taking absolute values, we see that

$$\left|\frac{g(x)e^{-iAx}}{w(x)} - \sum_k \frac{g(a_k)e^{-iAa_k}}{w'(a_k)(x-a_k)}\right|$$
$$\leqslant \frac{e}{\pi}\int_{-\infty}^{\infty}\left|\frac{g(t+(i/A))}{w(t+(i/A))}\right|\left|\frac{\sin A(x-t)}{x-t-(i/A)}\right| dt,$$

Q.E.D.

Corollary. *Let $w(z)$ be as in the lemma, and suppose that $f(z)$ is entire, of exponential type $\leqslant A/2$, and bounded on $\mathbb{R}$. Then there is a polynomial $P(z)$ of degree less than that of $w(z)$, such that*

$$\left|\frac{P(x) - e^{-iAx}\dfrac{\sin(Ax/2)}{(Ax/2)}f(x)}{w(x)}\right| \leqslant \frac{Ke}{\pi}\sup_{t\in\mathbb{R}}\left|\frac{f(t+(i/A))}{w(t+(i/A))}\right| \quad \textit{for } x\in\mathbb{R}.$$

Here K is an absolute numerical constant, whose value we do not bother to calculate.

Proof. Put $g(z) = (\sin(Az/2)/(Az/2))f(z)$; then $g(z)$ satisfies the hypothesis of the previous lemma, so, with the polynomial

$$P(x) = \sum_{k=1}^{N}\frac{w(x)g(a_k)e^{-iAa_k}}{w'(a_k)(x-a_k)},$$

we get, for $x\in\mathbb{R}$,

$$(\dagger) \qquad \left|\frac{P(x) - e^{-iAx}g(x)}{w(x)}\right| \leqslant \frac{e}{\pi}\sup_{t\in\mathbb{R}}\left|\frac{f(t+(i/A))}{w(t+(i/A))}\right|$$
$$\times\int_{-\infty}^{\infty}\left|\frac{\sin\dfrac{A}{2}\left(t+\dfrac{i}{A}\right)}{\dfrac{A}{2}\left(t+\dfrac{i}{A}\right)}\right|\left|\frac{\sin A(x-t)}{x-t-(i/A)}\right| dt.$$

In the integral on the right, make the substitutions $At/2 = \tau$, $Ax/2 = \xi$. That integral then becomes

$$2\int_{-\infty}^{\infty}\left|\frac{\sin(\tau+(i/2))}{\tau+(i/2)}\right|\left|\frac{\sin 2(\xi-\tau)}{2(\xi-\tau)-i}\right| d\tau.$$

By Schwarz, this last is

$$\leqslant 2\sqrt{\left(\int_{-\infty}^{\infty}\left|\frac{\sin(\tau+(i/2))}{\tau+(i/2)}\right|^2 d\tau\cdot\int_{-\infty}^{\infty}\frac{\sin^2 2(\tau-\xi)}{4(\tau-\xi)^2}d\tau\right)},$$

a *finite quantity* – call it K – *independent of* ξ, hence (clearly) *independent* of A and x.

The right-hand side of (†) is thus bounded above by

$$K\cdot\frac{\mathrm{e}}{\pi}\sup_{t\in\mathbb{R}}\left|\frac{f(t+(\mathrm{i}/A))}{w(t+(\mathrm{i}/A))}\right|,$$

and the corollary is established.

Theorem. *Let* $W(x)=\sum_0^\infty a_{2k}x^{2k}$, where the a_{2k} are all $\geqslant 0$, *with* $a_0\geqslant 1$ *and* $a_{2k}>0$ *for infinitely many values of k. Then* $\mathscr{C}_W(0)=\mathscr{C}_W(0+)$.

Remark. We require $a_0\geqslant 1$ because our weights $W(x)$ are supposed to be $\geqslant 1$. We require $a_{2k}>0$ for infinitely many k because $W(x)$ is supposed to go to ∞ faster than any polynomial as $x\to\pm\infty$.

Proof of theorem. Let $\varphi\in\mathscr{C}_W(0+)$. Then there are finite sums

$$f_n(x) \;=\; \sum_{-1/2n\leqslant\lambda\leqslant 1/2n} a_n(\lambda)\mathrm{e}^{\mathrm{i}\lambda x}$$

with

$$\|f_n-\varphi\|_W \;\underset{n}{\longrightarrow}\; 0.$$

We put $g_n(x)=(\sin(x/2n)/(x/2n))f_n(x)$, and set out to apply the above corollary with $f=f_n$ and suitable polynomials w. Note that f_n is entire, of exponential type $1/2n$, and bounded on the real axis. Since $\|f_n-\varphi\|_W \underset{n}{\longrightarrow} 0$, we also have

$$\|\mathrm{e}^{-\mathrm{i}x/n}g_n(x)-\varphi(x)\|_W \;\underset{n}{\longrightarrow}\; 0,$$

in view of the fact that $W(x)\to\infty$ for $x\to\pm\infty$.*

Choose any $\varepsilon>0$. The norms $\|f_n\|_W$ must be bounded; wlog $|f_n(x)/W(x)|\leqslant 1$, say, for $x\in\mathbb{R}$ and every n. For the function $\varphi\in\mathscr{C}_W(\mathbb{R})$ we of course have $\varphi(x)/W(x)\underset{n}{\longrightarrow}0$ for $x\to\pm\infty$, so, since $\|f_n-\varphi\|_W\underset{n}{\longrightarrow}0$, *there must be an* L (depending on ε) such that

$$\left|\frac{f_n(x)}{W(x)}\right| \;\leqslant\; \varepsilon \quad\text{for}\quad |x|\geqslant L$$

whenever n is sufficiently large. Take such an L and *fix* it.

Pick any n large enough for the previous relation to be true, and *fix* it *for the moment.* Our individual function $f_n(x)$ is *bounded* on $\mathbb{R}$ (true, with perhaps an enormous bound!), so, for some N_0, we will surely have

$$\left|f_n(x)\Big/\sum_0^{N_0}a_{2k}x^{2k}\right| \;<\; \varepsilon \quad\text{for}\quad |x|>A,\ \text{say},$$

* Note that $\|\mathrm{e}^{-\mathrm{i}x/n}(\sin(x/2n)/(x/2n))\varphi(x) \;-\; \varphi(x)\|_W \underset{n}{\longrightarrow} 0$ for any $\varphi\in\mathscr{C}_W(\mathbb{R})$.

where, wlog, $A > L$, the number chosen above. Also,

$$1 \leqslant \sum_0^N a_{2k}x^{2k} \underset{N}{\longrightarrow} W(x),$$

the sums on the left being monotone increasing with N ($a_{2k} \geqslant 0$!). Therefore,

$$\left| f_n(x) \Big/ \sum_0^N a_{2k}x^{2k} \right| \underset{N}{\longrightarrow} |f_n(x)/W(x)|$$

uniformly for $-A \leqslant x \leqslant A$, and, if $N \geqslant N_0$ is large enough, we have, in view of the previous inequality,

$$(\S) \qquad \left| f_n(x) \Big/ \sum_0^N a_{2k}x^{2k} \right| \leqslant 2 \quad \text{for} \quad x \in \mathbb{R}$$

(since $\|f_n\|_W \leqslant 1$), and also

$$(\dagger\dagger) \qquad \left| f_n(x) \Big/ \sum_0^N a_{2k}x^{2k} \right| \leqslant 2\varepsilon \quad \text{for} \quad |x| \geqslant L.$$

Fix such an N for the moment (it depends of course on n which we have already fixed!), and call

$$V(x) = \sum_0^N a_{2k}x^{2k}.$$

Because $V(x) \geqslant 1$ on $\mathbb{R}$, we can find another polynomial $w(x)$, *with all its zeros in* $\Im z < 0$, such that $|w(x)| = V(x)$, $x \in \mathbb{R}$.

There is no loss of generality in supposing that the zeros of w are *distinct*. There are, in any case, a finite number ($2N$) of them, lying in the open lower half plane. Separating each *multiple* zero (if there are any*) into a cluster of *simple* ones, *very close together*, will change $w(x)$ to a polynomial $\tilde{w}(x)$ having the new zeros, and such that

$$(1-\delta)|w(x)| \leqslant |\tilde{w}(x)| \leqslant (1+\delta)|w(x)|$$

on $\mathbb{R}$, with $\delta > 0$ *as small as we like.* One may then run through the following argument with $\tilde{w}$ in place of w; the effect of this will merely be to render the final inequality worse by a harmless factor of $(1+\delta)/(1-\delta)$.

Let us proceed, then, assuming that the zeros of w *are* simple. Desiring, as we do, to use the above corollary, we need an estimate for

$$\sup_{t \in \mathbb{R}} \left| \frac{f_n(t+in)}{w(t+in)} \right|.$$

The function $e^{iz/2n} f_n(z)/w(z)$ is *analytic and bounded* for $\Im z > 0$, and continuous up to $\mathbb{R}$. Therefore we can use Poisson's formula (lemma of

* and there *are*! – *all* zeros of $w(z)$ are of *even order*!

