larly $\phi(x-0)=\phi(x)$ for all values of x. Hence the function is continuous.

If we put $x=\frac{1}{2}(x_1+x_2)$ in (1), we obtain

$$\phi(\tfrac{1}{2}x_1+\tfrac{1}{2}x_2)<\tfrac{1}{2}\{\phi(x_1)+\phi(x_2)\}. \tag{2}$$

This is sometimes taken as the definition of convexity* instead of (1). It is less restrictive than the definition adopted here, and does not involve continuity.

A sufficient condition for $\phi(x)$ to be convex is that $\phi''(x)>0$; for then $\phi'(x)$ is increasing, and

$$\frac{1}{x-x_1}\int_{x_1}^{x}\phi'(t)\,dt<\phi'(x)<\frac{1}{x_2-x}\int_{x}^{x_2}\phi'(t)\,dt \qquad (x_1<x<x_2),$$

which gives (1).

5.32. The three-circles theorem as a convexity theorem. Hadamard's three-circles theorem may be expressed by saying that $\log M(r)$ *is a convex function of* $\log r$. For we may write it in the form

$$\log M(r_2)\leqslant\frac{\log r_3-\log r_2}{\log r_3-\log r_1}\log M(r_1)+\frac{\log r_2-\log r_1}{\log r_3-\log r_1}\log M(r_3),$$

and the sign of equality occurs only if the function is a constant multiple of a power of z.

5.4. Mean values of $|f(z)|$. The mean values

$$I_1(r)=\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|\,d\theta, \qquad I_2(r)=\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^2\,d\theta,$$

have properties similar to those of $M(r)$.

5.41. *$I_2(r)$ increases steadily with r, and $\log I_2(r)$ is a convex function of $\log r$.*

Let
$$f(z)=\sum_{n=0}^{\infty}a_nz^n.$$

The fact that $I_2(r)$ is steadily increasing is then obvious from the formula

$$I_2(r)=\sum_{n=0}^{\infty}|a_n|^2r^{2n}$$

of § 2.5.

* e.g. in Pólya and Szegö, *Aufgaben*, 1, p. 52.

To prove convexity, let $u = \log r$, and let I_2', I_2'' denote derivatives with respect to u. Then

$$\frac{d^2}{du^2}(\log I_2) = \frac{I_2 I_2'' - I_2'^2}{I_2^2},$$

and by Schwarz's inequality

$$I_2'^2 = (\sum |a_n|^2 2ne^{2nu})^2 \leqslant (\sum |a_n|^2 e^{2nu})(\sum |a_n|^2 4n^2 e^{2nu}) = I_2 I_2''.$$

Hence the result.

5.42. *$I_1(r)$ increases steadily with r, and $\log I_1(r)$ is a convex function of $\log r$.*

It is possible to prove this in the same sort of way as the previous theorem,* but the proof is not so easy, since there is no simple expression for I_1 in terms of the coefficients a_n. So we adopt an entirely different method.†

Let $0 < r_1 < r_2 < r_3$, and let $k(\theta)$ and $F(z)$ be defined by

$$k(\theta) f(r_2 e^{i\theta}) = |f(r_2 e^{i\theta})| \qquad (0 \leqslant \theta \leqslant 2\pi),$$

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} f(ze^{i\theta}) k(\theta)\, d\theta.$$

Then $F(z)$ is regular for $|z| \leqslant r_3$, and attains its maximum in this circle on the boundary, say at $z = r_3 e^{i\lambda}$. Hence

$$I_1(r_2) = F(r_2) \leqslant |F(r_3 e^{i\lambda})| \leqslant I_1(r_3),$$

which proves the first part.

Now choose α so that

$$r_1^\alpha I_1(r_1) = r_3^\alpha I_1(r_3).$$

Then

$$r_2^\alpha I_1(r_2) = r_2^\alpha F(r_2) \leqslant \max_{r_1 \leqslant |z| \leqslant r_3} |z^\alpha F(z)| \leqslant r_1^\alpha I_1(r_1) = r_3^\alpha I_1(r_3),$$

and the result follows as in Hadamard's three-circles theorem.

5.5 Theorem of Borel and Carathéodory.‡ This result enables us to deduce an upper bound for the modulus of a function on a circle $|z| = r$, from bounds for its real or imaginary parts on a larger concentric circle $|z| = R$.

Let $f(z)$ be an analytic function regular for $|z| \leqslant R$, and let $M(r)$

* See Hardy (8), and Landau, *Ergebnisse der Funktionentheorie*, § 23.

† Pólya and Szegö, *Aufgaben*, Dritter Abschnitt, No. 308.

‡ See Borel, *Acta M.* 20, and Landau, *Ergebnisse*, § 24.

and $A(r)$ denote, as usual, the maxima of $|f(z)|$ and $\mathbf{R}\{f(z)\}$ on $|z|=r$. Then for $0<r<R$

$$M(r) \leqslant \frac{2r}{R-r}A(R)+\frac{R+r}{R-r}|f(0)|.$$

The result is obvious if $f(z)$ is a constant. If $f(z)$ is not constant, suppose first that $f(0)=0$. Then $A(R)>A(0)=0$.

Let
$$\phi(z)=\frac{f(z)}{2A(R)-f(z)}.$$
Then $\phi(z)$ is regular for $|z|\leqslant R$, since the real part of the denominator does not vanish; $\phi(0)=0$; and, if $f(z)=u+iv$,

$$|\phi(z)|^2=\frac{u^2+v^2}{\{2A(R)-u\}^2+v^2}\leqslant 1$$

since $-2A(R)+u\leqslant u\leqslant 2A(R)-u$. Hence Schwarz's lemma gives
$$|\phi(z)|\leqslant\frac{r}{R}.$$

Hence
$$|f(z)|=\left|\frac{2A(R)\phi(z)}{1+\phi(z)}\right|\leqslant\frac{2A(R)r}{R-r},$$
and the result stated follows.

If $f(0)$ is not zero, we apply the result already obtained to $f(z)-f(0)$. Thus

$$|f(z)-f(0)|\leqslant\frac{2r}{R-r}\max_{|z|=R}\mathbf{R}\{f(z)-f(0)\}\leqslant\frac{2r}{R-r}\{A(R)+|f(0)|\},$$

and the result again follows. If $A(R)\geqslant 0$, we deduce

$$M(r)\leqslant\frac{R+r}{R-r}\{A(R)+|f(0)|\}.$$

By arguing with $-f(z)$, or with $\pm if(z)$ we obtain similar results in which $A(r)$ is replaced by $\min\mathbf{R}\{f(z)\}$, $\max\mathbf{I}\{f(z)\}$, or $\min\mathbf{I}\{f(z)\}$.

The inequality is thus proved. The form of the right-hand side may be varied to a certain extent. It must, however, contain, besides $A(R)$, a term involving $f(0)$, or we could falsify the result by replacing $f(z)$ by $f(z)+ik$, where k is a sufficiently large real number. Also it must contain a factor, such as $1/(R-r)$, which tends to infinity as $r\to R$. To show this, consider the function $f(z)=-i\log(1-z)$, and let $0<r<R<1$. Then $A(R)<\frac{1}{2}\pi$, however near R is to 1; and $f(0)=0$. But $M(r)\to\infty$ as $r\to 1$.

5.51. The same principle can be extended to the derivatives of $f(z)$. *Under the conditions of the above theorem, with* $A(R) \geqslant 0$,

$$\max_{|z|=r} |f^{(n)}(z)| \leqslant \frac{2^{n+2} n! R}{(R-r)^{n+1}} \{A(R) + |f(0)|\}.$$

For
$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w-z)^{n+1}}\, dw, \tag{1}$$
where C is the circle with centre $w = z$ and radius $\delta = \frac{1}{2}(R-r)$. On this circle
$$|w| \leqslant r + \tfrac{1}{2}(R-r) = \tfrac{1}{2}(R+r),$$
so that Carathéodory's theorem gives
$$\max |f(w)| \leqslant \frac{R + \frac{1}{2}(R+r)}{R - \frac{1}{2}(R+r)} \{A(R) + |f(0)|\} < \frac{4R}{R-r} \{A(R) + |f(0)|\}.$$
Hence, by (1),
$$|f^{(n)}(z)| \leqslant \frac{n!}{\delta^n} \frac{4R}{(R-r)} \{A(R) + |f(0)|\} = \frac{2^{n+2} n! R}{(R-r)^{n+1}} \{A(R) + |f(0)|\}.$$

5.6. The theorems of Phragmén and Lindelöf.* The following important extension of the maximum-modulus theorem was given by Phragmén and Lindelöf:

Let C be a simple closed contour, and let $f(z)$ be regular inside and on C, except at one point P of C. Let $|f(z)| \leqslant M$ on C, except at P.

Suppose further that there is a function $\omega(z)$, regular and not zero in C, such that $|\omega(z)| \leqslant 1$ inside C, and such that, if ϵ is any given positive number, we can find a system of curves, arbitrarily near to P and connecting the two sides of C round P, on which
$$|\{\omega(z)\}^{\epsilon} f(z)| \leqslant M.$$
Then $|f(z)| \leqslant M$ at all points inside C.

To prove this, consider the function
$$F(z) = \{\omega(z)\}^{\epsilon} f(z),$$
which is regular in C. If z_0 is any point inside C, we can, by the hypothesis about $\omega(z)$, find a curve surrounding z_0 on which
$$|F(z)| \leqslant M.$$
Hence $\quad |F(z_0)| \leqslant M,$

and so $\quad |f(z_0)| \leqslant M |\omega(z_0)|^{-\epsilon}.$

Making $\epsilon \to 0$, $\quad |f(z_0)| \leqslant M.$

This proves the theorem.

* Phragmén and Lindelöf-(1).

It is not difficult to see that the exceptional point P may be replaced by any finite number, or even by an infinity, of points, provided that functions $\omega(z)$ corresponding to them with suitable properties can be found.

In the following sections we give a number of theorems of this type. Instead of actually using the above theorem, it is usually simpler to start again with a special auxiliary function adapted to the region considered. In practice the exceptional point P is always at infinity.

5.61. The above theorem gives many important results about the behaviour of a function in the neighbourhood of an essential singularity. By making a preliminary transformation, we can always suppose that the exceptional point is at infinity. The fundamental theorem then takes the following form:

Let $f(z)$ be an analytic function of $z = re^{i\theta}$, regular in the region D between two straight lines making an angle π/α at the origin, and on the lines themselves. Suppose that

$$|f(z)| \leqslant M \tag{1}$$

on the lines, and that, as $r \to \infty$,

$$f(z) = O(e^{r^\beta}), \tag{2}$$

where $\beta < \alpha$, uniformly in the angle. Then actually the inequality (1) *holds throughout the region D.*

We may suppose without loss of generality that the two lines are $\theta = \pm\frac{1}{2}\pi/\alpha$. Let

$$F(z) = e^{-\epsilon z^\gamma} f(z),$$

where $\beta < \gamma < \alpha$ and $\epsilon > 0$. Then

$$|F(z)| = e^{-\epsilon r^\gamma \cos\gamma\theta} |f(z)|. \tag{3}$$

On the lines $\theta = \pm\frac{1}{2}\pi/\alpha$, $\cos\gamma\theta > 0$, since $\gamma < \alpha$. Hence on these lines

$$|F(z)| \leqslant |f(z)| \leqslant M.$$

Also on the arc $|\theta| \leqslant \frac{1}{2}\pi/\alpha$ of the circle $|z| = R$,

$$|F(z)| \leqslant e^{-\epsilon R^\gamma \cos\frac{1}{2}\gamma\pi/\alpha} |f(z)| < A e^{R^\beta - \epsilon R^\gamma \cos\frac{1}{2}\gamma\pi/\alpha},$$

and the right-hand side tends to 0 as $R \to \infty$. Hence, if R is sufficiently large, $|F(z)| \leqslant M$ on this arc also. Hence, by the maximum-modulus theorem, $|F(z)| \leqslant M$ throughout the interior of the region $|\theta| \leqslant \frac{1}{2}\pi/\alpha$, $r \leqslant R$; i.e., since R is arbitrarily

large, throughout the region D. Hence, by (3),

$$|f(z)| \leqslant Me^{\epsilon r^{\gamma}}$$

in D; and making $\epsilon \to 0$ the result stated follows.

It is evidently unnecessary to suppose that the function $f(z)$ is regular in the region $|z| \leqslant r_0$, if there is an arc $|z| = r_1 > r_0$ on which (1) is satisfied. With this extension the theorem is significant for $\alpha < \frac{1}{2}$, the angle including part of the plane more than once, and the function not being necessarily one-valued. We can also replace the straight lines of the theorem by curves extending to infinity; the reader should have no difficulty in supplying the details of such extensions.

5.62. It is important to notice the relation between the 'angle' of the theorem, and the order of $f(z)$ at infinity. The wider the angle is, the smaller the order of $f(z)$ must be for the theorem to be true.

In the following theorem, the order is just not small enough for the previous proof to apply, and a more subtle argument is required.

The conclusion of the previous theorem still holds, if we are only given that

$$f(z) = O(e^{\delta r^{\alpha}})$$

for every positive δ, *uniformly in the angle.*

As before we take the angle to be $-\frac{1}{2}\pi/\alpha \leqslant \theta \leqslant \frac{1}{2}\pi/\alpha$. Let

$$F(z) = e^{-\epsilon z^{\alpha}} f(z).$$

Then $F(z)$ tends to zero on the real axis, and so has an upper bound M' on the real axis. Let

$$M'' = \max(M, M').$$

We may now apply the previous theorem to each of the two angles $(-\frac{1}{2}\pi/\alpha, 0)$ and $(0, \frac{1}{2}\pi/\alpha)$, and we thus find that

$$|F(z)| \leqslant M''$$

throughout the whole given angle.

But in fact $M' \leqslant M$; for $|F(z)|$ attains the value M' at a point of the real axis; hence, if $M' = M''$, $F(z)$ must reduce to a constant, and $M'' = M$. Otherwise $M' < M''$, so that $M'' = M$ in any case. It therefore follows that

$$|F(z)| \leqslant M.$$

Hence
$$|f(z)| \leqslant M|e^{-\epsilon z^\alpha}|,$$
and the result follows on making $\epsilon \to 0$.

5.63. *If $f(z) \to a$ as $z \to \infty$ along two straight lines, and $f(z)$ is regular and bounded in the angle between them, then $f(z) \to a$ uniformly in the whole angle.*

We may suppose without loss of generality that the limit a is 0. We may also suppose that the angle between the two lines is less than π, since the general case can be reduced to this by a substitution of the form $z = w^k$. We may thus suppose that the lines are $\theta = \pm\theta'$, where $\theta' < \frac{1}{2}\pi$.

Let
$$F(z) = \frac{z}{z+\lambda} f(z),$$
where $\lambda > 0$. Then
$$|F(z)| = \frac{r}{\sqrt{(r^2+2r\lambda\cos\theta+\lambda^2)}}|f(z)| < \frac{r}{\sqrt{(r^2+\lambda^2)}}|f(z)|.$$
Now $|f(z)| \leqslant M$, say, everywhere, and $|f(z)| < \epsilon$ for $r > r_1 = r_1(\epsilon)$ and $\theta = \pm\theta'$. Let $\lambda = r_1 M/\epsilon$. Then for $r \leqslant r_1$
$$|F(z)| < \frac{r}{\lambda} M < \epsilon$$
and $|F(z)| < |f(z)| < \epsilon$ for $r > r_1$ and $\theta = \pm\theta'$. Hence, by the main Phragmén-Lindelöf theorem, $|F(z)| \leqslant \epsilon$ in the whole region. Hence
$$|f(z)| \leqslant \left(1+\frac{\lambda}{r}\right)|F(z)| < 2\epsilon$$
if $r > \lambda$. This proves the theorem.

5.64. *If $f(z) \to a$ as $z \to \infty$ along a straight line, and $f(z) \to b$ as $z \to \infty$ along another straight line, and $f(z)$ is regular and bounded in the angle between, then $a = b$, and $f(z) \to a$ uniformly in the angle.*

Let $f(z) \to a$ along $\theta = \alpha$, and $f(z) \to b$ along $\theta = \beta$, where $\alpha < \beta$. The function
$$\{f(z) - \tfrac{1}{2}(a+b)\}^2$$
is regular and bounded in the angle, and tends to $\frac{1}{4}(a-b)^2$ on each of the straight lines. Hence it tends to this limit uniformly in the angle; that is,
$$\{f(z) - \tfrac{1}{2}(a+b)\}^2 - \tfrac{1}{4}(a-b)^2 = \{f(z)-a\}\{f(z)-b\}$$

tends uniformly to zero. Thus to any ϵ corresponds an arc on which

$$|\{f(z)-a\}\{f(z)-b\}| \leqslant \epsilon.$$

At every point of this arc either $|f(z)-a| \leqslant \sqrt{\epsilon}$ or $|f(z)-b| \leqslant \sqrt{\epsilon}$ (or both), and we may suppose that the former inequality holds at $\theta = \alpha$, the latter at $\theta = \beta$; let θ_0 be the upper bound of values of θ for which the former holds; then θ_0 is a limit of points where the former holds, and is either a point where the latter holds, or a limit of such points; hence, since $f(z)$ is continuous, both inequalities hold at θ_0. Taking z to be this point, we have

$$|a-b| \leqslant |f(z)-a|+|f(z)-b| \leqslant 2\sqrt{\epsilon},$$

and, making $\epsilon \to 0$, it follows that $a = b$. Finally $f(z) \to a$ uniformly, by the previous theorem.

These theorems have obvious affinities with Montel's theorem (§ 5.23). But in Montel's theorem the line along which the function tends to a limit must be interior to the region of boundedness, so that these theorems become corollaries of Montel's only if we assume a slightly wider region of boundedness.

5.65. The Phragmén-Lindelöf theorem for other regions. The angle of the above theorem may be transformed into other regions, for example into a strip.

Take, for example, the theorem of § 5.61, applied to the region $r \geqslant 1$, $|\theta| \leqslant \frac{1}{2}\pi/\alpha$, and put $s = i \log z$, $f(z) = \phi(s)$. If $s = \sigma + it$, the lines $\arg z = \pm\frac{1}{2}\pi/\alpha$ become parallel lines $\sigma = \pm\frac{1}{2}\pi/\alpha$, and $t = \log|z|$. Hence, *if* $|\phi(s)| \leqslant M$ *on the upper half of the two parallel lines and on the segment of the real axis joining them, while*

$$\phi(\sigma+it) = O(e^{e^{\rho t}}) \qquad (\rho < \alpha) \tag{1}$$

in the strip between them, then actually $|\phi(s)| \leqslant M$ *throughout the strip.*

Another theorem of this type, which we shall require in the theory of Dirichlet series, is as follows:

If $\phi(s)$ *is regular and* $O(e^{\epsilon|t|})$, *for every positive* ϵ, *in the strip* $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, *and*

$$\phi(\sigma_1+it) = O(|t|^{k_1}), \qquad \phi(\sigma_2+it) = O(|t|^{k_2}),$$

then

$$\phi(\sigma+it) = O(|t|^{k(\sigma)})$$

uniformly for $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, $k(\sigma)$ *being the linear function of* σ *which takes the values* k_1, k_2 *for* $\sigma = \sigma_1$, σ_2.

The result is true more generally if $\phi(s)$ satisfies a condition

of the form (1). With the given condition it may be proved directly as follows.

Suppose first that $k_1 = 0$, $k_2 = 0$, so that $\phi(s)$ is bounded for $\sigma = \sigma_1$, $\sigma = \sigma_2$. Let M be the upper bound of $\phi(s)$ on these two lines and on the segment of the real axis between σ_1 and σ_2. Let

$$g(s) = e^{\epsilon s i}\phi(s).$$

Then
$$|g(s)| = e^{-\epsilon t}|\phi(s)| \leqslant |\phi(s)| \leqslant M$$

for $\sigma = \sigma_1$, $\sigma = \sigma_2$. Also $|g(s)| \to 0$ as $t \to \infty$ for $\sigma_1 \leqslant \sigma \leqslant \sigma_2$; and so, if T is large enough, $|g(s)| \leqslant M$ on $t = T$, $\sigma_1 \leqslant \sigma \leqslant \sigma_2$. Hence $|g(s)| \leqslant M$ at all points of the rectangle (σ_1, σ_2), $(0, T)$. Hence $|g(s)| \leqslant M$ at all points in the half-strip, i.e.

$$|\phi(s)| \leqslant e^{\epsilon t}M.$$

Making $\epsilon \to 0$, it follows that $|\phi(s)| \leqslant M$ for $t > 0$, and similarly for $t < 0$. This proves the theorem in the particular case considered.

In the general case, let

$$\psi(s) = (-is)^{k(s)} = e^{k(s)\log(-is)},$$

where the logarithm has its principal value. This function is regular for $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, $t \geqslant 1$; also, if $k(s) = as+b$,

$$\begin{aligned}\mathbf{R}\{k(s)\log(-is)\} &= \mathbf{R}[\{k(\sigma)+iat\}\log(t-i\sigma)] \\ &= k(\sigma)\log t + O(1).\end{aligned}$$

Hence
$$|\psi(s)| = t^{k(\sigma)}e^{O(1)}.$$

The function $\Phi(s) = \phi(s)/\psi(s)$ therefore satisfies the same conditions as $\phi(s)$ did in the first part. Hence $\Phi(s)$ is bounded in the strip, and
$$\phi(s) = O\{|\psi(s)|\} = O(t^{k(\sigma)}).$$

5.7. The Phragmén-Lindelöf function $h(\theta)$. In several of the preceding theorems we have been considering the way in which a function behaves as z tends to infinity in different directions. We shall now make a more systematic study of this question.

Consider first the function

$$f(z) = e^{(a+ib)z^\rho}.$$

Then
$$|f(z)| = e^{r^\rho(a\cos\rho\theta - b\sin\rho\theta)}.$$

The behaviour of $\log|f(z)|$ depends in the first place on the factor r^ρ, which is independent of θ. The different behaviour

in different directions is determined by the factor

$$h(\theta) = a\cos\rho\theta - b\sin\rho\theta = r^{-\rho}\log|f(z)|.$$

This is of course a very special case; but the general case is not so different from it as might be expected.

We shall suppose throughout the following sections that $f(z)$ is regular for $\alpha < \theta < \beta$, $|z| \geqslant r_0$, and that $f(z)$ is 'of order ρ' in this angle, i.e. that

$$\overline{\lim}\,\frac{\log|f(re^{i\theta})|}{r^{\rho+\epsilon}} = 0$$

uniformly in θ, for every positive value of ϵ, but not for any negative value. (For example, the above function is of order ρ.)

We define $h(\theta)$ in general as

$$h(\theta) = \overline{\lim_{r\to\infty}}\,\frac{\log|f(re^{i\theta})|}{V(r)},$$

where $V(r)$ depends on the function considered. We should naturally choose $V(r)$ so that $h(\theta)$ is finite and not identically zero. Here we shall consider the simplest case $V(r) = r^{\rho}$; but our argument would apply almost unchanged to any function such as

$$r^{\rho}(\log r)^{p}(\log\log r)^{q}\ldots.$$

5.701. It is convenient to introduce at this point an expression containing the word 'infinity', or the symbol ∞, which is not used in elementary analysis. We shall use $\lim\phi_n = \infty$ to mean the same thing as $\phi_n \to \infty$; and we shall say that $\phi(x)$ has an infinite value, or $\phi(x) = \infty$, if, and only if, $\phi(x)$ is defined as the limit of a sequence $\phi_n(x)$, and the sequence diverges to infinity for the particular value of x in question. We use $-\infty$ in the same way. For example, we might write

$$\int_0^1 \frac{dt}{t} = \infty$$

if the left-hand side is defined as $\lim\limits_{\epsilon\to 0}\int_{\epsilon}^{1}$; and $h(\theta) = \infty$ means that $r^{-\rho}\log|f(re^{i\theta})|$ takes arbitrarily large values as $r \to \infty$.

The novelty consists in writing '$= \infty$', as if we had defined

a number '∞'; but it should be remembered that we have not done so, and that 'infinity' remains an incomplete symbol.*

5.71. *Let $\alpha < \theta_1 < \theta_2 < \beta$, and $\theta_2 - \theta_1 < \pi/\rho$, and let*

$$h(\theta_1) \leqslant h_1, \qquad h(\theta_2) \leqslant h_2.$$

Let $H(\theta)$ be the function of the form $a\cos\rho\theta + b\sin\rho\theta$ which takes the values h_1, h_2 at θ_1, θ_2. Then

$$h(\theta) \leqslant H(\theta) \qquad (\theta_1 \leqslant \theta \leqslant \theta_2).$$

It is easily seen that

$$H(\theta) = \frac{h_1 \sin\rho(\theta_2 - \theta) + h_2 \sin\rho(\theta - \theta_1)}{\sin\rho(\theta_2 - \theta_1)},$$

but we do not require this expression in the proof.

Let $$H_\delta(\theta) = a_\delta \cos\rho\theta + b_\delta \sin\rho\theta$$

be the H-function which is equal to $h_1 + \delta$, $h_2 + \delta$ $(\delta > 0)$ for $\theta = \theta_1$, $\theta = \theta_2$ respectively. Let

$$F(z) = f(z) e^{-(a_\delta - ib_\delta) z^\rho}.$$

Then $$|F(z)| = |f(z)| e^{-H_\delta(\theta) r^\rho}, \tag{1}$$

and so, if r is large enough,

$$|F(re^{i\theta_1})| = O(e^{(h_1+\delta) r^\rho - H_\delta(\theta_1) r^\rho}) = O(1).$$

A similar result holds for $F(re^{i\theta_2})$. Hence, by the theorem of § 5.61, $F(z)$ is bounded in the angle (θ_1, θ_2). Hence, by (1),

$$f(z) = O(e^{H_\delta(\theta) r^\rho}) \tag{2}$$

uniformly in the angle. Hence $h(\theta) \leqslant H_\delta(\theta)$ for $\theta_1 \leqslant \theta \leqslant \theta_2$. Since $H_\delta(\theta) \to H(\theta)$ as $\delta \to 0$, the result follows.

5.711. As a particular case of the above theorem, one or both of $h(\theta_1)$, $h(\theta_2)$, may be $-\infty$. The conclusion is then that $h(\theta) = -\infty$ for $\theta_1 < \theta < \theta_2$. The same proof still applies, one or both of the numbers h_1, h_2 now being arbitrarily large and negative.

