Entire functions of exponential type and the Riemann zeta-function

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Preface

These are lectures notes for a mini-course at the VI ENAMA - Encontro Nacional de Análise Matemática e Aplicações - held in Aracaju, Brazil, in November 2012. I would like to thank the organizing and scientific committees of the VI ENAMA, and the colleagues from the Federal University of Sergipe for their hospitality. The purpose of these notes is to present briefly some recent advances in problems in approximation theory and its applications to the theory of the Riemann zeta-function.

The first chapter covers foundational material in harmonic analysis related to the theory of functions of exponential type. We start with Poisson summation formula and quickly move to two celebrated results: the Paley-Wiener theorem and the Plancherel-Polya theorem. These are then applied to obtain useful interpolation formulas and Bernstein's inequality.

The second chapter introduces the reader to a classical problem in the interface of approximation theory and harmonic analysis, the so called Beurling-Selberg extremal problem. In this setting the goal is to approximate (minimizing the $L^1(\mathbb{R})$-norm) a given real function by an entire function of prescribed exponential type. We have no ambition to cover in its full the vast material related to this topic. Our goal is to present some of the recent advances in this theory, in the form of a general method (which we call the Gaussian subordination method) to generate the solution of this problem for a wide class of even, odd and truncated real functions.

The third chapter describes three applications of these extremal functions to the theory of the Riemann zeta-function. Specifically, our goal is to provide the best (up to date) bounds, under the assumption of the Riemann hypothesis, for three objects related to $\zeta(s)$, namely, the size of $\zeta(s)$ in the critical line, the argument function $S(t)$, and its antiderivative $S_1(t)$.

I would like to thank in particular the outstanding group of mathematicians with whom I have collaborated in these research projects: Vorrapan Chandee (Univ. of Montreal), Friedrich Littmann (North Dakota State Univ.), Micah Milinovich (Univ. of Mississipi) and Jeffrey D. Vaaler (The Univ. of Texas at Austin). Chapters 2 and 3 basically summarize our recent joint projects.

Rio de Janeiro, October 2012,

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Chapter 1

Harmonic analysis tools

1.1 The Poisson summation formula

If $f \in L^1(\mathbb{R})$ we define its Fourier transform by

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \xi} f(x)\,\mathrm{d}x.$$

In an analogous manner, if we identify periodic functions (of period 1) with functions defined on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and if $g \in L^1(\mathbb{T})$, we define its Fourier coefficients by

$$\widehat{g}(k) = \int_{\mathbb{T}} e^{-2\pi i x k} g(x)\,\mathrm{d}x,$$

where $k \in \mathbb{Z}$. Given a measurable function f on $\mathbb{R}$, we consider its periodization

$$P_f(x) = \sum_{m \in \mathbb{Z}} f(x+m). \tag{1.1}$$

Naturally, if we do not impose any decay on f, one can not infer any sort of convergence for the sum (1.1). However, if we assume $f \in L^1(\mathbb{R})$, we will have P_f with period 1 and

$$\begin{aligned}
\|P_f\|_{L^1(\mathbb{T})} &= \int_0^1 \left| \sum_{m \in \mathbb{Z}} f(x+m) \right| \mathrm{d}x \le \int_0^1 \sum_{m \in \mathbb{Z}} |f(x+m)|\ \mathrm{d}x \\
&= \sum_{m \in \mathbb{Z}} \int_0^1 |f(x+m)|\ \mathrm{d}x = \sum_{m \in \mathbb{Z}} \int_m^{m+1} |f(y)|\ \mathrm{d}y = \|f\|_{L^1(\mathbb{R})}.
\end{aligned}$$

Therefore $P_f \in L^1(\mathbb{T})$ and we can calculate its Fourier coefficients. An application of Fubini's theorem gives us

$$\begin{aligned}\widehat{P_f}(k) &= \int_0^1 e^{-2\pi i x k} \left(\sum_{m\in\mathbb{Z}} f(x+m) \right) \mathrm{d}x = \sum_{m\in\mathbb{Z}} \int_0^1 e^{-2\pi i x k} \, f(x+m)\, \mathrm{d}x \\ &= \sum_{m\in\mathbb{Z}} \int_m^{m+1} e^{-2\pi i y k}\, f(y)\, \mathrm{d}y = \int_{\mathbb{R}} e^{-2\pi i y k}\, f(y)\, \mathrm{d}y = \widehat{f}(k).\end{aligned} \tag{1.2}$$

If, for instance, we have the decay estimates

$$|f(x)| \leq \frac{C}{(1+|x|)^{1+\delta}} \quad \text{and} \quad |\widehat{f}(\xi)| \leq \frac{C}{(1+|\xi|)^{1+\delta}}, \tag{1.3}$$

for some $\delta > 0$ and some constant $C > 0$, the series (1.1) defining P_f will be absolutely convergent, thus defining a continuous function. Moreover, from (1.2) and (1.3), the Fourier inversion for the continuous periodic function P_f will hold pointwise, i.e.

$$P_f(x) = \sum_{k\in\mathbb{Z}} \widehat{P_f}(k)\, e^{2\pi i x k}\,,$$

and this is equivalent to

$$\sum_{m\in\mathbb{Z}} f(x+m) = \sum_{k\in\mathbb{Z}} \widehat{f}(k)\, e^{2\pi i x k} \tag{1.4}$$

for all $x \in \mathbb{R}$. Expression (1.4) is known as the *Poisson summation formula.*

Our objective now is to weaken the decay conditions (1.3) in a way that we can still obtain (1.4) pointwise. Recall that a function $f : \mathbb{R} \to \mathbb{C}$ of bounded variation is said to be normalized if

$$f(x) = \frac{1}{2}\big\{ f(x^+) + f(x^-) \big\}$$

for all $x \in \mathbb{R}$, where $f(x^{\pm})$ are the lateral limits at the point x. From the basic theory of Fourier series (see for instance [13, Theorem 8.43]), we know that if $P : \mathbb{T} \to \mathbb{C}$ is a normalized function of bounded variation then the Fourier inversion holds pointwise , i.e.

$$P(x) = \lim_{N\to\infty} \sum_{k=-N}^{N} \widehat{P}(k)\, e^{2\pi i x k} \tag{1.5}$$

for all $x \in \mathbb{T}$.

Theorem 1.1 (Poisson summation formula)**.** *Let $f \in L^1(\mathbb{R})$ be a normalized function of bounded variation. Then we have*

$$\lim_{N\to\infty} \sum_{m=-N}^{N} f(x+m) = \lim_{N\to\infty} \sum_{k=-N}^{N} \widehat{f}(k)\, e^{2\pi i k x}$$

for all $x \in \mathbb{R}$.

Proof. As before, let $P_f(x) = \sum_{m\in\mathbb{Z}} f(x+m)$. We know that $P_f \in L^1(\mathbb{T})$, since $f \in L^1(\mathbb{R})$. Let $x_0 \in [-1/2, 1/2)$ be a point where the sum is absolutely convergent (in particular $|P_f(x_0)| < \infty$). For any other point $x \in [-1/2, 1/2)$, say $x > x_0$, the difference $|f(x+m) - f(x_0+m)|$ is less than or equal to the variation of f on the interval $[x_0+m, x+m]$, and therefore the sum of these increments is less than or equal to the total variation of f (let us call it Vf). Therefore

$$\begin{aligned}\sum_{m\in\mathbb{Z}} |f(x+m)| &\leq \sum_{m\in\mathbb{Z}} |f(x_0+m)| + \sum_{m\in\mathbb{Z}} |f(x+m) - f(x_0+m)| \\ &\leq \sum_{m\in\mathbb{Z}} |f(x_0+m)| + Vf < \infty\,,\end{aligned}$$

from which we conclude that the sum is absolutely convergent for each $x \in \mathbb{T}$. Now observe that for each partition $-1/2 = a_0 < a_1 < \ldots < a_k = 1/2$, we have

$$\begin{aligned}\sum_{i=1}^{k} |P_f(a_i) - P_f(a_{i-1})| &= \sum_{i=1}^{k} \left|\sum_{m\in\mathbb{Z}} \big\{f(a_i+m) - f(a_{i-1}+m)\big\}\right| \\ &\leq \sum_{i=1}^{k} \sum_{m\in\mathbb{Z}} \big|f(a_i+m) - f(a_{i-1}+m)\big| \leq Vf,\end{aligned}$$

and therefore P_f has bounded variation. Finally, since f is normalized, we have for each point $x \in [-1/2, 1/2)$

$$\begin{aligned}\lim_{\epsilon\to 0} \frac{1}{2}\big\{P_f(x+\epsilon) + P_f(x-\epsilon)\big\} &= \lim_{\epsilon\to 0} \frac{1}{2} \sum_{m\in\mathbb{Z}} \big\{f(x+\epsilon+m) + f(x-\epsilon+m)\big\} \\ &= \sum_{m\in\mathbb{Z}} \frac{1}{2} \lim_{\epsilon\to 0} \big\{f(x+\epsilon+m) + f(x-\epsilon+m)\big\} \\ &= \sum_{m\in\mathbb{Z}} f(x+m) = P_f(x),\end{aligned}$$