§H.1, Chapter III), getting

$$\frac{e^{i(t+in)/2n}f_n(t+in)}{w(t+in)} = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{n}{(t-x)^2+n^2}\cdot\frac{e^{ix/2n}f_n(x)}{w(x)}dx.$$

Since $|w(x)| = V(x)$, we see, by (§) and (††), that the integral on the right is in absolute value

$$\leqslant \frac{2}{\pi}\int_{-L}^{L}\frac{n}{(t-x)^2+n^2}dx + \frac{2\varepsilon}{\pi}\int_{|x|\geqslant L}\frac{n}{(t-x)^2+n^2}dx.$$

This is in turn

$$\leqslant \frac{2}{\pi}\left(\arctan\frac{t+L}{n} - \arctan\frac{t-L}{n}\right) + 2\varepsilon \leqslant \frac{4}{\pi}\frac{L}{n} + 2\varepsilon,$$

so we have

$$\left|\frac{e^{-1/2}f_n(t+in)}{w(t+in)}\right| \leqslant \frac{4}{\pi}\frac{L}{n} + 2\varepsilon, \quad t\in\mathbb{R}.$$

Apply now the corollary with $f = f_n$ and $A = 1/n$. According to it and to the inequality just proved, there is a *polynomial* $P_n(x)$ (depending, of course, partly on our $w(x)$ whose choice *also* depended on the n we have taken!) such that, for $x\in\mathbb{R}$,

$$\left|\frac{P_n(x) - e^{-ix/n}g_n(x)}{w(x)}\right| \leqslant \frac{Ke}{\pi}\cdot e^{1/2}\left(\frac{4}{\pi}\frac{L}{n} + 2\varepsilon\right),$$

where (as we recall),

$$g_n(x) = \frac{\sin(x/2n)}{(x/2n)}f_n(x).$$

Therefore, since $|w(x)| = V(x) \leqslant W(x)$ (!),

$$\left|\frac{P_n(x) - e^{-ix/n}g_n(x)}{W(x)}\right| \leqslant \frac{Ke^{3/2}}{\pi}\left(\frac{4}{\pi}\frac{L}{n} + 2\varepsilon\right), \quad x\in\mathbb{R}.$$

Our number L depended *only* on ε, and the intermediate partial sum

$$V(x) = \sum_0^N a_{2k}x^{2k}$$

(which depended on n) is now *gone. The only restriction on n* (which was kept fixed during the above argument) was that it be *sufficiently large* (*how* large depended on L). For *fixed* L, then, there is, for *each* sufficiently large n, a polynomial $P_n(x)$ satisfying the above relation. If such an n is also $> 2L/\pi\varepsilon$, we will thus certainly have

$$\left|\frac{P_n(x) - e^{-ix/n}g_n(x)}{W(x)}\right| \leqslant 4\varepsilon\frac{Ke^{3/2}}{\pi}$$

for $x \in \mathbb{R}$.

Let us return to our function $\varphi \in \mathscr{C}_W(0+)$. For each *sufficiently large n*, we have a *polynomial* $P_n(x)$ with

$$\|P_n - \varphi\|_W \leqslant \|\varphi(x) - e^{-ix/n}g_n(x)\|_W + \|e^{-ix/n}g_n(x) - P_n(x)\|_W,$$

which, according to the inequality we have finally established, is

$$\leqslant 4\varepsilon\frac{Ke^{3/2}}{\pi} + \|e^{-ix/n}g_n(x) - \varphi(x)\|_W.$$

However, $\|e^{-ix/n}g_n(x) - \varphi(x)\|_W \underset{n}{\longrightarrow} 0$ by choice of our functions f_n. Therefore $\|P_n - \varphi\|_W \leqslant 8\varepsilon(Ke^{3/2}/\pi)$ for all sufficiently large n. Since $\varepsilon > 0$ was arbitrary, we have, then, $\|P_n - \varphi\|_W \underset{n}{\longrightarrow} 0$, and $\varphi \in \mathscr{C}_W(0)$.

This proves that $\mathscr{C}_W(0+) \subseteq \mathscr{C}_W(0)$. Since the reverse inclusion is always true, we are done.

Remark. An analogous result holds for approximation in the norms $\|\ \ \|_{W,p}$, $1 < p < \infty$. There, a much *easier* proof can be given, based on duality and the fact that the Hilbert transform is a bounded operator on $L_p(\mathbb{R})$ for $1 < p < \infty$. The reader is encouraged to try to work out such a proof.

We can apply the technique of convex logarithmic regularisation developed in Chapter IV together with the theorem just proved so as to obtain another result in which a *regularity condition* on $W(x)$ replaces the explicit representation for it figuring above.

Theorem. *Let* $W(x) \geqslant 1$ *be* even, *with* $\log W(x)$ *a* convex *function of* $\log x$ *for* $x > 0$. *Suppose that for* each $\Lambda > 1$ *there is a constant* C_Λ *such that*

$$x^2 W(x) \leqslant C_\Lambda W(\Lambda x), \quad x \in \mathbb{R}.$$

Then $\mathscr{C}_W(0) = \mathscr{C}_W(0+)$.

Remark. Speaking, as we are, of $\mathscr{C}_W(0)$, we of course require that $x^n/W(x) \longrightarrow 0$ for $x \to \pm\infty$ and all $n \geqslant 0$, so $W(x)$ must tend to ∞ fairly rapidly as $x \to \pm\infty$. But one *cannot derive* the condition involving numbers $\Lambda > 1$ from this fact and the convexity of $\log W(x)$ in $\log|x|$. Nor have I been able to *dispense* with that ungainly condition.

Proof of theorem. Let us first show that, if $\varphi \in \mathscr{C}_W(\mathbb{R})$ and we write $\varphi_\lambda(x) = \varphi(\lambda^2 x)$ for $\lambda < 1$, then $\|\varphi - \varphi_\lambda\|_W \longrightarrow 0$ as $\lambda \to 1$.

We know that $\log W(x)$ tends to ∞ as $x \to \pm\infty$. Hence, since that function is *convex* in $\log x$ for $x > 0$, it must be *increasing* in x for all *sufficiently large* x. Take any $\varphi \in \mathscr{C}_W(\mathbb{R})$; since φ is continuous on $\mathbb{R}$ we certainly have $|\varphi(x) - \varphi_\lambda(x)| \longrightarrow 0$ uniformly on *any* interval $[-M, M]$ as $\lambda \to 1$. Also, $|\varphi(x)/W(x)| < \varepsilon$ for $|x|$ sufficiently large. Choose M big enough so that this inequality holds for $|x| \geqslant M/4$ and also $W(x)$ increases for $x \geqslant M/4$. Then, if $\frac{1}{2} < \lambda < 1$ and $|x| \geqslant M$,

$$\left|\frac{\varphi_\lambda(x)}{W(x)}\right| = \left|\frac{\varphi(\lambda^2 x)}{W(x)}\right| \leqslant \left|\frac{\varphi(\lambda^2 x)}{W(\lambda^2 x)}\right| < \varepsilon,$$

as well as $|\varphi(x)/W(x)| < \varepsilon$, so

$$\left|\frac{\varphi(x) - \varphi_\lambda(x)}{W(x)}\right| < 2\varepsilon$$

for $|x| \geqslant M$ and $\frac{1}{2} < \lambda < 1$. Making λ close enough to 1, we get the quantity on the left $< 2\varepsilon$ for $-M \leqslant x \leqslant M$ also, so $\|\varphi - \varphi_\lambda\|_W < 2\varepsilon$.

Take now any $\varphi \in \mathscr{C}_W(0+)$. We have to show that φ also belongs to $\mathscr{C}_W(0)$, and, by what we have just proved, this will follow if we establish that $\varphi_\lambda \in \mathscr{C}_W(0)$ for *each* $\lambda < 1$. We proceed to verify that fact.

We may, wlog, assume that $W(x) \equiv 1$ for $|x| \leqslant 1$ and *increases* for $x \geqslant 1$. For $n = 0, 1, 2, \ldots$, put

$$S_n = \sup_{r>0} \frac{r^n}{W(r)}$$

and, then, for $r > 0$, write

$$T(r) = \sup_{n \geqslant 0} \frac{r^{2n}}{S_{2n}}.$$

Since $\log W(r)$ increases for $r \geqslant 1$, the *proof* of the *second* lemma from §D of Chapter IV shows that

$$\frac{W(r)}{r^2} \leqslant T(r) \leqslant W(r) \quad \text{for} \quad r \geqslant 1$$

(cf. proof of second theorem in §D, this chapter). Take now

$$(\S\S) \qquad S(x) = 1 + \sum_{n=0}^{\infty} \frac{x^{2n+2}}{S_{2n}}.$$

Then, by the preceding inequalities, for $|x| \geqslant 1$,

$$S(x) \geqslant x^2 T(|x|) \geqslant W(x)$$

whilst, for any λ, $0 < \lambda < 1$,

$$S(x) \;=\; 1 + x^2 \sum_0^\infty \lambda^{2n} \frac{(x/\lambda)^{2n}}{S_{2n}} \;\leqslant\; 1 + \frac{x^2}{1-\lambda^2} T\left(\frac{|x|}{\lambda}\right)$$

$$\leqslant\; 1 + \frac{x^2}{1-\lambda^2} W\left(\frac{x}{\lambda}\right).$$

The first of these relations* clearly also holds for $|x| \leqslant 1$, because $W(x) \equiv 1$ there. *So does the second.* For, the inequality between its last two members is true for $|x| \geqslant \lambda$, while $T(|x|/\lambda)$ is, by its definition, *increasing* when $0 < |x| < \lambda$, and $W(x/\lambda)$ *constant* for such x. We thus have

$$W(x) \;\leqslant\; S(x) \;\leqslant\; 1 + \frac{x^2}{1-\lambda^2} W\left(\frac{x}{\lambda}\right)$$

for all x.

According to the hypothesis, there is a constant K_λ for each $\lambda < 1$ with

$$\frac{x^2}{1-\lambda^2} W\left(\frac{x}{\lambda}\right) \;\leqslant\; K_\lambda W\left(\frac{x}{\lambda^2}\right).$$

We may of course take $K_\lambda \geqslant 1$, and thus get finally

$$(\ddagger) \qquad W(x) \;\leqslant\; S(x) \;\leqslant\; 2K_\lambda W\left(\frac{x}{\lambda^2}\right), \quad x \in \mathbb{R}.$$

Given our function $\varphi \in \mathscr{C}_W(0+)$, we have a sequence of functions f_n, $f_n \in \mathscr{E}_{1/n}$, with

$$\|\varphi - f_n\|_W \;\xrightarrow[n]{}\; 0.$$

Thence, by ($\ddagger$), *a fortiori*,

$$\|\varphi - f_n\|_S \;\xrightarrow[n]{}\; 0,$$

so $\varphi \in \mathscr{C}_S(0+)$ as well. Now, however, $S(x)$ has the form (§§), so *we may apply the previous theorem, getting* $\varphi \in \mathscr{C}_S(0)$. There is thus a sequence of *polynomials* $P_n(x)$ with

$$\|\varphi - P_n\|_S \;\xrightarrow[n]{}\; 0.$$

From this we see, by ($\ddagger$) again, that

$$\sup_{x \in \mathbb{R}} \left| \frac{\varphi(x) - P_n(x)}{W(x/\lambda^2)} \right| \;\xrightarrow[n]{}\; 0,$$

* i.e., that between $S(x)$ and $W(x)$

i.e.,

$$\sup_{x\in\mathbb{R}} \left| \frac{\varphi(\lambda^2 x) - P_n(\lambda^2 x)}{W(x)} \right| \xrightarrow[n]{} 0$$

for each λ, $0 < \lambda < 1$.

