through z'' – and then allowing ζ to approach z'', we see that the 60° opening of the sector $\widehat{z''zz'}$ *cannot be made larger* if the inequality in question is to hold for all $\zeta \in E_6$.)

Again, for $1 \leqslant k \leqslant 6$,

$$\int_{E_k} \log^+ \frac{\rho}{|z_k - \zeta|}\, d\mu(\zeta) \quad \leqslant \quad \int_E \log^+ \frac{\rho}{|z_k - \zeta|}\, d\mu(\zeta).$$

But the right-hand integral is $< \delta$ by choice of ρ since $z_k \in \bar{E}_k \subseteq E$! Thence, going back to the previous relations, we find that

$$\int_E \log^+ \frac{\rho}{|z - \zeta|}\, d\mu(\zeta) \quad < \quad 6\delta$$

as we set out to show.

As explained above, this implies that $U(z) < M' + 7\delta$ in a suitably small neighborhood of any $z_0 \in E$ and thus finally, that $U(z) \leqslant M'$ in $\mathbb{C} \sim E$, after squeezing δ. From that, however, our result follows as we saw at the beginning of this proof. We are done.

Problem 50

With K a *compact* subset of the open (*sic*!) unit disk Δ and μ a positive measure supported on K, put

$$V(z) \quad = \quad \int_K \log\left|\frac{1-\bar{\zeta}z}{z-\zeta}\right| d\mu(\zeta), \qquad |z| \leqslant 1.$$

Suppose that V is *finite* at each point of K. Show then that if $W(z)$ is superharmonic and $\geqslant 0$ in Δ, and satisfies

$$W(z) \;\geqslant\; V(z)$$

for $z \in K$, we have $V(z) \leqslant W(z)$ in Δ.

Remark. The finiteness of V at the points of μ's support cannot be dispensed with here. Consider, for example,

$$V(z) \quad = \quad \log\frac{1}{|z|}$$

and

$$W(z) \quad = \quad \frac{1}{2}\log\frac{1}{|z|} \quad !$$

(Hint: Argue first as in the above proof to get, for any given $\varepsilon > 0$, a compact subset E of K with

$$\mu(K \sim E) \quad < \quad \varepsilon$$

and

$$\int_E \log^+ \frac{\rho}{|z-\zeta|}\, d\mu(\zeta) \longrightarrow 0$$

uniformly for $z \in E$ as $\rho \longrightarrow 0$.

Put

$$U(z) \quad = \quad \int_E \log\left|\frac{1-z\bar{\zeta}}{z-\zeta}\right| d\mu(\zeta);$$

here, $U(z) \leqslant V(z)$, so in particular $U(z) \leqslant W(z)$ on $E \subseteq K$. For any fixed $z \in \Delta \sim K$, $V(z) - U(z)$ is *small* if ε is, so it is enough to show that $U(z) \leqslant W(z)$ at each $z \in \Delta \sim E$.

The difference $W(z) - U(z)$ is superharmonic in $\Delta \sim E$; the last relation therefore holds (by a corollary from article 1) provided that

$$\liminf_{z \to z_0} (W(z) - U(z)) \quad \geqslant \quad 0$$

for each $z_0 \in \partial(\Delta \sim E)$.

When $|z_0| = 1$, this is manifest, W being $\geqslant 0$ in Δ with (here) $V(z)$ and $U(z)$ *continuous* and *zero* at z_0. It is hence only necessary to look at the behaviour near points $z_0 \in E$.

Fix any such z_0, and take any $\delta > 0$. Reasoning as in the above proof, show that

$$U(z) \quad < \quad U(z_0) \; + \; 7\delta$$

in a sufficiently small neighborhood of z_0. Since $W(z_0) \geqslant U(z_0)$, we therefore have

$$W(z) - U(z) \quad > \quad -8\delta$$

in such a neighborhood.)

We come now to the result about continuity spoken of at the beginning of this article.

Theorem (due independently to Evans and to Vasilesco). *Given a positive measure μ supported on a compact set K, put*

$$V(z) \quad = \quad \int_K \log \frac{1}{|z-\zeta|}\, d\mu(\zeta).$$

If the restriction of V to K *is continuous at a point* $z_0 \in K$, $V(z)$ *(as a function defined in* $\mathbb{C}$*) is continuous at* z_0.

Proof. Given $\varepsilon > 0$, there is an $\eta > 0$ (which we *fix*) such that

$$|V(z) - V(z_0)| \leqslant \varepsilon \quad \text{for } z \in K \text{ with } |z - z_0| \leqslant \eta.$$

Consider, on the *compact* set

$$K_\eta = K \cap \{|z - z_0| \leqslant \eta\}$$

the *continuous* functions

$$F_\rho(z) = \min\left\{V(z_0) - 2\varepsilon, \int_K \min\left(\log\frac{1}{|z-\zeta|}, \log\frac{1}{\rho}\right) d\mu(\zeta)\right\},$$

defined for each $\rho > 0$. When ρ *diminishes* towards 0, $F_\rho(z)$ *increases* for each fixed z, tending, moreover, to $\min(V(z_0) - 2\varepsilon, V(z))$, equal to the *constant* $V(z_0) - 2\varepsilon$ for $z \in K_\eta$. According to *Dini's theorem*, the convergence must then be *uniform* on K_η, so, for all sufficiently small $\rho > 0$, we have

$$\int_K \min\left(\log\frac{1}{|z-\zeta|}, \log\frac{1}{\rho}\right) d\mu(\zeta) > V(z_0) - 3\varepsilon, \qquad z \in K_\eta.$$

The integral on the left is, however,

$$\leqslant \int_K \log\frac{1}{|z-\zeta|}\, d\mu(\zeta),$$

which is in turn $\leqslant V(z_0) + \varepsilon$ for $z \in K_\eta$; subtraction thus yields

$$\int_K \log^+\frac{\rho}{|z-\zeta|}\, d\mu(\zeta) < 4\varepsilon, \qquad z \in K_\eta,$$

for $\rho > 0$ sufficiently small.

Fix any such $\rho < \eta/2$. We desire to use Maria's theorem so as to take advantage of the relation just found, but the appearance of $\log^+$ in the integrand instead of the logarithm gives rise to a slight difficulty.

Taking a new parameter λ with $1 < \lambda < 2$, we bring in the set

$$K_{\lambda\rho} = K \cap \{|z - z_0| \leqslant \lambda\rho\}.$$

Since $\lambda\rho < 2\rho < \eta$, we have $K_{\lambda\rho} \subseteq K_\eta$, so surely

$$\int_K \log^+\frac{\rho}{|z-\zeta|}\, d\mu(\zeta) < 4\varepsilon$$

for $z \in K_{\lambda\rho}$, whence, *a fortiori*,

$$\int_{K_{\lambda\rho}} \log\frac{\rho}{|z-\zeta|}\, d\mu(\zeta) \leqslant \int_{K_{\lambda\rho}} \log^+\frac{\rho}{|z-\zeta|}\, d\mu(\zeta) < 4\varepsilon$$

when z is in $K_{\lambda\rho}$.

Thence, applying Maria's result to the integral

$$\int_{K_{\lambda\rho}} \log \frac{\rho}{|z-\zeta|}\, d\mu(\zeta)$$

(which differs by but an additive constant from

$$\left. \int_{K_{\lambda\rho}} \log \frac{1}{|z-\zeta|}\, d\mu(\zeta) \right),$$

we see that it is in fact $\leqslant 4\varepsilon$ for *all* z. From this we will now deduce that

$$\int_{K_{\lambda\rho}} \log^+ \frac{\rho}{|z-\zeta|}\, d\mu(\zeta) \quad < \quad 5\varepsilon$$

(with $\log^+$ again and not log!) whenever z is sufficiently close to z_0, provided that $\lambda > 1$ is taken near enough to 1.

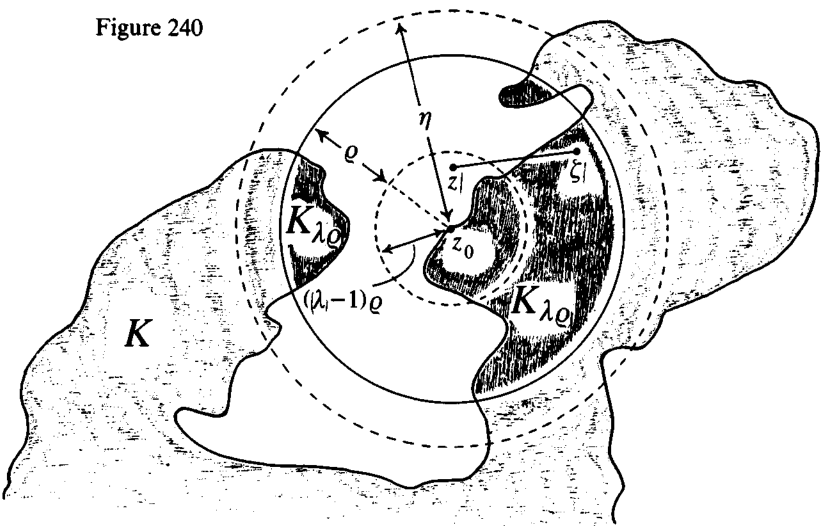
We have

$$\log^+ \frac{\rho}{|z-\zeta|} \quad = \quad \log \frac{\rho}{|z-\zeta|} \quad + \quad \log^- \frac{\rho}{|z-\zeta|}.$$

Here, when $\zeta \in K_{\lambda\rho}$ and $|z - z_0| \leqslant (\lambda - 1)\rho$, we are assured that $|z-\zeta| \leqslant (2\lambda - 1)\rho$, making

$$\log^- \frac{\rho}{|z-\zeta|} \quad \leqslant \quad \log(2\lambda - 1).$$

Figure 240



Therefore, for $|z-z_0| \leqslant (\lambda-1)\rho$,

$$\int_{K_{\lambda\rho}} \log^+ \frac{\rho}{|z-\zeta|}\,d\mu(\zeta) \;\leqslant\; \int_{K_{\lambda\rho}} \log \frac{\rho}{|z-\zeta|}\,d\mu(\zeta) \;+\; \mu(K_{\lambda\rho})\log(2\lambda-1).$$

By choosing (and then fixing) $\lambda > 1$ *close enough to* 1, we ensure that the second term on the right is $<\ \varepsilon$; since, then, the first is $\leqslant\ 4\varepsilon$ as we have seen, we get

$$\int_{K_{\lambda\rho}} \log^+ \frac{\rho}{|z-\zeta|}\,d\mu(\zeta) \;<\; 5\varepsilon \qquad \text{for } |z-z_0| \;\leqslant\; (\lambda-1)\rho.$$

Now, when $|z-z_0| \leqslant (\lambda-1)\rho$ and $\zeta \in K \sim K_{\lambda\rho}$, making $|\zeta-z_0| > \lambda\rho$, we have (see the preceding picture)

$$|z-\zeta| \;>\; \rho,$$

so

$$\log^+ \frac{\rho}{|z-\zeta|} \;=\; 0.$$

For $|z-z_0| \leqslant (\lambda-1)\rho$, the integral in the last relation is thus equal to

$$\int_K \log^+ \frac{\rho}{|z-\zeta|}\,d\mu(\zeta),$$

which is hence $<\ 5\varepsilon$ then!

Let us return to $V(z)$, which can be expressed as

$$\int_K \min\left(\log\frac{1}{|z-\zeta|},\ \log\frac{1}{\rho}\right)d\mu(\zeta) \;+\; \int_K \log^+ \frac{\rho}{|z-\zeta|}\,d\mu(\zeta).$$

When z is close enough to z_0 the *second* term is $<\ 5\varepsilon$ as we have just shown; the *first* term, however, is *continuous* in z, and hence tends to

$$\int_K \min\left(\log\frac{1}{|z_0-\zeta|},\ \log\frac{1}{\rho}\right)d\mu(\zeta) \;\leqslant\; V(z_0)$$

as $z \longrightarrow z_0$. Therefore,

$$V(z) \;<\; V(z_0) \;+\; 6\varepsilon$$

for z sufficiently close to z_0.

At the same time, V is superharmonic, so by property (i) (!),

$$\liminf_{z\to z_0} V(z) \;\geqslant\; V(z_0).$$

Thus,

$$V(z) \longrightarrow V(z_0) \quad \text{as } z \longrightarrow z_0$$

since $\varepsilon > 0$ was arbitrary; the function V is thus continuous at z_0.

Q.E.D.

Corollary. *Let $U(z)$ be superharmonic in the unit disk, Δ, and* harmonic *in the open subset Ω thereof. If $z_0 \in \Delta \sim \Omega$ and the* restriction *of U to $\Delta \sim \Omega$ is continuous at z_0, $U(z)$ is continuous at z_0.*

Proof. Pick any r with $|z_0| < r < 1$; then, by the Riesz representation theorem from the preceding article,

$$U(z) \quad = \quad \int_{|\zeta| \leqslant r} \log \frac{1}{|z-\zeta|} \, d\mu(\zeta) \quad + \quad H(z)$$

for $|z| < r$, where $H(z)$ is harmonic for such z and μ is a positive measure. We know also from the *last* theorem of that article that

$$\mu(\Omega \cap \{|\zeta| < r\}) \quad = \quad 0;$$

taking, then, the compact set

$$K \quad = \quad (\{|\zeta| \leqslant r\} \cap \sim \Omega) \cup \{|\zeta| = r\},$$

we can write

$$U(z) \quad = \quad \int_K \log \frac{1}{|z-\zeta|} \, d\mu(\zeta) \quad + \quad H(z), \qquad |z| < r.$$

Here, since $|z_0| < r$, H is continuous at z_0, and the *restriction* of $U(z) - H(z)$ to K is also, by hypothesis. We thus arrive at the desired result by applying the theorem to

$$\int_K \log \frac{1}{|z-\zeta|} \, d\mu(\zeta).$$

Done.

Problem 51

Let μ be a positive measure supported on K, a compact subset of the open unit disk, and suppose that

$$V(z) \quad = \quad \int_K \log \left| \frac{1 - \bar{\zeta} z}{z - \zeta} \right| d\mu(\zeta)$$

is finite at each point of K. Show that there is a sequence of positive measures μ_n supported on K for which:

(i) $d\mu_n(\zeta) \leqslant d\mu_{n+1}(\zeta) \leqslant d\mu(\zeta)$ for each n;

(ii) $\mu(K) - \mu_n(K) \underset{n}{\longrightarrow} 0$;

(iii) each of the functions

$$V_n(z) = \int_K \log\left|\frac{1-\bar{\zeta}z}{z-\zeta}\right| d\mu_n(\zeta)$$

is continuous on $\bar{\Delta}$;

(iv) $V_n(z) \underset{n}{\longrightarrow} V(z)$ for each $z \in \bar{\Delta}$.

(Hint: Start by arguing as in the proof of Maria's theorem, getting compact subsets K_n of K with $\mu(K \sim K_n) < 1/n$, on each of which the convergence of

$$\int_K \min\left(\log\left|\frac{1-\bar{\zeta}z}{z-\zeta}\right|,\ \log\frac{|1-\bar{\zeta}z|}{\rho}\right) d\mu(\zeta)$$

to $V(z)$ for ρ tending to zero is *uniform*. This makes the restriction of V to each K_n continuous thereon.

Arranging matters so as to have $K_n \subseteq K_{n+1}$ for each n, define μ_n by putting $\mu_n(E) = \mu(E \cap K_n)$ for $E \subseteq K$. Each of the differences $\sigma_n = \mu - \mu_n$ is also a positive measure on K.

We have (with V_n as in (iii)),

$$V_n(z) = V(z) - \int_K \log\left|\frac{1-\bar{\zeta}z}{z-\zeta}\right| d\sigma_n(\zeta),$$

where the integral on the right (*without* the $-$ sign) is superharmonic in Δ. Hence, since V, restricted to any of the K_n, is continuous thereon, we see that

$$\limsup_{\substack{z \to z_0 \\ z \in K_n}} V_n(z) \leqslant V_n(z_0) \qquad \text{for } z_0 \in K_n.$$

On the other hand,

$$\liminf_{z \to z_0} V_n(z) \geqslant V_n(z_0).$$

The restriction of V_n to K_n, the support of μ_n, is thus continuous. Now apply the preceding theorem.

Observe finally that

$$d\mu_n(\zeta) = \chi_{K_n}(\zeta)\, d\mu(\zeta)$$

with, for each fixed $z_0 \in \bar{\Delta}$,

$$\log\left|\frac{1-\bar{\zeta}z_0}{z_0-\zeta}\right| \chi_{K_n}(\zeta) \longrightarrow \log\left|\frac{1-\bar{\zeta}z_0}{z_0-\zeta}\right| \qquad \text{a.e. } (\mu)$$

on K as $n \longrightarrow \infty$. This makes $V_n(z_0) \xrightarrow[n]{} V(z_0)$ by monotone convergence.)

B. Relation of the existence of multipliers to the finiteness of a superharmonic majorant

1. Discussion of a certain regularity condition on weights

We return to the question formulated somewhat loosely at the beginning of §A in the last chapter, which was to find the conditions a weight $W(x) \geqslant 1$ must fulfill beyond the necessary one that

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2}\,dx \quad < \quad \infty,$$

in order to ensure the existence of entire functions $\varphi(z) \not\equiv 0$ of arbitrarily small exponential type making $W(x)\varphi(x)$ bounded (for instance) on $\mathbb{R}$. These must be conditions pertaining to the regularity of $W(x)$. Although an explicit minimal description of the needed regularity is not available as I write this, it seems likely that two separate requirements are involved.

One, not particularly bound up with the matter now under discussion, would serve to rule out the purely local idiosyncrasies in W's behaviour that could spoil existence of the above mentioned functions φ making $W(x)\varphi(x)$ bounded on $\mathbb{R}$ when such φ with, for example,

$$\int_{-\infty}^{\infty} |W(x)\varphi(x)|^p\,dx \quad < \quad \infty \qquad \text{(for some } p > 0)$$

were forthcoming. A very simple illustration helps to clarify this idea.

Consider any weight $W_0(x) \geqslant 1$ for which a non-zero entire function φ_0 of exponential type $A < \pi$ with $W_0(x)|\varphi_0(x)| \leqslant 1$ on $\mathbb{R}$ is known to exist.

Unless $W_0(x)$ is already bounded (a case without interest for us here!), $\varphi_0(z)$ must differ from a pure exponential and hence have infinitely many zeros (coming from its Hadamard factorization). Dividing out any two of those then gives us a new non-zero entire function φ, also of exponential type A, for which $W_0(x)|\varphi(x)| \leqslant \text{const.}/(1+x^2)$, $x \in \mathbb{R}$, so that

$$\int_{-\infty}^{\infty} W_0(x)|\varphi(x)|\,dx \quad < \quad \infty.$$

Taking the new weight

$$W(x) \quad = \quad |\sin \pi x|^{-1/2}\, W_0(x)$$

which becomes infinite at each integer, we still have

$$\int_{-\infty}^{\infty} W(x)|\varphi(x)|\,dx \quad < \quad \infty.$$

There is, however, no longer any entire $\psi(z) \not\equiv 0$ of exponential type $< \pi$ for which $W(x)\psi(x)$ is bounded on $\mathbb{R}$. Indeed, such a function ψ would have to be bounded on $\mathbb{R}$ and hence satisfy $|\psi(z)| \leqslant \text{const.}\exp(B|\Im z|)$, with $B < \pi$, by the third Phragmén–Lindelöf theorem in §C of Chapter III. In the present circumstances, ψ would also have to vanish at each integer, but then the usual application of Jensen's formula would show that it must vanish identically. Starting with a weight W_0 for which entire $\varphi_0 \not\equiv 0$ of *arbitrarily small exponential type* $A > 0$ with $\varphi_0(x)W_0(x)$ bounded on $\mathbb{R}$ are available, we thus find ourselves in a situation where – adopting the language of §A, Chapter X – the related weight $W(x)$ *admits multipliers in* $L_1(\mathbb{R})$ *but not in* $L_\infty(\mathbb{R})$.

By such rather artificial and almost trivial constructions one obtains various weights W from the original W_0 that admit multipliers in some spaces $L_p(\mathbb{R})$ but not in others. This seems to have nothing to do with the real reason (whatever it may be) for W_0 to have admitted multipliers (in $L_\infty(\mathbb{R})$) to begin with. That must also be the reason why the weights W admit multipliers in certain of the $L_p(\mathbb{R})$, and thus probably involves *some property of behaviour common to* $W_0(x)$ *and all of the* $W(x)$, independent of the special irregularities introduced in passing from the former to the latter. If this is so, it is natural to think of that behaviour property as *the essential one* governing admittance of multipliers, and the *second regularity condition* for weights would be *that they possess it*. By the *first regularity condition*, weights like $|\sin \pi x|^{-1/2} W_0(x)$ would be ruled out.

From this point of view, a search for the presumed essential second condition appears to be of primary importance. In order to be unhindered in that search, one is motivated to start by imposing on the weights W some imperfect version of the first condition, stronger than needed*, rather than seeking to express the latter in minimal form. That is how we will proceed here.

Such a version of the first condition should be both simple and sufficiently general. One, given in Beurling and Malliavin's 1962 paper, is very mild but rather elaborate. Discussion of it is postponed to the

* even at the cost of then arriving at a less than fully general version of the second condition

scholium at the end of this article. The following simpler variant seems adequate for most purposes; it is easy to work with and still applicable to a broad class of weights.

Regularity requirement. *There are three strictly positive constants, L, C and α such that, for each $x \in \mathbb{R}$, one has a real interval J_x of length L containing x with*

$$W(t) \quad \geqslant \quad C(W(x))^\alpha \qquad \text{for } t \in J_x.$$

(Unless $W(x)$ is bounded – a case without interest for us here – the parameter α figuring in the condition must obviously be $\leqslant 1$.) *Much of the work in the present chapter will be limited to the weights W that meet this requirement.**

What our condition does is impose a weak kind of *uniform semicontinuity* on $\log^+ \log W(x)$. It implies, for instance, a certain boundedness property on finite intervals.

Lemma. *A weight $W(x)$ meeting the regularity requirement is either identically infinite on some interval of length L or else bounded above on every finite interval.*

Proof. Suppose that $-M \leqslant x_n \leqslant M$ and $W(x_n) \xrightarrow[n]{} \infty$. Wlog, let

$$x_n \xrightarrow[n]{} x_0.$$

To each x_n is associated an interval J_{x_n} of length L containing it, on which

$$W(x) \quad \geqslant \quad C(W(x_n))^\alpha.$$

For infinitely many values of n, J_{x_n} must extend *to the same side of* x_n (either to the right or to the left) by a distance $\geqslant L/2$. Assuming, wlog, that we have infinitely many such intervals extending by that amount *to the right* of the corresponding points x_n, we see that

$$x_0' \quad = \quad x_0 + \frac{L}{4}$$

lies in *infinitely many of them.* The preceding relation therefore makes $W(x_0') = \infty$. Then, however, $W(x) = \infty$ for the x belonging to the interval $J_{x_0'}$ of length L.

* Regarding its partial elimination, see Remark 5 near the end of §E.2.

Here are some of the ways in which weights fulfilling the regularity requirement arise.

Lemma. *If* $\Omega(t) \geqslant 0$, *the average*

$$W(x) \quad = \quad \frac{1}{2L}\int_{-L}^{L} \Omega(x+t)\,\mathrm{d}t$$

satisfies the requirement with parameters L, $C = 1/2$ *and* $\alpha = 1$.

Proof. Given any x, we have

$$\frac{1}{2L}\int_J \Omega(t)\,\mathrm{d}t \quad \geqslant \quad \frac{1}{2}W(x)$$

for an interval J equal to *one* of the two segments $[x-L,\ x]$, $[x,\ x+L]$. Taking that interval J as J_x, we then have

$$W(\xi) \quad = \quad \frac{1}{2L}\int_{-L}^{L} \Omega(\xi+t)\,\mathrm{d}t \quad \geqslant \quad \frac{1}{2L}\int_{J_x} \Omega(s)\,\mathrm{d}s \quad \geqslant \quad \frac{1}{2}W(x)$$

for each $\xi \in J_x$.

In like manner, one verifies:

Lemma. *If* $\Omega(t) \geqslant 0$ *and* $p > 0$ *(sic!)*,

$$W(x) \quad = \quad \left(\frac{1}{2L}\int_{-L}^{L} (\Omega(x+t))^p\,\mathrm{d}t\right)^{1/p}$$

satisfies the requirement with parameters L, $C = 2^{-1/p}$ *and* $\alpha = 1$.

Lemma. *If* $\Omega(t) \geqslant 1$, *the weight*

$$W(x) \quad = \quad \exp\left\{\frac{1}{2L}\int_{-L}^{L} \log\Omega(x+t)\,\mathrm{d}t\right\}$$

satisfies the requirement with parameters L, $C = 1$ *and* $\alpha = 1/2$.

Weights meeting the requirement are also obtained by use of the Poisson kernel:

Lemma. *Let* $\Omega(t) \geqslant 1$ *be such that*

$$\int_{-\infty}^{\infty} \frac{\log\Omega(t)}{1+t^2}\,\mathrm{d}t \quad < \quad \infty.$$

Then, for fixed $y > 0$, the weight

$$W(x) \quad = \quad \exp\left\{\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y\log\Omega(t)}{(x-t)^2+y^2}\mathrm{d}t\right\}$$

fulfills the requirement with parameters L, $C = 1$ and $\alpha \;=\; \mathrm{e}^{-L/2y}$.

Proof. Since $\log\Omega(t) \;\geqslant\; 0$, we have (Harnack!)

$$\left|\frac{\mathrm{d}\log W(x)}{\mathrm{d}x}\right| \quad\leqslant\quad \frac{1}{y}\log W(x),$$

so that

$$\log W(\xi) \quad\geqslant\quad (\log W(x))\mathrm{e}^{-|\xi-x|/y}.$$

Take $J_x \;=\; [x - L/2,\; x + L/2]$.

A weight meeting the regularity requirement and also admitting multipliers has a $\mathscr{C}_\infty$ *majorant* with the same properties.

Theorem. *Let $W(x) \;\geqslant\; 1$ fulfill the requirement with parameters L, C, and α, and suppose that*

$$\int_{-\infty}^{\infty}\frac{\log W(t)}{1+t^2}\mathrm{d}t \quad<\quad \infty.$$

There is then an infinitely differentiable weight $W_1(x) \;\geqslant\; W(x)$ also meeting the requirement such that, corresponding to any entire function $\varphi(z) \;\not\equiv\; 0$ of exponential type $\leqslant\; A$ making $W(x)|\varphi(x)| \;\leqslant\; 1$ on $\mathbb{R}$, one has an entire $\psi(z) \;\not\equiv\; 0$ of exponential type $\leqslant\; mA$ with $W_1(x)|\psi(x)| \;\leqslant$ const., $x \in \mathbb{R}$. Here, for m we can take any integer $\geqslant\; 4/\alpha$.

Remark. As we know, the integral condition on $\log W$ follows from the existence of just *one* entire function φ having the properties in question.

