Also
$$\int_0^\lambda |\phi_k(ixe^{-i\delta})|^2\,dx = \int_{1/\lambda}^\infty \left|\phi_k\!\left(\frac{1}{ixe^{-i\delta}}\right)\right|^2 \frac{dx}{x^2},$$
and by the above formula this should be approximately
$$\frac{(2\pi)^{k/(k-1)}}{k-1}\times$$
$$\times\int_{1/\lambda}^\infty \left|\sum_{n=1}^\infty \frac{d_k(n)}{n^{(\frac{1}{2}k-1)/(k-1)}}\exp\{-(k-1)i(2\pi)^{k/(k-1)}(nx)^{1/(k-1)}e^{-i\delta/(k-1)}\}\right|^2 \frac{dx}{x^{2-k/(k-1)}}.$$
Putting $x = \xi^{k-1}$, this is
$$(2\pi)^{k/(k-1)}\times$$
$$\times\int_{\lambda^{-1/(k-1)}}^\infty \left|\sum_{n=1}^\infty \frac{d_k(n)}{n^{(\frac{1}{2}k-1)/(k-1)}}\exp\{-(k-1)i(2\pi)^{k/(k-1)}n^{1/(k-1)}\xi e^{-i\delta/(k-1)}\}\right|^2 d\xi,$$
and we can integrate as before. We obtain
$$K\sum_{m=1}^\infty\sum_{n=1}^\infty \frac{d_k(m)d_k(n)}{(mn)^{(\frac{1}{2}k-1)/(k-1)}}\times$$
$$\times\frac{\begin{array}{r}\exp[(k-1)(2\pi)^{k/(k-1)}\{(n^{1/(k-1)}-m^{1/(k-1)})i\cos\delta/(k-1)-\\ -(m^{1/(k-1)}+n^{1/(k-1)})\sin\delta/(k-1)\}\lambda^{-1/(k-1)}]\end{array}}{(n^{1/(k-1)}-m^{1/(k-1)})i\cos\delta/(k-1)-(m^{1/(k-1)}+n^{1/(k-1)})\sin\delta/(k-1)},$$
where K depends on k only.

The terms with $m = n$ are
$$O\left\{\frac{1}{\delta}\sum_{n=1}^\infty \frac{d_k^2(n)}{n}\exp(-K\delta n^{1/(k-1)}\lambda^{-1/(k-1)})\right\} = O\left\{\frac{1}{\delta}\frac{1}{(\lambda\delta)^\epsilon}\right\}.$$

The rest are
$$O\left\{\sum_{m>n}\sum \frac{1}{(mn)^{(\frac{1}{2}k-1)/(k-1)}}\frac{\exp(-K\delta m^{1/(k-1)}\lambda^{-1/(k-1)})}{m^{1/(k-1)}-n^{1/(k-1)}}\right\}.$$
Now
$$\sum_{n=1}^{m-1}\frac{1}{n^{(\frac{1}{2}k-1)/(k-1)}(m^{1/(k-1)}-n^{1/(k-1)})}$$
$$= O\left\{\sum_{n=1}^{\frac{1}{2}m}\frac{1}{n^{(\frac{1}{2}k-1)/(k-1)}m^{1/(k-1)}}+\sum_{\frac{1}{2}m}^{m-1}\frac{1}{m^{(\frac{1}{2}k-1)/(k-1)+1/(k-1)-1}(m-n)}\right\}$$
$$= O(m^{1-(\frac{1}{2}k-1)/(k-1)-1/(k-1)+\epsilon}) = O(m^{(\frac{1}{2}k-1)/(k-1)+\epsilon}).$$

Hence we obtain
$$O\left\{\sum_{m=2}^\infty m^\epsilon\exp(-K\delta m^{1/(k-1)}\lambda^{-1/(k-1)})\right\} = O\left\{\int_0^\infty x^\epsilon\exp(-K\delta x^{1/(k-1)}\lambda^{-1/(k-1)})\,dx\right\}$$
$$= O\left\{\left(\frac{\lambda}{\delta^{k-1}}\right)^{1+\epsilon}\right\}.$$
Altogether
$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^{2k}e^{-2\delta t}\,dt = O\left\{\frac{1}{(\lambda\delta)^{1+\epsilon}}\right\}+O\left\{\left(\frac{\lambda}{\delta^{k-1}}\right)^{1+\epsilon}\right\},$$
and taking $\lambda = \delta^{\frac{1}{2}k-1}$, we obtain
$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^{2k}e^{-2\delta t}\,dt = O(\delta^{-\frac{1}{2}k-\epsilon}) \quad (k\geqslant 2).$$
This index is what we should obtain from the approximate functional equation.

7.19. The attempt to obtain a non-trivial upper bound for
$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^{2k}e^{-\delta t}\,dt$$
for $k > 2$ fails. But we can obtain a lower bound† for it which may be somewhere near the truth; for in this problem we can ignore $\phi_k(ixe^{-i\delta})$ for small x, since by (7.13.5)
$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^{2k}e^{-2\delta t}\,dt > \int_1^\infty |\phi_k(ixe^{-i\delta})|^2\,dx+O(1), \tag{7.19.1}$$
and we can approximate to the right-hand side by the method already used.

If k is any positive integer, and $\sigma > 1$,
$$\zeta^k(s) = \prod_p\left(1-\frac{1}{p^s}\right)^{-k} = \prod_p\sum_{m=0}^\infty\frac{(k+m-1)!}{(k-1)!\,m!}\frac{1}{p^{ms}} = \sum_{n=1}^\infty\frac{d_k(n)}{n^s}.$$
If we replace the coefficient of each term p^{-ms} by its square, the coefficient of each n^{-s} is replaced by its square. Hence if
$$F_k(s) = \sum_{n=1}^\infty\frac{d_k^2(n)}{n^s},$$
then
$$F_k(s) = \prod_p\sum_{m=0}^\infty\left\{\frac{(k+m-1)!}{(k-1)!\,m!}\right\}^2\frac{1}{p^{ms}} = \prod_p f_k(p^{-s}),$$

† Titchmarsh (4).

say. Thus
$$f_k\left(\frac{1}{p^s}\right) = 1+\frac{k^2}{p^s}+\dots,$$
and
$$\left(1-\frac{1}{p^s}\right)^{k^2} f_k\left(\frac{1}{p^s}\right) = \left(1-\frac{k^2}{p^s}+\dots\right)\left(1+\frac{k^2}{p^s}+\dots\right) = 1+O\left(\frac{1}{p^{2\sigma}}\right).$$
Hence the product
$$\prod_p \left(1-\frac{1}{p^s}\right)^{k^2} f_k\left(\frac{1}{p^s}\right)$$
is absolutely convergent for $\sigma > \frac{1}{2}$, and so represents an analytic function, $g(s)$ say, regular for $\sigma > \frac{1}{2}$, and bounded in any half-plane $\sigma \geqslant \frac{1}{2}+\delta$; and
$$F_k(s) = \zeta^{k^2}(s)g(s).$$
Now
$$\sum_{n=1}^{\infty} d_k^2(n)e^{-2n\sin\delta} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Gamma(s)F_k(s)(2\sin\delta)^{-s}\,ds.$$
Moving the line of integration just to the left of $\sigma = 1$, and evaluating the residue at $s = 1$, we obtain in the usual way
$$\sum_{n=1}^{\infty} d_k^2(n)e^{-2n\sin\delta} \sim \frac{C_k'}{\delta}\log^{k^2-1}\frac{1}{\delta}.$$
Similarly
$$\sum_{n=1}^{\infty} \frac{d_k^2(n)}{n}e^{-2n\sin\delta} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Gamma(s)F_k(s+1)(2\sin\delta)^{-s}\,ds \sim C_k\log^{k^2}\frac{1}{\delta},$$
since here there is a pole of order k^2+1 at $s = 0$.

We can now prove

THEOREM 7.19. *For any fixed integer k, and* $0 < \delta \leqslant \delta_0 = \delta_0(k)$,
$$\int_0^{\infty} |\zeta(\tfrac{1}{2}+it)|^{2k}e^{-\delta t}\,dt \geqslant \frac{C_k}{\delta}\log^{k^2}\frac{1}{\delta}.$$

The integral on the right of (7.19.1) is equal to (7.18.1) with $\lambda = 1$; and
$$\Sigma_1 \sim \frac{C_k}{2\delta}\log^{k^2}\frac{1}{\delta},$$
while
$$\Sigma_2+\Sigma_3 = O\left(\frac{1}{\delta}\log^{k^2-1}\frac{1}{\delta}\right).$$
The result therefore follows.

NOTES FOR CHAPTER 7

7.20. When applied (with care) to a general Dirichlet polynomial, the proof of the first lemma of §7.2 leads to
$$\int_0^T \left|\sum_1^N a_n n^{-it}\right|^2 dt = \sum_1^N |a_n|^2\{T+O(n\log 2n)\}.$$
However Montgomery and Vaughan [1] have given a superior result, namely
$$\int_0^T \left|\sum_1^N a_n n^{-it}\right|^2 dt = \sum_1^N |a_n|^2\{T+O(n)\}. \tag{7.20.1}$$
Ramachandra [2] has given an alternative proof of this result. Both proofs are more complicated than the argument leading to (7.2.1). However (7.20.1) has the advantage of dealing with the mean value of $\zeta(s)$ uniformly for $\sigma \geqslant \frac{1}{2}$. Suppose for example that $\sigma = \frac{1}{2}$. One takes $x = 2T$ in Theorem 4.11, whence
$$\zeta(\tfrac{1}{2}+it) = \sum_{n\leqslant 2T} n^{-\frac{1}{2}-it} + O(T^{-\frac{1}{2}}) = Z+O(T^{-\frac{1}{2}}),$$
say, for $T \leqslant t \leqslant 2T$. Then
$$\int_T^{2T} |Z|^2\,dt = \sum_{n\leqslant 2T} n^{-1}\{T+O(n)\} = T\log T+O(T).$$
Moreover $Z \ll T^{\frac{1}{2}}$, whence
$$\int_T^{2T} |Z|\,T^{-\frac{1}{2}}\,dt \ll T.$$
Then, since
$$\int_T^{2T} O(T^{-\frac{1}{2}})^2\,dt = O(1),$$
we conclude that
$$\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^2\,dt = T\log T+O(T),$$

and Theorem 7.3 follows (with error term $O(T)$) on summing over $\frac{1}{2}T, \frac{1}{4}T, \ldots$. In particular we see that Theorem 4.11 is sufficient for this purpose, contrary to Titchmarsh's remark at the beginning of §7.3.

We now write

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^2\,dt = T\log\left(\frac{T}{2\pi}\right) + (2\gamma-1)\,T + E(T).$$

Much further work has been done concerning the error term $E(T)$. It has been shown by Balasubramanian [1] that $E(T) \ll T^{\frac{1}{3}+\varepsilon}$. A different proof was given by Heath-Brown [4]. The estimate may be improved slightly by using exponential sums, and Ivic [3; Corollary 15.4] has sketched the argument leading to the exponent $\frac{35}{108}+\varepsilon$, using a lemma due to Kolesnik [4]. It is no coincidence that this is twice the exponent occurring in Kolesnik's estimate for $\mu(\frac{1}{2})$, since one has the following result.

LEMMA 7.20. *Let k be a fixed positive integer and let $t \geqslant 2$. Then*

$$\zeta(\tfrac{1}{2}+it)^k \ll (\log t)\left(1 + \int_{-\log^2 t}^{\log^2 t} |\zeta(\tfrac{1}{2}+it+iu)|^k e^{-|u|}\,du\right). \tag{7.20.2}$$

This is a trivial generalization of Lemma 3 of Heath-Brown [2], which is the case $k = 2$. It follows that

$$\zeta(\tfrac{1}{2}+it)^2 \ll (\log t)^4 + (\log t)\max E\{t \pm (\log t)^2\}. \tag{7.20.3}$$

Thus, if μ is the infimum of those α for which $E(T) \ll T^{\alpha}$, then $\mu(\frac{1}{2}) \leqslant \frac{1}{2}\mu$. On the other hand, an examination of the initial stages of the process for estimating $\zeta(\frac{1}{2}+it)$ by van der Corput's method shows that one is, in effect, bounding the mean square of $\zeta(\frac{1}{2}+it)$ over a short range $(t-\Delta, t+\Delta)$. Thus it appears that one can hope for nothing better for $\mu(\frac{1}{2})$, by this method, than is given by (7.20.3).

The connection between estimates for $\zeta(\frac{1}{2}+it)$ and those for $E(T)$ should not be pushed too far however, for Good [1] has shown that $E(T) = \Omega(T^{\frac{1}{4}})$. Indeed Heath-Brown [1] later gave the asymptotic formula

$$\int_0^T E(t)^2\,dt = \tfrac{2}{3}(2\pi)^{-\frac{1}{2}}\frac{\zeta(\frac{3}{2})^4}{\zeta(3)}T^{\frac{3}{2}} + O(T^{\frac{5}{4}}\log^2 T) \tag{7.20.4}$$

from which the above Ω-result is immediate. It is perhaps of interest to note that the error term of (7.20.4) must be $\Omega\{T^{\frac{3}{4}}(\log T)^{-1}\}$, since any estimate $O\{F(T)\}$ readily yields $E(T) \ll \{F(T)\log T\}^{\frac{1}{3}}$, by an argument analogous to that used in the proof of Lemma α in 14.13. It would be nice to reduce the error term in (7.20.4) to $O(T^{1+\varepsilon})$ so as to include Balasubramanian's bound $E(T) \ll T^{\frac{1}{4}+\varepsilon}$.

Higher mean-values of $E(T)$ have been investigated by Ivic [1] who showed, for example, that

$$\int_0^T E(t)^8\,dt \ll T^{3+\varepsilon}. \tag{7.20.5}$$

This readily implies the estimate $E(T) \ll T^{\frac{3}{8}+\varepsilon}$.

The mean-value theorems of Heath-Brown and Ivic depend on a remarkable formula for $E(T)$ due to Atkinson [1]. Let $0 < A < A'$ be constants and suppose $AT \leqslant N \leqslant A'T$. Put

$$N' = N'(T) = \frac{T}{2\pi} + \frac{N}{2} - \left(\frac{NT}{2\pi} + \frac{N^2}{4}\right)^{\frac{1}{2}}.$$

Then $E(T) = \Sigma_1 + \Sigma_2 + O(\log^2 T)$, where

$$\Sigma_1 = 2^{-\frac{1}{2}}\sum_{n\leqslant N}(-1)^n d(n)\left(\frac{nT}{2\pi} + \frac{n^2}{4}\right)^{-\frac{1}{4}}\left\{\sinh^{-1}\left(\frac{\pi n}{2T}\right)^{\frac{1}{2}}\right\}^{-1}\sin f(n)$$

with (7.20.6)

$$f(n) = \tfrac{1}{4}\pi + 2T\sinh^{-1}\left(\frac{\pi n}{2T}\right)^{\frac{1}{2}} + (\pi^2 n^2 + 2\pi nT)^{\frac{1}{2}}, \tag{7.20.7}$$

and

$$\Sigma_2 = 2\sum_{n\leqslant N'} d(n)\,n^{-\frac{1}{2}}\left(\log\frac{T}{2\pi n}\right)^{-1}\sin g(n)$$

where

$$g(n) = T\log\frac{T}{2\pi n} - T - \tfrac{1}{4}\pi.$$

Atkinson loses a minus sign on [1; p 375]. This is corrected above. In applications of the above formula one can usually show that Σ_2 may be ignored. On the Lindelöf hypothesis, for example, one has

$$\sum_{n\leqslant x} d(n)\,n^{-\frac{1}{2}-iT} \ll T^{\varepsilon}$$

for $x \ll T$, so that $\Sigma_2 \ll T^{\varepsilon}$ by partial summation; and in general one finds $\Sigma_2 \ll T^{2\mu(\frac{1}{2})+\varepsilon}$. The sum Σ_1 is closely analogous to that occuring in the explicit formula (12.4.4) for $\Delta(x)$ in Dirichlet's divisor problem. Indeed, if $n = o(T^{\frac{1}{3}})$ then the summands of (7.20.6) are

$$(-1)^n\left(\frac{2T}{\pi}\right)^{\frac{1}{4}}\frac{d(n)}{n^{\frac{3}{4}}}\cos\left(2\surd(2\pi nT)-\frac{\pi}{4}\right)+o\left(T^{\frac{1}{4}}\frac{d(n)}{n^{\frac{3}{4}}}\right).$$

7.21. Ingham's result has been improved by Heath-Brown [4] to give

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^4\,dt = \sum_{n=0}^{4} c_n T(\log T)^n + O(T^{\frac{7}{8}+\varepsilon}) \tag{7.21.1}$$

where $c_4 = (2\pi^2)^{-1}$ and

$$c_3 = 2\{4\gamma - 1 - \log(2\pi) - 12\zeta'(2)\pi^{-2}\}\pi^{-2}.$$

The proof requires an asymptotic formula for

$$\sum_{n \leqslant N} d(n)\,d(n+r)$$

with a good error term, uniform in r. Such estimates are obtained in Heath-Brown [4] by applying Weil's bound for the Kloosterman sum (see §7.24).

7.22. Better estimates for σ_k are now available. In particular we have $\sigma_3 \leqslant \frac{7}{12}$ and $\sigma_4 \leqslant \frac{5}{8}$. The result on σ_4 is due to Heath-Brown [8]. To deduce the estimate for σ_3 one merely uses Gabriel's convexity theorem (see § 9.19), taking $\alpha = \frac{1}{2}$, $\beta = \frac{5}{8}$, $\lambda = \frac{1}{4}$, $\mu = \frac{1}{8}$, and $\sigma = \frac{7}{12}$.

The key ingredient required to obtain $\sigma_4 \leqslant \frac{5}{8}$ is the estimate

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^{12}\,dt \ll T^2(\log T)^{17} \tag{7.22.1}$$

of Heath-Brown [2]. According to (7.20.2) this implies the bound $\mu(\frac{1}{2}) \leqslant \frac{1}{6}$. In fact, in establishing (7.22.1) it is shown that, if $|\zeta(\frac{1}{2}+it_r)| \geqslant V\,(>0)$ for $1 \leqslant r \leqslant R$, where $0 < t_r \leqslant T$ and $t_{r+1} - t_r \geqslant 1$, then

$$R \ll T^2 V^{-12}(\log T)^{16},$$

and, if $V \geqslant T^{\frac{2}{13}}(\log T)^2$, then

$$R \ll TV^{-6}(\log T)^8.$$

Thus one sees not only that $\zeta(\frac{1}{2}+it) \ll t^{\frac{1}{6}}(\log t)^{\frac{4}{3}}$, but also that the number

of points at which this bound is close to being attained is very small. Moreover, for $V \geqslant T^{\frac{2}{13}}(\log T)^2$, the behaviour corresponds to the, as yet unproven, estimate

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^6\,dt \ll T^{1+\varepsilon}.$$

To prove (7.22.1) one uses Atkinson's formula for $E(T)$ (see §7.20) to show that

$$\int_{T-G}^{T+G} |\zeta(\tfrac{1}{2}+it)|^2\,dt \ll G\log T +$$

$$G\sum_K (TK)^{-\frac{1}{4}}\left(|S(K)| + K^{-1}\int_0^K |S(x)|\,dx\right)e^{-G^2K/T}, \tag{7.22.2}$$

where K runs over powers of 2 in the range $T^{\frac{1}{3}} \leqslant K \leqslant TG^{-2}\log^3 T$, and

$$S(x) = S(x, K, T) = \sum_{K < n \leqslant K+x} (-1)^n d(n) e^{if(n)}$$

with $f(n)$ as in (7.20.7). The bound (7.22.2) holds uniformly for $\log^2 T \leqslant G \leqslant T^{\frac{4}{12}}$. In order to obtain the estimate (7.22.1) one proceeds to estimate how often the sum $S(x, K, T)$ can be large, for varying T. This is done by using a variant of Halász's method, as described in §9.28.

By following similar ideas, Graham, in work in the process of publication, has obtained

$$\int_0^T |\zeta(\tfrac{5}{7}+it)|^{196}\,dt \ll T^{14}(\log T)^{425}. \tag{7.22.3}$$

Of course there is no analogue of Atkinson's formula available here, and so the proof is considerably more involved. The result (7.22.3) contains the estimate $\mu(\frac{5}{7}) \leqslant \frac{1}{14}$ (which is the case $l = 4$ of Theorem 5.14) in the same way that (7.22.1) implies $\mu(\frac{1}{2}) \leqslant \frac{1}{6}$.

7.23. As in §7.9, one may define σ_k, for all positive real k, as the infimum of those σ for which (7.9.1) holds, and σ'_k similarly, for (7.9.2).

Then it is still true that $\sigma_k = \sigma'_k$, and that

$$\int_1^T |\zeta(\sigma+it)|^{2k}\,dt = T\sum_1^\infty d_k(n)^2 n^{-2\sigma} + O(T^{1-\delta})$$

for $\sigma > \sigma_k$, where $\delta = (\sigma, k) > 0$ may be explicitly determined. This may be proved by the method of Haselgrove [1]; see also Turganaliev [1]. In particular one may take $\delta(\sigma, \frac{1}{2}) = \frac{1}{2}(\sigma - \frac{1}{2})$ for $\frac{1}{2} < \sigma < 1$ (Ivic [3; (8.111)] or Turganaliev [1]). For some quite general approaches to these fractional moments the reader should consult Ingham (4) and Bohr and Jessen (4).

Mean values for $\sigma = \frac{1}{2}$ are far more difficult, and in no case other than $k = 1$ or 2 is an asymptotic formula for

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt = I_k(T),$$

say, known, even assuming the Riemann hypothesis. However Heath-Brown [7] has shown that

$$T(\log T)^{k^2} \ll I_k(T) \ll T(\log T)^{k^2} \quad \left(k = \frac{1}{n}\right),$$

Ramachandra [3], [4] having previously dealt with the case $k = \frac{1}{2}$. Jutila [4] observed that the implied constants may be taken to be independent of k. We also have

$$I_k(T) \gg T(\log T)^{k^2}$$

for any positive rational k. This is due to Ramachandra [4] when k is half an integer, and to Heath-Brown [7] in the remaining cases. (Titchmarsh [1; Theorem 29] states such a result for positive integral k, but the reference given there seems to yield only Theorem 7.19, which is weaker.) When k is irrational the best result known is Ramachandra's estimate [5]

$$I_k(T) \gg T(\log T)^{k^2}(\log\log T)^{-k^2}.$$

If one assumes the Riemann hypothesis one can obtain the better results

$$I_k(T) \ll T(\log T)^{k^2} \quad (0 \leqslant k \leqslant 2)$$

and

$$I_k(T) \gg T(\log T)^{k^2} \quad (k \geqslant 0), \tag{7.23.1}$$

for which see Ramachandra [4] or Heath-Brown [7]. Conrey and Ghosh [1] have given a particularly simple proof of (7.23.1) in the form

$$I_k(T) \geqslant \{C_k + o(1)\}\, T(\log T)^{k^2},$$

with

$$C_k = \{\Gamma(k^2+1)\}^{-1}\prod_p\left\{\left(1-\frac{1}{p}\right)^{k^2}\sum_{m=0}^\infty\left(\frac{\Gamma(k+m)}{m!\,\Gamma(k)}\right)^2 p^{-m}\right\}.$$

They suggest that this relation may even hold with equality (as it does when $k = 1$ or 2).

7.24. The work of Atkinson (2) alluded to at the end of §7.16 is of special historical interest, since it contains the first occurence of Kloosterman sums in the subject. These sums are defined by

$$S(q; a, b) = \sum_{\substack{n=1\\(n,q)=1}}^{q} \exp\left(\frac{2\pi i}{q}(an + b\bar{n})\right), \tag{7.24.1}$$

where $n\bar{n} \equiv 1 \pmod q$. Such sums have been of great importance in recent work, notably that of Heath-Brown [4] mentioned in §7.21, and of Iwaniec [1] and Deshouillers and Iwaniec [2], [3] referred to later in this section. The key fact about these sums is the estimate

$$|S(q; a, b)| \leqslant d(q)q^{\frac{1}{2}}(q, a, b)^{\frac{1}{2}}, \tag{7.24.2}$$

which indicates a very considerable amount of cancellation in (7.24.1). This result is due to Weil [1] when q is prime (the most important case) and to Estermann [2] in general. Weil's proof uses deep methods from algebraic geometry. It is possible to obtain further cancellations by averaging $S(q; a, b)$ over q, a and b. In order to do this one employs the theory of non-holomorphic modular forms, as in the work of Deshouillers and Iwaniec [1]. This is perhaps the most profound area of current research in the subject.

One way to see how Kloosterman sums arise is to use (7.15.2). Suppose for example one considers

$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^2 \left|\sum_{u\leqslant U} u^{-it}\right|^2 e^{-t/T}\,dt. \tag{7.24.3}$$

Applying (7.15.2) with $2\delta = 1/T + i\log(v/u)$ one is led to examine

$$\sum_{n=1}^\infty d(n)\exp\left(\frac{2\pi i n u}{v}e^{i/T}\right).$$

One may now replace $e^{i/T}$ by $1+(i/T)$ with negligible error, producing

$$\sum_{n=1}^{\infty} d(n)\exp\left(\frac{2\pi inu}{v}\right)\exp\left(-\frac{2\pi nu}{vT}\right)=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Gamma(s)\left(\frac{Tv}{2\pi u}\right)^{s} D\left(s,\frac{u}{v}\right)ds$$

where

$$D\left(s,\frac{u}{v}\right)=\sum_{n=1}^{\infty} d(n)\exp\left(\frac{2\pi inu}{v}\right)n^{-s}.$$

This Dirichlet series was investigated by Estermann [1], using the function $\zeta(s,a)$ of §2.17. It has an analytic continuation to the whole complex plane, and satisfies the functional equation

$$D\left(s,\frac{u}{v}\right)=2v^{1-2s}\frac{\Gamma(1-s)^2}{(2\pi)^{2-2s}}\left\{D\left(1-s,\frac{\bar u}{v}\right)-\cos(\pi s)\,D\left(1-s,-\frac{\bar u}{v}\right)\right\}$$

providing that $(u,v)=1$. To evaluate our original integral (7.24.3) it is necessary to average over u and v, so that one is led to consider

$$\sum_{\substack{u,v\leqslant U\\(u,v)=1}} D\left(1-s,\frac{\bar u}{v}\right)=\sum_{v\leqslant U}\sum_{n=1}^{\infty} d(n)n^{s-1}\sum_{\substack{u\leqslant U\\(u,v)=1}}\exp\left(\frac{2\pi in\bar u}{v}\right),$$

for example. In order to get a sharp bound for the innermost sum on the right one introduces the Kloosterman sum:

$$\begin{aligned}\sum_{\substack{u\leqslant U\\(u,v)=1}}\exp\left(\frac{2\pi in\bar u}{v}\right)&=\sum_{\substack{m=1\\(m,v)=1}}^{v}\exp\left(\frac{2\pi in\bar m}{v}\right)\sum_{\substack{u\leqslant U\\u\equiv m(\mathrm{mod}\ v)}}1\\&=\sum_{\substack{m=1\\(m,v)=1}}^{v}\exp\left(\frac{2\pi in\bar m}{v}\right)\sum_{u\leqslant U}\left\{\frac{1}{v}\sum_{a=1}^{v}\exp\left(\frac{2\pi ia(m-u)}{v}\right)\right\}\\&=\frac{1}{v}\sum_{a=1}^{v}S(v;a,n)\sum_{u\leqslant U}\exp\left(-\frac{2\pi iau}{v}\right),\end{aligned}$$

and one can now get a significant saving by using (7.24.2). Notice also that $S(v;a,n)$ is averaged over v, a and n, so that estimates for averages of Kloosterman sums are potentially applicable.

By pursuing such ideas and exploiting the connection with non-holomorphic modular forms, Iwaniec [1] showed that

$$\sum_{1}^{R}\int_{t_r}^{t_r+\Delta}|\zeta(\tfrac12+it)|^4\,dt\ll(R\Delta+TR^{\frac12}\Delta^{-\frac12})\,T^{\epsilon}$$

for $0\leqslant t_r\leqslant T$, $t_{r+1}-t_r\geqslant\Delta\geqslant T^{\frac12}$. In particular, taking $R=1$, one has

$$\int_{T}^{T+T^{\frac23}}|\zeta(\tfrac12+it)|^4\,dt\ll T^{\frac23+\epsilon}\tag{7.24.4}$$

which again implies $\mu(\frac12)\leqslant\frac16$, by (7.20.2). Moreover, by a suitable choice of the points t_r one can deduce (7.22.1), with $T^{2+\epsilon}$ on the right.

Mean-value theorems involving general Dirichlet polynomials and partial sums of the zeta function are of interest, particularly in connection with the problems considered in Chapters 9 and 10. Such results may be proved by the methods of this chapter, but sharper estimates can be obtained by using Kloosterman sums and their connection with modular forms. Thus Deshouillers and Iwaniec [2], [3] established the bounds

$$\int_0^T|\zeta(\tfrac12+it)|^4\left|\sum_{n\leqslant N}a_n n^{it}\right|^2 dt\ll T^{\epsilon}(T+T^{\frac12}N^2+T^{\frac34}N^{\frac54})\sum_{n\leqslant N}|a_n|^2\tag{7.24.5}$$

and

$$\int_0^T|\zeta(\tfrac12+it)|^2\left|\sum_{m\leqslant M}a_m m^{it}\right|^2\left|\sum_{n\leqslant N}b_n n^{it}\right|^2 dt$$

$$\ll T^{\epsilon}(T+T^{\frac12}M^{\frac32}N+T^{\frac12}MN^{\frac12}+M^{\frac74}N^{\frac32})\left(\sum_{m\leqslant M}|a_m|^2\right)\left(\sum_{n\leqslant N}|b_n|^2\right)\tag{7.24.6}$$

for $N\leqslant M$. In a similar vein Balasubramanian, Conrey, and Heath-Brown [1] showed that

$$\int_0^T|\zeta(\tfrac12+it)|^2\left|\sum_{m\leqslant M}\mu(m)F(m)m^{-\frac12-it}\right|^2 dt=CT+O_A\{T(\log T)^{-A}\},\tag{7.24.7}$$

$$C=\sum_{m,n\leqslant M}\frac{\mu(m)\mu(n)}{mn}F(m)\overline{F(n)}\,(m,n)\left(\log\frac{T(m,n)^2}{2\pi mn}+2\gamma-1\right)$$

for $M\leqslant T^{\frac{9}{17}-\epsilon}$, where A is any positive constant, and the function F satisfies $F(x)\ll1$, $F'(x)\ll x^{-1}$. The proof requires Weil's estimate for the Kloosterman sum, if $T^{\frac12}\leqslant M\leqslant T^{\frac{9}{17}-\epsilon}$.

VIII

Ω-THEOREMS

8.1. Introduction. The previous chapters have been largely concerned with what we may call O-theorems, i.e. results of the form

$$\zeta(s) = O\{f(t)\}, \qquad 1/\zeta(s) = O\{g(t)\}$$

for certain values of σ.

In this chapter we prove a corresponding set of Ω-theorems, i.e. results of the form

$$\zeta(s) = \Omega\{\phi(t)\}, \qquad 1/\zeta(s) = \Omega\{\psi(t)\},$$

the Ω symbol being defined as the negation of o, so that $F(t) = \Omega\{\phi(t)\}$ means that the inequality $|F(t)| > A\phi(t)$ is satisfied for some arbitrarily large values of t.

If, for a given function $F(t)$, we have both

$$F(t) = O\{f(t)\}, \qquad F(t) = \Omega\{f(t)\},$$

we may say that the order of $F(t)$ is determined, and the only remaining question is that of the actual constants involved.

For $\sigma > 1$, the problems of $\zeta(\sigma+it)$ and $1/\zeta(\sigma+it)$ are both solved. For $\frac{1}{2} \leqslant \sigma \leqslant 1$ there remains a considerable gap between the O-results of Chapters V–VI and the Ω-results of the present chapter. We shall see later that, on the Riemann hypothesis, it is the Ω-results which represent the real truth, and the O-results which fall short of it. We are always more successful with Ω-theorems. This is perhaps not surprising, since an O-result is a statement about all large values of t, an Ω-result about some indefinitely large values only.

8.2. The first Ω results were obtained by means of Diophantine approximation, i.e. the approximate solution in integers of given equations. The following two theorems are used.

Dirichlet's Theorem. *Given N real numbers $a_1, a_2,\ldots, a_N$, a positive integer q, and a positive number t_0, we can find a number t in the range*

$$t_0 \leqslant t \leqslant t_0 q^N, \tag{8.2.1}$$

and integers $x_1, x_2,\ldots, x_N$, such that

$$|ta_n - x_n| \leqslant 1/q \quad (n = 1, 2,\ldots, N). \tag{8.2.2}$$

The proof is based on an argument which was introduced and employed extensively by Dirichlet. This argument, in its simplest form, is that, if there are $m+1$ points in m regions, there must be at least one region which contains at least two points.

Consider the N-dimensional unit cube with a vertex at the origin and edges along the coordinate axes. Divide each edge into q equal parts, and thus the cube into q^N equal compartments. Consider the q^N+1 points, in the cube, congruent (mod 1) to the points $(ua_1, ua_2,\ldots, ua_N)$, where $u = 0, t_0, 2t_0,\ldots, q^N t_0$. At least two of these points must lie in the same compartment. If these two points correspond to $u = u_1$, $u = u_2$ $(u_1 < u_2)$, then $t = u_2 - u_1$ clearly satisfies the requirements of the theorem.

The theorem may be extended as follows. Suppose that we give u the values $0, t_0, 2t_0,\ldots, mq^N t_0$. We obtain mq^N+1 points, of which one compartment must contain at least $m+1$. Let these points correspond to $u = u_1,\ldots, u_{m+1}$. Then $t = u_2-u_1,\ldots, u_m-u_1$, all satisfy the requirements of the theorem.

We conclude that the interval $(t_0, mq^N t_0)$ contains at least m solutions of the inequalities (8.2.2), any two solutions differing by at least t_0.

8.3. Kronecker's Theorem. *Let $a_1, a_2,\ldots, a_N$ be linearly independent real numbers, i.e. numbers such that there is no linear relation*

$$\lambda_1 a_1 + \ldots + \lambda_N a_N = 0$$

in which the coefficients $\lambda_1,\ldots$ are integers not all zero. Let $b_1,\ldots, b_N$ be any real numbers, and q a given positive number. Then we can find a number t and integers $x_1,\ldots, x_N$, such that

$$|ta_n - b_n - x_n| \leqslant 1/q \quad (n = 1, 2,\ldots, N). \tag{8.3.1}$$

If all the numbers b_n are 0, the result is included in Dirichlet's theorem. In the general case, we have to suppose the a_n linearly independent; for example, if the a_n are all zero, and the b_n are not all integers, there is in general no t satisfying (8.3.1). Also the theorem assigns no upper bound for the number t such as the q^N of Dirichlet's theorem. This makes a considerable difference to the results which can be deduced from the two theorems.

Many proofs of Kronecker's theorem are known.† The following is due to Bohr (15).

We require the following lemma

Lemma. *If $\phi(x)$ is positive and continuous for $a \leqslant x \leqslant b$, then*

$$\lim_{n\to\infty} \left\{ \int_a^b \{\phi(x)\}^n \, dx \right\}^{1/n} = \max_{a\leqslant x\leqslant b} \phi(x).$$

A similar result holds for an integral in any number of dimensions.

† Bohr (15), (16), Bohr and Jessen (3), Estermann (3), Lettenmeyer (1).

Let $M = \max \phi(x)$. Then

$$\left\{\int_a^b \{\phi(x)\}^n\,dx\right\}^{1/n} \leqslant \{(b-a)M^n\}^{1/n} = (b-a)^{1/n}M.$$

Also, given ϵ, there is an interval, (α, β) say, throughout which

$$\phi(x) \geqslant M-\epsilon.$$

Hence

$$\left\{\int_a^b \{\phi(x)\}^n\,dx\right\}^{1/n} \geqslant \{(\beta-\alpha)(M-\epsilon)^n\}^{1/n} = (\beta-\alpha)^{1/n}(M-\epsilon),$$

and the result is clear. A similar proof holds in the general case.

Proof of Kronecker's theorem. It is sufficient to prove that we can find a number t such that each of the numbers

$$e^{2\pi i(a_n t-b_n)} \quad (n = 1, 2,..., N)$$

differs from 1 by less than a given ϵ; or, if

$$F(t) = 1+\sum_{n=1}^{N} e^{2\pi i(a_n t-b_n)},$$

that the upper bound of $|F(t)|$ for real values of t is $N+1$. Let us denote this upper bound by L. Clearly $L \leqslant N+1$.

Let
$$G(\phi_1, \phi_2,..., \phi_N) = 1+\sum_{n=1}^{N} e^{2\pi i\phi_n},$$

where the numbers ϕ_1, ϕ_2,..., ϕ_N are independent real variables, each lying in the interval $(0, 1)$. Then the upper bound of $|G|$ is $N+1$, this being the value of $|G|$ when $\phi_1 = \phi_2 = ... = \phi_N = 0$.

We consider the polynomial expansions of $\{F(t)\}^k$ and $\{G(\phi_1,..., \phi_N)\}^k$, where k is an arbitrary positive integer; and we observe that each of these expansions contains the same number of terms. For, the numbers a_1, a_2,..., a_N being linearly independent, no two terms in the expansion of $\{F(t)\}^k$ fall together. Also the moduli of corresponding terms are equal. Thus if

$$\{G(\phi_1,..., \phi_N)\}^k = 1+\sum C_q\, e^{2\pi i(\lambda_{q,1}\phi_1+...+\lambda_{q,N}\phi_N)},$$

then
$$\{F(t)\}^k = 1+\sum C_q\, e^{2\pi i(\lambda_{q,1}(a_1 t-b_1)+...+\lambda_{q,N}(a_N t-b_N))}$$
$$= 1+\sum C_q\, e^{2\pi i(\alpha_q t-\beta_q)},$$

say. Now the mean values

$$F_k = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |F(t)|^{2k}\,dt$$

and
$$G_k = \int_0^1\int_0^1 ... \int_0^1 |G(\phi_1,..., \phi_N)|^{2k}\,d\phi_1... d\phi_N$$

are equal, each being equal to

$$1+\sum C_q^2.$$

This is easily seen in each case on expressing the squared modulus as a product of conjugates and integrating term by term.

Since $N+1$ is the upper bound of $|G|$, the lemma gives

$$\lim_{k\to\infty} G_k^{1/2k} = N+1.$$

Hence also
$$\lim_{k\to\infty} F_k^{1/2k} = N+1.$$

But plainly
$$F_k^{1/2k} \leqslant L$$

for all values of k. Hence $L \geqslant N+1$, and so in fact $L = N+1$. This proves the theorem.

8.4. THEOREM 8.4. *If $\sigma > 1$, then*

$$|\zeta(s)| \leqslant \zeta(\sigma) \tag{8.4.1}$$

for all values of t, while

$$|\zeta(s)| \geqslant (1-\epsilon)\zeta(\sigma) \tag{8.4.2}$$

for some indefinitely large values of t.

We have
$$|\zeta(s)| = \left|\sum_{n=1}^{\infty} n^{-s}\right| \leqslant \sum_{n=1}^{\infty} n^{-\sigma} = \zeta(\sigma),$$

so that the whole difficulty lies in the second part. To prove this we use Dirichlet's theorem. For all values of N

$$\zeta(s) = \sum_{n=1}^{N} n^{-\sigma}e^{-it\log n} + \sum_{n=N+1}^{\infty} n^{-\sigma-it},$$

and hence (the modulus of the first sum being not less than its real part)

$$|\zeta(s)| \geqslant \sum_{n=1}^{N} n^{-\sigma}\cos(t\log n) - \sum_{n=N+1}^{\infty} n^{-\sigma}. \tag{8.4.3}$$

By Dirichlet's theorem there is a number t ($t_0 \leqslant t \leqslant t_0 q^N$) and integers x_1,..., x_N, such that, for given N and q ($q \geqslant 4$),

$$\left|\frac{t\log n}{2\pi} - x_n\right| \leqslant \frac{1}{q} \quad (n = 1, 2,..., N).$$

Hence $\cos(t\log n) \geqslant \cos(2\pi/q)$ for these values of n, and so

$$\sum_{n=1}^{N} n^{-\sigma}\cos(t\log n) \geqslant \cos(2\pi/q)\sum_{n=1}^{N} n^{-\sigma} > \cos(2\pi/q)\zeta(\sigma) - \sum_{n=N+1}^{\infty} n^{-\sigma}.$$

Hence by (8.4.3)

$$|\zeta(s)| \geqslant \cos(2\pi/q)\zeta(\sigma) - 2\sum_{N+1}^{\infty} n^{-\sigma}.$$

Now
$$\zeta(\sigma) = \sum_{n=1}^{\infty} n^{-\sigma} > \int_1^{\infty} u^{-\sigma}\, du = \frac{1}{\sigma-1},$$

and
$$\sum_{N+1}^{\infty} n^{-\sigma} < \int_N^{\infty} u^{-\sigma}\, du = \frac{N^{1-\sigma}}{\sigma-1}.$$

Hence
$$|\zeta(s)| \geqslant \{\cos(2\pi/q) - 2N^{1-\sigma}\}\zeta(\sigma), \tag{8.4.4}$$

and the result follows if q and N are large enough.

THEOREM 8.4 (A). *The function $\zeta(s)$ is unbounded in the open region*

$$\sigma > 1,\ t > \delta > 0.$$

This follows at once from the previous theorem, since the upper bound $\zeta(\sigma)$ of $\zeta(s)$ itself tends to infinity as $\sigma \to 1$.

THEOREM 8.4 (B). *The function $\zeta(1+it)$ is unbounded as $t \to \infty$.*

This follows from the previous theorem and the theorem of Phragmén and Lindelöf. Since $\zeta(2+it)$ is bounded, if $\zeta(1+it)$ were also bounded $\zeta(s)$ would be bounded throughout the half-strip $1 \leqslant \sigma \leqslant 2$, $t > \delta$; and this is false, by the previous theorem.

8.5. Dirichlet's theorem also gives the following more precise result.†

THEOREM 8.5. *However large t_1 may be, there are values of s in the region $\sigma > 1$, $t > t_1$, for which*

$$|\zeta(s)| > A \log\log t. \tag{8.5.1}$$

Also
$$\zeta(1+it) = \Omega(\log\log t). \tag{8.5.2}$$

Take $t_0 = 1$ and $q = 6$ in the proof of Theorem 8.4. Then (8.4.4) gives

$$|\zeta(s)| \geqslant (\tfrac{1}{2} - 2N^{1-\sigma})/(\sigma-1) \tag{8.5.3}$$

for a value of t between 1 and 6^N. We choose N to be the integer next above $8^{1/(\sigma-1)}$. Then

$$|\zeta(s)| \geqslant \frac{1}{4(\sigma-1)} \geqslant \frac{\log(N-1)}{4\log 8} > A\log N \tag{8.5.4}$$

for a value of t such that $N > A\log t$. The required inequality (8.5.1) then follows from (8.5.4). It remains only to observe that the value of t in question must be greater than any assigned t_1, if $\sigma - 1$ is sufficiently small; otherwise it would follow from (8.5.3) that $\zeta(s)$ was unbounded

† Bohr and Landau (1).

in the region $\sigma > 1$, $1 < t \leqslant t_1$; and we know that $\zeta(s)$ is bounded in any such region.

The second part of the theorem now follows from the first by the Phragmén–Lindelöf method. Consider the function

$$f(s) = \frac{\zeta(s)}{\log\log s},$$

the branch of $\log\log s$ which is real for $s > 1$, and is restricted to $|s| > 1$, $\sigma > 0$, $t > 0$ being taken. Then $f(s)$ is regular for $1 \leqslant \sigma \leqslant 2$, $t > \delta$. Also $|\log\log s| \sim \log\log t$ as $t \to \infty$, uniformly with respect to σ in the strip. Hence $f(2+it) \to 0$ as $t \to \infty$, and so, if $f(1+it) \to 0$, $f(s) \to 0$ uniformly in the strip.† This contradicts (8.5.1), and so (8.5.2) follows.

It is plain that arguments similar to the above may be applied to all Dirichlet series, with coefficients of fixed sign, which are not absolutely convergent on their line of convergence. For example, the series for $\log\zeta(s)$ and its differential coefficients are of this type. The result for $\log\zeta(s)$ is, however, a corollary of that for $\zeta(s)$, which gives at once

$$|\log\zeta(s)| > \log\log\log t - A$$

for some indefinitely large values of t in $\sigma > 1$. For the nth differential coefficient of $\log\zeta(s)$ the result is that

$$\left|\left(\frac{d}{ds}\right)^n \log\zeta(s)\right| > A_n(\log\log t)^n$$

for some indefinitely large values of t in $\sigma > 1$.

8.6. We now turn to the corresponding problems‡ for $1/\zeta(s)$. We cannot apply the argument depending on Dirichlet's theorem to this function, since the coefficients in the series

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

are not all of the same sign; nor can we argue similarly with Kronecker's theorem, since the numbers $(\log n)/2\pi$ are not linearly independent. Actually we consider $\log\zeta(s)$, which depends on the series $\sum p^{-s}$, to which Kronecker's theorem can be applied.

THEOREM 8.6. *The function $1/\zeta(s)$ is unbounded in the open region $\sigma > 1$, $t > \delta > 0$.*

We have for $\sigma \geqslant 1$

$$\left|\log\zeta(s) - \sum_p \frac{1}{p^s}\right| = \left|\sum_p \sum_{m=2}^{\infty} \frac{1}{mp^{ms}}\right| \leqslant \sum_p \sum_{m=2}^{\infty} \frac{1}{p^m} = \sum_p \frac{1}{p(p-1)} = A.$$

† See e.g. my *Theory of Functions*, § 5.63, with the angle transformed into a strip.
‡ Bohr and Landau (7).

Now

$$\mathbf{R}\Big(\sum_p \frac{1}{p^s}\Big) = \sum_{n=1}^{\infty} \frac{\cos(t\log p_n)}{p_n^\sigma} \leqslant \sum_{n=1}^{N} \frac{\cos(t\log p_n)}{p_n^\sigma} + \sum_{n=N+1}^{\infty} \frac{1}{p_n^\sigma}.$$

Also *the numbers* $\log p_n$ *are linearly independent.* For it follows from the theorem that an integer can be expressed as a product of prime factors in one way only, that there can be no relation of the form

$$p_1^{\lambda_1} p_2^{\lambda_2} \dots p_N^{\lambda_N} = 1,$$

where the λ's are integers, and therefore no relation of the form

$$\lambda_1 \log p_1 + \dots + \lambda_N \log p_N = 0.$$

Hence also the numbers $(\log p_n)/2\pi$ are linearly independent. It follows therefore from Kronecker's theorem that we can find a number t and integers $x_1,\dots, x_N$ such that

$$\left| t\frac{\log p_n}{2\pi} - \tfrac{1}{2} - x_n \right| \leqslant \tfrac{1}{6} \quad (n = 1, 2,\dots, N),$$

or

$$|t\log p_n - \pi - 2\pi x_n| \leqslant \tfrac{1}{3}\pi \quad (n = 1, 2,\dots, N).$$

Hence for these values of n

$$\cos(t\log p_n) = -\cos(t\log p_n - \pi - 2\pi x_n) \leqslant -\cos\tfrac{1}{3}\pi = -\tfrac{1}{2},$$

and hence

$$\mathbf{R}\Big(\sum_p \frac{1}{p^s}\Big) \leqslant -\frac{1}{2}\sum_{n=1}^{N} \frac{1}{p_n^\sigma} + \sum_{n=N+1}^{\infty} \frac{1}{p_n^\sigma}.$$

Since $\sum p_n^{-1}$ is divergent, we can, if H is any assigned positive number, choose σ so near to 1 that $\sum p_n^{-\sigma} > H$. Having fixed σ, we can choose N so large that

$$\sum_{n=1}^{N} p_n^{-\sigma} > \tfrac{3}{4}H, \qquad \sum_{n=N+1}^{\infty} p_n^{-\sigma} < \tfrac{1}{4}H.$$

Then

$$\mathbf{R}\Big(\sum_p p^{-s}\Big) < -\tfrac{3}{8}H + \tfrac{1}{4}H = -\tfrac{1}{8}H.$$

Since H may be as large as we please, it follows that $\mathbf{R}(\sum p^{-s})$, and so $\log|\zeta(s)|$, takes arbitrarily large negative values. This proves the theorem.

THEOREM 8.6 (A). *The function* $1/\zeta(1+it)$ *is unbounded as* $t \to \infty$.

This follows from the previous theorem in the same way as Theorem 8.4 (B) from Theorem 8.4 (A).

We cannot, however, proceed to deduce an analogue of Theorem 8.5 for $1/\zeta(s)$. In proving Theorem 8.5, each of the numbers $\cos(t\log n)$ has to be made as near as possible to 1, and this can be done by Dirichlet's theorem. In Theorem 8.6, each of the numbers $\cos(t\log p_n)$ has to be made as near as possible to -1, and this requires Kronecker's theorem.

Now Theorem 8.5 depends on the fact that we can assign an upper limit to the number t which satisfies the conditions of Dirichlet's theorem. Since there is no such upper limit in Kronecker's theorem, the corresponding argument for $1/\zeta(s)$ fails. We shall see later that the analogue of Theorem 8.5 is in fact true, but it requires a much more elaborate proof.

8.7. Before proceeding to these deeper theorems, we shall give another method of proving some of the above results.† This method deals directly with integrals of high powers of the functions in question, and so might be described as a short cut which avoids explicit use of Diophantine approximation.

We write

$$M\{|f(s)|^2\} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(\sigma+it)|^2\,dt,$$

and prove the following lemma.

LEMMA. *Let*

$$g(s) = \sum_{m=1}^{\infty} \frac{b_m}{m^s}, \qquad h(s) = \sum_{n=1}^{\infty} \frac{c_n}{n^s}$$

be absolutely convergent for a given value of σ, *and let every* m *with* $b_m \neq 0$ *be prime to every* n *with* $c_n \neq 0$. *Then for such* σ

$$M\{|g(s)h(s)|^2\} = M\{|g(s)|^2\}M\{|h(s)|^2\}.$$

By Theorem 7.1

$$M\{|g(s)|^2\} = \sum_{m=1}^{\infty} \frac{|b_m|^2}{m^{2\sigma}}, \qquad M\{|h(s)|^2\} = \sum_{n=1}^{\infty} \frac{|c_n|^2}{n^{2\sigma}}.$$

Now

$$g(s)h(s) = \sum_{r=1}^{\infty} \frac{d_r}{r^s},$$

where each term $d_r r^{-s}$ is the product of two terms $b_m m^{-s}$ and $c_n n^{-s}$. Hence

$$M\{|g(s)h(s)|^2\} = \sum_{r=1}^{\infty} \frac{|d_r|^2}{r^{2\sigma}} = \sum\sum \frac{|b_m c_n|^2}{(mn)^{2\sigma}} = M\{|g(s)|^2\}M\{|h(s)|^2\}.$$

We can now prove the analogue for $1/\zeta(s)$ of Theorem 8.4.

THEOREM 8.7. *If* $\sigma > 1$, *then*

$$\left|\frac{1}{\zeta(s)}\right| \leqslant \frac{\zeta(\sigma)}{\zeta(2\sigma)} \tag{8.7.1}$$

for all values of t, *while*

$$\left|\frac{1}{\zeta(s)}\right| \geqslant (1-\epsilon)\frac{\zeta(\sigma)}{\zeta(2\sigma)} \tag{8.7.2}$$

for some indefinitely large values of t.

† Bohr and Landau (7).

We have, for $\sigma > 1$,

$$\left|\frac{1}{\zeta(s)}\right| = \left|\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}\right| \leqslant \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^\sigma}.$$

Since
$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p \left(1-\frac{1}{p^s}\right)$$
we have also

$$\sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^\sigma} = \prod_p \left(1+\frac{1}{p^\sigma}\right) = \prod_p \left(\frac{1-p^{-2\sigma}}{1-p^{-\sigma}}\right) = \frac{\zeta(\sigma)}{\zeta(2\sigma)},$$

and the first part follows.

To prove the second part, write

$$\frac{1}{\zeta(s)} = \prod_{n=1}^{N} \left(1-\frac{1}{p_n^s}\right)\eta_N(s),$$

$$\frac{1}{\{\zeta(s)\}^k} = \prod_{n=1}^{N} \left(1-\frac{1}{p_n^s}\right)^k \{\eta_N(s)\}^k.$$

By repeated application of the lemma it follows that

$$M\left\{\frac{1}{|\zeta(s)|^{2k}}\right\} = \prod_{n=1}^{N} M\left\{\left|\left(1-\frac{1}{p_n^s}\right)\right|^{2k}\right\} M\{|\eta_N(s)|^{2k}\}.$$

Now, for every p,

$$M\left\{\left|1-\frac{1}{p^s}\right|^{2k}\right\} = \frac{\log p}{2\pi} \int_0^{2\pi/\log p} \left|1-\frac{1}{p^s}\right|^{2k} dt,$$

since the integrand is periodic with period $2\pi/\log p$; and

$$M\{|\eta_N(s)|^{2k}\} \geqslant 1,$$

since the Dirichlet series for $\{\eta_N(s)\}^k$ begins with $1+\dots$. Hence

$$M\left\{\frac{1}{|\zeta(s)|^{2k}}\right\} \geqslant \prod_{n=1}^{N} \frac{\log p_n}{2\pi} \int_0^{2\pi/\log p_n} \left|1-\frac{1}{p_n^s}\right|^{2k} dt.$$

Now
$$\lim_{k\to\infty} \left(\int_0^{2\pi/\log p} \left|1-\frac{1}{p^s}\right|^{2k} dt\right)^{1/2k} = \max_{0\leqslant t\leqslant 2\pi/\log p} \left|1-\frac{1}{p^s}\right| = 1+\frac{1}{p^\sigma}.$$

Hence
$$\varliminf_{k\to\infty} \left[M\left\{\frac{1}{|\zeta(s)|^{2k}}\right\}\right]^{1/2k} \geqslant \prod_{n=1}^{N} \left(1+\frac{1}{p_n^\sigma}\right).$$

Since the left-hand side is independent of N, we can make $N \to \infty$ on the right, and obtain

$$\varliminf_{k\to\infty} \left[M\left\{\frac{1}{|\zeta(s)|^{2k}}\right\}\right]^{1/2k} \geqslant \frac{\zeta(\sigma)}{\zeta(2\sigma)}.$$

Hence to any ϵ corresponds a k such that

$$\left[M\left\{\frac{1}{|\zeta(s)|^{2k}}\right\}\right]^{1/2k} > (1-\epsilon)\frac{\zeta(\sigma)}{\zeta(2\sigma)},$$

and (8.7.2) now follows.

Since $\zeta(\sigma)/\zeta(2\sigma) \to \infty$ as $\sigma \to 1$, this also gives an alternative proof of Theorem 8.6

It is easy to see that a similar method can be used to prove Theorem 8.4 (A). It is also possible to prove Theorems 8.4 (B) and 8.6 (A) directly by this method without using the Phragmén–Lindelöf theorem. This, however, requires an extension of the general mean-value theorem for Dirichlet series.

8.8. THEOREM 8.8.† *However large t_0 may be, there are values of s in the region $\sigma > 1$, $t > t_0$ for which*

$$\left|\frac{1}{\zeta(s)}\right| > A \log\log t.$$

Also
$$\frac{1}{\zeta(1+it)} = \Omega(\log\log t).$$

As in the case of Theorem 8.5, it is enough to prove the first part. We first prove some lemmas. The object of these lemmas is to supply, for the particular case in hand, what Kronecker's theorem lacks in the general case, viz. an upper bound for the number t which satisfies the conditions (8.3.1).

LEMMA α. *If m and n are different positive integers,*

$$\left|\log\frac{m}{n}\right| > \frac{1}{\max(m,n)}.$$

For if $m < n$

$$\log\frac{n}{m} \geqslant \log\frac{n}{n-1} = \frac{1}{n}+\frac{1}{2}\frac{1}{n^2}+\dots > \frac{1}{n}.$$

LEMMA β. *If $p_1,\dots, p_N$ are the first N primes, and $\mu_1,\dots, \mu_N$ are integers, not all 0 (not necessarily positive), then*

$$\left|\log \prod_{n=1}^{N} p_n^{\mu_n}\right| > p_N^{-\mu N} \quad (\mu = \max|\mu_n|).$$

For $\prod_{n=1}^{N} p_n^{\mu_n} = u/v$, where

$$u = \prod_{\mu_n>0} p_n^{\mu_n}, \qquad v = \prod_{\mu_n<0} p_n^{-\mu_n},$$

† Bohr and Landau (7).

and u and v, being mutually prime, are different. Also

$$\max(u,v) \leqslant \prod_{n=1}^{N} p_n^{\mu} \leqslant p_N^{N\mu},$$

and the result follows from Lemma α.

LEMMA γ. *The number of solutions in positive or zero integers of the equation*

$$\nu_0+\nu_1+\ldots+\nu_N = k$$

does not exceed $(k+1)^N$.

For $N = 1$ the number of solutions is $k+1$, so that the theorem holds. Suppose that it holds for any given N. Then for given ν_{N+1} the number of solutions of

$$\nu_0+\nu_1+\ldots+\nu_N = k-\nu_{N+1}$$

does not exceed $(k-\nu_{N+1}+1)^N \leqslant (k+1)^N$; and ν_{N+1} can take $k+1$ values. Hence the total number of solutions is $\leqslant (k+1)^{N+1}$, whence the result.

LEMMA δ. *For $N > A$, there exits a t satisfying $0 \leqslant t \leqslant \exp(N^6)$ for which*

$$\cos(t\log p_n) < -1+\frac{1}{N} \quad (n \leqslant N).$$

Let $N > 1$, $k > 1$. Then

$$\left(\sum_{n=0}^{N} x_n\right)^k = \sum c(\nu_0,\ldots,\nu_N) x_0^{\nu_0}\ldots x_N^{\nu_N},$$

where

$$c(\nu_0,\ldots,\nu_N) = \frac{k!}{\nu_0!\ldots\nu_N!}, \qquad \sum \nu_n = k.$$

The number of distinct terms in the expansion is at most $(k+1)^N < k^{2N}$, by Lemma γ. Hence

$$\left(\sum c\right)^2 \leqslant \sum c^2 \sum 1 < k^{2N} \sum c^2,$$

so that

$$\sum c^2 > k^{-2N}\left(\sum c\right)^2 = k^{-2N}(N+1)^{2k}.$$

Let

$$F(t) = 1-\sum_{n=1}^{N} e^{it\log p_n},$$

so that

$$\{F(t)\}^k = \sum c(\nu_0,\ldots,\nu_N)(-1)^{\nu_1+\ldots+\nu_N}\exp\left(it\sum_{1}^{N}\nu_n\log p_n\right),$$

$$|F(t)|^{2k} = \sum_{\nu,\nu'}\sum cc'(-1)^{\Sigma\nu_n+\Sigma\nu_n'}\exp\left\{it\sum_{n}(\nu_n-\nu_n')\log p_n\right\}$$

$$= \Sigma_1+\Sigma_2,$$

where Σ_1 is taken over values of (ν,ν') for which $\nu_1 = \nu_1'$, $\nu_2 = \nu_2'$,..., and

Σ_2 over the remainder. Now

$$\frac{1}{T}\int_0^T e^{i\alpha t}\,dt = 1 \quad (\alpha = 0),$$

$$\left|\frac{1}{T}\int_0^T e^{i\alpha t}\,dt\right| = \left|\frac{e^{i\alpha T}-1}{i\alpha T}\right| \leqslant \frac{2}{|\alpha|T} \quad (\alpha \neq 0).$$

Hence

$$\frac{1}{T}\int_0^T |F(t)|^{2k}\,dt \geqslant \Sigma_1 c^2 - \Sigma_2 \frac{2cc'}{|\sum(\nu_n-\nu_n')\log p_n|T}.$$

By Lemma β, since the numbers $\nu_n-\nu_n'$ are not all 0,

$$\left|\sum(\nu_n-\nu_n')\log p_n\right| = \left|\log\prod_{1}^{N} p_n^{(\nu_n-\nu_n')}\right| > p_N^{-N\max|\nu_n-\nu_n'|} \geqslant p_N^{-kN}.$$

Hence

$$\frac{1}{T}\int_0^T |F(t)|^{2k}\,dt \geqslant \sum c^2 - \frac{2p_N^{kN}}{T}\sum\sum cc'$$

$$\geqslant k^{-2N}\left(\sum c\right)^2 - \frac{2p_N^{kN}}{T}\left(\sum c\right)^2$$

$$= \left(k^{-2N}-\frac{2p_N^{kN}}{T}\right)(N+1)^{2k}.$$

In this we take $k = N^4$, $T = e^{N^6}$, and obtain, for $N > A$,

$$k^{-2N}-\frac{2p_N^{kN}}{T} = N^{-8N}-2\left(\frac{p_N}{e^N}\right)^{kN} > e^{-N^3/(N+1)}.$$

Hence

$$\left(\frac{1}{T}\int_0^T |F(t)|^{2k}\,dt\right)^{1/2k} \geqslant (N+1)e^{-1/(2N(N+1))} > N+1-\frac{1}{2N}.$$

Hence there is a t in $(0, e^{N^6})$ such that

$$|F(t)| > N+1-\frac{1}{2N}.$$

Suppose, however, that $\cos(t\log p_n) \geqslant -1+1/N$ for some value of n. Then

$$|F(t)| \leqslant N-1+|1-e^{it\log p_n}| = N-1+\sqrt{2}(1-\cos t\log p_n)^{\frac{1}{2}}$$

$$\leqslant N-1+\sqrt{2}\left(2-\frac{1}{N}\right)^{\frac{1}{2}} \leqslant N+1-\frac{1}{2N},$$

a contradiction. Hence the result.

We can now prove Theorem 8.8. As in § 8.6, for $\sigma > 1$

$$\log\frac{1}{|\zeta(s)|} = -\sum \frac{\cos(t\log p_n)}{p_n^\sigma}+O(1).$$

Let now N be large, $t = t(N)$ the number of Lemma δ, $\delta = 1/\log N$, and $\sigma = 1+\delta$. Then

$$\log\frac{A}{|\zeta(s)|} \geqslant -\sum \frac{\cos(t\log p_n)}{p_n^\sigma} \geqslant \left(1-\frac{1}{N}\right)\sum_1^N \frac{1}{p_n^\sigma}-\sum_{N+1}^\infty \frac{1}{p_n^\sigma}$$

$$> \left(1-\frac{1}{N}\right)\sum \frac{1}{p^\sigma}-2\sum_{N+1}^\infty \frac{1}{p_n^\sigma} > \left(1-\frac{1}{N}\right)\{\log\zeta(\sigma)-A\}-2\sum_{N+1}^\infty \frac{1}{(An\log n)^\sigma}$$

$$> \left(1-\frac{1}{N}\right)\left(\log\frac{1}{\delta}-A\right)-\frac{A}{\log N}\sum_{N+1}^\infty \frac{1}{n^\sigma},$$

$$\log\frac{A\delta}{|\zeta(s)|} > -A-\frac{1}{N}\log\frac{1}{\delta}-\frac{A}{\log N}\frac{N^{1-\sigma}}{\sigma-1} > -A,$$

$$\frac{1}{|\zeta(s)|} > \frac{A}{\delta} = A\log N > A\log\log t.$$

The number $t = t(N)$ evidently tends to infinity with N, since $1/\zeta(s)$ is bounded in $|t| \leqslant A$, $\sigma \geqslant 1$, and the proof is completed.

8.9. In Theorems 8.5 and 8.8 we have proved that each of the inequalities

$$|\zeta(1+it)| > A\log\log t, \qquad 1/|\zeta(1+it)| > A\log\log t$$

is satisfied for some arbitrarily large values of t, if A is a suitable constant. We now consider the question how large the constant can be in the two cases.