But this means that $\varphi_\lambda \in \mathscr{C}_W(0)$ for each such λ, the fact we had to verify. The theorem is proved.

Remark. *Some* regularity in $W(x)$ is *necessary* for the equality of $\mathscr{C}_W(0)$ and $\mathscr{C}_W(0+)$; exactly *what kind* is not yet known. In the next article we give an example showing that the behaviour of $W(x)$ *cannot depart too much* from that required in the above two theorems if we are to have $\mathscr{C}_W(0) = \mathscr{C}_W(0+)$. In the first part of the next chapter we will give another example, of an even weight $W(x)$, *increasing* for $x > 0$, such that $\mathscr{C}_W(0) \neq \mathscr{C}_W(0+) = \mathscr{C}_W(\mathbb{R})$.

3. Example of a weight W with $\mathscr{C}_W(0) \neq \mathscr{C}_W(0+) \neq \mathscr{C}_W(\mathbb{R})$

The idea for this example comes from a letter J.-P. Kahane sent me in 1963.

Take

$$S(z) = \prod_1^\infty \left(1 - \frac{z^2}{4^n}\right),$$

pick any fixed number λ_1, $1 < \lambda_1 < 2$, and write

$$C(z) = \left(1 - \frac{z^2}{\lambda_1^2}\right) \prod_2^\infty \left(1 - \frac{z^2}{4^n}\right).$$

This function $S(z)$ is the same as the one used in §C, and $C(z)$ differs from it *only* in that the two zeros, -2 and 2, of $S(z)$ *closest to the origin* have been *moved* slightly, the first towards -1 and the second towards 1.

Let us write $\lambda_{-1} = -\lambda_1$, and, for $|n| \geqslant 2$, $\lambda_n = (\operatorname{sgn} n)2^{|n|}$. Then

$$C(z) = \prod_1^\infty \left(1 - \frac{z^2}{\lambda_n^2}\right),$$

and

$$\sum_{-\infty}^{\infty}{}' \frac{\lambda_n S(\lambda_n)}{C'(\lambda_n)} = 2\frac{\lambda_1 S(\lambda_1)}{C'(\lambda_1)} < 0.$$

For large n, we clearly have

$$C'(\lambda_n) \sim \frac{4}{\lambda_1^2} S'(\lambda_n),$$

where $S'(\lambda_n)$ was studied in §C. There we found that

$$|S'(\lambda_n)| \;=\; |S'(2^n)| \;\sim\; \text{const.}2^{(n-1)^2}$$

for large n, so surely (in view of the evenness of $C(z)$),

$$\sum_{-\infty}^{\infty}{}' \frac{|\lambda_n|^p}{|C'(\lambda_n)|} \;<\; \infty \quad \text{for} \quad p = 0, 1, 2, 3, \ldots$$

Use of the Lagrange interpolation formula now shows, as in §C (where an analogous result was proved with $S'(\lambda_n)$ in place of $C'(\lambda_n)$), that

$$P(z) \;=\; C(z) \sum_{-\infty}^{\infty}{}' \frac{P(\lambda_n)}{(z-\lambda_n)C'(\lambda_n)}$$

for any *polynomial* P. Taking $P(z) = z^{p+1}$ and then putting $z = 0$ gives us

$$\sum_{-\infty}^{\infty}{}' \frac{\lambda_n^p}{C'(\lambda_n)} \;=\; 0, \quad p = 0, 1, 2, \ldots.$$

We are ready to construct our weight W. Taking a large constant K (chosen so as to make $W(x)$ come out $\geqslant 1$), put $W(x) = KS(1)$ for $|x| \leqslant 1$. For $|x| \geqslant 1$, make $W(x) = Kx^2|S(x)|$ when x lies *outside all the intervals*

$$[2^n(1-2^{-4n}), \quad 2^n(1+2^{-4n})].$$

Finally, if $2^n(1-2^{-4n}) \;\leqslant\; |x| \;\leqslant\; 2^n(1+2^{-4n})$ for some $n \geqslant 1$, define $W(x)$ as

$$\sup\{K\xi^2|S(\xi)|: \quad |\xi - 2^n| \leqslant 2^{-3n}\}.$$

We see first of all that $xS(x)/W(x) \longrightarrow 0$ for $x \to \pm\infty$, so $xS(x) \in \mathscr{C}_W(\mathbb{R})$. Hence, since $S(z)$, and therefore $zS(z)$, is of exponential type *zero* we have $xS(x) \in \mathscr{C}_W(0+)$ by the *first* theorem of article 1.

We need some information about the asymptotic behaviour of $S(x)$ for $x \to \infty$. This may be obtained by the method followed in §C. Suppose that $x = 2^n\alpha$ with $1/\sqrt{2} \;\leqslant\; \alpha \;\leqslant\; \sqrt{2}$. Then we have

$$\begin{aligned} |S(x)| \;&=\; \prod_{k=1}^{n-1}\left|1 - \frac{4^n\alpha^2}{4^k}\right| \times |1-\alpha^2| \times \prod_{k=n+1}^{\infty}\left|1 - \frac{4^n\alpha^2}{4^k}\right| \\ &=\; |1-\alpha^2|\prod_{k=1}^{n-1}\left(\frac{4^n\alpha^2}{4^k}\right)\prod_{l=1}^{n-1}\left(1 - \frac{1}{4^l\alpha^2}\right)\prod_{l=1}^{\infty}\left(1-\frac{\alpha^2}{4^l}\right), \end{aligned}$$

and this last is

$$\sim\; |1-\alpha^2|\frac{(4^n\alpha^2)^{n-1}}{2^{n(n-1)}}S\left(\frac{1}{\alpha}\right)S(\alpha) \;=\; |1-\alpha^2|(2^n\alpha^2)^{n-1}S\left(\frac{1}{\alpha}\right)S(\alpha).$$

Thence, for large n,

$$\sup\{|S(\xi)|: \ |\xi - 2^n| \leqslant 2^{-3n}\} \ \sim \ \text{const.}2^{n^2-5n},$$

so, for $2^n(1-2^{-4n}) \ \leqslant \ |x| \ \leqslant \ 2^n(1+2^{-4n})$,

$$W(x) \ \sim \ \text{const.}2^{n^2-3n},$$

and, in particular,

$$W(\lambda_n) \ \sim \ \text{const.}2^{n^2-3n}.$$

Comparing this with the relation $|C'(\lambda_n)| \sim \text{const.}2^{(n-1)^2}$, valid for large n, which we already know, we see that

$$\sum_{-\infty}^{\infty}{}' \frac{W(\lambda_n)}{|C'(\lambda_n)|} \ < \ \infty.$$

This permits us to define a *finite* signed Radon measure μ on the set of points λ_n, $n = \pm 1, \pm 2, \ldots$, by putting

$$\mu(\{\lambda_n\}) \ = \ \frac{W(\lambda_n)}{C'(\lambda_n)}.$$

Then, for $p = 0, 1, 2, \ldots$,

$$\int_{-\infty}^{\infty} \frac{x^p}{W(x)} \mathrm{d}\mu(x) \ = \ \sum_{-\infty}^{\infty}{}' \frac{\lambda_n^p}{C'(\lambda_n)},$$

which is *zero*, as we have seen, whilst

$$\int_{-\infty}^{\infty} \frac{xS(x)}{W(x)} \mathrm{d}\mu(x) \ = \ \sum_{-\infty}^{\infty}{}' \frac{\lambda_n S(\lambda_n)}{C'(\lambda_n)} \ < \ 0.$$

So $xS(x) \in \mathscr{C}_W(0+)$ *can't* be in $\mathscr{C}_W(0)$, and $\mathscr{C}_W(0) \neq \mathscr{C}_W(0+)$.

In the present example, $\mathscr{C}_W(0+)$ is a *proper subspace* of $\mathscr{C}_W(\mathbb{R})$. Indeed, this is almost immediate. By the above asymptotic computation of $S(x)$ we clearly have

$$W(x) \ = \ \text{const.}|x|^{\log_2|x| + \theta(x)}$$

with a quantity $\theta(x)$ *varying between two constants*. Therefore,

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2} \mathrm{d}x \ < \ \infty,$$

so we surely have $\mathscr{C}_W(A) \neq \mathscr{C}_W(\mathbb{R})$ for each A by an obvious extension of

T. Hall's theorem (p. 169). Hence $\mathscr{C}_W(0+) \neq \mathscr{C}_W(\mathbb{R})$. The construction of our example is finished.*

Remark. Let $\Omega(x) = \prod_1^\infty (1 + x^2/4^n)$. The asymptotic evaluation of $\Omega(x)$ for $x \to \infty$ can be made in fashion similar to that for $S(x)$, and is, in fact, *easier* than the latter. As is clear after a moment's thought, here one also obtains

$$\Omega(x) \sim \text{const.}|x|^{\log_2|x| + \varphi(x)} \quad \text{for} \quad |x| \to \infty$$

with a certain $\varphi(x)$ varying between two constants. Thus,

$$\frac{\Omega(x)}{W(x)} = \text{const.}|x|^{\psi(x)}, \quad x \in \mathbb{R},$$

where $A \leqslant \psi(x) \leqslant B$, say.

However, $\mathscr{C}_\Omega(0) = \mathscr{C}_\Omega(0+)$. This follows from the first theorem of the previous article, in view of the evident fact that $\Omega(x) = 1 + a_2x^2 + a_4x^4 + \cdots$ with $a_{2k} > 0$.

The difference in behaviour of $\Omega(x)$ and $W(x)$ is small in comparison to their size, and yet $\mathscr{C}_\Omega(0) = \mathscr{C}_\Omega(0+)$ although $\mathscr{C}_W(0) \neq \mathscr{C}_W(0+)$.

The question of *how* a weight W's *local behaviour* is related to the *equality* of $\mathscr{C}_W(0)$ and $\mathscr{C}_W(0+)$ *merits further study.*

* $W(x)$ has jump discontinuities among the points $\pm(2^n \pm 2^{-3n})$, $n \geqslant 1$, but a continuous weight with the same properties as W is furnished by an evident elaboration of the procedure in the text.

