Suppose $|\zeta(\frac{1}{2}+iT)| \ll \exp(\log(T)/f(T))$ and $m(1/2, \alpha r_{1/2}) \leq \log(T)/f(T) + O(1)$ for some $f(T)$. Then

$$d_\alpha(1/2, -1) \geq \log\left(\frac{f(T)+C_1}{C_2}\right).$$

For example, if $\zeta(1/2+iT) \ll \exp((\log T)^{1-\epsilon})$ and this holds in a neighbourhood of $1/2+iT$, we may set $f(T) = (\log T)^\epsilon$ and conclude that

$$d_\alpha(1/2, -1) \gg \log\log T.$$

Thus a rectangular strip in the zero-free region would have height $\ll \frac{1}{\log\log T}$, and assuming the Riemann Hypothesis would allow the use of a "bow-tie region" having minimal height at most $\ll (\log T)^{-C}$ for some C.

2.3 The case of general L-functions

To extend the above to the general case, we need a uniform upper bound on $L(s)$ for s close to the real axis with $\Re(s) \geqslant -7$, and to have a good lower bound on some $L(s)$ in this region. In this case the analytic conductor plays the role of T above, but the estimates become more involved as they need to work for small s, uniformly in all parameters, and must allow for a high density of trivial zeros in the region being studied. Because of this we will use the following more restricted definition of the Selberg Class, which is still sufficiently broad to cover all known examples, corresponding to all λ_i in the gamma factor being 1/2.

Consider a Dirichlet series $L(s) = \sum a_n n^{-s}$ that extends to an entire function (wlog a_1=1), with polynomial growth on vertical lines, and functional equation

$$\Lambda(s) \;\; := \;\; Q^{s/2}\gamma(s)\,L(s) = \omega\overline{\Lambda}(1-s),$$

where $|\omega| = 1$ and γ is the degree d gamma factor

$$\gamma(s) \;\; = \;\; \prod_{i=1}^{d} \Gamma\left(\frac{s}{2}+\mu_i\right),$$

and $\Re\mu_i \geqslant 0$ for $i = 1, \cdots, d$.

Assuming an Euler product and Ramanujan hypothesis, in the strip $20 > \Re s > 2$,

$$|\log L(s)| = |\sum \log(1 - \alpha_{p,i} p^{-s})| \leq \sum |\log(1 - p^{-2})| = \log \zeta(2).$$

This gives the bound $|\log L(s)| \ll a_2 d$, which is needed, where specifically $a_2 = \log \zeta(2) \approx 0.4977$. It suffices that for s_1, s_2 in this strip there is a uniform bound on $\frac{\log L(s_1)}{\log L(s_2)}$, but the given bound on $|\log L(s)|$ will be used here for simplicity. The dependence on the Ramanujan Hypothesis is unfortunate but while there are results of Molteni [73] and Li [64] that give unconditional bounds on $GL(n)$ L-functions at the edge of the critical strip, these have a dependence on the conductor that would weaken the result here.

Let $q = Q \prod (11 + |\mu_i|)^{1.5} e^{0.1\Re\mu_i}$

Theorem 4. *With the above definitions and conditions for $L(s)$, the lowest lying zero ρ_1 has $\Im\rho_1 \ll \frac{1}{\log\log\log\left(q^{1/d}\right)}$ where the implied constant is absolute, as long as $q^{1/d} > e^{\frac{1}{2}(a_2+0.308)} \sim 1.5$.*

In particular this means that given a sequence of L functions where $q \gg A^d$ for all A, the lowest lying zero converges to the real axis. This is the strongest result possible since $L(s, \chi_q)^d$ has conductor q^d so the family of these has growing conductor but the same zeros. It is still plausible that for all primitive L-functions the lowest lying zero converges to the real axis as $q \to \infty$. This should require a good way of recognizing primitive L-function and a way of seperating these from high powers.

Proof. Define rectangles in the complex plane $\Gamma_1, \Gamma_2, \Gamma_4$ with common center $9 + ih$ and having lower-left corners respectively at 2, -5, -19.

Consider the function $f(s) = \frac{\log L(s)}{d}$ and suppose $L(s)$ does not vanish inside Γ_4, so that f is well defined inside Γ_4. By assumption $|f(s)| \leqslant a_2$ on Γ_1.

First we use the functional equation and Stirling's approximation to find an effective upper bound when $-19 \leqslant \Re s \leqslant 0$ and $|\Im s| \leqslant 2h \ll 1$. Let $m = \sum_{i=1}^{d} \Re\mu_i + |\Im\mu_i + t/2|$. Then from the functional equation

$$|L(s)| = \left|\overline{L}(1-s)\right| Q^{(1-\sigma)/2} \prod_{i=1}^{d} \left| \frac{\Gamma\left(-s/2+1/2+\overline{\mu_i}\right)}{\Gamma\left(s/2+\mu_i\right)} \right|.$$

Note that if $A > B$ and $A + B > 0$ then $\left|\frac{\Gamma(A+iy)}{\Gamma(B-iy)}\right| < \left|\frac{\Gamma(A+iy)}{\Gamma(B-iy)}\right| \left(\frac{A+iy}{B-iy}\right) = \left|\frac{\Gamma(A+iy+1)}{\Gamma(B-iy+1)}\right|$ since $\left(\frac{A+iy}{B-iy}\right) > 1$. When $\Re s \geq -19$ we may repeat this 11 times to get

$$|L(s)| \leqslant \left|\overline{L}(1-s)\right| Q^{11} \prod_{i=1}^{d} \left| \frac{\Gamma\left(-s/2+1/2+\overline{\mu_i}+11\right)}{\Gamma\left(s/2+\mu_i+11\right)} \right|.$$

The gamma functions in the product are evaluated in the domain $\Re \geq 1.5$ and the relative error in Stirling's approximation there is less than 20%, therefore

$$|L(s)| \leqslant e^{a_2 d} Q^{11} \prod_{i=1}^{d} 2 \left| \frac{\left(-s/2+1/2+\overline{\mu_i}+11\right)^{-s/2+1/2+\overline{\mu_i}+11} e^{s-1/2} \sqrt{s/2+\mu_i+11}}{\left(s/2+\mu_i+11\right)^{s/2+\mu_i+11} \sqrt{-s/2+1/2+\mu_i+11}} \right|.$$

Now bounding all of the parameters gives

$$|L(s)| \leqslant e^{a_2 d} Q^{11} \prod_{i=1}^{d} 2 \left| \frac{\left|21+\mu_i\right|^{21+\Re\mu_i} \left|11+\mu_i\right|}{\left|3/2+\mu_i\right|^{11+\Re\mu_i} \left|11.5+\mu_i\right|} e^{\operatorname{Arg}\left(\frac{-s/2+1/2+\overline{\mu_i}+11}{s/2+\mu_i+11}\right)(\Im s/2+\Im\mu_i)} \right|,$$

which further simplifies to

$$|L(s)| \leqslant e^{a_2 d} Q^{11} 40^{10+\Re\sum\mu_i} \prod_{i=1}^{d} \left|21+\mu_i\right|^{10} e^{\pi|h|+\pi\Im\mu_i}.$$

This gives the following needed upper-bound

$$\begin{aligned} \Re f(s) = \frac{1}{d}\Re\log L(s) &\ll a_2 + \log Q^{1/d} + \frac{1}{d}\sum \log\left|10+\mu_i\right| \\ &\ll 1 + \log q^{1/d}. \end{aligned}$$

