the use of Fatou's lemma for property (i), and of the handy relation

$$\frac{1}{2\pi}\int_0^{2\pi} \log\frac{1}{|z+\rho e^{i\vartheta}-\zeta|}\,d\vartheta \;=\; \min\left(\log\frac{1}{|z-\zeta|},\ \log\frac{1}{\rho}\right)$$

(essentially the same as one appearing in the derivation of Jensen's formula, Chapter I!) for property (ii).

Integrals like the above one actually turn out to be practically capable of representing all superharmonic functions. In a sense made precise by Riesz' theorem, to be proved in article 2, the most general superharmonic function is equal to such an integral plus a harmonic function.

By such examples, one sees that superharmonic functions are far from being 'well behaved'. Consider, for instance

$$U(z) \;=\; \sum_n a_n \log\frac{1}{|z-z_n|},$$

formed with the z_n of modulus $< 1/2$ tending to 0 and numbers $a_n > 0$ chosen so as to make

$$\sum_n a_n \log\frac{1}{|z_n|} \;<\; \infty.$$

Here, $U(0) < \infty$ although U is infinite at each of the z_n. In more sophisticated versions of this construction, the z_n are *dense* in $\{|z| < 1/2\}$ and various sequences of $a_n > 0$ with $\sum_n a_n < \infty$ are used.

We now allow both properties from our definition to play their parts, (ii) as well as (i). In that way, we obtain the first general results pertaining specifically to superharmonic functions, among which the following *strong minimum principle* is probably the most important:

Lemma. *Let $U(z)$ be superharmonic in a domain $\mathscr{D}$. Then, if Ω is a (connected) domain with compact closure contained in $\mathscr{D}$,*

$$U(z) \;>\; \inf_{\zeta\in\partial\Omega} U(\zeta) \qquad \text{for } z\in\Omega$$

unless $U(z)$ is constant on $\bar{\Omega}$.

Proof. As we know, $U(z)$ attains its (finite) minimum, M, on $\bar{\Omega}$, and it is enough to show that if $U(z_0) = M$ at some $z_0\in\Omega$, we have $U(z) \equiv M$ on $\bar{\Omega}$. The reasoning here is like that followed in establishing the strong maximum principle for harmonic functions.

Assuming that there is such a z_0, we have, by property (ii),

$$M \quad = \quad U(z_0) \quad \geqslant \quad \frac{1}{2\pi}\int_0^{2\pi} U(z_0 + \rho e^{i\vartheta})\,d\vartheta$$

whenever $\rho > 0$ is sufficiently small. Here, $U(z_0 + \rho e^{i\vartheta}) \geqslant M$ and if, at any ϑ_0, we had $U(z_0 + \rho e^{i\vartheta_0}) > M$, $U(z_0 + \rho e^{i\vartheta})$ would be $> M$ for all ϑ belonging to some *open interval including* ϑ_0, by property (i). In that event, the above right-hand integral would also be $> M$, yielding a contradiction. We must therefore have $U(z_0 + \rho e^{i\vartheta}) \equiv M$ for small enough values of $\rho > 0$.

The rest of the proof is like that of the result for harmonic functions, with $E = \{z \in \Omega\colon\ U(z) = M\}$ closed in Ω's relative topology thanks to property (i). We are done.

Corollary. *Let $U(z)$ be superharmonic in a domain $\mathscr{D}$, and let $\mathscr{O}$ be an open set with compact closure lying in $\mathscr{D}$. Then, for $z \in \mathscr{O}$,*

$$U(z) \quad \geqslant \quad \inf_{\zeta \in \partial\mathscr{O}} U(\zeta).$$

Proof. Apply the lemma in each component of $\mathscr{O}$.

Corollary. *Let $U(z)$ be superharmonic in $\mathscr{D}$, a domain with compact closure. If* $\liminf_{z\to\zeta} U(z) \geqslant M$ *at each $\zeta \in \partial\mathscr{D}$, one has $U(z) \geqslant M$ in $\mathscr{D}$.*

Proof. Fix any $\varepsilon > 0$. Then, corresponding to each $\zeta \in \partial\mathscr{D}$ there is an r_ζ, $0 < r_\zeta < \varepsilon$, such that

$$U(z) \geqslant M - \varepsilon \quad \text{for } z \in \mathscr{D} \text{ and } |z - \zeta| \leqslant r_\zeta.$$

Here, $\partial\mathscr{D}$ is compact, so it can be covered by a *finite number* of the open disks

$$\{|z - \zeta| < r_\zeta\}.$$

Let $\mathscr{O}$ be the open set equal to the *complement*, in $\mathscr{D}$, of the *union of the closures* of *those particular disks.*

The closure $\bar{\mathscr{O}}$ is compact and contained in $\mathscr{D}$. If $z \in \partial\mathscr{O}$, we have $|z - \zeta| = r_\zeta$ for some $\zeta \in \partial\mathscr{D}$, so $U(z) \geqslant M - \varepsilon$. $U(z)$ is hence $\geqslant M - \varepsilon$ in $\mathscr{O}$ by the previous corollary. $\mathscr{O}$, however, certainly includes all points of $\mathscr{D}$ distant by more than ε from $\partial\mathscr{D}$. Our result thus follows on making $\varepsilon \longrightarrow 0$.

From these results we can deduce a useful *characterization* of superharmonic functions.

Theorem. *If $U(z)$ is $> -\infty$ and enjoys property* (i) *in a domain $\mathscr{D}$, it is superharmonic there provided that for each $z_0 \in \mathscr{D}$ and every disk Δ of sufficiently small radius with centre at z_0, one has*

$$U(z_0) \geqslant h(z_0)$$

for every function $h(z)$ harmonic in Δ and continuous up to $\partial\Delta$, satisfying

$$h(\zeta) \leqslant U(\zeta)$$

on $\partial\Delta$.

Conversely, if $U(z)$ is superharmonic in $\mathscr{D}$ and Ω is any domain having compact closure $\subseteq \mathscr{D}$, every function $h(z)$ harmonic in Ω and continuous up to $\partial\Omega$ is $\leqslant U(z)$ in Ω provided that $h(\zeta) \leqslant U(\zeta)$ on $\partial\Omega$.

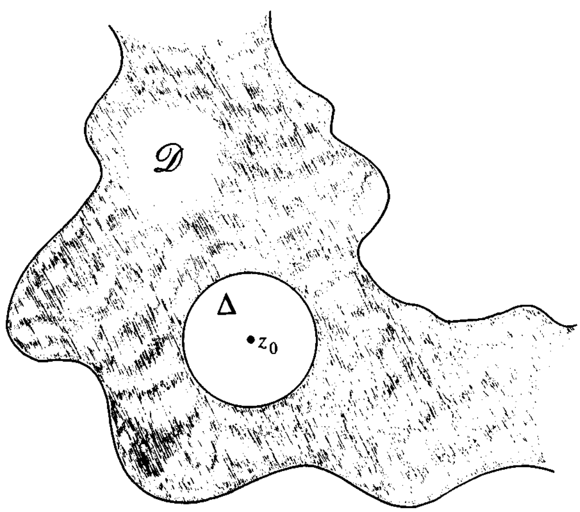

Figure 232

Proof. For the first part, we take any $z_0 \in \mathscr{D}$ and verify property (ii) for U there, assuming the hypothesis concerning disks Δ about z_0.

Let then $0 < r < \operatorname{dist}(z_0, \partial\mathscr{D})$. By the first lemma of this article, there is an *increasing sequence* of functions $u_n(\vartheta)$, *continuous* and of period 2π, such that

$$u_n(\vartheta) \xrightarrow[n]{} U(z_0 + r\mathrm{e}^{\mathrm{i}\vartheta}), \qquad 0 \leqslant \vartheta \leqslant 2\pi.$$

Put $h_n(z_0 + r\mathrm{e}^{\mathrm{i}\vartheta}) = u_n(\vartheta)$, and, for $0 \leqslant \rho < r$, take

$$h_n(z_0 + \rho\mathrm{e}^{\mathrm{i}\vartheta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - \rho^2}{r^2 + \rho^2 - 2r\rho\cos(\vartheta - \tau)} u_n(\tau)\,\mathrm{d}\tau.$$

Then each function $h_n(z)$ is harmonic in the disk Δ of radius r about z_0 and continuous up to $\partial\Delta$, where we of course have

$$h_n(\zeta) \leqslant U(\zeta).$$

If $r > 0$ is small enough, our assumption thus tells us that

$$h_n(z_0) \leqslant U(z_0)$$

for every n. Now Lebesgue's monotone convergence theorem ensures that

$$h_n(z_0) \underset{n}{\longrightarrow} \frac{1}{2\pi}\int_0^{2\pi} U(z_0 + re^{i\tau})\,d\tau$$

as $n \longrightarrow \infty$. Hence

$$\frac{1}{2\pi}\int_0^{2\pi} U(z_0 + re^{i\tau})\,d\tau \leqslant U(z_0)$$

for all sufficiently small $r > 0$, and property (ii) holds.

The other part of the theorem is practically a restatement of the *second* of the above corollaries. Indeed, if $h(z)$, harmonic in Ω and continuous up to $\bar{\Omega} \subseteq \mathscr{D}$ satisfies $h(\zeta) \leqslant U(\zeta)$ on $\partial\Omega$, we certainly have

$$\liminf_{\substack{z\to\zeta \\ z\in\Omega}} (U(z) - h(z)) \geqslant 0$$

at each $\zeta \in \partial\Omega$ on account of property (i). At the same time, $U(z) - h(z)$ is superharmonic in Ω; it must therefore be $\geqslant 0$ there by the corollary in question.

This does it.

By combining the two arguments followed in the last proof, we immediately obtain the following inequality:

For $U(z)$ superharmonic in $\mathscr{D}$, $z_0 \in \mathscr{D}$, and $0 < r < \operatorname{dist}(z_0\ \partial\mathscr{D})$,

$$\boxed{U(z_0 + \rho e^{i\vartheta}) \geqslant \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - \rho^2}{r^2 + \rho^2 - 2r\rho\cos(\vartheta - \tau)} U(z_0 + re^{i\tau})\,d\tau, \quad 0 \leqslant \rho < r.}$$

This in turn gives us a result needed in article 2:

Lemma. *If $U(z)$ is superharmonic in a domain $\mathscr{D}$ and $z_0 \in \mathscr{D}$,*

$$\frac{1}{2\pi}\int_0^{2\pi} U(z_0 + re^{i\vartheta})\,d\vartheta$$

is a decreasing *function of r for* $0 < r < \operatorname{dist}(z_0,\ \partial\mathscr{D})$.

