiplicative characters, (α, β) underlies γ as in Section 2, and $g(\rho, \psi) = \sum_{t \in \mathbb{F}_q^\times} \rho(t)\psi(t)$ is the standard Gauss sum. The desired quantity is then given by a sum due to Katz [Kat90, p. 258]. Renormalizing to fit our conventions, it is

$$\mathrm{Tr}(\mathrm{Fr}_q|H(\gamma, t)) = \frac{q^{\phi_0}}{1 - q}\sum_\chi \frac{J(\gamma, \chi)}{J(\gamma, 1)}\chi(t). \qquad (10.1)$$

Here ϕ_0 is the vertical coordinate of a lowest point on the zigzag diagram of γ, e.g. $\phi_0 = -1$ in Figure 5.1.

General powers of a good prime p. The Gross-Koblitz formula lets one replace the above Gauss sums by values of the p-adic gamma function. This is both a computational improvement and extends the formula to all powers of any good prime. With this method, the desired integers $\mathrm{Tr}(\mathrm{Fr}_q|H(\gamma, t))$ are first approximated p-adically. Errors are under control and exact values are determined from sufficiently good approximations. See [BCM15] for a closely related approach to the essential numbers $\mathrm{Tr}(\mathrm{Fr}_q|H(\gamma, t))$ and references to earlier contributions.

11 Frobenius polynomials

Frobenius polynomials are a concise way of packaging the point counts of the preceding section. They play the leading role in the formula for L-functions of the next section. After saying what they are, this section explains several reasons why they are useful, even before one gets to L-functions.

Capturing point counts. Consider the numbers $c_{p,e} = \mathrm{Tr}(\mathrm{Fr}_p^e|M) \in \mathbb{Q}$ for a fixed motive $M \in \mathcal{M}(\mathbb{Q}, \mathbb{Q})$ of rank n, a fixed good prime p, and varying e. They can be captured in a single degree n polynomial $F_p(M, x) = \det(1 - \mathrm{Fr}_p x|M)$.

The relation, which comes from summing the geometric series belonging to each of the n eigenvalues, is

$$\exp\left(\sum_{e=1}^{\infty} \frac{c_{p,e}}{e}x^e\right) = \frac{1}{F_p(M, x)}. \qquad (11.1)$$

Write

$$F_p(M, x) = 1 + a_{p,1}x + \cdots + a_{p,n-1}x^{n-1} + a_{p,n}x^n.$$

Then the $c_{p,e}$ for $e \leq k$ determine $a_{p,k}$. Thus the $c_{p,e}$ for $e \leq n$ determine $F_p(M, x)$. But, even better, Poincaré duality on a source variety ultimately implies that one has $a_{p,e} = \epsilon(p)a_{p,n-e}p^{(n-2e)w/2}$ for a sign $\epsilon(p)$. For HGMs, this sign is known and in fact always 1 when w is odd. So $F_p(M, x)$ can be computed using only $c_{p,e}$ for $e \leq \lfloor n/2 \rfloor$.

Relation with Hodge vectors. Indexing by weight w, consider as examples the rank six family parameters

$$Q_0 = \frac{\Phi_3\Phi_{12}}{\Phi_1\Phi_2\Phi_8}, \qquad Q_1 = \frac{\Phi_3\Phi_{12}}{\Phi_1^2\Phi_8}, \qquad Q_5 = \frac{\Phi_3^3}{\Phi_1^6}. \qquad (11.2)$$

The first two are the families from Section 7, with Hodge vectors respectively (6) and $(3, 3)$; the last one has Hodge vector $(1, 1, 1, 1, 1, 1)$. Specializing at a randomly chosen common point gives motives $M_{6,w} = H(Q_w, 3/2)$.

