*completely independent* of the *particular* sum $S(z)$ under consideration. Therefore $W_{C,E}(\mathrm{i})$, the supremum of $|S(\mathrm{i})|$ for such sums $S$ with $\|S\|_{W,E} \leqslant 1$, is *finite*. This is what we needed to show in order to infer the proper inclusion of $\mathscr{C}_W(C,E)$ in $\mathscr{C}_W(E)$. The *first part* of our theorem is thus *proved*.

There remains the *second part*. *That*, however, is not new! Putting, as before, $W(x)$ equal to $\infty$ on $\mathbb{R} \sim E$, which makes $\mathscr{C}_W(C,E)$ coincide with the subspace $\mathscr{C}_W(C)$ of $\mathscr{C}_W(\mathbb{R})$ considered in Chapter VI, §E, we have, in the notation of that §,

$$\int_{-\infty}^{\infty} \frac{\log W_C(x)}{1+x^2}\,\mathrm{d}x \;<\; \infty$$

if $\mathscr{C}_W(C) \neq \mathscr{C}_W(\mathbb{R})$, according to *Akhiezer's theorem* (Chapter VI, §E.2). Our function $W_{C,E}(x)$ is simply $W_C(x)$. Hence, since $E \subseteq \mathbb{R}$ (!),

$$\int_E \frac{\log W_{C,E}(x)}{1+x^2}\,\mathrm{d}x \;<\; \infty$$

when $\mathscr{C}_W(C,E) \neq \mathscr{C}_W(E)$.

The theorem is completely proved. We are done.

***Remark.*** If we do not assume anything about the continuity of a weight $W(x) \geqslant 1$ defined on $E$, it is still possible to characterize the equality of $\mathscr{C}_W(C,E)$ with $\mathscr{C}_W(E)$ by an analogue of *Mergelian's second theorem* involving an integral over $E$. The establishment of such a result proceeds very much along the lines of the proof just finished, and is left to the reader.

**Problem 18**

Let $E$ be a closed set on $\mathbb{R}$ of the kind specified at the beginning of this §. Show that there are two constants $a$, $b$, depending on $E$, such that, for any entire function $f(z)$ of exponential type $\leqslant C$, bounded on $\mathbb{R}$, we have

$$\int_{-\infty}^{\infty} \frac{\log(1+|f(x)|^2)}{1+x^2}\,\mathrm{d}x \;\leqslant\; aC \;+\; b\int_E \frac{\log(1+|f(x)|^2)}{1+x^2}\,\mathrm{d}x.$$

(Hint: One may apply the *third* and *fourth* theorems from Chapter III, §G.3, and reason as in the above proof. Another procedure is to use the proof of the lemma in §E.1 of Chapter VI so as to first approximate $f(z)$ by finite sums $S(z)$ of the form considered above, having exponents in arithmetic progression.)

If $W(x)$, continuous and $\geqslant 1$ on $E$, is such that

$$\frac{x^n}{W(x)} \longrightarrow 0 \quad \text{for} \quad x \longrightarrow \pm\infty \quad \text{in} \quad E$$

and $n = 1, 2, 3, \ldots$, we denote by $W_{0,E}(z)$ the *supremum* of $|P(z)|$ for all *polynomials* $P$ with $\|P\|_{W,E} \leqslant 1$.

***Theorem.*** *Let* $E \subseteq \mathbb{R}$ *be a set of the kind specified at the beginning of this §, and let* $W(x)$, *continuous and* $\geqslant 1$ *on* $E$, *tend to* $\infty$ *faster than any power of* $x$ *as* $x \to \pm\infty$ *in* $E$.

*If, for* polynomials $P(z)$ *with* $\|P\|_{W,E} \leqslant 1$, *the integrals*

$$\int_E \frac{\log|P(x)|}{1+x^2}\,dx$$

*are* bounded above, *then* $\mathscr{C}_W(0,E)$ *is* properly contained *in* $\mathscr{C}_W(E)$.

*If* $\mathscr{C}_W(0,E)$ is *properly contained in* $\mathscr{C}_W(E)$, *then*

$$\int_E \frac{\log W_{0,E}(x)}{1+x^2}\,dx \;<\; \infty.$$

***Proof.*** The *second* part reduces (as at the end of the preceding demonstration) to a known result of Akhiezer (in this case from §B.1, Chapter VI) on putting $W(x) = \infty$ on $\mathbb{R} \sim E$. Hence only the *first* part requires discussion here.

According to *Pollard's theorem* (Chapter VI, §B.3), proper inclusion of $\mathscr{C}_W(0,E)$ in $\mathscr{C}_W(E)$ will certainly follow if the integrals

$$\int_{-\infty}^{\infty} \frac{\log(1+|P(x)|^2)}{1+x^2}\,dx$$

are *bounded above* for $P$ ranging over the *polynomials* with $\|P\|_{W,E} \leqslant 1$. It is therefore enough to *show* this, under the assumption that

$$\int_E \frac{\log|P(x)|}{1+x^2}\,dx \;\leqslant\; M, \text{ say,}$$

for *any* polynomial $P$ with $\|P\|_{W,E} \leqslant 1$.

We may, first of all, argue as in the proof of the above lemma to conclude that our assumption implies a seemingly stronger property: we have

$$\int_E \frac{\log(1+|P(x)|^2)}{1+x^2}\,dx \;\leqslant\; 2M + \pi\log 2$$

for the polynomials $P$ with $\|P\|_{W,E} \leqslant 1$. The proof will therefore be complete if we can verify that

$$\int_{-\infty}^{\infty} \frac{\log(1+|P(x)|^2)}{1+x^2}\,dx \;\leqslant\; b\int_E \frac{\log(1+|P(x)|^2)}{1+x^2}\,dx$$

for polynomials $P$, $b$ being a certain constant depending on the set $E$. This we do, using the result of *problem* 18.

Take any polynomial $P$, of degree $N$, say. With an arbitrary $\eta > 0$, put

$$f_\eta(z) = \left(\frac{\sin \eta z}{\eta z}\right)^N P(z);$$

$f_\eta(z)$ is then entire, of exponential type $N\eta$, and *bounded* on $\mathbb{R}$. By *problem* 18, we thus have

$$\int_{-\infty}^{\infty} \frac{\log(1+|f_\eta(x)|^2)}{1+x^2}dx \leqslant aN\eta + b\int_E \frac{\log(1+|f_\eta(x)|^2)}{1+x^2}dx.$$

Here, $|f_\eta(x)| \leqslant |P(x)|$ on $\mathbb{R}$ and $f_\eta(x) \longrightarrow P(x)$ as $\eta \to 0$, so the desired inequality follows on making $\eta \to 0$. We are done.

## 4. What happens when the set $E$ is sparse

The sets $E$ described at the beginning of this § have the property that

$$|E \cap I|/|I| \geqslant c > 0$$

for all intervals $I$ on $\mathbb{R}$ of length exceeding some $L$. In other words, their *lower uniform density* is *positive*. One suspects that the continual occurrence of the form $dx/(1+x^2)$ in the integrals over $E$ figuring in the preceding article is somehow connected with this positivity. As a first step towards finding out whether our hunch has any basis in fact, let us try to see what happens to the form $dx/(1+x^2)$ when $E$ becomes *sparse*. We do this in the special case where

$$E = \bigcup_{n=-\infty}^{\infty} [a_n - \delta,\ a_n + \delta]$$

with $a_n = |n|^p \operatorname{sgn} n$ and $p > 1$. This example was worked out by Benedicks (see his preprint), and all the material in the present article is due to him.

► In order that there may be no doubt, we point out that *the sets E now under consideration are no longer of the sort described at the beginning of the present* §.

***Lemma.*** *Let S be the square*

$$\{(x, y):\ -a < x < a \text{ and } -a < y < a\},$$

*and denote by H the* union *of its* two horizontal sides, *and by V the* union *of its* two vertical sides. *Then, if* $-a < x < a$,

$$\omega_S(H, x) \leqslant \omega_S(H, 0)$$

*and*

$$\omega_S(V, x) \geqslant \omega_S(V, 0),$$

*where, as usual,* $\omega_S(\ ,z)$ *denotes harmonic measure for* $S$.

***Proof*** (Benedicks). Let, wlog, $0 < x_0 < a$ and consider the harmonic function

$$\Delta(z) = \omega_S(H, z) - \omega_S(H, z + x_0)$$

defined in the rectangle

$$T = \{(x, y):\ -a < x < a - x_0 \text{ and } -a < y < a\}.$$

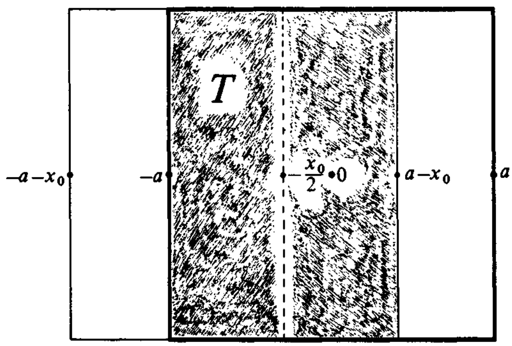



Figure 129

It is clear by symmetry that, for $z \in S$, $\omega_S(H, z) = \omega_S(H, \bar{z})$, and also $\omega_S(H, x + iy) = \omega_S(H, -x + iy)$. Therefore $\Delta(-\frac{1}{2}x_0 + iy) = 0$ on the vertical bisector of $T$ (see figure). Again, on $T$'s *right vertical side*,

$$\begin{aligned}\Delta(a - x_0 + iy) &= \omega_S(H, a - x_0 + iy) - \omega_S(H, a + iy) \\ &= \omega_S(H, a - x_0 + iy) \geqslant 0\end{aligned}$$

(and similarly, on the opposite side of $T$,

$$\Delta(-a + iy) = -\omega_S(H, -a + x_0 + iy) \leqslant 0).$$

It is clear on the other hand that $\Delta(z) = 0$ on the *top* and *bottom sides* of $T$ $(1 - 1 = 0)$. By the principle of maximum we thus have $\Delta(z) \geqslant 0$ in the *right half* of $T$; in particular,

$$\Delta(0) = \omega_S(H, 0) - \omega_S(H, x_0) \geqslant 0,$$

and $\omega_S(H, x_0) \leqslant \omega_S(H, 0)$, proving the *first* inequality asserted by the lemma.

The *second* inequality follows from the first one because

$$\omega_S(H,z) + \omega_S(V,z) \equiv 1$$

in $S$ and clearly $\omega_S(V,0) = \omega_S(H,0)$. We are done.

***Lemma*** (Benedicks). *Let $E \subseteq \mathbb{R}$ be any 'reasonable' closed set (for instance, a finite union of closed intervals), let $S$ be the square of the preceding lemma, and put*

$$\Omega = S \cap \sim E.$$

*If $H$ denotes the union of the two horizontal sides of $S$ and $V$ that of the vertical ones, we have*

$$\omega_\Omega(V,0) \leqslant \omega_\Omega(H,0)$$

*for the harmonic measure $\omega_\Omega(\ ,z)$ associated with the domain $\Omega$.*

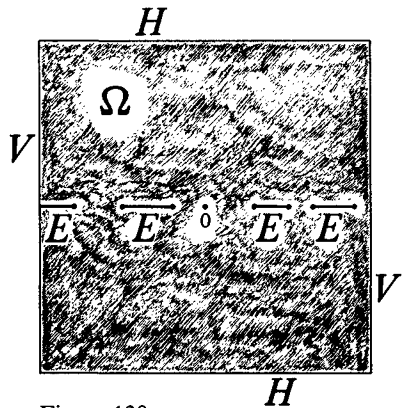



Figure 130

***Proof.*** By a formula derived near the end of §B.1, Chapter VII, for $z \in \Omega$,

$$\omega_\Omega(H,z) = \omega_S(H,z) - \int_E \omega_S(H,\xi)\,d\omega_\Omega(\xi,z)$$

$$\omega_\Omega(V,z) = \omega_S(V,z) - \int_E \omega_S(V,\xi)\,d\omega_\Omega(\xi,z).$$

From the previous lemma,

$$\omega_S(H,\xi) \leqslant \omega_S(H,0) = \omega_S(V,0) \leqslant \omega_S(V,\xi)$$

for real $\xi$ lying in $S$; in particular, for $\xi \in E$. Substituting this relation into the preceding ones and then making $z = 0$, we get $\omega_\Omega(V,0) \leqslant \omega_\Omega(H,0)$.

Q.E.D.

***Corollary.*** *In the above configuration,*

$$\omega_\Omega(\partial S, 0) \leqslant 2\omega_\Omega(H, 0).$$

***Proof.*** Clear.

***Lemma.*** *Let $p > 1$ and put*

$$E = \bigcup_{n=-\infty}^{\infty} [|n|^p \operatorname{sgn} n - \delta, \ |n|^p \operatorname{sgn} n + \delta],$$

*$\delta > 0$ being taken small enough so that the intervals figuring in the union do not intersect. With $x_0 > 0$, let $S_{x_0}$ be the square*

$$\left\{\frac{x_0}{2} < \Re z < \frac{3x_0}{2}, \ -\frac{x_0}{2} < \Im z < \frac{x_0}{2}\right\},$$

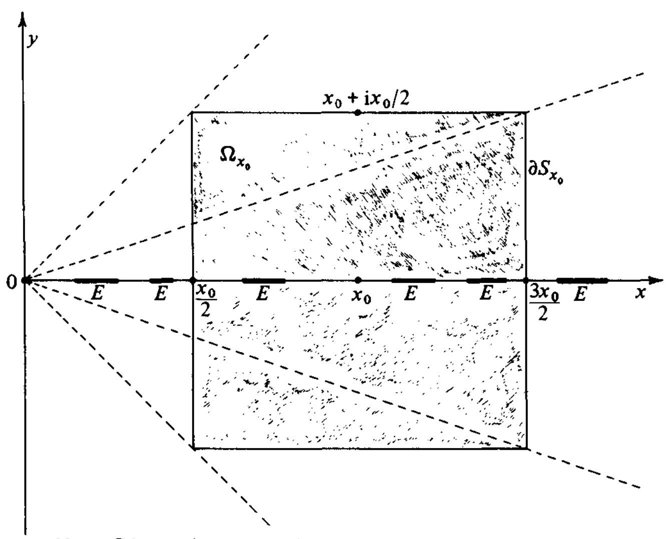
*and $\Omega_{x_0}$ the domain*

$$S_{x_0} \cap \sim E.$$

*For large $x_0$, the harmonic measure $\omega_{\Omega_{x_0}}(\quad, z)$ associated with $\Omega_{x_0}$ satisfies*

$$\omega_{\Omega_{x_0}}(\partial S_{x_0}, x_0) \leqslant \text{const.} \frac{\log x_0}{x_0^{1/p}}.$$

***Proof*** (Benedicks). By use of a test function and application of the preceding corollary.




Note: $E$ is not shown to scale

Figure 131

The function $z^{1/p}$ (taken as positive on the positive real axis) is analytic for $\Re z > 0$; so, therefore, is

$$\sin \pi z^{1/p}.$$

In $\Re z > 0$, this function vanishes only at the midpoints $n^p$ of the intervals making up $E$, and, at $x = n^p$,

$$\frac{\mathrm{d}(\sin \pi x^{1/p})}{\mathrm{d}x} = (-1)^n \frac{\pi}{pn^{p-1}}.$$

This means that, if we take $x_0 > 0$ *large* and put $C_0 = 1/x_0^{(p-1)/p}$, we have $|\sin \pi x^{1/p}| \geqslant kC_0\delta$ for $x$ *outside* $E$ on the interval $(x_0/2, 3x_0/2)$, $k > 0$ being a constant depending on $p$, but independent of $x_0$ and $\delta$. Recalling the behaviour of the Joukowski transformation

$$w \longrightarrow w + \sqrt{(w^2-1)},$$

we see that for a suitable definition of $\sqrt{\ }$, the function

$$v(z) = \log\left|\frac{\sin \pi z^{1/p}}{kC_0\delta} + \sqrt{\left(\frac{\sin^2 \pi z^{1/p}}{(kC_0\delta)^2} - 1\right)}\right|$$

is *positive and harmonic* in $\Omega_{x_0}$.

For this reason, when $x \in \mathbb{R} \cap \Omega_{x_0}$,

$$v(x) \geqslant \inf_{\zeta \in H} v(\zeta) \cdot \omega_{\Omega_{x_0}}(H, x),$$

$H$ denoting the union of the two *horizontal* sides of $\partial S_{x_0}$. However,

$$v(\zeta) \geqslant \text{const.} x_0^{1/p} \qquad \text{for } \zeta \in H$$

as is easily seen (almost without computation, if one refers to the above diagram). Also,

$$v(x) \leqslant \log \frac{2}{kC_0\delta} = (1 - 1/p)\log x_0 + O(1), \qquad x \in \mathbb{R} \cap \Omega_{x_0}.$$

Therefore

$$\omega_{\Omega_{x_0}}(H, x) \leqslant \text{const.} \frac{\log x_0}{x_0^{1/p}}, \qquad x \in \mathbb{R} \cap \Omega_{x_0}.$$

Since $x_0$ lies at the centre of the square $S_{x_0}$, the corollary to the previous lemma gives

$$\omega_{\Omega_{x_0}}(\partial S_{x_0}, x_0) \leqslant 2\omega_{\Omega_{x_0}}(H, x_0).$$

Combining this and the preceding relations, we obtain the desired result.

***Theorem*** (Benedicks). *Let $G$ be the Green's function for the domain*

$$\mathscr{D} \;=\; \mathbb{C} \sim E \;=\; \mathbb{C} \sim \bigcup_{n=-\infty}^{\infty} [|n|^p \operatorname{sgn} n - \delta,\; |n|^p \operatorname{sgn} n + \delta],$$

*where $p > 1$ and $\delta > 0$ is small enough so that the intervals in the union do not intersect.*

*Then, for real $x$ of large modulus,*

$$G(x, \mathrm{i}) \;\leqslant\; C\frac{\log|x|}{|x|^{(p+1)/p}},$$

*with a constant $C$ depending on $p$ and $\delta$.*

***Proof.*** $G(z, \mathrm{i})$ is certainly bounded above – by $M$ say – in the sector $\{0 \leqslant |\Im z| < \Re z\}$. Given $x_0 > 0$, the square $S_{x_0}$ considered in the previous lemma lies in that sector, so $G(\zeta, \mathrm{i}) \leqslant M$ on $\partial S_{x_0}$. $G(z, \mathrm{i})$ is, moreover, harmonic in $\Omega_{x_0} \subseteq \mathscr{D}$ and *zero on $E$*, whence $G(x_0, \mathrm{i}) \leqslant M \cdot \omega_{\Omega_{x_0}}(\partial S_{x_0}, x_0)$. By the last lemma we therefore have

$$(*) \qquad G(x_0, \mathrm{i}) \;\leqslant\; \text{const.}\frac{\log|x_0|}{|x_0|^{1/p}}$$

for large $x_0 > 0$.

Benedicks' idea is to now use Poisson's formula for the half plane, so as to improve $(*)$ by iteration. Take any fixed $\alpha$ with $0 < \alpha < 1/p$. Then $(*)$ certainly implies (by symmetry of $E$) that

$$G(x, \mathrm{i}) \;\leqslant\; \frac{\text{const.}}{|x|^\alpha + 1}, \qquad x \in \mathbb{R},$$

$G(x, \mathrm{i})$ being at any rate bounded on the real axis. The function $G(z, \mathrm{i})$ is in fact *bounded and harmonic in* $\Im z < 0$, so

$$G(z, \mathrm{i}) \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|}{|z - t|^2} G(t, \mathrm{i})\,\mathrm{d}t, \qquad \Im z < 0.$$

Plugging in the previous relation, we get

$$G(z, \mathrm{i}) \;\leqslant\; \text{const.}\int_{-\infty}^{\infty} \frac{|y|}{(x - t)^2 + y^2}\cdot\frac{\mathrm{d}t}{|t|^\alpha + 1}, \qquad y < 0.$$

Let, wlog, $x > 0$. Then, the integral just written can be broken up as $\int_{-x/2}^{x/2} + \int_{|t| \geqslant x/2}$. Since $\int_{-\infty}^{\infty} (|y|/((x - t)^2 + y^2))\mathrm{d}t = \pi$, the *second* of these terms is obviously $\mathrm{O}(x^{-\alpha})$ for large $x$. The *first*, on the other hand, is

$$\leqslant\; \text{const.}\frac{4|y|}{x^2 + 4y^2}\cdot x^{1-\alpha},$$

so, all in all,

$$G(z, \mathrm{i}) \leqslant \text{const.}|x|^{-\alpha}$$

for $0 > y > -|x|$, $|x|$ being large.

The inequality just found *remains true*, however, for $0 < y < |x|$, in spite of the logarithmic singularity that $G(z, \mathrm{i})$ has at i. This follows from the fact that $0 < \alpha < 1/p < 1$ and the relation

$$G(z, \mathrm{i}) - G(\bar{z}, \mathrm{i}) = \log\left|\frac{z+\mathrm{i}}{z-\mathrm{i}}\right|.$$

To verify the latter, just subtract the right side from the left. The difference is *harmonic* in $\Im z > 0$ and *bounded* there (the logarithmic poles at i cancel each other out). It is also clearly zero on $\mathbb{R}$, so hence zero for $\Im z > 0$. For large $|z|$, $\log|(z+i)/(z-i)| = O(1/|z|)$, and we see that

$$G(z, \mathrm{i}) \leqslant \text{const.}|x|^{-\alpha} \qquad \text{for } 0 \leqslant |y| < x$$

since this inequality is true for $0 > y > -|x|$.

Suppose that $x_0 > 0$ is large; we can use the previous lemma again. By what has just been shown,

$$G(\zeta, \mathrm{i}) \leqslant \text{const.}|x_0|^{-\alpha}, \qquad \zeta \in \partial S_{x_0}.$$

Arguing as at the beginning of this proof, we get

$$G(x_0, \mathrm{i}) \leqslant \text{const.}|x_0|^{-\alpha}\omega_{\Omega_{x_0}}(\partial S_{x_0}, x_0) \leqslant \text{const.}\frac{\log|x_0|}{|x_0|^{\alpha}|x_0|^{1/p}}.$$

Hence, since $0 < \alpha < 1/p$, we have

$$G(x, \mathrm{i}) \leqslant \frac{\text{const.}}{|x|^{2\alpha}+1}, \qquad x \in \mathbb{R}.$$

*The exponent $\alpha$ in the inequality we started with has been improved to $2\alpha$.*

If now $2\alpha < 1$, we may start from the inequality just obtained and *repeat* the above argument, ending with the relation

$$G(x, \mathrm{i}) \leqslant \frac{\text{const.}}{|x|^{3\alpha}+1}, \qquad x \in \mathbb{R}.$$

The process may evidently be continued so as to yield successively the estimates

$$G(x, \mathrm{i}) \leqslant \frac{\text{const.}}{|x|^{n\alpha}+1}, \qquad x \in \mathbb{R},$$

with $n = 3, 4, \ldots$, *as long as* $(n-1)\alpha < 1$. Choosing $\alpha$, $0 < \alpha < 1/p$, *to not* *be of the form* $1/m$, $m = 1, 2, 3, \ldots$, we arrive at an estimate

$$(\ast\ast) \qquad G(x, \mathrm{i}) \leqslant \frac{\text{const.}}{|x|^{n\alpha} + 1}, \qquad x \in \mathbb{R},$$

*where* $n\alpha$ *is the first integral multiple of* $\alpha$ *strictly* $> 1$.

Because the exponent $n\alpha$ in $(\ast\ast)$ is $> 1$, we have

$$\int_{-\infty}^{\infty} G(t, \mathrm{i})\,\mathrm{d}t < \infty.$$

As before, for $y < 0$, we can write

$$G(z, \mathrm{i}) = \frac{1}{\pi}\int_{-|x|/2}^{|x|/2} \frac{|y|}{|z-t|^2} G(t, \mathrm{i})\,\mathrm{d}t + \frac{1}{\pi}\int_{|t| \geqslant |x|/2} \frac{|y|}{|z-t|^2} G(t, \mathrm{i})\,\mathrm{d}t.$$

For $|t| \leqslant |x|/2$, $|y|/|z-t|^2 \leqslant 1/|x|$, so the *first* term on the right is $\leqslant \text{const.}/|x|$ in view of the preceding relation. The *second* is $\leqslant \text{const.}/|x|^{n\alpha} = o(1/|x|)$ by $(\ast\ast)$. Thence, for $|x|$ large,

$$G(z, \mathrm{i}) \leqslant \frac{\text{const.}}{|x|}, \qquad y < 0.$$

Using the relation

$$G(z, \mathrm{i}) - G(\bar{z}, \mathrm{i}) = \log\left|\frac{\mathrm{i}+z}{\mathrm{i}-z}\right|$$

as above, we find that in fact

$$G(z, \mathrm{i}) \leqslant \frac{\text{const.}}{|x|} \qquad \text{for } |z| \text{ large.}$$

Take this relation and apply the preceding lemma *one more time*. For large $x_0$, we have

$$G(\zeta, \mathrm{i}) \leqslant \text{const.}/x_0 \qquad \text{on } \partial S_{x_0}.$$

Therefore

$$G(x_0, \mathrm{i}) \leqslant \frac{\text{const.}}{x_0}\,\omega_{\Omega_{x_0}}(\partial S_{x_0}, x_0) \leqslant \frac{\text{const.} \log x_0}{x_0^{1+(1/p)}}.$$

This is what we wanted to prove.

We are done.

***Corollary.*** *A Phragmén–Lindelöf function* $Y(z)$ *exists for the domain*

$$\mathbb{C} \sim \bigcup_{n=-\infty}^{\infty} [|n|^p \operatorname{sgn} n - \delta,\ |n|^p \operatorname{sgn} n + \delta].$$

***Proof.*** By the theorem, we certainly have

$$\int_{-\infty}^{\infty} G(x,\mathrm{i})\,\mathrm{d}x \;<\; \infty.$$

The result then follows by the second theorem of article 2.

***Remark.*** Although the theorem tells us that, *on the real axis,*

$$G(x,\mathrm{i}) \;\leqslant\; \text{const.}\frac{\log|x|}{|x|^{1+1/p}}$$

when $|x|$ is large, the inequality

$$G(z,\mathrm{i}) \;\leqslant\; \frac{\text{const.}}{|x|},$$

valid for $|z|$ large, obtained near the end of the theorem's proof, *cannot be improved* in the sector $0 \leqslant |y| \leqslant |x|$.

Indeed, since $G(t,\mathrm{i}) \geqslant 0$ we have, for large $|x|$,

$$G(x - \mathrm{i}|x|,\ \mathrm{i}) \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|x|}{(x-t)^2 + x^2}\,G(t,\mathrm{i})\,\mathrm{d}t$$

$$\geqslant\; \frac{4}{13\pi|x|}\int_{-|x|/2}^{|x|/2} G(t,\mathrm{i})\,\mathrm{d}t \;\sim\; \frac{4}{13\pi|x|}\int_{-\infty}^{\infty} G(t,\mathrm{i})\,\mathrm{d}t.$$

A better bound on $G(z,\mathrm{i})$ *can* be obtained if $|y|$ is *much smaller* than $|x|$. The following result is used in the next exercise.

