sufficiently small values of $x-a$, since $D^+f(a) \geqslant 0$. Hence $\phi(x) = 0$ for some values of x between a and b. Let ξ be the greatest such value. Then $D^+\phi(\xi) \leqslant 0$, $D^+f(\xi)+\epsilon \leqslant 0$, contrary to hypothesis. Hence $f(b)-f(a) \geqslant -\epsilon(b-a)$ for every positive ϵ, and the result follows.]

(v) The derivates and incrementary ratios of a continuous function have the same bounds in any interval; i.e. if any one of the derivates satisfies $\alpha \leqslant Df \leqslant \beta$, then $\alpha \leqslant \{f(x_2)-f(x_1)\}/(x_2-x_1) \leqslant \beta$, and conversely.

[Consider $\phi(x) = f(x)-\alpha x$, and use the previous example.]

(vi) If one of the derivates of $f(x)$ is continuous at a certain point, then $f(x)$ has a differential coefficient at the point.

11.4. Functions of bounded variation. We say that $f(x)$ is of bounded variation in (a, b) if, in this interval, it can be expressed in the form $\phi(x)-\psi(x)$, where ϕ and ψ are non-decreasing bounded functions.

It is easily seen that the sum, difference, or product of two functions of bounded variation is also of bounded variation.

An alternative definition is obtained by assuming that, if the interval (a, b) is divided up by points $a = x_0 < x_1 < \ldots < x_n = b$, then

$$\sum_{\nu=0}^{n-1} |f(x_{\nu+1})-f(x_\nu)|$$

is less than a constant independent of the mode of division. The upper bound of these sums is called the total variation.

It is easily seen that, if the first condition holds, then so does the second. For

$$|f(x_{\nu+1})-f(x_\nu)| \leqslant \phi(x_{\nu+1})-\phi(x_\nu)+\psi(x_{\nu+1})-\psi(x_\nu),$$

so that

$$\sum_{\nu=0}^{n-1} |f(x_{\nu+1})-f(x_\nu)| \leqslant \phi(b)-\phi(a)+\psi(b)-\psi(a).$$

To prove the converse, let p be the sum of those differences $f(x_{\nu+1})-f(x_\nu)$ which are positive, $-n$ the sum of those which are negative. Then, if v is the sum $\sum |f(x_{\nu+1})-f(x_\nu)|$, we have

$$v = p+n, \qquad f(b)-f(a) = p-n,$$

and so $\qquad v = 2p+f(a)-f(b), \qquad v = 2n+f(b)-f(a).$

Hence, if v is bounded for all modes of division, so are p and n. Let V, P, and N be the upper bounds of v, p, and n. Then

$$V = 2P+f(a)-f(b), \qquad V = 2N+f(b)-f(a).$$

Let $V(x)$, $P(x)$, and $N(x)$ be the corresponding numbers for

the interval (a, x). They are obviously bounded non-decreasing functions of x; and

$$V(x) = 2P(x)+f(a)-f(x), \qquad V(x) = 2N(x)+f(x)-f(a),$$

so that
$$f(x) = f(a)+P(x)-N(x).$$

This is the required expression for $f(x)$.

The functions $V(x)$, $P(x)$, and $N(x)$ are called the total variation and the positive and negative variations of $f(x)$ in (a, x).

If $f(x)$ is continuous and of bounded variation, its variation $V(x)$ is continuous. We can find a mode of division of the interval (a, x), with a point of division x' as near x as we please, such that

$$v > V(x)-\epsilon$$

and also
$$|f(x)-f(x')| < \epsilon.$$

Let
$$v' = v-|f(x)-f(x')|.$$

Then v' is a sum corresponding to the interval (a, x'), and so

$$V(x') \geqslant v' > V(x)-2\epsilon.$$

Since $V(x')$ is non-decreasing, it follows that $V(x') \to V(x)$ as $x' \to x$ from below. Similarly $V(x') \to V(x)$ as $x' \to x$ from above. Hence $V(x)$ is continuous.

A continuous function of bounded variation is the difference between two continuous non-decreasing functions. For if $f(x)$ is continuous, so are $P(x)$ and $N(x)$.

11.41. The differential coefficient of a function of bounded variation. The object of the next three sections is to prove that *a function of bounded variation has a finite differential coefficient almost everywhere.*

Our proof depends on the following lemmas, due to Sierpinski.* They are of the same type as the Heine-Borel theorem, but apply to sets which need not even be measurable.

LEMMA 1. *Suppose that each point x of a set E in (a,b) is the left-hand end-point of one or more intervals $(x, x+h_x)$ of a family H. Then there is a finite non-overlapping set S of intervals of H which includes a sub-set E' of E such that $m_e(E') > m_e(E)-\epsilon$.*

Let E_n be the set of points of E which are associated with some $h_x > 1/n$. Then E is the outer limiting set of the sets E_n, we have $\lim m_e(E_n) = m_e(E)$ (§ 10.29), and we can take n so large that $m_e(E_n) > m_e(E)-\frac{1}{2}\epsilon$.

* Sierpinski (1). A similar lemma is given by W. H. and G. C. Young (1).

Let a_1 be the lower bound of E_n, b_1 its upper bound, and let $l = b_1 - a_1$. Let $\eta = \frac{1}{2}\epsilon/(nl+1)$. Then there is a point x_1 of E_n such that $a_1 \leqslant x_1 < a_1 + \eta$. Let $(x_1, x_1 + h_1)$ be an associated interval for which $h_1 > 1/n$.

If there are points of E_n to the right of $x_1 + h_1$, let a_2 be their lower bound. Then there is a point x_2 of E_n in $(a_2, a_2 + \eta)$. Let $(x_2, x_2 + h_2)$ be an associated interval with $h_2 > 1/n$.

Continuing the process, we reach b_1 in a finite number of steps, since each step takes us at least $1/n$ nearer to it. In fact, if there are N steps, then $(N-1)/n < l$, i.e. $N < nl + 1$.

Let S denote the set of intervals $(x_\nu, x_\nu + h_\nu)$ so constructed, and T the set of intervals $(x_\nu - \eta, x_\nu)$. Then $E_n < S + T$, and $m(T) < N\eta < \frac{1}{2}\epsilon$. Hence

$$m_e(E) - \tfrac{1}{2}\epsilon < m_e(E_n) \leqslant m_e(E_n S) + m_e(E_n T) < m_e(E_n S) + \tfrac{1}{2}\epsilon,$$

and the set $E' = E_n S$ has the required property.

LEMMA 2. *Suppose in addition that for every x there are arbitrarily small intervals $(x, x + h_x)$. Then we may conclude in addition that*

$$m(S) < m_e(E) + \epsilon.$$

The additional condition is necessary; we might, for example, take E to be a single point x, and associate with it the interval $(x, x+1)$. Then Lemma 1 would hold, but not Lemma 2.

Let O be an open set containing E, such that

$$m(O) < m_e(E) + \epsilon.$$

Let H_1 be the sub-class of the family of intervals H consisting of those intervals that lie in O. In view of the additional condition imposed in Lemma 2, every point of E is the left-hand end-point of one or more intervals of H_1. We can now apply Lemma 1 with H replaced by H_1. We obtain a new set of intervals S which has the same property as that constructed in the proof of Lemma 1. But now S is a set of non-overlapping intervals included in O. Hence

$$m(S) \leqslant m(O) < m_e(E) + \epsilon.$$

This proves the lemma.

In these lemmas the intervals of which S consists may be regarded as either open or closed, whichever is most convenient in any particular case. For if the result has been obtained with S consisting of closed intervals, we can replace them by open

intervals by removing a finite number of points, i.e. a set of measure zero. This clearly does not affect the result.

LEMMA 3. *We may suppose S in the above construction to be included in any given set of intervals G which contains E.*

For we may replace O by OG in the construction.

11.42. *If* $f(x)$ is non-decreasing in (a,b), it has almost everywhere in (a,b) a differential coefficient $f'(x)$.*

Let E be the set where $D_+f < D^+f$. We shall first prove that $m_e(E) = 0$.

Now E is the sum of the sets $E(u,v)$ where

$$D_+f < u < v < D^+f,$$

u and v running through all rational numbers $(u < v)$. Hence it is sufficient to prove that $m_e\{E(u,v)\} = 0$ for every pair of such numbers.

Suppose on the contrary that one of these sets $E(u,v)$ has a positive exterior measure, say μ. Every point x of it is the left-hand end-point of arbitrarily small intervals $(x, x+h)$ for which

$$f(x+h)-f(x) < hu.$$

Hence by Lemma 2 there is a finite set S of such intervals, containing a part E' of $E(u,v)$ such that $m_e(E') > \mu-\epsilon$, and such that $\sum_1 h < \mu+\epsilon$, where $\sum_1$ denotes a summation over S. Hence

$$\sum_1 \{f(x+h)-f(x)\} < u \sum_1 h < u(\mu+\epsilon).$$

Again, every point of E' is the left-hand end-point of intervals $(x, x+k)$ such that

$$f(x+k)-f(x) > kv,$$

and by Lemma 3 there is a finite set of these intervals, included in S and of measure greater than $m_e(E')-\epsilon > \mu-2\epsilon$. If $\sum_2$ denotes a summation over these intervals,

$$\sum_2 \{f(x+k)-f(x)\} > v \sum_2 k > v(\mu-2\epsilon).$$

But since $f(x)$ is non-decreasing, and the k-intervals are included in the h-intervals,

$$\sum_2 \{f(x+k)-f(x)\} \leqslant \sum_1 \{f(x+h)-f(x)\}.$$

Hence $v(\mu-2\epsilon) < u(\mu+\epsilon)$, which is false if ϵ is small enough. Hence $f'_+(x)$ (and similarly $f'_-(x)$) exists almost everywhere.

* This proof is due to Rajchman and Saks (1).

Further, we can argue in the above way with D^+ replaced by D^-; every point of E' is then the right-hand end-point of arbitrarily small intervals $(x-k, x)$ such that $f(x)-f(x-k) > kv$, and the conclusion follows as before. Hence almost everywhere $D_+f \geqslant D^-f$, i.e. almost everywhere $f'_+(x) \geqslant f'_-(x)$. Similarly we can prove the reversed inequality, and the result follows.

11.43. There is a more general theorem on the possible sets where $f'_-(x) \neq f'_+(x)$, and the result has nothing to do with monotony.

The set of points where the right-hand and left-hand derivatives of any function exist and are different is enumerable.

Let E be the set where $f'_-(x) < f'_+(x)$, and let all rational numbers be arranged in a sequence $r_1, r_2, \ldots$. Then if x is a point of E, there is a smallest integer k such that

$$f'_-(x) < r_k < f'_+(x).$$

There is then a smallest integer m such that $r_m < x$, and such that

$$\{f(\xi)-f(x)\}/(\xi-x) < r_k$$

for $r_m < \xi < x$; and a smallest integer n such that $r_n > x$, and

$$\{f(\xi)-f(x)\}/(\xi-x) > r_k$$

for $x < \xi < r_n$. The two inequalities together give

$$f(\xi)-f(x) > r_k(\xi-x) \qquad (r_m < \xi < r_n,\ \xi \neq x). \tag{1}$$

Thus to every x corresponds a unique triad of numbers (k, m, n); and no two values of x correspond to the same triad; for if x_1 and x_2 correspond to the same triad, we have, on putting $x = x_1$, $\xi = x_2$ in (1), $f(x_2)-f(x_1) > r_k(x_2-x_1)$, and, on putting $x = x_2$, $\xi = x_1$, the same inequality reversed.

Since the set of triads (k, m, n) is enumerable, it follows that E is enumerable or finite. This is the required result. Since the measure of an enumerable set is zero, this theorem can be used to give an alternative ending to the proof of the theorem of the previous section.

11.5. Integrals. A function which is the Lebesgue indefinite integral of another function is called *an integral*.

An integral is continuous. For if $F(x)$ is the integral of $f(x)$, then

$$F(x+h)-F(x) = \int_x^{x+h} f(t)\,dt,$$

which tends to 0 with h, by § 10.73 (v).

The integral of a positive function is a non-decreasing function. For if $f(x) \geqslant 0$, $h > 0$,

$$F(x+h)-F(x) = \int_x^{x+h} f(t)\,dt \geqslant 0.$$

An integral is a function of bounded variation. For let

$$F(x) = F(a) + \int_a^x f(t)\,dt,$$

and let $f_1(x) = f(x)$ where $f(x) \geqslant 0$, and $f_1(x) = 0$ elsewhere, and $-f_2(x) = f(x) - f_1(x)$. Then $f_1(x) \geqslant 0$, $f_2(x) \geqslant 0$, and

$$\begin{aligned} F(x) &= F(a) + \int_a^x f_1(t)\,dt - \int_a^x f_2(t)\,dt \\ &= F(a) + F_1(x) - F_2(x), \end{aligned}$$

where $F_1(x)$ and $F_2(x)$ are bounded non-decreasing functions.

11.51. Differentiation of the indefinite integral. Let $f(x)$ be integrable over (a, b), and let

$$F(x) = \int_a^x f(t)\,dt.$$

Since $F(x)$ is a function of bounded variation, it has a finite differential coefficient $F'(x)$ almost everywhere. Our next object is to prove that $F'(x) = f(x)$ almost everywhere.

11.52. The proof depends on the following lemma.

If $\phi(x)$ is integrable, and $\int_a^x \phi(t)\,dt = 0$ for all values of x in (a, b), then $\phi(x) = 0$ for almost all values of x in (a, b).

If this is not so, then either $\phi(x) > 0$ in a set of positive measure, or $\phi(x) < 0$ in a set of positive measure—suppose, for example, the former. Any set of positive measure contains a closed set of positive measure, since its complement can be included in an open set less than the whole interval. Hence $\phi(x) > 0$ in a closed set of positive measure—say E.

Now the integral of ϕ over any interval is zero; hence, by § 10.71, the integral over any open set is zero. Hence the integral over any closed set is zero, and in particular

$$\int_E \phi(x)\,dx = 0.$$

Hence, by § 10.73, $\phi(x) = 0$ almost everywhere in E, contrary to hypothesis. This proves the lemma.

11.53. *If $f(x)$ is bounded, and $F(x)$ is its integral, then $F'(x) = f(x)$ almost everywhere.*

Let $|f(x)| \leqslant M$. Then

$$\left|\frac{F(x+h)-F(x)}{h}\right| = \left|\frac{1}{h}\int_x^{x+h} f(t)\,dt\right| \leqslant M,$$

and
$$\lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = F'(x)$$

almost everywhere. Hence, by the theorem of bounded convergence,* as $h \to 0$,

$$\int_a^x \frac{F(t+h)-F(t)}{h}\,dt \to \int_a^x F'(t)\,dt.$$

But the left-hand side is equal to

$$\frac{1}{h}\int_{a+h}^{x+h} F(t)\,dt - \frac{1}{h}\int_a^x F(t)\,dt = \frac{1}{h}\int_x^{x+h} F(t)\,dt - \frac{1}{h}\int_a^{a+h} F(t)\,dt,$$

which tends to $F(x)-F(a)$, since F is continuous. Hence

$$\int_a^x F'(t)\,dt = F(x)-F(a), \tag{1}$$

i.e.
$$\int_a^x \{F'(t)-f(t)\}\,dt = 0, \tag{2}$$

for all values of x. The result now follows from the lemma.

11.54. To extend the theorem to unbounded functions, we require another lemma.

If $\phi(x)$ is continuous and non-decreasing in (a,b), then $\phi'(x)$ is integrable, and

$$\int_a^b \phi'(x)\,dx \leqslant \phi(b)-\phi(a).$$

For $\{\phi(x+h)-\phi(x)\}/h \geqslant 0$, and $\{\phi(x+h)-\phi(x)\}/h$ tends to

* To apply the theorem as given in § 10.5, we make $h \to 0$ through an enumerable sequence; so also in the next section.

$\phi'(x)$ almost everywhere as $h \to 0$. Hence, by Fatou's theorem (§ 10.81),

$$\varliminf_{h\to 0} \int_a^b \frac{\phi(x+h)-\phi(x)}{h}\,dx \geqslant \int_a^b \phi'(x)\,dx.$$

Also, since ϕ is continuous, the left-hand side is equal to $\phi(b)-\phi(a)$, as in the above proof. Hence the result.

11.55. *If $f(x)$ is any integrable function, $F'(x)=f(x)$ almost everywhere.*

We may as usual suppose that $f(x) \geqslant 0$. We define $\{f(x)\}_n$ as in § 10.7. Since $f(t)-\{f(t)\}_n \geqslant 0$, the function

$$\int_a^x [f(t)-\{f(t)\}_n]\,dt$$

is non-decreasing, so that its differential coefficient is never negative. Hence

$$\frac{d}{dx}\left\{\int_a^x f(t)\,dt\right\} \geqslant \frac{d}{dx}\left\{\int_a^x \{f(t)\}_n\,dt\right\}$$

wherever these differential coefficients exist. Hence, by the theorem for bounded functions, $F'(x) \geqslant \{f(x)\}_n$ almost everywhere. Making $n \to \infty$ we see that $F'(x) \geqslant f(x)$ almost everywhere. Hence

$$\int_a^b F'(x)\,dx \geqslant \int_a^b f(x)\,dx.$$

The above lemma, however, gives this inequality reversed. Hence in fact the two sides are equal, i.e.

$$\int_a^b \{F'(x)-f(x)\}\,dx = 0.$$

Since the integrand is never negative, it must be zero almost everywhere. This is the required result.

11.6. The Lebesgue set. The theorem that $F'(x)=f(x)$ almost everywhere was extended by Lebesgue as follows.

If $f(x)$ is integrable,

$$\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} |f(t)-\alpha|\,dt = |f(x)-\alpha|$$

for all values of α, except when x belongs to a set of measure zero; that is, $|f(x)-\alpha|$ is the derivative of its indefinite integral for all values of α and almost all values of x.

If α were fixed there would be nothing to prove, since $|f(x)-\alpha|$ is integrable, and the result follows from the above fundamental theorem.

Consider next all rational values of α, say $\alpha_1, \alpha_2, \ldots$. The sets in which the theorem is false for $\alpha_1, \alpha_2, \ldots$ are all of measure zero, and so their aggregate is of measure zero. Hence $|f(x)-\alpha|$ is the derivative of its integral for all rational values of α, except when x belongs to a set E of measure zero.

Now let x be a point not in E, α an irrational number, and β a rational number near to α. Since

$$\big||f(t)-\alpha|-|f(t)-\beta|\big| \leqslant |\beta-\alpha|.$$

we have

$$\left|\frac{1}{h}\int_x^{x+h} |f(t)-\alpha|\,dt - \frac{1}{h}\int_x^{x+h} |f(t)-\beta|\,dt\right| \leqslant |\beta-\alpha|.$$

But
$$\left|\frac{1}{h}\int_x^{x+h} |f(t)-\beta|\,dt - |f(x)-\beta|\right| \leqslant \epsilon$$

if $|h| < h_0(\beta, \epsilon)$. Hence

$$\begin{aligned}
&\left|\frac{1}{h}\int_x^{x+h} |f(t)-\alpha|\,dt - |f(x)-\alpha|\right| \\
&\qquad \leqslant \left|\frac{1}{h}\int_x^{x+h} |f(t)-\alpha|\,dt - \frac{1}{h}\int_x^{x+h} |f(t)-\beta|\,dt\right| + \\
&\qquad + \left|\frac{1}{h}\int_x^{x+h} |f(t)-\beta|\,dt - |f(x)-\beta|\right| + \big||f(x)-\beta|-|f(x)-\alpha|\big| \\
&\qquad \leqslant |\beta-\alpha|+\epsilon+|\beta-\alpha|,
\end{aligned}$$

which may be made as small as we please, by choice first of β and then of ϵ. Hence $|f(x)-\alpha|$ is also the derivative of its indefinite integral for all irrational α, if x is not a point of E. This proves the theorem.

We may, in particular, take $\alpha = f(x)$. Hence

$$\int_0^h |f(x+t)-f(x)|\,dt = o(h)$$

as $h \to 0$, for almost all values of x. The set where this holds is called the *Lebesgue set*.

All points of continuity are of course included in the Lebesgue set.

The interest of the Lebesgue set lies in the fact that many theorems which hold at all points of continuity are also found to hold at all points of the Lebesgue set, and so almost everywhere. We shall have examples of this in the chapter on Fourier series.

We note finally that if the modulus sign is omitted from the formula, the α disappears, and the result reduces to the previous theorem.

11.7. Absolutely continuous functions. *A function $f(x)$ is said to be absolutely continuous in an interval (a, b) if, given ϵ, we can find δ such that*

$$\sum_{\nu=1}^{n} |f(x_\nu+h_\nu)-f(x_\nu)| \leqslant \epsilon$$

for every set of non-overlapping intervals $(x_\nu, x_\nu+h_\nu)$ such that $\sum h_\nu \leqslant \delta$.

An absolutely continuous function is continuous, since we can take the above sum to consist of one term only.

An absolutely continuous function is of bounded variation, since its total variation over an interval of length δ is at most ϵ, and consequently its total variation over (a, b) is at most $(b-a)\epsilon/\delta$.

On the other hand, there are continuous functions of bounded variation which are not absolutely continuous. An example of such a function will be given in § 11.72.

11.71. *A necessary and sufficient condition that a function should be an integral is that it should be absolutely continuous.*

If $F(x)$ is the integral of $f(x)$,

$$\sum_{\nu=1}^{n} |F(x_\nu+h_\nu)-F(x_\nu)| \leqslant \sum_{\nu=1}^{n} \int_{x_\nu}^{x_\nu+h_\nu} |f(x)|\,dx = \int_E |f(x)|\,dx,$$

where E denotes the set of intervals $(x_\nu, x_\nu+h_\nu)$. The right-hand side tends to zero with $\sum h_\nu$, in the sense of the above definition, by § 10.73 (v). Hence $F(x)$ is absolutely continuous.

To prove the converse we require the following lemma.

If $\phi(x)$ is absolutely continuous in (a,b), and $\phi'(x)=0$ almost everywhere, then $\phi(x)$ is a constant.

Let E be the set where $\phi'(x)=0$. Every point x of E is the left-hand end-point of arbitrarily small intervals $(x, x+h)$, such that

$$|\phi(x+h)-\phi(x)|<\epsilon h.$$

By the lemmas of § 11.41, we can select a finite set S of these intervals which do not overlap, and which contain all E except a set of measure δ, and so all (a,b) except a set of measure δ.

Let $x_1, x_2, \ldots$ be the end-points of the intervals of S, and let $\sum_1$ denote a summation over the intervals of S, and $\sum_2$ over the complementary intervals. Then

$$|\phi(b)-\phi(a)| \leqslant \sum\nolimits_1 |\phi(x_{\nu+1})-\phi(x_\nu)| + \sum\nolimits_2 |\phi(x_{\nu+1})-\phi(x_\nu)|.$$

Now $\quad \sum_1 |\phi(x_{\nu+1})-\phi(x_\nu)| < \epsilon \sum_1 (x_{\nu+1}-x_\nu) < \epsilon(b-a).$

Also $\sum_2 (x_{\nu+1}-x_\nu) < \delta$, and so, by the property of absolute continuity,

$$\sum\nolimits_2 |\phi(x_{\nu+1})-\phi(x_\nu)|$$

tends to zero with δ. Hence, making $\delta \to 0$,

$$|\phi(b)-\phi(a)| \leqslant \epsilon(b-a).$$

Making $\epsilon \to 0$, it follows that $\phi(b)=\phi(a)$; and similarly $\phi(x)=\phi(a)$ for every value of x.

Suppose now that $F(x)$ is any absolutely continuous function. Then it is continuous and of bounded variation, and we may write

$$F(x)=F_1(x)-F_2(x),$$

where F_1 and F_2 are continuous non-decreasing functions. By the lemma of § 11.54, $F_1'(x)$ and $F_2'(x)$ are integrable, and hence so is $F'(x)$. Hence

$$\int_a^x F'(t)\,dt$$

is absolutely continuous, and so also is

$$\phi(x)=F(x)-\int_a^x F'(t)\,dt.$$

But $\phi'(x) = 0$ almost everywhere. Hence, by the lemma, $\phi(x)$ is a constant, i.e.

$$F(x) - \int_a^x F'(t)\,dt = F(a).$$

Thus $F(x)$ is the integral of $F'(x)$.

11.72. A continuous increasing function which is not an integral.* We can define a function of this type by means of Cantor's ternary set (§ 10.291).

Let a_n always take the values 0 or 2, and let $b_n = \frac{1}{2}a_n$, so that b_n is always 0 or 1. If

$$x = \cdot a_1 a_2 a_3 \ldots (3)$$

is a point of Cantor's set E, we define

$$f(x) = \cdot b_1 b_2 b_3 \ldots (2)$$

(in the scale of 2).

At the ends of an interval δ_{pk}, $f(x)$ therefore has the values

$$\cdot b_1 \ldots b_m 0111 \ldots (2), \qquad \cdot b_1 \ldots b_m 1000 \ldots (2),$$

and these are equal. We define $f(x)$ throughout the interval δ_{pk} to be equal to its value at the end-points.