After the required initialization of a variable x by `_<x>:=PolynomialRing(Integers())`, and after inputting Qw as in (2.4), *Magma* quickly gives some Frobenius polynomials via e.g. `EulerFactor(Q0,3/2,5)`:

$$F_5(M_{6,0}, x) = 1 - x - x^5 + x^6,$$
$$F_7(M_{6,0}, x) = 1 \qquad\qquad - x^6,$$

$$F_5(M_{6,1}, x) = 1 + x + 6x^2 + 16x^3 + \cdots,$$
$$F_7(M_{6,1}, x) = 1 - 2x + 12x^2 - 28x^3 + \cdots,$$

$$F_5(M_{6,5}, x) = 1 - 9x + 5 \cdot 156x^2 - 5^3 \cdot 2556x^3 + \cdots,$$
$$F_7(M_{6,5}, x) = 1 + 12x + 7 \cdot 888x^2 + 7^3 \cdot 1816x^3 + \cdots.$$

These displays illustrate a basic motivic principle: as weight increases, motives of a given rank n become more complicated. A more refined principle involves Hodge numbers and can be expressed by forming a weakly increasing vector $(s_1, \ldots, s_n) = (0, \ldots, w)$, where an entry i appears $h^{i,w-i}$ times. Then the Newton-over-Hodge inequality is $\mathrm{ord}_p(a_{p,k}) \geq \sum_{j=1}^{k} s_j$. For $k = 1, \ldots, 6$, these lower bounds from the Hodge vector $(1, 1, 1, 1, 1, 1)$ controlling $M_{6,5}$ are $(0, 1, 3, 6, 10, 15)$. For $(3, 0, 0, 0, 0, 3)$ the bounds $(0, 0, 0, 5, 10, 15)$ would be smaller, leaving more possibilities for Frobenius polynomials. In this sense, spread out Hodge vectors correspond to more complicated motives.

Congruences. Reduced to $\mathbb{F}_\ell$, the numbers $a_{p,k}$ for $p \neq \ell$ depend only on the mod ℓ Galois representation belonging to M. In our examples, suppose one kills $\ell = 2$ in (11.2) by replacing all $\Phi_{2^j a}$ by $\Phi_a^{\phi(2^j)}$. Then Q_0 and Q_1 both become Q_5. This agreement implies that $F_p(M_{6,w}, x) \in \mathbb{F}_\ell[x]$ is independent of w. This independence can be seen for the primes 5 and 7 in the displayed Frobenius polynomials. The analogous congruences hold for any ℓ, when one changes our Tate twist convention to make the weight of $H(Q, t)$ the number of integers among the α_j and β_j, minus one. This web of congruences, like the web corresponding to splicing considered in Section 7, makes it clear that HGMs constitute a natural collection of motives.

Finite Galois groups. Frobenius polynomials render Galois-theoretic aspects of the situation very concrete. As a warm-up, consider Q_0 as a representative of the relatively familiar case of ordinary Galois theory. Here the ℓ-adic representations all come from a single representation $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to W(E_6) \subset GL_6(\mathbb{Q})$. Let λ_p be the partition of 27 obtained by taking the degrees of the irreducible factors of $g(3/2, x)$ from (7.1). Then the twenty-five possibilities for the pair (λ_p, F_p) correspond to the twenty-five conjugacy classes in the finite group $W(E_6)$. If one can collect enough classes, then one can conclude that the image G is all of $W(E_6)$. In our example $t = 3/2$, the above primes 5 and 7 give $(5^5 1^2, 1 - x - x^5 + x^6)$ and $(6^4 3, 1 - x^6)$ respectively. In ATLAS notation, these are the classes $5A$ and $6I$. They do not quite suffice to prove $G = W(E_6)$. But the prime 11 gives the class $12C$ and since no maximal subgroup contains elements from $5A$, $6I$, and $12C$, indeed $G = W(E_6)$.

The Chebotarev density theorem says that each pair appears proportionally to the number of elements in its conjugacy class. For example, the classes $5A$, $6I$ and $12C$ occur with frequency $1/10$, $1/12$, and $1/12$ respectively.

Infinite Galois groups. The cases Q_1 and Q_5 are beyond classical Galois theory as the motivic Galois groups have positive dimension. But the situation remains quite similar. Consider for example odd weight motives of rank $n = 2r$ so that G is in the conformal symplectic group CSp_n. The Weyl group of CSp_n is the hyperoctahedral group $W(C_r)$ of signed permutation matrices, with order $2^r r!$. A separable $F_p(M, x)$, being conformally palindromic, has Galois group within $W(C_r)$. If it has Galois group all of $W(C_r)$ then G necessarily contains a certain twisted maximal torus. Suppose a second prime p' satisfies the same condition and moreover the joint Galois group of $F_p(M, x)F_{p'}(M, x)$ is all of $W(C_r) \times W(C_r)$. Then G contains two maximal tori which are sufficiently different to force $G = \mathrm{CSp}_n$, by the classification of subgroups containing a maximal torus.