Proof. Integrate both sides of the boxed inequality with respect to ϑ and then use Fubini's theorem on the right.

Along these same lines, we have, finally, the

Theorem. *Let $U(z)$ be superharmonic in a domain $\mathscr{D}$, and suppose that $z_0 \in \mathscr{D}$ and that $0 < R < \operatorname{dist}(z_0, \partial\mathscr{D})$. Denoting by Δ the disk $\{|z - z_0| < R\}$, put $V(z) = U(z)$ for $z \in \mathscr{D} \sim \Delta$. In Δ, take*

$$V(z_0 + re^{i\vartheta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos(\vartheta - \tau)} U(z_0 + Re^{i\tau})\,d\tau$$

(for $0 \leqslant r < R$). Then $V(z) \leqslant U(z)$ and $V(z)$ is superharmonic in $\mathscr{D}$.

Proof. For $z \in \mathscr{D} \sim \Delta$, the relation $V(z) \leqslant U(z)$ is manifest, and for $z \in \Delta$ it is a consequence of the above boxed inequality.

To verify property (ii) for V, suppose first of all that $z \in \mathscr{D} \sim \Delta$. Then, for sufficiently small $\rho > 0$,

$$V(z) = U(z) \geqslant \frac{1}{2\pi}\int_0^{2\pi} U(z + \rho e^{i\vartheta})\,d\vartheta.$$

By the relation just considered, the right-hand integral is in turn

$$\geqslant \frac{1}{2\pi}\int_0^{2\pi} V(z + \rho e^{i\vartheta})\,d\vartheta;$$

V thus enjoys property (ii) at z.

We must also look at the points $z \in \Delta$. On $\partial\Delta$, the function $U(\zeta)$ is *bounded below*, according to an early observation in this article. The Poisson integral used above to define $V(z)$ in Δ is therefore *either infinite for every r, $0 \leqslant r < R$, or else convergent for each such r*. In the former case, $V(z) \equiv \infty$ for $z \in \Delta$, and V (trivially) possesses property (ii) at those z. In the latter case, $V(z)$ is actually *harmonic* in Δ and hence, for any given z therein, *equal to the previous mean value when $\rho < \operatorname{dist}(z, \partial\Delta)$*. Here also, V has property (ii) at z.

Verifications of the relation $V(z) > -\infty$ and of property (i) remain. The first of these is clear; it is certainly true in $\mathscr{D} \sim \Delta$ where V coincides with U, and also true in Δ where, as a Poisson integral,

$$V(z) \geqslant \inf_{|\zeta - z_0| = R} U(\zeta)$$

with the right side $> -\infty$, as we know.

We have, then, to check property (i). The only points at which this can present any difficulty must lie on $\partial\Delta$, for, *inside* Δ, V is either *harmonic* and thus *continuous* or else *everywhere infinite*, and *outside* $\bar{\Delta}$, V coincides (in $\mathscr{D}$) with U, a function *having* the semicontinuity in question. Let therefore $|z - z_0| = R$. Then we surely have

$$\liminf_{\substack{\zeta \to z \\ \zeta \notin \Delta}} V(\zeta) \quad = \quad \liminf_{\substack{\zeta \to z \\ \zeta \notin \Delta}} U(\zeta) \quad \geqslant \quad U(z) \quad = \quad V(z),$$

so we need only examine the behaviour of $V(\zeta)$ for ζ tending to z *from within* Δ. The relation just written holds in particular, however, for $\zeta = z_0 + Re^{i\tau}$ tending to z *on* $\partial\Delta$. Since $U(z_0 + Re^{i\tau})$ is also bounded below on $\partial\Delta$, we see by the elementary properties of the Poisson kernel that

$$\liminf_{\substack{\zeta \to z \\ \zeta \in \Delta}} V(\zeta) \quad \geqslant \quad U(z) \quad = \quad V(z).$$

We thus have

$$\liminf_{\zeta \to z} V(z) \quad \geqslant \quad V(z)$$

for the points z on $\partial\Delta$, as well as at the other $z \in \mathscr{D}$, and V has property (i).

The theorem is proved.

Our work will involve the consideration of certain *families* of superharmonic functions. Concerning these, one has two main results.

Theorem. *Let the* $U_n(z)$ *be superharmonic in a domain* $\mathscr{D}$, *with*

$$U_1(z) \leqslant U_2(z) \leqslant U_3(z) \leqslant \cdots \leqslant U_n(z) \leqslant \cdots$$

there. Then

$$U(z) \quad = \quad \lim_{n \to \infty} U_n(z)$$

is superharmonic (*perhaps* $\equiv \infty$) *in* $\mathscr{D}$.

Proof. Since $U_1(z) > -\infty$ in $\mathscr{D}$, the same is true for $U(z)$.

Verification of property (i) is almost automatic. Given $z_0 \in \mathscr{D}$, let M be any number $< U(z_0)$. Then, for some particular n, $U_n(z_0) > M$, so, since U_n enjoys property (i), $U_n(z) > M$ in a neighborhood of z_0. *A fortiori*, $U(z) > M$ in that same neighborhood, and $\liminf_{z \to z_0} U(z) \geqslant U(z_0)$ on account of the arbitrariness of M.

Property (ii) is a consequence of Lebesgue's monotone convergence theorem. Let $z_0 \in \mathscr{D}$ and fix any $\rho < \operatorname{dist}(z_0, \partial\mathscr{D})$. Then, by the above boxed inequality,

$$U_n(z_0) \quad \geqslant \quad \frac{1}{2\pi}\int_0^{2\pi} U_n(z_0 + \rho e^{i\vartheta})\, d\vartheta$$

for each n. Here $U_1(z_0 + \rho e^{i\vartheta})$ is bounded below for $0 \leqslant \vartheta \leqslant 2\pi$, so the right-hand integral tends to

$$\frac{1}{2\pi}\int_0^{2\pi} U(z_0 + \rho e^{i\vartheta})\,d\vartheta$$

as $n \longrightarrow \infty$ by the monotone convergence. At the same time, $U_n(z_0) \xrightarrow[n]{} U(z_0)$, so property (ii) holds.

We are done.

A statement of opposite character is valid for *finite* collections of superharmonic functions. If, namely, $U_1(z), U_2(z), \ldots, U_N(z)$ *are superharmonic in a domain* $\mathscr{D}$, *so is* $\min_{1\leqslant k\leqslant N} U_k(z)$. This observation, especially useful when the functions $U_k(z)$ involved are *harmonic*, is easily verified directly.

WARNING. *The corresponding statement about* $\max_{1\leqslant k\leqslant N} U_k(z)$ *is* (in general) *false for superharmonic functions* U_k.

One has a version of the observation for *infinite* collections of superharmonic functions:

Theorem. *Let* $\mathscr{F}$ *be any family of functions superharmonic in a domain* $\mathscr{D}$. *For* $z \in \mathscr{D}$, *put*

$$W(z) = \inf\{U(z):\ U \in \mathscr{F}\},$$

and then let

$$V(z) = \liminf_{\zeta\to z} W(\zeta), \qquad z \in \mathscr{D}.$$

Then $V(z) \leqslant U(z)$ *in* $\mathscr{D}$ *for every* $U \in \mathscr{F}$, *and* (*especially*) *if* $V(z) > -\infty$ *in* $\mathscr{D}$, *it is superharmonic there.*

Remark. Something like the last condition is needed in order to avoid situations like the one where $\mathscr{D} = \mathbb{C}$ and $\mathscr{F}$ consists of the functions $n\Im z$, $n = 1, 2, 3, \ldots$. There, $V(z) = \Im z$ for $\Im z > 0$ but $V(z) = -\infty$ for $\Im z \leqslant 0$. Such functions V are not superharmonic.

Proof of theorem. First of all, $V(z) \leqslant W(z)$ in $\mathscr{D}$, i.e., $V(z) \leqslant U(z)$ there for each $U \in \mathscr{F}$. Indeed, since any such U is superharmonic in $\mathscr{D}$, $\liminf_{\zeta\to z} U(\zeta)$ is actually *equal* to $U(z)$ there, as observed earlier in this article (a result of playing properties (i) and (ii) against each other).

Therefore, whenever $U \in \mathscr{F}$,

$$V(z) \;=\; \liminf_{\zeta \to z} W(\zeta) \;\leqslant\; \liminf_{\zeta \to z} U(\zeta) \;=\; U(z), \qquad z \in \mathscr{D}.$$

Secondly, V has property (i) in $\mathscr{D}$. To see this, fix any $z_0 \in \mathscr{D}$ and pick* any $M < V(z_0)$; according to our definition of V, $W(z)$ is then $> M$ in some punctured open neighborhood of z_0 (i.e., an open neighborhood of z_0 with z_0 *deleted*). But this certainly makes $V(z) = \liminf_{\zeta \to z} W(\zeta) \geqslant M$ in that punctured neighborhood, so, since $M < V(z_0)$ was arbitrary, we have $\liminf_{z \to z_0} V(z) \geqslant V(z_0)$.

To complete verification of $V(z)$'s superharmonicity in $\mathscr{D}$ when that function is $> -\infty$ there, one may resort to the criterion provided by the *first* of the preceding theorems. According to the latter, it is enough to show that if $z_0 \in \mathscr{D}$ and Δ is any disk centred at z_0 with radius $< \operatorname{dist}(z_0, \partial\mathscr{D})$, we have $V(z_0) \geqslant h(z_0)$ for each function $h(z)$ *continuous* on $\bar{\Delta}$, *harmonic* in Δ, and satisfying $h(\zeta) \leqslant V(\zeta)$ on $\partial\Delta$. But for any such function h we certainly have $h(\zeta) \leqslant U(\zeta)$ on $\partial\Delta$ for every $U \in \mathscr{F}$, so, by the second part of the theorem referred to, $h(z) \leqslant U(z)$ in Δ for those U. Hence

$$h(z) \;\leqslant\; \inf_{U \in \mathscr{F}} U(z) \;=\; W(z)$$

in Δ, and finally, h being continuous at z_0 (the centre of Δ !),

$$h(z_0) \;=\; \lim_{z \to z_0} h(z) \;\leqslant\; \liminf_{z \to z_0} W(z) \;=\; V(z_0),$$

as required. We are done.

