value

$$\leqslant \quad \frac{1}{\tau^2}\int_{a\tau}^{b\tau} KCs\,\mathrm{d}s \quad \leqslant \quad \tfrac{1}{2}KCb^2.$$

The increasing function $\nu_\rho(\tau)$ thus has a *bounded derivative* in $(0,\infty)$.

We may at this point integrate

$$2\Re\int_0^\infty \frac{z^2}{z^2-\tau^2}\,\frac{\nu_\rho(\tau)}{\tau}\,\mathrm{d}\tau$$

by parts in the direction opposite to the one taken previously, to get

$$\int_0^\infty \log\left|1 - \frac{z^2}{\tau^2}\right|\mathrm{d}\nu_\rho(\tau);$$

this, then, is equal to

$$\int_a^b F(\xi z)\,\frac{\rho(\xi)}{\xi}\,\mathrm{d}\xi$$

by the above work. Making $y \longrightarrow 0$ now causes the first of these two integrals to tend to

$$\int_0^\infty \log\left|1 - \frac{x^2}{\tau^2}\right|\mathrm{d}\nu_\rho(\tau)$$

(that is especially easy to see here, where $0 \leqslant \mathrm{d}\nu_\rho(\tau) \leqslant \text{const.}\,\mathrm{d}\tau$). The integral just written is therefore equal to $F_\rho(x)$ according to what was observed initially. We are done.

Lemma. *Let $\nu(t)$ be odd and increasing, with $\nu(t)/t$ bounded on* $(0,\infty)$, *and put*

$$F(x) \quad = \quad \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right|\mathrm{d}\nu(t) \qquad \text{for } x\in\mathbb{R}.$$

Suppose that

$$\int_{-\infty}^\infty \frac{|F(x)|}{1+x^2}\,\mathrm{d}x \quad < \quad \infty.$$

Then, for the Hilbert transform

$$\tilde{F}(x) \quad = \quad \frac{1}{\pi}\int_{-\infty}^\infty \left(\frac{1}{x-t} + \frac{t}{t^2+1}\right)F(t)\,\mathrm{d}t$$

of F, we have, with a certain constant A,

$$\tilde{F}(x) \quad = \quad Ax - \pi\nu(x) \qquad \text{a.e.,} \quad x\in\mathbb{R}.$$

Proof. Write

$$F(z) \quad = \quad \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}\nu(t)$$

for complex z. The given conditions on ν then make

$$F(z) \;\leqslant\; F(\mathrm{i}|z|) \;\leqslant\; \text{const.}\,|z|.$$

Based on this relation and on the property of $|F(x)|$ assumed in the hypothesis we can, by an argument like one made during the proof of the *second* theorem in §B.1, establish for F the representation from §G.1 of Chapter III:

$$F(z) \quad = \quad A|\Im z| \quad + \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|\,F(t)}{|z-t|^2}\,\mathrm{d}t.$$

A *harmonic conjugate* $G(z)$ for $F(z)$ in the upper half plane is therefore given by the formula

$$G(z) \quad = \quad -A\Re z \quad + \quad \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\Re z - t}{|z-t|^2} + \frac{t}{t^2+1}\right)F(t)\,\mathrm{d}t.$$

This, then, must, to within an additive constant, agree with the *obvious* harmonic conjugate for $\Im z > 0$ of the original logarithmic potential defining $F(z)$. In other words,

$$G(z) \quad = \quad C + \int_0^\infty \arg\left(1 - \frac{z^2}{t^2}\right)\mathrm{d}\nu(t)$$

where, for $\Im z > 0$, we take the determination of the argument having $\arg 1 = 0$ in order to ensure convergence of the integral on the right.

For $\Im z > 0$ and $t > 0$, $1 - (z^2/t^2)$ lies in the following domain:

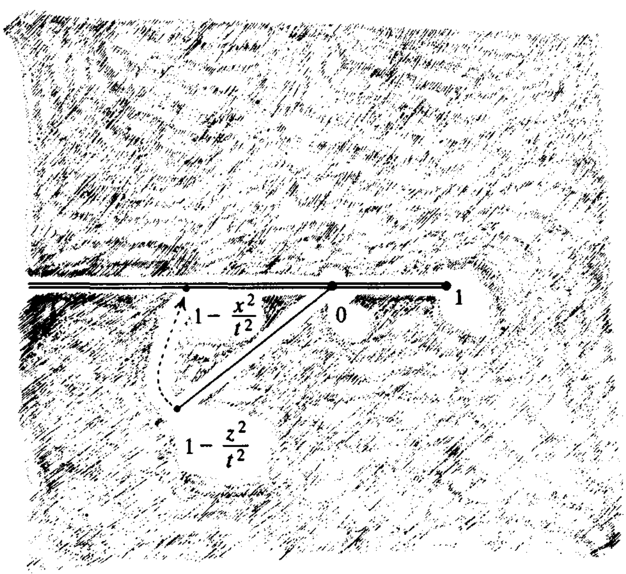

Figure 258

The branch of the argument we are using has in this domain a value between $-\pi$ and π. When z tends from the upper half plane to a given real $x > 0$, $\arg(1 - (z^2/t^2))$ tends to a *boundary value* equal to $-\pi$ for $0 < t < x$ (see the preceding figure) and to *zero* for $t > x$. As long, then, as $\nu(t)$ is *continuous* at such an x,

$$\int_0^\infty \arg\left(1 - \frac{z^2}{t^2}\right) d\nu(t)$$

will tend to $-\pi\nu(x)$ as $z \longrightarrow x$ from the upper half plane. At the same time, $G(z)$ will, at almost every such x, tend to

$$-Ax + \tilde{F}(x)$$

as we know (see the scholium in §H.1 of Chapter III and problem 25 at the end of §C.1, Chapter VIII). Thus,

$$-Ax + \tilde{F}(x) = C - \pi\nu(x) \quad \text{a.e.,} \quad x > 0.$$

On the negative real axis we find in the same way that

$$-Ax + \tilde{F}(x) = C + \pi\nu(|x|) \quad \text{a.e.;}$$

the right side is, however, equal to $C - \pi\nu(x)$ there, ν being odd. Hence,

$$-Ax + \tilde{F}(x) = C - \pi\nu(x) \quad \text{a.e.,} \quad x \in \mathbb{R}.$$

But $F(x)$ is even, making $\tilde{F}(x)$ odd, like $\nu(x)$. Therefore C must be zero, and

$$\tilde{F}(x) = Ax - \pi\nu(x) \quad \text{a.e. on } \mathbb{R},$$

Q.E.D.

Lemma. *Let $F(x)$, even and $\mathscr{C}_2$, satisfy the condition*

$$\int_{-\infty}^{\infty} \frac{|F(x)|}{1+x^2} dx < \infty,$$

and suppose that there is an increasing continuous odd function $\mu(x)$, $O(x)$ on $[0, \infty)$, such that $\tilde{F}(x) + \mu(x)$ is also increasing on $\mathbb{R}$, $\tilde{F}(x)$ being the Hilbert transform of F. Suppose, moreover, that $|\tilde{F}(x)/x|$ is bounded on $\mathbb{R}$. Then

$$F(x) = F(0) - \frac{1}{\pi}\int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\tilde{F}(t) \qquad \text{for } x \in \mathbb{R},$$

with the integral on the right (involving the signed *measure* $d\tilde{F}(t)$ *) absolutely convergent.*

Remark. Boundedness of $\tilde{F}(x)/x$ on $\mathbb{R}$ actually *follows* from the rest of the hypothesis. That is the conclusion of the next lemma.