To analyze a given motive, the necessary computations can be done using *Magma*'s `GaloisGroup` command. The order of the Galois group of $F_p(M_{6,1}, x)$ is 16, 16, 4, 48, 48 for $p = 5, 7, 11, 13, 17$, and the pair $(p, p') = (13, 17)$

satisfies the criterion. For $5 \leq p < 100$, all $F_p(M_{6,5}, x)$ have Galois group $W(C_3)$ except $p = 13$. Excluding 13, all $\binom{22}{2} = 231$ pairs (p, p') satisfy the criterion. In general, it becomes easier to establish genericity as the weight increases, a reflection of the growth in complexity discussed above.

Applying this two-prime technique to the special and semi HGMs of Section 9 suggests that almost always their motivic Galois groups are as big as possible. In particular, the exotic Hodge vectors with interior zeros arising there indeed come from irreducible motives. Details in the case (9.1) are given in [Rob19].

The Chebotarev density theorem extends to the full motivic setting if all the L-functions described below have their expected analytic properties. Readers wishing to see a glimpse of this theory can compute hundreds of $a_{p,1}/p^{w/2}$ for $M_{6,w}$ for $w = 1$ or $w = 5$. By all appearances, the data matches the Sato-Tate measure μ, meaning the pushforward of Haar measure on the compact group Sp_6 to $[-6, 6]$ via the defining character. One would have to compute thousands of $a_{p,1}/p^{w/2}$ before one could confidently distinguish this measure from the Gaussian measure of mean 0 and standard deviation 1.

12 L-functions

We now finally define L-functions and illustrate how everything works by some numeric computations.

Local invariants. Let $M \in \mathcal{M}(\mathbb{Q}, \mathbb{Q})$ be a motive of rank n and weight w, having bad reduction within a finite set S of primes. We have discussed two types of local invariants associated to M. Corresponding to the place ∞ of $\mathbb{Q}$ is the Hodge vector $h = (h^{w,0}, \ldots, h^{0,w})$ with total n, and also a signature σ. Corresponding to a prime $p \notin S$ is the degree n Frobenius polynomial $F_p(M, x)$. For primes $p \in S$, there is also a Frobenius polynomial $F_p(M, x)$, now of degree $\leq n$, and moreover a conductor exponent $c_p \geq n - \deg(F_p(M, x))$, both to be discussed shortly. The conductor of M, which can be viewed as quantifying the severity of its bad reduction, is the integer $N = \prod_{p \in S} p^{c_p}$.

Formal products. The local invariants can be combined into a holomorphic function in the right half-plane $\mathrm{Re}(s) > \frac{w}{2} + 1$, called the completed L-function of M:

$$\Lambda(M, s) = N^{s/2} \Gamma_{h,\sigma}(s) \prod_p \frac{1}{F_p(M, p^{-s})}. \qquad (12.1)$$

The product over primes alone is the L-function $L(M, s)$, while the remaining factors give the completion. The infinity factor is given by an explicit formula:

$$\Gamma_{h,\sigma}(s) = \Gamma_{\mathbb{R}}(s - \tfrac{w}{2})^{h_+} \Gamma_{\mathbb{R}}(s - \tfrac{w}{2} + 1)^{h_-} \prod_{p<q} \Gamma_{\mathbb{C}}(s - p)^{h^{p,q}}. \quad (12.2)$$

Here $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. The factors involving $h_{\pm} = (h^{w/2,w/2} \pm (-1)^{w/2}\sigma)/2$ only appear when w is even; in the common case that $\sigma = 0$, they can be replaced by $\Gamma_{\mathbb{C}}(s - \frac{w}{2})^{h^{w/2,w/2}/2}$, by the duplication formula.