It now remains to find a lower-bound on the maximum of f on Γ_2. This is very similar to finding the asymptotics as above, but for the zeros coming from the poles of Γ. These may be dealt with by instead analyzing a certain geometric smoothing of the function. For a function g and for $\delta \in \mathbb{R}$ denote $g_{m\delta} := \exp\left(\frac{1}{\delta}\int_s^{s+\delta} \log g(s)\, ds\right)$. Note $\Gamma_{m1}(s) = \sqrt{2\pi}\left(\frac{s}{e}\right)^s$. As before, the functional equation states that

$$|L(s)| = \left|\overline{L}(1-s)\right| Q^{(1-\sigma)/2} \prod_{i=1}^{d} \left|\frac{\Gamma(-s/2+1/2+\overline{\mu_i})}{\Gamma(s/2+\mu_i)}\right|.$$

For $-19 \leqslant \Re s \leqslant -3$ we bound L as before

$$|L_{m2}(s)| \geqslant e^{-a_2 d} Q^2 \prod_{i=1}^{d} \left|\frac{\Gamma_{m1}(-s/2-1/2+\overline{\mu_i})}{\Gamma_{m1}(s/2+\mu_i)}\right|.$$

Since Γ_{m1} simplifies, this simplifies to

$$e^{-a_2 d} Q^2 \prod_{i=1}^{d} \left|\frac{(-s/2-1/2+\overline{\mu_i})^{-s/2-1/2+\overline{\mu_i}+10}\, e^{s+1/2}}{(s/2+\mu_i)^{s/2+\mu_i}}\right|.$$

In the range $-19 \leq \Re s \leq -3$ this gives the bound

$$\begin{aligned} |L_{m2}(s)| &\geqslant e^{-a_2 d} Q^2 \prod_{i=1}^{d} \left|\frac{1\cdot}{|8.5+m|^{8.5+\Re\mu_i}} \frac{|11+m|^3}{e^{\mathrm{Arg}\left(\frac{-s/2+1/2+\overline{\mu_i}+10}{s/2+\mu_i+10}\right)(\Im s/2+\Im\mu_i)}} e^{-2.5}\right| \\ &\geqslant e^{-a_2 d} Q^2 1.29^{8.5d+\Re\sum\mu_i} \prod_{i=1}^{d} |11+m|^3\, e^{-2.5}. \end{aligned}$$

Taking the logarithm of both sides,

$$\begin{aligned} \frac{1}{d}\Re\log L_{m2}(s) &\geqslant -a_2 - 0.308 + \log Q^{2/d} + \frac{1}{d}\log\prod|11+m|^{3/d} + \frac{0.1}{d}\Re\sum\mu_i \\ &\geqslant -a_2 - 0.308 + 2\log q^{1/d}. \end{aligned}$$

Hence there is some $s \in [-3,1]$ such that $\Re f(s) \geqslant -a_2 - 0.308 + 2\log q^{1/d}$.

Applying lemma 1,

$$-a_2 - 0.308 + 2\log q^{1/d} \leqslant C_1\left(a_2 + \log q^{1/d}\right) e^v \left(\frac{C_2 a_2}{a_2 + \log q^{1/d}}\right)^{e^{-v}},$$

and rearranging the two sides,

$$\left(2\log q^{1/d} - a_2 - 0.308\right)\left(\log q^{1/d} + a_2\right)^{e^{-v}-1} \leqslant C_1\left(C_2 a_2\right)^{e^{-v}} e^v$$

Hence if $2\log q^{1/d} > a_2 + 0.308$ then $\left(\log q^{1/d}\right)^{e^{-v}} \ll e^v \left(C_2 a_2\right)^{e^{-v}}$, so $\log q^{1/d} \ll e^{ve^v}$ and we conclude that

$$\log\log\log q^{1/d} \ll v.$$

□

2.4 Results from constraints on functions on a disc

Several theorems about the rigidity of entire non-vanishing functions have direct analogues for functions on a disc that are well behaved near the center of the disc but grow very quickly. For example, Schottky's Theorem is the analogue of Little Picard.

Theorem 5. *(Schottky) If f is analytic on the unit disc and misses the values $0, 1$ in its range then*

$$\log|f(z)| \leq \frac{1+|z|}{1-|z|}(7 + \max(0, \log|f(0)|))$$

These theorems apply naturally to $\zeta(z+3/2+iT)$ for sufficiently large T assuming the Riemann Hypothesis and adds significant rigidity. However, since applications of these theorems require $f(s)$ to be very large near $\Re s = 1$ the effects only appear for very large T (around 10^{300}) and so we cannot see these curious functions directly. Some similar phenomena are studied by Ivić[47].

Theorem 6. *(Borel) If $f_1, \cdots, f_n$ are nowhere vanishing entire functions none of which is a scalar multiple of the other, then the sum $f_1 + \cdots + f_n$ cannot be identically zero*

This theorem shows that functions satisfying a Riemann Hypothesis are quite exceptional and must be linearly independent, which is known by more direct means in the Selberg class given an Euler product[51].

2.5 Riemann Hypothesis for "any" entire function

A question to ask at this point is whether there is any entire function that behaves asymptotically like $\zeta(s)$ (without requiring a Dirichlet series nor a functional equation) which satisfies some form of a Riemann Hypothesis. Namely I propose the following as a question of interest.

Question 3. *Is there an entire function $f : \mathbb{C}5 \to \mathbb{C}$ such that for some $\delta > 0$ the following are satisfied.*

1. *$f(z)$ is bounded when $\Re z > 1 + \delta$.*
2. *$f(z)$ is unbounded when $\Re z = 1$.*
3. *$f(z)$ grows polynomially in vertical strips: for all σ there exists $C_\sigma > 0$ so that $|f(\sigma + it)| \ll |t|^{C_\sigma}$.*
4. *(weak RH) $f(z)$ does not vanish when $\Re z > \frac{1}{2}$.*
5. *(strong RH) $f(z)$ vanishes only when $\Re(z) = \frac{1}{2}$ or for $\Im(z)$ in a finite set.*

Conjecturally all L-functions satisfy all 5 conditions, but this has not been proven for any L-function. Many explicit functions satisfy all but condition (2), for example $1 + e^{-z}$ or $\zeta(\frac{1}{2} + z) \cdot (z - \frac{1}{2})$s. If you drop condition (3) then you have as examples the Selberg zeta function, or $\exp(L(s, \chi))$. If you drop condition (1) then you have as an example $\prod_{n=1}^{\infty} \cosh(z/n^2)$.

The remainder of this section consists of an explicit construction of a function satisfying (1)-(4) (namely the weak but not the strong Riemann Hypothesis).

Proposition 1. *There is a function $f(z)$ satisfying properties (1)-(4) above.*

Proof. Consider the entire function $g(z) = \frac{1}{z}(1 - e^{-z}) = \int_0^1 e^{-tz} dt$. From the first expression, in any right half plane $g(z) = O(1/z)$. From the definite integral,

$$|g(z)| \leq \int_0^1 e^{-t\Re(z)} dt = g(\Re(z)).$$

Comparing the integrand termwise, $g(z)$ is dominated in $\Re(z) > 1$ strictly by $a := g(1) > 0$ and tends rapidly to 0 as $\Re(z)$ increases.