Remark. This theorem, together with the second of those preceding it, forms the basis for what is known as *Perron's method* of solution of the Dirichlet problem.

2. The Riesz representation of superharmonic functions

A superharmonic function can be approximated from below by others which are also infinitely differentiable. This is obvious for the function $U(z)$ identically infinite in a domain $\mathscr{D}$, that one being just the limit, as $n \longrightarrow \infty$, of the *constant* functions $U_n(z) = n$. We therefore turn to the construction of such approximations to functions $U(z)$ superharmonic and $\not\equiv \infty$ in $\mathscr{D}$.

Given such a U, one starts by forming the means

$$U_\rho(z) \;=\; \frac{1}{2\pi}\int_0^{2\pi} U(z + \rho e^{i\vartheta})\, d\vartheta;$$

* in the case where $V(z_0) > -\infty$; otherwise property (i) clearly does hold at z_0

when $\rho > 0$ is given, these are defined for the z in $\mathscr{D}$ with $\operatorname{dist}(z, \partial\mathscr{D}) > \rho$. According to property (ii) from our definition,

$$U_\rho(z) \leqslant U(z)$$

for all sufficiently small $\rho > 0$ (and in fact for *all* such $\rho < \operatorname{dist}(z, \partial\mathscr{D})$ by the boxed inequality near the end of the preceding article); on the other hand, $\liminf_{\rho\to 0} U_\rho(z) \geqslant U(z)$ by property (i). Thus, for each $z \in \mathscr{D}$,

$$U_\rho(z) \longrightarrow U(z) \quad \text{as } \rho \longrightarrow 0.$$

A lemma from the last article shows that this convergence is actually *monotone*; the $U_\rho(z)$ *increase* as ρ *diminishes* towards 0.

Concerning the U_ρ, we have the useful

Lemma. *If $U(z)$ is superharmonic in a (connected) domain $\mathscr{D}$ and not identically infinite there, the $U_\rho(z)$ are finite for $z \in \mathscr{D}$ and* $0 < \rho < \operatorname{dist}(z, \partial\mathscr{D})$.

Proof. Suppose that

$$U_r(z_0) = \frac{1}{2\pi}\int_0^{2\pi} U(z_0 + re^{i\tau})\,d\tau = \infty$$

for some $z_0 \in \mathscr{D}$ and an r with $0 < r < \operatorname{dist}(z_0, \partial\mathscr{D})$. It is claimed that then $U(z) \equiv \infty$ in $\mathscr{D}$.

By one of our first observations about superharmonic functions in the preceding article, $U(z_0 + re^{i\tau})$ is *bounded below* for $0 \leqslant \tau \leqslant 2\pi$. The above relation therefore makes the Poisson integrals occurring in the boxed inequality near the end of that article *infinite*, and we must have $U(z) \equiv \infty$ for $|z - z_0| < r$.

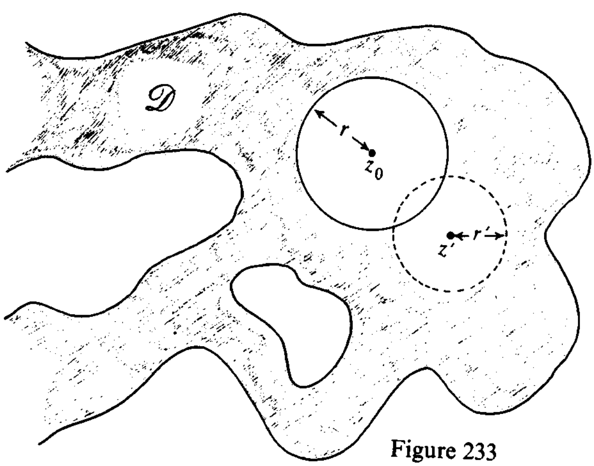

Figure 233

Let now z' be any point in $\mathscr{D}$ about which one can draw a circle – of radius r', say – *lying entirely* in $\mathscr{D}$ and also intersecting the open disk of radius r centred at z_0*. Since $U(z)$ is bounded below on that circle, we have $U_{r'}(z') = \infty$ so, by the argument just made, $U(z) \equiv \infty$ for $|z-z'| < r'$.

The process may evidently be continued indefinitely so as to gradually fill out the connected open region $\mathscr{D}$. In that way, one sees that $U(z) \equiv \infty$ therein, and the lemma is proved.

Corollary. *If $U(z)$ is superharmonic and not identically infinite in a (connected) domain $\mathscr{D}$, it is locally L_1 there (with respect to Lebesgue measure for $\mathbb{R}^2$).*

Proof. It is enough to verify that if $z_0 \in \mathscr{D}$ and $0 < r < \frac{1}{2}\operatorname{dist}(z_0, \partial\mathscr{D})$, we have

$$\iint_{r \leqslant |z-z_0| \leqslant 2r} |U(z)|\,\mathrm{d}x\,\mathrm{d}y \quad < \quad \infty,$$

for, since each point of $\mathscr{D}$ lies in the *interior* of some annulus like the one over which the integral is taken, any compact subset of $\mathscr{D}$ can be covered by a finite number of such annuli.

By the lower bound property already used so often, there is an $M < \infty$ such that $U(z) \geqslant -M$ when $|z-z_0| \leqslant 2r$. The preceding integral is therefore

$$\begin{aligned}
&\leqslant \quad \iint_{r \leqslant |z-z_0| \leqslant 2r} (\,U(z) + 2M\,)\,\mathrm{d}x\,\mathrm{d}y \\
&= \quad 6\pi r^2 M \quad + \quad \int_r^{2r}\int_0^{2\pi} U(z_0 + \rho e^{i\vartheta})\rho\,\mathrm{d}\vartheta\,\mathrm{d}\rho \\
&= \quad 6\pi r^2 M \;+\; 2\pi \int_r^{2r} U_\rho(z_0)\rho\,\mathrm{d}\rho.
\end{aligned}$$

$U_\rho(z_0)$ is, as noted above, a *decreasing* function of ρ; the last expression is thus

$$\leqslant \quad 6\pi r^2 M \;+\; 3\pi r^2 U_r(z_0).$$

This, however, is finite by the lemma.

We are done.

With the means $U_\rho(z)$ at hand, we continue our construction of superharmonic $\mathscr{C}_\infty$ approximations to a given $U(z)$ superharmonic and $\not\equiv \infty$ in a domain $\mathscr{D}$. For this purpose, one chooses any function $\varphi(\rho)$ infinitely

* with its interior

differentiable on $(0, \infty)$, *identically zero outside* $(1, 2)$ *and* > 0 *on that interval*, normalized so as to make

$$\int_1^2 \varphi(\rho)\rho\,d\rho \quad = \quad 1 :$$

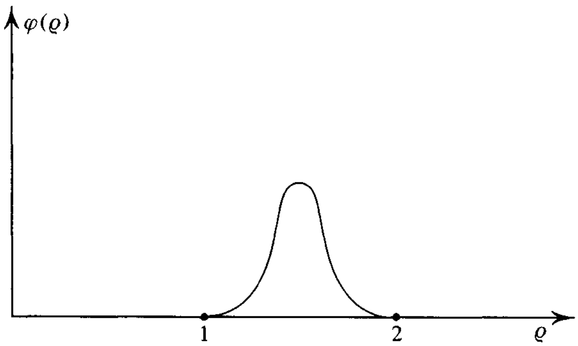

Figure 234

(As we shall see, it turns out to be convenient to work with a function $\varphi(\rho)$ *vanishing for the values of* ρ *near* 0 as well as for the large ones.) Using φ, one then forms *averages of the means* U_ρ :

$$(\Phi_r U)(z) \quad = \quad \frac{1}{r^2}\int_0^\infty U_\rho(z)\,\varphi(\rho/r)\rho\,d\rho;$$

for given $z \in \mathscr{D}$, these are defined when

$$0 \; < \; r \; < \; \tfrac{1}{2}\operatorname{dist}(z,\ \partial\mathscr{D}).$$

The $\Phi_r U$ are the approximations we set out to obtain; one has, namely, the

Theorem. *Given* $r > 0$, *denote by* $\mathscr{D}_r$ *the set of* $z \in \mathscr{D}$ *with* $\operatorname{dist}(z,\ \partial\mathscr{D}) \; > \; 2r$. *Let* U *be superharmonic in* $\mathscr{D}$, *then*:

$$(\Phi_r U)(z) \;\leqslant\; U(z) \quad \text{for } z \in \mathscr{D}_r;$$

$$(\Phi_r U)(z) \;\longrightarrow\; U(z) \quad \text{as } r \longrightarrow 0 \text{ for each } z \in \mathscr{D};$$

$$(\Phi_{2r} U)(z) \;\leqslant\; (\Phi_r U)(z) \quad \text{for } z \in \mathscr{D}_{2r}.$$

If also $U(z) \;\not\equiv\; \infty$ *in the (connected) domain* $\mathscr{D}$, *each* $(\Phi_r U)(z)$ *is infinitely differentiable in the corresponding* $\mathscr{D}_r$, *and superharmonic in each connected component thereof.*

Proof. The *first two* properties of the $\Phi_r U$ follow as direct consequences of the behaviour, noted above, of the $U_\rho(z)$ together with φ's normalization. The *third* is then assured by $\varphi(\rho)$'s being supported on the interval $(1, 2)$.