***Lemma.*** *For large* $|x|$,

$$G(z,\mathrm{i}) \;\leqslant\; \text{const.}\frac{\log|x|}{|x|^{1+1/p}}, \qquad 0 \leqslant |y| \leqslant |x|^{1-1/p}.$$

***Proof.*** Taking wlog $x > 0$, consider first the case where $y < 0$. By the theorem,

$$G(z,\mathrm{i}) \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|y|}{(x-t)^2 + y^2}\,G(t,\mathrm{i})\,\mathrm{d}t$$

$$\leqslant\; \text{const.}\int_{-\infty}^{\infty} \frac{|y|}{(x-t)^2 + y^2}\cdot\frac{\log^+|t| + 1}{|t|^{1+1/p} + 1}\,\mathrm{d}t.$$

As usual, we break up the right-hand integral into

$$\int_{-x/2}^{x/2} \;+\; \int_{|t|\geqslant x/2}$$

The first term is $\leqslant$ const.$|y|/x^2$ (because $1 + 1/p > 1$ !), and this is

$\leqslant$ const.$/x^{1+1/p}$ for $|y| \leqslant x^{1-1/p}$. The second term is clearly

$$\leqslant \frac{\text{const.}\log x}{x^{1+1/p}}.$$

This handles the case of negative $y$.

For $0 < y < x^{1-1/p}$, use the relation

$$G(z, \mathrm{i}) - G(\bar{z}, \mathrm{i}) = \log\left|\frac{z+\mathrm{i}}{z-\mathrm{i}}\right|$$

already applied in the proof of the theorem. Note that the *right hand* side is

$$\Re \log\left(\frac{1+(\mathrm{i}/z)}{1-(\mathrm{i}/z)}\right) = 2\frac{\Im z}{|z|^2} + \mathrm{O}\left(\frac{1}{|z|^3}\right)$$

for large $|z|$. For $0 \leqslant \Im z \leqslant |x|^{1-1/p}$, this is

$$\leqslant \text{const.}\frac{1}{|x|^{1+1/p}}.$$

The lemma thus follows because it is true for negative $y$.

In the following problem the reader is asked to work out the analogue, for our present sets $E$, of Benedicks' beautiful result about the ones with positive lower uniform density (Problem 16).

**Problem 19**

If $t$ is on the component $[n^p - \delta,\ n^p + \delta]$ of

$$\mathscr{D} = \mathbb{C} \sim \bigcup_{-\infty}^{\infty} [|k|^p \operatorname{sgn} k - \delta,\quad |k|^p \operatorname{sgn} k + \delta],$$

show that

$$\frac{\mathrm{d}\omega_{\mathscr{D}}(t, \mathrm{i})}{\mathrm{d}t} \leqslant \frac{\text{const.}}{t^{1+1/p} + 1} \cdot \frac{1}{\sqrt{(\delta^2 - (t-n^p)^2)}},$$

where $\omega_{\mathscr{D}}(\ , z)$ denotes harmonic measure for $\mathscr{D}$. Here, the constant depends only on $p > 1$ and $\delta > 0$.

***Remark.*** The result is due to Benedicks. We see that the factor $\log|t|$ in the estimate for $G(t, \mathrm{i})$ furnished by the above theorem *disappears* when we evaluate *harmonic measure.*

Hint for the problem: One proceeds as in the solution of Problem 16, here comparing $G(z, \mathrm{i})$ with

$$U(z) = \log\left|\frac{z-n^p}{\delta} + \sqrt{\left(\left(\frac{z-n^p}{\delta}\right)^2 - 1\right)}\right|$$

on the ellipse $\Gamma_n$ with foci at $n^p \pm \delta$ and semi-minor-axis equal to $n^{p-1}\delta$:

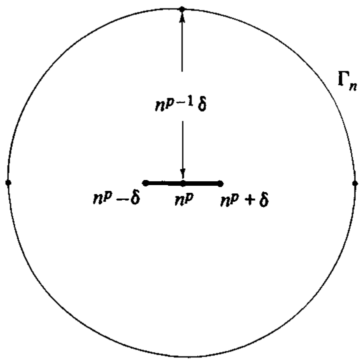



Figure 132

By Problem 19, we have, for the harmonic measure of the component

$$E_n \;=\; [|n|^p \operatorname{sgn} n - \delta,\; |n|^p \operatorname{sgn} n + \delta]$$

of $E = \partial\mathscr{D}$,

$$\boxed{\omega_{\mathscr{D}}(E_n, \mathrm{i}) \;\leqslant\; \frac{\text{const.}}{|n|^{p+1}+1}.}$$

Using this estimate, one can establish a result corresponding to the *first part* of the first theorem in article 3.

***Theorem.*** *Let $W(x) \geqslant 1$ be continuous on*

$$E \;=\; \bigcup_{-\infty}^{\infty} [|n|^p \operatorname{sgn} n - \delta,\; |n|^p \operatorname{sgn} n + \delta],$$

*and suppose that $W(x) \longrightarrow \infty$ for $x \to \pm\infty$ in $E$. If, for some $C > 0$, the supremum of*

$$\int_E \frac{\log|S(t)|}{1+|t|^{1+1/p}}\,\mathrm{d}t$$

*for $S$ ranging over all finite sums of the form*

$$S(t) \;=\; \sum_{-C \leqslant \lambda \leqslant C} a_\lambda e^{\mathrm{i}\lambda t}$$

*with $\|S\|_{W,E} \leqslant 1$ is* finite, *then $\mathscr{C}_W(E, C) \neq \mathscr{C}_W(E)$.*

***Proof.*** We have the above boxed estimate for the harmonic measure (in $\mathscr{D} = \mathbb{C} \sim E$) of the components of $E$, and a previous corollary gives us a Phragmén–Lindelöf function $Y(z)$ for $\mathscr{D}$. Using these facts, one proceeds exactly as in the proof of the first theorem of article 3.

***Remark.*** The sparsity of the set $E$ involved here has caused the form $\mathrm{d}t/(1+t^2)$ occurring in the result of article 3 to be replaced by $\mathrm{d}t/(1+|t|^{1+1/p})$.

***Remark.*** The statement of the above theorem goes in only one direction, unlike that of the corresponding one in article 3. There, since we were dealing with the restriction of the form $\mathrm{d}t/(1+t^2)$ to $E$, we were able to obtain a *converse* by simply appealing to Akhiezer's theorem from §E.2 of Chapter VI. In the present situation we can't do that, because we are dealing with $\mathrm{d}t/(1+|t|^{1+1/p})$ instead of $\mathrm{d}t/(1+t^2)$, and $1/p < 1$. *It would be interesting to see whether* (as seems likely) *the converse is true here.*

In case $W(x) \to \infty$ faster than any power of $|x|$ as $x \to \pm\infty$ in $E$, we can formulate a result like the above one for *polynomial approximation* on $E$ in the weight $W$. The statement of it is exactly like that of the *first part* of the *second* theorem in article 3, save that the integrals

$$\int_E \frac{\log|P(x)|}{1+x^2}\mathrm{d}x$$

figuring there are here replaced by

$$\int_E \frac{\log|P(x)|}{1+|x|^{1+1/p}}\mathrm{d}x.$$

The proof runs much like that of the result in article 3. Details are left to the reader.

## B. The set $E$ reduces to the integers

Consider the set

$$E_\rho = \bigcup_{n=-\infty}^{\infty} [n-\rho,\ n+\rho],$$

where $0 \leqslant \rho < \frac{1}{2}$. If $\rho > 0$, the results of §§A.1–A.3 apply to $E_\rho$, and there is, in particular, a constant $b_\rho$ such that the inequality

$$\int_{-\infty}^{\infty} \frac{\log(1+|P(x)|^2)}{1+x^2}\mathrm{d}x \leqslant b_\rho \int_{E_\rho} \frac{\log(1+|P(x)|^2)}{1+x^2}\mathrm{d}x,$$

used in proving the second theorem of §A.3, holds for polynomials $P$.

For this reason, given any $M$, the set of polynomials $P$ such that

$$\int_{E_\rho} \frac{\log^+|P(x)|}{1+x^2}\,dx \leqslant M$$

forms a normal family in the complex plane.

Suppose now that $\rho = 0$. Then $E_\rho = \mathbb{Z}$, and the proof in §A of the above inequality involving $b_\rho$, available when $\rho > 0$, cannot be made to work so as to yield a relation of the form

$$\int_{-\infty}^{\infty} \frac{\log(1+|P(x)|^2)}{1+x^2}\,dx \leqslant C\sum_{-\infty}^{\infty} \frac{\log(1+|P(n)|^2)}{1+n^2}$$

That proof depends on the properties of harmonic measure for $\mathscr{D}_\rho = \mathbb{C} \sim E_\rho$ worked out in §A.1 (for $\rho > 0$); there is, however, *no harmonic measure* for $\mathscr{D} = \mathbb{C} \sim \mathbb{Z}$. This makes it seem very unlikely that the set of polynomials $P$ satisfying

$$\sum_{-\infty}^{\infty} \frac{\log^+|P(n)|}{1+n^2} \leqslant M$$

for arbitrary given $M$ would form a normal family in the complex plane, and it is in fact easy to construct a counter example to such a claim.

Take simply

$$P_N(x) = (1-x^2)^{[N/\log N]} \prod_{k=1}^{N}\left(1-\frac{x^2}{k^2}\right)$$

for $N \geqslant 2$, with $[p]$ denoting the greatest integer $\leqslant p$ as usual. Then it is not hard to verify that

$$(*) \qquad \sum_{-\infty}^{\infty} \frac{\log^+|P_N(n)|}{1+n^2} \leqslant 20 \qquad \text{for } N \geqslant 8.$$

At the same time,

$$P_N(\mathrm{i}) \geqslant 2^{[N/\log N]} \xrightarrow[N]{} \infty.$$

**Problem 20**

Prove $(*)$.

$$\left(\text{Hint:} \sum_{1}^{\infty} \frac{1}{n^2}\log^+|P_N(n)| \leqslant \left[\frac{N}{\log N}\right] \sum_{n=N+1}^{\infty} \frac{\log(n^2-1)}{n^2}\right.$$

$$+ \sum_{n=N+1}^{\infty} \frac{1}{n^2}\left[\sum_{k=1}^{N} \log\left|\frac{n^2}{k^2}-1\right|\right]^+ .$$

After replacing the sums on the right by suitable integrals and doing some calculation, one obtains the upper bound

$$2 + \frac{2}{\log N} + 2 + 2\int_1^\infty \log\left(\frac{\xi+1}{\xi-1}\right)\frac{d\xi}{\xi}.$$

Here, the integral can be worked out by contour integration.)

This example, however, does not invalidate the analogue (with obvious *statement*) of Akhiezer's theorem for weighted polynomial approximation on $\mathbb{Z}$. In order to disprove such a conjecture, one would (at least) need similar examples with the number 20 standing on the right side of $(*)$ replaced by *arbitrarily small quantities* $> 0$. No matter how one tries to construct such examples, something always seems to go wrong. It seems impossible to diminish the number in $(*)$ to less than a certain strictly positive quantity without forcing boundedness of the $|P_N(\mathrm{i})|$. One comes in such fashion to believe in the existence of a number $C > 0$ such that the set of polynomials $P$ with

$$\sum_{-\infty}^{\infty} \frac{\log^+ |P(n)|}{1+n^2} \leqslant C$$

*does form a normal family* in the complex plane.

This partial extension of the result from §A.3 to the limiting case $E_\rho = \mathbb{Z}$ turns out to be *valid*. With its help one can establish the *complete analogue* of Akhiezer's theorem for weighted polynomial approximation on $\mathbb{Z}$; its interest is not, however, limited to that application. The extension is easily reduced to a special version of it for polynomials $P$ of the particular form

$$P(x) = \prod_k \left(1 - \frac{x^2}{x_k^2}\right)$$

with *real* roots $x_k > 0$, and most of the *real work* is involved in the treatment of this case, taking up all but the last two of the following articles. The investigation is straightforward but very laborious; although I have tried hard to simplify it, I have not succeeded too well.

The difficulties are what they are, and there is no point in stewing over them. It is better to just take hold of the traces and forge ahead.

## 1. Using certain sums as upper bounds for integrals corresponding to them

Our situation from now up to almost the end of the present § is as follows: we have a polynomial $P(z)$ of the special form

$$P(z) = \prod_k \left(1 - \frac{z^2}{x_k^2}\right),$$

where the $x_k$ are $> 0$ (in other words, $P(z)$ is *even*, with all of its zeros *real*, and $P(0) = 1$), and we are given an *upper bound* for the sum

$$\sum_{-\infty}^{\infty} \frac{\log^+ |P(m)|}{1 + m^2},$$

or, what amounts to the same thing *here*, for

$$\sum_{1}^{\infty} \frac{1}{m^2} \log^+ |P(m)|.$$

From this information we desire to *obtain* a bound on $|P(z)|$ for each complex $z$.

The first idea that comes to mind is to try to use our knowledge about the preceding *sum* in order to control *the integral*

$$\int_{-\infty}^{\infty} \frac{\log^+ |P(x)|}{1 + x^2} \mathrm{d}x;$$

we have indeed seen in Chapter VI, §B.1, how to deduce an upper bound on $|P(z)|$ from one for this integral. This plan, although probably too simple to be carried out as it stands, does suggest a start on the study of our problem. For *certain intervals* $I \subset (0, \infty)$,

$$\int_I \frac{\log |P(x)|}{x^2} \mathrm{d}x$$

*is* comparable with

$$\sum_{m \in I \cap \mathbb{Z}} \frac{\log^+ |P(m)|}{m^2}.$$

We have

$$\frac{\mathrm{d}^2 \log |P(x)|}{\mathrm{d}x^2} \;=\; -2 \sum_k \frac{x^2 + x_k^2}{(x^2 - x_k^2)^2} \;<\; 0,$$

so $\log |P(x)|$ is *concave* (downward) on any real interval *free of the zeros* $\pm x_k$ *of* $P$. This means that, if $a < b$ and $P$ has no *zeros* on $[a, b]$,

$$\int_a^b \log |P(x)| \mathrm{d}x \;\leqslant\; (b - a) \log |P(m)|$$

for the *midpoint* $m$ of $[a, b]$:

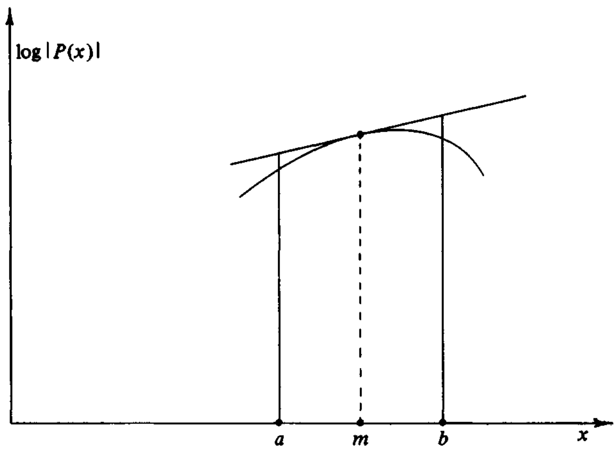



Figure 133

Of course $\int_a^b \log|P(x)|\,dx$ is not the integral we are dealing with here. If, however, $a > 0$ is large and $b - a$ not too big, the presence of the factor $1/x^2$ in front of $\log|P(x)|$ does not make much difference. A similar formula still holds, except that $m$ is no longer *exactly the midpoint* of $[a, b]$.

***Lemma.*** *Let* $0 < a < m < b$, *and suppose that* $P$ *has no zeros on* $[a, b]$. *Then*

$$\int_a^b \frac{\log|P(x)|}{x^2}\,dx \;\leqslant\; (\log|P(m)|)\int_a^b \frac{dx}{x^2},$$

*provided that*

$$\boxed{\log\frac{b}{a} = \frac{m}{a} - \frac{m}{b}.}$$

***Proof.*** Let $M$ denote the *slope* of the graph of $\log|P(x)|$ vs. $x$ at $x = m$. Then, since $\log|P(x)|$ is *concave* on $[a, b]$, we have there

$$\log|P(x)| \;\leqslant\; \log|P(m)| + M(x - m)$$

(see the previous figure). Hence

$$\int_a^b \frac{\log|P(x)|}{x^2}\,dx \;\leqslant\; (\log|P(m)|)\int_a^b \frac{dx}{x^2} + M\int_a^b \left(\frac{1}{x} - \frac{m}{x^2}\right)dx.$$

The second term on the right is

$$M\log\frac{b}{a} \;-\; M\left(\frac{m}{a}-\frac{m}{b}\right),$$

and this is zero if the boxed condition on $m$ is satisfied. Done.

We will be interested in situations where the number $m$ figuring in the above boxed relation is a *positive integer*, and where *one* of the two numbers $a$, $b$ ($a\leqslant m\leqslant b$) is to be *found*, the other being *given*. Regarding these, we have two estimates.

***Lemma.*** *If $m\geqslant 7$ and $m-1\leqslant a\leqslant m$, the number $b\geqslant m$ such that*

$$\log\frac{b}{a} \;=\; \frac{m}{a} \;-\; \frac{m}{b}$$

*is $\leqslant\; m+2$.*

***Proof.*** Write $\rho=a/b$; then $0<\rho\leqslant 1$, and the relation to be satisfied becomes $\log(1/\rho)=(m/a)(1-\rho)$. If $a=m$, this is obviously satisfied for $\rho=1$, i.e., $m=b$; otherwise $0<\rho<1$, and we have

$$\frac{m}{a} \;=\; \frac{\log\dfrac{1}{\rho}}{1-\rho}.$$

Now

$$\log\frac{1}{\rho} \;=\; 1-\rho+\tfrac{1}{2}(1-\rho)^2+\tfrac{1}{3}(1-\rho)^3+\cdots,$$

so the preceding relation implies that

$$\frac{m}{a} \;\geqslant\; 1+\tfrac{1}{2}(1-\rho),$$

i.e.,

$$1-\rho \;\leqslant\; 2\frac{m-a}{a},$$

and

$$\rho \;\geqslant\; \frac{3a-2m}{a}.$$

Therefore

$$b \;=\; \frac{a}{\rho} \;\leqslant\; \frac{a^2}{3a-2m},$$

and

$$b - m \leqslant \frac{(2m-a)(m-a)}{3a-2m} \leqslant \frac{m+1}{m-3}.$$

Here the right-hand side is $\leqslant 2$ for $m \geqslant 7$. We are done.

***Lemma.*** *If $m \geqslant 2$ and $m \leqslant b \leqslant m+1$, the number $a \leqslant m$ such that*

$$\log\frac{b}{a} = \frac{m}{a} - \frac{m}{b}$$

*is $> m-2$.*

***Proof.*** Put $\rho = a/b$ as in proving the preceding lemma; here, it is also convenient to write

$$y = \frac{m}{b}.$$

Then $0 < \rho \leqslant 1$ and $0 < y \leqslant 1$. In terms of $y$ and $\rho$, our equation becomes

$$\log\frac{1}{\rho} = \frac{y}{\rho} - y.$$

When $y < 1$, we must also have $\rho < 1$, and then

$$y = \frac{\rho\log(1/\rho)}{1-\rho}.$$

This yields, for $0 < \rho < 1$,

$$\frac{\mathrm{d}y}{\mathrm{d}\rho} = \frac{\log(1/\rho)-(1-\rho)}{(1-\rho)^2} = \tfrac{1}{2} + \tfrac{1}{3}(1-\rho) + \tfrac{1}{4}(1-\rho)^2 + \cdots \geqslant \tfrac{1}{2}.$$

Hence, since the value $y = 1$ corresponds to $\rho = 1$, we have, for $0 < y < 1$,

$$\tfrac{1}{2}(1-\rho) \leqslant 1 - y,$$

i.e.,

$$\rho \geqslant 1 - 2(1-y).$$

It was given that $m \leqslant b \leqslant m+1$, so

$$1 - y = \frac{b-m}{b} \leqslant \frac{1}{m+1}$$

(the middle term here is a monotone function of $b$). Therefore, by the previous relation,

$$\rho \geqslant 1 - \frac{2}{m+1},$$

and finally,

$$a = \rho b \geqslant \rho m \geqslant m - \frac{2m}{m+1} > m - 2.$$

We are done.

***Theorem.*** *Let $6 \leqslant a < b$. There is a number $b^*$, $b \leqslant b^* < b+3$, such that*

$$\int_a^{b^*} \frac{\log|P(x)|}{x^2}\,\mathrm{d}x \leqslant 5 \sum_{a<m<b^*} \frac{\log^+|P(m)|}{m^2},$$

*provided that $P$ has no zeros on $[a, b^*]$. The sum on the right is taken over the* integers *$m$ with $a < m < b^*$.*

► ***Definition.*** During the rest of this §, we will say that $b^*$ is *well disposed* with respect to $a$.

***Proof.*** By repeated application of the first two of the above lemmas.

Let the integer $m_1$ be such that $m_1 - 1 \leqslant a < m_1$; then $m_1 \geqslant 7$, so, by the *second* lemma, we can find a number $a_1$, $m_1 < a_1 \leqslant m_1 + 2$, with

$$\log\frac{a_1}{a} = \frac{m_1}{a} - \frac{m_1}{a_1}.$$

We have $a_1 \leqslant a + 3$, so, since $b > a$, $a_1 < b + 3$.

By the *first* lemma, if $P(x)$ is free of zeros on $[a, a_1]$,

$$\int_a^{a_1} \frac{\log|P(x)}{x^2}\,\mathrm{d}x \leqslant \log|P(m_1)| \int_a^{a_1} \frac{\mathrm{d}x}{x^2}.$$

Here, $a_1 - a \leqslant 3$ and $m_1/a \leqslant \frac{7}{6}$, so

$$\int_a^{a_1} \frac{\mathrm{d}x}{x^2} \leqslant \frac{5}{m_1^2}.$$

Therefore,

$$\int_a^{a_1} \frac{\log|P(x)|}{x^2}\,\mathrm{d}x \leqslant \frac{5\log^+|P(m_1)|}{m_1^2}.$$

If now $a_1 \geqslant b$, we simply put $b^* = a_1$ and the theorem is proved. Otherwise, $a_1 < b$ and we take the integer $m_2$ such that $m_2 - 1 \leqslant a_1 < m_2$. Since $a_1 > m_1$, $m_2 > m_1$, and we can find an $a_2$, $m_2 < a_2 \leqslant m_2 + 2$, with

$$\log\frac{a_2}{a_1} = \frac{m_2}{a_1} - \frac{m_2}{a_2}.$$

We have $a_2 \leqslant a_1 + 3 < b + 3$, and, by the first lemma,

$$\int_{a_1}^{a_2} \frac{\log|P(x)|}{x^2}\mathrm{d}x \leqslant \log|P(m_2)| \int_{a_1}^{a_2} \frac{\mathrm{d}x}{x^2} \leqslant \frac{5\log^+|P(m_2)|}{m_2^2}$$

just as in the preceding step, provided that $P$ has no zeros on $[a_1, a_2]$.

If $a_2 \geqslant b$, we put $b^* = a_2$. If not, we continue as above, getting numbers $a_3 > a_2$, $a_4 > a_3$, and so forth, $a_{k+1} \leqslant a_k + 3$, until we first reach an $a_l$ with $a_l \geqslant b$. We will then have $a_l < b + 3$, and we put $b^* = a_l$. There are integers $m_k$, $m_2 < m_3 < \cdots < m_l$, with $a_{k-1} < m_k < a_k$, $k = 3, \ldots, l$, and, as in the previous steps,

$$\int_{a_{k-1}}^{a_k} \frac{\log|P(x)|}{x^2}\mathrm{d}x \leqslant \frac{5\log^+|P(m_k)|}{m_k^2}$$

for $k = 3, \ldots, l$, as long as $P$ has no zeros on $[a_{k-1}, a_k]$.

Write $a_0 = a$. Then, if $P$ has no zeros on $[a, b^*] = [a_0, a_l]$,

$$\int_a^{b^*} \frac{\log|P(x)|}{x^2}\mathrm{d}x = \sum_{k=1}^{l} \int_{a_{k-1}}^{a_k} \frac{\log|P(x)|}{x^2}\mathrm{d}x$$

$$\leqslant \sum_{k=1}^{l} \frac{5\log^+|P(m_k)|}{m_k^2} \leqslant \sum_{\substack{a<m<b^* \\ m\in\mathbb{Z}}} \frac{5\log^+|P(m)|}{m^2}.$$

We are done.

In the result just proved, $a$ is kept fixed and we move from $b$ to a point $b^*$ well disposed with respect to $a$, lying between $b$ and $b+3$. One can obtain the same effect keeping $b$ fixed and moving downward from $a$.

***Theorem.*** *Let $10 \leqslant a < b$. There is an $a^*$, $a - 3 < a^* \leqslant a$, such that $b$ is well disposed with respect to $a^*$, i.e.,*

$$\int_{a^*}^{b} \frac{\log|P(x)|}{x^2}\mathrm{d}x \leqslant 5 \sum_{a^*<m<b} \frac{\log^+|P(m)|}{m^2},$$

*provided that $P(x)$ has no zeros on $[a^*, b]$.*

The proof uses the *first* and *third* of the above lemmas, and is otherwise very much like the one of the previous theorem. Its details are left to the reader.

## 2. Construction of certain intervals containing the zeros of $P(x)$

We have seen in the preceding article how certain intervals $I \subseteq (0, \infty)$ can be obtained for which

$$\int_I \frac{\log|P(x)|}{x^2}dx \leqslant 5\sum_{m\in I}\frac{\log^+|P(m)|}{m^2}$$

as long as they are free of zeros of $P$. Our next step is to split up $(0, \infty)$ into two kinds of intervals: zero-free ones of the sort just mentioned and then some residual ones which, together, contain all the positive zeros of $P(x)$. The latter are closely related to some intervals used earlier by Vladimir Bernstein (*not* the S. Bernstein after whom the weighted polynomial approximation problem is named) in his study of Dirichlet series, and it is to their construction we now turn.

▶ As is customary, we denote by $n(t)$ *the number of zeros* $x_k$ *of* $P(x)$ *in the interval* $[0, t]$ for $t \geqslant 0$ (*counting multiplicities as in Chapter* III). When $t < 0$, we take $n(t) = 0$. The function $n(t)$ is thus *integer-valued and increasing*. It is *zero* for all $t > 0$ *sufficiently close to* 0 (because the $x_k > 0$), and *constant* for sufficiently large $t$ ($P$ being a polynomial).

The graph of $n(t)$ vs. $t$ consists of some *horizontal portions* separated by *jumps*. At each jump, $n(t)$ increases by an integral multiple of unity. In this quantization must lie the essential difference between the behaviour of subharmonic functions of the *special form* $\log|P(x)|$ with $P$ a polynomial, and that of *general* ones having at most logarithmic growth at $\infty$, for which there holds no valid analogue of the theorem to be established in this §. (Just look at the subharmonic functions $\eta\log|P_N(z)|$, where $\eta > 0$ is arbitrarily small and the $P_N$ are the polynomials considered in Problem 20.) During the present article we will see precisely how the quantization affects matters.

For the following work we *fill in the vertical portions* of the *graph* of $n(t)$ vs. $t$. In other words, if $n(t)$ has a jump discontinuity at $t_0$, we consider the *vertical segment joining* $(t_0, n(t_0-))$ *to* $(t_0, n(t_0+))$ *as forming part of that graph*.

Our constructions are arranged in three stages.

### *First stage. Construction of the Bernstein intervals*

We begin by taking an arbitrary *small* number $p > 0$ (requiring, say, that $p < 1/20$). Once chosen, $p$ is kept *fixed* during most of the discussion of this and the following articles.