Consider $h_{n,N}(z) = (1 + g(z)e^{-n^2 z})^N$ where N is a large integer and $n := n(N)$ is such that $N = e^{n^2+n}/a$. For every half-plane $\Re(z) \geq \sigma$, both $g(z)$ and $e^{-n^2 z}$ are maximized at $z = \sigma$ and are positive there, hence $h_{n,N}(z)$ also has this property. Computing at $z = 1$ and at $z = 1+2/n$ yields $h_{n,N}(1) = (1+ae^{-n^2})^N = (1+e^n/N)^N \sim e^{e^n}$ and $h_{n,N}(1+2/n) < (1 + ae^{-n^2-2n})^N \sim (1 + e^{-n}/N)^N \sim 1 + e^{-n}$ respectively.

By construction, on any right half-plane $g(z) = O_{n,N}(1/z)$ (where the constant depends on the half-plane) and so $h_{n,N}(x + iy) = 1 + O_{n,N}(1/y)$. Thus we can pick sufficiently large $t_{n,N}$ such that $|\log|h_{n,N}(z + it_{n,N})|| < 2^{-n}\log(1 + |\Im(z)|)$ for $\Re(z) > -N$.

Now take our candidate function to be

$$f(z) := \prod_{N=1}^{\infty} h_{n,N}(z + it_{n,N}).$$

where the N form a sufficiently fast growing subsequence of the natural numbers so that $\sum e^{-n}$ converges and $e^{n^2/2} > g(1/2)$ for all n. This construction is courtesy of Fedja Nazarov.

At this point all that remains is to verify that $f(z)$ satisfies the 4 properties.

1. As long as $\Re(z) > 1 + 1/n$, the product for $f(z)$ is dominated by $\prod(1 + e^{-n})$. Since $\sum e^{-n}$ is absolutely convergent, the product converges absolutely to a bounded analytic function.

2. $h_{n,N}(1-it_{n,N}) \sim e^{e^n}$ and for sufficiently large $t_{*,*}$ the other terms in the product for $f(1 - it_{n,N})$ can be made arbitrarily close to 1. Thus we can ensure that $f(1 - it_{n,N}) \asymp e^{e^n}$ making f unbounded on $\Re(z) = 1$.

3. For $\Re(z) > -N_0$: $|f(z)| \leq \prod_{N=1}^{N_0} h_{n,N}(-N_0)\prod_{N>N_0}(1 + |\Im(z)|)^{2^{-N}} \ll_{N_0} (1 + \Im(z))^2$, giving absolute converge with polynomial growth on vertical lines, uniformly on half-planes.

4. The $\{n\}$ were chosen so that $|g(z)e^{-n^2 z}|$ is strictly bounded above by 1 when

$\Re(z) > 1/2$. This means that $h_{n,N}(z)$ has no zeros with $\Re(z) > 1/2$ and thus neither does $f(z)$.

This function, however, grows very rapidly to the left, and has a multitude of zeros in every half-plane $\Re(z) < \sigma < 1/2$.

$\square$

Chapter 3

Dirichlet Series with Analytic Continuation and Degree 1 growth rate

3.1 Introduction

The analytic study of Dirichlet series began with Riemann [84] who proved the functional equation of the zeta function. He gave two proofs of the functional equation, the latter has been greatly generalized into the theory of modular forms, whose further generalization into automorphic forms is a central area of study of modern number theory. The first proof is likewise very interesting, extends generally to Dirichlet L-functions, forms the basis for Shintani zeta functions (which can be used to develop the theory of Dedekind zeta functions) and will be central to the current work.

Hamburger[43] was the first to consider the problem of determining a function from such a functional equation, when he proved that a Dirichlet series with moderate growth constraints and the functional equation of $\zeta(s)$ must be $\zeta(s)$. Hecke[45][7] dealt with the case of degree two functional equation $\Gamma(s+k)\lambda^s$. He related precisely Dirichlet series with such a functional equation to analytic functions on the upper half plane modulo a triangle group, and thus was able to determine the dimension of the space of these functions for each set of parameters. Weil's converse theorem additionally states that if sufficiently many Dirichlet twists of the L-function have appropriate functional equations then the Dirichlet series comes from a modular form of some level. Maass[68] proved a Hamburger-type theorem for the functions $\zeta(s)^2$, $2^{-s}\zeta(s)^2$, and $(1+2^{1-2s})\zeta(s)^2$ by showing that any L-function with the same functional equation as these must arise as a specific Maass form. All other cases remain open. Moving to higher degree, the existing converse theorems for degree 3 require infinitely many Dirichlet twists along with the Dirichlet series factoring as an appropriate Euler product[36]. In general $GL(n-2)$ twists are used for degree n converse theorems and it is unclear whether $GL(n-2)$ or $GL(n/2)$ or $GL(1)$ are necessary and sufficient.

The interest in such problems is manifold. Hecke's theorem, while no long as popular to use directly, is the foundation on which Weil's Converse theorem is developed [48], so having an analogue for higher degree may open the way to a better understanding of higher converse theorems. There are many open questions about the behaviour of Dirichlet Series (Lindelof hypothesis, Riemann Hypothesis, Moments Conjecture). Certain results can be proven without any arithmetic content so having a large concrete family of purely analytic examples can help to test conjectures and also to see which results truly do or do not depend on arithmetic.

The general problem is to understand the Selberg class [87][75] of Dirichlet series with functional equations and Euler product (although all of the following work does not actually require an Euler Product). The most coarse measure of complexity for these is the degree d of the functional equation. Conrey and Ghosh [26] determined that there is nothing with degree strictly between 0 and 1 and that degree 0 consists only of Dirichlet polynomials. Kaczorowski and Perelli [54] determined everything of degree 1, and Soundararajan [90] has a particularly short and clean proof of this fact. More recently Kaczorowski and Perelli [57] have shown that there is nothing of degree strictly between 1 and 2.

In section 3.2 I outline a more general framework, which instead of a functional equation requires only a growth bound similar to that of a function in the Selberg class. Expanding on Kuznetsov[61] I determine all such functions for the equivalent of degree 1.

In section 3.3 I reintroduce the functional equation, showing how it relates to singularities of functions, and in corollary 2 construct all Dirichlet series with degree 1 functional equation.

Section 3.5 holds the proofs of the main theorems, and appendix A provides some useful lemmas for dealing with products of gamma functions.

Every result here, aside from corollaries 1 and 2, work also for generalized Dirichlet series $\sum a_n b_n^{-s}$ for increasing positive numbers $b_1, b_2, \cdots$. This will be used in the next chapter to prove corresponding results for higher degree.

3.2 Results not assuming Functional Equation

Before bringing in functional equations we make a definition of degree based only on growth rate.

Definition 6. *Given a Dirichlet series $L(s) = \sum a_n n^{-s}$ absolutely convergent in some right half-plane define the following degrees if $L(s)$ extends to an entire function with polynomial growth bounds on vertical lines.*

Horizontal Degree $\leq d$ (and conductor Q)

$|L(s)| \ll \Gamma(1 + d|s|)Q^{|s|}$ *for some Q.*

Vertical Degree $\leq d$

$|L(\sigma + iT)| = O_\sigma(T^{d\sigma+A})$ *for some A and all $\sigma < 0$*

If one of the degrees satisfies the above for given d but not for any strictly smaller d then we say that the degree equals d.

If $L(s)$ has a functional equation of degree d then its horizontal and vertical degrees are both d, so in this sense the above definition extends the standard notion of degree. They are named as such because horizontal degree gives a bound that is most useful on the negative real axis, and vertical degree bounds the behaviour on vertical lines in the complex plane.