Proof of theorem. Any entire function φ satisfying the conditions of the hypothesis must in particular have modulus $\leqslant\; 1$ on the real axis, so, by the second theorem of §G.2, Chapter III,

$$\log|\varphi(z)| \quad\leqslant\quad A\Im z \quad+\quad \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z\log|\varphi(t)|}{|z-t|^2}\mathrm{d}t$$

for $\Im z \;>\; 0$. Adding to both sides the finite quantity

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z\log W(t)}{|z-t|^2}\mathrm{d}t$$

we see, remembering the given relation

$$\log|\varphi(t)| \;+\; \log W(t) \;\leqslant\; 0, \qquad t \in \mathbb{R},$$

that

$$\log|\varphi(z)| \;+\; \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z \log W(t)}{|z-t|^2}\,\mathrm{d}t \;\leqslant\; A\Im z, \qquad \Im z \;>\; 0.$$

Put now $z = x + \mathrm{i}L$, and use the fact that

$$\log W(t) \;\geqslant\; \alpha \log W(x) \;+\; \log C$$

for t belonging to an interval of length L containing the point x. Since $\log W(t) \geqslant 0$, the integral on the left comes out

$$\geqslant\; \frac{1}{4}(\alpha \log W(x) \;+\; \log C),$$

and we find that

$$\frac{4}{\alpha}\log|\varphi(x+\mathrm{i}L)| \;+\; \log W_1(x) \;\leqslant\; \text{const.}, \qquad x \in \mathbb{R},$$

where

$$W_1(x) \;=\; C^{-1/\alpha}\exp\left\{\frac{4}{\pi\alpha}\int_{-\infty}^{\infty}\frac{L\log W(t)}{(x-t)^2 + L^2}\,\mathrm{d}t\right\}$$

is certainly $\geqslant W(x)$. This function is, on the other hand, infinitely differentiable, and it satisfies the regularity requirement by the last lemma.

At the same time,

$$W_1(x)|\varphi(x+\mathrm{i}L)|^{4/\alpha} \;\leqslant\; \text{const.}, \qquad x \in \mathbb{R}.$$

Because φ is bounded on the real axis, we know by the third Phragmén–Lindelöf theorem of §C, Chapter III that $\varphi(x + \mathrm{i}L)$ is also bounded for $x \in \mathbb{R}$. Hence, taking any integer $m \geqslant 4/\alpha$, we have

$$W_1(x)|\psi(x)| \;\leqslant\; \text{const.}, \qquad x \in \mathbb{R},$$

with the entire function

$$\psi(z) \;=\; (\varphi(z + \mathrm{i}L))^m,$$

obviously of exponential type $\leqslant mA$.

Done.

The elementary result just proved permits us to restrict our attention to *infinitely differentiable weights* when searching for the form of the 'essential' second condition that those meeting the regularity requirement must satisfy in order to admit multipliers. This observation will play a rôle in the last two §§ of the present chapter. But the main service rendered by the requirement is to make the property of admitting multipliers reduce to a more general one, easier to work with, for weights fulfilling it.

In order to explain what is meant by this, let us first consider the situation where an entire function $\varphi(z) \not\equiv 0$ of exponential type $\leqslant A$ with $W(x)|\varphi(x)| \leqslant$ const. on $\mathbb{R}$ is *known to exist*. If the weight $W(x)$ is *even*, some details of the following discussion may be skipped, making it *shorter* (although not really *easier*). One can in fact stick to just even weights (and even functions $\varphi(z)$) and still *get by* – see the remark following the last theorem in this article – and the reader is invited to make this simplification if he or she wants to. We treat the general case here in order to show that such investigations do not become *that much harder* when *evenness* is *abandoned*.

Assume that $W(x) \geqslant 1$ is either *continous*, or fulfills the *regularity requirement* (of course, one property does not imply the other). Then, since $\varphi(z) \not\equiv 0$, $W(x)$ cannot be identically infinite on any interval of length > 0. By the first of the above lemmas, this means that $W(x)$ is *bounded on finite intervals* under the second assumption. The same is of course true in the event of the first assumption.

The function $W(x)$ is, in particular, bounded near the origin, so if $\varphi(z)$ has a zero there – of order k, say – the product $W(x)\varphi(x)/x^k$ will still be bounded on $\mathbb{R}$. We can, in other words, assume wlog that $\varphi(0) \neq 0$, and hence that φ has a Hadamard factorization of the form

$$\varphi(z) \quad = \quad Ce^{\gamma z}\prod_{\lambda}\left(1-\frac{z}{\lambda}\right)e^{z/\lambda}.$$

Following a procedure already familiar to us, we construct from the product on the right a new entire function $\psi(z)$ *having only real zeros* (cf. §H.3 of Chapter III and the first half of the proof of the second Beurling–Malliavin theorem, §B.3, Chapter X).

Denote by Λ the set of zeros λ figuring in the above product with $\Re\lambda \neq 0$. For each $\lambda \in \Lambda$ we put

$$\frac{1}{\lambda'} \quad = \quad \Re\left(\frac{1}{\lambda}\right);$$

this gives us real numbers λ' with $|\lambda'| \geqslant |\lambda|$. (It is understood here that each λ' is to be taken with a multiplicity equal to the number of times that the corresponding $\lambda \in \Lambda$ figures as a zero of φ.) The *number* $N(r)$ of points λ' having modulus $\leqslant r$ (counting multiplicities) is thus *at most* equal to the *total numbers of zeros with such modulus that φ has* (again counting multiplicities). The latter quantity has, however, *asymptotic behaviour for large r governed by Levinson's theorem* (§H.3, Chapter III), because $\varphi(x)$ must be *bounded* on $\mathbb{R}$, $W(x)$ being $\geqslant 1$. In this way we see that that quantity is $\sim 2A'r/\pi$ for $r \longrightarrow \infty$, where A' is some positive number $\leqslant$ the *type* of φ, and hence $\leqslant A$.

We therefore have

$$\frac{N(r)}{r} \leqslant \frac{2A'}{\pi} + o(1)$$

for large r.

This being so, the product

$$e^{z\Re\gamma}\prod_{\lambda\in\Lambda}\left(1-\frac{z}{\lambda'}\right)e^{z/\lambda'}$$

(with each factor repeated according to the multiplicity of the corresponding λ') is *convergent* (see §A, Chapter III) and hence equal to some entire function $\psi(z)$. We know from §B of Chapter III, however, that the preceding relation involving $N(r)$ is insufficient to ensure ψ's being of exponential type. In order to show that, we resort to an indirect argument (cf. §H.3, Chapter III).

What the condition on $N(r)$ does give is the estimate $\log|\psi(z)| \leqslant O(|z|\log|z|)$, valid for large $|z|$ (§B, Chapter III); we thus have

$$\log|\psi(z)| \leqslant O(|z|^{1+\varepsilon})$$

(with arbitrary $\varepsilon > 0$) for z with large modulus. At the same time, $\psi(x)$ is bounded on the real axis. Indeed, for $\lambda\in\Lambda$,

$$\left|\left(1-\frac{x}{\lambda}\right)e^{x/\lambda}\right| \geqslant \left|1-\frac{x}{\lambda'}\right|e^{x/\lambda'}, \qquad x\in\mathbb{R},$$

whereas, for any *purely imaginary zero* λ of φ,

$$\left|\left(1-\frac{x}{\lambda}\right)e^{x/\lambda}\right| = \sqrt{\left(1+\frac{x^2}{|\lambda|^2}\right)} \geqslant 1, \qquad x\in\mathbb{R}.$$

Comparison of the above product equal to $\psi(z)$ with the Hadamard representation for φ thus shows at once that $|C\psi(x)| \leqslant |\varphi(x)|$ for $x\in\mathbb{R}$, yielding

$$|\psi(x)| \leqslant \text{const.}, \qquad x\in\mathbb{R},$$

since such a relation holds for $\varphi(x)$.

On the *imaginary axis*, the above estimate on $\log|\psi(z)|$ can be improved. We have:

$$\log|\psi(\mathrm{i}y)| \;=\; \frac{1}{2}\sum_{\lambda\in\Lambda}\log\left(1+\frac{y^2}{(\lambda')^2}\right) \;=\; \frac{1}{2}\int_0^\infty \log\left(1+\frac{y^2}{r^2}\right)\mathrm{d}N(r)$$

$$=\; |y|\int_0^\infty \frac{|y|}{t^2+y^2}\,\frac{N(r)}{r}\,\mathrm{d}r$$

(note that $N(r) \;=\; 0$ for $r>0$ close to zero). Plugging the above inequality for $N(r)$ into the last integral, we see immediately that

$$\limsup_{y\to\pm\infty}\frac{\log|\psi(\mathrm{i}y)|}{|y|} \;\leqslant\; A'.$$

Use this relation together with the two previous estimates on ψ to make a Phragmén–Lindelöf argument in each of the quadrants I, II, III and IV. One finds as in §H.3 of Chapter III that

$$|\psi(z)| \;\leqslant\; \text{const.}\,\mathrm{e}^{A'|\Im z|}.$$

Thus, since $A'\leqslant A$, $\psi(z)$ is *of exponential type* $\leqslant A$ (as our original function φ was).

This argument has been given at length because it will be used again later on. Then we will simply *refer* to it, omitting the details.

Let us return to our weight $W(x)$. Since, as we have seen, $|\psi(x)| \;\leqslant\; \text{const.}\,|\varphi(x)|$ on $\mathbb{R}$, it is true that

$$W(x)|\psi(x)| \;\leqslant\; \text{const.}, \qquad x\in\mathbb{R}.$$

Knowing, then, of the existence of *any* entire function $\varphi(z) \;\not\equiv\; 0$ having exponential type $\leqslant A$ and satisfying this relation, *we can construct a new one, $\psi(z)$, with only real zeros, that also satisfies it.* Moreover, as the above work shows, we can get such a ψ with $\psi(0) \;=\; 1$.

We now rewrite the last relation using a Stieltjes integral. As in §B of Chapter X, it is convenient to introduce an increasing function $n(t)$, equal, for $t>0$, to the *number of zeros λ' of ψ* (counting multiplicities) in $[0,\; t]$, and, for $t<0$, to the *negative of the number of such λ'* in $[t,\; 0)$. This function $n(t)$ (N.B. *it should not be confounded with $N(r)$* !) is *integer-valued* and, since $\psi(0) \;=\; 1$, *identically zero in a neighborhood of the origin.* Application of the Levinson theorem from §H.2 of Chapter III to the entire function $\psi(z)$ shows that the limits of $n(t)/t$ for $t\longrightarrow\pm\infty$ exist, both being equal to a number $\leqslant A/\pi$. Thus,

$$\frac{n(t)}{t} \;\leqslant\; \frac{A}{\pi} \;+\; \mathrm{o}(1) \qquad \text{for } t\longrightarrow\pm\infty.$$

Writing γ instead of $\Re\gamma$, the product representation for $\psi(z)$ can be put in the form

$$\log|\psi(z)| \;=\; \gamma\Re z \;+\; \int_{-\infty}^{\infty}\left(\log\left|1 - \frac{z}{t}\right| + \frac{\Re z}{t}\right)\mathrm{d}n(t).$$

The relation involving W and ψ can hence be expressed thus:

$$\gamma x + \int_{-\infty}^{\infty}\left(\log\left|1-\frac{x}{t}\right| + \frac{x}{t}\right)\mathrm{d}n(t) + \log W(x) \;\leqslant\; \text{const.}, \qquad x\in\mathbb{R}.$$

The existence of our original multiplier φ for W, of exponential type $\leqslant A$, has in this way enabled us to get an increasing integer-valued function $n(t)$ having the above properties and fulfilling the last relation.

If, on the other hand, one *has* an integer-valued increasing function $n(t)$ meeting these conditions, it is easy to construct an entire function ψ of exponential type $\leqslant A$ making $W(x)|\psi(x)| \leqslant$ const. on $\mathbb{R}$. All one need do is put

$$\psi(z) \;=\; \mathrm{e}^{\gamma z}\prod_{\lambda'}\left(1 - \frac{z}{\lambda'}\right)\mathrm{e}^{z/\lambda'}$$

with λ' running through the *discontinuities* of $n(t)$, each taken a number of times equal to the corresponding jump in $n(t)$. The boundedness of the product $W(x)\psi(x)$ then follows directly, and the Phragmén–Lindelöf argument used previously shows $\psi(z)$ to be of exponential type $\leqslant A$. *The existence of our multiplier* φ is, in other words, *equivalent to that of an increasing integer-valued function* $n(t)$ *satisfying the conditions just enumerated.*

Our regularity requirement is of course not *needed* for this equivalence, which holds for any weight bounded in a neighborhood of the origin. *What that requirement does is permit us,* when dealing with weights subject to it, *to drop* from the last statement *the condition that* $n(t)$ *be integer-valued.* The cost of this is that one ends with a multiplier φ of exponential type *several times larger than* A instead of one with type $\leqslant A$.

Some version of the lemma from §A.1 of Chapter X is needed for this reduction. If $W(x)$ were known to be *even* (with the increasing function involved odd!), the lemma could be used as it stands, and the proof of the next theorem made shorter (regarding this, the reader is again directed to the remark following the second of the next two theorems). The general situation requires a more elaborate form of that result. As in §B.2 of Chapter X, it is convenient to use $[p]$ to denote *the least integer* $\geqslant p$ *when* *p is negative, while maintaining the usual meaning of that symbol for $p \geqslant 0$.* The following variant of the lemma is then sufficient for our purposes:

Lemma. *Let $\nu(t)$ be increasing on $\mathbb{R}$, zero on $(-a, a)$, where $a > 0$, and $O(t)$ for $t \longrightarrow \pm\infty$. Then, for $\Im z \neq 0$, we have*

$$c\Re z \;+\; \int_{-\infty}^{\infty}\left(\log\left|1-\frac{z}{t}\right| + \frac{\Re z}{t}\right)(\mathrm{d}[\nu(t)] - \mathrm{d}\nu(t))$$

$$\leqslant\; \log^+\left|\frac{\Re z}{\Im z}\right| \;+\; \log\left|1 + \frac{|\Re z| + \mathrm{i}|\Im z|}{a}\right|,$$

c being a certain real constant depending on ν.

A proof of this estimate was already carried out for $a = 1$ and $\Im z = 1$ while establishing the *Little Multiplier Theorem* in §B.2 of Chapter X. The argument for the general case is not different from the one made there.*

We are now able to establish the promised reduction.

Theorem. *Let the weight $W(x) \geqslant 1$ meet our regularity requirement, with parameters L, C and α. Suppose there is an increasing function $\rho(t)$, zero on a neighborhood of the origin, with*

$$\frac{\rho(t)}{t} \;\leqslant\; \frac{A}{\pi} + o(1) \qquad \text{for } t \longrightarrow \pm\infty$$

and

$$\gamma x \;+\; \int_{-\infty}^{\infty}\left(\log\left|1-\frac{x}{t}\right| + \frac{x}{t}\right)\mathrm{d}\rho(t) \;+\; \log W(x) \;\leqslant\; \text{const.}$$

on the real axis, where γ is a real constant. Then there is a non-zero entire function $\psi(z)$ of exponential type $\leqslant 4A/\alpha$ with $W(x)\psi(x)$ bounded on $\mathbb{R}$.

Remark. The number 4 could be replaced by any other > 2 by refining one point in the following argument.

Proof of theorem. Put

$$U(z) \;=\; \gamma\Re z \;+\; \int_{-\infty}^{\infty}\left(\log\left|1-\frac{z}{t}\right| + \frac{\Re z}{t}\right)\mathrm{d}\rho(t);$$

our conditions on $\rho(t)$ make the right-hand integral have unambiguous

* By following the procedure indicated in the footnote on p. 186, one can, noting that $[\nu(t)] - \nu(t) \geqslant 0$ for $t < 0$, improve the upper bound furnished by the lemma to $\log|z/\Im z|$; this is *independent* of the size of the interval $(-a, a)$ on which $\nu(t)$ is known to vanish.

meaning for *all* complex z, taking, perhaps, the value $-\infty$ for some of these.* The *lack of evenness* of $W(x)$ and $U(z)$ will necessitate our attention to certain details.

$U(z)$ is *subharmonic* in the complex plane; it is, in other words, equal there to *the negative of a superharmonic function* having the properties taken up near the beginning of §A.1. According to the *first* of those we have in particular

$$\limsup_{z \to x_0} U(z) \leqslant U(x_0) \qquad \text{for } x_0 \in \mathbb{R}.$$

Our hypothesis, however, is that $U(x_0) + \log W(x_0) \leqslant K$, say, on $\mathbb{R}$, with $\log W(x_0) \geqslant 0$ there. Hence

$$\limsup_{z \to x_0} U(z) \leqslant K, \qquad x_0 \in \mathbb{R}.$$

Starting from this relation, one now *repeats* for $U(z)$ the Phragmén–Lindelöf argument made above for $\log|\psi(z)|$, using the properties of $\rho(t)$ in place of those of $N(r)$. In that way, it is found that

$$U(z) \leqslant K + A|\Im z|.$$

The function $U(z)$ is actually *harmonic*† for $\Im z > 0$, and we proceed to establish for it the Poisson representation

$$U(z) = A'\Im z + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z\, U(t)}{|z-t|^2}\,dt$$

in that half plane, with

$$A' = \limsup_{y \to \infty} \frac{U(iy)}{y} \leqslant A.$$

(This step could be avoided if $W(x)$ were known to be continuous; such continuity is, however, superfluous here.) Our *formula* for $U(z)$ shows $U(iy)$ to be $\geqslant 0$ for $y > 0$, so the quantity A' is certainly $\geqslant 0$. That it does not *exceed* A is guaranteed by the estimate on $U(z)$ just found. That estimate and the *fourth* theorem of §C, Chapter III, now show that in fact

$$U(z) \leqslant K + A'\Im z \qquad \text{for } \Im z > 0;$$

the function $U(z) - K - A'\Im z$ is thus *harmonic and* $\leqslant 0$ in the upper half plane.

* any such z must be *real* – $U(z)$ is *finite* for $\Im z \neq 0$

† and, in particular, *finite* (see preceding footnote) – the integral in the following Poisson representation is thus surely *convergent*.

By §F.1 of Chapter III we therefore have

$$U(z) \; - \; K \; - \; A'\Im z \quad = \quad -b\Im z \; - \; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \, \mathrm{d}\sigma(t)}{|z-t|^2}$$

for $\Im z \; > \; 0$, with a constant $b \geqslant 0$ and a certain *positive measure* σ on $\mathbb{R}$. It is readily verified that b must equal zero. Our desired Poisson representation for $U(z)$ will now follow from an argument like the one in §G.1 of Chapter III *if we verify absolute continuity of* σ.

For this purpose, it is enough to show that when $y \longrightarrow 0$,

$$\int_{-M}^{M} |U(x+\mathrm{i}y) - U(x)|\,\mathrm{d}x \quad \longrightarrow \quad 0$$

for each finite M. Given such an M, we can write

$$U(z) \quad = \quad \gamma\Re z \quad + \quad \left(\int_{-2M}^{2M} + \int_{|t|>2M}\right)\left(\log\left|1 - \frac{z}{t}\right| \; + \; \frac{\Re z}{t}\right)\mathrm{d}\rho(t).$$

The *second* of the two integrals involved here clearly tends *uniformly* to

$$\int_{|t|>2M}\left(\log\left|1 - \frac{x}{t}\right| \; + \; \frac{x}{t}\right)\mathrm{d}\rho(t)$$

as $z \; = \; x + \mathrm{i}y$ tends to x, when $-M \leqslant x \leqslant M$. Hence, since $\rho(t)$ is zero on a neighborhood of the origin, the matter at hand boils down to checking that

$$\int_{-M}^{M}\left|\int_{-2M}^{2M}\left(\log|x+\mathrm{i}y-t| \; - \; \log|x-t|\right)\mathrm{d}\rho(t)\right|\mathrm{d}x \quad \longrightarrow \quad 0$$

as $y \longrightarrow 0$. The inner integrand is already positive here, so the left-hand expression is just

$$\int_{-2M}^{2M}\int_{-M}^{M}\left(\log|x+\mathrm{i}y-t| \; - \; \log|x-t|\right)\mathrm{d}x\,\mathrm{d}\rho(t).$$

In this last, however, the inner integral is easily seen – by direct calculation, if need be – to tend to zero uniformly for $-2M \; \leqslant \; t \; \leqslant \; 2M$ as $y \longrightarrow 0$. (Incidentally, $\int_{-M}^{M} \log|w-x|\,\mathrm{d}x$ is the negative of a logarithmic potential generated by a *bounded* linear density on a finite segment, and therefore continuous everywhere in w.) The preceding relation therefore holds, so σ is absolutely continuous, giving us the desired Poisson representation for $U(z)$.

Once that representation is available, we have, for $\Im z > 0$,

$$U(z) \quad + \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log W(t)}{|z-t|^2}\,\mathrm{d}t$$
$$= \quad A'\Im z \quad + \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z\,(U(t)+\log W(t))}{|z \;-\; t|^2}\,\mathrm{d}t.$$

Since, however, $U(t) \;+\; \log W(t) \;\leqslant\; K$ on $\mathbb{R}$, the right side of this relation must be $\leqslant\; K + A'\Im z$, so we have

$$\gamma \Re z \quad + \quad \int_{-\infty}^{\infty}\left(\log\left|1-\frac{z}{t}\right| \;+\; \frac{\Re z}{t}\right)\mathrm{d}\rho(t)$$
$$+ \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log W(t)}{|z-t|^2}\,\mathrm{d}t \quad \leqslant \quad K \;+\; A'\Im z, \qquad \Im z \;>\; 0.$$

(Putting $z = \mathrm{i}$, we see by the way that $\int_{-\infty}^{\infty}(\log W(t)/(1+t^2))\,\mathrm{d}t \;<\; \infty$.)

By hypothesis, W meets our regularity requirement with parameters L, C, and α; this means that

$$\log W(t) \quad \geqslant \quad \alpha \log W(x) \;+\; \log C$$

for $t \in J_x$, an interval of length L containing x. Therefore, if

$$z \;=\; x + \mathrm{i}L,$$

the *second* integral on the left in the preceding relation is $\geqslant\; (\alpha/4)\log W(x) \;+\; (1/4)\log C$. After multiplying the latter through by $4/\alpha$ we thus find, recalling that $A' \leqslant A$,

$$\frac{4\gamma}{\alpha}x \quad + \quad \int_{-\infty}^{\infty}\left(\log\left|1-\frac{x+\mathrm{i}L}{t}\right| \;+\; \frac{x}{t}\right)\mathrm{d}(4\rho(t)/\alpha)$$
$$+ \quad \log W(x) \quad \leqslant \quad K', \qquad x \in \mathbb{R},$$

where

$$K' \quad = \quad \frac{4K + 4AL - \log C}{\alpha}.$$

It is at this point that we apply the last lemma, with

$$\nu(t) \quad = \quad \frac{4}{\alpha}\rho(t)$$

and $z \;=\; x + \mathrm{i}L$. If $\rho(t)$, and hence $\nu(t)$, vanishes on the neighborhood $(-a,\; a)$ of the origin, we see on combining that lemma with the preceding relation that

$$\beta x \quad + \quad \int_{-\infty}^{\infty}\left(\log\left|1-\frac{x+\mathrm{i}L}{t}\right| \;+\; \frac{x}{t}\right)\mathrm{d}[\nu(t)] \quad + \quad \log W(x)$$

$$\leqslant \quad K' \;+\; \log^+\left|\frac{x}{L}\right| \;+\; \log\left|1 \;+\; \frac{|x|+\mathrm{i}L}{a}\right|$$

on $\mathbb{R}$, with a certain constant β. From this we have, *a fortiori*,

$$\beta x \quad + \quad \int_{-\infty}^{\infty}\left(\log\left|1-\frac{x}{t}\right| \;+\; \frac{x}{t}\right)\mathrm{d}[\nu(t)] \quad + \quad \log W(x)$$

$$\leqslant \quad K'' \;+\; 2\log^+|x| \qquad \text{for } x\in\mathbb{R},$$

K'' being a new constant. The first two terms on the left add up, however, to $\log|\varphi(x)|$, where

$$\varphi(z) \quad = \quad \mathrm{e}^{\beta z}\prod_{\lambda}\left(1-\frac{z}{\lambda}\right)\mathrm{e}^{z/\lambda}$$

is the Hadamard product formed from the discontinuities λ of $[\nu(t)]$, each one taken with multiplicity equal to the height of the jump in that function corresponding to it. Since

$$\frac{\nu(t)}{t} \quad = \quad \frac{4\rho(t)}{\alpha t} \quad \leqslant \quad \frac{4A}{\alpha\pi} \;+\; \mathrm{o}(1)$$

for $t \longrightarrow \pm\infty$ (hypothesis!), that product is certainly convergent in the complex plane, and φ is an entire function. In terms of it, the previous relation can be rewritten as

$$W(x)|\varphi(x)| \quad \leqslant \quad \text{const.}\,(x^2+1), \qquad x\in\mathbb{R}.$$

It is now claimed that $\varphi(z)$ *must have infinitely many zeros* λ, *unless* $W(x)$ *is already bounded on* $\mathbb{R}$ (in which case our theorem is trivially true). Because those λ are the discontinuities of $[\nu(t)] \;=\; [4\rho(t)/\alpha]$, the presence of infinitely many of them is equivalent to the *unboundedness* of $\rho(t)$ (either above or below). It is thus enough to show that *if* $|\rho(t)|$ *is bounded*, $W(x)$ *is also bounded.*

We do this by proving that if $|\rho(t)|$ is *bounded, the function* $U(z)$ *used above must be equal to zero.* For real y, we have

$$U(\mathrm{i}y) \quad = \quad \frac{1}{2}\int_{-\infty}^{\infty}\log\left|1 \;+\; \frac{y^2}{t^2}\right|\mathrm{d}\rho(t) \quad = \quad \int_{-\infty}^{\infty}\frac{y^2}{y^2+t^2}\,\frac{\rho(t)}{t}\,\mathrm{d}t.$$

Here, $\rho(t)$ vanishes for $|t| \;<\; a$, so, if $|\rho(t)|$ is also *bounded*, the ratio $\rho(t)/t$

appearing in the last integral tends to *zero* for $t \longrightarrow \pm\infty$, besides being bounded on $\mathbb{R}$. That, however, makes

$$\int_{-\infty}^{\infty} \frac{y}{y^2+t^2}\,\frac{\rho(t)}{t}\,dt \quad\longrightarrow\quad 0 \qquad \text{for } y \longrightarrow \pm\infty,$$

as one readily sees on breaking up the integral into two appropriate pieces. We thus have

$$\frac{U(iy)}{|y|} \quad\longrightarrow\quad 0 \qquad \text{for } y \longrightarrow \pm\infty,$$

and *the quantity* A' figuring in the above examination of $U(z)$ *is equal to zero.* By the estimate obtained there, we must then have

$$U(z) \;\leqslant\; K$$

for $\Im z \geqslant 0$, and exactly the same reasoning (or the evident equality of $U(\bar{z})$ and $U(z)$) shows this to also hold for $\Im z \leqslant 0$. The subharmonic function $U(z)$ is, in other words, *bounded above in the complex plane if* $|\rho(t)|$ *is bounded.*

Such a subharmonic function is, however, necessarily constant. That is a general proposition, set below as problem 52. In the present circumstances, we can arrive at the same conclusion by a simple *ad hoc* argument. Since $\rho(t)/t \geqslant 0$, the previous formula for $U(iy)$ yields, for $y > 0$,

$$U(iy) \quad\geqslant\quad \int_{-y}^{y} \frac{y^2}{y^2+t^2}\,\frac{\rho(t)}{t}\,dt \quad\geqslant\quad \frac{1}{2}\int_{-y}^{y}\frac{\rho(t)}{t}\,dt.$$

If ever $\rho(t)$ is different from zero, there must be some k and y_0, both > 0, with either $\rho(t) \geqslant k$ for $y \geqslant y_0$ or $\rho(t) \leqslant -k$ for $y \leqslant -y_0$, and in both cases the last right-hand integral will be

$$\geqslant \quad \frac{k}{2}\log\frac{y}{y_0}$$

for $y \geqslant y_0$. This, however, would make $U(iy) \longrightarrow \infty$ for $y \longrightarrow \infty$, *contradicting the boundedness of* $U(z)$, so we must have $\rho(t) \equiv 0$. But then

$$\gamma x \quad=\quad U(x) \quad\leqslant\quad K \;-\; \log W(x), \qquad x \in \mathbb{R},$$

which contradicts our assumption that $W(x) \geqslant 1$ (either for $x \longrightarrow \infty$ or for $x \longrightarrow -\infty$) *unless* $\gamma = 0$. Finally, then, the boundedness of $\rho(t)$ forces $U(x)$ to *reduce to zero*, whence

$$\log W(x) \quad=\quad U(x) \;+\; \log W(x) \quad\leqslant\quad K, \qquad x \in \mathbb{R},$$

i.e., $W(x)$ *is bounded*, as we claimed.