Denote by $\mathcal{O}$ the set of points $t_0 \in \mathbb{R}$ with the property that *a straight line of slope* $p$ through $(t_0, n(t_0))$ *cuts or touches the graph of* $n(t)$ *vs.* $t$ *only once*. $\mathcal{O}$ is open and its *complement* in $\mathbb{R}$ consists of a *finite number* of closed intervals $B_0$, $B_1$, $B_2,\ldots$ called the Bernstein intervals for slope $p$ associated with the polynomial $P(x)$. (Together, the $B_k$ make up what V. Bernstein called a *neighborhood set* for the positive zeros of $P$ – see page 259 of his book on Dirichlet series. His construction of the $B_k$ is different from the one given here.) It is best to show the formation of the $B_k$ by a diagram:

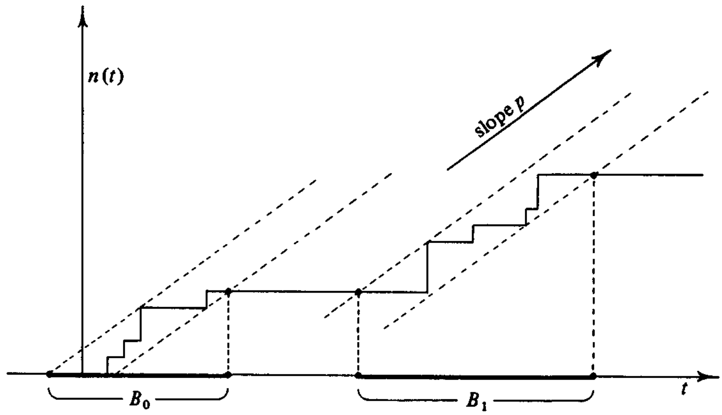



Figure 134

We see that all the positive zeros of $P$ (points of discontinuity of $n(t)$) are contained in the union of the $B_k$. Also, *taking any* $B_k$ *and denoting it by* $[a, b]$:

*The part of the graph of* $n(t)$ *vs.* $t$ *corresponding to the values of* $t$ *in* $B_k$ *lies between the two parallel lines of slope* $p$ *through the points* $(a, n(a))$ *and* $(b, n(b))$.

For a closed interval $I = [\alpha, \beta]$, say, let us write $n(I)$ for $n(\beta+) - n(\alpha-)$. The statement just made then implies that

$$n(B_k)/p|B_k| \leqslant 1$$

for each Bernstein interval. An inequality in the opposite sense is less apparent.

***Lemma*** (Bernstein). *For each of the* $B_k$,

$$n(B_k)/p|B_k| \geqslant 1/2.$$

***Proof.*** It is geometrically evident that a line of slope $p$ which cuts (or touches) the graph of $n(t)$ vs. $t$ *more than once* must come into contact with some *vertical portion* of it – let the reader make a diagram.

Take any interval $B_k$, denote it by $[a,b]$, and denote the portion of the graph of $n(t)$ vs. $t$ corresponding to the values $a \leqslant t \leqslant b$ by G. We indicate by $L$ and $M$ the lines of slope $p$ through the points $(a, n(a))$ and $(b, n(b))$ respectively. According to our definition, any line $N$ of slope $p$ *between* $L$ and $M$ must cut (or touch) the graph of $n(t)$ vs. $t$ *at least twice*, and hence come into contact with some *vertical portion* of that graph. Otherwise such a line $N$, which surely cuts $G$, would intersect the graph *only once*, at some point with abscissa $t_0 \in (a,b)$; $t_0$ would then belong to $\mathcal{O}$ and thus *not* to $B_k$. The line $N$ must in fact come into contact with a vertical portion of $G$, for, as a glance at the preceding figure shows, it can never touch any part of the graph that does not lie over $[a,b]$.

In order to prove the lemma, it is therefore enough to show that *if*

$$n(B_k)/p|B_k| \;<\; 1/2.$$

*there must be some line $N$ of slope $p$, lying between $L$ and $M$, that does not come into contact with any vertical portion of $G$.*

Let $V$ be the *union* of the *vertical portions* of $G$, and for $X \in V$, denote by $\Pi(X)$ *the downward projection, along a line with slope $p$*, of the point $X$ onto the horizontal line through $(a, n(a))$.

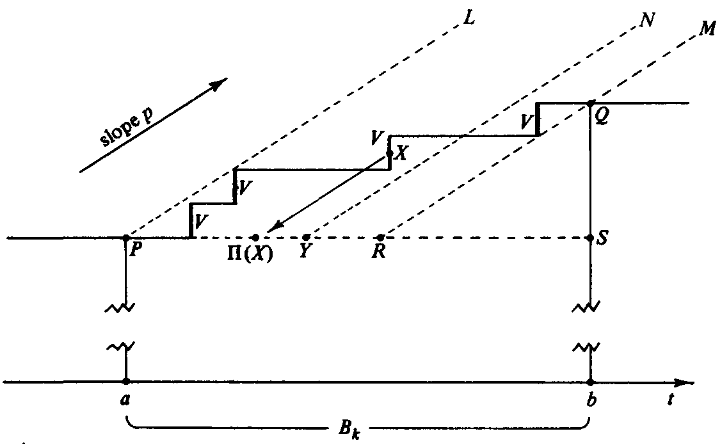



Figure 135

In this figure, $|B_k| = \overline{PS}$ and $n(B_k) = \overline{QS}$. The result, $\Pi(V)$, of applying $\Pi$ to all the points of $V$ is a certain closed subset of the segment $PR$, and, if we use $|\quad|$ to denote linear Lebesgue measure, it is clear that

$$|\Pi(V)| \;\leqslant\; |V|/p.$$

We have $p\cdot\overline{RS} = \overline{QS}$, so, if $n(B_k) = \overline{QS} < \frac{1}{2}p|B_k| = \frac{1}{2}p\cdot\overline{PS}$, $p\cdot\overline{RS} < \frac{1}{2}p\cdot\overline{PS}$, and therefore $\overline{PR} > \frac{1}{2}\overline{PS} > \overline{QS}/p = |V|/p$. With the preceding relation, this yields

$$|\Pi(V)| \;<\; \overline{PR}.$$

There is thus a point $Y$ on $PR$ *not belonging* to the projection $\Pi(V)$. If, then, $N$ is the line of slope $p$ through $Y$, $N$ cannot come into contact with $V$. This line $N$ lies between $L$ and $M$, so we are done.

***Second stage. Modification of the Bernstein intervals***

The Bernstein intervals $B_k$ just constructed include all the positive zeros of $P(x)$, and

$$\frac{1}{2} \;\leqslant\; \frac{n(B_k)}{p|B_k|} \;\leqslant\; 1.$$

We are going to modify them so as to obtain new closed intervals $I_k \subseteq (0,\infty)$ containing all the positive zeros of $P(x)$, positioned so as to make

$$\int_I \frac{\log|P(x)|}{x^2}\,\mathrm{d}x \;\leqslant\; 5\sum_{m\in I}\frac{\log^+|P(m)|}{m^2}$$

for each of the interval components $I$ of

$$(0,\infty) \sim \bigcup_k I_k.$$

(Note that $B_0$ need not even be contained in $[0,\infty)$.) For the calculations which come later on, it is also very useful to have *all the ratios* $n(I_k)/|I_k|$ *the same*, and we carry out the construction so as to ensure this.

Specifically, the intervals $I_k$, which we will write as $[\alpha_k, \beta_k]$ with $k = 0, 1, 2, \ldots$ and $0 < \alpha_0 < \beta_0 < \alpha_1 < \beta_1 < \cdots$, are to have the following properties:

(i) *All the positive zeros of* $P(x)$ *are contained in the union of the* $I_k$,

(ii) $n(I_k)/p|I_k| = \frac{1}{2}$, $\quad k = 0, 1, 2, \ldots,$

(iii) *For* $\alpha_0 \leqslant t \leqslant \beta_0$,

$$n(\beta_0) - n(t) \;\leqslant\; \frac{p}{1-3p}(\beta_0 - t),$$

*and, for* $\alpha_k \leqslant t \leqslant \beta_k$ *with* $k \geqslant 1$,

$$n(t) - n(\alpha_k) \;\leqslant\; \frac{p}{1-3p}(t - \alpha_k),$$

$$n(\beta_k) - n(t) \;\leqslant\; \frac{p}{1-3p}(\beta_k - t),$$

(recall that we are assuming $0 < p < \frac{1}{20}$),

(iv) *For* $k \geqslant 1$, $\alpha_k$ *is well disposed with respect to* $\beta_{k-1}$ (see the preceding article).

Denote the Bernstein intervals $B_k$, $k = 0, 1, 2, \ldots$, by $[a_k, b_k]$, arranging the indices so as to have $b_{k-1} < a_k$. We begin by constructing $I_0$. Take $\alpha_0$ as the *smallest* positive zero of $P(x)$; $\alpha_0$ is the first point of discontinuity of $n(t)$ and $a_0 < \alpha_0 < b_0$. We have

$$\frac{n([\alpha_0, b_0])}{p(b_0 - \alpha_0)} = \frac{n(B_0)}{p(b_0 - \alpha_0)} > \frac{n(B_0)}{p|B_0|} \geqslant \frac{1}{2}$$

by the lemma from the preceding (first) stage. For $\tau \geqslant b_0$, let $J_\tau$ be the interval $[\alpha_0, \tau]$. As we have just seen,

$$n(J_\tau)/p|J_\tau| > 1/2$$

for $\tau = b_0$. When $\tau$ increases from $b_0$ to $a_1$ (assuming that there *is* a Bernstein interval $B_1$; there need not be!) the numerator of the left-hand ratio remains equal to $n(B_0)$, while the denominator increases. The ratio itself therefore decreases when $\tau$ goes from $b_0$ to $a_1$, and either gets down to $\frac{1}{2}$ in $(b_0, a_1)$, or else remains $> \frac{1}{2}$ there. (In case there is no Bernstein interval $B_1$ we may take $a_1 = \infty$, and then the first possibility is realized.)

Suppose that we *do* have $n(J_\tau)/p|J_\tau| = \frac{1}{2}$ for some $\tau$, $b_0 < \tau < a_1$. Then we put $\beta_0$ *equal to that value of* $\tau$, and property (ii) certainly holds for $I_0 = [\alpha_0, \beta_0]$. Property (iii) does also. Indeed, by construction of the $B_k$, the line of slope $p$ through $(\beta_0, n(\beta_0))$ cuts the graph of $n(t)$ vs. $t$ *only once*, so the portion of the graph corresponding to values of $t < \beta_0$ *lies entirely to the left* of that line (look at the *first* of the diagrams in this article). That is,

$$n(\beta_0) - n(t) \leqslant p(\beta_0 - t), \qquad t \leqslant \beta_0,$$

whence, *a fortiori*,

$$n(\beta_0) - n(t) \leqslant \frac{p}{1 - 3p}(\beta_0 - t), \qquad t \leqslant \beta_0$$

(since $0 < p < 1/20$, $0 < 1 - 3p < 1$).

It may happen, however, that $n(J_\tau)/p|J_\tau|$ remains $> \frac{1}{2}$ for $b_0 < \tau < a_1$. Then that ratio is *still* $\geqslant \frac{1}{2}$ for $\tau = b_1$. This is true because $n(B_1)/p|B_1| \geqslant \frac{1}{2}$ (lemma from the preceding stage), and

$$n([\alpha_0, b_1]) = n(a_1 -) - n(\alpha_0 -) + n(B_1),$$

while

$$b_1 - \alpha_0 = a_1 - \alpha_0 + |B_1|.$$

Thus, in our present case, $n(J_\tau)/p|J_\tau|$ is $\geqslant \frac{1}{2}$ for $\tau = b_1$ and again decreases as $\tau$ moves from $b_1$ towards $a_2 > b_1$. (If there *is* no interval $B_2$ we may take $a_2 = \infty$.) If, for some $\tau \in [b_1, a_2)$, we have $n(J_\tau)/p|J_\tau| = \frac{1}{2}$, we take $\beta_0$ equal to that value of $\tau$, and property (ii) holds for $I_0 = [\alpha_0, \beta_0]$. Also, for $\beta_0 \in [b_1, a_2)$, the part of the graph of $n(t)$ vs. $t$ corresponding to the values $t \leqslant \beta_0$ lies *on or entirely to the left* of the line of slope $p$ through $(\beta_0, n(\beta_0))$, as in the situation already discussed. Therefore, $n(\beta_0) - n(t) \leqslant (p/(1-3p))(\beta_0 - t)$ for $t \leqslant \beta_0$ as before, and property (iii) holds for $I_0$.

In case $n(J_\tau)/p|J_\tau|$ still remains $> \frac{1}{2}$ for $b_1 \leqslant \tau < a_2$, we will have $n(J_\tau)/p|J_\tau| \geqslant \frac{1}{2}$ for $\tau = b_2$ by an argument like the one used above, and we look for $\beta_0$ in the interval $[b_2, a_3)$. The process continues in this way, and we either get a $\beta_0$ lying between two successive intervals $B_k$, $B_{k+1}$ (perhaps coinciding with the right endpoint of $B_k$), or else pass through the half open interval separating the *last two* of the $B_k$ without ever bringing the ratio $n(J_\tau)/p|J_\tau|$ down to $\frac{1}{2}$. If this second eventuality occurs, suppose that $B_l = [a_l, b_l]$ is the *last* $B_k$; then $n(J_\tau)/p|J_\tau| \geqslant \frac{1}{2}$ for $\tau = b_l$ by the reasoning already used. Here, $n(J_\tau)$ remains equal to $n([0, b_l])$ for $\tau \geqslant b_l$ while $|J_\tau|$ increases without limit, so a value $\beta_0$ of $\tau \geqslant b_l$ will make $n(J_\tau)/p|J_\tau| = \frac{1}{2}$. There is then only one interval $I_k$, namely, $I_0 = [\alpha_0, \beta_0]$, and our construction is finished, because properties (i) and (ii) obviously hold, while (iii) does by the above reasoning and (iv) is vacuously true.

In the event that the process gives us a $\beta_0$ *lying between two successive Bernstein intervals*, we have to construct $I_1 = [\alpha_1, \beta_1]$. In these circumstances we must first choose $\alpha_1$ so as to have it *well disposed* with respect to $\beta_0$, ensuring property (iv) for $k = 1$.

► ***It is here that we make crucial use of the property that each jump in*** $n(t)$ ***has height*** $\geqslant$ **1.**

Assume that $b_k \leqslant \beta_0 < a_{k+1}$. We have $p(\beta_0 - \alpha_0) = 2n(I_0) \geqslant 2$ with $0 < p < \frac{1}{20}$; therefore $\beta_0 > 40$ and there is by the *first* theorem of the preceding article a number $\alpha_1$, $a_{k+1} \leqslant \alpha_1 < a_{k+1} + 3$, which is *well disposed with respect to* $\beta_0$.

Now $\alpha_1$ may well lie to the right of $a_{k+1}$. It is nevertheless true that $n(\alpha_1) = n(a_{k+1} -)$, and moreover

$$n(t) - n(\alpha_1) \leqslant \frac{p}{1-3p}(t - \alpha_1) \qquad \text{for } t \geqslant \alpha_1.$$

The following diagram shows how these properties follow from two facts:

*that $n(t)$ increases by at least* 1 *at each jump*, and that $1/p > 3$:

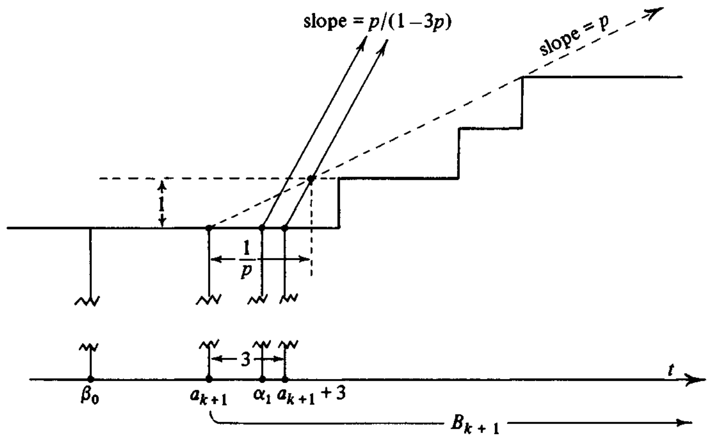



Figure 136

For this choice of $\alpha_1$, properties (i)–(iv) will hold, provided that $\beta_1$, $\alpha_2$ and so forth are correctly determined.

We go on to specify $\beta_1$. This is very much like the determination of $\beta_0$. Since

$$n(b_{k+1}) - n(\alpha_1) \;=\; n(b_{k+1}) - n(a_{k+1}-) \;=\; n(B_{k+1}),$$

we certainly have

$$\frac{n([\alpha_1, b_{k+1}])}{p(b_{k+1}-\alpha_1)} = \frac{n(B_{k+1})}{p(b_{k+1}-\alpha_1)} \geqslant \frac{n(B_{k+1})}{p|B_{k+1}|} \geqslant \frac{1}{2}$$

by the lemma from the preceding stage. For $\tau \geqslant b_{k+1}$, denote by $J'_\tau$ the interval $[\alpha_1, \tau]$; then $n(J'_\tau)/p|J'_\tau|$ is $\geqslant \frac{1}{2}$ for $\tau = b_{k+1}$ and *diminishes* as $\tau$ increases along $[b_{k+1}, a_{k+2})$. (If there *is* no $B_{k+2}$ we take $a_{k+2} = \infty$.) We may evidently proceed just as above to get a $\tau \geqslant b_{k+1}$, lying either in a half open interval separating two successive Bernstein intervals or else beyond all of the latter, such that $n(J'_\tau)/p|J'_\tau| = \frac{1}{2}$. *That value* of $\tau$ is taken as $\beta_1$. The part of the graph of $n(t)$ vs. $t$ corresponding to values of $t \leqslant \beta_1$ lies, as before, *on or to the left* of the line through $(\beta_1, n(\beta_1))$ with slope $p$. Hence, *a fortiori*,

$$n(\beta_1) - n(t) \;\leqslant\; \frac{p}{1-3p}(\beta_1 - t) \qquad \text{for } t \leqslant \beta_1.$$

We see that properties (ii) and (iii) hold for $I_0$ and $I_1 = [\alpha_1, \beta_1]$.

If $I_0 \cup I_1$ does not already include all of the $B_k$, $\beta_1$ must lie between two of them, and we may proceed to find an $\alpha_2$ in the way that $\alpha_1$ was found above. Then we can construct an $I_2$. Since there are only a finite number of $B_k$, the process will eventually stop, and we will end with a finite number of intervals $I_k = [\alpha_k, \beta_k]$ having properties (ii)–(iv). Property (i) will then also hold, since, when we finish, the union of the $I_k$ includes that of the $B_k$.

Here is a picture showing the relation of the intervals $I_k$ to the graph of $n(t)$ vs. $t$:

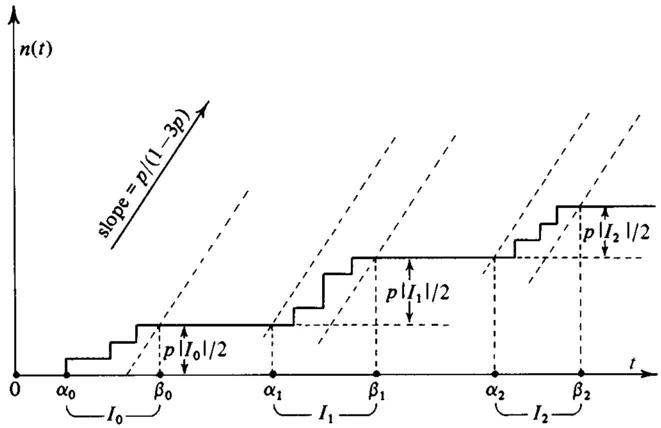



Figure 137

Let us check the statement made before starting the construction of the $I_k$, to the effect that

$$\int_I \frac{\log|P(x)|}{x^2}\,dx \;\leqslant\; 5\sum_{m\in I}\frac{\log^+|P(m)|}{m^2}$$

for each of the interval components $I$ of the *complement*

$$(0,\infty) \sim \bigcup_k I_k.$$

Since, for $k > 0$, $\alpha_k$ is well disposed with respect to $\beta_{k-1}$, this is certainly true for the components $I$ of the form $(\beta_{k-1}, \alpha_k)$, $k \geqslant 1$ (if there *are* any!), by the first theorem of the preceding article. This is also true, and trivially so, for $I = (0, \alpha_0)$, because

$$|P(x)| \;=\; \prod_k \left|1 - \frac{x^2}{x_k^2}\right| \;<\; 1$$

for $0 < x < \alpha_0$, all the positive zeros $x_k$ of $P(x)$ being $\geqslant \alpha_0$. Finally, if $I_l$ is the *last* of the $I_k$, our relation is true for $I = (\beta_l, \infty)$. This follows because we can obviously get *arbitrarily large* numbers $A > \beta_l$ which are well disposed with respect to $\beta_l$. We then have

$$\int_{\beta_l}^{A} \frac{\log|P(x)|}{x^2}\,dx \;\leqslant\; 5 \sum_{\beta_l < m < A} \frac{\log^+|P(m)|}{m^2}$$

for each such $A$ by the first theorem of the preceding article, and need only make $A$ tend to $\infty$.

### *Third stage. Replacement of the first few intervals $I_k$ by a single one if $n(t)/t$ is not always $\leqslant p/(1-3p)$*

Recall that the problem we are studying is as follows: we are presented with an unknown even polynomial $P(z)$ having only real roots and such that $P(0) = 1$, and *told* that

$$\sum_1^\infty \frac{\log^+|P(m)|}{m^2}$$

*is small.* We are *asked to obtain,* for $z \in \mathbb{C}$, a *bound on* $|P(z)|$ depending on that sum, but independent of $P$.

As a control on the size of $|P(z)|$ we will use the quantity

$$\sup_{t>0} \frac{n(t)}{t}.$$

A computation like the one at the end of §B, Chapter III, shows indeed that

$$\log|P(z)| \;\leqslant\; \pi|z| \sup_{t>0} \frac{n(t)}{t}.$$

We are therefore interested in obtaining an upper bound on $\sup_{t>0}(n(t)/t)$ from a suitable (small) one for

$$\sum_1^\infty \frac{\log^+|P(m)|}{m^2}.$$

Our procedure is to *work backwards,* assuming that $\sup_{t>0}(n(t)/t)$ is *not small* and thence deriving a *strictly positive lower bound for the sum.* We begin with the following simple

***Lemma.*** *If $\sup_{t>0}(n(t)/t) > p/(1-3p)$, we have $|I_0|/\beta_0 \geqslant \frac{2}{3}$ for the interval $I_0 = [\alpha_0, \beta_0]$ arrived at in the previous stage of our construction.*

***Proof.*** Let us examine carefully the *initial portion* of the last diagram given above:

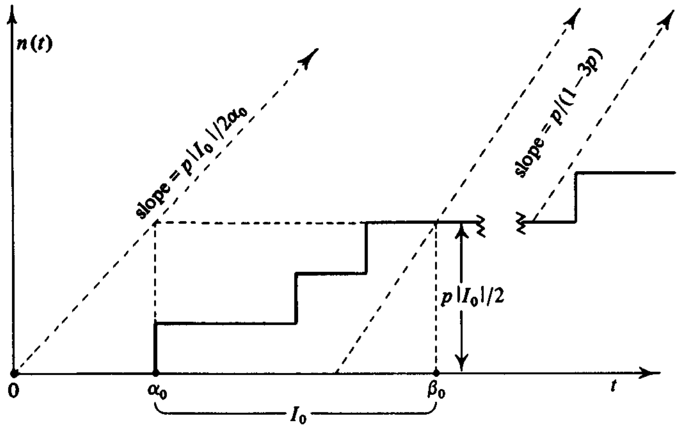



Figure 138

We see that, for $t > 0$,

$$\frac{n(t)}{t} \leqslant \max\left\{\frac{p}{1-3p}, \frac{p|I_0|}{2\alpha_0}\right\},$$

*whether or not* the first term in curly brackets is less than the second.

Here,

$$\frac{p|I_0|}{2\alpha_0} = \frac{p}{2}\cdot\frac{|I_0|/\beta_0}{1-|I_0|/\beta_0},$$

and this is $< p < p/(1-3p)$ (making the above maximum *equal* to $p/(1-3p)$) if $|I_0|/\beta_0 < \frac{2}{3}$. Done.

Our construction of the intervals $I_k$ involved the parameter $p$. We now bring in another quantity, $\eta$, which will continue to intervene during most of the articles of this §. For the time being, we require only that $0 < \eta < \frac{2}{3}$ and take the value of $\eta$ as fixed during the work that follows. From time to time we will obtain various intermediate results whose validity will depend on $\eta$'s having been chosen sufficiently small to begin with. A final decision about $\eta$'s size will be made when we put together those results.

In accordance with the above indication of our procedure, we assume henceforth that

$$\sup_{t>0}\frac{n(t)}{t} > \frac{p}{1-3p}.$$

By the lemma we then certainly have

$$|I_0|/\beta_0 > \eta,$$

since we are taking $0 < \eta < \frac{2}{3}$. This being the case, we replace the first few intervals $I_k$ by a single one, according to the following construction.

Let $\omega(x)$ be the continuous and piecewise linear function defined on $[0, \infty)$ which has slope 1 on each of the intervals $I_k$ and slope *zero* elsewhere, and vanishes at the origin:

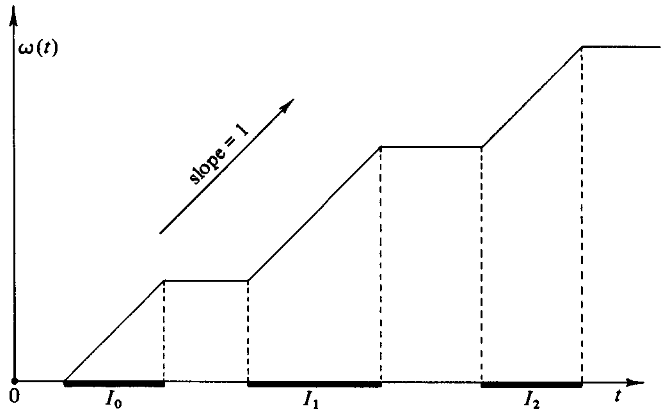



Figure 139

The ratio $\omega(t)/t$ is continuous and tends to zero as $t \to \infty$ since there are only a finite number of $I_k$. Clearly, $\omega(t)/t < 1$, so, if $t$ belongs to the *interior* of an $I_k$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\omega(t)}{t}\right) = \frac{1}{t} - \frac{\omega(t)}{t^2} > 0;$$

i.e., *$\omega(t)/t$ is strictly increasing on each $I_k$.*

We have

$$\omega(\beta_0)/\beta_0 = |I_0|/\beta_0 > \eta,$$

so, in view of what has just been said, there must be a *largest* value of $t$ $(>\beta_0)$ for which

$$\omega(t)/t = \eta$$

and that value *cannot lie in the interior, or be a left endpoint*, of any of the intervals $I_k$. Denote by $d$ that value of $t$. Then, since $d > \beta_0$, there must be a *last interval $I_k$* – call it $I_m$ – *lying entirely to the left* of $d$. If $I_m$ is also the last of the intervals $I_k$ we write

$$d_0 = d,$$
$$c_0 = (1-\eta)d,$$

and denote the interval $[c_0, d_0]$ by $J_0$. In this case all the positive zeros of $P(x)$ (discontinuities of $n(t)$) lie to the *left* of $d_0$.