Both the L-function and the completing factor are multiplicative in M so that $\Lambda(M_1 \oplus M_2, s) = \Lambda(M_1, s)\Lambda(M_2, s)$. Another simple aspect of the formalism is that Tate twists correspond to shifts: $\Lambda(M(j), s) = \Lambda(M, s + j)$.

Expected analytic properties. The L-function $L(\mathbb{Q}, s)$ of the unital motive $\mathbb{Q}$ is just the Riemann zeta function $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$, and the completing factor is $\Gamma_{\mathbb{R}}(s)$. Riemann established that $\Lambda(\mathbb{Q}, s)$ has a meromorphic continuation to the whole s-plane, with poles only at 0 and 1; moreover he proved that $\Lambda(\mathbb{Q}, 1 - s) = \Lambda(\mathbb{Q}, s)$. The product $\Lambda(M, s)$ is expected to have similar analytic properties. First, for M irreducible and not of the form $\mathbb{Q}(j)$, there should be an analytic continuation to the entire s-plane, bounded in vertical strips. Second, always

$$\Lambda(M, w + 1 - s) = \epsilon\Lambda(M, s), \qquad (12.3)$$

for some sign ϵ. For comparison with Section 14, note that most everything said in the last three sections generalizes to motives in $\mathcal{M}(\mathbb{Q}, E)$, with Frobenius polynomials being in $E[x]$. However (12.3) takes the more complicated form $\Lambda(\overline{M}, w + 1 - s) = \epsilon\Lambda(M, s)$, with $\overline{M}$ the complex conjugate motive and ϵ only on the unit circle.

Determining invariants at bad primes. One approach to the conductor exponents c_p and Frobenius polynomials $F_p(M, x)$ associated to bad primes p is to compute them directly by studying the bad reduction of an underlying variety. For an HGM $H(Q, t)$, *Magma* takes this approach for primes which are tame for (Q, t), as sketched in Section 13.

A very different approach uses the fact that the list of possible $(c_p, F_p(M, x))$ for a given prime p is finite, and the product (12.1) has the conjectured analytic properties for at most one member of the product list. The current state of HGMs for the wild primes of Q mixes the two approaches: we first greatly reduce the length of the lists by using proved and conjectural general facts. Then we search within the much smaller product list for the right quantities.

Our view is that numerical computations such as those that follow in this section and Section 15 admit only one plausible interpretation: the bad factors have been properly identified and the analytic properties indeed hold. However rigorous confirmation does not seem to be in sight at the moment, despite the progress described in Section 14.

A rank four example. For Q of degree ≤ 6 and $t = 1$, Watkins numerically identified all the bad quantities, so that the corresponding L-functions are immediately accessible on *Magma*. For example, take the family parameter to be $Q = \Phi_2^2\Phi_{12}/\Phi_{18}$, implemented as always by modifying (2.4). At the specialization point $t = 1$, the Hodge vector

is $(1, 1, 1, 1)$. The corresponding L-function, set up so that calculations are done with 10 digits of precision, is

```
L := LSeries(Q,1:Precision:=10);
```

The bad information stored in *Magma* is revealed by `EulerFactor(L,p)` and `Conductor(L)` to be $F_2(M, x) = 1 + 2x$, $F_3(M, x) = 1$, and $N = 2^6 3^9$. The sign ϵ is calculated numerically, with `Sign(L)` returning -1.000000000. So the order of vanishing of $L(M, s)$ at the central point $s = 2$ should be odd. This order is apparently three since

```
Evaluate(L,2:Derivative:=1);
```

returns zero to ten decimal places, but the same command with 1 replaced by 3 returns 51.72756346.

A rank six example. More typically, *Magma* does not know $F_p(M, x)$ and c_p for wild primes p and one needs to input this information. As an example, take $M = H(\Phi_3^4/\Phi_1^8, 1)$ with Hodge vector $(1, 1, 1, 0, 0, 1, 1, 1)$. The only prime bad for the data is $p = 3$. A good first guess is that $F_3(M, x)$ is just the constant 1. A short search over some possible c_3 is implemented after redefining Q by

```
[CFENew(LSeries(Q,1:Precision:=10),
 BadPrimes:=[<3,c,1>]): c in [6..10]];
```

The returned number for $c = 9$ is 0.0000000000, while the numbers for the other c are all at least 0.1. This information strongly suggests that indeed $F_3(M, x) = 1$ and $c_3 = 9$. After setting up L with `[<3,9,1>]`, analytic calculations can be done as before. For example, here the order of central vanishing is apparently 2. In the miraculous command CFENew, CFE stands for the *Magma* command CheckFunctionalEquation, implemented by Tim Dokchitser using his [Dok04]; New reflects subsequent improvements by Watkins.