As an example that any horizontal degree $d > 0$ is possible consider

$$L(s) := \sum_{n=1}^{\infty} \exp(-n^{1/d})n^{-s},$$

which is absolutely convergent everywhere, bounded by its value for real s, and for s real going to $-\infty$ is asymptotic to

$$\int_0^\infty \exp(-x^{1/d})x^{-s-1}dx = d\int_0^\infty e^{-x}x^{-ds-1}dx = d\Gamma(-ds).$$

Thus $L(s)$ has horizontal degree d and vertical degree 0.

We can now state the main results, whose proofs will come in section 3.5.

Proposition 2. *Vertical degree $\leq$ Horizontal degree, in the sense that if the horizontal degree is $\leq d$ then the vertical degree is $\leq d$.*

This proposition will be discussed further in Section 4.2 where we deal more explicitly with degree d.

Conjecture 2. *The vertical degree is always a non-negative integer.*

Theorem 7. *A Dirichlet series $L(s) := \sum_{n=1}^\infty a_n n^{-s}$ has horizontal degree 1 if and only if $p(x) := \sum_{n=1}^\infty a_n x^n$ has radius of convergence 1 and continues analytically in a neighbourhood of 1. Furthermore if this is the case then it also has vertical degree 1.*

More precisely, the following are equivalent, for given Q, and some A not necessarily the same in all cases:

1. *$L(s) := \sum a_n/n^s$ extends to an entire function satisfying $|L(s)| \ll \Gamma(1 + |s|)|s|^A Q^{|s|}$.*

2. *$L(s) := \sum a_n/n^s$ extends to an entire function satisfying $|L(s)|\Gamma(s)Q^s \ll |s|^A e^{-\pi i|Im(s)|/2}$ for s in a left half-plane, away from the poles.*

3. *$f(z) := \sum a_n e^{-nz}$ extends analytically in a $1/Q$ neighbourhood of 0, and $f(z) \ll \frac{1}{1-Q|z|}^A$ in this neighbourhood.*

And in this case f has Taylor expansion: $f(z) = \sum_{n=0}^\infty \frac{(-z)^n}{n!}L(-n)$.

Corollary 1. *If horizontal degree is strictly between 0 and 1, or horizontal degree is 1 and $Q < \frac{1}{\pi}$, then $\sum a_n n^{-s}$ is absolutely convergent everywhere and in particular has vertical degree 0.*

Proof. In either case the bound in theorem 7 is satisfied for some $Q > \frac{1}{\pi}$. The theorem concludes that $f(z)$ is analytic on $1/Q$ neighbourhoods of $2\pi i\mathbb{Z}$, and these cover some half-plane $\Re z > \delta$ where $\delta < 0$. This makes $p(x)$ have radius of convergence > 1 so the a_n decay exponentially. □

The result extends to meromorphic function in the sense that $L(s)$ having a simple pole at $s = 1$ corresponds to $f(z)$ having a simple pole at 1.

Note that this gives a vast family of Dirichlet series. For example consider

$$p(x) := e^{1/(x+1)} = \sum_{n=0}^{\infty} {}_1F_1(n+1;2;1)(-x)^n,$$

which leads to a Dirichlet series with horizontal and vertical degree 1

$$L(s) := \sum_{n=1}^{\infty} {}_1F_1(n+1;2;1)(-x)^n.$$

It would be interesting to see whether or not this function satisfies the Lindelof Hypothesis. In fact, we could go so far as to conjecture that "everything" satisfies a Lindelof Hypothesis:

Definition 7. *Consider $L(s)$ with finite horizontal degree (or simply entire continuation to a function of order 1). As in [95] define for each σ*

$$\mu(\sigma) := \inf\{\mu : |\zeta(\sigma + iT)| \ll |t|^{\mu}\}.$$

As a function of σ, $\mu(\sigma)$ is non-increasing and is 0 when σ is sufficiently large. By the Phragmen-Lindelof Principle it is convex upwards, and it has slope $\geq -d$.

Conjecture 3 (Analytic Lindelof Hypothesis)**.** *$\mu(\sigma)$ is piecewise linear with all linear parts having integer slope, necessarily increasing from $-d$ to 0.*

3.2.1 Explicit Construction of all with Horizontal degree ≤ 1

Consider the twisted zeta function $\zeta_a(s) := \sum_{n=1}^{\infty} e^{-na} n^{-s}$ where $-\pi \leq \Im a \leq \pi$ and $\Re a \geq 0$. The corresponding $p(x)$ is $p_a(x) = \sum_{n=1}^{\infty}(e^{-a}x)^n = \frac{x}{e^a - x}$. All of these are of horizontal degree 1 with parameter $Q = |a|$. These twisted zeta functions will in fact generate all such Dirichlet series.

Theorem 8. *Consider $L(s) = \sum a_n n^{-s}$ that has horizontal degree ≤ 1 with some given Q and A, and $a_n \ll n^\delta$ for some δ. Consider the union of the right half-plane, and the $1/Q$ disc around 0. Take $\mathcal{C}$ to be the portion of the boundary that lies in the strip $-\pi \leq \Im s \leq \pi$. Then for every $\delta' > \delta + 1/2$ there is some $f \in L^2(\mathcal{C})$ so that*

$$L(s) = \int_{\mathcal{C}} f(t)\zeta_{-t}(s+\delta)dt.$$

where the integral converges uniformly on compact subsets of $\mathcal{C}$.

Proof. Without loss of generality shift s sufficiently so that $A < -1/2$ and $\delta < -1/2$ so that we can take $\delta' = 0$. Then $p(x) = \sum a_n x^n$ is in the Hardy space H^2 of the disc and so converges almost everywhere on the boundary to an L^2 function that we also denote by $p(x)$.

The Cauchy integral formula says

$$p(x) = \frac{1}{2\pi i}\int_{|t|=1} \frac{p(t)}{t-x}dt.$$

Making the substitution $x = e^{-z}$ gives

$$f(z) = \frac{1}{2\pi i}\int_{-\pi i}^{\pi i} \frac{f(t)}{e^{z-t}-1}dt.$$

$f(t)$ on the imaginary axis is taken to be in $L^2(\mathcal{C}/(2\pi i))$, and by assumption is analytic in a $1/Q$ neighbourhood of 0. $f(t)$ is also L^2 on the boundary of this neighbourhood, since its power series expansion at 0 has coefficients growing like $Q^n n^A$. Thus we may shift the contour to $\mathcal{C}$

$$f(z) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f(t)}{e^{z-t}-1}dt.$$

Now take the Mellin transform of both sides

$$L(s)\Gamma(s) = \frac{1}{2\pi i}\int_0^\infty z^{s-1}\int_{\mathcal{C}} \frac{f(t)}{e^{z-t}-1}dt\ dz.$$

By absolute convergence we may interchange the two integrals to get

$$L(s)\Gamma(s) = \frac{1}{2\pi i}\int_{\mathcal{C}} f(t)\int_0^\infty \frac{z^{s-1}}{e^{z-t}-1}dz\ dt.$$

The interior integral gives the twisted zeta function

$$L(s)\Gamma(s) = \frac{1}{2\pi i}\int_{\mathcal{C}} f(t)\zeta_{-t}(s)\Gamma(s)dt,$$

and cancelling out $\Gamma(s)$ from both sides gives

$$L(s) = \frac{1}{2\pi i}\int_{\mathcal{C}} d(t)\zeta(-t)(s).$$

□

3.2.2 The case of vertical degree < 1

While the techniques above work only with strict horizontal control, vertical growth suffices.