It may be, however, that $I_m$ is *not* the last of the $I_k$; then there is an interval

$$I_{m+1} = [\alpha_{m+1}, \beta_{m+1}],$$

and we must have $d < \alpha_{m+1}$ according to the above observation. Since $d > \beta_0 > 2/p > 40$ (remember that we are taking $0 < p < \frac{1}{20}$), we can apply the *second* theorem of article 1 to conclude that there is a $d_0$,

$$d - 3 \;<\; d_0 \;\leqslant\; d,$$

such that $\alpha_{m+1}$ is *well disposed with respect to* $d_0$. We then put

$$c_0 \;=\; d_0 - \eta d$$

and denote by $J_0$ the interval $[c_0,\, d_0]$. The intervals $I_{m+1}, I_{m+2}, \ldots$ are also relabeled as follows:

$$I_{m+1} \;=\; J_1,$$
$$I_{m+2} \;=\; J_2,$$

and we write $\alpha_{m+1} = c_1$, $\beta_{m+1} = d_1$, $\alpha_{m+2} = c_2$, $\beta_{m+2} = d_2$, and so forth, so as to have the uniform notation

$$J_k \;=\; [c_k,\, d_k], \qquad k = 0, 1, 2, \ldots$$

In the present case, $\beta_m \leqslant d < \alpha_{m+1}$ (*sic*!) so, referring to the previous (second) stage of our construction, we see that the part of the graph of $n(t)$ vs. $t$ corresponding to values of $t \leqslant d$ lies entirely to the left of, or on, the line of slope $p$ (*sic*!) through $(d,\, n(d))$. By an argument very much like the one near the end of the second stage, based on the fact that $n(t)$ increases by *at least* 1 at each of its jumps, this implies that $d_0$, *although it may lie to the left of d, still lies to the right of all the zeros of* $P(x)$ *in* $I_0, \ldots, I_m$, *and that*

$$n(d_0) - n(t) \;\leqslant\; \frac{p}{1-3p}(d_0 - t) \qquad \text{for } t \leqslant d_0.$$

(The diagram used here is obtained by *rotating through 180°* the one from the argument just referred to.)

We have, in the first place, $c_0 \;\geqslant\; (1-\eta)d - 3 \;>\; 0$, because $\eta < \frac{2}{3}$ and $d > \beta_0 > 40$.

In the second place,

$$|J_0|/d_0 \;\geqslant\; |J_0|/d \;=\; \eta,$$

by choice of $d$. Also,

$$\frac{|J_0|}{d_0} \;<\; \frac{|J_0|}{d-3} \;=\; \frac{\eta d}{d-3} \;<\; \frac{40\eta}{37},$$

since $d > 40$.

Finally,

$$\frac{n(d_0)}{p|J_0|} = \frac{1}{2}.$$

Indeed, both $d_0$ and $d$ lie strictly between all the discontinuities of $n(t)$ in $I_0, I_1, \ldots, I_m$ and those in $I_{m+1}, I_{m+2}, \ldots$ (or to the *right* of the last $I_k$ if our construction yields *only one* interval $J_0$), so

$$n(d_0) = \sum_{k=0}^{m} n(I_k) = \frac{1}{2}\sum_{k=0}^{m} p|I_k|$$

by property (ii) of the $I_k$. And

$$\sum_{k=0}^{m} |I_k| = \omega(d) = \eta d = |J_0|$$

by the choice of $d$ and the definition of $J_0$. Thus $n(d_0) = \frac{1}{2}p|J_0|$, as claimed.

Denote by $J$ the union of the $J_k$, and put for the moment

$$\tilde{\omega}(t) = |[0,t]\cap J|.$$

The function $\tilde{\omega}(t)$ is similar to $\omega(t)$, considered above, and differs from the latter only in that it increases (with constant slope 1) on each of the $J_k$ instead of doing so on the $I_k$. The ratio $\tilde{\omega}(t)/t$ is therefore *increasing* on each $J_k$ (see above), so in particular

$$\frac{\tilde{\omega}(t)}{t} \leqslant \frac{\tilde{\omega}(d_0)}{d_0} = \frac{|J_0|}{d_0} < \frac{40\eta}{37}$$

for $t \in J_0 = [c_0, d_0]$. This inequality remains (trivially) true for $0 \leqslant t < c_0$, since $\tilde{\omega}(t) = 0$ there. It also remains true for $d_0 \leqslant t \leqslant d$, for $\tilde{\omega}(t)$ is constant on that interval. And finally,

$$\tilde{\omega}(t) = \omega(t) \qquad \text{for } t > d,$$

so $\tilde{\omega}(t)/t = \omega(t)/t < \eta$ for such $t$ by choice of $d$. Thus, we surely have

$$\frac{\tilde{\omega}(t)}{t} < 2\eta \qquad \text{for } t \geqslant 0.$$

The quantity on the left is, however, equal to $|J_0|/d_0 \geqslant \eta$ for $t = d_0$.

*The purpose of the constructions in this article has been to arrive at the intervals* $J_k$, and the remaining work of this § concerned with even polynomials having real zeros *deals exclusively with them.* The preceding discussions amount to a proof of the following

***Theorem.*** *Let $p$, $0 < p < \frac{1}{20}$, and $\eta$, $0 < \eta < \frac{2}{3}$, be given, and suppose that*

$$\sup_t \frac{n(t)}{t} > \frac{p}{1-3p}.$$

*Then there is a finite collection of intervals $J_k = [c_k, d_k]$, $k \geqslant 0$, lying in $(0, \infty)$, such that*

(i) *all the discontinuities of $n(t)$ lie in $(0, d_0) \cup \bigcup_{k \geqslant 1} J_k$;*

(ii) $$\frac{n(d_0)}{p|J_0|} = \frac{n(J_k)}{p|J_k|} = \frac{1}{2} \quad \textit{for } k \geqslant 1$$

*(if there are intervals $J_k$ with $k \geqslant 1$);*

(iii) *for $0 \leqslant t \leqslant d_0$,*

$$n(d_0) - n(t) \leqslant \frac{p}{1-3p}(d_0 - t),$$

*whilst, for $c_k \leqslant t \leqslant d_k$ when $k \geqslant 1$,*

$$n(t) - n(c_k) \leqslant \frac{p}{1-3p}(t - c_k)$$

*and*

$$n(d_k) - n(t) \leqslant \frac{p}{1-3p}(d_k - t);$$

(iv) *for $k \geqslant 1$, $c_k$ is well disposed with respect to $d_{k-1}$ (if there are $J_k$ with $k \geqslant 1$);*

(v) *for $t \geqslant 0$,*

$$\frac{1}{t}\left|[0, t] \cap \bigcup_{k \geqslant 0} J_k\right| < 2\eta,$$

*and the quantity on the left is $\geqslant \eta$ for $t = d_0$.*

***Remark 1.*** The $J_k$ with $k \geqslant 1$ (if there are any) are just certain of the $I_r$ from the *second stage*. So, for $k \geqslant 1$, the above property (ii) is just property (ii) for the $I_r$.

***Remark 2.*** By property (iv) and the theorems of article 1, we have

$$\int_{d_{k-1}}^{c_k} \frac{\log|P(x)|}{x^2} dx \leqslant 5 \sum_{d_{k-1} < m < c_k} \frac{\log^+|P(m)|}{m^2}$$

for each of the intervals $(d_{k-1}, c_k)$ with $k \geqslant 1$ (if there are any). And, if $J_1$

is the *last* of the $J_k$,

$$\int_{d_l}^{\infty} \frac{\log|P(x)|}{x^2}\,dx \;\leqslant\; 5 \sum_{d_l<m<\infty} \frac{\log^+|P(m)|}{m^2}.$$

See the end of the *second* stage of the preceding construction.

Here is a picture of the graph of $n(t)$ vs. $t$, showing the intervals $J_k$:

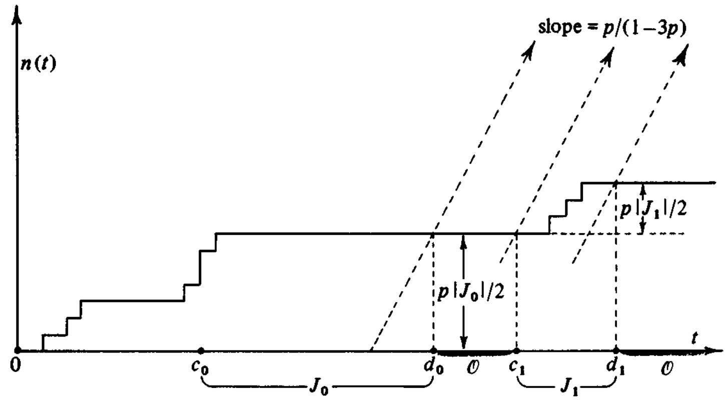



Figure 140

## 3. Replacement of the distribution $n(t)$ by a continuous one

Having chosen $p$, $0<p<1/20$, and $\eta$, $0<\eta<\frac{2}{3}$, we continue with our program, assuming that

$$\sup_{t>0} \frac{n(t)}{t} \;>\; \frac{p}{1-3p},$$

our aim being to obtain a *lower bound* for

$$\sum_1^{\infty} \frac{\log^+|P(m)|}{m^2}.$$

Our assumption makes it possible, by the work of the preceding article, to get the intervals

$$J_k \;=\; [c_k,\, d_k] \;\subset\; (0,\infty), \quad k=0,1,\ldots,$$

related to the (unknown) increasing function $n(t)$ in the manner described by the theorem at the end of that article.

Let $J_l$ be the *last* of those $J_k$; *during this article we will denote the union*

$$(d_0,\, c_1)\cup(d_1,\, c_2)\cup\cdots\cup(d_{l-1},\, c_l)\cup(d_l,\, \infty)$$

*by* $\mathcal{O}$ – see the preceding diagram. (Note that this is *not* the same set $\mathcal{O}$ as the one used at the beginning of article 2!) Our idea is to estimate

$$\sum_{m\in\mathcal{O}} \frac{\log^+|P(m)|}{m^2}$$

from below, this quantity being certainly *smaller* than the one we are interested in. According to Remark 2 following the theorem about the $J_k$, we have

$$\sum_{m\in\mathcal{O}} \frac{\log^+|P(m)|}{m^2} \geqslant \frac{1}{5}\int_{\mathcal{O}} \frac{\log|P(x)|}{x^2}\,dx.$$

What we want, then, is a *lower bound* for *the integral* on the right. This is the form that our initial simplistic plan of 'replacing' sums by integrals finally assumes.

In terms of $n(t)$,

$$\log|P(x)| = \sum_k \log\left|1-\frac{x^2}{x_k^2}\right| = \int_0^\infty \log\left|1-\frac{x^2}{t^2}\right| dn(t),$$

so the object of our interest is the expression

$$\frac{1}{5}\int_{\mathcal{O}}\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right| dn(t)\frac{dx}{x^2}.$$

Here, $n(t)$ is constant on each component of $\mathcal{O}$, and increases only on that set's complement.

We are now able to render our problem more tractable by replacing $n(t)$ with another increasing function $\mu(t)$ of much more simple and regular behaviour, continuous and piecewise linear on $\mathbb{R}$ and constant on each of the intervals complementary to the $J_k$. The slope $\mu'(t)$ will take only *two* values, 0 and $p/(1-3p)$, and, on each $J_k$, $\mu(t)$ will increase by $p|J_k|/2$. What we have to do is find such a $\mu(t)$ which makes

$$\frac{1}{5}\int_{\mathcal{O}}\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right| d\mu(t)\frac{dx}{x^2}$$

*smaller* than the expression written above, yet *still* (we hope) *strictly positive*.

Part of our requirement on $\mu(t)$ is that $\mu(t) = n(t)$ for $t\in\mathcal{O}$, so we will have

$$\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right| d\mu(t) - \int_0^\infty \log\left|1-\frac{x^2}{t^2}\right| dn(t)$$

$$= \int_0^{d_0} \log\left|1-\frac{x^2}{t^2}\right| d(\mu(t)-n(t))$$

$$+ \sum_{k\geqslant 1}\int_{c_k}^{d_k} \log\left|1-\frac{x^2}{t^2}\right| d(\mu(t)-n(t)).$$

We are interested in values of $x$ in $\mathcal{O}$, and for them, each of the above terms can be integrated by parts. Since $\mu(t) = n(t) = 0$ for $t$ near 0 and $\mu(d_0) = n(d_0)$, $\mu(c_k) = n(c_k)$ and $\mu(d_k) = n(d_k)$ for $k \geqslant 1$, we obtain in this way the expression

$$\int_0^{d_0} \frac{2x^2}{x^2 - t^2} \frac{\mu(t) - n(t)}{t} \mathrm{d}t \; + \; \sum_{k \geqslant 1} \int_{c_k}^{d_k} \frac{2x^2}{x^2 - t^2} \frac{\mu(t) - n(t)}{t} \mathrm{d}t.$$

Therefore

$$\int_{\mathcal{O}} \int_0^{\infty} \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}(\mu(t) - n(t)) \frac{\mathrm{d}x}{x^2}$$
$$= \int_0^{d_0} \int_{\mathcal{O}} \frac{2\mathrm{d}x}{x^2 - t^2} \cdot \frac{\mu(t) - n(t)}{t} \mathrm{d}t$$
$$+ \sum_{k \geqslant 1} \int_{c_k}^{d_k} \int_{\mathcal{O}} \frac{2\mathrm{d}x}{x^2 - t^2} \cdot \frac{\mu(t) - n(t)}{t} \mathrm{d}t,$$

and we desire to find a function $\mu(t)$ fitting our requirements, for which each of the terms on the right comes out *negative.*

Put

$$F(t) \; = \; 2 \int_{\mathcal{O}} \frac{\mathrm{d}x}{x^2 - t^2}$$

for $t \notin \mathcal{O}$. We certainly have $F(t) > 0$ for $0 < t < d_0$, so the first right-hand term, which equals

$$\int_0^{d_0} F(t) \frac{\mu(t) - n(t)}{t} \mathrm{d}t$$

is $\leqslant 0$ if $\mu(t) \leqslant n(t)$ on $[0, d_0]$. Referring to the diagram at the end of the previous article, we see that this will happen if, for $0 \leqslant t \leqslant d_0$, $\mu(t)$ has the form shown here:

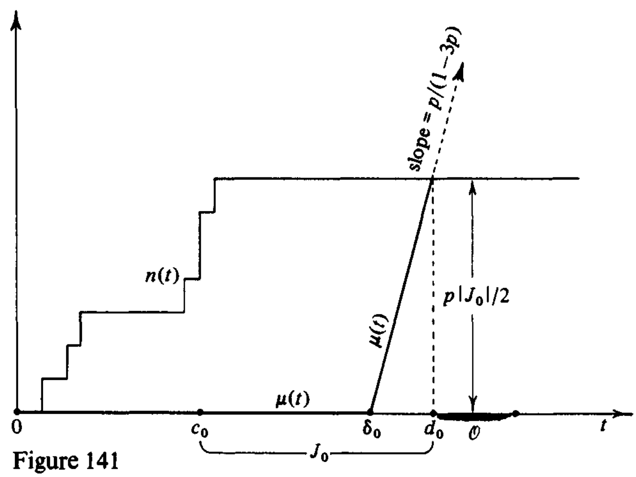



Figure 141

For $k \geqslant 1$, we need to define $\mu(t)$ on $[c_k, d_k]$ in a manner compatible with our requirements, so as to make

$$\int_{c_k}^{d_k} F(t) \frac{\mu(t) - n(t)}{t} \mathrm{d}t \ \leqslant \ 0.$$

Here, $\mathcal{O}$ includes intervals of the form

$$(c_k - \delta, \ c_k)$$

and

$$(d_k, \ d_k + \delta)$$

where $\delta > 0$, so, when $t \in (c_k, d_k)$, $F(t) \to -\infty$ for $t \to c_k$ and $F(t) \to \infty$ for $t \to d_k$. Moreover, for such $t$,

$$F'(t) \ = \ 4t \int_{\mathcal{O}} \frac{\mathrm{d}x}{(x^2 - t^2)^2} \ > \ 0,$$

so there is *precisely one point* $t_k \in (c_k, d_k)$ *where* $F(t)$ *vanishes*, and $F(t) < 0$ for $c_k < t < t_k$, while $F(t) > 0$ for $t_k < t < d_k$. We see that in order to make

$$\int_{c_k}^{d_k} F(t) \frac{\mu(t) - n(t)}{t} \mathrm{d}t \ \leqslant \ 0,$$

it is enough to define $\mu(t)$ so as to make

$$\mu(t) \ \geqslant \ n(t) \qquad \text{for } c_k \leqslant t < t_k$$

and

$$\mu(t) \ \leqslant \ n(t) \qquad \text{for } t_k \leqslant t \leqslant d_k.$$

The following diagram shows how to do this:

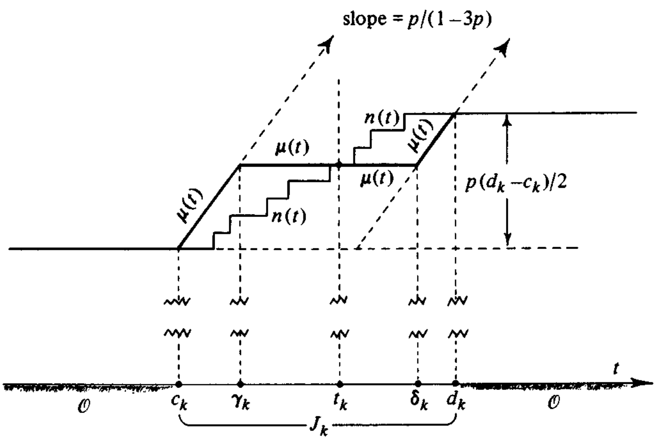



Figure 142

We carry out this construction on each of the $J_k$. When we are done we will have a function $\mu(t)$, defined for $t \geqslant 0$, with the following properties:

(i) $\mu(t)$ is piecewise linear and increasing, and constant on each interval component of

$$(0, \infty) \sim \bigcup_{k \geqslant 0} J_k;$$

(ii) on each of the intervals $J_k$, $\mu(t)$ increases by $p|J_k|/2$;

(iii) on $J_0$, $\mu(t)$ has *slope zero* for $c_0 < t < \delta_0$ and *slope* $p/(1-3p)$ for $\delta_0 < t < d_0$, where $(d_0 - \delta_0)/(d_0 - c_0) = (1-3p)/2$;

(iv) on each $J_k$, $k \geqslant 1$, $\mu(t)$ has *slope zero* for $\gamma_k < t < \delta_k$ and *slope* $p/(1-3p)$ in the intervals $(c_k, \gamma_k)$ and $(\delta_k, d_k)$, where $c_k < \gamma_k < \delta_k < d_k$ and

$$\frac{\gamma_k - c_k + d_k - \delta_k}{d_k - c_k} = \frac{1-3p}{2};$$

(v) $$\int_{\mathcal{O}} \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\mu(t) \frac{dx}{x^2} \leqslant \int_{\mathcal{O}} \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| dn(t) \frac{dx}{x^2}.$$

Here is a drawing of the graph of $\mu(t)$ vs. $t$ which the reader will do well to look at from time to time while reading the following articles:

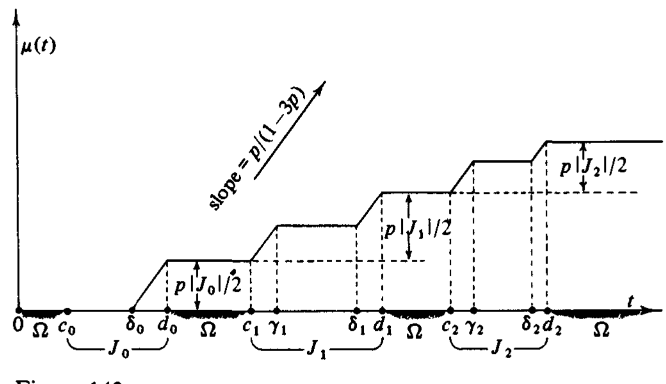



Figure 143

In what follows, we will in fact be working with integrals not over $\mathcal{O}$, but over the set $\Omega = (0, c_0) \cup \mathcal{O} = (0, \infty) \sim \bigcup_{k > 0} J_k$ (see the diagram). Since our function $\mu(t)$ is *zero* for $t \leqslant c_0$, we certainly have

$$\int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\mu(t) \leqslant 0$$

for $0 < t < c_0$. Hence, by property (v),

$$\int_\Omega \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\mu(t) \frac{dx}{x^2} \leqslant \int_{\mathscr{O}} \frac{\log|P(x)|}{x^2} dx$$

for our polynomial $P$. And, as we have seen at the beginning of this article, the right-hand integral is in turn

$$\leqslant 5\sum_1^\infty \frac{\log^+|P(m)|}{m^2}.$$

What we have here is a

***Theorem.*** *Let $0 < p < 1/20$ and $0 < \eta < \frac{2}{3}$, and suppose that*

$$\sup_{t>0} \frac{n(t)}{t} > \frac{p}{1-3p}.$$

*Then there are intervals $J_k \subset (0, \infty)$, $k \geqslant 0$, fulfilling the conditions enumerated in the theorem of the preceding article, and a piecewise linear increasing function $\mu(t)$, related to those $J_k$ in the manner just described, such that*

$$\int_\Omega \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\mu(t) \frac{dx}{x^2} \leqslant 5\sum_1^\infty \frac{\log^+|P(m)|}{m^2}$$

*for the polynomial $P(x)$.*

*Here,*

$$\Omega = (0, \infty) \sim \bigcup_{k \geqslant 0} J_k.$$

Our problem has thus boiled down to the *purely analytical one* of finding a positive lower bound for

$$\int_\Omega \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\mu(t) \frac{dx}{x^2}$$

when $\mu(t)$ has the very special form shown in the above diagram. Note that here $|J_0|/d_0 \geqslant \eta$ according to the theorem of the preceding article.

## 4. Some formulas

The problem, formulated at the end of the last article, to which we have succeeded in reducing our original one seems at first glance to be rather easy – one feels that one can just sit down and *compute*

$$\int_\Omega \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\mu(t) \frac{dx}{x^2}.$$

This, however, is far from being the case, and quite formidable difficulties still stand in our way. The trouble is that the intervals $J_k$ to which $\mu$ is related may be exceedingly numerous, and we have no control over their positions relative to each other, nor on their relative lengths. To handle our task, we are going to need all the formulas we can muster.

***Lemma.*** *Let $\nu(t)$ be increasing on $[0, \infty)$, with $\nu(0) = 0$ and $\nu(t) = O(t)$ for $t \to 0$ and for $t \to \infty$. Then, for $x \in \mathbb{R}$,*

$$\int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\nu(t) \;=\; -x\int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu(t)}{t}\right).$$

***Proof.*** Both sides are even functions of $x$ and zero for $x = 0$, so we may as well assume that $x > 0$. If $\nu(t)$ has a (jump) discontinuity at $x$, both sides are clearly equal to $-\infty$, so we may suppose $\nu(t)$ continuous at $x$.

We have

$$\int_0^x \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu(t)}{t}\right) \;=\; \int_0^x \frac{1}{t}\log\left|\frac{x+t}{x-t}\right| d\nu(t)$$
$$-\;\int_0^x \frac{1}{t^2}\log\left|\frac{x+t}{x-t}\right| \nu(t)\,dt.$$

Using the identity

$$\int \frac{1}{t^2}\log\left|\frac{x+t}{x-t}\right| dt \;=\; -\frac{1}{t}\log\left|\frac{x+t}{x-t}\right| - \frac{1}{x}\log\left|1 - \frac{x^2}{t^2}\right|,$$

we integrate the second term on the right by parts, obtaining for it the value

$$-\;\frac{2\nu(x)\log 2}{x} \;+\; \int_0^x \left(\frac{1}{t}\log\left|\frac{x+t}{x-t}\right| + \frac{1}{x}\log\left|1 - \frac{x^2}{t^2}\right|\right) d\nu(t),$$

taking into account the given behaviour of $\nu(t)$ near 0. Hence

$$\int_0^x \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu(t)}{t}\right) \;=\; \frac{2\nu(x)\log 2}{x} \;-\; \frac{1}{x}\int_0^x \log\left|1 - \frac{x^2}{t^2}\right| d\nu(t).$$

In the same way, we get

$$\int_x^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu(t)}{t}\right)$$
$$=\; -\left(\frac{2\nu(x)\log 2}{x}\right) \;-\; \frac{1}{x}\int_x^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\nu(t).$$

Adding these last two relations gives us the lemma.

***Corollary.*** *Let $\nu(t)$ be* increasing and bounded *on* $[0,\infty)$, *and* zero *for all $t$ sufficiently close to* 0. *Let $\omega(x)$ be increasing on* $[0,\infty)$, constant *for all sufficiently large $x$, and continuous at* 0. *Then*

$$\int_0^\infty\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right| d\nu(t)\frac{dx-d\omega(x)}{x^2}$$
$$= \int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu(t)}{t}\right)\frac{d\omega(x)}{x}.$$

***Proof.*** By the lemma, the left-hand side equals

$$\int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu(t)}{t}\right)\frac{d\omega(x)-dx}{x}.$$

Our condition on $\nu$ makes

$$\int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu(t)}{t}\right)\frac{dx}{x}$$

absolutely convergent, so we can change the order of integration. For $t>0$,

$$\int_0^\infty \log\left|\frac{x+t}{x-t}\right|\frac{dx}{x}$$

assumes a constant value (equal to $\pi^2/2$ as shown by contour integration – see Problem 20), so, since in our present circumstances

$$\int_0^\infty d\left(\frac{\nu(t)}{t}\right) = 0,$$

the previous double integral vanishes, and the corollary follows.*

In our application of these results we will take

$$\nu(t) = \frac{1-3p}{p}\mu(t),$$

$\mu(t)$ being the function constructed in the previous article. This function $\nu(t)$ *increases with constant slope* 1 on each of the intervals $[\delta_k, d_k]$, $k\geqslant 0$, and $[c_k, \gamma_k]$, $k\geqslant 1$, and is *constant* on each of the intervals *complementary* to those. Therefore, if

$$\tilde{\Omega} = (0,\infty)\sim\bigcup_{k\geqslant 1}[c_k,\gamma_k]\sim\bigcup_{k\geqslant 0}[\delta_k,d_k]$$

* The two sides of the relation established may both be infinite, e.g., when $\nu(t)$ and $\omega(t)$ have some coinciding jumps. But the meaning of the two iterated integrals in question is always unambiguous; in the second one, for instance, the outer integral of the *negative part* of the inner one converges.

(note that this set $\tilde{\Omega}$ *includes* our $\Omega$), we have

$$\int_{\tilde{\Omega}} \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\mu(t) \frac{dx}{x^2}$$

$$= \frac{p}{1-3p} \int_0^\infty \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\nu(t) \frac{dx - d\nu(x)}{x^2}.$$

The corollary shows that this expression (which we can think of as a first approximation to

$$\int_{\Omega} \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\mu(t) \frac{dx}{x^2} \quad )$$

is equal to

$$\frac{p}{1-3p} \int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu(t)}{t}\right) \frac{d\nu(x)}{x}.$$

This double integral can be given a symmetric form thanks to the

***Lemma.*** *Let $\nu(t)$ be continuous, increasing, and piecewise continuously differentiable on* $[0, \infty]$. *Suppose, moreover, that* $\nu(0) = 0$, *that $\nu(t)$ is constant for $t$ sufficiently large, and, finally*, that* $(d/dt)(\nu(t)/t)$ *remains bounded when* $t \to 0+$. *Then,*

$$\int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu(t)}{t}\right) \frac{\nu(x)}{x^2} dx \;=\; -\frac{\pi^2}{4}(\nu'(0))^2.$$

***Proof.*** Our assumptions on $\nu$ make reversal of the order of integrations in the left-hand expression legitimate, so it is equal to

$$\int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \frac{\nu(x)}{x^2} dx \, d\left(\frac{\nu(t)}{t}\right)$$

$$= \int_0^\infty \left( \int_0^\infty \log\left|\frac{\xi+1}{\xi-1}\right| \frac{\nu(t\xi)}{t\xi} \frac{d\xi}{\xi} \right) d\left(\frac{\nu(t)}{t}\right).$$

Since

$$\int_0^\infty \log\left|\frac{\xi+1}{\xi-1}\right| \frac{d\xi}{\xi} \;=\; \frac{\pi^2}{2}$$

(which may be verified by contour integration), we have

$$\int_0^\infty \log\left|\frac{\xi+1}{\xi-1}\right| \frac{\nu(t\xi)}{t\xi} \frac{d\xi}{\xi} \;\longrightarrow\; \frac{\pi^2}{2}\nu'(0)$$

for $t \to 0$, and integration by parts of the outer integral in the previous

* This last condition can be relaxed. See problem 28(b), p. 569.

expression yields the value

$$-\frac{\pi^2}{2}(v'(0))^2 - \int_0^\infty \frac{v(t)}{t}\frac{d}{dt}\left(\int_0^\infty \log\left|\frac{\xi+1}{\xi-1}\right|\frac{v(t\xi)}{t\xi}\frac{d\xi}{\xi}\right)dt.$$

Under the conditions of our hypothesis, the differentiation with respect to $t$ can be carried out under the inner integral sign. The last expression thus becomes

$$-\frac{\pi^2}{2}(v'(0))^2 - \int_0^\infty \frac{v(t)}{t}\int_0^\infty \log\left|\frac{\xi+1}{\xi-1}\right|\frac{d}{dt}\left(\frac{v(t\xi)}{t\xi}\right)\frac{d\xi}{\xi}dt$$

$$= -\frac{\pi^2}{2}(v'(0))^2 - \int_0^\infty \frac{v(t)}{t}\int_0^\infty \log\left|\frac{x+t}{x-t}\right|\frac{x}{t}\frac{d}{dx}\left(\frac{v(x)}{x}\right)\frac{dx}{x}dt.$$

In other words

$$\int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{v(t)}{t}\right)\frac{v(x)}{x^2}dx$$

$$= -\frac{\pi^2}{2}(v'(0))^2 - \int_0^\infty\int_0^\infty \log\left|\frac{t+x}{t-x}\right| d\left(\frac{v(x)}{x}\right)\frac{v(t)}{t^2}dt.$$

The second term on the right obviously equals the left-hand side, so the lemma follows.

