and

$$\gamma x \quad + \quad \int_{-\infty}^{\infty}\left(\log\left|1-\frac{x}{t}\right| \;+\; \frac{x}{t}\right)\mathrm{d}\rho(t) \quad + \quad \log W(x) \quad \leqslant \quad K \quad \text{for } x\in\mathbb{R}$$

with some constant γ. Hence, in keeping with the line of thought embarked on at the beginning of article 1, we regard the (hypothetical) *second* ('essential') *condition* for admittance of multipliers by a weight W as being *very close* (if not identical) to whatever requirement it must satisfy in order to guarantee existence of such increasing functions ρ. That requirement, and attempts to arrive at precise knowledge of it, will therefore be our main object of interest during the remainder of this chapter.

Suppose that for a given weight $W(x) \geqslant 1$ we *have* such a function $\rho(t)$ corresponding to some $A > 0$. The relation

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|\Im z|\log W(t)}{|z-t|^2}\mathrm{d}t \;-\; A|\Im z|$$
$$\leqslant \quad K \;-\; \gamma\Re z \;-\; \int_{-\infty}^{\infty}\left(\log\left|1-\frac{z}{t}\right| \;+\; \frac{\Re z}{t}\right)\mathrm{d}\rho(t)$$

(with the left side *interpreted* as $\log W(x)$ for $z = x\in\mathbb{R}$) then holds throughout the complex plane. For $\Im z > 0$, this has indeed already been verified while proving the second theorem of article 1 (near the beginning of the proof). That, however, is enough, since both sides are unchanged when z is replaced by $\bar{z}$.

Now the *right side* of the last relation is obviously a *superharmonic function* of z, *finite* for z off of the real axis. The *existence* of our function ρ thus leads (in almost trivial fashion) to that of a *superharmonic majorant* $\not\equiv \infty$ *for*

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|\Im z|\log W(t)}{|z-t|^2}\mathrm{d}t \;-\; A|\Im z|$$

(interpreted as $\log W(x)$ for $z = x\in\mathbb{R}$) *in the whole complex plane.* The key to the proof of the Beurling–Malliavin multiplier theorem given below in §C lies in the observation that *the converse of this statement is true,* at least for *continuous* weights $W(x)$. That fact (which, from a certain point of view, is nearly tautological) will be established in the next article. For this purpose and the later applications as well, we will need the *smallest superharmonic majorant* of a continuous function together with some of its properties, to whose examination we now proceed.

Let $F(z)$ be any function *real-valued and continuous* in the whole complex plane. (In our applications, we will use a function $F(z)$ equal to the preceding expression – interpreted as $\log W(x)$ for $z = x \in \mathbb{R}$ – where $W \geqslant 1$ is *continuous* and such that $\int_{-\infty}^{\infty} (\log W(x)/(1+x^2))\mathrm{d}x < \infty$.) We next take the *family* $\mathscr{F}$ of *functions superharmonic and* $\geqslant F$ (everywhere); our convention being to consider the function *identically equal to* $+\infty$ *as superharmonic* (see §A.1), $\mathscr{F}$ is *certainly not empty.* Then put

$$Q(z) \quad = \quad \inf\{\,U(z):\ U \in \mathscr{F}\}$$

for each complex z, and finally take

$$(\mathfrak{M}F)(z) \quad = \quad \liminf_{\zeta \to z} Q(\zeta);$$

$\mathfrak{M}F$ is the function we will be dealing with. (The reason for use of the symbol $\mathfrak{M}$ will appear in problems 55 and 56 below. $\mathfrak{M}F$ is a kind of *maximal function* for F.)

In our present circumstances, $Q(z)$ is $\geqslant$ the *continuous function* $F(z)$, so we must also have

$$(\mathfrak{M}F)(z) \quad \geqslant \quad F(z).$$

This certainly makes $(\mathfrak{M}F)(z) > -\infty$ everywhere, so $(\mathfrak{M}F)(z)$ is *itself superharmonic* (everywhere) by the last theorem of §A.1, *and must hence belong to* $\mathscr{F}$ in view of the relation just written. The same theorem also tells us, however, that $(\mathfrak{M}F)(z) \leqslant U(z)$ for *every* $U \in \mathscr{F}$; $\mathfrak{M}F$ is thus *a member of* $\mathscr{F}$ *and at the same time* $\leqslant$ *every member of* $\mathscr{F}$. $\mathfrak{M}F$ is, in other words, *the smallest superharmonic majorant of* F.

It may well happen, of course, that $(\mathfrak{M}F)(z) \equiv \infty$. However, if $\mathfrak{M}F$ is finite at just one point, it is finite everywhere. That is the meaning of the

Lemma. *If, for any* z_0, $(\mathfrak{M}F)(z_0) = \infty$, *we have* $(\mathfrak{M}F)(z) \equiv \infty$.

Proof. To simplify the writing, let us wlog consider the case where $z_0 = 0$. By continuity of F at 0, there is certainly some finite M such that

$$F(z) \leqslant M \quad \text{for } |z| \leqslant 1, \quad \text{say.}$$

Given, however, that $(\mathfrak{M}F)(0) = \infty$, there is an r, $0 < r < 1$, for which

$$(\mathfrak{M}F)(z) \geqslant M+1, \qquad |z| \leqslant r,$$

because the superharmonic function $\mathfrak{M}F$ has property (i) at 0 (§A.1).

It is now claimed that

$$\int_{-\pi}^{\pi} (\mathfrak{M}F)(re^{i\vartheta})\,\mathrm{d}\vartheta \quad = \quad \infty.$$

Reasoning by contradiction, *assume that the integral on the left is finite.* Then, since $(\mathfrak{M}F)(re^{i\vartheta})$ is *bounded below* for $0 \leqslant \vartheta \leqslant 2\pi$ (*here*, simply because $\mathfrak{M}F \geqslant F$, but see also the beginning of §A.1), we must have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{r^2-\rho^2}{r^2+\rho^2-2r\rho\cos(\varphi-\tau)}(\mathfrak{M}F)(re^{i\tau})\,d\tau \quad < \quad \infty$$

for $0 \leqslant \rho < r$ and $0 \leqslant \varphi \leqslant 2\pi$. Take now the function $V(z)$ *equal*, for $|z| \geqslant r$ *to* $(\mathfrak{M}F)(z)$ and, for $z = \rho e^{i\varphi}$ with $0 \leqslant \rho < r$, *to the Poisson integral just written. This function* $V(z)$ *is superharmonic* (everywhere) by the *second theorem* of §A.1.

We have

$$V(\rho e^{i\varphi}) \geqslant M+1 \quad \text{for } 0 \leqslant \rho < r,$$

since $(\mathfrak{M}F)(re^{i\tau}) \geqslant M+1$. At the same time,

$$F(\rho e^{i\varphi}) \leqslant M \quad \text{for } 0 \leqslant \rho < r$$

because $r < 1$, so $V(z) \geqslant F(z)$ for $|z| < r$. This, however, is also true for $|z| \geqslant r$, where $V(z) = (\mathfrak{M}F)(z)$. We thus have in $V(z)$ a *superharmonic majorant of* $F(z)$, so

$$V(z) \geqslant (\mathfrak{M}F)(z).$$

Thence,

$$(\mathfrak{M}F)(0) \quad \leqslant \quad V(0) \quad = \quad \frac{1}{2\pi}\int_{-\pi}^{\pi}(\mathfrak{M}F)(re^{i\tau})\,d\tau \quad < \quad \infty.$$

But it was given that $(\mathfrak{M}F)(0) = \infty$. This contradiction shows that the *integral* in the last relation must be *infinite*, as claimed.

Apply now the *first* lemma of §A.2 to the function $(\mathfrak{M}F)(z)$, superharmonic everywhere. We find that

$$(\mathfrak{M}F)(z) \equiv \infty$$

for all z. The proof is complete.

Corollary. *The function* $(\mathfrak{M}F)(z)$ *is either finite everywhere or infinite everywhere.*

Henceforth, to indicate that the *first alternative* of the corollary holds, we will simply say *that* $\mathfrak{M}F$ *is finite.*

Lemma. *If* $\mathfrak{M}F$ *is finite and* $F(z)$ *is harmonic in any open set* $\mathcal{O}$, $(\mathfrak{M}F)(z)$ *is also harmonic in* $\mathcal{O}$.

Proof. Let $z_0 \in \mathcal{O}$ and take $r > 0$ so small that the *closed* disk of radius r about z_0 lies in $\mathcal{O}$; it suffices to show that $(\mathfrak{M}F)(z)$ is harmonic for $|z - z_0| < r$.

Supposing wlog that $z_0 = 0$, we take the superharmonic function $V(z)$ used in the proof of the preceding lemma. From the second theorem of §A.1, we have

$$V(z) \leqslant (\mathfrak{M}F)(z).$$

Here, however, we are assuming that $(\mathfrak{M}F)(z) < \infty$, so the Poisson integral

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{r^2 - |z|^2}{|z - re^{i\tau}|^2}\,(\mathfrak{M}F)(re^{i\tau})\,d\tau,$$

equal, for $|z| < r$, *to* $V(z)$, must be *absolutely convergent* for such z, $(\mathfrak{M}F)(re^{i\tau})$ being *bounded below*, as we know. $V(z)$ is thus *harmonic* for $|z| < r$.