where we used dominated convergence to move the limit inside, since for $\epsilon < 1/2$ we have

$$|f(x+\epsilon+m)| \leq |f(x+m)| + Vf_{[x+m, x+m+1/2]}$$

and

$$|f(x-\epsilon+m)| \leq |f(x+m)| + Vf_{[x+m-1/2, x+m]}.$$

Therefore P_f is normalized and the result now follows from (1.5). $\square$

1.2 The Paley-Wiener theorem

In this section we investigate the relation between the growth of a function and the size of the support of its Fourier transform. For $\delta > 0$, we shall say that an entire function $F : \mathbb{C} \to \mathbb{C}$ has *exponential type* $2\pi\delta$, if for each $\epsilon > 0$ there exists a constant C_ϵ such that

$$|F(z)| \leq C_\epsilon \, e^{(2\pi\delta+\epsilon)|z|},$$

for all $z \in \mathbb{C}$. In other words, for each $\epsilon > 0$, we have

$$|F(z)| = O\big(e^{(2\pi\delta+\epsilon)|z|}\big).$$

The class of entire functions with exponential type at most $2\pi\delta$ will be called $E^{2\pi\delta}$. We shall say that an entire function $F \in L^p(\mathbb{R})$ if the restriction of F to the real axis (call it $F|_{\mathbb{R}}$) belongs to L^p, i.e. if

$$\int_{-\infty}^{\infty} |F(x)|^p \, \mathrm{d}x < \infty,$$

when $1 \leq p < \infty$, and $\sup_{x\in\mathbb{R}} |F(x)| < \infty$, if $p = \infty$.

Theorem 1.2 (Paley-Wiener)**.** *For an entire function $F \in L^2(\mathbb{R})$ the two conditions below are equivalent:*

(i) F has exponential type $2\pi\delta$.

(ii) The Fourier transform of $F|_{\mathbb{R}}$ is supported on $[-\delta, \delta]$.

Proof. Here we shall essentially follow [44, Chapter XVI]. A different proof is presented in [40, Chapter III].

Step 1: (ii) $\Rightarrow$ (i). If $f = \widehat{F|_{\mathbb{R}}}$ is supported on $[-\delta, \delta]$, then $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and

$$F(x) = \int_{-\delta}^{\delta} f(t)\, e^{2\pi i t x} \, \mathrm{d}t. \tag{1.6}$$

The right-hand side of (1.6) extends to an entire function and thus

$$F(z) = \int_{-\delta}^{\delta} f(t)\, e^{2\pi i t z} \, \mathrm{d}t. \tag{1.7}$$

If $z = x + iy$, from (1.7) it is clear that

$$|F(z)| \leq e^{2\pi\delta|y|} \int_{-\delta}^{\delta} |f(t)| \, \mathrm{d}t \, ,$$

which shows that F has exponential type $2\pi\delta$.

Step 2: (i) $\Rightarrow$ (ii). This is the deep part of the theorem. We must show that if $F \in E^{2\pi\delta} \cap L^2(\mathbb{R})$ then

$$f(t) = \int_{-\infty}^{\infty} F(\xi)\, e^{-2\pi i t \xi} \, \mathrm{d}\xi \tag{1.8}$$

is zero for almost all t outside $[-\delta, \delta]$, where the integral in (1.8) is meant as the L^2-limit of the truncated integrals $\int_{-R}^{R}$ as $R \to \infty$. Consider the following function

$$g(z;\theta) = \int_{\gamma_\theta} F(w)\, e^{-2\pi w z}\, \mathrm{d}w, \tag{1.9}$$

where the integral is taken over the ray $\gamma_\theta = \{\arg(w) = -\theta\}$, i.e. $w = \rho e^{-i\theta}$, with ρ varying from 0 to ∞.

Let $z = x + iy$. We first claim that the integral in (1.9) is absolutely and uniformly convergent on each closed half-plane contained in the open half-plane

$$x \cos\theta + y \sin\theta > \delta.$$

Note that this is the open half-plane (that does not contain the origin) delimited by the tangent to the circle $|w| = \delta$ at the point $\delta e^{i\theta}$. Let us call this open half-plane by H_θ. In fact, if z belongs to a closed half-plane Γ entirely contained in H_θ, we put $w = \rho\, e^{-i\theta}$ on the integrand of (1.9) to see that

$$\begin{aligned}\left|F(w)\, e^{-2\pi w z}\right| &= O\left\{e^{(2\pi\delta+\epsilon)\rho - \Re\left(2\pi(x+iy)\rho\, e^{-i\theta}\right)}\right\} \\ &= O\left\{e^{(2\pi\delta+\epsilon)\rho - 2\pi(x\cos\theta + y\sin\theta)\rho}\right\},\end{aligned}$$

for all $\epsilon > 0$. We can then choose ϵ sufficiently small such that the last expression decays exponentially fast with ρ, uniformly on Γ, thus proving our claim. In particular, by Morera's theorem, $z \mapsto g(z;\theta)$ is analytic in H_θ.

Secondly, we observe that if $0 < |\theta' - \theta''| < \pi$, the functions $g(z;\theta')$ and $g(z;\theta'')$ coincide on the intersection of the half-planes $H_{\theta'}$ and $H_{\theta''}$. In fact, let us suppose without loss of generality that $\theta' < \theta''$, and let $z \in H_{\theta'} \cap H_{\theta''}$. It is easy to see geometrically that $z \in H_\theta$ for any $\theta' \le \theta \le \theta''$, and that the integrand $G(w)$ of (1.9), considered as a function of w alone, decays exponentially to 0 as $|w| \to \infty$ in the angle $[-\theta'', -\theta']$ (since this decay rate depends only on the distance from z to $H_{\theta'}$ and $H_{\theta''}$). By Cauchy's theorem we can change the ray of integration and thus $g(z;\theta') = g(z;\theta'')$.

Let $g_0(z) = g(z;0)$ and $g_1(z) = g(z;\pi)$. We now observe that g_0 and g_1 are analytic in the half-planes $x > 0$ and $x < 0$, respectively, and are analytic continuations of each other across the segments $y > \delta$ and $y < -\delta$ of the imaginary axis. To see this, let $x \ge \epsilon > 0$ and note that

$$\begin{aligned}|g_0(z)| &= \left|\int_0^\infty F(w)\, e^{-2\pi w(x+iy)}\, \mathrm{d}w\right| \\ &\le \left(\int_0^\infty |F(w)|^2\, \mathrm{d}w\right)^{1/2} \left(\int_0^\infty e^{-4\pi\epsilon w}\, \mathrm{d}w\right)^{1/2}.\end{aligned}$$

Therefore, by Morera's theorem, g_0 is analytic for $x > 0$. In an analogous manner, one shows that g_1 is analytic for $x < 0$. Consider now $g_2 = g(z, \pi/2)$.

We know that g_2 is analytic in $H_{\frac{\pi}{2}}$ and from the previous paragraph we also know it agrees with g_0 in $H_0 \cap H_{\frac{\pi}{2}}$. Since g_0 is analytic in the whole half-plane $x > 0$, it follows that g_2 is the analytic continuation of g_0 across the segment $y > \delta$ of the imaginary axis. Similarly, g_2 is the analytic continuation of g_1 across the same segment. A similar argument holds for $y < -\delta$.

We are now able to conclude the proof. Recall that

$$g_0(x+iy) = \int_0^\infty F(\xi)\, e^{-2\pi\xi(x+yi)}\, \mathrm{d}\xi.$$

Since

$$\int_0^\infty \left|F(\xi)\right|^2 \left|1 - e^{-2\pi\xi x}\right|^2 \mathrm{d}\xi \to 0$$

as $x \to 0^+$, we see by Plancherel's theorem that $g_0(x+iy)$ tends to

$$\int_0^\infty F(\xi)\, e^{-2\pi\xi yi}\, \mathrm{d}\xi$$

in L^2 as $x \to 0^+$. In an analogous manner

$$g_1(x+iy) = -\int_{-\infty}^0 F(\xi)\, e^{-2\pi\xi(x+yi)}\, \mathrm{d}\xi$$

tends to

$$-\int_{-\infty}^0 F(\xi)\, e^{-2\pi\xi yi}\, \mathrm{d}\xi$$

in L^2 as $x \to 0^-$. We conclude that $g_0(x+iy) - g_1(-x+iy)$ tends to

$$f(y) = \int_{-\infty}^\infty F(\xi)\, e^{-2\pi\xi yi}\, \mathrm{d}\xi$$

as $x \to 0^+$. However we know that $g_0(x+iy) - g_1(-x+iy) \to 0$ pointwise when $x \to 0^+$, for all $|y| > \delta$. Hence $f \equiv 0$ on $[-\delta, \delta]^c$ and the proof is complete.

□

1.3 The Plancherel-Polya theorem

1.3.1 Statement and proof

The objective of this section is to prove the following result of Plancherel and Polya [33].