***Corollary.*** *Let $v(t)$ be increasing, continuous, and piecewise linear on* $[0,\infty)$, constant *for all sufficiently large $t$ and* zero *for $t$ near* 0. *Then*

$$\int_0^\infty\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right| dv(t)\frac{dx - dv(x)}{x^2}$$

$$= \int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{v(t)}{t}\right)d\left(\frac{v(x)}{x}\right).$$

***Proof.*** By the previous corollary, the left-hand expression equals

$$\int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{v(t)}{t}\right)\frac{dv(x)}{x}.$$

In the present circumstances, $v'(0)$ exists and equals *zero*. Therefore by the lemma

$$\int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{v(t)}{t}\right)\frac{v(x)}{x^2}dx = 0,$$

and the previous expression is equal to

$$\int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{v(t)}{t}\right)d\left(\frac{v(x)}{x}\right).$$

**Problem 21**

Prove the last lemma using contour integration. (Hint: For $\Im z > 0$, consider the analytic function

$$F(z) = \frac{1}{\pi}\int_0^\infty \log\left(\frac{z+t}{z-t}\right) d\left(\frac{\nu(t)}{t}\right),$$

and examine the boundary values of $\Re F(z)$ and $\Im F(z)$ on the real axis. Then look at $\int_\Gamma ((F(z))^2/z)\,dz$ for a suitable contour $\Gamma$.)

## 5. The energy integral

The expression, quadratic in $d(\nu(t)/t)$, arrived at near the end of the previous article, namely,

$$\int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu(t)}{t}\right) d\left(\frac{\nu(x)}{x}\right),$$

has a simple physical interpretation. Let us assume that a flat metal plate of infinite extent, perpendicular to the $z$-plane, intersects the latter along the $y$-axis. This plate we suppose grounded. Let electric charge be continuously distributed on a very large thin sheet, made of non-conducting material, and intersecting the $z$-plane perpendicularly along the positive $x$-axis. Suppose the charge density on that sheet to be constant along lines perpendicular to the $z$-plane, and that the total charge contained in any rectangle of height 2 thereon, bounded by two such lines intersecting the $x$-axis at $x$ and at $x + \Delta x$, is equal to the *net change* of $\nu(t)/t$ along $[x, x + \Delta x]$. This set-up will produce an *electric field* in the region lying to the right of the grounded metal plate; *near* the $z$-plane, the *potential function* for that field is equal, very nearly, to

$$u(z) = \int_0^\infty \log\left|\frac{z+t}{z-t}\right| d\left(\frac{\nu(t)}{t}\right).$$

The quantity

$$\int_0^\infty u(x) d\left(\frac{\nu(x)}{x}\right) = \int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu(t)}{t}\right) d\left(\frac{\nu(x)}{x}\right)$$

is then proportional to the *total energy* of the electric field generated by our distribution of electric charge (and inversely proportional to the height of the charged sheet). We therefore expect it to be *positive*, even though *charges of both sign* be present at different places on the non-conducting sheet, i.e., when $d(\nu(t)/t)/dt$ is *not of constant sign.*

Under quite general circumstances, the *positivity* of the quadratic form in question turns out to be *valid*, and plays a crucial rôle in the computations of the succeeding articles. In the present one, we derive two formulas, either of which makes that property evident.

The first formula is familiar from physics, and goes back to Gauss. It is convenient to write

$$\rho(t) = \frac{\nu(t)}{t}.$$

***Lemma.*** *Let $\rho(t)$ be continuous on $[0,\infty)$, piecewise $\mathscr{C}_3$ there (say), and differentiable at 0. Suppose furthermore that $\rho(t)$ is uniformly* Lip 1 *on $[0,\infty)$ and $t\rho(t)$* constant *for sufficiently large $t$.*

*If we write*

$$u(z) = \int_0^\infty \log\left|\frac{z+t}{z-t}\right| \mathrm{d}\rho(t),$$

*we have*

$$\boxed{\int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t)\,\mathrm{d}\rho(x) = \frac{1}{\pi}\int_0^\infty \int_0^\infty \{(u_x(z))^2 + (u_y(z))^2\}\,\mathrm{d}x\,\mathrm{d}y.}$$

***Remark 1.*** Note that we *do not require* that $\rho(t)$ *vanish* for $t$ *near zero*, although $\rho(t) = \nu(t)/t$ has this property when $\nu(t)$ is the function introduced in the previous article.

***Remark 2.*** The factor $1/\pi$ occurs on the right, and not $1/2\pi$ which one might expect from physics, because the right-hand integral is taken over the *first quadrant instead* of over the *whole right half plane* (where the 'electric field' is present). The right-hand expression is of course the *Dirichlet integral* of $u$ over the first quadrant.

***Remark 3.*** The function $u(z)$ is *harmonic* in each separate quadrant of the $z$-plane. Since

$$\log\left|\frac{z+\bar{w}}{z-w}\right|$$

is the Green's function for the right half plane, $u(z)$ is frequently referred to as the *Green potential of the charge distribution* $\mathrm{d}\rho(t)$ (for that half plane).

***Proof of lemma.*** For $y>0$, we have

$$u_y(z) = \int_0^\infty \left(\frac{y}{(x+t)^2+y^2} - \frac{y}{(x-t)^2+y^2}\right)\mathrm{d}\rho(t),$$

and, when $x>0$ is not a point of discontinuity for $\rho'(t)$, the right side tends to $-\pi\rho'(x)$ as $y \to 0+$ by the usual (elementary) approximate identity property of the Poisson kernel. Thus,

$$u_y(x + i0) \;=\; -\pi\rho'(x),$$

and

$$\int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\rho(t)\,d\rho(x) \;=\; -\frac{1}{\pi}\int_0^\infty u(x)u_y(x+i0)\,dx.$$

At the same time, $u(iy) = 0$ for $y > 0$, so the left-hand double integral from the previous relation is equal to

$$-\frac{1}{\pi}\int_0^\infty u(x)u_y(x+i0)\,dx \;-\; \frac{1}{\pi}\int_0^\infty u(iy)u_x(iy)\,dy.$$

We have here a line integral around the boundary of the first quadrant. Applying Green's theorem to it in cook-book fashion, we get the value

$$\frac{1}{\pi}\int_0^\infty \int_0^\infty \left(\frac{\partial}{\partial y}(u(z)u_y(z)) + \frac{\partial}{\partial x}(u(z)u_x(z))\right) dx\,dy,$$

which reduces immediately to

$$\frac{1}{\pi}\int_0^\infty \int_0^\infty ((u_y(z))^2 + (u_x(z))^2)\,dx\,dy$$

(proving the lemma), since $u$ is *harmonic* in the first quadrant, making $u\nabla^2 u = 0$ there.

We have, however, to justify our use of Green's theorem. The way to do that here is to adapt to our present situation the common 'non-rigorous' derivation of the theorem (using squares) found in books on engineering mathematics. Letting $\mathscr{D}_A$ denote the square with vertices at $0$, $A$, $A + iA$ and $iA$, we verify in that way without difficulty (and without any being created by the discontinuities of $\rho'(x) = -u_y(x + i0)/\pi$ ), that

$$\int_{\partial\mathscr{D}_A} (uu_x dy - uu_y dx) \;=\; \iint_{\mathscr{D}_A} (u_x^2 + u_y^2)\,dx\,dy.*$$

The line integral on the left equals

$$-\int_0^A u(x)u_y(x+i0)\,dx \;+\; \int_{\Gamma_A} (uu_x dy - uu_y dx),$$

where $\Gamma_A$ denotes the *right side* and *top* of $\mathscr{D}_A$:

* The simplest procedure is to take $h > 0$ and write the corresponding relation involving $u(z + ih)$ in place of $u(z)$, whose truth is certain here. Then one can make $h \to 0$. Cf the discussion on pp. 506–7.

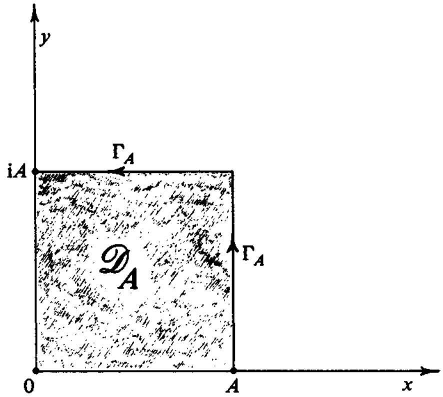



Figure 144

We will be done if we show that

$$\int_{\Gamma_A} (uu_x \mathrm{d}y - uu_y \mathrm{d}x) \longrightarrow 0 \qquad \text{for } A \longrightarrow \infty.$$

For this purpose, one may break up $u(z)$ as

$$\int_0^M \log\left|\frac{z+t}{z-t}\right| \mathrm{d}\rho(t) \;+\; \int_M^\infty \log\left|\frac{z+t}{z-t}\right| \mathrm{d}\rho(t),$$

$M$ being chosen large enough so as to have $\rho(t) = C/t$ on $[M, \infty)$. Calling the *first* of these integrals $u_1(z)$, we easily find, for $|z| > M$ (by expanding the logarithm in powers of $t/z$), that

$$|u_1(z)| \;\leqslant\; \frac{\text{const.}}{|z|}$$

and that the first partial derivatives of $u_1(z)$ are $O(1/|z|^2)$.

Denote by $u_2(z)$ the *second* of the above integrals, which, by choice of $M$, is actually equal to

$$-C\int_M^\infty \log\left|\frac{z+t}{z-t}\right|\frac{\mathrm{d}t}{t^2}.$$

The substitution $t = |z|\tau$ enables us to see after very little calculation that this expression is in modulus

$$\leqslant \text{const.}\frac{\log|z|}{|z|}$$

for large $|z|$.

To investigate the partial derivatives of $u_2(z)$ in the open first quadrant, we take the function

$$F(z) \;=\; \int_M^\infty \log\left(\frac{z+t}{z-t}\right)\frac{\mathrm{d}t}{t^2},$$

analytic in that region, and note that by the Cauchy–Riemann equations,

$$\frac{\partial u_2(z)}{\partial x} - i\frac{\partial u_2(z)}{\partial y} = -CF'(z)$$

there. Here,

$$F'(z) = \int_M^\infty \frac{dt}{t^2(z+t)} - \int_M^\infty \frac{dt}{t^2(z-t)}.$$

The first term on the right is obviously $O(1/|z|)$ in modulus when $\Re z$ and $\Im z > 0$. The second works out to

$$\int_M^\infty \left(\frac{1}{zt^2} + \frac{1}{z^2 t} + \frac{1}{z^2(z-t)}\right) dt = \frac{1}{zM} + \frac{1}{z^2}\log\left(\frac{z-M}{M}\right),$$

using a suitable determination of the logarithm. This is evidently $O(1/|z|)$ for large $|z|$, so $|F'(z)| = O(1/|z|)$ for $z$ with large modulus in the first quadrant. The same is thus true for the first partial derivatives of $u_2(z)$.

Combining the estimates just made on $u_1(z)$ and $u_2(z)$, we find for $u = u_1 + u_2$ that

$$|u(z)| \leqslant \text{const.}\frac{\log|z|}{|z|}$$

$$|u_x(z)| \leqslant \text{const.}\frac{1}{|z|}$$

$$|u_y(z)| \leqslant \text{const.}\frac{1}{|z|}$$

when $\Re z > 0$, $\Im z > 0$, $|z|$ being large. Therefore

$$\int_{\Gamma_A} (uu_x dy - uu_y dx) = O\left(\frac{\log A}{A}\right)$$

for large $A$, and the line integral tends to zero as $A \to \infty$. This is what was needed to finish the proof of the lemma. We are done.

***Corollary.*** *If $\rho(t)$ is* real *and satisfies the hypothesis of the lemma,*

$$\int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\rho(t)\, d\rho(x) \geqslant 0.$$

***Proof.*** Clear.

***Notation.*** We write

$$\boxed{E(d\rho(t), d\sigma(t)) = \int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\rho(t)\, d\sigma(x)}$$

for *real* measures $\rho$ and $\sigma$ on $[0,\infty)$ without point mass at the origin making both of the integrals

$$\int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t)\,\mathrm{d}\rho(x), \qquad \int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\sigma(t)\,\mathrm{d}\sigma(x)$$

*absolutely convergent*. (Vanishing of $\rho(\{0\})$ and $\sigma(\{0\})$ is required because $\log|(x+t)/(x-t)|$ cannot be defined at $(0,0)$ so as to be continuous there.)

Note that, in the case of functions $\rho(t)$ and $\sigma(t)$ satisfying the hypothesis of the above lemma, the integrals just written *do* converge absolutely. In terms of $E(\mathrm{d}\rho(t), \mathrm{d}\sigma(t))$, we can state the very important

***Corollary.*** *If $\rho(t)$ and $\sigma(t)$, defined and real valued on $[0,\infty)$, both satisfy the hypothesis of the lemma,*

$$|E(\mathrm{d}\rho(t), \mathrm{d}\sigma(t))| \;\leqslant\; \sqrt{(E(\mathrm{d}\rho(t), \mathrm{d}\rho(t)))}\cdot\sqrt{(E(\mathrm{d}\sigma(t), \mathrm{d}\sigma(t)))}.$$

***Proof.*** Use the preceding corollary and proceed as in the usual derivation of Schwarz' inequality.

***Remark.*** The result remains valid as long as $\rho$ and $\sigma$, with $\rho(\{0\})=\sigma(\{0\})=0$, are such that the abovementioned absolute convergence holds. We will see that at the end of the present article.

***Scholium and warning.*** The results just given should not mislead the reader into believing that the energy integral corresponding to the *ordinary logarithmic potential* is necessarily positive. Example:

$$\int_0^{2\pi}\int_0^{2\pi} \log\frac{1}{|2\mathrm{e}^{\mathrm{i}\vartheta}-2\mathrm{e}^{\mathrm{i}\varphi}|}\,\mathrm{d}\vartheta\,\mathrm{d}\varphi \;=\; \int_0^{2\pi} 2\pi\log\tfrac{1}{2}\,\mathrm{d}\varphi \;=\; -4\pi^2\log 2 \;\;!$$

It is strongly recommended that the reader find out exactly where the argument used in the proof of the lemma *goes wrong*, when one attempts to adapt it to the potential

$$u(z) \;=\; \int_0^{2\pi} \log\frac{1}{|2\mathrm{e}^{\mathrm{i}\vartheta}-z|}\,\mathrm{d}\vartheta.$$

For 'nice' real measures $\mu$ of compact support, *it is true that*

$$\int_{\mathbb{C}}\int_{\mathbb{C}} \log\frac{1}{|z-w|}\,\mathrm{d}\mu(z)\,\mathrm{d}\mu(w) \;\geqslant\; 0$$

*provided that* $\int_{\mathbb{C}}\mathrm{d}\mu(z)=0$. *The reader should verify this fact* by applying a suitable version of Green's theorem to the potential $\int_{\mathbb{C}}\log(1/|z-w|)\,\mathrm{d}\mu(w)$.

The formula for $E(\mathrm{d}\rho(t), \mathrm{d}\rho(t))$ furnished by the above lemma exhibits that quantity's positivity. The same service is rendered by an analogous relation involving the values of $\rho(t)$ on $[0, \infty)$. Such representations go back to Jesse Douglas; we are going to use one based on a beautiful identity of Beurling. In order to encourage the reader's participation, we set as a problem the derivation of Beurling's result.

**Problem 22**

(a) Let $m$ be a real measure on $\mathbb{R}$. Suppose that $h > 0$ and that $\int\int_{|\xi-\eta|\leqslant h} dm(\xi)\,dm(\eta)$ converges absolutely. Show that

$$\int_{-\infty}^{\infty} (m(x+h) - m(x))^2\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (h - |\xi - \eta|)^+\,dm(\xi)dm(\eta).$$

(Hint: Trick:

$$(m(x+h) - m(x))^2 \;=\; \int_x^{x+h}\int_x^{x+h} dm(\xi)dm(\eta).\;)$$

(b) Let $K(x)$ be even and positive, $\mathscr{C}_2$ and *convex* for $x > 0$, and such that $K(x) \to 0$ for $x \to \infty$. Show that, for $x \neq 0$,

$$K(x) \;=\; \int_0^{\infty} (h - |x|)^+ K''(h)\,dh.$$

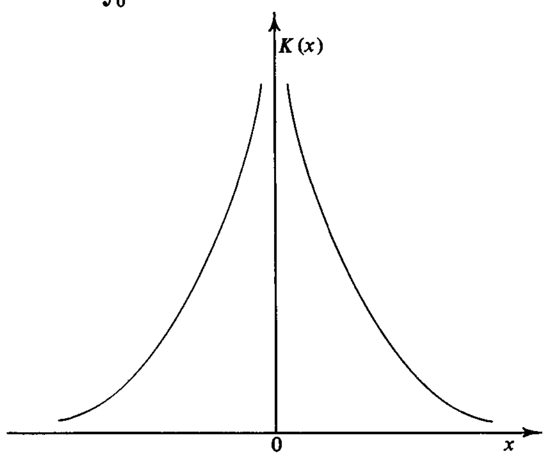



Figure 145

(Hint: First observe that $K'(x)$ must also $\to 0$ for $x \to \infty$.)

(c) If $K(x)$ is as in (b) and $m$ is a real measure on $\mathbb{R}$ with $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(|\xi-\eta|)\,dm(\xi)\,dm(\eta)$ absolutely convergent, that integral is equal to

$$\int_{-\infty}^{\infty}\int_0^{\infty} [m(x+h) - m(x)]^2 K''(h)\,dh\,dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [m(y) - m(x)]^2 K''(|x-y|)\,dy\,dx.$$

(Hint: The assumed absolute convergence guarantees that $m$ fulfills, for each $h > 0$, the condition required in part (a). The order of integration in

$$\int_{-\infty}^{\infty}\int_{0}^{\infty} K''(h)(m(x+h)-m(x))^2\,\mathrm{d}h\,\mathrm{d}x$$

may be reversed, yielding, by part (a), an iterated triple integral. Here, that triple integral is absolutely convergent and we may conclude by the help of part (b).)

***Lemma.*** *Let the real measure $\rho$ on $[0,\infty)$, without point mass at the origin, be such that*

$$\int_0^{\infty}\int_0^{\infty}\log\left|\frac{x+t}{x-t}\right|\mathrm{d}\rho(t)\,\mathrm{d}\rho(x)$$

*is absolutely convergent. Then*

$$\boxed{\int_0^{\infty}\int_0^{\infty}\log\left|\frac{x+t}{x-t}\right|\mathrm{d}\rho(t)\,\mathrm{d}\rho(x) = \int_0^{\infty}\int_0^{\infty}\left(\frac{\rho(x)-\rho(y)}{x-y}\right)^2\frac{x^2+y^2}{(x+y)^2}\,\mathrm{d}x\,\mathrm{d}y.}$$

***Proof.*** The left-hand double integral is of the form

$$\int_0^{\infty}\int_0^{\infty} k\left(\frac{x}{t}\right)\mathrm{d}\rho(x)\,\mathrm{d}\rho(t),$$

where

$$k(\tau) = \log\left|\frac{1+\tau}{1-\tau}\right| = k\left(\frac{1}{\tau}\right),$$

so we can reduce that integral to one figuring in Problem 22(c) by making the substitutions $x = \mathrm{e}^{\xi}$, $t = \mathrm{e}^{\eta}$, $\rho(x) = m(\xi)$, $\rho(t) = m(\eta)$, and

$$k\left(\frac{x}{t}\right) = K(\xi-\eta) = \log\left|\coth\left(\frac{\xi-\eta}{2}\right)\right|.$$

$K(h)$, besides being obviously even and positive, tends to zero for $h \to \infty$. Also

$$K'(h) = \frac{1}{2}\tanh\frac{h}{2} - \frac{1}{2}\coth\frac{h}{2},$$

and

$$K''(h) \;=\; \frac{1}{4}\operatorname{sech}^2\frac{h}{2} \;+\; \frac{1}{4}\operatorname{cosech}^2\frac{h}{2} \;>\; 0,$$

so $K(h)$ is convex for $h > 0$. The application of Beurling's formula from problem 22(c) is therefore legitimate, and yields

$$\int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t)\,\mathrm{d}\rho(x)$$

$$= \int_{-\infty}^\infty\int_{-\infty}^\infty K(|\xi-\eta|)\mathrm{d}m(\eta)\mathrm{d}m(\xi)$$

$$= \frac{1}{2}\int_{-\infty}^\infty\int_{-\infty}^\infty K''(|\xi-\eta|)[m(\xi)-m(\eta)]^2\,\mathrm{d}\xi\,\mathrm{d}\eta$$

(note that the first of these integrals, *and hence the second*, is absolutely convergent by hypothesis).

Here,

$$K''(|\xi-\eta|) \;=\; \frac{1}{4}\,\frac{\sinh^2\left(\dfrac{\xi-\eta}{2}\right)+\cosh^2\left(\dfrac{\xi-\eta}{2}\right)}{\sinh^2\left(\dfrac{\xi-\eta}{2}\right)\cosh^2\left(\dfrac{\xi-\eta}{2}\right)}$$

$$= \frac{\cosh(\xi-\eta)}{\sinh^2(\xi-\eta)} \;=\; 2\mathrm{e}^\xi\mathrm{e}^\eta\,\frac{\mathrm{e}^{2\xi}+\mathrm{e}^{2\eta}}{(\mathrm{e}^{2\xi}-\mathrm{e}^{2\eta})^2},$$

so the third of the above expressions reduces to

$$\int_{-\infty}^\infty\int_{-\infty}^\infty \frac{\mathrm{e}^{2\xi}+\mathrm{e}^{2\eta}}{(\mathrm{e}^\xi+\mathrm{e}^\eta)^2}\left(\frac{m(\xi)-m(\eta)}{\mathrm{e}^\xi-\mathrm{e}^\eta}\right)^2 \mathrm{e}^\xi\mathrm{e}^\eta\,\mathrm{d}\xi\,\mathrm{d}\eta$$

$$= \int_0^\infty\int_0^\infty \frac{x^2+t^2}{(x+t)^2}\left(\frac{\rho(x)-\rho(t)}{x-t}\right)^2 \mathrm{d}x\,\mathrm{d}t.$$

We are done.

***Remark.*** This certainly implies that the *first* of the above corollaries is true for *any real measure $\rho$ with $\rho(\{0\}) = 0$ rendering absolutely convergent* the double integral used to define $E(\mathrm{d}\rho(t), \mathrm{d}\rho(t))$. The *second* corollary is then *also true* for such real measures $\rho$ and $\sigma$.

The formula provided by this second lemma is one of the main ingredients in our treatment of the question discussed in the present §. It is the basis for the important calculation carried out in the next article.

## 6. A lower estimate for $\int_{\tilde{\Omega}}\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right|\mathrm{d}\mu(t)\,\frac{\mathrm{d}x}{x^2}$

We return to where we left off near the end of article 4, focusing our attention on the quantity

$$\int_{\tilde{\Omega}}\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right|\mathrm{d}\mu(t)\frac{\mathrm{d}x}{x^2},$$

where $\mu(t)$ is the function constructed in article 3 and

$$\tilde{\Omega} = (0, \infty) \sim \{x:\ \mu'(x)>0\}.$$

Before going any further, the reader should refer to the graph of $\mu(t)$ found near the end of article 3. As explained in article 4, we prefer to work not with $\mu(t)$, but with

$$\nu(t) = \frac{1-3p}{p}\mu(t);$$

the graph of $\nu(t)$ looks just like that of $\mu(t)$, save that its slanting portions all have slope 1, and not $p/(1-3p)$. Those slanting portions lie over certain intervals $[c_k,\ \gamma_k]$, $k\geqslant 1$, $[\delta_k,\ d_k]$, $k\geqslant 0$, contained in the $J_k=[c_k,\ d_k]$, and

$$\tilde{\Omega} = (0,\infty)\sim\bigcup_{k\geqslant 0}[\delta_k,d_k]\sim\bigcup_{k\geqslant 1}[c_k,\gamma_k].$$

This set $\tilde{\Omega}$ is obtained from the one $\Omega$ shown on the graph of $\mu(t)$ by adjoining to the latter the intervals $(c_0,\ \delta_0)\subseteq J_0$ and $(\gamma_k,\ \delta_k)\subseteq J_k$, $k\geqslant 1$.

By the corollary at the end of article 4,

$$\begin{aligned}
&\int_{\tilde{\Omega}}\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right|\mathrm{d}\mu(t)\frac{\mathrm{d}x}{x^2}\\
&= \frac{p}{1-3p}\int_0^\infty\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right|\mathrm{d}\nu(t)\frac{\mathrm{d}x-\mathrm{d}\nu(x)}{x^2}\\
&= \frac{p}{1-3p}\int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right|\mathrm{d}\left(\frac{\nu(t)}{t}\right)\mathrm{d}\left(\frac{\nu(x)}{x}\right),
\end{aligned}$$

and this is just

$$\frac{p}{1-3p}E\left(\mathrm{d}\left(\frac{\nu(t)}{t}\right),\mathrm{d}\left(\frac{\nu(x)}{x}\right)\right),$$

$E(\ \ ,\ \ )$ being the bilinear form defined and studied in the previous article.

This identification is a key step in our work. It, and the results of article

5, enable us to see that

$$\int_{\tilde{\Omega}}\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right| d\mu(t)\frac{dx}{x^2}$$

is at least *positive* (until now, we were not even sure of this). The second lemma of article 5 actually makes it possible for us to estimate that integral from below in terms of a sum,

$$\sum_{k\geqslant 1}\left(\frac{\gamma_k-c_k}{\gamma_k}\right)^2 \;+\; \sum_{k\geqslant 0}\left(\frac{d_k-\delta_k}{d_k}\right)^2,$$

like one which occurred previously in Chapter VII, §A.2. In our estimate, that sum is affected with a certain *coefficient.*

On account of the theorem of article 3, we are *really* interested in

$$\int_{\Omega}\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right| d\mu(t)\frac{dx}{x^2}$$

rather than the quantity considered here. It will turn out later on that the passage from integration over $\tilde{\Omega}$ to that over $\Omega$ involves a serious *loss*, in whose evaluation the sum just written again figures. For this reason we have to take care to get *a large enough numerical value* for the coefficient mentioned above. That circumstance requires us to be somewhat fussy in the computation made to derive the following result. ▶ *From now on, in order to make the notation more uniform, we will write*

$$\gamma_0 \;=\; c_0.$$

***Theorem.*** *If* $\nu(t) = ((1-3p)/p)\mu(t)$ *with the function* $\mu(t)$ *from article* 3, *and the parameter* $\eta > 0$ *used in the construction of the* $J_k$ *(see the theorem, end of article* 2) *is* sufficiently small, *we have*

$$E\left(d\left(\frac{\nu(t)}{t}\right), d\left(\frac{\nu(t)}{t}\right)\right)$$
$$\geqslant\; (\tfrac{3}{2}-\log 2-K\eta)\sum_{k\geqslant 0}\left\{\left(\frac{\gamma_k-c_k}{\gamma_k}\right)^2+\left(\frac{d_k-\delta_k}{d_k}\right)^2\right\}.$$

*Here, K is a purely numerical constant, independent of p or the configuration of the* $J_k$.