5.712. *If $\alpha < \theta_1 < \theta_2 < \theta_3 < \beta$, $\theta_2 - \theta_1 < \pi/\rho$, $\theta_3 - \theta_2 < \pi/\rho$; and $h(\theta_1)$, $h(\theta_2)$ are finite, and $H(\theta)$ is an H-function such that*

$$h(\theta_1) \leqslant H(\theta_1), \qquad h(\theta_2) = H(\theta_2),$$

then $$h(\theta_3) \geqslant H(\theta_3). \tag{1}$$

* *P.M.* § 55.

Choose θ_1' so that $\theta_3-\pi/\rho < \theta_1' < \theta_2$. Then $h(\theta_1') \leqslant H(\theta_1')$ by § 5.71. Hence, by § 5.711, $h(\theta_3)$ is not $-\infty$. If (1) is false, we can choose δ so that $h(\theta_3) \leqslant H(\theta_3)-\delta$. Let

$$H_\delta(\theta) = H(\theta)-\delta\frac{\sin\rho(\theta-\theta_1')}{\sin\rho(\theta_3-\theta_1')}.$$

Then

$$h(\theta_1') \leqslant H(\theta_1') = H_\delta(\theta_1'), \qquad h(\theta_3) \leqslant H(\theta_3-\delta) = H_\delta(\theta_3).$$

Hence
$$h(\theta_2) \leqslant H_\delta(\theta_2) < H(\theta_2),$$

contrary to hypothesis.

5.713. *If* $\theta_1 < \theta_2 < \theta_3$, $\theta_2-\theta_1 < \pi/\rho$, $\theta_3-\theta_2 < \pi/\rho$, *then*

$$h(\theta_1)\sin\rho(\theta_3-\theta_2)+h(\theta_2)\sin\rho(\theta_1-\theta_3)+h(\theta_3)\sin\rho(\theta_2-\theta_1) \geqslant 0.$$

For any $H(\theta)$

$$H(\theta_1)\sin\rho(\theta_3-\theta_2)+H(\theta_2)\sin\rho(\theta_1-\theta_3)+H(\theta_3)\sin\rho(\theta_2-\theta_1) = 0,$$

and choosing $H(\theta)$ so that $H(\theta_1) = h(\theta_1)$, $H(\theta_2) = h(\theta_2)$, and observing that, by the above theorem, $h(\theta_3) \geqslant H(\theta_3)$, we have the result stated.

The function $h(\theta)$ is continuous in any interval where it is finite.

Let $h(\theta)$ be finite in the interval $\theta_1 \leqslant \theta \leqslant \theta_3$, and let $\theta_1 < \theta_2 < \theta_3$. Let $H_{1,2}(\theta)$ be the h-function which takes the values $h(\theta_1)$, $h(\theta_2)$ at θ_1, θ_2; and define $H_{2,3}(\theta)$ similarly. Then by the above theorems

$$H_{2,3}(\theta) \leqslant h(\theta) \leqslant H_{1,2}(\theta) \qquad (\theta_1 \leqslant \theta \leqslant \theta_2)$$
$$H_{1,2}(\theta) \leqslant h(\theta) \leqslant H_{2,3}(\theta) \qquad (\theta_2 \leqslant \theta \leqslant \theta_3).$$

Hence, in whichever of these intervals θ lies,

$$\frac{H_{1,2}(\theta)-H_{1,2}(\theta_2)}{\theta-\theta_2} \leqslant \frac{h(\theta)-h(\theta_2)}{\theta-\theta_2} \leqslant \frac{H_{2,3}(\theta)-H_{2,3}(\theta_2)}{\theta-\theta_2}.$$

The extreme terms tend to limits as $\theta \to \theta_2$; hence the middle term is bounded, and so $h(\theta) \to h(\theta_2)$.

It also follows that $|f(re^{i\theta})| < \exp[r^\rho\{h(\theta)+\epsilon\}]$ uniformly for $r > r_0(\epsilon)$; (divide the θ-range into $n = n(\epsilon)$ parts).

5.72. Geometrical interpretation of the property of $h(\theta)$. In the case $\rho = 1$, the property of the function $h(\theta)$ has a simple geometrical interpretation.

For every value of θ in an interval where $h(\theta)$ is finite and positive, consider the radius vector of length $h(\theta)$ making an angle θ with an initial line, and the perpendicular to this radius

vector at its end. (Consider, for example, the cases $f(z) = \cosh z$, $f(z) = \cos z + \cosh z$.)

Let h_1, h_2, h_3 be the values of $h(\theta)$ at θ_1, θ_2, θ_3, where $\theta_1 < \theta_2 < \theta_3$. Then the three perpendiculars are

$$x\cos\theta_1 + y\sin\theta_1 = h_1,$$
$$x\cos\theta_2 + y\sin\theta_2 = h_2,$$
$$x\cos\theta_3 + y\sin\theta_3 = h_3.$$

The first and third meet at a point (X, Y) given by

$$X = \frac{h_1\sin\theta_3 - h_3\sin\theta_1}{\sin(\theta_3-\theta_1)}, \qquad Y = \frac{h_3\cos\theta_1 - h_1\cos\theta_3}{\sin(\theta_3-\theta_1)}.$$

Now the condition that (X, Y) should lie on the opposite side from the origin of the second perpendicular (or on it) is

$$X\cos\theta_2 + Y\sin\theta_2 - h_2 \geqslant 0,$$

or

$$(h_1\sin\theta_3 - h_3\sin\theta_1)\cos\theta_2 + \\ + (h_3\cos\theta_1 - h_1\cos\theta_3)\sin\theta_2 - h_2\sin(\theta_3-\theta_1) \geqslant 0,$$

or $$h_1\sin(\theta_3-\theta_2) + h_2\sin(\theta_1-\theta_3) + h_3\sin(\theta_2-\theta_1) \geqslant 0.$$

This is precisely the condition which the function $h(\theta)$ satisfies.

If the perpendiculars envelope a curve, then two tangents to it meet on the opposite side to the origin of any tangent at a point between them. It is easily seen geometrically that this means that *the curve is always concave to the origin.*

5.8. The following interesting applications of the Phragmén-Lindelöf principle are due to Carlson.*

Let $f(z)$ be regular and of the form $O(e^{k|z|})$ for $\mathbf{R}(z) \geqslant 0$; and let $f(z) = O(e^{-a|z|})$, where $a > 0$, on the imaginary axis. Then $f(z) = 0$ identically.

We apply the argument of § 5.71 to $f(z)$, with $\rho = 1$, $\theta_1 = 0$, $\theta_2 = \frac{1}{2}\pi$, $h_1 = k$, $h_2 = -a$; and here we can take $\delta = 0$ throughout the argument. Then § 5.71 (2) gives

$$f(z) = O\{e^{(k\cos\theta - a|\sin\theta|)r}\} \tag{1}$$

for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$; and a similar argument shows that (1) also holds for $-\frac{1}{2}\pi \leqslant \theta \leqslant 0$.

Let $$F(z) = e^{\omega z} f(z)$$

where ω is a (large) positive number. Then by (1) there is a

* In an Upsala thesis (1914). See M. Riesz (1), Hardy (14).

constant M, independent of ω, such that

$$|F(z)| \leqslant Me^{\{(k+\omega)\cos\theta - a|\sin\theta|\}r} \qquad (-\tfrac{1}{2}\pi \leqslant \theta \leqslant \tfrac{1}{2}\pi). \tag{2}$$

In particular we have

$$|F(z)| \leqslant M \tag{3}$$

for $\theta = \pm\frac{1}{2}\pi$ and $\theta = \pm\alpha$, where $\alpha = \arctan\{k+\omega)/a\}$.

We can now apply the theorem of § 5.61 to each of the three angles $(-\frac{1}{2}\pi, -\alpha)$, $(-\alpha, \alpha)$, and $(\alpha, \frac{1}{2}\pi)$. It follows that (3) actually holds for $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$.

Hence $|f(z)| \leqslant Me^{-\omega r\cos\theta}$, and making $\omega \to \infty$ it follows that $|f(z)| = 0$. This proves the theorem.

5.81. *If $f(z)$ is regular and of the form $O(e^{k|z|})$, where $k < \pi$, for $\mathbf{R}(z) \geqslant 0$, and $f(z) = 0$ for $z = 0, 1, 2, 3, \ldots$, then $f(z) = 0$ identically.*

Consider the function $F(z) = f(z)\operatorname{cosec}\pi z$. On the circles $|z| = n+\frac{1}{2}$, $\operatorname{cosec}\pi z$ is bounded. Hence $F(z) = O(e^{k|z|})$ on these circles, and also on the imaginary axis. Since $F(z)$ is regular it follows that, if $n-\frac{1}{2} < |z| < n+\frac{1}{2}$,

$$F(z) = O(e^{k(n+\frac{1}{2})}) = O(e^{k|z|}),$$

and so $F(z)$ is of this form throughout $\mathbf{R}(z) \geqslant 0$. Also

$$F(z) = O(e^{(k-\pi)|z|})$$

on the imaginary axis. The result therefore follows from the previous theorem.

MISCELLANEOUS EXAMPLES

1. A function $f(z)$ is regular inside and on a simple closed contour C, and $|f(z)| \leqslant M$ on C. Deduce from Cauchy's integral for $\{f(z)\}^n$ that, if z is inside C,

$$|f(z)|^n \leqslant KM^n,$$

where K is independent of n. Hence show that $|f(z)| \leqslant M$ inside C. [Landau.]

2. Use Poisson's integral to show that a function which is harmonic in a region cannot have a maximum at an interior point of the region.

3. If $f(z)$ is regular and $O(e^{r^{1-\epsilon}})$ for $\mathbf{R}(z) \geqslant 0$, $|f(z)| \leqslant M$ on the imaginary axis, and $f(1) = 0$, then for $x > 0$

$$|f(x+iy)| \leqslant \left\{\frac{(1-x)^2+y^2}{(1+x)^2+y^2}\right\}^{\frac{1}{2}} M.$$

[Consider $(1+z)/(1-z).f(z)$].

4. A function $f(z)$ is regular and satisfies the inequalities

$$e^{-r^{\rho+\epsilon}} < |f(z)| < e^{r^{\rho+\epsilon}}$$

in an angle $\theta_1 \leqslant \theta \leqslant \theta_2$, where $\theta_2 - \theta_1 < \pi/\rho$. As $r \to \infty$, $r^{-\rho}\log|f(z)|$ tends to the limits h_1 and h_2 for $\theta = \theta_1, \theta_2$. Let $H(\theta)$ be the function

of the form $a \cos \rho\theta + b \sin \rho\theta$ which takes the values h_1, h_2 for $\theta = \theta_1$, θ_2. Then $h(\theta) = H(\theta)$ throughout the interval $\theta_1 \leqslant \theta \leqslant \theta_2$.

5. Show that, if $f(z) = O(e^{r^{1+\epsilon}})$ in a given angle, the function

$$h(\theta) = \overline{\lim} \frac{\log|f(re^{i\theta})|}{r \log r}$$

has properties similar to those of the h-functions considered in the text.

Show that if $f(z) = 1/\Gamma(\frac{1}{2}+z)$, then $h(\theta) = -\cos\theta$ for all values of θ.

6. An analytic function $f(z)$ is regular and not zero in the half-strip defined by $a < x < b$, $y > 0$; $f(z) = O(y^A)$ as $y \to \infty$ uniformly in the strip, and $|\log f(z)|$ is bounded on the middle line $x = \frac{1}{2}(a+b)$. Prove that $\log f(z) = O(\log y)$ uniformly for $a+\delta < x < b-\delta$.

[Apply Carathéodory's theorem to $\log f(z)$ in a circle with centre at $\frac{1}{2}(a+b)+iy$.]

7. A function $f(z)$ is regular, and $|f(z)| \leqslant M$, for $\mathbf{R}(z) \geqslant 0$, and $f(z)$ has zeros at $z_1, z_2, \ldots$ in this half-plane. Prove that

$$|f(z)| \leqslant \left|\frac{z_1-z}{\bar{z}_1+z}\,\frac{z_2-z}{\bar{z}_2+z} \cdots \frac{z_n-z}{\bar{z}_n+z}\right| M$$

for $\mathbf{R}(z) > 0$; and deduce that, if $f(z)$ is not identically zero, the series

$$\sum \mathbf{R}\left(\frac{1}{z_n}\right)$$

is convergent. [See Pólya and Szegö, Absch. III, Nos. 295, 298.]

CHAPTER VI

CONFORMAL REPRESENTATION

6.1. Conformal representation. If w is an analytic function of z, then to values of z, which we represent as points in the z-plane, correspond values of w, which we represent as points in the w-plane. We also speak of the point in the w-plane representing its corresponding point in the z-plane; and of regions of the z-plane being represented, or mapped, on corresponding regions of the w-plane. The object of this chapter is to discuss in more detail the nature of this representation or mapping.

Let $w = f(z)$ be an analytic function of z, regular and one-valued in a region D of the z-plane. Let z_0 be an interior point of D; and let C_1 and C_2 be two continuous curves passing through z_0, and having definite tangents at this point, making angles α_1, α_2, say, with the real axis.

We have to discover what is the representation of this figure in the w-plane. Before we go any further, we shall make a restriction, the reason for which will appear in a moment. *We shall suppose that $f'(z_0)$ is not zero.*

Let z_1 and z_2 be points of the curves C_1 and C_2 near to z_0. We shall suppose that they are at the same distance r from z_0, so that we can write

$$z_1 - z_0 = re^{i\theta_1}, \qquad z_2 - z_0 = re^{i\theta_2}.$$

Then as $r \to 0$, $\theta_1 \to \alpha_1$, and $\theta_2 \to \alpha_2$.

The point z_0 corresponds to a point w_0 in the w-plane, and z_1 and z_2 correspond to points w_1 and w_2 which describe curves C_1' and C_2'. Let

$$w_1 - w_0 = \rho_1 e^{i\phi_1}, \qquad w_2 - w_0 = \rho_2 e^{i\phi_2}.$$

Then, by the definition of an analytic function,

$$\lim \frac{w_1 - w_0}{z_1 - z_0} = f'(z_0).$$

Since $f'(z_0)$ is not zero, we may write it in the form $Re^{i\delta}$. Then

$$\lim \frac{\rho_1 e^{i\phi_1}}{re^{i\theta_1}} = Re^{i\delta}.$$

Hence $\lim(\phi_1 - \theta_1) = \delta$, i.e.

$$\lim \phi_1 = \alpha_1 + \delta.$$

Hence the curve C_1' has a definite tangent at w_0, making an angle $\alpha_1+\delta$ with the real axis.

Similarly C_2' has a definite tangent at w_0, making an angle $\alpha_2+\delta$ with the real axis.

Hence the curves C_1', C_2' intersect at the same angle as the curves C_1, C_2. Also the angle between the curves has the same sense in the two figures.

Because of this property of the conservation of angles, an analytic representation is called 'conformal'. Any small figure in one plane corresponds to an approximately similar figure in the other plane, since all angles are approximately the same. To obtain one figure from the other we must rotate it through a certain angle—the angle $\delta = \arg\{f'(z_0)\}$ of the above notation —and subject it to a certain magnification, viz.

$$\lim \frac{\rho_1}{r} = R = |f'(z_0)|.$$

It is clear from the above analysis that the magnification is the same in all directions.

6.11. The case $f'(z) = 0$. Suppose now that $f'(z)$ has a zero of order n at the point z_0. Then in the neighbourhood of this point

$$f(z) = f(z_0)+a(z-z_0)^{n+1}+\dots$$

where $a \neq 0$. Hence

$$w_1-w_0 = a(z_1-z_0)^{n+1}+\dots,$$

i.e.

$$\rho_1 e^{i\phi_1} = |a|r^{n+1}e^{i\{\delta+(n+1)\theta_1\}}+\dots,$$

where $\delta = \arg a$. Hence

$$\begin{aligned}\lim \phi_1 &= \lim\{\delta+(n+1)\theta_1\}\\ &= \delta+(n+1)\alpha_1.\end{aligned}$$

Similarly

$$\lim \phi_2 = \delta+(n+1)\alpha_2.$$

Thus the curves C_1', C_2' still have definite tangents at w_0, but the angle between the tangents is

$$\lim(\phi_2-\phi_1) = (n+1)(\alpha_2-\alpha_1).$$

Also the linear magnification, $\lim \rho_1/r$, is zero. The conformal property therefore does not hold at such a point.

6.12. In the above conformal representations we have, not merely conservation of angles, but conservation of the sign of angles; if we get from C_1 to C_2 by a rotation through an angle

α in the positive sense, we also get from C_1' to C_2' by a rotation through α in the positive sense.

There are also conformal representations in which the magnitude of angles is conserved, but their sign is changed. Consider for example, the transformation

$$w = \bar{z},$$

where $\bar{z}$ is the complex number conjugate to z. This replaces every point by its reflection in the real axis. Hence angles are conserved, but their signs are changed. And this is true generally for every transformation of the form

$$w = f(\bar{z}),$$

where $f(z)$ is an analytic function of z; for this is the product of two transformations:

$$\text{(i)}\ Z = \bar{z}, \qquad \text{(ii)}\ w = f(Z).$$

In (i) angles are conserved, their signs changed. In (ii) angles and signs are conserved. Hence in the resulting transformation angles are conserved and their signs changed.

6.2. Linear* transformation. The function

$$w = \frac{az+b}{cz+d}$$

is called a linear function of z. We shall suppose that

$$ad - bc \neq 0,$$

for otherwise the numerator and denominator are proportional, and w is merely a constant.

To every value of z corresponds just one value of w. This is apparent except, in the case $c \neq 0$, for the value $z = -d/c$, which makes the denominator vanish. But as $z \to -d/c$, $|w| \to \infty$; and we may regard the point at infinity in the w-plane as corresponding to the point $z = -d/c$ in the z-plane.

If $c = 0$, then $$w = \frac{a}{d}z + \frac{b}{d}$$

and (since $a \neq 0$) the points at infinity in the two planes correspond.

Conversely $$z = \frac{dw-b}{-cw+a},$$

so that z is a linear function of w.

* Or bilinear.

Example. Prove that in general there are two values of z ('invariant points') for which $w = z$, but that there is one only if

$$(a-d)^2+4bc = 0.$$

Show that, if there are distinct invariant points p and q, the transformation may be put in the form

$$\frac{w-p}{w-q} = k\frac{z-p}{z-q};$$

and that, if there is only one invariant point p, the transformation may be put in the form

$$\frac{1}{w-p} = \frac{1}{z-p}+k.$$

6.21. Circles. The equation

$$|z-z_0| = \rho$$

represents a circle with centre z_0 and radius ρ.

Two points p, q are said to be inverse with respect to the circle if they are collinear with the centre and on the same side of it, and if the product of their distances from the centre is equal to ρ^2. Thus, if

$$p = z_0+le^{i\lambda},$$

then

$$q = z_0+\frac{\rho^2}{l}e^{i\lambda}.$$

If

$$z = z_0+\rho e^{i\theta}$$

is any point of the circle, then

$$\left|\frac{z-p}{z-q}\right| = \left|\frac{\rho e^{i\theta}-le^{i\lambda}}{\rho e^{i\theta}-\rho^2 l^{-1}e^{i\lambda}}\right| = \frac{l}{\rho}\left|\frac{\rho e^{i\theta}-le^{i\lambda}}{le^{i\theta}-\rho e^{i\lambda}}\right| = \frac{l}{\rho}.$$

This is therefore a new form of the equation of the circle.

Conversely, any equation

$$\left|\frac{z-p}{z-q}\right| = k \qquad (k \neq 1)$$

represents a circle* with respect to which p and q are inverse points. For the equation gives

$$|z|^2-2\mathbf{R}(\bar{p}z)+|p|^2 = k^2\{|z|^2-2\mathbf{R}(\bar{q}z)+|q|^2\},$$

or

$$|z|^2-2\frac{\mathbf{R}\{(\bar{p}-k^2\bar{q})z\}}{1-k^2}+\frac{|p|^2-k^2|q|^2}{1-k^2} = 0,$$

or

$$\left|z-\frac{p-k^2q}{1-k^2}\right|^2 = \frac{|p-k^2q|^2}{(1-k^2)^2}-\frac{|p|^2-k^2|q|^2}{1-k^2}.$$

Since

$$|p-k^2q|^2-(1-k^2)(|p|^2-k^2|q|^2) = k^2|p-q|^2,$$

* The 'circle of Apollonius'.

as is easily verified, we obtain

$$\left|z-\frac{p-k^2q}{1-k^2}\right|=\frac{k|p-q|}{|1-k^2|}.$$

The equation therefore represents a circle, with centre

$$z_0=\frac{p-k^2q}{1-k^2},$$

and radius

$$\rho=\frac{k|p-q|}{|1-k^2|}.$$

Also

$$p-z_0=\frac{k^2(q-p)}{1-k^2},\qquad q-z_0=\frac{q-p}{1-k^2},$$

so that $(p-z_0)/(q-z_0)$ is real and positive, and

$$|p-z_0||q-z_0|=\rho^2.$$

Hence p and q are inverse points.

In the particular case $k=1$, z is equidistant from the points p and q, and therefore lies on the perpendicular bisector of the line joining them.

6.22. Linear transformation of a circle. *In a linear transformation, a circle transforms into a circle, and inverse points transform into inverse points. In the particular case in which the circle becomes a straight line, inverse points become points symmetrical about the line.*

For let

$$\left|\frac{z-p}{z-q}\right|=k$$

be a circle (or straight line), with p and q as inverse points. Let

$$w=\frac{az+b}{cz+d},\qquad z=\frac{dw-b}{-cw+a}.$$

Then the circle transforms into

$$\left|\frac{dw-b-p(-cw+a)}{dw-b-q(-cw+a)}\right|=k$$

or

$$\left|\frac{w-\dfrac{ap+b}{cp+d}}{w-\dfrac{aq+b}{cq+d}}\right|=k\left|\frac{cq+d}{cp+d}\right|.$$

The result is obvious from this equation.

Example. Prove that the linear transformation in which only one point p is invariant may be considered as the result of (i) an inversion

in a circle (centre z_0, say), through the point p, (ii) an inversion in the circle with centre w_0 corresponding to z_0 in the transformation, and touching the previous circle at p.

6.23. *To find all linear transformations of the half-plane* $\mathbf{I}(z) \geqslant 0$ *into the unit circle* $|w| \leqslant 1$.

To points z, $\bar{z}$, symmetrical about the real z-axis correspond points w, $\dfrac{1}{\bar{w}}$, inverse with respect to the unit w-circle. In particular, the origin and the point at infinity in the w-plane correspond to conjugate values of z.

Let
$$w = \frac{az+b}{cz+d}$$
be the required transformation. Plainly $c \neq 0$, or the points at infinity would correspond. Now $w = 0$, $w = \infty$ correspond to $z = -b/a$, $-d/c$. Hence we may write
$$-\frac{b}{a} = \alpha, \qquad -\frac{d}{c} = \bar{\alpha},$$
and
$$w = \frac{a}{c}\,\frac{z-\alpha}{z-\bar{\alpha}}.$$
The point $z = 0$ must correspond to a point of the circle $|w| = 1$, so that
$$\left|\frac{a}{c}\cdot\frac{-\alpha}{-\bar{\alpha}}\right| = \left|\frac{a}{c}\right| = 1.$$
Hence we put
$$a = ce^{i\lambda},$$
where λ is real, and obtain
$$w = e^{i\lambda}\,\frac{z-\alpha}{z-\bar{\alpha}}. \tag{1}$$
Since $z = \alpha$ gives $w = 0$, α must be a point of the upper half-plane, i.e. $\mathbf{I}(\alpha) > 0$. With this condition the function (1) gives the required representation. For if z is real, obviously $|w| = 1$; and if $\mathbf{I}(z) > 0$, then z is nearer to α than to $\bar{\alpha}$, and $|w| < 1$.

There are three arbitrary constants in the transformation, λ, $\mathbf{R}(\alpha)$, and $\mathbf{I}(\alpha)$. We can therefore make three given points of the real axis correspond to three given points of the unit circle.

Example. The general linear transformation of the half-plane $\mathbf{R}(z) \geqslant 0$ on the circle $|w| \leqslant 1$ is
$$w = e^{i\lambda}\,\frac{z-\alpha}{z+\bar{\alpha}} \qquad (\mathbf{R}(\alpha) > 0).$$

6.24. *To find all linear transformations of the unit circle* $|z| \leqslant 1$ *into the unit circle* $|w| \leqslant 1$.

Let

$$w = \frac{az+b}{cz+d}.$$

Here $w=0$, $w=\infty$, must correspond to inverse points $z=\alpha$, $z=1/\bar{\alpha}$, where $|\alpha|<1$. Hence

$$-\frac{b}{a}=\alpha, \qquad -\frac{d}{c}=\frac{1}{\bar{\alpha}},$$

$$w=\frac{a}{c}\,\frac{z-\alpha}{z-1/\bar{\alpha}}=\frac{a\bar{\alpha}}{c}\cdot\frac{z-\alpha}{\bar{\alpha}z-1}.$$

The point $z=1$ corresponds to a point on $|w|=1$. Hence

$$\left|\frac{a\bar{\alpha}}{c}\cdot\frac{1-\alpha}{\bar{\alpha}-1}\right|=\left|\frac{a\bar{\alpha}}{c}\right|=1.$$

Hence
$$w=e^{i\lambda}\,\frac{z-\alpha}{\bar{\alpha}z-1},$$

where λ is real.

This is the required result; for if $z=e^{i\theta}$, $\alpha=be^{i\beta}$, then

$$|w|=\left|\frac{e^{i\theta}-be^{i\beta}}{be^{i(\theta-\beta)}-1}\right|=1.$$

If $z=re^{i\theta}$, where $r<1$, then

$$\begin{aligned}|z-\alpha|^2-|\bar{\alpha}z-1|^2 &= r^2-2rb\cos(\theta-\beta)+b^2-\{b^2r^2-2br\cos(\theta-\beta)+1\}\\ &= (r^2-1)(1-b^2)<0.\end{aligned}$$

Hence $|w|<1$.