Since neither $|\zeta(1+it)|/\log\log t$ nor $|\zeta(1+it)|^{-1}/\log\log t$ is known to be bounded, the question of the constants might not seem to be of much interest. But we shall see later that on the Riemann hypothesis they are both bounded; in fact if

$$\lambda = \overline{\lim_{t\to\infty}}\frac{|\zeta(1+it)|}{\log\log t}, \qquad \mu = \overline{\lim_{t\to\infty}}\frac{1/|\zeta(1+it)|}{\log\log t}, \tag{8.9.1}$$

then, on the Riemann hypothesis,

$$\lambda \leqslant 2e^\gamma, \qquad \mu \leqslant \frac{12}{\pi^2}e^\gamma, \tag{8.9.2}$$

where γ is Euler's constant.

There is therefore a certain interest in proving the following results.†

THEOREM 8.9 (A). $$\overline{\lim_{t\to\infty}}\frac{|\zeta(1+it)|}{\log\log t} \geqslant e^\gamma.$$

THEOREM 8.9 (B). $$\overline{\lim_{t\to\infty}}\frac{1/|\zeta(1+it)|}{\log\log t} \geqslant \frac{6}{\pi^2}e^\gamma.$$

Thus on the Riemann hypothesis it is only a factor 2 which remains in doubt in each case.

We first prove some identities and inequalities. As in § 7.19, if

$$F_k(s) = \sum_{n=1}^\infty \frac{d_k^2(n)}{n^s} \quad (\sigma > 1) \tag{8.9.3}$$

and

$$f_k(x) = \sum_{m=0}^\infty \left\{\frac{(k+m-1)!}{(k-1)!\,m!}\right\}^2 x^m, \tag{8.9.4}$$

then

$$F_k(s) = \prod_p f_k(p^{-s}). \tag{8.9.5}$$

Now for real x

$$f_k(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{m=0}^\infty \frac{(k+m-1)!}{(k-1)!\,m!}x^{\frac{1}{2}m}e^{im\phi}\right|^2 d\phi$$

$$= \frac{1}{\pi}\int_0^\pi \frac{d\phi}{|1-x^{\frac{1}{2}}e^{i\phi}|^{2k}} = \frac{1}{\pi}\int_0^\pi \frac{d\phi}{(1-2\sqrt{x}\cos\phi+x)^k}. \tag{8.9.6}$$

Using the familiar formula

$$P_n(z) = \frac{1}{\pi}\int_0^\pi \frac{d\phi}{\{z-\sqrt{(z^2-1)}\cos\phi\}^{n+1}} \tag{8.9.7}$$

for the Legendre polynomial of degree n, we see that

$$f_k(x) = \frac{1}{(1-x)^k}P_{k-1}\left(\frac{1+x}{1-x}\right). \tag{8.9.8}$$

Naturally this identity holds also for complex x; it gives

$$F_k(s) = \prod_p \frac{1}{(1-p^{-s})^k}P_{k-1}\left(\frac{1+p^{-s}}{1-p^{-s}}\right) = \zeta^k(s)\prod_p P_{k-1}\left(\frac{1+p^{-s}}{1-p^{-s}}\right). \tag{8.9.9}$$

A similar set of formulae holds for $1/\zeta(s)$. We have

$$\frac{1}{\{\zeta(s)\}^k} = \prod_p \left(1-\frac{1}{p^s}\right)^k = \prod_p \left(1-\frac{k}{p^s}+\frac{k(k-1)}{1.2}\frac{1}{p^{2s}}-\ldots+\frac{(-1)^k}{p^{ks}}\right).$$

† Littlewood (5), (6), Titchmarsh (4), (14).

Hence
$$\frac{1}{\zeta^k(s)} = \sum_{n=1}^{\infty} \frac{b_k(n)}{n^s}, \tag{8.9.10}$$
where the coefficients $b_k(n)$ are determined in an obvious way from the above product. They are integers, but are not all positive.

The form of these coefficients shows that
$$\sum_{n=1}^{\infty} \frac{|b_k(n)|}{n^s} = \prod_p \left(1+\frac{k}{p^s}+\ldots+\frac{1}{p^{ks}}\right) = \prod_p \left(1+\frac{1}{p^s}\right)^k$$
$$= \prod_p \left(1-\frac{1}{p^{2s}}\right)^k \left(1-\frac{1}{p^s}\right)^{-k} = \left\{\frac{\zeta(s)}{\zeta(2s)}\right\}^k. \tag{8.9.11}$$

Again, let
$$G_k(s) = \sum_{n=1}^{\infty} \frac{b_k^2(n)}{n^s}. \tag{8.9.12}$$
As in the case of $F_k(s)$,
$$G_k(s) = \prod_p \left(1+\frac{k^2}{p^s}+\frac{k^2(k-1)^2}{1^2.2^2}\frac{1}{p^{2s}}+\ldots+\frac{1}{p^{ks}}\right) = \prod_p g_k(p^{-s}), \tag{8.9.13}$$
say. Now, for real x,
$$g_k(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\sum_{m=0}^{k} \frac{k!}{m!(k-m)!} x^{\frac{1}{2}m} e^{im\phi}\right|^2 d\phi$$
$$= \frac{1}{\pi}\int_0^{\pi} |1+x^{\frac{1}{2}}e^{i\phi}|^{2k}\, d\phi = \frac{1}{\pi}\int_0^{\pi} (1+2x^{\frac{1}{2}}\cos\phi+x)^k\, d\phi.$$
Comparing this with the formula
$$P_n(z) = \frac{1}{\pi}\int_0^{\pi} \{z+\sqrt{(z^2-1)}\cos\phi\}^n\, d\phi$$
we see that†
$$g_k(x) = (1-x)^k P_k\left(\frac{1+x}{1-x}\right). \tag{8.9.14}$$
Hence
$$G_k(s) = \prod_p (1-p^{-s})^k P_k\left(\frac{1+p^{-s}}{1-p^{-s}}\right) = \frac{1}{\zeta^k(s)} \prod_p P_k\left(\frac{1+p^{-s}}{1-p^{-s}}\right).$$
We have also the identity
$$F_{k+1}(s) = \zeta^{2k+1}(s) G_k(s). \tag{8.9.15}$$

† This formula is, essentially, Murphy's well-known formula
$$P_k(\cos\theta) = \cos^{2k}\tfrac{1}{2}\theta\, F(-k, -k; 1; -\tan^2\tfrac{1}{2}\theta)$$
with $x = -\tan^2\frac{1}{2}\theta$; cf. Hobson, *Spherical and Ellipsoidal Harmonics*, pp. 22, 31.

Again for $0 < x < \frac{1}{2}$
$$f_k(x) > \frac{1}{\pi}\int_0^{\pi/k} \frac{d\phi}{(1-2\sqrt{x}\cos\phi+x)^k}$$
$$= \frac{1}{\pi(1-\sqrt{x})^{2k}}\int_0^{\pi/k} \left\{1-\frac{2\sqrt{x}(1-\cos\phi)}{1-2\sqrt{x}\cos\phi+x}\right\}^k d\phi$$
$$= \frac{1}{\pi(1-\sqrt{x})^{2k}}\int_0^{\pi/k} \left\{1+O\left(\frac{1}{k^2}\right)\right\}^k d\phi > \frac{1}{2k(1-\sqrt{x})^{2k}} \tag{8.9.16}$$
if k is large enough. Hence also
$$g_k(x) = (1-x)^{2k+1} f_{k+1}(x) > \frac{(1+\sqrt{x})^{2k+1}}{2k+2} > \frac{(1+\sqrt{x})^{2k}}{3k} \tag{8.9.17}$$
for k large enough; and
$$g_k(x) \leqslant \frac{1}{\pi}\int_0^{\pi} (1+\sqrt{x})^{2k}\, d\phi = (1+\sqrt{x})^{2k} \tag{8.9.18}$$
for all values of x and k.

8.10. Proof of Theorem 8.9 (A).

Let $\sigma > 1$. Then
$$\int_{-T}^{T} \left(1-\frac{|t|}{T}\right)|\zeta(\sigma+it)|^{2k}\, dt = \int_{-T}^{T} \left(1-\frac{|t|}{T}\right) \sum_{m=1}^{\infty} \frac{d_k(m)}{m^{\sigma+it}} \sum_{n=1}^{\infty} \frac{d_k(n)}{n^{\sigma-it}}\, dt$$
$$= \sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^{2\sigma}} \int_{-T}^{T} \left(1-\frac{|t|}{T}\right) dt + \sum_{m\neq n}\sum \frac{d_k(m)d_k(n)}{(mn)^{\sigma}} \int_{-T}^{T} \left(1-\frac{|t|}{T}\right)\left(\frac{n}{m}\right)^{it} dt$$
$$= T\sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^{2\sigma}} + \sum_{m\neq n}\sum \frac{d_k(m)d_k(n)}{(mn)^{\sigma}} \frac{4\sin^2\{\frac{1}{2}T\log(n/m)\}}{T\log^2(n/m)}$$
$$\geqslant T\sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^{2\sigma}} = TF_k(2\sigma). \tag{8.10.1}$$
Since (from its original definition) $f_k(p^{-2\sigma}) \geqslant 1$ for all values of p,
$$F_k(2\sigma) \geqslant \prod_{p\leqslant x} f_k(p^{-2\sigma}) \geqslant \prod_{p\leqslant x} \left\{\frac{1}{2k}\left(1-\frac{1}{p^{\sigma}}\right)^{-2k}\right\} \tag{8.10.2}$$
for any positive x and k large enough. Here the number of factors is $\pi(x) < Ax/\log x$. Hence if $x > \sqrt{k}$
$$\prod_{p\leqslant x} \frac{1}{2k} \geqslant \left(\frac{1}{2k}\right)^{Ax/\log x} = \exp\left(-\frac{Ax\log 2k}{\log x}\right) > e^{-Ax}. \tag{8.10.3}$$

Also

$$\log\prod_{p\leqslant x}\frac{1-p^{-\sigma}}{1-p^{-1}} = \sum_{p\leqslant x}\log\frac{1-p^{-\sigma}}{1-p^{-1}} = \sum_{p\leqslant x}O\left(\frac{1}{p^{\sigma}}-\frac{1}{p}\right)$$

$$= \sum_{p\leqslant x}O\left(\log p\int\limits_{1}^{\sigma}\frac{du}{p^{u}}\right) = O\left\{(\sigma-1)\sum_{p\leqslant x}\frac{\log p}{p}\right\} = O\{(\sigma-1)\log x\}. \tag{8.10.4}$$

Hence
$$F_k(2\sigma) > e^{-Ax-Ak(\sigma-1)\log x}\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)^{-2k},$$
and

$$\left\{\frac{1}{T}\int\limits_{-T}^{T}\left(1-\frac{|t|}{T}\right)|\zeta(\sigma+it)|^{2k}\,dt\right\}^{1/2k} > e^{-Ax/k-A(\sigma-1)\log x}\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)^{-1}$$

$$> \{e^{\gamma}+o(1)\}e^{-Ax/k-A(\sigma-1)\log x}\log x$$

as $x\to\infty$, by (3.15.2).

Let $x = \delta k$, where $k^{-\frac{1}{2}} < \delta < 1$, and $\sigma = 1+\eta/\log k$, where $0<\eta<1$. Then the right-hand side is greater than

$$\{e^{\gamma}+o(1)\}e^{-A\delta-A\eta}\left(\log k-\log\frac{1}{\delta}\right).$$

Also, if $m_{\sigma,T} = \max\limits_{1\leqslant|t|\leqslant T}|\zeta(\sigma+it)|$, the left-hand side does not exceed

$$\left\{\frac{2}{T}\int\limits_{0}^{1}\left(1-\frac{|t|}{T}\right)\left(\frac{2}{\sigma-1}\right)^{2k}dt\right\}^{1/2k}+\left\{\frac{2}{T}\int\limits_{1}^{T}\left(1-\frac{|t|}{T}\right)m_{\sigma,T}^{2k}\,dt\right\}^{1/2k}$$

$$< \left(\frac{2}{T}\right)^{1/2k}\frac{2}{\sigma-1}+2^{1/2k}m_{\sigma,T}.$$

Hence

$$m_{\sigma,T} > 2^{-1/2k}\{e^{\gamma}+o(1)\}e^{-A\delta-A\eta}\left(\log k-\log\frac{1}{\delta}\right)-\frac{2\log k}{T^{1/2k}\eta}.$$

Let $T = \eta^{-4k}$, so that

$$\log\log T = \log k+\log\left(4\log\frac{1}{\eta}\right).$$

Then

$$m_{\sigma,T} > 2^{-1/2k}\{e^{\gamma}+o(1)\}e^{-A\delta-A\eta}\left\{\log\log T-\log\left(4\log\frac{1}{\eta}\right)-\log\frac{1}{\delta}\right\}-$$

$$-2\eta\left\{\log\log T-\log\left(4\log\frac{1}{\eta}\right)\right\}.$$

Giving δ and η arbitrarily small values, and then making $k\to\infty$, i.e. $T\to\infty$, we obtain

$$\overline{\lim}\,\frac{m_{\sigma,T}}{\log\log T}\geqslant e^{\gamma},$$

where, of course, σ is a function of T.

The result now follows by the Phragmén–Lindelöf method. Let

$$f(s) = \frac{\zeta(s)}{\log\log(s+hi)}$$

where $h>4$, and let
$$\lambda = \overline{\lim}\,\frac{|\zeta(1+it)|}{\log\log t}.$$

We may suppose λ finite, or there is nothing to prove. On $\sigma = 1$, $t\geqslant 0$, we have

$$|f(s)|\leqslant\frac{|\zeta(s)|}{\log\log t}<\lambda+\epsilon \quad (t>t_0).$$

Also, on $\sigma = 2$, $\qquad |f(s)| = o(1) < \lambda+\epsilon \quad (t>t_1).$

We can choose h so that $|f(s)|<\lambda+\epsilon$ also on the remainder of the boundary of the strip bounded by $\sigma = 1$, $\sigma = 2$, and $t = 1$. Then, by the Phragmén–Lindelöf theorem, $|f(s)|<\lambda+\epsilon$ in the interior, and so

$$\overline{\lim}\,\frac{|\zeta(s)|}{\log\log t} = \overline{\lim}\,\frac{|\zeta(s)|}{\log\log(t+h)}\leqslant\lambda.$$

Hence $\lambda\geqslant e^{\gamma}$, the required result.

8.11. Proof of Theorem 8.9 (B). The above method depends on the fact that $d_k(n)$ is positive. Since $b_k(n)$ is not always positive, a different method is required in this case.

Let $\sigma>1$, and let N be any positive number. Then

$$\frac{1}{T}\int\limits_{0}^{T}\left|\sum_{n\leqslant N}\frac{b_k(n)}{n^{s}}\right|^2dt = \frac{1}{T}\int\limits_{0}^{T}\sum_{m\leqslant N}\frac{b_k(m)}{m^{\sigma+it}}\sum_{n\leqslant N}\frac{b_k(n)}{n^{\sigma-it}}\,dt$$

$$= \sum_{n\leqslant N}\frac{b_k^2(n)}{n^{2\sigma}}+\frac{1}{T}\sum\sum_{m\neq N}\frac{b_k(m)b_k(n)}{m^{\sigma}n^{\sigma}}\int\limits_{0}^{T}\left(\frac{n}{m}\right)^{it}dt$$

$$\geqslant\sum_{n\leqslant N}\frac{b_k^2(n)}{n^{2\sigma}}-\frac{1}{T}\sum\sum_{m\neq n}\frac{|b_k(m)b_k(n)|}{m^{\sigma}n^{\sigma}}\,\frac{2}{|\log n/m|}.$$

Now
$$\left|\log\frac{n}{m}\right|\geqslant\log\frac{n+1}{n}\geqslant\frac{1}{2n}\geqslant\frac{1}{2N},$$

so that the last sum does not exceed

$$\frac{4N}{T}\sum\sum_{m\neq n}\frac{|b_k(m)b_k(n)|}{m^{\sigma}n^{\sigma}} < \frac{4N}{T}\left(\sum_{n=1}^{\infty}\frac{|b_k(n)|}{n^{\sigma}}\right)^2 = \frac{4N}{T}\left\{\frac{\zeta(\sigma)}{\zeta(2\sigma)}\right\}^{2k}.$$

Since $\zeta(\sigma)\sim 1/(\sigma-1)$ as $\sigma\to 1$, and $\zeta(2)>1$, we have, if σ is sufficiently near to 1,

$$\frac{\zeta(\sigma)}{\zeta(2\sigma)}<\frac{1}{\sigma-1}.$$

Hence the above last sum is less than

$$\frac{4N}{T(\sigma-1)^{2k}}.$$

Also

$$\left|\frac{1}{\zeta^k(s)}-\sum_{n\leqslant N}\frac{b_k(n)}{n^s}\right| \leqslant \sum_{n>N}\frac{|b_k(n)|}{n^\sigma} < \frac{1}{N^{\frac{1}{2}\sigma-\frac{1}{2}}}\sum_{n>N}\frac{|b_k(n)|}{n^{\frac{1}{2}\sigma+\frac{1}{2}}}$$

$$< \frac{1}{N^{\frac{1}{2}\sigma-\frac{1}{2}}}\left\{\frac{\zeta(\frac{1}{2}\sigma+\frac{1}{2})}{\zeta(\sigma+1)}\right\}^k < \frac{1}{N^{\frac{1}{2}\sigma-\frac{1}{2}}}\left(\frac{2}{\sigma-1}\right)^k$$

for σ sufficiently near to 1. Since for $\sigma > 2$

$$G_k(\sigma) \leqslant \prod_p \left(1+\frac{1}{p^{\frac{1}{2}\sigma}}\right)^{2k} = \prod_p \left(\frac{1-p^{-\sigma}}{1-p^{-\frac{1}{2}\sigma}}\right)^{2k} = \left\{\frac{\zeta(\frac{1}{2}\sigma)}{\zeta(\sigma)}\right\}^{2k},$$

we have similarly

$$G_k(2\sigma)-\sum_{n\leqslant N}\frac{b_k^2(n)}{n^{2\sigma}} = \sum_{n>N}\frac{b_k^2(n)}{n^{2\sigma}} < \frac{1}{N^{\sigma-1}}\sum_{n>N}\frac{b_k^2(n)}{n^{\sigma+1}}$$

$$< \frac{G_k(\sigma+1)}{N^{\sigma-1}} < \frac{1}{N^{\sigma-1}}\left\{\frac{\zeta(\frac{1}{2}\sigma+\frac{1}{2})}{\zeta(\sigma+1)}\right\}^{2k} < \frac{1}{N^{\sigma-1}}\left(\frac{2}{\sigma-1}\right)^{2k}.$$

These two differences are therefore both bounded if

$$N = \left(\frac{2}{\sigma-1}\right)^{2k/(\sigma-1)}$$

With this value of N we have

$$\frac{1}{T}\int_0^T\left|\frac{1}{\zeta^k(s)}+O(1)\right|^2 dt = \frac{1}{T}\int_0^T\left|\sum_{n\leqslant N}\frac{b_k(n)}{n^s}\right|^2 dt$$

$$> G_k(2\sigma)-\frac{4N}{T(\sigma-1)^{2k}}+O(1)$$

$$> \prod_{p\leqslant x}\left\{\frac{1}{3k}\left(1+\frac{1}{p^\sigma}\right)^{2k}\right\}-\frac{4N}{T(\sigma-1)^{2k}}+O(1)$$

by (8.9.17). Now

$$\log\prod_{p\leqslant x}\frac{1+p^{-1}}{1+p^{-\sigma}} = O\{(\sigma-1)\log x\}$$

as in (8.10.4). Hence, as in (8.10.3) and (3.15.3),

$$\prod_{p\leqslant x}\left\{\frac{1}{3k}\left(1+\frac{1}{p^\sigma}\right)^{2k}\right\} > e^{-Ax-Ak(\sigma-1)\log x}\{b+o(1)\}^{2k}\log^{2k}x$$

where $b = 6e^\gamma/\pi^2$.

Choosing x and σ as in the last proof,

$$\frac{N}{(\sigma-1)^{2k}} < \left(\frac{2\log k}{\eta}\right)^{2k\log k/\eta+2k},$$

and we obtain

$$\frac{1}{T}\int_0^T\left|\frac{1}{\zeta^k(s)}+O(1)\right|^2 dt > e^{-A\delta k-A\eta k}\{b+o(1)\}^{2k}\log^{2k}\delta k -$$

$$-\frac{4}{T}\left(\frac{2\log k}{\eta}\right)^{2k\log k/\eta+2k}+O(1).$$

Finally, let

$$T = \left(\frac{2\log k}{\eta}\right)^{2k\log k/\eta+2k}.$$

Then

$$\log\log T = \log k+\log\left(\frac{2\log k}{\eta}+2\right)+\log\log\frac{2\log k}{\eta} < (1+\epsilon)\log k$$

for $k > k_1 = k_1(\epsilon, \eta)$. Hence

$$\frac{1}{T}\int_0^T\left|\frac{1}{\zeta^k(s)}+O(1)\right|^2 dt > e^{-A\delta k-A\eta k}\{b+o(1)\}^{2k}\left(\frac{\log\log T}{1+\epsilon}-\log\frac{1}{\delta}\right)^{2k}+O(1).$$

Let

$$M_{\sigma,T} = \max_{0\leqslant t\leqslant T}\frac{1}{|\zeta(\sigma+it)|}.$$

Since the first term on the right of the above inequality tends to infinity with k (for fixed δ, η, and ϵ) it is clear that $M_{\sigma,T}^k$ tends to infinity. Hence

$$\left|\frac{1}{\zeta^k(s)}+O(1)\right| < 2M_{\sigma,T}^k$$

if k is large enough, and we deduce that

$$4M_{\sigma,T}^{2k} > \tfrac{1}{2}e^{-A\delta k-A\eta k}\{b+o(1)\}^{2k}\left(\frac{\log\log T}{1+\epsilon}-\log\frac{1}{\delta}\right)^{2k}$$

for k large enough. Hence

$$M_{\sigma,T} > \frac{1}{8^{1/2k}}e^{-A\delta-A\eta}\{b+o(1)\}\left(\frac{\log\log T}{1+\epsilon}-\log\frac{1}{\delta}\right).$$

Giving δ, ϵ, and η arbitrarily small values, and then varying T, we obtain

$$\overline{\lim}\frac{M_{\sigma,T}}{\log\log T} \geqslant b.$$

The theorem now follows as in the previous case.

8.12. The above theorems are mainly concerned with the neighbourhood of the line $\sigma = 1$. We now penetrate further into the critical strip, and prove†

THEOREM 8.12. *Let σ be a fixed number in the range $\frac{1}{2} \leqslant \sigma < 1$. Then the inequality*

$$|\zeta(\sigma+it)| > \exp(\log^{\alpha} t)$$

is satisfied for some indefinitely large values of t, provided that

$$\alpha < 1-\sigma.$$

Throughout the proof k is supposed large enough, and δ small enough, for any purpose that may be required. We take $\frac{1}{2} < \sigma < 1$, and the constants $C_1, C_2,\ldots$, and those implied by the symbol O, are independent of k and δ, but may depend on σ, and on ϵ when it occurs. The case $\sigma = \frac{1}{2}$ is deduced finally from the case $\sigma > \frac{1}{2}$.

We first prove some lemmas.

LEMMA α. *Let*

$$\Gamma(s)\zeta^k(s) = \sum_{m=0}^{k-1} \frac{(-1)^m m!\, a_m^{(k)}}{(s-1)^{m+1}} + \ldots$$

in the neighbourhood of $s = 1$. Then

$$|a_m^{(k)}| < e^{C_1 k} \quad (1 \leqslant m \leqslant k).$$

The $a_m^{(k)}$ are the same as those of § 7.13. We have

$$\Gamma(s) = \sum_{n=0}^{\infty} c_n (s-1)^n, \qquad \zeta^k(s) = (s-1)^{-k} \sum_{n=0}^{\infty} e_n^{(k)} (s-1)^n,$$

where $\qquad |c_n| \leqslant C_2^n, \qquad |e_n^{(1)}| \leqslant C_3^n \quad (C_2 > 1,\, C_3 > 1).$

Hence $e_n^{(k)}$ is less than the coefficient of $(s-1)^n$ in

$$\left\{\sum_{n=0}^{\infty} C_3^n (s-1)^n\right\}^k = \{1-C_3(s-1)\}^{-k} = \sum_{n=0}^{\infty} \frac{(k+n-1)!}{(k-1)!\,n!} C_3^n (s-1)^n.$$

Hence

$$m!\,|a_m^{(k)}| = \left|\sum_{n=0}^{k-m-1} c_{k-m-n-1}\, e_n^{(k)}\right| < \sum_{n=0}^{k-m-1} C_2^{k-m-n-1} \frac{(k+n-1)!}{(k-1)!\,n!} C_3^n$$
$$< k C_2^k C_3^k \frac{(2k-2)!}{\{(k-1)!\}^2} < e^{C_1 k}.$$

LEMMA β.

$$\frac{1}{\pi} \int_{-\infty}^{\infty} |\Gamma(\sigma+it)\zeta^k(\sigma+it) e^{(\frac{1}{2}\pi-\delta)t}|^2\, dt$$
$$> \int_1^{\infty} \left|\sum_{n=1}^{\infty} d_k(n) \exp(-inxe^{-i\delta})\right|^2 x^{2\sigma-1}\, dx - \exp(C_4 k \log k).$$

† Titchmarsh (4).

By (7.13.3) the left-hand side is greater than

$$2\int_1^{\infty} |\phi_k(ixe^{-i\delta})|^2 x^{2\sigma-1}\, dx \geqslant \int_1^{\infty} \left|\sum_{n=1}^{\infty} d_k(n)\exp(-inxe^{-i\delta})\right|^2 x^{2\sigma-1}\, dx -$$
$$-2\int_1^{\infty} |R_k(ixe^{-i\delta})|^2 x^{2\sigma-1}\, dx.$$

Since $|\log(ixe^{-i\delta})| \leqslant \log x + \frac{1}{2}\pi$,

$$|R_k(ixe^{-i\delta})| \leqslant \frac{1}{x}\{|a_0^{(k)}| + |a_1^{(k)}|(\log x + \tfrac{1}{2}\pi) + \ldots + a_{k-1}^{(k)}(\log x + \tfrac{1}{2}\pi)^{k-1}\}$$
$$\leqslant \frac{ke^{C_1 k}(\log x + \frac{1}{2}\pi)^{k-1}}{x},$$

and

$$\int_1^{\infty} (\log x + \tfrac{1}{2}\pi)^{2k-2} x^{2\sigma-3}\, dx < \int_1^{\infty} (2\log x)^{2k-2} x^{2\sigma-3}\, dx + \int_1^{\infty} \pi^{2k-2} x^{2\sigma-3}\, dx$$
$$= \frac{\Gamma(2k-1)}{2(1-\sigma)^{2k-1}} + \frac{\pi^{2k-2}}{2-2\sigma}.$$

The result now clearly follows.

LEMMA γ.

$$\int_1^{\infty} \left|\sum_{n=1}^{\infty} d_k(n)\exp(-inxe^{-i\delta})\right|^2 x^{2\sigma-1}\, dx$$
$$> \frac{C_5}{\delta^{2\sigma}} \sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^{2\sigma}} e^{-2n\sin\delta} - C_6 \log\frac{1}{\delta} \sum_{n=1}^{\infty} d_k^2(n) e^{-n\sin\delta}.$$

The left-hand side is equal to

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} d_k(m) d_k(n) \int_1^{\infty} \exp(imxe^{i\delta} - inxe^{-i\delta}) x^{2\sigma-1}\, dx$$
$$= \sum_{m=n} + \sum_{m\neq n} = \Sigma_1 + \Sigma_2.$$

Now $\qquad \displaystyle\int_1^{\infty} e^{-2nx\sin\delta} x^{2\sigma-1}\, dx = (2n\sin\delta)^{-2\sigma} \int_{2n\sin\delta}^{\infty} e^{-y} y^{2\sigma-1}\, dy,$

and for $2n\sin\delta \leqslant 1$

$$\int_{2n\sin\delta}^{\infty} e^{-y} y^{2\sigma-1}\, dy \geqslant \int_1^{\infty} e^{-y} y^{2\sigma-1}\, dy = C_7 > C_7 e^{-2n\sin\delta},$$

while for $2n\sin\delta > 1$

$$\int_{2n\sin\delta}^{\infty} e^{-y} y^{2\sigma-1}\, dy > \int_{2n\sin\delta}^{\infty} e^{-y}\, dy = e^{-2n\sin\delta}.$$

Hence

$$\Sigma_1 = \sum_{n=1}^{\infty} d_k^2(n) \int\limits_1^{\infty} e^{-2nx\sin\delta} x^{2\sigma-1}\,dx > \frac{C_5}{\delta^{2\sigma}} \sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^{2\sigma}} e^{-2n\sin\delta}.$$

Also, using (7.14.4),

$$|\Sigma_2| < C_8 \sum_{m=2}^{\infty} \sum_{n=1}^{m-1} d_k(m)d_k(n)\frac{e^{-m\sin\delta}}{m-n}$$

$$= C_8 \sum_{r=1}^{\infty} \sum_{m=r+1}^{\infty} d_k(m)d_k(m-r)\frac{e^{-m\sin\delta}}{r}$$

$$= C_8 \sum_{r=1}^{\infty} \frac{e^{-\frac{1}{2}r\sin\delta}}{r} \sum_{m=r+1}^{\infty} d_k(m)e^{-\frac{1}{2}m\sin\delta}\, d_k(m-r)e^{-\frac{1}{2}(m-r)\sin\delta}$$

$$\leqslant C_8 \sum_{r=1}^{\infty} \frac{e^{-\frac{1}{2}r\sin\delta}}{r}\left\{\sum_{m=r+1}^{\infty} d_k^2(m)e^{-m\sin\delta} \sum_{m=r+1}^{\infty} d_k^2(m-r)e^{-(m-r)\sin\delta}\right\}^{\frac{1}{2}}$$

$$< C_8 \sum_{r=1}^{\infty} \frac{e^{-\frac{1}{2}r\sin\delta}}{r} \sum_{m=1}^{\infty} d_k^2(m)e^{-m\sin\delta} < C_8 \log\frac{1}{\delta} \sum_{m=1}^{\infty} d_k^2(m)e^{-m\sin\delta}.$$

This proves the lemma.

LEMMA δ. *For* $\sigma > 1$

$$\exp\left\{C_9\left(\frac{k}{\log k}\right)^{2/\sigma}\right\} < F_k(\sigma) < \exp(C_{10}k^{2/\sigma}).$$

It is clear from (8.9.6) that

$$f_k(x) \leqslant (1-\sqrt{x})^{-2k} \quad (0 < x < 1).$$

Also it is easily verified that

$$\left\{\frac{(k+m-1)!}{(k-1)!\,m!}\right\}^2 \leqslant \frac{(k^2+m-1)!}{(k^2-1)!\,m!}.$$

Hence, for $0 < x < 1$,

$$f_k(x) \leqslant \sum_{m=0}^{\infty} \frac{(k^2+m-1)!}{(k^2-1)!\,m!} x^m = (1-x)^{-k^2}.$$

Hence

$$\log F_k(\sigma) = \sum_{p^\sigma \leqslant k^2} \log f_k(p^{-\sigma}) + \sum_{p^\sigma > k^2} \log f_k(p^{-\sigma})$$

$$\leqslant 2k \sum_{p^\sigma \leqslant k^2} \log(1-p^{-\frac{1}{2}\sigma})^{-1} + k^2 \sum_{p^\sigma > k^2} \log(1-p^{-\sigma})^{-1}$$

$$= O\Big(k \sum_{p^\sigma \leqslant k^2} p^{-\frac{1}{2}\sigma}\Big) + O\Big(k^2 \sum_{p^\sigma > k^2} p^{-\sigma}\Big)$$

$$= O\{k(k^{2/\sigma})^{1-\frac{1}{2}\sigma}\} + O\{k^2(k^{2/\sigma})^{1-\sigma}\} = O(k^{2/\sigma}).$$

On the other hand, (8.10.2) gives

$$\log F_k(\sigma) > 2k \sum_{p<x} \log(1-p^{-\frac{1}{2}\sigma})^{-1} - \sum_{p<x} \log 2k$$

$$> 2k \sum_{p<x} p^{-\frac{1}{2}\sigma} - C_{11}\frac{x}{\log x}\log 2k$$

$$> C_{12}\, k\frac{x^{1-\frac{1}{2}\sigma}}{\log x} - C_{11}\frac{x}{\log x}\log 2k.$$

Taking

$$x = \left(\frac{C_{12}}{2C_{11}}\frac{k}{\log k}\right)^{2/\sigma}$$

the other result follows.

Proof of Theorem 8.12 *for* $\frac{1}{2} < \sigma < 1$. It follows from Lemmas β and γ and Stirling's theorem that

$$\int\limits_0^{\infty} |\zeta(\sigma+it)|^{2k} e^{-2\delta t} t^{2\sigma-1}\,dt > \frac{C_{13}}{\delta^{2\sigma}} \sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^{2\sigma}} e^{-2n\sin\delta} -$$

$$-C_{14}\log\frac{1}{\delta}\sum_{n=1}^{\infty} d_k^2(n)e^{-n\sin\delta} - C_{15}e^{C_4 k\log k}.$$

Now, if $0 < \epsilon < 2\sigma-1$,

$$\sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^{2\sigma}} e^{-2n\sin\delta} = F_k(2\sigma) - \sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^{2\sigma}}(1-e^{-2n\sin\delta})$$

$$> F_k(2\sigma) - C_{16}\sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^{2\sigma}}(n\delta)^\epsilon$$

$$= F_k(2\sigma) - C_{16}\,\delta^\epsilon F_k(2\sigma-\epsilon)$$

$$> \exp\left\{C_9\left(\frac{k}{\log k}\right)^{1/\sigma}\right\} - C_{16}\,\delta^\epsilon \exp\{C_{10}\,k^{2/(2\sigma-\epsilon)}\},$$

and

$$\sum_{n=1}^{\infty} d_k^2(n)e^{-n\sin\delta} < C_{17}\sum_{n=1}^{\infty} d_k^2(n)(n\delta)^{\epsilon-2\sigma} = C_{17}\,\delta^{\epsilon-2\sigma}F_k(2\sigma-\epsilon)$$

$$< C_{17}\,\delta^{\epsilon-2\sigma}\exp\{C_{10}\,k^{2/(2\sigma-\epsilon)}\}.$$

Let

$$\delta = \exp\left\{-\frac{C_{10}}{\epsilon}k^{2/(2\sigma-\epsilon)}\right\}.$$

Then

$$\int\limits_0^{\infty} |\zeta(\sigma+it)|^{2k} e^{-2\delta t} t^{2\sigma-1}\,dt > \frac{1}{\delta^{2\sigma}}\left[C_{13}\exp\left\{C_9\left(\frac{k}{\log k}\right)^{1/\sigma}\right\} - C_{13}C_{16} -\right.$$

$$\left. -C_{14}C_{17}\frac{C_{10}}{\epsilon}k^{2/(2\sigma-\epsilon)}\right] - C_{15}e^{C_4 k\log k}$$

$$> \frac{C_{18}}{\delta^{2\sigma}}\exp\left\{C_9\left(\frac{k}{\log k}\right)^{1/\sigma}\right\}.$$

Suppose now that

$$|\zeta(\sigma+it)| \leqslant \exp(\log^\alpha t) \quad (t \geqslant t_0)$$

where $0 < \alpha < 1$. Then

$$\int_0^\infty |\zeta(\sigma+it)|^{2k} e^{-2\delta t} t^{2\sigma-1}\,dt \leqslant C_{19}^{2k} + \int_1^\infty e^{2k\log^\alpha t - 2\delta t} t^{2\sigma-1}\,dt.$$

If $t > k^2/\delta^2$, $k > k_0$, then

$$\frac{k}{\delta} < \sqrt{t} < \frac{1}{2}\frac{t}{\log^\alpha t}.$$

Hence

$$\int_1^\infty e^{2k\log^\alpha t - 2\delta t} t^{2\sigma-1}\,dt \leqslant e^{2k\log^\alpha(k^2/\delta^2)} \int_1^{k^2/\delta^2} e^{-2\delta t} t^{2\sigma-1}\,dt + \int_{k^2/\delta^2}^\infty e^{-\delta t} t^{2\sigma-1}\,dt$$

$$< e^{2k\log^\alpha(k^2/\delta^2)} \frac{C_{20}}{\delta^{2\sigma}}.$$

Hence $$\left(\frac{k}{\log k}\right)^{1/\sigma} = O\left(k\log^\alpha\frac{k}{\delta}\right) = O(k^{1+(2\alpha)/(2\sigma-\epsilon)}).$$

Hence $$\frac{1}{\sigma} \leqslant 1 + \frac{2\alpha}{2\sigma-\epsilon},$$

and since ϵ may be as small as we please

$$\frac{1}{\sigma} \leqslant 1 + \frac{\alpha}{\sigma}, \qquad \alpha \geqslant 1-\sigma.$$

The case $\sigma = \frac{1}{2}$. Suppose that

$$\zeta(\tfrac{1}{2}+it) = O\{\exp(\log^\beta t)\} \quad (0 < \beta < \tfrac{1}{2}).$$

Then the function $$f(s) = \zeta(s)\exp(-\log^\beta s)$$

is bounded on the lines $\sigma = \frac{1}{2}$, $\sigma = 2$, $t > t_0$, and it is $O(t)$ uniformly in this strip. Hence by the Phragmén–Lindelöf theorem $f(s)$ is bounded in the strip, i.e.

$$\zeta(\sigma+it) = O\{\exp(\log^\beta t)\}$$

for $\frac{1}{2} < \sigma < 2$. Since this is not true for $\frac{1}{2} < \sigma < 1-\beta$, it follows that $\beta \geqslant \frac{1}{2}$.

NOTES FOR CHAPTER 8

8.13. Levinson [1] has sharpened Theorems 8.9(A) and 8.9(B) to show that the inequalities

$$|\zeta(1+it)| \geqslant e^\gamma \log\log t + O(1)$$

and

$$\frac{1}{|\zeta(1+it)|} \geqslant \frac{6e^\gamma}{\pi^2}(\log\log t - \log\log\log t) + O(1)$$

each hold for arbitrarily large t. Theorem 8.12 has also been improved, by Montgomery [3]. He showed that for any σ in the range $\frac{1}{2} < \sigma < 1$, and for any real ϑ, there are arbitrarily large t such that

$$\mathbf{R}\{e^{i\vartheta}\log\zeta(\sigma+it)\} \geqslant \tfrac{1}{20}(\sigma-\tfrac{1}{2})^{-1}(\log t)^{1-\sigma}(\log\log t)^{-\sigma}.$$

Here $\log\zeta(s)$ is, as usual, defined by continuous variation along lines parallel to the real axis, using the Dirichlet series (1.1.9) for $\sigma > 1$. It follows in particular that

$$\zeta(\sigma+it) = \Omega\left\{\exp\left(\frac{\frac{1}{20}}{\sigma-\frac{1}{2}}\frac{(\log t)^{1-\sigma}}{(\log\log t)^\sigma}\right)\right\} \quad (\tfrac{1}{2} < \sigma < 1),$$

and the same for $\zeta(\sigma+it)^{-1}$. For $\sigma = \frac{1}{2}$ the best result is due to Balasubramanian and Ramachandra [2], who showed that

$$\max_{T \leqslant t \leqslant T+H} |\zeta(\tfrac{1}{2}+it)| \geqslant \exp\left(\tfrac{3}{4}\frac{(\log H)^{\frac{1}{2}}}{(\log\log H)^{\frac{1}{2}}}\right)$$

if $(\log T)^\delta \leqslant H \leqslant T$ and $T \geqslant T(\delta)$, where δ is any positive constant. Their method is akin to that of §8.12, in that it depends on a lower bound for a mean value of $|\zeta(\frac{1}{2}+it)|^{2k}$, uniform in k. By constrast the method of Montgomery [3] uses the formula

$$\frac{4}{\pi}\int_{-(\log t)^2}^{(\log t)^2} e^{-i\vartheta}\log\zeta(\sigma+it+iy)\left(\frac{\sin\frac{1}{2}y}{y}\right)^2\{1+\cos(\vartheta+y\log x)\}dy$$

$$= \sum_{|\log n/x| \leqslant \frac{1}{2}} \frac{\Lambda(n)}{\log n} n^{-\sigma-it}\left(\tfrac{1}{2} - \left|\log\frac{n}{x}\right|\right) + O\{x(\log t)^{-2}\}. \quad (8.13.1)$$

This is valid for any real x and ϑ, providing that $\zeta(s) \neq 0$ for $\mathbf{R}(s) \geqslant \sigma$ and $|\mathbf{I}(s) - t| \leqslant 2(\log t)^2$. After choosing x suitably one may use the extended version of Dirichlet's theorem given in §8.2 to show that the real part of the sum on the right of (8.13.1) is large at points $t_1 < \ldots < t_N \leqslant T$, spaced at least $4(\log T)^2$ apart. One can arrange that N exceeds $N(\sigma, T)$, whence at least one such t_n will satisfy the condition that $\zeta(s) \neq 0$ in the corresponding rectangle.

IX

THE GENERAL DISTRIBUTION OF THE ZEROS

9.1. In § 2.12 we deduced from the general theory of integral functions that $\zeta(s)$ *has an infinity of complex zeros.* This may be proved directly as follows.

We have

$$\frac{1}{2^2}+\frac{1}{3^2}+\ldots<\frac{1}{2^2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots=\frac{1}{4}+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots=\frac{3}{4}.$$

Hence for $\sigma \geqslant 2$

$$|\zeta(s)| \leqslant 1+\frac{1}{2^\sigma}+\frac{1}{3^\sigma}+\ldots \leqslant 1+\frac{1}{2^2}+\ldots<\frac{7}{4}, \tag{9.1.1}$$

and
$$|\zeta(s)| \geqslant 1-\frac{1}{2^\sigma}-\ldots \geqslant 1-\frac{1}{2^2}-\ldots>\frac{1}{4}. \tag{9.1.2}$$

Also
$$\mathbf{R}\{\zeta(s)\}=1+\frac{\cos(t\log 2)}{2^\sigma}+\ldots \geqslant 1-\frac{1}{2^2}-\ldots>\frac{1}{4}. \tag{9.1.3}$$

Hence for $\sigma \geqslant 2$ we may write

$$\log\zeta(s)=\log|\zeta(s)|+i\arg\zeta(s),$$

where $\arg\zeta(s)$ is that value of $\arctan\{\mathbf{I}\zeta(s)/\mathbf{R}\zeta(s)\}$ which lies between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. It is clear that

$$|\log\zeta(s)|<A \quad (\sigma \geqslant 2). \tag{9.1.4}$$

For $\sigma<2$, $t \neq 0$, we define $\log\zeta(s)$ as the analytic continuation of the above function along the straight line $(\sigma+it, 2+it)$, provided that $\zeta(s) \neq 0$ on this segment of line.

Now consider a system of four concentric circles C_1, C_2, C_3, C_4, with centre $3+iT$ and radii 1, 4, 5, and 6 respectively. Suppose that $\zeta(s) \neq 0$ in or on C_4. Then $\log\zeta(s)$, defined as above, is regular in C_4. Let M_1, M_2, M_3 be its maximum modulus on C_1, C_2, and C_3 respectively.

Since $\zeta(s)=O(t^A)$, $\mathbf{R}\{\log\zeta(s)\}<A\log T$ in C_4, and the Borel–Carathéodory theorem gives

$$M_3 \leqslant \frac{2.5}{6-5}A\log T+\frac{6+5}{6-5}\log|\zeta(3+iT)|<A\log T.$$

Also $M_1<A$, by (9.1.4). Hence Hadamard's three-circles theorem, applied to the circles C_1, C_2, C_3, gives

$$M_2 \leqslant M_1^\alpha M_3^\beta<A\log^\beta T,$$

where
$$1-\alpha=\beta=\log 4/\log 5<1.$$

Hence
$$\zeta(-1+iT)=O\{\exp(\log^\beta T)\}=O(T^\epsilon).$$

But by (9.1.2), and the functional equation (2.1.1) with $\sigma=2$,

$$|\zeta(-1+iT)|>AT^{\frac{3}{2}}.$$

We have thus obtained a contradiction. Hence every such circle C_4 contains at least one zero of $\zeta(s)$, and so there are an infinity of zeros. The argument also shows that the gaps between the ordinates of successive zeros are bounded.

9.2. The function $N(T)$. Let $T>0$, and let $N(T)$ denote the number of zeros of the function $\zeta(s)$ in the region $0 \leqslant \sigma \leqslant 1$, $0<t \leqslant T$. The distribution of the ordinates of the zeros can then be studied by means of formulae involving $N(T)$.

The most easily proved result is

THEOREM 9.2. *As* $T \to \infty$
$$N(T+1)-N(T)=O(\log T). \tag{9.2.1}$$

For it is easily seen that

$$N(T+1)-N(T) \leqslant n(\surd 5),$$

where $n(r)$ is the number of zeros of $\zeta(s)$ in the circle with centre $2+iT$ and radius r. Now, by Jensen's theorem,

$$\int_0^3 \frac{n(r)}{r}\,dr=\frac{1}{2\pi}\int_0^{2\pi}\log|\zeta(2+iT+3e^{i\theta})|\,d\theta-\log|\zeta(2+iT)|.$$

Since $|\zeta(s)|<t^A$ for $-1 \leqslant \sigma \leqslant 5$, we have

$$\log|\zeta(2+iT+3e^{i\theta})|<A\log T.$$

Hence
$$\int_0^3 \frac{n(r)}{r}\,dr<A\log T+A<A\log T.$$

Since
$$\int_0^3 \frac{n(r)}{r}\,dr \geqslant \int_{\surd 5}^3 \frac{n(r)}{r}\,dr \geqslant n(\surd 5)\int_{\surd 5}^3 \frac{dr}{r}=An(\surd 5),$$

the result (9.2.1) follows.

Naturally it also follows that

$$N(T+h)-N(T)=O(\log T)$$

for any fixed value of h. In particular, the multiplicity of a multiple zero of $\zeta(s)$ in the region considered is at most $O(\log T)$.

9.3. The closer study of $N(T)$ depends on the following theorem.† If T is not the ordinate of a zero, let $S(T)$ denote the value of

$$\pi^{-1}\arg\zeta(\tfrac{1}{2}+iT)$$

obtained by continuous variation along the straight lines joining 2, $2+iT$, $\frac{1}{2}+iT$, starting with the value 0. If T is the ordinate of a zero, let $S(T)=S(T+0)$. Let

$$L(T)=\frac{1}{2\pi}T\log T-\frac{1+\log 2\pi}{2\pi}T+\frac{7}{8}. \tag{9.3.1}$$

THEOREM 9.3. *As* $T\to\infty$

$$N(T)=L(T)+S(T)+O(1/T). \tag{9.3.2}$$

The number of zeros of the function $\Xi(z)$ (see § 2.1) in the rectangle with vertices at $z=\pm T\pm\frac{3}{2}i$ is $2N(T)$, so that

$$2N(T)=\frac{1}{2\pi i}\int\frac{\Xi'(z)}{\Xi(z)}\,dz$$

taken round the rectangle. Since $\Xi(z)$ is even and real for real z, this is equal to

$$\frac{2}{\pi i}\left(\int\limits_{T}^{T+\frac{3}{2}i}+\int\limits_{T+\frac{3}{2}i}^{\frac{3}{2}i}\right)\frac{\Xi'(z)}{\Xi(z)}\,dz=\frac{2}{\pi i}\left(\int\limits_{2}^{2+iT}+\int\limits_{2+iT}^{\frac{1}{2}+iT}\right)\frac{\xi'(s)}{\xi(s)}\,ds$$

$$=\frac{2}{\pi}\Delta\arg\xi(s),$$

where Δ denotes the variation from 2 to $2+iT$, and thence to $\frac{1}{2}+iT$, along straight lines. Recalling that

$$\xi(s)=\tfrac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s),$$

we obtain

$$\pi N(T)=\Delta\arg s(s-1)+\Delta\arg\pi^{-\frac{1}{2}s}+\Delta\arg\Gamma(\tfrac{1}{2}s)+\Delta\arg\zeta(s).$$

Now

$$\Delta\arg s(s-1)=\arg(-\tfrac{1}{4}-T^2)=\pi,$$

$$\Delta\arg\pi^{-\frac{1}{2}s}=\Delta\arg e^{-\frac{1}{2}s\log\pi}=-\tfrac{1}{2}T\log\pi,$$

and by (4.12.1)

$$\begin{aligned}\Delta\arg\Gamma(\tfrac{1}{2}s)&=\mathbf{I}\log\Gamma(\tfrac{1}{4}+\tfrac{1}{2}iT)\\&=\mathbf{I}\{(-\tfrac{1}{4}+\tfrac{1}{2}iT)\log(\tfrac{1}{2}iT)-\tfrac{1}{2}iT+O(1/T)\}\\&=\tfrac{1}{2}T\log\tfrac{1}{2}T-\tfrac{1}{8}\pi-\tfrac{1}{2}T+O(1/T).\end{aligned}$$

Adding these results, we obtain the theorem, provided that T is not the ordinate of a zero. If T is the ordinate of a zero, the result follows from

† Backlund (2), (3).

the definitions and what has already been proved, the term $O(1/T)$ being continuous.

The problem of the behaviour of $N(T)$ is thus reduced to that of $S(T)$.

9.4. We shall now prove the following lemma.

LEMMA. *Let* $0\leqslant\alpha<\beta<2$. *Let* $f(s)$ *be an analytic function, real for real* s, *regular for* $\sigma\geqslant\alpha$ *except at* $s=1$; *let*

$$|\mathbf{R}f(2+it)|\geqslant m>0$$

and

$$|f(\sigma'+it')|\leqslant M_{\sigma,t}\quad(\sigma'\geqslant\sigma,\ 1\leqslant t'\leqslant t).$$

Then if T *is not the ordinate of a zero of* $f(s)$

$$|\arg f(\sigma+iT)|\leqslant\frac{\pi}{\log\{(2-\alpha)/(2-\beta)\}}\left(\log M_{\alpha,T+2}+\log\frac{1}{m}\right)+\frac{3\pi}{2} \tag{9.4.1}$$

for $\sigma\geqslant\beta$.

Since $\arg f(2)=0$, and

$$\arg f(s)=\arctan\left\{\frac{\mathbf{I}f(s)}{\mathbf{R}f(s)}\right\},$$

where $\mathbf{R}f(s)$ does not vanish on $\sigma=2$, we have

$$|\arg f(2+iT)|<\tfrac{1}{2}\pi.$$

Now if $\mathbf{R}f(s)$ vanishes q times between $2+iT$ and $\beta+iT$, this interval is divided into $q+1$ parts, throughout each of which $\mathbf{R}\{f(s)\}\geqslant 0$ or $\mathbf{R}\{f(s)\}\leqslant 0$. Hence in each part the variation of $\arg f(s)$ does not exceed π. Hence

$$|\arg f(s)|\leqslant(q+\tfrac{3}{2})\pi\quad(\sigma\geqslant\beta).$$

Now q is the number of zeros of the function

$$g(z)=\tfrac{1}{2}\{f(z+iT)+f(z-iT)\}$$

for $\mathbf{I}(z)=0$, $\beta\leqslant\mathbf{R}(z)\leqslant 2$; hence $q\leqslant n(2-\beta)$, where $n(r)$ denotes the number of zeros of $g(z)$ for $|z-2|\leqslant r$. Also

$$\int\limits_{0}^{2-\alpha}\frac{n(r)}{r}\,dr\geqslant\int\limits_{2-\beta}^{2-\alpha}\frac{n(r)}{r}\,dr\geqslant n(2-\beta)\log\frac{2-\alpha}{2-\beta},$$

and by Jensen's theorem

$$\int\limits_{0}^{2-\alpha}\frac{n(r)}{r}\,dr=\frac{1}{2\pi}\int\limits_{0}^{2\pi}\log|g\{2+(2-\alpha)e^{i\theta}\}|\,d\theta-\log|g(2)|$$

$$\leqslant\log M_{\alpha,T+2}+\log 1/m.$$

This proves the lemma.

We deduce

Theorem 9.4. *As* $T \to \infty$

$$S(T) = O(\log T), \tag{9.4.2}$$

i.e.
$$N(T) = \frac{1}{2\pi} T \log T - \frac{1+\log 2\pi}{2\pi} T + O(\log T). \tag{9.4.3}$$

We apply the lemma with $f(s) = \zeta(s)$, $\alpha = 0$, $\beta = \frac{1}{2}$, and (9.4.2) follows, since $\zeta(s) = O(t^A)$. Then (9.4.3) follows from (9.3.2).

Theorem 9.4 has a number of interesting consequences. It gives another proof of Theorem 9.2, since $(0 < \theta < 1)$

$$L(T+1) - L(T) = L'(T+\theta) = O(\log T).$$

We can also prove the following result.

If the zeros $\beta + i\gamma$ *of* $\zeta(s)$ *with* $\gamma > 0$ *are arranged in a sequence* $\rho_n = \beta_n + i\gamma_n$ *so that* $\gamma_{n+1} \geqslant \gamma_n$, *then as* $n \to \infty$

$$|\rho_n| \sim \gamma_n \sim \frac{2\pi n}{\log n}. \tag{9.4.4}$$

We have
$$N(T) \sim \frac{1}{2\pi} T \log T.$$

Hence
$$2\pi N(\gamma_n \pm 1) \sim (\gamma_n \pm 1)\log(\gamma_n \pm 1) \sim \gamma_n \log \gamma_n.$$

Also
$$N(\gamma_n - 1) \leqslant n \leqslant N(\gamma_n + 1).$$

Hence
$$2\pi n \sim \gamma_n \log \gamma_n.$$

Hence
$$\log n \sim \log \gamma_n,$$

and so
$$\gamma_n \sim \frac{2\pi n}{\log n}.$$

Also $|\rho_n| \sim \gamma_n$, since $\beta_n = O(1)$.

We can also deduce the result of § 9.1, that the gaps between the ordinates of successive zeros are bounded. For if $|S(t)| \leqslant C \log t$ $(t \geqslant 2)$,

$$N(T+H) - N(T) = \frac{1}{2\pi} \int_T^{T+H} \log \frac{t}{2\pi}\, dt + S(T+H) - S(T) + O\left(\frac{1}{T}\right)$$
$$\geqslant \frac{H}{2\pi} \log \frac{T}{2\pi} - C\{\log(T+H) + \log T\} + O\left(\frac{1}{T}\right),$$

which is ultimately positive if H is a constant greater than $4\pi C$.

The behaviour of the function $S(T)$ appears to be very complicated. It must have a discontinuity k where T passes through the ordinate of a zero of $\zeta(s)$ of order k (since the term $O(1/T)$ in the above theorem is in fact continuous). Between the zeros, $N(T)$ is constant, so that the

variation of $S(T)$ must just neutralize that of the other terms. In the formula (9.3.1), the term $\frac{7}{8}$ is presumably overwhelmed by the variations of $S(T)$. On the other hand, in the integrated formula

$$\int_0^T N(t)\, dt = \int_0^T L(t)\, dt + \int_0^T S(t)\, dt + O(\log T)$$

the term in $S(T)$ certainly plays a much smaller part, since, as we shall presently prove, the integral of $S(t)$ over $(0, T)$ is still only $O(\log T)$. Presumably this is due to frequent variations in the sign of $S(t)$. Actually we shall show that $S(t)$ changes sign an infinity of times.

9.5. A problem of analytic continuation.

The above theorems on the zeros of $\zeta(s)$ lead to the solution of a curious subsidiary problem of analytic continuation.† Consider the function

$$P(s) = \sum_p \frac{1}{p^s}. \tag{9.5.1}$$

This is an analytic function of s, regular for $\sigma > 1$. Now by (1.6.1)

$$P(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \zeta(ns). \tag{9.5.2}$$

As $n \to \infty$, $\log \zeta(ns) \sim 2^{-ns}$. Hence the right-hand side represents an analytic function of s, regular for $\sigma > 0$, except at the singularities of individual terms. These are branch-points arising from the poles and zeros of the functions $\zeta(ns)$; there are an infinity of such points, but they have no limit-point in the region $\sigma > 0$. Hence $P(s)$ is regular for $\sigma > 0$, except at certain branch-points.

Similarly, the function

$$Q(s) = -P'(s) = -\sum_{n=1}^{\infty} \mu(n) \frac{\zeta'(ns)}{\zeta(ns)} \tag{9.5.3}$$

is regular for $\sigma > 0$, except at certain simple poles.

We shall now prove that *the line* $\sigma = 0$ *is a natural boundary of the functions* $P(s)$ *and* $Q(s)$.

We shall in fact prove that every point of $\sigma = 0$ is a limit-point of poles of $Q(s)$. By symmetry, it is sufficient to consider the upper half-line. Thus it is sufficient to prove that for every $u > 0$, $\delta > 0$, the square

$$0 < \sigma < \delta, \qquad u < t \leqslant u + \delta \tag{9.5.4}$$

contains at least one pole of $Q(s)$.

† Landau and Walfisz (1).

As $p \to \infty$ through primes,

$$N\{p(u+\delta)\} \sim \frac{1}{2\pi}(u+\delta)p\log p, \qquad N(pu) \sim \frac{1}{2\pi}up\log p,$$

by Theorem 9.4. Hence for all $p \geqslant p_0(\delta, u)$

$$N\{p(u+\delta)\}-N(pu) > 0. \tag{9.5.5}$$

Also, by Theorem 9.2, the multiplicity $v(\rho)$ of each zero $\rho = \beta+i\gamma$ with ordinate $\gamma \geqslant 2$ is less than $A\log\gamma$, where A is an absolute constant.

Now choose $p = p(\delta, u)$ satisfying the conditions

$$p > \frac{1}{\delta}, \qquad p \geqslant \frac{2}{u}, \qquad p \geqslant p_0(\delta, u), \qquad p > A\log\{p(u+\delta)\}.$$

There is then, by (9.5.5), a zero ρ of $\zeta(s)$ in the rectangle

$$\tfrac{1}{2} \leqslant \sigma < 1, \qquad pu < t \leqslant p(u+\delta). \tag{9.5.6}$$

Since $\gamma > pu \geqslant 2$, its multiplicity $v(\rho)$ satisfies

$$v(\rho) < A\log\gamma \leqslant A\log\{p(u+\delta)\} < p,$$

and so is not divisible by p.

The point ρ/p belongs to the square (9.5.4). We shall show that this point is a pole of $Q(s)$. Let m run through the positive integers (finite in number) for which $\zeta(m\rho/p) = 0$. Then we have to prove that

$$\sum \frac{\mu(m)}{m} v\left(\frac{m\rho}{p}\right) \neq 0. \tag{9.5.7}$$

The term of this sum corresponding to $m = p$ is $-v(\rho)/p$. No other m occurring in the sum is divisible by p, since for $m \geqslant 2p$

$$\mathbf{R}\left(\frac{m\rho}{p}\right) = \frac{m\beta}{p} \geqslant \frac{2p \cdot \frac{1}{2}}{p} = 1.$$

Hence
$$\sum \frac{\mu(m)}{m} v\left(\frac{m\rho}{p}\right) = \frac{a}{b} - \frac{v(\rho)}{p},$$

where a and b are integers, and p is not a factor of b. Since p is also not a factor of $v(\rho)$, $ap \neq bv(\rho)$, and (9.5.7) follows.

There are various other functions with similar properties. For example,† let

$$f_{l,k}(s) = \sum_{n=1}^{\infty} \frac{\{d_k(n)\}^l}{n^s},$$

where k and l are positive integers, $k \geqslant 2$. By (1.2.2) and (1.2.10), $f_{l,k}(s)$ is a meromorphic function of s if $l = 1$, or if $l = 2$ and $k = 2$. *For all other values of l and k, $f_{l,k}(s)$ has $\sigma = 0$ as a natural boundary, and it has no singularities other than poles in the half-plane $\sigma > 0$.*

† Estermann (1).

9.6. An approximate formula for $\zeta'(s)/\zeta(s)$. The following approximate formula for $\zeta'(s)/\zeta(s)$ in terms of the zeros near to s is often useful.

THEOREM 9.6 (A). *If $\rho = \beta+i\gamma$ runs through zeros of $\zeta(s)$,*

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{|t-\gamma|\leqslant 1} \frac{1}{s-\rho} + O(\log t), \tag{9.6.1}$$

uniformly for $-1 \leqslant \sigma \leqslant 2$.

Take $f(s) = \zeta(s)$, $s_0 = 2+iT$, $r = 12$ in Lemma α of § 3.9. Then $M = A\log T$, and we obtain

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{|\rho-s_0|\leqslant 6} \frac{1}{s-\rho} + O(\log T) \tag{9.6.2}$$

for $|s-s_0| \leqslant 3$, and so in particular for $-1 \leqslant \sigma \leqslant 2$, $t = T$. Replacing T by t in the particular case, we obtain (9.6.2) with error $O(\log t)$, and $-1 \leqslant \sigma \leqslant 2$. Finally any term occurring in (9.6.2) but not in (9.6.1) is bounded, and the number of such terms does not exceed

$$N(t+6)-N(t-6) = O(\log t)$$

by Theorem 9.2. This proves (9.6.1).

Another proof depends on (2.12.7), which, by a known property of the Γ-function, gives

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{\rho} \left(\frac{1}{s-\rho}+\frac{1}{\rho}\right) + O(\log t).$$

Replacing s by $2+it$ and subtracting,

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{\rho} \left(\frac{1}{s-\rho}-\frac{1}{2+it-\rho}\right) + O(\log t),$$

since $\zeta'(2+it)/\zeta(2+it) = O(1)$.

Now
$$\sum_{|t-\gamma|\leqslant 1} \frac{1}{2+it-\rho} = \sum_{|t-\gamma|\leqslant 1} O(1) = O(\log t)$$

by Theorem 9.2. Also

$$\sum_{t+n<\gamma\leqslant t+n+1} \left(\frac{1}{s-\rho}-\frac{1}{2+it-\rho}\right) = \sum_{t+n<\gamma\leqslant t+n+1} \frac{2-\sigma}{(s-\rho)(2+it-\rho)}$$

$$= \sum_{t+n<\gamma\leqslant t+n+1} O\left\{\frac{1}{(\gamma-t)^2}\right\} = \sum_{t+n<\gamma\leqslant t+n+1} O\left(\frac{1}{n^2}\right) = O\left\{\frac{\log(t+n)}{n^2}\right\},$$

again by Theorem 9.2. Since

$$\sum_{n=1}^{\infty} \frac{\log(t+n)}{n^2} < \sum_{n\leqslant t} \frac{\log 2t}{n^2} + \sum_{n>t} \frac{\log 2n}{n^2} = O(\log t),$$

it follows that $$\sum_{\gamma>t+1}\left(\frac{1}{s-\rho}-\frac{1}{2+it-\rho}\right)=O(\log t).$$

Similarly $$\sum_{\gamma<t-1}\left(\frac{1}{s-\rho}-\frac{1}{2+it-\rho}\right)=O(\log t)$$

and the result follows again.

The corresponding formula for $\log\zeta(s)$ is given by

THEOREM 9.6 (B). *We have*

$$\log\zeta(s)=\sum_{|t-\gamma|\leqslant 1}\log(s-\rho)+O(\log t) \tag{9.6.3}$$

uniformly for $-1\leqslant\sigma\leqslant 2$, *where* $\log\zeta(s)$ *has its usual meaning, and* $-\pi<\mathbf{I}\log(s-\rho)\leqslant\pi$.

Integrating (9.6.1) from s to $2+it$, and supposing that t is not equal to the ordinate of any zero, we obtain

$$\log\zeta(s)-\log\zeta(2+it)=\sum_{|t-\gamma|\leqslant 1}\{\log(s-\rho)-\log(2+it-\rho)\}+O(\log t).$$

Now $\log\zeta(2+it)$ is bounded; also $\log(2+it-\rho)$ is bounded, and there are $O(\log t)$ such terms. Their sum is therefore $O(\log t)$. The result therefore follows for such values of t, and then by continuity for all values of s in the strip other than the zeros.