***Remark.*** Later on, we will need the numerical value

$$\boxed{\tfrac{3}{2}-\log 2 \;=\; 0.80685\ldots.}$$

***Proof of theorem.*** By the second lemma of article 5 and brute force. The lemma gives

$$E\left(d\left(\frac{v(t)}{t}\right), d\left(\frac{v(t)}{t}\right)\right)$$

$$= \int_0^\infty\int_0^\infty \left(\frac{\dfrac{v(x)}{x}-\dfrac{v(y)}{y}}{x-y}\right)^2 \frac{x^2+y^2}{(x+y)^2}\,dx\,dy$$

$$\geqslant \frac{1}{2}\sum_{k\geqslant 0}\int_{J_k}\int_{J_k}\left(\frac{\dfrac{v(x)}{x}-\dfrac{v(y)}{y}}{x-y}\right)^2 dx\,dy.$$

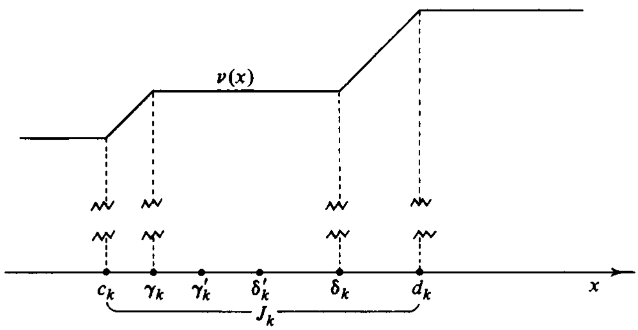



Figure 146

On each interval $J_k = [c_k, d_k]$ we take

$$\begin{aligned} \gamma_k' &= c_k + 2(\gamma_k - c_k) \\ \delta_k' &= d_k - 2(d_k - \delta_k) \end{aligned}$$

(see figure). Since

$$\frac{\gamma_k - c_k + d_k - \delta_k}{d_k - c_k} = \frac{1-3p}{2} < \frac{1}{2}$$

(properties (iii), (iv) of the description near the end of article 3) we have $\gamma_k' < \delta_k'$. Therefore, for each $k$,

$$\int_{J_k}\int_{J_k}\left(\frac{\dfrac{v(x)}{x}-\dfrac{v(y)}{x}}{x-y}\right)^2 dx\,dy$$

$$\geqslant \left\{\int_{c_k}^{\gamma_k'}\int_{c_k}^{\gamma_k'} + \int_{\delta_k'}^{d_k}\int_{\delta_k'}^{d_k}\right\}\left(\frac{\dfrac{v(x)}{x}-\dfrac{v(y)}{y}}{x-y}\right)^2 dx\,dy.$$

We estimate the *second* of the integrals on the right – the other one is handled similarly.

We begin by writing

$$\int_{\delta_k'}^{d_k}\int_{\delta_k'}^{d_k}\left(\frac{\dfrac{v(x)}{x}-\dfrac{v(x)}{y}}{x-y}\right)^2 \mathrm{d}x\,\mathrm{d}y$$

$$\geqslant \left\{\int_{\delta_k}^{d_k}\int_{\delta_k}^{d_k}+\int_{\delta_k'}^{\delta_k}\int_{\delta_k}^{d_k}+\int_{\delta_k}^{d_k}\int_{\delta_k'}^{\delta_k}\right\}\left(\frac{\dfrac{v(x)}{x}-\dfrac{v(y)}{y}}{x-y}\right)^2 \mathrm{d}x\,\mathrm{d}y.$$

Of the three double integrals on the right, the *first* is easiest to evaluate. Things being bad enough as they are, *let us lighten the notation by dropping, for the moment, the subscript k, putting*

$$\delta' \quad \text{for} \quad \delta_k',$$

$$\delta \quad \text{for} \quad \delta_k$$

and

$$d \quad \text{for} \quad d_k.$$

Since $v'(x) = 1$ for $\delta_k = \delta < x < d = d_k$,

$$\frac{v(x)}{x} = 1+\frac{v(\delta)-\delta}{x}, \qquad \delta \leqslant x \leqslant d.$$

Using this, we easily find that

$$\int_{\delta}^{d}\int_{\delta}^{d}\left(\frac{\dfrac{v(x)}{x}-\dfrac{v(y)}{y}}{x-y}\right)^2 \mathrm{d}x\,\mathrm{d}y = \left(1-\frac{v(\delta)}{\delta}\right)^2\left(\frac{d-\delta}{d}\right)^2.$$

In terms of

$$\tilde{J} = \bigcup_{k\geqslant 0}\left((c_k, \gamma_k)\cup(\delta_k, d_k)\right)$$

and

$$J = \bigcup_{k\geqslant 0} J_k,$$

we have clearly

$$v(t) = |[0, t]\cap\tilde{J}| \leqslant |[0, t]\cap J|, \qquad t>0.$$

The right-hand quantity is, however, $\leqslant 2\eta t$ by construction of the $J_k$ (property (v) in the theorem at the end of article 2). Therefore

$v(\delta)/\delta \;=\; v(\delta_k)/\delta_k \;\leqslant\; 2\eta$, and the integral just evaluated is

$$\geqslant\; (1-2\eta)^2\left(\frac{d-\delta}{d}\right)^2.$$

We pass now to the *second* of the three double integrals in question, continuing to omit the subscript $k$. To simplify the work, we make the changes of variable

$$x = \delta + s, \qquad y = \delta - t,$$

and denote $d-\delta = \delta-\delta'$ by $\Delta$. Then

$$\int_{\delta'}^{\delta}\int_{\delta}^{d}\left(\frac{\dfrac{v(x)}{x}-\dfrac{v(y)}{y}}{x-y}\right)^2 dx\,dy \;=\; \int_0^{\Delta}\int_0^{\Delta}\left(\frac{\dfrac{v(\delta)+s}{\delta+s}-\dfrac{v(\delta)}{\delta-t}}{s+t}\right)^2 ds\,dt,$$

since $v(y) = v(\delta)$ for $\delta' \leqslant y \leqslant \delta$ (see the above figure). The expression on the right simplifies to

$$\int_0^{\Delta}\int_0^{\Delta}\left(\frac{s}{(\delta+s)(t+s)}-\frac{v(\delta)}{(\delta-t)(\delta+s)}\right)^2 ds\,dt$$

which in turn is

$$\geqslant\; \frac{1}{d^2}\int_0^{\Delta}\int_0^{\Delta}\left(\frac{s}{t+s}\right)^2 dt\,ds \;-\; 2\frac{v(\delta)}{\delta}\cdot\frac{1}{\delta'\delta}\int_0^{\Delta}\int_0^{\Delta}\frac{s}{t+s}\,ds\,dt$$

$$\geqslant\; \frac{\Delta^2}{d^2}(1-\log 2)-\frac{4\eta\Delta^2}{\delta'\delta}$$

(we have again used the fact that $v(\delta)/\delta \leqslant 2\eta$). We have $v(d) \geqslant v(d)-v(\delta) = d-\delta = \Delta$, so, since $v(d)/d \leqslant 2\eta$,

$$\delta \;=\; d-\Delta \;\geqslant\; (1-2\eta)d$$

and

$$\delta' \;=\; d-2\Delta \;\geqslant\; (1-4\eta)d.$$

By the computation just made we thus have

$$\int_{\delta'}^{\delta}\int_{\delta}^{d}\left(\frac{\dfrac{v(x)}{x}-\dfrac{v(y)}{y}}{x-y}\right)^2 dx\,dy$$

$$\geqslant\; \left(1-\log 2-\frac{4\eta}{(1-2\eta)(1-4\eta)}\right)\left(\frac{d-\delta}{d}\right)^2.$$

For the *third* of our three double integrals we have exactly the same estimate. Hence, *restoring now the subscript k*,

$$\int_{\delta_k'}^{d_k}\int_{\delta_k'}^{d_k}\left(\frac{\frac{v(x)}{x}-\frac{v(y)}{y}}{x-y}\right)^2 \mathrm{d}x\,\mathrm{d}y \;\geqslant\; (3-2\log 2-15\eta)\left(\frac{d_k-\delta_k}{d_k}\right)^2,$$

as long as $\eta > 0$ is sufficiently small.

In the same way, one finds that

$$\int_{c_k}^{\gamma_k'}\int_{c_k}^{\gamma_k'}\left(\frac{\frac{v(x)}{x}-\frac{v(y)}{y}}{x-y}\right)^2 \mathrm{d}x\,\mathrm{d}y \;\geqslant\; (3-2\log 2-K\eta)\left(\frac{\gamma_k-c_k}{\gamma_k}\right)^2$$

for small enough $\eta > 0$, $K$ being a certain numerical constant. Adding this to the previous relation gives us a lower estimate for

$$\int_{J_k}\int_{J_k}\left(\frac{\frac{v(x)}{x}-\frac{v(y)}{y}}{x-y}\right)^2 \mathrm{d}x\,\mathrm{d}y;$$

adding these estimates and referring again to the relation at the beginning of this proof, we obtain the theorem.

Q.E.D.

From the initial discussion of this article, we see that the theorem has the following

***Corollary.*** *Let $\mu(t)$ be the function constructed in article 3 and $\tilde{\Omega}$ be the complement, in $(0,\infty)$, of the set on which $\mu(t)$ is increasing. Then, if the parameter $\eta > 0$ used in constructing the $J_k$ is sufficiently small,*

$$\int_{\tilde{\Omega}}\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right|\mathrm{d}\mu(t)\frac{\mathrm{d}x}{x^2}$$
$$\geqslant\; \frac{p}{1-3p}\left(\tfrac{3}{2}-\log 2-K\eta\right)\sum_{k\geqslant 0}\left(\left(\frac{\gamma_k-c_k}{\gamma_k}\right)^2+\left(\frac{d_k-\delta_k}{d_k}\right)^2\right).$$

Here $K$ is a numerical constant, independent of $p$ or of the particular configuration of the $J_k$.

In the following work, our guiding idea will be to show that $\int_\Omega\int_0^\infty \log|1-x^2/t^2|\mathrm{d}\mu(t)(\mathrm{d}x/x^2)$ is *not too much less* than the left-hand integral in the above relation, in terms of the sum on the right.

## 7. Effect of taking $x$ to be constant on each of the intervals $J_k$

We continue to write

$$\Omega \;=\; (0,\infty)\sim J,$$

where $J = \bigcup_{k\geq 0} J_k$ with $J_k = [c_k, d_k]$, and

$$\tilde{\Omega} = (0, \infty) \sim \tilde{J},$$

with

$$\tilde{J} = \bigcup_{k\geq 0} ((c_k, \gamma_k) \cup (\delta_k, d_k))$$

being the set on which $\mu(t)$ is increasing. The comparison of $\int_\Omega \int_0^\infty \log|1 - x^2/t^2| \mathrm{d}\mu(t)(\mathrm{d}x/x^2)$, object of our interest, with $\int_{\tilde{\Omega}} \int_0^\infty \log|1 - x^2/t^2| \mathrm{d}\mu(t)(\mathrm{d}x/x^2)$ is simplified by using two approximations to those quantities.

As in the previous article, we work in terms of

$$\nu(t) = \frac{1-3p}{p} \mu(t)$$

instead of $\mu(t)$. Put

$$\boxed{u(z) = \int_0^\infty \log\left|\frac{z+t}{z-t}\right| \mathrm{d}\left(\frac{\nu(t)}{t}\right).}$$

Then, by the corollary to the first lemma in article 4,

$$\int_\Omega \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\mu(t)\frac{\mathrm{d}x}{x^2} = \frac{p}{1-3p}\int_J u(x)\frac{\mathrm{d}x}{x}$$

and

$$\int_{\tilde{\Omega}} \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\mu(t)\frac{\mathrm{d}x}{x^2} = \frac{p}{1-3p}\int_{\tilde{J}} u(x)\frac{\mathrm{d}x}{x}.$$

Our approximation consists in the replacement of

$$\int_J u(x)\frac{\mathrm{d}x}{x} \quad \text{by} \quad \sum_{k\geq 0} \frac{1}{d_k}\int_{J_k} u(x)\mathrm{d}x$$

and of

$$\int_{\tilde{J}} u(x)\frac{\mathrm{d}x}{x} \quad \text{by} \quad \sum_{k\geq 0} \frac{1}{d_k}\left(\int_{c_k}^{\gamma_k} + \int_{\delta_k}^{d_k}\right) u(x)\mathrm{d}x.$$

To estimate the difference between the left-hand and right-hand quantities we use the positivity of the bilinear form $E(\ ,\ )$, proved in article 5.

***Theorem.*** *If the parameter $\eta > 0$ used in the construction of the $J_k$ is sufficiently small.*

$$\left| \int_J u(x)\frac{\mathrm{d}x}{x} - \sum_{k\geqslant 0}\frac{1}{d_k}\int_{J_k} u(x)\mathrm{d}x \right|$$

*and*

$$\left| \int_{\tilde{J}} u(x)\frac{\mathrm{d}x}{x} - \sum_{k\geqslant 0}\frac{1}{d_k}\left( \int_{c_k}^{\gamma_k} + \int_{\delta_k}^{d_k} \right) u(x)\mathrm{d}x \right|$$

*are both*

$$\leqslant C\eta^{\frac{1}{2}} E\left( \mathrm{d}\left(\frac{v(t)}{t}\right), \mathrm{d}\left(\frac{v(t)}{t}\right)\right),$$

*where $C$ is a purely numerical constant, independent of $p < \frac{1}{20}$ or the configuration of the $J_k$.*

***Remark.*** Here,

$$E\left( \mathrm{d}\left(\frac{v(t)}{t}\right), \mathrm{d}\left(\frac{v(t)}{t}\right)\right) = \int_{\tilde{J}} u(x)\frac{\mathrm{d}x}{x}$$

according to the corollary at the end of article 4.

***Proof.*** Let us treat the *second* difference; the first is handled similarly. Take

$$\varphi(x) = \begin{cases} \dfrac{1}{x} - \dfrac{1}{d_k}, & c_k < x < \gamma_k, \quad k \geqslant 1; \\ \dfrac{1}{x} - \dfrac{1}{d_k}, & \delta_k < x < d_k, \quad k \geqslant 0; \\ 0 & \text{elsewhere.} \end{cases}$$

(Recall that $\gamma_0 = c_0$, so $(c_0, \gamma_0)$ is empty.) The second of the expressions in question is then just the absolute value of

$$\int_0^\infty u(x)\varphi(x)\mathrm{d}x = \int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\left(\frac{v(t)}{t}\right)\varphi(x)\mathrm{d}x,$$

i.e., of $E(\mathrm{d}(v(t)/t), \varphi(t)\mathrm{d}t)$, in the notation of article 5. By the second corollary in that article and the remark at the end of it,

$$\left| E\left( \mathrm{d}\left(\frac{v(t)}{t}\right), \varphi(t)\mathrm{d}t\right)\right| \leqslant \sqrt{\left(E\left( \mathrm{d}\left(\frac{v(t)}{t}\right), \mathrm{d}\left(\frac{v(t)}{t}\right)\right)\right)}$$

$$\times \sqrt{(E(\varphi(t)\mathrm{d}t, \varphi(t)\mathrm{d}t))}.$$

The function $\varphi(x)$ is surely zero outside of the $J_k$, and, on $J_k$,

$$0 \leqslant \varphi(x) \leqslant \frac{d_k - x}{x d_k} \leqslant \frac{|J_k|}{x d_k}$$

with $|J_k|/d_k \leqslant 2\eta$ as in the proof of the theorem of article 6. Therefore,

$$0 < \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \varphi(t)\mathrm{d}t \leqslant 2\eta \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \frac{\mathrm{d}t}{t} = \pi^2\eta,$$

and

$$E(\varphi(t)\mathrm{d}t, \varphi(t)\mathrm{d}t) = \int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \varphi(t)\mathrm{d}t\, \varphi(x)\mathrm{d}x$$

$$\leqslant \pi^2\eta \sum_{k\geqslant 0} \int_{J_k} \frac{d_k - x}{x d_k}\mathrm{d}x \leqslant \tfrac{1}{2}\pi^2\eta \sum_{k \geqslant 0} \frac{|J_k|^2}{c_k d_k}.$$

We have $c_k = d_k - |J_k| \geqslant (1-2\eta)d_k$ (see above), and, according to property (iv) from the list near the end of article 3,

$$|J_k| = d_k - c_k = \frac{2}{1-3p}\{(\gamma_k - c_k) + (d_k - \delta_k)\}.$$

Since we are assuming (throughout this §) that $p < \frac{1}{20}$, this makes

$$|J_k|^2 < 2\left(\frac{40}{17}\right)^2 \{(\gamma_k - c_k)^2 + (d_k - \delta_k)^2\},$$

yielding, by the preceding relation,

$$\frac{|J_k|^2}{c_k d_k} \leqslant \frac{12}{1-2\eta}\left\{\left(\frac{\gamma_k - c_k}{\gamma_k}\right)^2 + \left(\frac{d_k - \delta_k}{d_k}\right)^2\right\}.$$

Substitute this inequality into the previous estimate and then apply the theorem from the preceding article. One obtains

$$E(\varphi(t)\mathrm{d}t, \varphi(t)\mathrm{d}t)$$

$$\leqslant \frac{6\pi^2\eta}{(1-2\eta)(\frac{3}{2} - \log 2 - K\eta)} E\left(\mathrm{d}\left(\frac{\nu(t)}{t}\right), \mathrm{d}\left(\frac{\nu(t)}{t}\right)\right).$$

Using this in the above inequality for $|E(\mathrm{d}(\nu(t)/t), \varphi(t)\,\mathrm{d}t)|$, we immediately arrive at the desired bound on the difference in question. We are done.

## 8. An auxiliary harmonic function

We desire to use the lower bound furnished by the theorem of article 6 for

$$\int_{\tilde{J}} u(x)\frac{\mathrm{d}x}{x} = E\left(\mathrm{d}\left(\frac{\nu(t)}{t}\right), \mathrm{d}\left(\frac{\nu(t)}{t}\right)\right)$$

in order to obtain one for $\int_J u(x)(\mathrm{d}x/x)$, the quantity of interest to us. Our plan is to pass from

$$\int_{\tilde{J}} u(x)\frac{\mathrm{d}x}{x} \qquad \text{to} \qquad \sum_{k\geqslant 0}\frac{1}{d_k}\left(\int_{c_k}^{\gamma_k} + \int_{\delta_k}^{d_k}\right)u(x)\,\mathrm{d}x$$

and from

$$\sum_{k\geqslant 0}\frac{1}{d_k}\int_{c_k}^{d_k} u(x)\,\mathrm{d}x \qquad \text{to} \qquad \int_J u(x)\frac{\mathrm{d}x}{x};$$

according to the result of the preceding article (whose notation we maintain here), this will entail only small losses (relative to $\int_{\tilde{J}}u(x)(\mathrm{d}x/x)$ ), if $\eta > 0$ is small. This procedure still requires us, however, to get from the *first* sum to the *second.*

The simplest idea that comes to mind is to just compare corresponding terms of the two sums. That, however, would not be quite right, for in $\int_{c_k}^{d_k} u(x)\,\mathrm{d}x$, the integration takes place over a *set with larger Lebesgue measure* than in $(\int_{c_k}^{\gamma_k} + \int_{\delta_k}^{d_k})u(x)\,\mathrm{d}x$. In order to correct for this discrepancy, one should take an *appropriate multiple* of the second integral and then match the result against the first. The factor to be used here is obviously

$$\frac{2}{1-3p},$$

since (article 3),

$$\frac{\gamma_k - c_k + d_k - \delta_k}{d_k - c_k} = \frac{1-3p}{2}.$$

We are looking, then, at

$$\int_{c_k}^{d_k} u(x)\mathrm{d}x \;-\; \frac{2}{1-3p}\left(\int_{c_k}^{\gamma_k} + \int_{\delta_k}^{d_k}\right)u(x)\mathrm{d}x$$

$$= \int_{\gamma_k}^{\delta_k} u(x)\mathrm{d}x \;-\; \frac{1+3p}{1-3p}\left(\int_{c_k}^{\gamma_k} + \int_{\delta_k}^{d_k}\right)u(x)\mathrm{d}x.$$

From now on, it will be convenient to write

$$\boxed{\lambda = \frac{1+3p}{1-3p};}$$

$\lambda$ is $>1$ and *very close to* 1 *if* $p > 0$ *is small.* It is also useful to split up each interval $(\gamma_k, \delta_k)$ into two pieces, associating the left-hand one with $(c_k, \gamma_k)$ and the other with $(\delta_k, d_k)$, and doing this in such a way that each piece has $\lambda$ *times the length* of the interval to which it is associated. This is of course possible because

$$\frac{\delta_k - \gamma_k}{\gamma_k - c_k + d_k - \delta_k} = \frac{1+3p}{1-3p} = \lambda;$$

we thus take $g_k \in (\gamma_k, \delta_k)$ with

$$g_k = \gamma_k + \lambda(\gamma_k - c_k)$$

(and hence also $g_k = \delta_k - \lambda(d_k - \delta_k)$ ), and look at each of the two differences

$$\int_{\gamma_k}^{g_k} u(x)\,\mathrm{d}x - \lambda\int_{c_k}^{\gamma_k} u(x)\,\mathrm{d}x, \qquad \int_{g_k}^{\delta_k} u(x)\,\mathrm{d}x - \lambda\int_{\delta_k}^{d_k} u(x)\,\mathrm{d}x$$

separately; what we want to show is that *neither comes out too negative*, for we are trying to obtain a *positive lower bound* on $\int_J u(x)(\mathrm{d}x/x)$.

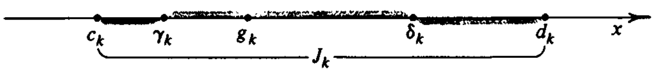



Figure 147

It is a fact that the two differences just written *can* be estimated in terms of $E(\mathrm{d}(\nu(t)/t), \mathrm{d}(\nu(t)/t))$.

**Problem 23**

(a) Show that for our function

$$u(z) = \int_0^\infty \log\left|\frac{z+t}{z-t}\right| \mathrm{d}\left(\frac{\nu(t)}{t}\right),$$

one has

$$E\left(\mathrm{d}\left(\frac{\nu(t)}{t}\right), \mathrm{d}\left(\frac{\nu(t)}{t}\right)\right) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{u(x)-u(y)}{x-y}\right)^2 \mathrm{d}x\,\mathrm{d}y.$$

This is Jesse Douglas' formula – I hope the coefficient on the right is correct. (Hint: Here, $u(x) = -(1/x)\int_0^\infty \log|1 - x^2/t^2|\,\mathrm{d}\nu(t)$ belongs to $L_2(-\infty, \infty)$ (it is *odd* on $\mathbb{R}$), so we can use Fourier–Plancherel transforms. In terms of

$$\hat{u}(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda t} u(t)\,\mathrm{d}t$$

we have

$$u(x + iy) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|\lambda|y} e^{-i\lambda x}\hat{u}(\lambda)\,\mathrm{d}\lambda$$

for $y > 0$ (the left side being just the Poisson harmonic extension of the function $u(x)$ to $\Im z > 0$), and

$$\frac{u(x+h)-u(x)}{h} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\frac{e^{-i\lambda h}-1}{h}\hat{u}(\lambda)\,d\lambda.$$

(All the right-hand integrals are to be understood in the l.i.m. sense.) Use Plancherel's theorem to express

$$\int_{-\infty}^{\infty}\left(\frac{u(x+h)-u(x)}{h}\right)^2 dx \qquad \text{and} \qquad \int_{-\infty}^{\infty}[(u_x(z))^2+(u_y(z))^2]\,dx$$

in terms of integrals involving $|\hat{u}(\lambda)|^2$, then integrate $h$ from $-\infty$ to $\infty$ and $y$ from 0 to $\infty$, and compare the results. Refer finally to the first lemma of article 5.)

(b) Show that

$$\left|\int_{\gamma_k}^{g_k} u(x)dx - \lambda\int_{c_k}^{\gamma_k} u(x)dx\right|$$
$$\leqslant \sqrt{\left(\frac{(1+\lambda)^4-1-\lambda^4}{12}\right)}\cdot(\gamma_k-c_k)\cdot\sqrt{\left(\int_{c_k}^{\gamma_k}\int_{\gamma_k}^{g_k}\left(\frac{u(x)-u(y)}{x-y}\right)^2 dy\,dx\right)},$$

and obtain a similar estimate for

$$\int_{g_k}^{\delta_k} u(x)dx - \lambda\int_{\delta_k}^{d_k} u(x)dx.$$

(Hint: *Trick*:

$$\int_{\gamma_k}^{g_k} u(x)dx - \lambda\int_{c_k}^{\gamma_k} u(x)dx = \frac{1}{\gamma_k-c_k}\int_{c_k}^{\gamma_k}\int_{\gamma_k}^{g_k}[u(y)-u(x)]\,dy\,dx.\ )$$

(c) Use the result of article 6 with those of (a) and (b) to estimate

$$\left|\sum_{k\geqslant 0}\frac{1}{d_k}\left(\int_{\gamma_k}^{\delta_k} u(x)dx - \lambda\left(\int_{c_k}^{\gamma_k}+\int_{\delta_k}^{d_k}\right)u(x)dx\right)\right|$$

in terms of $E(d(v(t)/t),\ d(v(t)/t))$.

By working the problem, one finds that the difference considered in part (c) is in absolute value $\leqslant C\int_{\mathfrak{J}} u(x)(dx/x)$ for a certain numerical constant $C$. The trouble is, however, that the value of $C$ obtained in this way comes out quite a bit *larger* than 1, so that the result cannot be used to yield a *positive* lower bound on $\int_J u(x)(dx/x)$, $\lambda$ being near 1. Too much is lost in following the simple reasoning of part (b); we need a more refined argument that will bring the value of $C$ down below 1.