The function $f(x)$ is non-decreasing. In proving this it is sufficient to consider points x of E, since $f(x)$ is constant in the intervals of CE. Let

$$x' = \cdot a_1' a_2' \ldots (3), \qquad x'' = \cdot a_1'' a_2'' \ldots (3)$$

be points of E, $x'' > x'$. Then there is a suffix n such that $a_m' = a_m''$ $(m < n)$, $a_n' < a_n''$. Hence

$$f(x') = \cdot b_1' \ldots b_{n-1}' b_n' \ldots (2) \leqslant \cdot b_1'' \ldots b_{n-1}'' b_n'' \ldots (2) = f(x'').$$

The function $f(x)$ is continuous. We have to prove that $f(x') \to f(x)$ as $x' \to x$, and again it is sufficient to consider points x, x' of E. Let

$$x = \cdot a_1 a_2 \ldots (3), \qquad x' = \cdot a_1' a_2' \ldots (3).$$

If now $x' \to x$, there will be a value of n, which tends to infinity as $x' \to x$, such that $a_m = a_m'$ $(m < n)$. Hence

$$f(x) - f(x') = \cdot 00 \ldots 0 b_n \ldots - \cdot 00 \ldots 0 b_n' \ldots \to 0.$$

On the other hand,

$$\int_0^1 f'(x)\,dx \neq f(1) - f(0).$$

* A detailed discussion of this function is given by Hille and Tamarkin (1),

For the right-hand side is 1, since $f(1) = \cdot 111\ldots(2) = 1, f(0) = 0$; but $f(x)$ is constant in the intervals δ_{pk}, so that $f'(x) = 0$ in the interior of any of these intervals. Hence $f'(x) = 0$ almost everywhere, and the left-hand side is 0.

It follows that $f(x)$ is not the integral of its differential coefficient, and so is not absolutely continuous. It is easy to see this directly. Consider the sum

$$\sum |f(\beta_k) - f(\alpha_k)|$$

taken over the intervals (α_k, β_k) which remain after the pth step of removing intervals δ_{pk}. It is equal to

$$\sum \{f(\beta_k) - f(\alpha_k)\} = f(1) - f(0) = 1.$$

But $$\sum (\beta_k - \alpha_k) = 1 - \frac{1}{3} - \frac{2}{9} - \ldots - \frac{2^{p-1}}{3^p} = \left(\frac{2}{3}\right)^p,$$

which tends to zero as $p \to \infty$. Hence $f(x)$ is not absolutely continuous.

11.8. Integration of a differential coefficient. If $f(x)$ has a differential coefficient almost everywhere, or even everywhere, in an interval (a, b), the formula

$$\int_a^x f'(t)\, dt = f(x) - f(a) \qquad (a \leqslant x \leqslant b) \tag{1}$$

is not necessarily true. It may fail in one or other of two ways. Consider, for example, the function

$$f(x) = x^2 \sin \frac{1}{x^2} \quad (x > 0), \qquad f(0) = 0,$$

already referred to in § 10.7. Here

$$f'(x) = 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2} \quad (x > 0), \qquad f'(0) = 0,$$

so that $f'(x)$ exists everywhere; but, as we saw in § 10.7, it is not integrable in the Lebesgue sense, so that (1), on the Lebesgue theory, has no meaning.

If we can imagine a function with this kind of singularity distributed everywhere in an interval, we shall obtain some idea of the nature of the problem of integrating a differential coefficient. The problem has been solved by means of the Denjoy integral. This is a highly general type of non-absolutely convergent integral, and it would take us too far to discuss its

properties here. The result is that, *if $f'(x)$ exists everywhere, the formula* (1) *is true, the integral being a Denjoy integral.*

If we do not assume that $f'(x)$ exists everywhere, but merely almost everywhere, the formula (1) may break down still more completely. The integral on the left may exist as a Lebesgue integral, but be unequal to the right-hand side. We have already had an example of this in § 11.72—in fact an example where $f'(x) = 0$ almost everywhere, without $f(x)$ being a constant.

In order to obtain the formula (1), the integral being a Lebesgue integral, we have therefore to impose further conditions on $f(x)$ or on $f'(x)$. There are several theorems, varying in difficulty according to what is assumed. Their common feature is that we suppose that $f'(x)$ exists everywhere. The example of § 11.72 shows that no set of conditions which is merely given almost everywhere is sufficient.

11.81. *If $f'(x)$ exists everywhere and is bounded, then* 11.8 (1) *is true.*

If $|f'(x)| \leqslant M$, then (*P.M.* § 125) there is a number θ between 0 and 1 such that

$$\left|\frac{f(x+h)-f(x)}{h}\right| = |f'(x+\theta h)| \leqslant M. \tag{1}$$

Hence $\{f(x+h)-f(x)\}/h$ converges boundedly to $f'(x)$, and the proof is now the same as that of 11.53 (1) (with $f(x)$ instead of $F(x)$).

Alternatively, we may observe that it follows from (1) that

$$\sum_{\nu=1}^{n} |f(x_\nu+h_\nu)-f(x_\nu)| \leqslant M \sum_{\nu=1}^{n} h_\nu.$$

Hence $f(x)$ is absolutely continuous, and the required result follows from § 11.71.

11.83. *If $f(x)$ is any function such that $f'(x)$ is finite everywhere and is integrable, then* 11.8 (1) *is true.*

This evidently shows in particular that 11.8 (1) holds if $f(x)$ is of bounded variation and $f'(x)$ is finite everywhere; for if $f(x)$ is of bounded variation, $f'(x)$ is integrable (see § 11.54 and

example 12 below).

The following proof is substantially that given by Schlesinger and Plessner.† It depends on the two following lemmas.

LEMMA 1. *Let E be any set in (a,b) of measure zero, ϵ a given positive number. Then there is a non-decreasing absolutely continuous function $\chi(x)$ such that $\chi'(x)=+\infty$ in E, and $\chi(b)-\chi(a)<\epsilon$.*

We can include E in a sequence of open sets $O_1 > O_2 > \ldots$ such that $m(O_n)<\epsilon_n$, $\epsilon_1+\epsilon_2+\ldots=\epsilon$. Let $f_n(x)$ be the characteristic function of the set O_n. Then

$$\int_a^b f_n(t)\,dt = m(O_n) < \epsilon_n.$$

Let
$$\phi_n(x) = f_1(x)+f_2(x)+\ldots+f_n(x).$$

Then $\phi_n(t)$ is non-decreasing as $n\to\infty$ for every t, and

$$\int_a^x \phi_n(t)\,dt < \epsilon_1+\epsilon_2+\ldots+\epsilon_n \leqslant \epsilon.$$

Hence by § 10.82 $\phi_n(t)$ tends to a finite limit $\phi(t)$ almost everywhere, and

$$\lim_{n\to\infty}\int_a^x \phi_n(t)\,dt = \int_a^x \phi(t)\,dt = \chi(x),$$

say.

This function $\chi(x)$ has the required properties. Since it is the integral of a non-negative function it is non-decreasing and absolutely continuous, and

$$\chi(b)-\chi(a) = \int_a^b \phi(t)\,dt < \epsilon.$$

Also
$$\frac{d}{dx}\int_a^x f_\nu(t)\,dt = 1$$

in O_ν, and so, if $\chi_n(x) = \int_a^x \phi_n(t)\,dt$,

$$\chi_n'(x) = \sum_{\nu=1}^{n}\frac{d}{dx}\int_a^x f_\nu(t)\,dt = n$$

† *Lebesguesche Integrale*, pp. 166–74.

in O_n. Hence

$$\frac{\chi(x+h)-\chi(x)}{h} \geqslant \frac{\chi_n(x+h)-\chi_n(x)}{h} > n-\delta$$

for $|h| < h_0(\delta)$ and x in O_n. Hence $D\chi \geqslant n$ for each of the four derivates and x in O_n. Since a point of E belongs to O_n for every n, it follows that $\chi'(x) = +\infty$ in E.

LEMMA 2. *If $f(x)$ is continuous in (a, b), and $D^+f \geqslant 0$ almost everywhere in the interval, and D^+f is nowhere $-\infty$, then $f(x)$ is a non-decreasing function.*

It is sufficient to prove that $f(b) \geqslant f(a)$, since the general result then follows by a similar argument.

Let E be the set of measure zero where $D^+f < 0$. By Lemma 1 there is an absolutely continuous function $\chi(x)$ such that $\chi'(x) = +\infty$ in E, and $\chi(b)-\chi(a) < \epsilon$.

Let $$g(x) = f(x)+\chi(x).$$

Then in E, $D^+g = +\infty$, since $D_+\chi = +\infty$ and D^+f is finite, and $D^+g \geqslant D_+\chi + D^+f$. Also in CE

$$D^+g \geqslant D^+f \geqslant 0$$

since χ is non-decreasing. Hence $D^+g \geqslant 0$ everywhere, and so, by § 11.3, ex. (iv), $g(b) \geqslant g(a)$. Hence

$$f(b)-f(a) \geqslant -\{\chi(b)-\chi(a)\} > -\epsilon,$$

and, making $\epsilon \to 0$, the result follows.

11.84. We can now prove the theorem stated in § 11.83. Let n be any positive number, and let

$$g_n(x) = \min\{f'(x), n\}, \qquad G_n(x) = \max\{f'(x), -n\}.$$

Then $g_n(x) \leqslant f'(x) \leqslant G_n(x)$, and, since $f'(x)$ is integrable, so are $g_n(x)$ and $G_n(x)$. Let

$$f_n(x) = \int_a^x g_n(t)\,dt, \qquad F_n(x) = \int_a^x G_n(t)\,dt.$$

Then $$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} F_n(x) = \int_a^x f'(t)\,dt = \phi(x),$$

say. Now $$D^+\{F_n(x)-f(x)\} \geqslant D^+F_n - D^+f.$$

This is almost everywhere equal to $G_n(x)-f'(x)$, i.e.

$$D^+\{F_n(x)-f(x)\} \geqslant 0$$

almost everywhere. Also

$$\frac{F_n(x+h)-F_n(x)}{h} \geqslant \frac{1}{h}\int_x^{x+h} (-n)\,dt = -n,$$

so that $D^+F_n \geqslant -n$, and so $D^+(F_n-f)$ is nowhere $-\infty$. Hence, by Lemma 2, $F_n(x)-f(x)$ is non-decreasing, i.e.

$$F_n(x)-f(x) \geqslant F_n(a)-f(a) = -f(a).$$

Making $n \to \infty$, we obtain

$$\phi(x) \geqslant f(x)-f(a).$$

A similar argument with $f_n(x)$ gives the reversed inequality, and this proves the theorem.

MISCELLANEOUS EXAMPLES

1. For $x = \frac{1}{3}$ the function $f(x)$ of § 11.23 has the derivative $+\infty$.

2. The density of a set E at a point x may be defined as

$$\lim_{h\to 0} \frac{m(EH)}{2h},$$

where H is the interval $(x-h, x+h)$.

Prove that the density of a set is 1 almost everywhere in the set, and 0 almost everywhere outside it.

[Consider the integral of the characteristic function of E.]

3. A set E in $(0, 1)$ is such that, if (α, β) is any interval, then

$$m\{E(\alpha, \beta)\} \geqslant \delta(\beta-\alpha)$$

where $\delta > 0$. Show that $m(E) = 1$.

4. If, as $h \to 0$,

$$\int_a^b |f(x+h)-f(x)|\,dx = o(h),$$

then $f(x)$ is almost everywhere equal to a constant.

[Consider $\int_{x_1}^{x_2} \{f(x+h)-f(x)\}\,dx$. See Titchmarsh (7), where, however, the proof is unnecessarily complicated.]

5. Let α and β be positive numbers, $f(x) = x^\alpha \sin x^{-\beta}$ $(0 < x \leqslant 1)$, and $f(0) = 0$. Then $f(x)$ is of bounded variation in $(0, 1)$ if $\alpha > \beta$, but not if $\alpha \leqslant \beta$.

6. A function $f(x)$, defined for $0 \leqslant x < 1$, is absolutely continuous in every interval $(0, \xi)$, where $\xi < 1$, and its total variation in $(0, \xi)$ is bounded as $\xi \to 1$. Show that $f(x)$ tends to a limit as $\xi \to 1$, and that, if we define $f(1)$ to be equal to this limit, then $f(x)$ is absolutely continuous in the whole interval $(0, 1)$.

[The point of this example is that the difference between 'continuity plus bounded variation' and absolute continuity is a property of a whole interval, and cannot be traced to the behaviour of the function in the neighbourhood of any one point.]

7. The theorem of § 11.83 remains true if $f'(x) = +\infty$ in an enumerable set.

8. A necessary and sufficient condition that a function should be convex in an interval (a, b), in the sense of § 5.31, is that it should be the integral of a bounded increasing function over any interval interior to (a, b).

9. If $f(x)$ is absolutely continuous, so is $|f(x)|^p$, where $p \geqslant 1$.

10. A necessary and sufficient condition that $f(x)$ should be almost everywhere equal to a function of bounded variation in (a, b) is that as $h \to 0$

$$\int_a^b |f(x+h)-f(x)|\,dx = O(h)$$

[where $f(x) = 0$, say, outside (a, b)].*

[If $f(x)$ is of bounded variation, we have $f(x) = \phi(x)-\psi(x)$, where ϕ and ψ are positive, non-decreasing and bounded in (a, b). Then, if $h > 0$,

$$\int_a^b |f(x+h)-f(x)|\,dx \leqslant \int_a^b \{\phi(x+h)-\phi(x)\}\,dx + \int_a^b \{\psi(x+h)-\psi(x)\}\,dx$$

$$= \int_b^{b+h} \phi(t)\,dt - \int_a^{a+h} \phi(t)\,dt + \int_b^{b+h} \psi(t)\,dt - \int_a^{a+h} \psi(t)\,dt = O(h),$$

so that the condition is necessary.

Suppose now that the condition is satisfied. Let

$$\phi_n(x) = n \int_x^{x+1/n} f(t)\,dt.$$

Then

$$\int_a^b |\phi_n(x+h)-\phi_n(x)|\,dx = n \int_a^b dx \left| \int_{x+h}^{x+h+1/n} - \int_x^{x+1/n} f(t)\,dt \right|$$

$$= n \int_a^b dx \left| \int_0^{1/n} \{f(x+t+h)-f(x+t)\}\,dt \right|$$

$$\leqslant n \int_a^b dx \int_0^{1/n} |f(x+t+h)-f(x+t)|\,dt$$

$$= n \int_0^{1/n} dt \int_a^b |f(x+t+h)-f(x+t)|\,dx = O(h).$$

* Hardy and Littlewood (5), pp. 599–601, and (6), p. 619.

If $(x_\nu, x_\nu+h_\nu)$ is any set of non-overlapping intervals,

$$\sum |\phi_n(x_\nu+h_\nu)-\phi_n(x_\nu)| = \sum \left| \int_{x_\nu}^{x_\nu+h_\nu} \phi_n'(x)\,dx \right|$$

$$\leqslant \sum \int_{x_\nu}^{x_\nu+h_\nu} |\phi_n'(x)|\,dx \leqslant \int_a^b |\phi_n'(x)|\,dx,$$

and, by Fatou's lemma and the above result,

$$\int_a^b |\phi_n'(x)|\,dx \leqslant \underline{\lim} \int_a^b \left| \frac{\phi_n(x+h)-\phi_n(x)}{h} \right| dx = O(1).$$

Hence $$\sum |\phi_n(x_\nu+h_\nu)-\phi_n(x_\nu)| = O(1).$$

But $\phi_n(x) \to f(x)$ almost everywhere. Hence

$$\sum |f(x_\nu+h_\nu)-f(x_\nu)| < A$$

if none of the points x_ν, $x_\nu+h_\nu$ belong to a certain set E of measure zero. If a does not belong to E, it follows as in § 11.4 that $f(x) = f(a)+P(x)-N(x)$ in CE, where $P(x)$ and $N(x)$ are bounded and non-decreasing in CE. In E we can define $P(x)$ as $\lim P(x')$, where $x' \to x$ from below through CE. The result follows without difficulty from this.]

11. In § 11.4 the existence of $f'(x)$ at a point does not imply that of $V'(x)$.

[Consider $f(x) = x^2 \cos x^{-\alpha}$ $(0 < x \leqslant 1)$, $f(0) = 0$, $1 < \alpha < 2$.]

12. In § 11.54 the condition that $\phi(x)$ is continuous can be omitted.

[The proof shows that if α and β are any two points of continuity

$$\int_\alpha^\beta \phi'(x)\,dx \leqslant \phi(\beta)-\phi(\alpha).$$

But for any non-decreasing function points of continuity are everywhere dense. Hence, making $\alpha \to a+0$, $\beta \to b-0$, through such points, we obtain

$$\int_a^b \phi'(x)\,dx \leqslant \phi(b-0)-\phi(a+0).]$$

13. The set consisting of the intervals $\left(\frac{1}{2n+1}, \frac{1}{2n}\right)$, $n = 1, 2, \ldots$, has density $\frac{1}{4}$ at $x = 0$.

14. A convergent series of non-decreasing functions can be differentiated term by term almost everywhere.

Fubini: see Rajchman and Saks (1).

[Let $$u_1(x)+u_2(x)+\ldots+u_n(x) = s_n(x) \to s(x) \quad (a \leqslant x \leqslant b).$$

Then $s(x)$ is non-decreasing; and

$$\frac{s(x+h)-s(x)}{h} = \sum_{n=1}^{\infty} \frac{u_n(x+h)-u_n(x)}{h} \geqslant \sum_{n=1}^{N} \frac{u_n(x+h)-u_n(x)}{h}$$

for every N. Making $h \to 0$, it follows that

$$s'(x) \geqslant \sum_{n=1}^{N} u_n'(x)$$

almost everywhere. Hence $\sum u_n'(x)$ converges almost everywhere, to $\phi(x)$ say, and $\phi(x) \leqslant s'(x)$.

Suppose that the set $E(u, v)$, where $\phi(x) < u < v < s'(x)$, has positive measure μ. Almost everywhere in $E(u, v)$

$$s_n'(x) < u < v < s'(x),$$

so that $$s_n(x+h)-s_n(x) < hu < hv < s(x+h)-s(x)$$

for sufficiently small h. This holds over a finite non-overlapping set of intervals of total length $l > \frac{1}{2}\mu > 0$. Summing over these intervals

$$\begin{aligned} l(v-u) &< \sum \{s(x+h)-s_n(x+h)\}-\{s(x)-s_n(x)\} \\ &\leqslant \{s(b)-s_n(b)\}-\{s(a)-s_n(a)\} \end{aligned}$$

since $s(x)-s_n(x)$ is non-decreasing. Making $n \to \infty$, $l \leqslant 0$, a contradiction. Hence $\phi(x) = s'(x)$ almost everywhere.]

15. If $f'(x)$ is finite everywhere, and equal to a continuous function almost everywhere, it is equal to it everywhere.

16. Show that $$\lim_{\delta \to 0} \int_{\delta}^{1} \frac{f(x+t)-f(x-t)}{t}\, dt$$

exists for every x if $f(x)$ is Weierstrass's non-differentiable function.

Show that the limit does not exist at $x = 0$ if $f(x)$ is the continuous function 0 $(x \leqslant 0)$, $1/\log(1/x)$ $(x > 0)$.

[This limit exists almost everywhere if $f(x)$ is any integrable function. See Titchmarsh, *Fourier Integrals*, Theorem 105.]

CHAPTER XII

FURTHER THEOREMS ON LEBESGUE INTEGRATION

12.1. In this chapter we adopt a slightly more practical point of view than in the two preceding ones. We have carried the general theory of definite and indefinite integrals as far as we shall require it, and we shall now prove a number of theorems which are useful in the manipulation of integrals.

12.11. Integration by parts. The formula of integration by parts in the Lebesgue theory is, of course, the same as the ordinary one: if $G(x)$ is an indefinite integral of $g(x)$, then

$$\int_a^b f(x)\,g(x)\,dx = [f(x)G(x)]_a^b - \int_a^b f'(x)G(x)\,dx.$$

The formula holds if $g(x)$ is any integrable function, and $f(x)$ is an integral.

The proof depends on the fact that *the product of two absolutely continuous functions is absolutely continuous.* For let $\phi(x)$ and $\psi(x)$ be absolutely continuous in (a, b), and let M and M' be the upper bounds of $|\phi(x)|$ and $|\psi(x)|$. Let $(x_\nu, x_\nu+h_\nu)$ be a set of non-overlapping intervals in (a, b). Then

$$\begin{aligned}\sum |\phi(x_\nu+h_\nu)\psi(x_\nu+h_\nu)-\phi(x_\nu)\psi(x_\nu)| \\ = \sum |\phi(x_\nu+h_\nu)\{\psi(x_\nu+h_\nu)-\psi(x_\nu)\}+\psi(x_\nu)\{\phi(x_\nu+h_\nu)-\phi(x_\nu)\}| \\ \leqslant M \sum |\psi(x_\nu+h_\nu)-\psi(x_\nu)|+M' \sum |\phi(x_\nu+h_\nu)-\phi(x_\nu)|.\end{aligned}$$

The last two sums tend to zero with $\sum h_\nu$, and so $\phi(x)\psi(x)$ is absolutely continuous.

In the given formula, $f(x)$ and $G(x)$ are absolutely continuous, and hence so is $f(x)G(x)$; and

$$\int_a^b \frac{d}{dx}\{f(x)G(x)\}\,dx = [f(x)G(x)]_a^b.$$

But
$$\frac{d}{dx}\{f(x)G(x)\} = f'(x)G(x)+f(x)g(x)$$

wherever $f'(x)$ and $G'(x)$ exist, and $G'(x) = g(x)$. Since this is true almost everywhere the result follows.

12.2. Approximation to an integrable function. The following theorem is often useful.

If $f(x)$ is measurable over a finite interval, then, given two positive numbers δ and ϵ, we can define an absolutely continuous function $\phi(x)$ such that $|f-\phi|<\delta$ except in a set of measure less than ϵ.

Suppose first that $f(x)$ is bounded. We may suppose without loss of generality that $f(x)\geqslant 0$. Divide up the interval of variation of $f(x)$ by the scale

$$0,\ \delta,\ 2\delta,\ldots,\ n\delta.$$

Let e_ν be the set where $\nu\delta\leqslant f(x)<(\nu+1)\delta$. Let $\psi_\nu(x)=\nu\delta$ in e_ν, and zero elsewhere. Then the function

$$\psi(x)=\psi_0(x)+\ldots+\psi_{n-1}(x)$$

differs from $f(x)$ by less than δ.

Let E_ν be an open set, including e_ν, of measure less than $m(e_\nu)+\epsilon/3n$. Let S_ν be the sum of a finite number of the intervals of E_ν, such that $m(E_\nu-S_\nu)<\epsilon/3n$. Let $\phi_\nu(x)=\nu\delta$ in S_ν, and zero elsewhere. Then $\phi_\nu=\psi_\nu$ except in a set of measure less than $2\epsilon/3n$; also ϕ_ν is discontinuous at a finite number of points, viz. the ends of the intervals of S_ν. To remove these discontinuities, we join the graph of the function to zero at the end of each interval by a straight line inclined so that the modifications all occur in a set of measure less than $\epsilon/3n$. Thus if ϕ'_ν is the modified function, ϕ'_ν is absolutely continuous, and $\phi'_\nu=\psi_\nu$ except in a set of measure ϵ/n.

Let
$$\phi(x)=\phi'_0+\phi'_1+\ldots+\phi'_{n-1}.$$

Then $\phi(x)$ is absolutely continuous, and $\phi(x)=\psi(x)$ except in a set of measure ϵ. Hence $\phi(x)$ has the required property.

If $f(x)$ is not bounded, let $\{f(x)\}_k=f(x)$ where $|f(x)|\leqslant k$, and $\{f(x)\}_k=0$ elsewhere. We can take k so large that $\{f(x)\}_k=f(x)$ except in a set of measure $\frac{1}{2}\epsilon$. By the first part, we can determine $\phi(x)$ so that $|\{f(x)\}_k-\phi(x)|<\delta$ except in a set of measure $\frac{1}{2}\epsilon$. Then $\phi(x)$ has the required property.

Notice that, if $f(x)$ is bounded, $\phi(x)$ can be constructed to lie between the same bounds as $f(x)$.

If $f(x)$ is integrable, we can construct $\phi(x)$ so that, in addition to the above properties,

$$\int_a^b |f(x)-\phi(x)|\,dx<\eta, \tag{1}$$

where η is arbitrarily small. If $f(x)$ is bounded, say $|f(x)| \leqslant M$, then $|\phi(x)| \leqslant M$, and

$$\int_a^b |f(x)-\phi(x)|\,dx \leqslant \delta(b-a)+2\epsilon M,$$

giving the required result. If $f(x)$ is unbounded, we define $\{f(x)\}_k$ as above, and then determine $\phi(x)$ so that $|\{f(x)\}_k-\phi(x)| < \delta$ except in a set of measure $\frac{1}{2}\epsilon/k$. Then

$$\int_a^b |f(x)-\phi(x)|\,dx \leqslant \int_a^b |f(x)-f\{(x)\}_k|\,dx + \int_a^b |\{f(x)\}_k-\phi(x)|\,dx.$$

The first term tends to zero as $k \to \infty$, and the second term does not exceed $\delta(b-a)+\epsilon$. Hence the result.