Thus, except for the latter trivial situation, $|\rho(t)|$ is unbounded and the entire function $\varphi(z)$ has infinitely many zeros. Dividing it by the factors $1 - z/\lambda$ corresponding to *any two such zeros*, we obtain a *new entire function*, $\psi(z)$, such that*

$$W(x)|\psi(x)| \leqslant \text{const.}, \qquad x \in \mathbb{R}.$$

We now repeat the Phragmén–Lindelöf argument applied previously to another function $\psi(z)$ and then, in the course of the present proof, to $U(z)$. Since

$$\frac{[\nu(t)]}{t} \leqslant \frac{\nu(t)}{t} = \frac{4\rho(t)}{\alpha t} \leqslant \frac{4A}{\alpha\pi} + o(1)$$

for $t \longrightarrow \pm\infty$, we find in that way that

$$|\psi(z)| \leqslant \text{const.}\, e^{4A|\Im z|/\alpha};$$

ψ is thus of exponential type $\leqslant 4A/\alpha$. We have $\psi(0) = 1$, so $\psi(z) \not\equiv 0$. Referring to the previous relation involving W and ψ, we see that the theorem is proved.

Let us now settle on a definite meaning for the notion of admitting multipliers, hitherto understood somewhat loosely, and agree to henceforth employ that term *only when actual boundedness on* $\mathbb{R}$ *is involved.*

Definition. A weight $W(x) \geqslant 1$ will be said to admit multipliers if there are entire functions $\varphi(z) \not\equiv 0$ of arbitrarily small exponential type for which

$$W(x)|\varphi(x)| \leqslant \text{const.}, \qquad x \in \mathbb{R}.$$

Combining the last theorem with the conclusion of the discussion preceding it, we then have the

Corollary. *A weight $W(x) \geqslant 1$ fulfilling our regularity requirement admits multipliers iff, corresponding to any $A > 0$, there is an increasing function $\rho(t)$, zero on some neighborhood of the origin, with*

$$\frac{\rho(t)}{t} \leqslant \frac{A}{\pi} + o(1) \qquad \text{for } t \longrightarrow \pm\infty$$

* $W(x)$ must be *bounded* in the neighborhood of each of the two zeros of φ just removed. Otherwise W would be identically infinite on an interval of length L by the first lemma in this article, and then the Poisson integral of $U(t) \leqslant K - \log W(t)$ would diverge. That, however, cannot happen, as we have already remarked in a footnote near the beginning of this proof.

and at the same time

$$\gamma x \quad + \quad \int_{-\infty}^{\infty}\left(\log\left|1 - \frac{x}{t}\right| \; + \; \frac{x}{t}\right)\mathrm{d}\rho(t) \quad + \quad \log W(x) \quad \leqslant \quad \text{const.}$$

on $\mathbb{R}$ *for some real constant* γ.

In the case where $W(x)$ is equal to $|F(x)|$ for some *entire function* $F(z)$ of *exponential type*, the results just given hold *without any additional special assumption about the regularity of* W.

Theorem. *Let* $F(z)$ *be entire and of exponential type, with* $|F(x)| \geqslant 1$ *on* $\mathbb{R}$. *Suppose there is an increasing function* $\rho(t)$, *zero on a neighborhood of the origin, such that*

$$\frac{\rho(t)}{t} \quad \leqslant \quad \frac{A}{\pi} + \mathrm{o}(1) \qquad \text{for } t \longrightarrow \pm\infty$$

and

$$\gamma x \quad + \quad \int_{-\infty}^{\infty}\left(\log\left|1 - \frac{x}{t}\right| \; + \; \frac{x}{t}\right)\mathrm{d}\rho(t) \quad + \quad \log|F(x)| \quad \leqslant \quad \text{const.}$$

on $\mathbb{R}$ *for some real constant* γ. *Then there is an entire function* $\psi(z) \not\equiv 0$ *of exponential type* $\leqslant A$ *(sic!) with*

$$|F(x)\psi(x)| \quad \leqslant \quad \text{const.}, \qquad x \in \mathbb{R}.$$

Proof. Writing $|F(x)| = W(x)$, one starts out and proceeds as in the demonstration of the proceeding theorem, up to the point where the relation

$$U(z) \quad + \quad \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z \log W(t)}{|z-t|^2}\mathrm{d}t \quad \leqslant \quad K \; + \; A\Im z$$

is obtained for $\Im z > 0$, with

$$U(z) \quad = \quad \gamma\Re z \quad + \quad \int_{-\infty}^{\infty}\left(\log\left|1 - \frac{z}{t}\right| \; + \; \frac{\Re z}{t}\right)\mathrm{d}\rho(t).$$

From this one sees in particular* that

$$\int_{-\infty}^{\infty}\frac{\log|F(t)|}{1+t^2}\mathrm{d}t \quad = \quad \int_{-\infty}^{\infty}\frac{\log W(t)}{1+t^2}\mathrm{d}t \quad < \quad \infty,$$

which enables us to use some results from Chapter III.

* cf. footnotes near beginning of proof of the preceding theorem.

We can, in the first place, assume that *all the zeros of* $F(z)$ *lie in the lower half plane*, according to the *second* theorem of §G.3 in Chapter III. Then, however, by §G.1 of that chapter,

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log W(t)}{|z-t|^2}\,dt \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log|F(t)|}{|z-t|^2}\,dt$$

$$= \quad \log|F(z)| \; - \; B\Im z \qquad \text{for } \Im z \; > \; 0,$$

where

$$B \quad = \quad \limsup_{y\to\infty} \frac{\log|F(\mathrm{i}y)|}{y}.$$

Our previous relation involving U and W thus becomes

$$U(z) \; + \; \log|F(z)| \quad \leqslant \quad K \; + \; (A+B)\Im z, \qquad \Im z \; > \; 0.$$

In this we put $z \; = \; x+\mathrm{i}$, getting

$$\gamma x \quad + \quad \int_{-\infty}^{\infty}\left(\log\left|1-\frac{x+\mathrm{i}}{t}\right| \; + \; \frac{x}{t}\right)d\rho(t)$$

$$+ \quad \log|F(x+\mathrm{i})| \quad \leqslant \quad \text{const.}, \qquad x\in\mathbb{R}.$$

Apply now the lemma used in the proof of the last theorem, but this time with

$$\nu(t) \quad = \quad \rho(t).$$

In that way one sees that

$$\beta x \quad + \quad \int_{-\infty}^{\infty}\left(\log\left|1-\frac{x+\mathrm{i}}{t}\right| \; + \; \frac{x}{t}\right)d[\rho(t)]$$

$$+ \quad \log|F(x+\mathrm{i})| \quad \leqslant \quad 2\log^+|x| \; + \; O(1), \qquad x\in\mathbb{R},$$

with a new real constant β. There is as before a certain entire function φ with $\log|\varphi(x+\mathrm{i})|$ equal to the *sum of the first two terms on the left*, and we have

$$|F(x+\mathrm{i})\varphi(x+\mathrm{i})| \quad \leqslant \quad \text{const.}(x^2+1), \qquad x\in\mathbb{R}.$$

It now follows as previously that $\varphi(z)$ *has infinitely many zeros*, unless $|F(x)|$ is *itself* bounded, in which case there is nothing to prove. Dividing out from $\varphi(z)$ the linear factors corresponding to *two* of those zeros gives us an entire function $\psi(z) \; \not\equiv \; 0$ with

$$|F(x+\mathrm{i})\psi(x+\mathrm{i})| \quad \leqslant \quad \text{const.}, \qquad x\in\mathbb{R}.$$

Here, our initial assumption that $|F(x)| \geqslant 1$ on $\mathbb{R}$ and the Poisson representation for $\log|F(z)|$ in $\{\Im z > 0\}$ already used imply that

$$|F(x+\mathrm{i})| \quad \geqslant \quad \text{const.} \quad > \quad 0 \qquad \text{for } x \in \mathbb{R},$$

so by the preceding relation we have in particular

$$|\psi(x+\mathrm{i})| \quad \leqslant \quad \text{const.}, \qquad x \in \mathbb{R}.$$

By hypothesis, we also have

$$\frac{[\rho(t)]}{t} \quad \leqslant \quad \frac{\rho(t)}{t} \quad \leqslant \quad \frac{A}{\pi} \; + \; \mathrm{o}(1)$$

for $t \longrightarrow \pm\infty$, permitting us to use once again the Phragmén–Lindelöf argument made three times already in this article. In that way we see that

$$|\psi(z+\mathrm{i})| \quad \leqslant \quad \text{const.}\, \mathrm{e}^{A|\Im z|},$$

meaning that ψ is of exponential type $\leqslant A$. The product $F(z+\mathrm{i})\psi(z+\mathrm{i})$ is then also of exponential type. Since that product is by the above relation *bounded for real z*, we have by the third theorem of §C in Chapter III, that

$$|F(x)\psi(x)| \quad \leqslant \quad \text{const.} \qquad \text{for } x \in \mathbb{R}.$$

Our function ψ thus has all the properties claimed by the theorem, and we are done.

Remark. Suppose that we know of an increasing function $\rho(t)$, zero on a neighborhood of the origin, satisfying the conditions assumed for the above results with some number $A > 0$ and a weight $W(x) \geqslant 1$. For the increasing function $\mu(t) = \rho(t) - \rho(-t)$, also zero on a neighborhood of the origin, we then have

$$\frac{\mu(t)}{t} \quad \leqslant \quad \frac{2A}{\pi} \; + \; \mathrm{o}(1) \qquad \text{for } t \longrightarrow \infty,$$

as well as

$$\int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\mu(t) \quad + \quad \log\{W(x)W(-x)\} \quad \leqslant \quad \text{const.}$$

for $x \in \mathbb{R}$. In this relation, both terms appearing on the left are *even*; that enables us to simplify the argument made in proving the first of the preceding two theorems when applying it in the present situation.

If the weight $W(x)$ meets our regularity requirement* with parameters

* see also Remark 5 near the end of §E.2.

L, C, and α, we do have

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{L\log\{W(t)W(-t)\}}{(x-t)^2+L^2}\,dt \;\geqslant\; \frac{\alpha}{4}\log\{W(x)W(-x)\} + \frac{\log C}{2} \quad \text{for } x\in\mathbb{R};$$

this one sees by writing the logarithm figuring in the left-hand member as a sum and then dealing separately with the two integrals thus obtained. The behaviour of the even subharmonic function

$$V(z) = \int_0^{\infty} \log\left|1 - \frac{z^2}{t^2}\right| d\mu(t)$$

is easier to investigate than that of the function $U(z)$ used in the above proofs (cf. §B of Chapter III). When $V(x+iL)$ has made its appearance, one may apply directly the lemma from §A.1 of Chapter X instead of resorting to the latter's more complicated variant given above.

By proceeding in this manner, one obtains an *even* entire function $\Psi(z)$ with

$$W(x)W(-x)|\Psi(x)| \leqslant \text{const.}, \qquad x\in\mathbb{R},$$

and thus, since $W(-x) \geqslant 1$ (!),

$$W(x)|\Psi(x)| \leqslant \text{const.}, \qquad x\in\mathbb{R}.$$

The function $\Psi(z)$ is of exponential type, but here that type turns out to be bounded above by $8A/\alpha$ rather than by $4A/\alpha$ as we found for the function $\psi(z)$ obtained previously.

Insofar as W's *admitting of multipliers* is concerned, the *extra factor of two* is of no importance. The reader may therefore prefer this approach (involving a preliminary reduction to the even case) which bypasses some fussy details of the one followed above, but yields less precise estimates for the exponential types of the multipliers obtained. Anyway, according to the remark following the statement of the first of the above two theorems, the estimate $4A/\alpha$ on the type of $\psi(z)$ is not very precise.

Problem 52

Show that a function $V(z)$ *superharmonic* in the whole complex plane and bounded *below* there is constant. (Hint: Referring to the first theorem of §A.2, take the means $(\Phi_r V)(z)$ considered there. Assuming wlog that $V(z) \not\equiv \infty$, each of those means is also superharmonic and bounded below in $\mathbb{C}$, and it is enough to establish the result *for them*. The Φ, V are also $\mathscr{C}_\infty$, so we may as well assume to begin with that $V(z)$ is $\mathscr{C}_\infty$.

That reduction made, observe that if $V(z)$ is actually *harmonic* in $\mathbb{C}$, the desired result boils down to Liouville's theorem, so it suffices to *establish* this harmonicity. For that purpose, fix any z_0 and look at the means

$$V_r(z_0) \quad = \quad \frac{1}{2\pi}\int_0^{2\pi} V(z_0 + re^{i\vartheta})\,d\vartheta.$$

Consult the proof of the *second* lemma in §A.2, and then show that

$$\frac{\partial V_r(z_0)}{\partial \log r}$$

is a *decreasing* function of r, so that $V_r(z_0)$ *either remains constant for all* $r > 0$ – and hence equal to $V(z_0)$ – *or else tends to* $-\infty$ *as* $r \longrightarrow \infty$. In the second case, V could not be bounded below in $\mathbb{C}$. Apply Gauss' theorem from §A.1.)

Scholium. The regularity requirement for weights given in the 1962 paper of Beurling and Malliavin is much less stringent than the one we have been using. A relaxed version of the former can be stated thus:

There are four constants $C > 0$, $\alpha > 0$, $\beta < 1$ and $\gamma < 1$ such that, to each $x \in \mathbb{R}$ corresponds an interval I_x of length $e^{-|x|^\gamma}$ (sic!) containing x with

$$W(t) \quad \geqslant \quad C e^{-|x|^\beta}(W(x))^\alpha \qquad \text{for } t \in I_x.$$

The point we wish to make here is that the exponentials in $|x|^\gamma$ and $|x|^\beta$ are *in a sense* red herrings; a close analogue of the first of the above two theorems, *with practically the same proof*, is valid for weights meeting the more general condition. The only new ingredient needed is the elementary Paley–Wiener multiplier theorem.

Problem 53

Suppose that $W(x) \geqslant 1$ fulfills the condition just formulated, and that there is an increasing function $\rho(t)$, zero on a neighborhood of the origin, with

$$\frac{\rho(t)}{t} \quad \leqslant \quad \frac{A}{\pi} + o(1) \qquad \text{for } t \longrightarrow \pm\infty$$

and

$$cx \quad + \quad \int_{-\infty}^{\infty}\left(\log\left|1 - \frac{x}{t}\right| + \frac{x}{t}\right)d\rho(t) \quad + \quad \log W(x) \quad \leqslant \quad \text{const.}$$

on $\mathbb{R}$, where c is a certain real constant. Show that for any $\eta > 0$ there is an entire function $\psi(z) \not\equiv 0$ of exponential type $< 4A/\alpha + \eta$ making

$$W(x)|\psi(x)| \leqslant \text{const.}, \qquad x \in \mathbb{R}.$$

(Hint: Follow exactly the proof of the result referred to until arriving at the relation

$$U(z) + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log W(t)}{|z-t|^2}\,dt \leqslant K + A'\Im z, \qquad \Im z > 0.$$

In this, substitute $z = x + ie^{-|x|^\gamma}$ (!) and invoke the condition, finding, for that value of z,

$$\frac{4c}{\alpha}x + \int_{-\infty}^{\infty}\left(\log\left|1-\frac{z}{t}\right| + \frac{x}{t}\right)d(4\rho(t)/\alpha)$$
$$+ \log W(x) \leqslant K' + \frac{4}{\alpha}|x|^\beta$$

with a new constant K'. Using the lemma (with z as above!) and continuing as before, we get an entire function $\varphi(z) \not\equiv 0$ such that

$$\frac{\log|\varphi(iy)|}{|y|} \leqslant \frac{4A}{\alpha} + o(1) \qquad \text{for } y \longrightarrow \pm\infty,$$

$$\log|\varphi(z)| \leqslant O(|z|^{1+\varepsilon})$$

for large $|z|$ ($\varepsilon > 0$ being arbitrary), and finally

$$W(x)|\varphi(x)| \leqslant \text{const.}(x^2+1)\exp\left(|x|^\gamma + \frac{4}{\alpha}|x|^\beta\right)$$

on the real axis. To the right side of the last relation, apply the theorem from §A.1 of Chapter X (and §D of Chapter IV!).).

2. The smallest superharmonic majorant

According to the results from the latter part of the preceding article (beginning with the second theorem therein), a weight $W(x) \geqslant 1$ having any one of various regularity properties *admits multipliers* if and only if, corresponding to any $A > 0$, there exists an increasing function $\rho(t)$, zero on some neighborhood of the origin, such that

$$\frac{\rho(t)}{t} \leqslant \frac{A}{\pi} + o(1) \qquad \text{for } t \longrightarrow \pm\infty$$

and

$$\gamma x \;+\; \int_{-\infty}^{\infty}\left(\log\left|1-\frac{x}{t}\right| + \frac{x}{t}\right)\mathrm{d}\rho(t) \;+\; \log W(x) \;\leqslant\; K \quad \text{for } x\in\mathbb{R}$$

with some constant γ. Hence, in keeping with the line of thought embarked on at the beginning of article 1, we regard the (hypothetical) *second* ('essential') *condition* for admittance of multipliers by a weight W as being *very close* (if not identical) to whatever requirement it must satisfy in order to guarantee existence of such increasing functions ρ. That requirement, and attempts to arrive at precise knowledge of it, will therefore be our main object of interest during the remainder of this chapter.

Suppose that for a given weight $W(x) \geqslant 1$ we *have* such a function $\rho(t)$ corresponding to some $A > 0$. The relation

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|\Im z|\log W(t)}{|z-t|^2}\mathrm{d}t \;-\; A|\Im z|$$
$$\leqslant\; K \;-\; \gamma\Re z \;-\; \int_{-\infty}^{\infty}\left(\log\left|1-\frac{z}{t}\right| + \frac{\Re z}{t}\right)\mathrm{d}\rho(t)$$

(with the left side *interpreted* as $\log W(x)$ for $z = x\in\mathbb{R}$) then holds throughout the complex plane. For $\Im z > 0$, this has indeed already been verified while proving the second theorem of article 1 (near the beginning of the proof). That, however, is enough, since both sides are unchanged when z is replaced by $\bar{z}$.

Now the *right side* of the last relation is obviously a *superharmonic function* of z, *finite* for z off of the real axis. The *existence* of our function ρ thus leads (in almost trivial fashion) to that of a *superharmonic majorant* $\not\equiv \infty$ *for*

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|\Im z|\log W(t)}{|z-t|^2}\mathrm{d}t \;-\; A|\Im z|$$

(interpreted as $\log W(x)$ for $z = x\in\mathbb{R}$) *in the whole complex plane.* The key to the proof of the Beurling–Malliavin multiplier theorem given below in §C lies in the observation that *the converse of this statement is true*, at least for *continuous* weights $W(x)$. That fact (which, from a certain point of view, is nearly tautological) will be established in the next article. For this purpose and the later applications as well, we will need the *smallest superharmonic majorant* of a continuous function together with some of its properties, to whose examination we now proceed.

Let $F(z)$ be any function *real-valued and continuous* in the whole complex plane. (In our applications, we will use a function $F(z)$ equal to the preceding expression – interpreted as $\log W(x)$ for $z = x \in \mathbb{R}$ – where $W \geqslant 1$ is *continuous* and such that $\int_{-\infty}^{\infty} (\log W(x)/(1+x^2))\mathrm{d}x < \infty$.) We next take the *family* $\mathscr{F}$ of *functions superharmonic and* $\geqslant F$ (everywhere); our convention being to consider the function *identically equal to* $+\infty$ *as superharmonic* (see §A.1), $\mathscr{F}$ is *certainly not empty*. Then put

$$Q(z) \quad = \quad \inf\{U(z):\ U \in \mathscr{F}\}$$

for each complex z, and finally take

$$(\mathfrak{M}F)(z) \quad = \quad \liminf_{\zeta \to z} Q(\zeta);$$

$\mathfrak{M}F$ is the function we will be dealing with. (The reason for use of the symbol $\mathfrak{M}$ will appear in problems 55 and 56 below. $\mathfrak{M}F$ is a kind of *maximal function* for F.)

In our present circumstances, $Q(z)$ is $\geqslant$ the *continuous function* $F(z)$, so we must also have

$$(\mathfrak{M}F)(z) \quad \geqslant \quad F(z).$$

This certainly makes $(\mathfrak{M}F)(z) > -\infty$ everywhere, so $(\mathfrak{M}F)(z)$ is *itself superharmonic* (everywhere) by the last theorem of §A.1, *and must hence belong to* $\mathscr{F}$ in view of the relation just written. The same theorem also tells us, however, that $(\mathfrak{M}F)(z) \leqslant U(z)$ for *every* $U \in \mathscr{F}$; $\mathfrak{M}F$ is thus *a member of* $\mathscr{F}$ *and at the same time* $\leqslant$ *every member of* $\mathscr{F}$. $\mathfrak{M}F$ is, in other words, *the smallest superharmonic majorant of* F.

It may well happen, of course, that $(\mathfrak{M}F)(z) \equiv \infty$. However, if $\mathfrak{M}F$ is finite at just one point, it is finite everywhere. That is the meaning of the

Lemma. *If, for any* z_0, $(\mathfrak{M}F)(z_0) = \infty$, *we have* $(\mathfrak{M}F)(z) \equiv \infty$.

Proof. To simplify the writing, let us wlog consider the case where $z_0 = 0$. By continuity of F at 0, there is certainly some finite M such that

$$F(z) \leqslant M \quad \text{for } |z| \leqslant 1, \quad \text{say.}$$

Given, however, that $(\mathfrak{M}F)(0) = \infty$, there is an r, $0 < r < 1$, for which

$$(\mathfrak{M}F)(z) \geqslant M + 1, \qquad |z| \leqslant r,$$

because the superharmonic function $\mathfrak{M}F$ has property (i) at 0 (§A.1).

It is now claimed that

$$\int_{-\pi}^{\pi} (\mathfrak{M}F)(re^{i\vartheta})\,\mathrm{d}\vartheta \quad = \quad \infty.$$

Reasoning by contradiction, *assume that the integral on the left is finite.* Then, since $(\mathfrak{M}F)(re^{i\vartheta})$ is *bounded below* for $0 \leqslant \vartheta \leqslant 2\pi$ (*here*, simply because $\mathfrak{M}F \geqslant F$, but see also the beginning of §A.1), we must have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{r^2-\rho^2}{r^2+\rho^2-2r\rho\cos(\varphi-\tau)}(\mathfrak{M}F)(re^{i\tau})\,d\tau \quad < \quad \infty$$

for $0 \leqslant \rho < r$ and $0 \leqslant \varphi \leqslant 2\pi$. Take now the function $V(z)$ *equal*, for $|z| \geqslant r$ *to* $(\mathfrak{M}F)(z)$ and, for $z = \rho e^{i\varphi}$ with $0 \leqslant \rho < r$, *to the Poisson integral just written. This function* $V(z)$ *is superharmonic* (everywhere) by the *second theorem* of §A.1.

We have

$$V(\rho e^{i\varphi}) \geqslant M+1 \quad \text{for } 0 \leqslant \rho < r,$$

since $(\mathfrak{M}F)(re^{i\tau}) \geqslant M+1$. At the same time,

$$F(\rho e^{i\varphi}) \leqslant M \quad \text{for } 0 \leqslant \rho < r$$

because $r < 1$, so $V(z) \geqslant F(z)$ for $|z| < r$. This, however, is also true for $|z| \geqslant r$, where $V(z) = (\mathfrak{M}F)(z)$. We thus have in $V(z)$ a *superharmonic majorant of* $F(z)$, so

$$V(z) \geqslant (\mathfrak{M}F)(z).$$

Thence,

$$(\mathfrak{M}F)(0) \quad \leqslant \quad V(0) \quad = \quad \frac{1}{2\pi}\int_{-\pi}^{\pi}(\mathfrak{M}F)(re^{i\tau})\,d\tau \quad < \quad \infty.$$

But it was given that $(\mathfrak{M}F)(0) = \infty$. This contradiction shows that the *integral* in the last relation must be *infinite*, as claimed.

Apply now the *first* lemma of §A.2 to the function $(\mathfrak{M}F)(z)$, superharmonic everywhere. We find that

$$(\mathfrak{M}F)(z) \equiv \infty$$

for all z. The proof is complete.

Corollary. *The function* $(\mathfrak{M}F)(z)$ *is either finite everywhere or infinite everywhere.*

Henceforth, to indicate that the *first alternative* of the corollary holds, we will simply say *that* $\mathfrak{M}F$ *is finite.*

Lemma. *If* $\mathfrak{M}F$ *is finite and* $F(z)$ *is harmonic in any open set* $\mathcal{O}$, $(\mathfrak{M}F)(z)$ *is also harmonic in* $\mathcal{O}$.

Proof. Let $z_0 \in \mathcal{O}$ and take $r > 0$ so small that the *closed* disk of radius r about z_0 lies in $\mathcal{O}$; it suffices to show that $(\mathfrak{M}F)(z)$ is harmonic for $|z-z_0| < r$.

Supposing wlog that $z_0 = 0$, we take the superharmonic function $V(z)$ used in the proof of the preceding lemma. From the second theorem of §A.1, we have

$$V(z) \leqslant (\mathfrak{M}F)(z).$$

Here, however, we are assuming that $(\mathfrak{M}F)(z) < \infty$, so the Poisson integral

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{r^2-|z|^2}{|z-re^{i\tau}|^2}\,(\mathfrak{M}F)(re^{i\tau})\,d\tau,$$

equal, for $|z| < r$, *to* $V(z)$, must be *absolutely convergent* for such z, $(\mathfrak{M}F)(re^{i\tau})$ being *bounded below*, as we know. $V(z)$ is thus *harmonic* for $|z| < r$.