If we are also given that $z=0$ corresponds to $w=0$, then $\alpha=0$, and the transformation becomes

$$w=e^{i\lambda}z.$$

If also $\dfrac{dw}{dz}=1$ at $z=0$, then

$$w=z.$$

Example. *The general linear transformation of the circle* $|z| \leqslant \rho$ *into the circle* $|w| \leqslant \rho'$ *is*

$$w=\rho\rho' e^{i\lambda}\,\frac{z-\alpha}{\bar{\alpha}z-\rho^2} \qquad (|\alpha|<\rho).$$

6.25. *If* $f(z)$ *is regular for* $|z|<1$, $\mathbf{R}\{f(z)\}>0$, *and* $f(0)=a>0$, *then* $|f'(0)| \leqslant 2a$.

A result of this type follows from Carathéodory's theorem

and its corollary (§§ 5.5–5.51). The following argument is essentially the same, but can now be put in a form which throws some light on the general method.

Suppose that we can find a linear transformation $g=\phi(f)$ such that $\mathbf{R}(f)=0$ corresponds to $|g|=1$, while $f=a$ corresponds to $g=0$. Then we shall have $|g(z)|<1$ for $\mathbf{R}\{f(z)\}>0$, i.e. for $|z|<1$, and $g(0)=0$. In this form the data are easy to deal with. We have

$$|g'(0)|=\left|\frac{1}{2\pi i}\int\limits_{|z|=1-\epsilon}\frac{g(z)}{z^2}\,dz\right|\leqslant\frac{1}{1-\epsilon},$$

and hence, making $\epsilon\to 0$, $|g'(0)|\leqslant 1$.

We find, as in § 6.23, that the required linear transformation is

$$g(z)=\frac{f(z)-a}{f(z)+a},$$

or

$$f(z)=a\frac{1+g(z)}{1-g(z)}.$$

Hence

$$f'(z)=\frac{2ag'(z)}{\{1-g(z)\}^2},$$

and

$$|f'(0)|=2a|g'(0)|\leqslant 2a.$$

6.3. Various transformations. We shall now consider some examples of functions which are not linear.

6.31. The function $w=z^2$. If $z=re^{i\theta}$ and $w=\rho e^{i\phi}$, then $\rho e^{i\phi}=r^2e^{2i\theta}$, so that

$$\rho=r^2,\qquad \phi=2\theta.$$

The distance from the origin is therefore squared, and the polar angle is doubled. An angular region $\alpha<\arg z<\beta$ is represented on an angular region $2\alpha<\arg w<2\beta$; if $\beta-\alpha>\pi$, the angular region in the w-plane covers part of this plane twice. The ambiguity arising from this is removed if we replace the w-plane by the Riemann surface described in § 4.3.

If $z=x+iy$, $w=u+iv$, then

$$u+iv=(x+iy)^2=x^2-y^2+2ixy,$$

or

$$u=x^2-y^2,\qquad v=2xy.$$

Hence the straight lines $u=a$, $v=b$ correspond to the rectangular hyperbolas

$$x^2-y^2=a,\qquad 2xy=b.$$

These cut at right angles except in the case $a=0$, $b=0$, when

they intersect at the origin at an angle $\frac{1}{4}\pi$. Since dw/dz has a simple zero at the origin, this is in accordance with the general theorems on the transformation of angles.

Examples. (i) Prove that the straight lines x = const., y = const., correspond to systems of confocal parabolas.

(ii) Consider in the same way the function $w = z^n$ for $n = 3, 4, \ldots$.

6.32. The function $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$. Here w becomes infinite at $z = 0$, while

$$\frac{dw}{dz} = \frac{1}{2}\left(1 - \frac{1}{z^2}\right),$$

which vanishes at $z = \pm 1$. These points may therefore be expected to play a special part in the transformation.

Putting $z = re^{i\theta}$, $w = u + iv$, we have

$$u = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos\theta, \qquad v = \frac{1}{2}\left(r - \frac{1}{r}\right)\sin\theta,$$

and, eliminating θ,

$$\frac{u^2}{\frac{1}{4}\left(r + \frac{1}{r}\right)^2} + \frac{v^2}{\frac{1}{4}\left(r - \frac{1}{r}\right)^2} = 1.$$

This is an ellipse in the w-plane, and it corresponds to each of the two circles $|z| = r$, $|z| = \frac{1}{r}$. As $r \to 1$, the major semi-axis of the ellipse tends to 1, while the minor semi-axis tends to zero. As $r \to 0$ or as $r \to \infty$, both semi-axes tend to infinity. From this it is clear that the inside and the outside of the unit circle in the z-plane both correspond to the whole w-plane, cut along the real axis from -1 to 1. The unit circle $|z| = 1$ itself corresponds to the straight line from -1 to 1 described twice.

On solving the equation for z, we see that the inverse function is a two-valued function of w. We can remove all ambiguity from the representation by replacing the w-plane by a Riemann surface of two sheets, each slit from -1 to 1, and joined crossways along the slit. If we pass round one or other of the points $w = \pm 1$, a different value of z is reached, but, if we pass round both, z returns to its original value.

Examples. To what curve in the w-plane does the line $x = 1$ correspond? Consider the result as an example of § 6.11.

6.33. The logarithmic function. *If $w = \log z$, the angular region $\alpha < \arg z < \beta$ corresponds to the infinite strip $\alpha < v < \beta$ in the w-plane.*

For if $z = re^{i\theta}$, then a value of w is

$$\log r + i\theta.$$

Hence
$$u = \log r, \qquad v = \theta.$$

As r goes from 0 to ∞, u goes from $-\infty$ to ∞, and the result follows.

If we consider the general value of $\log z$,

$$w = \log r + i(\theta + 2k\pi),$$

where k is any integer, we obtain, not one strip, but an infinity of strips in the w-plane, corresponding to the infinity of values of the logarithm.

On the other hand, a strip $\alpha < v < \beta$ corresponds to an angle in the z-plane; but, if $\beta - \alpha > 2\pi$, part of the plane will be covered more than once. We can, however, avoid any ambiguity by replacing the single z-plane by a Riemann surface consisting of an infinity of sheets, each cut along the real axis from 0 to $-\infty$, and the upper half-plane of each joined to the lower half-plane of the next along the slit. Then a strip of the w-plane of breadth 2π corresponds to one complete sheet of the Riemann surface, and every point of the Riemann surface corresponds to just one point of the w-plane.

Examples. (i) Investigate the properties of the transformation $w = \tan z$ by considering it as the result of the two transformations

$$w = \frac{1}{i}\frac{\zeta - 1}{\zeta + 1}, \qquad \zeta = e^{2iz},$$

and hence obtain a Riemann surface for the inverse function

$$z = \operatorname{arc\,tan} w.$$

[Hurwitz-Courant, *Funktionentheorie*, p. 293.]

(ii) Consider the properties of the transformation

$$w = z^{\alpha}$$

for general values of α.

[The function z^{α} is defined as $e^{\alpha \log z}$. Consider separately rational, irrational, and complex values of α.]

6.34. *If $w = \tan^2 \frac{1}{2} z$, the strip in the z-plane between the lines $x = 0$, $x = \frac{1}{2}\pi$ is represented on the interior of the unit circle in the w-plane, cut along the real axis from $w = -1$ to $w = 0$.*

We have $$w = \frac{1-\cos z}{1+\cos z}.$$

If $z = \frac{1}{2}\pi + iy$, then $\cos z = -i \sinh y$, and $|w| = 1$. It is easily seen that, as y goes from $-\infty$ to ∞, $\operatorname{am} w$ goes from $-\pi$ to π, so that w describes the whole unit circle once.

If $z = iy$, $\cos z = \cosh y$, and w is real. As y goes from $+\infty$ to 0, w goes from -1 to 0, and as y goes from 0 to $-\infty$, w retraces its path from 0 to -1.

The boundary of the strip therefore corresponds to the boundary of the cut circle, and there should be no difficulty in verifying that the interiors correspond.

Example. Prove that the line $x = \frac{1}{4}\pi$ corresponds to a loop of a closed curve, cutting the real axis at $w = -1$ and $w = 1/(3+2\sqrt{2})$.

6.4. Simple ('schlicht') functions.* We shall say that a function $f(z)$ is *simple* in a region D if it is analytic, one-valued, and does not take any value more than once in D.

The function $w = f(z)$ then represents the region D of the z-plane on a region D' of the w-plane, in such a way that there is a one-one correspondence between the points of the two regions.

If $f(z)$ is simple in D, $f'(z) \neq 0$ in D. For suppose, on the contrary, that $f'(z_0) = 0$. Then $f(z) - f(z_0)$ has a zero of order n $(n \geqslant 2)$ at z_0. Since $f(z)$ is not constant, we can find a circle $|z - z_0| = \delta$ on which $f(z) - f(z_0)$ does not vanish, and inside which $f'(z)$ has no zeros except z_0. Let m be the lower bound of $|f(z) - f(z_0)|$ on this circle. Then by Rouché's theorem, if $0 < |a| < m$, $f(z) - f(z_0) - a$ has n zeros in the circle (it cannot have a double zero, since $f'(z)$ has no other zeros in the circle). This is contrary to the hypothesis that $f(z)$ does not take any value more than once.

A simple function of a simple function is simple. If $f(z)$ is simple in D, and $F(w)$ in D', then $F\{f(z)\}$ is simple in D; for $F\{f(z_1)\} = F\{f(z_2)\}$ implies $f(z_1) = f(z_2)$, since F is simple; and this implies $z_1 = z_2$, since f is simple.

6.41. Inverse functions. In the above relationship, to every point of D' corresponds just one point of D. We can therefore consider z as a function of w, say $z = \phi(w)$. This is called the inverse function of $w = f(z)$.

* The German word is *schlicht*; Dienes, *The Taylor Series*, uses *biuniform*.

The inverse function is simple in D'. For it is one-valued; and it does not take any value more than once, since $f(z)$ is one-valued. Finally, it is analytic; for if $w_0 = f(z_0)$, then it is easily seen by considering $\int f'(z)/\{f(z)-w\}\,dz$ that, in any given neighbourhood of z_0, $f(z)$ takes every value w sufficiently near to w_0. Hence $z = \phi(w)$ is continuous, and

as $w \to w_0$, since $f'(z_0) \neq 0$.

6.42. Uniqueness of conformal transformation. *A simple function $w = f(z)$ which represents the unit circle on itself, so that the centre and a given direction through it remain unaltered, is the identical transformation $w = z$.*

We have $|f(z)| = 1$ for $|z| = 1$, and $f(0) = 0$. Hence, by Schwarz's lemma (§ 5.2),

$$|w| = |f(z)| \leqslant |z|.$$

But, applying Schwarz's lemma to the inverse function, we have $|z| \leqslant |w|$. Hence $|w| = |z|$, i.e.

$$|f(z)/z| = 1 \qquad (|z| \leqslant 1).$$

Since a function of constant modulus is itself constant, it follows that

$$f(z) = az,$$

where $|a| = 1$. The remaining conditions then show that $a = 1$.

Other functions, such as $w = z^2$, satisfy the conditions except that they are not simple.

A simple function which represents the unit circle on itself is a linear function.

If $w = f(z)$ represents the unit circle on itself, and $f(0) = w_0$, we can, by § 6.24, find a linear function $l(w)$ which represents the unit circle on itself, and is such that $l(w_0) = 0$. Then $l\{f(z)\}$ represents the unit circle on itself, and $l\{f(0)\} = 0$. Hence, by the above theorem, $l\{f(z)\} = az$. Since the inverse function of a linear function is linear, $f(z)$ is a linear function of z.

6.43. *Let $f(z)$ be regular at $z = 0$, and $f'(0) \neq 0$. Then $f(z)$ is simple in the immediate neighbourhood of $z = 0$; i.e. in the circle $|z| \leqslant \rho$, if ρ is small enough.*

We may suppose that $f(0) = 0$. Since $f'(0) \neq 0$, the origin is a zero of $f(z)$ of the first order. We can therefore find a circle

C, with centre $z=0$, on which $f(z) \neq 0$, and inside which $f(z)$ has no zero other than $z=0$. Let m be the lower bound of $|f(z)|$ on C.

Since $f(z)$ is continuous and vanishes at $z=0$, we can find a circle $|z|=\rho$ inside which $|f(z)|<m$. Then $w=f(z)$ is simple in this circle. For let w' be any number such that $|w'|<m$. Then by Rouché's theorem (§ 3.42) the number of zeros of $f(z)-w'$ in C is the same as the number of zeros of $f(z)$, that is, one. Hence there is just one point z' in C corresponding to each such value of w'. The region consisting of these values of z' is therefore represented simply on the circle $|w|<m$; and this region includes the circle $|z|=\rho$.

An alternative proof may be obtained by considering the power series

$$f(z)=\sum_{n=0}^{\infty} a_n z^n,$$

where $a_1 \neq 0$. If $f(z_1)=f(z_2)$,

$$\sum_{n=1}^{\infty} a_n(z_1^n-z_2^n)=0,$$

i.e. $$(z_1-z_2)\left\{a_1+\sum_{n=2}^{\infty} a_n(z_1^{n-1}+z_1^{n-2}z_2+\ldots+z_2^{n-1})\right\}=0.$$

If $|z_1|<\rho, |z_2|<\rho$, the modulus of the second factor is greater than

$$|a_1|-\sum_{n=2}^{\infty} n|a_n|\rho^{n-1},$$

which is positive if ρ is small enough. Hence $z_1=z_2$, and the result follows.

6.44. *The limit of a uniformly convergent sequence of simple functions is either simple or constant. More precisely, if $f_n(z)$ is simple in D for each value of n, and $f_n(z) \to f(z)$ uniformly in D, then $f(z)$ is simple in D, or is a constant.*

The possibility of the limit being constant is shown by the example $f_n(z)=z/n$.

In any case, $f(z)$ is analytic and one-valued in D. If it is not simple, there are two points z_1 and z_2 at which $w=f(z)$ takes the same value w_0. Describe, with z_1 and z_2 as centres, two circles which lie in D, do not overlap, and such that $f(z)-w_0$ does not vanish on either circumference (this is possible unless $f(z)$ is a constant). Let m be the lower bound of $|f(z)-w_0|$ on

the two circumferences. Then we can choose n so large that $|f(z)-f_n(z)|<m$ on the two circumferences. Hence, by Rouché's theorem, the function

$$f_n(z)-w_0=\{f(z)-w_0\}+\{f_n(z)-f(z)\}$$

has as many zeros in the circles as $f(z)-w_0$, that is, at least two. Hence $f_n(z)$ is not simple, contrary to hypothesis. This proves the theorem.

6.45. *Let C be a simple closed contour in the z-plane, enclosing a region D. Let $w=f(z)$ be an analytic function of z, regular in D and on C, and taking no value more than once on C. Then $f(z)$ is simple in D.*

The contour C corresponds to a contour C' in the w-plane. C' is closed, since $f(z)$ is one-valued; and it has no double points, since $f(z)$ does not take any value twice on C. Let D' be the region enclosed by C'.

We assume that $f(z)$ takes in D values other than those on C, say at z_0. Then, if Δ_C denotes variation round C,

$$\frac{1}{2\pi}\Delta_C \arg\{f(z)-f(z_0)\}$$

is equal to the number of zeros of $f(z)-f(z_0)$ in C; it is therefore a positive integer, since there is at least one such zero. But it is also equal to

$$\frac{1}{2\pi}\Delta_{C'} \arg(w-w_0),$$

where $w_0=f(z_0)$; and this is either 0, if w_0 is outside C', or ± 1, if w_0 is inside C', the sign depending on the direction in which C' is described. Hence it is equal to 1. Hence w_0 lies inside C', C' is described in the positive direction, and $f(z)$ takes the value w_0 just once in D. Thus D is represented simply on D'.

6.46. Extensions. The condition in the above analysis that the function $f(z)$ should be analytic on the contour can be relaxed to a certain extent. The state of affairs is not much altered if $f(z)$ is not analytic, but is continuous, at certain points of C. Suppose that there is a singularity at z_1 on the contour, and that C_1 is C with an indentation round z_1. Then the number of zeros of $f(z)-w_0$ inside C_1 is

$$\frac{1}{2\pi}\Delta_{C_1} \arg\{f(z)-w_0\}=\frac{1}{2\pi i}\int_{C_1}\frac{f'(z)}{f(z)-w_0}\,dz;$$

and, as the indentation is closed up, this tends to the corresponding integral round C, if $f(z)$ is continuous and $f'(z) = O(|z-z_1|^\alpha)$, where $\alpha > -1$. The argument of § 6.45 therefore still applies.

The function $f(z)$ may also have a pole on the contour; the region D' then extends to infinity. The theorem of § 6.45 still holds if the pole is of the first order, and the contour is a fairly ordinary one. Suppose that, by a preliminary change of variable, we take the pole at the origin, and that the direction can be taken so that $\mathbf{R}(z) \geqslant 0$ at all points of D. Let

$$w = f(z) = \frac{1}{z} + g(z),$$

where $g(z)$ is regular in D. Then, for z in D,

$$\mathbf{R}(w) \geqslant \min \mathbf{R}\{g(z)\} = a,$$

say. Let $b < a$. Then $|w-b| \geqslant a-b$ for z in D. Hence

$$\zeta = \frac{1}{w-b} = \frac{z}{1+zg(z)-bz}$$

is regular in D. The theorem applies directly to ζ, and so, since w is a simple function of ζ, to w.

The result is not necessarily true for poles of higher order.

Examples. (i) Let $$w = \frac{1}{i}\frac{z+1}{z-1}.$$

If $z = e^{i\theta}$, then $$w = \frac{1}{i}\frac{e^{i\theta}+1}{e^{i\theta}-1} = -\cot\tfrac{1}{2}\theta.$$

Hence, as z describes the unit circle from 0 to 2π, w describes the real axis from $-\infty$ to ∞. The only singularity on the z-boundary is a simple pole. Hence the unit circle in the z-plane is represented simply on the upper half w-plane. This of course is easily verified.

(ii) Let $$w = -\frac{1}{i}\left(\frac{z+1}{z-1}\right)^3.$$

Then, if $z = e^{i\theta}$, $$w = -\cot^3\tfrac{1}{2}\theta.$$

Hence there is a one-one correspondence between the unit circle in the z-plane and the real w-axis. But since there is a triple pole on the boundary the areas do not necessarily correspond. In fact

$$w = i\left(\frac{x+iy+1}{x+iy-1}\right)^3 = i\frac{(x^2+y^2-1)^3-12(x^2+y^2-1)y^2}{\{(x-1)^2+y^2\}^3} + \ldots,$$

and $v = 0$ corresponds to

$$(x^2+y^2-1)(x^2+y^2-2\sqrt{3}y-1)(x^2+y^2+2\sqrt{3}y-1) = 0,$$

i.e. the equation of three circles. Hence $v > 0$ if z is outside each of these circles, or outside one and inside the other two.

The reader should draw a figure showing the regions of the z-plane which correspond to the upper half of the w-plane.

6.5. The function $w = \int_0^z \frac{dt}{\sqrt{(1-t^2)}}$. We know that this function is equal to $\arcsin z$. Consider, however, what can be deduced from the integral about the representation of the z-plane on the w-plane.

Consider what part of the w-plane corresponds to the first quadrant in the z-plane. If $z = iy$ is purely imaginary, we have

$$w = \int_0^y \frac{i\,ds}{\sqrt{1+s^2}},$$

which is also purely imaginary; and as $y \to \infty$, so does w. The two imaginary axes therefore correspond.

Again, as z increases along the real axis from 0, so does w, until z reaches 1, and w reaches the value

$$\int_0^1 \frac{dt}{\sqrt{(1-t^2)}}.$$

Let us denote the value of this integral by I. Actually $I = \frac{1}{2}\pi$, but we need not assume that this is known.

We must now suppose that the path of integration passes above the point $z = 1$, say by a small semicircle. Then $\arg(1-t)$ decreases from 0 to $-\pi$, and so $\arg(1-t)^{-\frac{1}{2}}$ increases from 0 to $\frac{1}{2}\pi$; thus the integrand becomes purely imaginary, and we have

$$w = I + i \int_1^z \frac{dt}{\sqrt{(t^2-1)}}.$$

Finally, as z tends to infinity along the real axis, w tends to infinity along the line $u = I$.

Hence the boundary of the first quadrant in the z-plane corresponds to the boundary of the half-strip $0 < u < I$, $v > 0$ in the w-plane.

Secondly, the function is simple in this region. We cannot deduce this from § 6.45 without some further argument, since both the regions extend to infinity. But it is easily seen directly. For, if t lies in the first quadrant, the imaginary part of t^2-1

is positive, and $\arg\sqrt{(1-t^2)}$ lies between $-\frac{1}{2}\pi$ and 0. Hence

$$f(z_2)-f(z_1)=\int_{z_1}^{z_2}\frac{dt}{\sqrt{(1-t^2)}},$$

and, taking the integral along the straight line, it is of the form

$$k\int\frac{d\lambda}{\rho e^{i\phi}},$$

where λ is a real variable, $\rho>0$, $-\frac{1}{2}\pi<\phi<0$. Such an integral plainly cannot vanish. Here the function cannot take any value twice.

Also the quadrant of the circle $|z|=R$, where R is large, corresponds to a curve which (by the previous remark) has no double point, and which connects the two sides of the strip and lies entirely at a great distance from the real axis. Hence, by the theorem of § 6.45, the quadrant is represented simply on the whole strip.

The next problem is to continue the function beyond this limited region. This can be done by the method of reflection of § 4.51. In fact all the boundaries in each figure are straight lines.

In the first place, the imaginary axes correspond. Hence, reflecting in these lines, we see that the second quadrant in the z-plane corresponds to the half-strip $-I<u<0$, $v>0$, in the w-plane. Hence the upper half of the z-plane corresponds to the half-strip $-I<u<I$, $v>0$.

Next, reflect with respect to the segment (0, 1) of the real axis in the z-plane. We obtain the lower half of the z-plane. In the w-plane we obtain the half-strip $-I<u<I$, $v<0$.

Hence the whole strip $-I<u<I$ corresponds to the whole z-plane, but there are singularities at $z=\pm 1$ round which we must not pass; we may, for example, suppose the plane cut from $-\infty$ to -1 and from 1 to ∞.

Again, a reflection in the segment $(1,\infty)$ of the real z-axis corresponds to a reflection in the line $u=I$ in the w-plane. Hence the lower half of the z-plane (obtained by continuation to the right of $z=1$) corresponds to the half-strip $I<u<3I$, $v>0$.

It is plain that we can continue this process indefinitely. The

whole w-plane is divided up into strips of breadth $2I$, each of which corresponds to the whole z-plane.

If we reflect a point w_0 of the strip $-I < u < I$, first in $u = I$ and then in $u = 3I$, we obtain the point w_0+4I. Meanwhile the corresponding z_0, being reflected twice in the real axis, has returned to its original value. Then w_0 and w_0+4I correspond to the same z_0; i.e. if $z = g(w)$, then $g(w) = g(w+4I)$. The inverse function $g(w)$ is therefore periodic, with period $4I$.

Example. Prove that the function

$$w = \int_0^z \frac{dt}{\sqrt{\{(1-t^2)(1-k^2t^2)\}}} \qquad (0 < k < 1)$$

represents the upper half of the z-plane on the rectangle in the w-plane bounded by the lines $u = -K$, $u = K$, $v = 0$, $v = K'$, where

$$K = \int_0^1 \frac{dt}{\sqrt{\{(1-t^2)(1-k^2t^2)\}}}, \qquad K' = \int_1^{1/k} \frac{dt}{\sqrt{\{(t^2-1)(1-k^2t^2)\}}}.$$

Prove that the inverse function $z = g(w)$ has the two periods $4K$ and $2iK'$. [Hurwitz-Courant, *Funktionentheorie*, pp. 302–3.]

6.6. Representation of a polygon on a half-plane. The functions of the previous section are examples of the representation of a polygon on a half-plane. It is possible to do this with *any* polygon. The complete proof would take us too far, but we can show in a general way how it is to be done.

Consider a polygon in the w-plane with n sides and angles $\alpha_1\pi, \alpha_2\pi, \ldots, \alpha_n\pi$, where $\alpha_1+\ldots+\alpha_n = n-2$. If $\alpha_m < 1$ $(m = 1, 2, \ldots n)$ the polygon is convex. Some of the α's may be greater than 1, but the polygon must never cross itself. Suppose that the vertices of the polygon are to correspond to points $a_1, a_2, \ldots, a_n$ on the real axis in the z-plane. So long as z remains on the real axis without passing any of the points $a_1, \ldots, w$ remains on the same side of the polygon; hence the angle between the z-curve and the w-curve is constant, i.e. $\arg(dw/dz)$ is constant (see § 6.1).

If $$\frac{dw}{dz} = C(z-a_1)^{\alpha_1-1}(z-a_2)^{\alpha_2-1}\ldots(z-a_n)^{\alpha_n-1},$$

then dw/dz has this property. When z passes the point a_1 by a small circle above it, $\arg(z-a_1)$ decreases from π to 0, the amplitudes of the other factors returning to their original values. Hence $\arg(dw/dz)$ decreases by $\pi(\alpha_1-1)$. Hence the w-curve

turns through $\pi(1-\alpha_1)$ in the positive direction. This corresponds to an angle $\pi\alpha_1$ of the polygon.

The required function is therefore of the form

$$w = C\int_{z_0}^{z} (t-a_1)^{\alpha_1-1}(t-a_2)^{\alpha_2-1}\ldots(t-a_n)^{\alpha_n-1}\,dt.$$

The integrand is $O(1/|t|^2)$ as $|t|\to\infty$; hence the integral converges as $z\to\pm\infty$, and to the same value in each case, since the integral along a large semicircle above the real axis tends to zero. Hence, as z describes the real axis, w describes a closed curve, and in fact, from the construction, a polygon with the prescribed angles. By first considering the real z-axis as closed by a large semicircle above it, we can apply the theorems of §§ 6.45–6, and we find that the interior of the polygon is represented simply on the upper half-plane.

