Proof. We differentiate both sides of (2.18) with respect to w and obtain the identity

$$\begin{aligned}
&\frac{G_\lambda(z)}{(z-w)^2} - \frac{G_\lambda(w)}{(z-w)^2} - \frac{G'_\lambda(w)}{z-w} \\
&= 2\pi\lambda^{\frac{3}{2}} \int_{-\infty}^{0}\int_{-\infty}^{0} e^{-2\pi\lambda tu} G_\lambda(z-t) G'_\lambda(w-u)\ \mathrm{d}u\ \mathrm{d}t \\
&\quad - 2\pi\lambda^{\frac{3}{2}} \int_{0}^{\infty}\int_{0}^{\infty} e^{-2\pi\lambda tu} G_\lambda(z-t) G'_\lambda(w-u)\ \mathrm{d}u\ \mathrm{d}t.
\end{aligned} \tag{2.26}$$

Using integration by parts we get

$$\begin{aligned}
&\int_{-\infty}^{0} e^{-2\pi\lambda tu} G'_\lambda(w-u)\ \mathrm{d}u \\
&\qquad = 2\pi\lambda \int_{-\infty}^{0} t e^{-2\pi\lambda tu} \big\{G_\lambda(w) - G_\lambda(w-u)\big\}\ \mathrm{d}u,
\end{aligned} \tag{2.27}$$

and

$$\begin{aligned}
&\int_{0}^{\infty} e^{-2\pi\lambda tu} G'_\lambda(w-u)\ \mathrm{d}u \\
&\qquad = 2\pi\lambda \int_{0}^{\infty} t e^{-2\pi\lambda tu} \big\{G_\lambda(w) - G_\lambda(w-u)\big\}\ \mathrm{d}u.
\end{aligned} \tag{2.28}$$

The lemma now follows now by combining (2.26), (2.27) and (2.28). □

In order to apply the identities (2.11), (2.12) and (2.13), we require simple estimates for certain partial sums.

Lemma 2.6. *For all real u and positive integers N, we have*

$$\sum_{n=-N-1}^{N} (-1)^n G_\lambda(n+\tfrac{1}{2}-u) \ll_\lambda \min\{1,|u|\}, \tag{2.29}$$

$$\sum_{n=-N-1}^{N} \big\{G_\lambda(n+\tfrac{1}{2}) - G_\lambda(n+\tfrac{1}{2}-u)\big\} \ll_\lambda \min\{1,|u|\}, \tag{2.30}$$

$$\sum_{n=-N}^{N} \big\{G_\lambda(n) - G_\lambda(n-u)\big\} \ll_\lambda \min\{1,|u|\}, \tag{2.31}$$

where the constant implied by $\ll_\lambda$ depends on λ, but not on u or N. Moreover, if $S_{\lambda,N}(u)$ denotes the sum on the left of (2.29)*, then for $0 < t$ we have*

$$\left|\int_{0}^{\infty} e^{-2\pi\lambda tu} S_{\lambda,N}(u)\ \mathrm{d}u\right| \le \lambda^{-\frac{1}{2}}, \tag{2.32}$$

and for $t < 0$ we have

$$\left|\int_{-\infty}^{0} e^{-2\pi\lambda tu} S_{\lambda,N}(u)\ \mathrm{d}u\right| \le \lambda^{-\frac{1}{2}}. \tag{2.33}$$

Proof. For each positive integer N,

$$u \mapsto S_{\lambda,N}(u) = \sum_{n=-N-1}^{N} (-1)^n G_\lambda(n + \tfrac{1}{2} - u)$$

is an odd function of u. Hence its derivative is an even function of u. Therefore we get

$$\begin{aligned} \left|S_{\lambda,N}(u)\right| &= \left|\int_0^u S'_{\lambda,N}(v)\ \mathrm{d}v\right| \\ &\leq \int_0^{|u|} \left\{ \sum_{n=-\infty}^{\infty} \left|G'_\lambda(n + \tfrac{1}{2} - v)\right| \right\} \mathrm{d}v \\ &\leq C_\lambda |u|, \end{aligned}$$

where

$$C_\lambda = \sup_{v \in \mathbb{R}} \left\{ \sum_{n=-\infty}^{\infty} \left|G'_\lambda(n + \tfrac{1}{2} - v)\right| \right\}$$

is obviously finite. We also have

$$\left|S_{\lambda,N}(u)\right| \leq \sup_{v \in \mathbb{R}} \left\{ \sum_{n=-\infty}^{\infty} \left|G_\lambda(n + \tfrac{1}{2} - v)\right| \right\} < \infty,$$

and the bound (2.29) follows. The proofs of (2.30) and (2.31) are very similar.

Let $0 < t$ and $0 < u$. For positive integers N we define

$$R_{\lambda,N}(u) = \int_0^u S_{\lambda,N}(v)\ \mathrm{d}v.$$

Then it follows, using integration by parts, that

$$\int_0^\infty e^{-2\pi\lambda t u} S_{\lambda,N}(u)\ \mathrm{d}u = 2\pi\lambda t \int_0^\infty e^{-2\pi\lambda t u} R_{\lambda,N}(u)\ \mathrm{d}u. \tag{2.34}$$

For $\alpha < \beta$, let

$$\chi_{\alpha,\beta}(x) = \tfrac{1}{2}\operatorname{sgn}(\beta - x) + \tfrac{1}{2}\operatorname{sgn}(x - \alpha)$$

denote the normalized characteristic function of the real interval with endpoints α and β. Using the inequality

$$\left| \sum_{n=-N-1}^{N} (-1)^n \chi_{n+\frac{1}{2}-u,n+\frac{1}{2}}(x) \right| \leq 1,$$

we find that

$$\begin{aligned}
\big|R_{\lambda,N}(u)\big| &= \Big|\sum_{n=-N-1}^{N}(-1)^n\int_0^u G_\lambda(n+\tfrac{1}{2}-v)\ \mathrm{d}v\Big| \\
&= \Big|\int_{-\infty}^{\infty}\Big\{\sum_{n=-N-1}^{N}(-1)^n\chi_{n+\frac{1}{2}-u,n+\frac{1}{2}}(w)\Big\}G_\lambda(w)\ \mathrm{d}w\Big| \\
&\le \int_{-\infty}^{\infty}G_\lambda(w)\ \mathrm{d}w \\
&= \lambda^{-\frac{1}{2}}.
\end{aligned}$$

Then using (2.34) we get

$$\begin{aligned}
\Big|\int_0^\infty e^{-2\pi\lambda tu}S_{\lambda,N}(u)\ \mathrm{d}u\Big| &\le 2\pi\lambda t\int_0^\infty e^{-2\pi\lambda tu}\big|R_{\lambda,N}(u)\big|\ \mathrm{d}u \\
&\le 2\pi\lambda^{\frac{1}{2}}t\int_0^\infty e^{-2\pi\lambda tu}\ \mathrm{d}u \\
&= \lambda^{-\frac{1}{2}}.
\end{aligned}$$

This verifies (2.32), and (2.33) follows from (2.32) because $u\mapsto S_{\lambda,N}(u)$ is an odd function. □

Because $z\mapsto L_\lambda(z)$ interpolates both the value of $G_\lambda(z)$ and the value of its derivative $G'_\lambda(z)$ at each point of the coset $\mathbb{Z}+\frac{1}{2}$, the entire function

$$z\mapsto G_\lambda(z)-L_\lambda(z)$$

has a zero of multiplicity at least 2 at each point of $\mathbb{Z}+\frac{1}{2}$. It follows that

$$z\mapsto\Big(\frac{\pi}{\cos\pi z}\Big)^2\Big\{G_\lambda(z)-L_\lambda(z)\Big\}$$

is an entire function. In a similar manner, we find that

$$z\mapsto\Big(\frac{\pi}{\sin\pi z}\Big)^2\Big\{M_\lambda(z)-G_\lambda(z)\Big\}$$

is an entire function.

Lemma 2.7. *For all complex z we have*

$$\begin{aligned}
&\Big(\frac{\pi}{\cos\pi z}\Big)^2\Big\{G_\lambda(z)-L_\lambda(z)\Big\} \\
&= 2\pi^2\lambda^2\int_{-\infty}^{\infty}\frac{tG_\lambda(z-t)}{\sinh\pi\lambda t}\int_{-\frac{1}{2}}^{\frac{1}{2}}e^{-2\pi\lambda tu}\big\{\theta_3\big(u,i\lambda^{-1}\big)-\theta_3\big(\tfrac{1}{2},i\lambda^{-1}\big)\big\}\,\mathrm{d}u\,\mathrm{d}t,
\end{aligned}\tag{2.35}$$

and

$$\left(\frac{\pi}{\sin \pi z}\right)^2 \Big\{M_\lambda(z) - G_\lambda(z)\Big\} \\ = 2\pi^2\lambda^2 \int_{-\infty}^{\infty} \frac{tG_\lambda(z-t)}{\sinh \pi\lambda t} \int_{-\frac12}^{\frac12} e^{-2\pi\lambda t u}\big\{\theta_2\big(\tfrac12, i\lambda^{-1}\big) - \theta_2\big(u, i\lambda^{-1}\big)\big\}\, du\, dt. \tag{2.36}$$

Proof. In order to establish (2.35) we use the partial fraction expansion

$$\lim_{N\to\infty} \sum_{n=-N-1}^{N} \frac{1}{\big(z-n-\frac12\big)^2} = \left(\frac{\pi}{\cos \pi z}\right)^2, \tag{2.37}$$

which converges uniformly on compact subsets of $\mathbb{C} \setminus \big\{\mathbb{Z} + \frac12\big\}$. Then it follows from (2.3) and (2.37) that

$$\left(\frac{\pi}{\cos \pi z}\right)^2 \Big\{G_\lambda(z) - L_\lambda(z)\Big\} \\ = \lim_{N\to\infty} \sum_{n=-N-1}^{N} \left\{\frac{G_\lambda(z)}{(z-n-\frac12)^2} - \frac{G_\lambda(n+\frac12)}{(z-n-\frac12)^2} - \frac{G'_\lambda(n+\frac12)}{z-n-\frac12}\right\}. \tag{2.38}$$

Note that the limit on the right of (2.38) converges uniformly on compact subsets of $\mathbb{C}$. For positive integers N and all real u let

$$T_{\lambda,N}(u) = \sum_{n=-N-1}^{N} \big\{G_\lambda(n+\tfrac12) - G_\lambda(n+\tfrac12-u)\big\}.$$

From (2.12) we conclude that

$$\lim_{N\to\infty} T_{\lambda,N}(u) = \lambda^{-\frac12}\big\{\theta_2\big(0, i\lambda^{-1}\big) - \theta_2\big(u, i\lambda^{-1}\big)\big\}. \tag{2.39}$$