13 Bad primes

Fix a hypergeometric motive $M = H(Q, t)$ and a prime p. We now sketch how *Magma* computes the local invariants when p is tame for (Q, t), and describe some conjectural basic features for the case when p is wild for (Q, t).

Tame primes. When p is tame for (Q, t), the conductor exponent c_p is the codimension of the invariants of a power of a Levelt matrix h_τ from Section 2. When $\operatorname{ord}_p(t - 1) \geq 1$, the simple shape of h_1 gives a completely explicit formula: $c_p = 1$ except in the orthogonal case with $\operatorname{ord}_p(t - 1)$ even, where $c_p = 0$. When $\operatorname{ord}_p(t - 1) \leq 0$,

$$c_p = \operatorname{rank}(h_\tau^{|k|} - I). \qquad (13.1)$$

Here $k = \operatorname{ord}_p(t)$, $\tau = \infty$ if k is negative, and $\tau = 0$ if k is positive. So there is separate periodic behavior for $k < 0$ and $k > 0$, as illustrated by the top part of Figure 13.1. The example of this table comes from the case $(a, b) = (3, 5)$ of

(5.1), so the conductor there is very simply computed as the discriminant of the octic algebra $\mathbb{Q}[x]/(5x^8 + 8tx^5 + 3t^3)$.

Because ramification is at worst tame, the degree of $F_p(M, x)$ is $n - c_p$. When $\mathrm{ord}_p(t - 1)$ is positive, $F_p(M, x)$ is computed by slightly modifying the formulas for point counts sketched in Section 10. In the other cases, $F_p(M, x)$ comes from Jacobi motives as mentioned around (4.1), extracted from how the family $X_{Q,t}$ degenerates at the relevant cusp $\tau \in \{0, \infty\}$.

Wild primes. To simplify the overview, we just exclude the case where $\mathrm{ord}_p(t - 1) \geq 1$. Write specialization points as $t = vp^k$ with $k = \mathrm{ord}_p(t)$. The bottom part of Figure 13.1 shows right away that the situation is complicated.
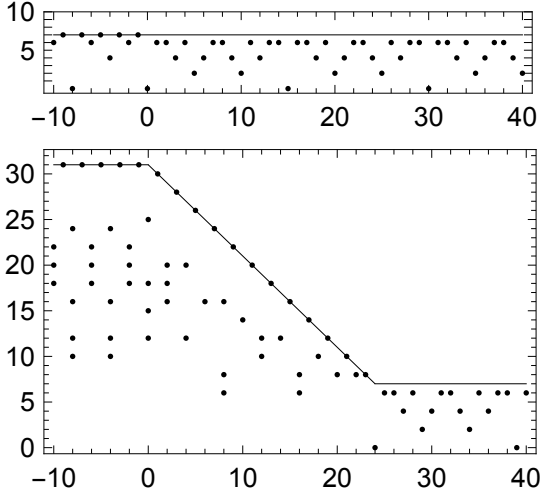

Figure 13.1: Pairs (k, c_p) where $k = \mathrm{ord}_p(t)$ and $c_p = \mathrm{ord}_p(\mathrm{Conductor}(H([-8, 3, 5], t)))$, compared with the graph of the corresponding σ. Top: The tame cases $p > 5$. Bottom: The wild case $p = 2$.