Passing to the *superharmonicity* of $\Phi_r U$, we first check property (ii) for that function in $\mathscr{D}_r$. This does not depend on the condition that $U(z) \not\equiv \infty$. Fix any $z \in \mathscr{D}_r$. For $0 < \sigma < \operatorname{dist}(z, \partial\mathscr{D}) - 2r$ we then have

$$\frac{1}{2\pi}\int_0^{2\pi} (\Phi_r U)(z + \sigma e^{i\psi})\, d\psi$$

$$= \quad \frac{1}{4\pi^2 r^2}\int_0^{2\pi}\int_0^{\infty}\int_0^{2\pi} U(z + \rho e^{i\vartheta} + \sigma e^{i\psi})\,\varphi(\rho/r)\rho\, d\vartheta\, d\rho\, d\psi.$$

Since $\varphi(\rho/r)$ vanishes for $\rho \geqslant 2r$, $z + \rho e^{i\vartheta}$ lies in $\mathscr{D}$ and has distance $> \sigma$ from $\partial\mathscr{D}$ for all the values of ρ actually involved in the second expression. The argument $z + \rho e^{i\vartheta} + \sigma e^{i\psi}$ of U thus ranges over a *compact subset* of $\mathscr{D}$ in that triple integral, and on such a subset U is *bounded below*, as we know. This makes it permissible for us to *perform first the integration with respect to* ψ. Doing that, and using the boxed inequality from the preceding article, we obtain a value

$$\leqslant \quad \frac{1}{2\pi r^2}\int_0^{\infty}\int_0^{2\pi} U(z + \rho e^{i\vartheta})\,\varphi(\rho/r)\rho\, d\vartheta\, d\rho \quad = \quad (\Phi_r U)(z),$$

showing that $\Phi_r U$ has property (ii) at z. Superharmonicity of $\Phi_r U$ in the components of $\mathscr{D}_r$ thus follows if it meets our definition's other two requirements there.

Satisfaction of the latter is, however, obviously guaranteed by the *infinite differentiability* of $\Phi_r U$ in $\mathscr{D}_r$, which we now proceed to verify for functions $U(z) \not\equiv \infty$ in $\mathscr{D}$.

The left-hand member of the last relation can be rewritten as

$$\frac{1}{2\pi r^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(z + \zeta)\,\varphi(|\zeta|/r)\, d\xi\, d\eta,$$

where, as usual, $\zeta = \xi + i\eta$. Putting $z + \zeta = \zeta' = \xi' + i\eta'$, this becomes

$$\frac{1}{2\pi r^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(\zeta')\,\varphi(|\zeta' - z|/r)\, d\xi'\, d\eta'.$$

Here, $\varphi(|\zeta' - z|/r)$ vanishes for $|\zeta' - z| \leqslant r$ and $|\zeta' - z| \geqslant 2r$. Looking, then, at values of z near some *fixed* $z_0 \in \mathscr{D}_r$ – to be definite, at those, say, with

$$|z - z_0| \quad < \quad \delta \quad = \quad \frac{1}{2}\min(r,\ \operatorname{dist}(z_0, \partial\mathscr{D}) \ - \ 2r),$$

we have

$$(\Phi_r U)(z) \quad = \quad \frac{1}{2\pi r^2} \iint_{r-\delta \leqslant |\zeta' - z_0| \leqslant 2r+\delta} U(\zeta')\, \varphi(|z-\zeta'|/r)\, \mathrm{d}\xi'\, \mathrm{d}\eta'.$$

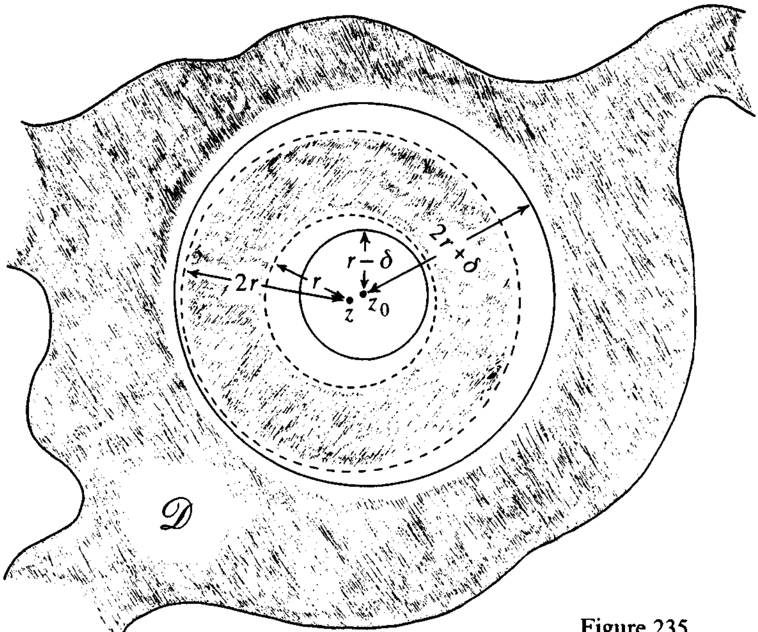

Figure 235

The annulus $K = \{\zeta' \colon\ r-\delta \leqslant |\zeta'-z_0| \leqslant 2r+\delta\}$ over which the integration is carried out here is a *compact* subset of $\mathscr{D}$ (independent of z !), so, for the function U under consideration we have

$$\iint_K |U(\zeta')|\, \mathrm{d}\xi'\, \mathrm{d}\eta' \quad < \quad \infty$$

by the above corollary. Infinite differentiability of $(\Phi_r U)(z)$ at z_0 can therefore be *read off by inspection* from the last formula, provided that $\varphi(|z-\zeta'|/r)$ enjoys the same property for *each* $\zeta' \in K$ (differentiation inside the integral signs). That, however, is indeed the case, as follows by the chain rule from infinite differentiability of φ and the fact that $|z_0 - \zeta'| \geqslant \delta > 0$ for each $\zeta' \in K$. (Here we have been helped by $\varphi(\rho)$'s vanishing for $0 < \rho < 1$.) $\Phi_r U$ is thus $\mathscr{C}_\infty$ in $\mathscr{D}_r$.

The theorem is proved.

The approximations $\Phi_r U$ to a given superharmonic function U are used in establishing the *Riesz representation* for the latter. That says essentially that a function $U(z)$ superharmonic and $\not\equiv \infty$ *in and on* a bounded domain $\mathscr{D}$ (i.e., in a domain *including* $\bar{\mathscr{D}}$) is given there by a *formula*

$$U(z) \quad = \quad \int_{\bar{\mathscr{D}}} \log\frac{1}{|z-\zeta|}\,\mathrm{d}\mu(\zeta) \quad + \quad H(z),$$

with μ a (finite) positive measure on $\bar{\mathscr{D}}$ and $H(z)$ *harmonic* in $\mathscr{D}$. (Conversely, expressions like the one on the right *are* always superharmonic in $\mathscr{D}$, according to the remarks following the first lemma of the preceding article.)

The representation is really of *local character*, for the restriction of the measure μ figuring in it to *any open disk* $\Delta \subseteq \mathscr{D}$ is *completely determined by the behaviour of* U *in* Δ (see problem 48 below), and at the same time, the function of z equal to

$$\int_{\mathscr{D}\sim\Delta} \log\frac{1}{|z-\zeta|}\;\mathrm{d}\mu(\zeta)$$

is certainly *harmonic* in Δ. The general form of the result can thus be obtained from *a special version of it for disks* by simply pasting some of those together so as to cover the given domain $\mathscr{D}$! In fact, *only the version for disks* will be required in the present chapter, so that is what we prove here. Passage from it to the more general form is left as an exercise to the reader (problem 49).

We proceed, then, to the derivation of the Riesz representation formula for disks. The idea is to first get it for $\mathscr{C}_\infty$ superharmonic functions by simple application of Green's theorem and then pass from those to the general ones with the help of the $\Phi_r U$. In this, an essential rôle is played by the classical

Lemma. *A function $V(z)$ infinitely differentiable in a domain $\mathscr{D}$ is superharmonic there if and only if*

$$\frac{\partial^2 V(z)}{\partial x^2} + \frac{\partial^2 V(z)}{\partial y^2} \quad \leqslant \quad 0 \quad \text{for } z \in \mathscr{D}.$$

Notation. The Laplacian $\partial^2/\partial x^2 \;+\; \partial^2/\partial y^2$ is denoted by ∇^2 (following earlier usage in this book).

Proof of lemma. Supposing that V is superharmonic in $\mathscr{D}$, we take any point z_0 therein. Then, by the third lemma of the preceding article,

$$\int_0^{2\pi} V(z_0 + \rho\mathrm{e}^{\mathrm{i}\vartheta})\,\mathrm{d}\vartheta$$

is a *decreasing* function of ρ for $0 \;<\; \rho \;<\; \operatorname{dist}(z_0,\; \partial\mathscr{D})$. The $\mathscr{C}_\infty$ character of V makes it possible for us to differentiate this expression under the integral sign with respect to ρ, so we have

$$\int_0^{2\pi} \frac{\partial V(z_0 + \rho e^{i\vartheta})}{\partial \rho} d\vartheta \quad \leqslant \quad 0$$

for small positive values of that parameter.

By Green's theorem, however,

$$\iint_{|z-z_0|<\rho} (\nabla^2 V)(z)\, dx\, dy \quad = \quad \int_0^{2\pi} \frac{\partial V(z_0 + \rho e^{i\vartheta})}{\partial \rho} \rho\, d\vartheta;$$

the left-hand integral is thus *negative*. Finally,

$$(\nabla^2 V)(z_0) \quad = \quad \lim_{\rho \to 0} \frac{1}{\pi \rho^2} \iint_{|z-z_0|<\rho} (\nabla^2 V)(z)\, dx\, dy,$$

showing that $\nabla^2 V \leqslant 0$ at z_0.

Assuming, on the other hand, that $\nabla^2 V \leqslant 0$ in $\mathscr{D}$, we see by the second of the above relations that

$$\int_0^{2\pi} V(z_0 + \rho e^{i\vartheta})\, d\vartheta$$

is a decreasing function of ρ for $0 < \rho < \operatorname{dist}(z_0, \partial\mathscr{D})$ when $z_0 \in \mathscr{D}$. At the same time,

$$V(z_0) \quad = \quad \lim_{\rho \to 0} \frac{1}{2\pi} \int_0^{2\pi} V(z_0 + \rho e^{i\vartheta})\, d\vartheta$$

in the present circumstances, so $V(z_0)$ must be $\geqslant$ each of the means figuring on the right for the values of ρ just indicated. This establishes property (ii) for V at z_0 and hence the superharmonicity of V in $\mathscr{D}$.

The lemma is proved.