We apply the identity (2.25) with $w = n + \frac12$ and sum over the integers n satisfying $-N-1 \le n \le N$. We get

$$\sum_{n=-N-1}^{N} \left\{\frac{G_\lambda(z)}{(z-n-\frac12)^2} - \frac{G_\lambda(n+\frac12)}{(z-n-\frac12)^2} - \frac{G'_\lambda(n+\frac12)}{z-n-\frac12}\right\} \\ = (2\pi)^2\lambda^{\frac52} \int_{-\infty}^{0}\int_{-\infty}^{0} te^{-2\pi\lambda t u} G_\lambda(z-t) T_{\lambda,N}(u)\ du\ dt \\ - (2\pi)^2\lambda^{\frac52} \int_{0}^{\infty}\int_{0}^{\infty} te^{-2\pi\lambda t u} G_\lambda(z-t) T_{\lambda,N}(u)\ du\ dt. \tag{2.40}$$

We now let $N \to \infty$ on both sides of (2.40). The limit on the left-hand side is determined by (2.38). On the right-hand side we use (2.30), the dominated convergence theorem and (2.39). In this way we obtain the identity

$$\begin{aligned}
&\left(\frac{\pi}{\cos \pi z}\right)^2 \Big\{G_\lambda(z) - L_\lambda(z)\Big\} \\
&= (2\pi\lambda)^2 \int_{-\infty}^0 \int_{-\infty}^0 t e^{-2\pi\lambda t u} G_\lambda(z-t)\big\{\theta_2\big(0, i\lambda^{-1}\big) - \theta_2\big(u, i\lambda^{-1}\big)\big\}\,\mathrm{d}u\,\mathrm{d}t \qquad (2.41)\\
&\quad - (2\pi\lambda)^2 \int_0^\infty \int_0^\infty t e^{-2\pi\lambda t u} G_\lambda(z-t)\big\{\theta_2\big(0, i\lambda^{-1}\big) - \theta_2\big(u, i\lambda^{-1}\big)\big\}\,\mathrm{d}u\,\mathrm{d}t.
\end{aligned}$$

If $0 < t$, using that $v \mapsto \theta_2(v, \tau)$ has period 1 and (2.10), we get

$$\begin{aligned}
&\int_0^\infty e^{-2\pi\lambda t u}\big\{\theta_2\big(0, i\lambda^{-1}\big) - \theta_2\big(u, i\lambda^{-1}\big)\big\}\,\mathrm{d}u \\
&\qquad = \sum_{m=0}^\infty \int_0^1 e^{-2\pi\lambda t(u+m)}\big\{\theta_2\big(0, i\lambda^{-1}\big) - \theta_2\big(u+m, i\lambda^{-1}\big)\big\}\,\mathrm{d}u \\
&\qquad = \big\{1 - e^{-2\pi\lambda t}\big\}^{-1} \int_0^1 e^{-2\pi\lambda t u}\big\{\theta_2\big(0, i\lambda^{-1}\big) - \theta_2\big(u, i\lambda^{-1}\big)\big\}\,\mathrm{d}u \qquad (2.42)\\
&\qquad = \big\{2\sinh \pi\lambda t\big\}^{-1} \int_{-\frac12}^{\frac12} e^{-2\pi\lambda t u}\big\{\theta_3\big(\tfrac12, i\lambda^{-1}\big) - \theta_3\big(u, i\lambda^{-1}\big)\big\}\,\mathrm{d}u.
\end{aligned}$$

If $t < 0$, in a similar manner, we find that

$$\begin{aligned}
&\int_{-\infty}^0 e^{-2\pi\lambda t u}\big\{\theta_2\big(0, i\lambda^{-1}\big) - \theta_2\big(u, i\lambda^{-1}\big)\big\}\,\mathrm{d}u \\
&\qquad = -\big\{2\sinh \pi\lambda t\big\}^{-1} \int_{-\frac12}^{\frac12} e^{-2\pi\lambda t u}\big\{\theta_3\big(\tfrac12, i\lambda^{-1}\big) - \theta_3\big(u, i\lambda^{-1}\big)\big\}\,\mathrm{d}u. \qquad (2.43)
\end{aligned}$$

The identity (2.35) follows now by combining (2.41), (2.42) and (2.43).

The proof of (2.36) proceeds along the same lines using (2.13) and (2.31). We leave the details to the reader. □

Corollary 2.8. *For all real values of x we have*

$$0 < \left(\frac{\pi}{\cos \pi x}\right)^2 \Big\{G_\lambda(x) - L_\lambda(x)\Big\}, \qquad (2.44)$$

and

$$0 < \left(\frac{\pi}{\sin \pi x}\right)^2 \Big\{M_\lambda(x) - G_\lambda(x)\Big\}. \qquad (2.45)$$

In particular, the inequality (2.5) *holds for all real x.*

Proof. For real u the periodic function $u \mapsto \theta_3\big(u, i\lambda^{-1}\big)$ takes its maximum value at $u = 0$ and its minimum values at $u = \frac12$. Therefore the function

$$t \mapsto \int_{-\frac12}^{\frac12} e^{-2\pi\lambda t u}\big\{\theta_3\big(u, i\lambda^{-1}\big) - \theta_3\big(\tfrac12, i\lambda^{-1}\big)\big\}\,\mathrm{d}u,$$

which appears in the integrand on the right of (2.35), is positive for all real values of t. This plainly verifies the inequality (2.44).

In a similar manner using (2.10), the periodic function $u \mapsto \theta_2\big(u, i\lambda^{-1}\big)$ takes its maximum value at $u = \frac{1}{2}$ and its minimum value at $u = 0$. Hence the function

$$t \mapsto \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi\lambda t u}\big\{\theta_2\big(\tfrac{1}{2}, i\lambda^{-1}\big) - \theta_2\big(u, i\lambda^{-1}\big)\big\}\ \mathrm{d}u,$$

which appears in the integrand on the right of (2.36), is positive for all real values of t. This establishes the inequality (2.45). $\square$

2.2.3 Proofs of Theorems 2.1 and 2.2

Let $F(z)$ be an entire function of exponential type at most 2π such that

$$F(x) \le G_\lambda(x) \tag{2.46}$$

for all real x. Clearly we may assume that $x \mapsto F(x)$ is integrable on $\mathbb{R}$, for if not then (2.14) is trivial. By Theorem 1.11 we know that $F'(x)$ is also integrable and thus F has bounded variation. By the Poisson summation formula, (2.12) and (2.46), we find that

$$\begin{aligned} \int_{-\infty}^{\infty} F(x)\ \mathrm{d}x &= \lim_{N\to\infty} \sum_{n=-N}^{N} F(n+v) \\ &\le \lim_{N\to\infty} \sum_{n=-N}^{N} G_\lambda(n+v) \\ &= \lambda^{-\frac{1}{2}}\theta_2\big(\tfrac{1}{2} - v, i\lambda^{-1}\big) \end{aligned} \tag{2.47}$$

for all real v. We have already noted that $v \mapsto \theta_2\big(\frac{1}{2} - v, i\lambda^{-1}\big)$ takes its minimum value at $v = \frac{1}{2}$. Hence (2.47) implies that

$$\int_{-\infty}^{\infty} F(x)\ \mathrm{d}x \le \lambda^{-\frac{1}{2}}\theta_2\big(0, i\lambda^{-1}\big),$$

and this proves (2.14).

In Corollary 2.8 we proved that $F(z) = L_\lambda(z)$ satisfies the inequality (2.46) for all real x. In this special case there is equality in the inequality (2.47) when $v = \frac{1}{2}$. Thus we have

$$\int_{-\infty}^{\infty} L_\lambda(x)\ \mathrm{d}x = \lambda^{-\frac{1}{2}}\theta_2\big(0, i\lambda^{-1}\big). \tag{2.48}$$

Now *assume* that $F(z)$ is an entire function of exponential type at most 2π that satisfies (2.46) for all real x, and assume that there is equality in the inequality (2.47) when $v = \frac{1}{2}$. Then we must have

$$F(n + \tfrac{1}{2}) = G_\lambda(n + \tfrac{1}{2})$$

for all integers n. Then from (2.46) we also get

$$F'(n + \tfrac{1}{2}) = G'_\lambda(n + \tfrac{1}{2})$$

for all integers n. Of course this shows that the entire function

$$z \mapsto F(z) - L_\lambda(z) \tag{2.49}$$

is integrable on $\mathbb{R}$, has exponential type at most 2π, vanishes at each point of $\mathbb{Z} + \frac{1}{2}$, and its derivative also vanishes at each point of $\mathbb{Z} + \frac{1}{2}$. By an application of Theorem 1.12 (with an appropriate shift of $\frac{1}{2}$) we conclude that the entire function (2.49) is identically zero. This proves Theorem 2.1, and Theorem 2.2 can be proved by the same sort of argument.

2.3 Distribution framework for even functions

2.3.1 The Paley-Wiener theorem for distributions

Let $\mathcal{D}(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R}) \subseteq \mathcal{E}(\mathbb{R})$ be the usual spaces of C^∞ functions on $\mathbb{R}$ as defined in the work of L. Schwartz [36], and let $\mathcal{E}'(\mathbb{R}) \subseteq \mathcal{S}'(\mathbb{R}) \subseteq \mathcal{D}'(\mathbb{R})$ be the corresponding dual spaces of distributions. Our notation and terminology for distributions follows that of [18], and precise definitions for these spaces are given in [18, Section 2.3]. We write $\varphi(x)$ for a generic element in the space $\mathcal{S}(\mathbb{R})$ of Schwartz functions. If T in $\mathcal{S}'(\mathbb{R})$ is a tempered distribution we write $T(\varphi)$ for the value of T at φ. Then the Fourier transform of T is the tempered distribution $\widehat{T}$ defined by

$$\widehat{T}(\varphi) = T(\widehat{\varphi}),$$

where

$$\widehat{\varphi}(y) = \int_{-\infty}^{\infty} \varphi(x) e(-yx)\, \mathrm{d}x$$

is the Fourier transform of the function φ. Functions $g : \mathbb{R} \to \mathbb{R}$ in any L^p class or with polynomial growth can be regarded as elements of $\mathcal{S}'(\mathbb{R})$ and we will usually make the identification

$$g(\varphi) = \int_{-\infty}^{\infty} g(x)\, \varphi(x)\, \mathrm{d}x$$

for all φ in $\mathcal{S}(\mathbb{R})$.

We recall the following form of the Paley-Wiener theorem for distributions, which is obtained by combining Theorem 1.7.5 and Theorem 1.7.7 in [21].

Theorem 2.9 (Paley-Wiener for distributions)**.** *Let $\delta > 0$, and let U be a tempered distribution in $\mathcal{S}'(\mathbb{R})$ with Fourier transform $\widehat{U}$ supported in the compact interval $[-\delta, \delta]$. Then $\widehat{U}$ belongs to $\mathcal{E}'(\mathbb{R})$, and*

$$z \mapsto F(z) = \widehat{U}_\xi\big(e(\xi z)\big)$$

defines an entire function of the complex variable $z = x + iy$ such that

$$\big|F(z)\big| \ll_B \big(1 + |z|\big)^B \exp\{2\pi\delta|y|\} \tag{2.50}$$

for some number $B \geq 0$ and all z in $\mathbb{C}$. Moreover, the entire function $F(z)$ satisfies the identity

$$U(\varphi) = \int_{-\infty}^{\infty} F(x)\,\varphi(x)\,\mathrm{d}x$$

for all φ in $\mathcal{S}(\mathbb{R})$.