To show that we can choose the constants so that a polygon with given sides, as well as given angles, can be represented, is more difficult. For a triangle, however, the result is easily obtained. Consider, for example, the triangle with vertices $w = i\sqrt{3}$, 0, 1 (and angles $\frac{1}{6}\pi$, $\frac{1}{2}\pi$, $\frac{1}{3}\pi$), and let the vertices correspond to $z = -1$, 0, 1. The above theory gives

$$w = C\int_{z_0}^{z} (t+1)^{-\frac{5}{6}}t^{-\frac{1}{2}}(t-1)^{-\frac{2}{3}}\,dt.$$

The origins correspond if $z_0 = 0$; and if we write the formula as

$$w = C'\int_{0}^{z} (t+1)^{-\frac{5}{6}}t^{-\frac{1}{2}}(1-t)^{-\frac{2}{3}}\,dt,$$

where C' is real and positive, the directions of the real axes correspond. Finally, if

$$1 = C'\int_{0}^{1} (t+1)^{-\frac{5}{6}}t^{-\frac{1}{2}}(1-t)^{-\frac{2}{3}}\,dt,$$

then $z = 1$ corresponds to $w = 1$, and the required representation is obtained.

Examples. (i) Prove that the function

$$w = \int_{0}^{z} \frac{dt}{(1-t^2)^{\frac{2}{3}}}$$

represents a half-plane on an equilateral triangle.

(ii) Prove that the function

$$w = \int_0^z \frac{dt}{\sqrt{(1-t^4)}}$$

represents the unit circle in the z-plane on a square in the w-plane. [Hurwitz-Courant. Put $z = (\zeta - i)/(\zeta + i)$.]

6.7. Representation of any region on a circle. A fundamental theorem of Riemann states that *any region with a suitable boundary can be represented on a circle by a simple analytic function.* It is beyond our scope to inquire exactly what forms of region are suitable. The region may be the interior of a closed curve; or one side of a curve which goes to infinity in both directions (e.g. a half-plane); or any form of strip between two such curves; or even the whole plane cut along a curve (e.g. along the real axis from 0 to infinity).

Let D be a region of one of the above types.

The function which represents any region simply on a bounded region must be simple and bounded. Let us first verify that there are such functions for D. Let a and b be two points on the boundary of D, and let

$$w = \sqrt{\left(\frac{z-a}{z-b}\right)}.$$

In D we can restrict ourselves to one branch of this function; this branch is simple, and the values taken by it cover a part only of the w-plane (since both branches together cover the whole w-plane once). Let w_0 be a point of the region not covered. Then $1/(w-w_0)$ is simple and bounded in D. Also

$$f(z) = \frac{p}{w-w_0} + q$$

is simple and bounded, and we can choose p and q so that, at a given point of D, $f(z) = 0$ and $f'(z) = 1$.

Consider all functions $f(z)$ which are simple and bounded in D, and such that $f(z) = 0$ and $f'(z) = 1$ at a given point P of D. Let $M(f)$ denote the maximum modulus of $f(z)$. Let ρ be the lower bound of $M(f)$ for all such functions.

There is then either a function $\phi(z)$ of the set such that $M(\phi) = \rho$; or a sequence $f_1, f_2, \ldots$ of functions of the set such that

$$\lim M(f_n) = \rho.$$

We shall show that the second alternative reduces to the first. Since the sequence $f_n(z)$ is bounded in D, we can, by § 5.22, select from it a partial sequence which tends uniformly to a limit in any region interior to D. Let $f_{n_1}(z)$, $f_{n_2}(z)$,... be such a sequence, and $\phi(z)$ its limit. Then $\phi(z)$ is also a function of the set; for it is bounded, and $\phi(z)=0$, $\phi'(z)=1$ at P, and $\phi(z)$ is simple (§ 6.44), being not constant since $\phi'(z)=1$. Also

$$M(\phi) \geqslant \rho$$

by definition of ρ; and

$$M(f_{n_\nu}) < \rho+\epsilon \qquad (\nu > \nu_0),$$

i.e.

$$|f_{n_\nu}(z)| < \rho+\epsilon \qquad (\nu > \nu_0).$$

Making $\nu \to \infty$, it follows that

$$|\phi(z)| < \rho+\epsilon,$$

i.e. $M(\phi) \leqslant \rho$.

This proves the existence of a function $\phi(z)$ of the set with $M(\phi)=\rho$; and since $\phi(z)$ is not constant, $\rho > 0$.

We shall show that the function $w=\phi(z)$ represents D simply on the circle $|w| < \rho$. In the proof, we may suppose that $\rho = 1$. Let Δ be the region of the w-plane on which $w=\phi(z)$ represents D. Since $M(\phi)=1$, Δ is included in $|w| \leqslant 1$, and reaches its circumference at one point at least.

If the theorem is not true, Δ has a boundary point α inside the circle ($|\alpha| < 1$). Then each branch of

$$w_1 = \sqrt{\left(\frac{w-\alpha}{\bar{\alpha}w-1}\right)}$$

is regular for w in Δ. Also $|w_1| \leqslant 1$ if $|w| \leqslant 1$ (§ 6.24), and $w_1(0) = \sqrt{\alpha}$. Let

$$w_2 = \frac{w_1-\sqrt{\alpha}}{\sqrt{\bar{\alpha}}w_1-1}.$$

Then $|w_2| \leqslant 1$ if $|w_1| \leqslant 1$. Also

$$\frac{dw_2}{dw} = \frac{dw_2}{dw_1}\cdot\frac{dw_1^2}{dw}\cdot\frac{1}{2w_1} = \frac{|\alpha|-1}{(\sqrt{\bar{\alpha}}w_1-1)^2}\,\frac{|\alpha|^2-1}{(\bar{\alpha}w-1)^2}\cdot\frac{1}{2w_1}$$

$$= \frac{|\alpha|-1}{(|\alpha|-1)^2}\cdot\frac{|\alpha|^2-1}{2\sqrt{\alpha}} = \frac{|\alpha|+1}{2\sqrt{\alpha}}$$

at $w=0$. The modulus of this is greater than unity. Hence

$$w_3 = \frac{2\sqrt{\alpha}}{|\alpha|+1}w_2$$

is a function of the set considered, and

$$M(w_3) = \frac{2|\sqrt{\alpha}|}{|\alpha|+1} < 1.$$

This gives a contradiction, and the theorem follows.

6.71. Uniqueness theorem. Let D be a region in the z-plane which is the interior of a simple closed contour, or which is of one of the other types considered in § 6.7. *Then there is a uniquely determined function $w = f(z)$ which represents D simply on the interior of the unit circle in the w-plane, and is such that, if z_0 is a given point in D, $f(z_0) = 0$ and $f'(z_0)$ is real and positive.*

It follows from § 6.7 that there is one such function, say $w = f(z)$. Let $z = F(w)$ be the inverse function. Suppose that there is another function $w = g(z)$ with the same properties. Then the function $W = g\{F(w)\}$ represents the unit circle simply on itself, the centre and the direction of the real axis through it remaining unaltered. Hence, by § 6.42, $g\{F(w)\} = w$, i.e. $g(z) = f(z)$.

6.8. Further properties of simple functions. The class of functions $f(z)$ which are simple for $|z| < 1$, and such that $f(0) = 0$, $f'(0) = 1$, has been studied in great detail. The function $w = z$ belongs to the class, and represents the unit circle on itself. For all functions of the class the 'map' of the unit circle is subject to certain limitations. For the details we may refer to Bieberbach, *Funktionentheorie*, ii. 82–94, Landau, *Ergebnisse* (ed. 2), pp. 107–14, or Dienes, *The Taylor Series*, Ch. VIII. We shall, however, obtain the simplest property of the map. *For any function of the class, no boundary point of the map of the unit circle is nearer to the origin than the point $\frac{1}{4}$.*

We deduce this from the two following theorems.

Let
$$w = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots$$
be simple for $|z| > 1$, and regular except for the pole at infinity. Then
$$\sum_{n=1}^{\infty} n|a_n|^2 \leqslant 1.$$

Since the function is simple, any circle $|z| = r > 1$ corresponds to a simple closed curve in the w-plane, which encloses a positive

area. If $w=u+iv$, $u=u(\theta)$, $v=v(\theta)$, on the curve, the area enclosed is

$$\int_0^{2\pi} u(\theta)v'(\theta)\,d\theta = \int_0^{2\pi} \frac{w(\theta)+\bar{w}(\theta)}{2}\cdot\frac{w'(\theta)-\bar{w}'(\theta)}{2i}\,d\theta$$

$$= \tfrac{1}{4}\int_0^{2\pi}\left\{re^{i\theta}+re^{-i\theta}+\sum_{n=1}^{\infty}\frac{a_n e^{-in\theta}+\bar{a}_n e^{in\theta}}{r^n}\right\}\times$$

$$\times\left\{re^{i\theta}+re^{-i\theta}-\sum_{n=1}^{\infty}\frac{na_n e^{-in\theta}+n\bar{a}_n e^{in\theta}}{r^n}\right\}d\theta$$

$$= \tfrac{1}{2}\pi\left\{\left(r+\frac{\bar{a}_1}{r}\right)\left(r-\frac{a_1}{r}\right)+\left(r+\frac{a_1}{r}\right)\left(r-\frac{\bar{a}_1}{r}\right)-\sum_{n=2}^{\infty}\frac{2na_n\bar{a}_n}{r^{2n}}\right\}$$

$$= \pi r^2-\pi\sum_{n=1}^{\infty} n|a_n|^2 r^{-2n}.$$

Since this is positive

$$\sum_{n=1}^{\infty} n|a_n|^2 r^{-2n} \leqslant r^2,$$

and making $r\to 1$ the result follows.

If $$w=f(z)=z+a_2z^2+\ldots$$
is simple in $|z|<1$, *then* $|a_2|\leqslant 2$.

The function

$$F(z)=\sqrt{\{f(z^2)\}}=z+\tfrac{1}{2}a_2z^3+\ldots$$

is also simple in $|z|<1$; for it is regular, since $f(z^2)$ does not vanish except at $z=0$, where it has a double zero; and if $F(z_1)=F(z_2)$, then $f(z_1^2)=f(z_2^2)$, and hence, since $f(z)$ is simple, $z_1^2=z_2^2$, i.e. $z_1=\pm z_2$. But $F(z)$ is an odd function, so that $z_1=-z_2$ gives $F(z_1)=-F(z_2)$. Hence the only solution of $F(z_1)=F(z_2)$ is $z_1=z_2$, i.e. $F(z)$ is simple.

It follows that

$$\left\{F\left(\frac{1}{z}\right)\right\}^{-1}=z-\frac{\frac{1}{2}a_2}{z}+\ldots$$

is simple for $|z|>1$. Hence by the previous theorem

$$\tfrac{1}{4}|a_2|^2+\ldots\leqslant 1,$$

and the result follows.

Now let $$w=f(z)=z+a_2z^2+\ldots$$

be a function of the class considered in the main theorem. Let c be a value which it does not take in the unit circle, i.e. a point outside the 'map' of the unit circle. Then

$$\frac{c\,f(z)}{c-f(z)} = z+\left(a_2+\frac{1}{c}\right)z^2+\ldots$$

is regular and simple for $|z| < 1$. Hence

$$\left|a_2+\frac{1}{c}\right| \leqslant 2,$$

$$\left|\frac{1}{c}\right| \leqslant 2+|a_2| \leqslant 4,$$

$$|c| \geqslant \tfrac{1}{4},$$

and the result follows.

Example. The function $z/(1-z)^2$ belongs to the above class. It has $a_2 = 2$, and it gives a map passing through $w = -\frac{1}{4}$.

[The only solution of

$$\frac{z}{(1-z)^2} = \frac{z'}{(1-z')^2}, \qquad |z| < 1, \qquad |z'| < 1,$$

is $z = z'$.]

MISCELLANEOUS EXAMPLES

1. In a given linear transformation, the point z_0 is such that there is some circle $|z-z_0| = R$ which transforms into a concentric circle $|w-z_0| = R'$. Show that the locus of z_0 is a rectangular hyperbola; and that to each point z_0 on the locus corresponds just one circle (real or imaginary) which transforms into a concentric circle.

2. Show that, if $$\frac{dz}{dw} = -2i\left(w-\frac{1}{w}\right),$$

and the constant of integration is properly chosen, the whole z-plane cut along the semi-infinite lines $x = \pm\pi$, $y \leqslant 0$, corresponds to the upper half of the w-plane.

3. Show that, if $$\frac{dz}{dw} = \frac{w}{\sqrt{(w^2-a^2)}},$$

and a and the constant of integration and the value of the square root are properly chosen, the upper half of the w-plane corresponds to the upper half of the z-plane, cut along the imaginary axis from $z = 0$ to a point $z = ik$.

4. If $f(z)$ is regular inside and on the unit circle, $|f(z)| \leqslant M$ on the circle, and $f(a) = 0$, where $|a| < 1$, then

$$|f(z)| \leqslant M\left|\frac{z-a}{\bar{a}z-1}\right|$$

inside the circle.

5. If $f(z)$ is regular inside and on the unit circle, $|f(z)| \leqslant M$ on the circle, and $f(0) = a$, where $0 < a < M$, then

$$|f(z)| \leqslant M \frac{M|z|+a}{a|z|+M}$$

inside the circle.

[Consider $F(z) = M\{f(z)-a\}/\{af(z)-M^2\}$.]

6. Either branch of the function

$$\frac{z}{\sqrt{(1-z)}}$$

is simple for $|z| < 1$.

7. Show that the function

$$\frac{z}{(1-z)^3}$$

is simple for $|z| < \frac{1}{2}$, but not in any larger circle with centre at the origin.

8. Show that the function

$$f(z) = z+a_2z^2+a_3z^3+\ldots$$

is simple for $|z| < 1$ if

$$\sum_{n=2}^{\infty} n|a_n| \leqslant 1.$$

CHAPTER VII

POWER SERIES WITH A FINITE RADIUS OF CONVERGENCE

7.1. The circle of convergence. We know that every power series has a circle of convergence, within which it converges, and outside which it diverges. The radius of this circle may, however, be infinite, so that the circle includes the whole plane. In this chapter we shall consider power series which have a finite radius of convergence.

The radius of convergence of a power series is determined by the moduli of the coefficients in the series.

The power series

$$\sum_{n=0}^{\infty} a_n z^n \tag{1}$$

has the radius of convergence

$$R = \varliminf_{n\to\infty} |a_n|^{-1/n}. \tag{2}$$

Suppose that R is defined by (2). If z is a point where the series (1) converges, $a_n z^n \to 0$ as $n \to \infty$. Hence, if n is sufficiently large,

$$|a_n z^n| < 1,$$

i.e.

$$|z| < |a_n|^{-1/n}.$$

Making $n \to \infty$, it follows that $|z| \leqslant R$. Hence the radius of convergence does not exceed R.

On the other hand, for sufficiently large values of n,

$$|a_n|^{-1/n} > R-\epsilon,$$

i.e.

$$|a_n| < (R-\epsilon)^{-n}.$$

Hence the series (1) is convergent if $\sum (R-\epsilon)^{-n}|z|^n$ is convergent, i.e. if $|z| < R-\epsilon$. Since ϵ is arbitrarily small, the series (1) is convergent if $|z| < R$. Thus the radius of convergence is at least equal to R. Putting together the two results, the theorem follows.

Examples. (i) Find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{2n!}{(n!)^2} z^n, \quad \sum_{n=1}^{\infty} \frac{n!}{n^n} z^n, \quad \sum_{n=0}^{\infty} n!\, z^{n^2}.$$

(ii) If $R = 1$, and the only singularities on the unit circle are simple poles, then a_n is bounded. [For

$$f(z) = \frac{a}{1-ze^{-i\alpha}} + \frac{b}{1-ze^{-i\beta}} + \dots + \frac{k}{1-ze^{-i\kappa}} + g(z),$$

where $g(z)$ is regular for $|z| < 1+\delta$ $(\delta > 0)$. Hence $g(z) = \sum b_n z^n$, where $b_n = o(1)$.]

(iii) If $R = 1$, and the only singularities on the unit circle are poles of order p, then $a_n = O(n^{p-1})$.

7.11. We also know from the Cauchy-Taylor theorem that the circle of convergence of the series passes through the singularity or singularities of the function which are nearest to the origin. *Hence the modulus of the nearest singularity can be determined from the moduli of the coefficients in the series.*

7.2. Position of the singularities. While the modulus of the nearest singularities is determined in quite a simple way, their exact position is not usually so easy to find. There are, however, some special cases in which we can identify a particular point as a singularity.

In the following theorems we shall take the radius of convergence to be unity; we can, of course, pass from this to the general case by a simple transformation.

7.21. *If $a_n \geqslant 0$ for all values of n, then $z = 1$ is a singular point.*

Suppose, on the contrary, that $z = 1$ is regular. Then, if we take a point ρ on the real axis between 0 and 1, there is a circle with centre ρ which includes the point 1, and in which the function is regular. If $f(z)$ is the function, the Taylor's series about ρ is

$$\sum_{\nu=0}^{\infty} \frac{f^{(\nu)}(\rho)}{\nu!}(z-\rho)^\nu, \tag{1}$$

and this converges at a point $z = 1+\delta$ $(\delta > 0)$. Now

$$f^{(\nu)}(\rho) = \sum_{n=\nu}^{\infty} n(n-1)\dots(n-\nu+1)a_n \rho^{n-\nu}, \tag{2}$$

and so the above series is

$$\sum_{\nu=0}^{\infty} \frac{(z-\rho)^\nu}{\nu!} \sum_{n=\nu}^{\infty} n(n-1)\dots(n-\nu+1)a_n \rho^{n-\nu}.$$

This is a double series of positive terms, convergent for $z = 1+\delta$.

Hence we may invert the order of the summations, and we obtain

$$\sum_{n=0}^{\infty} a_n \sum_{\nu=0}^{n} \frac{n(n-1)\ldots(n-\nu+1)}{\nu!}(z-\rho)^\nu \rho^{n-\nu}$$
$$= \sum_{n=0}^{\infty} a_n\{(z-\rho)+\rho\}^n = \sum_{n=0}^{\infty} a_n z^n.$$

Hence the original series is convergent for $z = 1+\delta$, contrary to the hypothesis that the radius of convergence is 1. This proves the theorem.

Another proof, due to Pringsheim, is as follows. There is at least one singularity, say $e^{i\alpha}$, on the unit circle. The Taylor's series about $\rho e^{i\alpha}$, where $0 < \rho < 1$, is

$$\sum_{\nu=0}^{\infty} \frac{f^{(\nu)}(\rho e^{i\alpha})}{\nu!}(z-\rho e^{i\alpha})^\nu,$$

and, since $e^{i\alpha}$ is a singularity, this has the radius of convergence $1-\rho$. But it is clear from (2) that, if $a_n \geqslant 0$ for all values of n,

$$|f^{(\nu)}(\rho e^{i\alpha})| \leqslant f^{(\nu)}(\rho).$$

Hence the radius of convergence of (1) does not exceed $1-\rho$. Hence $z = 1$ is a singularity.

7.22. *If a_n is real for all values of n, and $\sum a_n$ is properly divergent, i.e.*

$$s_n = a_0+a_1+\ldots+a_n \to \infty \text{ (or } \to -\infty),$$

then $z = 1$ is a singular point.

We have, for $|z| < 1$,

$$\frac{f(z)}{1-z} = \sum_{n=0}^{\infty} a_n z^n \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} s_n z^n. \tag{1}$$

by § 1.65, the series being absolutely convergent. Hence

$$f(z) = (1-z)\sum_{n=0}^{\infty} s_n z^n$$
$$= (1-z)\sum_{n=0}^{N} s_n z^n + (1-z)\sum_{n=N+1}^{\infty} s_n z^n = f_1(z)+f_2(z),$$

say. Suppose that $s_n \to \infty$. Then, given any positive number G, however large, we can choose N so that $s_n > G$ $(n > N)$. Then, if $0 < z < 1$,

$$f_2(z) > (1-z)\sum_{n=N+1}^{\infty} G z^n = G z^{N+1}.$$

Having fixed N, we can choose z_0 so near to 1 that

$$z^{N+1} > \tfrac{1}{2}, \qquad |f_1(z)| < \tfrac{1}{4}G \qquad (z > z_0),$$

since $z^{N+1} \to 1$ and $f_1(z) \to 0$. Hence

$$f(z) > \tfrac{1}{4}G \qquad (z > z_0),$$

i.e. $f(z) \to \infty$ as $z \to 1$. This proves the theorem.

If we merely know that $|s_n| \to \infty$, we cannot deduce that $z = 1$ is a singularity. For example,

$$\frac{1}{(1+z)^3} = 1 - 3z + \ldots + (-1)^n \tfrac{1}{2}(n+1)(n+2)z^n + \ldots,$$

and here $|s_n| \sim \frac{1}{4}n^2$, though the function is regular at $z = 1$.

7.23. General tests for singular points. If we consider any particular point on the circle of convergence, we can devise a test to determine whether it is a singularity or not; but it is not one which lends itself to simple calculations.

We may suppose that the radius of convergence is 1, and, by a preliminary transformation, we may bring the point to be considered to $z = 1$.

The principle to be used is that, if we expand $f(z)$ about a point on the real axis between 0 and 1, the circle of convergence includes $z = 1$ if $f(z)$ is regular at this point, and not otherwise. But we can make a transformation which brings the formula into a simpler form than the direct application of the principle would give.

Let
$$F(w) = \frac{1}{1-w} f\left(\frac{w}{1-w}\right).$$

Then $F(w)$ is regular for $\mathbf{R}(w) < \frac{1}{2}$, since $\mathbf{R}(w) < \frac{1}{2}$ gives $|w| < |1-w|$. Now

$$F(w) = \sum_{m=0}^{\infty} \frac{a_m w^m}{(1-w)^{m+1}} = \sum_{m=0}^{\infty} a_m w^m \sum_{r=0}^{\infty} \frac{(m+r)!}{m!\,r!} w^r$$

$$= \sum_{n=0}^{\infty} w^n \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} a_m.$$

Let
$$b_n = \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} a_m.$$

Then a necessary and sufficient condition that $z = 1$ should

be a singularity of $f(z)$, i.e. that $w = \frac{1}{2}$ should be a singularity of $F(w)$, is that

$$\varliminf_{n\to\infty} |b_n|^{-\frac{1}{n}} = \tfrac{1}{2}.$$

For then $F(w)$ has a singularity on $|w| = \frac{1}{2}$, and every point other than $w = \frac{1}{2}$ is known to be regular.

By using other transformations we obtain a variety of other equivalent conditions.

Example. Prove that every point on the unit circle is a singularity of

$$f(z) = \sum_{n=0}^{\infty} z^{2^n}.$$

[For the point $z = e^{i\theta}$, we have to consider

$$b_n = \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} a_m,$$

where $a_m = e^{i2^r\theta}$ if $m = 2^r$, and $a_m = 0$ otherwise. Clearly

$$|b_n| \leqslant \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} = 2^n.$$

Also
$$b_{2^n} = \sum_{m=0}^{n} \frac{2^n!}{2^m!(2^n-2^m)!} e^{i2^m\theta}.$$

The modulus of the term $m = n-1$ is asymptotic to $A2^{2^n-\frac{1}{2}n}$, by Stirling's theorem. Also, if u_m denotes the general term, and $0 < m \leqslant n-2$,

$$\left|\frac{u_m}{u_{m+1}}\right| = \frac{(2^m+1)\ldots 2^{m+1}}{(2^n-2^{m+1}+1)\ldots(2^n-2^m)} < \left(\frac{2^{m+1}}{2^n-2^m}\right)^{2^m} < \left(\frac{2}{3}\right)^{2^m} < \frac{4}{9},$$

and the remainder is easily seen to be negligible. Hence

$$\lim |b_{2^n}|^{\frac{1}{2^n}} = 2.]$$

7.3. Convergence of the series and regularity of the function. It will be noticed that we have not used the convergence or divergence of the original series as a test for regularity or singularity of the function. In general no such test is possible, for all possible relations can occur. If

$$f(z) = \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n} = \log \frac{1}{1+z},$$

the series is convergent, and the function regular, at $z = 1$; and the series is divergent, and the function singular, at $z = -1$.

On the other hand, if

$$f(z) = \sum_{n=0}^{\infty} (-1)^n z^n = \frac{1}{1+z}$$

the series is divergent, but the function is regular, at $z = 1$; while if

$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} = \int_0^z \frac{1}{w} \log \frac{1}{1-w}\, dw$$

the series is convergent at $z = 1$, but $f(z)$ has a singularity.

7.31. There is, however, one case in which divergence of the series indicates a singularity of the function; the case where $a_n \to 0$. This follows from the following theorem.

If

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

and $a_n \to 0$, the series is convergent at every point of the unit circle where the function is regular.

Two proofs of this theorem have been given. One, due to M. Riesz, is essentially a 'complex variable' method, and is given by Landau, *Ergebnisse*, § 18. The following proof, due to W. H. Young (7), is of Fourier-series type. In some respects it is not so simple as Riesz's, but it can easily be adapted to give more general results.

We may without loss of generality take the point in question to be $z = 1$; and we may suppose that $f(1) = 0$. We have then to prove that $s_n \to 0$.

It follows from § 7.22 (1) that

$$s_n = \frac{1}{2\pi i} \int \frac{f(z)\ \ dz}{1-z\, z^{n+1}}.$$

Taking the contour to be the circle $|z| = r < 1$, we have

$$s_n = \frac{1}{2\pi r^n} \int_{-\pi}^{\pi} \frac{f(re^{i\theta})}{1-re^{i\theta}} e^{-in\theta}\, d\theta.$$

Let $0 < \delta < \pi$, and let $\phi(\theta) = \phi(\theta, \delta, r)$ be such that

(i) $\phi(\theta) = 1/(1-re^{i\theta})$ for $-\pi < \theta < -\delta$ and $\delta < \theta < \pi$;

(ii) $\phi(\theta)$ and $\phi'(\theta)$ are continuous for $-\pi < \theta < \pi$;

(iii) $|\phi(\theta)| < K$, $|\phi'(\theta)| < K$, $|\phi''(\theta)| < K$, for $-\pi < \theta < \pi$,

where K depends on δ but not on r. For example, if

$$\phi(\theta) = a\theta^3 + b\theta^2 + c\theta + d \qquad (-\delta \leqslant \theta \leqslant \delta),$$

we can determine the coefficients so that

$$\phi(\pm\delta) = \frac{1}{1-re^{\pm i\delta}}, \quad \phi'(\pm\delta) = \frac{ire^{\pm i\delta}}{(1-re^{\pm i\delta})^2}.$$

Then (ii) is satisfied; and a, b, c, and d are linear functions of $\phi(\pm\delta)$ and $\phi'(\pm\delta)$, the moduli of these not exceeding $\frac{1}{2}\operatorname{cosec}\frac{1}{2}\delta$ and $\frac{1}{4}\operatorname{cosec}^2\frac{1}{2}\delta$ respectively. Hence (iii) is satisfied.