Here is the version of Riesz' result that we will be using. It is most convenient to obtain a representation differing slightly in appearance from the one written above, but equivalent to the latter. About this, more in the remark following the proof.

Theorem (F. Riesz). *Let $U(z)$ be superharmonic and $\not\equiv \infty$ in a domain $\mathscr{D}$, and suppose that $z_0 \in \mathscr{D}$ and $0 < r < \operatorname{dist}(z_0, \partial\mathscr{D})$. Then, for $|z - z_0| < r$, one has*

$$U(z) \quad = \quad \int_{|\zeta - z_0| \leqslant r} \log \left| \frac{r^2 - \overline{(\zeta - z_0)}(z - z_0)}{r(z - \zeta)} \right| d\mu(\zeta) \quad + \quad h(z),$$

where μ is a finite positive measure on the closed disk $\{|z - z_0| \leqslant r\}$, and $h(z)$ a function harmonic for $|z - z_0| < r$.

Remark. In the integrand we simply have the *Green's function* associated with the disk $\{|z-z_0| < r\}$. The integral is therefore frequently referred to as a *pure Green potential* for that disk – 'pure' because the *measure* μ is *positive.*

Proof of theorem. To simplify the writing, we take $z_0 = 0$ and $r = 1$ – *that also frees the letter r for another use during this proof*! For some $R > 1$, the closed disk

$$\bar{\Delta} \;=\; \{|z| \leqslant R\}$$

lies in $\mathscr{D}$, and the averages $\Phi_r U$ introduced previously are hence *defined, infinitely differentiable* and *superharmonic in and on* $\bar{\Delta}$ when the parameter r (not to be confounded with the radius of the disk for which our representation is being derived!) is small enough. We *fix* such an r, and *denote* $\Phi_r U$ *by* V *for the time being* (again to help keep the notation clear).

Fix also any z, $|z| < R$, for the moment. The Green's function

$$\log\left|\frac{R^2-\zeta\bar{z}}{R(\zeta-z)}\right|$$

is *harmonic* in ζ for $|\zeta| < R$ and $\zeta \neq z$; it is, besides, *zero* when $|\zeta| = R$. From this we see by applying Green's theorem in the region $\{\zeta\colon |\zeta-z| > \rho$ and $|\zeta| < R\}$ and afterwards causing ρ to tend to zero (cf. beginning of the proof of symmetry of the Green's function, end of

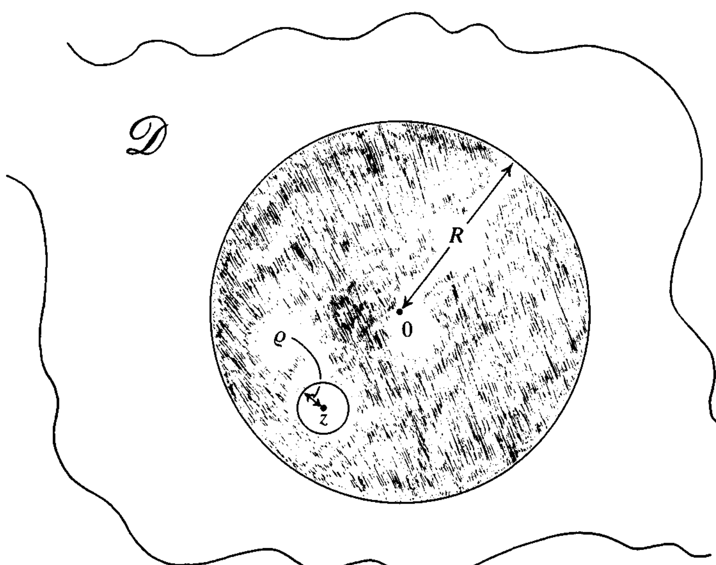

Figure 236

§A.2, Chapter VIII), that

$$V(z) \;=\; -\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial}{\partial\sigma}\log\left|\frac{R^2-\sigma e^{i\vartheta}\bar{z}}{R(\sigma e^{i\vartheta}-z)}\right|\right)_{\sigma=R} V(Re^{i\vartheta})R\,d\vartheta$$

$$-\;\frac{1}{2\pi}\iint_{|\zeta|<R}\log\left|\frac{R^2-\zeta\bar{z}}{R(\zeta-z)}\right|(\nabla^2 V)(\zeta)\,d\xi\,d\eta.$$

Working out the partial derivative in the first integral on the right, we get

$$V(z) \;=\; \frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-|z|^2}{|z-Re^{i\vartheta}|^2}\,V(Re^{i\vartheta})\,d\vartheta$$

$$-\;\frac{1}{2\pi}\iint_{|\zeta|<R}\log\left|\frac{R^2-\zeta\bar{z}}{R(\zeta-z)}\right|(\nabla^2 V)(\zeta)\,d\xi\,d\eta;$$

this, then, holds for each z of modulus $< R$.

Here, in the first integrand, we recognize the *Poisson kernel* for the disk $\{|z| < R\}$ (that's where the kernel *comes* from!); the *first* right-hand term is hence equal to a function *harmonic in that disk.* In the *second* term on the right, $(\nabla^2 V)(\zeta)$ is *negative according to the preceding lemma*, V being superharmonic in and on $\{|z| \leqslant R\}$. The last relation is therefore a *formula of the kind we are seeking to establish*, representing, however, the $\mathscr{C}_\infty$ superharmonic approximaton $V = \Phi_r U$ to our original superharmonic function U instead of U itself. We wish now to arrive at the desired formula for U by making $r \longrightarrow 0$.

With that in mind, we rearrange the preceding relation, writing $\Phi_r U$ in place of V:

$$-\frac{1}{2\pi}\iint_{|\zeta|<R}(\nabla^2\Phi_r U)(\zeta)\,\log\left|\frac{R^2-\zeta\bar{z}}{R(z-\zeta)}\right|d\xi\,d\eta$$

$$=\;(\Phi_r U)(z)\;-\;\frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-|z|^2}{|z-Re^{i\vartheta}|^2}(\Phi_r U)(Re^{i\vartheta})\,d\vartheta,\qquad |z|\;<\;R.$$

From this, we proceed to deduce the *boundedness of*

$$-\iint_{|\zeta|\leqslant 1}(\nabla^2\Phi_r U)(\zeta)\,d\xi\,d\eta$$

for r tending to zero.

Here we use the *first* lemma of this article, according to which there are points z *arbitrarily close to* 0 *for which* $U(z) < \infty$, it having been given that $U \not\equiv \infty$ in $\mathscr{D}$. *Fixing* such a z, of modulus $< 1/2$, say, and denoting it by the letter c, we have, from the preceding theorem,

$$(\Phi_r U)(c) \;\leqslant\; U(c),$$

so, by the last formula,

$$\frac{1}{2\pi}\iint_{|\zeta|<R}(-\nabla^2\Phi_r U)(\zeta)\log\left|\frac{R^2-\zeta\bar{c}}{R(\zeta-c)}\right|\,d\xi\,d\eta$$

$$\leqslant \quad U(c) \quad - \quad \frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-|c|^2}{|Re^{i\vartheta}-c|^2}(\Phi_r U)(Re^{i\vartheta})\,d\vartheta$$

for sufficiently small $r > 0$.

Now $(\Phi_r U)(Re^{i\vartheta})$ is by our construction an *average* of $U(\zeta)$ over the little annulus $r < |\zeta - Re^{i\vartheta}| < 2r$, and the union of these annuli for $0 \leqslant \vartheta \leqslant 2\pi$ is contained in the disk $\{|\zeta| \leqslant R + 2r\}$. The latter, in turn, is contained in a *fixed* disk $\{|\zeta| \leqslant R'\}$ slightly larger than $\{|\zeta| \leqslant R\}$ for values of $r < (R'-R)/2$. R, however, was chosen so as to make the disk $\{|\zeta| \leqslant R\}$ lie in $\mathscr{D}$; we may thus take $R' > R$ close enough to R to ensure that $\{|\zeta| \leqslant R'\}$ is also in $\mathscr{D}$. Once this is done, we know there is a finite M with $U(\zeta) \geqslant -M$ for $|\zeta| \leqslant R'$; this, then, holds in particular on the little annuli first mentioned when $2r < R' - R$. For the averages $\Phi_r U$ corresponding to those values of r we therefore have

$$(\Phi_r U)(Re^{i\vartheta}) \quad \geqslant \quad -M, \qquad 0 \leqslant \vartheta \leqslant 2\pi.$$

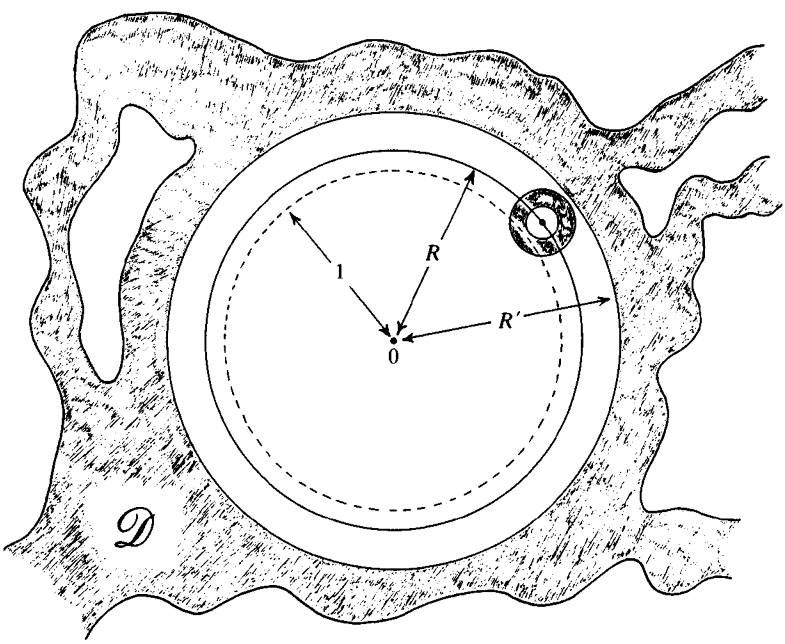

Figure 237

At the same time $R > 1$, so the expression

$$\log\left|\frac{R^2 - \zeta\bar{c}}{R(\zeta - c)}\right|,$$

positive for $|\zeta| < R$, is actually $\geqslant$ some $k > 0$ for $|\zeta| \leqslant 1$; meanwhile, $(-\nabla^2\Phi_r U)(\zeta) \geqslant 0$ for $|\zeta| \leqslant R$ as we know, when $r > 0$ is sufficiently small. Use these relations in the *left side* of the above inequality, and plug the previous one into the *right-hand* integral figuring in the latter. It is found that

$$\frac{k}{2\pi}\iint_{|\zeta|\leqslant 1}(-\nabla^2\Phi_r U)(\zeta)\,d\xi\,d\eta \quad\leqslant\quad U(c) + M,$$

a finite quantity, for $r > 0$ small enough. The integral on the left thus does remain bounded as $r \longrightarrow 0$.