Proof of lemma. The given properties of $F(x)$ ensure* that $\tilde{F}(x)$ is at least $\mathscr{C}_1$, and that

$$f(z) \quad = \quad \frac{\mathrm{i}}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{z-t} \; + \; \frac{t}{t^2+1}\right)F(t)\,\mathrm{d}t,$$

analytic for $\Im z \; > \; 0$, is continuous up to the real axis, where it takes the boundary value

$$f(x) \quad = \quad F(x) \; + \; \mathrm{i}\tilde{F}(x).$$

Since $F(x)$ is even, $\tilde{F}(x)$ is odd. Put

$$\nu(t) \quad = \quad \tilde{F}(t) \; + \; \mu(t),$$

then $\nu(t)$, like $\mu(t)$, is odd and continuous and, by hypothesis, increasing and $\mathrm{O}(t)$ on $[0, \infty)$. For $\Im z \; \neq \; 0$ we can thus form the function

$$V(z) \quad = \quad \int_0^{\infty} \log\left|1 \; - \; \frac{z^2}{t^2}\right| \mathrm{d}\nu(t);$$

it is harmonic in both the upper and the lower half planes. The same is true for

$$U(z) \quad = \quad \int_0^{\infty} \log\left|1 \; - \; \frac{z^2}{t^2}\right| \mathrm{d}\mu(t).$$

These functions have, in $\Im z \; > \; 0$, the harmonic conjugates

$$\tilde{V}(z) \quad = \quad \int_0^{\infty} \arg\left(1 \; - \; \frac{z^2}{t^2}\right) \mathrm{d}\nu(t),$$

$$\tilde{U}(z) \quad = \quad \int_0^{\infty} \arg\left(1 \; - \; \frac{z^2}{t^2}\right) \mathrm{d}\mu(t)$$

* To show that $\tilde{F}(x)$ is $\mathscr{C}_1$ for $|x| \; < \; A$, say, take an even $\mathscr{C}_\infty$ function $\varphi(t)$ equal to 1 for $|t| \; \leqslant \; A$ and to 0 for $|t| \; \geqslant \; 2A$. Then, since $F(t)$ is also even, we have, for $|x| \; < \; A$,

$$\tilde{F}(x) \quad = \quad (2x/\pi)\int_A^{\infty} ((1-\varphi(t))F(t)/(x^2-t^2))\,\mathrm{d}t \quad + \quad (1/\pi)⨍_{-2A}^{2A} (\varphi(t)F(t)/(x-t))\,\mathrm{d}t.$$

The *first* expression on the right is clearly $\mathscr{C}_\infty$ in x for $|x| \; < \; A$. To the *second*, we apply the partial integration technique used often in this book, and get $(1/\pi)\int_{-2A}^{2A}\log|x-t| \; \mathrm{d}(\varphi(t)F(t))$. Reason now as in the footnote to the theorem of §D.3. Since $\mathrm{d}(\varphi(t)F(t))/\mathrm{d}t$ is $\mathscr{C}_1$, the integral is also $\mathscr{C}_1$ (in x) for $|x| \; < \; A$.

(where the argument is determined so as to make arg 1 $=$ 0). Here, where $\nu(t)$ and $\mu(t)$ are continuous, we can argue as in the proof of the last lemma to show that $\tilde{V}(z)$ and $\tilde{U}(z)$ are continuous up to the real axis, where they take the boundary values

$$\tilde{V}(x) \;=\; -\pi\nu(x), \qquad \tilde{U}(x) \;=\; -\pi\mu(x).$$

Thus,

$$\tilde{V}(x) \;-\; \tilde{U}(x) \;=\; -\pi\tilde{F}(x).$$

Our assumptions on $\mu(x)$ are not strong enough to yield as much information about the behaviour of $U(z)$ (or of $V(z)$) at the points of $\mathbb{R}$. Consider, however, the *difference*

$$G(z) \;=\; V(z) \;-\; U(z).$$

Since $\tilde{F}(t) = \nu(t) - \mu(t)$, we can write

$$G(z) \;=\; \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| d\tilde{F}(t)$$

for $\Im z \neq 0$; the integral on the right is, however, absolutely convergent *even when z is real.* To check this, take any $R > |z|$ and break up that integral into two pieces, the first over $[0, 2R]$ and the second over $[2R, \infty)$. Regarding the *first* portion, note that $d\tilde{F}(t) = \tilde{F}'(t)\,dt$ with $\tilde{F}'(t)$ continuous and hence *bounded* on finite intervals ($\tilde{F}(t)$ being $\mathscr{C}_1$); for the *second*, just use $|d\tilde{F}(t)| \leqslant d\nu(t) + d\mu(t)$. In this way we also verify without difficulty that $G(z)$ is continuous up to (and *on*) $\mathbb{R}$, and takes there the boundary value

$$G(x) \;=\; \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\tilde{F}(t).$$

By this observation and the one preceding it we see that the function

$$g(z) \;=\; G(z) \;+\; i(\tilde{V}(z) - \tilde{U}(z)),$$

analytic for $\Im z > 0$, is continuous up to the real axis where it has the boundary value

$$g(x) \;=\; G(x) \;-\; i\pi\tilde{F}(x).$$

Bringing in now the function $f(z)$ described earlier, we can conclude that $\pi f(z) + g(z)$, analytic in $\Im z > 0$, is continuous up to $\mathbb{R}$ and assumes there the boundary value

$$\pi f(x) \;+\; g(x) \;=\; \pi F(x) \;+\; G(x).$$

The right side is obviously *real*, so we may use Schwarz reflection to continue $\pi f(z) + g(z)$ analytically across $\mathbb{R}$ and thus obtain an *entire function*. The latter's *real part*, $H(z)$, is hence *everywhere harmonic*, with

$$H(x) \quad = \quad \pi F(x) \; + \; G(x) \qquad \text{on } \mathbb{R}.$$

For $\Im z \neq 0$, we have $H(z) = H(\bar{z})$, so*

$$H(z) \quad = \quad \int_{-\infty}^{\infty} \frac{|\Im z| F(t)}{|z-t|^2} \mathrm{d}t \; + \; V(z) \; - \; U(z).$$

It is now claimed that $H(z)$ is a *linear function* of $\Re z$ and $\Im z$; this we verify by estimating the integrals $\int_{-\pi}^{\pi} (H(re^{i\vartheta}))^+ \mathrm{d}\vartheta$ for certain large values of r.