Conversely, suppose that $F(z)$ is an entire function of the complex variable z that satisfies the inequality (2.50) *for some numbers $B \geq 0$ and $\delta > 0$. Then there exists a tempered distribution V in $\mathcal{S}'(\mathbb{R})$ such that $\widehat{V}$ belongs to $\mathcal{E}'(\mathbb{R})$, $\widehat{V}$ is supported on the compact interval $[-\delta, \delta]$,*

$$F(z) = \widehat{V}_\xi\big(e(\xi z)\big),$$

and

$$V(\varphi) = \int_{-\infty}^{\infty} F(x)\,\varphi(x)\,\mathrm{d}x$$

for all φ in $\mathcal{S}(\mathbb{R})$.

Here we write $\widehat{U}_\xi$ to indicate that the distribution $\widehat{U}$ is acting on the function $\xi \mapsto \big(e(\xi z)\big)$.

2.3.2 Integrating the free parameter

Our goal now is to be able to integrate the parameter λ with respect to a suitable non-negative Borel measure ν on $[0, \infty)$ and obtain the solution of the extremal problem for a different function. One might first guess that the class of suitable measures ν on $[0, \infty)$ would consist of those measures for which the function

$$g(x) = \int_0^\infty G_\lambda(x)\,\mathrm{d}\nu(\lambda)$$

is well defined, and that this would be the function to be approximated. Such a method was carried out in [8], [9] and [19] with the Gaussian replaced by exponential functions. It turns out that this condition is unnecessarily restrictive, and in order to find the very minimal conditions to be imposed on the measure ν one must look at things on the Fourier transform side.

We will illustrate what this condition should be in the minorant case. Define the difference function

$$D_\lambda(x) = G_\lambda(x) - L_\lambda(x) \geq 0.$$

The minimal integral corresponds to

$$\int_{-\infty}^{\infty} \{G_\lambda(x) - L_\lambda(x)\}\,\mathrm{d}x = \widehat{D}_\lambda(0).$$

If we succeed in our attempt to integrate the parameter λ, we will end up solving an extremal problem for which the value of the minimal integral is given by (and thus we want to impose this finiteness condition)

$$\int_0^\infty \int_{-\infty}^\infty \{G_\lambda(x) - L_\lambda(x)\}\, \mathrm{d}x\, \mathrm{d}\nu(\lambda) = \int_0^\infty \widehat{D}_\lambda(0)\, \mathrm{d}\nu(\lambda) < \infty. \qquad (2.51)$$

We will show that this is also a sufficient condition, provided we can define appropriately the real function to be minorized.

Suppose ν is a non-negative Borel measure on $[0,\infty)$ satisfying (2.51). Since

$$|\widehat{D}_\lambda(t)| \le \widehat{D}_\lambda(0)$$

for all $t \in \mathbb{R}$, we observe that the function

$$t \mapsto \int_0^\infty \widehat{D}_\lambda(t)\, \mathrm{d}\nu(\lambda)$$

is well defined. In particular, from the classical Paley-Wiener theorem, the Fourier transform $t \mapsto \widehat{L}_\lambda(t)$ is supported on $[-1,1]$, and therefore

$$\int_0^\infty \widehat{D}_\lambda(t)\, \mathrm{d}\nu(\lambda) = \int_0^\infty \widehat{G}_\lambda(t)\, \mathrm{d}\nu(\lambda)$$

for $|t| \ge 1$. We are now in position to state the main results of this section. In the following theorems we write

$$[\alpha,\beta]^c = (-\infty,\alpha) \cup (\beta,\infty)$$

for the complement in $\mathbb{R}$ of a closed interval $[\alpha,\beta]$. Recall also that the Fourier transform of the Gaussian is given by

$$\widehat{G}_\lambda(t) = \lambda^{-\frac{1}{2}} e^{-\pi\lambda^{-1}t^2}.$$

Theorem 2.10 (Distribution Theorem - Minorant)**.** *Let ν be a non-negative Borel measure on $[0,\infty)$ satisfying*

$$\int_0^\infty \int_{-\infty}^\infty \{G_\lambda(x) - L_\lambda(x)\}\, \mathrm{d}x\, \mathrm{d}\nu(\lambda) < \infty.$$

Let $g : \mathbb{R} \to \mathbb{R}$ be a function of polynomial growth (thus an element of $\mathcal{S}'(\mathbb{R})$) that is continuous on $\mathbb{R}/\{0\}$, differentiable on $\mathbb{R}/\{0\}$, and such that

$$\widehat{g}(\varphi) = \int_{-\infty}^\infty \left\{ \int_0^\infty \widehat{G}_\lambda(t)\, \mathrm{d}\nu(\lambda) \right\} \varphi(t)\, \mathrm{d}t$$

for all Schwartz functions φ supported on $[-1,1]^c$. Then there exists a unique extremal minorant $l(z)$ of exponential type 2π for $g(x)$. The function $l(x)$ interpolates the values of $g(x)$ at $\mathbb{Z} + \frac{1}{2}$ and satisfies

$$\int_{-\infty}^\infty \{g(x) - l(x)\}\, \mathrm{d}x = \int_0^\infty \int_{-\infty}^\infty \{G_\lambda(x) - L_\lambda(x)\}\, \mathrm{d}x\, \mathrm{d}\nu(\lambda).$$

Theorem 2.11 (Distribution Theorem - Majorant)**.** *Let ν be a non-negative Borel measure on $[0,\infty)$ satisfying*

$$\int_0^\infty \int_{-\infty}^\infty \{M_\lambda(x) - G_\lambda(x)\}\,\mathrm{d}x\,\mathrm{d}\nu(\lambda) < \infty.$$

Let $g : \mathbb{R} \to \mathbb{R}$ be a function of polynomial growth (thus an element of $\mathcal{S}'(\mathbb{R})$) that is continuous on $\mathbb{R}$, differentiable on $\mathbb{R}/\{0\}$, and such that

$$\widehat{g}(\varphi) = \int_{-\infty}^\infty \left\{\int_0^\infty \widehat{G}_\lambda(t)\,\mathrm{d}\nu(\lambda)\right\}\varphi(t)\,\mathrm{d}t$$

for all Schwartz functions φ supported on $[-1,1]^c$. Then there exists a unique extremal majorant $m(z)$ of exponential type 2π for $g(x)$. The function $m(x)$ interpolates the values of $g(x)$ at $\mathbb{Z}$ and satisfies

$$\int_{-\infty}^\infty \{m(x) - g(x)\}\,\mathrm{d}x = \int_0^\infty \int_{-\infty}^\infty \{M_\lambda(x) - G_\lambda(x)\}\,\mathrm{d}x\,\mathrm{d}\nu(\lambda).$$

Similar results can be stated for the problem of majorizing or minorizing by functions of exponential type $2\pi\delta$. It is a matter of changing the interpolation points to $\delta\mathbb{Z}$ or $\delta(\mathbb{Z}+\frac{1}{2})$, and changing the support intervals to $[-\delta,\delta]^c$. For simplicity, we will proceed in our exposition only with type 2π.

The condition

$$\widehat{g}(\varphi) = \int_{-\infty}^\infty \left\{\int_0^\infty \widehat{G}_\lambda(t)\,\mathrm{d}\nu(\lambda)\right\}\varphi(t)\,\mathrm{d}t$$

for all Schwartz functions φ supported on $[-\delta,\delta]^c$, that appears on the statements of the theorems, asserts that the Fourier transform $\widehat{g}$, which is a tempered distribution, is actually given by a function

$$t \mapsto \int_0^\infty \widehat{G}_\lambda(t)\,\mathrm{d}\nu(\lambda)$$

outside the interval $[-\delta,\delta]$. This is a typical behavior of functions with polynomial growth, that might have the Fourier transform given by a singular part supported on the origin plus an additional component given by a function outside the origin (e.g. the Fourier transform of $-\log|x|$ is given by $(2|t|)^{-1}$ away from the origin). It is clear in this context that the only information relevant for the Beurling-Selberg extremal problem is knowledge of the Fourier transform of the original function outside a compact interval.

Finally, we shall see that this method is quite powerful, producing most of the previously known examples in the literature, and a wide class of new ones. In particular, we will be able solve the extremal problem for functions such as

$$\log|x|, \quad |x|^\sigma, \quad -\log\left(\frac{x^2+\alpha^2}{x^2+\beta^2}\right) \quad \text{and} \quad 1 - x\arctan\left(\frac{1}{x}\right).$$

where $\sigma > -1$ and $0 \le \alpha < \beta$. The last two of these functions will play an important role in the applications to the theory of the Riemann zeta-function developed in the next chapter.

2.3.3 Proofs of Theorems 2.10 and 2.11

Here we give a detailed proof of Theorem 2.10. The proof of Theorem 2.11 follows the same general method. First we construct the extreme minorant. Recall that

$$D_\lambda(x) = G_\lambda(x) - L_\lambda(x) \geq 0.$$

Then for each $x \in \mathbb{R}$ we define the non-negative valued function

$$d(x) = \int_0^\infty D_\lambda(x)\,\mathrm{d}\nu(\lambda). \tag{2.52}$$

It may happen that the value of $d(x)$ is $+\infty$ at some points x. However, the function $x \mapsto d(x)$ is integrable on $\mathbb{R}$, because

$$\int_{-\infty}^\infty d(x)\,\mathrm{d}x = \int_0^\infty \int_{-\infty}^\infty D_\lambda(x)\,\mathrm{d}x\,\mathrm{d}\nu(\lambda) = \int_0^\infty \widehat{D}_\lambda(0)\,\mathrm{d}\nu(\lambda) < \infty,$$

by the hypotheses of our theorem. Hence the Fourier transform $\widehat{d}(t)$ is a continuous function given by

$$\begin{aligned}\widehat{d}(t) = \int_{-\infty}^\infty d(x)\,e(-tx)\,\mathrm{d}x &= \int_{-\infty}^\infty \int_0^\infty D_\lambda(x)\,e(-tx)\,\mathrm{d}\nu(\lambda)\,\mathrm{d}x \\ &= \int_0^\infty \int_{-\infty}^\infty D_\lambda(x)\,e(-tx)\,\mathrm{d}x\,\mathrm{d}\nu(\lambda) = \int_0^\infty \widehat{D}_\lambda(t)\,\mathrm{d}\nu(\lambda),\end{aligned} \tag{2.53}$$

and for $|t| \geq 1$ we have

$$\widehat{d}(t) = \int_0^\infty \widehat{G}_\lambda(t)\,\mathrm{d}\nu(\lambda). \tag{2.54}$$