9.7. As an application of Theorem 9.6 (B) we shall prove the following theorem on the minimum value of $\zeta(s)$ in certain parts of the critical strip. We know from Theorem 8.12 that $|\zeta(s)|$ is sometimes large in the critical strip, but we can prove little about the distribution of the values of t for which it is large. The following result† states a much weaker inequality, but states it for many more values of t.

THEOREM 9.7. *There is a constant A such that each interval $(T, T+1)$ contains a value of t for which*

$$|\zeta(s)|>t^{-A}\quad(-1\leqslant\sigma\leqslant 2). \tag{9.7.1}$$

Further, if H is any number greater than unity, then

$$|\zeta(s)|>T^{-AH} \tag{9.7.2}$$

for $-1\leqslant\sigma\leqslant 2$, $T\leqslant t\leqslant T+1$, except possibly for a set of values of t of measure $1/H$.

Taking real parts in (9.6.3),

$$\begin{aligned}\log|\zeta(s)|&=\sum_{|t-\gamma|\leqslant 1}\log|s-\rho|+O(\log t)\\&\geqslant\sum_{|t-\gamma|\leqslant 1}\log|t-\gamma|+O(\log t).\end{aligned} \tag{9.7.3}$$

† Valiron (1), Landau (8), (18), Hoheisel (3).

Now

$$\begin{aligned}\int_T^{T+1}\sum_{|t-\gamma|\leqslant 1}\log|t-\gamma|\,dt&=\sum_{T-1\leqslant\gamma\leqslant T+2}\int_{\max(\gamma-1,T)}^{\min(\gamma+1,T+1)}\log|t-\gamma|\,dt\\&\geqslant\sum_{T-1\leqslant\gamma\leqslant T+2}\int_{\gamma-1}^{\gamma+1}\log|t-\gamma|\,dt\\&=\sum_{T-1\leqslant\gamma\leqslant T+2}(-2)>-A\log T.\end{aligned}$$

Hence $$\sum_{|t-\gamma|\leqslant 1}\log|t-\gamma|>-A\log T$$

for some t in $(T, T+1)$.

Hence $\log|\zeta(s)|>-A\log T$ for some t in $(T, T+1)$ and all σ in $-1\leqslant\sigma\leqslant 2$; and

$$\log|\zeta(s)|>-AH\log T$$

except in a set of measure $1/H$. This proves the theorem.

The exceptional values of t are, of course, those in the neighbourhood of ordinates of zeros of $\zeta(s)$.

9.8. Application to a formula of Ramanujan.† Let a and b be positive numbers such that $ab=\pi$, and consider the integral

$$\frac{1}{2\pi i}\int a^{-2s}\frac{\Gamma(s)}{\zeta(1-2s)}\,ds=\frac{1}{2\pi i}\int\frac{b^{2s}}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{2}-s)}{\zeta(2s)}\,ds$$

taken round the rectangle $(1\pm iT, -\frac{1}{2}\pm iT)$. The two forms are equivalent on account of the functional equation.

Let $T\to\infty$ through values such that $|T-\gamma|>\exp(-A_1\gamma/\log\gamma)$ for every ordinate γ of a zero of $\zeta(s)$. Then by (9.7.3)

$$\log|\zeta(\sigma+iT)|\geqslant-\sum_{|T-\gamma|\leqslant 1}A_1\gamma/\log\gamma+O(\log T)>-A_2T$$

where $A_2<\frac{1}{4}\pi$ if A_1 is small enough, and $T>T_0$. It now follows from the asymptotic formula for the Γ-function that the integrals along the horizontal sides of the contour tend to zero as $T\to\infty$ through the above values. Hence by the theorem of residues‡

$$\frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}a^{-2s}\frac{\Gamma(s)}{\zeta(1-2s)}\,ds-\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\frac{b^{2s}}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{2}-s)}{\zeta(2s)}\,ds=-\frac{1}{2\sqrt{\pi}}\sum_{\rho}b^{\rho}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}\rho)}{\zeta'(\rho)}.$$

† Hardy and Littlewood (2), 156–9.

‡ In forming the series of residues we have supposed for simplicity that the zeros of $\zeta(s)$ are all simple.

The first term on the left is equal to

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\frac{1}{2\pi i}\int\limits_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}\left(\frac{n}{a}\right)^{2s}\Gamma(s)\,ds = -\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\{1-e^{-(a/n)^2}\}$$
$$= \sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-(a/n)^2}.$$

Evaluating the other integral in the same way, and multiplying through by $\sqrt{a}$, we obtain Ramanujan's result

$$\sqrt{a}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-(a/n)^2}-\sqrt{b}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-(b/n)^2} = -\frac{1}{2\sqrt{b}}\sum b^{\rho}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}\rho)}{\zeta'(\rho)}. \tag{9.8.1}$$

We have, of course, not proved that the series on the right is convergent in the ordinary sense. We have merely proved that it is convergent if the terms are bracketed in such a way that two terms for which

$$|\gamma-\gamma'| < \exp(-A_1\gamma/\log\gamma)+\exp(-A_1\gamma'/\log\gamma')$$

are included in the same bracket. Of course the zeros are, on the average, much farther apart than this, and it is quite possible that the series may converge without any bracketing. But we are unable to prove this, even on the Riemann hypothesis.

9.9. We next prove a general formula concerning the zeros of an analytic function in a rectangle.† Suppose that $\phi(s)$ is meromorphic in and upon the boundary of a rectangle bounded by the lines $t = 0$, $t = T$, $\sigma = \alpha$, $\sigma = \beta$ $(\beta > \alpha)$, and regular and not zero on $\sigma = \beta$. The function $\log\phi(s)$ is regular in the neighbourhood of $\sigma = \beta$, and here, starting with any one value of the logarithm, we define $F(s) = \log\phi(s)$. For other points s of the rectangle, we define $F(s)$ to be the value obtained from $\log\phi(\beta+it)$ by continuous variation along $t =$ constant from $\beta+it$ to $\sigma+it$, provided that the path does not cross a zero or pole of $\phi(s)$; if it does, we put

$$F(s) = \lim_{\epsilon\to+0} F(\sigma+it+i\epsilon).$$

Let $\nu(\sigma', T)$ denote the excess of the number of zeros over the number of poles in the part of the rectangle for which $\sigma > \sigma'$, including zeros or poles on $t = T$, but not those on $t = 0$.

Then
$$\int F(s)\,ds = -2\pi i\int\limits_{\alpha}^{\beta}\nu(\sigma,T)\,d\sigma, \tag{9.9.1}$$

the integral on the left being taken round the rectangle in the positive direction.

† Littlewood (4).

We may suppose $t = 0$ and $t = T$ to be free from zeros and poles of $\phi(s)$; it is easily verified that our conventions then ensure the truth of the theorem in the general case.

We have

$$\int F(s)\,ds = \int\limits_{\alpha}^{\beta}F(\sigma)\,d\sigma-\int\limits_{\alpha}^{\beta}F(\sigma+iT)\,d\sigma+\int\limits_{0}^{T}\{F(\beta+it)-F(\alpha+it)\}\,i\,dt. \tag{9.9.2}$$

The last term is equal to

$$\int\limits_{0}^{T}i\,dt\int\limits_{\alpha}^{\beta}\frac{\phi'(\sigma+it)}{\phi(\sigma+it)}\,d\sigma = \int\limits_{\alpha}^{\beta}d\sigma\int\limits_{\sigma}^{\sigma+iT}\frac{\phi'(s)}{\phi(s)}\,ds,$$

and by the theorem of residues

$$\int\limits_{\sigma}^{\sigma+iT}\frac{\phi'(s)}{\phi(s)}\,ds = \left(\int\limits_{\sigma}^{\beta}+\int\limits_{\beta}^{\beta+iT}-\int\limits_{\sigma+iT}^{\beta+iT}\right)\frac{\phi'(s)}{\phi(s)}\,ds-2\pi i\nu(\sigma,T)$$
$$= F(\sigma+iT)-F(\sigma)-2\pi i\nu(\sigma,T).$$

Substituting this in (9.9.2), we obtain (9.9.1).

We deduce

THEOREM 9.9. *If*
$$S_1(T) = \int\limits_{0}^{T}S(t)\,dt,$$
then
$$S_1(T) = \frac{1}{\pi}\int\limits_{\frac{1}{2}}^{2}\log|\zeta(\sigma+iT)|\,d\sigma+O(1). \tag{9.9.3}$$

Take $\phi(s) = \zeta(s)$, $\alpha = \frac{1}{2}$, in the above formula, and take the real part. We obtain

$$\int\limits_{\frac{1}{2}}^{\beta}\log|\zeta(\sigma)|\,d\sigma-\int\limits_{0}^{T}\arg\zeta(\beta+it)\,dt-\int\limits_{\frac{1}{2}}^{\beta}\log|\zeta(\sigma+iT)|\,d\sigma+$$
$$+\int\limits_{0}^{T}\arg\zeta(\tfrac{1}{2}+it)\,dt = 0, \tag{9.9.4}$$

the term in $\nu(\sigma, T)$, being purely imaginary, disappearing. Now make $\beta\to\infty$. We have

$$\log\zeta(s) = \log\left(1+\frac{1}{2^s}+\ldots\right) = O(2^{-\sigma})$$

as $\sigma\to\infty$, uniformly with respect to t. Hence $\arg\zeta(s) = O(2^{-\sigma})$, so that the second integral tends to 0 as $\beta\to\infty$. Also the first integral is a constant, and

$$\int\limits_{2}^{\beta}\log|\zeta(\sigma+iT)|\,d\sigma = \int\limits_{2}^{\beta}O(2^{-\sigma})\,d\sigma = O(1).$$

Hence the result.

THEOREM 9.9 (A). $\qquad S_1(T) = O(\log T).$

By Theorem 9.6 (B)

$$\int_{\frac{1}{2}}^{2} \log|\zeta(s)|\,d\sigma = \sum_{|t-\gamma|\leqslant 1} \int_{\frac{1}{2}}^{2} \log|s-\rho|\,d\sigma + O(\log t).$$

The terms of the last sum are bounded, since

$$\tfrac{3}{2}\log(\tfrac{9}{4}+1) \geqslant \int_{\frac{1}{2}}^{2} \log\{(\sigma-\beta)^2+(\gamma-t)^2\}\,d\sigma \geqslant 2\int_{\frac{1}{2}}^{2} \log|\sigma-\beta|\,d\sigma > -A.$$

Hence
$$\int_{\frac{1}{2}}^{2} \log|\zeta(s)|\,d\sigma = O(\log t), \tag{9.9.5}$$

and the result follows from the previous theorem.

It was proved by F. and R. Nevanlinna (1), (2) that

$$\int_{0}^{T} \frac{S(t)}{t}\,dt = A + O\left(\frac{\log T}{T}\right). \tag{9.9.6}$$

This follows from the previous result by integration by parts; for

$$\int_{1}^{T} \frac{S(t)}{t}\,dt = \left[\frac{S_1(t)}{t}\right]_1^T + \int_{1}^{T} \frac{S_1(t)}{t^2}\,dt = A + \frac{S_1(T)}{T} - \int_{T}^{\infty} \frac{S_1(t)}{t^2}\,dt.$$

Since $S_1(T) = O(\log T)$, the middle term is $O(T^{-1}\log T)$, and the last term is

$$O\left(\int_{T}^{\infty} \frac{\log t}{t^2}\,dt\right) = O\left(-\left[\frac{\log t}{t}\right]_T^{\infty} + \int_{T}^{\infty} \frac{dt}{t^2}\right) = O\left(\frac{\log T}{T}\right).$$

Hence the result follows. A similar result clearly holds for

$$\int_{1}^{T} \frac{S(t)}{t^{\alpha}}\,dt \quad (0<\alpha<1).$$

It has recently been proved by A. Selberg (5) that

$$S(t) = \Omega_{\pm}\{(\log t)^{\frac{1}{3}}(\log\log t)^{-\frac{7}{3}}\} \tag{9.9.7}$$

with a similar result for $S_1(t)$; and that

$$S_1(t) = \Omega_{+}\{(\log t)^{\frac{1}{2}}(\log\log t)^{-4}\}. \tag{9.9.8}$$

9.10. THEOREM 9.10.† *$S(t)$ has an infinity of changes of sign.*

Consider the interval (γ_n, γ_{n+1}) in which $N(t) = n$. Let $l(t)$ be the

† Titchmarsh (17).

linear function of t such that $l(\gamma_n) = S(\gamma_n)$, $l(\gamma_{n+1}) = S(\gamma_{n+1}-0)$. Then for $\gamma_n < t < \gamma_{n+1}$

$$\begin{aligned} l(t)-S(t) &= \{S(\gamma_{n+1}-0)-S(\gamma_n)\}\frac{t-\gamma_n}{\gamma_{n+1}-\gamma_n} - \{S(t)-S(\gamma_n)\} \\ &= -\{L(\gamma_{n+1})-L(\gamma_n)\}\frac{t-\gamma_n}{\gamma_{n+1}-\gamma_n} + \{L(t)-L(\gamma_n)\} + O\left(\frac{1}{\gamma_n}\right), \end{aligned}$$

using (9.3.2) and the fact that $N(t)$ is constant in the interval. The first two terms on the right give

$$\begin{aligned} &-L'(\xi)(t-\gamma_n) + L'(\eta)(t-\gamma_n) \quad (\gamma_n<\eta<t,\ \gamma_n<\xi<\gamma_{n+1}) \\ &= L''(\xi_1)(\eta-\xi)(t-\gamma_n) \quad (\xi_1 \text{ between } \xi \text{ and } \eta) \\ &= O(1/\gamma_n) \end{aligned}$$

since $\gamma_{n+1}-\gamma_n = O(1)$. Hence

$$\begin{aligned} \int_{\gamma_n}^{\gamma_{n+1}} S(t)\,dt &= \int_{\gamma_n}^{\gamma_{n+1}} l(t)\,dt + O\left(\frac{\gamma_{n+1}-\gamma_n}{\gamma_n}\right) \\ &= \tfrac{1}{2}(\gamma_{n+1}-\gamma_n)\{S(\gamma_n)+S(\gamma_{n+1}-0)\} + O\left(\frac{\gamma_{n+1}-\gamma_n}{\gamma_n}\right). \end{aligned}$$

Suppose that $S(t) \geqslant 0$ for $t > t_0$. Then

$$N(\gamma_n) \geqslant N(\gamma_n-0)+1$$

gives
$$S(\gamma_n) \geqslant S(\gamma_n-0)+1 \geqslant 1.$$

Hence
$$\begin{aligned} \int_{\gamma_n}^{\gamma_{n+1}} S(t)\,dt &\geqslant \tfrac{1}{2}(\gamma_{n+1}-\gamma_n) + O\left(\frac{\gamma_{n+1}-\gamma_n}{\gamma_n}\right) \\ &\geqslant \tfrac{1}{4}(\gamma_{n+1}-\gamma_n) \quad (n \geqslant n_0). \end{aligned}$$

Hence
$$\int_{\gamma_{n_0}}^{\gamma_N} S(t)\,dt \geqslant \tfrac{1}{4}(\gamma_N-\gamma_{n_0}),$$

contrary to Theorem 9.9 (A). Similarly the hypothesis $S(t) \leqslant 0$ for $t > t_0$ can be shown to lead to a contradiction.

It has been proved by A. Selberg (5) that $S(t)$ changes sign at least

$$T(\log T)^{\frac{1}{3}}e^{-A\sqrt{\log\log T}}$$

times in the interval $(0, T)$.

9.11. At the present time no improvement on the result

$$S(T) = O(\log T)$$

is known. But it is possible to prove directly some of the results which would follow from such an improvement. We shall first prove†

THEOREM 9.11. *The gaps between the ordinates of successive zeros of $\zeta(s)$ tend to 0.*

† Littlewood (3).

This would follow at once from (9.3.2) if it were possible to prove that $S(t) = o(\log t)$.

The argument given in § 9.1 shows that the gaps are bounded. Here we have to apply a similar argument to the strip $T-\delta \leqslant t \leqslant T+\delta$, where δ is arbitrarily small, and it is clear that we cannot use four concentric circles. But the ideas of the theorems of Borel–Carathéodory and Hadamard are in no way essentially bound up with sets of concentric circles, and the difficulty can be surmounted by using suitable elongated curves instead.

Let D_4 be the rectangle with centre $3+iT$ and a corner at $-3+i(T+\delta)$, the sides being parallel to the axes. We represent D_4 conformally on the unit circle D_4' in the z-plane, so that its centre $3+iT$ corresponds to $z = 0$. By this representation a set of concentric circles $|z| = r$ inside D_4' will correspond to a set of convex curves inside D_4, such that as $r \to 0$ the curve shrinks upon the point $3+iT$, while as $r \to 1$ it tends to coincidence with D_4. Let D_1', D_2', D_3' be circles (independent, of course, of T) for which the corresponding curves D_1, D_2, D_3 in the s-plane pass through the points $2+iT$, $-1+iT$, $-2+iT$ respectively.

The proof now proceeds as before. We consider the function

$$f(z) = \log \zeta\{s(z)\},$$

where $s = s(z)$ is the analytic function corresponding to the conformal representation; and we apply the theorems of Borel–Carathéodory and Hadamard in the same way as before.

9.12. We shall now obtain a more precise result of the same kind.†

THEOREM 9.12. *For every large positive T, $\zeta(s)$ has a zero $\beta+i\gamma$ satisfying*

$$|\gamma-T| < \frac{A}{\log\log\log T}.$$

This was first proved by Littlewood by a detailed study of the conformal representation used in the previous proof. This involves rather complicated calculations with elliptic functions. We shall give here two proofs which avoid these calculations.

In the first, we replace the rectangles by a succession of circles. Let T be a large positive number, and suppose that $\zeta(s)$ has no zero $\beta+i\gamma$ such that $T-\delta \leqslant \gamma \leqslant T+\delta$, where $\delta < \frac{1}{2}$. Then the function

$$f(s) = \log \zeta(s),$$

where the logarithm has its principal value for $\sigma > 2$, is regular in the rectangle

$$-2 \leqslant \sigma \leqslant 3, \qquad T-\delta \leqslant t \leqslant T+\delta.$$

† Littlewood (3); proofs given here by Titchmarsh (13), Kramaschke (1).

Let c_ν, C_ν, $\mathbf{C}_\nu$, Γ_ν be four concentric circles, with centre $2-\frac{1}{4}\nu\delta+iT$, and radii $\frac{1}{8}\delta$, $\frac{1}{2}\delta$, $\frac{5}{8}\delta$, and δ respectively. Consider these sets of circles for $\nu = 0, 1,\ldots, n$, where $n = [12/\delta]+1$, so that $2-\frac{1}{4}n\delta \leqslant -1$, i.e. the centre of the last circle lies on, or to the left of, $\sigma = -1$. Let m_ν, M_ν, and $\mathbf{M}_\nu$ denote the maxima of $|f(s)|$ on c_ν, C_ν, and $\mathbf{C}_\nu$ respectively.

Let $A_1, A_2,\ldots$ denote absolute constants (it is convenient to preserve their identity throughout the proof). We have $\mathbf{R}\{f(s)\} < A_1 \log T$ on all the circles, and $|f(2+iT)| < A_2$. Hence the Borel–Carathéodory theorem for the circles $\mathbf{C}_0$ and Γ_0 gives

$$\mathbf{M}_0 < \frac{\delta+\frac{5}{8}\delta}{\delta-\frac{5}{8}\delta}(A_1 \log T+A_2) = 7(A_1 \log T+A_2),$$

and in particular

$$|f(2-\tfrac{1}{4}\delta+iT)| < 7(A_1 \log T+A_2).$$

Hence, applying the Borel–Carathéodory theorem to $\mathbf{C}_1$ and Γ_1,

$$\mathbf{M}_1 < 7\{A_1 \log T+|f(2-\tfrac{1}{4}\delta+iT)|\} < (7+7^2)A_1 \log T+7^2 A_2.$$

So generally $\quad \mathbf{M}_\nu < (7+\ldots+7^{\nu+1})A_1 \log T+7^{\nu+1}A_2,$

or, say,

$$\mathbf{M}_\nu < 7^\nu A_3 \log T. \tag{9.12.1}$$

Now by Hadamard's three-circles theorem

$$M_\nu \leqslant m_\nu^a \mathbf{M}_\nu^b,$$

where a and b are positive constants such that $a+b = 1$; in fact $a = \log\frac{5}{4}/\log 3$, $b = \log 2/\log 3$. Also, since the circle $C_{\nu-1}$ includes the circle c_ν, $m_\nu \leqslant M_{\nu-1}$. Hence

$$M_\nu \leqslant M_{\nu-1}^a \mathbf{M}_\nu^b \quad (\nu = 1, 2,\ldots, n).$$

Thus $\qquad M_1 \leqslant M_0^a \mathbf{M}_1^b, \qquad M_2 \leqslant M_1^a \mathbf{M}_2^b \leqslant M_0^{a^2}\mathbf{M}_1^{ab}\mathbf{M}_2^b,$

and so on, giving finally

$$M_n \leqslant M_0^{a^n}\mathbf{M}_1^{a^{n-1}b}\mathbf{M}_2^{a^{n-2}b}\ldots\mathbf{M}_n^b.$$

Hence, by (9.12.1),

$$M_n \leqslant M_0^{a^n} 7^{a^{n-1}b+2a^{n-2}b+\ldots+nb}(A_3 \log T)^{a^{n-1}b+a^{n-2}b+\ldots+b}.$$

Now

$$a^{n-1}b+2a^{n-2}b+\ldots+nb < n^2,$$

$$a^{n-1}b+a^{n-2}b+\ldots+b = b(1-a^n)/(1-a) = 1-a^n.$$

Hence $\qquad M_n \leqslant M_0^{a^n} 7^{n^2}(A_3 \log T)^{1-a^n} < A_4\, 7^{n^2}(\log T)^{1-a^n},$

since M_0 is bounded as $T \to \infty$.

But $|\zeta(s)| > t^{A_5}$ for $\sigma \leqslant -1$, $t > t_0$, so that $M_n > A_5 \log T$. Hence

$$A_5 < A_4\, 7^{n^2}(\log T)^{-a^n},$$

$$\log\log T < \left(\frac{1}{a}\right)^n \left(n^2 \log 7-\log\frac{A_5}{A_4}\right),$$

$$\log\log\log T < n \log\frac{1}{a}+A_6 \log n,$$

so that
$$\delta < \frac{12}{n-1} < \frac{A}{\log\log\log T},$$
and the result follows.

9.13. Second Proof. Consider the angular region in the s-plane with vertex at $s = -3+iT$, bounded by straight lines making angles $\pm\frac{1}{2}\alpha\,(0 < \alpha < \pi)$ with the real axis.

Let
$$w = (s+3-iT)^{\pi/\alpha}.$$
Then the angular region is mapped on the half-plane $\mathbf{R}(w) \geqslant 0$. The point $s = 2+iT$ corresponds to
$$w = 5^{\pi/\alpha}.$$
Let
$$z = \frac{w-5^{\pi/\alpha}}{w+5^{\pi/\alpha}}.$$
Then the angular region corresponds to the unit circle in the z-plane, and $s = 2+iT$ corresponds to its centre $z = 0$. If $s = \sigma+iT$ corresponds to $z = -r$, then
$$(\sigma+3)^{\pi/\alpha} = w = 5^{\pi/\alpha}\frac{1-r}{1+r},$$
i.e.
$$r = \left\{1-\left(\frac{\sigma+3}{5}\right)^{\pi/\alpha}\right\}\Big/\left\{1+\left(\frac{\sigma+3}{5}\right)^{\pi/\alpha}\right\}.$$

Suppose that $\zeta(s)$ has no zeros in the angular region, so that $\log\zeta(s)$ is regular in it.

Let $s = \frac{3}{2}+iT,\ -1+iT,\ -2+iT$ correspond to $z = -r_1,\ -r_2,\ -r_3$ respectively. Let M_1, M_2, M_3 be the maxima of $|\log\zeta(s)|$ on the s-curves corresponding to $|z| = r_1, r_2, r_3$. Then Hadamard's three-circles theorem gives
$$\log M_2 \leqslant \frac{\log r_3/r_2}{\log r_3/r_1}\log M_1 + \frac{\log r_2/r_1}{\log r_3/r_1}\log M_3.$$

It is easily verified that, on the curve corresponding to $|z| = r_1$, $\sigma \geqslant \frac{3}{2}$. For if $w = \xi+i\eta$, then
$$\sigma = -3+(\xi^2+\eta^2)^{\alpha/2\pi}\cos\left(\frac{\alpha}{\pi}\arctan\frac{\eta}{\xi}\right),$$
which is a minimum at $\eta = 0$, for given ξ, if $0 < \alpha < \frac{1}{2}\pi$; and the minimum is $-3+\xi^{\alpha/\pi}$, which, as a function of ξ, is a minimum when ξ is a minimum, i.e. when $z = -r_1$. It therefore follows that $\log M_1 < A$.

Since $\mathbf{R}\{\log\zeta(s)\} < A\log T$ in the angle, it follows from the Borel–Carathéodory theorem that
$$M_3 < \frac{2}{1-r_3}(A\log T+A) < \frac{A\log T}{1-r_3}.$$

Hence
$$\log M_2 \leqslant A + \frac{\log r_2/r_1}{\log r_3/r_1}\log\left(\frac{A\log T}{1-r_3}\right).$$
Now if r_1, r_2, and r_3 are sufficiently near to 1, i.e. if α is sufficiently small,
$$\frac{\log r_2/r_1}{\log r_3/r_1} = \frac{\log\left(1+\dfrac{r_2-r_1}{r_1}\right)}{\log\left(1+\dfrac{r_3-r_1}{r_1}\right)} \leqslant \left(\frac{r_2-r_1}{r_3-r_1}\right)^{\frac{1}{2}},$$
and
$$\frac{r_2-r_1}{r_3-r_1} = \frac{\dfrac{1-r_1}{1+r_1}-\dfrac{1-r_2}{1+r_2}}{\dfrac{1-r_1}{1+r_1}-\dfrac{1-r_3}{1+r_3}}\,\frac{1+r_2}{1+r_3} < \frac{(\frac{9}{10})^{\pi/\alpha}-(\frac{2}{5})^{\pi/\alpha}}{(\frac{9}{10})^{\pi/\alpha}-(\frac{1}{5})^{\pi/\alpha}}$$
$$< 1-A(\tfrac{4}{9})^{\pi/\alpha}.$$
Hence
$$\frac{\log r_2/r_1}{\log r_3/r_1} < 1-A(\tfrac{4}{9})^{\pi/\alpha}.$$
Also
$$1/(1-r_3) < A5^{\pi/\alpha}.$$
Hence
$$\log M_2 < A+\{1-A(\tfrac{4}{9})^{\pi/\alpha}\}\left\{\log\log T+\frac{\pi}{\alpha}\log 5+A\right\}.$$

Let $\alpha = \pi/(c\log\log\log T)$. Then
$$\log M_2 < A+\{1-A(\log\log T)^{-c\log\frac{9}{4}}\}\{\log\log T+c\log 5\log\log\log T+A\}$$
$$< \log\log T-(\log\log T)^{\frac{1}{2}}$$
if $c\log\frac{9}{4} < \frac{1}{2}$ and T is large enough. Hence
$$M_2 < \log T\,e^{-(\log\log T)^{\frac{1}{2}}} < \epsilon\log T \quad (T > T_0(\epsilon)).$$
In particular
$$\log|\zeta(-1+iT)| < \epsilon\log T,$$
$$|\zeta(-1+iT)| < T^{\epsilon}.$$
But
$$|\zeta(-1+iT)| = |\chi(-1+iT)\zeta(2-iT)| > KT^{\frac{3}{2}}.$$
We thus obtain a contradiction, and the result follows.

9.14. Another result† in the same order of ideas is

THEOREM 9.14. *For any fixed h, however small,*
$$N(T+h)-N(T) > K\log T$$
for $K = K(h)$, $T > T_0$.

This result is not a consequence of Theorem 9.4 if h is less than a certain value.

Consider the same angular region as before, with a new α such that

† Not previously published.

$\tan\alpha \leqslant \frac{1}{2}$, and suppose now that $\zeta(s)$ has zeros $\rho_1, \rho_2, \ldots, \rho_n$ in the angular region. Let

$$F(s) = \frac{\zeta(s)}{(s-\rho_1)\ldots(s-\rho_n)}.$$

Let C be the circle with centre $\frac{1}{2}+iT$ and radius 3. Then $|s-\rho_\nu| \geqslant 1$ on C. Hence

$$|F(s)| \leqslant |\zeta(s)| < T^A$$

on C, and so also inside C.

Let $f(s) = \log F(s)$. Then $f(s)$ is regular in the angle, and

$$\mathbf{R}f(s) < A\log T.$$

Also

$$f(2+iT) = \log\zeta(2+iT) - \sum_{\nu=1}^{n}\log(2+iT-\rho_\nu)$$
$$= O(1)+\sum_{\nu=1}^{n}O(1) = O(n).$$

Let M_1, M_2, and M_3 now denote the maxima of $|f(s)|$ on the three s-curves. Then

$$M_3 < \frac{A}{1-r_3}(\log T+n).$$

Also $M_1 < An$, as for $f(2+iT)$. Hence

$$\log|f(-1+iT)| \leqslant \log M_2$$
$$< \frac{\log r_3/r_2}{\log r_3/r_1}(A+\log n)+\frac{\log r_2/r_1}{\log r_3/r_1}\log\left\{\frac{A(n+\log T)}{1-r_3}\right\}$$
$$< A+\log n+\frac{\log r_2/r_1}{\log r_3/r_1}\left\{\log\frac{1}{1-r_3}+\log\left(\frac{\log T}{n}\right)\right\}$$
$$< A+\log n+\{1-A(\tfrac{4}{9})^{\pi/\alpha}\}\left\{\frac{\pi}{\alpha}\log 5+\log\left(\frac{\log T}{n}\right)\right\}$$

as before. But

$$|f(-1+iT)| = \left|\log\zeta(-1+iT)-\sum_{\nu=1}^{n}\log(-1+iT-\rho_\nu)\right|$$
$$\geqslant \log|\zeta(-1+iT)|-\sum_{\nu=1}^{n}O(1)$$
$$> A_1\log T-A_2 n,$$

say. If $n > \frac{1}{2}(A_1/A_2)\log T$ the theorem follows at once. Otherwise

$$|f(-1+iT)| > \tfrac{1}{2}A_1\log T,$$

and we obtain

$$\log\left(\frac{\log T}{n}\right) < A+\{1-A(\tfrac{4}{9})^{\pi/\alpha}\}\left\{\frac{\pi}{\alpha}\log 5+\log\left(\frac{\log T}{n}\right)\right\},$$

$$A(\tfrac{4}{9})^{\pi/\alpha}\log\left(\frac{\log T}{n}\right) < A+\{1-A(\tfrac{4}{9})^{\pi/\alpha}\}\frac{\pi}{\alpha}\log 5,$$

and hence
$$\log\log\left(\frac{\log T}{n}\right) < \frac{\pi}{\alpha}\log\frac{9}{4}+\log\frac{1}{\alpha}+A < \frac{A}{\alpha},$$
$$n > e^{-e^{A/\alpha}}\log T.$$

This proves the theorem.

9.15. The function $N(\sigma, T)$. We define $N(\sigma, T)$ to be the number of zeros $\beta+i\gamma$ of the zeta-function such that $\beta > \sigma$, $0 < t \leqslant T$. For each T, $N(\sigma, T)$ is a non-increasing function of σ, and is 0 for $\sigma \geqslant 1$. On the Riemann hypothesis, $N(\sigma, T) = 0$ for $\sigma > \frac{1}{2}$. Without any hypothesis, all that we can say so far is that

$$N(\sigma, T) \leqslant N(T) < AT\log T$$

for $\frac{1}{2} < \sigma < 1$.

The object of the next few sections is to improve upon this inequality for values of σ between $\frac{1}{2}$ and 1.

We return to the formula (9.9.1). Let $\phi(s) = \zeta(s)$, $\alpha = \sigma_0$, $\beta = 2$, and this time take the imaginary part. We have

$$\nu(\sigma, T) = N(\sigma, T) \quad (\sigma < 1), \qquad \nu(\sigma, T) = 0 \quad (\sigma \geqslant 1).$$

We obtain, if T is not the ordinate of a zero,

$$2\pi\int_{\sigma_0}^{1} N(\sigma, T)\,d\sigma = \int_0^T \log|\zeta(\sigma_0+it)|\,dt - \int_0^T \log|\zeta(2+it)|\,dt +$$
$$+\int_{\sigma_0}^{2}\arg\zeta(\sigma+iT)\,d\sigma+K(\sigma_0),$$

where $K(\sigma_0)$ is independent of T. We deduce†

THEOREM 9.15. *If* $\frac{1}{2} \leqslant \sigma_0 \leqslant 1$, *and* $T \to \infty$,

$$2\pi\int_{\sigma_0}^{1} N(\sigma, T)\,d\sigma = \int_0^T \log|\zeta(\sigma_0+it)|\,dt+O(\log T).$$

We have

$$\int_0^T \log|\zeta(2+it)|\,dt = \mathbf{R}\sum_{n=2}^{\infty}\frac{\Lambda_1(n)}{n^2}\frac{n^{-iT}-1}{-i\log n} = O(1).$$

Also, by § 9.4, $\arg\zeta(\sigma+iT) = O(\log T)$ uniformly for $\sigma \geqslant \frac{1}{2}$, if T is not the ordinate of a zero. Hence the integral involving $\arg\zeta(\sigma+iT)$ is $O(\log T)$. The result follows if T is not the ordinate of a zero, and this restriction can then be removed from considerations of continuity.

† Littlewood (4).

THEOREM 9.15 (A).† *For any fixed σ greater than $\frac{1}{2}$,*

$$N(\sigma, T) = O(T).$$

For any non-negative continuous $f(t)$

$$\frac{1}{b-a}\int_a^b \log f(t)\,dt \leqslant \log\left\{\frac{1}{b-a}\int_a^b f(t)\,dt\right\}.$$

Thus, for $\frac{1}{2} < \sigma < 1$,

$$\int_0^T \log|\zeta(\sigma+it)|\,dt = \tfrac{1}{2}\int_0^T \log|\zeta(\sigma+it)|^2\,dt$$

$$\leqslant \tfrac{1}{2}T\log\left\{\frac{1}{T}\int_0^T |\zeta(\sigma+it)|^2\,dt\right\} = O(T)$$

by Theorem 7.2. Hence, by Theorem 9.15,

$$\int_{\sigma_0}^1 N(\sigma, T)\,d\sigma = O(T)$$

for $\sigma_0 > \frac{1}{2}$. Hence, if $\sigma_1 = \frac{1}{2}+\frac{1}{2}(\sigma_0-\frac{1}{2})$,

$$N(\sigma_0, T) \leqslant \frac{1}{\sigma_0-\sigma_1}\int_{\sigma_1}^{\sigma_0} N(\sigma, T)\,d\sigma \leqslant \frac{2}{\sigma_0-\frac{1}{2}}\int_{\sigma_1}^1 N(\sigma, T)\,d\sigma = O(T),$$

the required result.

From this theorem, and the fact that $N(T) \sim AT\log T$, it follows that *all but an infinitesimal proportion of the zeros of $\zeta(s)$ lie in the strip $\frac{1}{2}-\delta < \sigma < \frac{1}{2}+\delta$, however small δ may be.*

9.16. We shall next prove a number of theorems in which the $O(T)$ of Theorem 9.15 (A) is replaced by $O(T^\theta)$, where $\theta < 1$.‡ We do this by applying the above methods, not to $\zeta(s)$ itself, but to the function

$$\zeta(s)M_X(s) = \zeta(s)\sum_{n<X}\frac{\mu(n)}{n^s}.$$

The zeros of $\zeta(s)$ are zeros of $\zeta(s)M_X(s)$. If $\sigma > 1$, $M_X(s) \to 1/\zeta(s)$ as $X \to \infty$, so that $\zeta(s)M_X(s) \to 1$. On the Riemann hypothesis this is also true for $\frac{1}{2} < \sigma \leqslant 1$. Of course we cannot prove this without any hypothesis; but we can choose X so that the additional factor neutralizes to a certain extent the peculiarities of $\zeta(s)$, even for values of σ less than 1.

Let $$f_X(s) = \zeta(s)M_X(s)-1.$$

† Bohr and Landau (4), Littlewood (4).

‡ Bohr and Landau (5), Carlson (1), Landau (12), Titchmarsh (5), Ingham (5).

We shall first prove

THEOREM 9.16. *If for some $X = X(\sigma, T)$, $T^{1-l(\sigma)} \leqslant X < T^A$,*

$$\int_{\frac{1}{2}T}^T |f_X(s)|^2\,dt = O(T^{l(\sigma)}\log^m T)$$

as $T \to \infty$, uniformly for $\sigma \geqslant \alpha$, where $l(\sigma)$ is a positive non-increasing function with a bounded derivative, and m is a constant $\geqslant 0$, then

$$N(\sigma, T) = O(T^{l(\sigma)}\log^{m+1}T)$$

uniformly for $\sigma \geqslant \alpha+1/\log T$.

We have $$f_X(s) = \zeta(s)\sum_{n<X}\frac{\mu(n)}{n^s}-1 = \sum\frac{a_n(X)}{n^s},$$

where $a_1(X) = 0$,

$$a_n(X) = \sum_{d|n}\mu(d) = 0 \quad (n < X),$$

and $$|a_n(X)| = \left|\sum_{\substack{d|n \\ d<X}}\mu(d)\right| \leqslant d(n)$$

for all n and X.

Let $$1-f_X^2 = \zeta M_X(2-\zeta M_X) = \zeta(s)g(s) = h(s)$$

say, where $g(s) = g_X(s)$ and $h(s) = h_X(s)$ are regular except at $s = 1$. Now for $\sigma \geqslant 2$, $X > X_0$,

$$|f_X(s)|^2 \leqslant \left(\sum_{n\geqslant X}\frac{d(n)}{n^2}\right)^2 = O(X^{2\epsilon-2}) < \frac{1}{2X} < \frac{1}{2},$$

so that $h(s) \neq 0$. Applying (9.9.1) to $h(s)$, and writing

$$\nu(\sigma, T_1, T_2) = \nu(\sigma, T_2)-\nu(\sigma, T_1),$$

we obtain

$$2\pi\int_{\sigma_0}^2 \nu(\sigma, \tfrac{1}{2}T, T)\,d\sigma = \int_{\frac{1}{2}T}^T \{\log|h(\sigma_0+it)|-\log|h(2+it)|\}\,dt +$$

$$+\int_{\sigma_0}^2 \{\arg h(\sigma+iT)-\arg h(\sigma+\tfrac{1}{2}iT)\}\,d\sigma.$$

Now $$\log|h(s)| \leqslant \log\{1+|f_X(s)|^2\} \leqslant |f_X(s)|^2,$$

so that, if $\sigma_0 \geqslant \alpha$,

$$\int_{\frac{1}{2}T}^T \log|h(\sigma_0+it)|\,dt \leqslant \int_{\frac{1}{2}T}^T |f_X(\sigma_0+it)|^2\,dt = O(T^{l(\sigma_0)}\log^m T).$$

Next

$$-\log|h(2+it)| \leqslant -\log\{1-|f_X(2+it)|^2\} \leqslant 2|f_X(2+it)|^2 < X^{-1}$$

so that
$$-\int_{\frac12 T}^{T} \log|h(2+it)|\,dt < \frac{T}{2X} = O(T^{l(\sigma_0)}).$$

Also we can apply the lemma of § 9.4 to $h(s)$, with $\alpha = 0$, $\beta \geqslant \frac12$, $m \geqslant \frac12$, and $M_{\sigma,t} = O(X^A T^A)$. We obtain
$$\arg h(s) = O(\log X + \log t)$$
for $\sigma \geqslant \frac12$. Hence
$$\int_{\sigma_0}^{\frac12} \{\arg h(\sigma+iT) - \arg h(\sigma+\tfrac12 iT)\}\,d\sigma = O(\log X+\log T) = O(\log T).$$

Hence
$$\int_{\sigma_0}^{2} \nu(\sigma, \tfrac12 T, T)\,d\sigma = O(T^{l(\sigma_0)}\log^m T).$$

Also
$$\int_{\sigma_0}^{2} \nu(\sigma, \tfrac12 T, T)\,d\sigma \geqslant \int_{\sigma_0}^{2} N(\sigma, \tfrac12 T, T)\,d\sigma \geqslant (\sigma_1-\sigma_0) N(\sigma_1, \tfrac12 T, T)$$
if $\sigma_0 < \sigma_1 \leqslant 2$. Taking $\sigma_1 = \sigma_0 + 1/\log T$, we have
$$T^{l(\sigma_0)} = T^{l(\sigma_1)+O(\sigma_1-\sigma_0)} = O(T^{l(\sigma_1)}).$$

Hence
$$N(\sigma_1, \tfrac12 T, T) = O(T^{l(\sigma_1)}\log^{m+1} T).$$

Replacing T by $\frac12 T$, $\frac14 T$,... and adding, the result follows.

9.17. The simplest application is

THEOREM 9.17. *For any fixed σ in $\frac12 < \sigma < 1$,*
$$N(\sigma, T) = O(T^{4\sigma(1-\sigma)+\epsilon}).$$

We use Theorem 4.11 with $x = T$, and obtain
$$f_X(s) = \sum_{m<T} \frac{1}{m^s} \sum_{n<X} \frac{\mu(n)}{n^s} - 1 + O(T^{-\sigma}|M_X(s)|)$$
$$= \sum \frac{b_n(X)}{n^s} + O(T^{-\sigma}X^{1-\sigma}), \tag{9.17.1}$$
where, if $X < T$, $b_n(X) = 0$ for $n < X$ and for $n > XT$; and, as for a_n, $|b_n(X)| \leqslant d(n) = O(n^\epsilon)$. Hence
$$\int_{\frac12 T}^{T} \left|\sum \frac{b_n(X)}{n^s}\right|^2 dt = \tfrac12 T \sum \frac{|b_n(X)|^2}{n^{2\sigma}} + \sum\sum \frac{b_m b_n}{(mn)^\sigma} \int_{\frac12 T}^{T} \left(\frac{n}{m}\right)^{it} dt$$
$$= O\left(T \sum_{n \geqslant X} \frac{1}{n^{2\sigma-\epsilon}}\right) + O\left(\sum_{n<m<XT}\sum \frac{1}{(mn)^{\sigma-\epsilon}\log m/n}\right)$$
$$= O(TX^{1-2\sigma+\epsilon}) + O\{(XT)^{2-2\sigma+\epsilon}\}$$

by (7.2.1). These terms are of the same order (apart from ϵ's) if $X = T^{2\sigma-1}$, and then
$$\int_{\frac12 T}^{T} \left|\sum \frac{b_n(X)}{n^s}\right|^2 dt = O(T^{4\sigma(1-\sigma)+\epsilon}).$$

The O-term in (9.17.1) gives
$$O(T^{1-2\sigma}X^{2-2\sigma}) = O(T^{1-2\sigma}X) = O(1).$$
The result therefore follows from Theorem 9.16.

9.18. The main instrument used in obtaining still better results for $N(\sigma, T)$ is the convexity theorem for mean values of analytic functions proved in § 7.8. We require, however, some slight extensions of the theorem. If the right-hand sides of (7.8.1) and (7.8.2) are replaced by finite sums
$$\sum C(T^a+1), \qquad \sum C'(T^b+1),$$
then the right-hand side of (7.8.3) is clearly to be replaced by
$$K\sum\sum (CT^a)^{(\beta-\sigma)/(\beta-\alpha)}(C'T^b)^{(\sigma-\alpha)/(\beta-\alpha)}.$$

In one of the applications a term $T^a\log^4 T$ occurs in the data instead of the above T^a. This produces the same change in the result. The only change in the proof is that, instead of the term
$$\int_0^\infty \left(\frac{u}{\delta}\right)^{a+2\alpha-1} e^{-2u}\,du = \frac{K}{\delta^{a+2\alpha-1}},$$
we obtain a term
$$\int_0^\infty \left(\frac{u}{\delta}\right)^{a+2\alpha-1} \log^4\frac{u}{\delta}\, e^{-2u}\,du$$
$$= \int_0^\infty \left(\frac{u}{\delta}\right)^{a+2\alpha-1} \left\{\log^4\frac{1}{\delta} + 4\log^3\frac{1}{\delta}\log u + \ldots\right\} e^{-2u}\,du < \frac{K}{\delta^{a+2\alpha-1}}\log^4\frac{1}{\delta}.$$

THEOREM 9.18. *If $\zeta(\frac12+it) = O(t^c\log^{c'} t)$, where $c' \leqslant \frac32$, then*
$$N(\sigma, T) = O(T^{2(1+2c)(1-\sigma)}\log^5 T)$$
uniformly for $\frac12 \leqslant \sigma \leqslant 1$.

If $0 < \delta < 1$,
$$\int_0^T |f_X(1+\delta+it)|^2\,dt = \sum_{m \geqslant X}\sum_{n \geqslant X} \frac{a_X(m)a_X(n)}{m^{1+\delta}n^{1+\delta}} \int_0^T \left(\frac{m}{n}\right)^{it} dt$$
$$= T\sum_{n \geqslant X} \frac{a_X^2(n)}{n^{2+2\delta}} + 2\sum_{X \leqslant m<n}\sum \frac{a_X(m)a_X(n)}{m^{1+\delta}n^{1+\delta}} \frac{\sin(T\log m/n)}{\log m/n}$$
$$\leqslant T\sum_{n \geqslant X} \frac{d^2(n)}{n^{2+2\delta}} + 2\sum_{X \leqslant m<n}\sum \frac{d(m)d(n)}{m^{1+\delta}n^{1+\delta}}.$$

Now† $\sum_{n\leqslant x} d^2(n) < Ax\log^3 x, \qquad \sum\sum_{m<n\leqslant x} \frac{d(m)d(n)}{(mn)^{\frac12}\log n/m} < Ax\log^3 x.$

Hence

$$\sum_{n\geqslant X}\frac{d^2(n)}{n^{1+\xi}} = \sum_{n\geqslant X} d^2(n)\int_n^\infty \frac{1+\xi}{x^{2+\xi}}\,dx = \int_X^\infty \frac{1+\xi}{x^{2+\xi}}\sum_{X\leqslant n\leqslant x} d^2(n)\,dx$$

$$< \int_X^\infty \frac{(1+\xi)A\log^3 x}{x^{1+\xi}}\,dx = \frac{A(1+1/\xi)}{X^\xi}\int_1^\infty \frac{\log^3(Xy^{1/\xi})}{y^2}\,dy$$

(putting $x = Xy^{1/\xi}$) $\qquad < \frac{A}{\xi X^\xi}\left(\log X+\frac1\xi\right)^3.$

Hence $$\sum_{n\geqslant X}\frac{d^2(n)}{n^{2+2\delta}} < \frac{A\log^3 X}{X^{1+2\delta}} < \frac{A}{X\delta^3}$$

since $X^{2\delta} = e^{2\delta\log X} > \frac16(2\delta\log X)^3$.

Also, since $\qquad 1<\log\lambda+\lambda^{-1}<\log\lambda+\lambda^{-\frac12}$

for $\lambda>1$,

$$\sum\sum_{X\leqslant m<n}\frac{d(m)d(n)}{(mn)^{1+\xi}\log n/m} < \sum\sum_{X\leqslant m<n}\frac{d(m)d(n)}{(mn)^{1+\xi}} + \sum\sum_{X\leqslant m<n}\frac{d(m)d(n)}{m^\xi n^{1+\xi}(mn)^{\frac12}\log n/m}$$

$$< \left(\sum_{n=1}^\infty \frac{d(n)}{n^{1+\xi}}\right)^2 + \sum\sum_{1\leqslant m<n}\frac{d(m)d(n)}{(mn)^{\frac12}\log n/m}\int_n^\infty\frac{1+\xi}{x^{2+\xi}}\,dx$$

$$< \zeta^4(1+\xi) + \int_1^\infty \frac{1+\xi}{x^{2+\xi}}\sum\sum_{m<n\leqslant x}\frac{d(m)d(n)}{(mn)^{\frac12}\log n/m}\,dx$$

$$< \zeta^4(1+\xi)+\int_1^\infty\frac{(1+\xi)A\log^3 x}{x^{1+\xi}}\,dx < \frac{A}{\xi^4}.$$

Hence $$\int_0^T |f_X(1+\delta+it)|^2\,dt < A\left(\frac TX+1\right)\delta^{-4}. \qquad (9.18.1)$$

For $\sigma=\frac12$ we use the inequalities

$$|f_X|^2 \leqslant 2(|\zeta|^2|M_X|^2+1),$$

$$\int_0^T |M_X(\tfrac12+it)|^2\, dt \leqslant T\sum_{n<X}\frac{\mu^2(n)}{n} + 2\sum\sum_{m<n<X}\frac{|\mu(m)\mu(n)|}{(mn)^{\frac12}\log n/m}$$

$$\leqslant T\sum_{n<X}\frac1n + 2\sum\sum_{m<n<X}\frac{1}{(mn)^{\frac12}\log n/m}$$

$$< A(T+X)\log X,$$

by (7.2.1).

† The first result follows easily from (7.16.3); for the second, see Ingham (1); the argument of § 7.21, and the first result, give an extra $\log x$.

Hence $$\int_0^T |f_X(\tfrac12+it)|^2\, dt < AT^{2c}(T+X)\log^{2c'}(T+2)\log X. \qquad (9.18.2)$$

The convexity theorem therefore gives

$$\int_{\frac12 T}^T |f_X(\sigma+it)|^2\, dt$$

$$= O\left\{\left(\frac TX+1\right)\delta^{-4}\right\}^{(\sigma-\frac12)/(\frac12+\delta)}\{T^{2c}(T+X)\log^{2c'}(T+2)\log X\}^{(1+\delta-\sigma)/(\frac12+\delta)}$$

$$= O\left\{\frac{T+X}{\delta^4}\,\frac{T^{4c(1-\sigma)}}{X^{2\sigma-1}}(XT^{2c})^{\{(2\sigma-1)\delta\}/(\frac12+\delta)}(\delta^4\log^3(T+2)\log X)^{(1+\delta-\sigma)/(\frac12+\delta)}\right\}.$$

Taking $\delta = 1/\log(T+X)$, we obtain

$$O\{(T+X)T^{4c(1-\sigma)}X^{1-2\sigma}\log^4(T+X)\}.$$

If $X=T$, the result follows from Theorem 9.16.

For example, by Theorem 5.5 we may take $c=\frac16$, $c'=\frac32$. Hence

$$N(\sigma,T)=O(T^{\frac83(1-\sigma)}\log^5 T). \qquad (9.18.3)$$

This is an improvement on Theorem 9.17 if $\sigma>\frac23$.

On the unproved Lindelöf hypothesis that $\zeta(\frac12+it)=O(t^\epsilon)$, Theorem 9.18 gives

$$N(\sigma,T)=O(T^{2(1-\sigma)+\epsilon}).$$

9.19. An improvement on Theorem 9.17 for all values of σ in $\frac12<\sigma<1$ is effected by combining (9.18.3) with

THEOREM 9.19(A). $N(\sigma,T)=O(T^{\frac32-\sigma}\log^5 T)$.

We have

$$\int_0^T|f_X(\tfrac12+it)|^2\,dt < A\int_0^T|\zeta(\tfrac12+it)|^2|M_X(\tfrac12+it)|^2\,dt + AT$$

$$< A\left\{\int_0^T|\zeta(\tfrac12+it)|^4\,dt\int_0^T|M_X(\tfrac12+it)|^4\,dt\right\}^{\frac12}+AT.$$

Now $$M_X^2(s)=\sum_{n<X^2}\frac{c_n}{n^s}, \qquad |c_n|\leqslant d(n).$$

Hence

$$\int_0^T|M_X(\tfrac12+it)|^4\,dt\leqslant T\sum_{n<X^2}\frac{d^2(n)}{n}+2\sum\sum_{m<n<X^2}\frac{d(m)d(n)}{(mn)^{\frac12}\log n/m}$$

$$< AT\log^4X+AX^2\log^3X.$$

Hence $$\int_0^T|f_X(\tfrac12+it)|^2\,dt < AT^{\frac12}(T+X^2)^{\frac12}\log^2(T+2)\log^2 X. \qquad (9.19.1)$$

From (9.18.1), (9.19.1), and the convexity theorem, we obtain

$$\int_{\frac{1}{2}T}^{T} |f_X(\sigma+it)|^2\,dt$$
$$= O\left\{\left(\frac{T}{X}+1\right)\delta^{-4}\right\}^{(\sigma-\frac{1}{2})/(\frac{1}{2}+\delta)}\{T^{\frac{1}{2}}(T+X^2)^{\frac{1}{2}}\log^2(T+2)\log^2 X\}^{(1+\delta-\sigma)/(\frac{1}{2}+\delta)}.$$

If $X = T^{\frac{1}{2}}$, $\delta = 1/\log(T+2)$, the result follows as before.

This is an improvement on Theorem 9.17 if $\frac{1}{2} < \sigma < \frac{3}{4}$.

Various results of this type have been obtained,† the most successful‡ being

Theorem 9.19 (B). $N(\sigma, T) = O(T^{3(1-\sigma)/(2-\sigma)}\log^5 T)$.

This depends on a two-variable convexity theorem;§ if

$$J(\sigma,\lambda) = \left\{\int_0^T |f(\sigma+it)|^{1/\lambda}\,dt\right\}^{\lambda},$$

then $$J(\sigma, p\lambda+q\mu) = O\{J^p(\alpha,\lambda)J^q(\beta,\mu)\} \quad (\alpha<\sigma<\beta),$$

where $$p = \frac{\beta-\sigma}{\beta-\alpha}, \qquad q = \frac{\sigma-\alpha}{\beta-\alpha}.$$

We have

$$\int_0^T |f_X(\tfrac{1}{2}+it)|^{\frac{4}{3}}\,dt < A\int_0^T |\zeta(\tfrac{1}{2}+it)|^{\frac{4}{3}}|M_X(\tfrac{1}{2}+it)|^{\frac{4}{3}}\,dt + AT$$
$$< A\left\{\int_0^T |\zeta(\tfrac{1}{2}+it)|^4\,dt\right\}^{\frac{1}{3}}\left\{\int_0^T |M_X(\tfrac{1}{2}+it)|^2\,dt\right\}^{\frac{2}{3}} + AT$$
$$< A\{T\log^4(T+2)\}^{\frac{1}{3}}\{(T+X)\log X\}^{\frac{2}{3}} + AT$$
$$< A(T+X)\log^2(T+X). \tag{9.19.2}$$

In the two-variable convexity theorem, take $\alpha = \frac{1}{2}$, $\beta = 1+\delta$, $\lambda = \frac{3}{4}$, $\mu = \frac{1}{2}$, and use (9.18.1) and (9.19.2). We obtain

$$\int_0^T |f_X(\sigma+it)|^{1/K}\,dt$$
$$< A\{(T+X)\log^2(T+X)\}^{\frac{3}{2}(1-\sigma+\delta)/(1-\frac{1}{2}\sigma+\frac{3}{2}\delta)}\left\{\left(\frac{T}{X}+1\right)\delta^{-4}\right\}^{(\sigma-\frac{1}{2})/(1-\frac{1}{2}\sigma+\frac{3}{2}\delta)},$$

where $K = p\lambda+q\mu$ lies between $\frac{1}{2}$ and $\frac{3}{4}$. Taking $X = T$, $\delta = 1/\log T$, we obtain

$$\int_0^T |f_X(\sigma+it)|^{1/K}\,dt < AT^{3(1-\sigma)/(2-\sigma)}\log^4 T.$$

† Titchmarsh (5), Ingham (5), (6). ‡ Ingham (6). § Gabriel (1).

The result now follows from a modified form of Theorem 9.16, since

$$\log|1-f_X^2| \leqslant \log(1+|f_X|^2) < A|f_X|^{1/K}.$$

A. Selberg† has recently proved

Theorem 9.19 (C). $N(\sigma, T) = O(T^{1-\frac{1}{4}(\sigma-\frac{1}{2})}\log T)$

uniformly for $$\tfrac{1}{2} \leqslant \sigma \leqslant 1.$$

This is an improvement on the previous theorem if σ is a function of T such that $\sigma-\frac{1}{2}$ is sufficiently small.

9.20. The corresponding problems with σ equal or nearly equal to $\frac{1}{2}$ are naturally more difficult. Here the most interesting question is that of the behaviour of

$$\int_{\frac{1}{2}}^{1} N(\sigma, T)\,d\sigma \tag{9.20.1}$$

as $T\to\infty$. If the zeros of $\zeta(s)$ are $\beta+i\gamma$, this is equal to

$$\int_{\frac{1}{2}}^{1}\Big(\sum_{\beta>\sigma,0<\gamma\leqslant T} 1\Big)\,d\sigma = \sum_{\beta>\frac{1}{2},0<\gamma\leqslant T}\int_{\frac{1}{2}}^{\beta} d\sigma = \sum_{\beta>\frac{1}{2},0<\gamma\leqslant T}(\beta-\tfrac{1}{2}).$$

Hence an equivalent problem is that of the sum

$$\sum_{0<\gamma\leqslant T}|\beta-\tfrac{1}{2}|. \tag{9.20.2}$$

There are some immediate results.‡ If we apply the above argument, but use Theorem 7.2 (A) instead of Theorem 7.2, we obtain at once

$$\int_{\sigma_0}^{1} N(\sigma, T)\,d\sigma < AT\log\left\{\min\left(\log T, \log\frac{1}{\sigma_0-\frac{1}{2}}\right)\right\} \tag{9.20.3}$$

for $\frac{1}{2} \leqslant \sigma_0 \leqslant 1$; and in particular

$$\int_{\frac{1}{2}}^{1} N(\sigma, T)\,d\sigma = O(T\log\log T). \tag{9.20.4}$$

These, however, are superseded by the following analysis, due to A. Selberg (2), the principal result of which is that

$$\int_{\frac{1}{2}}^{1} N(\sigma, T)\,d\sigma = O(T). \tag{9.20.5}$$

We consider the integral

$$\int_T^{T+U} |\zeta(\tfrac{1}{2}+it)\psi(\tfrac{1}{2}+it)|^2\,dt,$$

† Selberg (5). ‡ Littlewood (4).

where $0 < U \leqslant T$ and ψ is a function to be specified later. We use the formulae of §4.17. Since

$$e^{i\vartheta} = \{\chi(\tfrac{1}{2}+it)\}^{-\frac{1}{2}} = \left(\frac{t}{2\pi e}\right)^{\frac{1}{2}it} e^{-\frac{1}{8}\pi i}\left\{1+O\left(\frac{1}{t}\right)\right\},$$

we have

$$Z(t) = z(t)+\bar{z}(t)+O(t^{-\frac{1}{4}}), \tag{9.20.6}$$

where

$$z(t) = \left(\frac{t}{2\pi e}\right)^{\frac{1}{2}it} e^{-\frac{1}{8}\pi i} \sum_{n \leqslant x} n^{-\frac{1}{2}-it}$$

and $x = (t/2\pi)^{\frac{1}{2}}$. Let $T \leqslant t \leqslant T+U$, $\tau = (T/2\pi)^{\frac{1}{2}}$, $\tau' = \{(T+U)/2\pi\}^{\frac{1}{2}}$. Let

$$z_1(t) = \left(\frac{t}{2\pi e}\right)^{\frac{1}{2}it} e^{-\frac{1}{8}\pi i} \sum_{n \leqslant \tau} n^{-\frac{1}{2}-it}.$$

Proceeding as in § 7.3, we have

$$\int_T^{T+U} |z(t)-z_1(t)|^2\,dt = O\left(U \sum_{\tau<n\leqslant\tau'} \frac{1}{n}\right)+O(T^{\frac{1}{2}}\log T)$$

$$= O\left(U\frac{\tau'-\tau}{\tau}\right)+O(T^{\frac{1}{2}}\log T)$$

$$= O(U^2/T)+O(T^{\frac{1}{2}}\log T). \tag{9.20.7}$$

9.21. LEMMA 9.21. *Let m and n be positive integers, $(m, n) = 1$, $M = \max(m, n)$. Then*

$$\int_T^{T+U} z_1(t)\bar{z}_1(t)\left(\frac{n}{m}\right)^{it} dt = \frac{U}{(mn)^{\frac{1}{2}}} \sum_{r<\tau/M} \frac{1}{r}+O\{T^{\frac{1}{2}}M^2\log(MT)\}.$$

The integral is
$$\sum_{\mu\leqslant\tau}\sum_{\nu\leqslant\tau} \frac{1}{(\mu\nu)^{\frac{1}{2}}} \int_T^{T+U} \left(\frac{n\nu}{m\mu}\right)^{it} dt.$$

The terms with $m\mu = n\nu$ contribute

$$U\sum_{m\mu=n\nu}\sum \frac{1}{(\mu\nu)^{\frac{1}{2}}} = U \sum_{rn\leqslant\tau, rm\leqslant\tau} \frac{1}{(rn\,.\,rm)^{\frac{1}{2}}} = \frac{U}{(mn)^{\frac{1}{2}}} \sum_{r\leqslant\tau/M} \frac{1}{r}.$$

The remaining terms are

$$O\left\{\sum_{m\mu\neq n\nu}\sum \frac{1}{(\mu\nu)^{\frac{1}{2}}|\log(n\nu/m\mu)|}\right\} = O\left\{\sum_{m\mu\neq n\nu}\sum \frac{M}{(m\mu n\nu)^{\frac{1}{2}}|\log(n\nu/m\mu)|}\right\}$$

$$= O\left\{M \sum_{\kappa\leqslant M\tau, \lambda\leqslant M\tau}\sum \frac{1}{(\kappa\lambda)^{\frac{1}{2}}|\log\lambda/\kappa|}\right\} = O\{M^2\tau\log(M\tau)\},$$

and the result follows.

9.22. LEMMA 9.22. *Defining m, n, M as before, and supposing*

$$T^{\frac{4}{5}} < U \leqslant T,$$

$$\int_T^{T+U} z_1^2(t)\left(\frac{n}{m}\right)^{it} dt = \frac{U}{(mn)^{\frac{1}{2}}} \sum_{\tau/m\leqslant r\leqslant\tau/n} \frac{1}{r}+O(MT^{\frac{1}{2}})+O(U^2/T)+O(T^{\frac{9}{10}}) \tag{9.22.1}$$

if $n \leqslant m$. If $m < n$, the first term on the right-hand side is to be omitted.