We can then write

$$2\pi r^n s_n = \int_{-\delta}^{\delta} \frac{f(re^{i\theta})}{1-re^{i\theta}} e^{-in\theta}\, d\theta + \int_{-\pi}^{\pi} f(re^{i\theta})\phi(\theta)e^{-in\theta}\, d\theta - \\ - \int_{-\delta}^{\delta} f(re^{i\theta})\phi(\theta)e^{-in\theta}\, d\theta$$

$$= I_1 + I_2 - I_3.$$

Since $f(z)$ is regular at $z = 1$, and $f(1) = 0$, we have

$$f(re^{i\theta}) = O(|1-re^{i\theta}|)$$

in an interval $|\theta| \leqslant \theta_0$, uniformly for $r_0 \leqslant r \leqslant 1$. Hence

$$I_1 = \int_{-\delta}^{\delta} O(1)\, d\theta = O(\delta).$$

Suppose now that δ is fixed. We have

$$I_2 = \sum_{m=1}^{\infty} a_m r^m \int_{-\pi}^{\pi} e^{i(m-n)\theta}\phi(\theta)\, d\theta,$$

by uniform convergence; and integrating by parts twice each integral except the nth,

$$I_2 = a_n r^n \int_{-\pi}^{\pi} \phi(\theta)\, d\theta - \sum_{m \neq n} \frac{a_m r^m}{(m-n)^2} \int_{-\pi}^{\pi} e^{i(m-n)\theta}\phi''(\theta)\, d\theta,$$

all the integrated terms cancelling. Let $\epsilon_\nu = \max\limits_{m > \nu}(|a_m|)$, so that $\epsilon_\nu \to 0$. Then

$$|I_2| \leqslant 2\pi K \left\{ \epsilon_n + \epsilon_0 \sum_{m \leqslant \frac{1}{2}n} \frac{1}{(m-n)^2} + \epsilon_{\frac{1}{2}n} \sum_{m > \frac{1}{2}n} \frac{1}{(m-n)^2} \right\}$$

$$= O(\epsilon_n) + O(1/n) + O(\epsilon_{\frac{1}{2}n}).$$

Finally

$$I_3 = \left[\frac{f\phi e^{-in\theta}}{-in}\right]_{\delta}^{-\delta} + \frac{1}{in}\int_{-\delta}^{\delta} (f'\phi + f\phi')e^{-in\theta}\,d\theta = O\left(\frac{1}{n}\right).$$

Given ϵ, we can choose δ so that $|I_1| < \frac{1}{3}\epsilon$ for all values of n; and, having fixed δ, we can choose $n_0 = n_0(\epsilon)$ so large that $|I_2| < \frac{1}{3}\epsilon$ and $|I_3| < \frac{1}{3}\epsilon$ for $n > n_0$. Hence

$$2\pi r^n |s_n| < \epsilon \qquad (n > n_0).$$

Making $r \to 1$, it follows that $2\pi|s_n| < \epsilon$ $(n > n_0)$, i.e. $s_n \to 0$.

The reader will notice that we have not used the full force of the hypothesis '$f(z)$ is regular at $z = 1$'; and the proof would hold with little change if e.g. $f(z) = O(|1-z|^{\alpha})$, where $\alpha > 0$. For the more general form of the theorem we must refer the reader to Young's paper.

7.4. Over-convergence.* We know that, at every point outside the circle of convergence of a power series, the series is divergent. But if, instead of considering the whole sequence of partial sums of the series, we consider particular sequences of these sums, it is sometimes possible to obtain a convergent sequence. This is shown by the following example.

Let
$$f(z) = \sum_{n=1}^{\infty} \frac{\{z(1-z)\}^{4^n}}{p_n},$$

where p_n is the maximum coefficient in the polynomial $\{z(1-z)\}^{4^n}$. Then in each of the polynomials

$$\frac{\{z(1-z)\}^{4^n}}{p_n}$$

the moduli of the coefficients do not exceed 1, and one of them is actually equal to 1. Also the highest term in this polynomial is of degree 2.4^n, whereas the lowest term in the next polynomial is of degree 4^{n+1}. Hence, if we expand $f(z)$ in powers of z, each term is a single term of one of the above polynomials. The radius of convergence of this series is 1, since $|a_n| \leqslant 1$ for all n, while $a_n = 1$ for an infinity of values of n.

In particular, the above series of polynomials is convergent for $|z| < 1$. But, since it is formally unchanged by the substitution $z = 1-w$, it is also convergent for $|w| < 1$, i.e. for $|1-z| < 1$.

* Ostrowski (1), Zygmund (1), Estermann (2).

The special sequence of partial sums obtained by taking each polynomial as a whole is therefore convergent in a region which lies partly outside the unit circle.

A power series which has a sequence of partial sums convergent outside the circle of convergence of the series is said to be 'over-convergent'. Of course a power series can only be over-convergent in the neighbourhood of a point of the circle where the function is regular. We shall next define a class of functions which have this property of over-convergence in the neighbourhood of every point of the circle where the function is regular.

7.41. *Suppose that the power series*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

has the radius of convergence 1, *and that there are an infinite number of gaps in the sequence of coefficients, i.e. there are sequences of suffixes* p_k, q_k, *such that* $a_n = 0$ *for* $p_k < n < q_k$; *and* $q_k \geqslant (1+\vartheta)p_k$, *with a fixed positive* ϑ.

Then the sequence of the corresponding partial sums

$$s_{p_k}(z) = \sum_{n=0}^{p_k} a_n z^n$$

is convergent in a region of which every regular point of $f(z)$ *on the circle of convergence is an interior point.*

To prove this it is sufficient to consider the point $z = 1$. Suppose that $f(z)$ is regular at $z = 1$. Then, if δ is small enough, it is regular in and on the circle with centre $\frac{1}{2}$ and radius $\frac{1}{2}+\delta$.

We apply Hadamard's three-circles theorem to the function

$$\phi(z) = f(z) - s_{p_k}(z),$$

and the circles with centre $\frac{1}{2}$ and radii $\frac{1}{2}-\delta$, $\frac{1}{2}+\epsilon$, $\frac{1}{2}+\delta$, where $0 < \epsilon < \delta$. If M_1, M_2, M_3 are the maximum moduli of $\phi(z)$ on these circles, then

$$M_2^{\log\frac{1+2\delta}{1-2\delta}} \leqslant M_1^{\log\frac{1+2\delta}{1+2\epsilon}} M_3^{\log\frac{1+2\epsilon}{1-2\delta}}. \tag{1}$$

In order to prove that $s_{p_k}(z) \to f(z)$ in a region including $z = 1$, it is sufficient to show that we can take ϵ so small that $M_2 \to 0$ when $p_k \to \infty$. The idea of the proof is that, while M_3 is substantially of the order $(1+\delta)^{p_k}$, M_1 behaves like $(1-\delta)^{q_k}$, and

so, since q_k is greater than p_k, the right-hand side of (1) is small when p_k is large.

To every positive η (say with $\eta < \frac{1}{2}\delta$) corresponds a K such that

$$|a_n| < K(1-\eta)^{-n}.$$

Hence, as $k \to \infty$,

$$M_1 \leqslant |a_{q_k} z^{q_k}| + |a_{q_k+1} z^{q_k+1}| + \ldots$$

$$< K \frac{\left(\frac{1-\delta}{1-\eta}\right)^{q_k}}{1-\frac{1-\delta}{1-\eta}} = O\left\{\left(\frac{1-\delta}{1-\eta}\right)^{q_k}\right\} = O\left\{\left(\frac{1-\delta}{1-\eta}\right)^{(1+\vartheta)p_k}\right\}.$$

Also, if M is the maximum modulus of $f(z)$ on the outer circle,

$$M_3 \leqslant M + |a_0| + \ldots + |a_{p_k} z^{p_k}|$$

$$\leqslant M + K\left\{1 + \frac{1+\delta}{1-\eta} + \ldots + \left(\frac{1+\delta}{1-\eta}\right)^{p_k}\right\} = O\left\{\left(\frac{1+\delta}{1-\eta}\right)^{p_k}\right\}.$$

Hence the right-hand side of (1) is

$$O\left[\left\{\left(\frac{1-\delta}{1-\eta}\right)^{(1+\vartheta)\log\frac{1+2\delta}{1+2\epsilon}} \left(\frac{1+\delta}{1-\eta}\right)^{\log\frac{1+2\epsilon}{1-2\delta}}\right\}^{p_k}\right].$$

When $\epsilon \to 0$, $\eta \to 0$, the expression in brackets tends to

$$(1-\delta)^{(1+\vartheta)\log(1+2\delta)}(1+\delta)^{-\log(1-2\delta)},$$

which is less than 1 if δ is small enough; for its logarithm $\sim -2\vartheta\delta^2$ as $\delta \to 0$, and so is negative for small δ. Hence we may take ϵ and η so small that the original expression is less than 1; and the result then follows.

7.42. The occurrence of gaps in the series is not merely a useful device for producing over-convergence. It has an essential connexion with it. This is shown by the following theorem, which is a sort of converse of the preceding one.

If a sequence $s_{p_k}(z)$ of partial sums of the series $f(z) = \sum a_n z^n$, with radius of convergence 1, *is uniformly convergent in the neighbourhood of a point on the unit circle, then*

$$f(z) = g(z) + r(z),$$

where the power series $g(z)$ has an infinite number of gaps p_k, q_k, where $q_k > (1+\vartheta)p_k$, and the radius of convergence of the power series $r(z)$ is greater than 1.

We shall not give the proof, which is more difficult than that of the direct theorem.

7.43. Hadamard's gap theorem. *If, in the power series*

$$f(z)=\sum_{n=0}^{\infty} a_n z^n,$$

$a_n = 0$ *except when* n *belongs to a sequence* n_k *such that* $n_{k+1} > (1+\vartheta)n_k$, *where* $\vartheta > 0$, *then the circle of convergence of the series is a natural boundary of the function.*

This is an almost immediate corollary of the theorem on over-convergence. For, if $f(z)$ were regular at any point of the circle, the series would be over-convergent at that point, i.e. the sequence

$$s_{n_k}(z)=\sum_{n=1}^{n_k} a_n z^n$$

would be convergent at a point outside the circle. But for a series of the given form this sequence of partial sums is the same as the whole sequence of partial sums. Hence over-convergence is impossible, and consequently every point of the circle of convergence is a singularity of $f(z)$.

7.44. Mordell's proof of the theorem.* This is a very simple direct proof. Suppose that the radius of convergence is 1. Let $z = aw^p + bw^{p+1}$, where $0 < a < 1$, $a+b=1$, and p is a positive integer. Clearly $|z| \leqslant 1$ if $|w| \leqslant 1$; and it is easily seen that $|z| < 1$ if $|w| \leqslant 1$, except that $z = 1$ if $w = 1$. Let

$$\begin{aligned}\phi(w) = f(z) &= \sum a_n(aw^p + bw^{p+1})^n \\ &= \sum a_n(a^n w^{pn} + \ldots + b^n w^{(p+1)n}) = \sum b_n w^n.\end{aligned}$$

Then $\phi(w)$ is regular for $|w| \leqslant 1$, except possibly at $w = 1$. We shall show that the radius of convergence of the power series for $\phi(w)$ is 1, and hence that $w = 1$ is a singularity of $\phi(w)$.

We observe that, in the last expression but one for $\phi(w)$, no power of w occurs twice if

$$(p+1)n_k < pn_{k+1},$$

i.e.

$$p\left(\frac{n_{k+1}}{n_k} - 1\right) > 1$$

throughout the series; and this is true if $p > 1/\vartheta$. The expression $\sum b_n w^n$ is then obtained by simply omitting the brackets in the previous expression.

* Mordell (1).

If the series for $\phi(w)$ had a radius of convergence greater than 1, it would be convergent for a real $w > 1$, and therefore the series for $f(z)$ would be convergent for a real $z > 1$, which is false. This proves the theorem.

There is still another proof,* depending on the criterion of § 7.23.

The theorem of § 7.41 can be proved in a similar way.† Let the series for $f(z)$ satisfy the condition of § 7.41. Then $\phi(w)$ can have no singularity for $|w| \leqslant 1$ except possibly at $w = 1$. Hence if $f(z)$ is regular at $z = 1$, $\phi(w)$ is regular at $w = 1$, and so in $|w| < 1+\delta$ for some positive δ. Hence $\sum b_n w^n$ converges for $|w| < 1+\delta$, and in particular $\sum\limits_{n=0}^{(p+1)p_k} b_n w^n$ converges for $|w| < 1+\delta$. Hence $\sum\limits_{n=0}^{p_k} a_n z^n$ converges in a region of which $z = 1$ is an interior point.

7.5. Asymptotic behaviour near the circle of convergence. If the coefficients in the power series satisfy a sufficiently simple law as $n \to \infty$, we can deduce an asymptotic expression for the function $f(z)$ as z approaches the circle of convergence along a radius vector. The simplest case of this process is given by the following theorem.

Let $$f(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad g(x) = \sum_{n=0}^{\infty} b_n x^n,$$

where $a_n \geqslant 0$, $b_n \geqslant 0$, and the series converge for $0 < x < 1$ and diverge for $x = 1$. If, as $n \to \infty$,

$$a_n \sim C b_n, \tag{1}$$

then as $x \to 1$ $$f(x) \sim C g(x). \tag{2}$$

Given ϵ, we can find N such that

$$|a_n - C b_n| < \epsilon b_n \qquad (n > N).$$

* See Landau, *Ergebnisse*, § 19.

† Pointed out by Mr. M. M. Crum.

Then

$$\begin{aligned}|f(x)-Cg(x)| &= \left|\sum_{n=0}^{\infty}(a_n-Cb_n)x^n\right| \\ &\leqslant \left|\sum_{n=0}^{N}(a_n-Cb_n)x^n\right|+\left|\sum_{n=N+1}^{\infty}(a_n-Cb_n)x^n\right| \\ &\leqslant \sum_{n=0}^{N}|a_n-Cb_n|+\epsilon\sum_{n=N+1}^{\infty}b_nx^n \\ &\leqslant \sum_{n=0}^{N}|a_n-Cb_n|+\epsilon g(x).\end{aligned}$$

Having fixed N, we can, since $g(x)\to\infty$, choose δ so that

$$\sum_{n=0}^{N}|a_n-Cb_n|<\epsilon g(x) \qquad (x>1-\delta).$$

Then $$|f(x)-Cg(x)|<2\epsilon g(x) \qquad (x>1-\delta),$$

which proves the theorem.

The same result, however, holds under more general conditions. *Let the series converge for* $0<x<1$; *let*

$$s_n=a_0+a_1+\ldots+a_n, \qquad t_n=b_0+b_1+\ldots+b_n,$$

and let s_n *and* t_n *be positive, and* $\sum s_n$ *and* $\sum t_n$ *divergent, and let*

$$s_n\sim Ct_n. \tag{3}$$

Then (2) *is still true.*

For as in § 7.22, for $0<x<1$

$$f(x)=(1-x)\sum_{n=0}^{\infty}s_nx^n, \qquad g(x)=(1-x)\sum_{n=0}^{\infty}t_nx^n,$$

and by the previous theorem

$$\sum_{n=0}^{\infty}s_nx^n\sim C\sum_{n=0}^{\infty}t_nx^n.$$

Hence the result.

In particular, *if* $s_n\sim Cn$, *then*

$$f(x)\sim\frac{C}{1-x}.$$

Examples. (i) If $p<1$, as $x\to 1$

$$\sum_{n=1}^{\infty}\frac{x^n}{n^p}\sim\frac{\Gamma(1-p)}{(1-x)^{1-p}}.$$

[We have $$(1-x)^{p-1}=\frac{1}{\Gamma(1-p)}\sum_{n=0}^{\infty}\frac{\Gamma(n-p+1)}{\Gamma(n+1)}x^n,$$

and we can use the lemma of § 1.87.]

(ii) Show that if

$$F(\alpha, \beta, \gamma, x) = 1 + \frac{\alpha.\beta}{1.\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2 + \ldots,$$

then, as $x \to 1$,

$$F(\alpha, \beta, \gamma, x) \sim \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \frac{1}{(1-x)^{\alpha+\beta-\gamma}}$$

if $\alpha+\beta > \gamma$; and that

$$F(\alpha, \beta, \alpha+\beta, x) \sim \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \log \frac{1}{1-x}.$$

7.51. The converse problem. It is easily seen that there is no general converse of the above theorems; from the asymptotic behaviour of $f(x)$ we cannot deduce that of a_n, or even of s_n. Consider, for example, the function

$$f(x) = \frac{1}{(1+x)^2(1-x)} = (1-x)\sum_{n=0}^{\infty}(n+1)x^{2n}$$

$$= \sum_{n=0}^{\infty}(n+1)(x^{2n}-x^{2n+1}).$$

Here $s_{2m+1} = 0$, while $s_{2m} = m+1$; hence s_n oscillates infinitely, though $f(x) \sim \frac{1}{4}/(1-x)$.

The coefficients in this example are, of course, not all positive; and this is, in a sense, the cause of the failure of the converse theorem. If we assume that all the coefficients are positive, we can state a precise converse of the last result of the previous section.

If $a_n \geqslant 0$ for all values of n, and as $x \to 1$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \sim \frac{1}{1-x},$$

then as $n \to \infty$

$$s_n = \sum_{\nu=0}^{n} a_\nu \sim n.$$

This theorem is due to Hardy and Littlewood.* We shall give an extremely elegant proof which has recently been obtained by Karamata.†

7.52. In order to appreciate the point of the proof, it may be well to see what can be proved by fairly obvious arguments. In the first place

$$f(x) \geqslant \sum_{\nu=0}^{n} a_\nu x^\nu \geqslant x^n s_n$$

* Hardy and Littlewood (2). † Karamata (1).

for all values of x and n. Taking $x = e^{-1/n}$, we obtain, since $f(x) < A/(1-x)$,

$$e^{-1}s_n \leqslant \frac{A}{1-x} = \frac{A}{1-e^{-1/n}} < An,$$

say
$$s_n < A_1 n. \tag{1}$$

On the other hand, using (1), we have

$$\begin{aligned} f(x) &= (1-x)\sum_{m=0}^{\infty} s_m x^m \\ &< (1-x)s_n \sum_{m=0}^{n-1} x^m + A_1(1-x)\sum_{m=n}^{\infty} m x^m \\ &< s_n + A_1 n x^n + \frac{A_1 x^{n+1}}{1-x}. \end{aligned}$$

Taking $x = e^{-\lambda/n}$, we obtain, since $f(x) > A/(1-x) > An/\lambda$ if $n > 2\lambda$,

$$\frac{An}{\lambda} < s_n + Ane^{-\lambda} + \frac{Ane^{-\lambda}}{\lambda}.$$

Hence, if λ is sufficiently large,

$$s_n > A_2 n. \tag{2}$$

What we have to show is that A_1 and A_2 can be replaced by $1+\epsilon$ and $1-\epsilon$ respectively. The above argument is too crude to do this, and the method actually used is far from being an obvious one.

7.53. Karamata's proof. The proof depends on the well-known theorem of Weierstrass, that we can approximate uniformly to any continuous function by a sequence of polynomials.* Let $g(x)$ be continuous in $(0, 1)$, and ϵ a given positive number. Then there are polynomials $p(x)$, $P(x)$, such that

$$p(x) \leqslant g(x) \leqslant P(x), \tag{1}$$

and
$$\int_0^1 \{g(x)-p(x)\}\,dx \leqslant \epsilon, \qquad \int_0^1 \{P(x)-g(x)\}\,dx \leqslant \epsilon. \tag{2}$$

This is obviously true if $p(x)$ and $P(x)$ differ by at most $\frac{1}{2}\epsilon$ from $g(x)-\frac{1}{2}\epsilon$ and $g(x)+\frac{1}{2}\epsilon$ respectively.

If $g(x)$ has a discontinuity of the first kind in the interval, say at $x = c$, we can still construct polynomials satisfying (1) and (2). Suppose, for example, that $g(c-0) < g(c+0)$. Let

* A proof is given in § 13.33. For another proof see Goursat, *Cours d'Analyse*, t. 1, § 206.

$\phi(x) = g(x) + \frac{1}{2}\epsilon$ for $x < c - \delta$ and for $x > c$; and, for $c - \delta \leqslant x \leqslant c$, let $\phi(x) = \max\{l(x), g(x) + \frac{1}{4}\epsilon\}$, where $l(x)$ is the linear function of x such that $l(c-\delta) = g(c-\delta) + \frac{1}{2}\epsilon$, $l(c) = g(c+0) + \frac{1}{2}\epsilon$. Then $\phi(x)$ is continuous, and $\phi(x) > g(x)$. It is easily seen that, if δ is small enough, a polynomial $P(x)$ which approximates sufficiently closely to $\phi(x)$ has the required properties. Similarly we may construct $p(x)$.

To prove the theorem of Hardy and Littlewood, we first prove that

$$\lim_{x \to 1}(1-x) \sum_{n=0}^{\infty} a_n x^n P(x^n) = \int_0^1 P(t)\, dt \tag{3}$$

for any polynomial $P(x)$. It is clearly sufficient to consider the case $P(x) = x^k$. Then the left-hand side is

$$(1-x) \sum_{n=0}^{\infty} a_n x^{n+kn} = \frac{1-x}{1-x^{k+1}} \left\{(1-x^{k+1}) \sum_{n=0}^{\infty} a_n (x^{k+1})^n\right\}$$

$$\to \frac{1}{k+1} = \int_0^1 x^k\, dx,$$

and the result follows.

Next, we have

$$\lim_{x \to 1}(1-x) \sum_{n=0}^{\infty} a_n x^n g(x^n) = \int_0^1 g(t)\, dt \tag{4}$$

if $g(t)$ is continuous, or has a discontinuity of the first kind. For let $p(x)$ and $P(x)$ be polynomials satisfying (1) and (2). Then, since $g(x) \leqslant P(x)$, and the coefficients are positive,

$$\overline{\lim}(1-x) \sum_{n=0}^{\infty} a_n x^n g(x^n) \leqslant \overline{\lim}(1-x) \sum_{n=0}^{\infty} a_n x^n P(x^n)$$

$$= \int_0^1 P(t)\, dt < \int_0^1 g(t)\, dt + \epsilon.$$

Making $\epsilon \to 0$, it follows that

$$\overline{\lim}(1-x) \sum_{n=0}^{\infty} a_n x^n g(x^n) \leqslant \int_0^1 g(t)\, dt.$$

Similarly, arguing with $p(x)$, we obtain

$$\underline{\lim}(1-x) \sum_{n=0}^{\infty} a_n x^n g(x^n) \geqslant \int_0^1 g(t)\, dt,$$

and (4) follows.

Now let

$$g(t) = 0 \ (0 \leqslant t < e^{-1}), \qquad = 1/t \ (e^{-1} \leqslant t \leqslant 1).$$

Then
$$\int_0^1 g(t)\,dt = \int_{1/e}^1 \frac{dt}{t} = 1. \tag{5}$$

Let $x = e^{-1/N}$. Then

$$\sum_{n=0}^{\infty} a_n x^n g(x^n) = \sum_{n \leqslant 1/\log(1/x)} a_n = \sum_{n=0}^{N} a_n = s_N,$$

and so, by (4) and (5), $s_N \sim 1/(1-x) \sim N$. This proves the theorem.

7.6. Abel's theorem and its converse. In this section we return to a subject already discussed in Chapter I. In § 1.22 we proved Abel's theorem for real power series: *if the series*

$$\sum_{n=0}^{\infty} a_n$$

converges to the sum s, then

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \to s$$

as $x \to 1$ through real values. In § 1.23 we proved Tauber's theorem, that *the converse deduction holds, provided that* $a_n = o(1/n)$. We shall now consider a number of generalizations of these theorems.*

7.61. *If*
$$\sum_{n=0}^{\infty} a_n = s, \tag{1}$$

then
$$f(z) = \sum_{n=0}^{\infty} a_n z^n \to s \tag{2}$$

as $z \to 1$ along any path lying between two chords of the unit circle which pass through $z = 1$.

As in § 1.22, it is sufficient to show that the power series is uniformly convergent, but now we must prove uniform convergence in a region included between two chords through $z = 1$, and a sufficiently small circle with centre at $z = 1$.

We have to adapt the argument used to prove Abel's lemma (§ 1.131) to the present conditions. Let

$$s_{n,p} = a_n + a_{n+1} + \ldots + a_p,$$

* Landau, *Ergebnisse*, Ch. III, and Hardy and Littlewood (1), (2), (3), (4).

so that $|s_{n,p}| < \epsilon$ $(n_0 \leqslant n < p)$. Then

$$\sum_{\nu=n}^{m} a_\nu z^\nu = s_{n,n}z^n + (s_{n,n+1} - s_{n,n})z^{n+1} + \ldots + (s_{n,m} - s_{n,m-1})z^m$$
$$= s_{n,n}(z^n - z^{n+1}) + \ldots + s_{n,m-1}(z^{m-1} - z^m) + s_{n,m}z^m.$$

Hence for $n \geqslant n_0$

$$\left|\sum_{\nu=n}^{m} a_\nu z^\nu\right| \leqslant \epsilon\left\{\sum_{\nu=n}^{m-1} |z^\nu - z^{\nu+1}| + |z|^m\right\}$$
$$< \epsilon\left\{|1-z| \sum_{\nu=0}^{\infty} |z|^\nu + 1\right\}$$
$$= \epsilon\left\{\frac{|1-z|}{1-|z|} + 1\right\}.$$

The result now follows as in the previous case, provided that

$$\frac{|1-z|}{1-|z|}$$

is bounded as $z \to 1$ on the path considered. It is this that makes it necessary to restrict the path, for this function can be made large by taking z near to 1, but still nearer to the circumference.