Let $U \in \mathcal{S}'(\mathbb{R})$ be the tempered distribution defined by

$$U(\varphi) = \int_{-\infty}^\infty \{g(x) - d(x)\}\,\varphi(x)\,\mathrm{d}x. \tag{2.55}$$

We shall prove that the Fourier transform $\widehat{U}$ is supported on $[-1,1]$. In fact, for any $\varphi \in \mathcal{S}(\mathbb{R})$ with support in $[-1,1]^c$ we have

$$\begin{aligned}\widehat{U}(\varphi) &= \widehat{g}(\varphi) - \widehat{d}(\varphi) \\ &= \int_{-\infty}^\infty \left\{\int_0^\infty \widehat{G}_\lambda(t)\,\mathrm{d}\nu(\lambda)\right\}\varphi(t)\,\mathrm{d}t - \int_{-\infty}^\infty \widehat{d}(t)\,\varphi(t)\,\mathrm{d}t = 0\,,\end{aligned}$$

by (2.54) and the hypotheses of the theorem. By the Paley-Wiener theorem for distributions we find that $\widehat{U} \in \mathcal{E}'(\mathbb{R})$, and therefore

$$z \mapsto l(z) = \widehat{U}_\xi\left(e(\xi z)\right)$$

defines an entire function of exponential type 2π such that

$$U(\varphi) = \int_{-\infty}^\infty l(x)\,\varphi(x)\,\mathrm{d}x \tag{2.56}$$

for all $\varphi \in \mathcal{S}(\mathbb{R})$. From (2.55) and (2.56) we conclude that

$$d(x) = g(x) - l(x) \geq 0 \tag{2.57}$$

for almost all $x \in \mathbb{R}$. In particular, we get

$$\begin{aligned}\int_{-\infty}^{\infty} \{g(x) - l(x)\} \, \mathrm{d}x &= \int_{-\infty}^{\infty} d(x) \, \mathrm{d}x = \int_{0}^{\infty} \widehat{D}_\lambda(0) \, \mathrm{d}\nu(\lambda) \\ &= \int_{0}^{\infty} \int_{-\infty}^{\infty} \{G_\lambda(x) - L_\lambda(x)\} \, \mathrm{d}x \, \mathrm{d}\nu(\lambda) < \infty.\end{aligned}$$

Next we consider the interpolation points. The Poisson summation formula can be applied pointwise to D_λ, since it holds for the Gaussian G_λ and for the minorant L_λ, which is a continuous integrable function of bounded variation. This gives us

$$\lim_{N\to\infty} \sum_{n=-N}^{N} D_\lambda(x+n) = \lim_{N\to\infty} \sum_{k=-N}^{N} \widehat{D}_\lambda(k) \, e(xk). \tag{2.58}$$

Since the minorant L_λ interpolates the Gaussian G_λ at $\mathbb{Z} + \frac{1}{2}$, we have $D_\lambda\big(n + \frac{1}{2}\big) = 0$ for all $n \in \mathbb{Z}$. Therefore we apply (2.58) at $x = \frac{1}{2}$, and use the classical Paley-Wiener theorem. In this way we arrive at the identity

$$\widehat{D}_\lambda(0) = -\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} (-1)^k \, \widehat{G}_\lambda(k). \tag{2.59}$$

Now we define the function

$$d_1(x) = g(x) - l(x).$$

We note that $d_1(x)$ is a non-negative, continuous function on $\mathbb{R}/\{0\}$ that is equal almost everywhere to $d(x)$ defined in (2.52), and thus in $L^1(\mathbb{R})$. Define a periodic function $p : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^+ \cup \{\infty\}$ by

$$p(x) = \sum_{n\in\mathbb{Z}} d_1(n+x).$$

An application of Fubini's theorem provides

$$\int_{\mathbb{R}/\mathbb{Z}} p(x) \, \mathrm{d}x = \int_{-\infty}^{\infty} d_1(x) \, \mathrm{d}x < \infty\,,$$

and therefore $p(x) \in L^1(\mathbb{R}/\mathbb{Z})$. Moreover, the Fourier coefficients of $p(x)$ satisfy

$$\widehat{p}(k) = \widehat{d}_1(k) = \widehat{d}(k)$$

for all $k \in \mathbb{Z}$. Convolution with the smoothing Féjer kernel

$$F_N(x) = \frac{1}{N+1}\left(\frac{\sin \pi(N+1)x}{\sin \pi x}\right)^2$$

produces the pointwise identity

$$\begin{aligned}
p * F_N(x) &= \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N}\right) \widehat{p}(k)\, e(xk) \\
&= \widehat{d}(0) + \sum_{\substack{k=-N \\ k\neq 0}}^{N} \left(1 - \frac{|k|}{N}\right) \widehat{d}(k)\, e(xk) \\
&= \widehat{d}(0) + \sum_{\substack{k=-N \\ k\neq 0}}^{N} \left(1 - \frac{|k|}{N}\right) \int_0^\infty \widehat{G}_\lambda(k)\, \mathrm{d}\nu(\lambda)\, e(xk) \\
&= \widehat{d}(0) + \int_0^\infty \left\{ \sum_{\substack{k=-N \\ k\neq 0}}^{N} \left(1 - \frac{|k|}{N}\right) \widehat{G}_\lambda(k)\, e(xk) \right\} \mathrm{d}\nu(\lambda)\,,
\end{aligned}$$

where we have used (2.54). In particular, at $x = \frac{1}{2}$ we obtain

$$\widehat{d}(0) = p * F_N\left(\tfrac{1}{2}\right) + \int_0^\infty \left\{ \sum_{\substack{k=-N \\ k\neq 0}}^{N} (-1)^{k+1} \left(1 - \frac{|k|}{N}\right) \widehat{G}_\lambda(k) \right\} \mathrm{d}\nu(\lambda). \qquad (2.60)$$

Note that the integrand in (2.60) in non-negative since $\widehat{G}_\lambda$ is radially decreasing and we can group the terms in consecutive pairs. Moreover, it converges absolutely to (2.59) as $N \to \infty$. Therefore, an application of Fatou's lemma together with (2.53) gives us

$$\begin{aligned}
\widehat{d}(0) &\geq \liminf_{N\to\infty} p * F_N\left(\tfrac{1}{2}\right) \\
&\qquad + \liminf_{N\to\infty} \int_0^\infty \left\{ \sum_{\substack{k=-N \\ k\neq 0}}^{N} (-1)^{k+1} \left(1 - \frac{|k|}{N}\right) \widehat{G}_\lambda(k) \right\} \mathrm{d}\nu(\lambda) \\
&\geq \liminf_{N\to\infty} p * F_N\left(\tfrac{1}{2}\right) \\
&\qquad + \int_0^\infty \liminf_{N\to\infty} \left\{ \sum_{\substack{k=-N \\ k\neq 0}}^{N} (-1)^{k+1} \left(1 - \frac{|k|}{N}\right) \widehat{G}_\lambda(k) \right\} \mathrm{d}\nu(\lambda) \\
&= \liminf_{N\to\infty} p * F_N\left(\tfrac{1}{2}\right) + \int_0^\infty \widehat{D}_\lambda(0)\, \mathrm{d}\nu(\lambda)
\end{aligned}$$

$$= \liminf_{N\to\infty} p * F_N\left(\tfrac{1}{2}\right) + \widehat{d}(0)\,,$$

and since $p * F_N(x)$ is non-negative we conclude that

$$\liminf_{N\to\infty} p * F_N\left(\tfrac{1}{2}\right) = 0.$$

We now use the definition of $p(x)$, Fubini's theorem and Fatou's lemma again to arrive at

$$\begin{aligned} 0 = \liminf_{N\to\infty} p * F_N\left(\tfrac{1}{2}\right) &= \liminf_{N\to\infty} \int_0^1 p(y)\, F_N\left(\tfrac{1}{2} - y\right) \mathrm{d}y \\ &= \liminf_{N\to\infty} \int_0^1 \Big\{ \sum_{n\in\mathbb{Z}} d_1(n+y) \Big\} F_N\left(\tfrac{1}{2} - y\right) \mathrm{d}y \\ &= \liminf_{N\to\infty} \sum_{n\in\mathbb{Z}} \left\{ \int_0^1 d_1(n+y)\, F_N\left(\tfrac{1}{2} - y\right) \mathrm{d}y \right\} \\ &\geq \sum_{n\in\mathbb{Z}} \liminf_{N\to\infty} \int_0^1 d_1(n+y)\, F_N\left(\tfrac{1}{2} - y\right) \mathrm{d}y \\ &= \sum_{n\in\mathbb{Z}} d_1\left(n + \tfrac{1}{2}\right), \end{aligned} \tag{2.61}$$

where the last equality follows from the fact that $d_1(x)$ is continuous at the points $n + \frac{1}{2}$, $n \in \mathbb{Z}$. From (2.61) and the non-negativity of $d_1(x)$ we arrive at the implication

$$d_1\left(n + \tfrac{1}{2}\right) = 0 \Rightarrow g\left(n + \tfrac{1}{2}\right) = l\left(n + \tfrac{1}{2}\right) \tag{2.62}$$

for all $n \in \mathbb{Z}$. From (2.57) and the fact that $g(x)$ is differentiable on $\mathbb{R}/\{0\}$ (by hypothesis) we also have

$$g'\left(n + \tfrac{1}{2}\right) = l'\left(n + \tfrac{1}{2}\right)$$

for all $n \in \mathbb{Z}$.

Finally, we show that the integral is minimal and we establish uniqueness. Assume that $\widetilde{l}(z)$ is a real entire function of exponential type 2π such that

$$\widetilde{l}(x) \leq g(x) \tag{2.63}$$

for all $x \in \mathbb{R}$, and suppose that $\{g(x) - \widetilde{l}(x)\}$ is integrable. In this case the function

$$j(z) = l(z) - \widetilde{l}(z)$$

has exponential type 2π and is integrable on $\mathbb{R}$. Thus it has bounded variation and we can apply Poisson summation, together with (2.62) and (2.63), to get

$$\begin{aligned}\widehat{j}(0) &= \lim_{N\to\infty} \sum_{n=-N}^{N} j\big(n+\tfrac{1}{2}\big) \\ &= \lim_{N\to\infty} \sum_{n=-N}^{N} \Big(g\big(n+\tfrac{1}{2}\big) - \widetilde{l}\big(n+\tfrac{1}{2}\big)\Big) \geq 0\,.\end{aligned} \tag{2.64}$$

This plainly verifies that

$$\int_{-\infty}^{\infty} \{g(x) - \widetilde{l}(x)\}\,\mathrm{d}x \geq \int_{-\infty}^{\infty} \{g(x) - l(x)\}\,\mathrm{d}x\,,$$

and establishes the minimality of the integral. If equality occurs in (2.64) we must have

$$\widetilde{l}\big(n+\tfrac{1}{2}\big) = g\big(n+\tfrac{1}{2}\big) = l\big(n+\tfrac{1}{2}\big) \tag{2.65}$$

for all $n \in \mathbb{Z}$. From (2.63) we also have

$$\widetilde{l}'\big(n+\tfrac{1}{2}\big) = g'\big(n+\tfrac{1}{2}\big) = l'\big(n+\tfrac{1}{2}\big) \tag{2.66}$$

for all $n \in \mathbb{Z}$. The interpolation conditions (2.65) and (2.66) imply that

$$j\big(n+\tfrac{1}{2}\big) = j'\big(n+\tfrac{1}{2}\big) = 0$$

for all $n \in \mathbb{Z}$. By an application of Theorem 1.12 (with an appropriate shift of $\frac{1}{2}$), we conclude that the entire function $j(z)$ is identically zero. This proves the uniqueness of the extremal minorant $l(z)$, and completes the proof.