The left-hand side is

$$e^{-\frac{1}{4}\pi i} \sum_{\mu\leqslant\tau}\sum_{\nu\leqslant\tau} \frac{1}{(\mu\nu)^{\frac{1}{2}}} \int_T^{T+U} \left(\frac{t}{2\pi e}\frac{n}{\mu\nu m}\right)^{it} dt.$$

The integral is of the form considered in § 4.6, with

$$F(t) = t\log\frac{t}{ec}, \qquad c = \frac{2\pi\mu\nu m}{n}.$$

Hence by (4.6.5), with $\lambda_2 = (T+U)^{-1}$, $\lambda_3 = (T+U)^{-2}$, it is equal to

$$(2\pi c)^{\frac{1}{2}}e^{\frac{1}{4}\pi i-ic}+O(T^{\frac{2}{5}})+O\left\{\min\left(\frac{1}{|\log c/T|}, T^{\frac{1}{2}}\right)\right\}$$

$$+O\left\{\min\left(\frac{1}{\log|(T+U)/c|}, T^{\frac{1}{2}}\right)\right\}, \tag{9.22.2}$$

with the leading term present only when $T \leqslant c \leqslant T+U$. We therefore obtain a main term

$$2\pi\left(\frac{m}{n}\right)^{\frac{1}{2}} \sum_{\mu\leqslant\tau}\sum_{\nu\leqslant\tau} e^{-2\pi i\mu\nu m/n} \tag{9.22.3}$$

where μ and ν also satisfy

$$\tau^2 n/m \leqslant \mu\nu \leqslant \tau'^2 n/m.$$

The double sum is clearly zero unless $n \leqslant m$, as we now suppose. The ν-summation runs over the range $\nu_1 \leqslant \nu \leqslant \nu_2$, where $\nu_1 = \tau^2 n/m\mu$ and $\nu_2 = \min(\tau'^2 n/m\mu, \tau)$, and μ runs over $\tau n/m \leqslant \mu \leqslant \tau$. The inner sum is therefore $\nu_2-\nu_1+O(n)$ if $n|\mu$, and $O(n)$ otherwise. The error term $O(n)$ contributes $O\{(mn)^{\frac{1}{2}}\tau\} = O(MT^{\frac{1}{2}})$ in (9.22.1). On writing $\mu = nr$ we are left with

$$2\pi\left(\frac{m}{n}\right)^{\frac{1}{2}} \sum_{\tau/m\leqslant r\leqslant\tau/n} (\nu_2-\nu_1).$$

Let $\nu_3 = \tau'^2/mr$. Then $\nu_2 = \nu_3$ unless $r < \tau'^2/m\tau$. Hence the error on replacing v_2 by v_3 is

$$O\left\{\left(\frac{m}{n}\right)^{\frac{1}{2}} \sum_{\tau/m \leqslant r < \tau'^2/m\tau} \left(\frac{\tau'^2}{mr} - \tau\right)\right\} = O\left\{\left(\frac{m}{n}\right)^{\frac{1}{2}} \left(\frac{\tau'^2}{m\tau} - \frac{\tau}{m} + 1\right)\left(\frac{\tau'^2}{\tau} - \tau\right)\right\}$$
$$= O\left\{(mn)^{-\frac{1}{2}} \left(\frac{\tau'^2 - \tau^2}{\tau}\right)^2\right\} + O\left\{\left(\frac{m}{n}\right)^{\frac{1}{2}} \left(\frac{\tau'^2 - \tau^2}{\tau}\right)\right\}$$
$$= O(U^2 T^{-1}) + O(M^{\frac{1}{2}} U T^{-\frac{1}{2}}).$$

Finally there remains

$$2\pi\left(\frac{m}{n}\right)^{\frac{1}{2}} \sum_{\tau/m \leqslant r \leqslant \tau/n} (v_3 - v_1) = 2\pi\left(\frac{m}{n}\right)^{\frac{1}{2}} \sum_{\tau/m \leqslant r \leqslant \tau/n} \left(\frac{\tau'^2}{mr} - \frac{\tau^2}{mr}\right)$$

$$= \frac{U}{(mn)^{\frac{1}{2}}} \sum_{\tau/m \leqslant r \leqslant \tau/n} \frac{1}{r}.$$

Now consider the O-terms arising from (9.22.2). The term $O(T^{\frac{2}{5}})$ gives

$$O\left\{T^{\frac{2}{5}} \sum_{\mu \leqslant \tau} \sum_{\nu \leqslant \tau} \frac{1}{(\mu\nu)^{\frac{1}{2}}}\right\} = O(T^{\frac{2}{5}}\tau) = O(T^{\frac{9}{10}}).$$

Next

$$\sum_{\mu \leqslant \tau} \sum_{\nu \leqslant \tau} \frac{1}{(\mu\nu)^{\frac{1}{2}}} \min\left(\frac{1}{|\log(2\pi\mu\nu m/nT)|}, T^{\frac{1}{2}}\right)$$
$$= O\left\{T^{\epsilon} \sum_{r \leqslant \tau^2} \frac{1}{r^{\frac{1}{2}}} \min\left(\frac{1}{|\log(rm/n\tau^2)|}, T^{\frac{1}{2}}\right)\right\}.$$

Suppose, for example, that $n < m$. Then the terms with $r < \frac{1}{2}n\tau^2/m$ or $r > 2n\tau^2/m$ are

$$O\left(T^{\epsilon} \sum_{r \leqslant \tau^2} \frac{1}{r^{\frac{1}{2}}}\right) = O(T^{\epsilon}\tau) = O(T^{\frac{1}{2}+\epsilon}).$$

In the other terms, let $r = [n\tau^2/m] - r'$. We obtain

$$O\left\{T^{\epsilon} \sum_{r'} \frac{1}{(n\tau^2/m)^{\frac{1}{2}}} \frac{1}{|r' - \theta|/(n\tau^2/m)}\right\} \quad (|\theta| < 1)$$
$$= O\left\{T^{\epsilon}\left(\frac{n\tau^2}{m}\right)^{\frac{1}{2}} \log T\right\} = O(T^{\frac{1}{2}+\epsilon}),$$

omitting the terms $r' = -1, 0, 1$; and these are $O(T^{\frac{1}{2}+\epsilon})$.
A similar argument applies in the other cases.

9.23. LEMMA 9.23. *Let $(m,n) = 1$ with $m, n \leqslant X \leqslant T^{\frac{1}{5}}$. If $T^{\frac{14}{15}} \leqslant U \leqslant T$, then*

$$\int_T^{T+U} Z^2(t)\left(\frac{n}{m}\right)^{it} dt = \frac{U}{(mn)^{\frac{1}{2}}}\left\{\log\frac{T}{2\pi mn} + 2\gamma\right\} + O(U^{\frac{3}{2}} T^{-\frac{1}{2}} \log T).$$

Let $Z(t) = z_1(t) + \overline{z_1(t)} + e(t)$. Then

$$\int_T^{T+U} \{z_1(t) + \overline{z_1(t)}\}^2 \left(\frac{n}{m}\right)^{it} dt$$
$$= \int_T^{T+U} Z(t)^2 \left(\frac{n}{m}\right)^{it} dt + O\left(\int_T^{T+U} |Z(t)e(t)|\,dt\right) + O\left(\int_T^{T+U} |e(t)|^2\,dt\right).$$

We have

$$\int_T^{T+U} |e(t)|^2\,dt = O(U^2/T) + O(T^{\frac{1}{2}}\log T) = O(U^2/T)$$

by (9.20.7), and

$$\int_T^{T+U} |Z(t)|^2\,dt = O(U\log T) + O(T^{\frac{1}{2}+\epsilon}) = O(U\log T),$$

by Theorem 7.4. Hence

$$\int_T^{T+U} |Z(t)e(t)|\,dt = O\{(U^2/T)^{\frac{1}{2}}(U\log T)^{\frac{1}{2}}\}$$

by Cauchy's inequality. It follows that

$$\int_T^{T+U} Z(t)^2 \left(\frac{n}{m}\right)^{it} dt$$
$$= \int_T^{T+U} \{z_1(t)^2 + \overline{z_1(t)^2} + 2z_1(t)\overline{z_1(t)}\}\left(\frac{n}{m}\right)^{it} dt + O(U^{\frac{3}{2}} T^{-\frac{1}{2}} \log^{\frac{1}{2}} T).$$

By Lemmas 9.21 and 9.22 the main integral on the right is

$$\frac{U}{(mn)^{\frac{1}{2}}}\left(\sum_{r \leqslant \tau/n} \frac{1}{r} + \sum_{r \leqslant \tau/m} \frac{1}{r}\right) + O\{T^{\frac{1}{2}} X^2 \log(XT)\} + O(XT^{\frac{1}{2}}) +$$
$$+ O(U^2/T) + O(T^{\frac{9}{10}})$$

whether $n \leqslant m$ or not. The result then follows, since

$$\sum_{r \leqslant \tau/n} \frac{1}{r} + \sum_{r \leqslant \tau/m} \frac{1}{r} = \log \frac{\tau^2}{mn} + 2\gamma + O\left(\frac{X}{\tau}\right),$$

and since the error terms $O\{T^{\frac{1}{2}}X^2 \log(XT)\}$, $O(XT^{\frac{1}{2}})$, $O(U^2/T)$, $O(T^{\frac{9}{10}})$ and $O(UXT^{-\frac{1}{2}})$ are all $O(U^{\frac{3}{2}}T^{-\frac{1}{2}}\log T)$.

9.24. THEOREM 9.24.

$$\int\limits_{\frac{1}{2}}^{1} N(\sigma, T)\, d\sigma = O(T). \tag{9.24.1}$$

Consider the integral

$$I = \int\limits_{T}^{T+U} |\zeta(\tfrac{1}{2}+it)\psi(\tfrac{1}{2}+it)|^2\, dt = \int\limits_{T}^{T+U} Z^2(t)|\psi(\tfrac{1}{2}+it)|^2\, dt,$$

where
$$\psi(s) = \sum_{r<X} \delta_r r^{1-s}$$

and
$$\delta_r = \frac{\sum\limits_{\rho r<X} \mu(\rho r)\mu(\rho)/\phi(\rho r)}{\sum\limits_{\rho<X} \mu^2(\rho)/\phi(\rho)} = \frac{\mu(r)}{\phi(r)} \frac{\sum\limits_{\rho r<X, (\rho,r)=1} \mu^2(\rho)/\phi(\rho)}{\sum\limits_{\rho<X} \mu^2(\rho)/\phi(\rho)}.$$

Clearly
$$|\delta_r| \leqslant \frac{1}{\phi(r)}$$

for all values of r. Now

$$I = \sum_{q<X} \sum_{r<X} \delta_q \delta_r q^{\frac{1}{2}} r^{\frac{1}{2}} \int\limits_{T}^{T+U} Z^2(t) \left(\frac{n}{m}\right)^{it} dt,$$

where $m = q/(q,r)$, $n = r/(q,r)$. Using Lemma 9.23, the main term contributes to this

$$\sum_{q<X} \sum_{r<X} \delta_q \delta_r q^{\frac{1}{2}} r^{\frac{1}{2}} \frac{U}{(mn)^{\frac{1}{2}}} \log \frac{Te^{2\gamma}}{2\pi mn} = U \sum_{q<X} \sum_{r<X} \delta_q \delta_r (q,r) \log \frac{Te^{2\gamma}(q,r)^2}{2\pi qr}$$

$$= U \log \frac{Te^{2\gamma}}{2\pi} \sum_{q<X} \sum_{r<X} \delta_q \delta_r (q,r) - 2U \sum_{q<X} \sum_{r<X} \delta_q \delta_r (q,r) \log q +$$
$$+ 2U \sum_{q<X} \sum_{r<X} \delta_q \delta_r (q,r) \log(q,r).$$

For a fixed $q < X$,

$$\sum_{r<X} (q,r)\, \delta_r = \left\{ \sum_{\rho<X} \frac{\mu^2(\rho)}{\phi(\rho)} \right\}^{-1} \sum_{r<X, \rho r<X} \sum \frac{(q,r)\mu(\rho r)\mu(\rho)}{\phi(\rho r)}.$$

Now
$$(q,r) = \sum_{\nu|(q,r)} \phi(\nu) = \sum_{\nu|q, \nu|r} \phi(\nu).$$

Hence the second factor on the right is

$$\sum_{r<X, \rho r<X} \sum \frac{\mu(\rho r)\mu(\rho)}{\phi(\rho r)} \sum_{\nu|q, \nu|r} \phi(\nu) = \sum_{\nu|q} \phi(\nu) \sum_{\substack{r<X, \rho r<X \\ \nu|r}} \sum \frac{\mu(\rho r)\mu(\rho)}{\phi(\rho r)}.$$

Put $\rho r = l$. Then $\rho\nu|\rho r$, $\rho\nu|l$, i.e. $\rho|(l/\nu)$. Hence we get

$$\sum_{\nu|q} \phi(\nu) \sum_{\substack{l<X \\ \nu|l}} \frac{\mu(l)}{\phi(l)} \sum_{\rho|(l/\nu)} \mu(\rho).$$

The ρ-sum is 0 unless $l = \nu$, when it is 1. Hence we get

$$\sum_{\nu|q} \phi(\nu) \frac{\mu(\nu)}{\phi(\nu)} = \sum_{\nu|q} \mu(\nu) = \begin{cases} 1 & (q=1), \\ 0 & (q>1). \end{cases}$$

Hence

$$\sum_{q<X} \sum_{r<X} \delta_q \delta_r (q,r) = \left\{ \sum_{\rho<X} \frac{\mu^2(\rho)}{\phi(\rho)} \right\}^{-1} \delta_1 = \left\{ \sum_{\rho<X} \frac{\mu^2(\rho)}{\phi(\rho)} \right\}^{-1}$$

and
$$\sum_{q<X} \sum_{r<X} \delta_q \delta_r (q,r) \log q = \left\{ \sum_{\rho<X} \frac{\mu^2(\rho)}{\phi(\rho)} \right\}^{-1} \delta_1 \log 1 = 0.$$

Let $\phi_a(n)$ be defined by

$$\sum_{n=1}^{\infty} \frac{\phi_a(n)}{n^s} = \frac{\zeta(s-a-1)}{\zeta(s)},$$

so that
$$\phi_a(n) = n^{1+a} \sum_{m|n} \frac{\mu(m)}{m^{1+a}} = n^{1+a} \prod_{p|n} \left(1 - \frac{1}{p^{1+a}}\right).$$

Let $\psi(n)$ be defined by

$$\sum_{n=1}^{\infty} \frac{\psi(n)}{n^s} = -\frac{\zeta'(s-1)}{\zeta(s)}.$$

Then
$$-\zeta'(s-1) = \zeta(s) \sum_{n=1}^{\infty} \frac{\psi(n)}{n^s},$$

and hence
$$n \log n = \sum_{d|n} \psi(d).$$

Hence
$$(q,r)\log(q,r) = \sum_{d|q, d|r} \psi(d)$$

and
$$\sum_{q<X} \sum_{r<X} \delta_q \delta_r (q,r) \log(q,r) = \sum_{d<X} \psi(d) \sum_{\substack{d|q, d|r \\ q<X, r<X}} \delta_q \delta_r$$
$$= \sum_{d<X} \psi(d) \Big(\sum_{d|q, q<X} \delta_q \Big)^2.$$

Now $$\psi(n) = \left[\frac{\partial}{\partial a}\phi_a(n)\right]_{a=0} = \phi(n)\left(\log n + \sum_{p|n}\frac{\log p}{p-1}\right),$$
$$\psi(n) \leqslant \phi(n)\left(\log n + \sum_{p|n}\log p\right) \leqslant 2\phi(n)\log n.$$
Also
$$\sum_{\substack{d|q\\ q<X}}\delta_q = \left\{\sum_{\rho<X}\frac{\mu^2(\rho)}{\phi(\rho)}\right\}^{-1}\sum_{\substack{\rho q<X\\ d|q}}\sum\frac{\mu(\rho q)\mu(\rho)}{\phi(\rho q)}$$
$$= \left\{\sum_{\rho<X}\frac{\mu^2(\rho)}{\phi(\rho)}\right\}^{-1}\sum_{\substack{n<X\\ d|n}}\frac{\mu(n)}{\phi(n)}\sum_{\rho|n/d}\mu(\rho) = \left\{\sum_{\rho<X}\frac{\mu^2(\rho)}{\phi(\rho)}\right\}^{-1}\frac{\mu(d)}{\phi(d)}.$$
Hence $$\sum_{q<X}\sum_{r<X}\delta_q\delta_r(q,r)\log(q,r) \leqslant 2\log X\left\{\sum_{\rho<X}\frac{\mu^2(\rho)}{\phi(\rho)}\right\}^{-1}.$$
Since
$$\sum_{n=1}^{\infty}\frac{\mu^2(n)}{\phi(n)n^s} = \prod_p\left(1+\frac{\mu^2(p)}{\phi(p)p^s}\right) = \prod_p\left(1+\frac{1}{(p-1)p^s}\right)$$
$$= \zeta(s+1)\prod_p\left(1-\frac{1}{p^{s+1}}\right)\left(1+\frac{1}{(p-1)p^s}\right),$$
we have $$\sum_{\rho<X}\frac{\mu^2(\rho)}{\phi(\rho)} \sim A\log X.$$
The contribution of all the above terms to I is therefore
$$O\left(U\frac{\log T}{\log X}\right)+O(U) = O(U)$$
on taking, say, $X = T^{\frac{1}{100}}$.

The O-term in Lemma 9.23 gives
$$O(U^{\frac{3}{2}}T^{-\frac{1}{2}}\log T)\sum_{q<X}\sum_{r<X}\frac{q^{\frac{1}{2}}r^{\frac{1}{2}}}{\phi(q)\phi(r)}$$
$$= O(U^{\frac{3}{2}}T^{-\frac{1}{2}}\log T)O(X)$$
$$= O(U^{\frac{3}{2}}T^{-\frac{49}{100}}\log T).$$
Taking say $U = T^{\frac{14}{15}}$, this is $O(U)$. Hence $I = O(U)$.

By an argument similar to that of § 9.16, it follows that
$$\int_{\frac{1}{2}}^{1}\{N(\sigma,T+U)-N(\sigma,T)\}\,d\sigma = O(U).$$

Replacing T by $T+U$ $T+2U$,... and adding, $O(T/U)$ terms, we obtain
$$\int_{\frac{1}{2}}^{1}\{N(\sigma,2T)-N(\sigma,T)\}\,d\sigma = O(T).$$
Replacing T by $\frac{1}{2}T$, $\frac{1}{4}T$,... and adding, the theorem follows.

It also follows that, if $\frac{1}{2} < \sigma \leqslant 1$,
$$N(\sigma,T) = \frac{2}{\sigma-\frac{1}{2}}\int_{\frac{1}{2}\sigma+\frac{1}{4}}^{\sigma}N(\sigma',T)\,d\sigma'$$
$$\leqslant \frac{2}{\sigma-\frac{1}{2}}\int_{\frac{1}{2}}^{1}N(\sigma,T)\,d\sigma = O\left(\frac{T}{\sigma-\frac{1}{2}}\right). \tag{9.24.2}$$

Lastly, *if $\phi(t)$ is positive and increases to infinity with t, all but an infinitesimal proportion of the zeros of $\zeta(s)$ in the upper half-plane lie in the region*
$$|\sigma-\tfrac{1}{2}| < \frac{\phi(t)}{\log t}.$$
The curved boundary of the region
$$\sigma = \tfrac{1}{2}+\frac{\phi(t)}{\log t}, \qquad T^{\frac{1}{2}} < t < T$$
lies to the right of $$\sigma = \sigma_1 = \tfrac{1}{2}+\frac{\phi(T^{\frac{1}{2}})}{\log T},$$
and $$N(\sigma_1,T) = O\left(\frac{T}{\sigma_1-\frac{1}{2}}\right) = O\left(\frac{T\log T}{\phi(T^{\frac{1}{2}})}\right) = o(T\log T).$$
Hence the number of zeros outside the region specified is $o(T\log T)$, and the result follows.

NOTES FOR CHAPTER 9

9.25. The mean value of $S(t)$ has been investigated by Selberg (5). One has
$$\int_0^T|S(t)|^{2k}\,dt \sim \frac{(2k)!}{k!(2\pi)^{2k}}T(\log\log T)^k \tag{9.25.1}$$
for every positive integer k. Selberg's earlier conditional treatment (4) is discussed in §§ 14.20–24, the key feature used in (5) to deal with zeros off the critical line being the estimate given in Theorem 9.19(C). Selberg (5) also gave an unconditional proof of Theorem 14.19, which had previously been established on the Riemann hypothesis by Littlewood.

These results have been investigated further by Fujii [1], [2] and Ghosh [1], [2], who give results which are uniform in k.

It follows in particular from Fujii [1] that

$$\int_0^T |S(t+h)-S(t)|^2\,dt = \pi^{-2}T\log(3+h\log T)+O[T\{\log(3+h\log T)\}^{\frac{1}{2}}] \tag{9.25.2}$$

and

$$\int_0^T |S(t+h)-S(t)|^{2k}\,dt \ll T\{Ak^4\log(3+h\log T)\}^k \tag{9.25.3}$$

uniformly for $0 \leqslant h \leqslant \frac{1}{2}T$. One may readily deduce that

$$N_j(T) \ll N(T)\,e^{-A\sqrt{j}},$$

where $N_j(T)$ denotes the number of zeros $\beta+i\gamma$ of multiplicity exactly j, in the range $0<\gamma\leqslant T$. Moreover one finds that

$$\#\{n: 0<\gamma_n\leqslant T, \gamma_{n+1}-\gamma_n \geqslant \lambda/\log T\} \ll N(T)\exp\{-A\lambda^{\frac{1}{2}}(\log\lambda)^{-\frac{1}{4}}\},$$

uniformly for $\lambda \geqslant 2$, whence, in particular,

$$\sum_{0<\gamma_n\leqslant T} (\gamma_{n+1}-\gamma_n)^k \ll \frac{N(T)}{(\log T)^k}, \tag{9.25.4}$$

for any fixed $k \geqslant 0$. Fujii [2] also states that there exist constants $\lambda>1$ and $\mu<1$ such that

$$\frac{\gamma_{n+1}-\gamma_n}{2\pi/\log\gamma_n} \geqslant \lambda \tag{9.25.5}$$

and

$$\frac{\gamma_{n+1}-\gamma_n}{2\pi/\log\gamma_n} \leqslant \mu \tag{9.25.6}$$

each hold for a positive proportion of n (i.e. the number of n for which $0<\gamma_n\leqslant T$ is at least $AN(T)$ if $T\geqslant T_0$). Note that $2\pi/\log\gamma_n$ is the average spacing between zeros. The possibility of results such as (9.25.5) and (9.25.6) was first observed by Selberg [1].

9.26. Since the deduction of the results (9.25.5) and (9.25.6) is not obvious, we give a sketch. If M is a sufficiently large integer constant,

then (9.25.2) and (9.25.3) yield

$$\int_T^{2T} |S(t+h)-S(t)|^2 dt \gg T$$

and

$$\int_T^{2T} |S(t+h)-S(t)|^4\,dt \ll T$$

uniformly for

$$\frac{2\pi M}{\log T} \leqslant h \leqslant \frac{4\pi M}{\log T}.$$

By Hölder's inequality we have

$$\int_T^{2T} |S(t+h)-S(t)|^2 dt \leqslant \left(\int_T^{2T} |S(t+h)-S(t)|\,dt\right)^{\frac{2}{3}} \times \left(\int_T^{2T} |S(t+h)-S(t)|^4\,dt\right)^{\frac{1}{3}},$$

so that

$$\int_T^{2T} |S(t+h)-S(t)|\,dt \gg T.$$

We now observe that

$$S(t+h)-S(t) = N(t+h)-N(t)-\frac{h\log T}{2\pi}+O\left(\frac{1}{\log T}\right),$$

for $T\leqslant t\leqslant 2T$, whence

$$\int_T^{2T} \left|N(t+h)-N(t)-\frac{h\log T}{2\pi}\right| dt \gg T.$$

We proceed to write $h = 2\pi M\lambda/\log T$ and

$$\delta(t,\lambda) = N\left(t+\frac{2\pi\lambda}{\log T}\right)-N(t)-\lambda,$$

so that

$$N(t+h)-N(t)-\frac{h\log T}{2\pi}=\sum_{m=0}^{M-1}\delta\left(t+\frac{2\pi m\lambda}{\log T},\lambda\right).$$

Thus

$$T \ll \sum_{m=0}^{M-1}\int_{T+2\pi m\lambda/\log T}^{2T+2\pi m\lambda/\log T}|\delta(t,\lambda)|dt$$

$$= M\int_T^{2T}|\delta(t,\lambda)|dt+O(1),$$

and hence

$$\int_T^{2T}|\delta(t,\lambda)|dt \gg T \tag{9.26.1}$$

uniformly for $1\leqslant\lambda\leqslant 2$, since M is constant.

Now, if I is the subset of $[T, 2T]$ on which $N\left(t+\frac{2\pi\lambda}{\log T}\right)=N(t)$, then

$$|\delta(t,\lambda)|\leqslant\begin{cases}\delta(t,\lambda)+2\lambda & (t\in I),\\ \delta(t,\lambda)+2\lambda-2 & (t\in[T,2T]-I),\end{cases}$$

so that (9.26.1) yields

$$T \ll \int_T^{2T}\delta(t,\lambda)dt+(2\lambda-2)T+2m(I),$$

where $m(I)$ is the measure of I. However

$$\int_T^{2T}\delta(t,\lambda)\,dt=O\left(\frac{T}{\log T}\right),$$

whence $m(I)\gg T$, if $\lambda>1$ is chosen sufficiently close to 1. Thus, if

$$S=\left\{n: T\leqslant\gamma_n\leqslant 2T,\ \gamma_{n+1}-\gamma_n\geqslant\frac{2\pi\lambda}{\log T}\right\},$$

then

$$T\ll m(I)\ll\sum_{n\in S}(\gamma_{n+1}-\gamma_n)+O(1),$$

so that

$$T^2\ll\left\{\sum_{n\in S}(\gamma_{n+1}-\gamma_n)\right\}^2\leqslant(\# S)\left(\sum_{n\in S}(\gamma_{n+1}-\gamma_n)^2\right)$$

$$\ll\# S\frac{T}{\log T},$$

by (9.25.4) with $k=2$. It follows that

$$\# S\gg N(T), \tag{9.26.2}$$

proving that (9.25.5) holds for a positive proportion of n.

Now suppose that μ is a constant in the range $0<\mu<1$, and put

$$U=\{n: T\leqslant\gamma_n\leqslant 2T\},$$

and

$$V=\left\{n\in U: \gamma_{n+1}-\gamma_n\leqslant\frac{2\pi\mu}{\log T}\right\},$$

whence $\# U=\frac{T}{2\pi}\log T+O(T)$. Then

$$T=\sum_{n\in U}(\gamma_{n+1}-\gamma_n)+O(1)$$

$$\geqslant\sum_{n\in U-V}(\gamma_{n+1}-\gamma_n)+O(1)$$

$$\geqslant\frac{2\pi\mu}{\log T}(\# U-\# V-\# S)+\frac{2\pi\lambda}{\log T}S+O(1)$$

$$=\frac{2\pi\mu}{\log T}\left(\frac{T}{2\pi}\log T-\# V\right)+\frac{2\pi(\lambda-\mu)}{\log T}\# S+O\left(\frac{T}{\log T}\right).$$

If the implied constant in (9.26.2) is η, it follows that $\# V\gg N(T)$, on taking $\mu=1-\nu$, with $0<\nu<\eta(\lambda-1)/(1-\eta)$. Thus (9.25.6) also holds for a positive proportion of n.

9.27. Ghosh [1] was able to sharpen the result of Selberg mentioned at the end of §9.10, to show that $S(t)$ has at least

$$T(\log T)\exp\left(-\frac{A\log\log T}{(\log\log\log T)^{\frac{1}{2}-\delta}}\right)$$

sign changes in the range $0\leqslant t\leqslant T$, for any positive δ, and $A=A(\delta)$, $T\geqslant T(\delta)$. He also proved (Ghosh [2]) that the asymptotic formula (9.25.1) holds for any positive real k, with the constant on the right hand side replaced by $\Gamma(2k+1)/\Gamma(k+1)(2\pi)^{2k}$. Moreover he showed (Ghosh [2]) that

$$\frac{|S(t)|}{\sqrt{(\log\log t)}} = f(t),$$

say, has a limiting distribution

$$P(\sigma) = 2\pi^{\frac{1}{2}} \int_0^\sigma e^{-\pi^2 z^2} dz,$$

in the sense that, for any $\sigma > 0$, the measure of the set of $t \in [0, T]$ for which $f(t) \leqslant \sigma$, is asymptotically $TP(\sigma)$. (A minor error in Ghosh's statement of the result has been corrected here.)

9.28. A great deal of work has been done on the 'zero-density estimates' of §§9.15–19, using an idea which originates with Halász [1]. However it is not possible to combine this with the method of §9.16, based on Littlewood's formula (9.9.1). Instead one argues as follows (Montgomery [1; Chapter 12]). Let

$$M_X(s)\zeta(s) = \sum_1^\infty a_n n^{-s},$$

so that $a_n = 0$ for $2 \leqslant n \leqslant X$. If $\zeta(\rho) = 0$, where $\rho = \beta + i\gamma$ and $\beta > \frac{1}{2}$, then we have

$$e^{-1/Y} + \sum_{n>X} a_n n^{-\rho} e^{-n/Y} = \sum_{n=1}^\infty a_n n^{-\rho} e^{-n/Y}$$

$$= \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} M_X(s+\rho)\zeta(s+\rho)\Gamma(s) Y^s ds,$$

by the lemma of §7.9. On moving the line of integration to $\mathbf{R}(s) = \frac{1}{2} - \beta$ this yields

$$M_X(1)\Gamma(1-\rho)Y^{1-\rho} +$$

$$+\frac{1}{2\pi i} \int_{-\infty}^{\infty} M_X(\tfrac{1}{2}+it)\zeta(\tfrac{1}{2}+it)\Gamma(\tfrac{1}{2}-\beta+i(t-\gamma))Y^{\frac{1}{2}-\beta+i(t-\gamma)} dt,$$

since the pole of $\Gamma(s)$ at $s = 0$ is cancelled by the zero of $\zeta(s+\rho)$. If we now assume that $\log^2 T \leqslant \gamma \leqslant T$, and that $\log T \ll \log X$, $\log Y \ll \log T$, then $e^{-1/Y} \gg 1$ and

$$M_X(1)\Gamma(1-\rho)Y^{1-\rho} = o(1),$$

whence either

$$\left|\sum_{n>X} a_n n^{-\rho} e^{-n/Y}\right| \gg 1$$

or

$$\int_{-\infty}^{\infty} |M_X(\tfrac{1}{2}+it)\zeta(\tfrac{1}{2}+it)\Gamma(\tfrac{1}{2}-\beta+i(t-\gamma))|\, dt \gg Y^{\beta-\frac{1}{2}}.$$

In the latter case one has

$$|M_X(\tfrac{1}{2}+it_\rho)\zeta(\tfrac{1}{2}+it_\rho)| \gg (\beta-\tfrac{1}{2})\, Y^{\beta-\frac{1}{2}}$$

for some t_ρ in the range $|t_\rho - \gamma| \leqslant \log^2 T$. The problem therefore reduces to that of counting discrete points at which one of the Dirichlet series $\Sigma a_n n^{-s} e^{-n/Y}$, $M_X(s)$, and $\zeta(s)$ is large. In practice it is more convenient to take finite Dirichlet polynomials approximating to these.

The methods given in §§9.17–19 correspond to the use of a mean-value bound. Thus Montgomery [1; Chapter 7] showed that

$$\sum_{r=1}^R \left|\sum_{n=1}^N a_n n^{-s_r}\right|^2 \ll (T+N)(\log N)^2 \sum_{n=1}^N |a_n|^2 n^{-2\sigma} \qquad (9.28.1)$$

for any points s_r satisfying

$$\mathbf{R}(s_r) \geqslant \sigma, \qquad |\mathbf{I}(s_r)| \leqslant T, \qquad \mathbf{I}(s_{r+1} - s_r) \geqslant 1, \qquad (9.28.2)$$

and any complex a_n. Theorems 9.17, 9.18, 9.19(A), and 9.19(B) may all be recovered from this (except possibly for worse powers of log T). However one may also use Halász's lemma. One simple form of this (Montgomery [1; Theorem 8.2]) gives

$$\sum_{r=1}^R \left|\sum_{n=1}^N a_n n^{-s_r}\right|^2 \ll (N+RT^{\frac{1}{2}})(\log T) \sum_{n=1}^N |a_n|^2 n^{-2\sigma} \quad (9.28.3)$$

for any points s_r satisfying (9.28.2). Under suitable circumstances this implies a sharper bound for R than does (9.28.1). Under the Lindelöf hypothesis one may replace the term $RT^{\frac{1}{2}}$ in (9.28.3) by $RT^{\epsilon}N^{\frac{1}{2}}$, which is superior, since one invariably takes $N \leqslant T$ in applying the Halász lemma. (If $N \geqslant T$ it would be better to use (9.28.1).) Moreover Montgomery [1; Chapter 9] makes the conjecture (the Large Values

Conjecture):

$$\sum_{r=1}^{R}\left|\sum_{n=1}^{N} a_n n^{-s_r}\right|^2 \ll (N+RT^{\varepsilon})\sum_{n=1}^{N}|a_n|^2 n^{-2\sigma}$$

for points s_r satisfying (9.28.2). Using the Halász lemma with the Lindelöf hypothesis one obtains

$$N(\sigma, T) \ll T^{\varepsilon}, \qquad \tfrac{3}{4}+\varepsilon \leqslant \sigma \leqslant 1, \tag{9.28.4}$$

(Halász and Turán [1], Montgomery [1; Theorem 12.3]). If the Large Values Conjecture is true then the Lindelöf hypothesis gives the wider range $\frac{1}{2}+\varepsilon \leqslant \sigma \leqslant 1$ for (9.28.4)

9.29. The picture for unconditional estimates is more complex. At present it seems that the Halász method is only useful for $\sigma \geqslant \frac{3}{4}$. Thus Ingham's result, Theorem 9.19(B), is still the best known for $\frac{1}{2} < \sigma \leqslant \frac{3}{4}$. Using (9.28.3), Montgomery [1; Theorem 12.1] showed that

$$N(\sigma, T) \ll T^{2(1-\sigma)/\sigma}(\log T)^{14} \qquad (\tfrac{4}{5} \leqslant \sigma \leqslant 1),$$

which is superior to Theorem 9.19(B). This was improved by Huxley [1] to give

$$N(\sigma, T) \ll T^{3(1-\sigma)/(3\sigma-1)}(\log T)^{44} \quad (\tfrac{3}{4} \leqslant \sigma \leqslant 1). \tag{9.29.1}$$

Huxley used the Halász lemma in the form

$$R \ll \left\{NV^{-2}\sum_{n=1}^{N}|a_n|^2 n^{-2\sigma} + TNV^{-6}\left(\sum_{n=1}^{N}|a_n|^2 n^{-2\sigma}\right)^3\right\}(\log T)^2,$$

for points s_r satisfying (9.28.2) and the condition

$$\left|\sum_{n=1}^{N} a_n n^{-s_r}\right| \geqslant V.$$

In conjunction with Theorem 9.19(B), Huxley's result yields

$$N(\sigma, T) \ll T^{12/5(1-\sigma)}(\log T)^{44} \quad (\tfrac{1}{2} \leqslant \sigma \leqslant 1),$$

(c.f. (9.18.3)). A considerable number of other estimates have been given, for which the interested reader is referred to Ivic [3; Chapter 11]. We mention only a few of the most significant. Ivic [2] showed that

$$N(\sigma, T) \ll \begin{cases} T^{(3-3\sigma)/(7\sigma-4)+\varepsilon} & (\frac{3}{4} \leqslant \sigma \leqslant \frac{10}{13}) \\ T^{(9-9\sigma)/(8\sigma-2)+\varepsilon} & (\frac{10}{13} \leqslant \sigma \leqslant 1), \end{cases}$$

which supersede Huxley's result (9.29.1) throughout the range $\frac{3}{4} < \sigma < 1$. Jutila [1] gave a more powerful, but more complicated, result,

which has a similar effect. His bounds also imply the 'Density hypothesis' $N(\sigma, T) \ll T^{2-2\sigma+\varepsilon}$, for $\frac{11}{14} \leqslant \sigma \leqslant 1$. Heath-Brown [6] improved this by giving

$$N(\sigma, T) \ll T^{(9-9\sigma)/(7\sigma-1)+\varepsilon} \quad (\tfrac{11}{14} \leqslant \sigma \leqslant 1).$$

When σ is very close to 1 one can use the Vinogradov–Korobov exponential sum estimates, as described in Chapter 6. These lead to

$$N(\sigma, T) \ll T^{A(1-\sigma)^{3/2}}(\log T)^{A'},$$

for suitable numerical constants A and A', (see Montgomery [1; Corollary 12.5], who gives $A = 1334$, after correction of a numerical error).

Selberg's estimate given in Theorem 9.19(C) has been improved by Jutila [2] to give

$$N(\sigma, T) \ll T^{1-(1-\delta)(\sigma-\frac{1}{2})}\log T$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 1$, for any fixed $\delta > 0$.

9.30. Of course Theorem 19.24 is an immediate consequence of Theorem 19.9(C), but the proof is a little easier. The coefficients δ_r used in §9.24 are essentially

$$\mu(r)r^{-1}\frac{\log X/r}{\log X},$$

and indeed a more careful analysis yields

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^2\left|\sum_{r\leqslant X}\mu(r)\frac{\log X/r}{\log X}r^{-\frac{1}{2}-it}\right|^2 dt \sim T\left(1+\frac{\log T}{\log X}\right).$$

Here one can take $X \leqslant T^{\frac{1}{2}-\varepsilon}$ using fairly standard techniques, or $X \leqslant T^{\frac{8}{17}-\varepsilon}$ by employing estimates for Kloosterman sums (see Balasubramanian, Conrey and Heath-Brown [1]). The latter result yields (9.24.1) with the implied constant 0·0845.

X

THE ZEROS ON THE CRITICAL LINE

10.1. General discussion. The memoir in which Riemann first considered the zeta-function has become famous for the number of ideas it contains which have since proved fruitful, and it is by no means certain that these are even now exhausted. The analysis which precedes his observations on the zeros is particularly interesting. He obtains, as in § 2.6, the formula

$$\Gamma(\tfrac{1}{2}s)\pi^{-\frac{1}{2}s}\zeta(s) = \frac{1}{s(s-1)}+\int\limits_1^\infty \psi(x)(x^{\frac{1}{2}s-1}+x^{-\frac{1}{2}-\frac{1}{2}s})\,dx,$$

where $$\psi(x) = \sum_{n=1}^\infty e^{-n^2\pi x}.$$

Multiplying by $\frac{1}{2}s(s-1)$, and putting $s = \frac{1}{2}+it$, we obtain

$$\Xi(t) = \tfrac{1}{2}-(t^2+\tfrac{1}{4})\int\limits_1^\infty \psi(x)x^{-\frac{3}{4}}\cos(\tfrac{1}{2}t\log x)\,dx. \tag{10.1.1}$$

Integrating by parts, and using the relation

$$4\psi'(1)+\psi(1) = -\tfrac{1}{2},$$

which follows at once from (2.6.3), we obtain

$$\Xi(t) = 4\int\limits_1^\infty \frac{d}{dx}\{x^{\frac{3}{2}}\psi'(x)\}x^{-\frac{1}{4}}\cos(\tfrac{1}{2}t\log x)\,dx. \tag{10.1.2}$$

Riemann then observes:

'Diese Function ist für alle endlichen Werthe von t endlich, und lässt sich nach Potenzen von tt in eine sehr schnell convergirende Reihe entwickeln. Da für einen Werth von s, dessen reeller Bestandtheil grösser als 1 ist, $\log\zeta(s) = -\sum\log(1-p^{-s})$ endlich bleibt, und von den Logarithmen der übrigen Factoren von $\Xi(t)$ dasselbe gilt, so kann die Function $\Xi(t)$ nur verschwinden, wenn der imaginäre Theil von t zwischen $\frac{1}{2}i$ und $-\frac{1}{2}i$ liegt. Die Anzahl der Wurzeln von $\Xi(t) = 0$, deren reeller Theil zwischen 0 und T liegt, ist etwa

$$= \frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi};$$

denn dass Integral $\int d\log\Xi(t)$ positive um den Inbegriff der Werthe von t erstreckt, deren imaginäre Theil zwischen $\frac{1}{2}i$ und $-\frac{1}{2}i$, und deren reeller Theil zwischen 0 und T liegt, ist (bis auf einen Bruchtheil von der Ordnung der Grösse $1/T$) gleich $\{T\log(T/2\pi)-T\}i$; dieses Integral aber ist gleich der Anzahl der in diesem Gebiet liegenden Wurzeln von $\Xi(t) = 0$, multiplicirt mit $2\pi i$. Man findet nun in der That etwa so viel reelle Wurzeln innerhalb dieser Grenzen, und es ist sehr wahrscheinlich, dass alle Wurzeln reelle sind.'

This statement, that all the zeros of $\Xi(t)$ are real, is the famous 'Riemann hypothesis', which remains unproved to this day. The memoir goes on:

'Hiervon wäre allerdings ein strenger Beweis zu wünschen; ich habe indess die Aufsuchung desselben nach einigen flüchtigen vergeblichen Versuchen vorläufig bei Seite gelassen, da er für den nächsten Zweck meiner Untersuchung [i.e. the explicit formula for $\pi(x)$] entbehrlich schien.'

In the approximate formula for $N(T)$, Riemann's $1/T$ may be a mistake for $\log T$; for, since $N(T)$ has an infinity of discontinuities at least equal to 1, the remainder cannot tend to zero. With this correction, Riemann's first statement is Theorem 9.4, which was proved by von Mangoldt many years later.

Riemann's second statement, on the real zeros of $\Xi(t)$, is more obscure, and his exact meaning cannot now be known. It is, however, possible that anyone encountering the subject for the first time might argue as follows. We can write (10.1.2) in the form

$$\Xi(t) = 2\int\limits_0^\infty \Phi(u)\cos ut\,du, \tag{10.1.3}$$

where $$\Phi(u) = 2\sum_{n=1}^\infty (2n^4\pi^2e^{\frac{9}{2}u}-3n^2\pi e^{\frac{5}{2}u})e^{-n^2\pi e^{2u}}. \tag{10.1.4}$$

This series converges very rapidly, and one might suppose that an approximation to the truth could be obtained by replacing it by its first term; or perhaps better by

$$\Phi^*(u) = 2\pi^2\cosh\tfrac{9}{2}u\,e^{-2\pi\cosh 2u},$$

since this, like $\Phi(u)$, is an even function of u, which is asymptotically equivalent to $\Phi(u)$. We should thus replace $\Xi(t)$ by

$$\Xi^*(t) = 4\pi^2\int\limits_0^\infty \cosh\tfrac{9}{2}u\,e^{-2\pi\cosh 2u}\cos ut\,du.$$

The asymptotic behaviour of $\Xi^*(t)$ can be found by the method of steepest descents. To avoid the calculation we shall quote known Bessel-function formulae. We have†

$$K_z(a) = \int\limits_0^\infty e^{-a\cosh u}\cosh zu\,du,$$

and hence $$\Xi^*(t) = \pi^2\{K_{\frac{9}{4}+\frac{1}{2}it}(2\pi)+K_{\frac{9}{4}-\frac{1}{2}it}(2\pi)\}.$$

For fixed z, as $\nu\to\infty$

$$I_\nu(z) \sim (\tfrac{1}{2}z)^\nu/\Gamma(\nu+1).$$

† Watson, *Theory of Bessel Functions*, 6.22 (5).

Hence

$$I_{-\frac{9}{4}-\frac{1}{2}it}(2\pi) \sim \frac{\pi^{-\frac{9}{4}-\frac{1}{2}it}}{\Gamma(-\frac{5}{4}-\frac{1}{2}it)} \sim \frac{1}{\pi\sqrt{2}}\, e^{\frac{1}{4}\pi t}\left(\frac{t}{2\pi}\right)^{\frac{7}{4}}\left(\frac{t}{2\pi e}\right)^{\frac{1}{2}it} e^{-\frac{7}{8}i\pi},$$

$$I_{\frac{9}{4}+\frac{1}{2}it}(2\pi) \sim \frac{\pi^{\frac{9}{4}+\frac{1}{2}it}}{\Gamma(\frac{13}{4}+\frac{1}{2}it)} = O(e^{\frac{1}{4}\pi t}t^{-\frac{11}{4}}),$$

$$K_{\frac{9}{4}+\frac{1}{2}it}(2\pi) = \tfrac{1}{2}\pi \operatorname{cosec}\pi(\tfrac{9}{4}+\tfrac{1}{2}it)\{I_{-\frac{9}{4}-\frac{1}{2}it}(2\pi) - I_{\frac{9}{4}+\frac{1}{2}it}(2\pi)\}$$

$$\sim \frac{1}{\sqrt{2}}\, e^{-\frac{1}{4}\pi t}\left(\frac{t}{2\pi}\right)^{\frac{7}{4}}\left(\frac{t}{2\pi e}\right)^{\frac{1}{2}it} e^{\frac{7}{8}i\pi}.$$

Hence
$$\Xi^*(t) \sim \pi^{\frac{1}{4}}2^{-\frac{5}{4}}t^{\frac{7}{4}}e^{-\frac{1}{4}\pi t}\cos\left(\tfrac{1}{2}t\log\frac{t}{2\pi e}+\tfrac{7}{8}\pi\right).$$

The right-hand side has zeros at

$$\tfrac{1}{2}t\log\frac{t}{2\pi e}+\tfrac{7}{8}\pi = (n+\tfrac{1}{2})\pi,$$

and the number of these in the interval $(0, T)$ is

$$\frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi}+O(1).$$

The similarity to the formula for $N(T)$ is indeed striking.

However, if we try to work on this suggestion, difficulties at once appear. We can write

$$\Xi(t)-\Xi^*(t) = \int_{-\infty}^{\infty}\{\Phi(u)-\Phi^*(u)\}e^{iut}\,du.$$

To show that this is small compared with $\Xi(t)$ we should want to move the line of integration into the upper half-plane, at least as far as $\mathbf{I}(u) = \frac{1}{4}\pi$; and this is just where the series for $\Phi(u)$ ceases to converge. Actually
$$|\Xi(t)| > At^{\frac{7}{4}}e^{-\frac{1}{4}\pi t}|\zeta(\tfrac{1}{2}+it)|,$$
and $|\zeta(\frac{1}{2}+it)|$ is unbounded, so that the suggestion that $\Xi^*(t)$ is an approximation to $\Xi(t)$ is false, at any rate if it is taken in the most obvious sense.

10.2. Although every attempt to prove the Riemann hypothesis, that all the complex zeros of $\zeta(s)$ lie on $\sigma = \frac{1}{2}$, has failed, it is known *that* $\zeta(s)$ *has an infinity of zeros on* $\sigma = \frac{1}{2}$. This was first proved by Hardy in 1914. We shall give here a number of different proofs of this theorem.

First method.† We have
$$\Xi(t) = -\tfrac{1}{2}(t^2+\tfrac{1}{4})\pi^{-\frac{1}{4}-\frac{1}{2}it}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}it)\zeta(\tfrac{1}{2}+it),$$
where $\Xi(t)$ is an even integral function of t, and is real for real t. A zero of $\zeta(s)$ on $\sigma = \frac{1}{2}$ therefore corresponds to a real zero of $\Xi(t)$, and it is a question of proving that $\Xi(t)$ has an infinity of real zeros.

† Hardy (1).

Putting $x = -i\alpha$ in (2.16.2), we have

$$\frac{2}{\pi}\int_0^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}}\cosh\alpha t\,dt = e^{-\frac{1}{2}i\alpha}-2e^{\frac{3}{2}i\alpha}\psi(e^{2i\alpha})$$
$$= 2\cos\tfrac{1}{2}\alpha-2e^{\frac{3}{2}i\alpha}\{\tfrac{1}{2}+\psi(e^{2i\alpha})\}. \qquad (10.2.1)$$

Since $\zeta(\frac{1}{2}+it) = O(t^A)$, $\Xi(t) = O(t^Ae^{-\frac{1}{4}\pi t})$, and the above integral may be differentiated with respect to α any number of times provided that $\alpha < \frac{1}{4}\pi$. Thus

$$\frac{2}{\pi}\int_0^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}}t^{2n}\cosh\alpha t\,dt = \frac{(-1)^n\cos\frac{1}{2}\alpha}{2^{2n-1}}-2\left(\frac{d}{d\alpha}\right)^{2n}e^{\frac{3}{2}i\alpha}\{\tfrac{1}{2}+\psi(e^{2i\alpha})\}.$$

We next prove that the last term tends to 0 as $\alpha \to \frac{1}{4}\pi$, for every fixed n. The equation (2.6.3) gives at once the functional equation

$$x^{-\frac{1}{4}}-2x^{\frac{1}{4}}\psi(x) = x^{\frac{1}{4}}-2x^{-\frac{1}{4}}\psi\left(\frac{1}{x}\right),$$

or
$$\psi(x) = x^{-\frac{1}{2}}\psi\left(\frac{1}{x}\right)+\tfrac{1}{2}x^{-\frac{1}{2}}-\tfrac{1}{2}.$$

Hence
$$\psi(i+\delta) = \sum_1^\infty e^{-n^2\pi(i+\delta)} = \sum_1^\infty(-1)^ne^{-n^2\pi\delta}$$
$$= 2\psi(4\delta)-\psi(\delta)$$
$$= \frac{1}{\sqrt{\delta}}\psi\left(\frac{1}{4\delta}\right)-\frac{1}{\sqrt{\delta}}\psi\left(\frac{1}{\delta}\right)-\frac{1}{2}.$$

It is easily seen from this that $\frac{1}{2}+\psi(x)$ and all its derivatives tend to zero as $x \to i$ along any route in an angle $|\arg(x-i)| < \frac{1}{2}\pi$.

We have thus proved that

$$\lim_{\alpha\to\frac{1}{4}\pi}\int_0^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}}t^{2n}\cosh\alpha t\,dt = \frac{(-1)^n\pi\cos\frac{1}{8}\pi}{2^{2n}}. \qquad (10.2.2)$$

Suppose now that $\Xi(t)$ were ultimately of one sign, say, for example, positive for $t \geqslant T$. Then

$$\lim_{\alpha\to\frac{1}{4}\pi}\int_T^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}}t^{2n}\cosh\alpha t\,dt = L,$$

say. Hence
$$\int_T^{T'} \frac{\Xi(t)}{t^2+\frac{1}{4}}t^{2n}\cosh\alpha t\,dt \leqslant L$$

for all $\alpha < \frac{1}{4}\pi$ and $T' > T$. Hence, making $\alpha \to \frac{1}{4}\pi$,

$$\int_T^{T'} \frac{\Xi(t)}{t^2+\frac{1}{4}} t^{2n} \cosh \tfrac{1}{4}\pi t \, dt \leqslant L.$$

Hence the integral $$\int_0^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}} t^{2n} \cosh \tfrac{1}{4}\pi t \, dt$$

is convergent. The integral on the left of (10.2.2) is therefore uniformly convergent with respect to α for $0 \leqslant \alpha \leqslant \frac{1}{4}\pi$, and it follows that

$$\int_0^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}} t^{2n} \cosh \tfrac{1}{4}\pi t \, dt = \frac{(-1)^n \pi \cos \frac{1}{8}\pi}{2^{2n}}$$

for every n.

This, however, is impossible; for, taking n odd, the right-hand side is negative, and hence

$$\int_T^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}} t^{2n} \cosh \tfrac{1}{4}\pi t \, dt < -\int_0^T \frac{\Xi(t)}{t^2+\frac{1}{4}} t^{2n} \cosh \tfrac{1}{4}\pi t \, dt$$
$$< KT^{2n},$$

where K is independent of n. But by hypothesis there is a positive $m = m(T)$ such that $\Xi(t)/(t^2+\frac{1}{4}) \geqslant m$ for $2T \leqslant t \leqslant 2T+1$. Hence

$$\int_T^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}} t^{2n} \cosh \tfrac{1}{4}\pi t \, dt \geqslant \int_{2T}^{2T+1} m t^{2n} \, dt \geqslant m(2T)^{2n}.$$

Hence $$m 2^{2n} < K,$$

which is false for sufficiently large n. This proves the theorem.

10.3. A variant of the above proof depends on the following theorem of Fejér:†

Let n be any positive integer. Then the number of changes in sign in the interval $(0, a)$ of a continuous function $f(x)$ is not less than the number of changes in sign of the sequence

$$f(0), \quad \int_0^a f(t)\, dt, \quad \ldots, \quad \int_0^a f(t) t^n \, dt. \tag{10.3.1}$$

We deduce this from the following theorem of Fekete:‡

† Fejér (1). ‡ Fekete (1).

The number of changes in sign in the interval $(0, a)$ of a continuous function $f(x)$ is not less than the number of changes in sign of the sequence

$$f(a), \quad f_1(a), \quad \ldots, \quad f_n(a), \tag{10.3.2}$$

where

$$f_\nu(x) = \int_0^x f_{\nu-1}(t)\, dt \qquad (\nu = 1, 2, \ldots, n), \qquad f_0(x) = f(x).$$

To prove Fekete's theorem, suppose first that $n = 1$. Consider the curve $y = f_1(x)$. Now $f_1(0) = 0$, and, if $f(a)$ and $f_1(a)$ have opposite signs, y is positive decreasing or negative increasing at $x = a$. Hence $f(x)$ has at least one zero.

Now assume the theorem for $n-1$. Suppose that there are k changes of sign in the sequence $f_1(x), \ldots, f_n(x)$. Then $f_1(x)$ has at least k changes of sign. We have then to prove that

(i) if $f(a)$ and $f_1(a)$ have the same sign, $f(x)$ has at least k changes of sign,

(ii) if $f(a)$ and $f_1(a)$ have opposite signs, $f(x)$ has at least $k+1$ changes of sign.

Each of these cases is easily verified by considering the curve $y = f_1(x)$. This proves Fekete's theorem.

To deduce Fejér's theorem, we have

$$f_\nu(x) = \frac{1}{(\nu-1)!} \int_0^x (x-t)^{\nu-1} f(t)\, dt,$$

and hence

$$f_\nu(a) = \frac{1}{(\nu-1)!} \int_0^a (a-t)^{\nu-1} f(t)\, dt = \frac{1}{(\nu-1)!} \int_0^a f(a-t) t^{\nu-1}\, dt.$$

We may therefore replace the sequence (10.3.2) by the sequence

$$f(a), \quad \int_0^a f(a-t)\, dt, \quad \ldots, \quad \int_0^a f(a-t) t^{n-1}\, dt. \tag{10.3.3}$$

Since the number of changes of sign of $f(t)$ is the same as the number of changes of sign of $f(a-t)$, we can replace $f(t)$ by $f(a-t)$. This proves Fejér's theorem.

To prove that there are an infinity of zeros of $\zeta(s)$ on the critical line, we prove as before that

$$\lim_{\alpha \to \frac{1}{4}\pi} \int_0^\infty \frac{\Xi(t)}{t^2+\frac{1}{4}} t^{2n} \cosh \alpha t \, dt = \frac{(-1)^n \pi \cos \frac{1}{8}\pi}{2^{2n}}.$$

Hence $$\int_0^a \frac{\Xi(t)}{t^2+\frac{1}{4}} t^{2n} \cosh \alpha t \, dt$$

has the same sign as $(-1)^n$ for $n = 0, 1,..., N$, if $a = a(N)$ is large enough and $\alpha = \alpha(N)$ is near enough to $\frac{1}{4}\pi$. Hence $\Xi(t)$ has at least N changes of sign in $(0, a)$, and the result follows.†

10.4. Another method‡ is based on Riemann's formula (10.1.2).

Putting $x = e^{2u}$ in (10.1.2), we have

$$\Xi(t) = 4\int_0^\infty \frac{d}{du}\{e^{3u}\psi'(e^{2u})\}e^{-\frac{1}{2}u}\cos ut\,du$$

$$= 2\int_0^\infty \Phi(u)\cos ut\,du,$$

say. Then, by Fourier's integral theorem,

$$\Phi(u) = \frac{1}{\pi}\int_0^\infty \Xi(t)\cos ut\,dt,$$

and hence also

$$\Phi^{(2n)}(u) = \frac{(-1)^n}{\pi}\int_0^\infty \Xi(t)t^{2n}\cos ut\,dt.$$

Since $\psi(x)$ is regular for $\mathbf{R}(x) > 0$, $\Phi(u)$ is regular for $-\frac{1}{4}\pi < \mathbf{I}(u) < \frac{1}{4}\pi$. Let

$$\Phi(iu) = c_0 + c_1 u^2 + c_2 u^4 + \dots \quad (|u| < \tfrac{1}{4}\pi).$$

Then

$$(2n)!\,c_n = (-1)^n\Phi^{(2n)}(0) = \frac{1}{\pi}\int_0^\infty \Xi(t)t^{2n}\,dt.$$

Suppose now that $\Xi(t)$ is of one sign, say $\Xi(t) > 0$, for $t > T$. Then $c_n > 0$ for $n > n_0$, since

$$\int_0^\infty \Xi(t)t^{2n}\,dt > \int_{T+1}^{T+2} \Xi(t)t^{2n}\,dt - \int_0^T |\Xi(t)|t^{2n}\,dt$$

$$> (T+1)^{2n}\int_{T+1}^{T+2} \Xi(t)\,dt - T^{2n}\int_0^T |\Xi(t)|\,dt.$$

It follows that $\Phi^{(n)}(iu)$ increases steadily with u if $n > 2n_0$. But in fact $\Phi(u)$ and all its derivatives tend to 0 as $u \to \frac{1}{4}i\pi$ along the imaginary axis, by the properties of $\psi(x)$ obtained in § 10.2. The theorem therefore follows again.

10.5. The above proofs of Hardy's theorem are all similar in that they depend on the consideration of 'moments' $\int f(t)t^n\,dt$. The following

† Fekete (2). ‡ Pólya (3).

method† depends on a contrast between the asymptotic behaviour of the integrals

$$\int_T^{2T} Z(t)\,dt, \qquad \int_T^{2T} |Z(t)|\,dt,$$

where $Z(t)$ is the function defined in § 4.17. If $Z(t)$ were ultimately of one sign, these integrals would be ultimately equal, apart possibly from sign. But we shall see that in fact they behave quite differently.

Consider the integral

$$\int \{\chi(s)\}^{-\frac{1}{2}}\zeta(s)\,ds,$$

where the integrand is the function which reduces to $Z(t)$ on $\sigma = \frac{1}{2}$, taken round the rectangle with sides $\sigma = \frac{1}{2}$, $\sigma = \frac{5}{4}$, $t = T$, $t = 2T$. This integral is zero, by Cauchy's theorem. Now

$$\int_{\frac{1}{2}+iT}^{\frac{1}{2}+2iT} \{\chi(s)\}^{-\frac{1}{2}}\zeta(s)\,ds = i\int_T^{2T} Z(t)\,dt.$$

Also by (4.12.3)

$$\{\chi(s)\}^{-\frac{1}{2}} = \left(\frac{t}{2\pi}\right)^{\frac{1}{2}\sigma-\frac{1}{4}+\frac{1}{2}it} e^{-\frac{1}{2}it-\frac{1}{8}i\pi}\left\{1+O\left(\frac{1}{t}\right)\right\}.$$

Hence, by (5.1.2) and (5.1.4),

$$\{\chi(s)\}^{-\frac{1}{2}}\zeta(s) = O(t^{\frac{1}{2}\sigma-\frac{1}{4}}.\,t^{\frac{1}{2}-\frac{1}{2}\sigma+\epsilon}) = O(t^{\frac{1}{4}+\epsilon}) \quad (\tfrac{1}{2} \leqslant \sigma \leqslant 1),$$

$$= O(t^{\frac{1}{2}\sigma-\frac{1}{4}+\epsilon}) = O(t^{\frac{3}{8}+\epsilon}) \quad (1 < \sigma \leqslant \tfrac{5}{4}).$$

The integrals along the sides $t = T$, $t = 2T$ are therefore $O(T^{\frac{3}{8}+\epsilon})$.

The integral along the right-hand side is

$$\int_T^{2T} \left(\frac{t}{2\pi}\right)^{\frac{3}{8}+\frac{1}{2}it} e^{-\frac{1}{2}it-\frac{1}{8}i\pi}\left\{1+O\left(\frac{1}{t}\right)\right\}\zeta(\tfrac{5}{4}+it)i\,dt.$$

The contribution of the O-term is

$$\int_T^{2T} O(t^{-\frac{5}{8}})\,dt = O(T^{\frac{3}{8}}).$$

The other term is a constant multiple of

$$\sum_{n=1}^\infty n^{-\frac{5}{4}}\int_T^{2T} \left(\frac{t}{2\pi}\right)^{\frac{3}{8}+\frac{1}{2}it} e^{-\frac{1}{2}it-it\log n}\,dt.$$

Now

$$\frac{d^2}{dt^2}\left(\tfrac{1}{2}t\log\frac{t}{2\pi} - \tfrac{1}{2}t - t\log n\right) = \frac{1}{2t}.$$

Hence, by Lemma 4.5, the integral in the above sum is $O(T^{\frac{7}{8}})$, uniformly with respect to n, so that the whole sum is also $O(T^{\frac{7}{8}})$.

† See Landau, *Vorlesungen*, ii. 78–85.

Combining all these results, we obtain

$$\int_T^{2T} Z(t)\,dt = O(T^{\frac{7}{8}}). \tag{10.5.1}$$

On the other hand,

$$\int_T^{2T} |Z(t)|\,dt = \int_T^{2T} |\zeta(\tfrac{1}{2}+it)|\,dt \geqslant \left|\int_T^{2T} \zeta(\tfrac{1}{2}+it)\,dt\right|.$$

But

$$i\int_T^{2T} \zeta(\tfrac{1}{2}+it)\,dt = \int_{\frac{1}{2}+iT}^{\frac{1}{2}+2iT} \zeta(s)\,ds = \int_{\frac{1}{2}+iT}^{2+iT} + \int_{2+iT}^{2+2iT} + \int_{2+2iT}^{\frac{1}{2}+2iT}$$

$$= \left[s - \sum_{n=2}^{\infty} \frac{1}{n^s \log n}\right]_{2+iT}^{2+2iT} + \int_{\frac{1}{2}}^{2} O(T^{\frac{1}{2}})\,d\sigma = iT + O(T^{\frac{1}{2}}).$$

Hence

$$\int_T^{2T} |Z(t)|\,dt > AT. \tag{10.5.2}$$

Hardy's theorem now follows from (10.5.1) and (10.5.2).

Another variant of this method is obtained by starting again from (10.2.1). Putting $\alpha = \frac{1}{4}\pi - \delta$, we obtain

$$\int_0^{\infty} \frac{\Xi(t)}{t^2+\frac{1}{4}} \cosh\{(\tfrac{1}{4}\pi-\delta)t\}\,dt = O(1) + O\left\{\sum_{n=1}^{\infty} \exp(-n^2\pi i e^{-2i\delta})\right\}$$

$$= O(1) + O\left(\sum_{n=1}^{\infty} e^{-n^2\pi \sin 2\delta}\right) = O(1) + O\left(\int_0^{\infty} e^{-x^2\pi \sin 2\delta}\,dx\right) = O(\delta^{-\frac{1}{2}})$$

as $\delta \to 0$. If, for example, $\Xi(t) > 0$ for $t > t_0$, it follows that for $T > t_0$

$$\int_T^{2T} |Z(t)|\,dt = \left|\int_T^{2T} Z(t)\,dt\right| < A \int_T^{2T} \frac{\Xi(t)}{t^2+\frac{1}{4}} t^{\frac{1}{4}} e^{\frac{1}{4}\pi t}\,dt$$

$$< AT^{\frac{1}{4}} \int_T^{2T} \frac{\Xi(t)}{t^2+\frac{1}{4}} e^{\frac{1}{4}\pi t - \frac{1}{2}t/T}\,dt < AT^{\frac{1}{4}} \int_{t_0}^{\infty} \frac{\Xi(t)}{t^2+\frac{1}{4}} \cosh\left\{\left(\frac{1}{4}\pi - \frac{1}{2T}\right)t\right\}\,dt$$

$$= O(T^{\frac{1}{4}}.T^{\frac{1}{2}}) = O(T^{\frac{3}{4}}).$$

This is inconsistent with (10.5.2), so that the theorem again follows.