Let $|z| < r$. Then, since $\{|z| \leqslant r\} \subseteq \mathcal{O}$, where $F(z)$ is given to be harmonic,

$$F(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{r^2 - |z|^2}{|z - re^{i\tau}|^2}\,F(re^{i\tau})\,d\tau,$$

and the integral on the right is $\leqslant$ the preceding one, $\mathfrak{M}F$ being a *majorant* of F. Thus,

$$F(z) \leqslant V(z) \quad \text{for } |z| < r.$$

This, however, also holds for $|z| \geqslant r$ where $V(z) = (\mathfrak{M}F)(z)$. We see as in the proof of the last lemma that $V(z)$ is a *superharmonic majorant* of $F(z)$. Hence

$$V(z) \geqslant (\mathfrak{M}F)(z).$$

But the reverse inequality was already noted above. Therefore,

$$V(z) = (\mathfrak{M}F)(z).$$

Since $V(z)$ is harmonic for $|z| < r$, we are done.

Let us now look at the set E on which

$$(\mathfrak{M}F)(z) = F(z)$$

for some given continuous function F. E may, of course, be *empty*; *it is, in any event, closed.* Suppose, indeed, that we have a sequence of points $z_k \in E$ and that $z_k \xrightarrow[k]{} z_0$. Then, since $\mathfrak{M}F$ enjoys property (i) (§A.1), we have

$$(\mathfrak{M}F)(z_0) \leq \liminf_{k\to\infty}(\mathfrak{M}F)(z_k) = \liminf_{k\to\infty}F(z_k) = F(z_0),$$

F being continuous at z_0. Because $\mathfrak{M}F$ is a majorant of F, we also have $(\mathfrak{M}F)(z_0) \geq F(z_0)$, and thus finally $(\mathfrak{M}F)(z_0) = F(z_0)$, making $z_0 \in E$.

This means that the set of z for which $(\mathfrak{M}F)(z) > F(z)$ is open. Regarding it, we have the important

Lemma. *$(\mathfrak{M}F)(z)$, if finite, is harmonic in the open set where it is $> F(z)$.*

Note. I became aware of this result while walking in Berkeley and thinking about a conversation I had just had with L. Dubins on the material of the present article, especially on the notions developed in problems 55 and 56 below. Dubins thus gave me considerable help with this work.

Proof of lemma. Is much like those of the two previous ones. Let us show that if $(\mathfrak{M}F)(z_0) > F(z_0)$ with $\mathfrak{M}F$ finite, then $(\mathfrak{M}F)(z)$ is harmonic in some small disk about z_0.

We can, wlog, take $z_0 = 0$; suppose, then, that

$$(\mathfrak{M}F)(0) > F(0) + 2\eta, \quad \text{say},$$

where $\eta > 0$. Property (i) then gives us an $r > 0$ such that

$$(\mathfrak{M}F)(z) > F(0) + \eta$$

for $|z| \leq r$, and the continuity of F makes it possible for us to choose this r small enough so that we also have

$$F(z) < F(0) + \eta \quad \text{for } |z| \leq r.$$

Form now the superharmonic function $V(z)$ used in the proofs of the last two lemmas. As in the second of those, we certainly have

$$V(z) \leq (\mathfrak{M}F)(z),$$

according to our theorem from §A.1. In the present circumstances, for $|z| < r$,

$$V(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{r^2 - |z|^2}{|z - re^{i\tau}|^2}(\mathfrak{M}F)(re^{i\tau})\,d\tau$$

is $> F(0) + \eta$, whereas $F(z) < F(0) + \eta$ there; $V(z)$ is thus $\geq F(z)$ for $|z| < r$. When $|z| \geq r$, $V(z) = (\mathfrak{M}F)(z)$ is also $\geq F(z)$, so V is again a superharmonic majorant of F. Hence

$$V(z) \geq (\mathfrak{M}F)(z),$$

and we see finally that

$$V(z) = (\mathfrak{M}F)(z),$$

with the left side *harmonic* for $|z| < r$, just as in the proof of the preceding lemma. Done.

Lemma. *If $\mathfrak{M}F$ is finite, it is everywhere continuous.*

Proof. Depends on the Riesz representation for superharmonic functions.

Take the sets

$$E = \{z: (\mathfrak{M}F)(z) = F(z)\}$$

and

$$\mathcal{O} = \mathbb{C} \sim E;$$

as we have already observed, E is closed and $\mathcal{O}$ is open. By the preceding lemma, $(\mathfrak{M}F)(z)$ is *harmonic* in $\mathcal{O}$ and thus surely *continuous* therein. We therefore need only check continuity of $\mathfrak{M}F$ at the points of E.

Let, then, $z_0 \in E$ and consider any open disk Δ centered at z_0, say the one of radius $= 1$. In the open set $\Omega = \Delta \cap \mathcal{O}$ the function $(\mathfrak{M}F)(z)$ is *harmonic*, as just remarked and *on* $\Delta \sim \Omega = \Delta \cap E$, $(\mathfrak{M}F)(z) = F(z)$ *depends continuously on* z. The *restriction* of $\mathfrak{M}F$ to E is, in particular, *continuous* at the centre, z_0, of Δ.

The corollary to the *Evans-Vasilesco theorem* (at the end of §A.3) can now be invoked, thanks to the superharmonicity of $(\mathfrak{M}F)(z)$. After translating z_0 to the origin (and Δ to a disk about 0), we see by that result that $(\mathfrak{M}F)(z)$ is continuous at z_0. This does it.

Remark. These last two lemmas will enable us to use harmonic estimation to examine the function $(\mathfrak{M}F)(z)$ in §C.

It is a good idea at this point to exhibit two processes which generate $(\mathfrak{M}F)(z)$ when applied to a given continuous function F, although we will not make direct use of either in this book. These are described in problems 55 and 56. The first of those depends on

Problem 54

Let $U(z)$, defined and $> -\infty$ in a domain $\mathscr{D}$, satisfy

$$\liminf_{\zeta \to z} U(\zeta) \geqslant U(z)$$

for $z \in \mathscr{D}$. Show that $U(z)$ is then superharmonic in $\mathscr{D}$ iff, at each z therein, one has

$$\frac{1}{\pi r^2}\iint_{|\zeta - z| < r} U(\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta \quad \leqslant \quad U(z)$$

for all $r > 0$ sufficiently small. (As usual, $\zeta = \xi + i\eta$.)
(Hint: For the if part, the first theorem of §A.1 may be used.)

Problem 55

For Lebesgue measurable functions $F(z)$ defined on $\mathbb{C}$ and bounded below on each compact set, put

$$(MF)(z) \quad = \quad \sup_{r>0} \frac{1}{\pi r^2}\iint_{|\zeta - z| < r} F(\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

Then, starting with any F *continuous* on $\mathbb{C}$, form successively the functions $F^{(0)}(z) = F(z)$, $F^{(1)}(z) = (MF)(z)$, $F^{(2)}(z) = (MF^{(1)})(z)$, and so forth.

(a) Show that $F^{(0)}(z) \leqslant F^{(1)}(z) \leqslant F^{(2)}(z) \leqslant \cdots$.

(b) Show that $\lim_{n\to\infty} F^{(n)}(z) \leqslant (\mathfrak{M}F)(z)$.

(c) Show that $\lim_{n\to\infty} F^{(n)}(z)$ is superharmonic.
(Hint: For this, use problem 54.)

(d) Hence show that $\lim_{n\to\infty} F^{(n)}(z) = (\mathfrak{M}F)(z)$.

Remark. The function $\mathfrak{M}F$ was originally brought into the study of multiplier theorems through this construction.

The next problem involves *Jensen measures* (on $\mathbb{C}$). That term is used here to denote the *positive Radon measures μ of compact support* such that

$$\int_{\mathbb{C}} U(\zeta)\,\mathrm{d}\mu(\zeta) \quad \leqslant \quad U(0)$$

for *each function U superharmonic on* $\mathbb{C}$. (Any such function $U(z)$ is certainly *Borel measurable*, for, by the *first* theorem of §A.2, it is the pointwise limit of an *increasing sequence* of $\mathscr{C}_\infty$ superharmonic functions.) Some simple Jensen measures are the ν_r given by

$$\mathrm{d}\nu_r(\zeta) \quad = \quad \begin{cases} \dfrac{1}{\pi r^2}\,\mathrm{d}\xi\,\mathrm{d}\eta, & |\zeta| < r, \\ 0, & |\zeta| \geqslant r \end{cases}$$

(refer to problem 54 !).

For reasons which will soon become apparent, *we denote the collection of Jensen measures by* $\mathfrak{M}$. If U is any function superharmonic on $\mathbb{C}$, so are its translates, so, whenever $\mu \in \mathfrak{M}$ and $z \in \mathbb{C}$,

$$\int_{\mathbb{C}} U(z+\zeta)\,\mathrm{d}\mu(\zeta) \quad \leqslant \quad U(z).$$

Problem 56

The purpose here is to show that if F is continuous on $\mathbb{C}$,

$$(\mathfrak{M}F)(z) \quad = \quad \sup_{\mu \in \mathfrak{M}} \int_{\mathbb{C}} F(z+\zeta)\,\mathrm{d}\mu(\zeta).$$

(a) Show that $\int_{\mathbb{C}} F(z+\zeta)\,\mathrm{d}\mu(\zeta) \leqslant (\mathfrak{M}F)(z)$ for each $\mu \in \mathfrak{M}$.

Denote now the set of Jensen measures *absolutely continuous with respect to two-dimensional Lebesgue measure by* $\mathfrak{L}$. As examples of some measures in $\mathfrak{L}$, we have, for instance, the ν_r described above. $\mathfrak{L}$ is of course a subset of $\mathfrak{M}$.