VII

How small can the Fourier transform of a rapidly decreasing non-zero function be?

Let us consider functions $F(x) \in L_1(\mathbb{R})$ whose modulus goes to zero *rapidly* as $x \to \infty$, in such fashion that

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} \log^{-}\left(\int_{x}^{\infty} |F(t)|\,\mathrm{d}t \right) \mathrm{d}x \quad = \quad \infty .$$

The general theme of this chapter is that, for such a function F, the Fourier transform

$$\hat{F}(\lambda) \quad = \quad \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\lambda x} F(x)\,\mathrm{d}x$$

cannot be too small anywhere unless F vanishes identically.

The first result of this kind (obtained by Levinson) said that if (for such an F) *$\hat{F}$ vanishes throughout an interval of positive length*, then $F \equiv 0$. This was refined by Beurling, who proved that $\hat{F}(\lambda)$ *cannot even vanish on a set of positive measure unless* $F \equiv 0$. Analogues of these theorems hold for measures as well as functions F; they, and the methods used to establish them, have various important consequences, some of which apply to material already taken up in the present book.

These things have been known for more than 20 years. Until recently, the only developments since the sixties in the subject matter of this chapter had to do mainly with aspects of its presentation. That state of affairs was changed in 1982 by the appearance of a remarkable result, due to A.L. Volberg, which says that if

$$f(\vartheta) \quad = \quad \sum_{-\infty}^{\infty} \hat{f}(n)^{\mathrm{i}n\vartheta}$$

has

$$|\hat{f}(n)| \quad \leqslant \quad \mathrm{e}^{-M(n)} \quad \text{for} \quad n > 0$$

with $M(n)$ sufficiently regular and increasing, and if

$$\sum_1^\infty \frac{M(n)}{n^2} = \infty,$$

then

$$\int_{-\pi}^{\pi} \log|f(\vartheta)|\mathrm{d}\vartheta > -\infty$$

unless $f(\vartheta) \equiv 0$. The proof of this uses new ideas (coming from the study of weighted *planar* approximation by polynomials) and is very long; its inclusion has necessitated a considerable extension of the present chapter. I still do not completely understand the result's meaning; it applies to the *unit circle* and seems to *not have* a natural analogue for the *real line* which would generalize Levinson's and Beurling's theorems.

There are not too many easily accessible references for this chapter. The earliest results are in Levinson's book; material relating to them can also be found in the book by de Branges (some of it being set as problems). The main source for the first two §§ of this chapter consists, however, of the famous mimeographed notes for Beurling's Standford lectures prepared by P. Duren; those notes came out around 1961. Volberg published his theorem in a 6-page (!) *Doklady* note at the beginning of 1982. That paper is quite difficult to get through on account of its being so condensed.

A. The Fourier transform vanishes on an interval. Levinson's result

Levinson originally proved his theorem by means of a complicated argument, involving contour integration, which figured later on as one of the main ingredients in Beurling's proof of his deeper result. Beurling observed that Levinson's theorem (and others related to it) could be obtained more easily by the use of test functions, and then de Branges simplified that treatment by bringing Akhiezer's first theorem from §E.2 of Chapter VI into it. I follow this procedure in the present §. The particularly convenient and elegant test function used here (which has several other applications, by the way) was suggested to me by my reading of a paper of H. Widom.

1. Some shop math

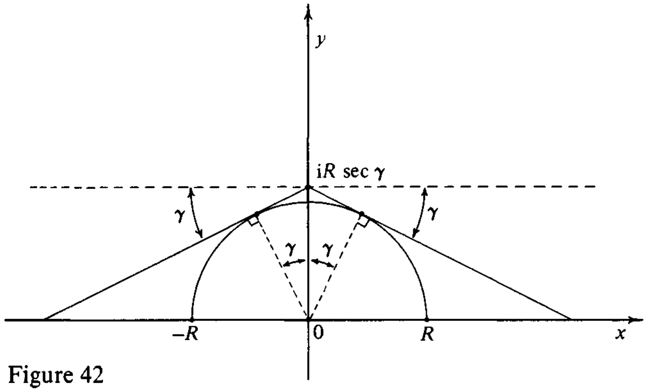

Figure 42

The circle of radius R about 0 lies *under* the two straight lines of slopes $\pm\tan\gamma$ passing through the point $iR\sec\gamma$. Therefore, if $A > 0$,

$$A\sqrt{(R^2 - x^2)} \;\leqslant\; AR\sec\gamma - (A\tan\gamma)|x|, \qquad -R \leqslant x \leqslant R.$$

Consider any function $\omega(x) \geqslant 0$ such that

$$|\omega(x) - \omega(x')| \;\leqslant\; (A\tan\gamma)|x - x'|.$$

If we adjust R so as to make $AR\sec\gamma = \omega(0)$, we have, for $-R \leqslant x \leqslant R$,

$$\omega(x) \;\geqslant\; \omega(0) - (A\tan\gamma)|x|,$$

which, by the above, is $\geqslant A\sqrt{(R^2 - x^2)}$. The function $\cos(A\sqrt{(x^2 - R^2)})$ is, however, in modulus $\leqslant 1$ for $|x| > R$, and for $-R \leqslant x \leqslant R$ it equals $\cosh(A\sqrt{(R^2 - x^2)}) \leqslant \exp(A\sqrt{(R^2 - x^2)})$. Therefore, for $x \in \mathbb{R}$,

$$\omega(x) \;\geqslant\; \log|\cos(A\sqrt{(x^2 - R^2)})|$$

when

$$R \;=\; \frac{\omega(0)}{A\sec\gamma}.$$

Let us apply these considerations to a function $W(x) \geqslant 1$ defined on $\mathbb{R}$ and satisfying

$$|\log W(x) - \log W(x')| \;\leqslant\; C|x - x'|$$

there. Taking any fixed $A > 0$, we determine an acute angle γ such that $A\tan\gamma = C$. Suppose $x_0 \in \mathbb{R}$ is given. Then we translate x_0 to the origin, using the above calculation with

$$\omega(x - x_0) \;=\; \log W(x).$$

We see that

$$|\cos(A\sqrt{((x - x_0)^2 - R^2)})| \;\leqslant\; W(x) \quad \text{for} \quad x \in \mathbb{R},$$

where

$$R = \frac{\log W(x_0)}{A \sec \gamma} = \frac{\log W(x_0)}{\sqrt{(A^2 + C^2)}}.$$

Here, $\cos(A\sqrt{((z-x_0)^2 - R^2)})$ *is an entire function of* z because the Taylor development of $\cos w$ about the origin contains only *even powers* of w. It is clearly *of exponential type* A, and, for $z = x_0$, has the value

$$\cosh AR \geqslant \tfrac{1}{2}e^{AR} = \tfrac{1}{2}(W(x_0))^{A/\sqrt{(A^2+C^2)}}.$$

Recall now the definition of the Akhiezer function $W_A(x)$ given in Chapter VI, §E.2, namely

$$W_A(x) = \sup\{|f(x)|\colon f \text{ entire of exponential type} \leqslant A, \\ \text{bounded on } \mathbb{R} \text{ and } |f(t)/W(t)| \leqslant 1 \text{ on } \mathbb{R}\}.$$

In terms of W_A, we have, by the computation just made, the

Theorem. *Let* $W(x) \geqslant 1$ *on* $\mathbb{R}$, *with*

$$|\log W(x) - \log W(x')| \leqslant C|x - x'|$$

for x *and* $x' \in \mathbb{R}$. *Then, if* $A > 0$,

$$W_A(x) \geqslant \tfrac{1}{2}(W(x))^{A/\sqrt{(A^2+C^2)}}, \quad x \in \mathbb{R}.$$

Corollary. *Let* $W(x) \geqslant 1$, *with* $\log W(x)$ *uniformly* Lip 1 *on* $\mathbb{R}$. *Then, if*

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2}\,dx = \infty,$$

we have

$$\int_{-\infty}^{\infty} \frac{\log W_A(x)}{1+x^2}\,dx = \infty$$

for each $A > 0$.

According to Akhiezer's first theorem (Chapter VI, §E.2), this in turn implies the

Theorem. *Let* $W(x) \geqslant 1$, *with* $\log W(x)$ *uniformly* Lip 1 *on* $\mathbb{R}$, *and* $W(x)$ *tending to* ∞ *as* $x \longrightarrow \pm\infty$. *If*

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2}\,dx = \infty,$$

linear combinations of $e^{i\lambda x}$, $-A \leqslant \lambda \leqslant A$, *are, for each* $A > 0$, $\|\ \ \|_W$*-dense in* $\mathscr{C}_W(\mathbb{R})$.

2. Beurling's gap theorem

As a first application of the above fairly easy result, let us prove the following beautiful proposition of Beurling:

Theorem. *Let μ be a totally finite complex Radon measure on $\mathbb{R}$ with $|d\mu(t)| = 0$ on each of the disjoint intervals (a_n, b_n), $0 < a_1 < b_1 < a_2 < b_2 < \cdots$, and suppose that*

$$(*) \qquad \sum_1^\infty \left(\frac{b_n - a_n}{a_n}\right)^2 = \infty.$$

If $\hat{\mu}(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda x} d\mu(x)$ vanishes identically on some real interval of positive length, then $\mu \equiv 0$.

Remark. This is not the only time we shall encounter the condition $(*)$ in the present book.

Proof of theorem (de Branges). We start by taking an *even* function $T(x) \geqslant 1$ whose logarithm is uniformly Lip 1 on $\mathbb{R}$, and which *increases to ∞ so slowly as $|x| \to \infty$*, that

$$\int_{-\infty}^{\infty} T(x)|d\mu(x)| < \infty.$$

(Construction of such a function T is in terms of the given measure μ, and is left to the reader as an easy exercise.)

For each n, let b'_n be *the lesser* of b_n and $2a_n$. Then, given that $(*)$ holds, we also have

$$\sum_n \left(\frac{b'_n - a_n}{a_n}\right)^2 = \infty.$$

Indeed, this sum certainly diverges if the one in $(*)$ does, when $(b'_n - a_n)/a_n$ differs from $(b_n - a_n)/a_n$ for only *finitely many n*. But the sum in question *also* diverges when *infinitely many* of its terms differ from the corresponding ones in $(*)$, since $(b'_n - a_n)/a_n = 1$ when $b'_n = 2a_n$.