Theorem 9. *Let $L(s) = \sum a_n n^{-s}$ be a Dirichlet series absolutely convergent for some s that analytically continues to an entire function, and has vertical degree less than 1, namely for each $\sigma < 0$ $|L(\sigma + iT)| = O_\sigma(T^{C-d\sigma})$ for some $0 \le d < 1$ and some C. Then the series $\sum a_n n^{-s}$ is absolutely convergent everywhere, and in particular the vertical degree is exactly 0.*

Proof. As before take $\Re z > 0$ and consider

$$f(z) = \sum a_n e^{-nz} = \frac{1}{2\pi i}\int_c L(s)\Gamma(s)z^{-s}ds.$$

Recall that for $s = \sigma+iT$, $\Gamma(s)\sin(\pi/2s) = O_\sigma(T^{-1/2+\sigma})$, so that $L(s)\Gamma(s)\sin(\pi/2s) = O_\sigma(T^{C-1/2+(1-d)\sigma})$. Thus we may shift the contour sufficiently far to the left so that on the the line of integration $L(s)\Gamma(s)\sin(\pi/2s) = O(\Im(s)^{-2})$ is absolutely integrable and hence:

$$f(z) = \sum_{k=0}^{K} L(-k)(-z)^k/k! + \int_{-K-1/2} L(s)\Gamma(s)z^{-s}ds = O(|z|^{K+1/2}).$$

This bound is uniform in the right half plane and extends continuously to $\Re(z) = 0$. In addition, $f(z)$ is periodic with period $2\pi i$, so is therefore bounded on the imaginary axis. The Fourier coefficients $a_n = \int_0^1 f(2\pi iz)e^{2\pi inz}$ also carry the same bound, so the series $\sum a_n n^s$ is absolutely convergent at $\Re s > 2$. Repeating the argument for $L(s+N)$ proves absolute convergence at $\Re s > 2-N$ for all N, and the result follows.

□

3.3 Results Assuming a Functional Equation

Definition 8. *Consider two Dirichlet series $L_a(s) = \sum a_n n^{-s}$ and $L_b(s) = \sum b_n n^{-s}$ absolutely convergent in some right half-plane, where $a_n, b_n \in \mathbb{C}$. Define Gamma-factors*

$$\begin{aligned} \gamma_a(s) &:= \prod_{i=1}^{r_a} \pi\Gamma(\lambda_{a,i}s + \mu_{a,i}) \\ \gamma_b(s) &:= \prod_{i=1}^{r_b} \Gamma(\lambda_{b,i}s + \mu_{b,i}) \end{aligned}$$

and suppose that $L_a(s)$ and $L_b(s)$ have analytic continuation to entire function with polynomial growth on vertical lines, and are related by a functional equation

$$\begin{aligned} \Lambda(s) &:= L_a(s)\gamma_a(s)Q_a^s \\ \Lambda(1-s) &= L_b(s)\gamma_b(s)Q_b^{s-1}, \end{aligned}$$

where $\lambda_{a,i}, \lambda_{b,i}, Q_a, Q_b$ all exceed zero.

The functional equation can also be written in the non-symmetric form

$$\begin{aligned} L_a(s) &= L_b(1-s)\frac{\gamma_b(1-s)}{\gamma_a(s)}Q^s \\ &= L_b(1-s)\,\gamma(1-s)\,\theta(1-s)\,Q^s, \end{aligned}$$

where $Q = Q_b/Q_a$, *there is the nonsymmetric gamma factor*

$$\begin{aligned} \gamma(s) &:= \prod_{i=1}^{r}\Gamma(\lambda_i s+\mu_i) \\ &= \prod_{i=1}^{r_b}\Gamma(\lambda_{b,i}s+\mu_{b,i})\prod_{i=1}^{r_a}\Gamma(\lambda_{a,i}s+1-\lambda_{a,i}-\mu_{a,i}), \end{aligned}$$

and a product of sines and cosines that we denote

$$\begin{aligned} \theta(s) &:= \prod_{i=1}^{r_a}\sin(\pi(s\lambda_i-\mu_i-\lambda_i)/2) \\ &:= \sum \alpha_j e(\omega_j s/4), \end{aligned}$$

where $\{\omega_j\} = \{\pm\lambda_1 \pm \lambda_2 \pm \cdots\}$. *The degree of the functional equation is defined to be* $d := \sum \lambda_i = 2\sum \lambda_{a,i} = 2\sum \lambda_{b,i}$.

Let $q^{-1} = Q\prod_{i=1}^{r}\lambda_i^{\lambda_i}$ *and let* $\mu = \sum_{i=1}^{r_a}(\mu_i - 1/2)$.

Conjecture 4. *The following is a list of conjectures in strictly increasing order of strength:*

1. *Selberg's degree conjecture [87][75] that* $d \in \mathbb{N}$.
2. *All* ω_j *are in* $\mathbb{N}$ *and have equal parity.*
3. *All* $\lambda_{a,i}$ *an* $\lambda_{b,i}$ *may be taken to be 1/2.*
4. *All* λ_i *can be taken to be 1.*

The main result in this chapter is that $L(s)$ having a functional equation leads to $f(z)$ being entire aside from isolated singularities on a finite number of rays

Theorem 10.

I *If* $L_a(s) := \sum a_n n^{-s}$ *has analytic continuation and a functional equation as above relating it to* $L_b(s)$ *then* $f(z) := \sum_{n=1}^{\infty} a_n \exp(-nz)$ *has analytic continuation to* $\mathbb{C}$ *aside from singularities at the points* $-nqe\left(\frac{\omega_j}{4}\right)$ *for* $n \in \mathbb{N}$.

II *More precisely, the singularities are of the same type and have "residues"* b_n. *Up to a polynomial* $p(z)$,

$$f(z) = p(z) + \sum_{n=1}^{\infty} \sum_{j=1}^{r} db_n \alpha_j G(ze(\omega_j s/4)q/n) z^{-1},$$

where $G(z)z^{-1}$ *is holomorphic aside from a branch cut singularity at* -1.

III *By shifting and possibly replacing* $L_a(s)$ *with some* $L_a(s + A)$, *we may wolog assume* $\mu = 0$. *Then around* $z = 1$ *when* $|\operatorname{Arg}(z)| < \pi$,

$$G(z/q)z^{-1} = \frac{B_0}{z+1} + B_1 \log(z+1) + O(1)$$

for some B_0, B_1.

We also assume that $L_a(s)$ cannot be expressed as $L_{\alpha(s)}/n_a^s$ for $n_a = 2, 3, \cdots$ and L_α also a Dirichlet series. Similarly we assume that $L_b(s)$ cannot be expressed as $L_{\beta(s)}/n_b^s$. Otherwise we could replace $L_a(s)$ with $L_a(s)n_a^s$, replace $L_b(s)$ with $L_b(s)n_b^s$, and replace Q with $\frac{n_a}{n_b}Q$.

Corollary 2. *If* $L_b(s) = \sum a_n n^{-s}$ *has degree 1, then* $q \in 2\pi/\mathbb{N}$ *and for some* A *the sequence* $a_n n^A$ *is periodic mod* $2\pi/q$. *Hence it is a linear combination of* $L(s + A, \chi)$ *for various* χ *(not necessarily primitive) of modulus* $2\pi/q$.

Proof. It has been shown that the singularities of $f(z)$ are evenly spread out along rays with angles $\omega_j/4$. But $f(z)$ is periodic which leaves only the possibility $(\omega_j)_{j=1}^r = (-1, 1)$ so that $(e(\omega j s_4)) = (-i, i)$. The location of the singularities is periodic mod 2π, and the singularities are supported on a lattice with spacing $2\pi/q$, therefore $2\pi/q \in \mathbb{N}$.