(b) Show that $\mathfrak{L}$ has a countable subset $\{\mu_k\}$, *dense therein with respect to L_1 convergence with bounded support*. This means that given any $\mu \in \mathfrak{L}$, with say $\mathrm{d}\mu(\zeta) = \varphi(\zeta)\mathrm{d}\xi\,\mathrm{d}\eta$, where $\varphi(\zeta) = 0$ a.e. for $|\zeta| \geqslant$ some integer N, we can find a *subsequence* $\{\mu_{k_j}\}$ of the μ_k such that, if we write $\mathrm{d}\mu_{k_j}(\zeta) = \varphi_{k_j}(\zeta)\mathrm{d}\xi\,\mathrm{d}\eta$, we also have $\varphi_{k_j}(\zeta) = 0$ a.e. for $|\zeta| \geqslant N$, and moreover

$$\iint_{|\zeta| \leqslant N} |\varphi(\zeta) - \varphi_{k_j}(\zeta)|\,\mathrm{d}\xi\,\mathrm{d}\eta \quad \longrightarrow \quad 0 \quad \text{as } j \longrightarrow \infty.$$

(Hint: For the open subsets of each of the spaces $L_1(|z| \leqslant N)$, $N = 1, 2, 3, \ldots$, there is a countable base, some of whose members contain densities belonging to measures from $\mathfrak{L}$. Select.)

(c) Taking the measures μ_k from (b), put

$$V_N(z) \quad = \quad \max_{1 \leqslant k \leqslant N} \int_{\mathbb{C}} F(z+\zeta)\,\mathrm{d}\mu_k(\zeta)$$

for our given continuous function F. Fix any $z \in \mathbb{C}$ and $R > 0$. Show that there is a $\nu \in \mathfrak{L}$ (depending, in general, on z, R and N) such that

$$\frac{1}{2\pi}\int_0^{2\pi} V_N(z + R\mathrm{e}^{\mathrm{i}\vartheta})\,\mathrm{d}\vartheta \quad = \quad \int_{\mathbb{C}} F(z+\zeta)\,\mathrm{d}\nu(\zeta).$$

(Hint: First show how to get a Borel function $k(\vartheta)$ taking the values $1, 2; 3, \ldots, N$ such that

$$V_N(z + R\mathrm{e}^{\mathrm{i}\vartheta}) \quad = \quad \int_{\mathbb{C}} F(z + R\mathrm{e}^{\mathrm{i}\vartheta} + \zeta)\,\mathrm{d}\mu_{k(\vartheta)}(\zeta).$$

Then define ν by the formula

$$\int_{\mathbb{C}} G(\zeta)\,\mathrm{d}\nu(\zeta) \quad = \quad \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{C}} G(\zeta + R\mathrm{e}^{\mathrm{i}\vartheta})\,\mathrm{d}\mu_{k(\vartheta)}(\zeta)\,\mathrm{d}\vartheta$$

and verify that it belongs to $\mathfrak{L}$.)

(d) Hence show that

$$V(z) \quad = \quad \sup_{\mu\in\mathfrak{L}} \int_{\mathbb{C}} F(z+\zeta)\,\mathrm{d}\mu(\zeta) \quad (sic!)$$

is superharmonic.
(Hint: Since F is continuous, we also have $V(z) \;=\; \sup_k \int_{\mathbb{C}} F(z+\zeta)\mathrm{d}\mu_k(\zeta)$ with the μ_k from (b). That is, $V(z) \;=\; \lim_{N\to\infty} V_N(z)$, where the V_N are the functions from (c). But by (c),

$$\frac{1}{2\pi}\int_0^{2\pi} V_N(z + R\mathrm{e}^{\mathrm{i}\vartheta})\mathrm{d}\vartheta \quad \leqslant \quad V(z)$$

for each N. Use monotone convergence.)

(e) Show that

$$\sup_{\mu\in\mathfrak{M}} \int_{\mathbb{C}} F(z+\zeta)\mathrm{d}\mu(\zeta) \quad = \quad (\mathfrak{M}F)(z).$$

(Hint: The left side is surely $\geqslant$ the function $V(z)$ from (d). Observe that $V(z) \;\geqslant\; F(z)$; for this the measures ν_r specified above may be used. This makes V a *superharmonic majorant* of F ! Refer to (a) and to the definition of $\mathfrak{M}F$.)

Remark. The last problem exhibits $\mathfrak{M}F$ as a *maximal function* formed from F by using the *Jensen measures.*

Each $\mu\in\mathfrak{M}$ acts as a *reproducing measure for functions harmonic on* $\mathbb{C}$. We have, in other words,

$$\int_{\mathbb{C}} H(z+\zeta)\mathrm{d}\mu(\zeta) \quad = \quad H(z), \qquad z\in\mathbb{C},$$

for every function H harmonic on $\mathbb{C}$ and every Jensen measure μ. It is important to realize that *not every positive measure μ of compact support having this reproducing property is a Jensen measure.* The following example was shown to me by T. Lyons:

Take

$$\mathrm{d}\mu(\zeta) \quad = \quad \varphi(\zeta)\mathrm{d}\xi\,\mathrm{d}\eta \;+\; \tfrac{1}{4}\mathrm{d}\delta_1(\zeta),$$

where δ_1 is the unit mass concentrated at the point 1 and

$$\varphi(\zeta) = \begin{cases} \dfrac{1}{4\pi}, & |\zeta| \leqslant 2 \text{ and } |\zeta - 1| \geqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

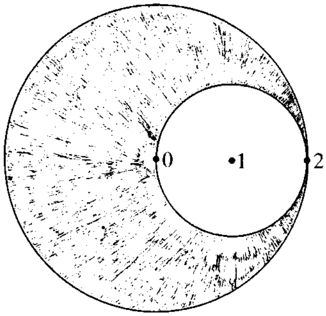

Figure 241

Then, since

$$\frac{1}{4}H(1) = \frac{1}{4\pi}\iint_{|\zeta-1|<1} H(\zeta)\mathrm{d}\xi\,\mathrm{d}\eta$$

for functions H harmonic on $\mathbb{C}$, we have

$$\int_{\mathbb{C}} H(\zeta)\mathrm{d}\mu(\zeta) = \frac{1}{4\pi}\iint_{|\zeta|\leqslant 2} H(\zeta)\mathrm{d}\xi\,\mathrm{d}\eta = H(0),$$

and similarly, by translation,

$$\int_{\mathbb{C}} H(z+\zeta)\mathrm{d}\mu(\zeta) = H(z)$$

for such functions H.

However, $U(z) = \log(1/|z-1|)$ is superharmonic in $\mathbb{C}$, and, with the present μ,

$$\int_{\mathbb{C}} U(\zeta)\mathrm{d}\mu(\zeta) = \infty \quad \text{although} \quad U(0) = 0.$$

This measure μ is therefore *not* in $\mathfrak{M}$.

The reader interested in a general treatment of the matters taken up in this article should consult a recent book by Gamelin (with *Jensen measures* in its title).

3. How $\mathfrak{M}F$ gives us a multiplier if it is finite

Starting now with a continuous* weight $W(x) \geqslant 1$ for which $\int_{-\infty}^{\infty}(\log W(t)/(1+t^2))\mathrm{d}t < \infty$, we choose and fix an $A > 0$ and form the function

$$F(z) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|\Im z|\log W(t)}{|z-t|^2}\mathrm{d}t - A|\Im z|,$$

the expression on the right being interpreted as $\log W(x)$ when $z = x \in \mathbb{R}$. This function F is then continuous and the material of the preceding article applies to it; the smallest superharmonic majorant, $\mathfrak{M}F$, of F is thus at our disposal.

Our object in the present article is to establish a *converse* to the observation made near the beginning of the last one. This amounts to showing that if $\mathfrak{M}F$ is finite, one actually *has* an increasing function ρ, zero on a neighborhood of the origin, such that

$$\frac{\rho(t)}{t} \leqslant \frac{A}{\pi} + o(1) \qquad \text{for } t \longrightarrow \pm\infty$$

and that

$$\log W(x) + \gamma x + \int_{-\infty}^{\infty}\left(\log\left|1-\frac{x}{t}\right| + \frac{x}{t}\right)\mathrm{d}\rho(t) \leqslant \text{const.} \quad \text{for } x \in \mathbb{R}$$

with a certain constant γ. We will do that by deriving a *formula*,

$$(\mathfrak{M}F)(z) = (\mathfrak{M}F)(0) - \gamma\Re z - \int_{-\infty}^{\infty}\left(\log\left|1-\frac{z}{t}\right| + \frac{\Re z}{t}\right)\mathrm{d}\rho(t),$$

involving an increasing function ρ with (subject to an unimportant auxiliary condition on W) the first of the properties in question, and then by simply *using* the fact that $(\mathfrak{M}F)(x)$ is a *majorant* of $F(x) = \log W(x)$. That necessitates our making a preliminary examination of $\mathfrak{M}F$ for the present function F.

Lemma. *If, for $F(z)$ given by the above formula, $\mathfrak{M}F$ is finite, we have*

$$\int_{-\infty}^{\infty}\frac{(\mathfrak{M}F)(t)}{1+t^2}\mathrm{d}t < \infty,$$

* The regularity requirement for weights discussed in article 1 does not, *in itself*, imply their continuity. Nevertheless, in treating weights meeting the requirement, further restriction to the continuous ones (or even to those of class $\mathscr{C}_{\infty}$) does not constitute a serious limitation. See the first theorem of article 1.

and then

$$(\mathfrak{M}F)(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|(\mathfrak{M}F)(t)}{|z-t|^2}\mathrm{d}t \quad - \quad A|\Im z|$$

for $z \notin \mathbb{R}$.