In the proof of uniqueness in the majorant case, we will obtain

$$j'(n) = 0$$

for all $n \neq 0$, since the original function $g(x)$ is not assumed to be differentiable at the origin. An application of Theorem 1.12 shows that $j'(0) = 0$ (since j must be integrable), and this leads to uniqueness.

2.3.4 Asymptotic analysis of the admissible measures

Recall that we are working with the family of Gaussian functions

$$G_\lambda(x) = e^{-\pi\lambda x^2}\,,$$

where $\lambda > 0$ is a parameter. The Fourier transform $t \mapsto \widehat{G}_\lambda(t)$ is given by

$$\widehat{G}_\lambda(t) = \lambda^{-\frac{1}{2}} e^{-\pi\lambda^{-1}t^2}.$$

In Theorems 2.1 and 2.2 we constructed, for each $\lambda > 0$, the extremal minorant $L_\lambda(z)$ and the extremal majorant $M_\lambda(z)$ for $G_\lambda(x)$. The values of the minimal integrals are given by

$$\begin{aligned}\int_{-\infty}^{\infty} &\{G_\lambda(x) - L_\lambda(x)\}\,\mathrm{d}x \\ &= \lambda^{-\frac{1}{2}}\Big(1 - \theta_2\big(0, i\lambda^{-1}\big)\Big) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (-1)^{n+1}\widehat{G}_\lambda(n)\,,\end{aligned} \tag{2.67}$$

and

$$\begin{aligned}\int_{-\infty}^{\infty} &\{M_\lambda(x) - G_\lambda(x)\}\,\mathrm{d}x \\ &= \lambda^{-\frac{1}{2}}\Big(\theta_3\big(0, i\lambda^{-1}\big) - 1\Big) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \widehat{G}_\lambda(n).\end{aligned} \tag{2.68}$$

From the two expressions above and the transformation formulas (2.12) and (2.13) we obtain the estimates

$$\int_{-\infty}^{\infty} \{G_\lambda(x) - L_\lambda(x)\}\,\mathrm{d}x = \begin{cases} O\big(\lambda^{-\frac{1}{2}}e^{-\pi\lambda^{-1}}\big) \text{ as } \lambda \to 0, \\ O\big(\lambda^{-\frac{1}{2}}\big) \text{ as } \lambda \to \infty, \end{cases} \tag{2.69}$$

and

$$\int_{-\infty}^{\infty} \{M_\lambda(x) - G_\lambda(x)\}\,\mathrm{d}x = \begin{cases} O\big(\lambda^{-\frac{1}{2}}e^{-\pi\lambda^{-1}}\big) \text{ as } \lambda \to 0, \\ O(1) \text{ as } \lambda \to \infty. \end{cases} \tag{2.70}$$

In order to apply Theorems 2.10 and 2.11 we require that the integrals with respect to ν of the functions of λ appearing in (2.67) and (2.68) are finite. The estimates (2.69) and (2.70) show that this is a wide class of measures because of the very fast decay when $\lambda \to 0$. One should compare this class of measures with the ones used in [8], [9] and [19], to fully notice the improvement and power of the Gaussian subordination method.

2.3.5 Examples

Positive definite functions

As a first application we present the following result.

Corollary 2.12. *Let ν be a finite non-negative Borel measure on $[0,\infty)$ and consider the function $g : \mathbb{R} \to \mathbb{R}$ given by*

$$g(x) = \int_0^{\infty} e^{-\pi\lambda x^2}\,\mathrm{d}\nu(\lambda)\,. \tag{2.71}$$

(i) *There exists a unique extremal minorant $l(z)$ of exponential type 2π for $g(x)$. The function $l(x)$ interpolates the values of $g(x)$ at $\mathbb{Z}+\frac{1}{2}$ and satisfies*

$$\int_{-\infty}^{\infty} \{g(x) - l(x)\}\, \mathrm{d}x = \int_0^{\infty} \left\{ \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (-1)^{n+1} \widehat{G}_\lambda(n) \right\} \mathrm{d}\nu(\lambda).$$

(ii) *There exists a unique extremal majorant $m(z)$ of exponential type 2π for $g(x)$. The function $m(x)$ interpolates the values of $g(x)$ at $\mathbb{Z}$ and satisfies*

$$\int_{-\infty}^{\infty} \{m(x) - g(x)\}\, \mathrm{d}x = \int_0^{\infty} \left\{ \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \widehat{G}_\lambda(n) \right\} \mathrm{d}\nu(\lambda).$$

Due to a classical result of Schoenberg [35, Theorems 2 and 3], a function $g : \mathbb{R} \to \mathbb{R}$ admits the representation (2.71) if and only if its radial extension to $\mathbb{R}^N$ is positive definite, for all $N \in \mathbb{N}$, or equivalently if the function $g\big(|x|^{1/2}\big)$ is completely monotone. Recall that a function $f(t)$ is *completely monotone* for $t \geq 0$ if

$$(-1)^n f^{(n)}(t) \geq 0 \quad \text{for} \quad 0 < t < \infty, \quad \text{and} \quad n = 1, 2, 3, \dots,$$

and

$$f(0) = f(0+).$$

The last condition expresses the continuity of $f(t)$ at the origin. Using this characterization we arrive at the following interesting examples contemplated by our Corollary 2.12.

Example 1. $g(x) = e^{-\alpha|x|^{2r}}$, $\alpha > 0$, and $0 < r \leq 1$.

Example 2. $g(x) = \big(x^2 + \alpha^2\big)^{-\beta}$, $\alpha > 0$ and $\beta > 0$.

The first example shows that we can recover all the theory for the exponential function $g(x) = e^{-\lambda|x|}$ developed in [8], [9] and [19], from the family of Gaussian functions and the distribution theorems. The second example includes the Poisson kernel $g(x) = 2\lambda/(\lambda^2 + 4\pi^2x^2)$, $\lambda > 0$. Another application of Corollary 2.12 yields the following example.

Example 3. $g(x) = -\log\left(\dfrac{x^2+\alpha^2}{x^2+\beta^2}\right)$, for $0 \leq \alpha < \beta$.

Indeed, for $0 \leq \alpha < \beta$ consider the non-negative measure

$$\mathrm{d}\nu(\lambda) = \frac{\left\{e^{-\pi\lambda\alpha^2} - e^{-\pi\lambda\beta^2}\right\}}{\lambda}\, \mathrm{d}\lambda\,,$$

and observe that

$$-\log\left(\frac{x^2+\alpha^2}{x^2+\beta^2}\right) = \int_0^{\infty} e^{-\pi\lambda x^2} \frac{\left\{e^{-\pi\lambda\alpha^2} - e^{-\pi\lambda\beta^2}\right\}}{\lambda}\, \mathrm{d}\lambda\,. \tag{2.72}$$

When $0 < \alpha < \beta$ this is a finite measure and we fall under the scope of Corollary 2.12. When $\alpha = 0$, we still have $g(x)$ integrable and thus its Fourier transform (in the classical sense) is given by

$$\widehat{g}(t) = \int_0^\infty \widehat{G}_\lambda(t) \, \frac{\left\{1 - e^{-\pi\lambda\beta^2}\right\}}{\lambda} \, \mathrm{d}\lambda \, .$$

Thus we still fall under the hypothesis of Theorem 2.10 to obtain an extremal minorant in this case (note that an extremal majorant does not exist due to the singularity at the origin). In particular, the values of the minimal integrals in the one-sided approximations are given by

$$\int_{-\infty}^{\infty} \left\{ -\log\left(\frac{x^2+\alpha^2}{x^2+\beta^2}\right) - l_{\alpha,\beta}(x) \right\} \mathrm{d}x = 2\log\left(\frac{1+e^{-2\pi\alpha}}{1+e^{-2\pi\beta}}\right) \, ,$$

if $0 \leq \alpha$, and

$$\int_{-\infty}^{\infty} \left\{ m_{\alpha,\beta}(x) + \log\left(\frac{x^2+\alpha^2}{x^2+\beta^2}\right) \right\} \mathrm{d}x = 2\log\left(\frac{1-e^{-2\pi\beta}}{1-e^{-2\pi\alpha}}\right) \, ,$$

if $0 < \alpha$.

Example 4. $g(x) = 1 - x\arctan\left(\frac{1}{x}\right)$.

One can consider the non-negative and finite measure given by

$$\mathrm{d}\nu(\lambda) = \int_{1/2}^{3/2} \left\{ \frac{e^{-\pi\lambda(\sigma-1/2)^2} - e^{-\pi\lambda}}{2\lambda} \right\} \mathrm{d}\sigma \, \mathrm{d}\lambda.$$

Using the fact that

$$g(x) = 1 - x\arctan\left(\frac{1}{x}\right) = \frac{1}{2}\int_{1/2}^{3/2} \log\left(\frac{x^2+1}{x^2+\left(\sigma-\frac{1}{2}\right)^2}\right) \mathrm{d}\sigma,$$

together with (2.72), we arrive at

$$g(x) = 1 - x\arctan\left(\frac{1}{x}\right) = \int_0^\infty e^{-\pi\lambda x^2} \, \mathrm{d}\nu(\lambda).$$

This was observed in [4] and shall be used when we consider bounds for $S_1(t)$ (the antiderivative of the argument function $S(t)$) under the Riemann hypothesis, in the next chapter.

Power functions

In this subsection we write $s = \sigma + it$ for a complex variable, and we define the meromorphic function $s \mapsto \gamma(s)$ by

$$\gamma(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right).$$

The function $\gamma(s)$ is analytic on $\mathbb{C}$ except for simple poles at the points $s = 0, -2, -4, \ldots$. It also occurs in the functional equation

$$\gamma(s)\zeta(s) = \gamma(1-s)\zeta(1-s), \tag{2.73}$$

where $\zeta(s)$ is the Riemann zeta-function.