Theorem 1.3 (Plancherel-Polya)**.** *Let F be an entire function of exponential type $2\pi\delta$ such that its restriction to the real axis belongs to $L^p(\mathbb{R})$, for some p*

with $0 < p < \infty$. Given $\Delta > 0$ let $\{\lambda_m\}_{m\in\mathbb{Z}}$ be a sequence of real numbers such that $|\lambda_m - \lambda_n| \geq \Delta$, for all $m, n \in \mathbb{Z}$. Then

$$\sum_{m\in\mathbb{Z}} |F(\lambda_m)|^p \leq C \int_{-\infty}^{\infty} |F(x)|^p \,\mathrm{d}x,$$

where $C = C(p, \delta, \Delta)$.

Proof. We start by noticing that, since $z \mapsto |F(z)|^p$ is a subharmonic function, the mean value property gives us

$$\begin{aligned} |F(\lambda_m)|^p &\leq \frac{1}{m(B(\Delta/2))} \int_{(x,y)\in B(\Delta/2)} |F(\lambda_m + x + iy)|^p \,\mathrm{d}x\,\mathrm{d}y \\ &\leq \frac{1}{m(B(\Delta/2))} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} |F(\lambda_m + x + iy)|^p \,\mathrm{d}x\,\mathrm{d}y, \end{aligned}$$

where $B(r)$ is the ball of radius r centered at the origin, and $m(B(r))$ is its volume. If we sum the last expression over $m \in \mathbb{Z}$, and use the fact that the λ_m's are at a distance Δ apart of each other, we find

$$\sum_{m\in\mathbb{Z}} |F(\lambda_m)|^p \leq \frac{1}{m(B(\Delta/2))} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \int_{-\infty}^{\infty} |F(x + iy)|^p \,\mathrm{d}x\,\mathrm{d}y. \tag{1.10}$$

We will show that, for any $y \in \mathbb{R}$, we have

$$\int_{-\infty}^{\infty} |F(x + iy)|^p \,\mathrm{d}x \leq e^{2\pi\delta p|y|} \int_{-\infty}^{\infty} |F(x)|^p \,\mathrm{d}x. \tag{1.11}$$

Clearly, the combination of (1.10) and (1.11) finish the proof of the theorem. The hard part is indeed the proof of (1.11) that shall make use of three lemmas as follows. $\square$

1.3.2 Auxiliary lemmas

We keep denoting $z = x + iy$. Let $G(z)$ be an analytic function on the half-plane $y > 0$, that is continuous on $y \geq 0$. Let a be a positive number and define the function $\Psi(z)$ by

$$\Psi(z) = \int_{-a}^{a} |G(z + s)|^p \,\mathrm{d}s,$$

where the path of integration is the segment $[-a, a]$. Observe that $\Psi(z)$ is defined and continuous for $y \geq 0$.

Lemma 1.4. *Let $\mathcal{D}$ be a bounded and closed domain contained in the half-plane $y \geq 0$. Then the maximum of $\Psi(z)$ is attained in the boundary of $\mathcal{D}$.*

Proof. From the fact that $|G(z)|^p$ is subharmonic we have

$$|G(\zeta)|^p \leq \frac{1}{2\pi} \int_0^{2\pi} \left|G\big(\zeta + re^{i\varphi}\big)\right|^p \mathrm{d}\varphi,$$

where the circle $|z - \zeta| \leq r$ is contained in the half-plane $y \geq 0$. Therefore, for $z = x + iy$ and $r \leq y$ we have

$$\begin{aligned}
\Psi(z) &= \int_{-a}^{a} |G(z+s)|^p \,\mathrm{d}s \\
&\leq \int_{-a}^{a} \left(\frac{1}{2\pi} \int_0^{2\pi} \left|G\big(z + s + re^{i\varphi}\big)\right|^p \mathrm{d}\varphi \right) \mathrm{d}s \\
&= \frac{1}{2\pi} \int_0^{2\pi} \Psi\big(z + re^{i\varphi}\big)\,\mathrm{d}\varphi.
\end{aligned}$$

Thus Ψ is subharmonic on the half-plane $y \geq 0$ and the result now follows from the maximum principle. $\square$

Lemma 1.5. *Let*

$$M = \sup_{x \in \mathbb{R}} \Psi(x)$$

and

$$N = \sup_{y \geq 0} \Psi(iy).$$

Suppose that M and N are finite and that $G(z)$ has exponential type on the half-plane $y \geq 0$. Then

$$\Psi(z) \leq \max\{M, N\}$$

for all z in the half-plane $y \geq 0$.

Proof. The hypothesis that $G(z)$ has exponential type on the half-plane $y \geq 0$ guarantees the existence of positive numbers B and b such that

$$|G(z)| \leq Be^{b|z|} \tag{1.12}$$

if $z = x + iy$, with $y \geq 0$. Let $\epsilon > 0$ and define

$$\begin{aligned}
G_\epsilon(z) &= G(z)\, e^{-\epsilon\left[(z+a)\, e^{-\pi i/4}\right]^{3/2}}; \\
\Psi_\epsilon(z) &= \int_{-a}^{a} \left|G_\epsilon(z+s)\right|^p \mathrm{d}s.
\end{aligned} \tag{1.13}$$

Above we choose the branch of $\left[(z+a)\, e^{-\pi i/4}\right]^{3/2}$ with positive real part when $x > -a$ and $y \geq 0$. From (1.12) and (1.13) we have

$$\left|G_\epsilon(z)\right| \leq B\, e^{b|z| - \epsilon\gamma|z+a|^{3/2}} \tag{1.14}$$

and

$$\left|G_\epsilon(z)\right| \le \left|G(z)\right|$$

for $x > -a$ and $y \ge 0$, where $\gamma = \cos\frac{3\pi}{8}$. Therefore we have

$$\left|\Psi_\epsilon(z)\right| \le \left|\Psi(z)\right|$$

when $x \ge 0$ and $y \ge 0$, and in particular

$$\left|\Psi_\epsilon(x)\right| \le M \tag{1.15}$$

for $x \ge 0$, and

$$\left|\Psi_\epsilon(iy)\right| \le N \tag{1.16}$$

for $y \ge 0$. Let z_0 be a point on the quadrant $x > 0$, $y > 0$. We now apply Lemma 1.4 to Ψ_ϵ and the domain $\mathcal{D} = \{z = x + iy;\, x \ge 0,\, y \ge 0,\, |z| \le R\}$. Assume that the radius R is sufficiently large so that $z_0 \in \mathcal{D}$ and that the maximum over the curved part of the boundary is at most $\max\{M, N\}$ (this can be done by (1.14)). By Lemma 1.4, (1.15) and (1.16) we arrive at

$$\left|\Psi_\epsilon(z_0)\right| \le \max\{M, N\}.$$

This reasoning holds for any $\epsilon > 0$. By considering $\epsilon \to 0^+$ we find that

$$\left|\Psi(z)\right| \le \max\{M, N\}$$

for any z in the first quadrant $x \ge 0$ and $y \ge 0$. The proof for the quadrant $x \le 0$ and $y \ge 0$ is analogous. □

Lemma 1.6. *In addition to the hypotheses of Lemma 1.5 assume that*

$$\lim_{y\to\infty} G(x + iy) = 0 \tag{1.17}$$

uniformly on the strip $-a \le x \le a$. Then $N \le M$ and therefore, for $y \ge 0$, we have

$$\int_{-a}^{a} |G(z+s)|^p \, \mathrm{d}s = \Psi(z) \le M.$$

Proof. Assume that $G(z)$ is not identically zero (otherwise the result is obviously true). Due to (1.17) we have $\Psi(iy) \to 0$ as $y \to \infty$, and thus the supremum N over the imaginary axis must be attained at a certain point iy_0. If $y_0 = 0$ we have

$$N = \Psi(iy_0) = \Psi(0) \le M.$$

If $y_0 > 0$ and $N > M$, we would have, by Lemma 1.5, the subharmonic function Ψ attaining its maximum over the half-plane $y \ge 0$ at an interior point, and thus Ψ would have to be constant, giving us a contradiction. Therefore we must have $N \le M$. □

Proof of the key inequality (1.11). After going through these three auxiliary lemmas, we are now in position to prove inequality (1.11) and complete the proof of the Plancherel-Polya theorem. It suffices to prove (1.11) in the case $y > 0$. Let $\epsilon > 0$ be given. We will apply Lemmas 1.5 and 1.6 to the function

$$G(z) = F(z)\, e^{i(2\pi\delta+\epsilon)z}.$$

Observe that $G(z)$ satisfies all the hypotheses of Lemmas 1.5 and 1.6. The number M defined in Lemma 1.5 is finite since

$$M \leq \int_{-\infty}^{\infty} |G(x)|^p \,\mathrm{d}x = \int_{-\infty}^{\infty} |F(x)|^p \,\mathrm{d}x.$$

Applying the conclusion of Lemma 1.6 to $z = iy$ we have

$$\begin{aligned} e^{-p(2\pi\delta+\epsilon)y} \int_{-a}^{a} |F(s+iy)|^p \,\mathrm{d}s &= \int_{-a}^{a} |G(s+iy)|^p \,\mathrm{d}s \\ &\leq \int_{-\infty}^{\infty} |F(x)|^p \,\mathrm{d}x. \end{aligned}$$

Since this result holds for any $a > 0$ and any $\epsilon > 0$ we obtain the desired result by making $a \to \infty$ and $\epsilon \to 0^+$. The proof is complete. □

1.4 Interpolation formulas

1.4.1 Basic results

We shall see here that entire functions of a prescribed exponential type (with an L^p condition on the real axis) are completely determined by their values on a set of well-spaced points. Recall that an entire function $F \in L^p(\mathbb{R})$, $1 \leq p < \infty$, if

$$\int_{-\infty}^{\infty} |F(x)|^p \,\mathrm{d}x < \infty.$$

In case $F \in L^2(\mathbb{R})$, its Fourier transform is defined as

$$\widehat{F}(t) = \lim_{R\to\infty} \int_{-R}^{R} F(x)\, e^{-2\pi i x t} \,\mathrm{d}t,$$

where the limit is taken in the L^2-sense. By the Paley-Wiener theorem, we know that F has exponential type $2\pi\delta$ and belongs to $L^2(\mathbb{R})$ if and only if $\widehat{F}$ is supported on $[-\delta, \delta]$, and so

$$F(z) = \int_{-\delta}^{\delta} \widehat{F}(t)\, e^{2\pi i t z} \,\mathrm{d}t \tag{1.18}$$

for all $z \in \mathbb{C}$.