Suppose, then, that

$$|1-z| \leqslant k(1-|z|) \qquad (k > 1). \tag{3}$$

This inequality is satisfied in a region bounded by the curve

$$|1-z| = k(1-|z|).$$

Putting $1-z = \rho e^{i\phi}$, the equation becomes

$$\rho = k - k|1-\rho e^{i\phi}|,$$

i.e.
$$(\rho-k)^2 = k^2(1-2\rho\cos\phi+\rho^2),$$

i.e.
$$\rho = 2\frac{k^2\cos\phi - k}{k^2-1}.$$

This represents a curve with two branches through $z = 1$, each making an angle $\arccos(1/k)$ with the real axis. By choosing k sufficiently large we can make the curve include any region of the required type. Since (3) is satisfied inside the curve, the theorem now follows.

7.62. We can also obtain a similar extension of Tauber's theorem.

If $f(z) \to s$ as $z \to 1$ along a path satisfying the same conditions as before, and $a_n = o(1/n)$, then $\sum a_n$ converges to the sum s.

In view of the above analysis, the proof given in § 1.23 now requires little modification. We have to prove that

$$S_1 - S_2 = \sum_{n=N+1}^{\infty} a_n z^n - \sum_{n=0}^{N} a_n(1-z^n) \to 0,$$

where $N = [1/(1-|z|)]$. As before, if $|na_n| < \epsilon$ for $n > N$,

$$|S_1| = \left|\sum_{N+1}^{\infty} na_n \cdot \frac{z^n}{n}\right| < \frac{\epsilon}{N+1} \sum_{N+1}^{\infty} |z|^n < \frac{\epsilon}{(N+1)(1-|z|)} < \epsilon.$$

Now $\quad |1-z^n| = |(1-z)(1+z+\ldots+z^{n-1})| \leqslant |1-z|n.$

Hence, if 7.61 (3) is satisfied,

$$|S_2| \leqslant \sum_{n=1}^{N} |na_n(1-z)| \leqslant k(1-|z|) \sum_{n=1}^{N} n|a_n| \leqslant \frac{k}{N} \sum_{n=1}^{N} n|a_n|,$$

and this tends to zero, by the lemma of § 1.23. This proves the theorem.

7.63. Tauber's theorem for regular paths. It is not possible to extend Abel's theorem, at any rate in its obvious form, to paths which touch the unit circle; for example, it is known* that the series

$$\sum_{n=1}^{\infty} n^{-b} e^{in^a} \qquad (0 < a < 1)$$

is convergent if $b > 1-a$; but, if $b < 1-\frac{1}{2}a$, the function

$$f(z) = \sum_{n=1}^{\infty} n^{-b} e^{in^a} z^n$$

does not tend to a limit as $z \to 1$ along an arc of a circle touching the unit circle at $z = 1$.

On the other hand, we can obtain an extension of Tauber's theorem to paths which touch the circle, provided that they are sufficiently regular.

A path will be called 'regular' if it is defined by equations $x = x(t)$, $y = y(t)$, where $x'(t)$ and $y'(t)$ are continuous and never both 0, so that there is a definite tangent at each point.

If $f(z) \to s$ as $z \to 1$ along a regular path inside the circle, and $a_n = o(1/n)$, then $\sum a_n$ converges to the sum s.

We may suppose without loss of generality that $s = 0$ Let

* See Hardy and Littlewood (3), p. 207.

C be the path in question. Then the integral

$$\int_z^1 f(w)\,dw,$$

taken along C, exists; and it is $o(|1-z|)$ as $z \to 1$. For, given ϵ, we have $|f(w)| \leqslant \epsilon$ for w sufficiently near to 1 on C. Hence

$$\left|\int_z^1 f(w)\,dw\right| \leqslant \epsilon\, l(z),$$

where $l(z)$ is the length of C from z to 1. But $l(z) \sim |1-z|$ as $z \to 1$; for if $t=0$ corresponds to $z=1$,

$$\frac{l(z)}{t} = \frac{1}{t}\int_0^t [\{x'(u)\}^2+\{y'(u)\}^2]^{\frac{1}{2}}\,du \to [\{x'(0)\}^2+\{y'(0)\}^2]^{\frac{1}{2}},$$

$x'(u)$ and $y'(u)$ being continuous; and

$$\frac{x-1}{t} \to x'(0), \qquad \frac{y}{t} \to y'(0).$$

Hence
$$\int_z^1 f(w)\,dw = o(|1-z|). \tag{1}$$

Now if z and z' are points on C,

$$\int_z^{z'} f(w)\,dw = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z'^{n+1}-z^{n+1}).$$

This series converges uniformly with respect to z' for $|z'| \leqslant 1$ (since $a_n = o(1/n)$). Hence, making $z' \to 1$,

$$\int_z^1 f(w)\,dw = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(1-z^{n+1}). \tag{2}$$

Let $N = [1/|1-z|]$. Then

$$\int_z^1 f(w)\,dw = \sum_{n=0}^{N} + \sum_{N+1}^{\infty} \frac{a_n}{n+1}(1-z^{n+1}) = \Sigma_1 + \Sigma_2,$$

and
$$\Sigma_2 = \sum_{N+1}^{\infty} o\left(\frac{1}{n^2}\right) = o\left(\frac{1}{N}\right) = o(|1-z|). \tag{3}$$

Also

$$\begin{aligned} 1-z^{n+1} &= (1-z)(1+z+\ldots+z^n) \\ &= (1-z)(n+1)-(1-z)^2\{n+(n-1)z+\ldots+z^{n-1}\} \\ &= (1-z)(n+1)+O(|1-z|^2n^2). \end{aligned}$$

Hence

$$\begin{aligned}\sum_1 &= (1-z)\sum_{n=0}^{N} a_n + O\Big(|1-z|^2 \sum_{n=1}^{N} n|a_n|\Big)\\ &= (1-z)s_N + o(|1-z|^2 N)\\ &= (1-z)s_N + o(|1-z|). \qquad (4)\end{aligned}$$

From (1), (3), and (4) it follows that $s_N = o(1)$, and this proves the theorem.

7.64. Littlewood's extension of Tauber's theorem. We now pass to an extension of quite a different kind. In all the forms of Tauber's theorem so far considered, the condition $a_n = o(1/n)$ has played an apparently essential part. It was, however, discovered by Littlewood that it can be replaced by the more general condition $a_n = O(1/n)$. Here we shall restrict ourselves for the sake of simplicity to the real axis, though it is possible to prove the theorem for complex paths.

7.65. We use the following lemma:

If $f(x)$ is a real function with differential coefficients of the first two orders for $0 \leqslant x < 1$, and, as $x \to 1$,

$$f(x) = o(1), \qquad f''(x) = O\left\{\frac{1}{(1-x)^2}\right\},$$

then

$$f'(x) = o\left(\frac{1}{1-x}\right).$$

Let $x' = x + \delta(1-x)$, where $0 < \delta < \frac{1}{2}$. Then

$$f(x') = f(x) + \delta(1-x)f'(x) + \tfrac{1}{2}\delta^2(1-x)^2 f''(\xi),$$

where $x < \xi < x'$. Hence

$$\begin{aligned}(1-x)f'(x) &= \frac{f(x')-f(x)}{\delta} + \tfrac{1}{2}\delta(1-x)^2 f''(\xi)\\ &= \frac{f(x')-f(x)}{\delta} + O(\delta), \qquad (1)\end{aligned}$$

since $\quad f''(\xi) = O\left\{\frac{1}{(1-\xi)^2}\right\} = O\left\{\frac{1}{(1-x')^2}\right\} = O\left\{\frac{1}{(1-x)^2}\right\}.$

By first choosing δ sufficiently small, and then x sufficiently near to 1, the right-hand side of (1) can be made as small as we please. This proves the lemma.

7.66. Littlewood's theorem. *If $f(x) = \sum a_n x^n \to s$ as $x \to 1$, and $a_n = O(1/n)$, then $\sum a_n$ converges to the sum s.*

The proof * depends on the theorem of § 7.51, and in proving

* The original proof, Littlewood (3), was different.

that theorem we have really overcome the most serious difficulties. We may obviously suppose, without loss of generality, that the limit s is zero. Then

$$f(x)=\sum_{n=0}^{\infty} a_n x^n = o(1)$$

as $x \to 1$. Also, since $a_n = O(1/n)$,

$$f''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = O\left\{\sum_{n=2}^{\infty}(n-1)x^{n-2}\right\} = O\left\{\frac{1}{(1-x)^2}\right\}.$$

Hence, by the lemma,

$$f'(x)=\sum_{n=1}^{\infty} na_n x^{n-1} = o\left(\frac{1}{1-x}\right).$$

Suppose that $|na_n| \leqslant c$. Then

$$\sum_{n=1}^{\infty}\left(1-\frac{na_n}{c}\right)x^{n-1} = \frac{1}{1-x}-\frac{f'(x)}{c} \sim \frac{1}{1-x}.$$

But the coefficients in this series are all positive, and so, by the theorem of § 7.51,

$$\sum_{\nu=1}^{n}\left(1-\frac{\nu a_\nu}{c}\right) \sim n,$$

or

$$\sum_{\nu=1}^{n} \nu a_\nu = o(n). \tag{1}$$

This is an asymptotic formula for a finite sum, and so is a considerable step in the right direction. To get exactly the required result, still another argument is required.

Let w_n denote the left-hand side of (1) if $n > 0$, and let $w_0 = 0$. Then

$$\begin{aligned} f(x)-a_0 &= \sum_{n=1}^{\infty}\frac{w_n-w_{n-1}}{n}x^n = \sum_{n=1}^{\infty} w_n\left(\frac{x^n}{n}-\frac{x^{n+1}}{n+1}\right) \\ &= \sum_{n=1}^{\infty} w_n\left\{\frac{x^n-x^{n+1}}{n+1}+\frac{x^n}{n(n+1)}\right\} \\ &= (1-x)\sum_{n=1}^{\infty}\frac{w_n}{n+1}x^n + \sum_{n=1}^{\infty}\frac{w_n}{n(n+1)}x^n. \end{aligned}$$

Since $w_n = o(n)$, the first term on the right is $o(1)$ as $x \to 1$. Hence, since $f(x) \to 0$,

$$\sum_{n=1}^{\infty}\frac{w_n}{n(n+1)}x^n \to -a_0.$$

But $w_n/\{n(n+1)\} = o(1/n)$, and so, by the ordinary form of Tauber's theorem,

$$\sum_{n=1}^{\infty} \frac{w_n}{n(n+1)} = -a_0.$$

The left-hand side is

$$\lim_{N\to\infty} \sum_{n=1}^{N} \frac{w_n}{n(n+1)} = \lim_{N\to\infty} \sum_{n=1}^{N} w_n \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \lim_{N\to\infty} \left\{\sum_{n=1}^{N} \frac{w_n - w_{n-1}}{n} - \frac{w_N}{N+1}\right\} = \lim_{N\to\infty} \sum_{n=1}^{N} a_n,$$

and the theorem is therefore proved.

7.7. Partial sums of a power series.* The study of the partial sums of a power series is facilitated by the use of the formulae of the theory of Fourier series. We shall use some of these formulae, and quote them from Chapter XIII; but in each case where they are used here the proof is an immediate consequence of uniform convergence.

Let $$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (|z| < 1),$$

and $$s_n(z) = a_0 + a_1 z + \ldots + a_n z^n.$$

Let

$$k(r,\theta) = \frac{1-r^2}{2(1-2r\cos\theta + r^2)} = \tfrac{1}{2} + r\cos\theta + r^2\cos 2\theta + \ldots,$$

and $$k_n(r,\theta) = \frac{1-r^2-2r^{n+1}\{\cos(n+1)\theta - r\cos n\theta\}}{2(1-2r\cos\theta+r^2)}$$

$$= \tfrac{1}{2} + \text{[illegible]}\cos\theta + \ldots + r^n\cos n\theta.$$

Then

$$s_n(re^{i\theta}) = \frac{1}{\pi}\int_0^{2\pi} f(\rho e^{i(\theta-\phi)}) k_n\left(\frac{r}{\rho}, \phi\right) d\phi \qquad (0 < r < \rho < 1). \quad (1)$$

This may be proved directly by term-by-term integration. It is a case of Parseval's formula (§ 13.54).

Also, by Dirichlet's integral (§ 13.2),

$$k_n(r,\theta) = \frac{1}{\pi}\int_0^{2\pi} \frac{\sin(n+\frac{1}{2})(\theta-\phi)}{2\sin\frac{1}{2}(\theta-\phi)} k(r,\phi)\, d\phi. \qquad (2)$$

* Landau (2), (3), (4), and *Ergebnisse*, Ch. I.

We can thus express s_n as a repeated integral involving f and k,

$$s_n(re^{i\theta}) = \frac{1}{2\pi^2}\int_0^{2\pi} f(\rho e^{i(\theta-\phi)})\,d\phi \int_0^{2\pi} \frac{\sin(n+\frac{1}{2})(\phi-\psi)}{\sin\frac{1}{2}(\phi-\psi)}\,k\left(\frac{r}{\rho},\psi\right)d\psi. \quad (3)$$

We consider also the arithmetic means of the partial sums,

$$\sigma_n(z) = \{s_0(z)+s_1(z)+\ldots+s_{n-1}(z)\}/n.$$

By (1),
$$\sigma_n(re^{i\theta}) = \frac{1}{\pi}\int_0^{2\pi} f(\rho e^{i(\theta-\phi)})K_n\left(\frac{r}{\rho},\phi\right)d\phi, \quad (4)$$

where
$$K_n(r,\theta) = \frac{1}{n}\sum_{\nu=0}^{n-1} k_\nu(r,\theta);$$

and, by Fejér's integral (§ 13.31),

$$K_n(r,\theta) = \frac{1}{2n\pi}\int_0^{2\pi} \frac{\sin^2\frac{1}{2}n(\theta-\phi)}{\sin^2\frac{1}{2}(\theta-\phi)}\,k(r,\phi)\,d\phi. \quad (5)$$

7.71. Bounded power series. Suppose now that $f(z)$ is bounded in the unit circle.

If $|f(z)| \leqslant M$ for $|z| < 1$, then $|\sigma_n(z)| \leqslant M$ for all values of n and $|z| < 1$; and, conversely, if $|\sigma_n(z)| \leqslant M$ for all n and $|z| < 1$, then $|f(z)| \leqslant M$.

It is clear from the above formulae that $k(r,\theta)$ and $K_n(r,\theta)$ are positive for $r < 1$. Hence, if $|f(z)| \leqslant M$, it follows from (4) that

$$|\sigma_n(re^{i\theta})| \leqslant \frac{1}{\pi}\int_0^{2\pi} MK_n\left(\frac{r}{\rho},\phi\right)d\phi.$$

But the right-hand side is what σ_n reduces to in the case $f(z) = M$, viz. M. This proves the first part.

Again $s_n(z)$, and so also $\sigma_n(z)$, tends to $f(z)$ as $n \to \infty$. The second part follows at once from this.

7.72. The corresponding results for $s_n(z)$ are not so simple. This is due to the fact that k_n, unlike K_n, is not always positive. It is not necessarily true that $|s_n(z)| \leqslant M$ for all values of n and z. In fact it is known* that the upper bound of $|s_n(z)|$, for all functions $f(z)$ such that $|f(z)| \leqslant M$, tends to infinity with n. We have, however, the following result:

* Landau, *Ergebnisse*, § 2.

There is an absolute constant A such that

$$|s_n(z)| < AM \log n$$

for all functions $f(z)$ such that $|f(z)| \leqslant M$.

If $|f(z)| \leqslant M$, by § 7.7 (3),

$$|s_n(re^{i\theta})| \leqslant \frac{M}{2\pi^2} \int_0^{2\pi} k\left(\frac{r}{\rho}, \psi\right) d\psi \int_0^{2\pi} \left|\frac{\sin(n+\frac{1}{2})(\phi-\psi)}{\sin\frac{1}{2}(\phi-\psi)}\right| d\phi.$$

The inner integral is equal to

$$2\int_0^{\pi} \left|\frac{\sin(n+\frac{1}{2})\alpha}{\sin\frac{1}{2}\alpha}\right| d\alpha < 2\int_0^{1/(n+\frac{1}{2})} \frac{(n+\frac{1}{2})\alpha}{\sin\frac{1}{2}\alpha}\, d\alpha + 2\int_{1/(n+\frac{1}{2})}^{\pi} \frac{d\alpha}{\sin\frac{1}{2}\alpha}$$

$$= O(1) + O(\log n);$$

and, putting $n = 0$ in 7.7 (2),

$$\frac{1}{2\pi}\int_0^{2\pi} k\left(\frac{r}{\rho}, \psi\right) d\psi = \tfrac{1}{2}.$$

This proves the theorem.

7.73. It is easily seen that $s_n(z)$ is bounded in a circle of radius r' less than 1; for $k_n(r, \theta)$ is obviously bounded in such a circle. The upper bound for $s_n(z)$ depends on M and on r'. What is not so obvious is that we can choose r', independent of M, so that the upper bound is exactly M.

*If $|f(z)| \leqslant M$, then $|s_n(z)| \leqslant M$ for $|z| \leqslant \frac{1}{2}$.**

It is clear that

$$k_n(r, \phi) \geqslant \frac{1 - r^2 - 2r^{n+1}(1+r)}{2(1-r)^2},$$

and if $r \leqslant \frac{1}{2}$, $n \geqslant 1$, the numerator is not less than

$$1 - \tfrac{1}{4} - 2 . \tfrac{1}{4}(1 + \tfrac{1}{2}) = 0.$$

Hence $k_n(r, \phi) \geqslant 0$ for $r \leqslant \frac{1}{2}$, and we can now proceed as in § 7.71. We have

$$|s_n(re^{i\theta})| \leqslant \frac{1}{\pi}\int_0^{2\pi} M k_n\left(\frac{r}{\rho}, \phi\right) d\phi \qquad (r \leqslant \tfrac{1}{2}\rho),$$

and the right-hand side is what $s_n(z)$ reduces to when $f(z) = M$, viz. M. This proves the theorem.

* Fejér (5).

The number $\frac{1}{2}$ is the greatest number with this property. For consider the function

$$f(z) = \frac{z-a}{az-1} \qquad (0 < a < 1).$$

Then $|f(e^{i\theta})| = 1$, so that $|f(z)| \leqslant 1$ for $|z| \leqslant 1$. Also

$$s_1(z) = a + (a^2-1)z,$$

$$s_1\left(-\frac{1}{2a}\right) = \frac{a^2+1}{2a} > 1,$$

and the point $-\frac{1}{2}/a$ where $s_1(z) > 1$ is arbitrarily near to $|z| = \frac{1}{2}$, since a is arbitrarily near to 1.

7.8. The zeros of partial sums.* Let

$$f(z) = a_0 + a_1 z + \ldots \qquad (a_0 \neq 0),$$

be a power series with radius of convergence 1, and let

$$s_n(z) = a_0 + a_1 z + \ldots + a_n z^n.$$

Then $s_n(z)$, being a polynomial of degree n, has n zeros.

If $f(z)$ has zeros inside the circle of convergence, then by Hurwitz's theorem (§ 3.45) every such zero is a limit-point of zeros of the polynomials $s_n(z)$.

Now consider the simplest function of the above type,

$$f(z) = \frac{1}{1-z} = 1 + z + z^2 + \ldots.$$

Here $$s_n(z) = 1 + z + \ldots + z^n = \frac{1-z^{n+1}}{1-z}.$$

Hence $s_n(z)$ has zeros distributed evenly round the circle, and it is plain that every point of the circle is a limit-point of such zeros.

It is somewhat remarkable that the general case is so nearly like this simple case. This was discovered by Jentzsch, who proved that, *for every power series, every point of the circle of convergence is a limit-point of zeros of partial sums.*

We shall deduce this from some quite simple ideas depending on the theory of equations.

Let δ be a given positive number, n a number such that

$$|a_n| > \frac{|a_0|}{(1+\delta)^n}. \tag{1}$$

* Jentzsch (1).

This is true for arbitrarily large values of n, or the radius of convergence would be greater than 1.

Let $z_1, z_2, \ldots, z_n$ be the zeros of the corresponding $s_n(z)$. Then

$$z_1 z_2 \ldots z_n = (-1)^n a_0/a_n,$$

and so

$$|z_1 z_2 \ldots z_n| < (1+\delta)^n.$$

Let $z_1, \ldots, z_k$ be the zeros of $s_n(z)$ in the circle $|z| \leqslant 1-\delta$. By Hurwitz's theorem (§ 3.45), k is constant for sufficiently large values of n, and

$$|z_1 z_2 \ldots z_k| > K,$$

where K depends on δ only.

Let $z_{n-p+1}, \ldots, z_n$ be the zeros for which $|z| > 1+\epsilon$. Then

$$(1+\epsilon)^p < |z_{n-p+1} \ldots z_n| = \left| \frac{z_1 z_2 \ldots z_n}{z_1 \ldots z_k \cdot z_{k+1} \ldots z_{n-p}} \right| < \frac{(1+\delta)^n}{K(1-\delta)^{n-p}}.$$

Hence

$$p < \frac{n\{\log(1+\delta) - \log(1-\delta)\} - \log K}{\log(1+\epsilon) - \log(1-\delta)} < \frac{An\delta - \log K}{A\epsilon}.$$

By choosing first ϵ, then δ, and then n, we can make p/n arbitrarily small.

Hence, for given δ, ϵ, and η, the number of zeros in the circle $|z| \leqslant 1+\epsilon$ is greater than $n(1-\eta)$, if n is a sufficiently large integer for which (1) *is true.*

7.81. It is clear from the above result that *the zeros of partial sums have at least one limit-point on the circle of convergence.*

We can obtain a little more information by considering the sum

$$\sum_{\nu=1}^{n} \frac{1}{z_\nu} = -\frac{a_1}{a_0}.$$

Putting $z_\nu = r_\nu e^{i\theta_\nu}$, we have

$$\sum_{\nu=1}^{n} \frac{\cos\theta_\nu}{r_\nu} = -\mathbf{R}\left(\frac{a_1}{a_0}\right). \tag{1}$$

If $\theta_\nu > \frac{1}{2}\pi + \alpha$, or $\theta_\nu < -\frac{1}{2}\pi - \alpha$, where $\alpha > 0$, for every ν, the left-hand side is less than

$$\frac{-n(1-\eta)\sin\alpha}{1+\epsilon}$$

by the above theorem. This is inconsistent with (1), if n is large enough. Hence there must be zeros in any angle including

$(-\frac{1}{2}\pi, \frac{1}{2}\pi)$. Similarly there must be zeros in any angle greater than π.

To prove Jentzsch's theorem we have to replace such an angle by an arbitrarily small one. This is done by using a conformal transformation which magnifies the effect of the zeros in the immediate neighbourhood of the point considered.

7.82. Let

$$w = \frac{\cos\lambda - z}{z\cos\lambda - 1}, \qquad z = \frac{w + \cos\lambda}{1 + w\cos\lambda}, \tag{1}$$

where $0 < \lambda < \frac{1}{2}\pi$, and where $f(\cos\lambda) \neq 0$. This transforms the unit circle in the z-plane into the unit circle in the w-plane. The point $z = 1$ becomes $w = 1$. The point $z = e^{i\lambda}$ becomes

$$w = \frac{\frac{1}{2}(e^{-i\lambda} - e^{i\lambda})}{\frac{1}{2}(e^{2i\lambda} - 1)} = -e^{-i\lambda} = e^{i(\pi-\lambda)},$$

and similarly $z = e^{-i\lambda}$ becomes $w = e^{-i(\pi-\lambda)}$. Thus, if $z = re^{i\theta}$, $w = \rho e^{i\phi}$, the arc $-\lambda \leqslant \theta \leqslant \lambda$ of the unit circle is transformed into the arc $-\pi + \lambda \leqslant \phi \leqslant \pi - \lambda$.

The zeros z_ν of $s_n(z)$ are transformed into the zeros $w_\nu = \rho_\nu e^{i\phi_\nu}$ of the function

$$\begin{aligned}(1 + w\cos\lambda)^n s_n\left(\frac{w + \cos\lambda}{1 + w\cos\lambda}\right) \\ = s_n(\cos\lambda) + w\{n\cos\lambda\, s_n(\cos\lambda) + \sin^2\lambda\, s_n'(\cos\lambda)\} + \ldots \\ = b_0 + b_1 w + \ldots,\end{aligned}$$

say; and corresponding to § 7.81 (1) we have

$$\sum_{\nu=1}^{n} \frac{\cos\phi_\nu}{\rho_\nu} = -\mathbf{R}\left(\frac{b_1}{b_0}\right) = -n\cos\lambda - \sin^2\lambda\, \mathbf{R}\left\{\frac{s_n'(\cos\lambda)}{s_n(\cos\lambda)}\right\}. \tag{2}$$

The last term tends to a limit as $n \to \infty$, since $s_n(\cos\lambda) \to f(\cos\lambda)$, which we have supposed is not 0, and $s_n'(\cos\lambda) \to f'(\cos\lambda)$. Hence as $n \to \infty$

$$\sum_{\nu=1}^{n} \frac{\cos\phi_\nu}{\rho_\nu} \sim -n\cos\lambda. \tag{3}$$

Suppose now that the region of the w-plane

$$1 - \epsilon < \rho < 1 + \epsilon, \qquad -(\pi - \lambda + \alpha) < \phi < \pi - \lambda + \alpha, \tag{4}$$

where $0 < \epsilon < 1$, $0 < \alpha < \lambda$, is free from zeros. Put

$$\sum_{\nu=1}^{n} \frac{\cos\phi_\nu}{\rho_\nu} = \sum_{\rho_\nu \leqslant 1-\epsilon} + \sum_{1-\epsilon < \rho_\nu < 1+\epsilon} + \sum_{\rho_\nu \geqslant 1+\epsilon} = \Sigma_1 + \Sigma_2 + \Sigma_3. \tag{5}$$

Since $\rho = 1$ corresponds to $r = 1$, it follows from considerations

of continuity that the circles $\rho = 1-\epsilon$, $\rho = 1+\epsilon$ correspond to curves (in fact circles) inside and outside $r = 1$ respectively, which can be made as near to it as we please by taking ϵ small enough.