Any such refinement that works seems to involve bringing in (by use of Green's theorem, for instance) certain double integrals taken over portions of the first quadrant, in which the partial derivatives of $u$ occur. Let us see how this comes about, considering the difference

$$\int_{\gamma_k}^{g_k} u(x)\,\mathrm{d}x \;-\; \lambda\int_{c_k}^{\gamma_k} u(x)\,\mathrm{d}x.$$

The latter can be rewritten as

$$\int_0^{(1+\lambda)\Delta_k} u(c_k + x)s_k(x)\,\mathrm{d}x,$$

where $\Delta_k = \gamma_k - c_k$, and

$$s_k(x) \;=\; \begin{cases} -\lambda, & 0 < x < \Delta_k, \\ \;\;1, & \Delta_k < x < (1+\lambda)\Delta_k. \end{cases}$$

Suppose that we can find a function $V(z) = V_k(z)$, *harmonic* in the half-strip

$$S_k \;=\; \{z\colon\; 0 < \Re z < (1+\lambda)\Delta_k \;\text{ and }\; \Im z > 0\}$$

and having the following boundary behaviour:

$$V_y(x + \mathrm{i}0) \;=\; -\,s_k(x), \qquad 0 < x < (1+\lambda)\Delta_k$$

($V_y(x + \mathrm{i}0)$ will be discontinuous at $x = \Delta_k$),

$$V_x(\mathrm{i}y) \;=\; 0, \qquad y > 0,$$

$$V_x(\mathrm{i}y + (1+\lambda)\Delta_k) \;=\; 0, \qquad y > 0.$$

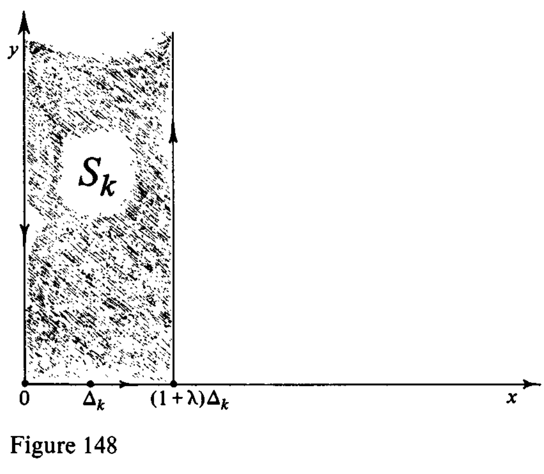



Figure 148

Then the previous integral becomes

$$\int_{\partial S_k} (-u(c_k + z)V_y(z)\,dx \quad + \quad u(c_k + z)V_x(z)\,dy),$$

$\partial S_k$ being oriented in the usual counterclockwise sense. Application of Green's theorem, if legitimate (which is easily shown to be the case here, as we shall see in due time), converts the line integral to

$$\iint_{S_k} (u_y(c_k + z)V_y(z) \quad + \quad u_x(c_k + z)V_x(z))\,dx\,dy$$

$$+ \quad \iint_{S_k} u(c_k + z)[V_{yy}(z) + V_{xx}(z)]\,dx\,dy.$$

The harmonicity of $V$ in $S_k$ will make the *second* integral vanish, and finally the *difference under consideration* will be equal to the *first* one. Referring to the first lemma of article 5, we see that the successful use of this procedure in order to get what we want necessitates our actually *obtaining* such a harmonic function $V = V_k$ and then *computing* (at least) its *Dirichlet integral*

$$\iint_{S_k} (V_x^2 + V_y^2)\,dx\,dy.$$

We will in fact need to know a little more than that. Let us proceed with the necessary calculations.

Our harmonic function $V_k(z)$ (assuming, of course, that there *is* one) will depend on *two* parameters, $\Delta_k$ and $\lambda = (1 + 3p)/(1 - 3p)$. The dependence on the *first* of these is nothing but a kind of homogeneity. Let $v(z, \lambda)$ be the function $V(z)$ corresponding to the special value $\pi/(1 + \lambda)$ of $\Delta_k$, using the value of $\lambda$ figuring in $V_k(z)$; $v(z, \lambda)$ is, in other words, to be harmonic in the half-strip

$$S \quad = \quad \{z\colon\ 0 < \Re z < \pi \ \text{ and } \ \Im z > 0\}$$

*with* $v_x(z, \lambda) = 0$ on the *vertical sides* of $S$ and

$$v_y(x + i0,\ \lambda) \quad = \quad \begin{cases} \lambda, & 0 < x < \dfrac{\pi}{1 + \lambda}, \\[2ex] -1, & \dfrac{\pi}{1 + \lambda} < x < \pi. \end{cases}$$

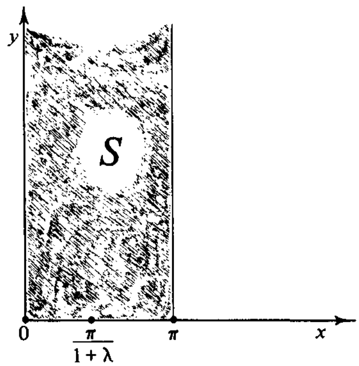



Figure 149

On the half-strip $S_k$ of width $(1+\lambda)\Delta_k$ shown previously, the function

$$\frac{1}{\pi}(1+\lambda)\Delta_k v(\pi z/(1+\lambda)\Delta_k,\ \lambda)$$

is harmonic, and its partial derivatives clearly satisfy the boundary conditions on those of $V_k(z)$ stipulated above. We may therefore take

$$\boxed{V_k(z) \;=\; \frac{1}{\pi}(1+\lambda)\Delta_k v(\pi z/(1+\lambda)\Delta_k,\ \lambda)\,;}$$

this permits us to do all our calculations with the standard function $v$. Note that we will have, by simple change of variables,

$$\iint_{S_k}\left(\frac{\partial V_k}{\partial x}\right)^2 \mathrm{d}x\,\mathrm{d}y \;=\; \left(\frac{(1+\lambda)\Delta_k}{\pi}\right)^2 \iint_S (v_x(z,\lambda))^2\,\mathrm{d}x\,\mathrm{d}y$$

and

$$\iint_{S_k}\left[\left(\frac{\partial V_k}{\partial y}\right)^+\right]^2 \mathrm{d}x\,\mathrm{d}y \;=\; \left(\frac{(1+\lambda)\Delta_k}{\pi}\right)^2 \iint_S [(v_y(z,\lambda))^+]^2\,\mathrm{d}x\,\mathrm{d}y,$$

while

$$\iint_{S_k}\left|\frac{\partial V_k}{\partial y}\right| \mathrm{d}x\,\mathrm{d}y \;=\; \left(\frac{(1+\lambda)\Delta_k}{\pi}\right)^2 \iint_S |v_y(z,\lambda)|\,\mathrm{d}x\,\mathrm{d}y.$$

***Lemma.*** *Given $\lambda \geqslant 1$, we can find a function $v(z,\lambda)$ harmonic in $S$ whose partial derivatives satisfy the boundary conditions specified above. If $\varepsilon > 0$ is* *given, we have, for all $\lambda \geq 1$ sufficiently close to* 1,

$$\pi \iint_S (v_x(z,\lambda))^2 \mathrm{d}x\,\mathrm{d}y \;<\; 4\left(1 + \frac{1}{3^3} + \frac{1}{5^3} + \cdots + \varepsilon\right),$$

$$\pi \iint_S [(v_y(z,\lambda))^+]^2 \mathrm{d}x\,\mathrm{d}y \;<\; 2\left(1 + \frac{1}{3^3} + \frac{1}{5^3} + \cdots + \varepsilon\right),$$

*and*

$$\iint_S |v_y(z,\lambda)| \mathrm{d}x\,\mathrm{d}y \;\leq\; C,$$

*C being a numerical constant, whose value we do not bother to calculate.*

***Remark.*** In the next article we will need the *numerical approximation*

$$\boxed{\frac{4}{\pi^2}\left(1 + \frac{1}{3^3} + \frac{1}{5^3} + \cdots\right) \;<\; 0.4268.}$$

***Proof of lemma.*** The method followed here (plain old 'separation of variables' from engineering mathematics) was suggested to me by Cedric Schubert. We look for a function $v$ represented in the form

$$v(z,\lambda) \;=\; \sum_1^\infty A_n(\lambda)\mathrm{e}^{-ny}\cos nx$$

The series on the right, if convergent, *will* represent a function harmonic in $S$ (each of its *terms* is harmonic!), and, for $y > 0$,

$$v_x(z,\lambda) \;=\; -\sum_1^\infty nA_n(\lambda)\mathrm{e}^{-ny}\sin nx$$

*will vanish* for $x = 0$ and $x = \pi$, for the exponentially decreasing factors $\mathrm{e}^{-ny}$ will make the series absolutely convergent.

For $y = 0$, by Abel's theorem,

$$v_y(x + \mathrm{i}0,\; \lambda) \;=\; -\sum_1^\infty nA_n(\lambda)\cos nx$$

at each $x$ for which the series on the right is convergent. Let us choose the $A_n(\lambda)$ so as to make the right side the *Fourier cosine series* of the function

$$s(x,\lambda) \;=\; \begin{cases} \lambda, & 0 < x < \dfrac{\pi}{1+\lambda}, \\[2ex] -1, & \dfrac{\pi}{1+\lambda} < x < \pi. \end{cases}$$

We know from the very rudiments of Fourier series theory that this is accomplished by taking

$$-nA_n(\lambda) = \frac{2}{\pi}\int_0^\pi s(x,\lambda)\cos nx\,dx,$$

and that the resulting cosine series *does* converge to $s(x,\lambda)$ for $0 < x < \pi/(1+\lambda)$ and for $\pi/(1+\lambda) < x < \pi$. We can therefore *get* in this way a function $v(z\,\lambda)$ meeting all of our requirements.

Let us continue as long as we can without resorting to explicit computations. For fixed $y > 0$, Parseval's formula yields

$$\int_0^\pi (v_y(z,\lambda))^2 dx = \frac{\pi}{2}\sum_1^\infty n^2(A_n(\lambda))^2 e^{-2ny},$$

and, in like manner,

$$\int_0^\pi [v_y(z,\lambda) - v_y(z,1)]^2 dx = \frac{\pi}{2}\sum_1^\infty n^2(A_n(\lambda) - A_n(1))^2 e^{-2ny}.$$

Integrating both sides of this last relation with respect to $y$, we find that

$$\int_0^\infty\int_0^\pi (v_y(z,\lambda) - v_y(z,1))^2 dx\,dy = \frac{\pi}{4}\sum_1^\infty n[A_n(\lambda) - A_n(1)]^2.$$

By Parseval's formula, we have, however,

$$\sum_1^\infty n^2[A_n(\lambda) - A_n(1)]^2 = \frac{2}{\pi}\int_0^\pi [s(x,\lambda) - s(x,1)]^2 dx,$$

and it is evident that the right-hand integral tends to *zero* as $\lambda \to 1$. Hence, by the preceding relation,

$$\int_0^\infty\int_0^\pi [v_y(z,\lambda) - v_y(z,1)]^2 dx\,dy \longrightarrow 0$$

for $\lambda \to 1$.

Now clearly

$$|(v_y(z,\lambda))^+ - (v_y(z,1))^+| \leqslant |v_y(z,\lambda) - v_y(z,1)|;$$

the result just obtained therefore implies that

$$\int_0^\infty\int_0^\pi [(v_y(z,\lambda))^+]^2 dx\,dy \longrightarrow \int_0^\infty\int_0^\pi [(v_y(z,1))^+]^2 dx\,dy$$

as $\lambda \to 1$. We see in the same fashion that

$$\int_0^\infty\int_0^\pi (v_x(z,\lambda))^2 dx\,dy = \frac{\pi}{4}\sum_1^\infty n(A_n(\lambda))^2,$$

which $\longrightarrow$ $(\pi/4)\sum_1^\infty n(A_n(1))^2$ as $\lambda \to 1$.

For our purpose, it thus suffices to make the calculations for the limiting case $\lambda = 1$. Here,

$$-nA_n(1) \;=\; \frac{2}{\pi}\left(\int_0^{\pi/2} - \int_{\pi/2}^{\pi}\right)\cos nx\,\mathrm{d}x \;=\; \frac{4\sin\frac{\pi}{2}n}{\pi n},$$

so

$$\frac{\pi}{4}\sum_1^\infty n(A_n(1))^2 \;=\; \frac{4}{\pi}\left(1 + \frac{1}{3^3} + \frac{1}{5^3} + \cdots\right),$$

whence, if $\lambda \geqslant 1$ is sufficiently close to 1,

$$\pi\int_0^\infty\int_0^\pi (v_x(z,\lambda))^2\,\mathrm{d}x\,\mathrm{d}y \;<\; 4\left(1 + \frac{1}{3^3} + \frac{1}{5^3} + \cdots + \varepsilon\right).$$

Again

$$v_y(x + \mathrm{i}0,\ 1) \;=\; \begin{cases} 1, & 0 < x < \dfrac{\pi}{2}, \\ -1, & \dfrac{\pi}{2} < x < \pi, \end{cases}$$

so by symmetry, for $y > 0$,

$$v_y\left(\frac{\pi}{2} - h + \mathrm{i}y,\ 1\right) \;=\; -v_y\left(\frac{\pi}{2} + h + \mathrm{i}y,\ 1\right) \;>\; 0, \qquad 0 < h < \frac{\pi}{2}.$$

Hence,

$$\int_0^\infty\int_0^\pi [(v_y(z,1))^+]^2\,\mathrm{d}x\,\mathrm{d}y \;=\; \int_0^\infty\int_0^{\pi/2} (v_y(z,1))^2\,\mathrm{d}x\,\mathrm{d}y$$

$$= \frac{1}{2}\int_0^\infty\int_0^\pi (v_y(z,1))^2\,\mathrm{d}x\,\mathrm{d}y \;=\; \frac{\pi}{8}\sum_1^\infty n(A_n(1))^2$$

$$= \frac{2}{\pi}\left(1 + \frac{1}{3^3} + \frac{1}{5^3} + \cdots\right).$$

Therefore, by the above observation,

$$\pi\int_0^\infty\int_0^\pi [(v_y(z,\lambda))^+]^2\,\mathrm{d}x\,\mathrm{d}y \;<\; 2\left(1 + \frac{1}{3^3} + \frac{1}{5^3} + \cdots + \varepsilon\right)$$

for $\lambda \geqslant 1$ close enough to 1.

We are left with the integral $\int_0^\infty\int_0^\pi |v_y(z,\lambda)|\,\mathrm{d}x\,\mathrm{d}y$. This, by Schwarz' inequality, is

$$\begin{aligned}
&\leqslant \int_0^\infty \left(\pi \int_0^\pi (v_y(z,\lambda))^2\,dx\right)^{\frac{1}{2}} dy \\
&= \int_0^\infty \left(\frac{\pi^2}{2}\sum_1^\infty n^2(A_n(\lambda))^2 e^{-2ny}\right)^{\frac{1}{2}} dy \\
&= \int_0^\infty e^{-y/2}\left(\frac{\pi^2}{2}\sum_1^\infty n^2(A_n(\lambda))^2 e^{-(2n-1)y}\right)^{\frac{1}{2}} dy \\
&\leqslant \sqrt{\left(\int_0^\infty e^{-y}\,dy \cdot \int_0^\infty \frac{\pi^2}{2}\sum_1^\infty n^2(A_n(\lambda))^2 e^{-(2n-1)y}\,dy\right)} \\
&= \sqrt{\left(\frac{\pi^2}{2}\sum_1^\infty \frac{n^2}{2n-1}(A_n(\lambda))^2\right)} \leqslant \frac{\pi}{\sqrt{2}}\sqrt{\left(\sum_1^\infty n(A_n(\lambda))^2\right)}.
\end{aligned}$$

We have already seen that the sum inside the radical in the last of these terms tends to a definite (finite) limit as $\lambda \to 1$. So

$$\int_0^\infty \int_0^\pi |v_y(z,\lambda)|\,dx\,dy$$

is certainly *bounded* for $\lambda \geqslant 1$ near 1. The lemma is proved.

Referring to the remarks made just before the lemma and to the boxed numerical estimate immediately following its statement, we obtain, regarding our original functions $V_k$, the following

***Corollary.*** *Given $\lambda \geqslant 1$ there is, for each k, a function $V_k(z)$ (depending on $\lambda$), harmonic in $S_k = \{z\colon\ 0 < \Re z < (1+\lambda)\Delta_k$ and $\Im z > 0\}$, with $\partial V_k(z)/\partial x = 0$ on the* vertical sides *of $S_k$ and, on the latter's* base, *$\partial V_k/\partial y$ taking the boundary values $\lambda$ and $-1$ along $(0, \Delta_k)$ and $(\Delta_k, (1+\lambda)\Delta_k)$ respectively.*

*If $\lambda \geqslant 1$* is close enough to 1, *we have.*

$$\pi \iint_{S_k} \left(\frac{\partial V_k}{\partial x}\right)^2 dx\,dy \leqslant 0.44(1+\lambda)^2\Delta_k^2,$$

$$\pi \iint_{S_k} \left[\left(\frac{\partial V_k}{\partial y}\right)^+\right]^2 dx\,dy \leqslant 0.22(1+\lambda)^2\Delta_k^2,$$

*and*

$$\iint_{S_k} \left|\frac{\partial V_k}{\partial y}\right| dx\,dy \leqslant \alpha(1+\lambda)^2\Delta_k^2,$$

*$\alpha$ being a certain numerical constant.*

## 9. Lower estimate for $\int_\Omega \int_0^\infty \log|1-x^2/t^2|\,d\mu(t)(dx/x^2)$

We return to the termwise comparison of

$$\sum_{k\geqslant 0}\frac{1}{d_k}\int_{c_k}^{d_k} u(x)\,dx \quad\text{and}\quad \frac{2}{1-3p}\sum_{k\geqslant 0}\frac{1}{d_k}\left(\int_{c_k}^{\gamma_k}+\int_{\delta_k}^{d_k}\right)u(x)\,dx,$$

which, as we saw in the first half of the preceding article, leads to the task of estimating

$$\int_{\gamma_k}^{g_k} u(x)\,dx \; - \; \lambda\int_{c_k}^{\gamma_k} u(x)\,dx$$

and

$$\int_{g_k}^{\delta_k} u(x)\,dx \; - \; \lambda\int_{\delta_k}^{d_k} u(x)\,dx$$

from below. The notation of the previous two articles is maintained here.

Following the idea of the last article, we use the harmonic function $V_k(z)$ described there to express the *first* of the above differences as a *line integral*

$$\int_{\partial S_k}\left(-u(c_k+z)\frac{\partial V_k(z)}{\partial y}dx \; + \; u(c_k+z)\frac{\partial V_k(z)}{\partial x}dy\right)$$

around the vertical half-strip $S_k$ whose base is the segment $[0,\ (1+\lambda)\Delta_k] = [0,\ (1+\lambda)(\gamma_k-c_k)]$ of the real axis. By use of Green's theorem, this line integral is converted to

$$\iint_{S_k}\left(u_x(c_k+z)\frac{\partial V_k(z)}{\partial x} \; + \; u_y(c_k+z)\frac{\partial V_k(z)}{\partial y}\right)dx\,dy,$$

thanks to the harmonicity of $V_k(z)$ in $S_k$. The justification of the present application of Green's theorem proceeds as follows.

We have

$$\int_0^{(1+\lambda)(\gamma_k-c_k)} u(c_k+x)(V_k)_y(x+i0)\,dx$$

$$= \lim_{h\to 0}\int_0^{(1+\lambda)(\gamma_k-c_k)} u(c_k+x+ih)(V_k)_y(x+ih)\,dx,$$

because $u(z)$ is continuous up to the real axis, and, as one verifies by referring to the computations with $v$ and $v_y$ near the end of the previous article,

$$\int_0^{(1+\lambda)(\gamma_k-c_k)} [(V_k)_y(x+ih) \; - \; (V_k)_y(x+i0)]^2\,dx \longrightarrow 0$$

for $h\to 0$.

However, for $h>0$ and $0<x<(1+\lambda)(\gamma_k-c_k)$,

$$\int_h^\infty \frac{\partial}{\partial y}\left(u(c_k+z)\frac{\partial V_k(z)}{\partial y}\right)\mathrm{d}y \;=\; -u(c_k+x+\mathrm{i}h)(V_k)_y(x+\mathrm{i}h),$$

since

$$|u(c_k+z)| \;\leqslant\; \text{const.}\frac{\log|z|}{|z|}, \qquad z\in S_k,$$

by an estimate used in proving the first lemma of article 5, and $V_k(z)$, together with its partial derivatives, tends (exponentially) to zero as $z\to\infty$ in $S_k$ (see the calculations at end of the previous article).

Again, since $\partial V_k(z)/\partial x = 0$ on the vertical sides of $S_k$,

$$\int_0^{(1+\lambda)(\gamma_k-c_k)} \frac{\partial}{\partial x}\left(u(c_k+z)\frac{\partial V_k(z)}{\partial x}\right)\mathrm{d}x \;=\; 0, \qquad y>0.$$

By integrating $y$ in this formula from $h$ to $\infty$ and $x$ in the previous one from 0 to $(1+\lambda)(\gamma_k-c_k)$, and then adding the results, we express

$$-\int_0^{(1+\lambda)(\gamma_k-c_k)} u(c_k+x+\mathrm{i}h)(V_k)_y(x+\mathrm{i}h)\,\mathrm{d}x$$

as the sum of two iterated integrals. For $h>0$, both of the latter are *absolutely convergent*, and the order of integration in *one of them* may be *reversed*. Doing this and remembering that $\nabla^2 V_k = 0$ in $S_k$, we see that the sum in question boils down to

$$\int_h^\infty\int_0^{(1+\lambda)(\gamma_k-c_k)}\left(u_x(c_k+z)\frac{\partial V_k(z)}{\partial x} \;+\; u_y(c_k+z)\frac{\partial V_k(z)}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y.$$

Making $h\to 0$ in this expression finally gives us the corresponding double integral over $S_k$ (whose absolute convergence readily follows from the first lemma in article 5 and the work at the end of the previous one by Schwarz' inequality).

This, together with our initial observation, shows that the double integral over $S_k$ is equal to

$$-\int_0^{(1+\lambda)(\gamma_k-c_k)} u(c_k+x)(V_k)_y(x+\mathrm{i}0)\,\mathrm{d}x,$$

a quantity clearly identical with the above line integral around $\partial S_k$.* In this way, we see that our use of Green's theorem is legitimate.

The line integral is, as we recall (and as we see by glancing at the preceding expression), the same as

$$\int_{\gamma_k}^{g_k} u(x)\,\mathrm{d}x \;-\; \lambda\int_{c_k}^{\gamma_k} u(x)\,\mathrm{d}x.$$

* and actually coinciding with the original expression on p. 499 (the second one displayed there) from which the line integral was elaborated

That difference is therefore equal to

$$\iint_{S_k}\left(u_x(c_k+z)\frac{\partial V_k(z)}{\partial x} \;+\; u_y(c_k+z)\frac{\partial V_k(z)}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y.$$

What we want is a *lower bound* for the difference, and that means we have to *find one for this double integral.*

Our intention is to express such a lower bound as a certain portion of $E(\mathrm{d}(v(t)/t), \mathrm{d}(v(t)/t))$, the hope being that when all these portions are added (and also all the ones corresponding to the differences

$$\int_{g_k}^{\delta_k} u(x)\,\mathrm{d}x \;-\; \lambda\int_{\delta_k}^{d_k} u(x)\,\mathrm{d}x\;\;),$$

we will end with a multiple of $E(\mathrm{d}(v(t)/t), \mathrm{d}(v(t)/t))$ that is not too large. In view, then, of the first lemma of article 5, we are interested in getting a lower bound in terms of

$$\frac{1}{\pi}\iint_{S_k}[(u_x(c_k+z))^2 \;+\; (u_y(c_k+z))^2]\,\mathrm{d}x\,\mathrm{d}y.$$

The present situation allows for very little leeway, and we have to be quite careful.

We start by writing

$$\iint_{S_k} u_x(c_k+z)\frac{\partial V_k(z)}{\partial x}\mathrm{d}x\,\mathrm{d}y$$
$$\geqslant -\sqrt{\left(\pi\iint_{S_k}\left(\frac{\partial V_k(x)}{\partial x}\right)^2\mathrm{d}x\,\mathrm{d}y\right)}$$
$$\times\sqrt{\left(\frac{1}{\pi}\iint_{S_k}(u_x(c_k+z))^2\,\mathrm{d}x\,\mathrm{d}y\right)}.$$

According to the corollary at the end of the last article, the right side is in turn

$$\geqslant\; -(0.44)^{\frac{1}{2}}(1+\lambda)(\gamma_k-c_k)\sqrt{\left(\frac{1}{\pi}\iint_{S_k}(u_x(c_k+z))^2\,\mathrm{d}x\,\mathrm{d}y\right)},$$

provided that $\lambda = (1+3p)/(1-3p)$ is close enough to 1 (recall that the $\Delta_k$ of the previous article equals $\gamma_k - c_k$).

For the estimation of

$$\iint_{S_k} u_y(c_k+z)\frac{\partial V_k(z)}{\partial y}\mathrm{d}x\,\mathrm{d}y,$$

we split up $S_k$ into *two pieces*,

$$S_k^+ \;=\; \left\{ z \in S_k : \;\; \frac{\partial V_k(z)}{\partial y} > 0 \right\}$$

and

$$S_k^- \;=\; S_k \sim S_k^+ .$$

We have

$$\iint_{S_k^+} u_y(c_k + z)\frac{\partial V_k(z)}{\partial y}\,\mathrm{d}x\,\mathrm{d}y$$
$$\geqslant -\sqrt{\left(\pi \iint_{S_k^+}\left(\frac{\partial V_k(z)}{\partial y}\right)^2 \mathrm{d}x\,\mathrm{d}y\right)} \times \sqrt{\left(\frac{1}{\pi}\iint_{S_k^+}(u_y(c_k+z))^2\,\mathrm{d}x\,\mathrm{d}y\right)},$$

which, by the corollary of the preceding article, is

$$\geqslant \; -(0.22)^{\frac{1}{2}}(1+\lambda)(\gamma_k - c_k)\sqrt{\left(\frac{1}{\pi}\iint_{S_k^+}(u_y(c_k+z))^2\,\mathrm{d}x\,\mathrm{d}y\right)}$$

for $\lambda$ close enough to 1. In this last expression, the integral involving $u_y$ may, if we wish, be replaced by one over $S_k$, yielding a worse result.

We are left with

$$\iint_{S_k^-} u_y(c_k + z)\frac{\partial V_k(z)}{\partial y}\,\mathrm{d}x\,\mathrm{d}y,$$

in which $\partial V_k(z)/\partial y \;\leqslant\; 0$. To handle this integral, we recall that

$$u(z) \;=\; \int_0^\infty \log\left|\frac{z+t}{z-t}\right| \mathrm{d}\left(\frac{\nu(t)}{t}\right),$$

which makes

$$u_y(z) \;=\; \int_0^\infty \left[\frac{y}{(x+t)^2+y^2} - \frac{y}{(x-t)^2+y^2}\right]\mathrm{d}\left(\frac{\nu(t)}{t}\right),$$

with the quantity in brackets obviously *negative* for $x, y$ and $t > 0$. Since $\nu(t)/t \leqslant 2\eta$ by our construction of the intervals $J_k$, we have

$$\mathrm{d}\left(\frac{\nu(t)}{t}\right) \;=\; \frac{\mathrm{d}\nu(t)}{t} \;-\; \frac{\nu(t)\,\mathrm{d}t}{t^2} \;\geqslant\; -2\eta\frac{\mathrm{d}t}{t},$$

and therefore, for $x$ and $y > 0$,

$$\begin{aligned} u_y(z) &\leqslant 2\eta\int_0^\infty\left(\frac{y}{(x-t)^2+y^2}-\frac{y}{(x+t)^2+y^2}\right)\frac{\mathrm{d}t}{t} \\ &= 2\eta\lim_{\delta\to 0}\int_{-\infty}^{\infty}\frac{t}{t^2+\delta^2}\,\frac{y}{(x-t)^2+y^2}\mathrm{d}t \\ &= 2\eta\pi\lim_{\delta\to 0}\frac{x}{x^2+(y+\delta)^2} \;=\; 2\pi\eta\frac{x}{x^2+y^2}. \end{aligned}$$

(We have simply used the Poisson representation for the function $\Re(1/(z+\mathrm{i}\delta))$, harmonic in $\Im z > 0$.) Thus,

$$u_y(c_k+z) \leqslant \frac{2\pi\eta}{c_k}, \qquad z\in S_k,$$

whence

$$\iint_{S_k^-} u_y(c_k+z)\frac{\partial V_k(z)}{\partial y}\mathrm{d}x\,\mathrm{d}y \geqslant -\frac{2\pi\eta}{c_k}\iint_{S_k^-}\left|\frac{\partial V_k(z)}{\partial y}\right|\mathrm{d}x\,\mathrm{d}y.$$

For $\lambda$ close to 1, the right side is

$$\geqslant -2\pi\alpha\eta\frac{(1+\lambda)^2(\gamma_k-c_k)^2}{c_k}$$

by the corollary from the previous article, $\alpha$ being a numerical constant.