Lemma 2.13. *Let $0 < \delta$ and let $\varphi(t)$ be a Schwartz function supported on $[-\delta, \delta]^c$. Then*

$$s \mapsto \int_{-\infty}^{\infty} |t|^{-s-1}\varphi(t)\ \mathrm{d}t \tag{2.74}$$

defines an entire function of s, and the identity

$$\gamma(s+1)\int_{-\infty}^{\infty} |t|^{-s-1}\varphi(t)\ \mathrm{d}t = \gamma(-s)\int_{-\infty}^{\infty} |x|^{s}\,\widehat{\varphi}(x)\ \mathrm{d}x \tag{2.75}$$

holds in the half-plane $\{s \in \mathbb{C} : -1 < \sigma\}$. In particular, the function on the right of (2.75) *is analytic at the points $s = 0, 2, 4, \ldots$.*

Proof. Because $\varphi(t)$ is supported in $[-\delta, \delta]^c$, the function $t \mapsto |t|^{-s-1}\varphi(t)$ is integrable on $\mathbb{R}$ for all complex values s. Hence by Morera's theorem the integral on the right of (2.74) defines an entire function. The identity (2.75) holds in the infinite strip $\{s \in \mathbb{C} : -1 < \sigma < 0\}$ by [39, Lemma 1, p. 117], and therefore it holds in the half-plane $\{s \in \mathbb{C} : -1 < \sigma\}$ by analytic continuation. The left-hand side of (2.75) is clearly analytic at each point of $\{s \in \mathbb{C} : -1 < \sigma\}$, hence the right-hand side of (2.75) is also analytic at each point of this half-plane. $\square$

Lemma 2.13 asserts that, for $-1 < \sigma$ and $\sigma \neq 0, 2, 4, \ldots$, the Fourier transform of the function $x \mapsto \gamma(-\sigma)|x|^{\sigma}$ is given by the function

$$t \mapsto \gamma(\sigma+1)|t|^{-\sigma-1}$$

outside the interval $[-\delta, \delta]$. We intend to apply the distribution theorems, and towards this end, we consider the non-negative Borel measure ν_σ on $[0, \infty)$ given by

$$\mathrm{d}\nu_\sigma(\lambda) = \lambda^{-\frac{\sigma}{2}-1}\,\mathrm{d}\lambda\,,$$

and observe that we have

$$\int_0^{\infty} \widehat{G}_\lambda(t)\,\mathrm{d}\nu_\sigma(\lambda) = \gamma(\sigma+1)|t|^{-\sigma-1}. \tag{2.76}$$

For $-1 < \sigma$, the measure ν_σ is admissible for the minorant problem according to the asymptotics (2.69). For the majorant problem we shall require that $0 < \sigma$, according to the asymptotics (2.70).

Theorems 2.10 and 2.11 now apply. The values of the integrals in the following corollary can be obtained using Theorems 2.1 and 2.2, and then applying termwise integration to the series (2.7) and (2.8).

Corollary 2.14. *Let $-1 < \sigma$ with $\sigma \neq 0, 2, 4, ...$ and let*

$$g_\sigma(x) = \gamma(-\sigma)|x|^\sigma.$$

(i) There exists a unique extremal minorant $l_\sigma(z)$ of exponential type 2π for $g_\sigma(x)$. The function $l_\sigma(x)$ interpolates the values of $g_\sigma(x)$ at $\mathbb{Z} + \frac{1}{2}$ and satisfies

$$\int_{-\infty}^{\infty} \{g_\sigma(x) - l_\sigma(x)\}\,\mathrm{d}x = \big(2 - 2^{1-\sigma}\big)\gamma(1+\sigma)\,\zeta(1+\sigma). \tag{2.77}$$

(ii) If $0 < \sigma$, there exists a unique extremal majorant $m_\sigma(z)$ of exponential type 2π for $g_\sigma(x)$. The function $m_\sigma(x)$ interpolates the values of $g_\sigma(x)$ at $\mathbb{Z}$ and satisfies

$$\int_{-\infty}^{\infty} \{m_\sigma(x) - g_\sigma(x)\}\,\mathrm{d}x = 2\,\gamma(1+\sigma)\,\zeta(1+\sigma). \tag{2.78}$$

Corollary 2.14 provides a complete description of the extreme minorants and extreme majorants associated to $x \mapsto |x|^\sigma$. For $\sigma \leq -1$ these functions are not integrable at the origin, and therefore no extremals exist, and for $\sigma = 2k, k \in \mathbb{Z}^+$, these functions are entire, have only polynomial growth, and therefore the extremal problem is trivial. Previous results had been obtained in [8] and [9] for the functions $x \mapsto |x|^\sigma$, $-1 < \sigma < 1$, and in [28] for the functions $x \mapsto |x|^{2k+1}$, with $k \in \mathbb{Z}^+$.

Logarithm

We complete our list of applications (in the even case) with one additional example that follows from the distribution theorems.

Corollary 2.15. *Let $\alpha \geq 0$ and consider*

$$x \mapsto \tau_\alpha(x) = -\log(x^2 + \alpha^2).$$

(i) There exists a unique extremal minorant l_α of exponential type 2π for τ_α. The function l_α interpolates the values of τ_α at $\mathbb{Z} + \frac{1}{2}$, and satisfies

$$\int_{-\infty}^{\infty} \{\tau_\alpha(x) - l_\alpha(x)\}\,\mathrm{d}x = 2\log\big(1 + e^{-2\pi\alpha}\big).$$

(ii) If $0 < \alpha$, there exists a unique extremal majorant m_α of exponential type 2π for τ_α. The function m_α interpolates the values of τ_α at $\mathbb{Z}$, and satisfies

$$\int_{-\infty}^{\infty} \{m_\alpha(x) - \tau_\alpha(x)\}\,\mathrm{d}x = -2\log\big(1 - e^{-2\pi\alpha}\big).$$

Proof. For $0 \le \alpha$ we have the identity

$$-\log(x^2+\alpha^2) = \int_0^\infty \frac{\left\{e^{-\pi\lambda(x^2+\alpha^2)} - e^{-\pi\lambda}\right\}}{\lambda}\,\mathrm{d}\lambda\,. \tag{2.79}$$

Let φ be a Schwartz function supported in $[-\delta,\delta]^c$. An application of Fubini's theorem leads to the identity

$$\begin{aligned}
&\int_{-\infty}^{\infty} -\log(x^2+\alpha^2)\,\widehat{\varphi}(x)\,\mathrm{d}x \\
&\qquad = \int_{-\infty}^{\infty}\left\{\int_0^\infty \frac{\left\{e^{-\pi\lambda(x^2+\alpha^2)} - e^{-\pi\lambda}\right\}}{\lambda}\mathrm{d}\lambda\right\}\widehat{\varphi}(x)\,\mathrm{d}x \\
&\qquad = \int_0^\infty\int_{-\infty}^{\infty} \frac{\left\{e^{-\pi\lambda(x^2+\alpha^2)} - e^{-\pi\lambda}\right\}}{\lambda}\,\widehat{\varphi}(x)\,\mathrm{d}x\,\mathrm{d}\lambda \qquad (2.80)\\
&\qquad = \int_0^\infty\left\{\int_{-\infty}^{\infty} \widehat{G}_\lambda(t)\,\varphi(t)\,\mathrm{d}t\right\}\frac{e^{-\pi\lambda\alpha^2}}{\lambda}\,\mathrm{d}\lambda \\
&\qquad = \int_{-\infty}^{\infty}\left\{\int_0^\infty \widehat{G}_\lambda(t)\,\frac{e^{-\pi\lambda\alpha^2}}{\lambda}\,\mathrm{d}\lambda\right\}\varphi(t)\,\mathrm{d}t.
\end{aligned}$$

Equation (2.80) provides the Fourier transform of $-\log(x^2+\alpha^2)$ outside a compact interval $[-\delta,\delta]$. We can therefore apply the distribution theorems (Theorems 2.10 and 2.11) with measure ν on $[0,\infty)$ given by

$$\mathrm{d}\nu(\lambda) = \frac{e^{-\pi\lambda\alpha^2}}{\lambda}\,\mathrm{d}\lambda.$$

According to the asymptotics (2.69) and (2.70), if $\alpha > 0$ we can treat the two one-sided approximation problems, and if $\alpha = 0$ we can only treat the minorant problem (which is in agreement with the fact that $-\log|x|$ is unbounded from above). The special case of $-\log|x|$ (when $\alpha = 0$) was first obtained in [8] and [9]. $\square$

2.4 The extremal problem for the truncated and odd Gaussians

We know move in the direction of developing the analogous extremal theory for the case of truncated and odd functions. This was carried out in the papers [5] and [6]. We present here briefly the developments of [5], that deal with the corresponding Gaussian subordination framework for the case of truncated and odd functions.

We keep the notation for the Gaussian

$$x \mapsto G_\lambda(x) = e^{-\pi\lambda x^2}$$

and the theta functions defined in (2.6), (2.7) and (2.8). We consider here the Beurling-Selberg extremal problem for the *truncated Gaussian* $x \mapsto G_\lambda^+(x)$ defined by

$$G_\lambda^+(x) = \begin{cases} G_\lambda(x) & \text{for } x > 0, \\ 1/2 & \text{for } x = 0, \\ 0 & \text{for } x < 0, \end{cases}$$

and the *odd Gaussian* $x \mapsto G_\lambda^o(x)$ defined by

$$G_\lambda^o(x) = \begin{cases} G_\lambda(x) & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -G_\lambda(x) & \text{for } x < 0. \end{cases}$$

Recall that the Fourier transform of the Gaussian $G_\lambda(x) = e^{-\pi\lambda x^2}$ is given by

$$\widehat{G}_\lambda(t) = \int_{-\infty}^{\infty} e^{-2\pi i t x} G_\lambda(x)\,\mathrm{d}x = \lambda^{-1/2} e^{-\pi\lambda^{-1}t^2},$$

and, via contour integration, the Fourier transform of the truncated Gaussian $G_\lambda^+(x)$ is shown to be

$$\widehat{G}_\lambda^+(t) = \frac{1}{2}\,\lambda^{-1/2} e^{-\pi\lambda^{-1}t^2} + \frac{t}{i\lambda}\int_0^1 e^{-\pi\lambda^{-1}t^2(1-y^2)}\,\mathrm{d}y. \tag{2.81}$$

Define the following two entire functions of exponential type

$$L_\lambda^+(z) = \frac{\sin^2 \pi z}{\pi^2} \sum_{n=1}^{\infty} \left\{ \frac{G_\lambda(n)}{(z-n)^2} + \frac{G_\lambda'(n)}{z-n} - \frac{G_\lambda'(n)}{z} \right\},$$

$$M_\lambda^+(z) = \frac{\sin^2 \pi z}{\pi^2} \sum_{n=1}^{\infty} \left\{ \frac{G_\lambda(n)}{(z-n)^2} + \frac{G_\lambda'(n)}{z-n} - \frac{G_\lambda'(n)}{z} \right\} + \frac{\sin^2 \pi z}{\pi^2 z^2}.$$

Note that L_λ^+ and M_λ^+ are entire functions of exponential type 2π that interpolate the values of G_λ^+ and its derivative at $\mathbb{Z}\backslash\{0\}$. The following two theorems provide the solution of the extremal problem for the truncated Gaussian.