By the last relation, we have

$$(H(z))^+ \quad \leqslant \quad \int_{-\infty}^{\infty} \frac{|\Im z| |F(t)|}{|z-t|^2} \mathrm{d}t \quad + \quad (V(z))^+ \quad + \quad (U(z))^-$$

for $\Im z \neq 0$. Consider first the *second* term on the right. Since $\nu(t)$ is increasing,

$$V(z) \quad \leqslant \quad \int_0^{\infty} \log\left(1 \; + \; \frac{|z|^2}{t^2}\right) \mathrm{d}\nu(t).$$

Here, $\nu(t) \leqslant \text{const.}\, t$ on $[0, \infty)$ by hypothesis, from which we deduce by the usual integration by parts that $V(z) \leqslant \text{const.}|z|$, and thence that

$$\int_{-\pi}^{\pi} (V(re^{i\vartheta}))^+ \,\mathrm{d}\vartheta \quad \leqslant \quad \text{const.}\, r.$$

To estimate the circular means of the *third* term on the right we use the formula

$$\int_{-\pi}^{\pi} (U(re^{i\vartheta}))^- \,\mathrm{d}\vartheta \quad = \quad \int_{-\pi}^{\pi} (U(re^{i\vartheta}))^+ \,\mathrm{d}\vartheta \quad - \quad \int_{-\pi}^{\pi} U(re^{i\vartheta}) \,\mathrm{d}\vartheta$$

together with the inequality

$$U(z) \quad \leqslant \quad \text{const.}\,|z|,$$

analogous to the one for $V(z)$ just mentioned. For this procedure, a *lower* bound on $\int_{-\pi}^{\pi} U(re^{i\vartheta}) \mathrm{d}\vartheta$ is needed, and that quantity is indeed $\geqslant 0$, as we now show†. When $0 < t \leqslant r$,

* the *integral* in the next formula is just $\pi \Re f(z)$ when $\Im z > 0$

† one may also just refer to the subharmonicity of $U(z)$

$|1-(re^{i\vartheta}/t)^2|^2 = (1-r^2/t^2)^2 + 4(r/t)^2\sin^2\vartheta \geqslant |1-e^{2i\vartheta}|^2$, so, since $\mu(t)$ increases,

$$\begin{aligned} U(re^{i\vartheta}) &= \int_0^\infty \log\left|1 - \frac{r^2e^{2i\vartheta}}{t^2}\right| d\mu(t) \\ &\geqslant \mu(r)\log|1-e^{2i\vartheta}| + \int_r^\infty \log\left|1 - \frac{r^2e^{2i\vartheta}}{t^2}\right| d\mu(t). \end{aligned}$$

Integration of the two right-hand terms from $-\pi$ to π now presents no difficulty (Fubini's theorem being applicable to the second one), and both of the results are zero. Thus, $\int_{-\pi}^{\pi} U(re^{i\vartheta})d\vartheta \geqslant 0$ which, substituted with $(U(re^{i\vartheta}))^+ \leqslant \text{const.}\, r$ into the previous relation, yields

$$\int_{-\pi}^{\pi} (U(re^{i\vartheta}))^- d\vartheta \leqslant \text{const.}\, r.$$

Examination of the *first* right-hand term in the above inequality for $(H(z))^+$ remains. To estimate the circular means of that term – call it $P(z)$ – one argues as in the proof of the first theorem from §B.3, leaning heavily on the convergence of $\int_{-\infty}^{\infty}(|F(t)|/(1+t^2))dt$ (without which, it is true, $P(z)$ would be infinite!). In that way, one finds that

$$\int_{-\pi}^{\pi} P(r_n e^{i\vartheta})d\vartheta \leqslant \text{const.}\, r_n$$

for a certain sequence of r_n tending to ∞.

Combining our three estimates, we get

$$\int_{-\pi}^{\pi} (H(r_n e^{i\vartheta}))^+ d\vartheta \leqslant \text{const.}\, r_n,$$

and from this we can deduce as in the proof just referred to that

$$H(z) = H(0) + A\Re z + B\Im z,$$

thus verifying the above claim.

For $x \in \mathbb{R}$, the last relation reduces to $\pi F(x) + G(x) = H(0) + Ax$. Here, $F(x)$ and $G(x)$ are both even, so $A = 0$, and

$$\pi F(x) = H(0) - \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\tilde{F}(t).$$

The integral on the right vanishes for $x = 0$, so $H(0) = \pi F(0)$, and finally

$$F(x) = F(0) - \frac{1}{\pi}\int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\tilde{F}(t)$$

for $x \in \mathbb{R}$, as required.

Lemma. *Let $F(x)$ be as in the preceding lemma, and suppose that for a certain continuous and increasing odd function $\mu(x)$, $O(x)$ on $[0, \infty)$, the sum $\tilde{F}(x) + \mu(x)$ is also increasing. Then $|\tilde{F}(x)/x|$ is bounded on $\mathbb{R}$.*

Proof. Is for the most part nothing but a crude version of the argument made in §H.2 of Chapter III.

It is really only the boundedness of $|\tilde{F}(x)/x|$ for *large* $x > 0$ that requires proof. That's because our assumptions on F make $\tilde{F}(x)$ odd and $\mathscr{C}_1$, and hence $\tilde{F}(x)/x$ even, and bounded near 0.

In order to see what happens for large values of x, we resort to Kolmogorov's theorem from Chapter III, §H.1, according to which

$$\int_{|\tilde{F}(x)|>\lambda} \frac{dx}{1+x^2} \leqslant \frac{K}{\lambda} \qquad \text{for } \lambda > 0,$$

K being a certain constant depending on F. In this relation, put $\lambda = 5K\cdot 2^n$ with $n \geqslant 1$; then, since

$$\int_{2^n}^{2^{n+1}} \frac{dx}{1+x^2} > \frac{1}{5\cdot 2^n},$$

there must, in each open interval $(2^n, 2^{n+1})$, be a point x_n with

$$|\tilde{F}(x_n)| \leqslant 5K\cdot 2^n.$$

By hypothesis, the functions $\mu(x)$ and $\tilde{F}(x) + \mu(x)$ are increasing, so for $x_n \leqslant x \leqslant x_{n+1}$ we have

$$-5K\cdot 2^n + \mu(2^n) \leqslant \tilde{F}(x_n) + \mu(x_n) \leqslant \tilde{F}(x) + \mu(x)$$
$$\leqslant \tilde{F}(x_{n+1}) + \mu(x_{n+1}) \leqslant 5K\cdot 2^{n+1} + \mu(2^{n+2}),$$

whence

$$-5K\cdot 2^n - \mu(x) \leqslant \tilde{F}(x) \leqslant 5K\cdot 2^{n+1} + \mu(2^{n+2}),$$

from which

$$-5K - \frac{\mu(x)}{x} \leqslant \frac{\tilde{F}(x)}{x} \leqslant 10K + 4\,\frac{\mu(2^{n+2})}{2^{n+2}}$$

in view of the relation $2^n < x_n < x_{n+1} < 2^{n+2}$.

It was also given that $\mu(t) \leqslant Ct$ on $[0, \infty)$. Thence,

$$\left|\frac{\tilde{F}(x)}{x}\right| \leqslant 10K + 4C$$

for $x_n \leqslant x \leqslant x_{n+1}$, and thus finally for all $x \geqslant x_1$.