Proof. Since $F(z) = F(\bar{z})$, we have $\min(U(z), U(\bar{z})) \geqslant F(z)$ for any superharmonic majorant U of F. By the next-to-the-last theorem of §A.1, the min just written is also superharmonic in z; it is, on the other hand, $\leqslant U(z)$, and does not change when z is replaced by $\bar{z}$. Therefore, for $\mathfrak{M}F$, the *smallest* superharmonic majorant of F, we have

$$(\mathfrak{M}F)(\bar{z}) \quad = \quad (\mathfrak{M}F)(z),$$

and for this reason it is necessary only to investigate $\mathfrak{M}F$ in the upper half plane.

The function $F(z)$ under consideration is *harmonic* for $\Im z > 0$, and thus, by the second lemma of the preceding article, $(\mathfrak{M}F)(z)$ *is too*, as long as it is *finite*. Because $(\mathfrak{M}F)(z) \geqslant F(z)$,

$$(\mathfrak{M}F)(z) \; + \; A\Im z \quad \geqslant \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log W(t)}{|z-t|^2}\mathrm{d}t,$$

a quantity $\geqslant 0$, for $\Im z > 0$ ($W(t)$ being $\geqslant 1$). The function on the left is hence *harmonic and positive in* $\Im z > 0$.

According to Chapter III, §F.1, we therefore have

$$(\mathfrak{M}F)(z) \; + \; A\Im z \quad = \quad \alpha\Im z \quad + \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \, \mathrm{d}\mu(t)}{|z-t|^2}$$

in $\{\Im z > 0\}$, where $\alpha \geqslant 0$ and μ is some *positive measure* on $\mathbb{R}$, with $\int_{-\infty}^{\infty}(1+t^2)^{-1}\mathrm{d}\mu(t) < \infty$. But $(\mathfrak{M}F)(z)$ is everywhere continuous by the fourth lemma of the last article; it is, in particular, continuous *up to the real axis*. Thus, $\mathrm{d}\mu(t) = (\mathfrak{M}F)(t)\mathrm{d}t$, and

$$(\mathfrak{M}F)(z) \quad = \quad (\alpha - A)\Im z \quad + \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z(\mathfrak{M}F)(t)}{|z-t|^2}\mathrm{d}t$$

for $\Im z > 0$. Using the symmetry of $(\mathfrak{M}F)(z)$ with respect to the x-axis just noted, we see that

$$(\mathfrak{M}F)(z) \quad = \quad (\alpha - A)|\Im z| \quad + \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|(\mathfrak{M}F)(t)}{|z-t|^2}\mathrm{d}t$$

(with the usual interpretation of the right side for $z \in \mathbb{R}$).

Here, $\alpha \geqslant 0$; it is claimed that α is in fact *zero*. Thanks to the sign of α,

$-\alpha|\Im z|$ is *superharmonic* (!), and the same is true of the difference

$$(\mathfrak{M}F)(z) \quad - \quad \alpha|\Im z|.$$

However, $(\mathfrak{M}F)(t) \geqslant F(t) = \log W(t)$, $\mathfrak{M}F$ being a majorant of F, so this difference must, by the preceding formula, be

$$\geqslant \quad -A|\Im z| \quad + \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|\log W(t)}{|z-t|^2}\,\mathrm{d}t \quad = \quad F(z).$$

$(\mathfrak{M}F)(z) - \alpha|\Im z|$ is thus a *superharmonic majorant* of $F(z)$, and therefore $\geqslant (\mathfrak{M}F)(z)$, the least such majorant. This makes $\alpha \leqslant 0$. Since $\alpha \geqslant 0$ as we know, we see that $\alpha = 0$, as claimed.

With $\alpha = 0$, the above formula for $(\mathfrak{M}F)(z)$ reduces to the desired representation. We are done.

Theorem. *Suppose that for a given continuous weight $W(x) \geqslant 1$ the function $\mathfrak{M}F$ corresponding to*

$$F(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|\log W(t)}{|z-t|^2}\,\mathrm{d}t \quad - \quad A|\Im z|$$

(where $A > 0$) is finite. *If $(\mathfrak{M}F)(z)$ is also harmonic in a neighborhood of the origin, we have*

$$\boxed{(\mathfrak{M}F)(z) \quad = \quad (\mathfrak{M}F)(0) \;-\; \gamma\Re z \;-\; \int_{-\infty}^{\infty}\left(\log\left|1-\frac{z}{t}\right| \;+\; \frac{\Re z}{t}\right)\mathrm{d}\rho(t),}$$

with a constant γ and a certain increasing function $\rho(t)$, zero on a neighborhood of the origin, such that

$$\frac{\rho(t)}{t} \longrightarrow \frac{A}{\pi} \qquad \text{for } t \longrightarrow \pm\infty.$$

Remark. The subsidiary requirement that $(\mathfrak{M}F)(z)$ be harmonic in a neighborhood of the origin serves merely to ensure $\rho(t)$'s vanishing in such a neighborhood; it can be lifted, but then the corresponding representation for $\mathfrak{M}F$ looks more complicated (see problem 57 below). Later on in this article, we will see that the harmonicity requirement does not really limit applicability of the boxed formula.

Proof of theorem. Is based on the Riesz representation from §A.2; to the superharmonic function $(\mathfrak{M}F)(z)$ we apply that representation as it is formulated in the remark preceding the last theorem of §A.2 (see the boxed formula there). For each $R > 0$ this gives us a positive measure μ_R on $\{|\zeta| \leqslant R\}$ and a function $H_R(z)$ harmonic *in the interior* of that disk, such that

$$(\mathfrak{M}F)(z) \quad = \quad \int_{|\zeta|\leqslant R} \log\frac{1}{|z-\zeta|}\,d\mu_R(\zeta) \quad + \quad H_R(z) \qquad \text{for } |z| \;<\; R.$$

By problem 48(c), the measures μ_R and $\mu_{R'}$ *agree* in $\{|\zeta| < R\}$ whenever $R' > R$; this means that we actually have a *single positive* (and in general *infinite*) *Borel measure* μ on $\mathbb{C}$ whose restriction to each open disk $\{|\zeta| < R\}$ is the corresponding μ_R (cf. problem 49). This enables us to rewrite the last formula as

$$(\mathfrak{M}F)(z) \quad = \quad \int_{|\zeta|\leqslant R} \log\frac{1}{|z-\zeta|}\,d\mu(\zeta) \quad + \quad H_R(z), \qquad |z| \;<\; R,$$

with, for each R, a certain function $H_R(z)$ (N.B. perhaps *not the same as the previous* $H_R(z)$ *!) harmonic in* $\{|z| < R\}$.

We see by the preceding lemma that $(\mathfrak{M}F)(z)$ is itself harmonic both in $\{\Im z > 0\}$ and in $\{\Im z < 0\}$, so, according to the last theorem of §A.2, μ cannot have any mass in either of those half planes. By the same token, μ has no mass in a certain neighborhood of the origin, $\mathfrak{M}F$ being, by hypothesis, harmonic in such a neighborhood. There is thus an increasing function $\rho(t)$, zero on a neighborhood of the origin, such that

$$\mu(E) \quad = \quad \int_{E\cap\mathbb{R}} d\rho(t)$$

for Borel sets $E \subseteq \mathbb{C}$, and we have

$$(\mathfrak{M}F)(z) \quad = \quad \int_{-R}^{R} \log\frac{1}{|z-t|}\,d\rho(t) \quad + \quad H_R(z) \qquad \text{for } |z| \;<\; R,$$

with H_R harmonic there. Our desired representation will be obtained by making $R \longrightarrow \infty$ in this relation. For that purpose, we need to know the asymptotic behaviour of $\rho(t)$ as $t \longrightarrow \pm\,\infty$.

It is claimed that the ratio

$$\frac{\rho(r) \;-\; \rho(-r)}{r}$$

(which is certainly positive) *remains bounded* when $r \longrightarrow \infty$. Fixing any R, let us consider values of $r < R$. Using the preceding formula and reasoning as in the proof of the last theorem in §A.2, we easily find that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} (\mathfrak{M}F)(re^{i\vartheta})\,d\vartheta \quad = \quad \int_{-R}^{R} \min\left(\log\frac{1}{|t|},\; \log\frac{1}{r}\right)d\rho(t) \quad + \quad H_R(0),$$

and thence, subtracting $(\mathfrak{M}F)(0)$ from both sides, that

$$-\int_{-R}^{R} \log^+ \frac{r}{|t|}\, d\rho(t) \quad = \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} (\mathfrak{M}F)(re^{i\vartheta})d\vartheta \quad - \quad (\mathfrak{M}F)(0)$$

for $0 < r < R$. Here, $\rho(t)$ vanishes on a neighborhood of the origin, so we can integrate the left side by parts to get

$$\int_0^r \frac{\rho(t) \; - \; \rho(-t)}{t}\, dt \quad = \quad (\mathfrak{M}F)(0) \quad - \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} (\mathfrak{M}F)(re^{i\vartheta})d\vartheta,$$

which is, of course, nothing but a version of Jensen's formula. In it, R no longer appears, so it is valid for all $r > 0$.