10.6. Still another method† depends on the formula (4.17.4), viz.

$$Z(t) = 2\sum_{n\leqslant x} \frac{\cos(\vartheta - t\log n)}{\sqrt{n}} + O(t^{-\frac{1}{4}}),$$

† Titchmarsh (11).

where $x = \sqrt{(t/2\pi)}$. Here $\vartheta = \vartheta(t)$ is defined by

$$\chi(\tfrac{1}{2}+it) = e^{-2i\vartheta(t)},$$

so that

$$\vartheta'(t) = -\frac{1}{2}\frac{\chi'(\frac{1}{2}+it)}{\chi(\frac{1}{2}+it)} = -\frac{1}{2}\left\{\log\pi - \frac{1}{2}\frac{\Gamma'(\frac{1}{4}-\frac{1}{2}it)}{\Gamma(\frac{1}{4}-\frac{1}{2}it)} - \frac{1}{2}\frac{\Gamma'(\frac{1}{4}+\frac{1}{2}it)}{\Gamma(\frac{1}{4}+\frac{1}{2}it)}\right\}$$

$$= -\tfrac{1}{2}\log\pi + \tfrac{1}{4}\log(\tfrac{1}{16}+\tfrac{1}{4}t^2) - \frac{1}{1+4t^2} - \mathbf{R}\int_0^{\infty} \frac{u\,du}{\{u^2+(\frac{1}{4}+\frac{1}{2}it)^2\}(e^{2\pi u}-1)},$$

and we have

$$\vartheta'(t) = \tfrac{1}{2}\log t - \tfrac{1}{2}\log 2\pi + O(1/t),$$

$$\vartheta(t) \sim \tfrac{1}{2}t\log t, \qquad \vartheta''(t) \sim \frac{1}{2t}.$$

The function $\vartheta(t)$ is steadily increasing for $t \geqslant t_0$. If ν is any positive integer ($\geqslant \nu_0$), the equation $\vartheta(t) = \nu\pi$ therefore has just one solution, say t_ν, and $t_\nu \sim 2\pi\nu/\log\nu$. Now

$$Z(t_\nu) = 2(-1)^\nu \sum_{n\leqslant x} \frac{\cos(t_\nu \log n)}{\sqrt{n}} + O(t_\nu^{-\frac{1}{4}}).$$

The sum

$$g(t_\nu) = \sum_{n\leqslant x} \frac{\cos(t_\nu\log n)}{\sqrt{n}} = 1 + \frac{\cos(t_\nu\log 2)}{\sqrt{2}} + \dots$$

consists of the constant term unity and oscillatory terms; and the formula suggests that $g(t_\nu)$ will usually be positive, and hence that $Z(t)$ will usually change sign in the interval $(t_\nu, t_{\nu+1})$.

We shall prove

THEOREM 10.6. *As* $N \to \infty$

$$\sum_{\nu=\nu_0}^{N} Z(t_{2\nu}) \sim 2N, \qquad \sum_{\nu=\nu_0}^{N} Z(t_{2\nu+1}) \sim -2N.$$

It follows at once that $Z(t_{2\nu})$ is positive for an infinity of values of ν, and that $Z(t_{2\nu+1})$ is negative for an infinity of values of ν; and the existence of an infinity of real zeros of $Z(t)$, and so of $\Xi(t)$, again follows.

We have

$$\sum_{\nu=M+1}^{N} g(t_{2\nu}) = \sum_{\nu=M+1}^{N} \sum_{n\leqslant\sqrt{(t_{2\nu}/2\pi)}} \frac{\cos(t_{2\nu}\log n)}{\sqrt{n}}$$

$$= N - M + \sum_{2\leqslant n\leqslant\sqrt{(t_{2N}/2\pi)}} \frac{1}{\sqrt{n}} \sum_{\tau\leqslant t_{2\nu}\leqslant t_{2N}} \cos(t_{2\nu}\log n),$$

where $\tau = \max(t_{2M+2}, 2\pi n^2)$. The inner sum is of the form

$$\sum \cos\{2\pi\phi(\nu)\},$$

where $$\phi(\nu) = \frac{t_{2\nu}\log n}{2\pi}.$$

We may define t_ν for all $\nu \geqslant \nu_0$ (not necessarily integral) by $\vartheta(t_\nu) = \nu\pi$. Then
$$\phi'(\nu) = \frac{\log n}{2\pi}\frac{dt_{2\nu}}{d\nu}, \qquad \vartheta'(t_{2\nu})\frac{dt_{2\nu}}{d\nu} = 2\pi,$$
so that $$\phi'(\nu) = \frac{\log n}{\vartheta'(t_{2\nu})}.$$

Hence $\phi'(\nu)$ is positive and steadily decreasing, and, if ν is large enough,
$$\phi''(\nu) = -2\pi\log n\frac{\vartheta''(t_{2\nu})}{\{\vartheta'(t_\nu)\}^3} \sim -\frac{8\pi\log n}{t_{2\nu}\log^3 t_{2\nu}} < -A\frac{\log n}{t_{2N}\log^3 t_{2N}}.$$

Hence, by Theorem 5.9,
$$\sum_{\tau\leqslant t_{2\nu}\leqslant t_{2N}} \cos(t_{2\nu}\log n) = O\left(t_{2N}\frac{\log^{\frac{1}{2}}n}{t_{2N}^{\frac{1}{2}}\log^{\frac{3}{2}}t_{2N}}\right)+O\left(\frac{t_{2N}^{\frac{1}{2}}\log^{\frac{3}{2}}t_{2N}}{\log^{\frac{1}{2}}n}\right)$$
$$= O(t_{2N}^{\frac{1}{2}}\log^{\frac{3}{2}}t_{2N}).$$

Hence
$$\sum_{2\leqslant n\leqslant\sqrt{(t_{2N}/2\pi)}}\frac{1}{\sqrt{n}}\sum_{\tau\leqslant t_{2\nu}\leqslant t_{2N}}\cos(t_{2\nu}\log n) = O(t_{2N}^{\frac{3}{4}}\log^{\frac{3}{2}}t_{2N})$$
$$= O(N^{\frac{3}{4}}\log^{\frac{3}{4}}N).$$

Hence $$\sum_{\nu=M+1}^{N} Z(t_{2\nu}) = 2N+O(N^{\frac{3}{4}}\log^{\frac{3}{4}}N),$$

and a similar argument applies to the other sum.

10.7. We denote by $N_0(T)$ the number of zeros of $\zeta(s)$ of the form $\frac{1}{2}+it$ $(0 < t \leqslant T)$. The theorem already proved shows that $N_0(T)$ tends to infinity with T. We can, however, prove much more than this.

THEOREM 10.7.† $N_0(T) > AT$.

Any of the above proofs can be put in a more precise form so as to give results in this direction. The most successful method is similar in principle to that of § 10.5, but is more elaborate. We contrast the behaviour of the integrals
$$I = \int_t^{t+H} \Xi(u)\frac{e^{\frac{1}{4}\pi u}}{u^2+\frac{1}{4}}e^{-u/T}\,du, \qquad J = \int_t^{t+H} |\Xi(u)|\frac{e^{\frac{1}{4}\pi u}}{u^2+\frac{1}{4}}e^{-u/T}\,du,$$
where $T \leqslant t \leqslant 2T$ and $T \to \infty$.

† Hardy and Littlewood (3).

We use the theory of Fourier transforms. Let $F(u), f(y)$ be functions related by the Fourier formulae
$$F(u) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f(y)e^{iyu}\,dy, \qquad f(y) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} F(u)e^{-iyu}\,du.$$

Integrating over $(t, t+H)$, we obtain
$$\int_t^{t+H} F(u)\,du = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f(y)\frac{e^{iyH}-1}{iy}e^{iyt}\,dy,$$
so that $$\int_t^{t+H} F(u)\,du, \qquad f(y)\frac{e^{iyH}-1}{iy}$$
are Fourier transforms. Hence the Parseval formula gives
$$\int_{-\infty}^{\infty}\left|\int_t^{t+H} F(u)\,du\right|^2 dt = \int_{-\infty}^{\infty} |f(y)|^2\frac{4\sin^2\frac{1}{2}Hy}{y^2}\,dy.$$

If $F(u)$ is real, $|f(y)|$ is even, and we have
$$\int_{-\infty}^{\infty}\left|\int_t^{t+H} F(u)\,du\right|^2 dt = 2\int_0^{\infty} |f(y)|^2\frac{4\sin^2\frac{1}{2}Hy}{y^2}\,dy$$
$$\leqslant 2H^2\int_0^{1/H} |f(y)|^2\,dy+8\int_{1/H}^{\infty}\frac{|f(y)|^2}{y^2}\,dy. \tag{10.7.1}$$

Now (2.16.2) may be written
$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Xi(t)}{t^2+\frac{1}{4}}e^{i\xi t}\,dt = \tfrac{1}{2}e^{\frac{1}{2}\xi}-e^{-\frac{1}{2}\xi}\psi(e^{-2\xi}).$$

Putting $\xi = -i(\frac{1}{4}\pi-\frac{1}{2}\delta)-y$, it is seen that we may take
$$F(t) = \frac{1}{\sqrt{(2\pi)}}\frac{\Xi(t)}{t^2+\frac{1}{4}}e^{(\frac{1}{4}\pi-\frac{1}{2}\delta)t}, \qquad f(y) = \tfrac{1}{2}e^{-\frac{1}{2}i(\frac{1}{4}\pi-\frac{1}{2}\delta)-\frac{1}{2}y}-$$
$$-e^{\frac{1}{2}i(\frac{1}{4}\pi-\frac{1}{2}\delta)+\frac{1}{2}y}\psi(e^{i(\frac{1}{2}\pi-\delta)+2y}).$$

Let $H \geqslant 1$. The contribution of the first term in $f(y)$ to (10.7.1) is clearly $O(H)$. Putting $y = \log x$, $G = e^{1/H}$, we therefore obtain
$$\int_{-\infty}^{\infty}\left|\int_t^{t+H} F(u)\,du\right|^2 dt = O\left\{H^2\int_1^G |\psi(e^{i(\frac{1}{2}\pi-\delta)}x^2)|^2\,dx\right\}+$$
$$+O\left\{\int_G^{\infty} |\psi(e^{i(\frac{1}{2}\pi-\delta)}x^2)|^2\frac{dx}{\log^2 x}\right\}+O(H). \tag{10.7.2}$$

Now

$$|\psi(e^{i(\frac{1}{4}\pi-\delta)}x^2)|^2 = \left|\sum_{n=1}^{\infty} e^{-n^2\pi x^2(\sin\delta+i\cos\delta)}\right|^2$$

$$= \sum_{n=1}^{\infty} e^{-2n^2\pi x^2\sin\delta} + \sum_{m\neq n}\sum e^{-(m^2+n^2)\pi x^2\sin\delta+i(m^2-n^2)\pi x^2\cos\delta}.$$

As in § 10.5, the first sum is $O(x^{-1}\delta^{-\frac{1}{2}})$, and its contribution to (10.7.2) is therefore

$$O\left(H^2\int_1^G x^{-1}\delta^{-\frac{1}{2}}\,dx\right)+O\left(\int_G^\infty \frac{\delta^{-\frac{1}{2}}\,dx}{x\log^2 x}\right)$$

$$= O\{H^2(G-1)\delta^{-\frac{1}{2}}\}+O(\delta^{-\frac{1}{2}}/\log G) = O(H\delta^{-\frac{1}{2}}).$$

The sum with $m \neq n$ contributes to the second term in (10.7.2) terms of the form

$$\int_G^\infty e^{-(m^2+n^2)\pi x^2\sin\delta+i(m^2-n^2)\pi x^2\cos\delta}\frac{dx}{\log^2 x} = O\left\{\frac{e^{-(m^2+n^2)\pi G^2\sin\delta}}{|m^2-n^2|\,G\log^2 G}\right\}$$

$$= O\left(\frac{H^2e^{-(m^2+n^2)\pi\sin\delta}}{|m^2-n^2|}\right)$$

by Lemma 4.3. Hence the sum is

$$O\left(H^2\sum_{m=2}^{\infty}\sum_{n=1}^{m-1}\frac{e^{-(m^2+n^2)\pi\sin\delta}}{m^2-n^2}\right) = O\left(H^2\sum_{m=2}^{\infty}\frac{e^{-m^2\pi\sin\delta}}{m}\sum_{n=1}^{m-1}\frac{1}{m-n}\right)$$

$$= O\left(H^2\sum_{m=2}^{\infty}\frac{\log m}{m}e^{-m^2\pi\sin\delta}\right) = O\left\{H^2\left(\sum_{m\leqslant 1/\delta}\frac{\log m}{m}+\sum_{m>1/\delta}e^{-m^2\pi\sin\delta}\right)\right\}$$

$$= O\left(H^2\log^2\frac{1}{\delta}\right) = O(H\delta^{-\frac{1}{2}})$$

for $\delta < \delta_0(H)$. The first integral in (10.7.2) may be dealt with in the same way. Hence

$$\int_{-\infty}^{\infty}\left|\int_t^{t+H}F(u)\,du\right|^2dt = O(H\delta^{-\frac{1}{2}}).$$

Taking $\delta = 1/T$ and $T > T_0(H)$, it follows that

$$\int_T^{2T}|I|^2\,dt = O(HT^{\frac{1}{2}}). \qquad (10.7.3)$$

10.8. We next prove that

$$J > (AH+\Psi)T^{-\frac{1}{4}}, \qquad (10.8.1)$$

where

$$\int_T^{2T}|\Psi|^2\,dt = O(T) \quad (0 < H < T). \qquad (10.8.2)$$

We have, if $s = \frac{1}{2}+it$, $T \leqslant t \leqslant 2T$,

$$T^{\frac{1}{4}}\,|\Xi(t)|\frac{e^{\frac{1}{4}\pi t}}{t^2+\frac{1}{4}} > A|\zeta(\tfrac{1}{2}+it)|.$$

Hence

$$T^{\frac{1}{4}}J > A\int_t^{t+H}|\zeta(\tfrac{1}{2}+iu)|\,du > A\left|\int_t^{t+H}\zeta(\tfrac{1}{2}+iu)\,du\right|$$

$$= A\left|\int_t^{t+H}\left\{\sum_{n<AT}\frac{1}{n^{\frac{1}{2}+iu}}+O(T^{-\frac{1}{2}})\right\}du\right|$$

$$= AH+O\left\{\left|\int_t^{t+H}\sum_{2\leqslant n<AT}\frac{1}{n^{\frac{1}{2}+iu}}\,du\right|\right\}+O(HT^{-\frac{1}{2}})$$

$$= AH+O\left\{\left|\sum_{2\leqslant n<AT}\left(\frac{1}{n^{\frac{1}{2}+i(t+H)}\log n}-\frac{1}{n^{\frac{1}{2}+it}\log n}\right)\right|\right\}+O(HT^{-\frac{1}{2}}).$$

It is now sufficient to prove that

$$\int_T^{2T}\left|\sum_{2\leqslant n<AT}\frac{1}{n^{\frac{1}{2}+it}\log n}\right|^2dt = O(T),$$

and the calculations are similar to those of § 7.3, but with an extra factor $\log m\log n$ in the denominator.

To prove Theorem 10.7, let S be the sub-set of the interval $(T, 2T)$ where $I = J$. Then

$$\int_S |I|\,dt = \int_S J\,dt.$$

Now

$$\int_S |I|\,dt \leqslant \int_T^{2T}|I|\,dt \leqslant \left(T\int_T^{2T}|I|^2\,dt\right)^{\frac{1}{2}} < AH^{\frac{1}{2}}T^{\frac{3}{4}}$$

by (10.7.3); and by (10.8.1) and (10.8.2)

$$\int_S J\,dt > T^{-\frac{1}{4}}\int_S(AH+\Psi)\,dt$$

$$> AT^{-\frac{1}{4}}Hm(S)-T^{-\frac{1}{4}}\int_T^{2T}|\Psi|\,dt$$

$$> AT^{-\frac{1}{4}}Hm(S)-T^{-\frac{1}{4}}\left(T\int_T^{2T}|\Psi|^2\,dt\right)^{\frac{1}{2}}$$

$$> AT^{-\frac{1}{4}}Hm(S)-AT^{\frac{3}{4}},$$

where $m(S)$ is the measure of S. Hence, for $H \geqslant 1$ and $T > T_0(H)$,

$$m(S) < ATH^{-\frac{1}{2}}.$$

Now divide the interval $(T, 2T)$ into $[T/2H]$ pairs of abutting intervals j_1, j_2, each, except the last j_2, of length H, and each j_2 lying to the right of the corresponding j_1. Then either j_1 or j_2 contains a zero of $\Xi(t)$ unless j_1 consists entirely of points of S. Suppose that the latter occurs for ν j_1's. Then

$$\nu H \leqslant m(S) < ATH^{-\frac{1}{2}}.$$

Hence there are, in $(T, 2T)$, at least

$$[T/2H]-\nu > \frac{T}{H}\left(\frac{1}{3}-\frac{A}{\sqrt{H}}\right) > \frac{T}{4H}$$

zeros if H is large enough. This proves the theorem.

10.9. For many years the above theorem of Hardy and Littlewood, that $N_0(T) > AT$, was the best that was known in this direction. Recently it has been proved by A. Selberg (2) that $N_0(T) > AT\log T$. This is a remarkable improvement, since it shows that a finite proportion of the zeros of $\zeta(s)$ lie on the critical line. On the Riemann hypothesis, of course,

$$N_0(T) = N(T) \sim \frac{1}{2\pi} T\log T.$$

The numerical value of the constant A in Selberg's theorem is very small.†

The essential idea of Selberg's proof is to modify the series for $\zeta(s)$ by multiplying it by the square of a partial sum of the series for $\{\zeta(s)\}^{-\frac{1}{2}}$. To this extent, it is similar to the proofs given in Chapter IX of theorems about the general distribution of the zeros.

We define α_ν by

$$\frac{1}{\sqrt{\zeta(s)}} = \sum_{\nu=1}^{\infty} \frac{\alpha_\nu}{\nu^s} \quad (\sigma > 1), \qquad \alpha_1 = 1.$$

It is seen from the Euler product that $\alpha_\mu \alpha_\nu = \alpha_{\mu\nu}$ if $(\mu, \nu) = 1$. Since the series for $(1-z)^{\frac{1}{2}}$ is majorized by that for $(1-z)^{-\frac{1}{2}}$, we see that, if

$$\sqrt{\zeta(s)} = \sum_{\nu=1}^{\infty} \frac{\alpha'_\nu}{\nu^s}, \qquad \alpha'_1 = 1,$$

then $|\alpha_\nu| \leqslant \alpha'_\nu \leqslant 1$.

Let
$$\beta_\nu = \alpha_\nu\left(1-\frac{\log \nu}{\log X}\right) \quad (1 \leqslant \nu < X).$$

Then
$$|\beta_\nu| \leqslant 1.$$

† It was calculated in an Oxford dissertation by S. H. Min.

All sums involving β_ν run over $[1, X]$ (or we may suppose $\beta_\nu = 0$ for $\nu \geqslant X$). Let

$$\phi(s) = \sum \frac{\beta_\nu}{\nu^s}.$$

10.10. Let†

$$\Phi(z) = \frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(\tfrac{1}{2}s)\pi^{-\frac{1}{2}s}\zeta(s)\phi(s)\phi(1-s)z^s\,ds$$

where $c > 1$. Moving the line of integration to $\sigma = \frac{1}{2}$, and evaluating the residue at $s = 1$, we obtain

$$\Phi(z) = \tfrac{1}{2}z\phi(1)\phi(0)+\frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma(\tfrac{1}{2}s)\pi^{-\frac{1}{2}s}\zeta(s)\phi(s)\phi(1-s)z^s\,ds$$

$$= \tfrac{1}{2}z\phi(1)\phi(0)-\frac{z^{\frac{1}{2}}}{2\pi}\int_{-\infty}^{\infty} \frac{\Xi(t)}{t^2+\frac{1}{4}}|\phi(\tfrac{1}{2}+it)|^2 z^{it}\,dt.$$

On the other hand,

$$\Phi(z) = \frac{1}{4\pi i}\sum_{n=1}^{\infty}\sum_{\mu}\sum_{\nu}\beta_\mu\beta_\nu\int_{c-i\infty}^{c+i\infty}\Gamma(\tfrac{1}{2}s)\pi^{-\frac{1}{2}s}\frac{z^s}{n^s\mu^s\nu^{1-s}}\,ds$$

$$= \sum_{n=1}^{\infty}\sum_{\mu}\sum_{\nu}\frac{\beta_\mu\beta_\nu}{\nu}\exp\left(-\frac{\pi n^2\mu^2}{z^2\nu^2}\right).$$

Putting $z = e^{-i(\frac{1}{4}\pi-\frac{1}{2}\delta)-y}$, it follows that the functions

$$F(t) = \frac{1}{\sqrt{(2\pi)}}\frac{\Xi(t)}{t^2+\frac{1}{4}}|\phi(\tfrac{1}{2}+it)|^2 e^{(\frac{1}{4}\pi-\frac{1}{2}\delta)t},$$

$$f(y) = \tfrac{1}{2}z^{\frac{1}{2}}\phi(1)\phi(0)-z^{-\frac{1}{2}}\sum_{n=1}^{\infty}\sum_{\mu}\sum_{\nu}\frac{\beta_\mu\beta_\nu}{\nu}\exp\left(-\frac{\pi n^2\mu^2}{z^2\nu^2}\right)$$

are Fourier transforms. Hence, as in § 10.7,

$$\int_{-\infty}^{\infty}\left\{\int_t^{t+h}F(u)\,du\right\}^2 dt \leqslant 2h^2\int_0^{1/h}|f(y)|^2\,dy+8\int_{1/h}^{\infty}|f(y)|^2y^{-2}\,dy \qquad (10.10.1)$$

where $h \leqslant 1$ is to be chosen later.

Putting $y = \log x$, $G = e^{1/h}$, the first integral on the right is equal to

$$\int_1^G\left|\frac{e^{-i(\frac{1}{4}\pi-\frac{1}{2}\delta)}}{2x}\phi(1)\phi(0)-\sum_{n=1}^{\infty}\sum_{\mu}\sum_{\nu}\frac{\beta_\mu\beta_\nu}{\nu}\exp\left(-\frac{\pi n^2\mu^2}{\nu^2}e^{i(\frac{1}{2}\pi-\delta)}x^2\right)\right|^2 dx.$$

† Titchmarsh (26).

Calling the triple sum $g(x)$, this is not greater than

$$2\int_1^G \frac{|\phi(1)\phi(0)|^2}{4x^2}\,dx+2\int_1^G |g(x)|^2\,dx < \tfrac{1}{2}|\phi(1)\phi(0)|^2+2\int_1^G |g(x)|^2\,dx.$$

Similarly the second integral in (10.10.1) does not exceed

$$\frac{|\phi(1)\phi(0)|^2}{2G\log^2 G}+2\int_G^\infty \frac{|g(x)|^2}{\log^2 x}\,dx.$$

10.11. We have to obtain upper bounds for these integrals as $\delta \to 0$, but it is more convenient to consider directly the integral

$$J(x,\theta) = \int_x^\infty |g(u)|^2 u^{-\theta}\,du \quad (0<\theta\leqslant \tfrac{1}{2},\ x\geqslant 1).$$

This is equal to

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \sum_{\kappa\lambda\mu\nu} \frac{\beta_\kappa\beta_\lambda\beta_\mu\beta_\nu}{\lambda\nu} \int_x^\infty \exp\left\{-\pi\left(\frac{m^2\kappa^2}{\lambda^2}+\frac{n^2\mu^2}{\nu^2}\right)u^2\sin\delta+\right.$$

$$\left.+i\pi\left(\frac{m^2\kappa^2}{\lambda^2}-\frac{n^2\mu^2}{\nu^2}\right)u^2\cos\delta\right\}\frac{du}{u^\theta}.$$

Let Σ_1 denote the sum of those terms in which $m\kappa/\lambda = n\mu/\nu$, and Σ_2 the remainder. Let $(\kappa\nu,\lambda\mu) = q$, so that

$$\kappa\nu = aq, \qquad \lambda\mu = bq, \qquad (a,b)=1.$$

Then, in Σ_1, $ma = nb$, so that $n = ra$, $m = rb$ $(r = 1, 2,\ldots)$. Hence

$$\Sigma_1 = \sum_{\kappa\lambda\mu\nu}\frac{\beta_\kappa\beta_\lambda\beta_\mu\beta_\nu}{\lambda\nu}\sum_{r=1}^\infty \int_x^\infty \exp\left(-2\pi\frac{r^2\kappa^2\mu^2}{q^2}u^2\sin\delta\right)\frac{du}{u^\theta}.$$

Now

$$\sum_{r=1}^\infty \int_x^\infty e^{-r^2u^2\eta}\frac{du}{u^\theta} = \eta^{\frac{1}{2}\theta-\frac{1}{2}}\sum_{r=1}^\infty \frac{1}{r^{1-\theta}}\int_{xr\sqrt{\eta}}^\infty e^{-y^2}\frac{dy}{y^\theta}$$

$$= \eta^{\frac{1}{2}\theta-\frac{1}{2}}\int_{x\sqrt{\eta}}^\infty \frac{e^{-y^2}}{y^\theta}\left(\sum_{r\leqslant y/(x\sqrt{\eta})}\frac{1}{r^{1-\theta}}\right)dy.$$

The last r-sum is of the form

$$\frac{1}{\theta}\left(\frac{y}{x\sqrt{\eta}}\right)^\theta-\frac{1}{\theta}+K(\theta)+O\left\{\left(\frac{y}{x\sqrt{\eta}}\right)^{\theta-1}\right\},$$

where $K(\theta)$, and later $K_1(\theta)$, are bounded functions of θ. Hence we obtain

$$\frac{1}{\theta x^\theta\eta^{\frac{1}{2}}}\left\{\int_0^\infty e^{-y^2}\,dy+O(x\sqrt{\eta})\right\}-\frac{\eta^{\frac{1}{2}\theta-\frac{1}{2}}}{\theta}\left[\int_0^\infty e^{-y^2}y^{-\theta}\,dy+O\{(x\sqrt{\eta})^{1-\theta}\}\right]+$$

$$+\eta^{\frac{1}{2}\theta-\frac{1}{2}}K(\theta)\left[\int_0^\infty e^{-y^2}y^{-\theta}\,dy+O\{(x\sqrt{\eta})^{1-\theta}\}\right]+O\{x^{1-\theta}\log(2+\eta^{-1})\}$$

$$= \frac{\sqrt{\pi}}{2\theta x^\theta\eta^{\frac{1}{2}}}+\frac{K_1(\theta)\eta^{\frac{1}{2}\theta-\frac{1}{2}}}{\theta}+O\left\{\frac{x^{1-\theta}}{\theta}\log(2+\eta^{-1})\right\}.$$

Putting $\eta = 2\pi\kappa^2\mu^2q^{-2}\sin\delta$, it follows that

$$\Sigma_1 = \frac{S(0)}{2(2\sin\delta)^{\frac{1}{2}}\theta x^\theta}+\frac{K_1(\theta)}{\theta}(2\pi\sin\delta)^{\frac{1}{2}\theta-\frac{1}{2}}S(\theta)+$$

$$+O\left\{\frac{x^{1-\theta}\log(2+\eta^{-1})}{\theta}\sum_{\kappa\lambda\mu\nu}\frac{|\beta_\kappa\beta_\lambda\beta_\mu\beta_\nu|}{\lambda\nu}\right\}, \tag{10.11.1}$$

where
$$S(\theta) = \sum_{\kappa\lambda\mu\nu}\left(\frac{q}{\kappa\mu}\right)^{1-\theta}\frac{\beta_\kappa\beta_\lambda\beta_\mu\beta_\nu}{\lambda\nu}.$$

Defining $\phi_a(n)$ as in § 9.24, we have

$$q^{1-\theta} = \sum_{\rho|q}\phi_{-\theta}(\rho) = \sum_{\rho|\kappa\nu,\,\rho|\lambda\mu}\phi_{-\theta}(\rho).$$

Hence
$$S(\theta) = \sum_{\rho<X^2}\phi_{-\theta}(\rho)\left(\sum_{\rho|\kappa\nu}\frac{\beta_\kappa\beta_\nu}{\kappa^{1-\theta}\nu}\right)^2.$$

Let d and d_1 denote positive integers whose prime factors divide ρ. Let $\kappa = d\kappa'$, $\nu = d_1\nu'$, where $(\kappa',\rho) = 1$, $(\nu',\rho) = 1$. Then

$$\sum_{\rho|\kappa\nu}\frac{\beta_\kappa\beta_\nu}{\kappa^{1-\theta}\nu} = \sum_{\rho|dd_1}\frac{1}{d^{1-\theta}d_1}\sum_{\kappa'}\frac{\beta_{d\kappa'}}{\kappa'^{1-\theta}}\sum_{\nu'}\frac{\beta_{d_1\nu'}}{\nu'}.$$

Now, for $(\kappa',\rho) = 1$, $\qquad \beta_{d\kappa'} = \dfrac{\alpha_d\alpha_{\kappa'}}{\log X}\log\dfrac{X}{d\kappa'}.$

Hence the above sum is equal to

$$\frac{1}{\log^2 X}\sum_{\rho|dd_1}\frac{\alpha_d\alpha_{d_1}}{d^{1-\theta}d_1}\sum_{\kappa'\leqslant X/d}\frac{\alpha_{\kappa'}}{\kappa'^{1-\theta}}\log\frac{X}{d\kappa'}\sum_{\nu'\leqslant X/d_1}\frac{\alpha_{\nu'}}{\nu'}\log\frac{X}{d_1\nu'}.$$

10.12. Lemma 10.12. *We have*

$$\sum_{\kappa'\leqslant X/d}\frac{\alpha_{\kappa'}}{\kappa'^{1-\theta}}\log\frac{X}{d\kappa'} = O\left\{\left(\frac{X}{d}\right)^\theta\log^{\frac{1}{2}}\frac{X}{d}\prod_{p|\rho}\left(1+\frac{1}{p}\right)^{\frac{1}{2}}\right\} \tag{10.12.1}$$

uniformly with respect to θ.

We may suppose that $X \geqslant 2d$, since otherwise the lemma is trivial.

Now
$$\frac{1}{2\pi i}\int\limits_{1-i\infty}^{1+i\infty}\frac{x^s}{s^2}\,ds = 0 \quad (0 < x \leqslant 1), \quad \log x \quad (x > 1).$$

Also
$$\sum_{(\kappa',\rho)=1}\frac{\alpha_{\kappa'}}{\kappa'^{1-\theta+s}} = \prod_{(p,\rho)=1}\left(1-\frac{1}{p^{1-\theta+s}}\right)^{\frac12} = \sum_{p|\rho}\left(1-\frac{1}{p^{1-\theta+s}}\right)^{-\frac12}\frac{1}{\sqrt{\zeta(1-\theta+s)}}.$$

Hence the left-hand side of (10.12.1) is equal to
$$\frac{1}{2\pi i}\int\limits_{1-i\infty}^{1+i\infty}\frac{1}{s^2}\left(\frac{X}{d}\right)^s\prod_{p|\rho}\left(1-\frac{1}{p^{1-\theta+s}}\right)^{-\frac12}\frac{ds}{\sqrt{\zeta(1-\theta+s)}}. \tag{10.12.2}$$

There are singularities at $s = 0$ and $s = \theta$. If $\theta \geqslant \{\log(X/d)\}^{-1}$, we can take the line of integration through $s = \theta$, the integral round a small indentation tending to zero. Now
$$\left|\frac{1}{\zeta(1+it)}\right| < A|t|$$
for all t (large or small). Also
$$\prod_{p|\rho}\left(1-\frac{1}{p^{1-\theta+s}}\right)^{-1} = O\left\{\left|\prod_{p|\rho}\left(1+\frac{1}{p^{1-\theta+s}}\right)\right|\right\} = O\left\{\prod_{p|\rho}\left(1+\frac{1}{p}\right)\right\}.$$

Hence (10.12.2) is
$$O\left\{\left(\frac{X}{d}\right)^\theta\prod_{p|\rho}\left(1+\frac1p\right)^{\frac12}\int\limits_{-\infty}^{\infty}\frac{|t|^{\frac12}\,dt}{\theta^2+t^2}\right\} = O\left\{\left(\frac{X}{d}\right)^\theta\prod_{p|\rho}\left(1+\frac1p\right)^{\frac12}\frac{1}{\theta^{\frac12}}\right\},$$
and the result stated follows.

If $\theta < \{\log(X/d)\}^{-1}$, we take the same contour as before modified by a detour round the right-hand side of the circle $|s| = 2\{\log(X/d)\}^{-1}$. On this circle
$$|(X/d)^s| \leqslant e^2,$$
the p-product goes as before, and
$$|\zeta(1-\theta+s)| > A\log(X/d).$$

Hence the integral round the circle is
$$O\left\{\log^{-\frac12}\frac{X}{d}\prod_{p|\rho}\left(1+\frac1p\right)^{\frac12}\int\left|\frac{ds}{s^2}\right|\right\} = O\left\{\log^{\frac12}\frac{X}{d}\prod_{p|\rho}\left(1+\frac1p\right)^{\frac12}\right\}.$$

The integral along the part of the line $\sigma = \theta$ above the circle is
$$O\left\{\left(\frac{X}{d}\right)^\theta\prod_{p|\rho}\left(1+\frac1p\right)^{\frac12}\int\limits_{A(\log X/d)^{-1}}^{\infty}\frac{dt}{t^{\frac32}}\right\} = O\left\{\left(\frac{X}{d}\right)^\theta\log^{\frac12}\frac{X}{d}\prod_{p|\rho}\left(1+\frac1p\right)^{\frac12}\right\}.$$

The lemma is thus proved in all cases.

10.13. Lemma 10.13.
$$\sum_{\rho|dd_1}\frac{|\alpha_d\alpha_{d_1}|}{dd_1} = O\left\{\frac1\rho\prod_{p|\rho}\left(1+\frac1p\right)\right\}.$$

Defining α'_d as in § 10.9, we have
$$\sum_{\rho|dd_1}\frac{|\alpha_d\alpha_{d_1}|}{dd_1} \leqslant \sum_{\rho|dd_1}\frac{\alpha'_d\alpha'_{d_1}}{dd_1} = \sum_{\rho|D}\frac1D,$$
where D is a number of the same class as d or d_1,
$$= \frac1\rho\prod_{p|\rho}\left(1-\frac1p\right)^{-1} = O\left\{\frac1\rho\prod_{p|\rho}\left(1+\frac1p\right)\right\}.$$

10.14. Lemma 10.14.
$$S(\theta) = O\left(\frac{X^{2\theta}}{\log X}\right)$$
uniformly with respect to θ. In particular
$$S(0) = O\left(\frac{1}{\log X}\right).$$

By the formulae of § 10.11, and the above lemmas,
$$\sum_{\rho|\kappa\nu}\frac{\beta_\kappa\beta_\nu}{\kappa^{1-\theta}\nu} = O\left\{\frac{1}{\log^2 X}\sum_{\rho|dd_1}\frac{|\alpha_d\alpha_{d_1}|}{d^{1-\theta}d_1}\left(\frac{X}{d}\right)^\theta\log^{\frac12}\frac{X}{d}\log^{\frac12}\frac{X}{d_1}\prod_{p|\rho}\left(1+\frac1p\right)\right\}$$
$$= O\left\{\frac{X^\theta}{\log X}\prod_{p|\rho}\left(1+\frac1p\right)\sum_{\rho|dd_1}\frac{|\alpha_d\alpha_{d_1}|}{dd_1}\right\}$$
$$= O\left\{\frac{X^\theta}{\rho\log X}\prod_{p|\rho}\left(1+\frac1p\right)^2\right\}.$$

Hence
$$S(\theta) = O\left\{\frac{X^{2\theta}}{\log^2 X}\sum_{\rho\leqslant X^2}\frac{\phi_{-\theta}(\rho)}{\rho^2}\prod_{p|\rho}\left(1+\frac1p\right)^4\right\}$$
$$= O\left\{\frac{X^{2\theta}}{\log^2 X}\sum_{\rho\leqslant X^2}\frac{1}{\rho^{1+\theta}}\prod_{p|\rho}\left(1+\frac1p\right)^4\right\}$$
$$= O\left\{\frac{X^{2\theta}}{\log^2 X}\sum_{\rho\leqslant X^2}\frac{1}{\rho^{1+\theta}}\sum_{n|\rho}\frac{1}{n^{\frac12}}\right\},$$
since
$$\prod_{p|\rho}\left(1+\frac1p\right)^4 = O\left\{\prod_{p|\rho}\left(1+\frac4p\right)\right\} = O\left\{\prod_{p|\rho}\left(1+\frac{1}{p^{\frac12}}\right)\right\} = O\left(\sum_{n|\rho}\frac{1}{n^{\frac12}}\right).$$

Hence
$$S(\theta) = O\left\{\frac{X^{2\theta}}{\log^2 X}\sum_{n\leqslant X^2}\sum_{\rho_1\leqslant X^2/n}\frac{1}{(n\rho_1)^{1+\theta}n^{\frac{1}{2}}}\right\}$$
$$= O\left\{\frac{X^{2\theta}}{\log^2 X}\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}+\theta}}\sum_{\rho_1\leqslant X^2/n}\frac{1}{\rho_1^{1+\theta}}\right\}$$
$$= O\left\{\frac{X^{2\theta}}{\log^2 X}\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}\sum_{\rho_1\leqslant X^2}\frac{1}{\rho_1}\right\}$$
$$= O\left(\frac{X^{2\theta}}{\log X}\right).$$

10.15. Estimation of Σ_1. From (10.11.1), Lemma 10.14, and the inequality $|\beta_\nu| \leqslant 1$, we obtain
$$\Sigma_1 = O\left(\frac{1}{\delta^{\frac{1}{2}}\theta x^\theta \log X}\right) + O\left\{\frac{(\delta^{\frac{1}{2}}xX^2)^\theta}{\delta^{\frac{1}{2}}\theta x^\theta\log X}\right\} + O\left\{\frac{x^{1-\theta}\log(X/\delta)}{\theta}X^2\log^2 X\right\}.$$
We shall ultimately take $X = \delta^{-c}$ and $h = (a\log X)^{-1}$, where a and c are suitable positive constants. Then $G = X^a = \delta^{-ac}$. If $x \leqslant G$, the last two terms can be omitted in comparison with the first if $GX^2 = O(\delta^{-\frac{1}{4}})$, i.e. if $(a+2)c \leqslant \frac{1}{4}$. We then have
$$\Sigma_1 = O\left(\frac{1}{\delta^{\frac{1}{2}}\theta x^\theta\log X}\right). \tag{10.15.1}$$

10.16. Estimation of Σ_2. If P and Q are positive, and $x \geqslant 1$,
$$\int_x^\infty e^{-Pu^2+iQu^2}\frac{du}{u^\theta} = \frac{1}{2}\int_{x^2}^\infty \frac{e^{-Pv}}{v^{\frac{1}{2}\theta+\frac{1}{2}}}e^{iQv}\,dv = O\left(\frac{e^{-P}}{x^\theta Q}\right),$$
e.g. by applying the second mean-value theorem to the real and imaginary parts. Hence
$$\Sigma_2 = O\left[\frac{1}{x^\theta}\sum_{\kappa\lambda\mu\nu}\frac{1}{\lambda\nu}\sum_{mn}{}'\left|\frac{m^2\kappa^2}{\lambda^2}-\frac{n^2\mu^2}{\nu^2}\right|^{-1}\exp\left\{-\pi\left(\frac{m^2\kappa^2}{\lambda^2}+\frac{n^2\mu^2}{\nu^2}\right)\sin\delta\right\}\right].$$
The terms with $m\kappa/\lambda > n\mu/\nu$ contribute to the m, n sum
$$O\left\{\sum_{m=1}^{\infty}e^{-\pi m^2\kappa^2\lambda^{-2}\sin\delta}\sum_{n<m\kappa\nu/\lambda\mu}\left(\frac{m^2\kappa^2}{\lambda^2}-\frac{n^2\mu^2}{\nu^2}\right)^{-1}\right\}.$$
Now
$$\frac{m^2\kappa^2}{\lambda^2}-\frac{n^2\mu^2}{\nu^2} \geqslant \frac{m\kappa}{\lambda}\left(\frac{m\kappa}{\lambda}-\frac{n\mu}{\nu}\right) = \frac{m\kappa(m\kappa\nu-n\lambda\mu)}{\lambda^2\nu},$$
and
$$\sum_n{}'\frac{1}{m\kappa\nu-n\lambda\mu} \leqslant 1+\frac{1}{\lambda\mu}+\frac{1}{2\lambda\mu}+\ldots = 1+O\left(\frac{\log mX}{\lambda\mu}\right).$$

Hence the m, n sum is
$$O\left\{\frac{\lambda^2\nu}{\kappa}\sum_{m=1}^{\infty}\left(\frac{1}{m}+\frac{\log(mX)}{m\lambda\mu}\right)e^{-\pi m^2\kappa^2\lambda^{-2}\sin\delta}\right\}$$
$$= O\left\{\frac{\lambda^2\nu}{\kappa}\left(1+\frac{\log X}{\lambda\mu}\right)\log\frac{X^2}{\delta}+\frac{\lambda\nu}{\kappa\mu}\log^2\frac{X^2}{\delta}\right\}$$
$$= O\left(\frac{\lambda^2\nu}{\kappa}\log\frac{1}{\delta}\right)+O\left(\frac{\lambda\nu}{\kappa\mu}\log^2\frac{1}{\delta}\right),$$
since, as in §10.15, we have $X = \delta^{-c}$, with $0 < c \leqslant \frac{1}{8}$. The remaining terms may be treated similarly. Hence
$$\Sigma_2 = O\left\{\frac{1}{x^\theta}\sum_{\kappa\lambda\mu\nu}\left(\frac{\lambda}{\kappa}\log\frac{1}{\delta}+\frac{1}{\kappa\mu}\log^2\frac{1}{\delta}\right)\right\} = O\left(\frac{X^4}{x^\theta}\log^2\frac{1}{\delta}\right). \tag{10.16.1}$$

10.17. Lemma 10.17. *Under the assumptions of* § 10.15
$$\int_{-\infty}^{\infty}\left|\int_t^{t+h}F(u)\,du\right|^2 dt = O\left(\frac{h}{\delta^{\frac{1}{2}}\log X}\right). \tag{10.17.1}$$

By (10.15.1) and (10.16.1),
$$J(x,\theta) = O\left(\frac{1}{\delta^{\frac{1}{2}}\theta x^\theta\log X}\right) \tag{10.17.2}$$
uniformly with respect to θ. Hence
$$\int_1^G |g(x)|^2\,dx = -\int_1^G x^\theta\frac{\partial J}{\partial x}\,dx = [-x^\theta J]_1^G+\theta\int_1^G x^{\theta-1}J\,dx$$
$$= O\left(\frac{1}{\delta^{\frac{1}{2}}\theta\log X}\right)+O\left(\theta\int_1^G\frac{dx}{\delta^{\frac{1}{2}}\theta x\log X}\right) = O\left(\frac{\log G}{\delta^{\frac{1}{2}}\log X}\right),$$
taking, for example, $\theta = \frac{1}{2}$. Also
$$\int_0^{\frac{1}{2}}\theta J(G,\theta)\,d\theta = \int_G^\infty |g(x)|^2\,dx\int_0^{\frac{1}{2}}\theta x^{-\theta}\,d\theta$$
$$= \int_G^\infty |g(x)|^2\left(\frac{1}{\log^2 x}-\frac{1}{2x^{\frac{1}{2}}\log x}-\frac{1}{x^{\frac{1}{2}}\log^2 x}\right)dx$$
$$\geqslant \int_G^\infty\frac{|g(x)|^2}{\log^2 x}\,dx-\frac{3}{2}\int_G^\infty\frac{|g(x)|^2}{x^{\frac{1}{2}}}\,dx$$

since $G = e^{1/h} \geqslant e$. Hence

$$\int_G^\infty \frac{|g(x)|^2}{\log^2 x}\,dx \leqslant \int_0^{\frac{1}{2}} \theta J(G,\theta)\,d\theta + \tfrac{3}{2}J(G,\tfrac{1}{2})$$

$$= O\left(\int_0^{\frac{1}{2}} \frac{d\theta}{\delta^{\frac{1}{2}} G^\theta \log X}\right) + O\left(\frac{1}{\delta^{\frac{1}{2}} G^{\frac{1}{2}} \log X}\right) = O\left(\frac{1}{\delta^{\frac{1}{2}} \log G \log X}\right).$$

Also $\phi(0) = O(X)$, $\phi(1) = O(\log X)$. The result therefore follows from the formulae of § 10.10.

10.18. So far the integrals considered have involved $F(t)$. We now turn to the integrals involving $|F(t)|$. The results about such integrals are expressed in the following lemmas.

LEMMA 10.18. $$\int_{-\infty}^\infty |F(t)|^2\,dt = O\left(\frac{\log 1/\delta}{\delta^{\frac{1}{2}} \log X}\right).$$

By the Fourier transform formulae, the left-hand side is equal to

$$2\int_0^\infty |f(y)|^2\,dy = 2\int_1^\infty \left|\frac{e^{-i(\frac{1}{4}\pi - \frac{1}{2}\delta)}}{2x}\phi(1)\phi(0) - g(x)\right|^2 dx$$

$$\leqslant 4\int_1^\infty |g(x)|^2\,dx + O(X^2\log^2 X).$$

Taking $x = 1$, $\theta = \{\log(1/\delta)\}^{-1}$ in (10.17.2), we have

$$\int_1^\infty |g(u)|^2 e^{-\log u/(\log 1/\delta)}\,du = O\left(\frac{\log 1/\delta}{\delta^{\frac{1}{2}} \log X}\right).$$

Hence $$\int_1^{\delta^{-2}} |g(u)|^2\,du = O\left(\frac{\log 1/\delta}{\delta^{\frac{1}{2}} \log X}\right).$$

We can estimate the integral over (δ^{-2}, ∞) in a comparatively trivial manner. As in § 10.11, this is less than

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \sum_{\kappa\lambda\mu\nu} \frac{|\beta_\kappa \beta_\lambda \beta_\mu \beta_\nu|}{\lambda\nu} \int_{\delta^{-2}}^\infty \exp\left\{-\pi\left(\frac{m^2\kappa^2}{\lambda^2} + \frac{n^2\mu^2}{\nu^2}\right)u^2 \sin\delta\right\} du.$$

Using, for example, $\kappa^2\lambda^{-2}\sin\delta > AX^{-2}\delta > A\delta^2$ (since $X = \delta^{-c}$ with $c < \frac{1}{2}$), and $|\beta_\nu| \leqslant 1$, this is

$$O\left\{X^2\log^2 X \sum_{m=1}^\infty \sum_{n=1}^\infty \int_{\delta^{-2}}^\infty e^{-A(m^2+n^2)\delta^2 u^2}\,du\right\}$$

$$= O\left(X^2\log^2 X \int_{\delta^{-2}}^\infty e^{-A\delta^2 u^2}\,du\right) = O(X^2 \log^2 X\, e^{-A/\delta^2}),$$

which is of the required form.

10.19. LEMMA 10.19.

$$\int_{-\infty}^\infty \left\{\int_t^{t+h} |F(u)|\,du\right\}^2 dt = O\left(\frac{h^2 \log 1/\delta}{\delta^{\frac{1}{2}} \log X}\right).$$

For the left-hand side does not exceed

$$\int_{-\infty}^\infty \left\{h\int_t^{t+h} |F(u)|^2\,du\right\} dt = h\int_{-\infty}^\infty |F(u)|^2\,du \int_{u-h}^u dt = h^2 \int_{-\infty}^\infty |F(u)|^2\,du,$$

and the result follows from the previous lemma.

10.20. LEMMA 10.20. *If* $\delta = 1/T$,

$$\int_0^T |F(t)|\,dt > AT^{\frac{3}{4}}.$$

We have

$$\left(\int_{\frac{1}{2}+i}^{2+i} + \int_{2+i}^{2+iT} + \int_{2+iT}^{\frac{1}{2}+iT} + \int_{\frac{1}{2}+iT}^{\frac{1}{2}+i}\right)\zeta(s)\phi^2(s)\,ds = 0.$$

Since $\phi(s) = O(X^{\frac{1}{2}})$ for $\sigma \geqslant \frac{1}{2}$, the first term is $O(X)$, and the third is $O(XT^{\frac{1}{2}})$. Also

$$\zeta(s)\phi^2(s) = 1 + \sum_{n=2}^\infty \frac{a_n}{n^s},$$

where $|a_n| \leqslant d_3(n)$. Hence

$$\int_{2+i}^{2+iT} \zeta(s)\phi^2(s)\,ds = i(T-1) + \sum_{n=2}^\infty a_n \int_{2+i}^{2+iT} \frac{ds}{n^s}$$

$$= i(T-1) + O\left(\sum_{n=2}^\infty \frac{d_3(n)}{n^2 \log n}\right)$$

$$= iT + O(1).$$

It follows that $$\int_0^T \zeta(\tfrac{1}{2}+it)\phi^2(\tfrac{1}{2}+it)\,dt \sim T.$$

Hence

$$\int_0^T |F(t)|\,dt > A\int_0^T t^{-\frac14}|\zeta(\tfrac12+it)\phi^2(\tfrac12+it)|\,dt$$
$$> AT^{-\frac14}\int_{\frac12 T}^T |\zeta(\tfrac12+it)\phi^2(\tfrac12+it)|\,dt$$
$$> AT^{-\frac14}\left|\int_{\frac12 T}^T \zeta(\tfrac12+it)\phi^2(\tfrac12+it)\,dt\right|$$
$$> AT^{\frac34}.$$

10.21. Lemma 10.21.

$$\int_0^T dt\int_t^{t+h}|F(u)|\,du > AhT^{\frac34}.$$

The left-hand side is equal to

$$\int_0^{T+h}|F(u)|\,du\int_{\max(0,u-h)}^{\min(T,u)}dt \geqslant \int_h^T |F(u)|\,du\int_{u-h}^u dt = h\int_h^T |F(u)|\,du,$$

and the result follows from the previous lemma.

10.22. Theorem 10.22.

$$N_0(T) > AT\log T.$$

Let E be the sub-set of $(0,T)$ where

$$\int_t^{t+h}|F(u)|\,du > \left|\int_t^{t+h}F(u)\,du\right|.$$

For such values of t, $F(u)$ must change sign in $(t,t+h)$, and hence so must $\Xi(u)$, and hence $\zeta(\frac12+iu)$ must have a zero in this interval.

Since the two sides are equal except in E,

$$\int_E dt\int_t^{t+h}|F(u)|\,du \geqslant \int_E\left\{\int_t^{t+h}|F(u)|\,du-\left|\int_t^{t+h}F(u)\,du\right|\right\}dt$$
$$= \int_0^T\left\{\int_t^{t+h}|F(u)|\,du-\left|\int_t^{t+h}F(u)\,du\right|\right\}dt$$
$$> AhT^{\frac34}-\int_0^T\left|\int_t^{t+h}F(u)\,du\right|dt.$$

The left-hand side is not greater than

$$\left(\int_E dt\int_E\left(\int_t^{t+h}|F(u)|\,du\right)^2dt\right\}^{\frac12} \leqslant \left\{m(E)\int_{-\infty}^{\infty}\left(\int_t^{t+h}|F(u)|\,du\right)^2dt\right\}^{\frac12}$$
$$< A\{m(E)\}^{\frac12}hT^{\frac14}\left(\frac{\log T}{\log X}\right)^{\frac12}$$

by Lemma 10.19 with $\delta = 1/T$. The second term on the right is not greater than

$$\left\{\int_0^T dt\int_0^T\left|\int_t^{t+h}F(u)\,du\right|^2dt\right\}^{\frac12} < \frac{Ah^{\frac12}T^{\frac34}}{\log^{\frac12}X}$$

by Lemma 10.17. Hence

$$\{m(E)\}^{\frac12} > A_1T^{\frac12}\left(\frac{\log X}{\log T}\right)^{\frac12}-A_2\frac{T^{\frac12}}{h^{\frac12}\log^{\frac12}T},$$

where A_1 and A_2 denote the particular constants which occur. Since $X = T^c$ and $h = (a\log X)^{-1} = (ac\log T)^{-1}$,

$$\{m(E)\}^{\frac12} > A_1c^{\frac12}T^{\frac12}-A_2(ac)^{\frac12}T^{\frac12}.$$

Taking a small enough, it follows that

$$m(E) > A_3T.$$

Hence, of the intervals $(0,h)$, $(h,2h)$,... contained in $(0,T)$, at least $[A_3T/h]$ must contain points of E. If $(nh,(n+1)h)$ contains a point t of E, there must be a zero of $\zeta(\frac12+iu)$ in $(t,t+h)$, and so in $(nh,(n+2)h)$. Allowing for the fact that each zero might be counted twice in this way, there must be at least

$$\tfrac12[A_3T/h] > AT\log T$$

zeros in $(0,T)$.

10.23. In this section we return to the function $\Xi^*(t)$ mentioned in § 10.1. In spite of its deficiencies as an approximation to $\Xi(t)$, it is of some interest to note that *all the zeros of* $\Xi^*(t)$ *are real.*†

A still better approximation to $\Phi(u)$ is

$$\Phi^{**}(u) = \pi(2\pi\cosh\tfrac92u-3\cosh\tfrac52u)e^{-2\pi\cosh 2u}.$$

This gives

$$\Xi^{**}(t) = 2\int_0^\infty \Phi^{**}(u)\cos ut\,du,$$

and we shall also prove that *all the zeros of* $\Xi^{**}(t)$ *are real.*

The function $K_z(a)$ is, for any value of a, an even integral function of z. We begin by proving that *if a is real all its zeros are purely imaginary.*

It is known that $w = K_z(a)$ satisfies the differential equation

$$\frac{d}{da}\left(a\frac{dw}{da}\right) = \left(a+\frac{z^2}{a}\right)w.$$

This is equivalent to the two equations

$$\frac{dw}{da} = \frac{W}{a}, \qquad \frac{dW}{da} = \left(a+\frac{z^2}{a}\right)w.$$

† Pólya (1), (2), (4).

These give $$\frac{d}{da}(W\bar{w}) = \frac{1}{a}\{|W|^2+(a^2+z^2)|w|^2\}.$$

It is also easily verified that w and W tend to 0 as $a \to \infty$. It follows that, if w vanishes for a certain z and $a = a_0 > 0$, then
$$\int_{a_0}^{\infty} \{|W|^2+(a^2+z^2)|w|^2\}\frac{da}{a} = 0.$$
Taking imaginary parts,
$$2ixy\int_{a_0}^{\infty} \frac{|w|^2}{a}\,da = 0.$$
Here the integral is not 0, and $K_z(a)$ plainly does not vanish for z real, i.e. $y = 0$. Hence $x = 0$, the required result.

We also require the following lemma.

Let c be a positive constant, $F(z)$ an integral function of genus 0 or 1, which takes real values for real z, and has no complex zeros and at least one real zero. Then all the zeros of
$$F(z+ic)+F(z-ic) \tag{10.23.1}$$
are also real.

We have $$F(z) = Cz^q e^{\alpha z}\prod_{n=1}^{\infty}\left(1-\frac{z}{\alpha_n}\right)e^{z/\alpha_n},$$
where C, α, α_1,... are real constants, $\alpha_n \neq 0$ for $n = 1, 2,...,$ $\sum \alpha_n^{-2}$ is convergent, q a non-negative integer. Let z be a zero of (10.23.1). Then
$$|F(z+ic)| = |F(z-ic)|,$$
so that
$$1 = \left|\frac{F(z-ic)}{F(z+ic)}\right|^2 = \left\{\frac{x^2+(y-c)^2}{x^2+(y+c)^2}\right\}^q \prod_{n=1}^{\infty}\frac{(x-\alpha_n)^2+(y-c)^2}{(x-\alpha_n)^2+(y+c)^2}.$$
If $y > 0$, every factor on the right is < 1; if $y < 0$, every factor is > 1. Hence in fact $y = 0$.

The theorem that the zeros of $\Xi^*(t)$ are all real now follows on taking
$$F(z) = K_{\frac{1}{2}iz}(2\pi), \qquad c = \tfrac{9}{2}.$$

10.24. For the discussion of $\Xi^{**}(t)$ we require the following lemma.

Let $|f(t)| < Ke^{-|t|^{2+\delta}}$ for some positive δ, so that
$$F(z) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f(t)e^{izt}\,dt$$

is an integral function of z. Let all the zeros of $F(z)$ be real. Let $\phi(t)$ be an integral function of t of genus 0 or 1, real for real t. Then the zeros of
$$G(z) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f(t)\phi(it)e^{izt}\,dt$$
are also all real.

We have $$\phi(t) = Ct^q e^{\alpha t}\prod_{m=1}^{\infty}\left(1-\frac{t}{\alpha_m}\right)e^{t/\alpha_m},$$
where the constants are all real, and $\sum \alpha_m^{-2}$ is convergent. Let
$$\phi_n(t) = Ct^q e^{\alpha t}\prod_{m=1}^{n}\left(1-\frac{t}{\alpha_m}\right)e^{t/\alpha_m}.$$
Then $\phi_n(t) \to \phi(t)$ uniformly in any finite interval, and (as in my *Theory of Functions*, § 8.25) $$|\phi_n(t)| < Ke^{|t|^{2+\epsilon}}$$
uniformly with respect to n. Hence
$$G(z) = \lim_{n\to\infty}\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f(t)\phi_n(it)e^{izt}\,dt = \lim_{n\to\infty} G_n(z),$$
say. It is therefore sufficient to prove that, for every n, the zeros of $G_n(z)$ are real.

Now it is easily verified that $F(z)$ is an integral function of order less than 2. Hence, if its zeros are real, so are those of
$$(D-\alpha)F(z) = e^{\alpha z}\frac{d}{dz}\{e^{-\alpha z}F(z)\}$$
for any real α. Applying this principle repeatedly, we see that all the zeros of
$$H(z) = D^q(D-\alpha_1)...(D-\alpha_n)F(z) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f(t)(it)^q(it-\alpha_1)...(it-\alpha_n)e^{izt}\,dt$$
are real. Since
$$G_n(z) = \frac{(-1)^nC}{\alpha_1...\alpha_n}H\left(z+\alpha+\frac{1}{\alpha_1}+...+\frac{1}{\alpha_n}\right)$$
the result follows.

Taking $$f(t) = 4\sqrt{(2\pi)}e^{-2\pi\cosh 2t}$$
we obtain $$F(z) = K_{\frac{1}{2}iz}(2\pi),$$
all of whose zeros are real. If
$$\phi(t) = \tfrac{1}{2}\pi^2\cos\tfrac{9}{2}t,$$

then $G(z) = \Xi^*(z)$, and it follows again that all the zeros of $\Xi^*(z)$ are real. If

$$\phi(t) = \tfrac{1}{2}\pi^2\left(\cos\frac{9}{2}t - \frac{3}{2\pi}\cos\frac{5}{2}t\right),$$

then $G(z) = \Xi^{**}(z)$. Hence all the zeros of $\Xi^{**}(z)$ are real.

10.25. By way of contrast to the Riemann zeta-function we shall now construct a function which has a similar functional equation, and for which the analogues of most of the theorems of this chapter are true; but which has no Euler product, and for which the analogue of the Riemann hypothesis is false.

We shall use the simplest properties of Dirichlet's L-functions (mod 5). These are defined for $\sigma > 1$ by

$$L_0(s) = \sum_{n=1}^{\infty} \frac{\chi_0(n)}{n^s} = \frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{6^s}+\dots,$$

$$L_1(s) = \sum_{n=1}^{\infty} \frac{\chi_1(n)}{n^s} = \frac{1}{1^s}+\frac{i}{2^s}-\frac{i}{3^s}-\frac{1}{4^s}+\frac{1}{6^s}+\dots,$$

$$L_2(s) = \sum_{n=1}^{\infty} \frac{\chi_2(n)}{n^s} = \frac{1}{1^s}-\frac{i}{2^s}+\frac{i}{3^s}-\frac{1}{4^s}+\frac{1}{6^s}+\dots,$$

$$L_3(s) = \sum_{n=1}^{\infty} \frac{\chi_3(n)}{n^s} = \frac{1}{1^s}-\frac{1}{2^s}-\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{6^s}+\dots.$$

Each $\chi(n)$ has the period 5. It is easily verified that in each case

$$\chi(m)\chi(n) = \chi(mn)$$

if m is prime to n; and hence that

$$L(s) = \prod_p \left\{1-\frac{\chi(p)}{p^s}\right\}^{-1} \quad (\sigma > 1).$$

It is also easily seen that

$$L_0(s) = \left(1-\frac{1}{5^s}\right)\zeta(s),$$

so that $L_0(s)$ is regular except for a simple pole at $s = 1$. The other three series are convergent for any real positive s, and hence for $\sigma > 0$. Hence $L_1(s)$, $L_2(s)$, and $L_3(s)$ are regular for $\sigma > 0$.

Now consider the function

$$\begin{aligned} f(s) &= \tfrac{1}{2}\sec\theta\{e^{-i\theta}L_1(s)+e^{i\theta}L_2(s)\} \\ &= \frac{1}{1^s}+\frac{\tan\theta}{2^s}-\frac{\tan\theta}{3^s}-\frac{1}{4^s}+\frac{1}{6^s}+\dots \\ &= \frac{1}{5^s}\{\zeta(s,\tfrac{1}{5})+\tan\theta\,\zeta(s,\tfrac{2}{5})-\tan\theta\,\zeta(s,\tfrac{3}{5})-\zeta(s,\tfrac{4}{5})\}, \end{aligned}$$

where $\zeta(s,a)$ is defined as in § 2.17.

By (2.17) $f(s)$ is an integral function of s, and for $\sigma < 0$ it is equal to

$$\begin{aligned} &\frac{2\Gamma(1-s)}{5^s(2\pi)^{1-s}}\Big\{\sin\tfrac{1}{2}\pi s \times \\ &\times \sum_{m=1}^{\infty}\left(\cos\frac{2m\pi}{5}+\tan\theta\cos\frac{4m\pi}{5}-\tan\theta\cos\frac{6m\pi}{5}-\cos\frac{8m\pi}{5}\right)\frac{1}{m^{1-s}}+ \\ &+\cos\tfrac{1}{2}\pi s\sum_{m=1}^{\infty}\left(\sin\frac{2m\pi}{5}+\tan\theta\sin\frac{4m\pi}{5}-\tan\theta\sin\frac{6m\pi}{5}-\sin\frac{8m\pi}{5}\right)\frac{1}{m^{1-s}}\Big\} \\ &= \frac{4\Gamma(1-s)\cos\frac{1}{2}\pi s}{5^s(2\pi)^{1-s}}\sum_{m=1}^{\infty}\left(\sin\frac{2m\pi}{5}+\tan\theta\sin\frac{4m\pi}{5}\right)\frac{1}{m^{1-s}}. \end{aligned}$$

If
$$\sin\frac{4\pi}{5}+\tan\theta\sin\frac{8\pi}{5} = \tan\theta\left(\sin\frac{2\pi}{5}+\tan\theta\sin\frac{4\pi}{5}\right), \tag{10.25.1}$$

this is equal to

$$\frac{4\Gamma(1-s)\cos\frac{1}{2}\pi s}{5^s(2\pi)^{1-s}}\left(\sin\frac{2\pi}{5}+\tan\theta\sin\frac{4\pi}{5}\right)f(1-s).$$

The equation (10.25.1) reduces to

$$\sin 2\theta = 2\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{2},$$

and we take θ to be the root of this between 0 and $\frac{1}{4}\pi$. We obtain

$$\tan\theta = \frac{\sqrt{(10-2\sqrt{5})}-2}{\sqrt{5}-1},$$

$$\sin\frac{2\pi}{5}+\tan\theta\sin\frac{4\pi}{5} = \frac{\sqrt{5}}{2},$$

and $f(s)$ satisfies the functional equation

$$f(s) = \frac{2\Gamma(1-s)\cos\frac{1}{2}s\pi}{5^{s-\frac{1}{2}}(2\pi)^{1-s}}f(1-s).$$

There is now no difficulty in extending the theorems of this chapter to $f(s)$. We can write the above equation as

$$\left(\frac{5}{\pi}\right)^{\frac{1}{2}s}\Gamma(\tfrac{1}{2}+\tfrac{1}{2}s)f(s) = \left(\frac{5}{\pi}\right)^{\frac{1}{2}-\frac{1}{2}s}\Gamma(1-\tfrac{1}{2}s)f(1-s),$$

and putting $s = \frac{1}{2}+it$ we obtain an even integral function of t analogous to $\Xi(t)$.