By this boundedness we see, keeping positivity of the functions $-\nabla^2\Phi_r U$ in mind, that there is a certain *positive measure* μ *on* $\{|\zeta| \leqslant 1\}$ such that, *on the closed unit disk,*

$$-\frac{1}{2\pi}(\nabla^2\Phi_r U)(\zeta)\,d\xi\,d\eta \quad\longrightarrow\quad d\mu(\zeta) \qquad w^*$$

as $r \longrightarrow 0$ through a certain sequence of values r_n (cf. §F.1 of Chapter III, where the same kind of argument is used). There is no loss of generality in our taking $r_{n+1} < r_n/2$; this will permit us to take advantage of the relation $\Phi_{2r}U \leqslant \Phi_r U$.

Let us now rewrite *for the unit disk* the representation of the $\Phi_r U$ derived above for $\{|z| < R\}$. That takes the form

$$(\Phi_r U)(z) \quad=\quad -\frac{1}{2\pi}\iint_{|\zeta|\leqslant 1}\log\left|\frac{1 - z\bar{\zeta}}{z - \zeta}\right|(\nabla^2\Phi_r U)(\zeta)\,d\xi\,d\eta$$

$$+\quad \frac{1}{2\pi}\int_0^{2\pi}\frac{1 - |z|^2}{|z - e^{i\vartheta}|^2}(\Phi_r U)(e^{i\vartheta})\,d\vartheta, \qquad |z| < 1$$

(assuming, of course, as always that $r > 0$ is sufficiently small). Fixing any z of modulus < 1, we let r tend to 0 through the sequence $\{r_n\}$. According to the preceding theorem, $(\Phi_r U)(z)$ will then tend to $U(z)$, and, since $r_n > 2r_{n+1}$, $(\Phi_r U)(e^{i\vartheta})$ will, for each ϑ, *increase monotonically*, tending to $U(e^{i\vartheta})$. The *second* integral on the right will thus tend to

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1 - |z|^2}{|z - e^{i\vartheta}|^2}\,U(e^{i\vartheta})\,d\vartheta$$

by the monotone convergence theorem. We desire at this point to deduce simultaneous convergence of the *first* term on the right to

$$\int_{|\zeta|\leqslant 1} \log\left|\frac{1-z\bar{\zeta}}{z-\zeta}\right| \mathrm{d}\mu(\zeta)$$

from the w^* convergence just described, since that would complete the proof.

That, however, involves a slight difficulty, for, as a function of ζ,

$$\log\left|\frac{1-\bar{\zeta}z}{\zeta-z}\right|$$

is discontinuous at $\zeta = z$. To deal with this, we first break up the preceding formula for $\Phi_r U$ in the following way:

$$\begin{aligned}(\Phi_r U)(z') \quad = \quad & -\frac{1}{2\pi}\iint_{|\zeta|\leqslant 1} \log\frac{1}{|z'-\zeta|}\,(\nabla^2\Phi_r U)(\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\ & - \frac{1}{2\pi}\iint_{|\zeta|\leqslant 1} \log|1-z'\bar{\zeta}|\,(\nabla^2\Phi_r U)(\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta \\ & + \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z'|^2}{|z'-e^{i\vartheta}|^2}\,(\Phi_r U)(e^{i\vartheta})\,\mathrm{d}\vartheta, \qquad |z'| < 1.\end{aligned}$$

Keeping z, of modulus < 1, fixed, we take z' in this relation equal to $z + \rho e^{i\psi}$, with $0 < \rho < 1-|z|$, and then integrate with respect to ψ on both sides, from 0 to 2π.

When $|\zeta| \leqslant 1$, $\log|1-z'\bar{\zeta}|$ is *harmonic* in z' for $|z'| < 1$; we thus have

$$\frac{1}{2\pi}\int_0^{2\pi} \log|1 - (z+\rho e^{i\psi})\bar{\zeta}|\,\mathrm{d}\psi \quad = \quad \log|1-z\bar{\zeta}|$$

for each such ζ and $0 < \rho < 1-|z|$. In like manner,

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1 - |z+\rho e^{i\psi}|^2}{|z + \rho e^{i\psi} - e^{i\vartheta}|^2}\,\mathrm{d}\psi \quad = \quad \frac{1-|z|^2}{|z-e^{i\vartheta}|^2}$$

for the indicated values of ρ. There is, finally, the elementary formula

$$\frac{1}{2\pi}\int_0^{2\pi} \log\frac{1}{|z+\rho e^{i\psi}-\zeta|}\,\mathrm{d}\psi \quad = \quad \min\left(\log\frac{1}{|z-\zeta|},\ \log\frac{1}{\rho}\right)$$

already mentioned in the last article.

With the help of these relations we find by integration of the parameter

ψ that

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}&(\Phi_r U)(z+\rho e^{i\psi})\,d\psi\\
&= -\frac{1}{2\pi}\iint_{|\zeta|\leqslant 1}\min\left(\log\frac{1}{|z-\zeta|},\ \log\frac{1}{\rho}\right)(\nabla^2\Phi_r U)(\zeta)\,d\xi\,d\eta\\
&\quad - \frac{1}{2\pi}\iint_{|\zeta|\leqslant 1}\log|1-z\bar{\zeta}|\,(\nabla^2\Phi_r U)(\zeta)\,d\xi\,d\eta\\
&\quad + \frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|z-e^{i\vartheta}|^2}(\Phi_r U)(e^{i\vartheta})\,d\vartheta\end{aligned}$$

for $|z| < 1$, $0 < \rho < 1-|z|$, and r sufficiently small.

Fix now such values of z and ρ, and make $r \longrightarrow 0$ through the sequence of values r_n. The *third* integral on the right in the present relation then tends to

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|z-e^{i\vartheta}|^2}U(e^{i\vartheta})\,d\vartheta$$

as observed above, and the *left side* tends to

$$\frac{1}{2\pi}\int_0^{2\pi}U(z+\rho e^{i\psi})\,d\psi \quad = \quad U_\rho(z)$$

for the same reason (monotone convergence). Here, the functions of ζ involved in the *first two integrals* on the right *are* continuous on the closed unit disk. This allows us to conclude from the w^* convergence described above that those integrals tend respectively to

$$\iint_{|\zeta|\leqslant 1}\min\left(\log\frac{1}{|z-\zeta|},\ \log\frac{1}{\rho}\right)d\mu(\zeta)$$

and to

$$\iint_{|\zeta|\leqslant 1}\log|1-z\bar{\zeta}|\ d\mu(\zeta).$$

Putting these observations together, we see that

$$\begin{aligned}U_\rho(z) \quad = \quad &\iint_{|\zeta|\leqslant 1}\min\left(\log\left|\frac{1-z\bar{\zeta}}{z-\zeta}\right|,\ \log\frac{|1-z\bar{\zeta}|}{\rho}\right)d\mu(\zeta)\\
&+ \frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|z-e^{i\vartheta}|^2}U(e^{i\vartheta})\,d\vartheta\end{aligned}$$

for $|z| < 1$ and $0 < \rho < 1-|z|$.

We finally let $\rho \longrightarrow 0$, continuing to hold z fixed. Then, as noted at the very beginning of this article, $U_\rho(z) \longrightarrow U(z)$. At the same time, the first right-hand integral in the formula just written tends to

$$\iint_{|\zeta|\leqslant 1} \log\left|\frac{1-z\bar{\zeta}}{z-\zeta}\right| d\mu(\zeta)$$

by monotone convergence! We therefore have

$$U(z) \quad = \quad \iint_{|\zeta|\leqslant 1} \log\left|\frac{1-z\bar{\zeta}}{z-\zeta}\right| d\mu(\zeta) \quad + \quad \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|z-e^{i\vartheta}|^2} U(e^{i\vartheta})\,d\vartheta$$

for $|z| < 1$. The function

$$h(z) \quad = \quad \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|z-e^{i\vartheta}|^2} U(e^{i\vartheta})\,d\vartheta$$

is *harmonic* in the open unit disk. Our theorem is thus proved.

Remark. The representation just obtained is frequently written differently. Taking, to simplify the notation, $z_0 = 0$, what we have so far reads

$$U(z) \quad = \quad \int_{|\zeta|\leqslant r} \log\left|\frac{r^2-z\bar{\zeta}}{r(z-\zeta)}\right| d\mu(\zeta) \quad + \quad h(z), \qquad |z| \leqslant r,$$

with $h(z)$ a certain function *harmonic* in $\{|z| < r\}$. Under the circumstances of the theorem (U superharmonic in a *slightly larger* disk), we even have an explicit formula for h,

$$h(z) \quad = \quad \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2-|z|^2}{|z-re^{i\vartheta}|^2} U(re^{i\vartheta})\,d\vartheta, \qquad |z| < r,$$

found at the end of the above proof.

The integral

$$\int_{|\zeta|\leqslant r} \log|r^2 - z\bar{\zeta}|\; d\mu(\zeta),$$

however, is itself a harmonic function of z for $|z| < r$. The preceding relation can be thus rewritten as

$$\boxed{U(z) \quad = \quad \int_{|\zeta|\leqslant r} \log\frac{1}{|z-\zeta|}\, d\mu(\zeta) \quad + \quad H(z), \qquad |z| < r,}$$

with

$$H(z) \quad = \quad h(z) \quad + \quad \int_{|\zeta| \leqslant r} \left(\log|r^2 - z\bar{\zeta}| \quad + \quad \log\frac{1}{r} \right) \mathrm{d}\mu(\zeta)$$

also harmonic in the disk $\{|z| < r\}$. Here we recognize on the right the familiar *logarithmic potential* (corresponding to the (here finite) positive measure μ) which has already played a rôle in §F of Chapter IX. The simplicity of this version in comparison with the original one is somewhat offset by a drawback: $H(z)$, unlike $h(z)$, is no longer determined by the boundary values $U(re^{i\vartheta})$ alone. It is often easier, nevertheless, to work with the former rather than the latter.