Theorem 1.7. *Let F be an entire function of exponential type π such that $F \in L^p(\mathbb{R})$ for some p with $1 \leq p < \infty$. Then*

$$F(z) = \frac{\sin \pi z}{\pi} \sum_{n \in \mathbb{Z}} (-1)^n \frac{F(n)}{(z-n)}, \tag{1.19}$$

where the expression on the right-hand side of (1.19) *converges uniformly on compact subsets of $\mathbb{C}$.*

If $p = 2$, the Fourier transform $\widehat{F}(t)$ occurring in (1.18) *has the following Fourier expansion (as functions in $L^2[-\frac{1}{2}, \frac{1}{2}]$)*

$$\widehat{F}(t) = \sum_{m \in \mathbb{Z}} F(m) e^{-2\pi i m t}. \tag{1.20}$$

If $p = 1$, the right-hand side of (1.20) *is absolutely convergent, and as $\widehat{F}$ is continuous, equality* (1.20) *holds pointwise. In particular*

$$\widehat{F}(0) = \sum_{m \in \mathbb{Z}} F(m)$$

and

$$0 = \widehat{F}\left(\tfrac{1}{2}\right) = \sum_{m \in \mathbb{Z}} (-1)^m F(m). \tag{1.21}$$

Proof. Let us start with the statements about the Fourier transforms. If $p = 2$, it is clear from (1.18) that $\{F(m)\}_{m \in \mathbb{Z}}$ are the Fourier coefficients of $\widehat{F}$ (thought as a periodic function of period 1). The Fourier expansion (1.20) is a consequence of this. If $p = 1$, the function $\widehat{F}$ will be a continuous function supported on $[-\frac{1}{2}, \frac{1}{2}]$ and again, by (1.18), its Fourier coefficients will be $\{F(m)\}_{m \in \mathbb{Z}}$. By the theorem of Plancherel and Polya, the sequence $\{|F(m)|\}_{m \in \mathbb{Z}}$ is summable and we conclude that (1.20) holds pointwise.

We now prove (1.19) when $p \leq 2$. In this case, there is no loss of generality in assuming $p = 2$ since, if $p < 2$, we have F bounded on the real axis (by the theorem of Plancherel and Polya) and thus $F \in L^2(\mathbb{R})$. Define

$$v_N(t) = \sum_{m=-N}^{N} F(m)\, e^{-2\pi i m t}.$$

We already know that $v_N \to \widehat{F}$ in $L^2[-\frac{1}{2}, \frac{1}{2}]$ when $N \to \infty$ (therefore we have convergence in $L^1[-\frac{1}{2}, \frac{1}{2}]$ as well). Thus

$$\begin{aligned} F(z) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \widehat{F}(t)\, e^{2\pi i t z}\, \mathrm{d}t = \lim_{N \to \infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} v_N(t)\, e^{2\pi i t z}\, \mathrm{d}t \\ &= \lim_{N \to \infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{m=-N}^{N} F(m)\, e^{2\pi i t (z-m)}\, \mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&= \lim_{N\to\infty} \sum_{m=-N}^{N} F(m) \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i t(z-m)}\,\mathrm{d}t \\
&= \lim_{N\to\infty} \sum_{m=-N}^{N} F(m)\, \frac{\sin \pi(z-m)}{\pi(z-m)}. \qquad (1.22)
\end{aligned}
$$

Since $\{F(m)\}_{m\in\mathbb{Z}}$ is square summable, it is not hard to check that the sum on the right-hand side of (1.22) converges absolutely and uniformly on compact subsets of $\mathbb{C}$ (apply Hölder's inequality and use that $|\sin(\pi z)/\pi z| \ll e^{\pi|\Im(z)|}/(1+|z|)$), thus defining an entire function. We have then established (1.19) in these cases.

To treat the case where $F \in L^p(\mathbb{R})$ with $2 < p < \infty$, we consider the entire function

$$
R(z) = \begin{cases} \dfrac{F(z)-F(0)}{z}, & \text{if } \ z \neq 0; \\[2ex] F'(0), & \text{if } \ z = 0. \end{cases} \qquad (1.23)
$$

It is clear that R is an entire function with the same exponential type as F. Moreover, since $F \in L^p(\mathbb{R})$, an application of Hölder's inequality will give $R \in L^1(\mathbb{R})$. We now use the interpolation formula for R (already established)

$$
\frac{F(z)-F(0)}{z} = \frac{\sin \pi z}{\pi} \left\{ \frac{F'(0)}{z} + \sum_{n\neq 0} (-1)^n \frac{F(n)-F(0)}{(z-n)n} \right\},
$$

together with (1.21)

$$
0 = \widehat{R}\big(\tfrac{1}{2}\big) = \sum_{n\in\mathbb{Z}} (-1)^n\, R(n),
$$

and the identity

$$
\frac{\pi}{\sin \pi z} = \frac{1}{z} + \sum_{n\neq 0} (-1)^n \left(\frac{1}{(z-n)} + \frac{1}{n} \right).
$$

We then get

$$
\begin{aligned}
F(z) = F(0) + \frac{\sin \pi z}{\pi} \Bigg\{ & F'(0) + \sum_{n\neq 0} (-1)^n F(n) \left(\frac{1}{(z-n)} + \frac{1}{n} \right) \\
& - F(0) \sum_{n\neq 0} (-1)^n \left(\frac{1}{(z-n)} + \frac{1}{n} \right) \Bigg\} \\
= \frac{\sin \pi z}{\pi} & \sum_{n\in\mathbb{Z}} (-1)^n \frac{F(n)}{(z-n)} \\
& + F(0) + \frac{\sin \pi z}{\pi} \left\{ \sum_{n\in\mathbb{Z}} (-1)^n R(n) - F(0) \frac{\pi}{\sin \pi z} \right\}
\end{aligned}
$$

$$= \frac{\sin \pi z}{\pi} \sum_{n \in \mathbb{Z}} (-1)^n \frac{F(n)}{(z-n)}.$$

Once more, it is clear that the last series converges absolutely and uniformly on compact subsets of $\mathbb{C}$ via Hölder's inequality, since $\sum_n |F(n)|^p$ converges and $|\sin(\pi z)/\pi z| \ll e^{\pi|\Im(z)|}/(1+|z|)$. $\square$

A careful reading of the last details of the previous proof gives us the following corollary.

Corollary 1.8. *Let F be an entire function of exponential type π such that the entire function R defined in* (1.23) *belongs to $L^p(\mathbb{R})$ for some p with $1 \leq p < \infty$. Then*

$$F(z) = \frac{\sin \pi z}{\pi} \left\{ F'(0) + \frac{F(0)}{z} + \sum_{n \neq 0} (-1)^n F(n) \left(\frac{1}{(z-n)} + \frac{1}{n} \right) \right\}, \qquad (1.24)$$

where the expression on the right-hand side of (1.24) *converges uniformly on compact subsets of $\mathbb{C}$.*

Observe in particular that we can apply Corollary 1.8 when F is an entire function of type π bounded on $\mathbb{R}$. Both Theorem 1.7 and Corollary 1.8 assume $F \in E^{\pi}$. Similar interpolation formulas can be obtained when $F \in E^{2\pi\delta}$ by considering $G(z) = F(z/2\delta)$. In this case, the interpolation points will be $(1/2\delta)\mathbb{Z}$. One can also consider the translation $H(z) = F(z-\alpha)$ to interpolate H at $\mathbb{Z} + \alpha$, when $H \in E^{\pi}$.

1.4.2 Bernstein's inequality

We start here with the following proposition.

Proposition 1.9. *Let F be a function of exponential type $2\pi\delta$ such that the entire function R defined by*

$$R(z) = \begin{cases} \dfrac{F(z) - F(0)}{z}, & \text{if } \ z \neq 0; \\[2ex] F'(0), & \text{if } \ z = 0, \end{cases}$$

is such that $R \in L^2(\mathbb{R})$. Then F' has exponential type $2\pi\delta$ as well.