We conclude that $f(s)$ has an infinity of zeros on the line $\sigma = \frac{1}{2}$, and that the number of such zeros between 0 and T is greater than AT.

On the other hand, we shall now prove that $f(s)$ *has an infinity of zeros in the half-plane* $\sigma > 1$.

If p is a prime, we define $\alpha(p)$ by

$$\alpha(p) = \tfrac{1}{2}(1+i)\chi_1(p)+\tfrac{1}{2}(1-i)\chi_2(p),$$

so that $$\alpha(p) = \pm 1 \quad \text{or} \quad \pm i.$$

For composite n, we define $\alpha(n)$ by the equation

$$\alpha(n_1 n_2) = \alpha(n_1)\alpha(n_2).$$

Thus $|\alpha(n)|$ is always 0 or 1. Let

$$M(s,\chi) = \sum_{n=1}^{\infty} \frac{\alpha(n)\chi(n)}{n^s} = \prod_p \left(1-\frac{\alpha(p)\chi(p)}{p^s}\right)^{-1},$$

where χ denotes either χ_1 or χ_2. Let

$$N(s) = \tfrac{1}{2}\{M(s,\chi_1)+M(s,\chi_2)\}.$$

Now
$$\alpha(p)\chi_1(p) = \tfrac{1}{2}(1+i)\chi_1^2+\tfrac{1}{2}(1-i)\chi_1\chi_2,$$
$$\alpha(p)\chi_2(p) = \tfrac{1}{2}(1+i)\chi_1\chi_2+\tfrac{1}{2}(1-i)\chi_2^2,$$

and these are conjugate since $\chi_1^2 = \bar{\chi}_2^2$ and χ_1^2 and $\chi_1\chi_2$ are real. Hence $M(s,\chi_1)$ and $M(s,\chi_2)$ are conjugate for real s, and $N(s)$ is real.

Let s be real, greater than 1, and $\to 1$. Then

$$\log M(s,\chi_1) = \sum_p \frac{\alpha(p)\chi_1(p)}{p^s}+O(1)$$
$$= \tfrac{1}{2}(1+i)\sum_p \frac{\chi_1^2(p)}{p^s}+\tfrac{1}{2}(1-i)\sum_p \frac{\chi_1(p)\chi_2(p)}{p^s}+O(1).$$

Now $\chi_1^2 = \chi_3$ and $\chi_1\chi_2 = \chi_0$. Hence

$$\sum_p \frac{\chi_1^2(p)}{p^s} = \sum_p \frac{\chi_3(p)}{p^s} = \log L_3(s)+O(1) = O(1),$$

$$\sum_p \frac{\chi_1(p)\chi_2(p)}{p^s} = \sum_p \frac{\chi_0(p)}{p^s} = \log L_0(s)+O(1) = \log\frac{1}{s-1}+O(1).$$

Hence
$$\log M(s,\chi_1) = \tfrac{1}{2}(1-i)\log\frac{1}{s-1}+O(1),$$

$$N(s) = \mathbf{R}M(s,\chi_1) = \frac{1}{\sqrt{(s-1)}}\cos\left(\tfrac{1}{2}\log\frac{1}{s-1}\right)e^{O(1)}.$$

It is clear from this formula that $N(s)$ *has a zero at each of the points* $s = 1+e^{-(2m+1)\pi}$ $(m = 1, 2,\ldots)$.

Now for $\sigma \geqslant 1+\delta$, and $\chi = \chi_1$ or χ_2,

$$\log L(s+i\tau,\chi)-\log M(s,\chi)$$
$$= \sum_{p\leqslant P}\left\{\log\left(1-\frac{\alpha(p)\chi(p)}{p^s}\right)-\log\left(\frac{1-p^{-i\tau}\chi(p)}{p^s}\right)\right\}+O\left(\frac{1}{P^\delta}\right)$$
$$= O\left\{\sum_{\substack{p\leqslant P\\ p\neq 5}}\frac{|\alpha(p)-p^{-i\tau}|}{p^\sigma}\right\}+O\left(\frac{1}{P^\delta}\right).$$

Let $\alpha(p) = e^{2\pi i\beta(p)}$. By Kronecker's theorem, given q, there is a number τ and integers x_p such that

$$\left|\tau\frac{\log p}{2\pi}+\beta(p)-x_p\right| \leqslant \frac{1}{q} \quad (p \leqslant P).$$

Then $$|\alpha(p)-p^{-i\tau}| = |e^{2\pi i(\beta(p)+(\tau\log p)/2\pi)}-1| \leqslant e^{2\pi/q}-1.$$

Hence $$\log L(s+i\tau,\chi)-\log M(s,\chi) = O\left(\frac{\log P}{q}\right)+O\left(\frac{1}{P^\delta}\right),$$

and we can make this as small as we please by choosing first P and then q. Using this with χ_1 and χ_2, it follows that, given $\epsilon > 0$ and $\delta > 0$, there is a τ such that

$$|f(s+i\tau)-N(s)| < \epsilon \quad (\sigma \geqslant 1+\delta).$$

Let $s_1 > 1$ be a zero of $N(s)$. For any $\eta > 0$ there exists an η_1 with $0 < \eta_1 < \eta$, $\eta_1 < s_1-1$, such that $N(s) \neq 0$ for $|s-s_1| = \eta_1$. Let

$$\epsilon = \min_{|s-s_1|=\eta_1} |N(s)|$$

and $\delta < s_1-\eta_1-1$. Then, by Rouché's theorem, $N(s)$ and

$$N(s)-\{N(s)-f(s+i\tau)\}$$

have the same number of zeros inside $|s-s_1| = \eta_1$, and so at least one. Hence $f(s)$ has at least one zero inside the circle $|s-s_1-i\tau| = \eta_1$.

A slight extension of the argument shows that the number of zeros of $f(s)$ in $\sigma > 1$, $0 < t \leqslant T$, exceeds AT as $T\to\infty$. For by the extension of Dirichlet's theorem (§ 8.2) the interval $(t_0, mq^P t_0)$ contains at least m values of t, differing by at least t_0, such that

$$\left|t\frac{\log p}{2\pi}-x_p'\right| \leqslant \frac{1}{q} \qquad (p \leqslant P).$$

The above argument then shows the existence of a zero in the neighbourhood of each point $s_1+i(\tau+t)$.

The method is due to Davenport and Heilbronn (1), (2); they proved that a class of functions, of which an example is

$$\sum_{m,n\neq 0,0}\sum \frac{1}{(m^2+5n^2)^s},$$

has an infinity of zeros for $\sigma > 1$. It has been shown by calculation† that this particular function has a zero in the critical strip, not on the critical line. The method throws no light on the general question of the occurrence of zeros of such functions in the critical strip, but not on the critical line.

NOTES FOR CHAPTER 10

10.26. In § 10.1 Titchmarsh's comment on Riemann's statement about the approximate formula for $N(T)$ is erroneous. It is clear that Riemann meant that the relative error $\{N(T)-L(T)\}/N(T)$ is $O(T^{-1})$.

10.27. Further work has been done on the problem mentioned at the end of § 10.25. Davenport and Heilbronn (1), (2) showed in general that if Q is any positive definite integral quadratic form of discriminant d, such that the class number $h(d)$ is greater than 1, then the Epstein Zeta-function

$$\zeta_Q(s) = \sum_{\substack{x,y=-\infty \\ (x,y)\neq(0,0)}}^{\infty} Q(x,y)^{-s} \qquad (\sigma > 1)$$

has zeros to the right of $\sigma = 1$. In fact they showed that the number of such zeros up to height T is at least of order T (and hence of exact order T). This result has been extended to the critical strip by Voronin [3], who proved that, for such functions $\zeta_Q(s)$, the number of zeros up to height T, for $\frac{1}{2} < \sigma_1 \leqslant \mathrm{I}(s) \leqslant \sigma_2 < 1$, is also of order at least T (and hence of exact order T). This answers the question raised by Titchmarsh at the end of § 10.25.

10.28. Much the most significant result on $N_0(T)$ is due to Levinson [2], who showed that

$$N_0(T) \geqslant \alpha N(T) \tag{10.28.1}$$

for large enough T, with $\alpha = 0{\cdot}342$. The underlying idea is to relate the distribution of zeros of $\zeta(s)$ to that of the zeros of $\zeta'(s)$. To put matters in

† Potter and Titchmarsh (1).

their proper perspective we first note that Berndt [1] has shown that

$$\#\{s=\sigma+it: 0<t\leqslant T, \zeta'(s)=0\} = \frac{T}{2\pi}\left(\log\frac{T}{4\pi}-1\right)+O(\log T),$$

and that Speiser (1) has proved that the Riemann Hypothesis is equivalent to the non-vanishing of $\zeta'(s)$ for $0<\sigma<\frac{1}{2}$. This latter result is related to the unconditional estimate

$$\begin{aligned}\#\{s=\sigma+it: -1<\sigma<\tfrac{1}{2}, T_1<t\leqslant T_2, \zeta'(s)=0\}\\ = \#\{s=\sigma+it: 0<\sigma<\tfrac{1}{2}, T_1<t\leqslant T_2, \zeta(s)=0\}\\ +O(\log T_2),\end{aligned} \tag{10.28.2}$$

zeros being counted according to multiplicity. This is due to Levinson and Montgomery [1], who also gave a number of other interesting results on the distribution of the zeros of $\zeta'(s)$.

We sketch the proof of (10.28.2). We shall make frequent reference to the logarithmic derivative of the functional equation (2.6.4), which we write in the form

$$\begin{aligned}\frac{\zeta'(s)}{\zeta(s)}+\frac{\zeta'(1-s)}{\zeta(1-s)} &= \log\pi-\tfrac{1}{2}\left(\frac{\Gamma'(\frac{1}{2}s)}{\Gamma(\frac{1}{2}s)}+\frac{\Gamma'(\frac{1}{2}-\frac{1}{2}s)}{\Gamma(\frac{1}{2}-\frac{1}{2}s)}\right)\\ &= -F(s),\end{aligned} \tag{10.28.3}$$

say. We note that $F(\frac{1}{2}+it)$ is always real, and that

$$F(s) = \log(t/2\pi)+O(1/t) \tag{10.28.4}$$

uniformly for $t\geqslant 1$ and $|\sigma|\leqslant 2$. To prove (10.28.2) it suffices to consider the case in which the numbers T_j are chosen so that $\zeta(s)$ and $\zeta'(s)$ do not vanish for $t=T_j$, $-1\leqslant\sigma\leqslant\frac{1}{2}$. We examine the change in argument in $\zeta'(s)/\zeta(s)$ around the rectangle with vertices $\frac{1}{2}-\delta+iT_1$, $\frac{1}{2}-\delta+iT_2$, $-1+iT_2$, and $-1+iT_1$, where δ is a small positive number. Along the horizontal sides we apply the ideas of § 9.4 to $\zeta(s)$ and $\zeta'(s)$ separately. We note that $\zeta(s)$ and $\zeta'(s)$ are each $O(t^A)$ for $-3\leqslant\sigma\leqslant 1$. Moreover we also have $|\zeta(-1+iT_j)| \gg T_j^{\frac{3}{2}}$, by the functional equation, and hence also

$$|\zeta'(-1+iT_j)| \gg T_j^{\frac{3}{2}}\left|\frac{\zeta'(-1+iT_j)}{\zeta(-1+iT_j)}\right| \gg T_j^{\frac{3}{2}}\log T_j,$$

by (10.28.3) and (10.28.4). The method of § 9.4 therefore shows that $\arg\zeta(s)$ and $\arg\zeta'(s)$ both vary by $O(\log T_2)$ on the horizontal sides of the rectangle. On the vertical side $\sigma = -1$ we have

$$\frac{\zeta'(s)}{\zeta(s)} = \log\left(\frac{t}{2\pi}\right) + O(1),$$

by (10.28.3) and (10.28.4), so that the contribution to the total change in argument is $O(1)$. For the vertical side $\sigma = \frac{1}{2} - \delta$ we first observe from (10.28.3) and (10.28.4) that

$$\mathbf{R}\left(-\frac{\zeta'(\frac{1}{2}+it)}{\zeta(\frac{1}{2}+it)}\right) \geqslant 1 \tag{10.28.5}$$

if $t \geqslant T_1$ with T_1 sufficiently large. It follows that

$$\mathbf{R}\left(-\frac{\zeta'(\frac{1}{2}-\delta+it)}{\zeta(\frac{1}{2}-\delta+it)}\right) \geqslant \tfrac{1}{2} \tag{10.28.6}$$

for $T_1 \leqslant t \leqslant T_2$, if $\delta = \delta(T_2)$ is small enough. To see this, it suffices to examine a neighbourhood of a zero $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$. Then

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{m}{s-\rho} + m' + O(|s-\rho|),$$

where $m \geqslant 1$ is the multiplicity of ρ. The choice $s = \frac{1}{2} + it$ with $t \to \gamma$ therefore yields $\mathbf{R}(m') \geqslant 1$, by (10.28.5). Hence, on taking $s = \frac{1}{2} - \delta + it$, we find that

$$\mathbf{R}\left(-\frac{\zeta'(s)}{\zeta(s)}\right) = \frac{m\delta}{|s-\rho|^2} + \mathbf{R}(m') + O(|s-\rho|) \geqslant \tfrac{1}{2}$$

for $|s-\rho|$ small enough. The inequality (10.28.6) now follows. We therefore see that $\arg \zeta'(s)/\zeta(s)$ varies by $O(1)$ on the vertical side $\mathbf{R}(s) = \frac{1}{2} - \delta$ of our rectangle, which completes the proof of (10.28.2).

If we write N for the quantity on the left of (10.28.2) it follows that

$$N_0(T_2) - N_0(T_1) = \{N(T_2) - N(T_1)\} - 2N + O(\log T_2), \tag{10.28.7}$$

so that we now require an upper bound for N. This is achieved by applying the 'mollifier method' of §§9.20–24 to $\zeta'(1-s)$. Let $\nu(\sigma, T_1, T_2)$ denote the number of zeros of $\zeta'(1-s)$ in the rectangle $\sigma \leqslant \mathbf{R}(s) \leqslant 2$, $T_1 < \mathbf{I}(s) < T_2$. The method produces an upper bound for

$$\int_u^2 \nu(\sigma, T_1, T_2)\, d\sigma, \tag{10.28.8}$$

which in turn yields an estimate $N \leqslant c\{N(T_2) - N(T_1)\}$ for large T_2. The constant c in this latter bound has to be calculated explicitly, and must

be less than $\frac{1}{2}$ for (10.28.7) to be of use. This is in contrast to (9.20.5), in which the implied constant was not calculated explicitly, and would have been relatively large. It is difficult to have much feel in advance for how large the constant c produced by the method will be. The following very loose argument gives one some hope that c will turn out to be reasonably small, and so it transpires in practice.

In using (10.28.8) to obtain a bound for N we shall take

$$u = \tfrac{1}{2} - a/\log T_2,$$

where a is a positive constant to be chosen later. The zeros $\rho' = \beta' + i\gamma'$ of $\zeta'(1-s)$ have an asymmetrical distribution about the critical line. Indeed Levinson and Montgomery [1] showed that

$$\sum_{0<\gamma'\leqslant T} (\tfrac{1}{2} - \beta') \sim \frac{T}{2\pi} \log\log T,$$

whence β' is $\frac{1}{2} - (\log\log \gamma')/\log \gamma'$ on average. Thus one might reasonably hope that a fair proportion of such zeros have $\beta' < u$, thereby making the integral (10.28.8) rather small.

We now look in more detail at the method. In the first place, it is convenient to replace $\zeta'(1-s)$ by

$$\zeta(s) + \frac{\zeta'(s)}{F(s)} = G(s),$$

say. If we write $h(s) = \pi^{-\frac{1}{2}s}\,\Gamma(\frac{1}{2}s)$ then (10.28.3), together with the functional equation (2.6.4), yields

$$\zeta'(1-s) = -\frac{F(s)\,h(s)\,G(s)}{h(1-s)},$$

so that $G(s)$ and $\zeta'(1-s)$ have the same zeros for t large enough. Now let

$$\psi(s) = \sum_{n<y} b_n n^{-s} \tag{10.28.9}$$

be a suitable 'mollifier' for $G(s)$, and apply Littlewood's formula (9.9.1) to the function $G(s)\psi(s)$ and the rectangle with vertices $u + iT_1$, $2 + iT_2$, $2 + iT_1$, $u + iT_2$. Then, as in §9.16, we find that

$$N \leqslant \frac{\log T_2}{a} \int_u^2 \nu(\sigma, T_1, T_2)\, d\sigma$$

$$\leqslant \frac{\log T_2}{2\pi a} \int_{T_1}^{T_2} \log|G(u+it)\,\psi(u+it)|\, dt + O(\log T_2).$$

Moreover, as in §9.16 we have

$$\int_{T_1}^{T_2} \log|G(u+it)\psi(u+it)|\,dt$$

$$\leqslant \tfrac{1}{2}(T_2-T_1)\log\left(\frac{1}{T_2-T_1}\int_{T_1}^{T_2}|G(u+it)\psi(u+it)|^2\,dt\right).$$

Hence, if we can show that

$$\int_{T_1}^{T_2}|G(u+it)\psi(u+it)|^2\,dt \sim c(a)\,(T_2-T_1) \qquad (10.28.10)$$

for suitable T_1, T_2, we will have

$$N \leqslant \left(\frac{\log c(a)}{2a}+o(1)\right)\{N(T_2)-N(T_1)\}, \qquad (10.28.11)$$

whence

$$N_0(T_2)-N_0(T_1) \geqslant \left(1-\frac{\log c(a)}{a}+o(1)\right)\{N(T_2)-N(T_1)\}$$

by (10.28.7).

The computation of the mean value (10.28.10) is the most awkward part of Levinson's argument. In [2] he takes $y = T_2{}^{\frac{1}{2}-\varepsilon}$ and

$$b_n = \mu(n)\,n^{u-\frac{1}{2}}\frac{\log y/n}{\log y}.$$

This leads eventually to (10.28.10) with

$$c(a) = e^{2a}\left(\frac{1}{2a^3}+\frac{1}{24a}\right)-\frac{1}{2a^3}-\frac{1}{a^2}-\frac{25}{24a}+\frac{7}{12}-\frac{a}{12}.$$

The optimal choice of a is roughly $a = 1{\cdot}3$, which produces (10.28.1) with $= 0{\cdot}342$.

The method has been improved slightly by Levinson [4], [5], Lou [1] and Conrey [1] and the best constant thus far is $\alpha = 0{\cdot}3658$ (Conrey [1]). The principal restriction on the method is that on the size of y in (10.28.9). The above authors all take $y = T_2{}^{\frac{1}{2}-\varepsilon}$, but there is some scope for improvement via the ideas used in the mean-value theorems (7.24.5), (7.24.6), and (7.24.7).

10.29. An examination of the argument just given reveals that the right hand side of (10.28.11) gives an upper bound for $N+N^*$, where

$$N^* = \#\{s=\tfrac{1}{2}+it\colon T_1<t\leqslant T_2,\ \zeta'(s)=0\},$$

(zeros being counted according to multiplicities). However it is clear from (10.28.3) and (10.28.4) that $\zeta'(\frac{1}{2}+it)$ can only vanish if $\zeta(\frac{1}{2}+it)$ does. Consequently, if we write $N^{(r)}$ for the number of zeros of $\zeta(s)$ of multiplicity r, on the line segment $s=\frac{1}{2}+it$, $T_1<t\leqslant T_2$, we will have

$$N^* = \sum_{r=2}^{\infty}(r-1)\,N^{(r)}.$$

Thus (10.28.7) may be replaced by

$$N^{(1)}-\sum_{r=3}^{\infty}(r-2)\,N^{(r)} = \{N(T_2)-N(T_1)\}-2(N+N^*)+O(\log T_2).$$

If we now define $N^{(r)}(T)$ in analogy to $N^{(r)}$, but counting zeros $\frac{1}{2}+it$ with $0<t\leqslant T$, we may deduce that

$$N^{(1)}(T)-\sum_{r=3}^{\infty}(r-2)\,N^{(r)}(T) \geqslant \alpha N(T), \qquad (10.29.1)$$

for large enough T, and $\alpha = 0{\cdot}342$. In particular at least a third of the non-trivial zeros of $\zeta(s)$ not only lie on the critical line, but are simple. This observation is due independently to Heath-Brown [5] and Selberg (unpublished). The improved constants α mentioned above do not all allow this refinement. However it has been shown by Anderson [1] that (10.29.1) holds with $\alpha = 0{\cdot}3532$.

10.30. Levinson's method can be applied equally to the derivatives $\xi^{(m)}(s)$ of the function $\xi(s)$ given by (2.1.12). One can show that the zeros of these functions lie in the critical strip, and that the number of them, $N_m(T)$ say, for $0<t\leqslant T$, is $N(T)+O_m(\log T)$. If the Riemann hypothesis holds then all these zeros must lie on the critical line. Thus it is of some interest to give unconditional estimates for

$$\liminf_{T\to\infty} N_m(T)^{-1}\,\#\{t\colon 0<t\leqslant T,\ \xi^{(m)}(\tfrac{1}{2}+it)=0\} = \alpha_m,$$

say. Levinson [3], [5] showed that $\alpha_1 \geqslant 0{\cdot}71$, and Conrey [1] improved and extended the method to give $\alpha_1 \geqslant 0{\cdot}8137$, $\alpha_2 \geqslant 0{\cdot}9584$ and in general $\alpha_m = 1+O(m^{-2})$.

XI

THE GENERAL DISTRIBUTION OF THE VALUES OF $\zeta(s)$

11.1. In the previous chapters we have been concerned almost entirely with the modulus of $\zeta(s)$, and the various values, particularly zero, which it takes. We now consider the problem of $\zeta(s)$ itself, and the values of s for which it takes any given value a.†

One method of dealing with this problem is to connect it with the famous theorem of Picard on functions which do not take certain values. We use the following theorem:‡

If $f(s)$ is regular and never 0 or 1 in $|s-s_0| \leqslant r$, and $|f(s_0)| \leqslant \alpha$, then $|f(s)| \leqslant A(\alpha,\theta)$ for $|s-s_0| \leqslant \theta r$, where $0<\theta<1$.

From this we deduce

THEOREM 11.1. *$\zeta(s)$ takes every value, with one possible exception, an infinity of times in any strip $1-\delta<\sigma\leqslant 1+\delta$.*

Suppose, on the contrary, that $\zeta(s)$ takes the distinct values a and b only a finite number of times in the strip, and so *never* above $t=t_0$, say. Let $T>t_0+1$, and consider the function $f(s)=\{\zeta(s)-a\}/(b-a)$ in the circles C, C', of radii $\frac{1}{2}\delta$ and $\frac{1}{4}\delta$ $(0<\delta<1)$, and common centre $s_0=1+\frac{1}{4}\delta+iT$. Then

$$|f(s_0)| \leqslant \alpha = \{\zeta(1+\tfrac{1}{4}\delta)+|a|\}/|b-a|,$$

and $f(s)$ is never 0 or 1 in C. Hence

$$|f(s)| < A(\alpha)$$

in C', and so $|\zeta(\sigma+iT)| < A(a,b,\alpha)$ for $1\leqslant\sigma\leqslant 1+\frac{1}{8}\delta$, $T>t_0+1$. Hence $\zeta(s)$ is bounded for $\sigma>1$, which is false, by Theorem 8.4(A). This proves the theorem.

We should, of course, expect the exceptional value to be 0.

If we assume the Riemann hypothesis, we can use a similar method inside the critical strip; but more detailed results independent of the Riemann hypothesis can be obtained by the method of Diophantine approximation. We devote the rest of the chapter to developments of this method.

† See Bohr (1)–(14), Bohr and Courant (1), Bohr and Jessen (1), (2), (5), Bohr and Landau (3), Borchsenius and Jessen (1), Jessen (1), van Kampen (1), van Kampen and Wintner (1), Kershner (1), Kershner and Wintner (1), (2), Wintner (1)–(4).

‡ See Landau's *Ergebnisse der Funktionentheorie*, § 24, or Valiron's *Integral Functions*, Ch. VI, § 3.

11.2. We restrict ourselves in the first place to the half-plane $\sigma>1$; and we consider, not $\zeta(s)$ itself, but $\log\zeta(s)$, viz. the function defined for $\sigma>1$ by the series

$$\log\zeta(s) = -\sum_p (p^{-s}+\tfrac{1}{2}p^{-2s}+\ldots).$$

We consider at the same time the function

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_p \log p(p^{-s}+p^{-2s}+\ldots).$$

We observe that both functions are represented by Dirichlet series, absolutely convergent for $\sigma>1$, and capable of being written in the form

$$F(s) = f_1(p_1^{-s})+f_2(p_2^{-s})+\ldots,$$

where $f_n(z)$ is a power-series in z whose coefficients do not depend on s. In fact

$$f_n(z) = -\log(1-z), \qquad f_n(z) = z\log p_n/(1-z)$$

in the above two cases. In what follows $F(s)$ denotes either of the two functions.

11.3. We consider first the values which $F(s)$ takes on the line $\sigma=\sigma_0$, where σ_0 is an arbitrary number greater than 1. On this line

$$F(s) = \sum_{n=1}^{\infty} f_n(p_n^{-\sigma_0}e^{-it\log p_n}),$$

and, as t varies, the arguments $-t\log p_n$ are, of course, all related. But we shall see that there is an intimate connexion between the set U of values assumed by $F(s)$ on $\sigma=\sigma_0$ and the set V of values assumed by the function

$$\Phi(\sigma_0,\theta_1,\theta_2,\ldots) = \sum_{n=1}^{\infty} f_n(p_n^{-\sigma_0}e^{2\pi i\theta_n})$$

of an infinite number of independent real variables θ_1, θ_2,....

We shall in fact show that *the set U, which is obviously contained in V, is everywhere dense in V, i.e. that corresponding to every value v in V (i.e. to every given set of values θ_1, θ_2,...) and every positive ϵ, there exists a t such that*

$$|F(\sigma_0+it)-v| < \epsilon.$$

Since the Dirichlet series from which we start is absolutely convergent for $\sigma=\sigma_0$, it is obvious that we can find $N=N(\sigma_0,\epsilon)$ such that

$$\left|\sum_{n=N+1}^{\infty} f_n(p_n^{-\sigma_0}e^{2\pi i\mu_n})\right| < \tfrac{1}{3}\epsilon \tag{11.3.1}$$

for any values of the μ_n, and in particular for $\mu_n=\theta_n$, or for

$$\mu_n = -(t\log p_n)/2\pi.$$

Now since the numbers $\log p_n$ are linearly independent, we can, by Kronecker's theorem, find a number t and integers $g_1, g_2,\ldots, g_N$ such that

$$|-t\log p_n - 2\pi\theta_n - 2\pi g_n| < \eta \quad (n = 1, 2,\ldots, N),$$

η being an assigned positive number. Since $f_n(p_n^{-\sigma_0}e^{2\pi i\theta})$ is, for each n, a continuous function of θ, we can suppose η so small that

$$\left|\sum_{n=1}^{N}\{f_n(p_n^{-\sigma_0}e^{2\pi i\theta_n}) - f_n(p_n^{-\sigma_0}e^{-it\log p_n})\}\right| < \tfrac{1}{3}\epsilon. \tag{11.3.2}$$

The result now follows from (11.3.1) and (11.3.2).

11.4. We next consider the set W of values which $F(s)$ takes 'in the immediate neighbourhood' of the line $\sigma = \sigma_0$, i.e. the set of all values of w such that the equation $F(s) = w$ has, for every positive δ, a root in the strip $|\sigma-\sigma_0| < \delta$.

In the first place, it is evident that U is contained in W. Further, it is easy to see that U is everywhere dense in W. For, for sufficiently small δ (e.g. for $\delta < \frac{1}{2}(\sigma_0-1)$),

$$|F'(s)| < K(\sigma_0)$$

for all values of s in the strip $|\sigma-\sigma_0| < \delta$, so that

$$|F(\sigma_0+it) - F(\sigma_1+it)| < K(\sigma_0)|\sigma_1-\sigma_0| \quad (|\sigma_1-\sigma_0| < \delta). \tag{11.4.1}$$

Now each value w in W is assumed by $F(s)$ either on the line $\sigma = \sigma_0$, in which case it is a u, or at points σ_1+it arbitrarily near the line, in which case, in virtue of (11.4.1), we can find a u such that

$$|w-u| < K(\sigma_0)|\sigma_1-\sigma_0| < \epsilon.$$

We now proceed to prove that W *is identical with* V. Since U is contained in and is everywhere dense in both V and W, it follows that each of V and W is everywhere dense in the other. It is therefore obvious that W is contained in V, if V is closed.

We shall see presently that much more than this is true, viz. that V consists of all points of an area, including the boundary. The following direct proof that V is closed is, however, very instructive.

Let v^* be a limit-point of V, and let v_ν ($\nu = 1, 2,\ldots$) be a sequence of v's tending to v^*. To each v_ν corresponds a point $P_\nu(\theta_{1,\nu}, \theta_{2,\nu},\ldots)$ in the space of an infinite number of dimensions defined by $0 \leqslant \theta_{n,\nu} < 1$ ($n = 1, 2,\ldots$), such that $\Phi(\sigma_0, \theta_{1,\nu},\ldots) = v_\nu$.

Now since (P_ν) is a bounded set of points (i.e. all the coordinates are bounded), it has a limit-point P^* $(\theta_1^*, \theta_2^*,\ldots)$, i.e. a point such that from (P_ν) we can choose a sequence (P_{ν_r}) such that each coordinate θ_{n,ν_r} of P_{ν_r} tends to the limit θ_n^* as $r\to\infty$.

It is now easy to prove that P^* corresponds to v^*, i.e. that

$$\Phi(\sigma_0, \theta_1^*,\ldots) = v^*,$$

so that v^* is a point of V. For the series for v_{ν_r}, viz.

$$\sum_{n=1}^{\infty} f_n(p_n^{-\sigma_0}e^{2\pi i\theta_{n,\nu_r}}),$$

is uniformly convergent with respect to r, since (by Weierstrass's M-test) it is uniformly convergent with respect to all the θ's; further, the nth term tends to $f_n(p_n^{-\sigma_0}e^{2\pi i\theta_n^*})$ as $r\to\infty$. Hence

$$v^* = \lim_{r\to\infty} v_{\nu_r} = \lim_{r\to\infty}\sum_{n=1}^{\infty} f_n(p_n^{-\sigma_0}e^{2\pi i\theta_{n,\nu_r}}) = \Phi(\sigma_0, \theta_1^*,\ldots),$$

which proves our result.

To establish the identity of V and W it remains to prove that V is contained in W. It is obviously sufficient (and also necessary) for this that W should be closed. But that W is closed does not follow, as might perhaps be supposed, from the mere fact that W is the set of values taken by a bounded analytic function in the immediate neighbourhood of a line. Thus e^{-z^2} is bounded and arbitrarily near to 0 in every strip including the real axis, but never actually assumes the value 0. The fact that W is closed (which we shall not prove directly) depends on the special nature of the function $F(s)$.

Let $v = \Phi(\sigma_0, \theta_1, \theta_2,\ldots)$ be an arbitrary value contained in V. We have to show that v is a member of W, i.e. that, in every strip

$$|\sigma-\sigma_0| < \delta,$$

$F(s)$ assumes the value v.

Let
$$G(s) = \sum_{n=1}^{\infty} f_n(p_n^{-s}e^{2\pi i\theta_n}),$$

so that $G(\sigma_0) = v$. We choose a small circle C with centre σ_0 and radius less than δ such that $G(s) \neq v$ on the circumference. Let m be the minimum of $|G(s)-v|$ on C.

Kronecker's theorem enables us to choose t_0 such that, for every s in C,

$$|F(s+it_0) - G(s)| < m.$$

The proof is almost exactly the same as that used to show that U is everywhere dense in V. The series for $F(s)$ and $G(s)$ are uniformly convergent in the strip, and, for each fixed N, $\sum_1^N f_n(p_n^{-\sigma}e^{2\pi i\mu_n})$ is a continuous function of $\sigma, \mu_1,\ldots, \mu_N$. It is therefore sufficient to show that we can choose t_0 so that the difference between the arguments of p_n^{-s} at $s = \sigma_0+it_0$ and $p_n^{-s}e^{2\pi i\theta_n}$ at $s = \sigma_0$, and consequently that between the respective arguments at every pair of corresponding points of the two circles is (mod 2π) arbitrarily small for $n = 1, 2,..., N$. The possibility of this choice follows at once from Kronecker's theorem.

We now have

$$F(s+it_0)-v = \{G(s)-v\}+\{F(s+it_0)-G(s)\},$$

and on the circumference of C

$$|F(s+it_0)-G(s)| < m \leqslant |G(s)-v|.$$

Hence, by Rouché's theorem, $F(s+it_0)-v$ has in C the same number of zeros as $G(s)-v$, and so at least one. This proves the theorem.

11.5. We now proceed to the study of the set V. Let V_n be the set of values taken by $f_n(p_n^{-s})$ for $\sigma = \sigma_0$, i.e. the set taken by $f_n(z)$ for $|z| = p_n^{-\sigma_0}$. Then V is the 'sum' of the sets of points $V_1, V_2,...$, i.e. it is the set of all values $v_1+v_2+...$, where v_1 is any point of V_1, v_2 any point of V_2, and so on. For the function $\log\zeta(s)$, V_n consists of the points of the curve described by $-\log(1-z)$ as z describes the circle $|z| = p_n^{-\sigma_0}$; for $\zeta'(s)/\zeta(s)$ it consists of the points of the curve described by

$$-(z\log p_n)/(1-z).$$

We begin by considering the function $\zeta'(s)/\zeta(s)$. In this case we can find the set V explicitly. Let

$$w_n = -\frac{z_n\log p_n}{1-z_n}.$$

As z_n describes the circle $|z_n| = p_n^{-\sigma_0}$, w_n describes the circle with centre

$$c_n = -\frac{p_n^{-2\sigma_0}\log p_n}{1-p_n^{-2\sigma_0}}$$

and radius $$\rho_n = \frac{p_n^{-\sigma_0}\log p_n}{1-p_n^{-2\sigma_0}}.$$

Let $$w_n = c_n+w_n' = c_n+\rho_n e^{i\phi_n},$$

and let $$c = \sum_{n=1}^{\infty} c_n = \frac{\zeta'(2\sigma_0)}{\zeta(2\sigma_0)}.$$

Then V is the set of all the values of

$$c+\sum_{n=1}^{\infty}\rho_n e^{i\phi_n}$$

for independent $\phi_1, \phi_2,...$. The set V' of the values of $\sum \rho_n e^{i\phi_n}$ is the 'sum' of an infinite number of circles with centre at the origin, whose radii $\rho_1, \rho_2,...$ form, as it is easy to see, a decreasing sequence. Let V_n' denote the nth circle.

Then $V_1'+V_2'$ is the area swept out by the circle of radius ρ_2 as its centre describes the circle with centre the origin and radius ρ_1. Hence, since $\rho_2 < \rho_1$, $V_1'+V_2'$ is the annulus with radii $\rho_1-\rho_2$ and $\rho_1+\rho_2$.

The argument clearly extends to any finite number of terms. Thus $V_1'+...+V_N'$ consists of all points of the annulus

$$\rho_1-\sum_{n=2}^{N}\rho_n \leqslant |w| \leqslant \sum_{n=1}^{N}\rho_n,$$

or, if the left-hand side is negative, of the circle

$$|w| \leqslant \sum_{n=1}^{N}\rho_n.$$

It is now easy to see that

(i) *if $\rho_1 > \rho_2+\rho_3+...$, the set V' consists of all points w of the annulus*

$$\rho_1-\sum_{n=2}^{\infty}\rho_n \leqslant |w| \leqslant \sum_{n=1}^{\infty}\rho_n;$$

(ii) *if $\rho_1 \leqslant \rho_2+\rho_3+...$, V' consists of all points w for which*

$$|w| \leqslant \sum_{n=1}^{\infty}\rho_n.$$

For example, in case (ii), let w_0 be an interior point of the circle. Then we can choose N so large that

$$\sum_{N+1}^{\infty}\rho_n < \sum_{n=1}^{N}\rho_n-|w_0|.$$

Hence $$w_1 = w_0-\sum_{N+1}^{\infty}\rho_n e^{i\phi_n}$$

lies within the circle $V_1'+...+V_N'$ for any values of the ϕ_n, e.g. for $\phi_{N+1} = ... = 0$. Hence

$$w_1 = \sum_{n=1}^{N}\rho_n e^{i\phi_n}$$

for some values of $\phi_1,..., \phi_n$, and so

$$w_0 = \sum_{n=1}^{\infty}\rho_n e^{i\phi_n}$$

as required. That V' also includes the boundary in each case is clear on taking all the ϕ_n equal.

The complete result is that there is an absolute constant $D = 2{\cdot}57...$, determined as the root of the equation

$$\frac{2^{-D}\log 2}{1-2^{-2D}} = \sum_{n=2}^{\infty}\frac{p_n^{-D}\log p_n}{1-p_n^{-2D}},$$

such that for $\sigma_0 > D$ we are in case (i), and for $1 < \sigma_0 \leqslant D$ we are in case (ii). The radius of the outer boundary of V' is

$$R = \frac{\zeta'(2\sigma_0)}{\zeta(2\sigma_0)} - \frac{\zeta'(\sigma_0)}{\zeta(\sigma_0)}$$

in each case; the radius of the inner boundary in case (i) is

$$r = 2\rho_1 - R = 2^{1-\sigma_0}\log 2/(1-2^{-2\sigma_0}) - R.$$

Summing up, we have the following results for $\zeta'(s)/\zeta(s)$.

THEOREM 11.5 (A). *The values which $\zeta'(s)/\zeta(s)$ takes on the line $\sigma = \sigma_0 > 1$ form a set everywhere dense in a region $R(\sigma_0)$. If $\sigma_0 > D$, $R(\sigma_0)$ is the annulus (boundary included) with centre c and radii r and R; if $\sigma_0 \leqslant D$, $R(\sigma_0)$ is the circular area (boundary included) with centre c and radius R; c, r, and R are continuous functions of σ_0 defined by*

$$c = \zeta'(2\sigma_0)/\zeta(2\sigma_0), \quad R = c - \zeta'(\sigma_0)/\zeta(\sigma_0), \quad r = 2^{1-\sigma_0}\log 2/(1-2^{-2\sigma_0}) - R.$$

Further, as $\sigma_0 \to \infty$,

$$\lim c = \lim r = \lim R = 0, \qquad \lim c/R = \lim (R-r)/R = 0;$$

as $\sigma_0 \to D$, $\lim r = 0$; and as $\sigma_0 \to 1$, $\lim R = \infty$, $\lim c = \zeta'(2)/\zeta(2)$.

THEOREM 11.5 (B). *The set of values which $\zeta'(s)/\zeta(s)$ takes in the immediate neighbourhood of $\sigma = \sigma_0$ is identical with $R(\sigma_0)$. In particular, since c tends to a finite limit and R to infinity as $\sigma_0 \to 1$, $\zeta'(s)/\zeta(s)$ takes all values infinitely often in the strip $1 < \sigma < 1+\delta$, for an arbitrary positive δ.*

The above results evidently enable us to study the set of points at which $\zeta'(s)/\zeta(s)$ takes the assigned value a. We confine ourselves to giving the result for $a = 0$; this is the most interesting case, since the zeros of $\zeta'(s)/\zeta(s)$ are identical with those of $\zeta'(s)$.

THEOREM 11.5 (C). *There is an absolute constant E, between 2 and 3, such that $\zeta'(s) \neq 0$ for $\sigma > E$, while $\zeta'(s)$ has an infinity of zeros in every strip between $\sigma = 1$ and $\sigma = E$.*

In fact it is easily verified that the annulus $R(\sigma_0)$ includes the origin if $\sigma_0 = 2$, but not if $\sigma_0 = 3$.

11.6. We proceed now to the study of $\log \zeta(s)$. In this case the set V consists of the 'sum' of the curves V_n described by the points

$$w_n = -\log(1-z_n)$$

as z_n describes the circle $|z_n| = p_n^{-\sigma_0}$.

In the first place, V_n is a convex curve. For if

$$u+iv = w = f(z) = f(x+iy),$$

and z describes the circle $|z| = r$, then

$$\frac{du}{dx} + i\frac{dv}{dx} = f'(z)\left(1+i\frac{dy}{dx}\right) = f'(z)\frac{x+iy}{iy}.$$

Hence $$\arctan\frac{dv}{du} = \arg\{zf'(z)\} - \tfrac{1}{2}\pi.$$

A sufficient condition that w should describe a convex curve as z describes $|z| = r$ is that the tangent to the path of w should rotate steadily through 2π as z describes the circle, i.e. that $\arg\{zf'(z)\}$ should increase steadily through 2π. This condition is satisfied in the case $f(z) = -\log(1-z)$; for $zf'(z) = z/(1-z)$ describes a circle enclosing the origin as z describes $|z| = r < 1$.

If $z = re^{i\theta}$, and $w = -\log(1-z)$, then

$$u = -\tfrac{1}{2}\log(1-2r\cos\theta+r^2), \qquad v = \arctan\frac{r\sin\theta}{1-r\cos\theta}.$$

The second equation leads to

$$r\cos\theta = \sin^2 v \pm \cos v(r^2-\sin^2 v)^{\frac{1}{2}}.$$

Hence, for real r and θ, $|v| \leqslant \arcsin r$. If $\cos\theta_1$ and $\cos\theta_2$ are the two values of $\cos\theta$ corresponding to a given v,

$$(1-2r\cos\theta_1+r^2)(1-2r\cos\theta_2+r^2) = (1-r^2)^2.$$

Hence if u_1 and u_2 are the corresponding values of u,

$$u_1+u_2 = -\log(1-r^2).$$

The curve V_n is therefore convex and symmetrical about the lines

$$u = -\tfrac{1}{2}\log(1-r^2) \quad \text{and} \quad v = 0.$$

Its diameters in the u and v directions are $\frac{1}{2}\log\{(1+r)/(1-r)\}$ and $\arcsin r$.

Let $$c_n = -\tfrac{1}{2}\log(1-p_n^{-2\sigma_0})$$

and $$w_n = c_n + w_n',$$

$$c = \sum_{n=1}^{\infty} c_n = \tfrac{1}{2}\log\zeta(2\sigma_0).$$

Then the points w_n' describe symmetrical convex figures with centre the origin. Let V' be the 'sum' of these figures.

It is now easy, by analogy with the previous case, to imagine the result. *The set V', which is plainly symmetrical about both axes, is either* (i) *the region bounded by two convex curves, one of which is entirely interior to the other, or* (ii) *the region bounded by a single convex curve. In each case the boundary is included as part of the region.*

This follows from a general theorem of Bohr on the 'summation' of a series of convex curves.

For our present purpose the following weaker but more obvious results will be sufficient. The set V' is included in the circle with centre the origin and radius

$$R = \sum_{n=1}^{\infty} \tfrac{1}{2}\log\frac{1+p_n^{-\sigma_0}}{1-p_n^{-\sigma_0}} = \tfrac{1}{2}\log\frac{\zeta^2(\sigma_0)}{\zeta(2\sigma_0)}.$$

If σ_0 is sufficiently large, V' lies entirely outside the circle of radius

$$\arcsin 2^{-\sigma_0} - \sum_{n=2}^{\infty} \tfrac{1}{2}\log\frac{1+p_n^{-\sigma_0}}{1-p_n^{-\sigma_0}} = \arcsin 2^{-\sigma_0} + \tfrac{1}{2}\log\frac{1+2^{-\sigma_0}}{1-2^{-\sigma_0}} - R.$$

If
$$\sum_{n=2}^{\infty} \arcsin p_n^{-\sigma_0} > \tfrac{1}{2}\log\frac{1+2^{-\sigma_0}}{1-2^{-\sigma_0}},$$

and so if σ_0 is sufficiently near to 1, V' includes all points inside the circle of radius

$$\sum_{n=1}^{\infty} \arcsin p_n^{-\sigma_0}.$$

In particular V' includes any given area, however large, if σ_0 is sufficiently near to 1.

We cannot, as in the case of circles, determine in all circumstances whether we are in case (i) or case (ii). It is not obvious, for example, whether there exists an absolute constant D' such that we are in case (i) or (ii) according as $\sigma_0 > D'$ or $1 < \sigma_0 \leqslant D'$. The discussion of this point demands a closer investigation of the geometry of the special curves with which we are dealing, and the question would appear to be one of considerable intricacy.

The relations between U, V, and W now give us the following analogues for $\log\zeta(s)$ of the results for $\zeta'(s)/\zeta(s)$.

THEOREM 11.6 (A). *On each line $\sigma = \sigma_0 > 1$ the values of $\log\zeta(s)$ are everywhere dense in a region $R(\sigma_0)$ which is either* (i) *the ring-shaped area bounded by two convex curves, or* (ii) *the area bounded by one convex curve. For sufficiently large values of σ_0 we are in case* (i), *and for values of σ_0 sufficiently near to* 1 *we are in case* (ii).

THEOREM 11.6 (B). *The set of values which $\log\zeta(s)$ takes in the immediate neighbourhood of $\sigma = \sigma_0$ is identical with $R(\sigma_0)$. In particular, since $R(\sigma_0)$ includes any given finite area when σ_0 is sufficiently near* 1, *$\log\zeta(s)$ takes every value an infinity of times in $1 < \sigma < 1+\delta$.*

As a consequence of the last result, we have

THEOREM 11.6 (C). *the function $\zeta(s)$ takes every value except* 0 *an infinity of times in the strip $1 < \sigma < 1+\delta$.*

This is a more precise form of Theorem 11.1.

11.7. We have seen above that $\log\zeta(s)$ takes any assigned value a an infinity of times in $\sigma > 1$. It is natural to raise the question *how often* the value a is taken, i.e. the question of the behaviour for large T of the number $M_a(T)$ of roots of $\log\zeta(s) = a$ in $\sigma > 1$, $0 < t < T$. This question is evidently closely related to the question as to how often, as $t \to \infty$, the point $(a_1 t, a_2 t,..., a_N t)$ of Kronecker's theorem, which, in virtue of the theorem, comes (mod 1) arbitrarily near every point in the N-dimensional unit cube, comes within a given distance of an assigned point $(b_1, b_2,..., b_N)$. The answer to this last question is given by the following theorem, which asserts that, roughly speaking, the point $(a_1 t,..., a_N t)$ comes near every point of the unit cube equally often, i.e. it does not give a preference to any particular region of the unit cube.

Let $a_1,..., a_N$ be linearly independent, and let γ be a region of the N-dimensional unit cube with volume Γ (in the Jordan sense). Let $I_\gamma(T)$ be the sum of the intervals between $t = 0$ and $t = T$ for which the point P $(a_1 t,..., a_N t)$ is (mod 1) *inside γ. Then*

$$\lim_{T\to\infty} I_\gamma(T)/T = \Gamma.$$

The region γ is said to have the volume Γ in the Jordan sense, if, given ϵ, we can find two sets of cubes with sides parallel to the axes, of volumes Γ_1 and Γ_2, included in and including γ respectively, such that

$$\Gamma_1 - \epsilon \leqslant \Gamma \leqslant \Gamma_2 + \epsilon.$$

If we call a point with coordinates of the form $(a_1 t,..., a_N t)$, mod 1, an 'accessible' point, Kronecker's theorem states that the accessible points are everywhere dense in the unit cube C. If now γ_1, γ_2 are two equal cubes with sides parallel to the axes, and with centres at accessible points P_1 and P_2, corresponding to t_1 and t_2, it is easily seen that

$$\lim I_{\gamma_1}(T)/I_{\gamma_2}(T) = 1.$$

For $(a_1 t,..., a_N t)$ will lie inside γ_2 when and only when $\{a_1(t+t_2-t_1),...\}$ lies inside γ_1.

Consider now a set of p non-overlapping cubes c, inside C, of side ϵ, each of which has its centre at an accessible point, and q of which lie inside γ; and a set of P overlapping cubes c', also centred on accessible points, whose union includes C and such that γ is included in a union of Q of them. Since the accessible points are everywhere dense, it is possible to choose the cubes such that q/P and Q/p are arbitrarily near to Γ. Now, denoting by $\sum_\gamma I_c(T)$ the sum of t-intervals in $(0, T)$ corresponding to the cubes c which lie in γ, and so on,

$$\sum_\gamma I_c(T) \Big/ \sum_C I_{c'}(T) \leqslant \frac{I_\gamma(T)}{T} \leqslant \sum_\gamma I_{c'}(T) \Big/ \sum_C I_c(T).$$

Making $T \to \infty$ we obtain

$$\frac{q}{P} \leqslant \overline{\lim_{T\to\infty}} \frac{I_\gamma(T)}{T} \leqslant \frac{Q}{p},$$

and the result follows.

11.8. We can now prove

THEOREM 11.8 (A). *If $\sigma = \sigma_0 > 1$ is a line on which $\log\zeta(s)$ comes arbitrarily near to a given number a, then in every strip $\sigma_0-\delta < \sigma < \sigma_0+\delta$ the value a is taken more than $K(a,\sigma_0,\delta)T$ times, for large T, in $0 < t < T$.*

To prove this we have to reconsider the argument of the previous sections, used to establish the existence of a root of $\log\zeta(s) = a$ in the strip, and use Kronecker's theorem in its generalized form. We saw that a sufficient condition that $\log\zeta(s) = a$ may have a root inside a circle with centre σ_0+it_0 and radius 2δ is that, for a certain N and corresponding numbers $\theta_1,\ldots,\theta_N$, and a certain $\eta = \eta(\sigma_0,\delta,\theta_1,\ldots,\theta_N)$

$$|-t_0\log p_n - 2\pi\theta_n - 2\pi g_n| < \eta \quad (n = 1, 2,\ldots, N).$$

From the generalized Kronecker's theorem it follows that the sum of the intervals between 0 and T in which t_0 satisfies this condition is asymptotically equal to $(\eta/2\pi)^N T$, and it is therefore greater than $\frac{1}{2}(\eta/2\pi)^N T$ for large T. Hence we can select more than $\frac{1}{8}(\eta/2\pi)^N T/\delta$ numbers t_0' in them, no two of which differ by less than 4δ. If now we describe circles with the points σ_0+it_0' as centres and radius 2δ, these circles will not overlap, and each of them will contain a zero of $\log\zeta(s)-a$. This gives the desired result.

We can also prove

THEOREM 11.8 (B). *There are positive constants $K_1(a)$ and $K_2(a)$ such that the number $M_a(T)$ of zeros of $\log\zeta(s)-a$ in $\sigma > 1$ satisfies the inequalities*

$$K_1(a)T < M_a(T) < K_2(a)T.$$

The lower bound follows at once from the above theorem. The upper bound follows from the more general result that *if b is any given constant, the number of zeros of $\zeta(s)-b$ in $\sigma > \frac{1}{2}+\delta$ $(\delta > 0)$, $0 < t < T$, is $O(T)$ as $T\to\infty$.*

The proof of this is substantially the same as that of Theorem 9.15 (A), the function $\zeta(s)-b$ playing the same part as $\zeta(s)$ did there. Finally the number of zeros of $\log\zeta(s)-a$ is not greater than the number of zeros of $\zeta(s)-e^a$, and so is $O(T)$.

11.9. We now turn to the more difficult question of the behaviour of $\zeta(s)$ in the critical strip. The difficulty, of course, is that $\zeta(s)$ is no

longer represented by an absolutely convergent Dirichlet series. But by a device like that used in the proof of Theorem 9.17, we are able to obtain in the critical strip results analogous to those already obtained in the region of absolute convergence.

As before we consider $\log\zeta(s)$. For $\sigma \leqslant 1$, $\log\zeta(s)$ is defined, on each line t = constant which does not pass through a singularity, by continuation along this line from $\sigma > 1$.

We require the following lemma.

LEMMA. *If $f(z)$ is regular for $|z-z_0| \leqslant R$, and*

$$\iint\limits_{|z-z_0|\leqslant R} |f(z)|^2\,dxdy = H,$$

then
$$|f(z)| \leqslant \frac{(H/\pi)^{\frac{1}{2}}}{R-R'} \quad (|z-z_0| \leqslant R' < R).$$

For if $|z'-z_0| \leqslant R'$,

$$\{f(z')\}^2 = \frac{1}{2\pi i}\int\limits_{|z-z'|=r} \frac{\{f(z)\}^2}{z-z'}\,dz = \frac{1}{2\pi}\int\limits_0^{2\pi} \{f(z'+re^{i\theta})\}^2\,d\theta.$$

Hence

$$|f(z')|^2 \int\limits_0^{R-R'} r\,dr \leqslant \frac{1}{2\pi}\int\limits_0^{R-R'}\int\limits_0^{2\pi} |f(z'+re^{i\theta})|^2 r\,drd\theta \leqslant \frac{H}{2\pi},$$

and the result follows.

THEOREM 11.9. *Let σ_0 be a fixed number in the range $\frac{1}{2} < \sigma \leqslant 1$. Then the values which $\log\zeta(s)$ takes on $\sigma = \sigma_0$, $t > 0$, are everywhere dense in the whole plane.*

Let
$$\zeta_N(s) = \zeta(s)\prod_{n=1}^{N}(1-p_n^{-s}).$$

This function is similar to the function $\zeta(s)M_X(s)$ of Chapter IX, but it happens to be more convenient here.

Let δ be a positive number less than $\frac{1}{2}(\sigma_0-\frac{1}{2})$. Then it is easily seen as in § 9.19 that for $N \geqslant N_0(\sigma_0,\epsilon)$, $T \geqslant T_0 = T_0(N)$,

$$\int\limits_1^T |\zeta_N(\sigma+it)-1|^2\,dt < \epsilon T$$

uniformly for $\sigma_0-\delta \leqslant \sigma \leqslant \sigma_1+\delta$ $(\sigma_1 > 1)$. Hence

$$\int\limits_1^T\int\limits_{\sigma_0-\delta}^{\sigma_1+\delta} |\zeta_N(\sigma+it)-1|^2\,d\sigma dt < (\sigma_1-\sigma_0+2\delta)\epsilon T.$$

Hence
$$\int\limits_{\nu-\frac{1}{2}}^{\nu+\frac{1}{2}}\int\limits_{\sigma_0-\delta}^{\sigma_1+\delta} |\zeta_N(\sigma+it)-1|^2\,d\sigma dt < (\sigma_1-\sigma_0+2\delta)\sqrt{\epsilon}$$

for more than $(1-\sqrt{\epsilon})T$ integer values of ν. Since this rectangle contains the circle with centre $s = \sigma+it$, where $\sigma_0 \leqslant \sigma \leqslant \sigma_1$, $\nu-\frac{1}{2}+\delta \leqslant t \leqslant \nu+\frac{1}{2}-\delta$, and radius δ, it is easily seen from the lemma that we can choose δ and ϵ so that *given* $0<\eta<1$, $0<\eta'<1$, *we have*

$$|\zeta_N(\sigma+it)-1| < \eta \quad (\sigma_0 \leqslant \sigma \leqslant \sigma_1) \tag{11.9.1}$$

for a set of values of t of measure greater than $(1-\eta')T$, *and for*

$$N \geqslant N_0(\sigma,\eta,\eta'), \qquad T \geqslant T_0(N).$$

Let
$$R_N(s) = -\sum_{N+1}^{\infty} \operatorname{Log}(1-p_n^{-s}) \quad (\sigma > 1),$$

where Log denotes the principal value of the logarithm. Then

$$\zeta_N(s) = \exp\{R_N(s)\}.$$

We want to show that $R_N(s) = \operatorname{Log}\zeta_N(s)$, i.e. that $|\mathbf{I}R_N(s)| < \frac{1}{2}\pi$, for $\sigma \geqslant \sigma_0$ and the values of t for which (11.9.1) holds. This is true for $\sigma = \sigma_1$ if σ_1 is sufficiently large, since $|R_N(s)| \to 0$ as $\sigma_1 \to \infty$. Also, by (11.9.1), $\mathbf{R}\zeta_N(s) > 0$ for $\sigma_0 \leqslant \sigma \leqslant \sigma_1$, so that $\mathbf{I}R_N(s)$ must remain between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ for all values of σ in this interval. This gives the desired result.

We have therefore

$$|R_N(s)| = |\operatorname{Log}[1+\{\zeta_N(s)-1\}]| < 2\,|\zeta_N(s)-1| < 2\eta$$

for $\sigma_0 \leqslant \sigma \leqslant \sigma_1$, $N \geqslant N_0(\sigma_0,\eta,\eta')$, $T \geqslant T_0(N)$, in a set of values of t of measure greater than $(1-\eta')T$.

Now consider the function

$$F_N(\sigma_0+it) = -\sum_{n=1}^{N} \log(1-p_n^{-\sigma_0-it}),$$

and in conjunction with it the function of N independent variables

$$\Phi_N(\theta_1,\ldots,\theta_N) = -\sum_{n=1}^{N} \log(1-p_n^{-\sigma_0}e^{2\pi i\theta_n}).$$

Since $\sum p_n^{-\sigma_0}$ is divergent, it is easily seen from our previous discussion of the values taken by $\log\zeta(s)$ that the set of values of Φ_N includes any given finite region of the complex plane if N is large enough. In particular, if a is any given number, we can find a number N and values of the θ's such that
$$\Phi_N(\theta_1,\ldots,\theta_N) = a.$$

We can then, by Kronecker's theorem, find a number t such that $|F_N(\sigma_0+it)-a|$ is arbitrarily small. But this in itself is not sufficient to prove the theorem, since this value of t does not necessarily make $|R_N(s)|$ small. An additional argument is therefore required.

Let
$$\Phi_{M,N} = -\sum_{n=M+1}^{N} \log(1-p_n^{-\sigma_0}e^{2\pi i\theta_n}) = \sum_{n=M+1}^{N}\sum_{m=1}^{\infty} \frac{p_n^{-m\sigma_0}e^{2\pi im\theta_n}}{m}.$$

Then, expressing the squared modulus of this as the product of conjugates, and integrating term by term, we obtain

$$\int_0^1\int_0^1\cdots\int_0^1 |\Phi_{M,N}|^2\,d\theta_{M+1}\ldots d\theta_N = \sum_{n=M+1}^{N}\sum_{m=1}^{\infty}\frac{p_n^{-2m\sigma_0}}{m^2}$$
$$< \sum_{n=M+1}^{N} p_n^{-2\sigma_0}\sum_{m=1}^{\infty}\frac{1}{m^2} < A\sum_{n=M+1}^{\infty} p_n^{-2\sigma_0},$$

which can be made arbitrarily small, by choice of M, for all N. It therefore follows from the theory of Riemann integration of a continuous function that, given ϵ, we can divide up the $(N-M)$-dimensional unit cube into sub-cubes q_ν, each of volume λ, in such a way that

$$\lambda\sum_\nu \max_{q_\nu} |\Phi_{M,N}|^2 < \tfrac{1}{2}\epsilon^2.$$

Hence *for* $M \geqslant M_0(\epsilon)$ *and any* $N > M$, *we can find cubes of total volume greater than* $\frac{1}{2}$ *in which* $|\Phi_{M,N}| < \epsilon$.

We now choose our value of t as follows.

(i) Choose M so large, and give $\theta_1',\ldots,\theta_M'$ such values, that

$$\Phi_M(\theta_1',\ldots,\theta_M') = a.$$

It then follows from considerations of continuity that, given ϵ, we can find an M-dimensional cube with centre $\theta_1',\ldots,\theta_M'$ and side $d > 0$ throughout which
$$|\Phi_M(\theta_1,\ldots,\theta_M)-a| < \tfrac{1}{3}\epsilon.$$

(ii) We may also suppose that M has been chosen so large that, for any value of N, $|\Phi_{M,N}| < \frac{1}{3}\epsilon$ in certain $(N-M)$-dimensional cubes of total volume greater than $\frac{1}{2}$.

(iii) Having fixed M and d, we can choose N so large that, for $T > T_0(N)$, the inequality $|R_N(s)| < \frac{1}{3}\epsilon$ holds in a set of values of t of measure greater than $(1-\frac{1}{2}d^M)T$.

(iv) Let $I(T)$ be the sum of the intervals between 0 and T for which the point
$$\{-(t\log p_1)/2\pi,\ldots,-(t\log p_N)/2\pi\}$$
is (mod 1) inside one of the N-dimensional cubes, of total volume greater than $\frac{1}{2}d^M$, determined by the above construction. Then by the extended Kronecker's theorem, $I(T) > \frac{1}{2}d^MT$ if T is large enough. There are

therefore values of t for which the point lies in one of these cubes, and for which at the same time $|R_N(s)| < \frac{1}{3}\epsilon$. For such a value of t

$$\begin{aligned}|\log\zeta(s)-a| &\leqslant |F_N(s)-a|+|R_N(s)| \\ &\leqslant |\Phi_M(\theta_1,\ldots,\theta_M)-a|+|\Phi_{M,N}|+|R_N(s)| \\ &< \tfrac{1}{3}\epsilon+\tfrac{1}{3}\epsilon+\tfrac{1}{3}\epsilon = \epsilon,\end{aligned}$$

and the result follows.

11.10. THEOREM 11.10. *Let $\frac{1}{2} < \alpha < \beta < 1$, and let a be any complex number. Let $M_{a,\alpha,\beta}(T)$ be the number of zeros of $\log\zeta(s)-a$ (defined as before) in the rectangle $\alpha < \sigma < \beta$, $0 < t < T$. Then there are positive constants $K_1(a,\alpha,\beta)$, $K_2(a,\alpha,\beta)$ such that*

$$K_1(a,\alpha,\beta)T < M_{a,\alpha,\beta}(T) < K_2(a,\alpha,\beta)T \quad (T > T_0).$$

We first observe that, for suitable values of the θ's, the series

$$-\sum_{n=1}^{\infty}\log(1-p_n^{-s}e^{2\pi i\theta_n})$$

is uniformly convergent in any finite region to the right of $\sigma = \frac{1}{2}$. This is true, for example, if $\theta_n = \frac{1}{2}n$ for sufficiently large values of n; for then

$$\sum_{n>n_0} p_n^{-s}e^{2\pi i\theta_n} = \sum_{n>n_0}(-1)^n p_n^{-s},$$

which is convergent for real $s > 0$, and hence uniformly convergent in any finite region to the right of the imaginary axis; and for any θ's $\sum |p_n^{-s}e^{2\pi i\theta_n}|^2 = \sum p_n^{-2\sigma}$ is uniformly convergent in any finite region to the right of $\sigma = \frac{1}{2}$.

If a is any given number, and the θ's have this property, we can choose n_1 so large that

$$\left|-\sum_{n=n_1+1}^{\infty}\log(1-p_n^{-s}e^{2\pi i\theta_n})\right| < \epsilon \quad (\sigma = \tfrac{1}{2}(\alpha+\beta)),$$

and at the same time so that the set of values of

$$-\sum_{n=1}^{n_1}\log(1-p_n^{-\frac{1}{2}\alpha-\frac{1}{2}\beta}e^{2\pi i\theta_n})$$

includes the circle with centre the origin and radius $|a|+|\epsilon|$. Hence by choosing first $\theta_{n_1+1},\ldots$, and then $\theta_1,\ldots,\theta_{n_1}$, we can find values of the θ's, say θ_1', $\theta_2',\ldots$, such that the series

$$G(s) = -\sum_{n=1}^{\infty}\log(1-p_n^{-s}e^{2\pi i\theta_n'})$$

is uniformly convergent in any finite region to the right of $\sigma = \frac{1}{2}$, and

$$G(\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta) = a.$$

We can then choose a circle C of centre $\frac{1}{2}\alpha+\frac{1}{2}\beta$ and radius $\rho < \frac{1}{4}(\beta-\alpha)$ on which $G(s) \neq a$.