As they stand, the two forms of the representation are *equivalent*, with the above relation between the harmonic functions h and H serving to pass from one to the other. As long as the (finite) measure μ is *positive*, and the function $H(z)$ *harmonic* in $\{|z| < r\}$, the boxed formula does give us a function $U(z)$ superharmonic there according to observations in the preceding article; this is also true of the other formula under the same circumstances regarding μ and h. Concerning, however, such a function U, with $H(z)$, say, known *only* to be harmonic for $|z| < r$, we can say nothing about the boundary values $U(re^{i\vartheta})$ (not even as regards their existence), and thus lose the above representation for the function $h(z)$ corresponding to H as a Poisson integral in $\{|z| < r\}$. In order to have the latter, some additional information about U is necessary, its superharmonicity in a *larger* disk, for instance (this in turn implied by harmonicity of H in such a disk).

Regarding the measure μ appearing in either version of Riesz' result one has the important

Theorem. *In the representation*

$$U(z) \quad = \quad \int_{|\zeta - z_0| \leqslant r} \log\frac{1}{|z-\zeta|}\,\mathrm{d}\mu(\zeta) \quad + \quad H(z), \qquad |z - z_0| \; < \; r,$$

of a function $U(z)$ superharmonic in and on $|z - z_0| \leqslant r$ (with μ positive on that disk and $H(z)$ harmonic in its interior), the measure μ has no mass in any open subset of the disk where $U(z)$ is harmonic.

Proof. Let $|z - z_0| < r$ and suppose that $U(z')$ is harmonic in and on the *closed* disk $|z' - z| \leqslant \rho$, where $0 < \rho < r - |z - z_0|$. By the mean value property we then have

$$\frac{1}{2\pi}\int_0^{2\pi} U(z+\rho e^{i\psi})\,d\psi \quad = \quad U(z).$$

$H(z)$, however, has also the mean value property. Hence, using once again the formula

$$\frac{1}{2\pi}\int_0^{2\pi} \log\frac{1}{|z+\rho e^{i\psi}-\zeta|}\,d\psi \quad = \quad \min\left(\log\frac{1}{|z-\zeta|},\ \log\frac{1}{\rho}\right)$$

together with the given representation for U, we see that the left-hand integral in the preceding relation equals

$$\int_{|\zeta-z_0|\leqslant r} \min\left(\log\frac{1}{|z-\zeta|},\ \log\frac{1}{\rho}\right)d\mu(\zeta) \quad + \quad H(z).$$

Subtracting $U(z)$ from this, we get

$$\int_{|\zeta-z_0|\leqslant r} \log^+\frac{\rho}{|\zeta-z|}\,d\mu(\zeta) \quad = \quad 0.$$

Therefore $\mu(\{|\zeta-z| \ < \ \rho\}) \ = \ 0$, μ being positive. This does it.

Problem 48

(a) In the Riesz representation

$$U(z) \quad = \quad \int_{|\zeta-z_0|\leqslant r} \log\frac{1}{|z-\zeta|}\,d\mu(\zeta) \quad + \quad H(z), \qquad |z-z_0| \ < \ r,$$

of a function $U(z)$ superharmonic in and on $|\zeta-z_0| \leqslant r$ (with the measure μ positive and $H(z)$ harmonic for $|\zeta-z_0| < r$), the *restriction of μ to the open disk $|\zeta-z_0| < r$ is unique*. (Hint: If $F(z)$ is any continuous function supported on a *compact subset* of the open disk in question, we have

$$F(\zeta) \quad = \quad \lim_{\rho\to 0}\frac{2}{\pi\rho^2}\iint_{|z-z_0|<r} F(z)\log^+\frac{\rho}{|\zeta-z|}\,dx\,dy$$

uniformly for $|\zeta-z_0| \ < \ r$.)

(b) In the representation for U written in (a), the function $H(z)$ harmonic for $|z-z_0| < r$ is *not unique* – it can be *altered* by letting μ have *more mass on the circle* $|\zeta-z_0| = r$. *Show uniqueness* for the function $h(z)$, harmonic in $\{|z-z_0| < r\}$, figuring in the *original form* of the Riesz representation of U in that disk:

$$U(z) = \int_{|\zeta - z_0| \leq r} \log \left| \frac{r^2 - (z - z_0)\overline{(\zeta - z_0)}}{r(z - \zeta)} \right| d\mu(\zeta) + h(z).$$

(c) Let $U(z)$, superharmonic in a domain $\mathscr{D}$, have the Riesz representation

$$U(z) = \int_{|\zeta - z_0| \leq r_0} \log \frac{1}{|z - \zeta|} d\mu_0(\zeta) + H_0(z),$$

$$U(z) = \int_{|\zeta - z_1| \leq r_1} \log \frac{1}{|z - \zeta|} d\mu_1(\zeta) + H_1(z),$$

with $H_0(z)$ and $H_1(z)$ harmonic, in the respective disks $\{|z - z_0| < r_0\}$, $\{|z - z_1| < r_1\}$, whose *closures* lie in $\mathscr{D}$. Show that the positive measures μ_0 and μ_1 *agree* on the *intersection* of those *open disks* as long as it is non-empty. (Hint: The method followed in part (a) may be used.)

The last part of this problem gives us a procedure for extending the Riesz representation from disks to more general domains – the pasting argument referred to earlier.

Problem 49

Let $U(z)$ be superharmonic in a domain $\mathscr{D}$, and let Ω be any smaller domain with *compact closure* lying in $\mathscr{D}$. Corresponding to each open disk Δ whose closure lies in $\mathscr{D}$ we have, by the Riesz representation, a positive measure μ_Δ on $\bar{\Delta}$ and a function H_Δ harmonic in Δ such that

$$U(z) = \int_{\bar{\Delta}} \log \frac{1}{|z - \zeta|} d\mu_\Delta(\zeta) + H_\Delta(z)$$

for $z \in \Delta$.

(a) Show that there is a (finite) positive measure μ on $\bar{\Omega}$ agreeing in each intersection $\bar{\Omega} \cap \Delta$ with the corresponding measure μ_Δ. (Hint: Use a finite covering of $\bar{\Omega}$ by some of the disks Δ and then refer to the result from problem 48(c).)

(b) Hence show that

$$U(z) = \int_{\bar{\Omega}} \log \frac{1}{|z - \zeta|} d\mu(\zeta) + H(z)$$

for $z \in \Omega$, where $H(z)$ is harmonic in that domain and μ is the measure obtained in (a). (Hint: It suffices to show that for each $z_0 \in \Omega$,

$$U(z) - \int_{\Delta_0} \log \frac{1}{|z - \zeta|} d\mu(\zeta)$$

is harmonic in Δ_0, a disk contained in Ω with centre at z_0.)

3. A maximum principle for pure logarithmic potentials. Continuity of such a potential when its restriction to generating measure's support has that property

Consider a function $U(z)$ superharmonic in and on $\{|z| \leqslant 1\}$ and thus having a Riesz representation

$$U(z) \quad = \quad \int_{|\zeta| \leqslant 1} \log \frac{1}{|z-\zeta|}\, d\mu(\zeta) \quad + \quad H(z)$$

(with μ positive and $H(z)$ harmonic) in the *open* unit disk. If U is actually *harmonic* in an open subset $\mathcal{O}$ of the latter, μ is in fact supported on the compact set

$$K \quad = \quad \{|\zeta| \leqslant 1\} \sim \mathcal{O}$$

according to the last theorem of the preceding article.

One is frequently interested in the *continuity* of $U(z)$ for $|z| < 1$. Because $H(z)$ is even harmonic for such z, the property in question is governed by the continuity of

$$\int_K \log \frac{1}{|z-\zeta|}\, d\mu(\zeta)$$

there. An important result of Evans and Vasilesco given in the present article guarantees the continuity of such a logarithmic potential (everywhere!), provided that *its restriction to the support K of μ* enjoys that property. This enables one to *exclude from consideration the open set $\mathcal{O}$ in which U is known to be harmonic* when checking for that function's continuity in the open unit disk.

The result referred to is based on a *version of the maximum principle*, of considerable interest in its own right.

Maria's theorem. *Let the (finite) positive measure μ be supported on a compact set K, and suppose that*

$$V(z) \quad = \quad \int_K \log \frac{1}{|z-\zeta|}\, d\mu(\zeta).$$

Then, if $V(z) \leqslant M$ at each $z \in K$, one has $V(z) \leqslant M$ in $\mathbb{C}$.

Remark. $V(z)$ *is*, of course *harmonic* in $\Omega = \mathbb{C} \sim K$ (and tends to $-\infty$ as $z \longrightarrow \infty$, unless $\mu \equiv 0$), but the theorem *does not follow without further work* from the ordinary maximum principle for harmonic functions. For $\zeta \in \partial\Omega \subseteq K$, *all* that the *elementary properties* of superharmonic functions tell us *directly* is that

$$\liminf_{z\to\zeta} V(z) \quad = \quad V(\zeta) \quad \leqslant \quad M.$$

If we only had limsup *on the left instead of* liminf, there would be no problem, but that's not what stands there! Such pitfalls abound in this subject.