As in the final remark of the above theorem, wlog shift so that $\mu = 0$. Then to first order the singularity at nqi behaves as a simple pole with residue $B_0 d b_n \alpha$ and we conclude that the b_n are periodic mod $2\pi/q$ as claimed. □

3.4 Results assuming an Euler product

Schwarz conjectured in 1978 that a power series with multiplicative coefficients has a natural boundary on its circle of convergence or is a rational function, but progress has only been made in the case where the mean value of the coefficients exists and is non-zero [67]. In our context the question becomes: aside from $L(s,\chi)$ is there a Dirichlet series of horizontal degree 1 with an Euler product.

Subject to extra conditions this can be answered in the negative. Suppose that $g(x) = \sum a_n x^n$ has completely multiplicative coefficients, radius of convergence 1, and only finitely many singularities on the unit circle.[1] Let m be the number of singularities.

Since the coefficients are bounded, $|a_p| \leqslant 1$ for all p. Also, by gap theorems, only finitely many of the a_p can be 0. In fact, it is necessary that $|a_p| = 1$ for all but finitely many p, since the coefficients need to satisfy an almost linear-recurrence. Without loss of generality, when $|a_p| \neq 1$ we can make $a_p = 0$ without changing the hypotheses.

For every prime p with $a_p \neq 0$: $\sum_{n=0}^{p-1} f(ze(n/p)) = pa_p f(z^p)$. Hence if z is a singularity of f on the unit circle then at least one of the $z^{1/p}e(n/p)$ is also a singularity. Repeating this argument gives a sequence of related singularities. Since there are only finitely many singularities this sequence has to repeat at some point, after at most m terms, which implies that $z = e(r)$ for some $r \in \mathbb{Q}$.

Take some p not dividing any of the denominators of these r. Then for $z = e(r)$, exactly one of the $z^{1/p}e(n/p) = e((r+n)/p)$ has denominator relatively prime to p, hence this must be the singularity. Thus for this n, $pa_p f(z) = f(z^{1/p}e(n/p))$ up to

[1] This is the case that often comes up when you have extra structure to the L-function such as a functional equation. In addition, the Hadamard Multiplication Theorem gives you Rankin-Selberg convolution for this class of functions.

a quantity that is bounded in a neighbourhood of z. Repeating this as before, after $m!$ steps we get back to $z = e(r)$, so $p^{m!}a_p^{m!}f(e(r+x)) = f(e(r+x^{1/p^{m!}}))$ up to a bounded quantity around $x = 0$. Similarly if one takes another prime q satisfying the hypotheses. Since the group generated by p and q is dense in $\mathbb{R}^*$ (eg: one may find natural numbers a, b such that $p^a q^{-b}$ is arbitrarily close to 1) one gets that $a_p^{m!} = p^z$ and $a_q^{m!} = q^z$ for some z that works for all such primes. By the Hadamard Multiplication Theorem $\sum a_n x^n$ and $\sum a_n n^{-z/m!} x^n$ have the same singularities, so without loss of generality we may assume that $z = 0$.

But this implies that all a_n are either $m!$-th roots of unity or 0. Szego's theorem then implies that $g(z)$ is a rational function.

The same argument works for other Euler products by applying the appropriate Hecke operators in place of $\sum_{n=0}^{p-1} f(ze(n/p)) - pa_p f(z^p)$.

3.5 Proofs of Theorems and Propositions

3.5.1 Theorem 7, the forward direction

This is essentially Lemma 3 in [61], which comes from a Mittag-Leffler expansion of $L(s)\Gamma(s)$ and some careful estimates of sums and integrals.

First we want to turn the weaker growth bound into the stronger growth bound.

Consider the following two functions, for $N > A$ sufficiently large so that $L(s)$ is bounded on $\Re(s) = N - 1$:

$$M(s) := L(s)\Gamma(s)\sin(\pi s)(s+i)^{-N}Q^s,$$

$$M_+(s) := M(s)e^{i\pi s/2}.$$

By the multiplication formula we know that $\Gamma(s)\Gamma(1-s)\sin(\pi s) = \pi$ and so when $s \in \mathbb{R}_{<0}$,

$$M_+(s) \ll \Gamma(1-s)\Gamma(s)(s+i)^{-N}\sin(\pi s)Q^s Q^{-s} \ll 1.$$

For $s = N - 1 + iT$ and $T > 0$, $L(s)$ is bounded, so the function $M_+(s)$ is also bounded, since both $\Gamma(s)e^{-\pi i s/2}s^{-N}$ and $\sin(\pi s)e^{i\pi s}$ are bounded.

$M_+(s)$ is of order 1 since it is a product of order 1 functions, so by Phragmen-Lindelof it follows that $M_+(s)$ is bounded on the quadrant $\Im s \geq 0$ and $\Re s \leq \sigma$.

Similarly $M(s)e^{-i\pi s/2}$ is bounded for $\Im(s) < 0$ and $\Re(s) < \sigma$, so we conclude that $M(s) \ll e^{i\pi|\Im(s)|/2}$. This is also the exact bound needed to show that the vertical degree is ≤ 1.

From the integral formula for $\Gamma(s)$,

$$\Gamma(s)/n^s = \int_0^\infty z^{s-1}e^{-nz}dz,$$

and so summing over n with linear coefficients a_n gives

$$\Gamma(s)L(s) = \int_0^\infty z^{s-1}f(z)dz.$$

Hence by Mellin inversion, when $\mathrm{Re}(z) > 0$:

$$f(z) = \frac{1}{2\pi i}\int_{(c)} L(s)\Gamma(s)z^{-s}ds$$

When $Q|z| < 1$ shift the contour to $-\infty$ to get the sum over residues:

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} z^n \operatorname{Res}_{z=-n}(L(s)\Gamma(s))) \\ &= \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} L(-n). \end{aligned}$$

By assumption $L(-n) \ll n!n^AQ^n$ so this sum converges to an analytic function in this region, bounded by $\sum_{n=0}^\infty \frac{|z|^n}{n!}Q^n n^A \ll \frac{1}{1-Q|z|}^{1+A}$.

3.5.2 Theorem 7, the backward direction

Take $f(z) = \sum_{n=1}^\infty a_n e^{-nz}$, which extends analytically in a small neighbourhood of 0. We wish to define and bound the analytic continuation of $L(s) = \sum_{n=1}^\infty a_n n^{-s}$.

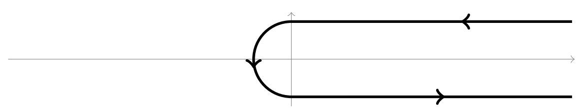
As above we may use the integral for the Gamma function

$$\Gamma(s)n^{-s} = \int_0^\infty x^{s-1}e^{-x}dx$$

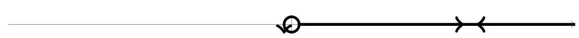
to get the expression as a Mellin transform when $\Re(s) > 1$:

$$\Gamma(s)L(s) = \int_0^\infty x^{s-1}f(x)dx.$$

(a) Circling the positive real axis at distance δ.



(b) Following the positive real axis.



(c) Following the positive real axis and circling the origin at distance q.

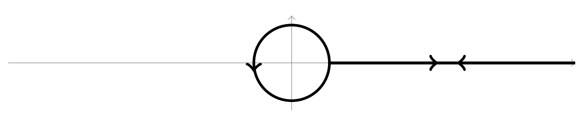


Figure 3.1: The contours used

Following Riemann consider an integral along the Hankel contour (a), which wraps around the positive real axis in the positive direction, while staying distance δ away from the real axis,

$$I(s) := \int_{+\infty+i\delta}^{+\infty-i\delta} (-z)^s f(z)dz/z.$$

This defines an entire function of s.