Let $\omega(x)$ be *zero* outside the intervals (a_n, b'_n), and *on each one of those intervals let the graph of $\omega(x)$ vs x be a 45° triangle with base on (a_n, b'_n).*

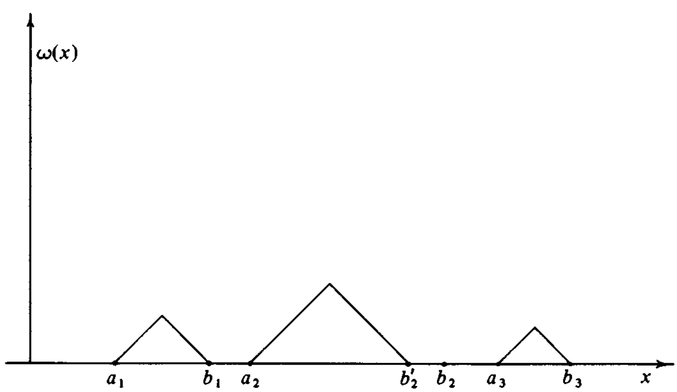

Figure 43

The function $\omega(x)$ is clearly uniformly Lip 1 on $\mathbb{R}$.

Put $W(x) = e^{\omega(x)}T(x)$. Then $W(x) \geqslant 1$ and $\log W(x)$ is uniformly Lip 1 on $\mathbb{R}$; also, $W(x) \to \infty$ for $x \to \pm\infty$. Since $|d\mu(x)| = 0$ throughout each interval (a_n, b_n') and $\omega(x)$ is *zero* outside those intervals,

$$\int_{-\infty}^{\infty} W(x)|d\mu(x)| \;=\; \int_{-\infty}^{\infty} T(x)|d\mu(x)| \;<\; \infty.$$

The complex Radon measure ν with

$$d\nu(x) = W(x)d\mu(x)$$

is therefore *totally finite*.

Suppose now that $\hat{\mu}(\lambda)$ vanishes on some *interval*; say, wlog, that

$$\int_{-\infty}^{\infty} e^{i\lambda x}d\mu(x) \;=\; 0 \quad \text{for} \quad -A \leqslant \lambda \leqslant A.$$

This can be rewritten as

$$(\ast) \qquad \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{W(x)}d\nu(x) \;=\; 0, \quad -A \leqslant \lambda \leqslant A.$$

However, $\log W(x) \;=\; \omega(x) + \log T(x) \;\geqslant\; \omega(x)$, and

$$\int_{a_1}^{\infty} \frac{\omega(x)}{x^2}dx \;\geqslant\; \sum_1^{\infty}\left(\frac{1}{b_n'}\right)^2\left(\frac{b_n' - a_n}{2}\right)^2 \;\geqslant\; \frac{1}{4}\sum_1^{\infty}\left(\frac{b_n' - a_n}{2a_n}\right)^2,$$

which is *infinite*, as we saw above. Therefore

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1 + x^2}dx \;=\; \infty,$$

and *linear combinations* of the $e^{i\lambda x}$, $-A \leqslant \lambda \leqslant A$, are $\|\ \|_W$*-dense* in $\mathscr{C}_W(\mathbb{R})$ by the *second* theorem of the preceding article.

Referring to $(\ast)$, we thence see that $\nu \equiv 0$, i.e., $d\nu(x) = W(x)d\mu(x) \equiv 0$ and $\mu = 0$. Q.E.D.

Problem 11

Let μ be a finite complex measure on $\mathbb{R}$, and put

$$e^{-\sigma(x)} \;=\; \int_{-\infty}^{\infty} e^{-|x-t|}|d\mu(t)|.$$

Suppose that $\int_{-\infty}^{\infty}(\sigma(x)/(1 + x^2))dx = \infty$. Then, if $\hat{\mu}(\lambda)$ vanishes identically on any interval, $\mu \equiv 0$ (Beurling). (Hint. Wlog, $\int_{-\infty}^{\infty}|d\mu(t)| \leqslant 1$ so that

$\sigma(x) \geqslant 0$. Assuming that $\hat{\mu}(\lambda) \equiv 0$ for $-A \leqslant \lambda \leqslant A$, write the relation

$$\int_{-\infty}^{\infty} e^{\sigma(x)-|x-t|}|d\mu(t)| \;=\; 1,$$

and use the picture

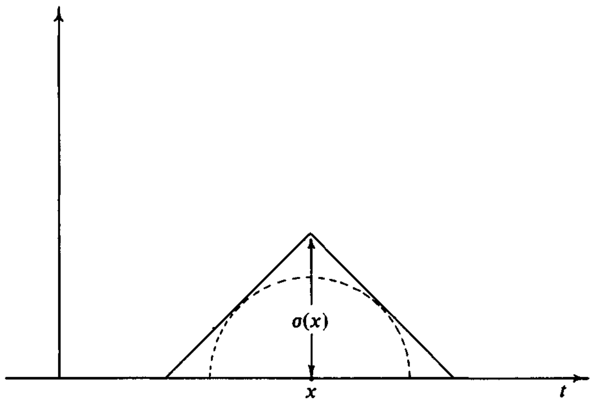

Figure 44

to estimate the supremum of $|f(x)|$ for entire functions f of exponential type $\leqslant A$, bounded on $\mathbb{R}$, and such that

$$\int_{-\infty}^{\infty} |f(t)|\;|d\mu(t)| \;\leqslant\; 1.)$$

Remark. Beurling generalized the result of problem 11 to complex Radon measures μ *which are not necessarily totally finite.* This extension will be taken up in Chapter X.

3. Weights which increase along the positive real axis

Lemma. *Let* $T(x) \geqslant 1$ *be defined and* increasing *for* $x \geqslant 0$, *and denote by* $\mathcal{T}(x)$ *the* largest minorant *of* $T(x)$ *with the property that* $|\log \mathcal{T}(x) - \log \mathcal{T}(x')| \leqslant |x - x'|$ *for* x *and* $x' \geqslant 0$. *If* $\int_1^{\infty} (\log T(x)/x^2)dx = \infty$, *then also* $\int_1^{\infty} (\log \mathcal{T}(x)/x^2)dx = \infty$.

Proof. The graph of $\log \mathcal{T}(x)$ vs x is obtained from that of $\log T(x)$ by means of the following construction:

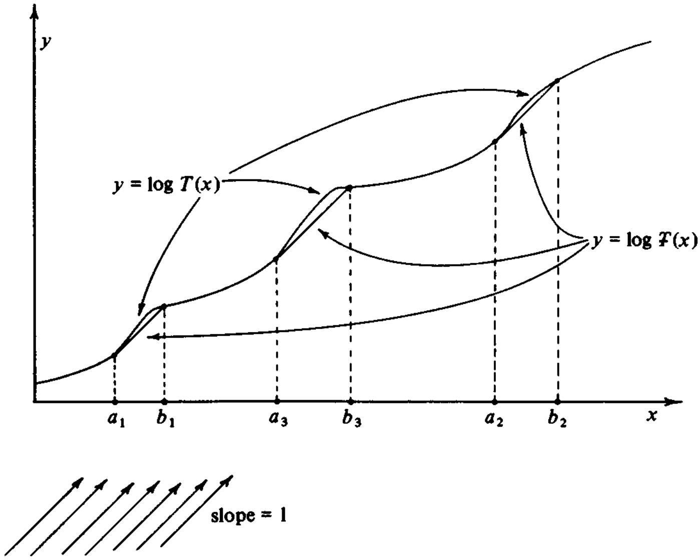

Figure 45

One imagines rays of light of slope 1 shining upwards *underneath* the graph of $\log T(x)$ vs x. The graph of $\log \mathcal{T}(x)$ is made up of the portions of the *former* one which are *illuminated* by those rays of light and some straight segments of slope 1. Those segments lie over certain intervals $[a_n, b_n]$ on the x-axis, of which there are generally countably many, that cannot necessarily be indexed in such fashion that $b_n \leqslant a_{n+1}$ for all n. The *open* intervals (a_n, b_n) are *disjoint*, and on *any one of them* we have

$$\log \mathcal{T}(x) \;=\; \log T(a_n) + (x - a_n).$$

On $[0, \infty) \sim \bigcup_n (a_n, b_n)$, $\mathcal{T}(x)$ and $T(x)$ are *equal*.

In order to prove the lemma, let us *assume* that $\int_1^\infty (\log \mathcal{T}(x)/x^2)\mathrm{d}x < \infty$ and then *show* that $\int_1^\infty (\log T(x)/x^2)\mathrm{d}x < \infty$. If, in the first place, (a_n, b_n) is any of the aforementioned intervals with $1 \leqslant a_n < b_n/2$, we have, since $\log T(a_n) \geqslant 0$,

$$\int_{a_n}^{b_n} \frac{\log \mathcal{T}(x)}{x^2}\mathrm{d}x \;\geqslant\; \int_{a_n}^{b_n} \frac{x - a_n}{x^2}\mathrm{d}x \;>\; \int_1^2 \frac{\xi - 1}{\xi^2}\mathrm{d}\xi$$
$$=\; \log 2 - \tfrac{1}{2}. >\; 0.$$

We can therefore only have *finitely many intervals* (a_n, b_n) with $b_n > 2a_n$ and $a_n \geqslant 1$ if $\int_1^\infty (\log \mathcal{T}(x)/x^2)\mathrm{d}x$ is *finite*.

This being granted, consider any *other* of the intervals (a_n, b_n) with $a_n \geqslant 1$.