Proof of theorem. We need only consider the situation where $M < \infty$, since otherwise the result is trivial. In that event, the quantities

$$V_\rho(z) \quad = \quad \int_K \min\left(\log\frac{1}{|z-\zeta|},\ \log\frac{1}{\rho}\right) d\mu(\zeta)$$

increase, for *each* $z \in K$, to the *finite* limit $V(z)$ as $\rho \longrightarrow 0$. Given $\varepsilon > 0$, there is thus by *Egorov's theorem* a compact $E \subseteq K$ such that

$$V_\rho(z) \longrightarrow V(z) \quad \textit{uniformly} \text{ for } z \in E$$

as $\rho \longrightarrow 0$, and

$$\mu(K \sim E) \quad < \quad \varepsilon.$$

Since $|z-\zeta| \leqslant \operatorname{diam} K$ for z and ζ in K, the *second* condition makes

$$\int_E \log\frac{1}{|z-\zeta|}\, d\mu(\zeta) \quad \leqslant \quad \int_K \log\frac{1}{|z-\zeta|}\, d\mu(\zeta) \quad + \quad (\log\operatorname{diam} K)\,\mu(K\sim E)$$

$$\leqslant \quad V(z) \ + \ \varepsilon\log\operatorname{diam} K \quad \leqslant \quad M \ + \ \varepsilon\log\operatorname{diam} K$$

for $z \in K$, hence certainly for $z \in E$. By choosing $\varepsilon > 0$ small enough, we can ensure that the last quantity on the right, $M + \varepsilon \log\operatorname{diam} K$, *denoted henceforth by* M', is as close as we like to M.

At the same time, when $z \notin K$,

$$\int_{K\sim E} \log\frac{1}{|z-\zeta|}\, d\mu(\zeta)$$

lies between

$$\varepsilon \log \frac{1}{\operatorname{dist}(z,K) + \operatorname{diam} K} \qquad \text{and} \qquad \varepsilon\log\frac{1}{\operatorname{dist}(z,K)}.$$

For any such fixed z, then,

$$\int_E \log\frac{1}{|z-\zeta|}\, d\mu(\zeta)$$

will be arbitrarily close to $V(z)$ when $\varepsilon > 0$ is sufficiently small (depending on z). We see, $\varepsilon > 0$ being arbitrary, that we will have $V(z) \leqslant M$ at each

$z \notin K$ (thus proving the theorem) *if we can deduce that*

$$\int_E \log \frac{1}{|z-\zeta|} \, d\mu(\zeta) \quad \leqslant \quad M'$$

outside E knowing that this holds everywhere on E.

The last implication looks just like the one affirmed by the theorem, so it may seem as though nothing has been gained. We nevertheless have more of a toehold here on account of the *first* condition on our set E, according to which

$$\int_K \log^+ \frac{\rho}{|z-\zeta|} \, d\mu(\zeta) \quad = \quad \int_K \log \frac{1}{|z-\zeta|} \, d\mu(\zeta)$$
$$- \quad \int_K \min\left(\log \frac{1}{|z-\zeta|}, \; \log \frac{1}{\rho} \right) d\mu(\zeta)$$

tends to zero *uniformly for* $z \in E$ as $\rho \longrightarrow 0$. Thence, *a fortiori* (!),

$$\int_E \log^+ \frac{\rho}{|z-\zeta|} \, d\mu(\zeta) \quad \longrightarrow \quad 0 \quad \textit{uniformly for} \quad z \in E$$

as $\rho \longrightarrow 0$. This uniformity plays an essential rôle in the following argument.

It will be convenient to write

$$U(z) \quad = \quad \int_E \log \frac{1}{|z-\zeta|} \, d\mu(\zeta).$$

The proof of our theorem has boiled down to showing that if

$$U(z) \; \leqslant \; M' \quad \text{for} \quad z \in E,$$

then $U(z)$ is *also* $\leqslant \; M'$ at each $z \notin E$.

This is where we use the maximum principle for harmonic functions. In $\mathbb{C} \sim E$, $\quad U$ is harmonic; also,

$$U(z) \quad \longrightarrow \quad -\infty \quad \text{as} \quad z \longrightarrow \infty$$

unless $\mu(E) \; = \; 0$, in which case the desired conclusion is obviously true. The principle of maximum will therefore make $U(z) \; \leqslant \; M'$ in $\mathbb{C} \sim E$ provided that $\limsup_{z \to z_0} U(z) \; \leqslant \; M'$ for each $z_0 \in E$ (cf. second corollary to the second lemma in article 1).

Take any $\delta > 0$; we wish to show that at each $z_0 \; \in \; E$,

$$U(z) \; < \; M' \; + \; 7\delta$$

for the points z in a neighborhood of z_0. Thanks to the uniformity arrived at in the preceding construction, we can *fix* a $\rho > 0$ such that

$$\int_E \log^+ \frac{\rho}{|z-\zeta|}\, d\mu(\zeta) \quad < \quad \delta$$

whenever $z \in E$. With such a ρ, which we can also take to be < 1, we have

$$U(z) \quad = \quad \int_E \min\left(\log\frac{1}{|z-\zeta|},\ \log\frac{1}{\rho}\right) d\mu(\zeta) \quad + \quad \int_E \log^+ \frac{\rho}{|z-\zeta|}\, d\mu(\zeta).$$

The first integral on the right is $\leqslant U(z)$ and hence $\leqslant M'$ for $z \in E$; it is, moreover, *continuous* in z. *That integral is therefore* $< M' + \delta$ *whenever* z *is sufficiently close to any* $z_0 \in E$; our task thus reduces to verifying that

$$\int_E \log^+ \frac{\rho}{|z-\zeta|}\, d\mu(\zeta) \quad < \quad 6\delta$$

for z close enough to such a z_0. The last relation holds in fact at *all* points z, as we now proceed to show with the help of an ingenious device used in Carleson's little book. The latter has the advantage of being applicable when the logarithmic potential kernel $\log(1/|z-\zeta|)$ is replaced by fairly general ones of the form $k(|z-\zeta|)$, and it can be used in $\mathbb{R}^n$ for $n > 2$ as well as in $\mathbb{R}^2$.

Fix any z. If $z \in E$, the integral in question is even $< \delta$ by choice of ρ, so we may suppose that $z \notin E$. Then, using z as vertex, we partition the complex plane into six sectors, each of $60°$ opening, and denote by $E_1, E_2, \ldots, E_6$ the respective intersections of E with those sectors (so as to have $E_1 \cup E_2 \cup \cdots \cup E_6 = E$).

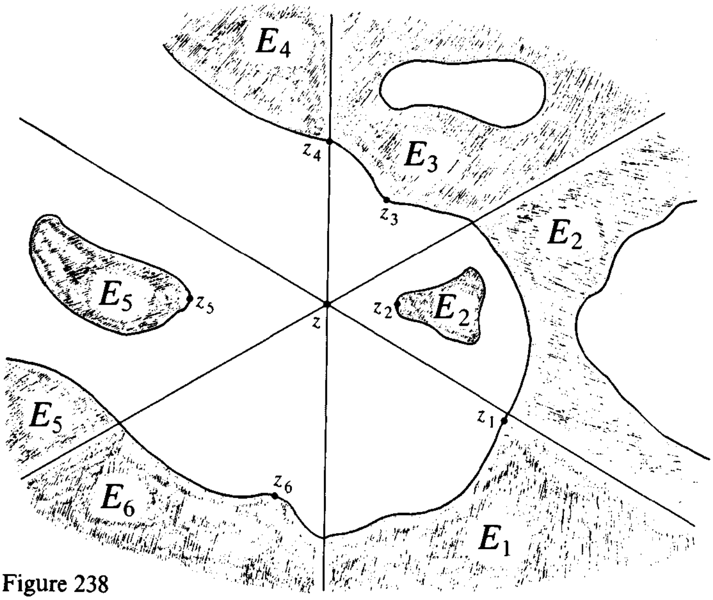

Figure 238

In each non-empty closure $\bar{E}_k$, $k = 1, 2, \ldots, 6$, pick a point z_k for which

$$|z_k - z| \quad = \quad \operatorname{dist}(z,\ E_k).$$

We have

$$\int_E \log^+ \frac{\rho}{|z-\zeta|}\, d\mu(\zeta) \quad \leqslant \quad \sum_{k=1}^{6} \int_{E_k} \log^+ \frac{\rho}{|z-\zeta|}\, d\mu(\zeta)$$

(with $\leqslant$ here and not $=$, because the E_k may intersect along the edges of the sectors*). However, for each k,

$$\int_{E_k} \log^+ \frac{\rho}{|z-\zeta|}\, d\mu(\zeta) \quad \leqslant \quad \int_{E_k} \log^+ \frac{\rho}{|z_k-\zeta|}\, d\mu(\zeta),$$

since

$$|z-\zeta| \ \geqslant \ |z_k - \zeta| \quad \text{when } \zeta \in E_k,$$

as one sees from the following diagram, drawn for $k = 6$:

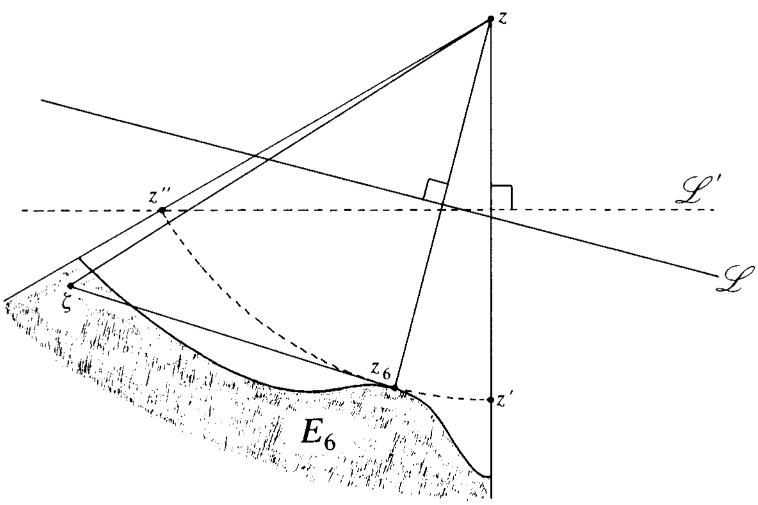

Figure 239

Here, $\mathscr{L}$ is the perpendicular bisector of the segment $[z,\ z_6]$ and $\mathscr{L}'$ that of $[z,\ z']$. Any point ζ in E_6 lies *on the same side* of $\mathscr{L}$ as z_6 and *on the opposite side thereof* from z, so the last inequality must hold. (By imagining z_6 to coincide with z' – in which case $\mathscr{L}$, coinciding with $\mathscr{L}'$, would pass

* if, for instance, we work with *closed* 60° sectors (which we may just as well do), in which case the sets E_k are *already* closed.