When $s > 0$ the integrand is continuous to the real axis, aside from a branch cut discontinuity, so we may shift to contour (b) to coincide with the real axis traversed twice wrapping around the origin, and pick up on factors of $(-1)^s$ from the branch-cut of $(-z)^s$,

$$\begin{aligned} I(s) &= \int_{+\infty+0i}^{+\infty-0i} (-z)^s f(z) dz/z \\ &= \int_{\infty}^{0} z^s e^{-\pi i s} f(z) dz/z + \int_0^\infty z^s e^{\pi i s} f(z) dz/z. \end{aligned}$$

Simplifying the expression, pulling out the factors of $e^{\pm \pi i s}$,

$$I(s) \quad = \quad (e^{\pi i s} - e^{-\pi i s}) \int_0^\infty z^{s-1} f(z) dz = 2i \sin(\pi s)\Gamma(s) L(s),$$

which is further simplified by the multiplication formula, to get

$$I(s) = 2\pi i L(s)/\Gamma(1-s),$$

and we conclude that

$$L(s) = \frac{\Gamma(1-s)}{2\pi i} I(s).$$

The function $I(s)$ is entire and, by Cauchy's formula, $I(s)$ is zero when $s = 1, 2, 3, \cdots$. This cancels out the poles of $\Gamma(1-s)$, thus proving that $L(s)$ is entire.

Move the contour to (c) to circle around the origin at distance $q = \frac{1-\epsilon}{Q}$ and otherwise follow the positive real axis. The integral defining $I(s)$ is then bounded as

$$\begin{aligned} I(s) &= (e^{\pi i s} - e^{-\pi i s}) \int_q^\infty z^s f(z) dz/z + \int_{|z|=q} (-z)^s f(z) dz/z \\ &\ll e^{\pi |\Im t} \int_q^\infty z^{\Re s} e^{-z} dz/z + q^s \epsilon^A. \end{aligned}$$

When $\Re s > 0$ the integral is bounded by $\Gamma(\Re s)$, when $\Re s < 1$ it is bounded by $q^{\Re(s)-1}$. Take $\epsilon \sim 1/|s|$ so that $q^s = \frac{(1-\epsilon)^s}{Q^s} \asymp Q^{-s}$.

Thus if $\Re(s) \geq 0.5$ then

$$L(s) \ll \Gamma(1-s)\Gamma(\Re s) Q^{|s|} + \Gamma(1-s) Q^{-s} |s|^A \ll \Gamma(1+|s|) Q^{|s|} |s|^A,$$

and if $\Re(s) \leq 0.5$ then

$$L(s) \ll \Gamma(1-s)Q^{-s}|s|^{A},$$

which combine to give the required (weaker) form of the bound.

3.5.3 Theorem 10

By substituting the functional equation into the expression

$$f(z) = \frac{1}{2\pi i}\int_{(c)} L(s)\Gamma(s)z^{-s}ds,$$

we end up with a sum of scaled Meijer G functions. Rather than work explicitly with these we follow Hardy and Titchmarsh [44] in showing that $\gamma(s)$ for all intents and purposes is $\Gamma(s)$ in our formulae (lemma 5), which will reduce all computations to the case of $\zeta(s)$. This approach is also expanded on in Braaksma [9].

Take c sufficiently large so that $L_a(c)$ and $L_b(c)$ are absolutely convergent series, and as before define

$$f(z) \quad := \quad \sum_{n=1}^{\infty} a_n e^{-nz} = \frac{1}{2\pi i}\int_{(c)} L_a(s)\,\Gamma(s)\,z^{-s}ds.$$

Shift the contour to $1-c$, noting that the poles of $\Gamma(s)$ contribute a polynomial $p(z)$, showing

$$f(z) + p(z) = \frac{1}{2\pi i}\int_{(1-c)} L_a(s)\,\Gamma(s)\,z^{-s}ds.$$

Now substitute $1-s$ for s to get

$$f(z) + p(z) = \frac{1}{2\pi i}\int_{(c)} L_a(1-s)\,\Gamma(1-s)\,z^{s-1}ds,$$

and apply the functional equation to see that

$$f(z) + p(z) = \frac{1}{2\pi i}\int_{(c)} L_b(s)\,\gamma(s)\,\theta(s)\,Q^s\Gamma(1-s)\,z^{s-1}ds.$$

Assuming that the interchange of summation and integration is justified, as is about to proven, expand out L_b to get

$$f(z)+p(z)=\frac{1}{2\pi i}\sum_{n=1}^{\infty}\sum_{j}b_n\alpha_j\int_{(c)}\gamma\left(s\right)n^{-s}e\left(\omega_j s/4\right)Q^s\Gamma\left(1-s\right)z^{s-1}ds.$$

We denote the inner integral as the function G,

$$G\left(z\right):=\frac{1}{2\pi i}\int_{(c)}\gamma\left(s\right)\Gamma\left(1-s\right)z^s ds,$$

so that

$$f(z)+p(z)=\sum_{n=1}^{\infty}\sum_{j}b_n\alpha_j G\left(ze\left(\omega_j s/4\right)\left(Q/n\right)\right)z^{-1}.$$

The result now follows from the following analysis of $G\left(z\right)$.

Lemma 3. *$G\left(z\right)z^{-1}$ is holomorphic aside from a branch cut singularity at $-q_1^{-1}$ where $q_1=\prod\lambda_i^{\lambda_i}$.*

Proof. This proof is done in three stages, dealing with the cases of $\mathrm{Arg}\left(z\right)<\pi$, $|z|<q_1$, and $|z|>q_1$.

Case 1. Letting $s=\sigma+iT$, $\gamma(s)$ is bounded like $e^{-|T|(\pi/2-\epsilon)}$, $\Gamma\left(1-s\right)$ is bounded like $e^{-|T|(\pi/2-\epsilon)}$, z^{s-1} is bounded as $e^{(\mathrm{Arg}(z)-\epsilon)|T|}$, so the integral is absolutely convergent as long as $|\mathrm{Arg}\left(z\right)|<\pi$.

Case 2. For $\mathrm{Re}\left(s\right)$ sufficiently large, lemma 5 in appendix A says that

$$\gamma\left(s\right)\Gamma\left(1-s\right)z^s \quad = \quad \left(zq_1\right)^s s^{\mu}\left(C_1+O\left(1/s\right)\right)\frac{\pi}{\sin\left(-\pi s+\pi d\right)}.$$

Hence when $|zq_1|<1$ you may shift the contour to the right as both the integrand (away from poles) and the residues at $\mathbb{Z}$ are exponentially decaying, to get z times an absolutely convergent power series in z.

Case 3. For $\mathrm{Re}\left(s\right)$ sufficiently negative, lemma 5 upon substituting $-s$ says that

$$\gamma\left(s\right)\Gamma\left(1-s\right)z^s \quad = \quad \left(zq_1\right)^s s^{\mu}\left(C_2+O\left(1/s\right)\right)\prod_{i=1}^{r}\frac{\pi}{\sin\left(-\pi\lambda_i s+\pi\mu_i\right)}$$

Hence when $|zq_1| > 1$ you may shift the contour to the left as both the integrand (away from poles) and the residues at $\bigcup_{i=1}^{r}\left(\mu_i + \frac{\mathbb{Z}}{\lambda_i}\right)$ are exponentially decaying, to get an absolutely convergent generalized power series, on the Riemann surface for $\log z$. $\square$

It can also be useful to understand the behaviour of $G(z)$ around $-q_1^{-1}$.