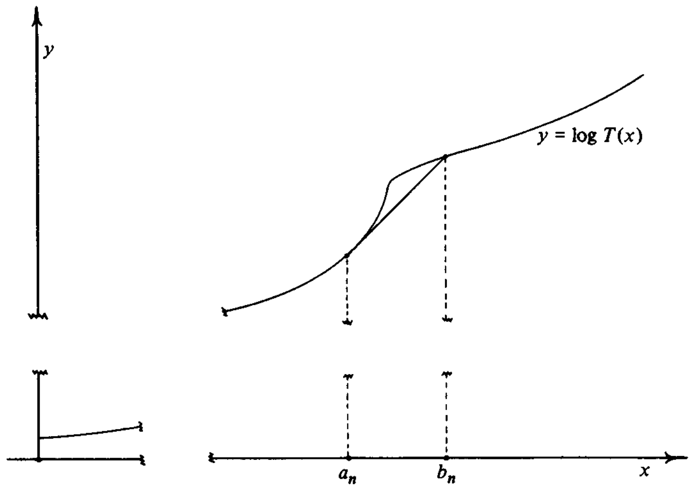

Figure 46

By shop math,

$$\int_{a_n}^{b_n} \frac{1}{x^2} \log \bar{T}(x) \mathrm{d}x \;\geqslant\; \frac{1}{b_n^2} \int_{a_n}^{b_n} \log \bar{T}(x) \mathrm{d}x$$

$$= \frac{1}{b_n^2} \cdot \frac{\log T(a_n) + \log T(b_n)}{2} \cdot (b_n - a_n) \;\geqslant\; \frac{(b_n - a_n) \log T(b_n)}{2b_n^2}.$$

At the same time, since $T(x)$ *increases*,

$$\int_{a_n}^{b_n} \frac{1}{x^2} \log T(x) \mathrm{d}x \;\leqslant\; \frac{(b_n - a_n) \log T(b_n)}{a_n^2} \;\leqslant\; 8 \cdot \frac{(b_n - a_n) \log T(b_n)}{2b_n^2}$$

when $b_n \leqslant 2a_n$. Therefore, for all the intervals (a_n, b_n) with $a_n \geqslant 1$ and $b_n \leqslant 2a_n$, hence, certainly, *for all save a finite number of the* (a_n, b_n) *contained in* $[1, \infty)$, we have

$$\int_{a_n}^{b_n} \frac{1}{x^2} \log T(x) \mathrm{d}x \;\leqslant\; 8 \int_{a_n}^{b_n} \frac{1}{x^2} \log \bar{T}(x) \mathrm{d}x.$$

The *sum* of the integrals $\int_{a_n}^{b_n} (1/x^2) \log T(x) \mathrm{d}x$ for the *remaining finite number of* (a_n, b_n) in $[1, \infty)$ *is surely finite* – note that *none* of those intervals can have *infinite length*, for such a one would be of the form (a_l, ∞), and in that case we would have

$$\int_{a_l}^{\infty} \frac{1}{x^2} \log \bar{T}(x) \mathrm{d}x \;\geqslant\; \int_{a_l}^{\infty} \frac{x - a_l}{x^2} \mathrm{d}x \;=\; \infty,$$

contrary to our assumption on $\mathcal{F}(x)$. We see that

$$\sum_{a_n \geqslant 1} \int_{a_n}^{b_n} \frac{\log T(x)}{x^2} \mathrm{d}x \; < \; \infty,$$

since

$$\sum_{\substack{a_n \geqslant 1 \\ 2b_n \leqslant a_n}} 8 \int_{a_n}^{b_n} \frac{\log \mathcal{F}(x)}{x^2} \mathrm{d}x$$

is finite.

On the complement

$$E = [1, \infty) \cap \sim \bigcup_n (a_n, b_n),$$

$T(x) = \mathcal{F}(x)$ by our construction. Hence

$$\int_E \frac{\log T(x)}{x^2} \mathrm{d}x \; = \; \int_E \frac{\log \mathcal{F}(x)}{x^2} \mathrm{d}x \; < \; \infty.$$

The whole half line $[1, \infty)$ can differ from the union of E and the (a_n, b_n) with $a_n \geqslant 1$ by *at most* an interval of the form $[1, b_m)$, which happens when there is an m such that $a_m < 1 < b_m$. If there *is* such an m, however, b_m must be *finite* (see above), and then

$$\int_1^{b_m} \frac{\log T(x)}{x^2} \mathrm{d}x \; < \; \infty.$$

Putting everything together, we see that

$$\int_1^{\infty} \frac{\log T(x)}{x^2} \mathrm{d}x \; < \; \infty,$$

which is what we had to show. We are done.

Corollary. *Let* $W(x) \geqslant 1$ *be defined on* $\mathbb{R}$ *and* increasing *for* $x \geqslant 0$. *If*

$$\int_1^{\infty} \frac{\log W(x)}{x^2} \mathrm{d}x \; = \; \infty,$$

we have

$$\int_1^{\infty} \frac{\log W_A(x)}{x^2} \mathrm{d}x \; = \; \infty$$

for each of the Akhiezer functions W_A, $A > 0$ (Chapter VI, §E.2).

Proof. Let, for $x \geqslant 0$, $\mathcal{F}(x)$ be the *largest minorant* of $W(x)$ on $[0, \infty)$ with

$$|\log \mathcal{F}(x) - \log \mathcal{F}(x')| \; \leqslant \; |x - x'|$$

there, and put $\mathcal{T}(x) = \mathcal{T}(0)$ for $x < 0$. By the lemma, $\int_1^\infty (\log W(x)/x^2)\mathrm{d}x = \infty$ implies that $\int_1^\infty (\log \mathcal{T}(x)/x^2)\mathrm{d}x = \infty$. Here, $\log \mathcal{T}(x)$ is certainly uniformly Lip 1 (and $\geqslant 0$) on $\mathbb{R}$, so, by the corollary of article 1, we see that

$$\int_1^\infty \frac{\log \mathcal{T}_A(x)}{x^2}\mathrm{d}x \;=\; \infty$$

for each $A > 0$.

We have $\mathcal{T}(x) \leqslant W(x) + \mathcal{T}(0)$ (the term $\mathcal{T}(0)$ on the right being perhaps needed for negative x). Therefore

$$\mathcal{T}_A(x) \;\leqslant\; (1 + \mathcal{T}(0))W_A(x),$$

and

$$\int_1^\infty \frac{\log W_A(x)}{x^2}\mathrm{d}x \;=\; \infty$$

for each $A > 0$ by the previous relation. Q.E.D.

From this, Akhiezer's first theorem (Chapter VI, §E.2) gives, without further ado, the following

Theorem. *Let $W(x) \geqslant 1$ on $\mathbb{R}$, with $W(x) \to \infty$ for $x \longrightarrow \pm\infty$. Suppose that $W(x)$ is* monotone *on* one *of the two half lines $(-\infty, 0]$, $[0, \infty)$, and that the* integral *of* $\log W(x)/(1 + x^2)$, *taken over whichever of those half lines on which monotoneity holds,* diverges. *Then $\mathscr{C}_W(A) = \mathscr{C}_W(\mathbb{R})$ for* every *$A > 0$, so $\mathscr{C}_W(0+) = \mathscr{C}_W(\mathbb{R})$.*

Remark. The notation is that of §E.2, Chapter VI. This result is due to Levinson. It is remarkable because only the monotoneity of $W(x)$ on a *half line* figures in it.

4. Example on the comparison of weighted approximation by polynomials and that by exponential sums

If $W(x) \geqslant 1$ tends to ∞ as $x \to \pm\infty$, we know that $\mathscr{C}_W(A)$ is *properly contained* in $\mathscr{C}_W(\mathbb{R})$ for each $A > 0$ in the case that

$$\int_{-\infty}^\infty \frac{\log W(x)}{1 + x^2}\mathrm{d}x \;<\; \infty.$$

(See Chapter VI, §E.2 and also the beginning of §D.) The theorem of the previous article shows that *mere monotoneity of $W(x)$ on $[0, \infty)$ without any additional regularity,* when accompanied by the condition

$$\int_0^\infty \frac{\log W(x)}{1 + x^2}\mathrm{d}x \;=\; \infty,$$

already guarantees the equality of $\mathscr{C}_W(A)$ and $\mathscr{C}_W(\mathbb{R})$ for each $A > 0$.

The question arises as to whether this also works for $\mathscr{C}_W(0)$, the $\| \quad \|_W$-closure of the *polynomials* in $\mathscr{C}_W(\mathbb{R})$. (Here, of course we must assume that $x^n/W(x) \longrightarrow 0$ as $x \longrightarrow \pm\infty$ for all $n \geqslant 0$.) *The following example will show that the answer to this question is NO.*

We start with a *very rapidly increasing* sequence of numbers λ_n. It will be sufficient to take

$$\lambda_1 = 2,$$
$$\lambda_2 = e^{\lambda_1},$$

and, in general, $\lambda_n = e^{\lambda_{n-1}}$. Let us check that $\lambda_n > \lambda_{n-1}^2$ for $n > 1$. We have $e^2 > 2^2 = 4$, and $(d/dx)(e^x - x^2) = e^x - 2x$ is >0 for $x = 2$. Also $(d^2/dx^2)(e^x - x^2) = e^x - 2 > 0$ for $x \geqslant 2$, so $e^x - x^2$ continues to increase strictly on $[2, \infty)$. Therefore $e^x > x^2$ for $x \geqslant 2$, so $\lambda_n = e^{\lambda_{n-1}} > \lambda_{n-1}^2$. We note that λ_{n-1}^2 is turn $\geqslant 2\lambda_{n-1}$, since the numbers λ_{n-1} are $\geqslant 2$.

We proceed to the construction of the weight W. For $0 \leqslant x \leqslant \lambda_1$, take $\log W(x) = 0$, and for $2\lambda_{n-1} \leqslant x \leqslant \lambda_n$ with $n > 1$ put $\log W(x) = n\lambda_{n-1}/2$ (by the computation just made we do have $2\lambda_{n-1} < \lambda_n$). We then specify $\log W(x)$ on the segments $[\lambda_{n-1}, 2\lambda_{n-1}]$ by making it *linear on each of them*, and finally define $W(x)$ for negative x by putting $W(-x) = W(x)$.