Combining the three estimates just obtained, we find with the help of Schwarz' inequality that

$$\begin{aligned} &\iint_{S_k}\left(u_x(c_k+z)\frac{\partial V_k(z)}{\partial x} + u_y(c_k+z)\frac{\partial V_k(z)}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y \\ &\geqslant -(0.44)^{\frac{1}{2}}(1+\lambda)(\gamma_k-c_k)\sqrt{\left(\frac{1}{\pi}\iint_{S_k}(u_x(c_k+z))^2\,\mathrm{d}x\,\mathrm{d}y\right)} \\ &\quad -(0.22)^{\frac{1}{2}}(1+\lambda)(\gamma_k-c_k)\sqrt{\left(\frac{1}{\pi}\iint_{S_k}(u_y(c_k+z))^2\,\mathrm{d}x\,\mathrm{d}y\right)} \\ &\quad -2\pi\alpha\eta\frac{(1+\lambda)^2(\gamma_k-c_k)^2}{c_k} \\ &\geqslant -(0.66)^{\frac{1}{2}}(1+\lambda)(\gamma_k-c_k) \\ &\qquad\times\sqrt{\left(\frac{1}{\pi}\iint_{S_k}((u_x(c_k+z))^2+(u_y(c_k+z))^2)\mathrm{d}x\,\mathrm{d}y\right)} \\ &\qquad\qquad -2\pi\alpha\eta\frac{(1+\lambda)^2(\gamma_k-c_k)^2}{c_k}, \end{aligned}$$

provided that $\lambda$ is close enough to 1. The double integral on the left is nothing but a complicated expression for the first of the two differences with which we are concerned – that was, indeed, our reason for bringing the function $V_k(z)$ into this work. Hence the relation just proved can be rewritten

$$\int_{\gamma_k}^{g_k} u(x)\,\mathrm{d}x \;-\; \lambda\int_{c_k}^{\gamma_k} u(x)\,\mathrm{d}x$$
$$\geqslant\; -\;(0.66)^{\frac{1}{2}}(1+\lambda)(\gamma_k - c_k)\sqrt{\left(\frac{1}{\pi}\int_0^\infty\int_{c_k}^{g_k}((u_x(z))^2+(u_y(z))^2)\,\mathrm{d}x\,\mathrm{d}y\right)}$$
$$-\;2\pi\alpha\eta\,\frac{(1+\lambda)^2(\gamma_k-c_k)^2}{c_k}.$$

The difference $\int_{g_k}^{\delta_k}u(x)\,\mathrm{d}x - \lambda\int_{\delta_k}^{d_k}u(x)\,\mathrm{d}x$ can also be estimated by the method of this and the preceding articles. One finds in exactly the same way as above that

$$\int_{g_k}^{\delta_k} u(x)\,\mathrm{d}x \;-\; \lambda\int_{\delta_k}^{d_k} u(x)\,\mathrm{d}x$$
$$\geqslant\; -\;(0.66)^{\frac{1}{2}}(1+\lambda)(d_k - \delta_k)\sqrt{\left(\frac{1}{\pi}\int_0^\infty\int_{g_k}^{d_k}((u_x(z))^2+(u_y(z))^2)\mathrm{d}x\,\mathrm{d}y\right)}$$
$$-\;2\pi\alpha\eta\,\frac{(1+\lambda)^2(d_k-\delta_k)^2}{g_k}$$

for $\lambda$ close enough to 1. The following diagram shows the regions over which the double integrals involved in this and the previous inequalities are taken:

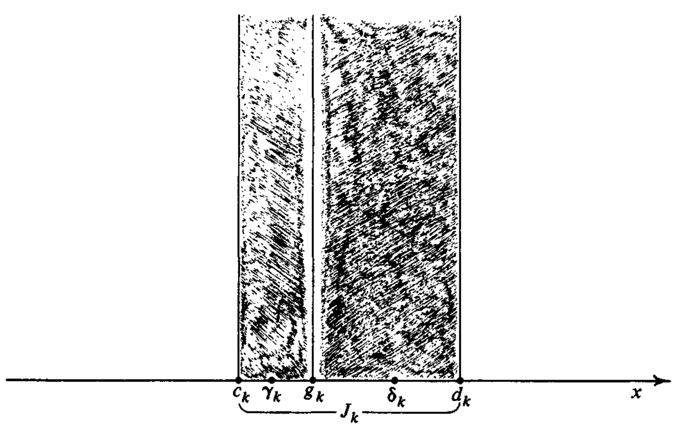



Figure 150

We now add the two relations just obtained. After dividing by $d_k$ and using Schwarz' inequality again together with the fact that

$$c_k \leqslant \gamma_k \leqslant g_k < \delta_k < d_k < (1+2\eta)c_k,$$

we get, recalling that $\lambda = (1+3p)/(1-3p)$,

$$\frac{1}{d_k}\int_{c_k}^{d_k} u(x)\,dx \;-\; \frac{2}{1-3p}\cdot\frac{1}{d_k}\left(\int_{c_k}^{\gamma_k} + \int_{\delta_k}^{d_k}\right)u(x)\,dx$$

$$\geqslant\; -\;(0.66)^{\frac{1}{2}}\frac{2}{1-3p}(1+2\eta)\sqrt{\left(\left(\frac{\gamma_k - c_k}{\gamma_k}\right)^2 + \left(\frac{d_k-\delta_k}{d_k}\right)^2\right)}$$

$$\times\sqrt{\left(\frac{1}{\pi}\int_0^\infty\int_{c_k}^{d_k}(u_x^2+u_y^2)\,dx\,dy\right)}$$

$$-\;\frac{8\pi\alpha(1+2\eta)}{(1-3p)^2}\eta\left[\left(\frac{\gamma_k-c_k}{\gamma_k}\right)^2 + \left(\frac{d_k-\delta_k}{d_k}\right)^2\right]$$

for $\lambda$ close enough to 1, *in other words, for $p>0$ close enough to zero.*

We have now carried out the program explained in the first half of article 8 and at the beginning of the present one. Summing the preceding relation over $k$ and using Schwarz' inequality once more, we obtain, for small $p>0$,

$$\sum_{k\geqslant 0}\frac{1}{d_k}\int_{c_k}^{d_k} u(x)\,dx \;-\; \frac{2}{1-3p}\sum_{k\geqslant 0}\frac{1}{d_k}\left(\int_{c_k}^{\gamma_k} + \int_{\delta_k}^{d_k}\right)u(x)\,dx$$

$$\gtrsim\; -\;(0.66)^{\frac{1}{2}}(1+2\eta)\frac{2}{1-3p}\sqrt{\sum_{k\geqslant 0}\left(\left(\frac{\gamma_k - c_k}{\gamma_k}\right)^2 + \left(\frac{d_k-\delta_k}{d_k}\right)^2\right)}$$

$$\times\sqrt{\left(\frac{1}{\pi}\sum_{k\geqslant 0}\int_0^\infty\int_{J_k}(u_x^2+u_y^2)\,dx\,dy\right)}$$

$$-\;\frac{8\pi\alpha(1+2\eta)}{(1-3p)^2}\eta\sum_{k\geqslant 0}\left(\left(\frac{\gamma_k-c_k}{\gamma_k}\right)^2 + \left(\frac{d_k-\delta_k}{d_k}\right)^2\right).$$

To the right-hand expression we apply the theorem of article 6 together with its remark and the first lemma of article 5. In this way, we find that the right side is

$$\geqslant\; -\;\frac{2}{1-3p}\sqrt{\left(\frac{0.66}{0.80-K\eta}\right)}\cdot(1+2\eta)E\left(d\left(\frac{v(t)}{t}\right), d\left(\frac{v(t)}{t}\right)\right)$$

$$-\;K'\eta E\left(d\left(\frac{v(t)}{t}\right), d\left(\frac{v(t)}{t}\right)\right)$$

for small enough positive values of $\eta$ and $p$, $K$ and $K'$ being certain numerical constants independent of $p$ and of the configuration of the $J_k$.

According to the theorem of article 7, the left-hand difference in the above relation is within

$$\frac{3-3p}{1-3p}C\eta^{\frac{1}{2}}E\left(\mathrm{d}\left(\frac{\nu(t)}{t}\right),\ \mathrm{d}\left(\frac{\nu(t)}{t}\right)\right)$$

of

$$\int_J u(x)\frac{\mathrm{d}x}{x} \ - \ \frac{2}{1-3p}\int_{\tilde{J}} u(x)\frac{\mathrm{d}x}{x}$$

for small enough $\eta > 0$, where $C$ is a numerical constant independent of $p$ or the configuration of the $J_k$. So, since

$$\int_{\tilde{J}} u(x)\frac{\mathrm{d}x}{x} \ = \ E\left(\mathrm{d}\left(\frac{\nu(t)}{t}\right),\mathrm{d}\left(\frac{\nu(t)}{t}\right)\right)$$

(see remark to the theorem of article 7), what we have boils down, for small enough $p$ and $\eta > 0$, to

$$\int_J u(x)\frac{\mathrm{d}x}{x} \ \geqslant \ \frac{2}{1-3p}\left(1-\sqrt{\left(\frac{0.66}{0.80}\right)}-A\eta-B\sqrt{\eta}\right)$$
$$\times\, E\left(\mathrm{d}\left(\frac{\nu(t)}{t}\right),\mathrm{d}\left(\frac{\nu(t)}{t}\right)\right)$$

with numerical constants $A$ and $B$ independent of $p$ and the configuration of the $J_k$. Here,

$$\sqrt{\left(\frac{0.66}{0.80}\right)} \ = \ 0.9083^-,$$

so, the coefficient on the right is

$$\geqslant \ \frac{2}{1-3p}(0.0917-A\eta-B\sqrt{\eta}).$$

***Not much at all, but still enough!***

We have finally arrived at the point where a *value* for the parameter $\eta$ must be chosen. This quantity, independent of $p$, was introduced during the *third stage* of the long construction in article 2, where it was necessary to take $0 < \eta < \frac{2}{3}$. Aside from that requirement, we were free to assign any value we liked to it. *Let us now choose, once and for all, a numerical value* $> 0$ *for* $\eta$, *small enough to ensure that all the estimates of articles* 6, 7 *and the present one hold good, and that besides*

$$0.0917-A\eta-B\sqrt{\eta} \ > \ 1/20.$$

*That value is henceforth fixed.* This matter having been settled, the relation finally obtained above reduces to

$$\int_J u(x)\frac{\mathrm{d}x}{x} \geqslant \frac{1}{10(1-3p)}E\left(\mathrm{d}\left(\frac{\nu(t)}{t}\right),\mathrm{d}\left(\frac{\nu(t)}{t}\right)\right).$$

To get a lower bound on the right-hand member, we use again the inequality

$$E\left(\mathrm{d}\left(\frac{\nu(t)}{t}\right),\ \mathrm{d}\left(\frac{\nu(t)}{t}\right)\right)$$
$$\geqslant (0.80 - K\eta)\sum_{k\geqslant 0}\left(\left(\frac{\gamma_k - c_k}{\gamma_k}\right)^2 + \left(\frac{d_k-\delta_k}{d_k}\right)^2\right)$$

(valid for *our* fixed value of $\eta$!), furnished by the theorem of article 6. In article 2, the intervals $J_k$ were constructed so as to make $d_0 - c_0 = |J_0| \geqslant \eta d_0$ (see property (v) in the description near the end of that article), and in the construction of the function $\mu(t)$ we had

$$\frac{d_0-\delta_0}{d_0-c_0} = \frac{1-3p}{2}$$

(property (iii) of the specification near the end of article 3). Therefore

$$\frac{d_0-\delta_0}{d_0} \geqslant \frac{1-3p}{2}\eta,$$

which, substituted into the previous inequality, yields

$$E\left(\mathrm{d}\left(\frac{\nu(t)}{t}\right),\mathrm{d}\left(\frac{\nu(t)}{t}\right)\right) \geqslant (0.80 - K\eta)\left(\frac{1-3p}{2}\right)^2\eta^2.$$

We substitute this into the relation written above, and get

$$\boxed{\int_J u(x)\frac{\mathrm{d}x}{x} \geqslant (1-3p)c}$$

with a certain purely numerical constant $c$. (We see that it is finally just the ratio $|J_0|/d_0$ associated with the *first* of the intervals $J_k$ that enters into these last calculations. If only we had been able to avoid consideration of the other $J_k$ in the above work!) In terms of the function $\mu(t) = (p/(1-3p))\nu(t)$ constructed in article 3, we have, as at the beginning of article 7,

$$\int_\Omega\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right|d\mu(t)\frac{dx}{x^2} \;=\; \frac{p}{1-3p}\int_J u(x)\frac{dx}{x}.$$

By the preceding boxed formula and the work of article 3 we therefore have the

***Theorem.*** *If $p \geqslant 0$ is small enough and if, for our original polynomial $P(x)$, the zero counting function $n(t)$ satisfies*

$$\sup_{t>0}\frac{n(t)}{t} \;>\; \frac{p}{1-3p},$$

*then, for the function $\mu(t)$ constructed in article 3, we have*

$$\int_\Omega\int_0^\infty \log\left|1-\frac{x^2}{t^2}\right|d\mu(t)\frac{dx}{x^2} \;\geqslant\; pc,$$

*$c$ being a numerical constant independent of $P(x)$. Here,*

$$\Omega \;=\; (0,\infty)\sim\bigcup_{k\geqslant 0} J_k,$$

*where the $J_k$ are the intervals constructed in article 2.*

In this way the task described at the very end of article 3 has been carried out, and the main work of the present § completed.

***Remark.*** One reason why the present article's estimations have had to be so delicate is the *smallness* of the lower bound on

$$E\left(d\left(\frac{\nu(t)}{t}\right), d\left(\frac{\nu(t)}{t}\right)\right)$$

obtained in article 6. If we could be sure that this quantity was considerably *larger*, a much simpler procedure could be used to get from $\int_{\tilde{J}}u(x)(dx/x)$ to $\int_J u(x)(dx/x)$; the one of problem 23 (article 8) for instance.

It is possible that $E(d(\nu(t)/t), d(\nu(t)/t))$ *is* quite a bit larger than the lower bound we have found for it. One can write

$$E\left(d\left(\frac{\nu(t)}{t}\right), d\left(\frac{\nu(t)}{t}\right)\right) \;=\; \int_{\tilde{J}}\int_{\tilde{J}}\frac{1}{x^2}\log\left|\frac{1}{1-x^2/t^2}\right|dt\,dx.$$

If the intervals $J_k$ are *very far apart from each other* (so that the cross terms

$$\int_{J_k\cap\tilde{J}}\int_{J_l\cap\tilde{J}}\frac{1}{x^2}\log\left|\frac{1}{1-x^2/t^2}\right|dt\,dx, \qquad k\neq l,$$

are all *very small*), the right-hand integral behaves like a constant multiple of

$$\sum_{k\geqslant 0}\left(\frac{|J_k|}{d_k}\right)^2\log\left(\frac{d_k}{|J_k|}\right).$$

When $\eta>0$ is taken to be *small*, this, on account of the inequality $|J_k|/d_k\leqslant 2\eta$, is much *larger* than the bound furnished by the theorem of article 6, which is essentially a *fixed* constant multiple of

$$\sum_{k\geqslant 0}\left(\frac{|J_k|}{d_k}\right)^2.$$

I have not been able to verify that the first of the above sums can be used to give a lower bound for $E(\mathrm{d}(\nu(t)/t),\ \mathrm{d}(\nu(t)/t))$ when the $J_k$ are *not* far apart. That, however, is perhaps still worth trying.

## 10. Return to polynomials

Let us now combine the theorem from the end of article 3 with the one finally arrived at above. We obtain, without further ado, the

***Theorem.*** *If $p>0$ is sufficiently small and $P(x)$ is any polynomial of the form*

$$\prod_k\left(1-\frac{x^2}{x_k^2}\right)$$

*with the $x_k>0$, the condition*

$$\sup_{t>0}\frac{n(t)}{t} > \frac{p}{1-3p}$$

*for $n(t)$ = number of $x_k$ (counting multiplicities) in $[0,t]$ implies that*

$$\sum_1^\infty\frac{\log^+|P(m)|}{m^2} \geqslant \frac{cp}{5}.$$

*Here, $c>0$ is a numerical constant* independent *of $p$ and of $P(x)$.*

***Corollary.*** *Let $Q(z)$ be any even polynomial (with, in general, complex zeros) such that $Q(0)=1$. There is an absolute constant $k$, independent of $Q$, such that, for all $z$,*

$$\frac{\log|Q(z)|}{|z|} \leqslant k\sum_1^\infty\frac{\log^+|Q(m)|}{m^2},$$

*provided that the sum on the right is less than some number $\gamma>0$, also independent of $Q$.*

***Proof.*** We can write

$$Q(z) = \prod_k \left(1 - \frac{z^2}{\zeta_k^2}\right).$$

Put $x_k = |\zeta_k|$ and then let

$$P(z) = \prod_k \left(1 - \frac{z^2}{x_k^2}\right);$$

we have $|P(x)| \leqslant |Q(x)|$ on $\mathbb{R}$, so

$$\sum_1^\infty \frac{\log^+ |P(m)|}{m^2} \leqslant \sum_1^\infty \frac{\log^+ |Q(m)|}{m^2}.$$

To $P(x)$ we apply the theorem, which clearly implies that

$$\sup_{t>0} \frac{n(t)}{t} \leqslant \frac{10}{c} \sum_1^\infty \frac{\log^+ |P(m)|}{m^2}$$

for $n(t)$, the number of $x_k$ in $[0, t]$, whenever the sum on the right is small enough. For $z \in \mathbb{C}$,

$$\log |Q(z)| \leqslant \sum_k \log\left(1 + \frac{|z|^2}{|\zeta_k|^2}\right) = \int_0^\infty \log\left(1 + \frac{|z|^2}{t^2}\right) \mathrm{d}n(t),$$

and partial integration converts the last expression to

$$\int_0^\infty \frac{n(t)}{t} \frac{2|z|^2}{|z|^2 + t^2} \mathrm{d}t \leqslant \pi |z| \sup_{t>0} \frac{n(t)}{t}.$$

In view of our initial relation, we therefore have

$$\frac{\log |Q(z)|}{|z|} \leqslant \frac{10\pi}{c} \sum_1^\infty \frac{\log^+ |Q(m)|}{m^2}$$

whenever the right-hand sum is small enough. Done.

***Remark 1.*** These results hold for objects more general than polynomials. Instead of $|Q(z)|$, we can consider any *finite product of the form*

$$\prod_k \left|1 - \frac{z^2}{\zeta_k^2}\right|^{\lambda_k}$$

*where the exponents* $\lambda_k$ *are all* $\geqslant$ *some fixed* $\alpha > 0$. Taking $|P(x)|$ as

$$\prod_k \left|1 - \frac{x^2}{x_k^2}\right|^{\lambda_k}$$

with $x_k = |\zeta_k|$, and writing

$$n(t) \;=\; \sum_{x_k \in [0,t]} \lambda_k$$

(so that each 'zero' $x_k$ is counted with 'multiplicity' $\lambda_k$), we easily convince ourselves that the arguments and constructions of articles 1 and 2 go through for *these* functions $|P(x)|$ and $n(t)$ without essential change. What was important there is the property, valid here, that $n(t)$ *increase by at least some fixed amount* $\alpha > 0$ *at each of its jumps*, crucial use having been made of this during the second and third stages of the construction in article 2. The work of articles 3–8 can thereafter be taken over *as is*, and we end with analogues of the above results for our present functions $|P(x)|$ and $|Q(z)|$.

Thus, in the case of *polynomials* $P(z)$, *it is not so much the single-valuedness* of the *analytic function* with *modulus* $|P(z)|$ as the *quantization of the point masses* associated with the *subharmonic function* $\log|P(z)|$ that is essential in the preceding development.

***Remark 2.*** The specific arithmetic character of $\mathbb{Z}$ plays no rôle in the above work. Analogous results hold if we replace the sums

$$\sum_1^\infty \frac{\log^+|P(m)|}{m^2}, \qquad \sum_1^\infty \frac{\log^+|Q(m)|}{m^2},$$

by others of the form

$$\sum_{\lambda\in\Lambda} \frac{\log^+|P(\lambda)|}{\lambda^2}, \qquad \sum_{\lambda\in\Lambda} \frac{\log^+|Q(\lambda)|}{\lambda^2},$$

$\Lambda$ being any fixed set of points in $(0,\infty)$ having at least one element in each interval of length $\geqslant h$ with $h > 0$ and fixed. This generalization requires some rather self-evident modification of the work in article 1. The reasoning in articles 2–8 then applies with hardly any change.

## Problem 24

Consider entire functions $F(z)$ *of very small exponential type* $\alpha$ having the special form

$$F(z) \;=\; \prod_k \left(1 - \frac{z^2}{x_k^2}\right)$$

where the $x_k$ are $> 0$, and such that

$$\int_{-\infty}^{\infty} \frac{\log^+|F(x)|}{1+x^2}\,dx \;<\; \infty.$$

*Investigate the possibility of adapting the development of this § to such functions $F(z)$* (instead of *polynomials $P(z)$*).

Here, if the small numbers $2\eta$ and $p$ are both several times larger than $\alpha$, the constructions of article 2 can be made to work (by problem 1(a), Chapter I!), yielding an infinite number of intervals $J_k$. The statement of the *second lemma* from article 4 has to be modified.

*I have not worked through this problem.*

We now come to the *principal result* of this whole §, an extension of the above corollary to *general polynomials.* To establish it, we need a simple

***Lemma.*** *Let $\alpha > 0$ be given. There is a number $M_\alpha$ depending on $\alpha$ such that, for* any *real valued function $f$ on $\mathbb{Z}$ satisfying*

$$\sum_{-\infty}^{\infty} \frac{\log^+|f(n)|}{1+n^2} \leqslant \alpha,$$

*we have*

$$\sum_1^\infty \frac{1}{n^2} \log\left(1 + \frac{n^2(f(n)+f(-n))^2}{M_\alpha^2}\right) \leqslant 6\alpha$$

*and*

$$\sum_1^\infty \frac{1}{n^2} \log\left(1 + \frac{(f(n)-f(-n))^2}{M_\alpha^2}\right) \leqslant 6\alpha$$

***Proof.*** When $q \geqslant 0$, the function $\log(1+q) - \log^+ q$ assumes its maximum for $q = 1$. Hence

$$\log(1+q) \leqslant \log 2 + \log^+ q, \qquad q \geqslant 0.$$

Also,

$$\log^+(qq') \leqslant \log^+ q + \log^+ q', \qquad q,\ q' \geqslant 0.$$

Therefore, if $M \geqslant 1$,

$$\begin{aligned}
&\log\left(1 + \frac{n^2(f(n)+f(-n))^2}{M^2}\right) \\
&\leqslant \log 2 + 2\log^+ n + 2\log^+(|f(n)| + |f(-n)|) \\
&\leqslant 3\log 2 + 2\log n + 2\max(\log^+|f(n)|,\ \log^+|f(-n)|)
\end{aligned}$$

for $n \geqslant 1$.

Given $\alpha > 0$, choose (and then *fix*) an $N$ sufficiently large to make

$$\sum_{n>N} \frac{3\log 2 + 2\log n}{n^2} < \alpha$$

Then, if $f$ is any real valued function with

$$\sum_{-\infty}^{\infty} \frac{\log^+|f(n)|}{1+n^2} \leqslant \alpha,$$

we will surely have

$$\sum_{n>N} \frac{1}{n^2} \log\left(1 + \frac{n^2(f(n)+f(-n))^2}{M^2}\right) < 5\alpha$$

by the previous relation, as long as $M \geqslant 1$. Similarly,

$$\sum_{n>N} \frac{1}{n^2} \log\left(1 + \frac{(f(n)-f(-n))^2}{M^2}\right) < 5\alpha$$

for such $f$, if $M \geqslant 1$.

Our condition on $f$ certainly implies that

$$\log^+|f(n)| \leqslant \alpha(1+n^2),$$

so

$$|f(n)| + |f(-n)| \leqslant 2e^{(1+N^2)\alpha}$$

for $1 \leqslant n \leqslant N$. Choosing $M_\alpha \geqslant 1$ sufficiently large so as to have

$$\sum_{1}^{N} \frac{1}{n^2} \log\left(1 + \frac{4n^2 e^{2\alpha(1+N^2)}}{M_\alpha^2}\right) < \alpha$$

will thus ensure that

$$\sum_{1}^{N} \frac{1}{n^2} \log\left(1 + \frac{n^2(f(n)+f(-n))^2}{M_\alpha^2}\right) < \alpha$$

and

$$\sum_{1}^{N} \frac{1}{n^2} \log\left(1 + \frac{(f(n)-f(-n))^2}{M_\alpha^2}\right) < \alpha.$$

Adding each of these relations to the corresponding one obtained above, we have the lemma.

***Theorem.*** *There are numerical constants $\alpha_0 > 0$ and $k$ such that, for any polynomial $p(z)$ with*

$$\sum_{-\infty}^{\infty} \frac{\log^+|p(n)|}{1+n^2} = \alpha \leqslant \alpha_0,$$

*we have, for all $z$,*

$$|p(z)| \leqslant K_\alpha e^{3k\alpha|z|},$$

*where $K_\alpha$ is a constant depending only on $\alpha$ (and not on $p$).*

***Proof.*** Given a polynomial $p$, we may as well assume to begin with that $p(x)$ is *real for real* $x$ – otherwise we just work separately with the polynomials $(p(z)+\overline{p(\bar{z})})/2$ and $(p(z)-\overline{p(\bar{z})})/2i$ which both have that property.

Considering, then, $p$ to be real on $\mathbb{R}$ and assuming that it satisfies the condition in the hypothesis, we take the number $M_\alpha$ furnished by the lemma and form each of the *polynomials*

$$Q_1(z) \;=\; 1+\frac{z^2(p(z)+p(-z))^2}{M_\alpha^2},$$

$$Q_2(z) \;=\; 1+\frac{(p(z)-p(-z))^2}{M_\alpha^2}.$$

The polynomials $Q_1$ and $Q_2$ are both *even*, and

$$Q_1(0) \;=\; Q_2(0) \;=\; 1.$$

By the lemma,

$$\sum_1^\infty \frac{1}{n^2}\log^+|Q_1(n)| \;\leqslant\; 6\alpha$$

and

$$\sum_1^\infty \frac{1}{n^2}\log^+|Q_2(n)| \;\leqslant\; 6\alpha,$$

since (here) $Q_1(x)\geqslant 1$ and $Q_2(x)\geqslant 1$ on $\mathbb{R}$.

If $\alpha>0$ is *small enough*, these inequalities imply *by the above corollary* that $(\log|Q_1(z)|)/|z|$ and $(\log|Q_2(z)|)/|z|$ are both $\leqslant 6k\alpha$, $k$ being a certain numerical constant. Therefore

$$\left|1+\frac{z^2(p(z)+p(-z))^2}{M_\alpha^2}\right| \;\leqslant\; e^{6k\alpha|z|},$$

and

$$\left|1+\frac{(p(z)-p(-z))^2}{M_\alpha^2}\right| \;\leqslant\; e^{6k\alpha|z|}.$$

From these relations we get

$$|z^2(p(z)+p(-z))^2| \;\leqslant\; M_\alpha^2(1+e^{6k\alpha|z|}) \;\leqslant\; 2M_\alpha^2 e^{6k\alpha|z|},$$

whence

$$|p(z)+p(-z)| \;\leqslant\; \sqrt{2}M_\alpha e^{3k\alpha|z|} \qquad \text{for } |z|\geqslant 1,$$