Proof. By the Paley-Wiener theorem we have $\widehat{R}$ with support in $[-\delta, \delta]$ and

$$R(z) = \int_{-\delta}^{\delta} \widehat{R}(t)\, e^{2\pi i t z}\, \mathrm{d}t,$$

for all $z \in \mathbb{C}$. Differentiating we have

$$\frac{F'(z)z - F(z) + F(0)}{z^2} = R'(z) = \int_{-\delta}^{\delta} 2\pi i t\, \widehat{R}(t)\, e^{2\pi i t z}\, \mathrm{d}t,$$

and from here we see that R' has exponential type $2\pi\delta$ and so does F'. $\square$

We now want to show that if F has an L^p integrability when restricted to the real axis, so does F'. The case $p = \infty$ of this claim is known as Bernstein's inequality and is proved below.

Theorem 1.10 (Bernstein's inequality)**.** *Let F be an entire function of exponential type $2\pi\delta$ that is bounded on the real axis. Then*

$$\sup_{x\in\mathbb{R}} |F'(x)| \leq 2\pi\delta \sup_{x\in\mathbb{R}} |F(x)|.$$

Proof. We may suppose $\delta > 0$ since the case $\delta = 0$ follows by taking limits (hence for $\delta = 0$ the constants are the only admissible functions). We may also suppose $\delta = 1/2$, for otherwise we consider $G(z) = F(z/2\delta)$. From Corollary 1.8 we have

$$F(z) = \frac{\sin \pi z}{\pi} \left\{ F'(0) + \frac{F(0)}{z} + \sum_{n\neq 0} (-1)^n F(n) \left(\frac{1}{(z-n)} + \frac{1}{n} \right) \right\}. \tag{1.25}$$

The termwise differentiation of (1.25) leads to a series converging uniformly on compact subsets of $\mathbb{C}$. Therefore, denoting by $F_1(z)$ the function inside the curly brackets in (1.25) we have

$$F'(x) = \cos \pi x \, F_1(x) + \frac{\sin \pi x}{\pi} \sum_{n\in\mathbb{Z}} (-1)^{n+1} \frac{F(n)}{(x-n)^2}.$$

Taking $x = \frac{1}{2}$ in the last expression we have

$$F'\big(\tfrac{1}{2}\big) = \frac{4}{\pi} \sum_{n\in\mathbb{Z}} (-1)^{n+1} \frac{F(n)}{(2n-1)^2}, \tag{1.26}$$

and thus

$$\big|F'\big(\tfrac{1}{2}\big)\big| \leq \frac{4}{\pi} \left(\sum_{n\in\mathbb{Z}} \frac{1}{(2n-1)^2} \right) \sup_{x\in\mathbb{R}} |F(x)| = \pi \sup_{x\in\mathbb{R}} |F(x)|. \tag{1.27}$$

For the general case, we take $x_0 \in \mathbb{R}$ and consider $G(z) = F\big(x_0 + z - \frac{1}{2}\big)$. Applying (1.27) to G we have

$$\big|F'(x_0)\big| = \big|G'\big(\tfrac{1}{2}\big)\big| \leq \pi \sup_{x\in\mathbb{R}} |G(x)| = \pi \sup_{x\in\mathbb{R}} |F(x)|.$$

□

Note from the previous proof that, in fact, from (1.26) we have

$$F'(x) = \frac{4}{\pi} \sum_{n\in\mathbb{Z}} (-1)^{n+1} \frac{F\big(x + n - \frac{1}{2}\big)}{(2n-1)^2} \tag{1.28}$$

for any $x \in \mathbb{R}$. This formula is the source of many applications, in particular the next one.

Theorem 1.11. *Let F be an entire function of exponential type $2\pi\delta$ that is bounded on the real axis. Then for any $w : [0,\infty) \to [0,\infty)$ which is convex and non-decreasing we have*

$$\int_{-\infty}^{\infty} w\big((2\pi\delta)^{-1}|F'(x)|\big)\mathrm{d}x \le \int_{-\infty}^{\infty} w(|F(x)|)\,\mathrm{d}x.$$

In particular, putting $w(u) = u^p$ for $p \ge 1$, we have

$$\left(\int_{-\infty}^{\infty} |F'(x)|^p \mathrm{d}x\right)^{1/p} \le (2\pi\delta)\left(\int_{-\infty}^{\infty} |F(x)|^p\,\mathrm{d}x\right)^{1/p}. \tag{1.29}$$

Remark. Note that the limiting case of (1.29) (when $p = \infty$) is exactly Bernstein's inequality.

Proof. It suffices to consider the case $\delta = \frac{1}{2}$, since the general case follows by a simple change of variables. From formula (1.28) we know that

$$|F'(x_0)| \le \frac{4}{\pi}\sum_{n\in\mathbb{Z}} \frac{\big|F\big(x_0 + n - \frac{1}{2}\big)\big|}{(2n-1)^2},$$

for any $x_0 \in \mathbb{R}$. We consider the probability measure μ_{x_0} given by

$$\mu_{x_0} = \frac{4}{\pi^2}\sum_{n\in\mathbb{Z}} \frac{1}{(2n-1)^2}\,\delta_{\left(x_0+n-\frac{1}{2}\right)},$$

where δ_a is the Dirac delta supported at the point $x = a$. From Jensen's inequality we have

$$w\left(\int_{-\infty}^{\infty} |F(x)|\,\mathrm{d}\mu_{x_0}(x)\right) \le \int_{-\infty}^{\infty} w(|F(x)|)\,\mathrm{d}\mu_{x_0}(x).$$

We now integrate with respect to the variable x_0 and use the fact that w is non-decreasing to get

$$\begin{aligned}
\int_{-\infty}^{\infty} w\big(\pi^{-1}|F'(x_0)|\big)\,\mathrm{d}x_0 &\le \int_{-\infty}^{\infty} w\left(\int_{-\infty}^{\infty} |F(x)|\,\mathrm{d}\mu_{x_0}(x)\right)\mathrm{d}x_0 \\
&\le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w(|F(x)|)\,\mathrm{d}\mu_{x_0}(x)\,\mathrm{d}x_0 \\
&= \int_{-\infty}^{\infty}\left(\frac{4}{\pi^2}\sum_{n\in\mathbb{Z}} \frac{w\big(\big|F\big(x_0+n-\frac{1}{2}\big)\big|\big)}{(2n-1)^2}\right)\mathrm{d}x_0 \\
&= \sum_{n\in\mathbb{Z}} \frac{4}{\pi^2(2n-1)^2}\int_{-\infty}^{\infty} w\big(\big|F\big(x_0+n-\tfrac{1}{2}\big)\big|\big)\,\mathrm{d}x_0 \\
&= \int_{-\infty}^{\infty} w(|F(x)|)\,\mathrm{d}x,
\end{aligned}$$

and this concludes the proof. □

1.4.3 Interpolation formulas involving derivatives

We saw in the previous sections that a function of exponential type π (with mild decay on the real axis) is completely determined by its values at the integers. We shall see in this section that, if we move to the bigger class of exponential type 2π, we will need more information (say, at the integers) to completely determine our function. It turns out the the values of the function and its derivative at the integers are sufficient to recover the original function, as the following theorem of Vaaler [43, Theorem 9] shows.

Theorem 1.12. *Let F be an entire function of exponential type 2π such that $F \in L^p(\mathbb{R})$ for some p with $1 \leq p < \infty$. Then*

$$F(z) = \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{m \in \mathbb{Z}} \frac{F(m)}{(z-m)^2} + \sum_{n \in \mathbb{Z}} \frac{F'(n)}{(z-n)} \right\}, \tag{1.30}$$

where the expression on the right-hand side of (1.30) *converges uniformly on compact subsets of $\mathbb{C}$.*

If $p = 2$, the Fourier transform $\widehat{F}(t)$ occurring in (1.18) *has the form*

$$\widehat{F}(t) = (1 - |t|)\, u_F(t) + (2\pi i)^{-1} \operatorname{sgn}(t)\, v_F(t) \tag{1.31}$$

for almost all $t \in [-1, 1]$, where u_F and v_F are periodic functions in $L^2[0,1]$ with period 1 and Fourier series expansions

$$u_F(t) = \sum_{m \in \mathbb{Z}} F(m)\, e^{-2\pi i m t}, \tag{1.32}$$

and

$$v_F(t) = \sum_{n \in \mathbb{Z}} F'(n)\, e^{-2\pi i n t}. \tag{1.33}$$

If $p = 1$, then (1.32) *and* (1.33) *are absolutely convergent, u_F and v_F are continuous, and* (1.31) *holds for all $t \in [-1, 1]$. In particular*

$$\widehat{F}(0) = u_F(0) = \sum_{m \in \mathbb{Z}} F(m), \tag{1.34}$$

and

$$0 = v_F(0) = \sum_{n \in \mathbb{Z}} F'(n). \tag{1.35}$$

Proof. We consider first the case $1 \leq p \leq 2$. In this case, by the theorem of Plancherel and Polya, we know that F is bounded, and thus $F \in L^2(\mathbb{R})$. From the Paley-Wiener theorem we have that $\widehat{F}$ is supported on $[-1, 1]$ and belongs to $L^2[-1, 1]$. For $0 \leq t < 1$ we define

$$u_F(t) = \widehat{F}(t) + \widehat{F}(t-1) \tag{1.36}$$

and

$$v_F(t) = 2\pi i \left\{ t\widehat{F}(t) + (t-1)\widehat{F}(t-1) \right\}. \tag{1.37}$$