Let $|z| < r$. Then, since $\{|z| \leqslant r\} \subseteq \mathcal{O}$, where $F(z)$ is given to be harmonic,

$$F(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{r^2-|z|^2}{|z-re^{i\tau}|^2}\,F(re^{i\tau})\,d\tau,$$

and the integral on the right is $\leqslant$ the preceding one, $\mathfrak{M}F$ being a *majorant* of F. Thus,

$$F(z) \leqslant V(z) \quad \text{for } |z| < r.$$

This, however, also holds for $|z| \geqslant r$ where $V(z) = (\mathfrak{M}F)(z)$. We see as in the proof of the last lemma that $V(z)$ is a *superharmonic majorant* of $F(z)$. Hence

$$V(z) \geqslant (\mathfrak{M}F)(z).$$

But the reverse inequality was already noted above. Therefore,

$$V(z) = (\mathfrak{M}F)(z).$$

Since $V(z)$ is harmonic for $|z| < r$, we are done.

Let us now look at the set E on which

$$(\mathfrak{M}F)(z) = F(z)$$

for some given continuous function F. E may, of course, be *empty*; *it is, in any event, closed.* Suppose, indeed, that we have a sequence of points $z_k \in E$ and that $z_k \xrightarrow[k]{} z_0$. Then, since $\mathfrak{M}F$ enjoys property (i) (§A.1), we have

$$(\mathfrak{M}F)(z_0) \leq \liminf_{k\to\infty}(\mathfrak{M}F)(z_k) = \liminf_{k\to\infty} F(z_k) = F(z_0),$$

F being continuous at z_0. Because $\mathfrak{M}F$ is a majorant of F, we also have $(\mathfrak{M}F)(z_0) \geq F(z_0)$, and thus finally $(\mathfrak{M}F)(z_0) = F(z_0)$, making $z_0 \in E$.

This means that the set of z for which $(\mathfrak{M}F)(z) > F(z)$ is open. Regarding it, we have the important

Lemma. $(\mathfrak{M}F)(z)$, *if finite, is harmonic in the open set where it is $> F(z)$.*

Note. I became aware of this result while walking in Berkeley and thinking about a conversation I had just had with L. Dubins on the material of the present article, especially on the notions developed in problems 55 and 56 below. Dubins thus gave me considerable help with this work.

Proof of lemma. Is much like those of the two previous ones. Let us show that if $(\mathfrak{M}F)(z_0) > F(z_0)$ with $\mathfrak{M}F$ finite, then $(\mathfrak{M}F)(z)$ is harmonic in some small disk about z_0.

We can, wlog, take $z_0 = 0$; suppose, then, that

$$(\mathfrak{M}F)(0) > F(0) + 2\eta, \quad \text{say,}$$

where $\eta > 0$. Property (i) then gives us an $r > 0$ such that

$$(\mathfrak{M}F)(z) > F(0) + \eta$$

for $|z| \leq r$, and the continuity of F makes it possible for us to choose this r small enough so that we also have

$$F(z) < F(0) + \eta \quad \text{for } |z| \leq r.$$

Form now the superharmonic function $V(z)$ used in the proofs of the last two lemmas. As in the second of those, we certainly have

$$V(z) \leq (\mathfrak{M}F)(z),$$

according to our theorem from §A.1. In the present circumstances, for $|z| < r$,

$$V(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{r^2 - |z|^2}{|z - re^{i\tau}|^2}(\mathfrak{M}F)(re^{i\tau})d\tau$$

is $> F(0) + \eta$, whereas $F(z) < F(0) + \eta$ there; $V(z)$ is thus $\geq F(z)$ for $|z| < r$. When $|z| \geq r$, $V(z) = (\mathfrak{M}F)(z)$ is also $\geq F(z)$, so V is again a superharmonic majorant of F. Hence

$$V(z) \geq (\mathfrak{M}F)(z),$$

and we see finally that

$$V(z) = (\mathfrak{M}F)(z),$$

with the left side *harmonic* for $|z| < r$, just as in the proof of the preceding lemma. Done.

Lemma. *If $\mathfrak{M}F$ is finite, it is everywhere continuous.*

Proof. Depends on the Riesz representation for superharmonic functions.

Take the sets

$$E = \{z: (\mathfrak{M}F)(z) = F(z)\}$$

and

$$\mathcal{O} = \mathbb{C} \sim E;$$

as we have already observed, E is closed and $\mathcal{O}$ is open. By the preceding lemma, $(\mathfrak{M}F)(z)$ is *harmonic* in $\mathcal{O}$ and thus surely *continuous* therein. We therefore need only check continuity of $\mathfrak{M}F$ at the points of E.

Let, then, $z_0 \in E$ and consider any open disk Δ centered at z_0, say the one of radius $= 1$. In the open set $\Omega = \Delta \cap \mathcal{O}$ the function $(\mathfrak{M}F)(z)$ is *harmonic*, as just remarked and *on* $\Delta \sim \Omega = \Delta \cap E$, $(\mathfrak{M}F)(z) = F(z)$ *depends continuously on* z. The *restriction* of $\mathfrak{M}F$ to E is, in particular, *continuous* at the centre, z_0, of Δ.

The corollary to the *Evans-Vasilesco theorem* (at the end of §A.3) can now be invoked, thanks to the superharmonicity of $(\mathfrak{M}F)(z)$. After translating z_0 to the origin (and Δ to a disk about 0), we see by that result that $(\mathfrak{M}F)(z)$ is continuous at z_0. This does it.

Remark. These last two lemmas will enable us to use harmonic estimation to examine the function $(\mathfrak{M}F)(z)$ in §C.

It is a good idea at this point to exhibit two processes which generate $(\mathfrak{M}F)(z)$ when applied to a given continuous function F, although we will not make direct use of either in this book. These are described in problems 55 and 56. The first of those depends on

Problem 54

Let $U(z)$, defined and $> -\infty$ in a domain $\mathcal{D}$, satisfy

$$\liminf_{\zeta \to z} U(\zeta) \geqslant U(z)$$

for $z \in \mathcal{D}$. Show that $U(z)$ is then superharmonic in $\mathcal{D}$ iff, at each z therein, one has

$$\frac{1}{\pi r^2}\iint_{|\zeta - z| < r} U(\zeta)\mathrm{d}\xi\mathrm{d}\eta \quad \leqslant \quad U(z)$$

for all $r > 0$ sufficiently small. (As usual, $\zeta = \xi + \mathrm{i}\eta$.)
(Hint: For the if part, the first theorem of §A.1 may be used.)

Problem 55

For Lebesgue measurable functions $F(z)$ defined on $\mathbb{C}$ and bounded below on each compact set, put

$$(MF)(z) \quad = \quad \sup_{r>0} \frac{1}{\pi r^2}\iint_{|\zeta - z| < r} F(\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

Then, starting with any F *continuous* on $\mathbb{C}$, form successively the functions $F^{(0)}(z) = F(z)$, $F^{(1)}(z) = (MF)(z)$, $F^{(2)}(z) = (MF^{(1)})(z)$, and so forth.

(a) Show that $F^{(0)}(z) \leqslant F^{(1)}(z) \leqslant F^{(2)}(z) \leqslant \cdots$.

(b) Show that $\lim_{n\to\infty} F^{(n)}(z) \leqslant (\mathfrak{M}F)(z)$.

(c) Show that $\lim_{n\to\infty} F^{(n)}(z)$ is superharmonic.
(Hint: For this, use problem 54.)

(d) Hence show that $\lim_{n\to\infty} F^{(n)}(z) = (\mathfrak{M}F)(z)$.

Remark. The function $\mathfrak{M}F$ was originally brought into the study of multiplier theorems through this construction.

The next problem involves *Jensen measures* (on $\mathbb{C}$). That term is used here to denote the *positive Radon measures* μ *of compact support* such that

$$\int_{\mathbb{C}} U(\zeta)\mathrm{d}\mu(\zeta) \quad \leqslant \quad U(0)$$

for *each function* U *superharmonic on* $\mathbb{C}$. (Any such function $U(z)$ is certainly *Borel measurable*, for, by the *first* theorem of §A.2, it is the pointwise limit of an *increasing sequence* of $\mathscr{C}_\infty$ superharmonic functions.) Some simple Jensen measures are the ν_r given by

$$\mathrm{d}\nu_r(\zeta) \quad = \quad \begin{cases} \dfrac{1}{\pi r^2}\mathrm{d}\xi\,\mathrm{d}\eta, & |\zeta| < r, \\[2ex] 0, & |\zeta| \geqslant r \end{cases}$$

(refer to problem 54 !).

For reasons which will soon become apparent, *we denote the collection of Jensen measures by* $\mathfrak{M}$. If U is any function superharmonic on $\mathbb{C}$, so are its translates, so, whenever $\mu \in \mathfrak{M}$ and $z \in \mathbb{C}$,

$$\int_{\mathbb{C}} U(z+\zeta)\,\mathrm{d}\mu(\zeta) \quad \leqslant \quad U(z).$$

Problem 56

The purpose here is to show that if F is continuous on $\mathbb{C}$,

$$(\mathfrak{M}F)(z) \quad = \quad \sup_{\mu \in \mathfrak{M}} \int_{\mathbb{C}} F(z+\zeta)\,\mathrm{d}\mu(\zeta).$$

(a) Show that $\int_{\mathbb{C}} F(z+\zeta)\,\mathrm{d}\mu(\zeta) \;\leqslant\; (\mathfrak{M}F)(z)$ for each $\mu \in \mathfrak{M}$.

Denote now the set of Jensen measures *absolutely continuous with respect to two-dimensional Lebesgue measure by* $\mathfrak{L}$. As examples of some measures in $\mathfrak{L}$, we have, for instance, the ν_r described above. $\mathfrak{L}$ is of course a subset of $\mathfrak{M}$.

(b) Show that $\mathfrak{L}$ has a countable subset $\{\mu_k\}$, *dense therein with respect to* L_1 *convergence with bounded support*. This means that given any $\mu \in \mathfrak{L}$, with say $\mathrm{d}\mu(\zeta) = \varphi(\zeta)\mathrm{d}\xi\,\mathrm{d}\eta$, where $\varphi(\zeta) = 0$ a.e. for $|\zeta| \geqslant$ some integer N, we can find a *subsequence* $\{\mu_{k_j}\}$ of the μ_k such that, if we write $\mathrm{d}\mu_{k_j}(\zeta) = \varphi_{k_j}(\zeta)\mathrm{d}\xi\,\mathrm{d}\eta$, we also have $\varphi_{k_j}(\zeta) = 0$ a.e. for $|\zeta| \geqslant N$, and moreover

$$\iint_{|\zeta| \leqslant N} |\varphi(\zeta) - \varphi_{k_j}(\zeta)|\,\mathrm{d}\xi\,\mathrm{d}\eta \quad \longrightarrow \quad 0 \quad \text{as } j \longrightarrow \infty.$$

(Hint: For the open subsets of each of the spaces $L_1(|z| \leqslant N)$, $N = 1, 2, 3, \ldots$, there is a countable base, some of whose members contain densities belonging to measures from $\mathfrak{L}$. Select.)

(c) Taking the measures μ_k from (b), put

$$V_N(z) \quad = \quad \max_{1 \leqslant k \leqslant N} \int_{\mathbb{C}} F(z+\zeta)\,\mathrm{d}\mu_k(\zeta)$$

for our given continuous function F. Fix any $z \in \mathbb{C}$ and $R > 0$. Show that there is a $\nu \in \mathfrak{L}$ (depending, in general, on z, R and N) such that

$$\frac{1}{2\pi}\int_0^{2\pi} V_N(z + R\mathrm{e}^{\mathrm{i}\vartheta})\,\mathrm{d}\vartheta \quad = \quad \int_{\mathbb{C}} F(z+\zeta)\,\mathrm{d}\nu(\zeta).$$

(Hint: First show how to get a Borel function $k(\vartheta)$ taking the values $1, 2; 3, \ldots, N$ such that

$$V_N(z + R\mathrm{e}^{\mathrm{i}\vartheta}) \quad = \quad \int_{\mathbb{C}} F(z + R\mathrm{e}^{\mathrm{i}\vartheta} + \zeta)\,\mathrm{d}\mu_{k(\vartheta)}(\zeta).$$

Then define ν by the formula

$$\int_{\mathbb{C}} G(\zeta)\,\mathrm{d}\nu(\zeta) \quad = \quad \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{C}} G(\zeta + R\mathrm{e}^{\mathrm{i}\vartheta})\,\mathrm{d}\mu_{k(\vartheta)}(\zeta)\,\mathrm{d}\vartheta$$

and verify that it belongs to $\mathfrak{L}$.)

(d) Hence show that

$$V(z) \quad = \quad \sup_{\mu\in\mathfrak{L}} \int_{\mathbb{C}} F(z+\zeta)\mathrm{d}\mu(\zeta) \quad (sic!)$$

is superharmonic.
(Hint: Since F is continuous, we also have $V(z) \;=\; \sup_k \int_{\mathbb{C}} F(z+\zeta)\mathrm{d}\mu_k(\zeta)$ with the μ_k from (b). That is, $V(z) \;=\; \lim_{N\to\infty} V_N(z)$, where the V_N are the functions from (c). But by (c),

$$\frac{1}{2\pi}\int_0^{2\pi} V_N(z+R\mathrm{e}^{\mathrm{i}\vartheta})\mathrm{d}\vartheta \quad \leqslant \quad V(z)$$

for each N. Use monotone convergence.)

(e) Show that

$$\sup_{\mu\in\mathfrak{M}} \int_{\mathbb{C}} F(z+\zeta)\mathrm{d}\mu(\zeta) \quad = \quad (\mathfrak{M}F)(z).$$

(Hint: The left side is surely $\geqslant$ the function $V(z)$ from (d). Observe that $V(z) \;\geqslant\; F(z)$; for this the measures ν_r specified above may be used. This makes V a *superharmonic majorant* of F ! Refer to (a) and to the definition of $\mathfrak{M}F$.)

Remark. The last problem exhibits $\mathfrak{M}F$ as a *maximal function* formed from F by using the *Jensen measures.*

Each $\mu\in\mathfrak{M}$ acts as a *reproducing measure for functions harmonic on* $\mathbb{C}$. We have, in other words,

$$\int_{\mathbb{C}} H(z+\zeta)\mathrm{d}\mu(\zeta) \quad = \quad H(z), \qquad z\in\mathbb{C},$$

for every function H harmonic on $\mathbb{C}$ and every Jensen measure μ. It is important to realize that *not every positive measure μ of compact support having this reproducing property is a Jensen measure.* The following example was shown to me by T. Lyons:

Take

$$\mathrm{d}\mu(\zeta) \quad = \quad \varphi(\zeta)\mathrm{d}\xi\,\mathrm{d}\eta \;+\; \tfrac{1}{4}\mathrm{d}\delta_1(\zeta),$$

where δ_1 is the unit mass concentrated at the point 1 and

$$\varphi(\zeta) = \begin{cases} \dfrac{1}{4\pi}, & |\zeta| \leqslant 2 \text{ and } |\zeta - 1| \geqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

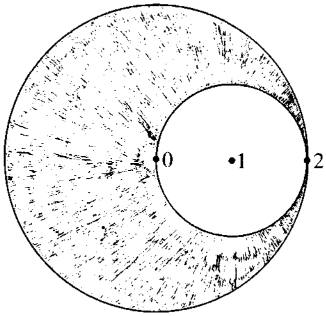


Figure 241

Then, since

$$\frac{1}{4}H(1) = \frac{1}{4\pi}\iint_{|\zeta-1|<1} H(\zeta)\mathrm{d}\xi\,\mathrm{d}\eta$$

for functions H harmonic on $\mathbb{C}$, we have

$$\int_{\mathbb{C}} H(\zeta)\mathrm{d}\mu(\zeta) = \frac{1}{4\pi}\iint_{|\zeta|\leqslant 2} H(\zeta)\mathrm{d}\xi\,\mathrm{d}\eta = H(0),$$

and similarly, by translation,

$$\int_{\mathbb{C}} H(z+\zeta)\mathrm{d}\mu(\zeta) = H(z)$$

for such functions H.

However, $U(z) = \log(1/|z-1|)$ is superharmonic in $\mathbb{C}$, and, with the present μ,

$$\int_{\mathbb{C}} U(\zeta)\mathrm{d}\mu(\zeta) = \infty \quad \text{although} \quad U(0) = 0.$$

This measure μ is therefore *not* in $\mathfrak{M}$.

The reader interested in a general treatment of the matters taken up in this article should consult a recent book by Gamelin (with *Jensen measures* in its title).

3. How $\mathfrak{M}F$ gives us a multiplier if it is finite

Starting now with a continuous* weight $W(x) \geqslant 1$ for which $\int_{-\infty}^{\infty}(\log W(t)/(1+t^2))\mathrm{d}t < \infty$, we choose and fix an $A > 0$ and form the function

$$F(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|\log W(t)}{|z-t|^2}\mathrm{d}t - A|\Im z|,$$

the expression on the right being interpreted as $\log W(x)$ when $z = x \in \mathbb{R}$. This function F is then continuous and the material of the preceding article applies to it; the smallest superharmonic majorant, $\mathfrak{M}F$, of F is thus at our disposal.

Our object in the present article is to establish a *converse* to the observation made near the beginning of the last one. This amounts to showing that if $\mathfrak{M}F$ is finite, one actually *has* an increasing function ρ, zero on a neighborhood of the origin, such that

$$\frac{\rho(t)}{t} \leqslant \frac{A}{\pi} + o(1) \qquad \text{for } t \longrightarrow \pm\infty$$

and that

$$\log W(x) + \gamma x + \int_{-\infty}^{\infty}\left(\log\left|1-\frac{x}{t}\right| + \frac{x}{t}\right)\mathrm{d}\rho(t) \leqslant \text{const.} \quad \text{for } x \in \mathbb{R}$$

with a certain constant γ. We will do that by deriving a *formula*,

$$(\mathfrak{M}F)(z) = (\mathfrak{M}F)(0) - \gamma\Re z - \int_{-\infty}^{\infty}\left(\log\left|1-\frac{z}{t}\right| + \frac{\Re z}{t}\right)\mathrm{d}\rho(t),$$

involving an increasing function ρ with (subject to an unimportant auxiliary condition on W) the first of the properties in question, and then by simply *using* the fact that $(\mathfrak{M}F)(x)$ is a *majorant* of $F(x) = \log W(x)$. That necessitates our making a preliminary examination of $\mathfrak{M}F$ for the present function F.

Lemma. *If, for $F(z)$ given by the above formula, $\mathfrak{M}F$ is finite, we have*

$$\int_{-\infty}^{\infty}\frac{(\mathfrak{M}F)(t)}{1+t^2}\mathrm{d}t < \infty,$$

* The regularity requirement for weights discussed in article 1 does not, *in itself*, imply their continuity. Nevertheless, in treating weights meeting the requirement, further restriction to the continuous ones (or even to those of class $\mathscr{C}_\infty$) does not constitute a serious limitation. See the first theorem of article 1.

and then

$$(\mathfrak{M}F)(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|(\mathfrak{M}F)(t)}{|z-t|^2}\,\mathrm{d}t \quad - \quad A|\Im z|$$

for $z \notin \mathbb{R}$.

Proof. Since $F(z) = F(\bar{z})$, we have $\min(U(z), U(\bar{z})) \geqslant F(z)$ for any superharmonic majorant U of F. By the next-to-the-last theorem of §A.1, the min just written is also superharmonic in z; it is, on the other hand, $\leqslant U(z)$, and does not change when z is replaced by $\bar{z}$. Therefore, for $\mathfrak{M}F$, the *smallest* superharmonic majorant of F, we have

$$(\mathfrak{M}F)(\bar{z}) \quad = \quad (\mathfrak{M}F)(z),$$

and for this reason it is necessary only to investigate $\mathfrak{M}F$ in the upper half plane.

The function $F(z)$ under consideration is *harmonic* for $\Im z > 0$, and thus, by the second lemma of the preceding article, $(\mathfrak{M}F)(z)$ *is too*, as long as it is *finite*. Because $(\mathfrak{M}F)(z) \geqslant F(z)$,

$$(\mathfrak{M}F)(z) \;+\; A\Im z \quad \geqslant \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log W(t)}{|z-t|^2}\,\mathrm{d}t,$$

a quantity $\geqslant 0$, for $\Im z > 0$ ($W(t)$ being $\geqslant 1$). The function on the left is hence *harmonic and positive in* $\Im z > 0$.

According to Chapter III, §F.1, we therefore have

$$(\mathfrak{M}F)(z) \;+\; A\Im z \quad = \quad \alpha\Im z \quad + \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z\,\mathrm{d}\mu(t)}{|z-t|^2}$$

in $\{\Im z > 0\}$, where $\alpha \geqslant 0$ and μ is some *positive measure* on $\mathbb{R}$, with $\int_{-\infty}^{\infty}(1+t^2)^{-1}\mathrm{d}\mu(t) < \infty$. But $(\mathfrak{M}F)(z)$ is everywhere continuous by the fourth lemma of the last article; it is, in particular, continuous *up to the real axis*. Thus, $\mathrm{d}\mu(t) = (\mathfrak{M}F)(t)\mathrm{d}t$, and

$$(\mathfrak{M}F)(z) \quad = \quad (\alpha - A)\Im z \quad + \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z\,(\mathfrak{M}F)(t)}{|z-t|^2}\,\mathrm{d}t$$

for $\Im z > 0$. Using the symmetry of $(\mathfrak{M}F)(z)$ with respect to the x-axis just noted, we see that

$$(\mathfrak{M}F)(z) \quad = \quad (\alpha - A)|\Im z| \quad + \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|(\mathfrak{M}F)(t)}{|z-t|^2}\,\mathrm{d}t$$

(with the usual interpretation of the right side for $z \in \mathbb{R}$).

Here, $\alpha \geqslant 0$; it is claimed that α is in fact *zero*. Thanks to the sign of α,

$-\alpha|\Im z|$ is *superharmonic* (!), and the same is true of the difference

$$(\mathfrak{M}F)(z) \;-\; \alpha|\Im z|.$$

However, $(\mathfrak{M}F)(t) \geqslant F(t) = \log W(t)$, $\mathfrak{M}F$ being a majorant of F, so this difference must, by the preceding formula, be

$$\geqslant \; -A|\Im z| \;+\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|\log W(t)}{|z-t|^2}\,dt \;=\; F(z).$$

$(\mathfrak{M}F)(z) - \alpha|\Im z|$ is thus a *superharmonic majorant* of $F(z)$, and therefore $\geqslant (\mathfrak{M}F)(z)$, the least such majorant. This makes $\alpha \leqslant 0$. Since $\alpha \geqslant 0$ as we know, we see that $\alpha = 0$, as claimed.

With $\alpha = 0$, the above formula for $(\mathfrak{M}F)(z)$ reduces to the desired representation. We are done.

Theorem. *Suppose that for a given continuous weight $W(x) \geqslant 1$ the function $\mathfrak{M}F$ corresponding to*

$$F(z) \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|\log W(t)}{|z-t|^2}\,dt \;-\; A|\Im z|$$

(where $A > 0$) is finite. *If $(\mathfrak{M}F)(z)$ is also harmonic in a neighborhood of the origin, we have*

$$\boxed{(\mathfrak{M}F)(z) \;=\; (\mathfrak{M}F)(0) \;-\; \gamma\Re z \;-\; \int_{-\infty}^{\infty}\left(\log\left|1-\frac{z}{t}\right| \;+\; \frac{\Re z}{t}\right)d\rho(t),}$$

with a constant γ and a certain increasing function $\rho(t)$, zero on a neighborhood of the origin, such that

$$\frac{\rho(t)}{t} \longrightarrow \frac{A}{\pi} \qquad \text{for } t \longrightarrow \pm\infty.$$

Remark. The subsidiary requirement that $(\mathfrak{M}F)(z)$ be harmonic in a neighborhood of the origin serves merely to ensure $\rho(t)$'s vanishing in such a neighborhood; it can be lifted, but then the corresponding representation for $\mathfrak{M}F$ looks more complicated (see problem 57 below). Later on in this article, we will see that the harmonicity requirement does not really limit applicability of the boxed formula.

Proof of theorem. Is based on the Riesz representation from §A.2; to the superharmonic function $(\mathfrak{M}F)(z)$ we apply that representation as it is formulated in the remark preceding the last theorem of §A.2 (see the boxed formula there). For each $R > 0$ this gives us a positive measure μ_R on $\{|\zeta| \leqslant R\}$ and a function $H_R(z)$ harmonic *in the interior* of that disk, such that

$$(\mathfrak{M}F)(z) = \int_{|\zeta|\leqslant R} \log\frac{1}{|z-\zeta|}\,d\mu_R(\zeta) + H_R(z) \qquad \text{for } |z| < R.$$

By problem 48(c), the measures μ_R and $\mu_{R'}$ *agree* in $\{|\zeta| < R\}$ whenever $R' > R$; this means that we actually have a *single positive* (and in general *infinite*) *Borel measure* μ on $\mathbb{C}$ whose restriction to each open disk $\{|\zeta| < R\}$ is the corresponding μ_R (cf. problem 49). This enables us to rewrite the last formula as

$$(\mathfrak{M}F)(z) = \int_{|\zeta|\leqslant R} \log\frac{1}{|z-\zeta|}\,d\mu(\zeta) + H_R(z), \qquad |z| < R,$$

with, for each R, a certain function $H_R(z)$ (N.B. perhaps *not the same as the previous* $H_R(z)$ *!) harmonic in* $\{|z| < R\}$.

We see by the preceding lemma that $(\mathfrak{M}F)(z)$ is itself harmonic both in $\{\Im z > 0\}$ and in $\{\Im z < 0\}$, so, according to the last theorem of §A.2, μ cannot have any mass in either of those half planes. By the same token, μ has no mass in a certain neighborhood of the origin, $\mathfrak{M}F$ being, by hypothesis, harmonic in such a neighborhood. There is thus an increasing function $\rho(t)$, zero on a neighborhood of the origin, such that

$$\mu(E) = \int_{E\cap\mathbb{R}} d\rho(t)$$

for Borel sets $E \subseteq \mathbb{C}$, and we have

$$(\mathfrak{M}F)(z) = \int_{-R}^{R} \log\frac{1}{|z-t|}\,d\rho(t) + H_R(z) \qquad \text{for } |z| < R,$$

with H_R harmonic there. Our desired representation will be obtained by making $R \longrightarrow \infty$ in this relation. For that purpose, we need to know the asymptotic behaviour of $\rho(t)$ as $t \longrightarrow \pm\infty$.