By the lemma, however,

$$-(\mathfrak{M}F)(z) \quad = \quad A|\Im z| \quad - \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|(\mathfrak{M}F)(t)}{|z-t|^2}\, dt,$$

a quantity $\leqslant A|\Im z|$, since $(\mathfrak{M}F)(t) \geqslant \log W(t) \geqslant 0$. Using this in the previous relation, we get

$$\int_0^r \frac{\rho(t) \; - \; \rho(-t)}{t}\, dt \quad \leqslant \quad (\mathfrak{M}F)(0) \; + \; \frac{2A}{\pi} r,$$

whence

$$\rho(r) \; - \; \rho(-r) \quad \leqslant \quad (\mathfrak{M}F)(0) \; + \; \frac{2A}{\pi} er$$

by the argument of problem 1(a) (!), $\rho(t)$ being increasing. Since $\rho(t)$ also vanishes in a neighborhood of 0, we see that

$$\frac{\rho(t)}{t} \quad \leqslant \quad \text{const.} \quad \text{on } \mathbb{R}.$$

Once this is known, it follows by reasoning like that of §A, Chapter III, that the integral

$$\int_{-\infty}^{\infty} \left(\log\left|1 - \frac{z}{t}\right| \; + \; \frac{\Re z}{t} \right) d\rho(t)$$

is convergent (*a priori*, to $-\infty$, possibly, when $z \in \mathbb{R}$) for all values of z – one needs here to again use the vanishing of $\rho(t)$ for t near 0. That integral, however, obviously differs from

$$\int_{-R}^{R} \log|z-t|\, d\rho(t)$$

by a function *harmonic* for $|z| < R$. Referring to the previous representation of $(\mathfrak{M}F)(z)$ in that disk, we see that

$$G(z) = (\mathfrak{M}F)(z) + \int_{-\infty}^{\infty} \left(\log\left|1 - \frac{z}{t}\right| + \frac{\Re z}{t} \right) d\rho(t)$$

must be *harmonic* for $|z| < R$, and hence finally for *all* z, since the parameter R no longer occurs on the right. Our local Riesz representations for $(\mathfrak{M}F)(z)$ in the disks $\{|z| < R\}$ thus have a *global* version,

$$(\mathfrak{M}F)(z) = -\int_{-\infty}^{\infty} \left(\log\left|1 - \frac{z}{t}\right| + \frac{\Re z}{t} \right) d\rho(t) + G(z),$$

valid for all z, with G harmonic everywhere.

We proceed to investigate $G(z)$'s behaviour for large $|z|$. The lemma gives, first of all,

$$(\mathfrak{M}F)(z) \leqslant \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|(\mathfrak{M}F)(t)}{|z-t|^2}\, dt,$$

$(\mathfrak{M}F)(t)$ being $\geqslant 0$. Therefore

$$[G(z)]^+ \leqslant \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|(\mathfrak{M}F)(t)}{|z-t|^2} dt + \left(\int_{-\infty}^{\infty} \left(\log\left|1 - \frac{z}{t}\right| + \frac{\Re z}{t} \right) d\rho(t) \right)^+ .$$

According to the discussion at the beginning of §B, Chapter III, our bound on the growth of $\rho(t)$ makes the *second* term on the right $\leqslant O(|z|\log|z|)$ for large values of $|z|$; we thus have

$$\int_{-\pi}^{\pi} [G(re^{i\vartheta})]^+ \, d\vartheta \leqslant \text{const.}\, r\log r + \frac{1}{\pi}\int_{-\pi}^{\pi}\int_{-\infty}^{\infty} \frac{r|\sin\vartheta|(\mathfrak{M}F)(t)}{r^2 + t^2 - 2rt\cos\vartheta}\, dt\, d\vartheta$$

when r is large, and, desiring to estimate the integral on the *left*, we must study the one figuring on the *right*. Changing the order of integration converts the latter to

$$\frac{2}{\pi}\int_{-\infty}^{\infty} \frac{1}{t}\log\left|\frac{r+t}{r-t}\right| (\mathfrak{M}F)(t)\, dt,$$

which we handle by resorting to a trick.

Take the *average* of the expression in question for $R \leqslant r \leqslant 2R$, say, where $R > 0$ is arbitrary. That works out to

$$\frac{2}{\pi R}\int_{-\infty}^{\infty}\int_{R}^{2R} \frac{1}{t}\log\left|\frac{r+t}{r-t}\right| (\mathfrak{M}F)(t)\, dr\, dt = \frac{2}{\pi R}\int_{-\infty}^{\infty} \Psi\left(\frac{R}{|t|}\right)(\mathfrak{M}F)(t)\, dt,$$

where

$$\Psi(u) \quad = \quad \int_u^{2u} \log\left|\frac{s+1}{s-1}\right| \mathrm{d}s.$$

The last integral can be directly evaluated, but here it is better to use power series and see how it acts when $u \longrightarrow 0$ and when $u \longrightarrow \infty$.

For $0 < u < 1$, expand the integrand in powers of s to get

$$\Psi(u) \quad = \quad 3u^2 \; + \; \mathrm{O}(u^4), \qquad 0 < u < 1.$$

For $u > 1$, we expand the integrand in powers of $1/s$ and find that

$$\Psi(u) \quad = \quad 2\log 2 \; + \; \mathrm{O}\left(\frac{1}{u^2}\right), \qquad u > 1.$$

$\Psi(R/|t|)/R$ thus behaves like $1/R$ for *small* values of $|t|/R$ and like R/t^2 for *large* ones, so, all in all,

$$\frac{2}{\pi R}\Psi\left(\frac{R}{|t|}\right) \quad \leqslant \quad \text{const.}\frac{R}{R^2+t^2} \qquad \text{for } t \in \mathbb{R}.$$

Substituting this into the previous relation, we see that

$$\frac{2}{\pi R}\int_R^{2R}\int_{-\infty}^{\infty}\frac{1}{t}\log\left|\frac{r+t}{r-t}\right|(\mathfrak{M}F)(t)\,\mathrm{d}t\,\mathrm{d}r \quad \leqslant \quad \text{const.}\int_{-\infty}^{\infty}\frac{R}{R^2+t^2}(\mathfrak{M}F)(t)\,\mathrm{d}t.$$

This, however, *implies the existence of an* r', $R \leqslant r' \leqslant 2R$, for which

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{1}{t}\log\left|\frac{r'+t}{r'-t}\right|(\mathfrak{M}F)(t)\,\mathrm{d}t \quad \leqslant \quad \text{const.}\int_{-\infty}^{\infty}\frac{R}{R^2+t^2}(\mathfrak{M}F)(t)\,\mathrm{d}t.$$

Here, the right side is $\leqslant$ const.R $\leqslant$ const.r' (and is even $\mathrm{o}(R)$) for large R, since $\int_{-\infty}^{\infty}((\mathfrak{M}F)(t)/(1+t^2))\,\mathrm{d}t \; < \; \infty$. Taking r equal to such an r' in our original relation involving G thus yields

$$\int_{-\pi}^{\pi}[G(r'e^{i\vartheta})]^+\,\mathrm{d}\vartheta \quad \leqslant \quad \text{const.}(r'\log r' \; + \; r')$$

when R, and hence r', is large.

Letting R take successively the values 2^n with $n = 1, 2, 3, \ldots$, we obtain in this way a certain sequence of numbers r_n tending to ∞ for which

$$\int_{-\pi}^{\pi}[G(r_n e^{i\vartheta})]^+\,\mathrm{d}\vartheta \quad \leqslant \quad \mathrm{O}(r_n \log r_n).$$

Since $G(z)$ is harmonic, we have on the other hand

$$\int_{-\pi}^{\pi}\left([G(r_n e^{i\vartheta})]^+ \; - \; [G(r_n e^{i\vartheta})]^-\right)\mathrm{d}\vartheta \quad = \quad \int_{-\pi}^{\pi}G(r_n e^{i\vartheta})\,\mathrm{d}\vartheta \quad = \quad 2\pi G(0),$$

so, subtracting this relation from *twice* the preceding, we get

$$\int_{-\pi}^{\pi} |G(r_n e^{i\vartheta})|\,d\vartheta \quad \leqslant \quad O(r_n \log r_n).$$

Now it follows that $G(z)$ must be of the form $A_0 + A_1 \Re z$. We have, indeed, $G(\bar{z}) = G(z)$, since $\mathfrak{M}F$ and the integral involving $d\rho$ have that property; the function $G(z)$, harmonic everywhere, is therefore given by a series development

$$G(re^{i\vartheta}) \quad = \quad \sum_{k=0}^{\infty} A_k r^k \cos k\vartheta.$$

For $k > 1$, we have

$$A_k \quad = \quad \frac{1}{\pi r^k} \int_{-\pi}^{\pi} G(re^{i\vartheta}) \cos k\vartheta \, d\vartheta.$$

Putting $r = r_n$ and making $n \longrightarrow \infty$, we see, using the estimate just found, that $A_k = 0$. The series thus boils down to *its first two terms.*

Going back to our global version of the Riesz representation for $(\mathfrak{M}F)(z)$ and using the description of G just found, we see that

$$(\mathfrak{M}F)(z) \quad = \quad A_0 \; + \; A_1 \Re z \; - \; \int_{-\infty}^{\infty} \left(\log\left|1 - \frac{z}{t}\right| \; + \; \frac{\Re z}{t} \right) d\rho(t).$$

Because $\rho(t)$ vanishes for t near 0, it is obvious that $A_0 = (\mathfrak{M}F)(0)$. Denoting A_1 by $-\gamma$, we now have the formula we set out to establish.