Example. If $f(x)$ is integrable over $(a-\epsilon, b+\epsilon)$, then

$$\lim_{h\to 0}\int_a^b |f(x+h)-f(x)|\,dx = 0.$$

12.21. Change of the independent variable. Here again the formula is familiar, but the conditions under which it holds are novel.

If $f(x)$ and $g(x)$ are integrable, $g(x) \geqslant 0$, and $G(x)$ is an indefinite integral of $g(x)$, $a = G(\alpha)$, $b = G(\beta)$, then

$$\int_a^b f(t)\,dt = \int_\alpha^\beta f\{G(x)\}g(x)\,dx,$$

where $f\{G(x)\}g(x)$ is defined as 0 if $g(x) = 0$.

The inverse function of $t = G(x)$, of which α and β are values, is not necessarily one-valued, since $G(x)$ may be constant in some intervals. But if more than one value of x corresponds to a given value of t, these values of x form a closed interval, and we can make the inverse function one-valued by taking x to be, say, the left-hand end-point of the interval.

We next observe that *if $F(x)$ and $G(x)$ are absolutely continuous functions, and $G(x)$ is monotonic, then $F\{G(x)\}$ is absolutely continuous.* For, since F is absolutely continuous,

$$\sum |F\{G(x_\nu+h_\nu)\}-F\{G(x_\nu)\}|$$

tends to zero with

$$\sum |G(x_\nu+h_\nu)-G(x_\nu)|,$$

and, since $G(x)$ is absolutely continuous, this tends to zero with $\sum h_\nu$.

It follows that, if $F(x)$ and $G(x)$ are integrals of $f(x)$ and $g(x)$, then $F\{G(x)\}$ has a finite differential coefficient for almost all values of x, and

$$\int_\alpha^\beta \frac{d}{dx}[F\{G(x)\}]\,dx = F\{G(\beta)\}-F\{G(\alpha)\} = \int_a^b f(t)\,dt.$$

The result will now follow if

$$\frac{d}{dx}[F\{G(x)\}] = f\{G(x)\}g(x) \tag{1}$$

for almost all values of x. But this is not obviously true. For

$$\frac{F\{G(x+h)\}-F\{G(x)\}}{h} = \frac{F\{G(x+h)\}-F\{G(x)\}}{G(x+h)-G(x)}\cdot\frac{G(x+h)-G(x)}{h},$$

and the second factor on the right tends to $g(x)$ for almost all values of x, while the first factor tends to $f\{G(x)\}$ for almost all values of $G(x)$; and the difficulty is that the exceptional set of values of $G(x)$, of measure zero, does not necessarily correspond to a set of values of x of measure zero.

Let $f(x)$ be bounded, say $|f(x)| \leqslant M$. Divide the interval (α, β) into sets $E_1,\ldots, E_4$ as follows. In E_1, $G'(x) = g(x) > 0$, and the first factor on the right tends to $f\{G(x)\}$; in E_2,

$$G'(x) = g(x) > 0$$

but the other condition is negatived; in E_3, $G'(x) = g(x)=0$; in E_4, $G'(x) \neq g(x)$. Clearly (1) holds in E_1; and it holds in E_3, since there

$$\left|\frac{F\{G(x+h)\}-F\{G(x)\}}{h}\right|$$

$$= \left|\frac{1}{h}\int_{G(x)}^{G(x+h)} f(t)\,dt\right| \leqslant M\left|\frac{G(x+h)-G(x)}{h}\right| \to 0$$

and each side of (1) is zero; $m(E_4) = 0$; and we have to prove that $m(E_2) = 0$.

Let $E_{2,n}$ be the part of E_2 in which $G'(x) > 1/n$. Enclose the corresponding t-set in an open set O of measure less than a given ϵ. With each x of $E_{2,n}$ associate an interval $(x, x+h_x)$ such that $G(x+h_x)-G(x) > h_x/n$, and such that the interval $G(x)$, $G(x+h_x)$ is in O. By Lemma 1 of § 11.41, there is a finite non-overlapping set S of the intervals $(x, x+h_x)$ such that

$$m_e(E_{2,n}) < m(S)+\epsilon = \sum_S h_x+\epsilon.$$

This is less than

$$n \sum_S \{G(x+h_x)-G(x)\}+\epsilon \leqslant nm(O)+\epsilon < (n+1)\epsilon.$$

Hence $m(E_{2,n}) = 0$, and, since E_2 is the outer limiting set of the sets $E_{2,n}$, $m(E_2) = 0$.

Lastly let $f(x)$ be any integrable function. We may suppose without loss of generality that it is positive. Defining $\{f(x)\}_n$ in the usual way, the theorem holds for $\{f(x)\}_n$, and it is sufficient to prove that

$$\lim \int_\alpha^\beta [f\{G(x)\}]_n g(x)\, dx = \int_\alpha^\beta f\{G(x)\} g(x)\, dx.$$

But $$\int_\alpha^\beta [f\{G(x)\}]_n g(x)\, dx = \int_a^b \{f(t)\}_n\, dt \leqslant \int_a^b f(t)\, dt.$$

The result therefore follows from the convergence theorem of § 10.82 (regarding $f\{G(x)\}g(x)$ as 0 if $f\{G(x)\} = \infty$, $g(x) = 0$).

12.3. The second mean-value theorem. *If $f(x)$ is integrable over (a, b), and $\phi(x)$ is positive, bounded, and non-increasing, then*

$$\int_a^b f(x)\phi(x)\, dx = \phi(a+0) \int_a^\xi f(x)\, dx,$$

where ξ is some number between a and b.

Let ϵ be a positive number less than $\phi(a+0)-\phi(b-0)$. Then there is a point x_1 such that

$$\begin{aligned} \phi(a+0)-\phi(x) &< \epsilon \qquad (a < x < x_1) \\ &\geqslant \epsilon \qquad (x > x_1). \end{aligned}$$

Similarly there are points $x_2, x_3, \ldots$ such that

$$\begin{aligned} \phi(x_{\nu-1}+0)-\phi(x) &< \epsilon \qquad (x_{\nu-1} < x < x_\nu) \\ &\geqslant \epsilon \qquad (x > x_\nu), \end{aligned}$$

so long as $\phi(x_{\nu-1}+0)-\phi(b-0) > \epsilon$. Otherwise we take $x_n = b$. The point b is thus reached in a finite number of steps, since the variation of $\phi(x)$ in each interval $(x_{\nu-1}, x_\nu)$ is at least ϵ.

Let $\psi(x) = \phi(x_\nu+0)$ in each interval $x_\nu \leqslant x < x_{\nu+1}$. Then $0 \leqslant \psi(x)-\phi(x) < \epsilon$ except possibly at the points $a = x_0$, x_1, $x_2, \ldots, b$, and

$$\int_a^b \psi(x)f(x)\,dx = \sum_{\nu=0}^{n-1} \phi(x_\nu+0) \int_{x_\nu}^{x_{\nu+1}} f(x)\,dx.$$

Let $F(x) = \int_a^x f(t)\,dt$; then, if m and M are the lower and upper bounds of $F(x)$, it follows from Abel's lemma (§ 1.131) that

$$m\phi(a+0) \leqslant \int_a^b \psi(x)f(x)\,dx \leqslant M\phi(a+0).$$

But

$$\left| \int_a^b \psi(x)f(x)\,dx - \int_a^b \phi(x)f(x)\,dx \right| \leqslant \epsilon \int_a^b |f(x)|\,dx,$$

which tends to zero with ϵ. Hence, making $\epsilon \to 0$, it follows that

$$m\phi(a+0) \leqslant \int_a^b \phi(x)f(x)\,dx \leqslant M\phi(a+0).$$

Since $F(x)$ is continuous, it takes every value between m and M, and so, at $x = \xi$ say, the value

$$\frac{1}{\phi(a+0)} \int_a^b \phi(x)f(x)\,dx.$$

This proves the theorem.

If $\phi(x)$ is positive and non-decreasing, the corresponding formula is

$$\int_a^b f(x)\phi(x)\,dx = \phi(b-0) \int_\xi^b f(x)\,dx,$$

where $a < \xi < b$.

If $\phi(x)$ is any monotonic function, there is a number ξ between a and b such that

$$\int_a^b f(x)\phi(x)\,dx = \phi(a+0)\int_a^\xi f(x)\,dx + \phi(b-0)\int_\xi^b f(x)\,dx.$$

This is obtained from the previous results by considering $\phi(x)-\phi(a+0)$ or $\phi(x)-\phi(b-0)$.

12.4. The Lebesgue class* L^p. We denote by $L^p(a,b)$ the class of functions $f(x)$ such that $f(x)$ is measurable, and $|f(x)|^p$, where $p>0$, is integrable over (a,b). If it is not necessary to specify the interval, we denote the class by L^p simply. The class L^1 is the class of functions integrable over (a,b), and is denoted simply by L.

We may classify functions defined over any set, or over an infinite interval, in the same way; for example, the function $(1+x)^{-\frac{1}{2}}$ belongs to $L^p(0,\infty)$ if $p>2$.

If $f(x)$ belongs to L^p, and $|g(x)|\leqslant|f(x)|$, then clearly $g(x)$ also belongs to L^p.

Examples. (i) A bounded function belongs to $L^p(a, b)$, where (a, b) is a finite interval, for all values of p.

(ii) If $f(x)$ belongs to $L^p(a, b)$, where (a, b) is a finite interval, then it also belongs to $L^q(a, b)$ for $q<p$.

(iii) If $f(x)$ belongs to $L^p(0, \infty)$ and to $L^q(0, \infty)$, where $p<q$, then it also belongs to $L^r(0, \infty)$ if $p<r<q$.

[Consider separately the sets where $|f(x)|\leqslant 1$ and $|f(x)|>1$.]

(iv) The sum of two functions of L^p also belongs to L^p.

[For $|f(x)+g(x)|^p\leqslant\max\{2^p|f(x)|^p, 2^p|g(x)|^p\}$.]

(v) The function $\{x\log^2 1/x\}^{-1}$ belongs to $L(0, \frac{1}{2})$, but not to any $L^p(0, \frac{1}{2})$ for $p>1$.

(vi) The function $\{x^{\frac{1}{2}}(1+|\log x|)\}^{-1}$ belongs to $L^2(0, \infty)$, but not to $L^p(0, \infty)$ for any other value of p.

12.41. Schwarz's inequality. *If $f(x)$ and $g(x)$ belong to L^2, then $f(x)g(x)$ belongs to L, and*

$$\left|\int f(x)g(x)\,dx\right|\leqslant\left\{\int|f(x)|^2\,dx\int|g(x)|^2\,dx\right\}^{\frac{1}{2}}.$$

The interval of integration may be finite or infinite.

* See in particular F. Riesz (2).

Since $2|fg| \leqslant f^2+g^2$, fg belongs to L. Hence the integral

$$\int \{\lambda f(x)+\mu g(x)\}^2\,dx$$
$$=\lambda^2 \int \{f(x)\}^2\,dx+2\lambda\mu \int f(x)g(x)\,dx+\mu^2 \int \{g(x)\}^2\,dx$$

exists for all values of λ and μ. It is evidently never negative. But the necessary and sufficient condition that $a\lambda^2+2h\lambda\mu+b\mu^2$ should be never negative is that $h^2 \leqslant ab$, $a \geqslant 0$, $b \geqslant 0$; and this gives the inequality stated.

Examples. (i) The case of equality in the above theorem occurs only if $f(x)/g(x)$ is almost everywhere equal to a constant.

(ii) If $f(x)$ and $g(x)$ belong to L^p, where $p > 2$, then $f(x)g(x)$ belongs to $L^{\frac{1}{2}p}$.

12.42. Hölder's inequality. This is a generalization of Schwarz's inequality.

If $f(x)$ belongs to L^p, and $g(x)$ to $L^{p/(p-1)}$, where $p > 1$, then $f(x)g(x)$ belongs to L, and

$$\left|\int f(x)g(x)\,dx\right| \leqslant \left\{\int |f(x)|^p\,dx\right\}^{1/p}\left\{\int |g(x)|^{p/(p-1)}\,dx\right\}^{1-1/p}. \quad (1)$$

The interval of integration may be finite or infinite.

Let E be the set where $|g(x)| \leqslant |f(x)|^{p-1}$. Then

$$|f(x)g(x)| \leqslant |f(x)|^p$$

in E; hence $f(x)g(x)$ is integrable over E. In the complementary set CE, $|f(x)| < |g(x)|^{1/(p-1)}$. Hence

$$|f(x)g(x)| < |g(x)|^{p/(p-1)}$$

in CE; hence $f(x)g(x)$ is integrable over CE, and so over the whole interval considered.

This argument can be used to obtain an inequality similar to (1), but with a factor 2 on the right-hand side. Let

$$I = \int_a^b |f(x)|^p\,dx, \qquad J = \int_a^b |g(x)|^{p/(p-1)}\,dx.$$

Then

$$\left|\int_a^b fg\,dx\right| \leqslant \int_E |fg|\,dx + \int_{CE} |fg|\,dx$$
$$\leqslant \int_E |f|^p\,dx + \int_{CE} |g|^{p/(p-1)}\,dx \leqslant I+J. \quad (2)$$

If we replace $f(x)$ and $g(x)$ in this inequality by

$$(J/I)^{(p-1)/p^2}f(x), \qquad (I/J)^{(p-1)/p^2}g(x),$$

respectively, the left-hand side is unchanged, and each term on the right-hand side is replaced by $I^{1/p}J^{1-1/p}$. Hence

$$\left|\int fg\,dx\right| \leqslant 2I^{1/p}J^{1-1/p}. \tag{3}$$

The inequality (1) can be deduced from the well-known inequality

$$x^m - 1 < m(x-1) \qquad (x > 1,\ 0 < m < 1). \tag{4}$$

Putting $x = a/b$ $(a > b)$, and multiplying by b,

$$a^m b^{1-m} < b + m(a-b).$$

Putting $m = \alpha$, $1-m = \beta$, so that $\alpha+\beta = 1$, this takes the form

$$a^\alpha b^\beta < a\alpha + b\beta, \tag{5}$$

and since this is symmetrical it holds if a and b are any unequal positive numbers. If $a = b$ it becomes an equality.

Using (5), we have, if $F(x) \geqslant 0$, $G(x) \geqslant 0$,

$$\int_a^b \left(\frac{F(x)}{\int_a^b F(t)\,dt}\right)^\alpha \left(\frac{G(x)}{\int_a^b G(t)\,dt}\right)^\beta dx \leqslant \int_a^b \left(\frac{\alpha F(x)}{\int_a^b F(t)\,dt} + \frac{\beta G(x)}{\int_a^b G(t)\,dt}\right) dx$$

$$= \alpha + \beta = 1,$$

i.e. $$\int_a^b \{F(x)\}^\alpha \{G(x)\}^\beta\,dx \leqslant \left\{\int_a^b F(x)\,dx\right\}^\alpha \left\{\int_a^b G(x)\,dx\right\}^\beta.$$

Finally, putting $\alpha = 1/p$, $F(x) = |f(x)|^p$, and $G(x) = |g(x)|^{p/(p-1)}$, the result (1) follows.*

Example. The case of equality occurs only if $|f(x)|^p/|g(x)|^{p/(p-1)}$ is almost everywhere equal to a constant.

12.421. Hölder's inequality for sums. This is

$$|\sum a_n b_n| \leqslant (\sum |a_n|^p)^{1/p} (\sum |b_n|^{p/(p-1)})^{1-1/p}.$$

The proof is similar to that of the integral inequality. We have

$$\sum \left\{\left(\frac{A_n}{\sum A_n}\right)^\alpha \left(\frac{B_n}{\sum B_n}\right)^\beta\right\} \leqslant \sum \left(\alpha \frac{A_n}{\sum A_n} + \beta \frac{B_n}{\sum B_n}\right)$$

$$= \alpha + \beta = 1,$$

i.e. $$\sum A_n^\alpha B_n^\beta \leqslant (\sum A_n)^\alpha (\sum B_n)^\beta,$$

and writing $\alpha = 1/p$, $A_n = |a_n|^p$, $B_n = |b_n|^{p/(p-1)}$, the result follows.

* This proof is given by Hardy (20).

12.43. Minkowski's inequality. *If $f(x)$ and $g(x)$ belong to L^p, where $p > 1$, then*

$$\left\{\int |f(x)+g(x)|^p\,dx\right\}^{1/p} \leqslant \left\{\int |f(x)|^p\,dx\right\}^{1/p} + \left\{\int |g(x)|^p\,dx\right\}^{1/p}. \quad (1)$$

For

$$\int |f+g|^p\,dx \leqslant \int |f|\cdot|f+g|^{p-1}\,dx + \int |g|\cdot|f+g|^{p-1}\,dx$$
$$\leqslant \left\{\int |f|^p\,dx\right\}^{1/p}\left\{\int |f+g|^p\,dx\right\}^{1-1/p} +$$
$$+\left\{\int |g|^p\,dx\right\}^{1/p}\left\{\int |f+g|^p\,dx\right\}^{1-1/p}$$

by Hölder's inequality. Dividing each side by

$$\left\{\int |f+g|^p\,dx\right\}^{1-1/p},$$

the result follows.

The corresponding inequality for sums

$$(\sum |a_n+b_n|^p)^{1/p} \leqslant (\sum |a_n|^p)^{1/p} + (\sum |b_n|^p)^{1/p} \quad (2)$$

can be proved in a similar way.

12.44. The integral of a function of L^p. We have seen in the previous chapter that a necessary and sufficient condition that a function should be an integral is that it should be absolutely continuous. There is a corresponding condition that a function should be an integral of a function of the class L^p.

A necessary and sufficient condition that a function $F(x)$ should be the integral of a function of the class L^p, where $p > 1$, is that the sum

$$\sum |F(x_\nu+h_\nu)-F(x_\nu)|^p h_\nu^{1-p},$$

taken over any system of non-overlapping intervals $(x_\nu, x_\nu+h_\nu)$, should be bounded.

If instead of 'should be bounded' we say 'should be bounded and tend to zero with $\sum h_\nu$', the theorem is still true, and in this form it is true for $p = 1$ also, and so includes the theorem on absolute continuity as a particular case. For $p > 1$ the two conditions, one of which appears to be more restrictive than the other, turn out to be equivalent.

To prove that the condition is necessary, suppose that

$$F(x) = F(a) + \int_a^x f(t)\,dt,$$

where $f(t)$ belongs to L^p. Then

$$|F(x_\nu+h_\nu)-F(x_\nu)| = \left|\int_{x_\nu}^{x_\nu+h_\nu} f(t)\,dt\right|$$

$$\leqslant \left\{\int_{x_\nu}^{x_\nu+h_\nu} |f(t)|^p\,dt\right\}^{1/p}\left\{\int_{x_\nu}^{x_\nu+h_\nu} dt\right\}^{1-1/p} = h_\nu^{1-1/p}\left\{\int_{x_\nu}^{x_\nu+h_\nu} |f(t)|^p\,dt\right\}^{1/p}.$$

Hence

$$\sum |F(x_\nu+h_\nu)-F(x_\nu)|^p h_\nu^{1-p} \leqslant \sum \int_{x_\nu}^{x_\nu+h_\nu} |f(t)|^p\,dt \leqslant \int_a^b |f(t)|^p\,dt,$$

so that the condition is necessary. Since

$$\sum \int_{x_\nu}^{x_\nu+h_\nu} |f(t)|^p\,dt$$

tends to zero with $\sum h_\nu$, the alternative condition is also necessary.

Suppose now that the condition is satisfied, and let M be the upper bound of the given sums. Then, by Hölder's inequality for sums,

$$\sum |F(x_\nu+h_\nu)-F(x_\nu)| = \sum |F(x_\nu+h_\nu)-F(x_\nu)| h_\nu^{1/p-1}.h_\nu^{1-1/p}$$
$$\leqslant \{\sum |F(x_\nu+h_\nu)-F(x_\nu)|^p h_\nu^{1-p}\}^{1/p}(\sum h_\nu)^{1-1/p} \leqslant M^{1/p}(\sum h_\nu)^{1-1/p},$$

which tends to zero with $\sum h_\nu$. Hence $F(x)$ is absolutely continuous, and so is an integral, say

$$F(x) = F(a) + \int_a^x f(t)\,dt.$$

It remains to prove that $f(t)$ belongs to L^p. Consider a sequence of finite sets of points in the interval, the mth set being $x_{m,1}, x_{m,2}, \ldots, x_{m,n}$, such that

$$\lim_{m\to\infty} \max_\nu (x_{m,\nu+1}-x_{m,\nu}) = 0.$$

For example, if the interval is $(0, 1)$ we may take $x_{m,\nu} = \nu/2^m$.

Let
$$f_m(x) = \frac{F(x_{m,\nu+1})-F(x_{m,\nu})}{x_{m,\nu+1}-x_{m,\nu}}$$

in each interval $x_{m,\nu} \leqslant x < x_{m,\nu+1}$. If x is not one of the points

$x_{m,\nu}$, and $F'(x)$ exists, and $x_{m,\nu} < x < x_{m,\nu+1}$, then

$$f_m(x) = \frac{F(x_{m,\nu+1})-F(x)}{x_{m,\nu+1}-x}\,\frac{x_{m,\nu+1}-x}{x_{m,\nu+1}-x_{m,\nu}} + \\ + \frac{F(x_{m,\nu})-F(x)}{x_{m,\nu}-x}\,\frac{x-x_{m,\nu}}{x_{m,\nu+1}-x_{m,\nu}}$$

$$= \{F'(x)+\delta_1\}\,\frac{x_{m,\nu+1}-x}{x_{m,\nu+1}-x_{m,\nu}} + \{F'(x)+\delta_2\}\,\frac{x-x_{m,\nu}}{x_{m,\nu+1}-x_{m,\nu}}$$

$$= F'(x)+\delta_3,$$

where $|\delta_3| \leqslant |\delta_1|+|\delta_2|$, and δ_1 and δ_2 tend to zero as

$$x_{m,\nu+1}-x_{m,\nu} \to 0.$$

Hence
$$\lim_{m\to\infty} f_m(x) = F'(x) = f(x)$$

almost everywhere. Also

$$\int_a^b |f_m(x)|^p\,dx = \sum |F(x_{m,\nu+1})-F(x_{m,\nu})|^p |x_{m,\nu+1}-x_{m,\nu}|^{1-p} \leqslant M.$$

Hence, by Fatou's theorem (§ 10.81), $f(x)$ belongs to L^p, and

$$\int_a^b |f(x)|^p\,dx \leqslant \underline{\lim} \int_a^b |f_m(x)|^p\,dx \leqslant M.$$

12.5. Mean convergence. If we are given a sequence of numbers, say s_n, we have usually to consider the behaviour of the difference $s_n - s$ between s_n and a given number s. In dealing with a sequence of functions, say $f_n(x)$, and a given function $f(x)$, it is often not the difference but the mean or average value of the difference which is important. This can be defined in various ways. If the functions belong to the class L^p, where $p \geqslant 1$, we consider the integral

$$\int_a^b |f_n(x)-f(x)|^p\,dx. \tag{1}$$

If this integral tends to zero as $n \to \infty$, we say that $f_n(x)$ converges in mean (*en moyenne*, *im Mittel*), to $f(x)$, with index p.

If
$$\int_a^b |f_m(x)-f_n(x)|^p\,dx \tag{2}$$

tends to zero as m and n tend independently to infinity, we say

that the sequence $f_n(x)$ converges in mean, with index p. Here the function $f(x)$ is not involved explicitly.

The fundamental theorem* of the subject is that *if the sequence $f_n(x)$ converges in mean, with index p, then there is a function $f(x)$ of the class L^p, defined uniquely apart from sets of measure zero, to which $f_n(x)$ converges in mean.*

The theorem is analogous to the 'general principle of convergence', that if $s_m - s_n \to 0$, then there is a number s to which s_n tends.

A word of explanation is necessary with regard to the 'uniqueness' of the limit-function $f(x)$. Suppose that we have found a function $f(x)$ which satisfies the given conditions. Then obviously any other function $g(x)$ which is equal to $f(x)$ almost everywhere also satisfies the conditions. So at any particular point the value of $f(x)$ is undetermined, though its general aggregate of values is in a sense determined. The function $f(x)$ should be regarded as a representative of a class of functions, any two of which are equal almost everywhere, and so all of which behave in the same way in integration.

The theorem for a finite interval and $p > 1$ may be proved as follows. To every integer ν corresponds a smallest positive integer n_ν such that

$$\int_a^b |f_m(x) - f_n(x)|^p \, dx < \frac{1}{3^\nu} \qquad (m \geqslant n_\nu,\ n \geqslant n_\nu).$$

In particular,

$$\int_a^b |f_{n_{\nu+1}}(x) - f_{n_\nu}(x)|^p \, dx < \frac{1}{3^\nu} \qquad (\nu = 1, 2, 3, \ldots).$$

If E_ν is the set where $|f_{n_{\nu+1}}(x) - f_{n_\nu}(x)| > 2^{-\nu/p}$, it follows that $m(E_\nu) < (\frac{2}{3})^\nu$. Hence the series

$$\sum_{\nu=1}^{\infty} |f_{n_{\nu+1}}(x) - f_{n_\nu}(x)|$$

is convergent, by comparison with $\sum 2^{-\nu/p}$, if, for some value of N, x does not belong to the set $E_{N+1} + E_{N+2} + \ldots$. Since the measure of this set tends to 0 as $N \to \infty$, it follows that the

* Fischer (1), F. Riesz (1), (2); W. H. and G. C. Young (2), where several alternative proofs are given; and Hobson (1).

above series is convergent for almost all values of x; hence so is

$$\sum_{\nu=1}^{\infty}\{f_{n_{\nu+1}}(x)-f_{n_\nu}(x)\},$$

i.e. there is a function $f(x)$ (defined almost everywhere) such that

$$\lim_{\nu\to\infty} f_{n_\nu}(x)=f(x)$$

almost everywhere.