Here is the picture:

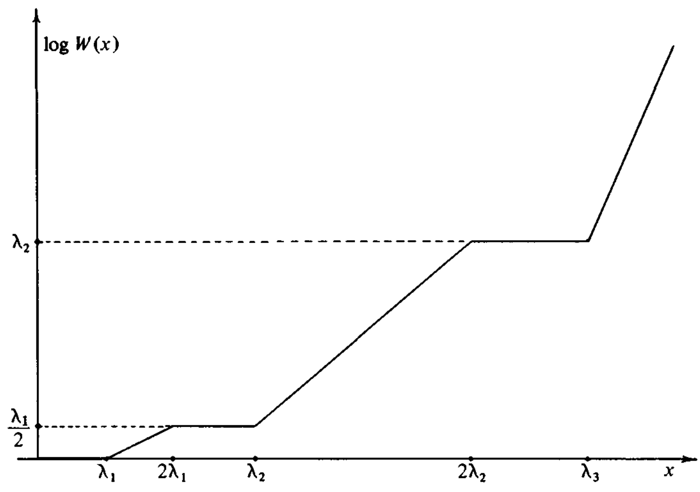

Figure 47

$W(x)$ is $\geqslant 1$ and *increasing* for $x \geqslant 0$, and, for large n,

$$\int_{2\lambda_n}^{\lambda_{n+1}} \frac{\log W(x)}{x^2} dx \;=\; \frac{(n+1)\lambda_n}{2}\left(\frac{1}{2\lambda_n} - \frac{1}{\lambda_{n+1}}\right)$$

is $\geqslant (n+1)\lambda_n/8\lambda_n = \frac{1}{8}(n+1)$. Therefore $\int_0^\infty (\log W(x)/(1+x^2))\mathrm{d}x = \infty$, so, *by the theorem of the previous article, $\mathscr{C}_W(A) = \mathscr{C}_W(\mathbb{R})$ for each $A > 0$ and $\mathscr{C}_W(0+) = \mathscr{C}_W(\mathbb{R})$.*

For $2\lambda_{n-1} \leqslant |x| \leqslant 2\lambda_n$,

$$W(x) \;\geqslant\; \mathrm{e}^{n\lambda_{n-1}/2} \;=\; \lambda_n^{n/2} \;\geqslant\; |x/2|^{n/2}.$$

Hence

$$\frac{x^p}{W(x)} \longrightarrow 0 \quad \text{as} \quad x \longrightarrow \pm\infty$$

for every $p \geqslant 0$, and it makes sense to talk about the space $\mathscr{C}_W(0)$. *It is claimed that $\mathscr{C}_W(0) \neq \mathscr{C}_W(\mathbb{R})$.*

To see this, take the entire function

$$C(z) \;=\; \prod_1^\infty \left(1 - \frac{z^2}{\lambda_n^2}\right).$$

Because the λ_n go to ∞ so rapidly, $C(z)$ is of *zero exponential type*. For $n > 1$,

$$|C'(\lambda_n)| \;=\; \frac{2}{\lambda_n}\left(\frac{\lambda_n}{\lambda_1}\right)^2\left(\frac{\lambda_n}{\lambda_2}\right)^2\cdots\left(\frac{\lambda_n}{\lambda_{n-1}}\right)^2$$
$$\times \prod_{k=1}^{n-1}\left(1 - \frac{\lambda_k^2}{\lambda_n^2}\right)\cdot\prod_{l=n+1}^{\infty}\left(1 - \frac{\lambda_n^2}{\lambda_l^2}\right).$$

Since the ratios λ_{j+1}/λ_j are always > 2 and $\to \infty$ as $j \to \infty$, the two products written with the sign $\prod$ on the right are both *bounded below* by *strictly positive constants* for $n > 1$ and indeed tend to 1 as $n \to \infty$. The product standing before them,

$$2\frac{\lambda_n}{\lambda_1^2}\cdot\frac{\lambda_n}{\lambda_2^2}\cdots\frac{\lambda_n}{\lambda_{n-1}^2}\cdot\lambda_n^{n-2},$$

far exceeds $2\lambda_n^{n-2}$ because $\lambda_j > \lambda_{j-1}^2$. Therefore we surely have

$$|C'(\lambda_n)| \;\geqslant\; \lambda_n^{n-2}$$

for large n.

At the same time, $W(\lambda_n) = \mathrm{e}^{n\lambda_{n-1}/2} = \lambda_n^{n/2}$, whence, for large n,

$$\frac{W(\lambda_n)}{|C'(\lambda_n)|} \;\leqslant\; \frac{\lambda_n^{n/2}}{\lambda_n^{n-2}} \;=\; \frac{1}{\lambda_n^{(n/2)-2}}.$$

Since the sequence $\{\lambda_n\}$ tends to ∞, we thus have

$$\sum_1^\infty \frac{W(\lambda_n)}{|C'(\lambda_n)|} \;<\; \infty.$$

For $n = 1, 2, 3, \ldots$ it is convenient to put $\lambda_{-n} = -\lambda_n$. Let us then define a discrete measure μ supported on the points λ_n, $n = \pm 1, \pm 2, \ldots$, by putting

$$\mu(\{\lambda_n\}) = \frac{W(\lambda_n)}{C'(\lambda_n)}.$$

The functions $W(x)$ and $C(x)$ are even, hence

$$\int_{-\infty}^{\infty} |d\mu(x)| < \infty$$

by the calculation just made.

We can now verify, just as in §H.3 of Chapter VI, that

$$(\dagger) \qquad \int_{-\infty}^{\infty} \frac{x^p}{W(x)} d\mu(x) = 0 \quad \text{for} \quad p = 0, 1, 2, \ldots.$$

The integral on the right is just the (absolutely convergent) sum

$$\sum_{-\infty}^{\infty}{}' \frac{\lambda_n^p}{C'(\lambda_n)},$$

and we have to show that this is zero for $p \geqslant 0$. Taking

$$C_N(z) = \prod_{n=1}^{N} \left(1 - \frac{z^2}{\lambda_n^2}\right)$$

(cf. §C, Chapter VI), we have the Lagrange interpolation formula

$$z^l = \sum_{-N}^{N}{}' \frac{\lambda_n^l C_N(z)}{(z - \lambda_n) C_N'(\lambda_n)},$$

valid for $0 \leqslant l < 2N$. Fix l. Clearly, $|C_N'(\lambda_n)| \geqslant |C'(\lambda_n)|$ for $-N \leqslant n \leqslant N$. Therefore, since $\sum_{-\infty}'^{\infty} |\lambda_n^l / C'(\lambda_n)| < \infty$, we can make $N \to \infty$ in the preceding relation and use dominated convergence to obtain

$$z^l = \sum_{-\infty}^{\infty}{}' \frac{\lambda_n^l C(z)}{(z - \lambda_n) C'(\lambda_n)}.$$

Putting $l = p + 1$ and specializing to $z = 0$, the desired result follows, and we have ($\dagger$).

Our measure μ is not zero. The strict inclusion of $\mathscr{C}_W(0)$ in $\mathscr{C}_W(\mathbb{R})$ is thus a consequence of ($\dagger$), and the construction of our example is completed.

Let us summarize what we have. We have found *an even weight $W(x) \geqslant 1$, increasing on $[0, \infty)$ at a rate faster than that of any power of x, such that $\mathscr{C}_W(0) \neq \mathscr{C}_W(\mathbb{R})$ but $\mathscr{C}_W(0+) = \mathscr{C}_W(\mathbb{R})$*. This was promised at the end of §H.2, Chapter VI. In §H.3 of that chapter we constructed an even weight W with $\mathscr{C}_W(0) \neq \mathscr{C}_W(0+)$ and $\mathscr{C}_W(0+) \neq \mathscr{C}_W(\mathbb{R})$.

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As the work of Chapter VI shows, the condition

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2}\mathrm{d}x \;<\; \infty$$

is sufficient to guarantee proper inclusion in $\mathscr{C}_W(\mathbb{R})$ of each of the spaces $\mathscr{C}_W(0)$ and $\mathscr{C}_W(A)$, $A > 0$ (for $\mathscr{C}_W(0)$ see §D of that chapter). The question is, *how much regularity do we have to impose on $W(x)$ in order that the contrary property*

$$(\dagger\dagger) \qquad \int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2}\mathrm{d}x \;=\; \infty$$

should imply that $\mathscr{C}_W(0) = \mathscr{C}_W(\mathbb{R})$ or that $\mathscr{C}_W(A) = \mathscr{C}_W(\mathbb{R})$ for $A > 0$?

As we saw in the previous article, *monotoneity* of $W(x)$ on $[0, \infty)$ is *enough* for ($\dagger\dagger$) to make $\mathscr{C}_W(A) = \mathscr{C}_W(\mathbb{R})$ when $A > 0$, in the case of *even* weights W. In §D, Chapter VI, it was also shown that ($\dagger\dagger$) implies $\mathscr{C}_W(0) = \mathscr{C}_W(\mathbb{R})$ for *even weights W with $\log W(x)$ convex in $\log|x|$*. The example just given shows that *logarithmic convexity cannot be replaced by monotoneity when weighted polynomial approximation is involved*, even though the *later* is *good enough* when we deal with *weighted approximation by exponential sums.*

We have here a *qualitative difference* between weighted polynomial approximation and that by linear combinations of the $\mathrm{e}^{\mathrm{i}\lambda x}$, $-A \leqslant \lambda \leqslant A$, and in fact the *first real distinction* we have seen between these two kinds of approximation. In Chapter VI, the study of the latter paralleled that of the former in almost every detail.

The reason for this difference is that (for weights W which are finite reasonably often) the $\|\quad\|_W$-density of polynomials in $\mathscr{C}_W(\mathbb{R})$ is governed by the *lower polynomial regularization* $W_*(x)$ of W, whereas that of $\mathscr{E}_A$ is determined by the lower regularization $W_A(x)$ of W based on the use of entire functions of exponential type $\leqslant A$. The latter are better than polynomials for getting at $W(x)$ from underneath. As the example shows, they are qualitatively better.

5. **Levinson's theorem**

There is one other easy application of the material in article 1 which should be mentioned. Although the result obtained in that way has been superseded by a deeper (and more difficult) one of Beurling, to be given in the next §, it is still worthwhile, and serves as a basis for Volberg's very refined work presented in the last § of this chapter.

Theorem (Levinson). *Let μ be a finite Radon measure on $\mathbb{R}$, and suppose that*

$$\int_0^\infty \frac{1}{1+x^2}\log\left(\frac{1}{\int_x^\infty |\mathrm{d}\mu(t)|}\right)\mathrm{d}x = \infty.$$

Then the Fourier–Stieltjes transform

$$\hat{\mu}(\lambda) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\lambda x}\,\mathrm{d}\mu(x)$$

cannot vanish identically over any interval of positive length unless $\mu \equiv 0$.