In order to complete this proof, we must still refine the estimate

$$\frac{\rho(t)}{t} \quad \leqslant \quad \text{const.}$$

obtained and used above to the asymptotic relation

$$\frac{\rho(t)}{t} \quad = \quad \frac{A}{\pi} \; + \; o(1), \qquad t \longrightarrow \pm\infty.$$

For this, some version of Levinson's theorem (the one from Chapter III) must be used.

Write

$$V(z) \quad = \quad \gamma \Re z \; + \; \int_{-\infty}^{\infty} \left(\log\left|1 - \frac{z}{t}\right| \; + \; \frac{\Re z}{t} \right) d\rho(t);$$

then, by the previous lemma and the representation formula just proved, we have

$$V(z) - (\mathfrak{M}F)(0) = -(\mathfrak{M}F)(z) = A|\Im z| - \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|(\mathfrak{M}F)(t)}{|z-t|^2}\mathrm{d}t.$$

From this, we readily see that

$$\frac{V(\mathrm{i}y)}{|y|} \longrightarrow A \quad \text{as } y \longrightarrow \pm\infty,$$

whilst

$$V(z) \leqslant (\mathfrak{M}F)(0) + A|\Im z|$$

for all z.

Take, as in the proofs of the last two theorems of article 1, an entire function φ such that

$$\log|\varphi(z)| = \beta\Re z + \int_{-\infty}^{\infty}\left(\log\left|1 - \frac{z}{t}\right| + \frac{\Re z}{t}\right)\mathrm{d}[\rho(t)]$$

where β is constant; according to a lemma from that article, we have, for suitable choice of β, the inequality

$$\log|\varphi(z)| \leqslant V(z) + \log^+\left|\frac{\Re z}{\Im z}\right| + \log^+|z| + \mathrm{O}(1).$$

Applying this first with $z = x \pm \mathrm{i}$ and using the preceding estimate for V, we see, taking account of the fact that $|\varphi(z)|$ *diminishes* when $|\Im z|$ does, that

$$|\varphi(z)| \leqslant \text{const.}(|z|^2 + 1), \qquad |\Im z| \leqslant 1.$$

We next find from the same relations that

$$|\varphi(z)| \leqslant \text{const.}(|z|^2 + 1)\mathrm{e}^{A|\Im z|}$$

when $|\Im z| > 1$; in view of the preceding inequality such an estimate (with perhaps a larger constant) must then hold *everywhere.* $\varphi(z)$ is thus of *exponential type.*

A computation like one near the end of the next-to-the-last theorem in article 1 now yields, for $y \in \mathbb{R}$,

$$\log|\varphi(\mathrm{i}y)| - V(\mathrm{i}y) = \int_{-\infty}^{\infty} \frac{y^2}{y^2+t^2}\,\frac{[\rho(t)] - \rho(t)}{t}\,\mathrm{d}t.$$

Since $[\rho(t)] - \rho(t)$ is *bounded* (above and below!) and zero on a neighborhood of the origin, the integral on the right is $\mathrm{o}(|y|)$ for

$y \longrightarrow \pm\infty$, and hence

$$\frac{\log|\varphi(\mathrm{i}y)|}{|y|} \longrightarrow A \quad \text{as } y \longrightarrow \pm\infty,$$

in view of the above similar relation for $V(\mathrm{i}y)$.

By the preceding estimates on $\varphi(z)$, we obviously have

$$\int_{-\infty}^{\infty} \frac{\log^+|\varphi(x)|}{1+x^2}\,\mathrm{d}x \quad < \quad \infty,$$

and the Levinson theorem from §H.2 of Chapter III can be applied to φ. Referring to the last of the above relations, we see in that way that

$$\frac{[\rho(t)]}{t} \longrightarrow \frac{A}{\pi} \quad \text{as } t \longrightarrow \pm\infty.$$

Therefore,

$$\frac{\rho(t)}{t} \longrightarrow \frac{A}{\pi} \qquad \text{for } t \longrightarrow \pm\infty.$$

Our theorem is proved.

Problem 57

If $(\mathfrak{M}F)(z)$ is finite, but not necessarily harmonic in a neighborhood of 0, find a representation for it analogous to the one furnished by the result just obtained.

As stated previously, the last theorem has quite general utility in spite of its harmonicity requirement. Any situation involving a finite function $\mathfrak{M}F$ can be reduced to one for which the corresponding $\mathfrak{M}F$ *is* harmonic near 0. The easiest way of doing that is to use the following

Lemma. *Let $W(t)$, continuous and $\geqslant 1$ on $\mathbb{R}$, be $\equiv 1$ for $-h < t < h$, where $h > 0$, and suppose that for $|x| < h$, we have*

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log W(t)}{(x-t)^2}\,\mathrm{d}t \quad > \quad A,$$

with the integral on the left convergent. Then the function

$$F(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|\log W(t)}{|z-t|^2}\,\mathrm{d}t \quad - \quad A|\Im z|$$

satisfies

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F(re^{i\vartheta})\,d\vartheta \quad > \quad F(0) \quad = \quad 0$$

for $0 < r < h$.

Proof. We have $F(z) = F(\bar{z})$, so

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F(re^{i\vartheta})\,d\vartheta \quad = \quad \frac{1}{\pi}\int_{0}^{\pi} F(re^{i\vartheta})\,d\vartheta.$$

It will be convenient to denote the right-hand integral by $J(r)$ and to work with the function

$$G(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log W(t)}{|z-t|^2}\,dt \quad - \quad A\,\Im z \quad (sic!)$$

instead of $F(z)$; we of course also have

$$J(r) \quad = \quad \frac{1}{\pi}\int_{0}^{\pi} G(re^{i\vartheta})\,d\vartheta.$$

In the present circumstances the function $G(z)$ is finite, and hence *harmonic*, in both the upper and the lower half planes. Moreover, since $\log W(t) \equiv 0$ for $|t| < h$, $G(z)$ (taken as *zero* on the real interval $(-h, h)$) is actually harmonic* in $\mathbb{C} \sim (-\infty, -h] \sim [h, \infty)$ and hence $\mathscr{C}_\infty$ in that region. There is thus no obstacle to differentiating under the integral sign so as to get

$$\frac{dJ(r)}{dr} \quad = \quad \frac{1}{\pi r}\int_{0}^{\pi} \frac{\partial G(re^{i\vartheta})}{\partial r}\,r\,d\vartheta, \qquad 0 < r < h.$$

Let $\mathscr{D}_r$ be the semi-circle of radius r lying in the upper half plane, having for diameter the real segment $[-r, r]$:

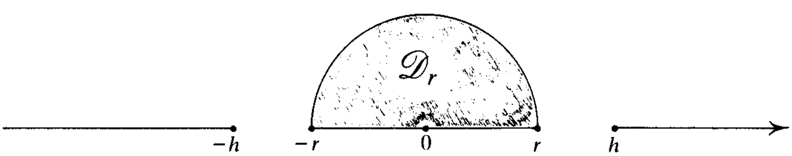

Figure 242

* by Schwarz' reflection principle, since $G(\bar{z}) = -G(z)$

When $r < h$, the function $G(z)$ is harmonic in a region including the *closure* of $\mathscr{D}_r$, so we can use Green's theorem to get

$$\int_{\partial\mathscr{D}_r} \frac{\partial G(\zeta)}{\partial n_\zeta} |\mathrm{d}\zeta| \quad = \quad \iint_{\mathscr{D}_r} (\nabla^2 G)(\zeta)\, \mathrm{d}\xi\, \mathrm{d}\eta \quad = \quad 0,$$

where $\partial/\partial n_\zeta$ denotes differentiation along the outward normal to $\partial\mathscr{D}_r$ at ζ. The left-hand expression is just

$$\int_0^\pi \frac{\partial G(r\mathrm{e}^{\mathrm{i}\vartheta})}{\partial r} r\,\mathrm{d}\vartheta \quad - \quad \int_{-r}^r G_y(x)\,\mathrm{d}x,$$

so the previous relation yields

$$J'(r) \quad = \quad \frac{1}{\pi r}\int_{-r}^r G_y(x)\,\mathrm{d}x.$$

Here, $G(z) = F(z)$ for $\Im z \geqslant 0$ with $G(x) = F(x) = \log W(x) = 0$ for $-h < x < h$, so, for such x,

$$G_y(x) \quad = \quad \lim_{\Delta y \to 0+} \frac{F(x + \mathrm{i}\Delta y)}{\Delta y},$$

which, by our formula for F, is equal to

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log W(t)}{(x-t)^2}\,\mathrm{d}t \quad - \quad A.$$

If, then, this expression is > 0 for $|x| < h$, we must, by the preceding formula, have

$$J'(r) > 0 \quad \text{for } 0 < r < h.$$

Obviously, $J(r) \longrightarrow F(0) = 0$ for $r \longrightarrow 0$. Therefore,

$$F(0) \quad < \quad J(r) \quad = \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} F(r\mathrm{e}^{\mathrm{i}\vartheta})\,\mathrm{d}\vartheta$$

when $0 < r < h$, given that the hypothesis holds. We are done.