Done.

We will need, finally, a simple extension of the Jensen formula for confocal ellipses derived in §C of Chapter IX.

Lemma. *Let $F(z)$ be subharmonic in and on a simply connected closed region $\bar{\Omega}$ containing the ellipse*

$$z = \frac{1}{2}\left(Re^{i\vartheta} + \frac{e^{-i\vartheta}}{R}\right), \qquad 0 \leqslant \vartheta \leqslant 2\pi,$$

in its interior, where $R > 1$. Suppose that μ is the positive measure on $\bar{\Omega}$ figuring in the Riesz representation of the superharmonic function $-F(z)$ in Ω, the interior of $\bar{\Omega}$, in other words, that

$$F(z) = \int_{\bar{\Omega}} \log|z-\zeta|\,d\mu(\zeta) + h(z) \qquad \text{for } z \in \Omega,$$

where $h(z)$ is harmonic *in Ω (see problem* 49, §A.2*). If, then, $M(r)$ denotes, for $1 < r \leqslant R$, the* total mass *μ has* inside or on the ellipse

$$z = \frac{1}{2}\left(re^{i\vartheta} + \frac{e^{-i\vartheta}}{r}\right),$$

we have

$$\int_1^R \frac{M(r)}{r}\,dr = \frac{1}{2\pi}\int_{-\pi}^{\pi} F\left(\frac{1}{2}\left(Re^{i\vartheta} + \frac{e^{-i\vartheta}}{R}\right)\right)d\vartheta - \frac{1}{\pi}\int_{-1}^{1} \frac{F(x)}{\sqrt{(1-x^2)}}\,dx.$$

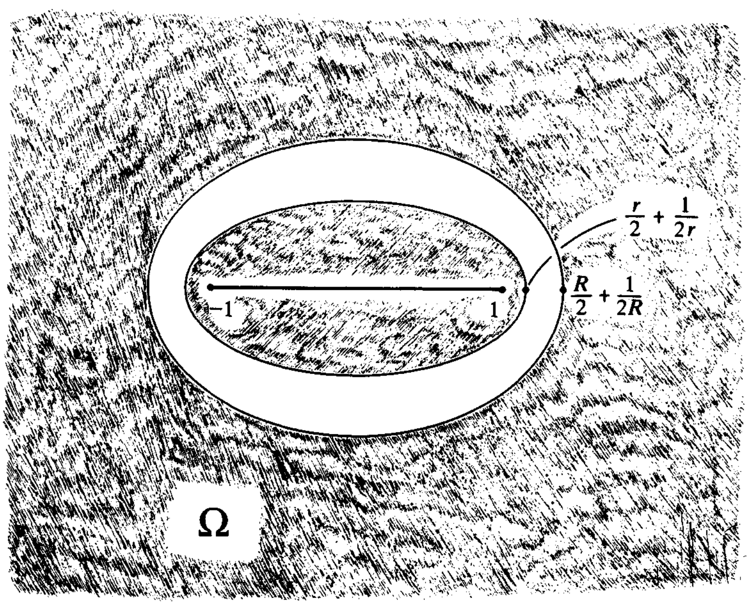

Figure 259

Proof. It is simplest to just *derive* this result from the theorem of Chapter IX, §C by double integration.

Fix, for the moment, any point $\zeta \in \bar{\Omega}$ with

$$\zeta \quad = \quad \frac{1}{2}\left(\rho e^{i\varphi} + \frac{e^{-i\varphi}}{\rho}\right)$$

where $\rho \geqslant 1$, and observe that (in case $\rho > 1$), we have

$$\rho \quad = \quad |\zeta + \sqrt{(\zeta^2 - 1)}|,$$

taking the proper determination of the square root for ζ outside the segment $[-1, 1]$.

Apply now the theorem referred to with the *analytic* function of z *equal to* $z - \zeta$ (!), getting

$$\int_{|\zeta+\sqrt{(\zeta^2-1)}|}^{R} \frac{dr}{r} \quad = \quad \int_{\rho}^{R} \frac{dr}{r} \quad = \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|\frac{1}{2}\left(Re^{i\vartheta} + \frac{e^{-i\vartheta}}{R}\right) - \zeta\right| d\vartheta$$
$$- \quad \frac{1}{\pi}\int_{-1}^{1} \frac{\log|x-\zeta|}{\sqrt{(1-x^2)}}\,dx,$$

the very first integral on the left being understood as zero for

$$|\zeta + \sqrt{(\zeta^2-1)}| \quad \geqslant \quad R.$$

Multiply the last relation by $d\mu(\zeta)$ and integrate over $\bar{\Omega}$. On the *left* we will get

$$\int_{\bar{\Omega}}\int_{|\zeta+\sqrt{(\zeta^2-1)}|}^{R} \frac{dr}{r}\,d\mu(\zeta) \quad = \quad \int_1^R \int_{1\leqslant|\zeta+\sqrt{(\zeta^2-1)}|\leqslant r} d\mu(\zeta)\,\frac{dr}{r} \quad = \quad \int_1^R \frac{M(r)}{r}\,dr,$$

and on the *right*, after changing the order of integration in both integrals,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi\left(\frac{1}{2}\left(Re^{i\vartheta} + \frac{e^{-i\vartheta}}{R}\right)\right) d\vartheta \quad - \quad \frac{1}{\pi}\int_{-1}^{1} \frac{\Phi(x)}{\sqrt{(1-x^2)}}\,dx,$$

where

$$\Phi(z) \quad = \quad \int_{\bar{\Omega}} \log|z-\zeta|\,d\mu(\zeta).$$

Our given subharmonic function $F(z)$ is equal to $\Phi(z) + h(z)$. Since $h(z)$ is harmonic in the *simply connected* region Ω, it has a harmonic conjugate $\tilde{h}(z)$ there, and the function

$$f(z) \quad = \quad \exp(h(z) + i\tilde{h}(z))$$

is *analytic and without zeros in* Ω. Apply the theorem of §C, Chapter IX, once more, this time to $f(z)$. Because $\log|f(z)| = h(z)$, we get

$$0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} h\left(\frac{1}{2}\left(Re^{i\vartheta} + \frac{e^{-i\vartheta}}{R}\right)\right)d\vartheta - \frac{1}{\pi}\int_{-1}^{1} \frac{h(x)}{\sqrt{(1-x^2)}}dx.$$

Adding the right side of this relation to the previous similar expression involving Φ equal, as we have seen, to

$$\int_1^R \frac{M(r)}{r}dr$$

now gives us the desired result.

Done.