We then extend the domain of u_F and v_F to $\mathbb{R}$ by requiring that both functions have period 1. Since $\widehat{F} \in L^2[-1,1]$, it is clear that both u_F and v_F are in $L^2[0,1]$. The identity (1.31) follows easily from (1.36), (1.37) and the periodicity of u_F and v_F. To obtain (1.32) and (1.33) we note that

$$F(n) = \int_0^1 \left\{ \widehat{F}(t) + \widehat{F}(t-1) \right\} e^{2\pi int}\,\mathrm{d}t = \int_0^1 u_F(t)\, e^{2\pi int}\,\mathrm{d}t$$

and

$$\begin{aligned} F'(n) &= \int_{-1}^1 2\pi it\,\widehat{F}(t)\, e^{2\pi int}\,\mathrm{d}t \\ &= \int_0^1 2\pi i \left\{ t\widehat{F}(t) + (t-1)\widehat{F}(t) \right\} e^{2\pi int}\,\mathrm{d}t \\ &= \int_0^1 v_F(t)\, e^{2\pi int}\,\mathrm{d}t \end{aligned}$$

for each integer n. Thus $F(n)$ and $F'(n)$ are the Fourier coefficients of u_F and v_F, respectively. Observe now that

$$\left(\frac{\sin \pi z}{\pi z}\right)^2 = \int_{-1}^1 (1-|t|)\, e^{2\pi itz}\,\mathrm{d}t$$

and

$$z\left(\frac{\sin \pi z}{\pi z}\right)^2 = \frac{1}{2\pi i}\int_{-1}^1 \operatorname{sgn}(t)\, e^{2\pi itz}\,\mathrm{d}t.$$

From these two identities we have, for each positive integer N,

$$\begin{aligned} &\left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{m=-N}^{N} \frac{F(m)}{(z-m)^2} + \sum_{n=-N}^{N} \frac{F'(n)}{(z-n)} \right\} \\ &\quad = \int_{-1}^1 \left\{ (1-|t|)\, u_F(t,N) + (2\pi i)^{-1} \operatorname{sgn}(t)\, v_F(t,N) \right\} e^{2\pi itz}\,\mathrm{d}t, \end{aligned} \tag{1.38}$$

where

$$u_F(t,N) = \sum_{m=-N}^{N} F(m)\, e^{-2\pi imt}$$

and

$$v_F(t,N) = \sum_{n=-N}^{N} F'(n)\, e^{-2\pi int}.$$

Since the sequences $\{F(m)\}_{m\in\mathbb{Z}}$ and $\{F'(n)\}_{n\in\mathbb{Z}}$ are square summable (by the theorem of Plancherel-Polya and Theorem 1.11), the left-hand side of (1.38) converges uniformly on compact subsets of $\mathbb{C}$ as $N \to \infty$. On the right-hand side of (1.38) we have $u_F(\cdot, N) \to u_F$ and $v_F(\cdot, N) \to v_F$ in L^2, and from this we establish (1.30).

If $p = 1$, by the theorem of Plancherel-Polya and Theorem 1.11, we know that $\{F(m)\}_{m\in\mathbb{Z}}$ and $\{F'(n)\}_{n\in\mathbb{Z}}$ are summable, thus u_F and v_F have absolutely convergent Fourier series and we may take u_F and v_F to be continuous periodic functions. Since $\widehat{F}(t)$ is now continuous and supported on $[-1, 1]$, the identity (1.31) must hold pointwise for all $t \in [-1, 1]$. If we let $t = 0$ we easily derive (1.34) and (1.35).

When $2 < p < \infty$ we make use of the entire function

$$R(z) = \begin{cases} \dfrac{F(z) - F(0)}{z}, & \text{if } \ z \neq 0; \\[2ex] F'(0), & \text{if } \ z = 0, \end{cases} \tag{1.39}$$

and its derivative

$$R'(z) = \begin{cases} \dfrac{zF'(z) - F(z) + F(0)}{z^2}, & \text{if } \ z \neq 0; \\[2ex] \frac{1}{2}F''(0), & \text{if } \ z = 0. \end{cases}$$

Since R has the same exponential type of F, and $R \in L^2(\mathbb{R})$, we have already established that

$$R(z) = \lim_{N\to\infty} \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{m=-N}^{N} \frac{R(m)}{(z-m)^2} + \sum_{n=-N}^{N} \frac{R'(n)}{(z-n)} \right\} \tag{1.40}$$

uniformly on compact subsets of $\mathbb{C}$. We multiply both sides of the expression (1.40) by z and use the definitions of R and R' to rewrite

$$\begin{aligned} F(z) - F(0) = \lim_{N\to\infty} \left(\frac{\sin \pi z}{\pi}\right)^2 \Bigg\{ & \sum_{m=-N}^{N} \frac{F(m)}{(z-m)^2} + \sum_{n=-N}^{N} \frac{F'(n)}{(z-n)} \\ & + \sum_{k=-N}^{N} R'(k) - F(0) \sum_{l=-N}^{N} \frac{1}{(z-l)^2} \Bigg\}. \end{aligned} \tag{1.41}$$

As the identity

$$\sum_{l\in\mathbb{Z}} \frac{1}{(z-l)^2} = \left(\frac{\pi}{\sin \pi z}\right)^2$$

is well known, all we have to show is that

$$\lim_{N\to\infty} \sum_{k=-N}^{N} R'(k) = 0. \tag{1.42}$$

By Theorem 1.11 we know that F and F' are in $L^p(\mathbb{R})$ and an application of Hölder's inequality shows us that $R \in L^1(\mathbb{R})$. In this case, identity (1.42) follows from (1.35) and this finishes the proof. $\square$

Our next result extends this interpolation formula for the case when F has exponential type 2π, and $R \in L^p(\mathbb{R})$ for some finite p. This has appeared in [43, Theorem 10].

Corollary 1.13. *Let F be an entire function of exponential type 2π such that R defined by (1.39) belongs to $L^p(\mathbb{R})$ for some p with $1 \leq p < \infty$. Then*

$$F(z) = \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{m \in \mathbb{Z}} \frac{F(m)}{(z-m)^2} + \frac{F'(0)}{z} + \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} F'(n) \left(\frac{1}{(z-n)} + \frac{1}{n}\right) + A_F \right\}, \tag{1.43}$$

where the expression on the right-hand side of (1.43) *converges uniformly on compact subsets of $\mathbb{C}$ and A_F is a constant given by*

$$A_F = \frac{1}{2}F''(0) + \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} \frac{F(0) - F(k)}{k^2}. \tag{1.44}$$

Proof. Since $R \in L^p(\mathbb{R})$ we may apply Theorem 1.12. As in the previous proof we find that (1.40) and (1.41) hold. We now reorganize (1.41) using the expression for the derivative $R'(z)$ as follows

$$F(z) = \lim_{N \to \infty} \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{m=-N}^{N} \frac{F(m)}{(z-m)^2} + \frac{F'(0)}{z} + \sum_{\substack{n=-N \\ n \neq 0}}^{N} F'(n) \left(\frac{1}{(z-n)} + \frac{1}{n}\right) + \frac{1}{2}F''(0) + \sum_{\substack{k=-N \\ k \neq 0}}^{N} \frac{F(0) - F(k)}{k^2} \right\}. \tag{1.45}$$

For a function of exponential type 2π, we have already seen that the fact that $F \in L^p(\mathbb{R})$ implies that $F \in L^q(\mathbb{R})$ if $p < q$. We may therefore assume without loss of generality that $1 < p < \infty$. It follows that

$$\begin{aligned} \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} |F(m)m^{-1}|^p &= \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} |R(m) + F(0)m^{-1}|^p \\ &\leq 2^p \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} \left\{ |R(m)|^p + |F(0)m^{-1}|^p \right\} < \infty, \end{aligned} \tag{1.46}$$

and

$$
\begin{aligned}
\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} |F'(n)n^{-1}|^p &= \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} |R'(n)+R(n)n^{-1}|^p \\
&\le 2^p \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \left\{|R'(n)|^p + |R(n)n^{-1}|^p\right\} < \infty.
\end{aligned}
\tag{1.47}
$$

From the series defining A_F we have

$$
\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \left|(F(0)-F(k))k^{-2}\right| = \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} |R(k)|\,|k|^{-1} < \infty. \tag{1.48}
$$

Estimates (1.46), (1.47) and (1.48), together with (1.45), show that the right-hand side of (1.43) converges uniformly on compacts subsets of $\mathbb{C}$ (easy application of Hölder's inequality), with A_F given by the absolutely convergent series (1.44). This concludes the proof. $\square$

Chapter 2

The Beurling-Selberg extremal problem

2.1 Introduction

After a brief review of some useful harmonic analysis tools in our first chapter, we now direct our interest to certain problems in approximation theory. Recall that an entire function $F : \mathbb{C} \to \mathbb{C}$ is of *exponential type* at most $2\pi\delta$ if for every $\epsilon > 0$ there exists a positive constant C_ϵ, such that the inequality

$$|F(z)| \leq C_\epsilon \, e^{(2\pi\delta+\epsilon)|z|}$$

holds for all complex numbers z.