It is claimed that the ratio

$$\frac{\rho(r) - \rho(-r)}{r}$$

(which is certainly positive) *remains bounded* when $r \longrightarrow \infty$. Fixing any R, let us consider values of $r < R$. Using the preceding formula and reasoning as in the proof of the last theorem in §A.2, we easily find that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} (\mathfrak{M}F)(re^{i\vartheta})d\vartheta = \int_{-R}^{R} \min\left(\log\frac{1}{|t|},\ \log\frac{1}{r}\right)d\rho(t) + H_R(0),$$

and thence, subtracting $(\mathfrak{M}F)(0)$ from both sides, that

$$-\int_{-R}^{R} \log^{+}\frac{r}{|t|}\,\mathrm{d}\rho(t) \quad = \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} (\mathfrak{M}F)(re^{i\vartheta})\mathrm{d}\vartheta \quad - \quad (\mathfrak{M}F)(0)$$

for $0 < r < R$. Here, $\rho(t)$ vanishes on a neighborhood of the origin, so we can integrate the left side by parts to get

$$\int_0^r \frac{\rho(t) \;-\; \rho(-t)}{t}\,\mathrm{d}t \quad = \quad (\mathfrak{M}F)(0) \quad - \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} (\mathfrak{M}F)(re^{i\vartheta})\mathrm{d}\vartheta,$$

which is, of course, nothing but a version of Jensen's formula. In it, R no longer appears, so it is valid for all $r > 0$.

By the lemma, however,

$$-(\mathfrak{M}F)(z) \quad = \quad A|\Im z| \quad - \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|(\mathfrak{M}F)(t)}{|z-t|^2}\,\mathrm{d}t,$$

a quantity $\leqslant A|\Im z|$, since $(\mathfrak{M}F)(t) \geqslant \log W(t) \geqslant 0$. Using this in the previous relation, we get

$$\int_0^r \frac{\rho(t) \;-\; \rho(-t)}{t}\,\mathrm{d}t \quad \leqslant \quad (\mathfrak{M}F)(0) \;+\; \frac{2A}{\pi}r,$$

whence

$$\rho(r) \;-\; \rho(-r) \quad \leqslant \quad (\mathfrak{M}F)(0) \;+\; \frac{2A}{\pi}er$$

by the argument of problem 1(a) (!), $\rho(t)$ being increasing. Since $\rho(t)$ also vanishes in a neighborhood of 0, we see that

$$\frac{\rho(t)}{t} \quad \leqslant \quad \text{const.} \quad \text{on } \mathbb{R}.$$

Once this is known, it follows by reasoning like that of §A, Chapter III, that the integral

$$\int_{-\infty}^{\infty}\left(\log\left|1 - \frac{z}{t}\right| \;+\; \frac{\Re z}{t}\right)\mathrm{d}\rho(t)$$

is convergent (*a priori*, to $-\infty$, possibly, when $z \in \mathbb{R}$) for all values of z – one needs here to again use the vanishing of $\rho(t)$ for t near 0. That integral, however, obviously differs from

$$\int_{-R}^{R} \log|z-t|\,\mathrm{d}\rho(t)$$

by a function *harmonic* for $|z| < R$. Referring to the previous representation of $(\mathfrak{M}F)(z)$ in that disk, we see that

$$G(z) \quad = \quad (\mathfrak{M}F)(z) \quad + \quad \int_{-\infty}^{\infty}\left(\log\left|1-\frac{z}{t}\right| + \frac{\Re z}{t}\right)\mathrm{d}\rho(t)$$

must be *harmonic* for $|z| < R$, and hence finally for *all* z, since the parameter R no longer occurs on the right. Our local Riesz representations for $(\mathfrak{M}F)(z)$ in the disks $\{|z| < R\}$ thus have a *global* version,

$$(\mathfrak{M}F)(z) \quad = \quad -\int_{-\infty}^{\infty}\left(\log\left|1-\frac{z}{t}\right| + \frac{\Re z}{t}\right)\mathrm{d}\rho(t) \quad + \quad G(z),$$

valid for all z, with G harmonic everywhere.

We proceed to investigate $G(z)$'s behaviour for large $|z|$. The lemma gives, first of all,

$$(\mathfrak{M}F)(z) \quad \leqslant \quad \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|\Im z|(\mathfrak{M}F)(t)}{|z-t|^2}\,\mathrm{d}t,$$

$(\mathfrak{M}F)(t)$ being $\geqslant 0$. Therefore

$$[G(z)]^+ \quad \leqslant \quad \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|\Im z|(\mathfrak{M}F)(t)}{|z-t|^2}\mathrm{d}t \;+\; \left(\int_{-\infty}^{\infty}\left(\log\left|1-\frac{z}{t}\right| + \frac{\Re z}{t}\right)\mathrm{d}\rho(t)\right)^+.$$

According to the discussion at the beginning of §B, Chapter III, our bound on the growth of $\rho(t)$ makes the *second* term on the right $\leqslant \mathrm{O}(|z|\log|z|)$ for large values of $|z|$; we thus have

$$\int_{-\pi}^{\pi}[G(re^{i\vartheta})]^+\,\mathrm{d}\vartheta \;\leqslant\; \mathrm{const.}\,r\log r \;+\; \frac{1}{\pi}\int_{-\pi}^{\pi}\int_{-\infty}^{\infty}\frac{r|\sin\vartheta|(\mathfrak{M}F)(t)}{r^2+t^2-2rt\cos\vartheta}\,\mathrm{d}t\,\mathrm{d}\vartheta$$

when r is large, and, desiring to estimate the integral on the *left*, we must study the one figuring on the *right*. Changing the order of integration converts the latter to

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{1}{t}\log\left|\frac{r+t}{r-t}\right|(\mathfrak{M}F)(t)\,\mathrm{d}t,$$

which we handle by resorting to a trick.

Take the *average* of the expression in question for $R \leqslant r \leqslant 2R$, say, where $R > 0$ is arbitrary. That works out to

$$\frac{2}{\pi R}\int_{-\infty}^{\infty}\int_{R}^{2R}\frac{1}{t}\log\left|\frac{r+t}{r-t}\right|(\mathfrak{M}F)(t)\,\mathrm{d}r\,\mathrm{d}t \quad = \quad \frac{2}{\pi R}\int_{-\infty}^{\infty}\Psi\left(\frac{R}{|t|}\right)(\mathfrak{M}F)(t)\,\mathrm{d}t,$$

where

$$\Psi(u) \quad = \quad \int_u^{2u} \log\left|\frac{s+1}{s-1}\right| \mathrm{d}s.$$

The last integral can be directly evaluated, but here it is better to use power series and see how it acts when $u \longrightarrow 0$ and when $u \longrightarrow \infty$.

For $0 < u < 1$, expand the integrand in powers of s to get

$$\Psi(u) \quad = \quad 3u^2 \; + \; \mathrm{O}(u^4), \qquad 0 < u < 1.$$

For $u > 1$, we expand the integrand in powers of $1/s$ and find that

$$\Psi(u) \quad = \quad 2\log 2 \; + \; \mathrm{O}\left(\frac{1}{u^2}\right), \qquad u > 1.$$

$\Psi(R/|t|)/R$ thus behaves like $1/R$ for *small* values of $|t|/R$ and like R/t^2 for *large* ones, so, all in all,

$$\frac{2}{\pi R}\Psi\left(\frac{R}{|t|}\right) \quad \leqslant \quad \text{const.}\frac{R}{R^2+t^2} \qquad \text{for } t \in \mathbb{R}.$$

Substituting this into the previous relation, we see that

$$\frac{2}{\pi R}\int_R^{2R}\int_{-\infty}^{\infty} \frac{1}{t}\log\left|\frac{r+t}{r-t}\right| (\mathfrak{M}F)(t)\,\mathrm{d}t\,\mathrm{d}r \quad \leqslant \quad \text{const.}\int_{-\infty}^{\infty} \frac{R}{R^2+t^2}(\mathfrak{M}F)(t)\,\mathrm{d}t.$$

This, however, *implies the existence of an* r', $R \leqslant r' \leqslant 2R$, for which

$$\frac{2}{\pi}\int_{-\infty}^{\infty} \frac{1}{t}\log\left|\frac{r'+t}{r'-t}\right| (\mathfrak{M}F)(t)\,\mathrm{d}t \quad \leqslant \quad \text{const.}\int_{-\infty}^{\infty} \frac{R}{R^2+t^2}(\mathfrak{M}F)(t)\,\mathrm{d}t.$$

Here, the right side is $\leqslant$ const. $R \leqslant$ const. r' (and is even $\mathrm{o}(R)$) for large R, since $\int_{-\infty}^{\infty}((\mathfrak{M}F)(t)/(1+t^2))\,\mathrm{d}t < \infty$. Taking r equal to such an r' in our original relation involving G thus yields

$$\int_{-\pi}^{\pi} [G(r'e^{i\vartheta})]^+\,\mathrm{d}\vartheta \quad \leqslant \quad \text{const.}(r'\log r' \; + \; r')$$

when R, and hence r', is large.

Letting R take successively the values 2^n with $n = 1, 2, 3, \ldots$, we obtain in this way a certain sequence of numbers r_n tending to ∞ for which

$$\int_{-\pi}^{\pi} [G(r_n e^{i\vartheta})]^+\,\mathrm{d}\vartheta \quad \leqslant \quad \mathrm{O}(r_n \log r_n).$$

Since $G(z)$ is harmonic, we have on the other hand

$$\int_{-\pi}^{\pi} \left([G(r_n e^{i\vartheta})]^+ \; - \; [G(r_n e^{i\vartheta})]^-\right)\mathrm{d}\vartheta \quad = \quad \int_{-\pi}^{\pi} G(r_n e^{i\vartheta})\,\mathrm{d}\vartheta \quad = \quad 2\pi G(0),$$

so, subtracting this relation from *twice* the preceding, we get

$$\int_{-\pi}^{\pi} |G(r_n e^{i\vartheta})|\,d\vartheta \quad \leqslant \quad O(r_n \log r_n).$$

Now it follows that $G(z)$ must be of the form $A_0 + A_1 \Re z$. We have, indeed, $G(\bar{z}) = G(z)$, since $\mathfrak{M}F$ and the integral involving $d\rho$ have that property; the function $G(z)$, harmonic everywhere, is therefore given by a series development

$$G(re^{i\vartheta}) \quad = \quad \sum_{k=0}^{\infty} A_k r^k \cos k\vartheta.$$

For $k > 1$, we have

$$A_k \quad = \quad \frac{1}{\pi r^k} \int_{-\pi}^{\pi} G(re^{i\vartheta}) \cos k\vartheta \, d\vartheta.$$

Putting $r = r_n$ and making $n \longrightarrow \infty$, we see, using the estimate just found, that $A_k = 0$. The series thus boils down to *its first two terms.*

Going back to our global version of the Riesz representation for $(\mathfrak{M}F)(z)$ and using the description of G just found, we see that

$$(\mathfrak{M}F)(z) \quad = \quad A_0 \;+\; A_1 \Re z \;-\; \int_{-\infty}^{\infty} \left(\log\left|1 - \frac{z}{t}\right| \;+\; \frac{\Re z}{t} \right) d\rho(t).$$

Because $\rho(t)$ vanishes for t near 0, it is obvious that $A_0 = (\mathfrak{M}F)(0)$. Denoting A_1 by $-\gamma$, we now have the formula we set out to establish.

In order to complete this proof, we must still refine the estimate

$$\frac{\rho(t)}{t} \quad \leqslant \quad \text{const.}$$

obtained and used above to the asymptotic relation

$$\frac{\rho(t)}{t} \quad = \quad \frac{A}{\pi} \;+\; o(1), \qquad t \longrightarrow \pm\infty.$$

For this, some version of Levinson's theorem (the one from Chapter III) must be used.

Write

$$V(z) \quad = \quad \gamma \Re z \;+\; \int_{-\infty}^{\infty} \left(\log\left|1 - \frac{z}{t}\right| \;+\; \frac{\Re z}{t} \right) d\rho(t);$$

then, by the previous lemma and the representation formula just proved, we have

$$V(z) - (\mathfrak{M}F)(0) = -(\mathfrak{M}F)(z) = A|\Im z| - \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|(\mathfrak{M}F)(t)}{|z-t|^2}\mathrm{d}t.$$

From this, we readily see that

$$\frac{V(\mathrm{i}y)}{|y|} \longrightarrow A \quad \text{as } y \longrightarrow \pm\infty,$$

whilst

$$V(z) \leqslant (\mathfrak{M}F)(0) + A|\Im z|$$

for all z.

Take, as in the proofs of the last two theorems of article 1, an entire function φ such that

$$\log|\varphi(z)| = \beta\Re z + \int_{-\infty}^{\infty}\left(\log\left|1 - \frac{z}{t}\right| + \frac{\Re z}{t}\right)\mathrm{d}[\rho(t)]$$

where β is constant; according to a lemma from that article, we have, for suitable choice of β, the inequality

$$\log|\varphi(z)| \leqslant V(z) + \log^+\left|\frac{\Re z}{\Im z}\right| + \log^+|z| + \mathrm{O}(1).$$

Applying this first with $z = x \pm \mathrm{i}$ and using the preceding estimate for V, we see, taking account of the fact that $|\varphi(z)|$ *diminishes* when $|\Im z|$ does, that

$$|\varphi(z)| \leqslant \text{const.}(|z|^2+1), \qquad |\Im z| \leqslant 1.$$

We next find from the same relations that

$$|\varphi(z)| \leqslant \text{const.}(|z|^2+1)e^{A|\Im z|}$$

when $|\Im z| > 1$; in view of the preceding inequality such an estimate (with perhaps a larger constant) must then hold *everywhere.* $\varphi(z)$ is thus of *exponential type.*

A computation like one near the end of the next-to-the-last theorem in article 1 now yields, for $y \in \mathbb{R}$,

$$\log|\varphi(\mathrm{i}y)| - V(\mathrm{i}y) = \int_{-\infty}^{\infty}\frac{y^2}{y^2+t^2}\,\frac{[\rho(t)]-\rho(t)}{t}\,\mathrm{d}t.$$

Since $[\rho(t)]-\rho(t)$ is *bounded* (above and below!) and zero on a neighborhood of the origin, the integral on the right is $\mathrm{o}(|y|)$ for

$y \longrightarrow \pm\infty$, and hence

$$\frac{\log|\varphi(iy)|}{|y|} \longrightarrow A \quad \text{as } y \longrightarrow \pm\infty,$$

in view of the above similar relation for $V(iy)$.

By the preceding estimates on $\varphi(z)$, we obviously have

$$\int_{-\infty}^{\infty} \frac{\log^+|\varphi(x)|}{1+x^2}\,dx \quad < \quad \infty,$$

and the Levinson theorem from §H.2 of Chapter III can be applied to φ. Referring to the last of the above relations, we see in that way that

$$\frac{[\rho(t)]}{t} \longrightarrow \frac{A}{\pi} \quad \text{as } t \longrightarrow \pm\infty.$$

Therefore,

$$\frac{\rho(t)}{t} \longrightarrow \frac{A}{\pi} \qquad \text{for } t \longrightarrow \pm\infty.$$

Our theorem is proved.

Problem 57

If $(\mathfrak{M}F)(z)$ is finite, but not necessarily harmonic in a neighborhood of 0, find a representation for it analogous to the one furnished by the result just obtained.

As stated previously, the last theorem has quite general utility in spite of its harmonicity requirement. Any situation involving a finite function $\mathfrak{M}F$ can be reduced to one for which the corresponding $\mathfrak{M}F$ *is* harmonic near 0. The easiest way of doing that is to use the following

Lemma. *Let $W(t)$, continuous and $\geqslant 1$ on $\mathbb{R}$, be $\equiv 1$ for $-h < t < h$, where $h > 0$, and suppose that for $|x| < h$, we have*

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log W(t)}{(x-t)^2}\,dt \quad > \quad A,$$

with the integral on the left convergent. Then the function

$$F(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|\log W(t)}{|z-t|^2}\,dt \quad - \quad A|\Im z|$$

satisfies

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F(re^{i\vartheta})\,d\vartheta \quad > \quad F(0) \quad = \quad 0$$

for $0 < r < h$.

Proof. We have $F(z) = F(\bar{z})$, so

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F(re^{i\vartheta})\,d\vartheta \quad = \quad \frac{1}{\pi}\int_{0}^{\pi} F(re^{i\vartheta})\,d\vartheta.$$

It will be convenient to denote the right-hand integral by $J(r)$ and to work with the function

$$G(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log W(t)}{|z-t|^2}\,dt \; - \; A\,\Im z \quad (\textit{sic!})$$

instead of $F(z)$; we of course also have

$$J(r) \quad = \quad \frac{1}{\pi}\int_{0}^{\pi} G(re^{i\vartheta})\,d\vartheta.$$

In the present circumstances the function $G(z)$ is finite, and hence *harmonic*, in both the upper and the lower half planes. Moreover, since $\log W(t) \equiv 0$ for $|t| < h$, $G(z)$ (taken as *zero* on the real interval $(-h, h)$) is actually harmonic* in $\mathbb{C} \sim (-\infty, -h] \sim [h, \infty)$ and hence $\mathscr{C}_\infty$ in that region. There is thus no obstacle to differentiating under the integral sign so as to get

$$\frac{dJ(r)}{dr} \quad = \quad \frac{1}{\pi r}\int_{0}^{\pi} \frac{\partial G(re^{i\vartheta})}{\partial r}\, r\,d\vartheta, \qquad 0 < r < h.$$

Let $\mathscr{D}_r$ be the semi-circle of radius r lying in the upper half plane, having for diameter the real segment $[-r, r]$:

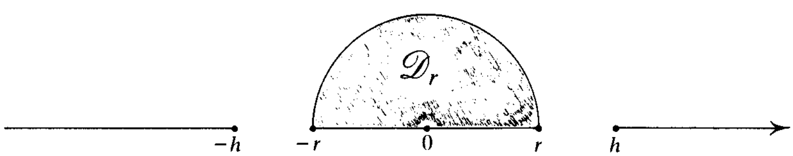


Figure 242

* by Schwarz' reflection principle, since $G(\bar{z}) = -G(z)$

When $r < h$, the function $G(z)$ is harmonic in a region including the *closure* of $\mathscr{D}_r$, so we can use Green's theorem to get

$$\int_{\partial \mathscr{D}_r} \frac{\partial G(\zeta)}{\partial n_\zeta} |\mathrm{d}\zeta| \quad = \quad \iint_{\mathscr{D}_r} (\nabla^2 G)(\zeta)\, \mathrm{d}\xi\, \mathrm{d}\eta \quad = \quad 0,$$

where $\partial/\partial n_\zeta$ denotes differentiation along the outward normal to $\partial \mathscr{D}_r$ at ζ. The left-hand expression is just

$$\int_0^\pi \frac{\partial G(r\mathrm{e}^{\mathrm{i}\vartheta})}{\partial r} r\, \mathrm{d}\vartheta \quad - \quad \int_{-r}^{r} G_y(x)\, \mathrm{d}x,$$

so the previous relation yields

$$J'(r) \quad = \quad \frac{1}{\pi r}\int_{-r}^{r} G_y(x)\, \mathrm{d}x.$$

Here, $G(z) = F(z)$ for $\Im z \geqslant 0$ with $G(x) = F(x) = \log W(x) = 0$ for $-h < x < h$, so, for such x,

$$G_y(x) \quad = \quad \lim_{\Delta y \to 0+} \frac{F(x + \mathrm{i}\Delta y)}{\Delta y},$$

which, by our formula for F, is equal to

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log W(t)}{(x-t)^2}\, \mathrm{d}t \quad - \quad A.$$

If, then, this expression is > 0 for $|x| < h$, we must, by the preceding formula, have

$$J'(r) \; > \; 0 \quad \text{for } 0 < r < h.$$

Obviously, $J(r) \longrightarrow F(0) = 0$ for $r \longrightarrow 0$. Therefore,

$$F(0) \quad < \quad J(r) \quad = \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} F(r\mathrm{e}^{\mathrm{i}\vartheta})\, \mathrm{d}\vartheta$$

when $0 < r < h$, given that the hypothesis holds. We are done.

Corollary. *Given $W(x)$ continuous and $\geqslant 1$ with $\int_{-\infty}^{\infty} (\log W(t)/(1+t^2))\, \mathrm{d}t < \infty$, and the number $A > 0$, form, for $h > 0$, the new weight*

$$W_h(x) \quad = \quad \begin{cases} 1, \quad |x| \leqslant h, \\ \mathrm{e}^{2\pi A h} W(x), \quad |x| \geqslant 2h, \\ \textit{linear for } -2h \leqslant x \leqslant -h \textit{ and for } h \leqslant x \leqslant 2h. \end{cases}$$

Put then

$$F_h(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|\log W_h(t)}{|z-t|^2}\,\mathrm{d}t \; - \; A|\Im z|.$$

If $(\mathfrak{M}F_h)(z)$ *is finite, it is harmonic in a neighborhood of the origin.*

Proof. When $-h < x < h$,

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log W_h(t)}{(x-t)^2}\,\mathrm{d}t \quad \geqslant \quad \frac{1}{\pi}\int_{2h}^{\infty}\left(\frac{2\pi Ah}{(t-x)^2} + \frac{2\pi Ah}{(t+x)^2}\right)\mathrm{d}t$$

$$= \quad \frac{8Ah^2}{4h^2-x^2} \quad \geqslant \quad 2A \quad > \quad A.$$

The lemma, applied to W_h and F_h, thus yields

$$F_h(0) \quad < \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} F_h(re^{i\vartheta})\,\mathrm{d}\vartheta$$

for $0 < r < h$. Since, however, $\mathfrak{M}F_h$ is a *superharmonic majorant* of F_h, the right-hand integral is

$$\leqslant \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} (\mathfrak{M}F_h)(re^{i\vartheta})\,\mathrm{d}\vartheta \quad \leqslant \quad (\mathfrak{M}F_h)(0),$$

i.e.,

$$F_h(0) \quad < \quad (\mathfrak{M}F_h)(0).$$

The corollary now follows by the third lemma of article 2.

The preceding results give us our desired converse to the statement from the last article.

Theorem. *Let* $W(x) \geqslant 1$ *be continuous, with*

$$\int_{-\infty}^{\infty} (\log W(t)/(1+t^2))\,\mathrm{d}t \quad < \quad \infty,$$

and put

$$F(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|\log W(t)}{|z-t|^2}\,\mathrm{d}t \; - \; A|\Im z|,$$

where $A > 0$, *interpreting the right side in the usual way when* $z \in \mathbb{R}$. *If the smallest superharmonic majorant,* $\mathfrak{M}F$, *of* F *is finite, there is an increasing* *function ρ, zero on a neighborhood of the origin, for which*

$$\log W(x) + \gamma x + \int_{-\infty}^{\infty}\left(\log\left|1-\frac{x}{t}\right| + \frac{x}{t}\right)\mathrm{d}\rho(t) \leqslant \text{const.}, \quad x\in\mathbb{R}$$

(with a certain constant γ), while

$$\frac{\rho(t)}{t} \longrightarrow \frac{A}{\pi} \quad \text{as } t\longrightarrow \pm\infty.$$

Proof. With $h>0$, form the functions W_h and F_h figuring in the preceding corollary. Since $\log W(t) \geqslant 0$, we have

$$\log W_h(t) \leqslant \log W(t) + 2\pi Ah,$$

whence

$$F_h(z) \leqslant F(z) + 2\pi Ah.$$

Thus, since $(\mathfrak{M}F)(z) \geqslant F(z)$,

$$F_h(z) \leqslant (\mathfrak{M}F)(z) + 2\pi Ah.$$

In the last relation, the right-hand member is *superharmonic*, and, of course, *finite* if $\mathfrak{M}F$ is. Then, however, $\mathfrak{M}F_h$, the *least superharmonic majorant of* F_h, must *also* be finite.

This, according to the corollary, implies that $(\mathfrak{M}F_h)(z)$ is harmonic in a neighborhood of the origin. Once that is known, the previous theorem gives us an increasing function ρ having the required properties, such that

$$(\mathfrak{M}F_h)(z) = (\mathfrak{M}F_h)(0) - \gamma\Re z - \int_{-\infty}^{\infty}\left(\log\left|1-\frac{z}{t}\right| + \frac{\Re z}{t}\right)\mathrm{d}\rho(t),$$

γ being a certain constant. Thus, since $(\mathfrak{M}F_h)(x) \geqslant F_h(x) = \log W_h(x)$,

$$\log W_h(x) + \gamma x + \int_{-\infty}^{\infty}\left(\log\left|1-\frac{x}{t}\right| + \frac{x}{t}\right)\mathrm{d}\rho(t)$$
$$\leqslant (\mathfrak{M}F_h)(0) \qquad \text{for } x\in\mathbb{R}.$$

Let now m_h denote the maximum of $W(x)$ for $-2h \leqslant x \leqslant 2h$. Then certainly

$$\log W(x) \leqslant \log m_h + \log W_h(x),$$

$W(x)$, and hence m_h, being $\geqslant 1$. This, substituted into the previous, yields finally

$$\log W(x) + \gamma x + \int_{-\infty}^{\infty}\left(\log\left|1-\frac{x}{t}\right| + \frac{x}{t}\right)\mathrm{d}\rho(t)$$
$$\leqslant (\mathfrak{M}F_h)(0) + \log m_h \qquad \text{for } x\in\mathbb{R}.$$

We are done.

The proof just given furnishes a more precise result which is sometimes useful.

Corollary. *If $W(x)$, satisfying the hypothesis of the theorem, is, in addition, 1 at the origin, and the function $\mathfrak{M}F$ corresponding to some given $A > 0$ is finite, we have, for any $\eta > 0$, an increasing function $\rho(t)$ with the properties affirmed by the theorem, such that*

$$\log W(x) + \gamma x + \int_{-\infty}^{\infty}\left(\log\left|1-\frac{x}{t}\right| + \frac{x}{t}\right)\mathrm{d}\rho(t) \leqslant (\mathfrak{M}F)(0) + \eta,\ x\in\mathbb{R}.$$

To verify this, we first observe that the continuity of $W(x)$ makes $m_h \longrightarrow 1$ and hence $\log m_h \longrightarrow 0$ when $h \longrightarrow 0$. On the other hand,

$$(\mathfrak{M}F_h)(0) \leqslant (\mathfrak{M}F)(0) + 2\pi Ah,$$

since $(\mathfrak{M}F)(z) + 2\pi Ah$ is a superharmonic majorant of $F_h(z)$, as remarked at the beginning of the proof. The desired relation involving ρ will therefore follow from the *last* one in the proof if we take $h > 0$ small enough so as to have

$$\log m_h + 2\pi Ah < \eta.$$

These results and the obvious converse noted in article 2 are used in conjunction with the material from article 1. Referring, for instance, to the corollary of the next-to-the-last theorem in article 1, we have the

Theorem. *Let $W(x)$, continuous and $\geqslant 1$ on the real axis, fulfill the regularity requirement formulated in article 1. In order that W admit multipliers, it is necessary and sufficient that*

$$\int_{-\infty}^{\infty}\frac{\log W(t)}{1+t^2}\mathrm{d}t < \infty$$

and that then, for each $A > 0$, the smallest superharmonic majorant *of*

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|\Im z|\log W(t)}{|z-t|^2}\mathrm{d}t - A|\Im z|$$

be finite.