Let
$$m = \min_{s \text{ on } C}|G(s)-a|.$$

Now let
$$\Phi_{M,N}(s) = -\sum_{n=M+1}^{N}\log(1-p_n^{-s}e^{2\pi i\theta_n}).$$

Then, as in the previous proof,

$$\int_0^1\cdots\int_0^1 \iint_{|s-\frac{1}{2}\alpha-\frac{1}{2}\beta|\leqslant\frac{1}{2}(\beta-\alpha)} |\Phi_{M,N}(s)|^2\, d\theta_{M+1}\ldots d\theta_N\, d\sigma dt < A\sum_{M+1}^{\infty}p_n^{-2\alpha}.$$

Hence for $M \geqslant M_0(\epsilon)$ and any $N > M$ we can find cubes of total volume greater than $\frac{1}{2}$ in which

$$\iint_{|s-\frac{1}{2}\alpha-\frac{1}{2}\beta|\leqslant\frac{1}{2}(\beta-\alpha)} |\Phi_{M,N}(s)|^2\, d\sigma dt < \epsilon$$

and so in which (by the lemma of § 11.9)

$$|\Phi_{M,N}(s)| < 2(\epsilon/\pi)^{\frac{1}{2}}(\beta-\alpha)^{-\frac{1}{2}} \quad (|s-\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta| \leqslant \tfrac{1}{4}(\beta-\alpha)).$$

We also want a little more information about $R_N(s)$, viz. that $R_N(s)$ is regular, and $|R_N(s)| < \eta$, throughout the rectangle

$$|\sigma-\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta| \leqslant \tfrac{1}{4}(\beta-\alpha), \quad t_0-\tfrac{1}{2} \leqslant t \leqslant t_0+\tfrac{1}{2},$$

for a set of values of t_0 of measure greater than $(1-\eta')T$. As before it is sufficient to prove this for $\zeta_N(s)-1$, and by the lemma it is sufficient to prove that

$$\phi(t_0) = \int_\alpha^\beta d\sigma \int_{t_0-1}^{t_0+1}|\zeta_N(s)-1|^2\, dt < \epsilon$$

for such t_0, by choice of N. Now

$$\begin{aligned}\int_1^T \phi(t_0)\, dt &= \int_\alpha^\beta d\sigma \int_1^T dt_0 \int_{t_0-1}^{t_0+1}|\zeta_N(s)-1|^2\, dt \\ &\leqslant \int_\alpha^\beta d\sigma \int_1^{T+1}|\zeta_N(s)-1|^2\, dt \int_{t-1}^{t+1} dt_0 = 2\int_\alpha^\beta d\sigma \int_1^{T+1}|\zeta_N(s)-1|^2\, dt < \epsilon T\end{aligned}$$

by choice of N as before. Hence the measure of the set where $\phi(t_0) > \surd\epsilon$ is less than $\surd\epsilon T$, and the desired result follows.

It now follows as before that there is a set of values of t_0 in $(0, T)$, of measure greater than KT, such that for $|s-\frac{1}{2}\alpha-\frac{1}{2}\beta| \leqslant \frac{1}{4}(\beta-\alpha)$

$$\left|\sum_{n=1}^{M}\log(1-p_n^{-s}e^{2\pi i\theta_n'})-\sum_{n=1}^{M}\log(1-p_n^{-s-it_0})\right| < \tfrac{1}{4}m,$$

$$|\Phi_{M,N}(s)| < \tfrac{1}{4}m,$$

and also $$|R_N(s+it_0)| < \tfrac{1}{4}m.$$

At the same time we can suppose that M has been taken so large that

$$\left|G(s)+\sum_{n=1}^{M}\log(1-p_n^{-s}e^{2\pi i\theta_n'})\right| < \tfrac{1}{4}m \quad (\sigma \geqslant \alpha).$$

Then $$|\log\zeta(s)-G(s)| < m$$

on the circle with centre $\frac{1}{2}\alpha+\frac{1}{2}\beta+it_0$ and radius ρ. Hence, as before, $\log\zeta(s)-a$ has at least one zero in such a circle. The number of such circles for $0 < t_0 < T$ which do not overlap is plainly greater than KT. The lower bound for $M_{a,\alpha,\beta}(T)$ therefore follows; the upper bound holds by the same argument as in the case $\sigma > 1$.

It has been proved by Bohr and Jensen, by a more detailed study of the situation, that there is a $K(a, \alpha, \beta)$ such that

$$M_{a,\alpha,\beta}(T) \sim K(a,\alpha,\beta)T.$$

An immediate corollary of Theorem 11.10 is that, *if $N_{a,\alpha,\beta}(T)$ is the number of points in the rectangle $\frac{1}{2} < \alpha < \sigma < \beta < 1$, $0 < t < T$ where $\zeta(s) = a$ $(a \neq 0)$, then*

$$N_{a,\alpha,\beta}(T) > K(a,\alpha,\beta)T \quad (T > T_0).$$

For $\zeta(s) = a$ if $\log\zeta(s) = \log a$, any one value of the right-hand side being taken. This result, in conjunction with Theorem 9.17, shows that the value 0 of $\zeta(s)$, if it occurs at all in $\sigma > \frac{1}{2}$, is at any rate quite exceptional, zeros being infinitely rarer than a-values for any value of a other than zero.

NOTES FOR CHAPTER 11

11.11. Theorem 11.9 has been generalized by Voronin [1], [2], who obtained the following 'universal' property for $\zeta(s)$. Let D_r be the closed disc of radius $r < \frac{1}{4}$, centred at $s = \frac{3}{4}$, and let $f(s)$ be any function continuous and non-vanishing on D_r, and holomorphic on the interior of D_r. Then for any $\varepsilon > 0$ there is a real number t such that

$$\max_{s\in D_r}|\zeta(s+it)-f(s)| < \varepsilon. \tag{11.11.1}$$

It follows that the curve

$$\gamma(t) = \left(\zeta(\sigma+it), \zeta'(\sigma+it), \ldots, \zeta^{(n-1)}(\sigma+it)\right)$$

is dense in $\mathbb{C}^n$, for any fixed σ in the range $\frac{1}{2} < \sigma < 1$. (In fact Voronin [1] establishes this for $\sigma = 1$ also.) To see this we choose a point $z = (z_0, z_1, \ldots, z_{n-1})$ with $z_0 \neq 0$, and take $f(s)$ to be a polynomial for which $f^{(m)}(\sigma) = z_m$ for $0 \leqslant m < n$. We then fix an R such that $0 < R < \frac{1}{4} - |\sigma - \frac{3}{4}|$, and such that $f(s)$ is nonvanishing on the closed disc $|s-\sigma| \leqslant R$. Thus, if $r = R + |\sigma - \frac{3}{4}|$, the disc D_r contains the circle $|s-\sigma| = R$, and hence (11.11.1) in conjunction with Cauchy's inequality

$$|g^{(m)}(z_0)| \leqslant \frac{m!}{R^m}\max_{|z-z_0|=R}|g(z)|,$$

yields

$$|\zeta^{(m)}(\sigma+it) - z_m| \leqslant \frac{m!}{R^m}\varepsilon \quad (0 \leqslant m < n).$$

Hence $\gamma(t)$ comes arbitrarily close to z. The required result then follows, since the available z are dense in $\mathbb{C}^n$.

Voronin's work has been extended by Bagchi [1] (see also Gonek [1]) so that D_r may be replaced by any compact subset D of the strip $\frac{1}{2} < \mathbf{R}(s) < 1$, whose complement in $\mathbb{C}$ is connected. The condition on f is then that it should be continuous and non-vanishing on D, and holomorphic on the interior (if any) of D. From this it follows that if Φ is any continuous function, and $h_1 < h_2 < \ldots < h_m$ are real constants, then $\zeta(s)$ cannot satisfy the differential–difference equation

$$\Phi\{\zeta(s+h_1), \zeta'(s+h_1), \ldots, \zeta^{(n_1)}(s+h_1), \zeta(s+h_2), \zeta'(s+h_2), \ldots, \zeta^{(n_2)}(s+h_2), \ldots\} = 0$$

unless Φ vanishes identically. This improves earlier results of Ostrowski [1] and Reich [1].

11.12. Levinson [6] has investigated further the distribution of the solutions $\rho_a = \beta_a + i\gamma_a$ of $\zeta(s) = a$. The principal results are that

$$\#\{\rho_a : 0 \leqslant \gamma_a \leqslant T\} = \frac{T}{2\pi}\log T + O(T)$$

and

$$\#\{\rho_a : 0 \leqslant \gamma_a \leqslant T, |\beta_a - \tfrac{1}{2}| \geqslant \delta\} = O_\delta(T) \quad (\delta > 0).$$

Thus (c.f. § 9.15) *all but an infinitesimal proportion of the zeros of $\zeta(s) - a$ lie in the strip $\frac{1}{2} - \delta < \sigma < \frac{1}{2} + \delta$, however small δ may be.*

In reviewing this work Montgomery (Math. Reviews **53** # 10737) quotes an unpublished result of Selberg, namely

$$\sum_{\substack{0 \leqslant \gamma_a \leqslant T \\ \beta_a \geqslant \frac{1}{2}}} (\beta_a - \tfrac{1}{2}) \sim \frac{1}{4\pi^{\frac{3}{2}}} T(\log\log T)^{\frac{1}{2}}. \tag{11.12.1}$$

This leads to a stronger version of the above principle, in which the infinite strip is replaced by the region

$$|\sigma - \tfrac{1}{2}| < \frac{\phi(t)(\log\log t)^{\frac{1}{2}}}{\log t},$$

where $\phi(t)$ is any positive function which tends to infinity with t. It should be noted for comparison with (11.12.1) that the estimate

$$\sum_{0 \leqslant \gamma_a \leqslant T} (\beta_a - \tfrac{1}{2}) = O(\log T)$$

is implicit in Levinson's work. It need hardly be emphasized that despite this result the numbers ρ_a are far from being symmetrically distributed about the critical line.

11.13. The problem of the distribution of values of $\zeta(\frac{1}{2}+it)$ is rather different from that of $\zeta(\sigma+it)$ with $\frac{1}{2} < \sigma < 1$. In the first place it is not known whether the values of $\zeta(\frac{1}{2}+it)$ are everywhere dense, though one would conjecture so. Secondly there is a difference in the rates of growth with respect to t. Thus, for a fixed $\sigma > \frac{1}{2}$, Bohr and Jessen (1), (2) have shown that there is a continuous function $F(z;\sigma)$ such that

$$\frac{1}{2T} m\{t \in [-T, T]: \log\zeta(\sigma+it) \in R\} \to \iint_R F(x+iy;\sigma)\,dx\,dy \quad (T \to \infty)$$

for any rectangle $R \subset \mathbb{C}$ whose sides are parallel to the real and imaginary axes. Here, as usual, m denotes Lebesgue measure, and $\log\zeta(s)$ is defined by continuous variation along lines parallel to the real axis, using (1.1.9) for $\sigma > 1$. By contrast, the corresponding result for $\sigma = \frac{1}{2}$ states that

$$\frac{1}{2T} m\left\{t \in [-T, T]: \frac{\log\zeta(\frac{1}{2}+it)}{\sqrt{[\frac{1}{2}\{\log\log(3+|t|)\}]}} \in R\right\} \to \frac{1}{2\pi}\iint_R e^{-(x^2+y^2)/2}\,dx\,dy$$
$$(T \to \infty).$$

(The right hand side gives a 2-dimensional distribution with mean 0 and variance 1.) This is an unpublished theorem of Selberg, which may be obtained via the method of Ghosh [**2**].

By using a different technique, based on the mean-value bounds of §7.23, Jutila [**4**] has obtained information on 'large deviations' of $\log|\zeta(\frac{1}{2}+it)|$. Specifically, he showed that there is a constant $A > 0$ such that

$$m\{t \in [0, T]: |\zeta(\tfrac{1}{2}+it)| \geqslant V\} \ll T\exp\left(-A\frac{\log^2 V}{\log\log T}\right),$$

uniformly for $1 \leqslant V \leqslant \log T$.

XII

DIVISOR PROBLEMS

12.1. The divisor problem of Dirichlet is that of determining the asymptotic behaviour as $x \to \infty$ of the sum

$$D(x) = \sum_{n \leqslant x} d(n),$$

where $d(n)$ denotes, as usual, the number of divisors of n. Dirichlet proved in an elementary way that

$$D(x) = x \log x + (2\gamma - 1)x + O(x^{\frac{1}{2}}). \tag{12.1.1}$$

In fact

$$\begin{aligned} D(x) &= \sum_{mn \leqslant x} \sum 1 = \sum_{m \leqslant \sqrt{x}} \sum_{n \leqslant \sqrt{x}} 1 + 2 \sum_{m \leqslant \sqrt{x}} \sum_{\sqrt{x} < n \leqslant x/m} 1 \\ &= [\sqrt{x}]^2 + 2 \sum_{m \leqslant \sqrt{x}} \left(\left[\frac{x}{m}\right] - [\sqrt{x}] \right) \\ &= 2 \sum_{m \leqslant \sqrt{x}} \left[\frac{x}{m}\right] - [\sqrt{x}]^2 \\ &= 2 \sum_{m \leqslant \sqrt{x}} \left\{ \frac{x}{m} + O(1) \right\} - \{\sqrt{x} + O(1)\}^2 \\ &= 2x\{\log \sqrt{x} + \gamma + O(x^{-\frac{1}{2}})\} + O(\sqrt{x}) - \{x + O(\sqrt{x})\}, \end{aligned}$$

and (12.1.1) follows. Writing

$$D(x) = x \log x + (2\gamma - 1)x + \Delta(x)$$

we thus have

$$\Delta(x) = O(x^{\frac{1}{2}}). \tag{12.1.2}$$

Later researches have improved this result, but the exact order of $\Delta(x)$ is still undetermined.

The problem is closely related to that of the Riemann zeta-function. By (3.12.1) with $a_n = d(n)$, $s = 0$, $T \to \infty$, we have

$$D(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta^2(w) \frac{x^w}{w}\, dw \quad (c > 1),$$

provided that x is not an integer. On moving the line of integration to the left, we encounter a double pole at $w = 1$, the residue being $x \log x + (2\gamma - 1)x$, by (2.1.16). Thus

$$\Delta(x) = \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \zeta^2(w) \frac{x^w}{w}\, dw \quad (0 < c' < 1).$$

The more general problem of

$$D_k(x) = \sum_{n \leqslant x} d_k(n),$$

where $d_k(n)$ is the number of ways of expressing n as a product of k factors, was also considered by Dirichlet. We have

$$D_k(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta^k(w) \frac{x^w}{w}\, dw \quad (c > 1).$$

Here there is a pole of order k at $w = 1$, and the residue is of the form $xP_k(\log x)$, where P_k is a polynomial of degree $k-1$. We write

$$D_k(x) = xP_k(\log x) + \Delta_k(x), \tag{12.1.3}$$

so that $\Delta_2(x) = \Delta(x)$.

The classical elementary theorem† of the subject is

$$\Delta_k(x) = O(x^{1-1/k} \log^{k-2} x) \quad (k = 2, 3, \ldots). \tag{12.1.4}$$

We have already proved this in the case $k = 2$. Now suppose that it is true in the case $k-1$. We have

$$\begin{aligned} D_k(x) &= \sum_{n_1 n_2 \ldots n_k \leqslant x} 1 = \sum_{mn \leqslant x} d_{k-1}(n) \\ &= \sum_{m \leqslant x^{1/k}} \sum_{n \leqslant x/m} d_{k-1}(n) + \sum_{x^{1/k} < m \leqslant x} \sum_{n \leqslant x/m} d_{k-1}(n) \\ &= \sum_{m \leqslant x^{1/k}} \sum_{n \leqslant x/m} d_{k-1}(n) + \sum_{n \leqslant x^{1-1/k}} d_{k-1}(n) \sum_{x^{1/k} < m \leqslant x/n} 1 \\ &= \sum_{m \leqslant x^{1/k}} D_{k-1}\left(\frac{x}{m}\right) + \sum_{n \leqslant x^{1-1/k}} \left\{ \frac{x}{n} - x^{1/k} + O(1) \right\} d_{k-1}(n) \\ &= \sum_{m \leqslant x^{1/k}} D_{k-1}\left(\frac{x}{m}\right) + x \sum_{n \leqslant x^{1-1/k}} \frac{d_{k-1}(n)}{n} - x^{1/k} D_{k-1}(x^{1-1/k}) + \\ &\qquad + O\{D_{k-1}(x^{1-1/k})\}. \end{aligned}$$

Let us denote by $p_k(z)$ a polynomial in z, of degree $k-1$ at most, not always the same one. Then

$$\sum_{m \leqslant \xi} \frac{\log^{k-2} m}{m} = p_k(\log \xi) + O\left(\frac{\log^{k-2} \xi}{\xi} \right).$$

Hence

$$\sum_{m \leqslant x^{1/k}} \frac{x}{m} P_{k-1}\left(\frac{x}{m}\right) = xp_k(\log x) + O(x^{1-1/k} \log^{k-2} x).$$

Also

$$\begin{aligned} \sum_{m \leqslant x^{1/k}} \Delta_{k-1}\left(\frac{x}{m}\right) &= O\left\{ x^{1-1/(k-1)} \log^{k-3} x \sum_{m \leqslant x^{1/k}} \frac{1}{m^{1-1/(k-1)}} \right\} \\ &= O\{x^{1-1/(k-1)} \log^{k-3} x \,.\, x^{1/\{k(k-1)\}}\} = O(x^{1-1/k} \log^{k-3} x). \end{aligned}$$

† See e.g. Landau (5).

The next term is

$$x\sum_{n\leqslant x^{1-1/k}}\frac{D_{k-1}(n)-D_{k-1}(n-1)}{n}=x\sum_{n\leqslant x^{1-1/k}}\frac{D_{k-1}(n)}{n(n+1)}+\frac{xD_{k-1}(N)}{N+1},$$

where $N=[x^{1-1/k}]$. Now

$$x\sum_{n\leqslant x^{1-1/k}}\frac{P_{k-1}(\log n)}{n+1}+x\frac{NP_{k-1}(\log N)}{N+1}=xp_k(\log x)+O(x^{1/k}\log^{k-2}x)$$

and

$$x\sum_{n\leqslant x^{1-1/k}}\frac{\Delta_{k-1}(n)}{n(n+1)}+\frac{x\Delta_{k-1}(N)}{N+1}=Cx-x\sum_{n>x^{1-1/k}}\frac{\Delta_{k-1}(n)}{n(n+1)}+\frac{x\Delta_{k-1}(N)}{N+1}$$

$$=Cx-x\sum_{n>x^{1-1/k}}O\left\{\frac{\log^{k-3}n}{n^{1+1/(k-1)}}\right\}+O(xN^{-1/(k-1)}\log^{k-3}N)$$

$$=Cx+O(x^{1-1/k}\log^{k-3}x).$$

Finally

$$x^{1/k}D_{k-1}(x^{1-1/k})=x^{1/k}\{x^{1-1/k}P_{k-1}(\log x^{1-1/k})+O(x^{(1-1/k)(1-1/(k-1))}\log^{k-3}x)\}$$

$$=xp_{k-1}(\log x)+O(x^{1-1/k}\log^{k-3}x).$$

This proves (12.1.4).

We may define the order α_k of $\Delta_k(x)$ as the least number such that

$$\Delta_k(x)=O(x^{\alpha_k+\epsilon})$$

for every positive ϵ. Thus it follows from (12.1.4) that

$$\alpha_k\leqslant\frac{k-1}{k}\quad(k=2,3,\ldots).\tag{12.1.5}$$

The exact value of α_k has not been determined for any value of k.

12.2. The simplest theorem which goes beyond this elementary result is

THEOREM 12.2.†

$$\alpha_k\leqslant\frac{k-1}{k+1}\quad(k=2,3,4,\ldots).$$

Take $a_n=d_k(n)$, $\psi(n)=n^\epsilon$, $\alpha=k$, $s=0$, and let x be half an odd integer, in Lemma 3.12. Replacing w by s, this gives

$$D_k(x)=\frac{1}{2\pi i}\int\limits_{c-iT}^{c+iT}\zeta^k(s)\frac{x^s}{s}\,ds+O\left(\frac{x^c}{T(c-1)^k}\right)+O\left(\frac{x^{1+\epsilon}}{T}\right)\quad(c>1).$$

† Voronoi (1), Landau (5).

Now take the integral round the rectangle $-a-iT$, $c-iT$, $c+iT$, $-a+iT$, where $a>0$. We have, by (5.1.1) and the Phragmén–Lindelöf principle,

$$\zeta(s)=O(t^{(a+\frac{1}{2})(c-\sigma)/(a+c)})$$

in the rectangle. Hence

$$\int\limits_{-a+iT}^{c+iT}\zeta^k(s)\frac{x^s}{s}\,ds=O\left(\int\limits_{-a}^{c}T^{k(a+\frac{1}{2})(c-\sigma)/(a+c)-1}x^\sigma\,d\sigma\right)$$

$$=O(T^{k(a+\frac{1}{2})-1}x^{-a})+O(T^{-1}x^c),$$

since the integrand is a maximum at one end or the other of the range of integration. A similar result holds for the integral over

$$(-a-iT,c-iT).$$

The residue at $s=1$ is $xP_k(\log x)$, and the residue at $s=0$ is

$$\zeta^k(0)=O(1).$$

Finally

$$\int\limits_{-a-iT}^{-a+iT}\zeta^k(s)\frac{x^s}{s}\,ds=\int\limits_{-a-iT}^{-a+iT}\chi^k(s)\zeta^k(1-s)\frac{x^s}{s}\,ds$$

$$=\sum_{n=1}^{\infty}d_k(n)\int\limits_{-a-iT}^{-a+iT}\frac{\chi^k(s)}{n^{1-s}}\frac{x^s}{s}\,ds$$

$$=ix^{-a}\sum_{n=1}^{\infty}\frac{d_k(n)}{n^{1+a}}\int\limits_{-T}^{T}\frac{\chi^k(-a+it)}{-a+it}(nx)^{it}\,dt.$$

For $1\leqslant t\leqslant T$,

$$\chi(-a+it)=Ce^{-it\log t+it\log 2\pi+it}t^{a+\frac{1}{2}}+O(t^{a-\frac{1}{2}})$$

and

$$\frac{1}{-a+it}=\frac{1}{it}+O\left(\frac{1}{t^2}\right).$$

The corresponding part of the integral is therefore

$$-iC^k\int\limits_{1}^{T}e^{ikt(-\log t+\log 2\pi+1)}(nx)^{it}t^{(a+\frac{1}{2})k-1}\,dt+O(T^{(a+\frac{1}{2})k-1}),$$

provided that $(a+\frac{1}{2})k>1$. This integral is of the form considered in Lemma 4.5, with

$$F(t)=kt(-\log t+\log 2\pi+1)+t\log nx.$$

Since

$$F''(t)=-\frac{k}{t}\leqslant-\frac{k}{T},$$

the integral is

$$O(T^{(a+\frac{1}{2})k-\frac{1}{2}}),$$

uniformly with respect to n and x. A similar result holds for the integral over $(-T,-1)$, while the integral over $(-1,1)$ is bounded. Hence

$$\Delta_k(x) = O\left(\frac{x^c}{T(c-1)^k}\right)+O\left(\frac{x^{1+\epsilon}}{T}\right)+O\left(\frac{T^{(a+\frac{1}{2})k-1}}{x^a}\right)+$$

$$+x^{-a}\sum_{n=1}^{\infty}\frac{d_k(n)}{n^{1+a}}O(T^{(a+\frac{1}{2})k-\frac{1}{2}})$$

$$= O\left(\frac{x^c}{T(c-1)^k}\right)+O\left(\frac{x^{1+\epsilon}}{T}\right)+O\left(\frac{T^{(a+\frac{1}{2})k-\frac{1}{2}}}{x^a}\right).$$

Taking $c = 1+\epsilon$, $a = \epsilon$, the terms are of the same order, apart from ϵ's, if

$$T = x^{2/(k+1)}.$$

Hence $$\Delta_k(x) = O(x^{(k-1)/(k+1)+\epsilon}).$$

The restriction that x should be half an odd integer is clearly unnecessary to the result.

12.3. By using some of the deeper results on $\zeta(s)$ we can obtain a still better result for $k \geqslant 4$.

THEOREM 12.3.† $$\alpha_k \leqslant \frac{k-1}{k+2} \quad (k = 4, 5,\ldots).$$

We start as in the previous theorem, but now take the rectangle as far as $\sigma = \frac{1}{2}$ only. Let us suppose that

$$\zeta(\tfrac{1}{2}+it) = O(t^{\lambda}).$$

Then $$\zeta(s) = O(t^{\lambda(c-\sigma)/(c-\frac{1}{2})})$$

uniformly in the rectangle. The horizontal sides therefore give

$$O\left(\int_{\frac{1}{2}}^{c} T^{k\lambda(c-\sigma)/(c-\frac{1}{2})-1}x^{\sigma}\,d\sigma\right) = O(T^{k\lambda-1}x^{\frac{1}{2}})+O(T^{-1}x^c).$$

Also $$\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\zeta^k(s)\frac{x^s}{s}\,ds = O(x^{\frac{1}{2}})+O\left(x^{\frac{1}{2}}\int_1^T|\zeta(\tfrac{1}{2}+it)|^k\frac{dt}{t}\right).$$

Now $$\int_1^T|\zeta(\tfrac{1}{2}+it)|^k\frac{dt}{t} \leqslant \max_{1\leqslant t\leqslant T}|\zeta(\tfrac{1}{2}+it)|^{k-4}\int_1^T|\zeta(\tfrac{1}{2}+it)|^4\frac{dt}{t}$$

$$= O\left\{T^{(k-4)\lambda}\int_1^T|\zeta(\tfrac{1}{2}+it)|^4\frac{dt}{t}\right\}.$$

† Hardy and Littlewood (4).

Also $$\phi(T) = \int_1^T|\zeta(\tfrac{1}{2}+it)|^4\,dt = O(T^{1+\epsilon}),$$

by (7.6.1), so that

$$\int_1^T|\zeta(\tfrac{1}{2}+it)|^4\frac{dt}{t} = \int_1^T\phi'(t)\frac{dt}{t} = \left[\frac{\phi(t)}{t}\right]_1^T+\int_1^T\frac{\phi(t)}{t^2}\,dt$$

$$= O(T^{\epsilon})+O\left(\int_1^T\frac{1}{t^{1-\epsilon}}\,dt\right) = O(T^{\epsilon}).$$

Hence $$\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\zeta^k(s)\frac{x^s}{s}\,ds = O(x^{\frac{1}{2}})+O(x^{\frac{1}{2}}T^{(k-4)\lambda+\epsilon}).$$

Altogether we obtain

$$\Delta_k(x) = O(T^{-1}x^c)+O(x^{\frac{1}{2}}T^{k\lambda-1})+O(x^{\frac{1}{2}}T^{(k-4)\lambda+\epsilon}).$$

The middle term is of smaller order than the last if $\lambda \leqslant \frac{1}{4}$. Taking $c = 1+\epsilon$, the other two terms are of the same order, apart from ϵ's, if

$$T = x^{1/(2(k-4)\lambda+2)}.$$

This gives $$\Delta_k(x) = O(x^{\{(2(k-4)\lambda+1)/(2(k-4)\lambda+2)\}+\epsilon}).$$

Taking $\lambda = \frac{1}{6}+\epsilon$ (Theorems 5.5, 5.12) the result follows. Further slight improvements for $k \geqslant 5$ are obtained by using the results stated in § 5.18.

12.4. The above method does not give any new result for $k = 2$ or $k = 3$. For these values slight improvements on Theorem 12.2 have been made by special methods.

THEOREM 12.4.† $$\alpha_2 \leqslant \frac{27}{82}.$$

The argument of § 12.2 shows that

$$\Delta(x) = \frac{1}{2\pi i}\sum_{n=1}^{\infty}d(n)\int_{-a-iT}^{-a+iT}\frac{\chi^2(s)}{n^{1-s}}\frac{x^s}{s}\,ds+O\left(\frac{T^{2a}}{x^a}\right)+O\left(\frac{x^c}{T}\right) \qquad (12.4.1)$$

where $a > 0$, $c > 1$. Let $T^2/(4\pi^2x) = N+\frac{1}{2}$, where N is an integer, and consider the terms with $n > N$. As before, the integral over $1 \leqslant t \leqslant T$ is of the form

$$\frac{1}{x^a n^{1+a}}\int_1^T e^{iF(t)}\{t^{2a}+O(t^{2a-1})\}\,dt, \qquad (12.4.2)$$

† van der Corput (4).

where
$$F(t) = 2t(-\log t+\log 2\pi+1)+t\log nx,$$
$$F'(t) = \log\frac{4\pi^2 nx}{t^2}.$$

Hence $F'(t) \geqslant \log\dfrac{n}{N+\frac{1}{2}}$, and (12.4.2) is
$$\frac{1}{x^a n^{1+a}}\left\{O\left(\frac{T^{2a}}{\log\{n/(N+\frac{1}{2})\}}\right)+O(T^{2a})\right\}.$$
For $n \geqslant 2N$ this contributes to (12.4.1)
$$O\left\{\frac{T^{2a}}{x^a}\sum_{n=2N}^{\infty}\frac{d(n)}{n^{1+a}}\right\} = O(N^{\epsilon}),$$
and for $N < n < 2N$ it contributes
$$O\left\{\frac{T^{2a}}{x^a}\sum_{n=N+1}^{2N}\frac{d(n)}{n^{1+a}\log\{n/(N+\frac{1}{2})\}}\right\} = O\left(N^{\epsilon}\sum_{m=1}^{N}\frac{1}{m}\right) = O(N^{\epsilon}).$$

Similarly for the integral over $-T \leqslant t \leqslant -1$; and the integral over $-1 < t < 1$ is clearly $O(x^{-a})$.

If $n \leqslant N$, we write
$$\int_{-a-iT}^{-a+iT} = \int_{-i\infty}^{i\infty} - \left(\int_{iT}^{i\infty} + \int_{-i\infty}^{-iT} + \int_{-iT}^{-a-iT} + \int_{-a+iT}^{iT}\right).$$
The first term is
$$\frac{1}{n}\int_{-i\infty}^{i\infty} 2^{2s}\pi^{2s-2}\sin^2\tfrac{1}{2}s\pi\,\Gamma^2(1-s)\frac{(nx)^s}{s}\,ds$$
$$= -\frac{1}{n\pi^2}\int_{1-i\infty}^{1+i\infty}\cos^2\tfrac{1}{2}w\pi\,\Gamma(w)\Gamma(w-1)\{2\pi\sqrt{(nx)}\}^{2-2w}\,dw$$
$$= -4i\sqrt{\left(\frac{x}{n}\right)}[K_1\{4\pi\sqrt{(nx)}\}+\tfrac{1}{2}\pi Y_1\{4\pi\sqrt{(nx)}\}]$$
in the usual notation of Bessel functions.†

The first integral in the bracket is
$$\int_{T}^{\infty} e^{iF(t)}\left(A+\frac{A'}{t}+O(t^{-2})\right)dt = O\left\{\frac{1}{\log\{(N+\frac{1}{2})/n\}}\right\},$$
which gives
$$\sum_{n=1}^{N}\frac{d(n)}{n\log\{(N+\frac{1}{2})/n\}} = O(N^{\epsilon})$$

† See, e.g., Titchmarsh, *Fourier Integrals*, (7.9.8), (7.9.11).

as before; and similarly for the second integral. The last two give
$$O\left\{\sum_{n=1}^{N}\frac{d(n)}{n}\int_{-a}^{0}\left(\frac{nx}{T^2}\right)^{\sigma}d\sigma\right\} = O\left\{\sum_{n=1}^{N}\frac{d(n)}{n}\left(\frac{T^2}{nx}\right)^{a}\right\} = O\left\{\left(\frac{T^2}{x}\right)^{a}\right\}.$$
Altogether we have now proved that
$$\Delta(x) = -\frac{2\sqrt{x}}{\pi}\sum_{n=1}^{N}\frac{d(n)}{\sqrt{n}}[K_1\{4\pi\sqrt{(nx)}\}+\tfrac{1}{2}\pi Y_1\{4\pi\sqrt{(nx)}\}]+O\left(\frac{T^{2a}}{x^a}\right)+O\left(\frac{x^c}{T}\right). \tag{12.4.3}$$
By the usual asymptotic formulae† for Bessel functions, this may be replaced by
$$\Delta(x) = \frac{x^{\frac{1}{4}}}{\pi\sqrt{2}}\sum_{n=1}^{N}\frac{d(n)}{n^{\frac{3}{4}}}\cos\{4\pi\sqrt{(nx)}-\tfrac{1}{4}\pi\}+O(x^{-\frac{1}{4}})+O\left(\frac{T^{2a}}{x^a}\right)+O\left(\frac{x^c}{T}\right). \tag{12.4.4}$$
Now
$$\sum_{n=1}^{N} d(n)e^{4\pi i\sqrt{(nx)}} = 2\sum_{m\leqslant\sqrt{N}}\ \sum_{n\leqslant N/m} e^{4\pi i\sqrt{(mnx)}} - \sum_{m\leqslant\sqrt{N}}\ \sum_{n\leqslant\sqrt{N}} e^{4\pi i\sqrt{(mnx)}}. \tag{12.4.5}$$
Consider the sum
$$\sum_{\frac{1}{2}N/m<n\leqslant N/m} e^{4\pi i\sqrt{(mnx)}}.$$
We apply Theorem 5.13, with $k = 5$, and
$$f(n) = 2\sqrt{(mnx)},\qquad f^{(5)}(n) = A(mx)^{\frac{1}{2}}n^{-\frac{9}{2}}.$$
Hence the sum is
$$O\left\{\frac{N}{m}\left(\frac{(mx)^{\frac{1}{2}}}{(N/m)^{\frac{9}{2}}}\right)^{\frac{1}{30}}\right\}+O\left\{\left(\frac{N}{m}\right)^{\frac{7}{8}}\left(\frac{(N/m)^{\frac{9}{2}}}{(mx)^{\frac{1}{2}}}\right)^{\frac{1}{30}}\right\}$$
$$= O\{(N/m)^{\frac{17}{20}}(mx)^{\frac{1}{60}}\}+O\{(N/m)^{\frac{41}{40}}(mx)^{-\frac{1}{60}}\}.$$
Replacing N by $\frac{1}{2}N$, $\frac{1}{4}N$,..., and adding, the same result holds for the sum over $1 \leqslant n \leqslant N/m$. Hence the first term on the right of (12.4.5) is
$$O\left(N^{\frac{17}{20}}x^{\frac{1}{60}}\sum_{m\leqslant\sqrt{N}}m^{-\frac{5}{6}}\right)+O\left(N^{\frac{41}{40}}x^{-\frac{1}{60}}\sum_{m\leqslant\sqrt{N}}m^{-\frac{25}{24}}\right) = O(N^{\frac{14}{15}}x^{\frac{1}{60}})+O(N^{\frac{41}{40}}x^{-\frac{1}{60}}).$$
Similarly the second inner sum is
$$O\{(\sqrt{N})^{\frac{17}{20}}(mx)^{\frac{1}{60}}\}+O\{(\sqrt{N})^{\frac{41}{40}}(mx)^{-\frac{1}{60}}\},$$
and the whole sum is
$$O\left(N^{\frac{17}{40}}x^{\frac{1}{60}}\sum_{m\leqslant\sqrt{N}}m^{\frac{1}{60}}\right)+O\left(N^{\frac{41}{80}}x^{-\frac{1}{60}}\sum_{m\leqslant\sqrt{N}}m^{-\frac{1}{60}}\right)$$
$$=O(N^{\frac{14}{15}}x^{\frac{1}{60}})+O(N^{\frac{241}{240}}x^{-\frac{1}{60}}).$$

† Watson, *Theory of Bessel Functions*, §§ 7.21, 7.23.

Hence, multiplying by $e^{-\frac{1}{4}i\pi}$ and taking the real part,

$$\sum_{n=1}^{N} d(n)\cos\{4\pi\sqrt{(nx)}-\tfrac{1}{4}\pi\} = O(N^{\frac{14}{15}}x^{\frac{1}{60}})+O(N^{\frac{41}{40}}x^{-\frac{1}{60}}).$$

Using this and partial summation, (12.4.4) gives

$$\begin{aligned}\Delta(x) &= O(N^{\frac{14}{15}-\frac{3}{4}}x^{\frac{1}{4}+\frac{1}{60}})+O(N^{\frac{41}{40}-\frac{3}{4}}x^{\frac{1}{4}-\frac{1}{60}})+O(N^{a})+O(N^{-\frac{1}{2}}x^{c-\frac{1}{2}})\\ &= O(N^{\frac{11}{60}}x^{\frac{4}{15}})+O(N^{\frac{11}{40}}x^{\frac{7}{30}})+O(N^{a})+O(N^{-\frac{1}{2}}x^{c-\frac{1}{2}}).\end{aligned}$$

Taking $a = \epsilon$, $c = 1+\epsilon$, the first and last terms are of the same order, apart from ϵ's, if

$$N = [x^{\frac{14}{41}}].$$

Hence

$$\Delta(x) = O(x^{\frac{27}{82}+\epsilon}),$$

the result stated.

A similar argument may be applied to $\Delta_3(x)$. We obtain

$$\Delta_3(x) = \frac{x^{\frac{1}{3}}}{\pi\sqrt{3}} \sum_{n<T^3/(8\pi^3 x)} \frac{d_3(n)}{n^{\frac{2}{3}}}\cos\{6\pi(nx)^{\frac{1}{3}}\}+O\left(\frac{x^{1+\epsilon}}{T}\right), \tag{12.4.6}$$

and deduce that

$$\alpha_3 \leqslant \tfrac{37}{75}.$$

The detailed argument is given by Atkinson (3).

If the series in (12.4.4) were absolutely convergent, or if the terms more or less cancelled each other, we should deduce that $\alpha_2 \leqslant \frac{1}{4}$; and it may reasonably be conjectured that this is the real truth. We shall see later that $\alpha_2 \geqslant \frac{1}{4}$, so that it would follow that $\alpha_2 = \frac{1}{4}$. Similarly from (12.4.6) we should obtain $\alpha_3 = \frac{1}{3}$; and so generally it may be conjectured that

$$\alpha_k = \frac{k-1}{2k}.$$

12.5. *The average order of* $\Delta_k(x)$. We may define β_k, the average order of $\Delta_k(x)$, to be the least number such that

$$\frac{1}{x}\int_0^x \Delta_k^2(y)\,dy = O(x^{2\beta_k+\epsilon})$$

for every positive ϵ. Since

$$\frac{1}{x}\int_0^x \Delta_k^2(y)\,dy = \frac{1}{x}\int_0^x O(y^{2\alpha_k+\epsilon})\,dy = O(x^{2\alpha_k+\epsilon}),$$

we have $\beta_k \leqslant \alpha_k$ for each k. In particular we obtain a set of upper bounds for the β_k from the above theorems.

As usual, the problem of average order is easier than that of order, and we can prove more about the β_k than about the α_k. We shall first prove the following theorem.†

† Titchmarsh (22).

THEOREM 12.5. *Let γ_k be the lower bound of positive numbers σ for which*

$$\int_{-\infty}^{\infty} \frac{|\zeta(\sigma+it)|^{2k}}{|\sigma+it|^2}\,dt < \infty. \tag{12.5.1}$$

Then $\beta_k = \gamma_k$; and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|\zeta(\sigma+it)|^{2k}}{|\sigma+it|^2}\,dt = \int_0^{\infty} \Delta_k^2(x)x^{-2\sigma-1}\,dx \tag{12.5.2}$$

provided that $\sigma > \beta_k$.

We have

$$D_k(x) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT} \frac{\zeta^k(s)}{s}x^s\,ds \quad (c > 1).$$

Applying Cauchy's theorem to the rectangle $\gamma-iT$, $c-iT$, $c+iT$, $\gamma+iT$, where γ is less than, but sufficiently near to, 1, and allowing for the residue at $s = 1$, we obtain

$$\Delta_k(x) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} \frac{\zeta^k(s)}{s}x^s\,ds. \tag{12.5.3}$$

Actually (12.5.3) holds for $\gamma_k < \gamma < 1$. For† $\zeta^k(s)/s \to 0$ uniformly as $t \to \pm\infty$ in the strip. Hence if we integrate the integrand of (12.5.3) round the rectangle $\gamma'-iT$, $\gamma-iT$, $\gamma+iT$, $\gamma'+iT$, where

$$\gamma_k < \gamma' < \gamma < 1,$$

and make $T \to \infty$, we obtain the same result with γ' instead of γ.

If we replace x by $1/x$, (12.5.3) expresses the relation between the Mellin transforms

$$f(x) = \Delta_k(1/x), \qquad \mathfrak{F}(s) = \zeta^k(s)/s,$$

the relevant integrals holding also in the mean-square sense. Hence Parseval's formula for Mellin transforms‡ gives

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|\zeta(\gamma+it)|^{2k}}{|\gamma+it|^2}\,dt = \int_0^{\infty} \Delta_k^2\left(\frac{1}{x}\right)x^{2\gamma-1}\,dx = \int_0^{\infty} \Delta_k^2(x)x^{-2\gamma-1}\,dx \tag{12.5.4}$$

provided that $\gamma_k < \gamma < 1$.

It follows that, if $\gamma_k < \gamma < 1$,

$$\int_{\frac{1}{2}X}^{X} \Delta_k^2(x)x^{-2\gamma-1}\,dx < K = K(k,\gamma),$$

$$\int_{\frac{1}{2}X}^{X} \Delta_k^2(x)\,dx < KX^{2\gamma+1},$$

† By an application of the lemma of § 11.9.

‡ See Titchmarsh, *Theory of Fourier Integrals*, Theorem 71.

and, replacing X by $\frac{1}{2}X, \frac{1}{4}X, \dots$, and adding,

$$\int_1^X \Delta_k^2(x)\,dx < KX^{2\gamma+1}.$$

Hence $\beta_k \leqslant \gamma$, and so $\beta_k \leqslant \gamma_k$.

The inverse Mellin formula is

$$\frac{\zeta^k(s)}{s} = \int_0^\infty \Delta_k\left(\frac{1}{x}\right)x^{s-1}\,dx = \int_0^\infty \Delta_k(x)x^{-s-1}\,dx. \tag{12.5.5}$$

The right-hand side exists primarily in the mean-square sense, for $\gamma_k < \sigma < 1$. But actually the right-hand side is uniformly convergent in any region interior to the strip $\beta_k < \sigma < 1$; for

$$\int_{\frac{1}{2}X}^X |\Delta_k(x)|x^{-\sigma-1}\,dx \leqslant \left\{\int_{\frac{1}{2}X}^X \Delta_k^2(x)\,dx \int_{\frac{1}{2}X}^X x^{-2\sigma-2}\,dx\right\}^{\frac{1}{2}}$$
$$= \{O(X^{2\beta_k+1+\epsilon})O(X^{-2\sigma-1})\}^{\frac{1}{2}} = O(X^{\beta_k-\sigma+\epsilon}),$$

and on putting $X = 2, 4, 8, \dots$, and adding we obtain

$$\int_1^\infty |\Delta_k(x)|x^{-\sigma-1}\,dx < K.$$

It follows that the right-hand side of (12.5.5) represents an analytic function, regular for $\beta_k < \sigma < 1$. The formula therefore holds by analytic continuation throughout this strip. Also (by the argument just given) the right-hand side of (12.5.4) is finite for $\beta_k < \gamma < 1$. Hence so is the left-hand side, and the formula holds. Hence $\gamma_k \leqslant \beta_k$, and so, in fact, $\gamma_k = \beta_k$. This proves the theorem.

12.6. THEOREM 12.6 (A).†

$$\beta_k \geqslant \frac{k-1}{2k} \quad (k = 2, 3, \dots).$$

If $\frac{1}{2} < \sigma < 1$, by Theorem 7.2

$$C_\sigma T < \int_{\frac{1}{2}T}^T |\zeta(\sigma+it)|^2\,dt \leqslant \left\{\int_{\frac{1}{2}T}^T |\zeta(\sigma+it)|^{2k}\,dt\right\}^{1/k}\left(\int_{\frac{1}{2}T}^T dt\right)^{1-1/k}.$$

Hence
$$\int_{\frac{1}{2}T}^T |\zeta(\sigma+it)|^{2k}\,dt \geqslant 2^{k-1}C_\sigma^k T.$$

† Titchmarsh (22).

Hence, if $0 < \sigma < \frac{1}{2}$, $T > 1$,

$$\int_{-\infty}^\infty \frac{|\zeta(\sigma+it)|^{2k}}{|\sigma+it|^2}\,dt > \int_{\frac{1}{2}T}^T \frac{|\zeta(\sigma+it)|^{2k}}{|\sigma+it|^2}\,dt > \frac{C'}{T^2}\int_{\frac{1}{2}T}^T |\zeta(\sigma+it)|^{2k}\,dt$$
$$> C''T^{k(1-2\sigma)-2}\int_{\frac{1}{2}T}^T |\zeta(1-\sigma-it)|^{2k}\,dt \quad \text{(by the functional equation)}$$
$$\geqslant C''2^{k-1}C_{1-\sigma}^k T^{k(1-2\sigma)-1}.$$

This can be made as large as we please by choice of T if $\sigma < \frac{1}{2}(k-1)/k$.

Hence
$$\gamma_k \geqslant \frac{k-1}{2k}$$

and the theorem follows.

THEOREM 12.6 (B).†

$$\alpha_k \geqslant \frac{k-1}{2k} \quad (k = 2, 3, \dots).$$

For $\alpha_k \geqslant \beta_k$.

Much more precise theorems of the same type are known. Hardy proved first that both

$$\Delta(x) > Kx^{\frac{1}{4}}, \qquad \Delta(x) < -Kx^{\frac{1}{4}}$$

hold for some arbitrarily large values of x, and then that $x^{\frac{1}{4}}$ may in each case be replaced by
$$(x\log x)^{\frac{1}{4}}\log\log x.$$

12.7. We recall that (§ 7.9) the numbers σ_k are defined as the lower bounds of σ such that

$$\frac{1}{T}\int_1^T |\zeta(\sigma+it)|^{2k}\,dt = O(1).$$

We shall next prove

THEOREM 12.7. *For each integer $k \geqslant 2$, a necessary and sufficient condition that*

$$\beta_k = \frac{k-1}{2k} \tag{12.7.1}$$

is that

$$\sigma_k \leqslant \frac{k+1}{2k}. \tag{12.7.2}$$

Suppose first that (12.7.2) holds. Then by the functional equation

$$\int_1^T |\zeta(\sigma+it)|^{2k}\,dt = O\left\{T^{k(1-2\sigma)}\int_1^T |\zeta(1-\sigma-it)|^{2k}\,dt\right\} = O(T^{k(1-2\sigma)+1})$$

† Hardy (2).

for $\sigma < \frac{1}{2}(k-1)/k$. It follows from the convexity of mean values that

$$\int_1^T |\zeta(\sigma+it)|^{2k}\,dt = O(T^{1+(\frac{1}{2}+1/2k+\epsilon/2k-\sigma)k})$$

for
$$\frac{k-1-\epsilon}{2k} < \sigma < \frac{k+1+\epsilon}{2k}.$$

The index of T is less than 2 if

$$\sigma > \frac{k-1+\epsilon}{2k}.$$

Then
$$\int_{\frac{1}{2}T}^{T} \frac{|\zeta(\sigma+it)|^{2k}}{|\sigma+it|^2}\,dt = O(T^{-\delta}) \quad (\delta > 0).$$

Hence (12.5.1) holds. Hence $\gamma_k \leqslant \frac{1}{2}(k-1)/k$. Hence $\beta_k \leqslant \frac{1}{2}(k-1)/k$, and so, by Theorem 12.6 (A), (12.7.1) holds.

On the other hand, if (12.7.1) holds, it follows from (12.5.2) that

$$\int_1^T |\zeta(\sigma+it)|^{2k}\,dt = O(T^2)$$

for $\sigma > \frac{1}{2}(k-1)/k$. Hence by the functional equation

$$\int_1^T |\zeta(\sigma+it)|^{2k}\,dt = O(T^{k(1-2\sigma)+2})$$

for $\sigma < \frac{1}{2}(k+1)/k$. Hence, by the convexity theorem, the left-hand side is $O(T^{1+\epsilon})$ for $\sigma = \frac{1}{2}(k+1)/k$; hence, in the notation of § 7.9, $\sigma_k' \leqslant \frac{1}{2}(k+1)/k$, and so (12.7.2) holds.

12.8. Theorem 12.8.†

$$\beta_2 = \tfrac{1}{4}, \qquad \beta_3 = \tfrac{1}{3}, \qquad \beta_4 \leqslant \tfrac{3}{7}.$$

By Theorem 7.7, $\sigma_k \leqslant 1-1/k$. Since

$$1-\frac{1}{k} \leqslant \frac{k+1}{2k} \quad (k \leqslant 3)$$

it follows that $\beta_2 = \frac{1}{4}$, $\beta_3 = \frac{1}{3}$.

The available material is not quite sufficient to determine β_4. Theorem 12.6 (A) gives $\beta_4 \geqslant \frac{3}{8}$. To obtain an upper bound for it, we observe that, by Theorem 5.5. and (7.6.1),

$$\int_1^T |\zeta(\tfrac{1}{2}+it)|^8\,dt = O\left(T^{\frac{2}{3}+\epsilon}\int_1^T |\zeta(\tfrac{1}{2}+it)|^4\,dt\right) = O(T^{\frac{5}{3}+\epsilon}),$$

† The value of β_2 is due to Hardy (3), and that of β_3 to Cramér (4); for β_4 see Titchmarsh (22).

and, since $\sigma_4 \leqslant \frac{7}{10}$ by Theorem 7.10,

$$\int_1^T |\zeta(\tfrac{3}{10}+it)|^8\,dt = O\left(T^{\frac{8}{5}}\int_1^T |\zeta(\tfrac{7}{10}-it)|^8\,dt\right) = O(T^{\frac{13}{5}+\epsilon}).$$

Hence by the convexity theorem

$$\int_1^T |\zeta(\sigma+it)|^8\,dt = O(T^{4-\frac{14}{3}\sigma+\epsilon})$$

for $\frac{3}{10} < \sigma < \frac{1}{2}$. It easily follows that $\gamma_4 \leqslant \frac{3}{7}$, i.e. $\beta_4 \leqslant \frac{3}{7}$.

NOTES FOR CHAPTER 12

12.9. For large k the best available estimates for α_k are of the shape $\alpha_k \leqslant 1 - Ck^{-\frac{2}{3}}$, where C is a positive constant. The first such result is due to Richert [**2**]. (See also Karatsuba [**1**], Ivic [**3**; Theorem 13.3] and Fujii [**3**].) These results depend on bounds of the form (6.19.2).

For the range $4 \leqslant k \leqslant 8$ one has $\alpha_k \leqslant \frac{3}{4} - 1/k$ (Heath-Brown [**8**]) while for intermediate values of k a number of estimates are possible (see Ivic [**3**; Theorem 13.2]). In particular one has $\alpha_9 \leqslant \frac{35}{54}$, $\alpha_{10} \leqslant \frac{41}{60}$, $\alpha_{11} \leqslant \frac{7}{10}$, and $\alpha_{12} \leqslant \frac{5}{7}$.

12.10. The following bounds for α_2 have been obtained.

$\frac{33}{100} = 0{\cdot}330000\ldots$	van der Corput (2),
$\frac{27}{82} = 0{\cdot}329268\ldots$	van der Corput (4),
$\frac{15}{46} = 0{\cdot}326086\ldots$	Chih [**1**], Richert [**1**],
$\frac{12}{37} = 0{\cdot}324324\ldots$	Kolesnik [**1**],
$\frac{346}{1067} = 0{\cdot}324273\ldots$	Kolesnik [**2**],
$\frac{35}{108} = 0{\cdot}324074\ldots$	Kolesnik [**4**],
$\frac{139}{429} = 0{\cdot}324009\ldots$	Kolesnik [**5**].

In general the methods used to estimate α_2 and $\mu(\frac{1}{2})$ are very closely related. Suppose one has a bound

$$\sum_{M<m\leqslant M_1}\sum_{N<n\leqslant N_1} \exp\left[2\pi i\{x(mn)^{\frac{1}{2}} + cx^{-1}(mn)^{\frac{3}{2}}\}\right] \ll (MN)^{\frac{3}{4}}x^{2\vartheta-\frac{1}{2}}, \tag{12.10.1}$$

for any constant c, uniformly for $M < M_1 \leqslant 2M$, $N < N_1 \leqslant 2N$, and $MN \leqslant x^{2-4\vartheta}$. It then follows that $\mu(\frac{1}{2}) \leqslant \frac{1}{2}\vartheta$, $\alpha_2 \leqslant \vartheta$, and $E(T) \ll T^{\vartheta+\varepsilon}$ (for $E(T)$ as in §7.20). In practice those versions of the van der Corput method used to tackle $\mu(\frac{1}{2})$ and α_2 also apply to (12.10.1), which explains the similarity between the table of estimates given above and that presented in §5.21 for $\mu(\frac{1}{2})$. This is just one manifestation of the close similarity exhibited by the functions $E(T)$ and $\Delta(x)$, which has its origin in the formulae (7.20.6) and (12.4.4). The classical lattice-point problem for the circle falls within the same area of ideas. Thus, if the bound (12.10.1) holds, along with its analogue in which the summation condition $m \equiv 1 \pmod 4$ is imposed, then one has

$$\#\{(m, n) \in \mathbb{Z}^2 \colon m^2 + n^2 \leqslant x\} = \pi x + O(x^{\theta+\varepsilon}).$$

Jutila [3] has taken these ideas further by demonstrating a direct connection between the size of $\Delta(x)$ and that of $\zeta(\frac{1}{2}+it)$ and $E(T)$. In particular he has shown that if $\alpha_2 = \frac{1}{4}$ then $\mu(\frac{1}{2}) \leqslant \frac{3}{20}$ and $E(T) \ll T^{\frac{5}{16}+\varepsilon}$.

Further work has also been done on the problem of estimating α_3. The best result at present is $\alpha_3 \leqslant \frac{43}{96}$, due to Kolesnik [3]. For α_4, however, no sharpening of the bound $\alpha_4 \leqslant \frac{1}{2}$ given by Theorem 12.3 has yet been found. This result, dating from 1922, seems very resistant to any attempt at improvement.

12.11. The Ω-results attributed to Hardy in §12.6 may be found in Hardy [1]. However Hardy's argument appears to yield only

$$\Delta(x) = \Omega_+((x \log x)^{\frac{1}{4}} \log\log x), \tag{12.11.1}$$

and not the corresponding Ω_- result. The reason for this is that Dirichlet's Theorem is applicable for Ω_+, while Kronecker's Theorem is needed for the Ω_- result. By using a quantitative form of Kronecker's Theorem, Corrádi and Kátai [1] showed that

$$\Delta(x) = \Omega_-\left\{x^{\frac{1}{4}} \exp\left(c \frac{(\log\log x)^{\frac{1}{4}}}{(\log\log\log x)^{\frac{3}{4}}}\right)\right\},$$

for a certain positive constant c. This improved earlier work of Ingham [1] and Gangadharan [1]. Hardy's result (12.11.1) has also been sharpened by Hafner [1] who obtained

$$\Delta(x) = \Omega_+[(x \log x)^{\frac{1}{4}} (\log\log x)^{\frac{1}{4}(3+2\log 2)} \exp\{-c(\log\log\log x)^{\frac{1}{2}}\}]$$

for a certain positive constant c. For $k \geqslant 3$ he also showed [2] that, for a suitable positive constant c, one has

$$\Delta_k(x) = \Omega_*[(x \log x)^{(k-1)/2k} (\log\log x)^a \exp\{-c(\log\log\log x)^{\frac{1}{2}}\}],$$

where

$$a = \frac{k-1}{2k}(k \log k + k + 1)$$

and Ω_* is Ω_+ for $k = 3$ and $\Omega_\pm$ for $k \geqslant 4$.

12.12. As mentioned in §7.2 we now have $\sigma_4 \leqslant \frac{5}{8}$, whence $\beta_4 = \frac{3}{8}$, (Heath-Brown [8]). For $k = 2$ and 3 one can give asymptotic formulae for

$$\int_0^x \Delta_k(y)^2 \, dy.$$

Thus Tong [1] showed that

$$\int_0^x \Delta_k(y)^2 \, dy = \frac{x^{(2k-1)/k}}{(4k-2)\pi^2} \sum_{n=1}^{\infty} d_k(n)^2 n^{-(k+1)/k} + R_k(x)$$

with $R_2(x) \ll x(\log x)^5$ and

$$R_k(x) \ll x^{c_k+\varepsilon}, \qquad c_k = 2 - \frac{3-4\sigma_k}{2k(1-\sigma_k)-1}, \qquad (k \geqslant 3).$$

Taking $\sigma_3 \leqslant \frac{7}{12}$ (see §7.22) yields $c_3 \leqslant \frac{14}{9}$. However the available information concerning σ_k is as yet insufficient to give $c_k < (2k-1)/k$ for any $k \geqslant 4$. It is perhaps of interest to note that Hardy's result (12.11.1) implies $R_2(x) = \Omega\{x^{\frac{3}{4}}(\log x)^{-\frac{1}{4}}\}$, since any estimate $R_2(x) \ll F(x)$ easily leads to a bound $\Delta_2(x) \ll \{F(x) \log x\}^{\frac{1}{4}}$, by an argument analogous to that given for the proof of Lemma α in §14.13.

Ivic [3; Theorems 13.9 and 13.10] has estimated the higher moments of $\Delta_2(x)$ and $\Delta_3(x)$. In particular his results imply that

$$\int_0^x \Delta_2(y)^8 \, dy \ll x^{3+\varepsilon}.$$

For $\Delta_3(x)$ his argument may be modified slightly to yield

$$\int_0^x |\Delta_3(y)|^3 \, dy \ll x^{2+\varepsilon}.$$

These results are readily seen to contain the estimates $\alpha_2 \leqslant \frac{1}{3}$, $\beta_2 \leqslant \frac{1}{4}$ and $\alpha_3 \leqslant \frac{1}{2}$, $\beta_3 \leqslant \frac{1}{3}$ respectively.

XIII

THE LINDELÖF HYPOTHESIS

13.1. THE Lindelöf hypothesis is that

$$\zeta(\tfrac{1}{2}+it) = O(t^\epsilon)$$

for every positive ϵ; or, what comes to the same thing, that

$$\zeta(\sigma+it) = O(t^\epsilon)$$

for every positive ϵ and every $\sigma \geqslant \frac{1}{2}$; for either statement is, by the theory of the function $\mu(\sigma)$, equivalent to the statement that $\mu(\sigma) = 0$ for $\sigma \geqslant \frac{1}{2}$. The hypothesis is suggested by various theorems in Chapters V and VII. It is also the simplest possible hypothesis on $\mu(\sigma)$, for on it the graph of $y = \mu(\sigma)$ consists simply of the two straight lines

$$y = \tfrac{1}{2}-\sigma \quad (\sigma \leqslant \tfrac{1}{2}), \qquad y = 0 \quad (\sigma \geqslant \tfrac{1}{2}).$$

We shall see later that the Lindelöf hypothesis is true if the Riemann hypothesis is true. The converse deduction, however, cannot be made —in fact (Theorem 13.5) the Lindelöf hypothesis is equivalent to a much less drastic, but still unproved, hypothesis about the distribution of the zeros.

In this chapter we investigate the consequences of the Lindelöf hypothesis. Most of our arguments are reversible, so that we obtain necessary and sufficient conditions for the truth of the hypothesis.

13.2. THEOREM 13.2.† *Alternative necessary and sufficient conditions for the truth of the Lindelöf hypothesis are*

$$\frac{1}{T}\int_1^T |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt = O(T^\epsilon) \quad (k = 1, 2,...); \tag{13.2.1}$$

$$\frac{1}{T}\int_1^T |\zeta(\sigma+it)|^{2k}\,dt = O(T^\epsilon) \quad (\sigma > \tfrac{1}{2},\ k = 1, 2,...); \tag{13.2.2}$$

$$\frac{1}{T}\int_1^T |\zeta(\sigma+it)|^{2k}\,dt \sim \sum_{n=1}^{\infty}\frac{d_k^2(n)}{n^{2\sigma}} \quad (\sigma > \tfrac{1}{2},\ k = 1, 2,...). \tag{13.2.3}$$

The equivalence of the first two conditions follows from the convexity theorem (§ 7.8), while that of the last two follows from the analysis of § 7.9. It is therefore sufficient to consider (13.2.1).

† Hardy and Littlewood (5).

The necessity of the condition is obvious. To prove that it is sufficient, suppose that $\zeta(\frac{1}{2}+it)$ is not $O(t^\epsilon)$. Then there is a positive number λ, and a sequence of numbers $\frac{1}{2}+it_\nu$, such that $t_\nu \to \infty$ with ν, and

$$|\zeta(\tfrac{1}{2}+it_\nu)| > Ct_\nu^\lambda \quad (C > 0).$$

On the other hand, on differentiating (2.1.4) we obtain, for $t \geqslant 1$,

$$|\zeta'(\tfrac{1}{2}+it)| < Et,$$

E being a positive absolute constant. Hence

$$|\zeta(\tfrac{1}{2}+it)-\zeta(\tfrac{1}{2}+it_\nu)| = \left|\int_{t_\nu}^{t}\zeta'(\tfrac{1}{2}+iu)\,du\right| < 2E|t-t_\nu|t_\nu < \tfrac{1}{2}Ct_\nu^\lambda$$

if $|t-t_\nu| \leqslant t_\nu^{-1}$ and ν is sufficiently large. Hence

$$|\zeta(\tfrac{1}{2}+it)| > \tfrac{1}{2}Ct_\nu^\lambda \quad (|t-t_\nu| \leqslant t_\nu^{-1}).$$

Take $T = \frac{2}{3}t_\nu$, so that the interval $(t_\nu-t_\nu^{-1}, t_\nu+t_\nu^{-1})$ is included in $(T, 2T)$ if ν is sufficiently large. Then

$$\int_T^{2T}|\zeta(\tfrac{1}{2}+it)|^{2k}\,dt > \int_{t_\nu-t_\nu^{-1}}^{t_\nu+t_\nu^{-1}}(\tfrac{1}{2}Ct_\nu^\lambda)^{2k}\,dt = 2(\tfrac{1}{2}C)^{2k}t_\nu^{2k\lambda-1},$$

which is contrary to hypothesis if k is large enough. This proves the theorem.

We could plainly replace the right-hand side of (13.2.1) by $O(T^A)$ without altering the theorem or the proof.

13.3. THEOREM 13.3. *A necessary and sufficient condition for the truth of the Lindelöf hypothesis is that, for every positive integer k and $\sigma > \frac{1}{2}$,*

$$\zeta^k(s) = \sum_{n\leqslant t^\delta}\frac{d_k(n)}{n^s} + O(t^{-\lambda}) \quad (t > 0), \tag{13.3.1}$$

where δ is any given positive number less than 1, and $\lambda = \lambda(k,\delta,\sigma) > 0$.

We may express this roughly by saying that, on the Lindelöf hypothesis, the behaviour of $\zeta(s)$, or of any of its positive integral powers, is dominated, throughout the right-hand half of the critical strip, by a section of the associated Dirichlet series whose length is less than any positive power of t, however small. The result may be contrasted with what we can deduce, without unproved hypothesis, from the approximate functional equation.