Given a real function $f : \mathbb{R} \to \mathbb{R}$ and $\delta > 0$, we address here the problem of finding an entire function K of exponential type $2\pi\delta$ such that

$$\int_{-\infty}^{\infty} |K(x) - f(x)| \, \mathrm{d}x \tag{2.1}$$

is minimized. Such a function is called a *best approximation* of f. This is a classical problem in harmonic analysis and approximation theory, considered by Bernstein, Akhiezer, Krein, Nagy and others, since at least 1938. In particular, Krein [24] in 1938 and Nagy [41] in 1939 published seminal papers solving this problem for a wide class of functions $f(x)$.

For applications to analytic number theory, it is convenient to consider an additional restriction: we ask that $K(z)$ is real on $\mathbb{R}$ and that $K(x) \geq f(x)$ for all $x \in \mathbb{R}$. In this case, a minimizer of the integral (2.1) is called an extremal majorant of $f(x)$ (or extremal upper one-sided approximation). Extremal minorants are defined analogously. Beurling started working on this one-sided extremal problem, independently, in the late 1930's, and obtained the solution for $f(x) = \mathrm{sgn}(x)$ and an inequality for almost periodic functions in an unpublished manuscript. The one-sided extremals for the signum function were later used by Selberg [37] to obtain the solution of the extremal problem for characteristic functions of intervals (of integer size, the general case was settled later, by B. Logan and alternatively by F. Littmann [30]) and a sharp form of the large sieve inequality. In these notes we are mostly interested in the one-sided version of this problem and, therefore, we shall be referring to it as the *Beurling-Selberg extremal problem*. A further discussion of the early development of this theory with many of its applications is presented in the excellent survey [43] by J. D. Vaaler.

The problem (2.1) is hard in the sense that there is no general known way to produce a solution given any $f : \mathbb{R} \to \mathbb{R}$. Besides the original examples $f(x) = \operatorname{sgn}(x)$ of Beurling and $f(x) = \chi_{[a,b]}(x)$ of Selberg, the solution for the exponential family $f(x) = e^{-\lambda|x|}$, $\lambda > 0$, was discovered by Graham and Vaaler in [19], with a first glimpse of the technique of integration on the free parameter λ to produce solutions for a family of even and odd functions. Later, the problem for $f(x) = x^n \operatorname{sgn}(x)$ and $f(x) = (x^+)^n$, where n is a positive integer, was considered by Littmann in [27, 28, 29]. Using the exponential subordination, Carneiro and Vaaler in [8, 9] extended the construction of extremal approximations for a class of even functions that includes $f(x) = \log|x|$, $f(x) = \log\big((x^2+1)/x^2\big)$ and $f(x) = |x|^\alpha$, with $-1 < \alpha < 1$. The analogous exponential subordination framework for truncated and odd functions was treated in [6].

Other classical applications of the solutions of these problems to analytic number theory include Hilbert-type inequalities [8, 19, 28, 32, 43], Erdös-Turán discrepancy inequalities [8, 25, 43], optimal approximations of periodic functions by trigonometric polynomials [2, 8, 9, 43] and Tauberian theorems [19]. The extremal problem in higher dimensions, with applications, is considered in [1, 20]. Approximations in L^p-norms with $p \neq 1$ are treated, for instance, in [15].

Our focus in this section is to present the recent advances in this field, in the form of a general method to produce extremal majorants and minorants for classes of even, odd and truncated functions subject to a certain Gaussian subordination, as developed in the works [5] and [7]. This is the most general method up to date to produce such special functions.

2.2 The extremal problem for the Gaussian

2.2.1 Statements of the main theorems

We consider the problem of majorizing and minorizing the Gaussian function

$$x \mapsto G_\lambda(x) = e^{-\pi\lambda x^2} \tag{2.2}$$

on $\mathbb{R}$ by entire functions of exponential type. Here $\lambda > 0$ is a parameter. We make use of classical interpolation techniques (as developed in Chapter 1) and integral representations to achieve this goal.

For each positive value of λ we define two entire functions of the complex variable z as follows:

$$L_\lambda(z) = \Big(\frac{\cos \pi z}{\pi}\Big)^2 \bigg\{ \sum_{m=-\infty}^{\infty} \frac{G_\lambda\big(m+\frac12\big)}{\big(z-m-\frac12\big)^2} + \sum_{n=-\infty}^{\infty} \frac{G'_\lambda\big(n+\frac12\big)}{\big(z-n-\frac12\big)} \bigg\}, \tag{2.3}$$

$$M_\lambda(z) = \Big(\frac{\sin \pi z}{\pi}\Big)^2 \bigg\{ \sum_{m=-\infty}^{\infty} \frac{G_\lambda(m)}{(z-m)^2} + \sum_{n=-\infty}^{\infty} \frac{G'_\lambda(n)}{(z-n)} \bigg\}. \tag{2.4}$$

The function $L_\lambda(z)$ is a real entire function of exponential type 2π which interpolates both the values of $G_\lambda(z)$ and the values of its derivative $G'_\lambda(z)$ on the coset $\mathbb{Z}+\frac12$. Similarly, the function $M_\lambda(z)$ is a real entire function of exponential type 2π which interpolates both the values of $G_\lambda(z)$ and the values of its derivative $G'_\lambda(z)$ on the integers $\mathbb{Z}$. We will show that these functions satisfy the basic inequality

$$L_\lambda(x) \le G_\lambda(x) \le M_\lambda(x) \tag{2.5}$$

for all real x. Moreover, we will show that the value of each of the two integrals

$$\int_{-\infty}^{\infty} \Big\{ G_\lambda(x) - L_\lambda(x) \Big\}\, \mathrm{d}x \quad \text{and} \quad \int_{-\infty}^{\infty} \Big\{ M_\lambda(x) - G_\lambda(x) \Big\}\, \mathrm{d}x,$$

is minimized.

In order to state a more precise form of our main results for the Gaussian function, we make use of the basic theta functions. Here v is a complex variable, τ is a complex variable with $\Im\{\tau\} > 0$, $q = e^{\pi i \tau}$, and $e(z) = e^{2\pi i z}$. Our notation for the theta functions is standard and follows that of Chandrasekharan [11]. Thus we define

$$\theta_1(v,\tau) = \sum_{n=-\infty}^{\infty} q^{(n+\frac12)^2} e\big((n+\tfrac12)v\big), \tag{2.6}$$

$$\theta_2(v,\tau) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e(nv), \tag{2.7}$$

$$\theta_3(v,\tau) = \sum_{n=-\infty}^{\infty} q^{n^2} e(nv). \tag{2.8}$$

We note that for a fixed value of τ with $\Im\{\tau\} > 0$, each of the functions $v \mapsto \theta_1(v,\tau)$, $v \mapsto \theta_2(v,\tau)$, and $v \mapsto \theta_3(v,\tau)$ is an *even* entire function of v. The function $v \mapsto \theta_1(v,\tau)$ is periodic with period 2, and satisfies the identity

$$\theta_1(v+1,\tau) = -\theta_1(v,\tau) \tag{2.9}$$

for all complex v. Both of the functions $v \mapsto \theta_2(v,\tau)$, and $v \mapsto \theta_3(v,\tau)$, are periodic with period 1. They are related by the identity

$$\theta_2(v+\tfrac12,\tau) = \theta_3(v,\tau). \tag{2.10}$$

The transformation formulas for the theta functions (see [11, Chapter V, Theorem 9, Corollary 1]) provide a connection with the Gaussian function $G_\lambda(z)$. In particular we have

$$\sum_{n=-\infty}^{\infty} (-1)^n G_\lambda(n-v) = \lambda^{-\frac{1}{2}} \theta_1\big(v, i\lambda^{-1}\big), \tag{2.11}$$

$$\sum_{n=-\infty}^{\infty} G_\lambda(n+\tfrac{1}{2}-v) = \lambda^{-\frac{1}{2}} \theta_2\big(v, i\lambda^{-1}\big), \tag{2.12}$$

$$\sum_{n=-\infty}^{\infty} G_\lambda(n-v) = \lambda^{-\frac{1}{2}} \theta_3\big(v, i\lambda^{-1}\big). \tag{2.13}$$

We consider now the problem of minorizing $G_\lambda(z)$ on $\mathbb{R}$ by a real entire function of exponential type at most 2π.