A function σ is graphed in both parts of Figure 13.1 and its general definition goes as follows. For d a positive integer, write

$$s(d) = \begin{cases} 1 & \text{if } \gcd(d, p) = 1, \\ 1 + \mathrm{ord}_p(d) + \frac{1}{p-1} & \text{else.} \end{cases}$$

Let

$$\sigma_\infty = \sum_{i=1}^{n} s(\mathrm{denom}(\alpha_i)), \qquad \sigma_0 = \sum_{i=1}^{n} s(\mathrm{denom}(\beta_i)).$$

Define $k_{\mathrm{crit}} = \sigma_\infty - \sigma_0 = -\sum_j \gamma_j \mathrm{ord}_p(\gamma_j)$ and transition points $k_\infty = \min(k_{\mathrm{crit}}, 0)$ and $k_0 = \max(k_{\mathrm{crit}}, 0)$. Then

$$\sigma(k) = \begin{cases} \sigma_\infty & \text{if } k \leq k_\infty, \\ \max(\sigma_\infty, \sigma_0) - |k| & \text{if } k_\infty \leq k \leq k_0, \\ \sigma_0 & \text{if } k \geq k_0. \end{cases}$$

In the tame case, σ is just the constant function n. In general, there are plateaus corresponding to the cusps ∞ and 0, and then a ramp of length $|k_{\mathrm{crit}}|$ between them.

We conjecture that

$$c_p \leq \sigma(k) - \mathrm{degree}(F_p(M, x)), \qquad (13.2)$$

with equality if k and p are relatively prime. The second statement is proved in [LNV84] in the general trinomial setting of (5.1). All of (13.2) has been computationally verified in many instances. As one passes from one family to another via mod ℓ congruences as in Section 11, wild ramification at p does not change. This fact and other theoretical stabilities give us confidence in (13.2). To make *Magma* more fully automatic, a key step would be to define a more complicated function $\sigma(k, v)$ with $c_p \leq \sigma(k, v) - \mathrm{degree}(F_p(M, x))$, and equality under broad circumstances.

At present, we understand a factor $f_p(M, x)$ of the Frobenius polynomial $F_p(M, x)$ as follows. For $k \neq k_{\mathrm{crit}}$, $f_p(M, x)$ comes from modifying the tame formulas; in particular its degree is given by replacing k by $k - k_{\mathrm{crit}}$ in (13.1). If $k = k_{\mathrm{crit}}$, corresponding to being at the bottom of the ramp, we use an *erasing* principle explained to us by Katz. Here one simply ignores all α_j and β_j that have denominator divisible by p. Let n_∞ and n_0 be respectively the number of α_j's and β_j's remaining. Then $n_\infty - n_0$ is a multiple of $p - 1$, so that the formulas described in Section 10 still make sense, as the choice of an auxiliary additive character on $\mathbb{F}_p$ again does not matter. The resulting $f_p(M, x)$ has degree $\max(n_\infty, n_0)$. We conjecture that the complementary factor $F_p(M, x)/f_p(M, x)$ is 1 whenever p and k are relatively prime. In practice, when $p \mid k$ it is usually 1 also, but not always.

14 Automorphy

One of the most exciting aspects of the theory of motives is its conjectured extremely tight connection to automorphic representations of adelic groups through the Langlands program.

Background. Let $\mathbb{A}$ be the adele ring of $\mathbb{Q}$; it is a restricted product of all the completions $\mathbb{Q}_p$, including $\mathbb{Q}_\infty = \mathbb{R}$. A cuspidal automorphic representation of $GL_n(\mathbb{A})$ has an L-function known to have an analytic continuation and functional equation. The main conjecture is that, after incorporating Tate twists to make normalizations match, the set of L-functions coming from irreducible rank n motives in $\mathcal{M}(\mathbb{Q}, \mathbb{C})$ is exactly the subset of automorphic L-functions for which the infinity factor has the form (12.2).

The case $n = 2$. For a motive $M \in \mathcal{M}(\mathbb{Q}, \mathbb{C})$ with nonvanishing Hodge numbers $h^{w,0} = h^{0,w} = 1$ and conductor N, one can switch to classical language. The desired automorphic representation is entirely given by a power series in $q = e^{2\pi i z}$ as in (1.6), but now this newform on $\Gamma_0(N)$ has weight $w + 1$.