Corollary. *Given $W(x)$ continuous and $\geqslant 1$ with $\int_{-\infty}^{\infty}(\log W(t)/(1+t^2))\,\mathrm{d}t < \infty$, and the number $A > 0$, form, for $h > 0$, the new weight*

$$W_h(x) \quad = \quad \begin{cases} 1, \quad |x| \leqslant h, \\ \mathrm{e}^{2\pi A h} W(x), \quad |x| \geqslant 2h, \\ \textit{linear for } -2h \leqslant x \leqslant -h \textit{ and for } h \leqslant x \leqslant 2h. \end{cases}$$

Put then

$$F_h(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|\log W_h(t)}{|z-t|^2}\,dt \ - \ A|\Im z|.$$

If $(\mathfrak{M}F_h)(z)$ *is finite, it is harmonic in a neighborhood of the origin.*

Proof. When $-h < x < h$,

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log W_h(t)}{(x-t)^2}\,dt \quad \geqslant \quad \frac{1}{\pi}\int_{2h}^{\infty}\left(\frac{2\pi A h}{(t-x)^2} + \frac{2\pi A h}{(t+x)^2}\right)dt$$

$$= \quad \frac{8Ah^2}{4h^2 - x^2} \quad \geqslant \quad 2A \quad > \quad A.$$

The lemma, applied to W_h and F_h, thus yields

$$F_h(0) \quad < \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} F_h(re^{i\vartheta})\,d\vartheta$$

for $0 < r < h$. Since, however, $\mathfrak{M}F_h$ is a *superharmonic majorant* of F_h, the right-hand integral is

$$\leqslant \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} (\mathfrak{M}F_h)(re^{i\vartheta})\,d\vartheta \quad \leqslant \quad (\mathfrak{M}F_h)(0),$$

i.e.,

$$F_h(0) \quad < \quad (\mathfrak{M}F_h)(0).$$

The corollary now follows by the third lemma of article 2.

The preceding results give us our desired converse to the statement from the last article.

Theorem. *Let* $W(x) \geqslant 1$ *be continuous, with*

$$\int_{-\infty}^{\infty} (\log W(t)/(1+t^2))\,dt \quad < \quad \infty,$$

and put

$$F(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|\log W(t)}{|z-t|^2}\,dt \ - \ A|\Im z|,$$

where $A > 0$, *interpreting the right side in the usual way when* $z \in \mathbb{R}$. *If the smallest superharmonic majorant,* $\mathfrak{M}F$, *of* F *is finite, there is an increasing* *function ρ, zero on a neighborhood of the origin, for which*

$$\log W(x) \;+\; \gamma x \;+\; \int_{-\infty}^{\infty}\left(\log\left|1-\frac{x}{t}\right| \;+\; \frac{x}{t}\right)\mathrm{d}\rho(t) \quad\leqslant\quad \text{const.}, \quad x\in\mathbb{R}$$

(with a certain constant γ), while

$$\frac{\rho(t)}{t} \;\longrightarrow\; \frac{A}{\pi} \quad \text{as } t\longrightarrow\pm\infty.$$

Proof. With $h > 0$, form the functions W_h and F_h figuring in the preceding corollary. Since $\log W(t) \geqslant 0$, we have

$$\log W_h(t) \;\leqslant\; \log W(t) \;+\; 2\pi Ah,$$

whence

$$F_h(z) \;\leqslant\; F(z) \;+\; 2\pi Ah.$$

Thus, since $(\mathfrak{M}F)(z) \geqslant F(z)$,

$$F_h(z) \;\leqslant\; (\mathfrak{M}F)(z) \;+\; 2\pi Ah.$$

In the last relation, the right-hand member is *superharmonic*, and, of course, *finite* if $\mathfrak{M}F$ is. Then, however, $\mathfrak{M}F_h$, the *least superharmonic majorant of* F_h, must *also* be finite.

This, according to the corollary, implies that $(\mathfrak{M}F_h)(z)$ is harmonic in a neighborhood of the origin. Once that is known, the previous theorem gives us an increasing function ρ having the required properties, such that

$$(\mathfrak{M}F_h)(z) \;=\; (\mathfrak{M}F_h)(0) \;-\; \gamma\Re z \;-\; \int_{-\infty}^{\infty}\left(\log\left|1-\frac{z}{t}\right| \;+\; \frac{\Re z}{t}\right)\mathrm{d}\rho(t),$$

γ being a certain constant. Thus, since $(\mathfrak{M}F_h)(x) \geqslant F_h(x) = \log W_h(x)$,

$$\log W_h(x) \;+\; \gamma x \;+\; \int_{-\infty}^{\infty}\left(\log\left|1-\frac{x}{t}\right| \;+\; \frac{x}{t}\right)\mathrm{d}\rho(t)$$
$$\leqslant\; (\mathfrak{M}F_h)(0) \qquad \text{for } x\in\mathbb{R}.$$

Let now m_h denote the maximum of $W(x)$ for $-2h \leqslant x \leqslant 2h$. Then certainly

$$\log W(x) \;\leqslant\; \log m_h \;+\; \log W_h(x),$$

$W(x)$, and hence m_h, being $\geqslant 1$. This, substituted into the previous, yields finally

$$\log W(x) \;+\; \gamma x \;+\; \int_{-\infty}^{\infty}\left(\log\left|1-\frac{x}{t}\right| \;+\; \frac{x}{t}\right)\mathrm{d}\rho(t)$$

$$\leqslant\; (\mathfrak{M}F_h)(0) \;+\; \log m_h \qquad \text{for } x\in\mathbb{R}.$$

We are done.

The proof just given furnishes a more precise result which is sometimes useful.

Corollary. *If $W(x)$, satisfying the hypothesis of the theorem, is, in addition,* 1 *at the origin, and the function $\mathfrak{M}F$ corresponding to some given $A > 0$ is finite, we have, for any $\eta > 0$, an increasing function $\rho(t)$ with the properties affirmed by the theorem, such that*

$$\log W(x) \;+\; \gamma x \;+\; \int_{-\infty}^{\infty}\left(\log\left|1-\frac{x}{t}\right| \;+\; \frac{x}{t}\right)\mathrm{d}\rho(t) \;\leqslant\; (\mathfrak{M}F)(0) \;+\; \eta,\; x\in\mathbb{R}.$$

To verify this, we first observe that the continuity of $W(x)$ makes $m_h \longrightarrow 1$ and hence $\log m_h \longrightarrow 0$ when $h \longrightarrow 0$. On the other hand,

$$(\mathfrak{M}F_h)(0) \;\leqslant\; (\mathfrak{M}F)(0) \;+\; 2\pi Ah,$$

since $(\mathfrak{M}F)(z) \;+\; 2\pi Ah$ is a superharmonic majorant of $F_h(z)$, as remarked at the beginning of the proof. The desired relation involving ρ will therefore follow from the *last* one in the proof if we take $h > 0$ small enough so as to have

$$\log m_h \;+\; 2\pi Ah \;<\; \eta.$$

These results and the obvious converse noted in article 2 are used in conjunction with the material from article 1. Referring, for instance, to the corollary of the next-to-the-last theorem in article 1, we have the

Theorem. *Let $W(x)$, continuous and $\geqslant 1$ on the real axis, fulfill the regularity requirement formulated in article* 1. *In order that W admit multipliers, it is necessary and sufficient that*

$$\int_{-\infty}^{\infty}\frac{\log W(t)}{1+t^2}\,\mathrm{d}t \;<\; \infty$$

and that then, for each $A > 0$, the smallest superharmonic majorant *of*

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|\Im z|\log W(t)}{|z-t|^2}\,\mathrm{d}t \;-\; A|\Im z|$$

be finite.

Looking at the *last* theorem of article 1 we see in the same way that such a result holds for any weight $W(x) \geqslant 1$ of the form $|F(x)|$, where F is *entire* and of *exponential type, without any additional assumption on the regularity of* W. This fact will be used in the next §.

The regularity requirement on W figuring in the above theorem may, of course, by replaced by the milder one discussed in the scholium to article 1.*

Let us hark back for a moment to the discussion at the beginning of article 1. Can one regard the condition that $(\mathfrak{M}F)(0)$ be *finite* for *each* of the functions

$$F(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|\log W(t)}{|z-t|^2}\,dt \;-\; A|\Im z|, \qquad A > 0,$$

as one of *regularity* to be satisfied by the weight W? *In a sense*, one *can* – see especially problem 55. Is *this*, then, the presumed *second* ('essential') kind of regularity a weight must have in order to admit multipliers?

C. Theorems of Beurling and Malliavin

We are going to apply the results from the end of the last § so as to obtain multiplier theorems for certain kinds of continuous weights W. Those are always assumed to be $\geqslant 1$ on the real axis, and only for the *unbounded* ones can there be any question about the existence of multipliers.

One can in fact work exclusively with weights $W(x)$ *tending to* ∞ *for* $x \longrightarrow \pm\infty$ without in any way lessening the generality of the results obtained. Suppose, indeed, that we are given an unbounded weight $W(x) \geqslant 1$; then

$$\Omega(x) \quad = \quad (1+x^2)W(x)$$

does tend to ∞ when $x \longrightarrow \pm\infty$, and it is claimed that there is a non-zero entire function of exponential type $\leqslant A$ *whose product with* Ω *is bounded on* $\mathbb{R}$ *if and only if* there is such an entire function whose *product with* W is bounded there.

It is clearly only the *if* part of this statement that requires checking. Consider, then, that we have an entire function $\varphi(z) \not\equiv 0$ of exponential type $\leqslant A$ making $\varphi(x)W(x)$ bounded on $\mathbb{R}$. Since $W(x)$ is unbounded, $|\varphi(x)|$ cannot be constant, so the Hadamard product for φ (Chapter III

* See also Remark 5 near the end of §E.2.

§A) must involve *linear factors* – there must in fact be *infinitely many* of those, for otherwise $|\varphi(x)|$ would *grow like a polynomial* in x when $|x| \longrightarrow \infty$. The function $\varphi(z)$ thus has infinitely many zeros, and, taking any two of them, say α and β, we can form a new entire function,

$$\psi(z) \quad = \quad \frac{\varphi(z)}{(z-\alpha)(z-\beta)},$$

also of exponential type $\leqslant A$, with $\psi(x)\Omega(x)$ bounded on the real axis.