2. Proof of the conjecture from §D.5

This book is coming to an end. Let us get on with the establishment of our conjecture which, after rephrasing, reads a bit more smoothly. One can, of course, write

$$\max(\tilde{\omega}'(x),\ K) - K$$

as

$$(\tilde{\omega}'(x) - K)^+.$$

Then the result we have in mind may be stated as follows:

Theorem. *Let $W(x) \geqslant 1$ be a weight meeting the local regularity requirement of* §B.1. *A necessary and sufficient condition for W to admit multipliers is that there exist an even $\mathscr{C}_\infty$ function $\omega(x)$, defined on $\mathbb{R}$, with*

$$\log W(x) \leqslant \omega(x)$$

there, such that

(i) $$\int_{-\infty}^{\infty} \frac{\omega(x)}{1+x^2}dx < \infty;$$

(ii) *the ($\mathscr{C}_\infty$) Hilbert transform $\tilde{\omega}(x)$ of ω has the following property:*

to any $\delta > 0$ there corresponds a K with

$$\tilde{\omega}'(x) \leqslant K$$

everywhere in $(0,\infty)$ outside a set of disjoint intervals $(a_n,\ b_n)$, $a_n > 0$, such that

$$\sum_n \left(\frac{b_n - a_n}{a_n}\right)^2 \quad < \quad \infty,$$

for each of which

$$\int_{a_n}^{b_n} (\tilde{\omega}'(x) \;-\; K)^+ \, \mathrm{d}x \quad \leqslant \quad \delta(b_n - a_n).$$

Remark. For any $\mathscr{C}_\infty$ function $\omega(x) \geqslant \log W(x)$ (and thus $\geqslant 0$) satisfying (i), we have

$$\tilde{\omega}'(x) \quad = \quad \frac{1}{\pi}\int_0^\infty \frac{2\omega(x) - \omega(x+t) - \omega(x-t)}{t^2}\,\mathrm{d}t.$$

This is practically immediate, for then the functions

$$\omega(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^\infty \frac{\Im z\;\omega(t)}{|z-t|^2}\,\mathrm{d}t,$$

$$\tilde{\omega}(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^\infty \left(\frac{\Re z - t}{|z-t|^2} \;+\; \frac{t}{t^2+1}\right)\omega(t)\,\mathrm{d}t,$$

both harmonic in $\Im z \;>\; 0$, have there partial derivatives *continuous up to* $\mathbb{R}$ (in the present circumstances). Thence, by the Cauchy–Riemann equations,

$$\tilde{\omega}'(x) \quad = \quad \lim_{y\to 0+} \frac{\partial\tilde{\omega}(x+\mathrm{i}y)}{\partial x} \quad = \quad -\lim_{y\to 0+} \frac{\partial\omega(x+\mathrm{i}y)}{\partial y}$$

$$= \quad \lim_{y\to 0+} \frac{\omega(x) - \omega(x+\mathrm{i}y)}{y},$$

and the last limit is clearly equal to the integral in question.

Proof of theorem

1^0 The necessity. As we saw at the very beginning of §C, there is no loss of generality incurred in taking

$$W(x) \;\longrightarrow \infty \quad \text{for } x \longrightarrow \pm\infty;$$

this property we henceforth assume.

If $W(x)$ *admits multipliers* there is, corresponding to *any* $a > 0$, a non-zero entire function $f(z)$ of exponential type $\leqslant\; a$ with

$$W(x)|f(x)| \;\leqslant\; \text{const.} \quad \text{for } x \in \mathbb{R}.$$

If peradventure $f(0) = 0$, the quotient $f(x)/x$ satisfies the same kind of relation, for $W(x)$ must be bounded on finite intervals by the first lemma of §B.1. We can thus first divide $f(z)$ by whatever power of z is needed to get an entire function different from zero at the origin, and may therefore just as well *assume to begin with that*

$$f(0) = 1.$$

The *even* entire function $g(z) = f(z)f(-z)$ is then also 1 at the origin. It has exponential type $\leqslant 2a$, and we have

$$W(x)\,W(-x)\,|g(x)| \leqslant \text{const.}, \qquad x \in \mathbb{R}.$$

We know by the discussion following the first theorem in §B.1 that $g(z)$ may be taken to have *all its zeros real*, thus being of the form

$$g(z) = \prod_1^\infty \left(1 - \frac{z^2}{\lambda_k^2}\right),$$

where the λ_k are certain numbers > 0. The preceding relation then reads

$$\log W(x) + \log W(-x) + \sum_1^\infty \log\left|1 - \frac{x^2}{\lambda_k^2}\right| \leqslant \text{const.}, \qquad x \in \mathbb{R};$$

it is from the expressions

$$\log|g(x)| = \sum_1^\infty \log\left|1 - \frac{x^2}{\lambda_k^2}\right|$$

corresponding to entire functions g having smaller and smaller exponential type that we will construct a function $\omega(x)$ having the properties affirmed by our theorem.

Take, then, a sequence of entire functions $g_n(z)$, each of the form just indicated, such that

$$\log W(x) + \log W(-x) + \log|g_n(x)| \leqslant \Gamma_n, \qquad \text{say,}$$

for $x \in \mathbb{R}$, while the exponential type α_n of g_n tends monotonically to zero as $n \longrightarrow \infty$. By passing to a subsequence if necessary, we arrange matters so as to also have

$$\sum_n \alpha_n < \infty.$$

Denoting by $\mu_n(t)$ the *number of zeros of* $g_n(z)$ on the segment $[0, t]$

(which makes $\mu_n(t) = O(t)$ on $[0, \infty)$ for each n), we have

$$\log W(x) + \log W(-x) + \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\mu_n(t) \leqslant \Gamma_n, \qquad x \in \mathbb{R}.$$

Here, $\log W(-x) \longrightarrow \infty$ for $x \longrightarrow \pm\infty$, so for each n there is a number $A_n > 0$ with

$$\log W(-x) \geqslant \Gamma_n \qquad \text{for } |x| \geqslant A_n.$$

It follows then from the last relation that

$$\log W(x) + \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\mu_n(t) \leqslant 0 \qquad \text{for } |x| \geqslant A_n.$$

We can evidently choose the A_n successively so as to have

$$A_{n+1} > 2A_n;$$

this property will be assumed to hold from now on.

For each n, let

$$\nu_n(t) = \begin{cases} 0, & 0 \leqslant t < A_n/\sqrt{2}, \\ \mu_n(t), & t \geqslant A_n/\sqrt{2}; \end{cases}$$

like the $\mu_n(t)$, the $\nu_n(t)$ are each *increasing* and $O(t)$ on $[0, \infty)$. We then put

$$F_n(x) = -\int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\nu_n(t) \qquad (\textit{sic}\,!),$$

and claim in the first place that

$$F_n(x) \geqslant 0 \qquad \text{for } x \in \mathbb{R}.$$

This is true when $|x| \leqslant A_n$, for then $|1 - (x^2/t^2)| \leqslant 1$ for the values of t (all $\geqslant A_n/\sqrt{2}$) that are actually involved in the preceding integral. For $|x| > A_n$, use the evident formula

$$F_n(x) = \int_0^{A_n/\sqrt{2}} \left\{\log\left(\frac{x^2}{t^2} - 1\right) - \log\left(\frac{2x^2}{A_n^2} - 1\right)\right\} d\mu_n(t)$$
$$- \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\mu_n(t).$$

The *first* integral on the right is here clearly $\geqslant 0$, and the *second* $\geqslant \log W(x) \geqslant 0$ by the above inequality. This establishes the claim, and shows besides that

$$F_n(x) \geqslant \log W(x) \qquad \text{for } |x| \geqslant A_n.$$

In order to get the function $\omega(x)$, we first smooth out each of the $F_n(x)$ by multiplicative convolution, relying on the fulfillment by $W(x)$ of the local regularity requirement*. According to the latter, there are three constants C, L and $k \geqslant 0$, the first two > 0, such that, for any $x \in \mathbb{R}$,

$$\log W(x) \leqslant C \log W(t) + k$$

for the t belonging to a certain interval of length L containing the point x.