To exhibit some matches between motivic and automorphic L-functions, consider the four reflexive parameters Q yielding motives $H(Q, 1)$ with Hodge vector $(1, 1, 0, 0, 1, 1)$:

Q	a_5	a_7	N'		b_5	b_7	N''	
Φ_2^6/Φ_1^3	-2	24	8	$\eta_2^4\eta_4^4$	74	24	8	
$\Phi_2^4\Phi_3/\Phi_1^4\Phi_6$	-18	8	12		54	-88	4	η_2^{12}
$\Phi_2^2\Phi_3^2/\Phi_1^2\Phi_6^2$	-6	-16	18		-66	176	6	
Φ_3^3/Φ_6^3	-16	-12	72		-16	12	72	

Magma computes automatically with these reducible motives, reporting their conductors to be $N = 64, 48, 108$, and 5184. However these computations do not see the decompositions $H(Q, 1) = M'(-1) \oplus M''$ analogous to (9.1), where now M' and M'' respectively have Hodge vectors $(1, 0, 0, 1)$ and $(1, 0, 0, 0, 0, 1)$. In the Frobenius polynomial

$$F_p(M(Q, 1), x) = (1 - pa_p x + p^5 x^2)(1 - b_p x + p^5 x^2),$$

the pa_p belonging to $M'(-1)$ can be distinguished from the b_p belonging to M'' whenever the latter is not a multiple of p. The reader might enjoy searching in the LMFDB's complete lists [LMF21] of modular forms to see that the a_p and b_p for $p = 5$ and $p = 7$ let one identify the relevant forms and in particular determine the above-displayed factorizations $N = N'N''$. Part of the further information given by the LMFDB is that two of the forms are expressible using the Dedekind eta function η_1, via $\eta_d = q^{d/24} \prod_{j=1}^{\infty}(1 - q^{dj})$.

Higher rank. For a given motive $M \in \mathcal{M}(\mathbb{Q}, \mathbb{C})$ with larger rank n, one can usually replace $GL_n(\mathbb{A})$ by the adelic points of a smaller group determined by the motivic Galois group G of M. In favorable cases, the representation sought again corresponds to a holomorphic form. For rank three orthogonal motives, classical modular forms are again relevant, but a symmetric square is now involved. In rank four, Hilbert modular forms are needed for orthogonal motives and Siegel modular forms are needed for symplectic motives. Numerical and sometimes proved matches have been found in these three settings. For example, [DPVZ20] treats some interesting rank four orthogonal cases.

Generally speaking, the Hodge numbers of central concern earlier in this survey continue to play a large role. In particular, motives for which all $h^{p,q}$ are 0 or 1 have theoretical advantages, and their motivic L-functions at least have a meromorphic continuation with the right functional equation [PT15].

15 Numerical computations

We promised in the introduction that we would equip the reader to numerically explore a large collection of motivic L-functions. We conclude this survey by giving sample computations in the context of two important topics, always

assuming that the expected analytic continuation and functional equation indeed hold. In both topics, we let $c = \frac{1}{2} + \frac{w}{2}$ be the center of the functional equation. The conductors N in our examples are small for their Hodge vectors h, allowing us to keep runtimes short and/or work to high precision.

Special values. If M is a motive in $\mathcal{M}(\mathbb{Q}, \mathbb{Q})$ then the numbers $L(M, k)$ for integers $k \leq c$ are mostly forced to be 0, because of poles in the infinity factor (12.2) and the functional equation. However, when $L(M, k)$ is nonzero it is expected to be arithmetically significant [Del79]. The arithmetic interpretation involves a determinant of periods like (2.1). To see the significance without entering into periods, one can look at the ratio $r_d = L(M \otimes \chi_d, k)/L(M, k)$, for d a positive quadratic discriminant. Then the periods cancel out so that r_d should be rational.

For a sample computation, take $M = H(\Phi_2^5/\Phi_1^5, 2^{10})$ and use (2.4) and $L:=LSeries(Q, 1024)$ to define its L-function, as usual. While 2 is wild for the family, it is unramified in M because because the exponent 10 is at the bottom of the ramp of Section 13. The erasing procedure from the end of that section applies, yielding

$$F_2(M, x) = (1 - 4x)(1 + 5x + 10x^2 + 80x^3 + 256x^4).$$

Since $t - 1 = 1023 = 3 \cdot 11 \cdot 31$ is squarefree, it is the conductor, by the recipe before (13.1). *Magma* gets all the bad factors right automatically. As a confirmation, $CFENew(L)$ quickly returns 0 to the default 30 digits.