This function $f(x)$ has the required property. For by Fatou's theorem

$$\varliminf_{\nu\to\infty}\int_a^b |f_{n_\mu}(x)-f_{n_\nu}(x)|^p\,dx \geqslant \int_a^b |f_{n_\mu}(x)-f(x)|^p\,dx;$$

but $$\int_a^b |f_{n_\mu}(x)-f_{n_\nu}(x)|^p\,dx<\epsilon \qquad (\mu>\nu_0,\ \nu>\nu_0).$$

Hence $$\int_a^b |f_{n_\mu}(x)-f(x)|^p\,dx\leqslant\epsilon \qquad (\mu>\nu_0),$$

i.e. $$\lim\int_a^b |f_{n_\mu}(x)-f(x)|^p\,dx=0,$$

i.e. the sub-sequence $f_{n_\mu}(x)$ converges in mean to $f(x)$.

Also, by Minkowski's inequality,

$$\left\{\int_a^b |f(x)-f_n(x)|^p\,dx\right\}^{1/p}$$
$$\leqslant\left\{\int_a^b |f(x)-f_{n_\mu}(x)|^p\,dx\right\}^{1/p}+\left\{\int_a^b |f_{n_\mu}(x)-f_n(x)|^p\,dx\right\}^{1/p},$$

which tends to zero as n and n_μ tend to infinity, by what has just been proved and the original hypothesis. Hence the whole sequence $f_n(x)$ converges in mean to $f(x)$.

Finally, suppose that $f_n(x)$ converges in mean to $f(x)$ and also to $g(x)$. Then

$$\left(\int_a^b |f-g|^p\,dx\right)^{1/p}\leqslant\left(\int_a^b |f-f_n|^p\,dx\right)^{1/p}+\left(\int_a^b |g-f_n|^p\,dx\right)^{1/p}\to 0.$$

Hence the left-hand side is 0, and so $f(x)=g(x)$ almost everywhere.

If $p=1$ the proof is simpler, since it is not necessary to use Minkowski's inequality.

12.51. The proof applies almost unchanged to an infinite interval. We find that the above series are convergent almost everywhere in (a, b) for every b, i.e. almost everywhere in (a, ∞); and Fatou's theorem holds for an infinite interval; for, taking the set E of § 10.81 to be the interval (a, b),

$$\int_a^b f(x)\,dx \leqslant \underline{\lim} \int_a^b f_n(x)\,dx \leqslant \underline{\lim} \int_a^\infty f_n(x)\,dx,$$

and, making $b \to \infty$, we obtain the required extension of Fatou's theorem. The proof for an infinite interval now follows.

12.52. *We have also (for a finite or infinite interval)*

$$\lim_{n\to\infty} \int |f_n(x)|^p\,dx = \int |f(x)|^p\,dx.$$

For by Minkowski's inequality

$$\left\{\int |f_n(x)|^p\,dx\right\}^{1/p} \leqslant \left\{\int |f(x)|^p\,dx\right\}^{1/p} + \left\{\int |f(x)-f_n(x)|^p\,dx\right\}^{1/p},$$

and also

$$\left\{\int |f(x)|^p\,dx\right\}^{1/p} \leqslant \left\{\int |f_n(x)|^p\,dx\right\}^{1/p} + \left\{\int |f(x)-f_n(x)|^p\,dx\right\}^{1/p}.$$

Hence $$\lim_{n\to\infty}\left\{\int |f_n(x)|^p\,dx\right\}^{1/p} = \left\{\int |f(x)|^p\,dx\right\}^{1/p},$$

and the result follows.

12.53. *If $f_n(x)$ converges in mean to $f(x)$ with index p, and $g(x)$ belongs to $L^{p/(p-1)}$, then*

$$\lim_{n\to\infty} \int f_n(x)g(x)\,dx = \int f(x)g(x)\,dx. \tag{1}$$

For

$$\left|\int \{f_n(x)-f(x)\}g(x)\,dx\right|$$
$$\leqslant \left\{\int |f_n(x)-f(x)|^p\,dx\right\}^{1/p}\left\{\int |g(x)|^{p/(p-1)}\,dx\right\}^{1-1/p},$$

which tends to zero.

In particular

$$\lim_{n\to\infty} \int_a^x f_n(t)\,dt = \int_a^x f(t)\,dt \tag{2}$$

for all values of x in the interval considered.

For the function $g(t) = 1$ $(a < t < x)$, $= 0$ $(t > x)$, belongs to $L^{p/(p-1)}$.

Examples. (i) If $f_n(x) \to f(x)$ boundedly over a finite interval, then $f_n(x)$ converges in mean to $f(x)$ with any index.

(ii) Consider the closed intervals $(0, \frac{1}{2})$, $(\frac{1}{2}, 1)$, $(0, \frac{1}{3})$, $(\frac{1}{3}, \frac{2}{3})$, $(\frac{2}{3}, 1)$, $(0, \frac{1}{4})$, etc. Let $f_n(x) = 1$ in the nth interval, and $f_n(x) = 0$ in the remainder of the interval $(0, 1)$. Then $f_n(x)$ converges in mean to zero in $(0, 1)$, with any index; but $f_n(x)$ does not tend to zero for any value of x.

(iii) If $f_n(x)$ converges in mean to $f(x)$, and $f_n(x) \to g(x)$ almost everywhere, then $f(x) = g(x)$ almost everywhere. [Use Egoroff's theorem.]

(iv) If $f_n(x)$ converges in mean to $f(x)$ with index p, and $g_n(x)$ to $g(x)$ with index $p/(p-1)$, then $\int f_n g_n \, dx \to \int fg \, dx$.

12.6. Repeated integrals. As in the elementary cases considered in § 1.8, the equation

$$\int_a^b dx \int_\alpha^\beta f(x,y)\, dy = \int_\alpha^\beta dy \int_a^b f(x,y)\, dx \tag{1}$$

is in general true in the Lebesgue theory. The general discussion of this, however, depends on the theory of the double integral

$$\int\int f(x,y)\, dxdy,$$

which in turn depends on the theory of two-dimensional sets of points. It would take us too far to carry this out in detail.

There is, however, a particular kind of repeated integral which includes many cases of interest, and which can be dealt with by the theory already developed.

Let $f(x)$ be integrable in the Lebesgue sense over (a, b), and $g(y)$ over (α, β), and let $k(x, y)$ be a continuous function of both variables, or, if it has discontinuities, let them be of the type described in § 1.82. *Then*

$$\int_a^b f(x)\, dx \int_\alpha^\beta g(y)k(x,y)\, dy = \int_\alpha^\beta g(y)\, dy \int_a^b f(x)k(x,y)\, dx. \tag{2}$$

Suppose first that $f(x)$ and $g(y)$ are bounded, say $|f(x)| \leqslant M$, $|g(y)| \leqslant M$. Let $|k(x,y)| \leqslant K$.

Let $\phi(x)$ be a continuous function satisfying 12.2 (1); and let $|\phi(x)| \leqslant M$. Let $\psi(y)$ be a continuous function related in the same way to $g(y)$.

Call the left-hand side of (2) I, and let

$$I' = \int_a^b \phi(x)\, dx \int_\alpha^\beta \psi(y)k(x,y)\, dy.$$

Then

$$I-I' = \int_a^b \{f(x)-\phi(x)\}\,dx \int_\alpha^\beta g(y)k(x,y)\,dy +$$

$$+ \int_a^b \phi(x)\,dx \int_\alpha^\beta \{g(y)-\psi(y)\}k(x,y)\,dy,$$

and hence

$$|I-I'| \leqslant \int_a^b |f(x)-\phi(x)|(\beta-\alpha)MK\,dx + (b-a)M\eta$$

$$\leqslant MK(\beta-\alpha+b-a)\eta.$$

Similarly, if the right-hand side of (2) is J, and

$$J' = \int_\alpha^\beta \psi(y)\,dy \int_a^b \phi(x)k(x,y)\,dx,$$

then $|J-J'|$ tends to 0 with η.

But, by the theorem of § 1.81, $I' = J'$, since $\phi(x)\psi(y)k(x,y)$ is continuous, or has discontinuities of the restricted type.

Hence $|I-J|$ tends to 0 with η, and so $I = J$.

The extension to unbounded functions may be left to the reader; we suppose first that $f(x)$ and $g(x)$ are positive, and argue with $\{f(x)\}_n$ and $\{g(x)\}_n$ in the usual manner.

12.61. *If $f(x)$ is integrable over $(0, 1)$, and $g(x)$ over $(0, 2)$, then the integral*

$$\int_0^1 f(x)g(x+t)\,dx$$

exists for almost all values of t in $(0, 1)$, and represents an integrable function of t.

It is sufficient to consider the case where f and g are positive. Define $\{f(x)\}_n$ as usual, and let

$$F_n(t) = \int_0^1 \{f(x)\}_n g(x+t)\,dx.$$

This integral exists for all values of t, and, for a given n, $F_n(t)$ is bounded, and for each value of t it is a non-decreasing function of n. Also

$$\int_0^1 F_n(t)\,dt = \int_0^1 dt \int_0^1 \{f(x)\}_n g(x+t)\,dx = \int_0^1 \{f(x)\}_n\,dx \int_0^1 g(x+t)\,dt, \tag{1}$$

if we may invert the order of integration.

To justify this, approximate to g by a continuous function ψ, as in the previous proof, and let

$$\chi(t) = \int_0^1 \{f(x)\}_n \psi(x+t)\, dx.$$

Then
$$\int_0^1 \chi(t)\, dt = \int_0^1 \{f(x)\}_n\, dx \int_0^1 \psi(x+t)\, dt, \tag{2}$$

this inversion being justified by the above theorem. Now

$$|F_n(t) - \chi(t)| \leqslant \int_0^1 \{f(x)\}_n |g(x+t) - \psi(x+t)|\, dx < n\eta,$$

so that the left-hand side of (1) differs from that of (2) by less than $n\eta$. Similarly the right-hand sides differ by less than $n\eta$. Hence, making $\eta \to 0$, we obtain (1).

Hence
$$\int_0^1 F_n(t)\, dt \leqslant \int_0^1 f(x)\, dx \int_0^2 g(y)\, dy.$$

Hence, as $n \to \infty$, $F_n(t)$ tends to a finite limit for almost all values of t (§ 10.82). The result now follows from the theorem of § 10.82.

12.62. Repeated infinite integrals. *If $f(x)$, $g(y)$, and $k(x, y)$ are positive, and the conditions of § 12.6 are satisfied for all values of $b > a$ and $\beta > \alpha$, then*

$$\int_a^\infty f(x)\, dx \int_\alpha^\infty g(y) k(x, y)\, dy = \int_\alpha^\infty g(y)\, dy \int_a^\infty f(x) k(x, y)\, dx \tag{1}$$

provided that either side is convergent.

The theorem is similar to that of § 1.85, but the supplementary conditions which appear there are now a consequence of the main hypothesis.

Suppose that the right-hand side of (1) is convergent. Since

$$\int_a^X f(x) k(x, y)\, dx \leqslant \int_a^\infty f(x) k(x, y)\, dx, \tag{2}$$

and the left-hand side of (2) is a measurable (in fact a continuous) function of y, it follows that

$$\int_\alpha^\infty g(y)\, dy \int_a^X f(x) k(x, y)\, dx$$

is convergent. Hence

$$\lim_{n\to\infty}\int_\alpha^n g(y)\,dy\int_a^X f(x)k(x,y)\,dx=\lim_{n\to\infty}\int_a^X f(x)\,dx\int_\alpha^n g(y)k(x,y)\,dy$$

is finite. Also

$$F_n(x)=f(x)\int_\alpha^n g(y)k(x,y)\,dy$$

is a non-decreasing function of n for each value of x. It therefore follows from § 10.82 that $F_n(x)$ tends to a finite limit, as $n\to\infty$, for almost all values of x in (a, X); i.e.

$$\int_\alpha^\infty g(y)k(x,y)\,dy$$

is convergent for almost all values of x in (a, X); and by § 10.82

$$\int_a^X f(x)\,dx\int_\alpha^\infty g(y)k(x,y)\,dy=\lim_{n\to\infty}\int_a^X f(x)\,dx\int_\alpha^n g(y)k(x,y)\,dy$$

$$=\lim_{n\to\infty}\int_\alpha^n g(y)\,dy\int_a^X f(x)k(x,y)\,dx=\int_\alpha^\infty g(y)\,dy\int_a^X f(x)k(x,y)\,dx. \quad (3)$$

By (2), the right-hand side of (3) is bounded as $X\to\infty$; hence so is the left-hand side, and therefore the left-hand side of (1) is convergent.

We can now prove in a similar way that the order of integration in

$$\int_\alpha^Y g(y)\,dy\int_a^\infty f(x)k(x,y)\,dx$$

may be inverted. The final result then follows as in § 1.85.

MISCELLANEOUS EXAMPLES.

1. If $f(x)$ is integrable over (a, b), and $a=x_0<x_1<x_2<\ldots<x_n=b$, then

$$\sum_{\nu=0}^{n-1}\left|\int_{x_\nu}^{x_{\nu+1}} f(t)\,dt\right|\to\int_a^b |f(t)|\,dt$$

as the greatest partial interval tends to zero.

[The proof is elementary for continuous functions; and then the general result may be deduced by means of the theorem of § 12.2.]

2. If $F(x)$ is absolutely continuous in (a, b), its total variation in the interval is

$$\int_a^b |F'(x)|\,dx.$$

[Use the result of the previous example.]

3. Show that, if $f(x)$ and $g(x)$ belong to L^2,

$$\int \{f(x)\}^2\,dx \int \{g(x)\}^2\,dx - \left\{\int f(x)g(x)\,dx\right\}^2$$
$$= \tfrac{1}{2}\int dy \int \{f(x)g(y)-f(y)g(x)\}^2\,dx,$$

and hence obtain another proof of Schwarz's inequality.

4. We use $\log' x$ to denote $\log x$ if $x \geqslant e$, and $\log' x = 1$ if $x < e$.

Show that if $\{f(x)\}^2 \log' f(x)$ and $\{g(x)\}^2/\log' g(x)$ are integrable over (a, b), then $f(x)g(x)$ is integrable over (a, b).

[Let E be the set where $f \leqslant g/\log' g$. Then

$$\int_E fg\,dx \leqslant \int_E \frac{\{g(x)\}^2}{\log' g(x)}\,dx.$$

In CE, $g \leqslant f \log' g$. If $g \leqslant e$ this gives

$$g \leqslant f \leqslant f \log' f.$$

If $g > e$, $\sqrt{g} < Ag/\log' g < Af$, $\log g < A \log f$, and hence again, $g < Af \log' f$. Hence

$$\int_{CE} fg\,dx \leqslant \int_{CE} \{f(x)\}^2 \log' f(x)\,dx.]$$

5. If $f^p(\log' f)^q$ and $g^{p/(p-1)}(\log' g)^{-q/(p-1)}$ are integrable, then $f(x)g(x)$ is integrable.

6. Prove that $$uv \leqslant u \log u + e^{v-1} \qquad (u > 1, v > 1).$$

Deduce that if $f(x)\log' f(x)$ and $e^{g(x)}$ are integrable, so is $f(x)g(x)$.

[W. H. Young (4). The inequality may be verified by putting $u = e^x$, $v = y+1$.]

7. If $\alpha > 0, \beta > 0, \gamma > 0, \alpha+\beta+\gamma = 1$,

$$\left|\int fgh\,dx\right| \leqslant \left(\int |f|^{1/\alpha}\,dx\right)^{\alpha}\left(\int |g|^{1/\beta}\,dx\right)^{\beta}\left(\int |h|^{1/\gamma}\,dx\right)^{\gamma}.$$

8. If $\lambda > 0$, $\mu > 0$, $\lambda\mu < 1$, and $f(x)$ and $g(x)$ belong to suitable L-classes, then

$$\left|\int fg\,dx\right|^{(1+\lambda)(1+\mu)/(1-\lambda\mu)}$$
$$\leqslant \int |f|^{1+\lambda}|g|^{1+\mu}\,dx \left(\int |f|^{1+\lambda}\,dx\right)^{\mu(1+\lambda)/(1-\lambda\mu)}\left(\int |g|^{1+\mu}\,dx\right)^{\lambda(1+\mu)/(1-\lambda\mu)}$$

[W. H. Young (2); the result may be obtained by suitable substitutions in ex. 7.]

9. If $F(x)$ is the integral of a function of the class L^p, where $p > 1$, then as $h \to 0$ $$F(x+h)-F(x) = o\,(h^{1-1/p}).$$

[If $F(x)$ is the integral of $f(x)$,

$$|F(x+h)-F(x)| = \left|\int_x^{x+h} f(t)\,dt\right|$$

$$\leqslant \left\{\int_x^{x+h} |f(t)|^p\,dt\right\}^{1/p}\left\{\int_x^{x+h} dt\right\}^{1-1/p} = h^{1-1/p}\left\{\int_x^{x+h} |f(t)|^p\,dt\right\}^{1-1/p},$$

and the last factor tends to zero with h.]

10. If $f(x)$ belongs to $L^p(0, \infty)$, where $p > 1$, the integral

$$\int_0^\infty f(x)\,\frac{\sin xy}{x}\,dx$$

is uniformly convergent in any finite interval.

11. If $f(x)$ belongs to L^p, where $p > 1$, and $\phi(y)$ is the integral defined in the previous example, then as $h \to 0$

$$\phi(y+h)-\phi(y) = o\,(h^{1/p}).$$

[For
$$\phi(y+h)-\phi(y) = \int_0^\infty \frac{f(x)}{x}\,[\sin\{x(y+h)\} - \sin xy]\,dx$$

$$= 2\int_0^\infty \frac{f(x)}{x}\,\sin\tfrac{1}{2}xh\cos x(y+\tfrac{1}{2}h)\,dx.$$

Hence

$$|\phi(y+h)-\phi(y)| \leqslant 2\int_0^\infty \left|f(x)\,\frac{\sin\frac{1}{2}xh}{x}\right| dx$$

$$\leqslant 2\left\{\int_0^\infty |f(x)|^p\,dx\right\}^{1/p}\left\{\int_0^\infty \left|\frac{\sin\frac{1}{2}xh}{x}\right|^{p/(p-1)} dx\right\}^{1-1/p}$$

The first factor is a constant, and the second factor, on putting $x = \xi/h$, is seen to be a multiple of $h^{1/p}$. This gives the required result with O instead of o. If, however, we apply the above argument to the integrals over $(0, \delta)$ and (Δ, ∞), where δ is arbitrarily small and Δ arbitrarily large, and notice that

$$\left|\int_\delta^\Delta \frac{f(x)}{x}\,\sin\tfrac{1}{2}xh\cos x(y+\tfrac{1}{2}h)\,dx\right| \leqslant \int_\delta^\Delta \frac{|f(x)|}{x}\,|\sin\tfrac{1}{2}xh|\,dx = O(h)$$

for fixed δ and Δ, the required result follows.]

12. Show that the integral

$$\phi(y) = \int_0^\infty f(x) \frac{\sin xy}{\sqrt{x}}\, dx$$

is absolutely convergent if $f(x)$ belongs to L^p, where $1 < p < 2$; and that, as $y \to 0$,

$$\phi(y) = o\,(y^{1/p-\frac{1}{2}}).$$

13. If $f(x)$ is uniformly continuous over $(0, \infty)$, and belongs to a class $L^p(0, \infty)$, then $f(x) \to 0$ as $x \to \infty$.

14. If $f(x)$ belongs to $L^p(0, \infty)$, where $p > 1$, so do the functions

$$\phi(x) = \frac{1}{x}\int_0^x f(t)\, dt, \qquad \psi(x) = \int_x^\infty \frac{f(t)}{t}\, dt.$$

[Hardy (17) and (19). Consider $\phi(x)$, for example. It is bounded except as $x \to 0$ or $x \to \infty$. Hence

$$\int_a^b |\phi(x)|^p\, dx$$

exists for $0 < a < b < \infty$; and it is sufficient to prove that this integral remains bounded as $a \to 0$ and $b \to \infty$.

We may suppose without loss of generality that $f(t) \geqslant 0$. Let

$$f_1(x) = \int_0^x f(t)\, dt.$$

Then $x^{1-p}\{f_1(x)\}^p$ tends to zero, both as $x \to 0$ and as $x \to \infty$; for

$$\{f_1(x)\}^p \leqslant \int_0^x \{f(t)\}^p\, dt \left(\int_0^x dt\right)^{p-1} = x^{p-1}\int_0^x \{f(t)\}^p\, dt,$$

whence the result for $x = 0$ follows. Again, if $x > \xi$, a similar argument shows that

$$f_1(x) \leqslant \int_0^\xi f(t)\, dt + \left\{x^{p-1}\int_\xi^x \{f(t)\}^p\, dt\right\}^{1/p},$$

and we can choose ξ so large that the last factor is arbitrarily small, for all $x > \xi$. This gives the result for $x \to \infty$.

We write
$$\int_a^b \{\phi(x)\}^p\, dx = \int_a^b \{f_1(x)\}^p x^{-p}\, dx,$$

and integrate by parts, obtaining

$$\left[\frac{\{f_1(x)\}^p x^{1-p}}{1-p}\right]_a^b + \frac{p}{p-1}\int_a^b \{f_1(x)\}^{p-1} f(x) x^{1-p}\, dx$$

$$= o\,(1) + \frac{p}{p-1}\int_a^b \{\phi(x)\}^{p-1} f(x)\, dx.$$

Hence

$$\int_a^b \{\phi(x)\}^p\,dx \leqslant o(1)+\frac{p}{p-1}\left\{\int_a^b \{\phi(x)\}^p\,dx\right\}^{1-1/p}\left\{\int_a^b \{f(x)\}^p\,dx\right\}^{1/p},$$

and dividing by the factor

$$\left\{\int_a^b \{\phi(x)\}^p\,dx\right\}^{1-1/p},$$

and making $a \to 0$, $b \to \infty$, we obtain

$$\left\{\int_0^\infty \{\phi(x)\}^p\,dx\right\}^{1/p} \leqslant \frac{p}{p-1}\left\{\int_0^\infty \{f(x)\}^p\,dx\right\}^{1/p}.$$

We leave the corresponding process for $\psi(x)$ to the reader.]

15. Prove that, with the hypotheses of the previous example, the integrals

$$\int_0^\infty |\phi(x)|^q x^{q/p-1}\,dx, \qquad \int_0^\infty |\psi(x)|^q x^{q/p-1}\,dx$$

are convergent for $q \geqslant p$.

16. If $f(x)$ belongs to $L^p(0, \infty)$, where $p > 1$, and

$$\phi(x) = \int_0^\infty e^{-xy} f(y)\,dy,$$

then $y^{1-2/p}\phi(y)$ belongs to L^p.

[For $$|\phi(x)| \leqslant \int_0^{1/x} |f(y)|\,dy + \int_{1/x}^\infty \frac{|f(y)|}{xy}\,dy,$$

and the result follows from ex. 14.]

17. If $f(x)$ belongs to $L^p(a, b)$, there is a continuous function $g(x)$ such that

$$\int_a^b |f(x)-g(x)|^p\,dx < \epsilon.$$

[The result for bounded $f(x)$ follows at once from § 12.2, and the general result may then be deduced from this.]

18. If $f(x)$ belongs to L^p over an interval including (a, b), then

$$\lim_{h\to 0}\int_a^b |f(x+h)-f(x)|^p\,dx = 0.$$

[The result is immediate for continuous functions. For the general result use the previous example.]

19. If $f(x)$ belongs to $L^p(-\infty, \infty)$, then

$$\lim_{h\to 0}\int_{-\infty}^\infty |f(x+h)-f(x)|^p\,dx = 0.$$

20. If $f(x)$ belongs to $L^p(-\infty, \infty)$, and $g(x)$ to $L^{p/(p-1)}(-\infty, \infty)$, then

$$F(t) = \int_{-\infty}^{\infty} f(x+t)\, g(x)\, dx$$

is a continuous function of t.

$$\left[\text{For } |F(t+h)-F(t)| \leqslant \left\{\int_{-\infty}^{\infty} |f(x+t+h)-f(x+t)|^p\, dx\right\}^{1/p} \left\{\int_{-\infty}^{\infty} |g(x)|^{p/(p-1)}\, dx\right\}^{1-1/p}.\right]$$

21. The function $F(t)$ of the previous example tends to zero at infinity.

$$\left[\text{Write } \int_{-\infty}^{\infty} = \int_{-\infty}^{-\frac{1}{2}t} + \int_{-\frac{1}{2}t}^{\infty}.\right]$$

22. If $f(x)$ belongs to $L^p(-\infty, \infty)$, then

$$F(x) = \int_{-\infty}^{\infty} \frac{f(t)}{1+(x-t)^2}\, dt$$

is continuous, and belongs to $L^p(-\infty, \infty)$.