The existence of multipliers for $W(x)$ is thus fully equivalent to existence thereof for $\Omega(x)$, a weight tending to ∞ for $x \longrightarrow \pm\infty$; that is fortunate, because weights having the latter property are easier to deal with. When working with a *given* weight W, it will sometimes be convenient to form from it the new one

$$\left(1 + \frac{x^2}{M^2}\right)W(x)$$

(using a *large* value of M) or

$$(1+x^2)^\eta W(x)$$

(taking for η a *small* value > 0), instead of dealing with the weight $\Omega(x)$ just looked at. Any of these weights $\tilde{W}$ fulfills the condition

$$\int_{-\infty}^{\infty} \frac{\log \tilde{W}(x)}{1+x^2}\mathrm{d}x \quad < \quad \infty$$

as long as

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2}\mathrm{d}x \quad < \quad \infty;$$

unless the latter holds W cannot, as we know, admit any multipliers.

In order to establish the existence of multipliers for a weight $\geqslant 1$ satisfying the last condition, we first form from it a new one according to one of the above recipes* if that is necessary to ensure our having a weight tending to ∞ with $|x|$. Then, choosing a number $A > 0$ and using the *new* weight W, we take the function

$$F(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|}{|z-t|^2}\log W(t)\,\mathrm{d}t \quad - \quad A|\Im z|$$

studied in §B.3. According to results obtained there, the question of our weight's *admittance of multipliers* reduces in large part to a simple decision

* the new weight obviously meets the local regularity requirement of §B.1 iff the original one does

about the *finiteness* of $\mathfrak{M}F$, the smallest superharmonic majorant of F, for various initial choices of the number A. We know by the first lemma of §B.2 that the latter property is *equivalent to the finiteness of* $\mathfrak{M}F$ *at any one point*, say that of $(\mathfrak{M}F)(0)$. To evaluate this quantity we will use harmonic estimation, guided by the knowledge that $\mathfrak{M}F$, *if* finite, must be *harmonic* in both the upper and lower half planes (first lemma of §B.3), and *also harmonic* across any real interval on which it is $> F$ (by the third lemma of §B.2).

1. Use of the domains from §C of Chapter VIII

Starting, then, with a continuous weight $W(x) \geqslant 1$ tending to ∞ for $x \longrightarrow \pm\infty$, we take (using some given $A > 0$) the function $F(z)$ whose formula has just been written, and look at its *smallest superharmonic majorant* $\mathfrak{M}F$, our aim being to see whether or not $(\mathfrak{M}F)(0) < \infty$. The idea is to get at $\mathfrak{M}F$ by using *other* superharmonic majorants whose qualitative behaviour is known.

For each $N > 1$, let

$$W_N(x) \quad = \quad \min(W(x),\ N)$$

and then form the function

$$F_N(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|}{|z-t|^2} \log W_N(t)\,\mathrm{d}t \quad - \quad A|\Im z|$$

corresponding to it in the way that $F(z)$ corresponds to W. Clearly,

$$F_N(z) \quad \underset{N}{\uparrow} \quad F(z);$$

it is claimed that also

$$(\mathfrak{M}F_N)(z) \quad \underset{N}{\uparrow} \quad (\mathfrak{M}F)(z).$$

For each N, we have, indeed,

$$(\mathfrak{M}F_{N+1})(z) \;\geqslant\; F_{N+1}(z) \;\geqslant\; F_N(z),$$

so $\mathfrak{M}F_{N+1}$ *is* a superharmonic *majorant of* F_N, and hence

$$(\mathfrak{M}F_{N+1})(z) \quad \geqslant \quad (\mathfrak{M}F_N)(z),$$

the *least* superharmonic majorant of $F_N(z)$. By the same token,

$$(\mathfrak{M}F)(z) \quad \geqslant \quad (\mathfrak{M}F_N)(z)$$

for each N, so we have

$$\lim_{N\to\infty} (\mathfrak{M}F_N)(z) \leqslant (\mathfrak{M}F)(z).$$

The sequence $\{\mathfrak{M}F_N\}$ is, as just shown, *increasing*, so $\lim_{N\to\infty} \mathfrak{M}F_N$ is *superharmonic* by the next-to-the-last theorem of §A.1. That limit must, however, be $\geqslant \lim_{N\to\infty} F_N = F$, so it is also a *majorant* for F. As a *superharmonic majorant* of F, $\lim_{N\to\infty} \mathfrak{M}F_N$ is therefore $\geqslant \mathfrak{M}F$. So, since the contrary relation holds, we in fact have *equality*, as asserted.

We thus have, in particular,

$$(\mathfrak{M}F)(0) = \lim_{N\to\infty} (\mathfrak{M}F_N)(0),$$

whence, in order to verify that $(\mathfrak{M}F)(0) < \infty$, *it suffices to obtain an upper bound independent of N on the values $(\mathfrak{M}F_N)(0)$.*

Each of the functions $\mathfrak{M}F_N$ is certainly *finite*. Indeed,

$$0 \leqslant \log W_N(t) \leqslant \log N,$$

so

$$F_N(z) \leqslant \log N - A|\Im z|.$$

But the right-hand expression in this last relation is *superharmonic*! Hence,

$$(\mathfrak{M}F_N)(z) \leqslant \log N - A|\Im z|.$$

Since, on the other hand,

$$F_N(z) \geqslant -A|\Im z|,$$

we also have

$$(\mathfrak{M}F_N)(z) \geqslant -A|\Im z|.$$

Thanks to our assumption that $W(x) \longrightarrow \infty$ for $x \longrightarrow \pm\infty$, there is a certain number L, depending on N, such that

$$W_N(x) = N \quad \text{for } |x| \geqslant L.$$

Therefore $F_N(x) = \log N$ for $|x| \geqslant L$, making $(\mathfrak{M}F_N)(x) \geqslant \log N$ for such x. By one of the previous relations, we have, however, $(\mathfrak{M}F_N)(x) \leqslant \log N$ on $\mathbb{R}$. Thus,

$$(\mathfrak{M}F_N)(x) = \log N = F_N(x) \qquad \text{for } |x| \geqslant L.$$

We see that *on the real axis*, $(\mathfrak{M}F_N)(x)$ (a *continuous* function by the fourth lemma of §B.2) can *strictly exceed* the (continuous) function $F_N(x)$ *only on an open subset $\mathcal{O}$ of $(-L, L)$*. The *first* lemma of §B.3 and the *third* one of §B.2 then ensure that $(\mathfrak{M}F_N)(z)$ is *harmonic* in the region

$$\{\Im z > 0\} \cup \{\Im z < 0\} \cup \mathcal{O};$$

it is, moreover, *continuous up to the boundary* of that region (indeed, continuous everywhere), again by the fourth lemma of §B.2. The boundary certainly includes the two infinite segments $(-\infty, -L]$ and $[L, \infty)$ of the real axis, on which $(\mathfrak{M}F_N)(x) = F_N(x)$.

The open subset $\mathcal{O}$ of $\mathbb{R}$ *might*, however, be so complicated as to raise doubts about our being able to solve the Dirichlet problem in the region (bounded by $\mathbb{R} \sim \mathcal{O}$) just described, and it is thus not clear that one can do harmonic estimation there. We get around this difficulty by means of a simple device.

The difference

$$(\mathfrak{M}F_N)(x) - F_N(x)$$

is continuous on $\mathbb{R}$ and *identically zero* on $\mathbb{R} \sim \mathcal{O}$. The latter set includes $(-\infty, -L] \cup [L, \infty)$, so our difference is actually *uniformly continuous* on $\mathbb{R}$, and, given any $\varepsilon > 0$, we can find a $\delta > 0$ such that

$$(\mathfrak{M}F_N)(x) - F_N(x) \leqslant \varepsilon \qquad \text{if} \quad \operatorname{dist}(x, \mathbb{R} \sim \mathcal{O}) \leqslant \delta.$$

The points x fulfilling the condition on the right make up a certain *closed* set

$$E_\delta = (\mathbb{R} \sim \mathcal{O}) + [-\delta, \delta]$$

possibly equal to $\mathbb{R}$ which, in any event, includes $(-\infty, -L+\delta] \cup [L-\delta, \infty)$ and *may*, in addition, contain some *disjoint closed intervals of length* $\geqslant 2\delta$ intersecting with $(-L+\delta, L-\delta)$. There can, of course, *be only finitely many of the latter* so E_δ, *if not identical with* $\mathbb{R}$, is simply a *finite union of disjoint closed intervals thereon including two of the form* $(-\infty, M]$, $[M', \infty)$. In the latter case, the *complement* $\mathbb{C} \sim E_\delta$ is *one of the domains* $\mathscr{D}$ *considered in* §*C of Chapter* VIII, and on $\partial\mathscr{D} = E_\delta$ we have

$$(\mathfrak{M}F_N)(x) \leqslant F_N(x) + \varepsilon$$

by construction.

Our object here is to estimate $(\mathfrak{M}F_N)(0)$. That quantity is of course $\geqslant F_N(0)$, and, as long as it is *equal* to $F_N(0)$, there is no problem, because $F_N(0) = \log W_N(0)$ is $\leqslant \log W(0)$ (is, in fact, equal to $\log W(0)$ for sufficiently large values of N). We thus need *only* look at the situation where

$$(\mathfrak{M}F_N)(0) > F_N(0).$$