Choose, for each n, a small number $\eta_n > 0$ less than both of the quantities

$$\frac{A_n}{2A_{n+1}} \quad \text{and} \quad \frac{L}{4A_{n+1}};$$

it is convenient to also have the η_n tend monotonically towards 0 as $n \longrightarrow \infty$. Take then a sequence of *infinitely differentiable functions* $\rho_n(\xi) \geqslant 0$ with ρ_n supported on the interval $[1-\eta_n,\ 1+\eta_n]$, such that

$$\int_{1-\eta_n}^{1} \frac{\rho_n(\xi)}{\xi}\,d\xi = \int_{1}^{1+\eta_n} \frac{\rho_n(\xi)}{\xi}\,d\xi = 1.$$

When

$$0 \leqslant x \leqslant \frac{L}{2\eta_n},$$

the points ξx with $1-\eta_n \leqslant \xi \leqslant 1$ are included in the segment $[x - L/2,\ x]$ and the ones with $1 \leqslant \xi \leqslant 1+\eta_n$ in the segment $[x,\ x + L/2]$. One of those segments surely lies in the interval of length L containing x on which the preceding relation involving $\log W(t)$ does hold. By that relation and the specifications for ρ_n we thus have

$$\log W(x) \leqslant C\int_0^{\infty} \log W(\xi x)\,\frac{\rho_n(\xi)}{\xi}\,d\xi + k \qquad \text{for } 0 \leqslant x \leqslant \frac{L}{2\eta_n}.$$

If, however, x is also $\geqslant 2A_n$, $\log W(\xi x)$ will, by the inequality found above, be $\leqslant F_n(\xi x)$ for $1-\eta_n \leqslant \xi \leqslant 1+\eta_n$, since then

* This is not really needed here. See next footnote.

$\xi x \geqslant 2A_n - 2A_{n+1}\eta_n \geqslant A_n$. The *right side* of the last relation is therefore

$$\leqslant \quad C\int_0^\infty F_n(\xi x)\,\frac{\rho_n(\xi)}{\xi}\,\mathrm{d}\xi \quad + \quad k$$

for such x. This expression is hence $\geqslant \log W(x)$ for $2A_n \leqslant x \leqslant 2A_{n+1}$ because $2A_{n+1} \leqslant L/2\eta_n$. Exactly the same reasoning* can be used for negative real x, and we have

$$\log W(x) \quad \leqslant \quad C\int_0^\infty F_n(\xi x)\,\frac{\rho_n(\xi)}{\xi}\,\mathrm{d}\xi \quad + \quad k \qquad \text{for } 2A_n \leqslant |x| \leqslant 2A_{n+1}.$$

Put now

$$G_n(x) \quad = \quad 2C\int_0^\infty F_n(\xi x)\,\frac{\rho_n(\xi)}{\xi}\,\mathrm{d}\xi.$$

We have $G_n(0) = 0$, and

$$G_n(x) \geqslant 0, \qquad x \in \mathbb{R}.$$

Also, since $W(x) \longrightarrow \infty$ for $x \longrightarrow \pm\infty$, the preceding relation implies that

$$\log W(x) \leqslant G_n(x) \qquad \text{for } 2A_n \leqslant |x| \leqslant 2A_{n+1}$$

as long as n exceeds a certain number n_0. We can, of course, arrange to have this inequality hold for *all* n by simply *throwing away* the G_n and A_n with $n \leqslant n_0$ and re-indexing the *remaining* ones. *This we henceforth suppose done.*

By the first lemma of article 1,

$$G_n(x) \quad = \quad -\int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right|\mathrm{d}\sigma_n(t)$$

with an increasing function $\sigma_n(t)$, *infinitely differentiable* in $(0,\infty)$, given by the formula

$$\sigma_n(t) \quad = \quad 2C\int_0^\infty \nu_n(\xi t)\,\frac{\rho_n(\xi)}{\xi}\,\mathrm{d}\xi.$$

* As long as $\log W(x)$ is *uniformly continuous* on $[A_n, 3A_{n+1}]$, say, the argument just made will go through (with $C = 1$) for small enough η_n. The necessity proof now under way *therefore works whenever* $W(x) \geqslant 1$ *is continuous on* $\mathbb{R}$. For this, we do not even *need* the entire functions $g_n(z)$ but *only* a sequence of (not necessarily integer-valued) functions $\mu_n(t)$ increasing on $[0,\infty]$ and $O(t)$ there, with $\limsup_{t\to\infty}(\mu_n(t)/t) = \alpha_n/\pi$ going to zero as $n \longrightarrow \infty$ and $\log W(x) + \log W(-x) + \int_0^\infty \log|1 - (x^2/t^2)|\mathrm{d}\mu_n(t)$ bounded above on $\mathbb{R}$ for each n.

In the present circumstances,

$$\nu_n(t) = 0 \qquad \text{for } 0 \leqslant t < A_n/\sqrt{2},$$

so

$$\sigma_n(t) = 0 \qquad \text{for } 0 \leqslant t \leqslant A_n/(\sqrt{2}\,(1+\eta_n)),$$

ρ_n being supported on $[1-\eta_n,\ 1+\eta_n]$. Therefore, if we extend $\sigma_n(t)$ to *the whole real axis* by making it *odd*, we get a function which is actually $\mathscr{C}_\infty$ on $\mathbb{R}$. It follows also by the lemma referred to that $\sigma_n'(t)$ is, for each n, *bounded in* $(0, \infty)$, indeed, *bounded on* $\mathbb{R}$ after $\sigma_n(t)$ is extended in the way just mentioned.

For our function $\omega(x)$ we will take the *sum of the* $G_n(x)$ and an additive constant. In order to verify that that function enjoys the properties it should, we shall need some bounds on the $G_n(x)$ and the $\sigma_n(t)$. To obtain those bounds, we must go back and look again at the entire functions $g_n(z)$ of exponential type α_n with which we started.