Remark 1. Of course, the same result holds if

$$\int_{-\infty}^0 \frac{1}{1+x^2}\log\left(\frac{1}{\int_{-\infty}^x |\mathrm{d}\mu(t)|}\right)\mathrm{d}x = \infty.$$

Remark 2. Beurling's theorem, to be proved in the next §, says that under the stated condition on $\log(\int_x^\infty |\mathrm{d}\mu(t)|)$, $\hat{\mu}(\lambda)$ cannot even vanish on a *set of positive measure* unless $\mu \equiv 0$.

Proof of theorem. It is enough, in the first place, to establish the result for *absolutely continuous* measures μ. Suppose, indeed, that μ is *any* measure satisfying the hypothesis; from it let us form the absolutely continuous measures μ_h, $h > 0$, having the densities

$$\frac{\mathrm{d}\mu_h(x)}{\mathrm{d}x} = \frac{1}{h}\int_x^{x+h} \mathrm{d}\mu(t).$$

Then

$$\hat{\mu}_h(\lambda) = \frac{1-\mathrm{e}^{-\mathrm{i}\lambda h}}{\mathrm{i}\lambda h}\,\hat{\mu}(\lambda),$$

so $\hat{\mu}_h(\lambda)$ *vanishes wherever $\hat{\mu}(\lambda)$ does.* Also,

$$\int_x^\infty |\mathrm{d}\mu_h(t)| \leqslant \int_x^\infty |\mathrm{d}\mu(t)|$$

for $x > 0$, so

$$\int_0^\infty \frac{1}{1+x^2}\log\left(\frac{1}{\int_x^\infty |\mathrm{d}\mu_h(t)|}\right)\mathrm{d}x = \infty$$

for each $h > 0$ by the hypothesis. Truth of our theorem for absolutely continuous measures would thus make the μ_h all zero if $\hat{\mu}(\lambda)$ vanishes on an interval of length > 0. But then $\mu \equiv 0$.

We may therefore take μ to be absolutely continuous. Assume, without loss of generality, that

$$\int_{-\infty}^{\infty} |d\mu(t)| \leqslant 1$$

and that $\hat{\mu}(\lambda) \equiv 0$ for $-A \leqslant \lambda \leqslant A$, $A > 0$.

For $x \geqslant 0$, write $W(x) = (\int_x^\infty |d\mu(t)|)^{-1/2}$, and, for $x < 0$, put

$$W(x) = \left(\int_{-\infty}^{x} |d\mu(t)|\right)^{-1/2}$$

The function $W(x)$ (perhaps discontinuous at 0) is $\geqslant 1$ and tends to ∞ as $x \to \pm\infty$. It is monotone on $(-\infty, 0)$ and on $[0, \infty)$, and *continuous* on each of those intervals (in the *extended sense*, as it *may take the value* ∞).

By integral calculus (!), we now find that

$$\int_0^\infty W(x)|d\mu(x)| = \int_0^\infty \frac{|d\mu(x)|}{\sqrt{\int_x^\infty |d\mu(t)|}} = 2\sqrt{\int_0^\infty |d\mu(t)|} < \infty,$$

and, in like manner,

$$\int_{-\infty}^{0} W(x)|d\mu(x)| < \infty.$$

The measure ν with $d\nu(x) = W(x)d\mu(x)$ is therefore *totally finite* on $\mathbb{R}$. (If $W(x)$ is infinite on any semi-infinite interval J, we of course must have $d\mu(x) \equiv 0$ on J, so $d\nu(x)$ is also zero there.) For $-A \leqslant \lambda \leqslant A$,

$$(\S) \qquad \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{W(x)} d\nu(x) = \hat{\mu}(\lambda) = 0.$$

However, by hypothesis,

$$\int_0^\infty \frac{\log W(x)}{1+x^2} dx = \frac{1}{2}\int_0^\infty \frac{1}{1+x^2} \log\left(\frac{1}{\int_x^\infty |d\mu(t)|}\right) dx = \infty,$$

so, since $W(x)$ is *increasing* on $[0, \infty)$, $\mathscr{C}_W(A)$ is $\| \quad \|_W$-dense in $\mathscr{C}_W(\mathbb{R})$ according to the theorem of article 3. Therefore, by (§),

$$\int_{-\infty}^{\infty} \varphi(x) d\mu(x) = \int_{-\infty}^{\infty} \frac{\varphi(x)}{W(x)} d\nu(x) = 0$$

for every continuous φ of compact support. This means that $\mu \equiv 0$. We are done.

The proof of Volberg's theorem uses the following

Corollary. Let $f(\vartheta) \sim \sum_{-\infty}^{\infty} \hat{f}(n) e^{in\vartheta}$ *belong to* $L_1(-\pi, \pi)$, *and suppose that* $f(\vartheta) = 0$ *a.e. on an interval* J *of positive length. If* $|\hat{f}(n)| \leqslant e^{-M(n)}$ *for*

$n > 0$ *with* $M(n)$ increasing, *and such that*

$$\sum_1^\infty \frac{M(n)}{n^2} = \infty,$$

then $f(\vartheta) \equiv 0, \; -\pi \leqslant \vartheta \leqslant \pi$.

Proof. Take any small $h > 0$ and form the convolution

$$f_h(\vartheta) = \frac{1}{h}\int_{-h}^{h}\left(1 - \frac{|t|}{h}\right) f(\vartheta - t)\,\mathrm{d}t.$$

If $h < \frac{1}{2}$ (length of J), $f_h(\vartheta)$ *also vanishes identically on an interval of positive length*

From the rudiments of Fourier series, we have

$$f_h(\vartheta) = \sum_{-\infty}^{\infty}\left(\frac{\sin(nh/2)}{nh/2}\right)^2 \hat{f}(n)\mathrm{e}^{\mathrm{i}n\vartheta}.$$

The sum on the right can be rewritten in evident fashion as $\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\vartheta x}\,\mathrm{d}\mu(x)$ with a (discrete) totally finite measure μ. Let $x > 0$ be given. If n is *the next integer* $\geqslant x$ we have, since $M(n)$ *increases*,

$$\int_x^\infty |\mathrm{d}\mu(t)| = \sum_{l \geqslant n}\left(\frac{\sin(lh/2)}{lh/2}\right)^2 |\hat{f}(l)|$$

$$\leqslant \mathrm{e}^{-M(n)}\sum_{l \geqslant n}\frac{4}{h^2 l^2} \leqslant \frac{\text{const.}}{h^2}\mathrm{e}^{-M(n)}.$$

Because $\sum_1^\infty M(n)/n^2 = \infty$, we see that

$$\int_0^\infty \frac{1}{1+x^2}\log^-\left(\int_x^\infty |\mathrm{d}\mu(t)|\right)\mathrm{d}x = \infty,$$

and conclude by the theorem that $f_h \equiv 0$. Making $h \to 0$, we see that $f \equiv 0$, Q.E.D.

B. The Fourier transform vanishes on a set of positive measure. Beurling's theorems

Beurling was able to extend considerably the theorem of Levinson given at the end of the preceding §. The main improvement in technique which made this extension possible involved the use of harmonic measure.

Harmonic measure will play an increasingly important rôle in the remaining chapters of this book. We therefore begin this § with a brief general discussion of what it is and what it does.

1. What is harmonic measure?

Suppose we have a finitely connected bounded domain $\mathscr{D}$ whose boundary, $\partial\mathscr{D}$, consists of several piecewise smooth Jordan curves. The *Dirichlet problem* for $\mathscr{D}$ requires us to find, for any given φ continuous on $\partial\mathscr{D}$, a function $U_\varphi(z)$ *harmonic in* $\mathscr{D}$ *and continuous up to* $\partial\mathscr{D}$ with $U_\varphi(\zeta) = \varphi(\zeta)$ for $\zeta\in\partial\mathscr{D}$. It is well known that the Dirichlet problem can always be solved for domains like those considered here. Many books on complex variable theory or potential theory contain proofs of this fact, which we henceforth take for granted.

Let us, however, tarry long enough to remind the reader of one particularly easy proof, available for *simply connected* domains $\mathscr{D}$. There, the Riemann mapping theorem provides us with a *conformal mapping* F of $\mathscr{D}$ onto the unit disk $\{|w| < 1\}$. Such a function F extends continuously up to $\partial\mathscr{D}$ and maps the latter in one–one fashion onto $\{|\omega| = 1\}$; this is true by a famous theorem of Carathéodory and can also be directly verified in many cases where $\partial\mathscr{D}$ has a simple explicit description (including all the ones to be met with in this book).

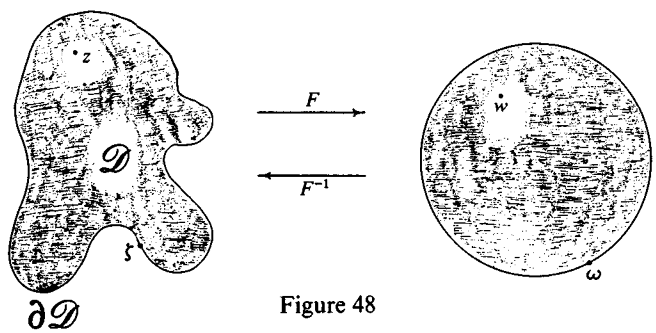

Figure 48

Denote by F^{-1} the inverse mapping to F. The function $\psi(\omega) = \varphi(F^{-1}(\omega))$ is then continuous on $\{|\omega| = 1\}$, and, if U_φ is the harmonic function sought which is to agree with φ on $\partial\mathscr{D}$, $V(w) = U_\varphi(F^{-1}(w))$ must be harmonic in $\{|w| < 1\}$ and continuous up to $\{|\omega| = 1\}$, where $V(\omega)$ must equal $\psi(\omega)$. A function V with these properties (there is only *one* such) can, however, be obtained from ψ by *Poisson's formula*:

$$V(w) \;=\; \frac{1}{2\pi}\int_{|\omega|=1} \frac{1-|w|^2}{|w-\omega|^2}\,\psi(\omega)|\mathrm{d}\omega|.$$

Going back to $\mathscr{D}$, and writing $z = F^{-1}(w)$, $\zeta = F^{-1}(\omega)$, we get

$$U_\varphi(z) \;=\; \frac{1}{2\pi}\int_{\partial\mathscr{D}} \frac{1-|F(z)|^2}{|F(z)-F(\zeta)|^2}\,\varphi(\zeta)|\mathrm{d}F(\zeta)|$$