Theorem 2.16 (Extremal minorant for the truncated Gaussian)**.** *The inequality*

$$L_\lambda^+(x) \le G_\lambda^+(x)$$

holds for all real x. Let $z \mapsto L(z)$ be an entire function of exponential type at most 2π which satisfies the inequality $L(x) \le G_\lambda^+(x)$ for all real x. Then

$$\int_{-\infty}^{\infty} \left\{ G_\lambda^+(x) - L(x) \right\} \mathrm{d}x \ge -\frac{\theta_3(0, i\lambda)}{2} + \frac{1}{2} + \frac{1}{2\sqrt{\lambda}}, \tag{2.82}$$

with equality if and only if $L = L_\lambda^+$.

Theorem 2.17 (Extremal majorant for the truncated Gaussian)**.** *The inequality*

$$G_\lambda^+(x) \le M_\lambda^+(x)$$

holds for all real x. Let $z \mapsto M(z)$ be an entire function of exponential type at most 2π which satisfies the inequality $G_\lambda^+(x) \le M(x)$ for all real x. Then

$$\int_{-\infty}^{\infty} \left\{ M(x) - G_\lambda^+(x) \right\} \mathrm{d}x \ge \frac{\theta_3(0, i\lambda)}{2} + \frac{1}{2} - \frac{1}{2\sqrt{\lambda}}\,, \tag{2.83}$$

with equality if and only if $M = M_\lambda^+$.

The strategy for the proofs of the two theorems above is a decomposition of these functions into integral representations analogous to those developed in Section 2.2.2 for the Gaussian. The integrands will involve certain truncated theta functions that turn out to be solutions of the heat equation, and the maximum principle for the heat operator is used to obtain the necessary inequalities. The uniqueness part will follow from the interpolation properties at $\mathbb{Z}$ as done in the proofs of Theorems 2.1 and 2.2. A simple dilation argument provides the optimal approximations of exponential type $2\pi\delta$ for any $\delta > 0$. Since the proofs of these results are rather lengthy and technical, we decided not to include them here, and instead refer the interested reader to the original source [5].

Once we have established the solution of the extremal problem for the truncated Gaussian as described in Theorems 2.16 and 2.17, we can easily derive the solution of this problem for the odd Gaussian $x \mapsto G_\lambda^o(x)$. Observe that

$$G_\lambda^o(x) = G_\lambda^+(x) - G_\lambda^+(-x)$$

and define the entire functions

$$\begin{aligned} L_\lambda^o(z) &= L_\lambda^+(z) - M_\lambda^+(-z), \\ M_\lambda^o(z) &= M_\lambda^+(z) - L_\lambda^+(-z). \end{aligned} \tag{2.84}$$

Theorems 2.16 and 2.17 imply that

$$L_\lambda^o(x) \le G_\lambda^o(x) \le M_\lambda^o(x).$$

These functions preserve the interpolation properties at $\mathbb{Z}$ and are the extremal minorant and majorant for the odd Gaussian, respectively. This follows by arguments analogous to the proofs of Theorems 2.1 and 2.2, and plainly guarantees the odd counterparts of all the results we present here for truncated functions.

2.5 Framework for truncated and odd functions

2.5.1 Integrating the free parameter

Having solved the Beurling-Selberg extremal problem for a family of functions with a free parameter $\lambda > 0$, we are now interested in integrating this parameter against a set of admissible non-negative Borel measures ν on $[0,\infty)$ to generate a new class of truncated (and odd) functions for which the extremal problem has a solution.

We now determine the set of admissible measures ν. For the minorant problem, the minimal condition we must impose on the measure ν is that the function on the right-hand side of (2.82) should be ν-integrable. The well-known asymptotics for the theta functions (given by the transformation formulas) lead us to consider non-negative Borel measures ν on $[0,\infty)$ satisfying

$$\int_0^\infty \frac{1}{1+\sqrt{\lambda}}\,\mathrm{d}\nu(\lambda) < \infty. \tag{2.85}$$

On the other hand, for the majorant problem, the minimal condition we must impose on the measure ν is that the function on the right-hand side of (2.83) should be ν-integrable. This is equivalent to the measure being finite, i.e.

$$\int_0^\infty \mathrm{d}\nu(\lambda) < \infty\,. \tag{2.86}$$

Define the truncation x_+^0 by

$$x_+^0 = \tfrac{1}{2}(1+\operatorname{sgn}(x))$$

and consider the truncated function $g:\mathbb{R}\to\mathbb{R}$ given by

$$g(x) = x_+^0 \int_0^\infty e^{-\pi\lambda x^2}\,\mathrm{d}\nu(\lambda)\,.$$

We are now able to state the two main results of this section.

Theorem 2.18 (Extremal minorant - general truncated case)**.** *Let ν satisfy (2.85). Then there exists a unique extremal minorant $z\mapsto l(z)$ of exponential type 2π for $x\mapsto g(x)$. The function l interpolates the values of g and its derivative at $\mathbb{Z}\backslash\{0\}$ and satisfies*

$$\int_{-\infty}^\infty \{g(x)-l(x)\}\,\mathrm{d}x = \int_0^\infty \left\{-\frac{\theta_3(0,i\lambda)}{2}+\frac{1}{2}+\frac{1}{2\sqrt{\lambda}}\right\}\mathrm{d}\nu(\lambda).$$

Theorem 2.19 (Extremal majorant - general truncated case)**.** *Let ν satisfy (2.86). Then there exists a unique extremal majorant $z\mapsto m(z)$ of exponential type 2π for $x\mapsto g(x)$. The function m interpolates the values of g and its derivative at $\mathbb{Z}\backslash\{0\}$ and satisfies*

$$\int_{-\infty}^\infty \{m(x)-g(x)\}\,\mathrm{d}x = \int_0^\infty \left\{\frac{\theta_3(0,i\lambda)}{2}+\frac{1}{2}-\frac{1}{2\sqrt{\lambda}}\right\}\mathrm{d}\nu(\lambda).$$

Observe that the class of measures allowed by (2.85) and (2.86) is more restrictive than the class we worked in the even Gaussian problem, thus making the method less powerful in this truncated/odd case (one might also see this from the fact that we did not have to appeal to the Fourier space for the definition of $g(x)$). When adapting Theorems 2.18 and 2.19 to the context of odd functions, we must ask for the more restrictive condition (2.86), due to the construction (2.84).

2.5.2 Proofs of Theorems 2.18 and 2.19

We start with the minorant case, where we have seen by Theorem 2.16 that

$$L_\lambda^+(z) = \frac{\sin^2 \pi z}{\pi^2} \sum_{n=1}^{\infty} \left\{ \frac{G_\lambda(n)}{(z-n)^2} + \frac{G'_\lambda(n)}{z-n} \right\} - \frac{\sin^2 \pi z}{\pi^2 z} \sum_{n=1}^{\infty} G'_\lambda(n)$$

satisfies

$$L_\lambda^+(x) \leq G_\lambda^+(x) \tag{2.87}$$

for all $x \in \mathbb{R}$, with

$$L_\lambda^+(n) = G_\lambda^+(n) \tag{2.88}$$

if $n \in \mathbb{Z}/\{0\}$, and

$$L_\lambda^+(0) = \lim_{x \to 0^-} G_\lambda^+(x) = 0. \tag{2.89}$$

We consider a non-negative Borel measure ν satisfying (2.85) and we need to show that

$$l(z) = \int_0^\infty L_\lambda^+(z)\, \mathrm{d}\nu(\lambda)$$

is a well defined entire function of exponential type at most 2π. If this is the case, by integrating expressions (2.87), (2.88) and (2.89) against ν, these properties will be carried on to $l(x)$ and $g(x) = \int_0^\infty G_\lambda^+(x)\, \mathrm{d}\nu(\lambda)$ making $l(x)$ the unique extremal minorant of exponential type at most 2π for $g(x)$ via the same arguments used in the proof of Theorem 2.1.

For this purpose we need to collect some estimates. For $n \in \mathbb{N}$ using (2.85) we have

$$\int_0^\infty G_\lambda(n)\, \mathrm{d}\nu(\lambda) = \int_0^1 G_\lambda(n)\, \mathrm{d}\nu(\lambda) + \int_1^\infty \sqrt{\lambda}\, G_\lambda(n)\, \frac{\mathrm{d}\nu(\lambda)}{\sqrt{\lambda}} \leq C_1 + \frac{C_2}{n}, \tag{2.90}$$

and

$$\begin{aligned} \int_0^\infty \left|G'_\lambda(n)\right| \mathrm{d}\nu(\lambda) &= 2\pi \int_0^1 \lambda\, n\, G_\lambda(n)\, \mathrm{d}\nu(\lambda) + 2\pi \int_1^\infty \lambda^{3/2}\, n\, G_\lambda(n)\, \frac{\mathrm{d}\nu(\lambda)}{\sqrt{\lambda}} \\ &\leq \frac{C_3}{n} + \frac{C_4}{n^2}, \end{aligned} \tag{2.91}$$

where C_1, C_2, C_3 and C_4 are positive constants depending exclusively on ν.

To analyze the remaining term observe that

$$\lambda^{1/2} \sum_{n=1}^{\infty} \left|G'_\lambda(n)\right| = \sum_{n=1}^{\infty} \frac{2\pi}{n^2}\, \lambda^{3/2}\, n^3\, G_\lambda(n) \leq C_5 \sum_{n=1}^{\infty} \frac{2\pi}{n^2},$$

which proves that $\sum_{n=1}^{\infty} \left|G'_\lambda(n)\right|$ is $\mathcal{O}\big(\lambda^{-1/2}\big)$ as $\lambda \to \infty$. On the other hand, using the arithmetic-geometric mean inequality and the fact that

$$\sum_{n=0}^{\infty} e^{-tn^2}\big(1 - 2tn^2\big) \geq \frac{1}{2},$$

obtained by differentiating both sides of the transformation formula (2.13), we arrive at

$$\begin{aligned}
\sum_{n=1}^{\infty}\left|G'_{\lambda}(n)\right| = \sum_{n=1}^{\infty} 2\pi\lambda\, n\, G_{\lambda}(n) &\le \sum_{n=1}^{\infty} \pi\left\{\lambda^{3/2}\, n^2 + \lambda^{1/2}\right\} G_{\lambda}(n)\\
&\le \frac{\lambda^{1/2}}{4} + \left(\tfrac{1}{2}+\pi\right)\, \lambda^{1/2} \sum_{n=1}^{\infty} G_{\lambda}(n)\\
&= \frac{\lambda^{1/2}}{4} + \left(\tfrac{1}{2}+\pi\right)\, \lambda^{1/2} \left(\frac{\theta_3(0,i\lambda)-1}{2}\right).
\end{aligned}$$

We know $\theta_3(0,i\lambda) \to \lambda^{-1/2}$ as $\lambda \to 0$, by the transformation formula (2.13). Therefore we may conclude that $\sum_{n=1}^{\infty}\left|G'_{\lambda}(n)\right|$ is $\mathcal{O}(1)$ as $\lambda \to 0$.