The number of terms in $\sum_1$ is less than $K = K(\delta, \epsilon, \lambda)$; and ρ_ν has a positive lower bound, since the zeros of $s_n(\cos\lambda)$ in question tend to zeros of $f(\cos\lambda)$. Hence

$$\sum_1 < K. \tag{6}$$

The number of terms in $\sum_3$ is, by § 7.8, less than ηn, where $\eta = \eta(n, \delta, \epsilon, \lambda)$ tends to 0 as $n \to \infty$ through a certain sequence of values. Hence

$$\sum_3 < \frac{\eta n}{1+\epsilon}. \tag{7}$$

In $\sum_2$ the number of terms exceeds $n(1-\eta)-K$, and by hypothesis $\cos\phi_\nu < -\cos(\lambda-\alpha)$ for each term of this sum. Hence

$$\sum_2 < -\frac{n(1-\eta)-K}{1+\epsilon}\cos(\lambda-\alpha). \tag{8}$$

From (5), (6), (7), (8) it follows that

$$\overline{\lim_{n\to\infty}}\, \frac{1}{n}\sum_{\nu=1}^{n} \frac{\cos\phi_\nu}{\rho_\nu} \leqslant -\frac{\cos(\lambda-\alpha)}{1+\epsilon}.$$

This contradicts (3) if $\alpha > 0$ and ϵ is small enough. There are therefore zeros in the region (4), and hence, since ϵ and α may be as small as we please, in any region containing the arc $\rho = 1$, $-\pi+\lambda < \phi < \pi-\lambda$. Hence, in the z-plane, there are zeros in any region containing the arc $r = 1$, $-\lambda < \theta < \lambda$. Finally, since λ may be as small as we please, it follows that $z = 1$ is a limit-point of zeros. Similarly every point on the unit circle is a limit-point of zeros.

MISCELLANEOUS EXAMPLES

1. If $|a_n/a_{n+1}| \to R$, then the radius of convergence of $\sum a_n z^n$ is R.

2. If
$$\left|\frac{a_n}{a_{n+1}}\right| = \left\{1 + \frac{c}{n} + o\left(\frac{1}{n}\right)\right\} R$$
where $c > 1$, then $\sum a_n z^n$ converges absolutely everywhere on its circle of convergence.

3. If $a_n/a_{n+1} \to 1$, then
$$\lim_{n\to\infty} \frac{s_n(z)}{a_n z^n} = \frac{z}{z-1}$$
uniformly for $|z| \geqslant 1+\delta > 1$. Hence show that all the limit-points of the zeros of partial sums are inside or on the unit circle. [S. Izumi (1).]

4. Show that as $x \to 1$

$$\sum_{n=0}^{\infty} x^{n^2} \sim \frac{1}{2}\sqrt{\left(\frac{\pi}{1-x}\right)},$$

and that, as $z \to e^{2i\pi p/q}$ along the radius vector,

$$\sum_{n=0}^{\infty} z^{n^2} \sim \frac{1}{2q}\sqrt{\left(\frac{\pi}{1-|z|}\right)} \sum_{r=0}^{q-1} e^{2i\pi pr^2/q}.$$

5. If $a_n \sim \log n$, then, as $x \to 1$,

$$\sum_{n=0}^{\infty} a_n x^n \sim \frac{1}{1-x}\log\frac{1}{1-x}.$$

$\left[\text{The right-hand side is } \sum\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)x^n\right].$

6. If $a_n \sim 1/\log n$, then, as $x \to 1$,

$$\sum_{n=0}^{\infty} a_n x^n \sim \frac{1}{(1-x)\log\{1/(1-x)\}}.$$

[If $\sum_p$ denotes a sum over the range $\epsilon p/\log(1/x) < n \leqslant \epsilon(p+1)/\log(1/x)$, then

$$\sum_p \frac{x^n}{\log n} < \frac{\epsilon e^{-\epsilon p}}{\log(1/x)\log\{1/\log(1/x)\}}, \text{ etc.}]$$

7. Show that if $a_n \geqslant 0$, and

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \sim \frac{1}{(1-x)^2},$$

then

$$s_n \sim \tfrac{1}{2}n^2.$$

[We have

$$f_1(x) = \int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1} \sim \frac{1}{1-x}.$$

Hence

$$\sum_{\nu=0}^{n} \frac{a_\nu}{\nu+1} \sim n,$$

and the result then follows by partial summation.]

8. Generally, if $a_n \geqslant 0$, and $f(x) \sim (1-x)^{-\alpha}$, where $\alpha > 1$, then

$$s_n \sim \frac{n^\alpha}{\Gamma(\alpha+1)}.$$

[We have

$$f_{\alpha-1}(x) = \frac{1}{\Gamma(\alpha-1)}\int_0^x (x-t)^{\alpha-2} f(t)\,dt = \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+1)}{\Gamma(n+\alpha)} x^{\alpha+n-1} \sim \sum_{n=1}^{\infty} \frac{a_n x^n}{n^{\alpha-1}},$$

and, on the other hand,

$$f_{\alpha-1}(x) \sim \frac{1}{\Gamma(\alpha-1)}\int_0^x (x-t)^{\alpha-2}(1-t)^{-\alpha}\,dt$$

$$= \frac{x^{\alpha-1}}{\Gamma(\alpha)(1-x)} \sim \frac{1}{\Gamma(\alpha)(1-x)}.$$

Hence
$$\sum_{\nu=1}^{n} \frac{a_\nu}{\nu^{\alpha-1}} \sim \frac{n}{\Gamma(\alpha)},$$
and the result now follows without difficulty.]

9. If $f(z)$ is regular in a region including the origin, and $f(0) = 1$, then $f(z)$ can be expanded in the form
$$f(z) = (1+a_1z)(1+a_2z^2)(1+a_3z^3)\ldots$$
for sufficiently small values of z.

[Ritt (1): Assuming an expansion of the above form, we write $f'(z)/f(z) = c_1+c_2z+\ldots$, and determine the numbers a_n in succession by equating coefficients in the equation
$$c_1+c_2z+\ldots = \sum_{n=1}^{\infty} \frac{na_nz^{n-1}}{1+a_nz^n}.$$
If $\mu_n = \max_{\nu\leqslant n}|a_\nu|^{1/\nu}$, we deduce from the recurrence relation that $\mu_n^n \leqslant n\mu_{1n}^n+|c_n|$. Hence μ_n is bounded, and the process can be justified.]

10. Show that the circle of convergence of the above product is the same as that of the series $\sum a_nz^n$, but that the power series for $f(z)$ may have a larger circle of convergence.

11. If each of the series
$$\sum_{n=0}^{\infty} a_nz^n, \qquad \sum_{n=0}^{\infty} b_nz^n, \qquad \sum_{n=0}^{\infty} a_nb_nz^n$$
has a radius of convergence equal to 1, then so have the series
$$\sum_{n=0}^{\infty} a_nb_n^2z^n, \qquad \sum_{n=0}^{\infty} a_n^2b_nz^n.$$

12. If each of the series
$$f(z) = \sum_{n=0}^{\infty} a_nz^n, \qquad g(z) = \sum_{n=0}^{\infty} b_nz^n, \qquad F(z) = \sum_{n=0}^{\infty} a_nb_nz^n$$
has a radius of convergence equal to 1, if $f(z)$ is regular on its circle of convergence except at $z = 1$, and $b_n \geqslant 0$ for all values of n, then $F(z)$ has a singularity at $z = 1$.

[Bohnenblust (1): the series
$$\phi(z) = \sum_{n=0}^{\infty} |a_n|^2b_nz^n$$
has the radius of convergence 1, and so by § 7.21 has a singularity at $z = 1$. By Hadamard's multiplication theorem (§ 4.6) the singularities of $\phi(z)$ are products of those of $F(z)$ and of
$$\bar{f}(z) = \sum_{n=0}^{\infty} \bar{a}_nz^n.$$
Thus $1 = \alpha\beta$, where α is a singularity of $F(z)$, β of $\bar{f}(z)$; and β must be 1. Hence $\alpha = 1$.]

13. If $f(z) = \sum a_n z^n$ is regular on its circle of convergence except at z_0, then every series which consists of a selection of terms from $\sum a_n z^n$, and which has the same radius of convergence, has a singularity at z_0.

14. Show that the theorem of § 7.21 is still true if the coefficients a_n are complex, provided that $|\arg a_n| < \alpha < \frac{1}{2}\pi$ for all values of n.

[We have $|a_n| \leqslant \sec\alpha\, \mathbf{R}a_n$.]

15. The function
$$\sum_{n=1}^{\infty} \frac{z^{2^n}}{n^2}$$
is continuous in and on the unit circle; but every point of the circle is a singularity.

16. If $f(z)$ is bounded in the unit circle, then $\sum |a_n|^2$ is convergent. [See § 2.5.]

The following examples are on the border-line between theory of power series and theory of real functions. It seems most convenient to insert them here, but some of them assume the theory of mean convergence given in § 12.5.

17. If $\sum |a_n|^2$ is convergent, then
$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})-f(r'e^{i\theta})|^2\, d\theta = \sum_{n=0}^{\infty} |a_n|^2 (r^n - r'^n)^2.$$
Hence, show that, as $r \to 1$, $f(re^{i\theta})$ converges in mean to a limit-function $F(\theta)$ of the class $L^2(0, 2\pi)$.

18. If $f(z) = u+iv$, $F(\theta) = U+iV$ in the previous example, show that Poisson's formulae
$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2}\, U(\phi)\, d\phi,$$
$$v(r,\theta)-v(0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{2r\sin(\theta-\phi)}{1-2r\cos(\theta-\phi)+r^2}\, U(\phi)\, d\phi$$
hold for $r < 1$.

19. Show that, in the above examples, $u(r,\theta) \to U(\theta)$ as $r \to 1$ for every value of θ in the Lebesgue set of $U(\theta)$. Deduce that $f(re^{i\theta}) \to F(\theta)$ as $r \to 1$ for almost all values of θ.

[The analysis is similar to that of § 13.34.]

20. Show that a bounded analytic function tends to a limit radially at almost all points of its circle of convergence.

21. If $U(\theta) \geqslant 0$ for all values of θ, then $u(r,\theta) \geqslant 0$ for all values of r and θ.

22. If $f(z)$ is regular and bounded for $|z| < 1$, and $f(z) \to 0$ as $r \to 1$ throughout an interval of values of θ, then $f(z)$ is identically zero.

[If $0 \leqslant \theta \leqslant 2\pi/p$ is part of the interval, consider the function

$$g(z) = f(z)f(ze^{2i\pi/p})\ldots f(ze^{2(p-1)i\pi/p}).]$$

23. More generally, if $f(z)$ is bounded and tends to zero radially for values of θ in a set of positive measure, then $f(z)$ is identically zero.

[See Bieberbach, ii, p. 156. Let E be the set where $f(z) \to 0$, and let $m(E) = \mu > 0$. Let $u_1(\theta) = \lambda/\mu$ in E, and $= -\lambda/(2\pi-\mu)$ in CE. Let $g(z)$ be the corresponding analytic function defined by the formulae of ex. 18. Then $g(0) = 0$. Let $h(z) = e^{g(z)}$, so that $h(0) = 1$. Then

$$f(0) = f(0)h(0) = \frac{1}{2\pi i}\int_{|z|=1} f(z)h(z)\frac{dz}{z} = \frac{1}{2\pi i}\int_{CE} f(z)h(z)\frac{dz}{z},$$

$$|f(0)| < Ae^{-\lambda/(2\pi-\mu)}.$$

Since λ may be as large as we please, $f(0) = 0$. Applying the same argument to $f(z)/z$, $f'(0) = 0$, etc.]

24. If $U(\theta)$ is any function integrable in the Lebesgue sense, and

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1+ze^{-i\phi}}{1-ze^{i\phi}}\,U(\phi)\,d\phi \qquad (|z| < 1),$$

then $f(z)$ tends to a limit as $r \to 1$ for almost all values of θ.

[Plessner (1): We may suppose without loss of generality that $U(\phi) \geqslant 0$. Then $\mathbf{R}\{f(z)\} \geqslant 0$. Hence the function $1/\{1+f(z)\}$ is bounded in the unit circle, and so tends to a limit for almost all values of θ. This limit is different from zero almost everywhere.]

CHAPTER VIII

INTEGRAL FUNCTIONS

8.1. Factorization of integral functions. An integral function is an analytic function which has no singularities except at infinity. The simplest such functions are polynomials. A polynomial $f(z)$ which has zeros at the points $z_1, z_2, \ldots, z_n$ can be factorized in the form

$$f(z) = f(0)\left(1-\frac{z}{z_1}\right)\left(1-\frac{z}{z_2}\right)\cdots\left(1-\frac{z}{z_n}\right).$$

The zeros of integral functions in general are equally important. An integral function which is not a polynomial may have an infinity of zeros z_n; and the product

$$\prod\left(1-\frac{z}{z_n}\right)$$

taken over these zeros may be divergent. So we cannot always factorize an integral function in the same way as a polynomial, and we have to consider less simple factors than $1-z/z_n$.

The expressions

$$E(u,0) = 1-u, \qquad E(u,p) = (1-u)e^{u+\frac{u^2}{2}+\cdots+\frac{u^p}{p}} \quad (p = 1, 2, \ldots),$$

are called *primary factors*. Each primary factor vanishes when $u = 1$; but the behaviour of $E(u,p)$ as $u \to 0$ depends on p. For $|u| < 1$,

$$\log E(u,p) = -\frac{u^{p+1}}{p+1} - \frac{u^{p+2}}{p+2} - \cdots.$$

Hence, if $k > 1$, and $|u| \leqslant 1/k$,

$$\begin{aligned} |\log E(u,p)| &\leqslant |u|^{p+1} + |u|^{p+2} + \cdots \\ &\leqslant |u|^{p+1}\left\{1+\frac{1}{k}+\frac{1}{k^2}+\cdots\right\} = \frac{k}{k-1}|u|^{p+1}. \end{aligned}$$

It is this inequality which determines the convergence of a product of primary factors.

8.11. The theorem of Weierstrass. If $f(z)$ is an integral function, what can we say about its zeros?

Since $f(z)$ is analytic except at infinity, the zeros can have no limit-point except at infinity. In general, this is all that we can say. This follows from the following theorem of Weierstrass.

Given any sequence of numbers $z_1, z_2, \ldots$ whose sole limiting-point

is at infinity, there is an integral function with zeros at these points, and these points only.

We may suppose the zeros arranged so that $|z_1| \leqslant |z_2| \leqslant \ldots$. Let $|z_n| = r_n$, and let $p_1, p_2, \ldots$ be a sequence of positive integers such that the series

$$\sum_{n=1}^{\infty} \left(\frac{r}{r_n}\right)^{p_n}$$

is convergent for all values of r. It is always possible to find such a sequence; for $r_n \to \infty$, since otherwise the zeros would have a limiting-point other than infinity; and we may take $p_n = n$, since

$$\left(\frac{r}{r_n}\right)^n < \frac{1}{2^n}$$

for $r_n > 2r$, and the series is therefore convergent.

Let
$$f(z) = \prod_{n=1}^{\infty} E\left(\frac{z}{z_n}, p_n - 1\right). \tag{1}$$

This function has the required property; for, if $|z_n| > 2|z|$,

$$\left|\log E\left(\frac{z}{z_n}, p_n - 1\right)\right| \leqslant 2\left(\frac{r}{r_n}\right)^{p_n}; \tag{2}$$

hence the series
$$\sum_{|z_n|>2R} \log E\left(\frac{z}{z_n}, p_n - 1\right)$$

is uniformly convergent for $|z| \leqslant R$, and hence* so is the product

$$\prod_{|z_n|>2R} E\left(\frac{z}{z_n}, p_n - 1\right).$$

Hence $f(z)$ is regular for $|z| \leqslant R$, and its only zeros in this region are those of

$$\prod_{|z_n|\leqslant 2R} E\left(\frac{z}{z_n}, p_n - 1\right),$$

i.e. the points $z_1, z_2, \ldots$. Since R may be as large as we please, this proves the theorem.

The function $f(z)$ is, of course, not uniquely determined by the zeros, since we have a wide choice of the numbers p_n.

8.12. It is possible to factorize any given integral function in the following way.

If $f(z)$ is an integral function, and $f(0) \neq 0$, then

$$f(z) = f(0)\, P(z)\, e^{g(z)}$$

* See § 1.43, end.

where $P(z)$ is a product of primary factors, and $g(z)$ is an integral function.

We form $P(z)$ as in the above theorem from the zeros of $f(z)$. Let

$$\phi(z) = \frac{f'(z)}{f(z)} - \frac{P'(z)}{P(z)}.$$

Then $\phi(z)$ is an integral function, since the poles of one term are cancelled by those of the other. Hence also

$$g(z) = \int_0^z \phi(t)\,dt = \log f(z) - \log f(0) - \log P(z)$$

is an integral function, and the result stated follows on taking exponentials.

If $f(z)$ has a zero of order p at $z = 0$, a factor z^p has to be inserted.

This factorization is not unique.

8.2. Functions of finite order. The general factorization theorem is not precise enough to be of much use; in general the numbers p_n increase indefinitely with n, and we can say little about the function $g(z)$. There is, however, one case in which we can put the theorem into a perfectly definite form, that of functions of finite order.

An integral function $f(z)$ is said to be of finite order if there is a positive number A such that, as $|z| = r \to \infty$,

$$f(z) = O(e^{r^A}).$$

The lower bound ρ of numbers A for which this is true is called the *order* of the function. Thus, if $f(z)$ is of order ρ,

$$f(z) = O(e^{r^{\rho+\epsilon}})$$

for every positive value of ϵ, but not for any negative value. In this, and similar statements throughout the chapter, ϵ is thought of as taking arbitrarily small values, and the constant implied in the O depends in general on ϵ. If it were independent of ϵ, we could replace ϵ by 0 in the formula.

Functions of finite order are, after polynomials, the simplest integral functions. A polynomial is of order zero; some of the properties of functions of small order are similar to those of polynomials.

Many familiar functions are easily seen to be of finite order; e^z is of order 1; so are $\sin z$ and $\cos z$; $\cos \sqrt{z}$ is an integral function

of order $\frac{1}{2}$; e^{z^k} is an integral function of order k, if k is a positive integer (if k is not an integer, it is not an integral function). The function e^{e^z} is of infinite order.

In what follows we shall suppose generally that $f(0)$ is not 0. This simplifies the analysis a little, and division by a factor z^k does not affect the order.

8.21. The function $n(r)$. Let $n(r)$ denote the number of zeros $z_1, z_2, \ldots$ of an integral function $f(z)$ for which $|z_n| \leqslant r$. Then $n(r)$ is a non-decreasing function of r which is constant in intervals; it is zero for $r < |z_1|$, if $f(0)$ is not zero.

This function is, as we have seen in § 3.61, connected with $f(z)$ by means of Jensen's formula. In fact

$$\int_0^r \frac{n(x)}{x}\,dx = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta - \log|f(0)|. \tag{1}$$

If $f(z)$ is an integral function, this holds for all values of r.

If $f(z)$ is of order ρ, then $n(r) = O(r^{\rho+\epsilon})$. For

$$\log|f(re^{i\theta})| < Kr^{\rho+\epsilon},$$

K depending on ϵ only. Hence, by (1),

$$\int_0^{2r} \frac{n(x)}{x}\,dx < Kr^{\rho+\epsilon}. \tag{2}$$

But, since $n(r)$ is non-decreasing,

$$\int_r^{2r} \frac{n(x)}{x}\,dx \geqslant n(r)\int_r^{2r} \frac{dx}{x} = n(r)\log 2.$$

Hence
$$n(r) \leqslant \frac{1}{\log 2}\int_0^{2r} \frac{n(x)}{x}\,dx < Kr^{\rho+\epsilon}$$

by (2).

We may thus say, roughly, that the higher the order of a function is, the more zeros it may have in a given region.

8.22. *If $r_1, r_2, \ldots$ are the moduli of the zeros of $f(z)$, then the series $\sum r_n^{-\alpha}$ is convergent if $\alpha > \rho$.*

Let β be a number between α and ρ. Then $n(r) < Ar^\beta$. Putting $r = r_n$, this gives

$$n < Ar_n^\beta.$$

Hence
$$r_n^{-\alpha} < An^{-\alpha/\beta},$$
and the result follows.

The lower bound of positive numbers α for which $\sum r_n^{-\alpha}$ is convergent is called the *exponent of convergence of the zeros*, and is denoted by ρ_1. What we have just proved is that $\rho_1 \leqslant \rho$. We may have $\rho_1 < \rho$; for example, if $f(z) = e^z$, $\rho = 1$; but there are no zeros, so that $\rho_1 = 0$.

Notice that $\rho_1 = 0$ for any function with a finite number of zeros; thus $\rho_1 > 0$ implies that there are an infinity of zeros.

8.23. Canonical products. An important consequence of the above theorem is that, if $f(z)$ is of finite order, then there is an integer p, independent of n, such that the product
$$\prod_{n=1}^{\infty} E\left(\frac{z}{z_n}, p\right) \tag{1}$$
is convergent for all values of z; for by 8.11 (1), with $p_n = p+1$, this product is convergent if
$$\sum \left(\frac{r}{r_n}\right)^{p+1} \tag{2}$$
is convergent;* and this is true for all values of r if $p+1 > \rho_1$, and so it is certainly true if $p+1 > \rho$.

If p is the smallest integer for which (2) is convergent, the product (1) is called the *canonical product* formed with the zeros of $f(z)$; and p is called its *genus*.

If ρ_1 is not an integer, then $p = [\rho_1]$; if ρ_1 is an integer, $p = \rho_1$ if $\sum r_n^{-\rho_1}$ is divergent, while $p = \rho_1 - 1$ if it is convergent. In any case
$$p \leqslant \rho_1 \leqslant \rho.$$

8.24. Hadamard's factorization theorem. *If $f(z)$ is an integral function of order ρ, with zeros $z_1, z_2, \ldots$ ($f(0) \neq 0$), then*
$$f(z) = e^{Q(z)} P(z),$$
where $P(z)$ is the canonical product formed with the zeros of $f(z)$, and $Q(z)$ is a polynomial of degree not greater than ρ.

We can now take the $P(z)$ of § 8.12 to be the canonical product. It follows from the factorization theorem of § 8.12 that there is an expression for $f(z)$ in the above form, in which $Q(z)$ is an integral function. What we have to prove is that in this case $Q(z)$ is a polynomial.

* Compare § 1.43, ex. (vii).

Let $\nu = [\rho]$, so that $p \leqslant \nu$. Taking logarithms and differentiating $\nu+1$ times, we obtain

$$\left(\frac{d}{dz}\right)^{\nu}\left\{\frac{f'(z)}{f(z)}\right\} = Q^{(\nu+1)}(z) - \nu! \sum_{n=1}^{\infty} \frac{1}{(z_n - z)^{\nu+1}}.$$

To prove that $Q(z)$ is a polynomial of degree ν at most, we have to prove that $Q^{(\nu+1)}(z) = 0$.

Let
$$g_R(z) = \frac{f(z)}{f(0)} \prod_{|z_n| \leqslant R} \left(1 - \frac{z}{z_n}\right)^{-1}.$$

Since $|1 - z/z_n| \geqslant 1$ for $|z| = 2R$, $|z_n| \leqslant R$, we have

$$|g_R(z)| \leqslant |f(z)/f(0)| = O(e^{(2R)^{\rho+\epsilon}}) \tag{1}$$

for $|z| = 2R$. Since $g_R(z)$ is an integral function, this holds for $|z| < 2R$ also.

Let $h_R(z) = \log g_R(z)$, the logarithm being determined so that $h_R(0) = 0$. Then $h_R(z)$ is regular for $|z| \leqslant R$, and, by (1),

$$\mathbf{R}\{h_R(z)\} < KR^{\rho+\epsilon}. \tag{2}$$

Hence, by § 5.51,

$$|h_R^{(\nu+1)}(z)| \leqslant \frac{2^{\nu+2}(\nu+1)!R}{(R-r)^{\nu+2}} KR^{\rho+\epsilon}$$

for $|z| = r < R$; and for $|z| = \frac{1}{2}R$ this gives

$$h_R^{(\nu+1)}(z) = O(R^{\rho+\epsilon-\nu-1}). \tag{3}$$

Hence
$$\begin{aligned} Q^{(\nu+1)}(z) &= h_R^{(\nu+1)}(z) + \nu! \sum_{|z_n|>R} \frac{1}{(z_n - z)^{\nu+1}} \\ &= O(R^{\rho+\epsilon-\nu-1}) + O\Big(\sum_{|z_n|>R} |z_n|^{-\nu-1}\Big) \end{aligned}$$

for $|z| = \frac{1}{2}R$, and so also for $|z| < \frac{1}{2}R$. The first term on the right tends to 0 as $R \to \infty$ if ϵ is small enough, since $\nu+1 > \rho$; and the second term tends to 0 since $\sum |z_n|^{-\nu-1}$ is convergent. Since the left-hand side is independent of R it must be zero, and the theorem follows.*

8.25. *The order of a canonical product is equal to the exponent of convergence of its zeros.*

We know that, for any function, $\rho_1 \leqslant \rho$. Hence we have to prove that, for a canonical product $P(z)$, $\rho \leqslant \rho_1$. Let $r_1, r_2, \ldots$

* Hadamard (2). This proof is due to Landau (5). For an alternative proof see § 8.72.

be the moduli of the zeros, and k a constant greater than unity. Let

$$\log|P(z)| = \sum_{r_n \leqslant kr} \log\left|E\left(\frac{z}{z_n}, p\right)\right| + \sum_{r_n > kr} \log\left|E\left(\frac{z}{z_n}, p\right)\right| = \Sigma_1 + \Sigma_2.$$

In Σ_2 we use the inequality 8.11 (2), and obtain

$$\Sigma_2 = O\left\{\sum_{r_n > kr} \left(\frac{r}{r_n}\right)^{p+1}\right\} = O\left\{r^{p+1} \sum_{r_n > kr} \frac{1}{r_n^{p+1}}\right\}.$$

If $p = \rho_1 - 1$, this is $O(r^{p+1}) = O(r^{\rho_1})$. Otherwise $\rho_1 + \epsilon < p+1$ if ϵ is small enough, and then

$$r^{p+1} \sum_{r_n > kr} r_n^{-p-1} = r^{p+1} \sum_{r_n > kr} r_n^{\rho_1+\epsilon-p-1} r_n^{-\rho_1-\epsilon}$$
$$< r^{p+1}(kr)^{\rho_1+\epsilon-p-1} \sum r_n^{-\rho_1-\epsilon} = O(r^{\rho_1+\epsilon}).$$

Again in Σ_1 we have terms involving $E(u, p)$, where $|u| \geqslant 1/k$, so that

$$\log|E(u,p)| \leqslant \log(1+|u|) + |u| + \ldots + \frac{|u|^p}{p} < K|u|^p,$$

where K depends on k only. Hence

$$\Sigma_1 < O\left(r^p \sum_{r_n \leqslant kr} r_n^{-p}\right) = O\left(r^p \sum_{r_n \leqslant kr} r_n^{\rho_1+\epsilon-p} r_n^{-\rho_1-\epsilon}\right)$$
$$= O\{r^p (kr)^{\rho_1+\epsilon-p} \sum r_n^{-\rho_1-\epsilon}\} = O(r^{\rho_1+\epsilon}).$$

Hence $$\log|P(z)| < O(r^{\rho_1+\epsilon}),$$

and the result follows.