Looking at the *last* theorem of article 1 we see in the same way that such a result holds for any weight $W(x) \geqslant 1$ of the form $|F(x)|$, where F is *entire* and of *exponential type, without any additional assumption on the regularity of* W. This fact will be used in the next §.

The regularity requirement on W figuring in the above theorem may, of course, by replaced by the milder one discussed in the scholium to article 1.*

Let us hark back for a moment to the discussion at the beginning of article 1. Can one regard the condition that $(\mathfrak{M}F)(0)$ be *finite* for *each* of the functions

$$F(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z| \log W(t)}{|z-t|^2}\, dt - A|\Im z|, \qquad A > 0,$$

as one of *regularity* to be satisfied by the weight W? *In a sense*, one *can* – see especially problem 55. Is *this*, then, the presumed *second* ('essential') kind of regularity a weight must have in order to admit multipliers?

C. Theorems of Beurling and Malliavin

We are going to apply the results from the end of the last § so as to obtain multiplier theorems for certain kinds of continuous weights W. Those are always assumed to be $\geqslant 1$ on the real axis, and only for the *unbounded* ones can there be any question about the existence of multipliers.

One can in fact work exclusively with weights $W(x)$ *tending to* ∞ *for* $x \longrightarrow \pm\infty$ without in any way lessening the generality of the results obtained. Suppose, indeed, that we are given an unbounded weight $W(x) \geqslant 1$; then

$$\Omega(x) = (1 + x^2) W(x)$$

does tend to ∞ when $x \longrightarrow \pm\infty$, and it is claimed that there is a non-zero entire function of exponential type $\leqslant A$ *whose product with* Ω *is bounded on* $\mathbb{R}$ *if and only if* there is such an entire function whose *product with* W is bounded there.

It is clearly only the *if* part of this statement that requires checking. Consider, then, that we have an entire function $\varphi(z) \not\equiv 0$ of exponential type $\leqslant A$ making $\varphi(x)W(x)$ bounded on $\mathbb{R}$. Since $W(x)$ is unbounded, $|\varphi(x)|$ cannot be constant, so the Hadamard product for φ (Chapter III

* See also Remark 5 near the end of §E.2.

§A) must involve *linear factors* – there must in fact be *infinitely many* of those, for otherwise $|\varphi(x)|$ would *grow like a polynomial* in x when $|x| \longrightarrow \infty$. The function $\varphi(z)$ thus has infinitely many zeros, and, taking any two of them, say α and β, we can form a new entire function,

$$\psi(z) \quad = \quad \frac{\varphi(z)}{(z-\alpha)(z-\beta)},$$

also of exponential type $\leqslant A$, with $\psi(x)\Omega(x)$ bounded on the real axis.

The existence of multipliers for $W(x)$ is thus fully equivalent to existence thereof for $\Omega(x)$, a weight tending to ∞ for $x \longrightarrow \pm\infty$; that is fortunate, because weights having the latter property are easier to deal with. When working with a *given* weight W, it will sometimes be convenient to form from it the new one

$$\left(1 + \frac{x^2}{M^2}\right)W(x)$$

(using a *large* value of M) or

$$(1+x^2)^\eta W(x)$$

(taking for η a *small* value > 0), instead of dealing with the weight $\Omega(x)$ just looked at. Any of these weights $\tilde{W}$ fulfills the condition

$$\int_{-\infty}^{\infty} \frac{\log \tilde{W}(x)}{1+x^2}\mathrm{d}x \quad < \quad \infty$$

as long as

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2}\mathrm{d}x \quad < \quad \infty;$$

unless the latter holds W cannot, as we know, admit any multipliers.

In order to establish the existence of multipliers for a weight $\geqslant 1$ satisfying the last condition, we first form from it a new one according to one of the above recipes* if that is necessary to ensure our having a weight tending to ∞ with $|x|$. Then, choosing a number $A > 0$ and using the *new* weight W, we take the function

$$F(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|}{|z-t|^2}\log W(t)\,\mathrm{d}t \quad - \quad A|\Im z|$$

studied in §B.3. According to results obtained there, the question of our weight's *admittance of multipliers* reduces in large part to a simple decision

* the new weight obviously meets the local regularity requirement of §B.1 iff the original one does

about the *finiteness* of $\mathfrak{M}F$, the smallest superharmonic majorant of F, for various initial choices of the number A. We know by the first lemma of §B.2 that the latter property is *equivalent to the finiteness of* $\mathfrak{M}F$ *at any one point*, say that of $(\mathfrak{M}F)(0)$. To evaluate this quantity we will use harmonic estimation, guided by the knowledge that $\mathfrak{M}F$, *if* finite, must be *harmonic* in both the upper and lower half planes (first lemma of §B.3), and *also harmonic* across any real interval on which it is $> F$ (by the third lemma of §B.2).

1. Use of the domains from §C of Chapter VIII

Starting, then, with a continuous weight $W(x) \geqslant 1$ tending to ∞ for $x \longrightarrow \pm\infty$, we take (using some given $A > 0$) the function $F(z)$ whose formula has just been written, and look at its *smallest superharmonic majorant* $\mathfrak{M}F$, our aim being to see whether or not $(\mathfrak{M}F)(0) < \infty$. The idea is to get at $\mathfrak{M}F$ by using *other* superharmonic majorants whose qualitative behaviour is known.

For each $N > 1$, let

$$W_N(x) \quad = \quad \min(W(x),\ N)$$

and then form the function

$$F_N(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|}{|z-t|^2}\log W_N(t)\,dt \quad - \quad A|\Im z|$$

corresponding to it in the way that $F(z)$ corresponds to W. Clearly,

$$F_N(z) \ \underset{N}{\uparrow} \ F(z);$$

it is claimed that also

$$(\mathfrak{M}F_N)(z) \ \underset{N}{\uparrow} \ (\mathfrak{M}F)(z).$$

For each N, we have, indeed,

$$(\mathfrak{M}F_{N+1})(z) \ \geqslant \ F_{N+1}(z) \ \geqslant \ F_N(z),$$

so $\mathfrak{M}F_{N+1}$ *is* a superharmonic *majorant of* F_N, and hence

$$(\mathfrak{M}F_{N+1})(z) \quad \geqslant \quad (\mathfrak{M}F_N)(z),$$

the *least* superharmonic majorant of $F_N(z)$. By the same token,

$$(\mathfrak{M}F)(z) \quad \geqslant \quad (\mathfrak{M}F_N)(z)$$

for each N, so we have

$$\lim_{N\to\infty} (\mathfrak{M}F_N)(z) \leqslant (\mathfrak{M}F)(z).$$

The sequence $\{\mathfrak{M}F_N\}$ is, as just shown, *increasing*, so $\lim_{N\to\infty}\mathfrak{M}F_N$ is *superharmonic* by the next-to-the-last theorem of §A.1. That limit must, however, be $\geqslant \lim_{N\to\infty} F_N = F$, so it is also a *majorant* for F. As a *superharmonic majorant* of F, $\lim_{N\to\infty}\mathfrak{M}F_N$ is therefore $\geqslant \mathfrak{M}F$. So, since the contrary relation holds, we in fact have *equality*, as asserted.

We thus have, in particular,

$$(\mathfrak{M}F)(0) = \lim_{N\to\infty} (\mathfrak{M}F_N)(0),$$

whence, in order to verify that $(\mathfrak{M}F)(0) < \infty$, *it suffices to obtain an upper bound independent of N on the values $(\mathfrak{M}F_N)(0)$.*

Each of the functions $\mathfrak{M}F_N$ is certainly *finite*. Indeed,

$$0 \leqslant \log W_N(t) \leqslant \log N,$$

so

$$F_N(z) \leqslant \log N - A|\Im z|.$$

But the right-hand expression in this last relation is *superharmonic*! Hence,

$$(\mathfrak{M}F_N)(z) \leqslant \log N - A|\Im z|.$$

Since, on the other hand,

$$F_N(z) \geqslant -A|\Im z|,$$

we also have

$$(\mathfrak{M}F_N)(z) \geqslant -A|\Im z|.$$

Thanks to our assumption that $W(x) \longrightarrow \infty$ for $x \longrightarrow \pm\infty$, there is a certain number L, depending on N, such that

$$W_N(x) = N \quad \text{for } |x| \geqslant L.$$

Therefore $F_N(x) = \log N$ for $|x| \geqslant L$, making $(\mathfrak{M}F_N)(x) \geqslant \log N$ for such x. By one of the previous relations, we have, however, $(\mathfrak{M}F_N)(x) \leqslant \log N$ on $\mathbb{R}$. Thus,

$$(\mathfrak{M}F_N)(x) = \log N = F_N(x) \qquad \text{for } |x| \geqslant L.$$

We see that *on the real axis*, $(\mathfrak{M}F_N)(x)$ (a *continuous* function by the fourth lemma of §B.2) can *strictly exceed* the (continuous) function $F_N(x)$ *only on an open subset $\mathcal{O}$ of $(-L, L)$*. The *first* lemma of §B.3 and the *third* one of §B.2 then ensure that $(\mathfrak{M}F_N)(z)$ is *harmonic* in the region

$$\{\Im z > 0\} \cup \{\Im z < 0\} \cup \mathcal{O};$$

it is, moreover, *continuous up to the boundary* of that region (indeed, continuous everywhere), again by the fourth lemma of §B.2. The boundary certainly includes the two infinite segments $(-\infty, -L]$ and $[L, \infty)$ of the real axis, on which $(\mathfrak{M}F_N)(x) = F_N(x)$.

The open subset $\mathcal{O}$ of $\mathbb{R}$ *might*, however, be so complicated as to raise doubts about our being able to solve the Dirichlet problem in the region (bounded by $\mathbb{R} \sim \mathcal{O}$) just described, and it is thus not clear that one can do harmonic estimation there. We get around this difficulty by means of a simple device.

The difference

$$(\mathfrak{M}F_N)(x) - F_N(x)$$

is continuous on $\mathbb{R}$ and *identically zero* on $\mathbb{R} \sim \mathcal{O}$. The latter set includes $(-\infty, -L] \cup [L, \infty)$, so our difference is actually *uniformly continuous* on $\mathbb{R}$, and, given any $\varepsilon > 0$, we can find a $\delta > 0$ such that

$$(\mathfrak{M}F_N)(x) - F_N(x) \leqslant \varepsilon \qquad \text{if} \quad \operatorname{dist}(x, \mathbb{R} \sim \mathcal{O}) \leqslant \delta.$$

The points x fulfilling the condition on the right make up a certain *closed* set

$$E_\delta = (\mathbb{R} \sim \mathcal{O}) + [-\delta, \delta]$$

possibly equal to $\mathbb{R}$ which, in any event, includes $(-\infty, -L+\delta] \cup [L-\delta, \infty)$ and *may*, in addition, contain some *disjoint closed intervals of length* $\geqslant 2\delta$ intersecting with $(-L+\delta, L-\delta)$. There can, of course, *be only finitely many of the latter* so E_δ, *if not identical with* $\mathbb{R}$, is simply a *finite union of disjoint closed intervals thereon including two of the form* $(-\infty, M]$, $[M', \infty)$. In the latter case, the *complement* $\mathbb{C} \sim E_\delta$ is *one of the domains* $\mathscr{D}$ *considered in* §C *of Chapter* VIII, and on $\partial\mathscr{D} = E_\delta$ we have

$$(\mathfrak{M}F_N)(x) \leqslant F_N(x) + \varepsilon$$

by construction.

Our object here is to estimate $(\mathfrak{M}F_N)(0)$. That quantity is of course $\geqslant F_N(0)$, and, as long as it is *equal* to $F_N(0)$, there is no problem, because $F_N(0) = \log W_N(0)$ is $\leqslant \log W(0)$ (is, in fact, equal to $\log W(0)$ for sufficiently large values of N). We thus need *only* look at the situation where

$$(\mathfrak{M}F_N)(0) > F_N(0).$$

Then, however, 0 *belongs* to our original open subset $\mathcal{O}$ of $\mathbb{R}$, and, given $\varepsilon > 0$, it is possible to take the $\delta > 0$ corresponding to it *small enough*, in our construction, so as to *still have*

$$0 \notin (\mathbb{R} \sim \mathcal{O}) + [-\delta, \delta] = E_\delta.$$

Doing so, we see that E_δ really is properly included in $\mathbb{R}$, making

$$\mathscr{D} = \mathbb{C} \sim E_\delta$$

a domain of the kind studied in Chapter VIII, §C, with $0 \in \mathscr{D}$. (We write $\mathscr{D}$ instead of the more logical $\mathscr{D}_\delta$ in order to *avoid* having to use subscripts of subscripts later on.)

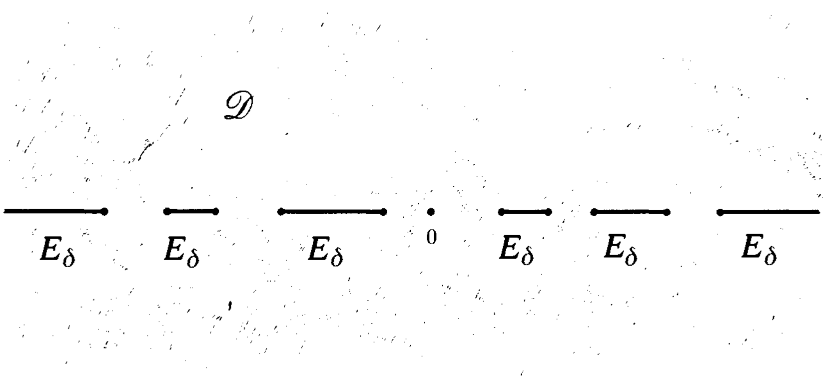


Figure 243

In the present circumstances, $\mathfrak{M}F_N$ is *harmonic* in $\mathscr{D}$ and *continuous* up to $\partial\mathscr{D}$, with

$$(\mathfrak{M}F_N)(t) \leqslant F_N(t) + \varepsilon = \log W_N(t) + \varepsilon \qquad \text{for } t \in \partial\mathscr{D},$$

and also, in view of the previous estimates,

$$(\mathfrak{M}F_N)(z) = O(1) - A|\Im z|$$

(everywhere). These facts make it possible for us to follow the procedure indicated at the very beginning of §C, Chapter VIII, and in that way carry out the harmonic estimation of $(\mathfrak{M}F_N)(z)$ in $\mathscr{D}$.

Let us, as in Chapter VIII, §C, denote the Phragmén–Lindelöf function for $\mathscr{D}$ by $Y_{\mathscr{D}}(z)$ and harmonic measure for that domain by $\omega_{\mathscr{D}}(\ \ , z)$.

Then, by the last relations we have

$$(\mathfrak{M}F_N)(0) \;\leqslant\; \int_{\partial\mathscr{D}} (\mathfrak{M}F_N)(t)\,\mathrm{d}\omega_{\mathscr{D}}(t,\,0) \;-\; AY_{\mathscr{D}}(0)$$
$$\leqslant\; \int_{\partial\mathscr{D}} (\log W_N(t) \;+\; \varepsilon)\,\mathrm{d}\omega_{\mathscr{D}}(t,\,0) \;-\; AY_{\mathscr{D}}(0).$$

(The first two members in this chain of inequalities are in fact equal, $\mathfrak{M}F_N$ being harmonic in $\mathscr{D}$.) Because $\log W_N(t) \leqslant \log W(t)$ and $\omega_{\mathscr{D}}(\partial\mathscr{D},\,0) = 1$, the estimate just written implies that

$$\boxed{(\mathfrak{M}F_N)(0) \;\leqslant\; \varepsilon + \int_{\partial\mathscr{D}} \log W(t)\,\mathrm{d}\omega_{\mathscr{D}}(t,\,0) \;-\; AY_{\mathscr{D}}(0);}$$

this, then, *must hold, whenever* $(\mathfrak{M}F_N)(0)$ *is not simply equal to* $\log W_N(0)$ *and hence* $\leqslant \log W(0)$.

In the boxed formula (derived, we remind the reader, *under the assumption* that $W(t) \longrightarrow \infty$ for $t \longrightarrow \pm\infty$), the *integrand* appearing in the right-hand integral *no longer depends on* N. But the *right side as a whole certainly involves* N (and ε as well!) *through the domain* $\mathscr{D}$, whose very construction depended on our knowing that $W(x) \geqslant N$ for $|x|$ sufficiently large! *In principle*, it does not generally seem possible to actually *know* $\mathscr{D}$ *precisely*, because such knowledge would depend on information about the function $(\mathfrak{M}F_N)(z)$ which we are in fact *trying to estimate* (really, *to find*) by using $\mathscr{D}$.

The formula is useful nevertheless, on account of the results found in §§C.4 and C.5 of Chapter VIII. As we saw there, when dealing with *certain kinds* of weights W, one can, by using quantities involving *only* W, express the *entire dependence of*

$$\int_{\partial\mathscr{D}} \log W(t)\,\mathrm{d}\omega_{\mathscr{D}}(t,\,0)$$

on the domain $\mathscr{D}$ *in terms of* $Y_{\mathscr{D}}(0)$. That is the basis for the following applications.

2. **Weight is the modulus of an entire function of exponential type**

We come to one of the main results of this chapter – indeed, of the present book. The proof, based on the matters discussed above and in §C of Chapter VIII, uses also a refinement of the Riesz–Fejér factorization theorem which has never been explicitly formulated up to now, although it is essentially contained in the material of Chapters III and VI. Here it is:

Lemma. *Let $P(z)$, entire and of exponential type* 2B, *satisfy the condition*

$$\int_{-\infty}^{\infty} \frac{\log^+ |P(x)|}{1+x^2} dx \quad < \quad \infty,$$

and suppose that $P(x) \geqslant 0$ on $\mathbb{R}$. Then there is an entire function $g(z)$ of exponential type B having all its zeros in $\Im z \leqslant 0$ and such that

$$g(z)\overline{g(\bar{z})} \quad = \quad P(z).$$

Proof. Except for the specification of the exponential type of g, this is just a restatement of the Riesz–Fejér result (the *third* theorem of §G.3, Chapter III). There is thus an entire function $g_0(z)$ having the stipulated properties, but we do not know its type.

In particular, $g_0(z)\overline{g_0(\bar{z})} = P(z)$. As long as λ is *real*, we then have

$$g_\lambda(z)\overline{g_\lambda(\bar{z})} \quad = \quad P(z)$$

for the function

$$g_\lambda(z) \quad = \quad e^{i\lambda z} g_0(z).$$

However, $\log|g_\lambda(iy)| = -\lambda y + \log|g_0(iy)|$, so we can evidently adjust the real parameter λ so as to make

$$\limsup_{y\to\infty} \frac{\log|g_\lambda(iy)|}{y} \qquad \text{and} \qquad \limsup_{y\to-\infty} \frac{\log|g_\lambda(iy)|}{|y|}$$

equal. Do this, and denote the common value of the two limsups by A, taking, then, $g(z)$ as $g_\lambda(z)$ for that particular choice of λ.

Since P is of exponential type $2B$, we have

$$\limsup_{y\to\infty} \frac{\log|P(iy)|}{y} \quad \leqslant \quad 2B.$$

At the same time, $|P(iy)| = |g(iy)||g(-iy)|$ for real y, so

$$\frac{\log|g(iy)|}{y} + \frac{\log|g(-iy)|}{y} = \frac{\log|P(iy)|}{y}.$$

On the real axis, $|g(x)|^2 = P(x)$, whence

$$\int_{-\infty}^{\infty} \frac{\log^+ |g(x)|}{1+x^2} dx \quad < \quad \infty$$

by hypothesis. Also, $g(z)$, like $g_0(z)$, has all its zeros in $\Im z \leqslant 0$; the remark at the end of §G.1, Chapter III, thus applies to it, and we actually have

$$\frac{\log|g(iy)|}{y} \longrightarrow A$$

as $y \longrightarrow \infty$. Now make $y \longrightarrow \infty$ through a sequence of values along which $\log|g(-iy)|/|y|$ *also* tends to A; referring to the previous relation we see that

$$A + A \leqslant \limsup_{y\to\infty} \frac{\log|P(iy)|}{y} \leqslant 2B,$$

so $A \leqslant B$.

By Chapter III, §E, we have

$$\log|g(z)| \leqslant A|\Im z| + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|\log^+|g(t)|}{|z-t|^2}\,dt,$$

and from this it follows easily as in the *fourth* theorem of §E.2, Chapter VI (see also §B.2 of that chapter), that g *is of exponential type* A. Since $g(z)\overline{g(\bar{z})} = P(z)$, P must be of exponential type $\leqslant 2A$, i.e., $B \leqslant A$. We have, however, just shown that $A \leqslant B$. Thus, $A = B$, and we are done.

Now we can give the

Theorem on the Multiplier (Beurling and Malliavin, 1961). *If $f(z)$, entire and of exponential type, is such that*

$$\int_{-\infty}^{\infty} \frac{\log^+|f(x)|}{1+x^2}\,dx < \infty,$$

there are entire functions $\varphi(z) \not\equiv 0$ of arbitrarily small exponential type with

$$(1 + |f(x)|)\varphi(x)$$

bounded on the real axis.

Proof. Given $A > 0$, we wish to find a non-zero entire φ of exponential type $\leqslant A$ having the desired property. Our plan is to invoke the second theorem from §B.3, referring to the last theorem in §B.1. This involves our showing that $(\mathfrak{M}F)(0) < \infty$ where

$$F(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|}{|z-t|^2}\log W(t)\,dt - A|\Im z|,$$

with $W(t) \geqslant 1$ an appropriate weight formed from f. For that purpose, the boxed formula at the end of the preceding article will be applied.

We proceed as at the beginning of §C.5, Chapter VIII, forming from f a new entire function $g_M(z)$ with $|g_M(x)| = |g_M(-x)| \geqslant 1$ on $\mathbb{R}$ and $g_M(0) = 1$. Our present construction differs slightly from the one made there.

Taking a large number M (whose value will depend on the type A of the multiplier we are seeking), we form the entire function

$$P_M(z) = \left(1 + \frac{z^2}{M^2}\right)\left(1 + \frac{z^2}{M^2}(f(z)\overline{f(\bar{z})} + f(-z)\overline{f(-\bar{z})})\right),$$

the purpose of the first factor on the right being to ensure that

$$P_M(x) \longrightarrow \infty \quad \text{for } x \longrightarrow \pm\infty.$$

Given that $f(z)$ is of exponential type B, $P_M(z)$ will be entire, and of exponential type $\leqslant 2B$. It is clear that $P_M(z)$ is *even*, that

$$P_M(x) \geqslant 1 \quad \text{for } x \in \mathbb{R},$$

with

$$P_M(0) = 1,$$

and that

$$P_M(x) \geqslant |f(x)|^2/M^2 \qquad \text{for } |x| \geqslant 1.$$

From the hypothesis it follows also that

$$\int_{-\infty}^{\infty} \frac{\log P_M(x)}{1+x^2}\,dx < \infty;$$

we can, indeed, choose M so as to make

$$\int_{-\infty}^{\infty} \frac{\log P_M(x)}{x^2}\,dx \qquad (\textit{sic}\,!)$$

as small > 0 as we like. Here, the behaviour of the integrand near the origin is alright, because (for large M)

$$|\log P_M(x)| \leqslant \text{const. } x^2/M^2 \qquad \text{for } |x| < 1$$

with a constant independent of M.

For any particular M, the lemma now gives us an entire function $g_M(z)$, of exponential type $\leqslant B$ (*half* that of P_M), having (here) all its zeros in the lower half plane, with

$$g_M(z)\overline{g_M(\bar{z})} = P_M(z).$$

Thence, in particular,

$$|g_M(x)| = \sqrt{P_M(x)} \geqslant |f(x)|/M$$

for real x of modulus $\geqslant 1$, so, since $P_M(x) \geqslant 1$ on $\mathbb{R}$,

$$1 + |f(x)| \leqslant C_M |g_M(x)|, \qquad x \in \mathbb{R},$$

with a constant C_M depending on M. Our result will thus be established if, for a suitable value of M, we can find an entire $\varphi(z) \not\equiv 0$ of exponential type A with $\varphi(x)g_M(x)$ bounded on $\mathbb{R}$. To do this, we follow the procedure explained in the last article.

Fixing a value of M (in a way to be described shortly) we take the weight $W(x) = |g_M(x)|$ and then use it in the formula written at the beginning of this proof so as to obtain a function F. According to the last theorem of §B.1 and the second one of §B.3, a function φ having the desired properties exists provided that $(\mathfrak{M}F)(0) < \infty$. It is now claimed that for proper choice of M *we in fact have*

$$(\mathfrak{M}F)(0) = 0.$$

To see this we verify (in the notation of the last article) that $(\mathfrak{M}F_N)(0) = 0$ for every $N \geqslant 1$. In the present circumstances, $W(0) = 1$, so for $N \geqslant 1$,

$$F_N(0) = F(0) = \log W(0) = 0,$$

and it is enough to show that assuming

$$(\mathfrak{M}F_N)(0) > \varepsilon$$

for some $\varepsilon > 0$ leads to a contradiction.

In case the last relation holds, it is certainly true that

$$(\mathfrak{M}F_N)(0) > \log W(0),$$

so the boxed formula from the end of the preceding article is valid, $W(x) = \sqrt{P_M(x)}$ having been ensured by our construction to tend to ∞ for $x \longrightarrow \pm\infty$. Thus,

$$(\mathfrak{M}F_N)(0) \leqslant \varepsilon + \int_{\partial\mathscr{D}} \log W(t)\, d\omega_{\mathscr{D}}(t, 0) - AY_{\mathscr{D}}(0),$$

where $\mathscr{D}$ is a certain (unknown) domain of the kind studied in §C of Chapter VIII.

Now *the second theorem of* §C.5 in Chapter VIII *can be used to estimate the quantity*

$$\int_{\partial\mathscr{D}} \log W(t)\, d\omega_{\mathscr{D}}(t, 0) = \int_{\partial\mathscr{D}} \log|g_M(t)|\, d\omega_{\mathscr{D}}(t, 0).$$

Our function g_M is of exponential type $\leqslant B$ and has otherwise the properties of the function G figuring in that theorem. Therefore,

$$\int_{\partial\mathscr{D}} \log|g_M(t)|\,d\omega_{\mathscr{D}}(t,\ 0) \quad\leqslant\quad Y_{\mathscr{D}}(0)\{J \ + \ \sqrt{(2eJ(J+\pi B/4))}\},$$

where

$$J \quad = \quad \int_0^\infty \frac{\log|g_M(x)|}{x^2}\,dx \quad = \quad \frac{1}{4}\int_{-\infty}^\infty \frac{\log P_M(x)}{x^2}\,dx.$$

As observed above, the right-hand integral will, for large enough M, *be as small as we like*. We can hence choose (and *fix*) *a value of M for which J* is *small enough to render*

$$J \ + \ \sqrt{(2eJ(J+\pi B/4))} \quad < \quad A.$$

This having been done, the previous relations yield

$$(\mathfrak{M}F_N)(0) \quad \leqslant \quad \varepsilon$$

(whatever N may be), contradicting our assumption that $(\mathfrak{M}F_N)(0) > \varepsilon$. Thus, $(\mathfrak{M}F_N)(0) = 0$, so, since this holds for every N, we have $(\mathfrak{M}F)(0) = 0$ as claimed, and the theorem is proved.