[Use the inequality

$$|F(x)|^p \leqslant \int_{-\infty}^{\infty} \frac{|f(t)|^p\, dt}{\{1+(x-t)^2\}^{\frac{1}{2}p}} \left\{\int_{-\infty}^{\infty} \frac{dt}{\{1+(x-t)^2\}^{\frac{1}{2}p/(p-1)}}\right\}^{p-1}.]$$

23. If $f(x)$ is integrable, the integral

$$f_\alpha(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt \qquad (\alpha > 0)$$

exists almost everywhere, and $f_\alpha(x)$ is integrable.

[The function $f_\alpha(x)$ is the integral of $f(x)$ of order α; for some properties of such integrals see Hardy (12) and Hardy and Littlewood (5).]

24. If $f(x)$ belongs to $L^p(p > 1)$, show by the method of ex. 20 that $f_\alpha(x)$ is continuous if $\alpha > 1/p$.

25. If $f_{\alpha,\beta}(x)$ denotes the integral of order β of $f_\alpha(x)$, then

$$f_{\alpha,\beta}(x) = f_{\alpha+\beta}(x) \qquad (\alpha > 0, \beta > 0),$$

wherever the right-hand side exists.

[We have to invert the repeated integral

$$\int_0^x (x-t)^{\beta-1}\, dt \int_0^t (t-u)^{\alpha-1} f(u)\, du$$

and use Ch. I, ex. 18. The integral $\int_\delta^x \int_0^{t-\delta}$ may be inverted by § 12.6. We can then make $\delta \to 0$ and use the theorem of § 10.82.]

CHAPTER XIII

FOURIER SERIES

13.1. Trigonometrical series and Fourier series. A trigonometrical series is a series of the form

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{1}$$

where the coefficients a_0, a_1, b_1,... are independent of x. The problem of representing a given function $f(x)$ by a series of this form was first encountered by Fourier in a problem of the conduction of heat. Subsequently it was found that these series play an important part in the theory of functions of a real variable, and it is from this point of view that we shall consider them here.

We naturally begin by trying to find formulae for the coefficients a_n, b_n, in terms of the given function $f(x)$. Suppose that the series converges uniformly, or even boundedly, to $f(x)$; we may then multiply by $\cos mx$, where m is a positive integer, and integrate term by term over the interval $(0, 2\pi)$. Since

$$\int_0^{2\pi} \cos mx \cos nx \, dx = \pi \ (n = m), \qquad = 0 \ (n \neq m)$$

and

$$\int_0^{2\pi} \cos mx \sin nx \, dx = 0$$

for all values of n, we obtain the result

$$a_m = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos mx \, dx. \tag{2}$$

The same formula also gives a_0; and similarly, multiplying by $\sin mx$ and integrating term by term,

$$b_m = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin mx \, dx. \tag{3}$$

The formulae (2) and (3) are known as the Euler-Fourier formulae for the coefficients.

There is, however, no *a priori* reason for supposing that a given function can be expanded in a boundedly convergent trigonometrical series. The above process is therefore not a

proof that the coefficients necessarily have the above form. What it really suggests is that we should adopt a different point of view. Instead of starting with the series, and assuming that it has a certain property, we start from the function, and define the coefficients by the above formulae. We then consider the properties of the series so formed.

Suppose, then, that we are given a function $f(x)$, integrable in the sense of Lebesgue over the interval $(0, 2\pi)$. Then the integrals (2) and (3) exist, and the numbers a_m, b_m defined by them are called the *Fourier coefficients of* $f(x)$. The trigonometrical series of the form (1), with these coefficients, is called the Fourier series of $f(x)$.

The scheme of the chapter is as follows. We first try to determine conditions under which the Fourier series converges to $f(x)$. A number of these conditions are found, but they are all rather special ones (§§ 13.11–13.25). We next consider a generalized kind of convergence (summability $(C, 1)$), and find that it enables us to put the theory into a more systematic form (§§ 13.3–13.35). In the following sections we consider some problems of term-by-term integration; and this leads us to consider properties of the Fourier coefficients themselves, apart from the Fourier series. In §§ 13.8–13.86 we return to the question of the relation between Fourier series in particular and trigonometrical series in general. Lastly we give some of the corresponding theory of Fourier integrals.

13.11. The convergence problem. The first problem which we have to consider is whether the series formed in the above manner converges, and, if it does, whether its sum is $f(x)$.

At the time when Fourier series first came into use, there seemed to many mathematicians to be something paradoxical in saying that an 'arbitrary' function could be represented by a series of functions, each of which is continuous and periodic. The reader who has examined the peculiarities of some of the series in Chapter I is perhaps prepared to believe that even this is possible; and we shall show that the series does, substantially, do what is required of it. We must not, however, expect too much.

In the first place, every term of the series has the period 2π; hence the sum of the series, if there is one, also has the period 2π. We therefore define the function $f(x)$ first in the interval

$0 \leqslant x < 2\pi$; outside this interval we define it by periodicity, i.e. by the equation

$$f(x+2\pi) = f(x).$$

Secondly, it is impossible that, whatever $f(x)$ is, the series should converge to the sum $f(x)$ for every value of x. Consider, for example, two functions $f(x)$ and $g(x)$ which differ at one point only. They have the same Fourier series, so that it cannot represent both functions at every point. More generally, two 'equivalent' functions, i.e. functions which are equal almost everywhere, have the same Fourier series, which therefore cannot represent them both if they differ anywhere.

Actually we shall see that the series does represent the function, provided that the function is not too complicated; and even in the most complicated cases, the series still represents in some sense the main features of the function.

13.12. Fourier series and Laurent series. There is a close formal connexion between a Fourier series and a Laurent series. Let $F(z)$ be a one-valued analytic function, regular for $R' < |z| < R$. Then

$$F(z) = \sum_{n=-\infty}^{\infty} c_n z^n,$$

where $$c_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{F(z)}{z^{n+1}}\, dz \qquad (R' < r < R).$$

Putting $z = re^{i\theta}$, we have

$$F(re^{i\theta}) = \sum_{n=-\infty}^{\infty} A_n e^{in\theta},$$

where $$A_n = \frac{1}{2\pi} \int_0^{2\pi} F(re^{i\phi}) e^{-in\phi}\, d\phi.$$

The expansion may also be written

$$F(re^{i\theta}) = A_0 + \sum_{n=1}^{\infty} \{(A_n + A_{-n})\cos n\theta + i(A_n - A_{-n})\sin n\theta\},$$

where

$$A_0 = \frac{1}{2\pi} \int_0^{2\pi} F(re^{i\phi})\, d\phi, \qquad A_n + A_{-n} = \frac{1}{\pi} \int_0^{2\pi} F(re^{i\phi}) \cos n\phi\, d\phi,$$

$$i(A_n - A_{-n}) = \frac{1}{\pi} \int_0^{2\pi} F(re^{i\phi}) \sin n\phi\, d\phi.$$

We have thus expressed the Laurent series in the form of a Fourier series. The fact that in this case the series represents the function, and indeed converges uniformly to it, follows from the theory of analytic functions. In general we assume much less about the function than that it is analytic, and the problem requires quite different methods.

13.2. Dirichlet's integral. Let $0 \leqslant x < 2\pi$, and let

$$s_n = s_n(x) = \tfrac{1}{2}a_0 + \sum_{m=1}^{n} (a_m \cos mx + b_m \sin mx). \tag{1}$$

This partial sum can be represented as a definite integral. We have

$$s_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt +$$

$$+\frac{1}{\pi}\sum_{m=1}^{n}\left\{\cos mx \int_0^{2\pi} f(t)\cos mt\,dt + \sin mx \int_0^{2\pi} f(t)\sin mt\,dt\right\}$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left\{\tfrac{1}{2} + \sum_{m=1}^{n}\cos m(x-t)\right\} f(t)\,dt = \frac{1}{2\pi}\int_0^{2\pi}\frac{\sin(n+\frac{1}{2})(x-t)}{\sin\frac{1}{2}(x-t)} f(t)\,dt.$$

Putting $t = x+u$, this becomes

$$s_n = \frac{1}{2\pi}\int_{-x}^{2\pi-x}\frac{\sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u} f(x+u)\,du,$$

or, since the integrand has the period 2π, and so takes the same values in $(2\pi-x, 2\pi)$ as in $(-x, 0)$,

$$s_n = \frac{1}{2\pi}\int_0^{2\pi}\frac{\sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u} f(x+u)\,du. \tag{2}$$

This formula is known as *Dirichlet's integral*. It may also be written in the form

$$s_n = \frac{1}{2\pi}\int_0^{\pi}\frac{\sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u}\{f(x+u)+f(x-u)\}\,du. \tag{3}$$

This is obtained by writing $u = -v$ in the range $(\pi, 2\pi)$, so that

this part of (2) becomes

$$\int_{-2\pi}^{-\pi} \frac{\sin(n+\frac{1}{2})v}{\sin\frac{1}{2}v} f(x-v)\,dv = \int_{0}^{\pi} \frac{\sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u} f(x-u)\,du$$

by periodicity.

Suppose, in particular, that $f(x)=1$ for all values of x. Then $a_0=2$, and all the rest of the Fourier coefficients are zero, so that $s_n=1$ for $n>0$. In this case the above formula becomes

$$1 = \frac{1}{2\pi}\int_{0}^{\pi} \frac{\sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u}\,2\,du.$$

Multiplying this by s, and subtracting from (3), we have

$$s_n - s = \frac{1}{2\pi}\int_{0}^{\pi} \frac{\sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u}\{f(x+u)+f(x-u)-2s\}\,du. \qquad (4)$$

A necessary and sufficient condition that the series should converge to the sum s is, therefore, that this integral should tend to zero. The 'convergence problem' is the problem of determining under what conditions the integral tends to zero, and, when it does so, whether $s=f(x)$. We may consider the convergence problem for one particular value of x, for all values of x, or for almost all values of x; or for some other set of values of x. We begin by considering one particular value of x.

13.21. The Riemann-Lebesgue theorem. The following theorem is fundamental in the theory.

If $f(x)$ is integrable over (a,b), then as $\lambda\to\infty$

$$\int_a^b f(x)\cos\lambda x\,dx \to 0, \qquad \int_a^b f(x)\sin\lambda x\,dx \to 0.$$

Consider, for example, the cosine integral. If $f(x)$ is an integral, we may integrate by parts, and obtain

$$\int_a^b f(x)\cos\lambda x\,dx = \left[f(x)\frac{\sin\lambda x}{\lambda}\right]_a^b - \frac{1}{\lambda}\int_a^b f'(x)\sin\lambda x\,dx.$$

The last integral is bounded, so that the whole is $O(1/\lambda)$.

In the general case, given ϵ, we can (§ 12.2) define an absolutely continuous function $\phi(x)$ such that

$$\int_a^b |f(x)-\phi(x)|\,dx < \epsilon.$$

Then $$\left|\int_a^b \{f(x)-\phi(x)\}\cos\lambda x\,dx\right| \leqslant \int_a^b |f(x)-\phi(x)|\,dx < \epsilon$$

for all values of λ; and, by the first part,

$$\left|\int_a^b \phi(x)\cos\lambda x\,dx\right| < \epsilon \qquad (\lambda > \lambda_0).$$

Hence $$\left|\int_a^b f(x)\cos\lambda x\,dx\right| < 2\epsilon \qquad (\lambda > \lambda_0),$$

the required result. A similar proof applies to the sine integral.

There is an alternative proof on the lines of § 13.72, using the example of § 12.2.

13.22. The Riemann-Lebesgue theorem has the following important consequences:

The Fourier coefficients of any integrable function tend to zero.

This is the particular case of the theorem where $\lambda = n$, and the limits are 0 and 2π.

The behaviour of the Fourier series for a particular value of x depends on the behaviour of the function in the immediate neighbourhood of this point only.

Let δ be a positive number less than π, and let $g(t) = f(t)$ in the interval $x-\delta < t < x+\delta$, and $g(t) = 0$ in the rest of the interval $(x-\pi, x+\pi)$. Let the partial sums of the Fourier series of $g(t)$ be denoted by S_n. Then

$$S_n = \frac{1}{2\pi}\int_0^\pi \frac{\sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u}\{g(x+u)+g(x-u)\}\,du$$

$$= \frac{1}{2\pi}\int_0^\delta \frac{\sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u}\{f(x+u)+f(x-u)\}\,du.$$

Hence

$$s_n - S_n = \frac{1}{2\pi}\int_\delta^\pi \frac{\sin(n+\frac{1}{2})u}{\sin\frac{1}{2}u}\{f(x+u)+f(x-u)\}\,du.$$

Now the function

$$\operatorname{cosec} \tfrac{1}{2}u\{f(x+u)+f(x-u)\}$$

is integrable over (δ, π) if $\delta > 0$; and hence, by the Riemann-Lebesgue theorem,

$$s_n - S_n \to 0.$$

Hence, however small δ may be, the behaviour of s_n depends on the nature of $f(t)$ in the interval $(x-\delta, x+\delta)$ only, and is not affected by the values which it takes outside this interval.

It is this property which makes it possible for the series to represent an arbitrary function; but the series only represents the function at the point x as a sort of limit of its average value over the interval $(x-\delta, x+\delta)$, and this will be equal to $f(x)$ only if the behaviour of the function is sufficiently simple. As we have already remarked in § 13.1, the value of $f(t)$ at the point $t = x$ itself does not determine or affect in any way the sum of the series.

13.23. Convergence tests. We first put the 'necessary and sufficient condition for convergence to the sum s' into a more convenient form. Let

$$\phi(u) = f(x+u)+f(x-u)-2s.$$

Then the condition, by § 13.2 (4), is

$$\lim_{n\to\infty} \int_0^\pi \frac{\sin(n+\frac{1}{2})u}{\sin \frac{1}{2}u} \phi(u)\, du = 0. \tag{1}$$

We may replace this by

$$\lim_{n\to\infty} \int_0^\delta \frac{\sin(n+\frac{1}{2})u}{\sin \frac{1}{2}u} \phi(u)\, du = 0, \tag{2}$$

where $0 < \delta \leqslant \pi$; for, by the Riemann-Lebesgue theorem, the difference between the integrals in (1) and (2) tends to zero. Next we may replace (2) by

$$\lim_{n\to\infty} \int_0^\delta \frac{\sin(n+\frac{1}{2})u}{u} \phi(u)\, du = 0; \tag{3}$$

for $(\operatorname{cosec} \frac{1}{2}u - 2/u)\phi(u)$ is integrable over $(0, \delta)$, and so, by the Riemann-Lebesgue theorem,

$$\lim_{n\to\infty} \int_0^\delta \sin(n+\tfrac{1}{2})u \left\{\frac{1}{\sin \frac{1}{2}u} - \frac{2}{u}\right\} \phi(u)\, du = 0.$$

We are now in a position to state some tests for convergence.

13.231. Dini's test. *If $\phi(u)/u$ is integrable over $(0, \delta)$, then the series converges to the sum s.*

This theorem is at once obvious from the above formula (3) and the Riemann-Lebesgue theorem. It should of course be remembered that the integrability of $\phi(u)/u$ in the Lebesgue sense implies 'absolute integrability'. The existence of

$$\lim_{\epsilon\to 0} \int_\epsilon^\delta \frac{\phi(u)}{u}\, du$$

is not a sufficient condition for convergence.

Examples. (i) At any point where $f(x)$ is differentiable, the series converges to the sum $f(x)$.

[At such a point, $\phi(u)/u$ is bounded.]

(ii) More generally, if $f(x)$ satisfies the 'Lipschitz condition' of order α, i.e.

$$f(x+h) - f(x) = O(|h|^\alpha) \qquad (0 < \alpha < 1),$$

then the series converges to the sum $f(x)$.

13.232. Jordan's test. *If $f(t)$ is of bounded variation in the neighbourhood of $t = x$, then the series converges to the sum*

$$\tfrac{1}{2}\{f(x+0) + f(x-0)\}.$$

Since 'bounded variation' means 'bounded variation over an interval', this condition is really one for convergence over an interval.

We know that, if $f(x)$ is of bounded variation, the limits $f(x+0)$ and $f(x-0)$ exist. Hence

$$\phi(u) = f(x+u) + f(x-u) - f(x+0) - f(x-0)$$

is of bounded variation in an interval to the right of $u = 0$, and $\phi(u) \to 0$ as $u \to 0$. Hence we may write

$$\phi(u) = \phi_1(u) - \phi_2(u),$$

where ϕ_1 and ϕ_2 are positive increasing functions of u; each of these functions tends to the same limit as $u \to 0$; and we may,

by subtracting a constant from each function, arrange that this limit shall be zero.

Suppose that δ is so small that $\phi(u)$ is of bounded variation in the interval $(0, \delta)$. Then

$$\int_0^\delta \frac{\sin(n+\frac{1}{2})u}{u}\,\phi(u)\,du$$

$$= \int_0^\delta \frac{\sin(n+\frac{1}{2})u}{u}\,\phi_1(u)\,du - \int_0^\delta \frac{\sin(n+\frac{1}{2})u}{u}\,\phi_2(u)\,du$$

$$= J_1 - J_2,$$

say. Consider the integral J_1. Given ϵ, choose η so small that $\phi_1(\eta) < \epsilon$. Then, by the second mean-value theorem,

$$\int_0^\eta \frac{\sin(n+\frac{1}{2})u}{u}\,\phi_1(u)\,du = \phi_1(\eta)\int_\xi^\eta \frac{\sin(n+\frac{1}{2})u}{u}\,du \quad (0 < \xi < \eta)$$

$$= \phi_1(\eta)\int_{(n+\frac{1}{2})\xi}^{(n+\frac{1}{2})\eta} \frac{\sin v}{v}\,dv.$$

The last integral is bounded for all values of n, ξ, and η, so that

$$\left|\int_0^\eta \frac{\sin(n+\frac{1}{2})u}{u}\,\phi_1(u)\,du\right| < A\epsilon.$$

Having fixed η, by the Riemann-Lebesgue theorem

$$\left|\int_\eta^\delta \frac{\sin(n+\frac{1}{2})u}{u}\,\phi_1(u)\,du\right| < \epsilon \qquad (n > n_0).$$

Hence $J_1 \to 0$; and similarly $J_2 \to 0$. This proves the theorem.

In particular *if $f(x)$ has only a finite number of maxima and minima and a finite number of discontinuities in the interval $(0, 2\pi)$, its Fourier series is convergent for all values of x to the sum* $\frac{1}{2}\{f(x+0)+f(x-0)\}$. For such a function is of bounded variation in the whole interval. These conditions are known as *Dirichlet's conditions*. They are, of course, satisfied in many cases; but they have the disadvantage that the sum of two functions which satisfy them does not itself necessarily satisfy them.

In connexion with Jordan's test, it is interesting to note that *if $f(x)$ is a function of bounded variation over $(0, 2\pi)$, its Fourier series is boundedly convergent.*

For let $0 \leqslant x < \pi$, and write Dirichlet's integral for $s_n(x)$ in the form

$$\frac{1}{2\pi}\int_{-\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{\sin(n+\frac{1}{2})(x-t)}{\sin\frac{1}{2}(x-t)} f(t)\,dt.$$

Since

$$\frac{1}{\sin\frac{1}{2}(x-t)} - \frac{1}{\frac{1}{2}(x-t)}$$

is bounded for $-\frac{3}{2}\pi < x-t < \frac{3}{2}\pi$, this differs by a bounded function from

$$\frac{1}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{\sin(n+\frac{1}{2})(x-t)}{x-t} f(t)\,dt.$$

Let $f(t) = f_1(t) - f_2(t)$, where f_1 and f_2 are positive non-decreasing in $(-\frac{1}{2}\pi, \frac{3}{2}\pi)$. Then, by the second mean-value theorem,

$$\int_{-\frac{1}{2}\pi}^{\frac{3}{2}\pi} \frac{\sin(n+\frac{1}{2})(x-t)}{x-t} f_1(t)\,dt = f_1(\tfrac{3}{2}\pi)\int_{\xi}^{\frac{3}{2}\pi} \frac{\sin(n+\frac{1}{2})(x-t)}{x-t}\,dt$$

$$(-\tfrac{1}{2}\pi < \xi < \tfrac{3}{2}\pi),$$

which is bounded for all n and x, as in the above proof. A similar result holds for f_2. Hence the series is boundedly convergent over $(0, \pi)$, and similarly over $(\pi, 2\pi)$.

13.233. de la Vallée-Poussin's test.* *If the function*

$$\psi(t) = \frac{1}{t}\int_0^t \phi(u)\,du$$

is of bounded variation in an interval to the right of $t = 0$, then the series is convergent. If s is so chosen that $\psi(t) \to 0$ as $t \to 0$, the sum of the series is s.

For

$$\phi(t) = \frac{d}{dt}\{t\psi(t)\} = \psi(t) + t\psi'(t).$$

Since $\psi(t)$ is of bounded variation and tends to zero, the part of the integral § 13.23 (3) involving it tends to zero, as in Jordan's test; and since $\psi'(t)$ is integrable (§ 11.54), the part involving $t\psi'(t)$ tends to zero, as in Dini's test.

* The same test was given previously by du Bois-Reymond, but of course with Riemann integrals.

13.24. Relations between the above tests.* Consider the function

$$f(x) = \frac{1}{\log 1/x} \quad (0 < x < \pi), \qquad = 0 \quad (\pi \leqslant x \leqslant 2\pi).$$

This function is bounded and monotonic in the neighbourhood of $x = 0$, so that Jordan's condition is satisfied, and the series converges. But Dini's condition is not satisfied, since the integral

$$\int_0^\delta \frac{dt}{t \log 1/t}$$

is divergent. Thus *Dini's condition does not include Jordan's.*

On the other hand, *Jordan's condition does not include Dini's.* For consider the function

$$f(x) = x^\alpha \sin 1/x \quad (0 < x < \pi), \qquad = 0 \quad (\pi \leqslant x \leqslant 2\pi),$$

where $0 < \alpha < 1$. Then Dini's condition for convergence at $x = 0$ is obviously satisfied. But the function is not of bounded variation (Ch. XI, ex. 5), i.e. Jordan's condition is not satisfied.

Lastly, *de la Vallée-Poussin's test includes both Dini's and Jordan's*, i.e. if either Dini's or Jordan's condition is fulfilled, then so is de la Vallée-Poussin's.

We first remark that *if $g(x)$ is of bounded variation in $(0, \delta)$, then so is*

$$G(x) = \frac{1}{x} \int_0^x g(t)\, dt.$$

For $g(x) = g_1(x) - g_2(x)$, where $g_1(x)$ and $g_2(x)$ are positive, non-decreasing, and bounded; and

$$G(x) = \frac{1}{x} \int_0^x g_1(x)\, dx - \frac{1}{x} \int_0^x g_2(x)\, dx = G_1(x) - G_2(x),$$

say, and it is easily seen that $G_1(x)$ and $G_2(x)$ are both positive, non-decreasing, and bounded. Hence $G(x)$ is of bounded variation.

The relation between Jordan's test and de la Vallée-Poussin's test follows at once; if $\phi(t)$ is of bounded variation, so is $\psi(t)$.

* For a detailed discussion of this question see Hardy (13).

Now consider Dini's test. If $\phi(u)/u$ is integrable,

$$\chi(t) = \int_0^t \frac{\phi(u)}{u}\,du$$

is a function of bounded variation; and

$$\psi(t) = \frac{1}{t}\int_0^t u\,\frac{d}{du}\{\chi(u)\}\,du = \chi(t) - \frac{1}{t}\int_0^t \chi(u)\,du,$$

which is also of bounded variation, by the above remark. Hence de la Vallée-Poussin's condition is satisfied.

13.25. Convergence throughout an interval. If one of the above conditions is fulfilled at all points of an interval, of course the series converges throughout the interval; and if the condition is fulfilled uniformly, the convergence is uniform. The simplest case is as follows.

The Fourier series of $f(x)$ converges uniformly to $f(x)$ in any interval interior to an interval where $f(x)$ is continuous and of bounded variation.

For in such an interval we can write $f(x) = f_1(x) - f_2(x)$, where $f_1(x)$ and $f_2(x)$ are continuous and non-decreasing. Then, by the property of uniform continuity, we can find η so that

$$|f_1(x+h) - f_1(x)| < \epsilon \qquad (|h| < \eta),$$

the choice of η depending only on ϵ and not on the value of x in the interval. It will be seen on referring to the proof of Jordan's test that this implies the uniform convergence of the integral dealt with in proving the test. We have also to show that the parts of Dirichlet's integral which have been shown to tend to 0, actually tend uniformly to 0; the reader should have no difficulty in verifying this.

The property of uniform convergence is, however, not so important as might be expected in the case of Fourier series, because questions of term-by-term integration can be dealt with under much more general conditions (§ 13.5).