8.26. *If ρ is not an integer, $\rho_1 = \rho$.*

We have in any case $\rho_1 \leqslant \rho$. Suppose that $\rho_1 < \rho$. Then $P(z)$ is of order ρ_1, i.e. of order less than ρ. Also, if $Q(z)$ is of degree q, $e^{Q(z)}$ is of order q; and $q \leqslant \rho$, and in this case $q < \rho$, since q is an integer and ρ is not. Hence $f(z)$ is the product of two functions, each of order less than ρ. Hence $f(z)$ is of order less than ρ, which gives a contradiction. Hence $\rho_1 = \rho$.

In particular, a function of non-integral order must have an infinity of zeros. In fact, if the order is not an integer, the function is dominated by the canonical product $P(z)$; whereas, if the order is an integer, $P(z)$ may reduce to a polynomial or a constant, and the order then depends entirely on the factor $e^{Q(z)}$.

In any case, since $P(z)$ is of order ρ_1, and $e^{Q(z)}$ of order q, we have

$$\rho = \max(q, \rho_1).$$

8.27. Genus. The genus of the integral function $f(z)$ is the greater of the two integers p and q, and is therefore an integer.

Since $p \leqslant \rho$ and $q \leqslant \rho$, the genus does not exceed the order. The actual determination of the genus of a given function is sometimes not easy.

Example. Prove that the genus is not less than $\rho - 1$.

8.3. The coefficients in the expansion of a function of finite order. *A necessary and sufficient condition that*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1}$$

should be an integral function of finite order ρ is that

$$\varliminf_{n\to\infty} \frac{\log(1/|a_n|)}{n \log n} = \frac{1}{\rho}.$$

The argument depends on the fact that $\sum |a_n z^n|$ does not differ very much from its greatest term, and that $|f(z)|$ lies between the two. This is further illustrated by the example which follows.

(i) Let
$$\varliminf \frac{\log(1/|a_n|)}{n \log n} = \mu,$$

where μ is 0, positive, or infinite. Then, for every positive ϵ,

$$\log(1/|a_n|) > (\mu - \epsilon) n \log n \qquad (n > n_0),$$

i.e.
$$|a_n| < n^{-n(\mu-\epsilon)}.$$

If $\mu > 0$, it follows that (1) converges for all values of z, so that $f(z)$ is an integral function. Also, if μ is finite,

$$|f(z)| < A r^{n_0} + \sum_{n=n_0+1}^{\infty} r^n n^{-n(\mu-\epsilon)} \qquad (r > 1).$$

Let $\sum_1$ denote the part of the last series for which $n \leqslant (2r)^{\frac{1}{\mu-\epsilon}}$, $\sum_2$ the remainder. Then in $\sum_1$

$$r^n \leqslant \exp\{(2r)^{\frac{1}{\mu-\epsilon}} \log r\},$$

so that

$$\sum\nolimits_1 < \exp\{(2r)^{\frac{1}{\mu-\epsilon}} \log r\} \sum n^{-n(\mu-\epsilon)} < K \exp\{(2r)^{\frac{1}{\mu-\epsilon}} \log r\}.$$

In $\sum_2$, $r n^{-(\mu-\epsilon)} < \frac{1}{2}$, so that

$$\sum\nolimits_2 < \sum (\tfrac{1}{2})^n < 1.$$

Hence
$$|f(z)| < K \exp\{(2r)^{\frac{1}{\mu-\epsilon}} \log r\},$$

i.e. $\rho \leqslant 1/(\mu - \epsilon)$. Making $\epsilon \to 0$, $\rho \leqslant 1/\mu$. In the case $\mu = \infty$,

the argument, with an arbitrarily large number instead of μ, shows that $\rho = 0$.

On the other hand, given ϵ, there is a sequence of values of n for which

$$\log(1/|a_n|) < (\mu+\epsilon)n \log n,$$

i.e.

$$|a_n| > n^{-n(\mu+\epsilon)},$$

i.e.

$$|a_n| r^n > \{rn^{-(\mu+\epsilon)}\}^n.$$

Taking $r = (2n)^{\mu+\epsilon}$, this gives

$$|a_n| r^n > 2^{(\mu+\epsilon)n} = \exp\left\{\tfrac{1}{2}(\mu+\epsilon)\log 2 \,.\, r^{\frac{1}{\mu+\epsilon}}\right\}.$$

Since by Cauchy's inequality $M(r) \geqslant |a_n| r^n$, it follows that, for a sequence of values of r tending to infinity,

$$M(r) > \exp\{Ar^{1/(\mu+\epsilon)}\}.$$

Hence $\rho \geqslant 1/(\mu+\epsilon)$, and, making $\epsilon \to 0$, $\rho \geqslant 1/\mu$. If $\mu = 0$, the argument shows that $f(z)$ is of infinite order.

(ii) Let $f(z)$ be a function of finite order ρ. Then $a_n \to 0$, so that μ, defined as before, is not negative. The argument then shows that $\mu = 1/\rho$.

8.4. Examples. (i) Prove that the order of the function

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^\alpha}$$

is $1/\alpha$.

[We may use the above theorem, or proceed more directly as follows. Suppose z real and positive. The terms of the series increase until n is approximately $z^{1/\alpha}$, and then decrease. Hence, if $z = n^\alpha$, we get a maximum term

$$\frac{n^{n\alpha}}{(n!)^\alpha} \sim \frac{n^{n\alpha}}{(n^{n+\frac{1}{2}}e^{-n}2^{\frac{1}{2}}\pi^{\frac{1}{2}})^\alpha} = \frac{e^{n\alpha}}{n^{\frac{1}{2}\alpha}(2^{\frac{1}{2}}\pi^{\frac{1}{2}})^\alpha} = \frac{e^{\alpha z^{1/\alpha}}}{z^{\frac{1}{2}}(2^{\frac{1}{2}}\pi)^\alpha}.$$

Since $|f(z)|$ is greater than this term, its order is at least $1/\alpha$.

On the other hand, $|f(z)| \leqslant f(|z|)$, and if z is real

$$\begin{aligned} f(z) &= \sum_{n=0}^{N} \frac{z^n}{(n!)^\alpha} + \sum_{N+1}^{\infty} \frac{z^n}{(n!)^\alpha} \\ &< \sum_{n=0}^{N} \frac{z^N}{(n!)^\alpha} + \sum_{N+1}^{\infty} \frac{z^n}{\{(N+1)!\,N^{n-N-1}\}^\alpha} \\ &< Az^N + \frac{z^{N+1}}{\{(N+1)!\}^\alpha(1-z/N^\alpha)}, \end{aligned}$$

provided that $N^\alpha > z$. Taking $N = [(2z)^{1/\alpha}]$, we obtain

$$f(z) = O(z^N) = O\{z^{(2z)^{1/\alpha}}\} = O(e^{z^{1/\alpha+\epsilon}}),$$

so that the order does not exceed $1/\alpha$. Hence $\rho = 1/\alpha$. (See Hardy's *Orders of Infinity*, ed. 1, p. 55.)]

(ii) Discuss in a similar way the function

$$\sum_{n=1}^{\infty} \frac{z^n}{n^{\alpha n}}.$$

(iii) If $\lambda \neq 0$, and $p(z)$ is a polynomial, $e^{\lambda z} - p(z)$ has an infinity of zeros.

[If not, $e^{\lambda z} - p(z) = e^{az+b}P(z)$, where $P(z)$ is a polynomial. By comparing rates of increase in various directions we find that $a = \lambda$, then $e^{\lambda z}$ = rational function.]

(iv) If $f(z)$ is of order ρ, and $g(z)$ of order $\rho' \leqslant \rho$, and the zeros of $g(z)$ are all zeros of $f(z)$, then $f(z)/g(z)$ is of order ρ at most.

[For $f(z) = P_1(z)e^{Q_1(z)}$, $g(z) = P_2(z)e^{Q_2(z)}$, and P_1/P_2 is either the canonical product formed with the zeros of f_1/f_2, or this product multiplied by an exponential factor of order not exceeding ρ. Hence the order of P_1/P_2 does not exceed ρ.]

(v) $\cos z$ and $\sin z$ are of order 1; the product formulae (§ 3.23) are cases of Hadamard's theorem.

(vi) $1/\Gamma(z)$ is of order 1; deduce the product formula (§ 4.41) from Hadamard's theorem.

[With the notation of § 4.41, $f(1-z) = -2i\pi/\Gamma(z)$, and

$$f(z) = O\left\{e^{\pi|z|}\left(1+\int_1^\infty t^{|z|}e^{-t}\,dt\right)\right\} = O\{e^{\pi|z|}(|z|+1)^{|z|+1}\}.\Big]$$

(vii) $\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s)\zeta(s)$ is an integral function with $\rho = 1$, $\rho_1 = 1$.

[To prove that $\rho \leqslant 1$, use § 4.43 (3) and ex. (iv); and $\rho \geqslant 1$ since $\log\zeta(s) \sim 2^{-s}$, $\log\xi(s) \sim \frac{1}{2}s\log s$ as $s \to \infty$ by real values. Next the functional equation gives $\xi(s) = \xi(1-s)$. Hence $\Xi(z) = \xi(\frac{1}{2}+iz)$ is even, and $\Xi(\sqrt{z})$ is an integral function of order $\frac{1}{2}$, and so has convergence-exponent $\frac{1}{2}$.]

(viii) $z^{-\nu}J_\nu(z)$ is an integral function with $\rho = 1$, $\rho_1 = 1$. Verify the result of § 8.3 in this case. [See p. 60, ex. 5.]

(ix) $F_\alpha(z) = \int_0^\infty e^{-t^\alpha}\cos zt\,dt$ $(\alpha > 1)$ is of order $\alpha/(\alpha-1)$.

[Either directly from the integral, or from the power-series.]

(x) $\vartheta_1(z) = -i\sum_{n=-\infty}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}e^{(2n+1)iz}$, where $|q| < 1$, is an integral function with $\rho = 2$, $\rho_1 = 2$.

[If $\lambda = (2|z|+\log 2)/\log|1/q|-\frac{1}{2}$,

$$\vartheta_1(z) \leqslant 2\sum_{n\leqslant\lambda}|q|^{(n+\frac{1}{2})^2}e^{(2n+1)|z|}+2\sum_{n>\lambda}(\tfrac{1}{2})^{n+\frac{1}{2}} = O(e^{(2\lambda+1)|z|}) = O(e^{K|z|^2}).$$

$\vartheta_1(z)$ has simple zeros at $z = m\pi+n\pi\tau$, where m and n run through all integers (see e.g. Whittaker and Watson, *Modern Analysis*, § 21.12). Hence $\rho_1 = 2$.]

(xi) $\vartheta_1(z)$ is an integral function of $\sin z$ of order 0.

[If $2\sin z = w$, $\vartheta_1(z) = g(w)$, then

$$g(w) = \sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2}\{w^{2n+1}-(2n+1)w^{2n-1}+\ldots\}$$

$$= O\left\{\sum_{n=0}^{\infty}|q|^{(n+\frac{1}{2})^2}(|w|+1)^{2n+1}\right\} = O\{e^{K\log^2(|w|+1)}\}.$$

It was proved by Pólya (2) that if g and h are integral functions, and $g\{h(z)\}$ of finite order, then either h is a polynomial and g of finite order, or h is not a polynomial but of finite order, and g of zero order.]

(xii) If $$f(z) = \prod_1^\infty\left(1+\frac{z}{r_n}\right) \qquad (r_n > 0)$$

is of order ρ, $0 < \rho < 1$, then for $\rho < \sigma < 1$

$$\int_0^\infty \frac{\log f(x)}{x^{\sigma+1}}\,dx = \frac{\pi}{\sigma \sin \pi\sigma} \sum_1^\infty \frac{1}{r_n^\sigma}, \qquad \int_0^\infty \frac{\log|f(-x)|}{x^{\sigma+1}}\,dx = \frac{\pi}{\sigma \tan \pi\sigma} \sum_1^\infty \frac{1}{r_n^\sigma}.$$

[We have

$$\int_0^\infty \frac{\log(1+x)}{x^{\sigma+1}}\,dx = \frac{\pi}{\sigma \sin \pi\sigma}, \qquad \int_0^\infty \frac{\log|1-x|}{x^{\sigma+1}}\,dx = \frac{\pi}{\sigma \tan \pi\sigma}.\Big]$$

8.5. The derived function. Many of the properties of the derived function of an integral function are the same as those of the primitive function. The following theorems are examples of this.

8.51. *The derived function $f'(z)$ is of the same order as $f(z)$.* Let $M'(r) = \max_{|z|=r} |f'(z)|$. Then

$$\frac{M(r) - |f(0)|}{r} \leqslant M'(r) \leqslant \frac{M(R)}{R-r}. \tag{1}$$

For
$$f(z) = \int_0^z f'(t)\,dt + f(0),$$

the integral being taken along the straight line. Hence

$$M(r) \leqslant rM'(r) + |f(0)|.$$

On the other hand,

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z)^2}\,dw,$$

where C is the circle $|w-z| = R-r$ $(|z| = r < R)$. Hence, choosing z so that $|f'(z)| = M'(r)$, we have

$$M'(r) \leqslant \frac{M(R)}{R-r}.$$

The result stated now follows on taking, say, $R = 2r$ in (1).

8.52. The well-known theorem, that if $f(z)$ is a polynomial with all its roots real, then $f'(z)$ has the same property, can be

* Pólya (2).

extended to a certain class of integral functions. The result is expressed by the following theorem of Laguerre:

If $f(z)$ is an integral function, real for real z, of order less than 2, with real zeros, then the zeros of $f'(z)$ are also all real, and are separated from each other by the zeros of $f(z)$.

We have $$f(z) = cz^k e^{az} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{\frac{z}{z_n}},$$

where k is zero or a positive integer, and c, a, and $z_1, z_2, \ldots$ are all real. Taking logarithms and differentiating,

$$\frac{f'(z)}{f(z)} = \frac{k}{z} + a + \sum_{n=1}^{\infty} \left(\frac{1}{z - z_n} + \frac{1}{z_n}\right).$$

Hence, if $z = x + iy$,

$$\mathrm{I}\left\{\frac{f'(z)}{f(z)}\right\} = -iy\left\{\frac{k}{x^2+y^2} + \sum_{n=1}^{\infty} \frac{1}{(x-z_n)^2+y^2}\right\},$$

which is zero if $y = 0$ only. Hence $f'(z)$ cannot be zero except on the real axis.

Again, $$\frac{d}{dz}\left\{\frac{f'(z)}{f(z)}\right\} = -\frac{k}{z^2} - \sum_{n=1}^{\infty} \frac{1}{(z-z_n)^2},$$

which is real and negative if z is real. Hence $f'(z)/f(z)$ decreases steadily as z increases through real values from z_n to z_{n+1}, and so it cannot vanish more than once between z_n and z_{n+1}. Clearly it changes sign, and so vanishes just once in this interval. This proves the theorem.

It is clear from the above result that, if the zeros of $f'(z)$ are $z_1', z_2', \ldots$, then the series

$$\sum \frac{1}{|z_n|^\alpha}, \qquad \sum \frac{1}{|z_n'|^\alpha},$$

converge or diverge together. Hence the zeros of $f'(z)$ have the same exponent of convergence as those of $f(z)$. It may be shown further that $f(z)$ and $f'(z)$ have the same genus, but this is not quite so easy to prove (see ex. 16 at the end of the chapter).

Since $f'(z)$ is of the same order as $f(z)$, and has real zeros only, the theorem may now be applied to it, and we see that $f''(z)$ has real zeros only; and so for $f'''(z)$, etc.

The proof also applies to a function $f(z)$ of order 2, but of

genus 1. In this case, however, we cannot extend the result to $f''(z)$,... without considering the problem of the genus of $f'(z)$.

It is easily seen by means of examples that the theorem is not true for functions of genus 2. For example, in the case

$$f(z) = ze^{z^2}, \qquad f'(z) = (2z^2+1)e^{z^2},$$

the zeros of $f'(z)$ are complex; and in the case

$$f(z) = (z^2-4)e^{\frac{1}{3}z^2}, \qquad f'(z) = \tfrac{2}{3}z(z^2-1)e^{\frac{1}{3}z^2},$$

the zeros of $f'(z)$ are real, but are not separated by those of $f(z)$.

Example. The differential equation

$$y\frac{d^2y}{dt^2} = -\sin^2 t$$

has no real solution, other than $y = \pm\sin t$, which is an integral function of finite order.

[Suppose that y is a function of finite order ρ. Then

$$y = e^{Q(t)}P(t),$$

where $P(t)$ is a canonical product, and $Q(t)$ a polynomial of degree not greater than ρ. Since the zeros of $P(t)$ are zeros of $\sin^2 t$, $P(t)$ is of order 1 at most.

Now
$$\frac{dy}{dt} = e^{Q(t)}\{P'(t)+P(t)Q'(t)\},$$

$$\frac{d^2y}{dt^2} = e^{Q(t)}\{P''(t)+2P'(t)Q'(t)+P(t)Q'^2(t)+P(t)Q''(t)\} = e^{Q(t)}f(t),$$

where $f(t)$ is of order 1 at most. Hence

$$f(t) = e^{at+b}P_1(t),$$

where $P_1(t)$ is a canonical product. Hence

$$e^{2Q(t)+at+b}P(t)P_1(t) = -\sin^2 t,$$

i.e.
$$e^{2Q(t)+at+b}P(t) = -\sin^2 t/P_1(t)$$

is of order 1 at most (§ 8.4, ex. (iv)). Hence $P(t)$ is of order 1 and $Q(t)$ is linear.

Hence y is a function of order 1.

We can now use Laguerre's theorem. y is a function of order 1 with real zeros. The zeros of $\frac{dy}{dt}$ are separated by those of y, so that, as y has no triple zeros, all the zeros of $\frac{dy}{dt}$ are simple. So all the zeros of $\frac{d^2y}{dt^2}$ are simple. Hence y has zeros at *all* the zeros of $\sin t$. Suppose y has a double zero at $t = k\pi$. Then $\frac{dy}{dt}$ has a zero between $(k-1)\pi$ and $k\pi$, a zero at $k\pi$, and a zero between $k\pi$ and $(k+1)\pi$. Hence $\frac{d^2y}{dt^2}$ has two zeros between

$(k-1)\pi$ and $(k+1)\pi$, which is impossible. Hence y has all the zeros of $\sin t$ just once. Hence

$$y = e^{\alpha t+\beta}\sin t.$$

Inserting this in the differential equation, we obtain

$$(\alpha^2-1)\sin t+2\alpha\cos t = -e^{-2\alpha t-2\beta}\sin t.$$

Since the left-hand side is bounded for real t, so is the right-hand side, and hence $\alpha = 0$. Then $\beta = 0$ or πi.]

8.6. Functions with real zeros only. A number of important functions have no complex zeros; for example, all the zeros of $1/\Gamma(z)$ are real. On the other hand it is sometimes very difficult to decide whether the zeros are real or not; for example, it was conjectured by Riemann, in 1859, that all the zeros of the function $\Xi(z)$ of § 8.45 are real, but this has never been proved.

8.61. The theorems of Laguerre. In some cases the question can be decided by the following theorems of Laguerre.*

Let $f(z)$ be a polynomial,

$$f(z) = a_0+a_1z+\ldots+a_pz^p,$$

all of whose zeros are real; and let $\phi(w)$ be an integral function of genus 0 or 1, which is real for real w, and all the zeros of which are real and negative. Then the polynomial

$$g(z) = a_0\phi(0)+a_1\phi(1)z+\ldots+a_p\phi(p)z^p$$

has all its zeros real, and as many positive, zero and negative zeros as $f(z)$.

Let
$$\phi(w) = ae^{kw}\prod_{n=1}^{\infty}\left(1+\frac{w}{\alpha_n}\right)e^{-\frac{w}{\alpha_n}},$$

where $\alpha_n > 0$ for all values of n. Consider the function

$$\begin{aligned} g_1(z) &= f(z)+\frac{z}{\alpha_1}f'(z) \\ &= \frac{z^{1-\alpha_1}}{\alpha_1}\frac{d}{dz}\{z^{\alpha_1}f(z)\} \qquad (z>0) \\ &= a_0+a_1\left(1+\frac{1}{\alpha_1}\right)z+\ldots+a_p\left(1+\frac{p}{\alpha_1}\right)z^p. \end{aligned}$$

Obviously $g_1(z)$ has as many zeros at $z=0$ as $f(z)$; and the second expression for it shows, by Rolle's theorem, that it has

* *Œuvres*, t. 1, p. 200.

the same number of positive zeros as $f(z)$. Similarly it has the same number of negative zeros.

By repeating the argument, we can obtain the same result for the function

$$g_n(z) = a_0 + a_1 \phi_n(1) z + \ldots + a_p \phi_n(p) z^p,$$

where
$$\phi_n(w) = \left(1 + \frac{w}{\alpha_1}\right) \ldots \left(1 + \frac{w}{\alpha_n}\right).$$

Next, the transformation $z = e^{k_n z'}$, where $k_n = k - \sum_1^n 1/\alpha_\nu$, shows that the same result holds for

$$G_n(z) = a_0 \Phi_n(0) + a_1 \Phi_n(1) z + \ldots + a_p \Phi_n(p) z^p,$$

where
$$\Phi_n(w) = a e^{kw} \prod_{\nu=1}^{n} \left(1 + \frac{w}{\alpha_\nu}\right) e^{-\frac{w}{\alpha_\nu}} = a e^{k_n w} \phi_n(w).$$

Finally, $\Phi_n(w) \to \phi(w)$ uniformly in any finite region. Hence $G_n(z) \to g(z)$ uniformly in any finite region; by Hurwitz's theorem (§ 3.45) the zeros of $g(z)$ are the limits of the zeros of $G_n(z)$; it is clear that $g(z)$ has the same number of zeros at $z = 0$ as $f(z)$; and this completes the proof.

8.62. *Suppose that $\phi(w)$ satisfies the conditions of the previous theorem, and that $f(z)$ is an integral function of the form*

$$f(z) = e^{az+b} \prod_{n=1}^{\infty} \left(1 + \frac{z}{z_n}\right),$$

the numbers a and z_n being all positive. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Then
$$g(z) = \sum_{n=0}^{\infty} a_n \phi(n) z^n$$

is an integral function, all of whose zeros are real and negative.

In the first place, $g(z)$ is an integral function; for, since $(1+x)e^{-x} \leqslant 1$ for $x \geqslant 0$,

$$|\phi(n)| \leqslant |a| e^{kn},$$

and so the series for $g(z)$ is everywhere convergent.

Let
$$f_p(z) = e^b \left(1 + \frac{az}{p}\right)^p \prod_{n=1}^{p} \left(1 + \frac{z}{z_n}\right)$$
$$= \sum_{n=0}^{2p} a_{n,p} z^n.$$

All the zeros of this are real and negative, and hence, by the previous theorem, so are the zeros of

$$g_p(z) = \sum_{n=0}^{2p} a_{n,p}\phi(n)z^n.$$

Finally, $g_p(z) \to g(z)$ uniformly in any finite region. In fact it is clear from the expression

$$\left(1+\frac{az}{p}\right)^p = 1+az+\left(1-\frac{1}{p}\right)\frac{a^2z^2}{2!}+\ldots+\frac{a^pz^p}{p^p}$$

that $a_{n,p} \to a_n$ as $p \to \infty$ for every fixed n, while $|a_{n,p}| \leqslant a_n$ for all values of n and p. Hence, if $N < 2p$,

$$|g(z)-g_p(z)| \leqslant \left|\sum_{n=1}^{N}(a_n-a_{n,p})z^n\right| + \left|\sum_{N+1}^{\infty} a_nz^n\right| + \left|\sum_{N+1}^{2p} a_{n,p}z^n\right|$$

$$\leqslant \left|\sum_{n=1}^{N}(a_n-a_{n,p})z^n\right| + 2\sum_{N+1}^{\infty}|a_nz^n|.$$

We can now choose N so large that the second term is less than any given ϵ, and then, having fixed N, the first term tends to zero. Hence $g_p(z) \to g(z)$.

As in the previous proof, the result now follows from Hurwitz's theorem.

8.63. The simplest case is that of the function $f(z) = e^z$. From this we deduce that *if $\phi(w)$ satisfies the conditions of the previous theorems, then*

$$F(z) = \sum_{n=0}^{\infty} \frac{\phi(n)}{n!} z^n$$

is an integral function, and all of its zeros are real and negative.

Examples. (i) Let

$$\phi(w) = 1/\Gamma(w+\nu+1) \qquad (\nu > -1).$$

This is an integral function of genus 1, with zeros at $w = -\nu-1$, $-\nu-2,\ldots$. These are all real and negative, and hence the zeros of

$$\sum_{n=0}^{\infty} \frac{z^n}{n!\,\Gamma(n+\nu+1)} = \frac{J_\nu(2i\sqrt{z})}{(i\sqrt{z})^\nu}$$

are all real and negative. Hence the zeros of $J_\nu(z)$ are all real.

(ii) The function*

$$F_\alpha(z) = \int_0^\infty e^{-t^\alpha}\cos zt\,dt$$

* Pólya (1).