We are done.

Scholium. The multiplier $\varphi(z)$ of exponential type $\leqslant A$ obtained by going from the conclusion of the above argument to the *second* theorem in §B.3 and thence to the *last* one in §B.1 has real zeros only. This is immediately apparent on glancing at the description of the function φ appearing towards the end of the latter result's proof – *that* φ, by the way, is not the same as the multiplier we are talking about *here*, which, in the theorem referred to, was called ψ.

In their 1967 *Acta* paper, Beurling and Malliavin made the important observation that the zeros of the multiplier φ can also, in the present circumstances, be taken to be *uniformly separated*, in other words, *that any two of those zeros are distant by at least a certain amount $h > 0$*. This can be readily seen by putting together some of the above results and then using a simple measure-theoretic lemma.

Let us look again at the *least superharmonic majorant* $(\mathfrak{M}F)(z)$ of the function

$$F(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^\infty \frac{|\Im z|\log|g_M(t)|}{|z-t|^2}\,dt \ - \ A|\Im z|$$

formed from the weight $W(x) = |g_M(x)|$ used in the preceding proof. Here, $|g_M(t)| = |g_M(-t)|$, so $F(z) = F(-z)$ and therefore $(\mathfrak{M}F)(z) = (\mathfrak{M}F)(-z)$ (cf. beginning of proof of first lemma, §B.3). We know that $\mathfrak{M}F$ is finite, but here, since $(\mathfrak{M}F)(0) = F(0) = \log|g_M(0)| = 0$, we cannot affirm that $(\mathfrak{M}F)(z)$ is harmonic in a neighborhood of 0 and thus are not able to directly apply the first theorem from §B.3. An analogous result is nevertheless available by problem 57. In the present circumstances, with $(\mathfrak{M}F)(z)$ *even*, that result takes the form

$$(\mathfrak{M}F)(z) \quad = \quad C \; - \; \int_0^1 \log|z^2 - t^2|\,\mathrm{d}\rho(t) \; - \; \int_1^\infty \log\left|1 - \frac{z^2}{t^2}\right|\mathrm{d}\rho(t),$$

where ρ is a certain *positive* measure on $[0, \infty)$ with

$$\frac{\rho([0,t])}{t} \quad \longrightarrow \quad \frac{A}{\pi} \qquad \text{for } t \longrightarrow \infty.$$

Because $(\mathfrak{M}F)(0) < \infty$, we actually have

$$\int_0^1 \log(t^2)\,\mathrm{d}\rho(t) \quad > \quad -\infty,$$

so, after changing the value of the constant C, we can just as well write

$$(\mathfrak{M}F)(z) \quad = \quad C \; - \; \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right|\mathrm{d}\rho(t).$$

By the first lemma of §C.5, Chapter VIII (where the function corresponding to our present $g_M(z)$ was denoted by $G(z)$), we have

$$\log|g_M(z)| \quad = \quad \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right|\mathrm{d}\nu(t) \qquad \text{for } \Im z \geqslant 0,$$

$\nu(t)$ being a certain absolutely continuous (and *smooth*) increasing function. Taking the function $g_M(z)$ to be of exponential type exactly *equal* to B (so as not to bring in more letters !), we also have

$$\log|g_M(z)| \quad = \quad B\Im z \quad + \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log|g_M(t)|}{|z-t|^2}\,\mathrm{d}t$$

for $\Im z > 0$ by §G.1 of Chapter III.* Referring to the above formula for $F(z) = F(\bar{z})$, we see from the last two relations that

$$F(z) \quad = \quad \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right|\mathrm{d}\nu(t) \quad - \quad (A+B)|\Im z|.$$

* see also end of proof of lemma at beginning of this article

$(\mathfrak{M}F)(z)$ is, however, a *majorant of* $F(z)$. Hence,

$$F(z) \;-\; (\mathfrak{M}F)(z) \;=\; \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right|(\mathrm{d}\nu(t) + \mathrm{d}\rho(t))$$

$$-\;\; (A+B)|\Im z| \;-\; C \qquad \text{is} \;\leqslant\; 0.$$

Our statement about the zeros of $\varphi(z)$ will follow from this inequality.

The real line is the union of two disjoint subsets, an *open* one, Ω, on which

$$F(x) \;-\; (\mathfrak{M}F)(x) \;<\; 0,$$

and the *closed* set $E \;=\; \mathbb{R} \sim \Omega$, on which

$$F(x) \;-\; (\mathfrak{M}F)(x) \;=\; 0.$$

According to the *third* lemma of §B.2, $(\mathfrak{M}F)(z)$ is *harmonic* in a neighborhood of each $x_0 \in \Omega$, so the measure involved in the Riesz representation of $(\mathfrak{M}F)(z)$ can have no mass in such a neighborhood (last theorem, §A.2). This means that

$$\mathrm{d}\rho(t) \;=\; 0 \qquad \text{in } \Omega \cap [0, \infty).$$

It is now claimed that

$$\mathrm{d}\nu(t) \;+\; \mathrm{d}\rho(t) \;\leqslant\; \frac{A+B}{\pi}\mathrm{d}t \qquad \text{on } E \cap [0, \infty).$$

Once this is established, the separation of the zeros of our multiplier $\varphi(z)$ is immediate. That function is gotten by dividing out any two zeros from the even entire function $\varphi_1(z)$ given by

$$\log|\varphi_1(z)| \;=\; \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}[\rho(t)]$$

(as in the proof of the last theorem, §B.1). Because $\mathrm{d}\nu(t) \;\geqslant\; 0$, the preceding two relations will certainly make

$$\mathrm{d}\rho(t) \;\leqslant\; \frac{A+B}{\pi}\mathrm{d}t \qquad \text{for } t \;\geqslant\; 0,$$

and thus *any two zeros of* $\varphi_1(z)$ *will be distant by at least*

$$\frac{\pi}{A+B}$$

units, in conformity with Beurling and Malliavin's observation.

Verification of the claim remains, and it is there that we resort to the

Lemma. *Let μ be a finite positive measure on $\mathbb{R}$ without point masses. Then the derivative $\mu'(t)$ exists* (finite or infinite) *for all t save those belonging to a Borel set E_0 with $\mu(E_0) + |E_0| = 0$. If E is any compact subset of $\mathbb{R}$,*

$$|E| \quad = \quad \int_E \frac{1}{\mu'(t) + 1}(\mathrm{d}\mu(t) + \mathrm{d}t).$$

Proof. The initial statement is like that of Lebesgue's differentiation theorem which, however, only asserts the existence of a (finite) derivative $\mu'(t)$ almost everywhere (with respect to Lebesgue measure). The present result can nonetheless be deduced from the latter one by making a change of variable. Lest the reader feel that he or she is being hoodwinked by the juggling of notation, let us proceed somewhat carefully.

Put, as usual, $\mu(t) = \int_0^t \mathrm{d}\mu(\tau)$, making the standard interpretation of the integral for $t < 0$. By hypothesis, $\mu(t)$ is bounded, increasing and without jumps, so

$$S(t) \quad = \quad \mu(t) + t$$

is a *continuous, strictly increasing* map of $\mathbb{R}$ onto itself. S therefore has a *continuous* (and also strictly increasing) *inverse* which we denote by T:

$$T(\mu(t) + t) \quad = \quad t.$$

If $\varphi(s)$ is continuous and of compact support we have the elementary substitution formula

$$(*) \qquad \int_{-\infty}^{\infty} \varphi(s)\,\mathrm{d}s \quad = \quad \int_{-\infty}^{\infty} \varphi(S(t))(\mathrm{d}\mu(t) + \mathrm{d}t)$$

which is easily checked by looking at Riemann sums. The dominated convergence theorem shows that $(*)$ is valid as well for any function φ everywhere equal to the *pointwise limit* of a *bounded* sequence of continuous ones with *fixed* compact support. That is the case, in particular, for $\varphi = \chi_F$, the characteristic function of a compact set F, and we thus have

$$|F| \quad = \quad \int_{-\infty}^{\infty} \chi_F(s)\,\mathrm{d}s \quad = \quad \int_{-\infty}^{\infty} \chi_F(S(t))(\mathrm{d}\mu(t) + \mathrm{d}t) \quad = \quad \mu(T(F)) + |T(F)|.$$

The quantity $|T(F)|$ is, of course, nothing other than the Lebesgue–Stieltjes measure $\int_F \mathrm{d}T(s)$ generated by the increasing function $T(s)$ in the usual way. The last relation shows that

$$|T(F)| \quad \leqslant \quad |F|$$

for compact sets F; the measure on the left is thus *absolutely continuous* with respect to Lebesgue measure, and indeed

$$0 \leqslant \frac{T(s+h)-T(s)}{h} \leqslant 1 \qquad \text{for } h \neq 0.$$

By the theorem of Lebesgue already referred to, we know that the derivative

$$T'(s) = \lim_{h\to 0} \frac{T(s+h)-T(s)}{h}$$

exists for all s outside some Borel set F_0 with $|F_0| = 0$. The image $E_0 = T(F_0)$ is also Borel (T being one-one and continuous both ways) and, for any *compact subset* C of E_0, the previous identity yields

$$\mu(C) + |C| = |S(C)| \leqslant |F_0| = 0,$$

since $C = T(S(C))$ and $S(C) \subseteq S(E_0) = F_0$. Therefore

$$\mu(E_0) + |E_0| = 0.$$

Suppose that $t \notin E_0$. Then $s = \mu(t)+t$ (for which $T(s) = t$) cannot lie in F_0, T being one-one, and thus $T'(s)$ exists. For $\delta \neq 0$ and any such t (and corresponding s), write

$$h(s,\delta) = \mu(t+\delta) - \mu(t) + \delta.$$

We have $s + h(s,\delta) = \mu(t+\delta) + t + \delta$, so by definition of T, $T(s+h(s,\delta)) = t+\delta$, and

$$(\dagger) \qquad \frac{T(s+h(s,\delta)) - T(s)}{h(s,\delta)} = \frac{\delta}{\mu(t+\delta)-\mu(t)+\delta}.$$

The function $\mu(t)$ is in any event continuous, so at each s (and corresponding t),

$$h(s,\delta) \longrightarrow 0 \quad \text{as } \delta \longrightarrow 0.$$

Therefore, when $t \notin E_0$, $\lim_{\delta\to 0}\big((\mu(t+\delta)-\mu(t))/\delta + 1\big)^{-1}$ must, by (†), *exist* and equal $T'(s)$. This shows that $\mu'(t)$ exists for such t (being infinite in case $T'(s) = 0$).

Take now any continuous function $\psi(s)$ of compact support. Because of the *absolute continuity* of the measure $\int_F \mathrm{d}T(s)$ already noted, we have

$$\int_{-\infty}^{\infty} \psi(s)\,\mathrm{d}T(s) = \int_{-\infty}^{\infty} \psi(s)\,T'(s)\,\mathrm{d}s.$$

Here,

$$T'(s) \quad = \quad \lim_{\delta \to 0} \frac{T(s + h(s,\delta)) \; - \; T(s)}{h(s,\delta)} \qquad \text{a.e.}$$

where $h(s,\delta)$ is the quantity introduced above. The difference quotients on the right lie, however, between 0 and 1. Hence $\int_{-\infty}^{\infty} \psi(s)\, T'(s)\, \mathrm{d}s$ equals the limit, for $\delta \longrightarrow 0$, of

$$\int_{-\infty}^{\infty} \psi(s) \, \frac{T(s + h(s,\delta)) \; - \; T(s)}{h(s,\delta)} \, \mathrm{d}s$$

by dominated convergence. In this last expression the *integrand* is continuous and of compact support when $\delta \neq 0$. We may therefore use $(*)$ to make the substitution $s \; = \; \mu(t) + t$ therein; with the help of $(\dagger)$, that gives us

$$\int_{-\infty}^{\infty} \psi(\mu(t) + t) \, \frac{\delta}{\mu(t + \delta) - \mu(t) + \delta} \, (\mathrm{d}\mu(t) \; + \; \mathrm{d}t).$$

The quantity $\delta/(\mu(t + \delta) - \mu(t) + \delta)$ lies between 0 and 1 and, as we have just seen, tends to $1/(\mu'(t) + 1)$ for every t outside E_0 when $\delta \longrightarrow 0$, where $\mu(E_0) \; + \; |E_0| \; = \; 0$. Another application of the dominated convergence theorem thus shows the integral just written to tend to $\int_{-\infty}^{\infty} \psi(\mu(t) + t)(\mu'(t) + 1)^{-1}(d\mu(t) \; + \; \mathrm{d}t)$ as $\delta \longrightarrow 0$. In this way, we see that

$$\int_{-\infty}^{\infty} \psi(s)\, \mathrm{d}T(s) \quad = \quad \int_{-\infty}^{\infty} \frac{\psi(\mu(t) + t)}{\mu'(t) + 1} \, (\mathrm{d}\mu(t) \; + \; \mathrm{d}t)$$

when ψ is continuous and of compact support.

Extension of this formula to functions ψ of the form χ_F with F compact now proceeds as at the beginning of the proof. Given, then, any compact E, we put $F \; = \; S(E)$, making $T(F) \; = \; E$ and $\chi_F(\mu(t) + t) \; = \; \chi_F(S(t)) \; = \; \chi_E(t)$; using $\psi(s) \; = \; \chi_F(s)$ we thus find that

$$|E| \quad = \quad |T(F)| \quad = \quad \int_{-\infty}^{\infty} \chi_F(s)\, \mathrm{d}T(s) \quad = \quad \int_{-\infty}^{\infty} \chi_E(t) \, \frac{\mathrm{d}\mu(t) + \mathrm{d}t}{\mu'(t) + 1}.$$

The lemma is established.

We proceed to the claim.

Problem 58

(a) Show that in our present situation, neither $\nu(t)$ nor $\rho(t)$ can have any point masses. (Hint: Concerning $\rho(t)$, recall that $(\mathfrak{M}F)(x)$ is continuous!)

(b) We take $\nu(0) = \rho(0) = 0$ and then extend the increasing *functions* $\nu(t)$ and $\rho(t)$ from $[0, \infty)$ to $\mathbb{R}$ by making them *odd*. Show that for $x \in E$ (the set on which $F(x) - (\mathfrak{M}F)(x) = 0$), we have

$$\frac{1}{2}\int_{-\infty}^{\infty} \log\left(1 + \frac{y^2}{(x-t)^2}\right)(\mathrm{d}\nu(t) + \mathrm{d}\rho(t)) - (A+B)y \leqslant 0 \quad \text{for } y > 0.$$

(c) Writing $\mu(t) = \nu(t) + \rho(t)$, show that for fixed $y > 0$,

$$\frac{1}{2}\int_{-\infty}^{\infty} \log\left(1 + \frac{y^2}{(x-t)^2}\right)\mathrm{d}\mu(t) = \int_0^{\infty} \frac{y^2}{y^2+\tau^2}\,\frac{\mu(x+\tau)-\mu(x-\tau)}{\tau}\,\mathrm{d}\tau.$$

(Hint: Since $\mu(\{x\}) = 0$, the left hand integral is the limit, for $\delta \longrightarrow 0$, of

$$\frac{1}{2}\int_{|t-x|\geq\delta} \log\left(1 + \frac{y^2}{(t-x)^2}\right)\mathrm{d}\mu(t).$$

Here we may integrate by parts to get

$$\int_{\delta}^{\infty} \frac{y^2}{y^2+\tau^2}\,\frac{\mu(x+\tau)-\mu(x-\tau) - (\mu(x+\delta)-\mu(x-\delta))}{\tau}\,\mathrm{d}\tau.$$

Now make $\delta \longrightarrow 0$ and use monotone convergence.)

(d) Hence show that *for each* $x \in E$ *where* $\mu'(x)$ *exists*, we have

$$\pi\mu'(x) \leqslant A + B.$$

(Hint: Refer to (b).)

(e) Show then that if F is any compact subset of E,

$$\pi\mu(F) \leqslant (A+B)|F|,$$

whence

$$\nu(F) + \rho(F) \leqslant \frac{A+B}{\pi}|F|.$$

(Hint: Apply the lemma.)

The reader who prefers a more modern treatment yielding the result of part (e) may, in place of (d), establish that

$$\pi(\underline{D}\mu)(x) \leqslant A + B \quad \text{for } x \in E,$$

where

$$(\underline{D}\mu)(x) = \liminf_{\Delta x \to 0} \frac{\mu(x+\Delta x) - \mu(x)}{\Delta x}.$$

Then, instead of using the lemma to get part (e), a suitable version of Vitali's covering theorem can be applied.

Remark. The original proof of the theorem of Beurling and Malliavin is different from the one given in this article, and the reader interested in working seriously on the subject of the present chapter should study it.

The first exposition of that proof is contained in a famous (and very rare) Stanford University preprint written by Malliavin in 1961, and the final version is in his joint *Acta* paper with Beurling, published in 1962. Other presentations can be found in Kahane's Seminaire Bourbaki lecture for 1961–62, and in de Branges' book. But the clearest explanation of the proof's *idea* is in a much later paper of Malliavin appearing in the 1979 *Arkiv*. Although some details are omitted in that paper, it is probably the best place to start reading.

3. A quantitative version of the preceding result.

Theorem. *Let $\Phi(z)$ be entire and even, of exponential type B, with $\Phi(x) \geqslant 0$ on the real axis and*

$$\int_{-\infty}^{\infty} \frac{\log^+ \Phi(x)}{1+x^2}\,dx \quad < \quad \infty.$$

For $M > 0$, denote by J_M the quantity

$$\int_0^{\infty} \frac{1}{x^2} \log\left(1 \;+\; x^2\frac{\Phi(x)}{M}\right) dx.$$

Suppose that for some given $A > 0$, M is large enough to make

$$J_M \quad + \quad \frac{1}{\sqrt{\pi}}\sqrt{(J_M(J_M \;+\; \pi B))} \quad < \quad A.$$

Then there is an even entire function $\varphi(z)$ of exponential type A with

$$\varphi(0) \;=\; 1$$

and

$$|\varphi(x)|\,\Phi(x) \quad \leqslant \quad 2e^2(A+B)^2 M \qquad \text{for } x \in \mathbb{R}.$$

Remark. There are actually functions φ having all the stipulated properties and satisfying a relation like the last one with the coefficient 2 replaced by any number > 1 – hence indeed by 1, as follows by a normal family argument. By that kind of argument one also sees that there are such φ

corresponding to a value of M *for which*

$$J_M \;+\; \frac{1}{\sqrt{\pi}}\sqrt{(J_M(J_M \;+\; \pi B))} \;=\; A.$$

Such improvements are not very significant.

Proof of theorem. We argue as in the last article, working this time with the auxiliary entire function

$$P(z) \;=\; \left(1 \;+\; \frac{z^2}{R^2}\right)\left(1 \;+\; z^2\frac{\Phi(z)}{M}\right)$$

which involves a large constant R as well as the parameter M. An extra factor has again been introduced on the right in order to make sure that

$$P(x) \longrightarrow \infty \quad \text{for } x \longrightarrow \pm\infty.$$

Like $\Phi(z)$, the function $P(z)$ is even, entire, and of exponential type B; it is, moreover, $\geqslant 1$ on the real axis and we can use it as a *weight* thereon. As long as M fulfills the condition in the hypothesis,

$$J \;=\; \int_0^\infty \frac{1}{x^2}\log P(x)\,\mathrm{d}x$$

will satisfy the relation

$$J \;+\; \frac{1}{\sqrt{\pi}}\sqrt{(J(J \;+\; \pi B))} \;\leqslant\; A$$

for large enough R; we *choose and fix* such a value of that quantity.

Using the weight $W(x) = P(x)$ and the number A, we then form the function $F(z)$ and the sequence of $F_N(z)$ corresponding to it as in the previous two articles, and set out to show that $(\mathfrak{M}F)(0) = 0$.

This is done as before, by verifying that $(\mathfrak{M}F_N)(0) \leqslant 0$ for each N. Assuming, on the contrary, that some $(\mathfrak{M}F_N)(0)$ is *strictly larger* than some $\varepsilon > 0$, we have

$$(\mathfrak{M}F_N)(0) \;\leqslant\; \varepsilon \;+\; \int_{\partial\mathscr{D}} \log P(t)\,\mathrm{d}\omega_{\mathscr{D}}(t, 0) \;-\; AY_{\mathscr{D}}(0)$$

with a domain $\mathscr{D}$ of the sort considered in §C of Chapter VIII, since here

$$F_N(t) \;\leqslant\; F(t) \;=\; \log P(t), \qquad t \in \mathbb{R},$$

where $\log P(t) \longrightarrow \infty$ for $t \longrightarrow \pm\infty$. We proceed to estimate the integral on the right.

The entire function $P(z)$ is real and $\geqslant 1$ on R, so, by the lemma from the last article, we can get an entire function $g(z)$ of exponential type $B/2$ (*half* that of P), having all its zeros in the lower half plane, and such that

$$g(z)\overline{g(\bar{z})} = P(z).$$

For the entire function

$$G(z) = (g(z))^2$$

of exponential type B with all its zeros in $\Im z < 0$ we then have $|G(x)| = P(x)$ on $\mathbb{R}$, so that

$$\int_{\partial\mathscr{D}} \log P(t)\,\mathrm{d}\omega_{\mathscr{D}}(t,\ 0) = \int_{\partial\mathscr{D}} \log|G(t)|\,\mathrm{d}\omega_{\mathscr{D}}(t,\ 0).$$

Here, $G(z)$ satisfies the hypothesis of the second theorem in §C.5 of Chapter VIII, and that result can be used to get an upper bound for the last integral. We can, however, do somewhat better by first improving the theorem, using, at the very end of its proof, the estimate furnished by problem 28(c) in place of the one applied there. The effect of this is to replace the term $\sqrt{(2\mathrm{e}\,J(J+\pi B/4))}$ figuring in the theorem's conclusion by

$$\frac{1}{\sqrt{\pi}}\sqrt{(J(J + \pi B))}$$

with

$$J = \int_0^\infty \frac{1}{x^2}\log|G(x)|\,\mathrm{d}x = \int_0^\infty \frac{1}{x^2}\log P(x)\,\mathrm{d}x,$$

and in that way one finds that

$$\int_{\partial\mathscr{D}} \log|G(t)|\,\mathrm{d}\omega_{\mathscr{D}}(t,\ 0) \leqslant Y_{\mathscr{D}}(0)\left\{J + \frac{1}{\sqrt{\pi}}\sqrt{(J(J+\pi B))}\right\}.$$

Substituted into the above relation, this yields

$$(\mathfrak{M}F_N)(0) \leqslant \varepsilon,$$

a contradiction, thanks to our initial assumption about M and our choice of R. It follows that $(\mathfrak{M}F_N)(0) \leqslant 0$ for every N and thus that $(\mathfrak{M}F)(0) = 0$.

Knowing that, we can, since $P(0) = 1$, apply the *corollary* to the second theorem of §B.3. That gives us, corresponding to any $\eta > 0$, an increasing function $\rho(t)$, *zero* on a neighborhood of the origin, such that

$$\frac{\rho(t)}{t} \longrightarrow \frac{A}{\pi} \qquad \text{for } t \longrightarrow \pm\infty$$

and that

$$\log P(x) \quad + \quad \gamma x \quad + \quad \int_{-\infty}^{\infty} \left(\log\left|1 - \frac{x}{t}\right| \;+\; \frac{x}{t} \right) \mathrm{d}\rho(t) \quad \leqslant \quad \eta$$

on $\mathbb{R}$, γ being a certain real constant. In the present circumstances $P(x) = P(-x)$, so, taking the increasing function

$$\nu(t) \quad = \quad \tfrac{1}{2}(\rho(t) \; - \; \rho(-t))$$

(also zero on a neighborhood of the origin), we have simply

$$\log P(x) \quad + \quad \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\nu(t) \quad \leqslant \quad \eta \qquad \text{for } x \in \mathbb{R}.$$

Our given function Φ is $\geqslant 0$ on the real axis. Therefore $P(x) \geqslant x^2\Phi(x)/M$ there, and our last relation certainly implies that

$$\log\left(x^2 \frac{\Phi(x)}{M}\right) \quad + \quad \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\nu(t) \quad \leqslant \quad \eta$$

on $\mathbb{R}$. Denote for the moment

$$\log\left|z^2 \frac{\Phi(z)}{M}\right| \quad + \quad \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}\nu(t)$$

by $U(z)$; this function is *subharmonic*, and, since Φ is of exponential type B while

$$\frac{\nu(t)}{t} \longrightarrow \frac{A}{\pi} \qquad \text{for } t \longrightarrow \infty,$$

we have

$$U(z) \quad \leqslant \quad (B + A)|z| \; + \; \mathrm{o}(|z|)$$

for large $|z|$. Because $U(x) \leqslant \eta$ on $\mathbb{R}$, we see by the third Phragmén–Lindelöf theorem of §C, Chapter III, that

$$U(z) \quad \leqslant \quad \eta \; + \; (A + B)\Im z \qquad \text{for } \Im z \; \geqslant \; 0.$$

To the integral $\int_0^\infty \log|1 - (z^2/t^2)| \mathrm{d}\nu(t)$ we now apply the lemma of

§A.1, Chapter X, according to which

$$\int_0^\infty \log\left|1-\frac{z^2}{t^2}\right|(\mathrm{d}[\nu(t)]-\mathrm{d}\nu(t)) \quad\leqslant\quad \log\left\{\frac{\max(|x|,|y|)}{2|y|} + \frac{|y|}{2\max(|x|,|y|)}\right\}$$

(where, as usual, $z = x+\mathrm{i}y$). Used together with the preceding inequality for $U(z)$, this yields

$$\log\left|z^2\frac{\Phi(z)}{M}\right| \quad+\quad \int_0^\infty \log\left|1-\frac{z^2}{t^2}\right|\mathrm{d}[\nu(t)]$$

$$\leqslant\quad \eta \;+\; (A+B)y \;+\; \log\left\{\frac{\max(|x|,y)}{2y} + \frac{y}{2\max(|x|,y)}\right\}$$

for $\Im z = y > 0$.

There is clearly an entire function $\varphi(z)$ with

$$\log|\varphi(z)| \quad=\quad \int_0^\infty \log\left|1-\frac{z^2}{t^2}\right|\mathrm{d}[\nu(t)];$$

φ is *even* and $\varphi(0) = 1$. Moreover, in view of the asymptotic behaviour of $\nu(t)$ for large t, *$\varphi(z)$ is of exponential type A*. In terms of φ, the preceding relation becomes

$$|\Phi(z)\varphi(z)| \quad\leqslant\quad M\mathrm{e}^{(A+B)y+\eta}\,\frac{\{\max(|x|,y)/y \;+\; y/\max(|x|,y)\}}{2(x^2 \;+\; y^2)}, \qquad y>0.$$

The fraction on the right is just

$$\frac{1}{2y^2}\cdot\frac{\max(\xi,1) \;+\; 1/\max(\xi,1)}{\xi^2 \;+\; 1}$$

with $\xi = |x|/y$, and hence is $\leqslant 1/y^2$. Thus, putting $z = x+\mathrm{i}h$ with $h>0$, we see that

$$|\Phi(x+\mathrm{i}h)\varphi(x+\mathrm{i}h)| \quad\leqslant\quad \frac{M}{h^2}\mathrm{e}^{(A+B)h+\eta} \qquad \text{for } x\in\mathbb{R}.$$

Applying once more the third Phragmén–Lindelöf theorem of Chapter III, §C, this time to $\Phi(z)\varphi(z)$ (of exponential type $A+B$) in the half plane $\{\Im z \leqslant h\}$, we get finally

$$|\Phi(x)\varphi(x)| \quad\leqslant\quad \frac{M}{h^2}\mathrm{e}^{2(A+B)h+\eta}, \qquad x\in\mathbb{R},$$

and, putting $h = 1/(A+B)$, we have

$$\Phi(x)|\varphi(x)| \quad\leqslant\quad \mathrm{e}^{\eta}\mathrm{e}^2(A+B)^2M \qquad \text{on } \mathbb{R}.$$

The quantity $\eta > 0$ was arbitrary, so the desired result is established. We are done.