Taking $a_n = d_k(n)$ in Lemma 3.12, we have (if x is half an odd integer)

$$\sum_{n<x}\frac{d_k(n)}{n^s} = \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\zeta^k(s+w)\frac{x^w}{w}\,dw + O\left(\frac{x^c}{T(\sigma+c-1)^k}\right)$$

where $c > 1-\sigma+\epsilon$. Now let $0 < t < T-1$, and integrate round the rectangle $\frac{1}{2}-\sigma-iT$, $c-iT$, $c+iT$, $\frac{1}{2}-\sigma+iT$. We have

$$\frac{1}{2\pi i}\int\limits_{\text{rectangle}} \zeta^k(s+w)\frac{x^w}{w}\,dw = \zeta^k(s)+\frac{x^{1-s}}{1-s}P\left(\frac{1}{1-s},\, \log x\right)$$
$$= \zeta^k(s)+O(x^{1-\sigma+\epsilon}t^{-1+\epsilon}),$$

P being a polynomial in its arguments. Also

$$\left(\int\limits_{\frac{1}{2}-\sigma-iT}^{c-iT}+\int\limits_{c+iT}^{\frac{1}{2}-\sigma+iT}\right)\zeta^k(s+w)\frac{x^w}{w}\,dw = O(x^cT^{-1+\epsilon})$$

by the Lindelöf hypothesis; and

$$\int\limits_{\frac{1}{2}-\sigma-iT}^{\frac{1}{2}-\sigma+iT}\zeta^k(s+w)\frac{x^w}{w}\,dw = O\left\{x^{\frac{1}{2}-\sigma}\int\limits_{-T}^{T}\frac{|\zeta^k(\frac{1}{2}+it+iv)|}{|\frac{1}{2}+iv|}\,dv\right\}$$
$$= O(x^{\frac{1}{2}-\sigma}T^{\epsilon})$$

by the Lindelöf hypothesis. Hence

$$\zeta^k(s) = \sum_{n<x}\frac{d_k(n)}{n^s}+O\left\{\frac{x^c}{T(\sigma+c-1)^k}\right\}+O(x^{1-\sigma+\epsilon}t^{\epsilon-1})+$$
$$+O(x^cT^{-1+\epsilon})+O(x^{\frac{1}{2}-\sigma}T^{\epsilon}),$$

and (13.3.1) follows on taking $x = [t^\delta]+\frac{1}{2}$, $c = 2$, $T = t^3$.

Conversely, the condition is clearly sufficient, since it gives

$$\zeta^k(s) = O\Big(\sum_{n\leqslant t^\delta} n^{\epsilon-\sigma}\Big)+O(t^{-\lambda}) = O(t^{\delta(1-\epsilon-\sigma)}),$$

where δ is arbitrarily small.

The result may be used to prove the equivalence of the conditions of the previous section, without using the general theorems quoted.

13.4. Another set of conditions may be stated in terms of the numbers α_k and β_k of the previous chapter.

THEOREM 13.4. *Alternative necessary and sufficient conditions for the truth of the Lindelöf hypothesis are*

$$\alpha_k \leqslant \tfrac{1}{2} \qquad (k = 2, 3,...), \tag{13.4.1}$$

$$\beta_k \leqslant \tfrac{1}{2} \qquad (k = 2, 3,...), \tag{13.4.2}$$

$$\beta_k = \frac{k-1}{2k} \qquad (k = 2, 3,...). \tag{13.4.3}$$

As regards sufficiency, we need only consider (13.4.2), since the other

conditions are formally more stringent. Now (13.4.2) gives $\gamma_k \leqslant \frac{1}{2}$, and so

$$\int\limits_{\frac{1}{2}T}^{T}\frac{|\zeta(\sigma+it)|^{2k}}{|\sigma+it|^2}\,dt = O(1) \quad (\sigma > \tfrac{1}{2}),$$

$$\int\limits_{\frac{1}{2}T}^{T}|\zeta(\sigma+it)|^{2k}\,dt = O(T^2) \quad (\sigma > \tfrac{1}{2}).$$

The truth of the Lindelöf hypothesis follows from this, as in § 13.2.

Now suppose that the Lindelöf hypothesis is true. We have, as in § 12.2,

$$D_k(x) = \frac{1}{2\pi i}\int\limits_{2-iT}^{2+iT}\zeta^k(s)\frac{x^s}{s}\,ds+O\left(\frac{x^2}{T}\right).$$

Now integrate round the rectangle with vertices at $\frac{1}{2}-iT$, $2-iT$, $2+iT$, $\frac{1}{2}+iT$. We have

$$\int\limits_{\frac{1}{2}\pm iT}^{2\pm iT}\zeta^k(s)\frac{x^s}{s}\,ds = O(x^2T^{\epsilon-1}),$$

$$\int\limits_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\zeta^k(s)\frac{x^s}{s}\,ds = O\left\{x^{\frac{1}{2}}\int\limits_{-T}^{T}|\tfrac{1}{2}+it|^{\epsilon-1}\,dt\right\} = O(x^{\frac{1}{2}}T^{\epsilon}).$$

The residue at $s = 1$ accounts for the difference between $D_k(x)$ and $\Delta_k(x)$. Hence

$$\Delta_k(x) = O(x^{\frac{1}{2}}T^{\epsilon})+O(x^2T^{\epsilon-1}).$$

Taking $T = x^2$, it follows that $\alpha_k \leqslant \frac{1}{2}$. Hence also $\beta_k \leqslant \frac{1}{2}$. But in fact $\sigma_k \leqslant \frac{1}{2}$ on the Lindelöf hypothesis, so that, by Theorem 12.7, (13.4.3) also follows.

13.5. *The Lindelöf hypothesis and the zeros.*

THEOREM 13.5.† *A necessary and sufficient condition for the truth of the Lindelöf hypothesis is that, for every* $\sigma > \frac{1}{2}$,

$$N(\sigma, T+1)-N(\sigma, T) = o(\log T).$$

The necessity of the condition is easily proved. We apply Jensen's formula

$$\log\frac{r^n}{r_1 \dots r_n} = \frac{1}{2\pi}\int\limits_{0}^{2\pi}\log|f(re^{i\theta})|\,d\theta-\log|f(0)|,$$

where $r_1,...$ are the moduli of the zeros of $f(s)$ in $|s| \leqslant r$, to the circle with centre $2+it$ and radius $\frac{3}{2}-\frac{1}{4}\delta$, $f(s)$ being $\zeta(s)$. On the Lindelöf

† Backlund (4).

hypothesis the right-hand side is less than $o(\log t)$; and, if there are N zeros in the concentric circle of radius $\frac{3}{2}-\frac{1}{2}\delta$, the left-hand side is greater than

$$N\log\{(\tfrac{3}{2}-\tfrac{1}{4}\delta)/(\tfrac{3}{2}-\tfrac{1}{2}\delta)\}.$$

Hence the number of zeros in the circle of radius $\frac{3}{2}-\frac{1}{2}\delta$ is $o(\log t)$; and the result stated, with $\sigma = \frac{1}{2}+\delta$, clearly follows by superposing a number (depending on δ only) of such circles.

To prove the converse,† let C_1 be the circle with centre $2+iT$ and radius $\frac{3}{2}-\delta$ $(\delta > 0)$, and let Σ_1 denote a summation over zeros of $\zeta(s)$ in C_1. Let C_2 be the concentric circle of radius $\frac{3}{2}-2\delta$. Then for s in C_2

$$\psi(s) = \frac{\zeta'(s)}{\zeta(s)} - \sum\nolimits_1 \frac{1}{s-\rho} = O\left(\frac{\log T}{\delta}\right).$$

This follows from Theorem 9.6 (A), since for each term which is in one of the sums

$$\sum\nolimits_1 \frac{1}{s-\rho}, \qquad \sum_{|t-\gamma|<1} \frac{1}{s-\rho},$$

but not in the other, $|s-\rho| \geqslant \delta$; and the number of such terms is $O(\log T)$.

Let C_3 be the concentric circle of radius $\frac{3}{2}-3\delta$, C the concentric circle of radius $\frac{1}{2}$. Then $\psi(s) = o(\log T)$ for s in C, since each term is $O(1)$, and by hypothesis the number of terms is $o(\log T)$. Hence Hadamard's three-circles theorem gives, for s in C_3,

$$|\psi(s)| < \{o(\log T)\}^{\alpha}\{O(\delta^{-1}\log T)\}^{\beta}$$

where $\alpha+\beta = 1$, $0 < \beta < 1$, α and β depending on δ only. Thus in C_3

$$\psi(s) = o(\log T),$$

for any given δ.

Now

$$\int\limits_{\frac{1}{2}+3\delta}^{2} \psi(s)\,d\sigma = \log\zeta(2+it) - \log\zeta(\tfrac{1}{2}+3\delta+it) - \sum\nolimits_1 \{\log(2+it-\rho) - \log(\tfrac{1}{2}+3\delta+it-\rho)\}$$
$$= O(1) - \log\zeta(\tfrac{1}{2}+3\delta+it) + o(\log T) + \sum\nolimits_1 \log(\tfrac{1}{2}+3\delta+it-\rho),$$

since $\sum_1$ has $o(\log T)$ terms. Also, if $t = T$, the left-hand side is $o(\log T)$. Hence, putting $t = T$ and taking real parts,

$$\log|\zeta(\tfrac{1}{2}+3\delta+iT)| = o(\log T) + \sum\nolimits_1 \log|\tfrac{1}{2}+3\delta+iT-\rho|.$$

Since $|\frac{1}{2}+3\delta+iT-\rho| < A$ in C_1, it follows that

$$\log|\zeta(\tfrac{1}{2}+3\delta+iT)| < o(\log T),$$

i.e. the Lindelöf hypothesis is true.

† Littlewood (4).

13.6. Theorem 13.6 (A).† *On the Lindelöf hypothesis*

$$S(t) = o(\log t).$$

The proof is the same as Backlund's proof (§ 9.4) that, without any hypothesis, $S(t) = O(\log t)$, except that we now use $\zeta(s) = O(t^{\epsilon})$ where we previously used $\zeta(s) = O(t^{A})$.

Theorem 13.6 (B).‡ *On the Lindelöf hypothesis*

$$S_1(t) = o(\log t).$$

Integrating the real part of (9.6.3) from $\frac{1}{2}$ to $\frac{1}{2}+3\delta$,

$$\int\limits_{\frac{1}{2}}^{\frac{1}{2}+3\delta} \log|\zeta(s)|\,d\sigma = \sum_{|\gamma-t|<1} \int\limits_{\frac{1}{2}}^{\frac{1}{2}+3\delta} \log|s-\rho|\,d\sigma + O(\delta\log t),$$

where $\rho = \beta+i\gamma$ runs through zeros of $\zeta(s)$. Now

$$\int\limits_{\frac{1}{2}}^{\frac{1}{2}+3\delta} \log|s-\rho|\,d\sigma = \frac{1}{2}\int\limits_{\frac{1}{2}}^{\frac{1}{2}+3\delta} \log\{(\sigma-\beta)^2+(\gamma-t)^2\}\,d\sigma \leqslant \frac{3\delta}{2}\log 2$$

and

$$\geqslant \int\limits_{\frac{1}{2}}^{\frac{1}{2}+3\delta} \log|\sigma-\beta|\,d\sigma \geqslant \int\limits_{\frac{1}{2}}^{\frac{1}{2}+3\delta} \log|\sigma-\tfrac{1}{2}-\tfrac{3}{2}\delta|\,d\sigma = 3\delta(\log\tfrac{3}{2}\delta-1).$$

Hence

$$\int\limits_{\frac{1}{2}}^{\frac{1}{2}+3\delta} \log|\zeta(s)|\,d\sigma = \sum_{|\gamma-t|<1} O\left(\delta\log\frac{1}{\delta}\right) + O(\delta\log t)$$
$$= O(\delta\log 1/\delta\,.\log t).$$

Also, as in the proof of Theorem 13.5,

$$\log\zeta(s) = \sum\nolimits_1 \log(s-\rho) + o(\log t) \quad (\tfrac{1}{2}+3\delta \leqslant \sigma \leqslant 2).$$

Hence

$$\int\limits_{\frac{1}{2}+3\delta}^{2} \log|\zeta(s)|\,d\sigma = \sum\nolimits_1 \int\limits_{\frac{1}{2}+3\delta}^{2} \log|s-\rho|\,d\sigma + o(\log t)$$
$$= \sum\nolimits_1 O(1) + o(\log t)$$
$$= o(\log t).$$

Hence, by Theorem 9.9,

$$S_1(t) = \frac{1}{\pi}\int\limits_{\frac{1}{2}}^{2} \log|\zeta(s)|\,d\sigma + O(1)$$
$$= O(\delta\log 1/\delta\,.\log t) + o(\log t) + O(1),$$

and the result follows on choosing first δ and then t.

† Cramér (1), Littlewood (4). ‡ Littlewood (4).

NOTES FOR CHAPTER 13

13.7. Since the proof of Theorem 13.6(A) is not quite straightforward we give the details. Let

$$g(z) = \tfrac{1}{2}\{\zeta(z+2+iT)+\zeta(z+2-iT)\}$$

and define $n(r)$ to be the number of zeros of $g(z)$ in the disc $|z| \leqslant r$. As in § 9.4 one finds that $S(T) \ll n(\tfrac{3}{2})+1$. Moreover, by Jensen's Thorem, one has

$$\int_0^R \frac{n(r)}{r}\,dr = \frac{1}{2\pi}\int_0^{2\pi} \log|g(Re^{i\vartheta})|\,d\vartheta - \log|g(0)|. \tag{13.7.1}$$

With our choice of $g(z)$ we have $\log|g(0)| = \log|\mathbf{R}\zeta(2+iT)| = O(1)$. We shall take $R = \tfrac{3}{2}+\delta$. Then, on the Lindelöf Hypothesis, one finds that

$$|\zeta(Re^{i\vartheta}+2\pm iT)| \leqslant T^{\varepsilon}$$

for $\cos\vartheta \geqslant -3/(2R)$ and T sufficiently large. The remaining range for ϑ is an interval of length $O(\delta^{\frac{1}{2}})$. Here we write $\mathbf{R}(Re^{i\vartheta}+2) = \sigma$, so that $\tfrac{1}{2}-\delta \leqslant \sigma \leqslant \tfrac{1}{2}$. Then, using the convexity of the μ function, together with the facts that $\mu(0) = \tfrac{1}{2}$ and, on the Lindelöf Hypothesis, that $\mu(\tfrac{1}{2}) = 0$, we have $\mu(\sigma) \leqslant \delta$. It follows that

$$|\zeta(Re^{i\vartheta}+2\pm iT)| \leqslant T^{\delta+\varepsilon}$$

for $\cos\vartheta \leqslant -3/2R$, and large enough T. We now see that the right-hand side of (13.7.1) is at most

$$O(\varepsilon\log T) + O\{\delta^{\frac{1}{2}}(\delta+\varepsilon)\log T\}.$$

Since

$$\frac{\delta}{R}\,n(\tfrac{3}{2}) \leqslant \int_0^R \frac{n(r)}{r}\,dr$$

we conclude that

$$n(\tfrac{3}{2}) = O\left\{\left(\frac{\varepsilon}{\delta}+\delta^{-\frac{1}{2}}(\delta+\varepsilon)\right)\log T\right\},$$

and on taking $\delta = \varepsilon^{\frac{2}{3}}$ we obtain $n(\tfrac{3}{2}) = O(\varepsilon^{\frac{1}{3}}\log T)$, from which the result follows.

13.8. It has been observed by Ghosh and Goldston (in unpublished

work) that the converse of Theorem 13.6(B) follows from Lemma 21 of Selberg (5).

THEOREM 13.8. *If $S_1(t) = o(\log t)$, then the Lindelöf hypothesis holds.*

We reproduce the arguments used by Selberg and by Ghosh and Goldston here. Let $\tfrac{1}{2} \leqslant \sigma \leqslant 2$, and consider the integral

$$\frac{1}{2\pi i}\int_{5-i\infty}^{5+i\infty} \frac{\log\zeta(s+iT)}{4-(s-\sigma)^2}\,ds.$$

Since $\log\zeta(s+iT) \ll 2^{-\mathbf{R}(s)}$ the integral is easily seen to vanish, by moving the line of integration to the right. We now move the line of integration to the left, to $\mathbf{R}(s) = \sigma$, passing a pole at $s = 2+\sigma$, with residue $-\tfrac{1}{4}\log\zeta(2+\sigma+iT) = O(1)$. We must make detours around $s = 1-iT$, if $\sigma < 1$, and around $s = \rho - iT$, if $\sigma < \beta$. The former, if present, will produce an integral contributing $O(T^{-2})$, and the latter, if present, will be

$$-\int_0^{\beta-\sigma} \frac{du}{4-\{u+i(\gamma-T)\}^2}.$$

It follows that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\log\zeta(\sigma+it+iT)}{4+t^2}\,dt - \sum_{\beta>\sigma}\int_0^{\beta-\sigma} \frac{du}{4-\{u+i(\gamma-T)\}^2} = O(1),$$

for $T \geqslant 1$. We now take real parts and integrate for $\tfrac{1}{2} \leqslant \sigma \leqslant 2$. Then by Theorem 9.9 we have

$$\tfrac{1}{2}\int_{-\infty}^{\infty} \frac{S_1(t+T)}{4+t^2}\,dt = \sum_{\beta>\frac{1}{2}}\int_0^{\beta-\frac{1}{2}} (\beta-\tfrac{1}{2}-u)\,\mathbf{R}\left(\frac{1}{4-\{u+i(\gamma-T)\}^2}\right)du + O(1). \tag{13.8.1}$$

By our hypothesis the integral on the left is $o(\log T)$. Moreover

$$\mathbf{R}\left(\frac{1}{4-\{u+i(\gamma-T)\}^2}\right) \geqslant \begin{cases} A\ (>0) & \text{if } |\gamma-T| \leqslant 1, \\ 0, & \text{otherwise.} \end{cases}$$

If $\sigma > \tfrac{1}{2}$ is given, then each zero counted by $N(\sigma, T+1) - N(\sigma, T)$ contributes at least $\tfrac{1}{2}(\sigma-\tfrac{1}{2})^2 A$ to the sum on the right of (13.8.1), whence $N(\sigma, T+1) - N(\sigma, T) = o(\log T)$. Theorem 13.8 therefore follows from Theorem 13.5.

XIV

CONSEQUENCES OF THE RIEMANN HYPOTHESIS

14.1. In this chapter we assume the truth of the unproved Riemann hypothesis, that all the complex zeros of $\zeta(s)$ lie on the line $\sigma = \frac{1}{2}$. It will be seen that a perfectly coherent theory can be constructed on this basis, which perhaps gives some support to the view that the hypothesis is true. A proof of the hypothesis would make the 'theorems' of this chapter essential parts of the theory, and would make unnecessary much of the tentative analysis of the previous chapters.

The Riemann hypothesis, of course, leaves nothing more to be said about the 'horizontal' distribution of the zeros. From it we can also deduce interesting consequences both about the 'vertical' distribution of the zeros and about the order problems. In most cases we obtain much more precise results with the hypothesis than without it. But even a proof of the Riemann hypothesis would not by any means complete the theory. The finer shades in the behaviour of $\zeta(s)$ would still not be completely determined.

On the Riemann hypothesis, the function $\log\zeta(s)$, as well as $\zeta(s)$, is regular for $\sigma > \frac{1}{2}$ (except at $s = 1$). This is the basis of most of the analysis of this chapter.

We shall not repeat the words 'on the Riemann hypothesis', which apply throughout the chapter.

14.2. THEOREM 14.2.† *We have*

$$\log\zeta(s) = O\{(\log t)^{2-2\sigma+\epsilon}\} \tag{14.2.1}$$

uniformly for $\frac{1}{2} < \sigma_0 \leqslant \sigma \leqslant 1$.

Apply the Borel–Carathéodory theorem to the function $\log\zeta(z)$ and the circles with centre $2+it$ and radii $\frac{3}{2}-\frac{1}{2}\delta$ and $\frac{3}{2}-\delta$ $(0 < \delta < \frac{1}{2})$. On the larger circle

$$\mathbf{R}\{\log\zeta(z)\} = \log|\zeta(z)| < A\log t.$$

Hence, on the smaller circle,

$$|\log\zeta(z)| \leqslant \frac{3-2\delta}{\frac{1}{2}\delta}A\log t+\frac{3-\frac{3}{2}\delta}{\frac{1}{2}\delta}|\log|\zeta(2+it)||$$
$$< A\delta^{-1}\log t. \tag{14.2.2}$$

† Littlewood (1).

Now apply Hadamard's three-circles theorem to the circles C_1, C_2, C_3 with centre σ_1+it $(1 < \sigma_1 \leqslant t)$, passing through the points $1+\eta+it$, $\sigma+it$, $\frac{1}{2}+\delta+it$. The radii are thus

$$r_1 = \sigma_1-1-\eta, \qquad r_2 = \sigma_1-\sigma, \qquad r_3 = \sigma_1-\tfrac{1}{2}-\delta.$$

If the maxima of $|\log\zeta(z)|$ on the circles are M_1, M_2, M_3, we obtain

$$M_2 \leqslant M_1^{1-a}M_3^a,$$

where

$$a = \log\frac{r_2}{r_1}\Big/\log\frac{r_3}{r_1} = \log\left(1+\frac{1+\eta-\sigma}{\sigma_1-1-\eta}\right)\Big/\log\left(1+\frac{\frac{1}{2}+\eta-\delta}{\sigma_1-1-\eta}\right)$$
$$= \frac{1+\eta-\sigma}{\frac{1}{2}+\eta-\delta}+O\left(\frac{1}{\sigma_1}\right) = 2-2\sigma+O(\delta)+O(\eta)+O\left(\frac{1}{\sigma_1}\right).$$

By (14.2.2), $M_3 < A\delta^{-1}\log t$; and, since

$$\log\zeta(s) = \sum_{n=2}^{\infty}\frac{\Lambda_1(n)}{n^s} \quad (\Lambda_1(n) \leqslant 1), \tag{14.2.3}$$

$$M_1 \leqslant \max_{x\geqslant 1+\eta}\left|\sum_{n=2}^{\infty}\frac{\Lambda_1(n)}{n^x}\right| \leqslant \sum_{n=2}^{\infty}\frac{1}{n^{1+\eta}} < \frac{A}{\eta}.$$

Hence

$$|\log\zeta(\sigma+it)| < \left(\frac{A}{\eta}\right)^{1-a}\left(\frac{A\log t}{\delta}\right)^a < \frac{A}{\eta^{1-a}\delta^a}(\log t)^{2-2\sigma+O(\delta)+O(\eta)+O(1/\sigma_1)}.$$

The result stated follows on taking δ and η small enough and σ_1 large enough. More precisely, we can take

$$\sigma_1 = \frac{1}{\delta} = \frac{1}{\eta} = \log\log t;$$

since $\qquad (\log t)^{O(\delta)} = e^{O(\delta\log\log t)} = e^{O(1)} = O(1),$

etc., we obtain

$$\log\zeta(s) = O\{\log\log t(\log t)^{2-2\sigma}\} \quad \left(\frac{1}{2}+\frac{1}{\log\log t} \leqslant \sigma \leqslant 1\right). \tag{14.2.4}$$

Since the index of $\log t$ in (14.2.1) is less than unity if ϵ is small enough, it follows that (with a new ϵ)

$$-\epsilon\log t < \log|\zeta(s)| < \epsilon\log t \quad (t > t_0(\epsilon)),$$

i.e. we have both

$$\zeta(s) = O(t^\epsilon), \tag{14.2.5}$$

$$\frac{1}{\zeta(s)} = O(t^\epsilon) \tag{14.2.6}$$

for every $\sigma > \frac{1}{2}$. In particular, *the truth of the Lindelöf hypothesis follows from that of the Riemann hypothesis.*

It also follows that *for every fixed* $\sigma > \frac{1}{2}$, *as* $T \to \infty$

$$\int_1^T \frac{dt}{|\zeta(\sigma+it)|^2} \sim \frac{\zeta(2\sigma)}{\zeta(4\sigma)} T.$$

For $\sigma > 1$ this follows from (7.1.2) and (1.2.7). For $\frac{1}{2} < \sigma \leqslant 1$ it follows from (14.2.6) and the analysis of § 7.9, applied to $1/\zeta(s)$ instead of to $\zeta^k(s)$.

14.3. The function† $\nu(\sigma)$. For each $\sigma > \frac{1}{2}$ we define $\nu(\sigma)$ as the lower bound of numbers a such that

$$\log \zeta(s) = O(\log^a t).$$

It is clear from (14.2.3) that $\nu(\sigma) \leqslant 0$ for $\sigma > 1$; and from (14.2.2) that $\nu(\sigma) \leqslant 1$ for $\frac{1}{2} < \sigma \leqslant 1$; and in fact from (14.2.1) that $\nu(\sigma) \leqslant 2-2\sigma$ for $\frac{1}{2} < \sigma \leqslant 1$.

On the other hand, since $\Lambda_1(2) = 1$, (14.2.3) gives

$$|\log \zeta(s)| \geqslant \frac{1}{2^\sigma} - \sum_{n=3}^{\infty} \frac{\Lambda_1(n)}{n^\sigma},$$

and hence $\nu(\sigma) \geqslant 0$ if σ is so large that the right-hand side is positive. Since

$$\sum_{n=3}^{\infty} \frac{\Lambda_1(n)}{n^\sigma} \leqslant \sum_{n=3}^{\infty} \frac{1}{n^\sigma} < \int_2^\infty \frac{dx}{x^\sigma} = \frac{2^{1-\sigma}}{\sigma-1}$$

this is certainly true for $\sigma \geqslant 3$. Hence $\nu(\sigma) = 0$ for $\sigma \geqslant 3$.

Now let $\frac{1}{2} < \sigma_1 < \sigma < \sigma_2 \leqslant 4$, and suppose that

$$\log \zeta(\sigma_1+it) = O(\log^a t), \qquad \log \zeta(\sigma_2+it) = O(\log^b t).$$

Let
$$g(s) = \log \zeta(s) \{\log(-is)\}^{-k(s)},$$

where $k(s)$ is the linear function of s such that $k(\sigma_1) = a$, $k(\sigma_2) = b$, viz.

$$k(s) = \frac{(s-\sigma_1)b+(\sigma_2-s)a}{\sigma_2-\sigma_1}.$$

Here
$$\{\log(-is)\}^{-k(s)} = e^{-k(s)\log\log(-is)},$$

where
$$\log(-is) = \log(t-i\sigma), \qquad \log\log(-is) \ (t > e)$$

denote the branches which are real for $\sigma = 0$. Thus

$$\log(-is) = \log t + \log\left(1-\frac{i\sigma}{t}\right) = \log t + O\left(\frac{1}{t}\right),$$

$$\log\log(-is) = \log\log t + \log\left\{1+O\left(\frac{1}{t\log t}\right)\right\}$$
$$= \log\log t + O(1/t).$$

† Bohr and Landau (3), Littlewood (5).

Hence

$$|\{\log(-is)\}^{-k(s)}| = e^{-\mathbf{R}\{k(s)\log\log(-is)\}} = e^{-k(\sigma)\log\log t + O(1/t)}$$
$$= (\log t)^{-k(\sigma)}\{1+O(1/t)\}.$$

Hence $g(s)$ is bounded on the lines $\sigma = \sigma_1$ and $\sigma = \sigma_2$; and it is $O(\log^K t)$ for some K uniformly in the strip. Hence, by the theorem of Phragmén and Lindelöf, it is bounded in the strip. Hence

$$\log \zeta(s) = O\{(\log t)^{k(\sigma)}\},$$

i.e.
$$\nu(\sigma) \leqslant k(\sigma) = \frac{(\sigma-\sigma_1)b+(\sigma_2-\sigma)a}{\sigma_2-\sigma_1}. \tag{14.3.1}$$

Taking $\sigma = 3$, $\sigma_2 = 4$, $\nu(3) = 0$, $b = 0$, we obtain $a \geqslant 0$. Hence $\nu(\sigma) \geqslant 0$ for $\sigma > \frac{1}{2}$. Hence $\nu(\sigma) = 0$ for $\sigma > 1$.

Since $\nu(\sigma)$ is finite for every $\sigma > \frac{1}{2}$, we can take $a = \nu(\sigma_1)+\epsilon$, $b = \nu(\sigma_2)+\epsilon$ in (14.3.1). Making $\epsilon \to 0$, we obtain

$$\nu(\sigma) \leqslant \frac{(\sigma-\sigma_1)\nu(\sigma_2)+(\sigma_2-\sigma)\nu(\sigma_1)}{\sigma_2-\sigma_1},$$

i.e. $\nu(\sigma)$ *is a convex function of* σ. Hence it is continuous, and it is non-increasing since it is ultimately zero.

We can also show that $\zeta'(s)/\zeta(s)$ *has the same* ν*-function as* $\log \zeta(s)$. Let $\nu_1(s)$ be the ν-function of $\zeta'(s)/\zeta(s)$. Since

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{2\pi i} \int_{|s-z|=\delta} \frac{\log \zeta(z)}{(s-z)^2}\, dz = O\left\{\frac{1}{\delta}(\log t)^{\nu(\sigma-\delta)+\epsilon}\right\},$$

we have
$$\nu_1(\sigma) \leqslant \nu(\sigma-\delta)$$

for every positive δ; and since $\nu(\sigma)$ is continuous it follows that

$$\nu_1(\sigma) \leqslant \nu(\sigma).$$

We can show, as in the case of $\nu(\sigma)$, that $\nu_1(\sigma)$ is non-increasing, and is zero for $\sigma \geqslant 3$. Hence for $\sigma < 3$

$$\log \zeta(s) = -\int_\sigma^3 \frac{\zeta'(x+it)}{\zeta(x+it)}\, dx - \log \zeta(3+it)$$
$$= O\left\{\int_\sigma^3 (\log t)^{\nu_1(x)+\epsilon}\, dx\right\} + O(1)$$
$$= O\{(\log t)^{\nu_1(\sigma)+\epsilon}\},$$

i.e.
$$\nu(\sigma) \leqslant \nu_1(\sigma).$$

The exact value of $\nu(\sigma)$ is not known for any value of σ less than 1. All we know is

THEOREM 14.3. *For* $\frac{1}{2} < \sigma < 1$,

$$1-\sigma \leqslant \nu(\sigma) \leqslant 2(1-\sigma).$$

The upper bound follows from Theorem 14.2 and the lower bound from Theorem 8.12. The same lower bound can, however, be obtained in another and in some respects simpler way, though this proof, unlike the former, depends essentially on the Riemann hypothesis. For the proof we require some new formulae.

14.4. THEOREM 14.4.† *As* $t \to \infty$,

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} e^{-\delta n} + \sum_{\rho} \delta^{s-\rho}\Gamma(\rho-s) + O(\delta^{\sigma-\frac{1}{2}} \log t), \quad (14.4.1)$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant \frac{9}{8}$, $e^{-\sqrt{t}} \leqslant \delta \leqslant 1$.

Taking $a_n = \Lambda(n)$, $f(s) = -\zeta'(s)/\zeta(s)$ in the lemma of § 7.9, we have

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} e^{-\delta n} = -\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma(z-s) \frac{\zeta'(z)}{\zeta(z)} \delta^{s-z}\, dz. \quad (14.4.2)$$

Now, by Theorem 9.6 (A),

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{|t-\gamma|<1} \frac{1}{s-\frac{1}{2}-i\gamma} + O(\log t),$$

and there are $O(\log t)$ terms in the sum. Hence

$$\frac{\zeta'(s)}{\zeta(s)} = O(\log t)$$

on any line $\sigma \neq \frac{1}{2}$. Also

$$\frac{\zeta'(s)}{\zeta(s)} = O\left(\frac{\log t}{\min|t-\gamma|}\right) + O(\log t)$$

uniformly for $-1 \leqslant \sigma \leqslant 2$. Since each interval $(n, n+1)$ contains values of t whose distance from the ordinate of any zero exceeds $A/\log n$, there is a t_n in any such interval for which

$$\frac{\zeta'(s)}{\zeta(s)} = O(\log^2 t) \quad (-1 \leqslant \sigma \leqslant 2,\ t = t_n).$$

† Littlewood (5), to the end of § 14.8.

By the theorem of residues,

$$\frac{1}{2\pi i}\left(\int_{2-it_n}^{2+it_n} + \int_{2+it_n}^{\frac{1}{4}+it_n} + \int_{\frac{1}{4}+it_n}^{\frac{1}{4}-it_n} + \int_{\frac{1}{4}-it_n}^{2-it_n}\right)\Gamma(z-s)\frac{\zeta'(z)}{\zeta(z)}\delta^{s-z}\, dz$$
$$= \frac{\zeta'(s)}{\zeta(s)} + \sum_{-t_n<\gamma<t_n} \Gamma(\rho-s)\delta^{s-\rho} - \Gamma(1-s)\delta^{s-1}.$$

The integrals along the horizontal sides tend to zero as $n \to \infty$, so that

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} e^{-\delta n} = -\frac{1}{2\pi i} \int_{\frac{1}{4}-i\infty}^{\frac{1}{4}+i\infty} \Gamma(z-s)\frac{\zeta'(z)}{\zeta(z)}\delta^{s-z}\, dz -$$
$$-\frac{\zeta'(s)}{\zeta(s)} - \sum_{\rho} \Gamma(\rho-s)\delta^{s-\rho} + \Gamma(1-s)\delta^{s-1}.$$

Since $\Gamma(z-s) = O(e^{-A|y-t|})$, the integral is

$$O\left\{\int_{-\infty}^{\infty} e^{-A|y-t|} \log(|y|+2)\delta^{\sigma-\frac{1}{4}}\, dy\right\}$$
$$= O\left\{\int_{0}^{2t} e^{-A|y-t|} \log(|2t|+2)\delta^{\sigma-\frac{1}{4}}\, dy\right\} +$$
$$+ O\left\{\left(\int_{-\infty}^{0} + \int_{2t}^{\infty}\right) e^{-\frac{1}{2}A|y|} \log(|y|+2)\delta^{\sigma-\frac{1}{4}}\, dy\right\}$$
$$= O(\delta^{\sigma-\frac{1}{4}}\log t) + O(\delta^{\sigma-\frac{1}{4}}) = O(\delta^{\sigma-\frac{1}{4}}\log t).$$

Also

$$\Gamma(1-s)\delta^{s-1} = O(e^{-At}\delta^{\sigma-1}) = O(e^{-At}\delta^{-\frac{1}{2}})$$
$$= O(e^{-At+\frac{1}{2}\sqrt{t}}) = O(e^{-At}) = O(\delta^{\sigma-\frac{1}{4}}\log t).$$

This proves the theorem.

14.5. We can now prove more precise results about $\zeta'(s)/\zeta(s)$ and $\log\zeta(s)$ than those expressed by the inequality $\nu(\sigma) \leqslant 2-2\sigma$.

THEOREM 14.5. *We have*

$$\frac{\zeta'(s)}{\zeta(s)} = O\{(\log t)^{2-2\sigma}\}, \quad (14.5.1)$$

$$\log\zeta(s) = O\left\{\frac{(\log t)^{2-2\sigma}}{\log\log t}\right\}, \quad (14.5.2)$$

uniformly for $\frac{1}{2} < \sigma_0 \leqslant \sigma \leqslant \sigma_1 < 1$.

We have

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| \leqslant \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma} e^{-\delta n} + \delta^{\sigma-\frac{1}{2}} \sum_{\rho} |\Gamma(\rho-s)| + O(\delta^{\sigma-\frac{1}{4}}\log t).$$

Now

$$\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{\sigma}}e^{-\delta n} = -\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Gamma(z-\sigma)\frac{\zeta'(z)}{\zeta(z)}\delta^{\sigma-z}\,dz = O(\delta^{\sigma-1}),$$

since we may move the line of integration to $\mathbf{R}(z) = \frac{3}{4}$, and the leading term is the residue at $z = 1$. Also

$$|\Gamma(\rho-s)| < Ae^{-A|\gamma-t|}$$

uniformly for σ in the above range. Hence

$$\sum|\Gamma(\rho-s)| < A\sum_{\gamma}e^{-A|t-\gamma|} = A\sum_{n=1}^{\infty}\sum_{n-1\leqslant|t-\gamma|<n}e^{-A|t-\gamma|}.$$

The number of terms in the inner sum is

$$O\{\log(t+n)\} = O(\log t)+O\{\log(n+1)\}.$$

Hence we obtain

$$O\Big[\sum_{n=1}^{\infty}e^{-An}\{\log t+\log(n+1)\}\Big] = O(\log t).$$

Hence $$\frac{\zeta'(s)}{\zeta(s)} = O(\delta^{\sigma-1})+O(\delta^{\sigma-\frac{1}{2}}\log t)+O(\delta^{\sigma-\frac{1}{4}}\log t),$$

and taking $\delta = (\log t)^{-2}$ we obtain the first result.

Again for $\sigma_0 \leqslant \sigma \leqslant \sigma_1$

$$\log\zeta(s) = \log\zeta(\sigma_1+it)-\int_{\sigma}^{\sigma_1}\frac{\zeta'(x+it)}{\zeta(x+it)}\,dx$$

$$= O\{(\log t)^{2-2\sigma_1+\epsilon}\}+O\left\{\int_{\sigma}^{\sigma_1}(\log t)^{2-2x}\,dx\right\}$$

$$= O\{(\log t)^{2-2\sigma_1+\epsilon}\}+O\left\{\frac{(\log t)^{2-2\sigma}}{\log\log t}\right\}.$$

If $\sigma \leqslant \sigma_2 < \sigma_1$ and $\epsilon < 2(\sigma_1-\sigma_2)$, this is of the required form; and since σ_1 and so σ_2 may be as near to 1 as we please, the second result (with σ_2 for σ_1) follows.

14.6. To obtain the alternative proof of the inequality $\nu(\sigma) \geqslant 1-\sigma$ we require an approximate formula for $\log\zeta(s)$.

THEOREM 14.6. *For fixed α and σ such that $\frac{1}{2} < \alpha < \sigma \leqslant 1$, and $e^{-\sqrt{t}} \leqslant \delta \leqslant 1$,*

$$\log\zeta(s) = \sum_{n=1}^{\infty}\frac{\Lambda_1(n)}{n^s}e^{-\delta n}+O\{\delta^{\sigma-\alpha}(\log t)^{\nu(\alpha)+\epsilon}\}+O(1).$$

Moving the line of integration in (14.4.2) to $\mathbf{R}(w) = \alpha$, we have

$$\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}e^{-\delta n} = -\frac{\zeta'(s)}{\zeta(s)}-\Gamma(1-s)\,\delta^{s-1}-\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\Gamma(z-s)\frac{\zeta'(z)}{\zeta(z)}\delta^{s-z}\,dz.$$

Since $\zeta'(s)/\zeta(s)$ has the ν-function $\nu(\sigma)$, the integral is of the form

$$O\left\{\delta^{\sigma-\alpha}\int_{-\infty}^{\infty}e^{-A|y-t|}\{\log(|y|+2)\}^{\nu(\alpha)+\epsilon}\,dy\right\} = O\{\delta^{\sigma-\alpha}(\log t)^{\nu(\alpha)+\epsilon}\};$$

and $\Gamma(1-s)\delta^{s-1}$ is also of this form, as in § 14.4. Hence

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}e^{-\delta n}+O\{\delta^{\sigma-\alpha}(\log t)^{\nu(\alpha)+\epsilon}\}.$$

This result holds uniformly in the range $[\sigma, \frac{9}{8}]$, and so we may integrate over this interval. We obtain

$$\log\zeta(s)-\sum_{n=1}^{\infty}\frac{\Lambda_1(n)}{n^s}e^{-\delta n}+O\{\delta^{\sigma-\alpha}(\log t)^{\nu(\alpha)+\epsilon}\}$$

$$= \log\zeta(\tfrac{9}{8}+it)-\sum_{n=1}^{\infty}\frac{\Lambda_1(n)}{n^{\frac{9}{8}+it}}e^{-\delta n} = O(1),$$

as required.

14.7. Proof that $\nu(\sigma) \geqslant 1-\sigma$. Theorem 14.6 enables us to extend the method of Diophantine approximation, already used for $\sigma > 1$, to values of σ between $\frac{1}{2}$ and 1. It gives

$$\log|\zeta(s)| = \sum_{n=1}^{\infty}\frac{\Lambda_1(n)}{n^{\sigma}}\cos(t\log n)e^{-\delta n}+O\{\delta^{\sigma-\alpha}(\log t)^{\nu(\alpha)+\epsilon}\}+O(1),$$

$$= \sum_{n=1}^{N}\frac{\Lambda_1(n)}{n^{\sigma}}\cos(t\log n)e^{-\delta n}+O\Big(\sum_{n=N+1}^{\infty}e^{-\delta n}\Big)+O\{\delta^{\sigma-\alpha}(\log t)^{\nu(\alpha)+\epsilon}\}+O(1)$$

for all values of N. Now by Dirichlet's theorem (§ 8.2) there is a number t in the range $2\pi \leqslant t \leqslant 2\pi q^N$, and integers $x_1,\ldots, x_N$, such that

$$\left|t\frac{\log n}{2\pi}-x_n\right| \leqslant \frac{1}{q}\quad (n = 1, 2,\ldots, N).$$

Let us assume for the moment that this number t satisfies the condition of Theorem 14.6 that $e^{-\sqrt{t}} \leqslant \delta$. It gives

$$\sum_{n=1}^{N}\frac{\Lambda_1(n)}{n^{\sigma}}\cos(t\log n)e^{-\delta n} \geqslant \sum_{n=1}^{N}\frac{\Lambda_1(n)}{n^{\sigma}}\cos\frac{2\pi}{q}e^{-\delta n}$$

$$= \sum_{n=1}^{N}\frac{\Lambda_1(n)}{n^{\sigma}}e^{-\delta n}+O\Big(\frac{1}{q}\Big)\sum_{n=1}^{N}\frac{1}{n^{\sigma}}.$$

Now

$$\sum_{n=1}^{N}\frac{\Lambda_1(n)}{n^\sigma}e^{-\delta n}\geqslant\frac{1}{\log N}\sum_{n=1}^{N}\frac{\Lambda(n)}{n^\sigma}e^{-\delta n}$$

$$\geqslant\frac{1}{\log N}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^\sigma}e^{-\delta n}+O\Big(\sum_{n=N+1}^{\infty}e^{-\delta n}\Big)$$

$$>\frac{K(\sigma)\delta^{\sigma-1}}{\log N}+O\Big(\frac{e^{-\delta N}}{\delta}\Big)$$

as in § 14.5. Hence

$$\log|\zeta(s)|>\frac{K(\sigma)\delta^{\sigma-1}}{\log N}+O\Big(\frac{e^{-\delta N}}{\delta}\Big)+O\Big(\frac{N^{1-\sigma}}{q}\Big)+O\{\delta^{\sigma-\alpha}(\log t)^{\nu(\alpha)+\epsilon}\}+O(1).$$

Take $q = N = [\delta^{-a}]$, where $a > 1$. The second and third terms on the right are then bounded. Also

$$\log t\leqslant N\log q+\log 2\pi\leqslant\frac{a}{\delta^a}\log\frac{1}{\delta}+\log 2\pi,$$

so that $$\delta\leqslant K(\log t)^{-1/a+\epsilon}.$$

Hence $$\log|\zeta(s)|>K(\log t)^{1-\sigma-\eta}+O\{(\log t)^{\alpha-\sigma+\nu(\alpha)+\eta'}\},$$

where η and η' are functions of a which tend to zero as $a\to 1$.

If the first term on the right is of larger order than the second, it follows at once that $\nu(\sigma)\geqslant 1-\sigma$. Otherwise

$$\alpha-\sigma+\nu(\alpha)\geqslant 1-\sigma,$$

and making $\alpha\to\sigma$ the result again follows.

We have still to show that the t of the above argument satisfies $e^{-\sqrt{t}}\leqslant\delta$. Suppose on the contrary that $\delta<e^{-\sqrt{t}}$ for some arbitrarily small values of δ. Now, by (8.4.4),

$$|\zeta(s)|\geqslant\Big(\cos\frac{2\pi}{q}-2N^{1-\sigma}\Big)\zeta(\sigma)>\frac{A}{\sigma-1}(\tfrac{1}{2}-2N^{1-\sigma})$$

for $\sigma>1$, $q\geqslant 6$. Taking $\sigma=1+\log 8/\log N$,

$$|\zeta(s)|>\frac{A}{\sigma-1}=A\log N>A\log\frac{1}{\delta}>At^{\frac{1}{2}}.$$

Since $|\zeta(s)|\to\infty$ and $t\geqslant 2\pi$, $t\to\infty$, and the above result contradicts Theorem 3.5. This completes the proof.

14.8. The function $\zeta(1+it)$. We are now in a position to obtain fairly precise information about this function. We shall first prove

THEOREM 14.8. *We have*

$$|\log\zeta(1+it)|\leqslant\log\log\log t+A. \tag{14.8.1}$$

In particular

$$\zeta(1+it)=O(\log\log t), \tag{14.8.2}$$

$$\frac{1}{\zeta(1+it)}=O(\log\log t). \tag{14.8.3}$$

Taking $\sigma=1$, $\alpha=\frac{3}{4}$ in Theorem 14.6, we have

$$|\log\zeta(1+it)|\leqslant\sum_{n=1}^{\infty}\frac{\Lambda_1(n)}{n}e^{-\delta n}+O(\delta^{\frac{1}{4}}\log t)+O(1)$$

$$\leqslant\sum_{n=1}^{N}\frac{\Lambda_1(n)}{n}+\sum_{n=N+1}^{\infty}e^{-\delta n}+O(\delta^{\frac{1}{4}}\log t)+O(1)$$

$$\leqslant\log\log N+O(e^{-\delta N}/\delta)+O(\delta^{\frac{1}{4}}\log t)+O(1)$$

by (3.14.4). Taking $\delta=\log^{-4}t$, $N=1+[\log^5 t]$, the result follows.

Comparing this result with Theorems 8.5 and 8.8, we see that, as far as the order of the functions $\zeta(1+it)$ and $1/\zeta(1+it)$ is concerned, the result is final. It remains to consider the values of the constants involved in the inequalities.

14.9. We define a function $\beta(\sigma)$ as

$$\beta(\sigma)=\frac{\nu(\sigma)}{2-2\sigma}.$$

By the convexity of $\nu(\sigma)$ we have, for $\frac{1}{2}<\sigma<\sigma'<1$,

$$\nu(\sigma')\leqslant\frac{(1-\sigma')\nu(\sigma)+(\sigma'-\sigma)\nu(1)}{1-\sigma}=\frac{1-\sigma'}{1-\sigma}\nu(\sigma),$$

i.e. $$\beta(\sigma')\leqslant\beta(\sigma).$$

Thus $\beta(\sigma)$ is non-increasing in $(\frac{1}{2},1)$. We write

$$\beta(\tfrac{1}{2})=\lim_{\sigma\to\frac{1}{2}+0}\beta(\sigma),\qquad \beta(1)=\lim_{\sigma\to 1-0}\beta(\sigma).$$

Then by Theorem 14.3, for $\frac{1}{2}<\sigma<1$,

$$\tfrac{1}{2}\leqslant\beta(1)\leqslant\beta(\sigma)\leqslant\beta(\tfrac{1}{2})\leqslant 1.$$

We shall now prove†

THEOREM 14.9. *As* $t\to\infty$

$$|\zeta(1+it)|\leqslant 2\beta(1)e^\gamma\{1+o(1)\}\log\log t, \tag{14.9.1}$$

$$\frac{1}{|\zeta(1+it)|}\leqslant 2\beta(1)\frac{6e^\gamma}{\pi^2}\{1+o(1)\}\log\log t. \tag{14.9.2}$$

† Littlewood (6).

We observe that the $O(1)$ in Theorem 14.6 is actually $o(1)$ if $\delta \to 0$. Also, taking $\sigma = 1$,

$$\delta^{1-\alpha}(\log t)^{\nu(\alpha)+\epsilon} = o(1)$$

if
$$\delta = (\log t)^{-2\beta(\alpha)-\eta} \quad (\eta > 0).$$

Hence, for such δ,

$$\log \zeta(1+it) = \sum_{n=1}^{\infty} \frac{\Lambda_1(n)}{n^{1+it}} e^{-\delta n} + o(1)$$
$$= \sum_{p,m} \frac{e^{-\delta p^m}}{mp^{m(1+it)}} + o(1)$$
$$= \sum_{p,m} \frac{e^{-\delta mp}}{mp^{m(1+it)}} + \sum_{p} \sum_{m>1} \frac{e^{-\delta p^m} - e^{-\delta mp}}{mp^{m(1+it)}} + o(1).$$

Now the modulus of the second double sum does not exceed

$$\sum_{p} \sum_{m>1} \frac{e^{-\delta p^m} - e^{-\delta mp}}{p^m}.$$

This is evidently uniformly convergent for $\delta \geqslant 0$, the summand being less than p^{-m}. Since each term tends to zero with δ the sum is $o(1)$. Hence

$$\log \zeta(1+it) = \sum_{p,m} \frac{e^{-\delta mp}}{mp^{m(1+it)}} + o(1)$$
$$= -\sum_{p} \log\left(1 - \frac{e^{-\delta p}}{p^{1+it}}\right) + o(1)$$
$$= -\sum_{p \leqslant \varpi} \log\left(1 - \frac{e^{-\delta p}}{p^{1+it}}\right) + O\left(\sum_{\varpi+1}^{\infty} e^{-\delta n}\right) + o(1).$$

The second term is $O(e^{-\delta\varpi}/\delta) = o(1)$ if $\varpi = [\delta^{-1-\epsilon}]$. Also

$$1 - \frac{1}{p} \leqslant \left|1 - \frac{e^{-\delta p}}{p^{1+it}}\right| \leqslant 1 + \frac{1}{p}.$$

Hence, by (3.15.2),

$$\log |\zeta(1+it)| \leqslant -\sum_{p \leqslant \varpi} \log\left(1 - \frac{1}{p}\right) + o(1)$$
$$= \log\log \varpi + \gamma + o(1),$$

or
$$|\zeta(1+it)| \leqslant e^{\gamma + o(1)} \log \varpi.$$

Now
$$\log \varpi \leqslant (1+\epsilon)\log\frac{1}{\delta} = (1+\epsilon)\{2\beta(\alpha)+\eta\}\log\log t,$$

and taking α arbitrarily near to 1, we obtain (14.9.1). Similarly, by (3.15.3),

$$\log \frac{1}{|\zeta(1+it)|} \leqslant \sum_{p \leqslant \varpi} \log\left(1 + \frac{1}{p}\right) + o(1)$$
$$= \log\log \varpi + \log \frac{6e^{\gamma}}{\pi^2} + o(1),$$

and (14.9.2) follows from this.

Comparing Theorem 14.9 with Theorems 8.9 (A) and (B), we see that, since we know only that $\beta(1) \leqslant 1$, in each problem a factor 2 remains in doubt. It is possible that $\beta(1) = \frac{1}{2}$, and if this were so each constant would be determined exactly.

14.10. The function $S(t)$. We shall next discuss the behaviour of this function on the Riemann hypothesis.

If $\frac{1}{2} < \alpha < \sigma < \beta$, $T < t < T'$, we have

$$\log \zeta(s) = \frac{1}{2\pi i}\left(\int_{\beta+iT}^{\beta+iT'} + \int_{\beta+iT'}^{\alpha+iT'} + \int_{\alpha+iT'}^{\alpha+iT} + \int_{\alpha+iT}^{\beta+iT}\right) \frac{\log \zeta(z)}{z-s}\, dz.$$

Let $\beta > 2$. By (14.2.2),

$$\int_{\alpha+iT}^{2+iT} \frac{\log \zeta(z)}{z-s}\, dz = O\left\{\frac{1}{t-T} \int_{\alpha}^{2} |\log \zeta(x+iT)|\, dx\right\} = O\left(\frac{\log T}{t-T}\right).$$

Also
$$\int_{2+iT}^{\beta+iT} \frac{\log \zeta(z)}{z-s}\, dz = \sum_{n=2}^{\infty} \Lambda_1(n) \int_{2+iT}^{\beta+iT} \frac{n^{-z}}{z-s}\, dz.$$

Now

$$\int_{2+iT}^{\beta+iT} \frac{n^{-z}}{z-s}\, dz = \left[\frac{-n^{-z}}{(z-s)\log n}\right]_{2+iT}^{\beta+iT} - \frac{1}{\log n} \int_{2+iT}^{\beta+iT} \frac{n^{-z}}{(z-s)^2}\, dz$$
$$= O\left\{\frac{1}{n^2(t-T)}\right\} + O\left\{\frac{1}{n^2} \int_{-\infty}^{\infty} \frac{dx}{(x-\sigma)^2+(t-T)^2}\right\} = O\left\{\frac{1}{n^2(t-T)}\right\}.$$

Hence
$$\int_{2+iT}^{\beta+iT} \frac{\log \zeta(z)}{z-s}\, dz = O\left(\frac{1}{t-T}\right),$$

and hence
$$\int_{\alpha+iT}^{\beta+iT} \frac{\log \zeta(z)}{z-s}\, dz = O\left(\frac{\log T}{t-T}\right)$$

uniformly with respect to β. Similarly for the integral over

$$(\beta+iT', \alpha+iT').$$

Also
$$\int_{\beta+iT}^{\beta+iT'} \frac{\log\zeta(z)}{z-s}\,dz = O\left(\frac{T'-T}{\beta-\sigma}\right).$$

Making $\beta\to\infty$, it follows that
$$\log\zeta(s) = \frac{1}{2\pi i}\int_{\alpha+iT}^{\alpha+iT'} \frac{\log\zeta(z)}{s-z}\,dz + O\left(\frac{\log T}{t-T}\right) + O\left(\frac{\log T'}{T'-t}\right). \tag{14.10.1}$$

A similar argument shows that, if $\mathbf{R}(s')<\frac{1}{2}$,
$$0 = \frac{1}{2\pi i}\int_{\alpha+iT}^{\alpha+iT'} \frac{\log\zeta(z)}{s'-z}\,dz + O\left(\frac{\log T}{t-T}\right) + O\left(\frac{\log T'}{T'-t}\right). \tag{14.10.2}$$

Taking $s' = 2\alpha-\sigma+it$, so that
$$s'-z = 2\alpha-\sigma+it-(\alpha+iy) = \alpha-iy-(\sigma-it),$$
and replacing (14.10.2) by its conjugate, we have
$$0 = \frac{1}{2\pi i}\int_{\alpha+iT}^{\alpha+iT'} \frac{\log|\zeta(z)|-i\arg\zeta(z)}{z-s}\,dz + O\left(\frac{\log T}{t-T}\right) + O\left(\frac{\log T'}{T'-t}\right). \tag{14.10.3}$$

From (14.10.1) and (14.10.3) it follows that
$$\log\zeta(s) = \frac{1}{\pi i}\int_{\alpha+iT}^{\alpha+iT'} \frac{\log|\zeta(z)|}{s-z}\,dz + O\left(\frac{\log T}{t-T}\right) + O\left(\frac{\log T'}{T'-t}\right) \tag{14.10.4}$$
and
$$\log\zeta(s) = \frac{1}{\pi}\int_{\alpha+iT}^{\alpha+iT'} \frac{\arg\zeta(z)}{s-z}\,dz + O\left(\frac{\log T}{t-T}\right) + O\left(\frac{\log T'}{T'-t}\right). \tag{14.10.5}$$

14.11. We can now show that each of the functions
$$\max\{\log|\zeta(s)|, 0\}, \qquad \max\{-\log|\zeta(s)|, 0\},$$
$$\max\{\arg\zeta(s), 0\}, \qquad \max\{-\arg\zeta(s), 0\}$$
has the same ν-function as $\log\zeta(s)$. Consider, for example,
$$\max\{\arg\zeta(s), 0\},$$
and let its ν-function be $\nu_1(\sigma)$. Since
$$|\arg\zeta(s)| \leqslant |\log\zeta(s)|$$
we have at once
$$\nu_1(\sigma) \leqslant \nu(\sigma).$$

Also (14.10.5) gives
$$\arg\zeta(s) = \frac{1}{\pi}\int_{T}^{T'} \frac{\sigma-\alpha}{(\sigma-\alpha)^2+(t-y)^2}\arg\zeta(\alpha+iy)\,dy + O\left(\frac{\log T}{t-T}\right) + O\left(\frac{\log T'}{T'-t}\right) \tag{14.11.1}$$
$$< A(\log T')^{\nu_1(\alpha)+\epsilon}\int_{T}^{T'} \frac{\sigma-\alpha}{(\sigma-\alpha)^2+(t-y)^2}\,dy + O\left(\frac{\log T}{t-T}\right) + O\left(\frac{\log T'}{T'-t}\right)$$
$$< A(\log t)^{\nu_1(\alpha)+\epsilon} + O(t^{-1}\log t),$$
taking, for example, $T=\frac{1}{2}t$, $T'=2t$.

It is clear from this that $\nu_1(\sigma)$ is non-increasing. Also the Borel–Carathéodory inequality, applied to circles with centre $2+it$ and radii $2-\alpha-\delta$, $2-\alpha-2\delta$, gives
$$|\log\zeta(\alpha+\delta+it)| < \frac{A}{\delta}\left\{(\log t)^{\nu_1(\alpha)+\epsilon} + \frac{\log t}{t}\right\} + \frac{A}{\delta}|\log|\zeta(2+it)||.$$

If $\alpha+\delta<1$, so that $\nu(\alpha+\delta)>0$, it follows that
$$\nu(\alpha+\delta) \leqslant \nu_1(\alpha)+\epsilon.$$

Since ϵ and δ may be as small as we please, and $\nu(\sigma)$ is continuous, it follows that
$$\nu(\alpha) \leqslant \nu_1(\alpha).$$

Hence
$$\nu_1(\sigma) = \nu(\sigma) \quad (\tfrac{1}{2}<\sigma<1).$$

Similarly all the ν-functions are equal.

14.12. *Ω-results*† *for $S(t)$ and $S_1(t)$.*

THEOREM 14.12 (A). *Each of the inequalities*
$$S(t) > (\log t)^{\frac{1}{2}-\epsilon}, \tag{14.12.1}$$
$$S(t) < -(\log t)^{\frac{1}{2}-\epsilon} \tag{14.12.2}$$
has solutions for arbitrarily large values of t.

Making $\alpha\to\frac{1}{2}$ in (14.11.1), by bounded convergence
$$\arg\zeta(s) = \int_{\frac{1}{2}t}^{2t} \frac{\sigma-\frac{1}{2}}{(\sigma-\frac{1}{2})^2+(t-y)^2}S(y)\,dy + O\left(\frac{\log t}{t}\right) \quad (\sigma>\tfrac{1}{2}). \tag{14.12.3}$$

If $S(t) < \log^\alpha t$ for all large t, this gives
$$\arg\zeta(s) < A\log^\alpha t\int_{\frac{1}{2}t}^{2t} \frac{\sigma-\frac{1}{2}}{(\sigma-\frac{1}{2})^2+(t-y)^2}\,dy + O\left(\frac{\log t}{t}\right)$$
$$< A\log^\alpha t + O(t^{-1}\log t).$$

† Landau (1), Bohr and Landau (3), Littlewood (5).

The above analysis shows that this is false if $a < \nu(\sigma)$, which is satisfied if $a < \frac{1}{2}$ and σ is near enough to $\frac{1}{2}$. This proves the first result, and the other may be proved similarly.

Theorem 14.12 (B).

$$S_1(t) = \Omega\{(\log t)^{\frac{1}{2}-\epsilon}\}.$$

From (14.10.5) with $\alpha \to \frac{1}{2}$ we have

$$\log\zeta(s) = i\int\limits_{\frac{1}{2}t}^{2t} \frac{S(y)}{s-\frac{1}{2}-iy}\,dy+O(1)$$

$$= i\left[\frac{S_1(y)}{s-\frac{1}{2}-iy}\right]_{\frac{1}{2}t}^{2t} + \int\limits_{\frac{1}{2}t}^{2t} \frac{S_1(y)}{(s-\frac{1}{2}-iy)^2}\,dy+O(1)$$

$$= \int\limits_{\frac{1}{2}t}^{2t} \frac{S_1(y)}{(s-\frac{1}{2}-iy)^2}\,dy+O(1) \qquad (14.12.4)$$

since $S_1(y) = O(\log y)$. The result now follows as before.

In view of the result of Selberg stated in § 9.9, this theorem is true independently of the Riemann hypothesis. In the case of $S(t)$, Selberg's method gives only an index $\frac{1}{3}$ instead of the index $\frac{1}{2}$ obtained on the Riemann hypothesis.

14.13. We now turn to results of the opposite kind.† We know that without any hypothesis

$$S(t) = O(\log t), \qquad S_1(t) = O(\log t),$$

and that on the Lindelöf hypothesis, and *a fortiori* on the Riemann hypothesis, each O can be replaced by o. On the Riemann hypothesis we should expect something more precise. The result actually obtained is

Theorem 14.13.

$$S(t) = O\left(\frac{\log t}{\log\log t}\right), \qquad (14.13.1)$$

$$S_1(t) = O\left(\frac{\log t}{(\log\log t)^2}\right). \qquad (14.13.2)$$

We first prove three lemmas.

Lemma α. *Let*

$$\phi(t) = \max_{1\leqslant u\leqslant t}|S_1(u)|,$$

so that $\phi(t)$ is non-decreasing, and $\phi(t) = O(\log t)$. Then

$$S(t) = O[\{\phi(2t)\log t\}^{\frac{1}{2}}].$$

† Landau (11), Cramér (1), Littlewood (4), Titchmarsh (3).

This is independent of the Riemann hypothesis. We have

$$N(t) = L(t)+R(t),$$

where $L(t)$ is defined by (9.3.1), and $R(t) = S(t)+O(1/t)$. Now

$$N(T+x)-N(T) \geqslant 0 \quad (0 < x < T).$$

Hence

$$R(T+x)-R(T) \geqslant -\{L(T+x)-L(T)\} > -Ax\log T.$$

Hence

$$\int\limits_T^{T+x} R(t)\,dt = xR(T)+\int\limits_0^x \{R(T+u)-R(T)\}\,du$$

$$> xR(T)-A\int\limits_0^x u\log T\,du$$

$$> xR(T)-Ax^2\log T.$$

Hence

$$R(T) < \frac{1}{x}\int\limits_T^{T+x} R(t)\,dt+Ax\log T$$

$$= \frac{S_1(T+x)-S_1(T)}{x}+O\left(\frac{1}{T}\right)+Ax\log T$$

$$= O\left\{\frac{\phi(2T)}{x}\right\}+O\left(\frac{1}{T}\right)+Ax\log T.$$

Taking $x = \{\phi(2T)/\log T\}^{\frac{1}{2}}$, the upper bound for $S(T)$ follows. Similarly by considering integrals over $(T-x, T)$ we obtain the lower bound.

Lemma β. *Let $\sigma \leqslant 1$, and let*

$$F(T) = \max|\log\zeta(s)|+\log^{\frac{1}{4}}T \quad \left(\sigma-\tfrac{1}{2} \geqslant \frac{1}{\log\log T}, \quad 4 \leqslant t \leqslant T\right).$$

Then

$$\log\zeta(s) = O\{F(T+1)e^{-A(\sigma-\frac{1}{2})\log\log T}\}$$

$$\left(\frac{1}{2}+\frac{1}{\log\log T} \leqslant \sigma \leqslant 2, \quad 4 \leqslant t \leqslant T\right).$$

We apply Hadamard's three-circles theorem as in § 14.2, but now take

$$\sigma_1 = \frac{3}{2}+\frac{1}{\log\log T}, \quad \eta = \tfrac{1}{4}, \quad \delta = \frac{1}{\log\log T}, \quad \sigma \leqslant \tfrac{5}{4}.$$

We obtain

$$M_2 < AM_3^a = AM_3(1/M_3)^{1-a},$$

where

$$M_3 \leqslant F(T+1),$$

and

$$1-a = \log\frac{r_3}{r_2}\Big/\log\frac{r_3}{r_1} = \log\left(1+\frac{\sigma-\frac{1}{2}-\delta}{\sigma_1-\sigma}\right)\Big/\log\left(\frac{\sigma_1-\frac{1}{2}-\delta}{\sigma_1-1-\eta}\right)$$

$$> A(\sigma-\tfrac{1}{2}-\delta).$$