$Evaluate(L, 2)$ gives $0.4278180899 \cdots$. Twisting by a d with $\gcd(d, 1023) = 1$ makes the conductor go up by a factor of d^5 and precision needs to be reduced.

```
Evaluate(LSeries(Q,1024:
QuadraticTwist:=5,Precision:=10),2);
```

takes six minutes to give its answer of 35.04685793. This ratio and then two others are apparently

$$r_5 = \frac{2^{11}}{5^2}, \qquad r_8 = 2^6\, 5, \qquad r_{13} = \frac{2^{10}\, 251}{13^2}.$$

The two L-functions appearing in r_d are completely different analytically, and so the apparent fact that quotients are rational is very remarkable.

Readers wanting to work out their own examples might want to begin with M having odd weight. Then if $L(M, c) \neq 0$, one has conjecturally rational quotients r_d for $k = c$. The lateral argument $k = c - j$ fits into the theory only in the rare case that the $2j$ most central entries of the Hodge vector are 0. In the even weight case, one needs to have $h_+ = 0$ to make $k = c - \frac{1}{2}$ fit into the theory, as in our example.

Critical zeros. For a weight w motive M, all the zeros of the completed L-function $\Lambda(M, s)$ lie in the critical strip $c - \frac{1}{2} \leq \mathrm{Re}(s) \leq c + \frac{1}{2}$. The Riemann hypothesis for M then predicts that all the zeros lie on the critical line $\mathrm{Re}(s) = c$.

We now show by examples that numerical identification of low-lying zeros is possible in modestly high rank.

For the examples, take $M_{10,w} = H(Q_w, 1)$ with $Q_{10} = \Phi_4\Phi_2^9/\Phi_1^{11}$ and $Q_7 = \Phi_4^4\Phi_2^4/\Phi_8^2\Phi_1^4$. So $M_{10,10}$ is orthogonal with Hodge vector $(1,1,1,1,1,0,1,1,1,1,1)$ while $M_{10,7}$ is symplectic with Hodge vector $(1,1,2,1,1,2,1,1)$.

The only bad prime in each case is 2. A search says that $F_2(M_{10,10}, x) = 1 + 32x$ and $c_2 = 11$. For $M_{10,7}$, $k = k_{\text{crit}} = 0$ so erasing applies, yielding $1 + 4x + 96x^2 + 512x^3 + 16384x^4$ as a factor of $F_2(M_{10,7}, x)$. A short search says that this factor is all of $F_2(M_{10,7}, x)$ and $c_2 = 18$.

In general, the Hardy Z-function of a motive M is

$$Z(M, t) = \epsilon^{1/2} \frac{N^{s/2}\Gamma_{h,\sigma}(s)}{|N^{s/2}\Gamma_{h,\sigma}(s)|} L(M, s),$$

with $s = c + it$. It is a real-valued function of the real variable t, even or odd depending on whether the sign ϵ is 1 or -1.
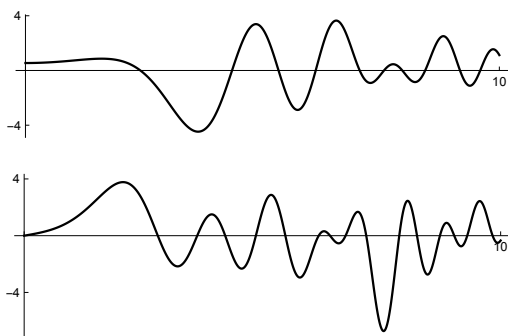

Figure 15.1: Graphs of $Z(M_{10,10}, t)$ and $Z(M_{10,7}, t)$

Figure 15.1 was computed via many calls to `Evaluate` at points of the form $c + it$. The signs in the two cases are 1 and -1, and the orders of central vanishing are the minimum possible, 0 and 1. On both plots, all local maxima are above the axis and all local minima are beneath the axis. Zeros off the critical line would likely cause a disruption of this pattern; thus the plots not only identify zeros on the critical line, but suggest a lack of zeros off the critical line.

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