No simple restriction on $f(x)$ which ensures that the Fourier series shall be convergent *almost everywhere*, without obviously proving more than this, appears to be known. It might, for example, be conjectured that continuity would be such a con-

dition; but no result of the kind suggested has been proved. On the other hand, a condition bearing not on the function itself, but on the Fourier coefficients, has been given: *the Fourier series is convergent almost everywhere if the series*

$$\sum (a_n^2+b_n^2)\log n$$

*is convergent.**

13.3. Summation of series by arithmetic means. If a series $u_1+u_2+\ldots$ is not convergent, i.e. if $s_n = u_1+\ldots+u_n$ does not tend to a limit, it is sometimes possible to associate with the series a 'sum' in a less direct way. The simplest such method is 'summation by arithmetic means'. We take the arithmetic mean

$$\sigma_n = \frac{s_1+s_2+\ldots+s_n}{n}$$

of the partial sums of the given series. If $s_n \to s$, then also $\sigma_n \to s$; for if $s_n = s+\delta_n$, then

$$\sigma_n = s+\frac{\delta_1+\delta_2+\ldots+\delta_n}{n},$$

and the last term tends to zero if $\delta_n \to 0$, by the lemma of § 1.23.

But σ_n may tend to a limit even though s_n does not. Consider, for example, the series

$$1-1+1-1+\ldots.$$

Here the partial sums s_n are alternately 1 and 0, and it is easily seen that $\sigma_n \to \frac{1}{2}$.

A series for which σ_n tends to a limit is said to be summable by arithmetic means, or by Cesàro's means of the first order, or $(C, 1)$.

Examples. (i) The series $1+0-1+1+0-1+\ldots$ is summable $(C, 1)$ to the sum $\frac{2}{3}$.

(ii) The series $\sin x + \sin 2x + \sin 3x + \ldots$ is summable $(C, 1)$ for all values of x; the sum is $\frac{1}{2}\cot\frac{1}{2}x$ if x is not an even multiple of π, and otherwise is 0.

(iii) The series $\frac{1}{2} + \cos x + \cos 2x + \cos 3x + \ldots$ is summable $(C, 1)$ to the sum zero if x is not an even multiple of π.

(iv) If $\sum u_n$ is summable $(C, 1)$, $s_n = o(n)$.

[For $s_n = n\sigma_n-(n-1)\sigma_{n-1}$.]

(v) Let $t_n = u_1+2u_2+\ldots+nu_n$. If $\sum u_n$ is summable $(C, 1)$, a necessary and sufficient condition that it should be convergent is $t_n = o(n)$.

[For $t_n = (n+1)s_n-n\sigma_n$.]

* Plessner (2).

(vi) If $\sum u_n$ is summable $(C, 1)$, and $u_n = o(1/n)$, then $\sum u_n$ is convergent.

[For $t_n = o(n)$, by the lemma of § 1.23. The result is analogous to Tauber's theorem.]

(vii) A necessary and sufficient condition that $\sum u_n$ should be summable $(C, 1)$ is that

$$\sum \frac{t_n}{n(n+1)}$$

should be convergent.

$$\left[\text{For} \qquad \sum_{n=1}^{N} \frac{t_n}{n(n+1)} = \frac{N}{N+1}\sigma_N.\right]$$

(viii) If $\sum u_n$ is summable $(C, 1)$, and $u_n = O(1/n)$, then $\sum u_n$ is convergent.

[Hardy; this is analogous to Littlewood's extension of Tauber's theorem. If $\sum u_n$ is not convergent, then $t_N > A_1 N$, or $t_N < -A_1 N$, for an infinity of values of N—say e.g. the former. Since

$$t_{n+1} = t_n + (n+1)u_n > t_n - A_2,$$

we have $$t_{N+\nu} > \tfrac{1}{2}A_1 N \qquad (0 \leqslant \nu < \tfrac{1}{2}NA_1/A_2).$$

Hence $$\sum_{n=N}^{N+\frac{1}{2}NA_1/A_2} \frac{t_n}{n(n+1)} > A,$$

and by (vii) the series is not summable $(C, 1)$.]

(ix) A series of positive terms is summable $(C, 1)$ only if it is convergent.

[If $s_n \to \infty$, then $\sigma_n \to \infty$.]

13.31. Summability of Fourier series. It was discovered by Fejér* that the method of summation by arithmetic means applies particularly well to Fourier series. We write

$$\sigma_n = \frac{s_0 + s_1 + \ldots + s_{n-1}}{n},$$

where s_n is given by § 13.21 (3). Hence

$$\sigma_n = \frac{1}{2n\pi}\int_0^\pi \frac{\sin\frac{1}{2}u + \sin\frac{3}{2}u + \ldots + \sin(n-\frac{1}{2})u}{\sin\frac{1}{2}u}\{f(x+u)+f(x-u\}\,du$$

$$= \frac{1}{2n\pi}\int_0^\pi \frac{\sin^2\frac{1}{2}nu}{\sin^2\frac{1}{2}u}\{f(x+u)+f(x-u)\}\,du. \tag{1}$$

This formula is known as *Fejér's integral*. Its importance is due to the fact that the factor $\sin^2\frac{1}{2}nu/\sin^2\frac{1}{2}u$ is positive. This makes

* Fejér (1).

it much easier to deal with Fejér's integral than with Dirichlet's, in which the corresponding factor, $\sin(n+\frac{1}{2})u/\sin\frac{1}{2}u$, oscillates between positive and negative values.

In the particular case where $f(x)=1$, the formula becomes

$$1=\frac{1}{2n\pi}\int_0^\pi \frac{\sin^2\frac{1}{2}nu}{\sin^2\frac{1}{2}u}\,2\,du,$$

since now $\sigma_n=1$ for $n>0$. Hence, multiplying by s and subtracting,

$$\sigma_n-s=\frac{1}{2n\pi}\int_0^\pi \frac{\sin^2\frac{1}{2}nu}{\sin^2\frac{1}{2}u}\{f(x+u)+f(x-u)-2s\}\,du. \qquad (2)$$

A necessary and sufficient condition that the series should be summable $(C,1)$ to the sum s is, therefore, that the integral (2) should tend to zero.

As in the convergence problem, we can simplify the condition. We write

$$\phi(u)=f(x+u)+f(x-u)-2s$$

as before. Then, if δ is any positive number less than π, a necessary and sufficient condition that the series should be summable $(C,1)$ to s is

$$\lim_{n\to\infty}\frac{1}{n}\int_0^\delta \frac{\sin^2\frac{1}{2}nu}{\sin^2\frac{1}{2}u}\,\phi(u)\,du=0; \qquad (3)$$

for

$$\left|\frac{1}{n}\int_\delta^\pi \frac{\sin^2\frac{1}{2}nu}{\sin^2\frac{1}{2}u}\,\phi(u)\,du\right|\leqslant\frac{1}{n}\int_\delta^\pi \frac{|\phi(u)|}{\sin^2\frac{1}{2}u}\,du,$$

which plainly tends to zero. Finally, the condition may be put in the form

$$\lim_{n\to\infty}\frac{1}{n}\int_0^\delta \frac{\sin^2\frac{1}{2}nu}{u^2}\,\phi(u)\,du=0; \qquad (4)$$

for

$$\left|\frac{1}{n}\int_0^\delta \sin^2\tfrac{1}{2}nu\left\{\frac{1}{\sin^2\frac{1}{2}u}-\frac{1}{(\frac{1}{2}u)^2}\right\}\phi(u)\,du\right|$$

$$\leqslant\frac{1}{n}\int_0^\delta \left\{\frac{1}{\sin^2\frac{1}{2}u}-\frac{1}{(\frac{1}{2}u)^2}\right\}|\phi(u)|\,du,$$

which tends to zero.

13.32. Fejér's theorem. *The Fourier series of $f(x)$ is summable $(C, 1)$ to the sum*

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\}$$

for every value of x for which this expression has a meaning. In particular, the series is summable $(C, 1)$ to the sum $f(x)$ at every point where $f(x)$ is continuous.

We now put $s = \frac{1}{2}\{f(x+0)+f(x-0)\}$ in the above formulae. Then $\phi(u) \to 0$ with u, and we have to prove that 13.31 (4) is true. Suppose that $|\phi(u)| \leqslant \epsilon$ for $u \leqslant \eta$. Then

$$\left|\frac{1}{n}\int_0^\delta \frac{\sin^2\frac{1}{2}nu}{u^2}\,\phi(u)\,du\right| \leqslant \frac{1}{n}\int_0^\eta \frac{\sin^2\frac{1}{2}nu}{u^2}\,\epsilon\,du + \frac{1}{n}\int_\eta^\delta \frac{\sin^2\frac{1}{2}nu}{u^2}\,|\phi(u)|\,du$$

$$\leqslant \frac{\epsilon}{n}\int_0^\eta \frac{\sin^2\frac{1}{2}nu}{u^2}\,du + \frac{1}{n}\int_\eta^\delta \frac{|\phi(u)|}{u^2}\,du$$

$$= I_1 + I_2,$$

say. Now

$$\frac{1}{n}\int_0^\eta \frac{\sin^2\frac{1}{2}nu}{u^2}\,du = \frac{1}{2}\int_0^{\frac{1}{2}n\eta} \frac{\sin^2 v}{v^2}\,dv < \frac{1}{2}\int_0^\infty \frac{\sin^2 v}{v^2}\,dv,$$

which is a constant. Hence $I_1 < A\epsilon$. Having fixed η, it is clear that $I_2 \to 0$ as $n \to \infty$. This proves the theorem.

13.33. Summability throughout an interval. The following theorem is an almost immediate consequence of Fejér's theorem.

The Fourier series of $f(x)$ is uniformly summable in any interval included in an interval where $f(x)$ is continuous.

For $f(x)$ is uniformly continuous in any such interval, and so, in the above proof, the choice of η depends only on ϵ and not on x. The result follows at once from this.

Weierstrass's approximation theorem. *If $f(x)$ is continuous in (a, b), and ϵ is a given positive number, there is a polynomial $p(x)$ such that*

$$|f(x)-p(x)| < \epsilon \qquad (a \leqslant x \leqslant b).$$

We can make a preliminary transformation so that the interval considered lies within $(0, 2\pi)$. Then, by the above theorem, there is a 'trigonometrical polynomial' $\sigma_n(x)$ such that

$|f(x)-\sigma_n(x)| < \frac{1}{2}\epsilon$ throughout the interval. If we replace each sine and cosine in $\sigma_n(x)$ by a sufficiently large number of terms in its power series, we obtain a polynomial $p(x)$ such that $|\sigma_n(x)-p(x)| < \frac{1}{2}\epsilon$ throughout the interval. This proves the theorem.

13.34. Almost everywhere summability. As long as we restrict ourselves to ordinary convergence, we cannot show that the Fourier series of a function represents the function in general, without imposing some rather heavy restriction on the function. The theory of summability removes this defect.

The Fejér-Lebesgue theorem. *The Fourier series of $f(x)$ is summable $(C, 1)$ to the sum $f(x)$, for every value of x for which*

$$\int_0^t |f(x+u)-f(x)|\,du = o(t). \tag{1}$$

In particular, it is summable $(C, 1)$ to $f(x)$ almost everywhere.

We have shown in § 11.6 that the condition (1) is satisfied for almost all values of x, for any integrable function. The second part of the theorem therefore follows at once from the first.

Let x be a point where (1) is satisfied, and take $s = f(x)$ in the formulae of § 13.31. Then

$$\int_0^t |\phi(u)|\,du = \int_0^t |f(x+u)+f(x-u)-2f(x)|\,du$$

$$\leqslant \int_0^t |f(x+u)-f(x)|\,du + \int_0^t |f(x-u)-f(x)|\,du = o(t).$$

Let
$$\Phi(t) = \int_0^t |\phi(u)|\,du,$$

and, given ϵ, choose η so that $\Phi(t) < \epsilon t$ for $t \leqslant \eta$. We suppose that $n > 1/\eta$, and write

$$\int_0^\delta \frac{\sin^2 \frac{1}{2}nu}{u^2}\phi(u)\,du = \int_0^{1/n} + \int_{1/n}^{\eta} + \int_{\eta}^{\delta} = J_1+J_2+J_3.$$

Then, since $\sin^2\theta \leqslant \theta^2$,

$$|J_1| \leqslant (\tfrac{1}{2}n)^2 \int_0^{1/n} |\phi(u)|\,du < \tfrac{1}{4}\epsilon n,$$

$$|J_2| \leqslant \int_{1/n}^{\eta} \frac{|\phi(u)|}{u^2}\,du = \frac{\Phi(\eta)}{\eta^2} - n^2\Phi\left(\frac{1}{n}\right) + 2\int_{1/n}^{\eta} \frac{\Phi(u)}{u^3}\,du$$

$$< \frac{\epsilon}{\eta} + 2\epsilon \int_{1/n}^{\eta} \frac{du}{u^2} < \epsilon/\eta + 2\epsilon n < 3\epsilon n,$$

and obviously $$|J_3| < \frac{A}{\eta^2}.$$

Hence $$\left|\frac{1}{n}\int_0^{\delta} \frac{\sin^2 \frac{1}{2}nu}{u^2}\phi(u)\,du\right| < \tfrac{1}{4}\epsilon + 3\epsilon + \frac{A}{n\eta^2},$$

and the required result follows on choosing first ϵ, then η, and then n.

13.35. An immediate corollary is that *a trigonometrical series cannot be the Fourier series of two functions which differ in a set of positive measure.* For if it is the Fourier series of $f(x)$ and of $g(x)$, it is summable $(C, 1)$ both to $f(x)$ and to $g(x)$ almost everywhere. Hence $f(x) = g(x)$ almost everywhere.

13.4. A continuous function with a divergent Fourier series. While we have seen that the continuity of a function is a sufficient condition for its Fourier series to be summable $(C, 1)$, for convergence we have had to assume other conditions. That this is really in accordance with the facts is shown by the following example, due to Fejér,* of a Fourier series which is divergent at a point, although the function which gives rise to it is continuous.

13.41. We first require a lemma.

The sum

$$\phi(n, r, x) = \frac{\cos(r+1)x}{2n-1} + \frac{\cos(r+2)x}{2n-3} + \dots + \frac{\cos(r+n)x}{1} -$$

$$- \frac{\cos(r+n+1)x}{1} - \frac{\cos(r+n+2)x}{3} - \dots - \frac{\cos(r+2n)x}{2n-1}$$

is bounded for all values of n, r, and x.

* Fejér (2), (3), (4).

We have

$$\phi(n,r,x) = \sum_{\nu=1}^{n} \frac{\cos(r+n-\nu+1)x}{2\nu-1} - \sum_{\nu=1}^{n} \frac{\cos(r+n+\nu)}{2\nu-1}$$

$$= 2\sin(r+n+\tfrac{1}{2})x \sum_{\nu=1}^{n} \frac{\sin(\nu-\frac{1}{2})x}{2\nu-1}$$

$$= 2\sin(r+n+\tfrac{1}{2})x \left\{ \sum_{\lambda=1}^{2n} \frac{\sin\frac{1}{2}\lambda x}{\lambda} - \frac{1}{2}\sum_{\mu=1}^{n} \frac{\sin\mu x}{\mu} \right\},$$

and each of the sums in the bracket is bounded (§ 1.76).

13.42. Let G_n denote the group of $2n$ numbers

$$\frac{1}{2n-1}, \frac{1}{2n-3}, \ldots, \frac{1}{3}, 1, -1, -\frac{1}{3}, \ldots, -\frac{1}{2n-1}.$$

Let $\lambda_1, \lambda_2, \ldots$ denote an increasing sequence of integers. Take the numbers of the groups $G_{\lambda_1}, G_{\lambda_2}, \ldots$ in order, and multiply each of the numbers of the group G_{λ_ν} by ν^{-2}. We obtain the sequence

$$\frac{1}{1^2}\frac{1}{2\lambda_1-1}, \ldots, -\frac{1}{1^2}\frac{1}{2\lambda_1-1}, \frac{1}{2^2(2\lambda_2-1)}, \frac{1}{2^2(2\lambda_2-3)}, \ldots,$$

say $\alpha_1, \alpha_2, \ldots$.

Now consider the series

$$\sum_{n=1}^{\infty} \alpha_n \cos nx. \tag{1}$$

Suppose first that the terms corresponding to each group G_{λ_ν} are bracketed together. The bracketed series is

$$\sum_{n=1}^{\infty} \frac{\phi(\lambda_n, 2\lambda_1+2\lambda_2+\ldots+2\lambda_{n-1}, x)}{n^2}, \tag{2}$$

which is absolutely and uniformly convergent, by the lemma. The sum of the series (2), say $f(x)$, is therefore a continuous function.

We next observe that the series (1) is the Fourier series of $f(x)$. For since (2) is uniformly convergent, we may multiply it by $\cos mx$ or $\sin mx$ and integrate term by term. The integral of each term is zero, except that of the one containing the term $\alpha_m \cos mx$; and from this we obtain

$$\int_0^{2\pi} f(x)\cos mx\, dx = \pi\alpha_m.$$

The numbers α_m are therefore the Fourier cosine coefficients of $f(x)$.

We show finally that the numbers λ_ν can be chosen so that the series (1) is divergent at the point $x=0$, i.e. that the series $\alpha_1+\alpha_2+\ldots$ is divergent. Let s_n be its nth partial sum. Then

$$s_{2\lambda_1+2\lambda_2+\ldots+2\lambda_{\nu-1}+\lambda_\nu}=\frac{1}{\nu^2}\left(\frac{1}{2\lambda_\nu-1}+\frac{1}{2\lambda_\nu-3}+\ldots+\frac{1}{3}+1\right)\sim\frac{\log\lambda_\nu}{2\nu^2}.$$

If the numbers λ_ν tend to infinity sufficiently rapidly, e.g. if $\lambda_\nu=\nu^{\nu^2}$, it follows that $s_n\to\infty$ as $n\to\infty$ through a certain sequence of values. Hence the series is divergent.

13.43. Fejér's example, together with a simple argument depending on Dirichlet's integral, enables us to say how large the partial sums s_n of a Fourier series of a continuous function can be.

If $f(x)$ is continuous, then

$$s_n=o(\log n);$$

and no more is true, since, if $\psi(n)$ is a function which decreases steadily to zero, however slowly, there is a Fourier series of a continuous function for which

$$s_n>\psi(n)\log n$$

for arbitrarily large values of n.

For the first part, we have to prove that

$$\int_0^\delta \frac{\sin(n+\frac{1}{2})u}{u}\phi(u)\,du=o(\log n)$$

if $\phi(u)\to 0$ as $u\to 0$. Suppose that $|\phi(u)|<\epsilon$ for $u\leqslant\eta$; and, if $n+\frac{1}{2}>1/\eta$, put

$$\int_0^\delta \frac{\sin(n+\frac{1}{2})u}{u}\phi(u)\,du=\int_0^{1/(n+\frac{1}{2})}+\int_{1/(n+\frac{1}{2})}^{\eta}+\int_\eta^\delta=I_1+I_2+I_3.$$

Then
$$|I_1|\leqslant(n+\tfrac{1}{2})\int_0^{1/(n+\frac{1}{2})}|\phi(u)|\,du<\epsilon,$$

$$|I_2|\leqslant\int_{1/(n+\frac{1}{2})}^{\eta}\frac{\epsilon}{u}\,du<\epsilon\log(n+\tfrac{1}{2}),$$

and
$$|I_3| \leqslant \frac{1}{\eta} \int_{\eta}^{\delta} |\phi(u)|\, du.$$

The result clearly follows from these inequalities.

The second part is obtained by taking λ_ν sufficiently large in Fejér's example. Suppose that $\lambda_\nu > 2\nu$, and let
$$n = 2\lambda_1 + 2\lambda_2 + \ldots + 2\lambda_{\nu-1} + \lambda_\nu.$$
Then
$$\lambda_\nu < n < 2\nu\lambda_\nu < \lambda_\nu^2.$$
Now $s_n > \psi(n)\log n$ for sufficiently large values of ν, if
$$\frac{\log \lambda_\nu}{2\nu^2} > \psi(n)\log n;$$
and since $\psi(n)\log n < \psi(\lambda_\nu)\log \lambda_\nu^2$, this is true if
$$\psi(\lambda_\nu) < \frac{1}{4\nu^2};$$
and this will be so if the numbers λ_ν tend to infinity rapidly enough.

13.5. Integration of Fourier series. *Any Fourier series, whether convergent or not, may be integrated term by term between any limits; that is, the sum of the integrals of the separate terms is the integral of the function of which the series is the Fourier series.*

Let $f(x)$ have the Fourier coefficients a_n, b_n, and let
$$F(x) = \int_0^x \{f(t) - \tfrac{1}{2}a_0\}\, dt.$$
Then $F(x)$ is periodic, continuous, and of bounded variation. Hence it can be expanded in a Fourier series, say
$$F(x) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx),$$
convergent for all values of x. Here
$$A_n = \frac{1}{\pi} \int_0^{2\pi} F(x)\cos nx\, dx$$
$$= \frac{1}{\pi}\left[F(x)\frac{\sin nx}{n}\right]_0^{2\pi} - \frac{1}{n\pi}\int_0^{2\pi} \{f(x) - \tfrac{1}{2}a_0\}\sin nx\, dx$$

$$= -\frac{1}{n\pi}\int_0^{2\pi} f(x)\sin nx\,dx = -\frac{b_n}{n};$$

and $$B_n = \frac{1}{\pi}\int_0^{2\pi} F(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[-F(x)\frac{\cos nx}{n}\right]_0^{2\pi} + \frac{1}{n\pi}\int_0^{2\pi}\{f(x)-\tfrac{1}{2}a_0\}\cos nx\,dx$$

$$= \frac{1}{n\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{a_n}{n},$$

the integrated terms vanishing since $F(2\pi) = F(0) = 0$. Hence

$$F(x) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty}\frac{a_n\sin nx - b_n\cos nx}{n}.$$

Putting $x = 0$, we obtain

$$\tfrac{1}{2}A_0 = \sum_{n=1}^{\infty}\frac{b_n}{n},$$

and, adding,

$$F(x) = \sum_{n=1}^{\infty}\frac{a_n\sin nx + b_n(1-\cos nx)}{n}.$$

This proves the theorem.

13.51. An interesting particular case is that *the series*

$$\sum_{n=1}^{\infty}\frac{b_n}{n}$$

is convergent. This remark enables us to write down convergent trigonometrical series which are not Fourier series. A simple example is

$$\sum_{n=2}^{\infty}\frac{\sin nx}{\log n}.$$

This is convergent for all values of x, but it cannot be the Fourier series of its sum, since the series

$$\sum\frac{1}{n\log n}$$

is divergent. Actually the sum of this trigonometrical series is not integrable in the sense of Lebesgue, and it is easy to prove

directly that the sum of the integrated series $\sum \cos nx/n \log n$ tends to infinity as $x \to 0$.

13.52. The following alternative proof of the above integration theorem is also interesting. We know that the series

$$\frac{\sin(x-t)}{1}+\frac{\sin 2(x-t)}{2}+\ldots=\phi(t)$$

is boundedly convergent. Hence we may multiply by $f(t)/\pi$ and integrate term by term over $(0, 2\pi)$. On the left we obtain

$$\sum_{n=1}^{\infty}\frac{1}{n\pi}\int_0^{2\pi}\sin n(x-t)f(t)\,dt=\sum_{n=1}^{\infty}\frac{a_n\sin nx-b_n\cos nx}{n},$$

the integrated series. On the right we get

$$\frac{1}{\pi}\int_0^{2\pi}\phi(t)f(t)\,dt=\frac{1}{\pi}\int_{x-2\pi}^{x}\phi(t)f(t)\,dt=\frac{1}{\pi}\int_{x-2\pi}^{x}\tfrac{1}{2}(\pi-x+t)f(t)\,dt$$

$$=\frac{1}{2\pi}\left[(\pi-x+t)F(t)\right]_{x-2\pi}^{x}-\frac{1}{2\pi}\int_{x-2\pi}^{x}F(t)\,dt=F(x)-\frac{1}{2\pi}\int_0^{2\pi}F(t)\,dt,$$

since $F(x-2\pi)=F(x)$. The result now follows as before.

13.53. A similar method leads to the following more general integration theorem.

A Fourier series may be multiplied by any function of bounded variation and integrated term by term between any finite limits.

Let $$g(x)=\tfrac{1}{2}\alpha_0+\sum_{n=1}^{\infty}(\alpha_n\cos nx+\beta_n\sin nx)$$

be a function of bounded variation. The series being boundedly convergent (§ 13.232), we may multiply by any integrable function $f(x)$ and integrate term by term over $(0, 2\pi)$. We obtain

$$\frac{1}{\pi}\int_0^{2\pi}f(x)g(x)\,dx=\tfrac{1}{2}a_0\alpha_0+\sum_{n=1}^{\infty}(a_n\alpha_n+b_n\beta_n), \tag{1}$$

where a_n, b_n, are the Fourier coefficients of $f(x)$. This is the same result as we should have obtained by multiplying the Fourier series for $f(x)$ by $g(x)$ and integrating term by term over $(0, 2\pi)$.

A similar result may be obtained for other ranges of integration by replacing $g(x)$ by 0 outside the required range.

13.54. Parseval's theorem. If $f(x)$ is of bounded variation, we may put $g(x)=f(x)$ in 13.53 (1), and obtain

$$\frac{1}{\pi}\int_0^{2\pi}\{f(x)\}^2\,dx=\tfrac{1}{2}a_0^2+\sum_{n=1}^{\infty}(a_n^2+b_n^2).$$

This is known as Parseval's theorem. We shall show in § 13.63 that it is true under much more general conditions than those we have so far assumed.