Lemma 4. *Without loss of generality, translate L_a so that $\mu = 0$. Then around $z = -1$, when $|\mathrm{Arg}(z)| < \pi$,*

$$G(z/q_1)\, z^{-1} = \frac{B_0}{z+1} + B_1 \log(z+1) + O(1)$$

for some B_0, B_1.

Proof. By definition of G, the left hand side is

$$G(z/q_1)\, z^{-1} = \frac{1}{2\pi i}\int_{(c)} \gamma(s)\,\Gamma(1-s)\,(z/q_1)^s\, z^{-1} ds.$$

Lemma 5 shows that $\gamma(s)$, up to correction terms $A_0 + A_1/s$, is essentially $\Gamma(s)$, ergo

$$G(z/q_1)\, z^{-1} = \frac{1}{2\pi i}\int_{(c)} z^{s-1}\left(A_0 + A_1/s + O\left(1/s^2\right)\right)\frac{\pi}{\sin(-\pi s+\pi)} ds.$$

Replacing s with $s+1$, for some new constants B_0 and B_1,

$$G(z/q_1)\, z^{-1} = \frac{1}{2\pi i}\int_{(c)} z^{s}\left(B_0 + B_1/s + O\left(1/s^2\right)\right)\frac{\pi}{\sin(\pi s)}.$$

Bringing the error term outside the integral then gives us

$$G(z/q_1)\, z^{-1} = \frac{1}{2\pi i}\int_{(c)} z^{s}\left(B_0 + B_1/s\right)\frac{\pi}{\sin(\pi s)} + O(1),$$

and computing the integral we conclude that

$$G(z/q_1)\, z^{-1} = \frac{B_0}{1+z} + B_1 \log(1+z) + O(1)$$

□

Chapter 4

Results on degree 2 and higher

4.1 Introduction

The theory of degree 1 Dirichlet series in the previous chapter is relatively straightforward and satisfying, particularly since we can understand these in terms of periodic functions, there's a natural linear space of these, and we have a natural inner product that allows us to decompose in terms of known building blocks. In higher degree each of these things break down in profound ways, and we have to pick our tradeoffs.

First of all, the standard $GL(d)$ L-function is formed by taking the Fourier-Whittaker coefficients $A(n_1, \cdots, n_d)$ of a Maass form and then using a subset of these coefficients as Dirichlet coefficients to form $\sum A(n_1, 1, 1, \cdots, 1)n^{-s}$. The process of expressing these as a zeta integral generally involves restricting to some subspace and then integrating against a known lower-dimensional objects, expressing the L-function as a type of matrix coefficient. Multiple Dirichlet series can be formed using the full set of Fourier coefficients, but past $n = 3$ there seem to be fundamental difficulties to getting the meromorphic continuation that one would ideally have[16][18][10].

A very different route is to interpret the Dirichlet coefficients as Hecke eigenvalues for simultaneous eigenfunctions of the Hecke operators $T(n)$ on an appropriate space. As with the Fourier coefficients this is just a narrow subset of the full amount of information in the original object, namely the behaviour under the full Hecke algebra. This however can be recovered from the Euler product using the $T(n)$ to generate the algebra, and utilizing a factorisation of $\sum T(n)n^{-s}$ as something of the form $\prod_p(1-T(p,1,1,\cdots)p^{-s}+T(p,p,1,1,\cdots)p^{-2s}-\cdots\pm T(p,p,\cdots,p)p^{-ds})$. This relies heavily on the Euler product, and one may not take linear combinations of eigenvalues in the same way that one takes linear combinations of Dirichlet series or Fourier series.

Like in the degree 1 case, linear twists $\sum a_n e(n\alpha)n^{-s}$ generally make sense but only for rational α. The naive Rankin-Selberg convolution $\sum a_n b_n n^{-s}$ stops having analytic continuation past $d=2$ and while there is a convolution on degree $d>2$ it requires first recovering a higher dimensional structure by, for example, factoring the Euler product into the Langlands parameters.

The Hurwitz zeta functions that figured prominently in the previous chapter have natural analogue in the Shintani zeta functions[46], which have analytic continuation, and can be used as a basis for Dedekind zeta functions and of Eisenstein series. A particular special case of these are the Witten zeta functions, which are expressed as sums over root systems of a semi-simple Lie group [101][103]. These do not however provide a good analogue to the continuous family of linear twists in degree 1.

The presence of the natural numbers in the Dirichlet series crucial since, as Kaczorowski and Perelli point out, if you extend to generalized Dirichlet series $\sum a_n b_n^{-s}$ every gamma factor and conductor is possible.

Currently what is known of the general theory is that degree strictly between 1 and 2 is impossible[57], degree 2 Hecke-type functional equations are fully understood[7], certain small cases of Maass-type functional equations with conductor 1 or 2 are understood[68], and sporadic cases in degree 2 where it is known that a certain functional equation, such as $\Gamma(s/3)\Gamma(2s/3)$ and conductor 1, is impossible.

4.2 Degree d Functional Equations

We revisit the results from chapter 3, both to apply it to higher degree, and because here we will need more precise information on the dependence on s in order to apply more general functional equations.

Theorem 11. *A Dirichlet series $L(s) := \sum_{n=1}^{\infty} a_n n^{-s}$ has horizontal degree d if and only if $f(z) := \sum_{n=1}^{\infty} a_n e^{-n^{1/d}z}$, well defined for $\Re z > 0$, continues analytically in a* *neighbourhood of 0. Furthermore if this is the case then it also has vertical degree d.*

More precisely, the following are equivalent, for given Q, and some A not necessarily the same in all cases:

1. *$L(s/d)$ extends to an entire function satisfying $|L(s/d)| \ll \Gamma(1+|s|)|s|^A Q^{|s|}$.*
2. *$L(s/d)$ extends to an entire function satisfying $|L(s/d)|\Gamma(s)Q^s \ll |s|^A e^{-\pi i|\Im(s)|/2}$ for s in a left half-plane, away from the poles.*
3. *$f(z)$ extends analytically in a $1/Q$ neighbourhood of 0, and $f(z) \ll \left(\frac{1}{1-Q|z|}\right)^A$ in this neighbourhood.*

And in this case f has Taylor expansion: $f(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} L(-n/d)$.

Proof. The proof is identical to that of Theorem 11, applying it to the generalized Dirichlet series $L(s/d)$. Note that this also shows that vertical degree is no greater than the horizontal degree, as claimed in Proposition 2. $\square$

For applying to functional equations recall the most general definition from the introduction.

Definition 9. *(Most general functional equation) $L(s) = \sum a_n n^{-s}$ satisfies for some set Spec on the Riemann surface for* log*, some complex b_x bounded by some power of $|x|$, and some uniformly Gamma-like $\gamma_x(s)$,*

$$L(s) = \sum_{x \in Spec} b_x x^{s-1} \gamma_x(1 - ds + \mu).$$

By translating s we may without loss of generality take $\mu = 0$.

Theorem 12. *Suppose $L(s) = \sum a_n n^{-s}$ satisfies for some set Spec and Gamma-like functions γ_x,*

$$L(s/d) = \sum_{x \in Spec} b_x x^{1-s} \gamma_x(1 - s).$$

Then $f(z) := \sum a_n e^{-n^{1/d} z}$ has analytic continuation to all of $\mathbb{C}$ aside from some singularities (potentially branch cut discontinuities) for $z \in -Spec$. The singularities