Because $W(x) \geqslant 1$, we certainly have

$$|g_n(x)| \leqslant e^{\Gamma_n} \qquad \text{for } x \in \mathbb{R},$$

and Levinson's theorem from §H.2 of Chapter III can be applied to the g_n to yield

$$\frac{\mu_n(t)}{t} \longrightarrow \frac{\alpha_n}{\pi}, \qquad t \longrightarrow \infty.$$

(One could in fact make do with a less elaborate result here.) Hence, since $\nu_n(t) = \mu_n(t)$ for $t \geqslant A_n/\sqrt{2}$,

$$\frac{\nu_n(t)}{t} \longrightarrow \frac{\alpha_n}{\pi} \qquad \text{for } t \longrightarrow \infty$$

and thence, by definition of σ_n,

$$\limsup_{t\to\infty} \frac{\sigma_n(t)}{t} \leqslant \frac{4}{\pi} C(1+\eta_n)\alpha_n,$$

in view of ρ_n's vanishing outside of $[1-\eta_n,\ 1+\eta_n]$ and the condition that

$$\int_0^\infty \frac{\rho_n(\xi)}{\xi}\,d\xi = 2.$$

Let us now extend the definition of our function G_n to the whole complex plane by taking

$$G_n(z) \quad = \quad -\int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| d\sigma_n(t);$$

this function is harmonic both in $\{\Im z > 0\}$ and in $\{\Im z < 0\}$, and, thanks to the smoothness of $\sigma_n(t)$, *continuous* up to the real axis. The previous relation for $\sigma_n(t)/t$ makes

$$G_n(z) \quad \geqslant \quad -4C(1+\eta_n)\alpha_n|z| \; - \; o(|z|)$$

for large $|z|$, so, putting

$$\beta_n \quad = \quad \limsup_{y\to\infty}\left(\frac{-G_n(iy)}{y}\right),$$

we have

$$\beta_n \quad \leqslant \quad 4C(1+\eta_n)\alpha_n.$$

As we know, $G_n(x) \geqslant 0$ on the real axis. This, together with the other properties of $G_n(z)$, ensures that an analogue of the representation from §G.1 of Chapter III is valid for that function, viz.,

$$G_n(z) \quad = \quad -\beta_n \Im z \quad + \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z\, G_n(t)}{|z-t|^2}\,dt, \qquad \Im z \; > \; 0.$$

(Cf. proof of the first lemma in §B.3. The more elaborate argument made during the proof of the second theorem in §B.1 is not needed here.)

According to the above observations about $\sigma_n(t)$, that function is certainly *zero* in a neighborhood of the origin. That, however, makes

$$G_n(z) \; = \; O(|z|^2) \qquad \text{for } z \longrightarrow 0,$$

whence

$$\frac{G_n(iy)}{y} \longrightarrow 0 \qquad \text{as } y \longrightarrow 0.$$

Referring to the preceding Poisson representation, we see from this that

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{G_n(t)}{t^2}\,dt \quad = \quad \beta_n$$

for the *positive* function $G_n(t)$.

It is now easy to get a *uniform* bound on $\sigma_n(t)/t$. Since $G_n(x) \geqslant 0$ on the real axis and $G_n(z) = G_n(\bar{z})$, the above Poisson representation tells us in particular that $G_n(z) \geqslant -\beta_n|\Im z|$, in other words, that

$$\int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}\sigma_n(t) \quad \leqslant \quad \beta_n|\Im z|.$$

Using this relation we can now show by a computation like the one involved in the proof of the *second* lemma from §C.5, Chapter VIII ($\sigma_n(t)$ being increasing) that

$$\frac{\sigma_n(t)}{t} \quad \leqslant \quad \frac{\mathrm{e}\beta_n}{\pi} \qquad \text{for } t > 0.$$

We proceed to the construction and examination of our function $\omega(x)$. By the first lemma of §B.1, $\log W(x)$ is *bounded above* for $|x| \leqslant 2A_1$; we *fix* any such bound and use it as our value for $\omega(0)$. Then we put

$$\omega(x) \quad = \quad \omega(0) \; + \; \sum_1^\infty G_n(x);$$

because the $G_n(x)$ are $\geqslant 0$ on $\mathbb{R}$ with $\log W(x) \leqslant G_n(x)$ for $2A_n \leqslant |x| \leqslant 2A_{n+1}$, and $\log W(x) \leqslant \omega(0)$ for $|x| \leqslant 2A_1$, we certainly have

$$\log W(x) \;\leqslant\; \omega(x), \qquad x \in \mathbb{R}.$$

The function $\omega(x)$ is clearly even. Since $\beta_n \leqslant 4C(1+\eta_n)\alpha_n$ with $\eta_n \longrightarrow 0$ for $n \longrightarrow \infty$ and the sum of the α_n convergent, we have

$$\sum_1^\infty \beta_n \;<\; \infty,$$

whence, by the previous estimate of the integrals of the $G_n(t)/t^2$ and monotone convergence,

$$\int_{-\infty}^\infty \frac{\omega(x)}{1+x^2}\mathrm{d}x \quad < \quad \infty.$$

Property (i) *thus holds for* ω.

To verify *infinite differentiability* for $\omega(x)$, we take

$$\sigma(t) \quad = \quad \sum_1^\infty \sigma_n(t)$$

and look at the increasing function $\sigma(t)$. The series on the right is surely convergent; we have indeed

$$\frac{\sigma(t)}{t} \quad \leqslant \quad \frac{\mathrm{e}}{\pi}\sum_1^\infty \beta_n,$$

a finite quantity, for $t > 0$, thanks to the uniform bounds on the ratios $\sigma_n(t)/t$ found above. The function $\sigma(t)$ is actually *zero* for $0 \leqslant t \leqslant A_1/(\sqrt{2}\,(1+\eta_1))$, since all of the $\sigma_n(t)$ are (here is where we use the property that the η_n *decrease*). The summand $\sigma_n(t)$ is moreover different from zero only when $|t| \geqslant A_n/(\sqrt{2}\,(1+\eta_n))$, *a large number for large n.* Hence, since for any *given* real x, $\log|1-(x^2/t^2)|$ is $\leqslant 0$ for $t \geqslant \sqrt{2}\,|x|$, we see – again by monotone convergence – that

$$\sum_1^\infty G_n(x) \quad = \quad -\int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\sigma(t), \qquad x \in \mathbb{R}.$$

Therefore

$$\omega(x) \quad = \quad \omega(0) - \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\sigma(t).$$

Here, however, $\sigma(t)$ is infinitely differentiable on $\mathbb{R}$ and odd there; this is so because each individual $\sigma_n(t)$ is odd and $\mathscr{C}_\infty$ as noted above, and, on any given finite interval, *only finitely many* of the $\sigma_n(t)$ can be different from zero. We may therefore conclude that $\omega(x)$ is also $\mathscr{C}_\infty$ by invoking the result proved in the footnote to the theorem of §D.3.

Verification of property (ii) for the function $\omega(x)$ remains; that is more involved. What we have to do is look at the *size* of $\tilde{\omega}'(x)$ for $x > 0$.

We have

$$\omega(0) - \omega(x) \quad = \quad \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\sigma(t).$$

The ratio $\sigma(t)/t$ is bounded, and property (i) holds for ω. Therefore the *second* lemma from article 1 applies here, and tells us that

$$\tilde{\omega}(x) \quad = \quad \pi\sigma(x) \quad = \quad \pi\sum_1^\infty \sigma_n(x).$$

Thus,

$$\tilde{\omega}'(x) \quad = \quad \pi\sum_1^\infty \sigma_n'(x)$$

(with, on any *bounded* interval, only *finitely many terms* actually appearing in the sum on the right).