13.6. Functions of the class L^2: Bessel's inequality. Let $f(x)$ be a function of the class $L^2(0, 2\pi)$, with Fourier coefficients a_n, b_n. Then

$$\phi(x)=f(x)-\tfrac{1}{2}a_0-\sum_{m=1}^{n}(a_m\cos mx+b_m\sin mx)$$

also belongs to L^2; and

$$\frac{1}{\pi}\int_0^{2\pi}\{\phi(x)\}^2\,dx=\frac{1}{\pi}\int_0^{2\pi}\{f(x)\}^2\,dx+\tfrac{1}{2}a_0^2+\sum_{m=1}^{n}(a_m^2+b_m^2)-$$

$$-\frac{a_0}{\pi}\int_0^{2\pi}f(x)\,dx-\frac{2}{\pi}\sum_{m=1}^{n}\int_0^{2\pi}(a_m\cos mx+b_m\sin mx)f(x)\,dx$$

$$=\frac{1}{\pi}\int_0^{2\pi}\{f(x)\}^2\,dx-\tfrac{1}{2}a_0^2-\sum_{m=1}^{n}(a_m^2+b_m^2),$$

by the Euler-Fourier formulae. Since the left-hand side is not negative, it follows that

$$\tfrac{1}{2}a_0^2+\sum_{m=1}^{n}(a_m^2+b_m^2)\leqslant\frac{1}{\pi}\int_0^{2\pi}\{f(x)\}^2\,dx \tag{1}$$

for all values of n. This result is known as Bessel's inequality.

Since the right-hand side of (1) is independent of n, it follows that *the series*

$$\tfrac{1}{2}a_0^2+\sum_{m=1}^{\infty}(a_m^2+b_m^2) \tag{2}$$

is convergent. Also

$$\tfrac{1}{2}a_0^2+\sum_{m=1}^{\infty}(a_m^2+b_m^2)\leqslant\frac{1}{\pi}\int_0^{2\pi}\{f(x)\}^2\,dx. \tag{3}$$

13.61. Parseval's theorem for continuous functions. We have seen that, for functions of bounded variation, the above inequality becomes an equality, viz. Parseval's theorem. The same result for continuous functions may be proved as follows. If $f(x)$ is continuous, $\sigma_n(x)$ tends uniformly to $f(x)$, and hence

$$\lim_{n\to\infty} \frac{1}{\pi} \int_0^{2\pi} \{f(x)-\sigma_n(x)\}f(x)\,dx = 0.$$

Now $$\sigma_n(x) = \tfrac{1}{2}a_0 + \sum_{m=1}^{n-1} (a_m \cos mx + b_m \sin mx)\left(1-\frac{m}{n}\right),$$

and, evaluating the integral as in the previous section, we obtain

$$\frac{1}{\pi}\int_0^{2\pi} \{f(x)\}^2\,dx - \tfrac{1}{2}a_0^2 - \sum_{m=1}^{n-1} (a_m^2+b_m^2)\left(1-\frac{m}{n}\right) \to 0.$$

Parseval's formula therefore holds if the series is summed $(C, 1)$. Since by § 13.6 the series is convergent, it follows from § 13.3 that it holds in the ordinary sense.*

There is no difficulty in extending this proof to functions which have simple discontinuities. Actually Parseval's theorem holds for all functions of the class L^2. We shall prove this as a corollary of the theorem of the next section.

13.62. The Riesz-Fischer theorem. Let

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

be any trigonometrical series with coefficients such that the series 13.6 (2) is convergent. Nothing that we have proved so far in this chapter enables us to decide whether such a series is a Fourier series. The problem is solved by means of the theory of mean convergence (§ 12.5). This theory was in fact originally constructed to deal with this very problem.

The following theorem was proved almost simultaneously by F. Riesz and Fischer.†

If the numbers a_n, b_n are such that the series 13.6 (2) *is convergent, then the series* (1) *is the Fourier series of a function $f(x)$*

* A number of different proofs under various conditions are given by Julia, *Exercices d'analyse*, 180–6.

† F. Riesz (1), Fischer (1).

of the class L^2. The partial sums of the series converge in mean to $f(x)$.

Denoting the nth partial sum of (1) by $s_n(x)$, we have

$$\int_0^{2\pi} \{s_n(x)-s_m(x)\}^2\,dx = \int_0^{2\pi} \Big\{\sum_{\nu=m+1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x)\Big\}^2\,dx$$

$$= \pi \sum_{\nu=m+1}^{n} (a_\nu^2+b_\nu^2),$$

all the product terms disappearing on integration. The right-hand side tends to zero when m and n tend independently to infinity. Hence $s_n(x)$ converges in mean to a function, $f(x)$ say, of the class L^2.

Also, by § 12.53,

$$\lim_{n\to\infty} \int_0^{2\pi} s_n(x)\cos \nu x\,dx = \int_0^{2\pi} f(x)\cos \nu x\,dx.$$

But the integral on the left is equal to πa_ν, if $n \geqslant \nu$. Hence

$$a_\nu = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos \nu x\,dx,$$

i.e. a_ν is the νth Fourier cosine coefficient of $f(x)$. Similarly b_ν is the νth sine coefficient. Hence the given trigonometrical series is the Fourier series of the function $f(x)$.

It is important to observe that it is here that the Lebesgue integral first plays an indispensable part in the theory. Most of the previous analysis is true for Riemann integrals and elementary generalized absolutely convergent integrals. Here the result shows that the extension to Lebesgue integrals is really necessary.

13.63. Parseval's theorem for functions of the class L^2. Let $f(x)$ be any function belonging to $L^2(0, 2\pi)$, and let its Fourier series have the usual form. Then the series 13.6 (2) is convergent. Hence, by the Riesz-Fischer theorem, the partial sums $s_n(x)$ converge in mean to a function $g(x)$, of which the given series is the Fourier series. Hence, by § 13.35, $g(x) = f(x)$ almost everywhere. Also, by § 12.52,

$$\lim_{n\to\infty} \int_0^{2\pi} \{s_n(x)\}^2\,dx = \int_0^{2\pi} \{f(x)\}^2\,dx,$$

and on evaluating the integral on the left-hand side we obtain Parseval's formula.

The more general formula 13.53 (1) *also holds if* $f(x)$ *and* $g(x)$ *are any two functions of the class* L^2. For Parseval's formula holds for the functions $f(x)+g(x)$ and $f(x)-g(x)$, and the result stated follows on subtraction.

13.7. Properties of Fourier coefficients. Originally the Fourier coefficients were merely the material out of which the Fourier series was constructed. But the coefficients have some interesting properties of their own. In fact Bessel's inequality, and the theorems of Parseval and Riesz-Fischer call attention to the problem of the behaviour of the Fourier coefficients of given classes of functions and give some important information about it.

The first theorem of this kind (§ 13.22) is that *the Fourier coefficients of any integrable function tend to zero.* On the other hand, they do not tend to zero in any definite order; that is, any theorem such as '$a_n = O(1/\log n)$ for all integrable functions' is certainly false. For consider the series

$$\sum_{n=1}^{\infty} \frac{\cos k_n x}{n^2},$$

where k_n denotes a sequence of positive integers which tends to infinity rapidly as $n \to \infty$. The series is uniformly convergent, and so is the Fourier series of its sum; and

$$a_{k_n} = \frac{1}{n^2},$$

which falsifies any theorem of the kind suggested, if k_n tends to infinity rapidly enough.

13.71. Suppose next that $f(x)$ belongs to the class L^2. This does not enable us to prove any more about the order of the coefficients; in fact the function defined by the above series is clearly continuous, and so belongs to L^2. But we do obtain a definite result about the *average* order, viz. that

$$\sum (a_n^2 + b_n^2)$$

is convergent (§ 13.6).

This result has been generalized so as to apply to other

Lebesgue classes; *if $f(x)$ belongs to L^p, where $1 < p \leqslant 2$, then the series*

$$\sum (|a_n|^{p/(p-1)} + |b_n|^{p/(p-1)})$$

is convergent.

The proof of this theorem is, however, too long to be given here.*

There is also a corresponding extension of the Riesz-Fischer theorem: *if the series* $\sum (|a_n|^p + |b_n|^p)$,

where $1 < p \leqslant 2$, is convergent, then the numbers a_n, b_n are the Fourier coefficients of a function of the class $L^{p/(p-1)}$.

Both these theorems cease to be true if $p > 2$, so that they are not converses of each other unless $p = 2$.

13.72. If we make still more special assumptions about the function, we obtain new results about the coefficients. *Suppose that $f(x)$ satisfies a Lipschitz condition of order α, i.e. as $h \to 0$*

$$f(x+h) - f(x) = O(|h|^\alpha) \qquad (0 < \alpha \leqslant 1)$$

uniformly with respect to x. Then

$$a_n = O(n^{-\alpha}), \qquad b_n = O(n^{-\alpha}).$$

For

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx = -\frac{1}{\pi} \int_{-\pi/n}^{2\pi - \pi/n} f\left(\frac{\pi}{n} + t\right) \cos nt \, dt$$

$$= -\frac{1}{\pi} \int_0^{2\pi} f\left(\frac{\pi}{n} + t\right) \cos nt \, dt,$$

and hence also

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} \left\{ f(x) - f\left(\frac{\pi}{n} + x\right) \right\} \cos nx \, dx$$

$$= \int_0^{2\pi} O\left(\frac{1}{n^\alpha}\right) dx = O\left(\frac{1}{n^\alpha}\right);$$

and similarly for b_n.

13.73. The next result of this kind is that *if $f(x)$ is of bounded variation, then* $a_n = O(1/n)$, $\quad b_n = O(1/n)$.

* W. H. Young (2), (3), (5), (6), Hausdorff (1), F. Riesz (4).

For $f(x)=f_1(x)-f_2(x)$, where $f_1(x)$ and $f_2(x)$ are positive and non-decreasing. Hence, by the second mean-value theorem,

$$\int_0^{2\pi} f_1(x)\cos nx\,dx = f_1(2\pi)\int_\xi^{2\pi}\cos nx\,dx \qquad (0<\xi<2\pi)$$

$$= -f_1(2\pi)\frac{\sin n\xi}{n} = O\left(\frac{1}{n}\right),$$

and a similar result holds for the other integral.

An alternative proof of Jordan's theorem (§ 13.232) can be deduced from this result. If $f(x)$ is of bounded variation, its Fourier series is summable $(C, 1)$ to the sum $\frac{1}{2}\{f(x+0)+f(x-0)\}$, by § 13.32. Since $a_n = O(1/n)$, $b_n = O(1/n)$, the series actually converges to this sum (§ 13.3, ex. (viii)).

If $f(x)$ is an integral, and has the period 2π, then

$$a_n = o(1/n), \qquad b_n = o(1/n).$$

For if $$f(x)=f(0)+\int_0^x \phi(t)\,dt \qquad (x\geqslant 0),$$

then $$\pi a_n = \left[f(x)\frac{\sin nx}{n}\right]_0^{2\pi} - \frac{1}{n}\int_0^{2\pi}\phi(x)\sin nx\,dx,$$

$$\pi b_n = \left[-f(x)\frac{\cos nx}{n}\right]_0^{2\pi} + \frac{1}{n}\int_0^{2\pi}\phi(x)\cos nx\,dx.$$

The integrated terms are zero, since $f(2\pi)=f(0)$ and the integrals on the right tend to zero, by the Riemann-Lebesgue theorem. This proves the theorem.

If $f'(x)$ satisfies special conditions, such as a Lipschitz condition, still further results of the same kind can, of course, be obtained.

13.8. Uniqueness of trigonometrical series. At the beginning of the chapter we associated with an integrable function a particular trigonometrical series, viz. the Fourier series of the function; and we have shown that the Fourier series does, in various ways, represent the function. The reader might, however, still contend that we had attached undue importance to Fourier series, and that there might be other types of trigonometrical series in which a given function could be expanded.

It is difficult to give a complete solution of this problem. If, however, we assume enough about the set of points where the series converge, we can show that, if a trigonometrical series converges to a given function, it is the only such series which does so; and therefore that, if the function can be expanded in a convergent Fourier series, it cannot be expanded in a convergent trigonometrical series of any other form.

The theory is due to Riemann, du Bois-Reymond, and Cantor. The theorem which we shall prove is as follows.

If two trigonometrical series converge to the same sum in the interval $(0, 2\pi)$, *with the possible exception of a finite number of points, then corresponding coefficients in the two series are equal, i.e. the series are identical.*

This is not all that is known, and more general theorems will be found, e.g., in Hobson's *Theory of Functions*, §§ 420–50. But some extensions which might naturally be suggested are not true; if we say 'are summable $(C, 1)$' instead of 'converge', the theorem becomes false, as is shown by § 13.3, ex. (iii).

The question whether a given trigonometrical series is a Fourier series is really a problem of integral equations. We are given numbers a_0, a_1, $b_1, \ldots$, and it is required to determine whether there is an integrable function $f(x)$ such that the Euler-Fourier formulae § 13.1 (2), (3), are true. The question is not settled by mere convergence, since a trigonometrical series may be everywhere convergent without being a Fourier series (§ 13.51). But if it converges uniformly, or boundedly, or in mean with index p $(p \geqslant 1)$, then it is a Fourier series; and the theorems of §§ 13.62–13.71 enable us to state conditions for mean convergence, with $p \geqslant 2$.

Another theorem which would naturally suggest itself is that if a trigonometrical series converges almost everywhere to an integrable function, then it is the Fourier series of the function; but this is not necessarily true, and the state of affairs is rather complicated.

The proof of the theorem stated above depends on a number of lemmas.

13.81. Cantor's lemma. *If* $a_n \cos nx + b_n \sin nx$ *tends to* 0 *for all values of* x *in an interval, then* a_n *and* b_n *tend to* 0.

Suppose that $a_n \cos nx + b_n \sin nx \to 0$ in the interval (α, β).

If the lemma is false, there is a constant A and a sequence of values of n for which $a_n^2+b_n^2>A$. Hence, as $n\to\infty$ through this sequence, the function

$$f_n(x)=\frac{(a_n\cos nx+b_n\sin nx)^2}{a_n^2+b_n^2}$$

converges boundedly to 0 in (α,β). Hence, by the theorem of bounded convergence,

$$\int_\alpha^\beta f_n(x)\,dx\to 0.$$

But, evaluating the integral, we find that

$$\int_\alpha^\beta f_n(x)\,dx=\tfrac{1}{2}(\beta-\alpha)+O\left(\frac{1}{n}\right),$$

and this gives a contradiction. This proves the lemma.

13.82. Suppose now that the series

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) \tag{1}$$

converges to the sum $f(x)$ in $(0,2\pi)$, except possibly at a finite number of points. Let

$$F(x)=\tfrac{1}{4}a_0x^2-\sum_{n=1}^{\infty}\frac{a_n\cos nx+b_n\sin nx}{n^2}. \tag{2}$$

Since by Cantor's lemma a_n and b_n tend to zero, this series is uniformly convergent, and $F(x)$ is continuous, for all values of x. If we could differentiate twice term by term, we should have $F''(x)=f(x)$. We cannot necessarily do this, and instead have to proceed as follows.

Riemann's First Theorem. *If*

$$G(x,h)=\frac{F(x+2h)+F(x-2h)-2F(x)}{4h^2}, \tag{3}$$

then $G(x,h)\to f(x)$ as $h\to 0$, for all values of x for which the series (1) *converges to $f(x)$.*

We have

$$\cos n(x+2h)+\cos n(x-2h)-2\cos nx=-4\cos nx\sin^2 nh,$$
$$\sin n(x+2h)+\sin n(x-2h)-2\sin nx=-4\sin nx\sin^2 nh.$$

and hence

$$G(x,h) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \frac{\sin^2 nh}{n^2h^2}. \qquad (4)$$

The nth term of (4) tends to the nth term of (1) as $h \to 0$. Hence it is sufficient to prove that the series (4) converges uniformly with respect to h. Let r_n denote the remainder of the series (1) after the term in $\sin nx$. Then $r_n \to 0$, say $|r_n| < \epsilon$ for $n \geqslant N$. Hence

$$\sum_{n=N}^{\infty} (a_n \cos nx + b_n \sin nx) \frac{\sin^2 nh}{n^2h^2} = \sum_{n=N}^{\infty} (r_n - r_{n+1}) \left(\frac{\sin nh}{nh}\right)^2$$

$$= r_N \left(\frac{\sin Nh}{Nh}\right)^2 - \sum_{N+1}^{\infty} r_n \left[\left(\frac{\sin nh}{nh}\right)^2 - \left\{\frac{\sin(n+1)h}{(n+1)h}\right\}^2\right],$$

and the modulus of this does not exceed

$$\epsilon + \epsilon \sum_{N+1}^{\infty} \int_{nh}^{(n+1)h} \left|\frac{d}{dt}\left(\frac{\sin^2 t}{t^2}\right)\right| dt < \epsilon + \epsilon \int_0^{\infty} \left|\frac{d}{dt}\left(\frac{\sin^2 t}{t^2}\right)\right| dt,$$

the last integral being convergent. Hence (4) is uniformly convergent, and the result follows.

13.83. Riemann's Second Theorem. *If a_n and b_n tend to zero, then*

$$\lim_{h \to 0} \frac{F(x+2h) + F(x-2h) - 2F(x)}{2h} = 0.$$

for all values of x.

We have to prove that

$$a_0 h + 2 \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \frac{\sin^2 nh}{n^2 h}$$

tends to zero. Given ϵ, we have

$$|a_n \cos nx + b_n \sin nx| < \epsilon \qquad (n > N).$$

Since $\sin^2 nh \leqslant n^2h^2$ for $n \leqslant 1/h$, the modulus of the sum does not exceed

$$A|h| + 2\sum_{n=1}^{N} A|h| + 2 \sum_{N < n \leqslant 1/h} \epsilon h + 2 \sum_{n > 1/h} \frac{\epsilon}{n^2 h}$$

$$< AN|h| + 2\epsilon + \frac{2\epsilon}{h} \int_{1/h}^{\infty} \frac{du}{(u-1)^2} < AN|h| + A\epsilon,$$

and the result follows by choosing first ϵ and then h sufficiently small.

13.84. Schwarz's theorem. *If $F(x)$ is continuous in an interval (a, b), and*

$$\lim_{h\to 0}\frac{F(x+h)+F(x-h)-2F(x)}{h^2}=0$$

for all values of x in the interval, then $F(x)$ is a linear function.

The expression on the left is called the generalized second derivative of $F(x)$. If $F(x)$ has an ordinary second derivative, the generalized second derivative is equal to it, and the result follows at once.

To prove the theorem, consider the function

$$\phi(x)=F(x)-F(a)-\frac{x-a}{b-a}\{F(b)-F(a)\}.$$

We have $\phi(a)=0$ and $\phi(b)=0$. If $\phi(x)=0$ for all values of x, the result follows. Otherwise it takes values different from zero, say, for example, positive values. Suppose that $\phi(c)>0$. Let

$$\psi(x)=\phi(x)-\tfrac{1}{2}\epsilon(x-a)(b-x),$$

where ϵ is positive and so small that $\psi(c)>0$. Then $\psi(x)$ has a positive upper bound, say at $x=\xi$, which it attains, since it is continuous. Hence

$$\psi(\xi+h)+\psi(\xi-h)-2\psi(\xi)\leqslant 0.$$

But

$$\frac{\psi(\xi+h)+\psi(\xi-h)-2\psi(\xi)}{h^2}=\frac{F(\xi+h)+F(\xi-h)-2F(\xi)}{h^2}+\epsilon,$$

and the right-hand side tends to ϵ as $h\to 0$. This gives a contradiction. Similarly the supposition that $\phi(x)$ takes negative values leads to a contradiction. Hence $\phi(x)=0$ for all values of x, which is the desired result.

13.85. The proof of the main theorem now follows from Schwarz's theorem. It is sufficient to prove that, if a trigonometrical series converges to zero except at a finite number of points, then it must vanish identically. If the series 13.82 (1) has this property, the function $F(x)$ is continuous, and its generalized second derivative is zero except at a finite number of points. Hence $F(x)$ is linear in the interval between any two

exceptional points, and the straight lines which form the graph join at the exceptional points. Now, taking x in Riemann's second theorem to be an exceptional point, it follows from the lemma that the slopes of the lines on the two sides of the exceptional point must be the same. Hence $F(x)$ is linear throughout the whole interval $(0, 2\pi)$, say

$$F(x) = ax+b.$$

Hence
$$\sum_{n=1}^{\infty} \frac{a_n \cos nx + b_n \sin nx}{n^2} = \tfrac{1}{4}a_0x^2 - ax - b.$$

Since the sum of the series is periodic, a_0 and a must be zero. Then, the series being uniformly convergent, we may multiply by $\cos mx$ or $\sin mx$ and integrate term by term; and we obtain

$$\frac{\pi a_m}{m^2} = -b \int_0^{2\pi} \cos mx\, dx = 0, \qquad \frac{\pi b_m}{m^2} = -b \int_0^{2\pi} \sin mx\, dx = 0,$$

for $m > 0$. This completes the proof.

13.9. Fourier series for any range. All our series so far have represented functions with the period 2π. A series of the form

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\frac{nx}{\lambda} + b_n \sin\frac{nx}{\lambda}\right)$$

represents a function with the period $2\pi\lambda$. Formulae for the coefficients may be calculated as before; we obtain

$$a_n = \frac{1}{\pi\lambda} \int_{-\pi\lambda}^{\pi\lambda} f(t)\cos\frac{nt}{\lambda}\, dt, \qquad b_n = \frac{1}{\pi\lambda} \int_{-\pi\lambda}^{\pi\lambda} f(t)\sin\frac{nt}{\lambda}\, dt.$$

Naturally the whole theory can be applied to series of this kind.

13.91. Fourier's integral formula. The above expansion may be written

$$f(x) = \frac{1}{2\pi\lambda} \int_{-\pi\lambda}^{\pi\lambda} f(t)\, dt + \sum_{n=1}^{\infty} \frac{1}{\pi\lambda} \int_{-\pi\lambda}^{\pi\lambda} f(t)\cos\frac{n(x-t)}{\lambda}\, dt.$$

Suppose now that $\lambda \to \infty$. Then the series on the right behaves very much like one of the sums by which a Riemann integral is defined. In fact, if we write $u_n = n/\lambda$, it is

$$\sum_{n=1}^{\infty} (u_{n+1} - u_n)\phi(u_n),$$

where $$\phi(u) = \frac{1}{\pi} \int_{-\pi\lambda}^{\pi\lambda} f(t) \cos u(x-t)\, dt.$$

If, therefore, we make $\lambda \to \infty$, and ignore such difficulties as the fact that $\phi(u)$ depends on λ, and that the approximating sum is an infinite series, we obtain

$$f(x) = \frac{1}{\pi} \int_0^\infty du \int_{-\infty}^\infty \cos u(x-t) f(t)\, dt.$$

This is Fourier's integral formula. It represents a function defined over $(-\infty, \infty)$ in the same way that a Fourier series represents a function with a finite period.

The difficulty of justifying a proof on these lines would be considerable. A direct consideration of the formula suggested is comparatively easy.

13.92. Suppose that $f(x)$ is integrable in the Lebesgue sense over $(-\infty, \infty)$. Then the integral

$$\int_{-\infty}^\infty \cos u(x-t) f(t)\, dt$$

converges uniformly with respect to u over any finite range. We may therefore integrate with respect to u over $(0, U)$, and invert the order of integration. Thus

$$\int_0^U du \int_{-\infty}^\infty \cos u(x-t) f(t)\, dt = \int_{-\infty}^\infty \frac{\sin U(x-t)}{x-t} f(t)\, dt.$$

Given ϵ, we can choose T so large that

$$\int_{-\infty}^{-T} |f(t)|\, dt < \epsilon, \qquad \int_T^\infty |f(t)|\, dt < \epsilon,$$

and we may suppose that $T > |x|+1$, x being supposed fixed. Then

$$\left| \int_{-\infty}^{-T} \frac{\sin U(x-t)}{x-t} f(t)\, dt \right| < \epsilon, \qquad \left| \int_T^\infty \frac{\sin U(x-t)}{x-t} f(t)\, dt \right| < \epsilon,$$

for all values of U. Having fixed T, the integrals

$$\int_{-T}^{x-\delta} \frac{\sin U(x-t)}{x-t} f(t)\, dt, \qquad \int_{x+\delta}^T \frac{\sin U(x-t)}{x-t} f(t)\, dt$$

tend to zero as $U \to \infty$, by the Riemann-Lebesgue lemma. Hence

$$\frac{1}{\pi}\int_0^U du \int_{-\infty}^{\infty} \cos u(x-t) f(t)\, dt = \frac{1}{\pi}\int_{x-\delta}^{x+\delta} \frac{\sin U(x-t)}{x-t} f(t)\, dt + o(1)$$

$$= \frac{1}{\pi}\int_0^{\delta} \frac{\sin Ut}{t}\{f(x+t)+f(x-t)\}\, dt + o(1).$$

The value of the limit, as $U \to \infty$, therefore depends only on the behaviour of $f(t)$ in the immediate neighbourhood of $t = x$; and the problem has been reduced to the discussion of an integral similar to Dirichlet's. Any of the convergence criteria of §§ 13.231–3 apply equally well to this problem. *In particular*

$$\lim_{U\to\infty} \frac{1}{\pi}\int_0^U du \int_{-\infty}^{\infty} \cos u(x-t) f(t)\, dt = \tfrac{1}{2}\{f(x+0)+f(x-0)\}$$

if $f(t)$ is integrable over $(-\infty, \infty)$, and of bounded variation in an interval including $t = x$.