Scholium. Let us try to understand the rôle played by the parameter M in the result just proved. As long as

$$\int_{-\infty}^{\infty} \frac{\log^+ \Phi(x)}{1+x^2} dx \quad < \quad \infty,$$

it is surely true that with

$$J_M \quad = \quad \int_0^{\infty} \frac{1}{x^2} \log\left(1 \ + \ x^2 \frac{\Phi(x)}{M}\right) dx,$$

the expression

$$J_M \quad + \quad \frac{1}{\sqrt{\pi}} \sqrt{(J_M(J_M \ + \ \pi B))}$$

eventually becomes less than any given $A > 0$ when M increases without limit; we *cannot*, however, tell *how large* M must be taken for that to happen *if only the value of the former integral and the type* B *of* Φ *are known*. Our result thus does not enable us to determine, *using that information alone*, how small $\sup_{x \in \mathbb{R}} \Phi(x)|\varphi(x)|$ can be rendered by taking a suitable even entire function φ of exponential type A with $\varphi(0) = 1$.

This is even the case for *polynomials* Φ (special kinds of functions of exponential type zero!).

Problem 59

Show that for the polynomials

$$\Phi_N(z) \quad = \quad \left(\frac{z^2}{2N^2} - 1\right)^{2N}$$

one has

$$\int_0^{\infty} \frac{1}{x^2} \log^+ \Phi_N(x) dx \quad \leqslant \quad \text{const.},$$

but that for $J > 0$ small enough, there is no value of M which will make

$$\int_0^{\infty} \frac{1}{x^2} \log\left(1 \ + \ x^2 \frac{\Phi_N(x)}{M}\right) dx \quad \leqslant \quad J$$

for all N simultaneously.

$$\left(\text{Hint: Look at the values of } \int_0^\infty \frac{1}{x^2}\log^+\left(\frac{\Phi_N(x)}{7^N}\right)dx.\right)$$

The parameter M, made to depend on A by requiring that

$$J_M \quad + \quad \frac{1}{\sqrt{\pi}}\sqrt{(J_M(J_M \; + \; \pi B))}$$

be $<$ A (or simply *equal to* A; see the remark following our theorem's statement) does nevertheless seem to be the *main factor governing how small*

$$\sup_{x\in\mathbb{R}} \Phi(x)\,|\varphi(x)|$$

can be for even entire functions φ of exponential type A with $\varphi(0) = 1$. The evidence for this is especially convincing when *entire functions* Φ of exponential type zero are concerned. Then the discrepancy between the above result and any best possible one essentially involves nothing more than *a constant factor affecting the type A of the multiplier φ* in question.

Problem 60

Suppose that $W(x) \geqslant 1$ is even and that there is an even entire function φ of exponential type A with $\varphi(0) = 1$ and

$$W(x)|\varphi(x)| \quad \leqslant \quad K \quad \text{for } x\in\mathbb{R}.$$

(a) Show that then

$$\int_{x_0}^\infty \frac{1}{x\sqrt{(x^2-x_0^2)}}\log\left(\frac{W(x)}{K}\right)dx \quad \leqslant \quad \frac{\pi}{2}A$$

for any $x_0 > 0$. (Hint: Use harmonic estimation in

$$\mathscr{D} \quad = \quad \mathbb{C} \sim (-\infty, -x_0] \sim [x_0, \infty)$$

to get an upper bound for $\log|\varphi(0)|$. Note that $\omega_{\mathscr{D}}(\quad, z)$ and $Y_{\mathscr{D}}(z)$ are explicitly available for this domain; for the latter, see, for instance, §A.2 of Chapter VIII).

Suppose now that φ, A and K are as in (a) *and that there is in addition an $x_0 > 0$ such that $W(x) \leqslant K$ for $|x| \leqslant x_0$ while $W(x) \geqslant K$ for $|x| \geqslant x_0$. Note that $W(x)$ need not be an increasing function of $|x|$ for this to hold for certain values of K*:

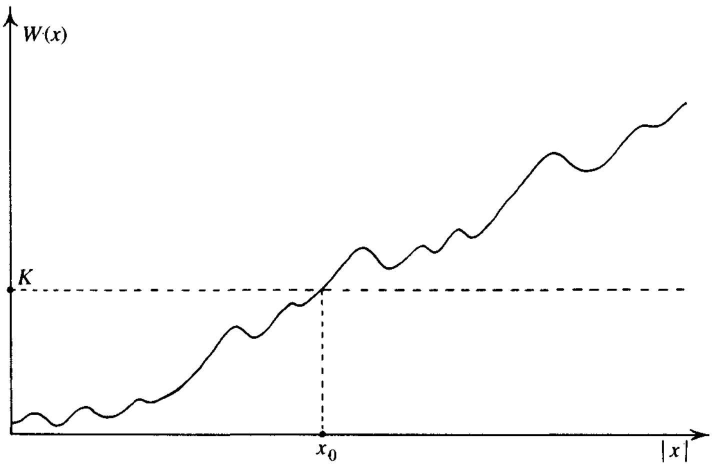


Figure 244

(b) Show that then, given any $\eta > 0$, there is a constant C depending only on η such that

$$\int_0^\infty \frac{1}{x^2} \log\left(1 + \frac{A^2x^2}{C}\frac{W(x)}{K}\right) dx \;<\; \left(\frac{\pi}{2} + \eta\right) A.$$

(Hint: Observe that

$$\log\left(1 + \frac{A^2x^2}{C}\frac{W(x)}{K}\right) \;\leqslant\; \log\left(1 + \frac{A^2x^2}{C}\right) \;+\; \log^+\left(\frac{W(x)}{K}\right).$$

Refer to part (a).)

By this problem we see in particular that if $W(x)$ *is the restriction to* $\mathbb{R}$ *of an entire function of exponential type zero* having, for some given K, the behaviour described therein, then, for

$$M \;=\; CK/A^2,$$

the integral J_M corresponding to W *satisfies the condition of our theorem pertaining to multipliers of exponential type*

$$A' \;=\; \frac{\sqrt{\pi}+1}{\sqrt{\pi}}\left(\frac{\pi}{2} + \eta\right)A$$

(rather than to those of exponential type A). For suitable choice of the constant C, the right side is

$$< \;\; 2.5A$$

Thus, subject to the above proviso regarding $W(x)$ and K, *the theorem will furnish* an even entire function ψ with $\psi(0) = 1$, *of exponential type* $2.5A$, for which

$$W(x)|\psi(x)| \leqslant 12.5e^2 CK \quad \text{on } \mathbb{R}$$

whenever the existence of such an entire φ, of *exponential type* A, with

$$W(x)|\varphi(x)| \leqslant K \quad \text{on } \mathbb{R}$$

is otherwise known.

There may be certain even entire functions Φ, $\geqslant 1$ on $\mathbb{R}$ and of exponential type zero, such that, for some *arbitrarily large values* of x_0, $\Phi(x) \leqslant \Phi(x_0)$ for $|x| \leqslant x_0$ and $\Phi(x) \geqslant \Phi(x_0)$ for $|x| \geqslant x_0$ with, in addition, *the graph of* $\log\Phi(x)$ vs $|x|$ *having a sizeable hump immediately to the right of each abscissa* x_0:

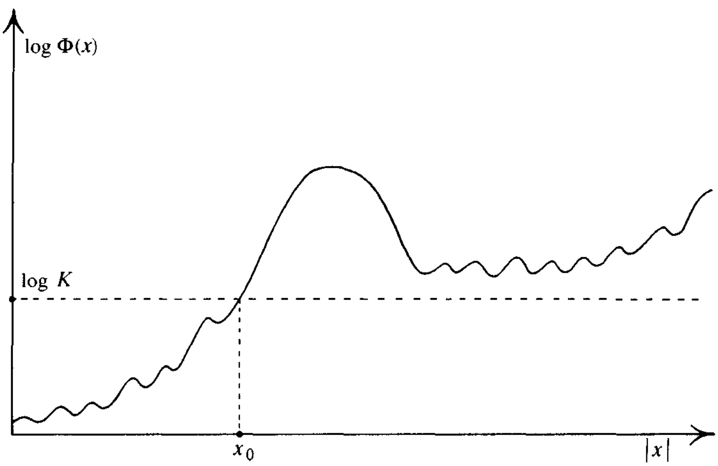


Figure 245

If we *have* such a function Φ and use the weight $W(x) = \Phi(x)$, the condition

$$\int_{x_0}^{\infty} \frac{1}{x\sqrt{(x^2 - x_0^2)}} \log\left(\frac{\Phi(x)}{K}\right) dx \leqslant \frac{\pi}{2} A$$

obtained in part (a) of the problem would, for $K = \Phi(x_0)$ with any of the x_0 just described, give us

$$\int_0^{\infty} \frac{1}{x^2} \log^+\left(\frac{\Phi(x)}{K}\right) dx \leqslant cA$$

where c is a number *definitely smaller* than $\pi/2$, and thus make it possible to *bring down the bound found in part* (b) *from* $(\pi/2 + \eta)A$ *to* $(c+\eta)A$. It is conceivable that one could construct such an entire function Φ with humps large enough to make

$$c \leqslant \frac{\sqrt{\pi}}{\sqrt{\pi}+1}$$

for a sequence of values of K tending to ∞ and values of A corresponding to them (through the *first* of the above two integral inequalities) tending to zero. Denoting the first sequence by $\{K_n\}$ and the second by $\{A_n\}$, we see that *for the function* Φ (if there is one!), the *upper bound provided by the theorem* would, for $A = A_n$, *be proportional to* K_n and hence *exceed the actual value in question by at most a constant factor* (for such A). Although I do not think the value of c can be diminished that much, the construction is perhaps worth *trying*. I have no time for that now; this book must go to press.

Our result seems farther from the truth when functions Φ of exponential type $B > 0$ are in question. For those, the condition on J_M figuring in the statement is essentially of the form

$$J_M \leqslant \text{const.}\ A^2$$

when A is small.

It is not terribly difficult to build even functions Φ of exponential type > 0 whose graphs (for real x) contain infinitely many *very long and practically flat plateaux*, e.g.,

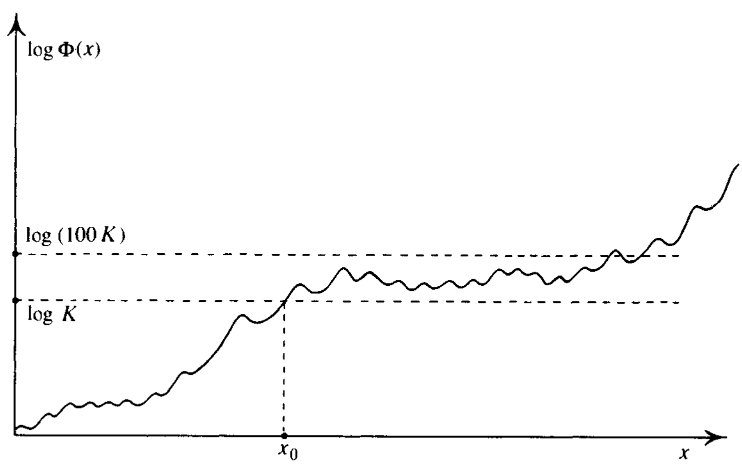


Figure 246

For this kind of function Φ one has arbitrarily large values of K (and of x_0 corresponding to them) such that

$$\int_0^\infty \frac{1}{x^2} \log^+\left(\frac{\Phi(x)}{100K}\right) dx$$

(say) *is exceedingly small* in comparison to

$$\int_{x_0}^\infty \frac{1}{x\sqrt{(x^2 - x_0^2)}} \log\left(\frac{\Phi(x)}{K}\right) dx;$$

putting the *latter* integral *equal* to $(\pi/2)A$, we can thus (for values of A corresponding to these particular ones of K) have the *former* integral $\leqslant$ const. A^2, and even much smaller. What brings

$$\int_0^\infty \frac{1}{x^2} \log\left(1 \ + \ A^2 x^2 \frac{\Phi(x)}{100CK}\right) dx$$

back up to a constant multiple of A in this situation is *not* the presence of $\Phi(x)/K$ in the integrand, but rather that of x^2 ! In order to reduce this last integral to a multiple of A^2, the A^2 figuring in the integrand must be replaced by A^4, making M *a constant multiple* of

$$K/A^4$$

if J_M is to satisfy the condition in the theorem. We thus find a *discrepancy involving the factor* $1/A^4$ between the *upper bound*

$$2e^2(A+B)^2 M \quad \cong \quad \text{const}\, K/A^4$$

furnished by our result (for *small* $A > 0$) and the *correct value, at least equal to* K when

$$A \quad = \quad \frac{2}{\pi} \int_{x_0}^\infty \frac{1}{x\sqrt{(x^2 - x_0^2)}} \log\left(\frac{\Phi(x)}{K}\right) dx.$$

It frequently turns out in actual examples that the K related to A in this way (and such that $\Phi(x) \leqslant K$ for $|x| < x_0$ while $\Phi(x) \geqslant K$ for $|x| \geqslant x_0$) goes to infinity quite rapidly as $A \longrightarrow 0$; one commonly finds that $K \sim \exp(\text{const.}/A)$. Compared with such behaviour, a few factors of $1/A$ more or less are practically of no account. Considering especially the approximate nature of the bound on $\int_{\partial\mathscr{D}} \log|G(t)| d\omega_{\mathscr{D}}(t, 0)$ that we have been using, it hardly seems possible to attain greater precision by the present method.

4. Still more about the energy. Description of the Hilbert space $\mathfrak{H}$ used in Chapter VIII, §C.5

Beginning with §B.5 of Chapter VIII, we have been denoting

$$\int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\rho(t)\,d\rho(x)$$

by $E(d\rho(t),\ d\rho(t))$ when dealing with real signed measures ρ on $[0,\ \infty)$ without point mass at the origin making the double integral absolutely convergent. In work with such measures ρ it is also convenient to write $U_\rho(x)$ for the Green potential

$$\int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\rho(t)\,,$$

at least in cases where the integral is well defined for $x > 0$. In the latter circumstance $U_\rho(x)$ *cannot*, as remarked at the end of §C.3, Chapter VIII, *be identically zero on* $(0,\ \infty)$ (or, for that matter, vanish a.e. with respect to $|d\rho|$ there) *unless the measure ρ vanishes.* It thus makes sense to regard

$$\sqrt{(E(d\rho(t),\ d\rho(t)))}$$

as a *norm*, $\|U_\rho\|_E$, for the functions $U_\rho(x)$ arising in such fashion. This norm comes from a *bilinear form* $\langle\ ,\ \rangle_E$ on those functions U_ρ, defined by putting

$$\langle U_\rho,\ U_\sigma\rangle_E \quad = \quad \int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\rho(t)\,d\sigma(x)$$

for any two of them, U_ρ and U_σ; the form's *positive definiteness* is a direct consequence of the results in §B.5, Chapter VIII. Since $\|U_\rho\|_E \ = \ \sqrt{(\langle U_\rho,\ U_\rho\rangle_E)}$, we obtain a certain *real Hilbert space* $\mathfrak{H}$ by forming the (abstract) *completion of the collection of functions U_ρ in the norm* $\|\ \ \|_E$.

The space $\mathfrak{H}$ was already used in the proof of the second theorem of §C.5, Chapter VIII. There, merely the *existence* of $\mathfrak{H}$ was needed, and we did not require any *concrete description* of its *elements.* One can indeed *make do* with just that existence and still proceed quite far. *Specific knowledge of* $\mathfrak{H}$ is, however, really necessary if one is to fully understand (and appreciate) the remaining work of this chapter. The present article is provided for that purpose.

It is actually better to use a wider collection of Green potentials U_ρ in forming the space $\mathfrak{H}$. One starts by showing that if ρ is a signed measure on $[0, \infty)$ without point mass at 0 making

$$\int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| |d\rho(t)||d\rho(x)| \quad < \quad \infty,$$

the integral,

$$\int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\rho(t)$$

is absolutely convergent at least *almost everywhere* (but perhaps not everywhere!) for $0 < x < \infty$. In these more general circumstances we will continue to denote that integral by $U_\rho(x)$; we will also have occasion to use the *extension* of that function to the *complex plane* given by the formula

$$U_\rho(z) \quad = \quad \int_0^\infty \log\left|\frac{z+t}{z-t}\right| d\rho(t).$$

It turns out to be true for these functions U_ρ that $E(d\rho(t), d\rho(t))$ is *determined* when $U_\rho(x)$ is *specified a.e.* on $\mathbb{R}$ (indeed, on $(0, \infty)$); we will in fact obtain a *formula* for the former quantity involving just the function $U_\rho(x)$. This will justify our writing

$$\|U_\rho\|_E \quad = \quad \sqrt{(E(d\rho(t), d\rho(t)))};$$

the space $\mathfrak{H}$ *will then be taken as the completion of the present class of functions* U_ρ *in the norm* $\|\ \ \|_E$. It will follow from our work* that the Hilbert space $\mathfrak{H}$ thus defined coincides with the one initially referred to in this article which, *a priori*, could be a proper subspace of it. That fact is pointed out now; we shall not, however, insist on it during our discussion for *as such* it will not be used.

We shall see in a moment that our space $\mathfrak{H}$ consists of *actual Lebesgue measurable odd functions defined* a.e. *on* $\mathbb{R}$; those will need to be *characterized.*

Let's get down to work.

Lemma. *If* ρ *is a real signed measure on* $[0, \infty)$ *without point mass at* 0, *such that*

$$\int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| |d\rho(t)||d\rho(x)| \quad < \quad \infty,$$

* see the last theorem in this article and the remark following it

the integral

$$\int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\rho(t)$$

is absolutely convergent for almost all real x, and equal a.e. on $\mathbb{R}$ *to* an odd Lebesgue measurable function which is locally L_1.

Proof. For x and $t > 0$, $\log|(x+t)/(x-t)| > 0$, and the left-hand expression is simply changed to its *negative* if, in it, x is replaced by $-x$. The whole lemma thus follows if we verify that

$$\int_0^a \int_0^\infty \log\left|\frac{x+t}{x-t}\right| |d\rho(t)|\, dx \quad < \quad \infty$$

for each finite a. We will use Schwarz' inequality for this purpose.

Fixing $a > 0$, we take the *restriction* λ *of ordinary Lebesgue measure to* $[0, a]$, and easily verify by direct calculation that

$$\int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\lambda(t)\, d\lambda(x) \quad < \quad \infty.$$

According, then, to the remark at the end of §B.5, Chapter VIII, the previous expression, nothing other than

$$E(|d\rho(t)|,\ d\lambda(t))$$

in the notation of that §, is

$$\leqslant \quad \sqrt{(E(|d\rho(t)|,\ |d\rho(t)|)\cdot E(d\lambda(t),\ d\lambda(t)))},$$

a finite quantity (by the hypothesis). Done.

By almost the same reasoning we can show that *the Hilbert space* $\mathfrak{H}$ must consist of Lebesgue measurable and locally integrable functions on $\mathbb{R}$. In the logical development of the present material, that statement should come somewhat later. Let us, however, strike while the iron is hot:

Theorem. *Suppose that the signed measures* ρ_n *on* $[0, \infty)$, *each without point mass at the origin, are such that*

$$\int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| |d\rho_n(t)|\,|d\rho_n(x)| \quad < \quad \infty$$

and that furthermore,

$$E\big(d\rho_n(t) - d\rho_m(t),\ d\rho_n(t) - d\rho_m(t)\big) \quad \xrightarrow[n,m]{} \quad 0.$$

Then, for each compact subset K of $\mathbb{R}$, the functions

$$U_n(x) \quad = \quad \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho_n(t)$$

(each defined a.e. by the lemma) form a Cauchy sequence in $L_1(K)$.

Proof. It is again sufficient to check this for sets $K = [0, a]$, where $a > 0$. Fixing any such a and focussing our attention on *some particular pair* (n, m), we take the function

$$\varphi(x) \quad = \quad \begin{cases} \operatorname{sgn}(U_n(x) - U_m(x)), & 0 \leqslant x \leqslant a, \\ 0 \text{ otherwise.} \end{cases}$$

Then,

$$\int_0^a |U_n(x) - U_m(x)| \, \mathrm{d}x \quad = \quad \int_0^\infty (U_n(x) - U_m(x))\, \varphi(x) \, \mathrm{d}x.$$

We have, however,

$$\int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| |\varphi(t)| \mathrm{d}t\, |\varphi(x)| \mathrm{d}x \quad \leqslant \quad \int_0^a \int_0^a \log\left|\frac{x+t}{x-t}\right| \mathrm{d}t \, \mathrm{d}x$$

with the right side finite, as already noted. Thence, by the remark at the end of §B.5, Chapter VIII,

$$\int_0^\infty (U_n(x) - U_m(x))\, \varphi(x) \, \mathrm{d}x \quad = \quad E(\mathrm{d}\rho_n(t) - \mathrm{d}\rho_m(t), \ \varphi(t)\, \mathrm{d}t)$$
$$\leqslant \quad \sqrt{\left(E(\mathrm{d}\rho_n(t) - \mathrm{d}\rho_m(t), \ \mathrm{d}\rho_n(t) - \mathrm{d}\rho_m(t)) \cdot E(\varphi(t)\mathrm{d}t, \ \varphi(t)\mathrm{d}t) \right)}.$$

Since $\log|(x+t)/(x-t)| \; > \; 0$ for x and $t \; > \; 0$,

$$E(\varphi(t)\, \mathrm{d}t, \ \varphi(t)\, \mathrm{d}t) \quad \leqslant \quad \int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| |\varphi(t)||\varphi(x)| \mathrm{d}t\, \mathrm{d}x$$

which, as we have just seen, is bounded above by a finite quantity – call it C_a – *depending on a but completely independent of n and m* ! The preceding relation thus boils down to

$$\int_0^a |U_n(x) - U_m(x)| \, \mathrm{d}x \quad \leqslant \quad \sqrt{\left(C_a E(\mathrm{d}\rho_n(t) - \mathrm{d}\rho_m(t), \ \mathrm{d}\rho_n(t) - \mathrm{d}\rho_m(t)) \right)},$$

and the theorem is proved.

Corollary. *Under the hypothesis of the theorem, a* subsequence *of the $U_n(x)$ converges a.e. to a locally integrable odd function $U(x)$ defined a.e. on $\mathbb{R}$. For* *any bounded measurable function φ of compact support in $[0, \infty)$, we have*

$$E(\mathrm{d}\rho_n(t),\ \varphi(t)\,\mathrm{d}t) \quad \xrightarrow[n]{} \quad \int_0^\infty U(x)\,\varphi(x)\,\mathrm{d}x.$$

Proof. The first part of the statement follows by elementary measure theory from the theorem. A standard application of Fatou's lemma then shows that

$$\int_0^a |U(x) - U_n(x)|\,\mathrm{d}x \quad \xrightarrow[n]{} \quad 0$$

for each finite a. Since the *left-hand* member of the limit relation to be proved is just

$$\int_0^\infty U_n(x)\,\varphi(x)\,\mathrm{d}x,$$

we are done.

Remark. Later on, an important generalization of the corollary will be given.

If we *only knew* that the measures $\varphi(t)\,\mathrm{d}t$ formed from *bounded* φ of *compact support* in $[0, \infty)$ were $\sqrt{(E(\ \ ,\ \ ))}$ *dense in the collection of signed measures* $\mathrm{d}\rho(t)$ satisfying the hypothesis of the above lemma, *it would follow* from the results just proved that *any element of that collection's abstract completion in said norm is determined by the measurable function $U(x)$ associated to the element in the way described by the corollary.* The density in question is indeed not too hard to verify; we will not, however, proceed in this manner. Instead, the statement just made will be established as a consequence of a formula to be derived below which, for other reasons, is needed in our work.

Given a measure ρ satisfying the hypothesis of our lemma, we will write

$$U_\rho(x) \quad = \quad \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t).$$

The function $U_\rho(x)$ is thus *odd*, and defined at least a.e. on $\mathbb{R}$. Concerning extension of the function U_ρ to the *complex plane*, we observe that the integral

$$\int_0^\infty \log\left|\frac{z+t}{z-t}\right| \mathrm{d}\rho(t)$$

converges *absolutely and uniformly* for z ranging over any compact subset of $\{\Im z > 0\}$ or of $\{\Im z < 0\}$.

It is sufficient to consider compact subsets K of the half-open quadrant

$$\{z\colon\ \Re z \geqslant 0\ (sic!)\ \text{and}\ \Im z > 0\}.$$

We have, by the lemma,

$$\int_0^\infty \log\left|\frac{x_0+t}{x_0-t}\right| |d\rho(t)| \quad < \quad \infty$$

for almost all $x_0 > 0$; *fixing any one of them* gives us a number C_K corresponding to the compact subset K such that

$$\log\left|\frac{z+t}{z-t}\right| \quad \leqslant \quad C_K \log\left|\frac{x_0+t}{x_0-t}\right| \qquad \text{for } t > 0 \text{ and } z \in K.$$

The affirmed uniform convergence is now manifest.

The integral

$$\int_0^\infty \log\left|\frac{z+t}{z-t}\right| d\rho(t)$$

is thus very well defined when z lies *off the real axis*; we denote that expression by $U_\rho(z)$. The uniform convergence just established makes $U_\rho(z)$ *harmonic* in both *the upper* and *the lower half planes*. It is, moreover, *odd*, and *vanishes on the imaginary axis*. At real points x where the integral used to define $U_\rho(x)$ is absolutely convergent, we have

$$U_\rho(x) \quad = \quad \lim_{y \to 0} U_\rho(x + iy),$$

so *on* $\mathbb{R}$, the function U_ρ can be regarded as the *boundary data* (existing a.e.) for the *harmonic function* $U_\rho(z)$ defined in *either* of the *half planes* bounded by $\mathbb{R}$.

We turn to the proof of the formula mentioned above which, for measures ρ meeting the conditions of the lemma, enables us to express $E(d\rho(t),\ d\rho(t))$ in terms of ρ's Green potential $U_\rho(x)$. We have the good fortune to *already know what that formula should be*, for, if the *behaviour* of

$$\rho(t) \quad = \quad \int_0^t d\rho(\tau)$$

is *nice enough*, problem 23(a), from the beginning of §B.8, Chapter VIII,