This shows that $\sum_{n=1}^{\infty}\left|G'_{\lambda}(n)\right|$ is ν-integrable, and together with (2.90) and (2.91) we can can move the integration inside the summation series since it converges absolutely to obtain

$$\begin{aligned}
l(z) &= \int_0^{\infty} L^{+}_{\lambda}(z)\,\mathrm{d}\nu(\lambda)\\
&= \frac{\sin^2 \pi z}{\pi^2}\sum_{n=1}^{\infty}\left\{\frac{\int_0^{\infty} G_{\lambda}(n)\,\mathrm{d}\nu(\lambda)}{(z-n)^2} + \frac{\int_0^{\infty} G'_{\lambda}(n)\,\mathrm{d}\nu(\lambda)}{z-n}\right\}\\
&\qquad\qquad - \frac{\sin^2 \pi z}{\pi^2 z}\int_0^{\infty}\sum_{n=1}^{\infty} G'_{\lambda}(n)\,\mathrm{d}\nu(\lambda).
\end{aligned}$$

An application of Morera's theorem shows that this is an entire function and the exponential type 2π is given by the main term $\sin^2 \pi z$. The proof of the majorizing case is analogous.

2.5.3 Examples

We highlight some interesting choices of non-negative Borel measures ν that can be applied in Theorems 2.18 and 2.19 . We will mainly present the truncated functions here. Similar examples can be given for the odd functions. The first of these examples considers $\nu = \delta$ (the Dirac delta). In this case we obtain the following.

Example 1. $g(x) = x_{+}^{0}$.

This reproves the classical extremal functions to the signum function contained in [43, Theorems 4 and 8]. In our setting the values of the minimal integrals can be found via the asymptotics of $\theta_3(0,i\lambda)$.

More generally, as in the case of even functions, one can consider any *finite* non-negative Borel measure ν on $[0,\infty)$. With the complete monotone characterization of the positive definite functions (see Section 2.3.5) we arrive at the following truncated and odd counterparts.

Example 2. $g(x) = x_+^0 \, e^{-\alpha |x|^{2r}}$, $\alpha > 0$ and $0 \le r \le 1$.

Example 3. $g(x) = x_+^0 \, (x^2 + \alpha^2)^{-\beta}$, $\alpha > 0$ and $\beta > 0$.

The family in Example 2 includes the truncated exponential $g(x) = x_+^0 e^{-\lambda |x|}$ treated in [19], while the family in Example 3 includes the truncated Poisson kernel $g(x) = x_+^0[2\lambda/(\lambda^2 + 4\pi^2 x^2)]$. Despite not knowing the exact expression of the measures ν that produce these families, one can arrive at the value of the minimal integral with the knowledge of the Fourier transforms of these functions via Poisson summation.

Observe that the non-negative measure

$$d\nu(\lambda) = \frac{\left\{ e^{-\pi\lambda\alpha^2} - e^{-\pi\lambda\beta^2} \right\}}{\lambda} \, d\lambda \,,$$

for $0 \le \alpha < \beta$ is a finite measure if $0 < \alpha$. If $\alpha = 0$, then ν still satisfies (2.85), and we can solve the minorant problem. This generates the following family.

Example 4. $g(x) = -x_+^0 \, \log(x^2 + \alpha^2)/(x^2 + \beta^2)\,$, $0 \le \alpha < \beta$.

Finally, recall the definition of the meromorphic function $s \mapsto \gamma(s)$ by

$$\gamma(s) = \pi^{-s/2} \, \Gamma\left(\frac{s}{2}\right),$$

which is analytic on $\mathbb{C}$ except for simple poles at the points $s = 0, -2 - 4, \ldots$ The family of measures

$$d\nu_\sigma(\lambda) = \lambda^{-\frac{\sigma}{2} - 1} \, d\lambda$$

satisfies (2.85) when $-1 < \sigma < 0$ and thus we can solve the minorant problem for the truncated power functions they produce.

Example 5. $g(x) = \gamma(-\sigma) \, x_+^0 \, |x|^\sigma$, $-1 < \sigma < 0$.

We close this section with a particular example of an odd function that will be relevant when we study the argument of the Riemann zeta-function in the next chapter.

Example 6. $g(x) = \arctan\left(\frac{1}{x}\right) - \frac{x}{1 + x^2}$.

In fact, it was observed in [4] that the measure

$$d\nu(\lambda) = \left\{ \int_0^\infty \frac{t}{2\sqrt{\pi\lambda^3}} \, e^{-\frac{t^2}{4\lambda}} \left(\frac{1}{t} \sin\left(\sqrt{\pi} t\right) - \sqrt{\pi} \cos\left(\sqrt{\pi} t\right) \right) dt \right\} d\lambda \tag{2.92}$$

is non-negative, finite and verifies

$$g(x) = \arctan\left(\frac{1}{x}\right) - \frac{x}{1 + x^2} = \operatorname{sgn}(x) \int_0^\infty e^{-\pi\lambda x^2} \, d\nu(\lambda). \tag{2.93}$$

Let us verify these facts. First we prove identity (2.93), for all $x > 0$. By making the change of variables $y = \sqrt{\pi} t$ and using Fubini's theorem, we see that the right-hand side of (2.93) is equal to

$$\int_0^\infty \left\{ \int_0^\infty \frac{e^{-\pi\lambda x^2 - \frac{y^2}{4\pi\lambda}}}{2\pi\lambda^{3/2}}\, y \,\mathrm{d}\lambda \right\} \left(\frac{\sin y}{y} - \cos y \right) \mathrm{d}y. \tag{2.94}$$

Call $W(x,y)$ the quantity inside the brackets in (2.94). To prove (2.93), it suffices to show that $W(x,y) = e^{-xy}$. For this, consider the change of variables $k = \frac{\sqrt{2\pi x\lambda}}{\sqrt{y}}$ which implies that

$$W(x,y) = \frac{\sqrt{2xy}}{\sqrt{\pi}}\, e^{-xy} \int_0^\infty \frac{e^{-\frac{xy}{2}\left(k-\frac{1}{k}\right)^2}}{k^2}\, \mathrm{d}k.$$

Now from the symmetry $k \to \frac{1}{k}$, we can rewrite the last expression as

$$W(x,y) = \frac{1}{2}\frac{\sqrt{2xy}}{\sqrt{\pi}}\, e^{-xy} \int_0^\infty e^{-\frac{xy}{2}\left(k-\frac{1}{k}\right)^2} \left(1 + \frac{1}{k^2}\right) \mathrm{d}k.$$

Finally, from the change of variables $w = k - \frac{1}{k}$, we arrive at

$$W(x,y) = \frac{1}{2}\frac{\sqrt{2xy}}{\sqrt{\pi}}\, e^{-xy} \int_{-\infty}^\infty e^{-\frac{xy}{2}w^2}\, \mathrm{d}w = e^{-xy}.$$

This proves (2.93).

We now prove that the measure ν given by (2.92) is non-negative. We do so by establishing that the density function

$$D(\lambda) = \int_0^\infty \frac{t}{2\sqrt{\pi\lambda^3}}\, e^{-\frac{t^2}{4\lambda}} \left(\frac{1}{t}\sin(\sqrt{\pi}t) - \sqrt{\pi}\cos(\sqrt{\pi}t) \right) \mathrm{d}t$$

is non-negative for all $\lambda > 0$. Again, we make the variable change $y = \sqrt{\pi}t$ and obtain that

$$D(\lambda) = \frac{1}{2\pi\lambda^{3/2}} \int_0^\infty e^{-\frac{y^2}{4\pi\lambda}} (\sin y - y\cos y)\, \mathrm{d}y.$$

Setting $\pi\lambda = a^2$, it suffices to prove that

$$\int_0^\infty e^{-\frac{y^2}{4a^2}} (\sin y - y\cos y)\, \mathrm{d}y \geq 0$$

for all $a > 0$. Using integration by parts and the Fourier transform of the odd Gaussian, we obtain that

$$\begin{aligned} \int_0^\infty e^{-\frac{y^2}{4a^2}} (\sin y - y\cos y)\, \mathrm{d}y &= \left\{ (1+2a^2) \int_0^\infty e^{-\frac{y^2}{4a^2}} \sin y \,\mathrm{d}y \right\} - 2a^2 \\ &= \left\{ (1+2a^2)\, 2a\, e^{-a^2} \int_0^a e^{w^2}\, \mathrm{d}w \right\} - 2a^2. \end{aligned}$$

We are left to prove that

$$h(a) = \int_0^a e^{w^2}\,\mathrm{d}w - \frac{a\,e^{a^2}}{1+2a^2} \geq 0$$

for all $a \geq 0$. This follows from observing that $h(0) = 0$ and

$$h'(a) = e^{a^2}\left(\frac{4a^2}{(1+2a^2)^2}\right) \geq 0$$

for all $a \geq 0$. This concludes the proof of the non-negativity of the measure.

Finally, we verify that ν is indeed a finite measure on $(0, \infty)$. In fact, note that (2.93) and the monotone convergence theorem imply

$$\int_0^\infty \mathrm{d}\nu(\lambda) = \lim_{x\to 0^+} \int_0^\infty e^{-\pi\lambda x^2}\,\mathrm{d}\nu(\lambda) = \lim_{x\to 0^+} \left\{\arctan\left(\frac{1}{x}\right) - \frac{x}{x^2+1}\right\} = \frac{\pi}{2},$$

and this concludes the verification of the original claims.

Chapter 3

Applications to the theory of the Riemann zeta-function

3.1 Bounds under the Riemann hypothesis

After a brief introduction to the Beurling-Selberg extremal problem and some of its recent advances in the previous chapter, our objective in this chapter is to provide some applications of these extremal tools. Here we will focus our attention on the interesting connection between some special functions and the theory of the Riemann zeta-function. These special functions are the following:

$$f(x) = \log\left(\frac{x^2+1}{x^2}\right), \tag{3.1}$$

$$g(x) = \arctan\left(\frac{1}{x}\right) - \frac{x}{1+x^2}, \tag{3.2}$$

and

$$h(x) = 1 - x\arctan\left(\frac{1}{x}\right). \tag{3.3}$$

Throughout this chapter we fix this notation for $f(x)$, $g(x)$ and $h(x)$, and recall that we obtained in the previous chapter the solution of the Beurling-Selberg extremal problem for these three functions.

Bernhard Riemann published his paper "*Über die Anzahl der Primzahlen unter einer gegebenen Grösse*" in the *Monatsberichte der Berliner Akademie* in November, 1859. There we find the statement that the function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

initially defined for $\text{Re}(s) > 1$, and then suitably extended meromorphically to the complex plane, "probably" has its complex zeros all aligned over the line $\Re(s) = 1/2$.