13.93. Fourier transforms. If $f(x)$ is an even function, Fourier's integral becomes

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \cos xu\, du \int_0^{\infty} \cos ut f(t)\, dt, \tag{1}$$

the term involving $\sin ut$ vanishing identically. This is Fourier's cosine formula. Similarly for an odd function we obtain Fourier's sine formula

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \sin xu\, du \int_0^{\infty} \sin ut f(t)\, dt. \tag{2}$$

If we write $$g(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^{\infty} \cos xt f(t)\, dt, \tag{3}$$

then (1) gives $$f(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^{\infty} \cos xt\, g(t)\, dt. \tag{4}$$

There is therefore a reciprocal relation between the functions $f(x)$ and $g(x)$; a pair of functions connected in one sense or

another by these formulae are known as *Fourier cosine transforms* of one another. Thus, for example, if $f(x)$ belongs to $L(0,\infty)$, and is of bounded variation in any finite interval, then (3) is absolutely convergent, and (4) holds in the sense that the integral converges (not necessarily absolutely) to

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\}.$$

Similarly from (2) we obtain the reciprocal formulae

$$h(x)=\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \sin xt f(t)\,dt, \qquad f(x)=\sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \sin xt\, h(t)\,dt, \quad (5)$$

and $f(x)$ and $h(x)$ are *Fourier sine transforms.*

13.94. Integration of Fourier integrals. It is convenient to notice at this point a theorem similar to that of § 13.5: *the formula obtained by integrating* 13.93 (1),

$$\int_0^\xi f(x)\,dx=\frac{2}{\pi}\int_0^\infty \frac{\sin\xi u}{u}\,du\int_0^\infty \cos ut f(t)\,dt,$$

holds for any function $f(t)$ integrable over $(0,\infty)$.

For

$$\int_0^U \frac{\sin\xi u}{u}\,du\int_0^\infty \cos ut f(t)\,dt=\int_0^\infty f(t)\,dt\int_0^U \frac{\sin\xi u\cos ut}{u}\,du$$

by uniform convergence; and the inner integral on the right is bounded for all U and t; for it is equal to

$$\frac{1}{2}\int_0^U \frac{\sin(\xi+t)u}{u}\,du+\frac{1}{2}\int_0^U \frac{\sin(\xi-t)u}{u}\,du$$

$$=\frac{1}{2}\int_0^{U(\xi+t)} \frac{\sin v}{v}\,dv\pm\frac{1}{2}\int_0^{U|\xi-t|} \frac{\sin v}{v}\,dv,$$

the sign being that of $\xi-t$, and

$$\int_0^V \frac{\sin v}{v}\,dv$$

is a bounded function of V. Hence, by Lebesgue's convergence

theorem, we may make $U \to \infty$ under the integral sign. Since

$$\int_0^\infty \frac{\sin \xi u \cos ut}{u}\, du = \tfrac{1}{2}\pi \quad (t < \xi), \qquad = 0 \quad (t > \xi),$$

the result now follows.

A similar result may be obtained from Fourier's sine formula.

13.95. Fourier transforms of the class L^2. The analysis of § 13.93 gives conditions under which the reciprocal formulae connecting Fourier transforms hold; but they suffer from the defect that, while the formulae are symmetrical in $f(x)$ and $g(x)$, the conditions which these functions satisfy are quite different. An alternative set of conditions, which has perfect symmetry, can be obtained by considering functions of the class L^2, and using the theory of mean convergence.*

Let $f(x)$ belong to the class $L^2(0, \infty)$. Then the formulae for cosine transforms hold in the sense that, as $a \to \infty$, the integral

$$g_a(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^a \cos xt f(t)\, dt \tag{1}$$

converges in mean to a function $g(x)$ of the class $L^2(0, \infty)$; and

$$f_a(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^a \cos xt\, g(t)\, dt \tag{2}$$

converges in mean to $f(x)$.

We prove this by a method suggested by the formal process of § 13.92. Let

$$a_n = \int_{n/\lambda}^{(n+1)/\lambda} f(x)\, dx \qquad (n = 1, 2, \ldots).$$

Then, as $\lambda \to \infty$, the sum

$$\Phi_{m,n} = \sum_{\nu=m+1}^{n} a_\nu \cos \frac{\nu x}{\lambda}$$

tends to the integral

$$\int_a^b \cos ux f(u)\, du,$$

if $0 \leqslant a < b$, and $m = [\lambda a]$, $n = [\lambda b] - 1$; for the difference is

* Plancherel (1), (2), (3); Titchmarsh (1), (3); Hardy (12); Pollard (1).

$$\sum_{\nu=m+1}^{n} \int_{\nu/\lambda}^{(\nu+1)/\lambda} \left(\cos ux - \cos\frac{\nu x}{\lambda}\right) f(u)\,du +$$

$$+ \int_{a}^{(m+1)/\lambda} \cos ux f(u)\,du + \int_{(n+1)/\lambda}^{b} \cos ux f(u)\,du,$$

and
$$\left|\cos ux - \cos\frac{\nu x}{\lambda}\right| \leqslant \frac{|x|}{\lambda},$$

so that the sum is $O(1/\lambda)$, while the last two integrals plainly tend to zero. Further, the convergence is clearly uniform with respect to x for $0 \leqslant x \leqslant X$.

Now we can apply to $\Phi_{m,n}$ an argument similar to that used in proving the Riesz-Fischer theorem. We have

$$a_n^2 \leqslant \int_{n/\lambda}^{(n+1)/\lambda} \{f(x)\}^2\,dx \int_{n/\lambda}^{(n+1)/\lambda} dx = \frac{1}{\lambda}\int_{n/\lambda}^{(n+1)/\lambda} \{f(x)\}^2\,dx;$$

and hence

$$\int_0^{\pi\lambda} \Phi_{m,n}^2\,dx = \tfrac{1}{2}\pi\lambda \sum_{\nu=m+1}^{n} a_\nu^2 \leqslant \tfrac{1}{2}\pi \int_{(m+1)/\lambda}^{(n+1)/\lambda} \{f(x)\}^2\,dx \leqslant \tfrac{1}{2}\pi \int_a^b \{f(x)\}^2\,dx,$$

and *a fortiori*
$$\int_0^X \Phi_{m,n}^2\,dx \leqslant \tfrac{1}{2}\pi \int_a^b \{f(x)\}^2\,dx$$

if $\pi\lambda > X$. Keeping X fixed and making $\lambda \to \infty$, we obtain

$$\int_0^X \{g_b(x) - g_a(x)\}^2\,dx \leqslant \int_a^b \{f(x)\}^2\,dx,$$

and then, making $X \to \infty$,

$$\int_0^\infty \{g_b(x) - g_a(x)\}^2\,dx \leqslant \int_a^b \{f(x)\}^2\,dx. \tag{3}$$

Since the right-hand side tends to zero as $a \to \infty$, $b \to \infty$, so does the left-hand side; that is, $g_a(x)$ converges in mean to a function, $g(x)$ say, of the class $L^2(0, \infty)$.

The same argument now shows that the integral (2) converges in mean, to a function $\phi(x)$, say. We have to prove that $\phi(x) = f(x)$ almost everywhere, and for this it is sufficient to show that

$$\int_0^\xi \phi(x)\,dx = \int_0^\xi f(x)\,dx \tag{4}$$

for all values of ξ. Now

$$\int_0^\xi \phi(x)\,dx = \lim_{a\to\infty} \int_0^\xi f_a(x)\,dx = \lim_{a\to\infty} \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\xi dx \int_0^a \cos xt\, g(t)\,dt$$

$$= \lim_{a\to\infty} \sqrt{\left(\frac{2}{\pi}\right)} \int_0^a \frac{\sin \xi t}{t}\, g(t)\,dt = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \frac{\sin \xi t}{t}\, g(t)\,dt.$$

On the other hand, for $0 < \xi < a$,

$$\int_0^\xi f(x)\,dx = \frac{2}{\pi} \int_0^\infty \frac{\sin \xi u}{u}\,du \int_0^a \cos ut f(t)\,dt = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \frac{\sin \xi u}{u}\, g_a(u)\,du,$$

by § 13.94, $f(x)$ being integrable over $(0, a)$. Making $a \to \infty$, and observing that $\sin \xi u/u$ belongs to L^2, we obtain, by § 12.53,

$$\int_0^\xi f(x)\,dx = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \frac{\sin \xi u}{u}\, g(u)\,du.$$

This proves (4), and completes the proof of the theorem.

There is, of course, a similar theorem for Fourier sine transforms.

13.96. We can also obtain a formula corresponding to Parseval's theorem. Putting $a = 0$ in 13.95 (3), we have

$$\int_0^\infty \{g_b(x)\}^2\,dx \leqslant \int_0^b \{f(x)\}^2\,dx \leqslant \int_0^\infty \{f(x)\}^2\,dx,$$

and making $b \to \infty$, by § 12.51

$$\int_0^\infty \{g(x)\}^2\,dx \leqslant \int_0^\infty \{f(x)\}^2\,dx.$$

But since the relation between $f(x)$ and $g(x)$ is reciprocal, the opposite inequality also holds. Hence in fact

$$\int_0^\infty \{g(x)\}^2\,dx = \int_0^\infty \{f(x)\}^2\,dx. \tag{1}$$

Finally, if $\phi(x)$ also belongs to L^2, and $\psi(x)$ is its transform, then $g(x)+\psi(x)$ is the transform of $f(x)+\phi(x)$. Hence

$$\int_0^\infty \{g(x)+\psi(x)\}^2\,dx = \int_0^\infty \{f(x)+\phi(x)\}^2\,dx,$$

and, subtracting (1) and the corresponding formula for ϕ and ψ, we obtain

$$\int_0^\infty g(x)\psi(x)\,dx = \int_0^\infty f(x)\phi(x)\,dx. \tag{2}$$

MISCELLANEOUS EXAMPLES

1. If $f(x)$ is first defined in $(0, \pi)$, then in $(-\pi, 0)$ by the equation $f(-x) = f(x)$, and elsewhere by periodicity, show that $f(x)$ has the Fourier cosine series

$$\tfrac{1}{2}a_0 + \sum_{n=1}^\infty a_n \cos nx,$$

where
$$a_n = \frac{2}{\pi}\int_0^\pi f(t)\cos nt\,dt.$$

Similarly, if $f(-x) = -f(x)$, then $f(x)$ has the Fourier sine series

$$\sum_{n=1}^\infty b_n \sin nx,$$

where
$$b_n = \frac{2}{\pi}\int_0^\pi f(t)\sin nt\,dt.$$

2. Show that

$$e^{ax} = \frac{e^{2\pi a}-1}{\pi}\left\{\frac{1}{2a} + \sum_{n=1}^\infty \frac{a\cos nx - n\sin nx}{a^2+n^2}\right\} \qquad (0 < x < 2\pi),$$

$$e^{ax} = \frac{e^{a\pi}-1}{a\pi} + \frac{2}{\pi}\sum_{n=1}^\infty \{(-1)^n e^{a\pi}-1\}\frac{a\cos nx}{a^2+n^2} \qquad (0 < x < \pi),$$

$$e^{ax} = \frac{2}{\pi}\sum_{n=1}^\infty \{1-(-1)^n e^{a\pi}\}\frac{n\sin nx}{a^2+n^2} \qquad (0 < x < \pi).$$

Find the sums of the series when $x = 0$.

3. Sum the series

$$\sum_{n=1}^\infty \frac{a\cos nx}{a^2+n^2}, \qquad \sum_{n=1}^\infty \frac{n\sin nx}{a^2+n^2} \qquad (0 < x < 2\pi).$$

4. Expand in Fourier series valid over $(0, 2\pi)$, and also in Fourier cosine and sine series valid over $(0, \pi)$, the functions

$$1, \quad x, \quad x^2, \quad x^3, \quad \cos ax, \quad \sin ax, \quad \cosh ax, \quad \sinh ax,$$
$$e^{ax}\cos bx, \quad e^{ax}\sin bx, \quad [x/\pi], \quad [2x/\pi].$$

Consider the values of x for which the series converge to a value different from the value of the function expanded.

5. Prove that, if $-1 < r < 1$,

$$\frac{1-r^2}{1-2r\cos\theta+r^2} = 1+2\sum_{n=1}^{\infty} r^n \cos n\theta$$

for all values of θ.

6. If a_n, b_n denote the Fourier coefficients of $f(x)$, then for $-1 < r < 1$

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)r^n = \frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1-2r\cos(x-t)+r^2}f(t)\,dt.$$

7. Prove that

$$\lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1-2r\cos(x-t)+r^2}f(t)\,dt = \tfrac{1}{2}\{f(x+0)+f(x-0)\}$$

for all values of x for which the right-hand side exists.

[The discussion is similar to that of Fejér's integral.]

8. Show that, if $f(x)$ is bounded, then

$$s_n = O(\log n).$$

9. Show that, if $m \leqslant f(x) \leqslant M$, then

$$m \leqslant \sigma_n(x) \leqslant M$$

for all values of n and x.

10. Show that, if $m \leqslant f(x) \leqslant M$, and

$$|a_n| \leqslant \frac{A_1}{n}, \qquad |b_n| \leqslant \frac{A_2}{n},$$

then

$$m-A_1-A_2 \leqslant s_n \leqslant M+A_1+A_2.$$

[Use the formula

$$s_n = \sigma_{n+1}-\frac{1}{n+1}\sum_{\nu=1}^{n}\nu(a_\nu\cos\nu x+b_\nu\sin\nu x).\Big]$$

11. Show that

$$\frac{\pi-x}{2} = \frac{\sin x}{1}+\frac{\sin 2x}{2}+\frac{\sin 3x}{3}+\ldots \qquad (0 < x < 2\pi),$$

and deduce that

$$\left|\frac{\sin x}{1}+\frac{\sin 2x}{2}+\ldots+\frac{\sin nx}{n}\right| \leqslant \tfrac{1}{2}\pi+1$$

for all values of n and x.

[Compare § 1.76. The actual upper bound of the partial sums is $\int_0^{\pi}\frac{\sin x}{x}\,dx = 1.85\ldots$; see Gronwall (1).]

12. Use Parseval's theorem to sum the series

$$\sum_{n=1}^{\infty}\frac{1}{n^2}, \qquad \sum_{n=1}^{\infty}\frac{1}{(a^2+n^2)^2}, \qquad \sum_{n=1}^{\infty}\frac{n^2}{(a^2+n^2)^2}.$$

13. A necessary and sufficient condition that

$$a_n = O(e^{-(k-\epsilon)n}), \qquad b_n = O(e^{-(k-\epsilon)n}),$$

where $k > 0$, for all positive values of ϵ, is that $f(x)$ should be almost everywhere equal to the value on the real axis of an analytic function $f(z)$, which is regular for $-k < y < k$, and has the period 2π.

14. Construct a Fourier series for which

$$s_n(0) > \frac{\log n}{\log\log n}$$

for arbitrarily large values of n.

15. Show that if, in the Fourier series of § 13.42, we substitute $\nu! x$ for x in the terms corresponding to the group of numbers G_{λ_ν}, we obtain a series which is also the Fourier series of a continuous function, and which diverges for all values of x such that x/π is a rational number.

16. Show that, if the series

$$\sum_{m=0}^{\infty} \alpha_m \cos(2^m x)$$

is a Fourier series, it is convergent for almost all values of x.

[In this case the formula used in example 10 becomes

$$\sigma_{2^k} - s_{2^k} = \frac{1}{2^k} \sum_{m=1}^{k-1} 2^m \alpha_m \cos(2^m x),$$

and since $\alpha_m \to 0$ the right-hand side tends to 0 for all values of x. Hence s_{2^k} tends to a limit wherever σ_{2^k} does, i.e. almost everywhere. See Kolmogoroff (1).]

17. If $f(x) = x^{-\alpha}$, where $0 < \alpha < 1$, for $0 < x \leqslant 2\pi$, show that, as $n \to \infty$,

$$a_n \sim \frac{n^{\alpha-1}}{2\Gamma(\alpha)\cos\frac{1}{2}\pi\alpha}, \qquad b_n \sim \frac{n^{\alpha-1}}{2\Gamma(\alpha)\sin\frac{1}{2}\pi\alpha}.$$

Show that $f(x)$ belongs to L^p if $p < 1/\alpha$, and that $\sum(|a_n|^q + |b_n|^q)$ is divergent if $q < 1/(1-\alpha)$.

[See Bromwich, *Infinite Series* (2nd ed.), § 174, Ex. 5, and Haslam-Jones (1). The result should be compared with the extended Riesz-Fischer theorem referred to in § 13.71. It shows that the exponent of convergence of the series of coefficients is the 'best possible'.]

18. A function $f(x)$ is equal to $\nu^\alpha \cos(\nu^2 x)$ in the intervals

$$\frac{\pi}{(\nu+1)^\beta} < x \leqslant \frac{\pi}{\nu^\beta}, \qquad \nu = 1, 2, \ldots,$$

where $0 < \alpha < \beta < 1$, and is defined in $(-\pi, 0)$ by the relation

$$f(-x) = -f(x).$$

Show that $f(x)$ is integrable in the Lebesgue sense, and that its Fourier sine coefficients satisfy

$$b_n = O(n^{\frac{1}{2}\alpha - \frac{1}{2}} \log n).$$

By taking α small enough and β/α near enough to 1, show that the convergence of $\sum |b_n|^q$, where $q > 2$, is not sufficient to ensure that $f(x)$ shall belong to L^p, where $p = p(q) > 1$.

[The point of the example is that if $q = 2$ the convergence of $\sum |b_n|^q$ *does* imply that $f(x)$ belongs to L^2, and there is an abrupt change in the state of affairs when q becomes greater than 2.

We have

$$b_n = \frac{1}{\pi} \sum_{\nu=1}^{\infty} \nu^\alpha \int_{\pi/(\nu+1)^\beta}^{\pi/\nu^\beta} \{\sin(n+\nu^2)x + \sin(n-\nu^2)x\}\, dx.$$

The terms for which $\sqrt{n}-2 \leqslant \nu \leqslant \sqrt{n}+2$ are

$$O(\nu^{\alpha-\beta-1}) = O(n^{\frac{1}{2}(\alpha-\beta-1)});$$

the terms for which $\nu \leqslant \sqrt{n}-2$ are

$$O\left(\sum_{\nu \leqslant \sqrt{n}-2} \frac{\nu^\alpha}{n-\nu^2}\right) = O\left(n^{\frac{1}{2}\alpha} \int_0^{\sqrt{n}-1} \frac{du}{n-u^2}\right)$$

$$= O\left(n^{\frac{1}{2}\alpha-\frac{1}{2}} \int_0^{1-n^{-\frac{1}{2}}} \frac{dv}{1-v^2}\right) = O(n^{\frac{1}{2}\alpha-\frac{1}{2}} \log n),$$

and a similar result holds for the remaining terms. See also Titchmarsh (2).]

19. Show that the function

$$f(x) = -x + \lim_{m\to\infty} \int_0^x (1+\cos t)(1+\cos 4t)\ldots(1+\cos 4^{m-1}t)\, dt$$

is continuous and of bounded variation, and has the period 2π; but that, if b_n is its nth Fourier sine coefficient, nb_n does not tend to zero so that $f(x)$ is not an integral.

[This example is due to F. Riesz (3). Let $\tau_m(x)$ denote the integrand. It is a cosine polynomial of order

$$1+4+\ldots+4^{m-1} = \tfrac{1}{3}(4^m-1).$$

On multiplying by $1+\cos 4^m x$, the first new term involves

$$\cos\{4^m - \tfrac{1}{3}(4^m-1)\}x = \cos \tfrac{1}{3}(2.4^m+1)x,$$

which is of higher order than any of the terms in $\tau_m(x)$. Hence $\tau_{m+1}(x)$ is obtained by adding new terms to $\tau_m(x)$ without altering the existing ones. Also it is easily seen that all the coefficients lie between 0 and 1.

Let α_m be the number of non-vanishing terms in τ_m. The recurrence relation $\alpha_{m+1} = 3\alpha_m - 1$ is easily verified. Hence $\alpha_{m+1}-\alpha_m = 3(\alpha_m-\alpha_{m-1})$, $\alpha_{m+1}-\alpha_m = 3^m$. Hence, if $0 < x \leqslant 2\pi$,

$$\left|\int_0^x \{\tau_{m+1}(t)-\tau_m(t)\}\, dt\right| \leqslant 2\pi \frac{3^m}{\frac{1}{3}(2.4^m+1)}.$$

Hence $\int_0^x \tau_m(t)\, dt$ tends uniformly to a limit, i.e. $f(x)$ is continuous. Also

$\tau_m(x)$ is non-decreasing, and so is its limit. Hence $f(x)$ is of bounded variation. Finally $b_{4^m} = 1/4^m$.]

20. Show that, if $a_n \cos nx + b_n \sin nx \to 0$ in a set of positive measure, then $a_n \to 0$ and $b_n \to 0$.

21. Show that the reciprocal formulae

$$F(x) = \int_{-\infty}^{\infty} e^{ixt} f(t)\, dt, \qquad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixt} F(t)\, dt,$$

hold under the same conditions as Fourier's integral.

22. Show that Mellin's inversion formulae

$$\phi(s) = \int_0^{\infty} x^{s-1} \psi(x)\, dx, \qquad \psi(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \phi(s) x^{-s}\, ds,$$

may, with suitable conditions, be deduced from the formulae of the previous example.

23. Show that the functions

$$x^{-\frac{1}{2}}, \qquad e^{-\frac{1}{2}x^2}, \qquad \operatorname{sech} x\sqrt{(\tfrac{1}{2}\pi)}$$

are their own Fourier cosine transforms, and that

$$x^{-\frac{1}{2}}, \qquad xe^{-\frac{1}{2}x^2}, \qquad \frac{1}{e^{x\sqrt{(2\pi)}}-1} - \frac{1}{x\sqrt{(2\pi)}}$$

are their own sine transforms.

24. Express $e^{-a|x|}$, where $a > 0$, as a Fourier integral. Verify the formula 13.96 (2) in the case where $f(x) = e^{-ax}$, $\phi(x) = e^{-bx}$.

25. Evaluate the integral

$$\int_0^{\infty} \frac{\sin ax \sin bx}{x^2}\, dx$$

by means of the formula 13.96 (2).

26. Let $f(x)$ belong to $L(0, \infty)$, and be continuous and steadily decreasing to zero as $x \to \infty$ (or be the difference between two functions of this type). Let $\alpha > 0$, $\alpha\beta = 2\pi$, and let $g(x)$ be the Fourier cosine transform of $f(x)$. Then

$$\sqrt{\alpha}\left\{\tfrac{1}{2}f(0) + \sum_{n=1}^{\infty} f(n\alpha)\right\} = \sqrt{\beta}\left\{\tfrac{1}{2}g(0) + \sum_{n=1}^{\infty} g(n\beta)\right\}.$$

[This is known as Poisson's formula. It is easily verified that

$$\sqrt{\beta}\left\{\tfrac{1}{2}g(0) + \sum_{m=1}^{n} g(m\beta)\right\}$$

$$= \frac{\sqrt{\alpha}}{2\pi} \int_0^{\pi} f\left(\frac{t}{\beta}\right) \frac{\sin(n+\frac{1}{2})t}{\sin \frac{1}{2}t}\, dt + \frac{\sqrt{\alpha}}{2\pi} \sum_{m=1}^{\infty} \int_{(2m-1)\pi}^{(2m+1)\pi} f\left(\frac{t}{\beta}\right) \frac{\sin(n+\frac{1}{2})t}{\sin \frac{1}{2}t}\, dt.$$

This differs from the left-hand side of Poisson's formula by

$$\frac{\sqrt{\alpha}}{2\pi}\int_0^{\pi}\left\{f\left(\frac{t}{\beta}\right)-f(0)\right\}\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt+$$

$$+\frac{\sqrt{\alpha}}{2\pi}\sum_{m=1}^{\infty}\int_{(2m-1)\pi}^{(2m+1)\pi}\left\{f\left(\frac{t}{\beta}\right)-f\left(\frac{2m\pi}{\beta}\right)\right\}\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\,dt.$$

The given conditions ensure that this series converges uniformly with respect to n; in fact, it is easily seen from the second mean-value theorem that the general term is $O[f\{(2m-1)\pi/\beta\}]$ independently of n; and each term tends to zero as $n\to\infty$ (as in the proof of Jordan's test), and the result follows.

For other conditions for the formula see Linfoot (1), Mordell (2).]

27. Verify Poisson's formula for the function $f(x)=1/(1+x^2)$. [The result is equivalent to that of § 3.22, ex. (iii).]

28. Deduce from Poisson's formula that if $x>0$

$$\sum_{n=-\infty}^{\infty}e^{-n^2x^2}=\frac{\sqrt{\pi}}{x}\sum_{n=-\infty}^{\infty}e^{-n^2\pi^2/x^2}.$$

29. Sum the series $\sum_{n=1}^{\infty}n^{-\nu}J_\nu(n\beta)$, where $\beta>0$, $\nu>\frac{1}{2}$, by means of Poisson's formula and the first result of Ch. 1, ex. 5.