Theorem 2.1 (Extremal minorant for the Gaussian)**.** *Let $F(z)$ be a real entire function of exponential type at most 2π such that*

$$F(x) \le G_\lambda(x)$$

for all real x. Then

$$\int_{-\infty}^{\infty} F(x)\ \mathrm{d}x \le \lambda^{-\frac{1}{2}} \theta_2\big(0, i\lambda^{-1}\big), \tag{2.14}$$

and there is equality in (2.14) *if and only if $F(z) = L_\lambda(z)$.*

Here is the analogous result for the problem of majorizing $G_\lambda(z)$ on $\mathbb{R}$ by a real entire function of exponential type at most 2π.

Theorem 2.2 (Extremal majorant for the Gaussian)**.** *Let $F(z)$ be a real entire function of exponential type at most 2π such that*

$$G_\lambda(x) \le F(x)$$

for all real x. Then

$$\lambda^{-\frac{1}{2}} \theta_3\big(0, i\lambda^{-1}\big) \le \int_{-\infty}^{\infty} F(x)\ \mathrm{d}x, \tag{2.15}$$

and there is equality in (2.15) *if and only if $F(z) = M_\lambda(z)$.*

By a simple change of variables, using Theorem 2.1 and Theorem 2.2, one can check that the real entire functions $z \mapsto L_{\lambda\delta^{-2}}(\delta z)$ and $z \mapsto M_{\lambda\delta^{-2}}(\delta z)$ are the unique extremal minorant and majorant, respectively, of exponential type $2\pi\delta$ for the function $G_\lambda(x)$.

The entire functions $L_\lambda(z)$ and $M_\lambda(z)$ have exponential type 2π, and the restrictions of these functions to $\mathbb{R}$ are both integrable. Hence their Fourier transforms

$$\widehat{L}_\lambda(t) = \int_{-\infty}^{\infty} L_\lambda(x)\, e(-xt)\ \mathrm{d}x, \quad \text{and} \quad \widehat{M}_\lambda(t) = \int_{-\infty}^{\infty} M_\lambda(x)\, e(-xt)\ \mathrm{d}x,$$

are both continuous, and both Fourier transforms are supported on the compact interval $[-1,1]$. These Fourier transforms can be given explicitly in terms of the theta functions, as a simple application of Theorem 1.12.

Theorem 2.3. *If $-1 \le t \le 1$ then the Fourier transforms $t \mapsto \widehat{L}_\lambda(t)$ and $t \mapsto \widehat{M}_\lambda(t)$ are given by*

$$\widehat{L}_\lambda(t) = (1-|t|)\theta_1(t,i\lambda) - (2\pi)^{-1}\lambda\,\mathrm{sgn}(t)\frac{\partial\theta_1}{\partial t}(t,i\lambda), \tag{2.16}$$

and

$$\widehat{M}_\lambda(t) = (1-|t|)\theta_3(t,i\lambda) - (2\pi)^{-1}\lambda\,\mathrm{sgn}(t)\frac{\partial\theta_3}{\partial t}(t,i\lambda). \tag{2.17}$$

2.2.2 Integral representations

In this section we establish several representations for combinations of Gaussian functions that will be used in the proofs of Theorems 2.1 and 2.2.

Lemma 2.4. *Let z and w be distinct complex numbers. Then we have*

$$\begin{aligned}\frac{G_\lambda(z)-G_\lambda(w)}{z-w} &= 2\pi\lambda^{\frac{3}{2}}\int_{-\infty}^{0}\int_{-\infty}^{0} e^{-2\pi\lambda tu}G_\lambda(z-t)G_\lambda(w-u)\,\mathrm{d}u\,\mathrm{d}t \\ &\quad - 2\pi\lambda^{\frac{3}{2}}\int_{0}^{\infty}\int_{0}^{\infty} e^{-2\pi\lambda tu}G_\lambda(z-t)G_\lambda(w-u)\,\mathrm{d}u\,\mathrm{d}t.\end{aligned} \tag{2.18}$$

Proof. It suffices to prove the identity (2.18) for $\lambda = 1$, then the general case will follow from an elementary change of variables. Therefore we simplify our notation and write $G(z) = G_1(z)$. We note that $G(z)$ satisfies the identity

$$G(z)^{-1} = \int_{-\infty}^{\infty} e^{2\pi zt}G(t)\ \mathrm{d}t \tag{2.19}$$

for all complex numbers z, and the identity

$$G(z)G(w)e^{2\pi zw} = G(z-w) \tag{2.20}$$

for all pairs of complex numbers z and w. From (2.19) we get

$$\begin{aligned}\frac{G(z)-G(w)}{z-w} &= G(z)G(w)\left\{\frac{G(w)^{-1}-G(z)^{-1}}{z-w}\right\} \\ &= G(z)G(w)(z-w)^{-1}\int_{-\infty}^{\infty}\left\{e^{2\pi wt}-e^{2\pi zt}\right\}G(t)\ \mathrm{d}t.\end{aligned} \tag{2.21}$$

Then using Fubini's theorem we find that

$$
\begin{aligned}
(z-w)^{-1} & \int_{-\infty}^{\infty} \left\{e^{2\pi wt} - e^{2\pi zt}\right\} G(t)\ \mathrm{d}t \\
&= 2\pi \int_{-\infty}^{0} \left\{ \int_{t}^{0} e^{2\pi(z-w)u}\ \mathrm{d}u \right\} e^{2\pi wt} G(t)\ \mathrm{d}t \\
&\qquad - 2\pi \int_{0}^{\infty} \left\{ \int_{0}^{t} e^{2\pi(z-w)u}\ \mathrm{d}u \right\} e^{2\pi wt} G(t)\ \mathrm{d}t \\
&= 2\pi \int_{-\infty}^{0} \left\{ \int_{-\infty}^{u} e^{2\pi wt} G(t)\ \mathrm{d}t \right\} e^{2\pi(z-w)u}\ \mathrm{d}u \\
&\qquad - 2\pi \int_{0}^{\infty} \left\{ \int_{u}^{\infty} e^{2\pi wt} G(t)\ \mathrm{d}t \right\} e^{2\pi(z-w)u}\ \mathrm{d}u \qquad (2.22) \\
&= 2\pi \int_{-\infty}^{0} \left\{ \int_{-\infty}^{0} e^{2\pi w(t+u)} G(t+u)\ \mathrm{d}t \right\} e^{2\pi(z-w)u}\ \mathrm{d}u \\
&\qquad - 2\pi \int_{0}^{\infty} \left\{ \int_{0}^{\infty} e^{2\pi w(t+u)} G(t+u)\ \mathrm{d}t \right\} e^{2\pi(z-w)u}\ \mathrm{d}u \\
&= 2\pi \int_{-\infty}^{0} \int_{-\infty}^{0} e^{2\pi(wt+zu)} G(t+u)\ \mathrm{d}t\ \mathrm{d}u \\
&\qquad - 2\pi \int_{0}^{\infty} \int_{0}^{\infty} e^{2\pi(wt+zu)} G(t+u)\ \mathrm{d}t\ \mathrm{d}u.
\end{aligned}
$$

Next we apply (2.20) twice and get

$$
\begin{aligned}
G(z)G(w)e^{2\pi(wt+zu)}G(t+u) &= G(z)G(w)G(u)G(t)e^{-2\pi tu+2\pi wt+2\pi zu} \\
&= G(z-u)G(w-t)e^{-2\pi tu}.
\end{aligned} \tag{2.23}
$$

Then we combine (2.21), (2.22) and (2.23) to obtain the special case

$$
\begin{aligned}
\frac{G(z)-G(w)}{z-w} &= 2\pi \int_{-\infty}^{0} \int_{-\infty}^{0} e^{-2\pi tu} G(z-t)G(w-u)\ \mathrm{d}u\ \mathrm{d}t \\
&\qquad - 2\pi \int_{0}^{\infty} \int_{0}^{\infty} e^{-2\pi tu} G(z-t)G(w-u)\ \mathrm{d}u\ \mathrm{d}t.
\end{aligned} \tag{2.24}
$$

The more general identity (2.18) follows by replacing z with $\lambda^{\frac{1}{2}}z$, by replacing w with $\lambda^{\frac{1}{2}}w$, and by making a corresponding change of variables in each integral on the right of (2.24). □

Lemma 2.5. *Let z and w be distinct complex numbers. Then we have*

$$
\begin{aligned}
&\frac{G_\lambda(z)}{(z-w)^2} - \frac{G_\lambda(w)}{(z-w)^2} - \frac{G_\lambda'(w)}{z-w} \\
&= (2\pi)^2\lambda^{\frac{5}{2}} \int_{-\infty}^{0} \int_{-\infty}^{0} te^{-2\pi\lambda tu} G_\lambda(z-t)\left\{G_\lambda(w) - G_\lambda(w-u)\right\}\ \mathrm{d}u\ \mathrm{d}t \qquad (2.25) \\
&\quad - (2\pi)^2\lambda^{\frac{5}{2}} \int_{0}^{\infty} \int_{0}^{\infty} te^{-2\pi\lambda tu} G_\lambda(z-t)\left\{G_\lambda(w) - G_\lambda(w-u)\right\}\ \mathrm{d}u\ \mathrm{d}t.
\end{aligned}
$$