Let $\delta > 0$ be given. Corresponding to it, we have an N such that

$$\mathrm{e}\sum_N^\infty \beta_n \;<\; \delta;$$

fixing such an N, we form the increasing function

$$s(t) \quad = \quad \sum_N^\infty \sigma_n(t)$$

and investigate its behaviour for $t \geqslant 0$.

That function is, in the first place, *zero* for $0 \;\leqslant\; t \;\leqslant\; A_N/(\sqrt{2}\,(1+\eta_N))$. Also,

$$\frac{s(t)}{t} \quad \leqslant \quad \frac{\mathrm{e}}{\pi}\sum_N^\infty \beta_n \quad < \quad \frac{\delta}{\pi} \qquad \text{for } t > 0$$

by the above uniform estimate on the ratios $\sigma_n(t)/t$. Therefore, if $x > 0$ is *sufficiently small*, the ratio

$$\frac{s(t) \;-\; s(x)}{t \;-\; x}$$

will be *zero* for $0 \;\leqslant\; t \;\leqslant\; A_N/(\sqrt{2}\,(1+\eta_N))$, and, for *larger* values of t,

$$\leqslant \quad \frac{A_N}{A_N \;-\; \sqrt{2}\,(1+\eta_N)x}\,\frac{s(t)}{t} \quad \leqslant \quad \frac{A_N}{A_N \;-\; \sqrt{2}\,(1+\eta_N)x}\cdot\frac{\mathrm{e}}{\pi}\sum_N^\infty \beta_n,$$

and hence $<\; \delta/\pi$. Doing, then, the F. Riesz construction on the graph of $s(x)$ vs. x for $x \geqslant 0$, and forming the open set

$$\mathcal{O} \quad = \quad \left\{x > 0\colon\; \frac{s(t) - s(x)}{t - x} \quad > \quad \frac{\delta}{\pi} \;\;\text{for } some \; t > x\right\},$$

we see that $\mathcal{O}$ *can contain no points to the left of a certain* $a_0 \;>\; 0$.

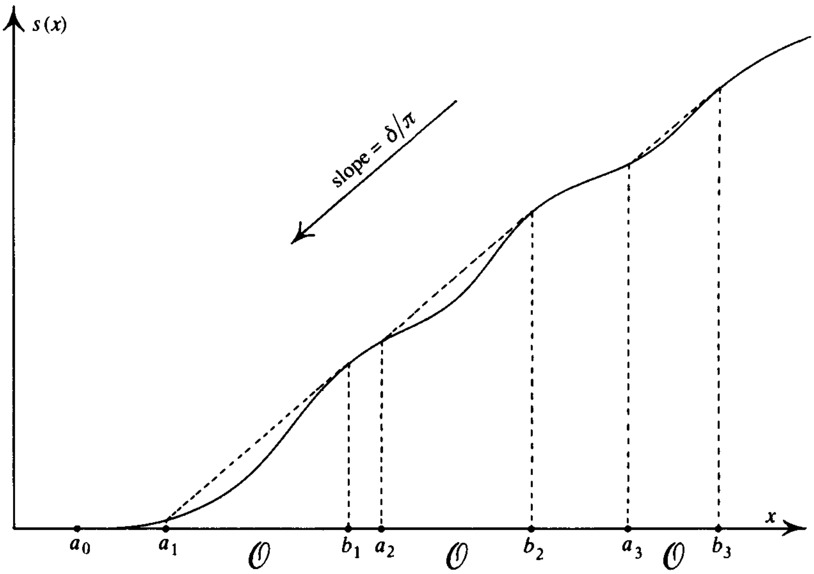

Figure 260

The set $\mathcal{O}$ is thus the union of a certain disjoint collection of intervals (a_k, b_k), $k = 1, 2, 3, \ldots$, with

$$a_k \geqslant a_0 > 0$$

for every k; these, of course, may be disposed in a much more complicated fashion than is shown in the diagram, there being no *a priori* lower limit to their lengths. *Every point $x > 0$ for which $s'(x) > \delta/\pi$ certainly belongs to $\mathcal{O}$*, so

$$s'(x) \leqslant \frac{\delta}{\pi} \qquad \text{for } x \in [0, \infty) \sim \mathcal{O}.$$

For each $k \geqslant 1$,

$$\frac{s(b_k) - s(a_k)}{b_k - a_k} = \frac{\delta}{\pi}$$

as is clear from the figure.

It is now claimed that

$$\sum_1^\infty \left(\frac{b_k - a_k}{a_k}\right)^2 < \infty;$$

this we will show by an argument essentially the same as the one made in §D.1 of Chapter IX, using, however, the fifth lemma from the preceding article in place of the theorem of §C in Chapter IX. We work with the *subharmonic* function

$$U(z) \quad = \quad -\sum_N^\infty G_n(z) \quad = \quad \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}s(t);$$

the two right-hand expressions are proved equal by what amounts to the reasoning used above in checking the analogous formula for $\omega(x)$ in terms of σ (monotone convergence).

Since the $G_n(x)$ are all $\geqslant 0$ on $\mathbb{R}$, we have

$$U(x) \quad \geqslant \quad \omega(0) - \omega(x), \qquad x \in \mathbb{R},$$

so, because ω has property (i),

$$\int_{-\infty}^{\infty} \frac{U(x)}{1+x^2} \mathrm{d}x \quad > \quad -\infty.$$

Writing

$$\beta \quad = \quad \sum_N^\infty \beta_n,$$

and recalling that $G_n(z) \geqslant -\beta_n|\Im z|$, we see, moreover, that

$$U(z) \quad \leqslant \quad \beta|\Im z|.$$

The convergence of the series $\sum_k((b_k - a_k)/a_k)^2$ will be deduced from the last two inequalities involving U and the fact that $\beta < \delta/\mathrm{e}$ due to our choice of N. (It would in fact be enough if we merely had $\beta < \delta$; our having been somewhat crude in the estimation of the $\sigma_n(t)/t$ has required us to work with an extra margin of safety expressed by the factor $1/\mathrm{e}$.)

Fixing our attention on any *particular* interval (a_k, b_k), let us denote its *midpoint* by c and its *length* by 2Δ, so as to have

$$(a_k, b_k) \quad = \quad (c - \Delta, c + \Delta).$$

The following discussion, corresponding to the one in §D.1 of Chapter IX, is actually quite simple; it may, however, at first appear complicated because of the changes of variable involved in it.

We take a certain quantity $R > 1$ (the same, in fact, for each of the intervals (a_k, b_k) – its exact *size* will be specified presently) and then, choosing a value for the parameter l,

$$\frac{2\Delta}{R + \dfrac{1}{R}} < l \leqslant \Delta,$$

apply the fifth lemma of article 1 to the subharmonic function

$$F(z) = U(lz + c).$$

Fix, for the moment, any number A large enough to ensure that

$$lA \gg \Delta,$$

and let us look at the Riesz representation for $F(z)$ (obtained by putting a minus sign in front of the one for the *superharmonic* function $-F(z)$!) in the disk $\{|z| < A\}$. In terms of the variable $w = lz + c$, we have this picture:

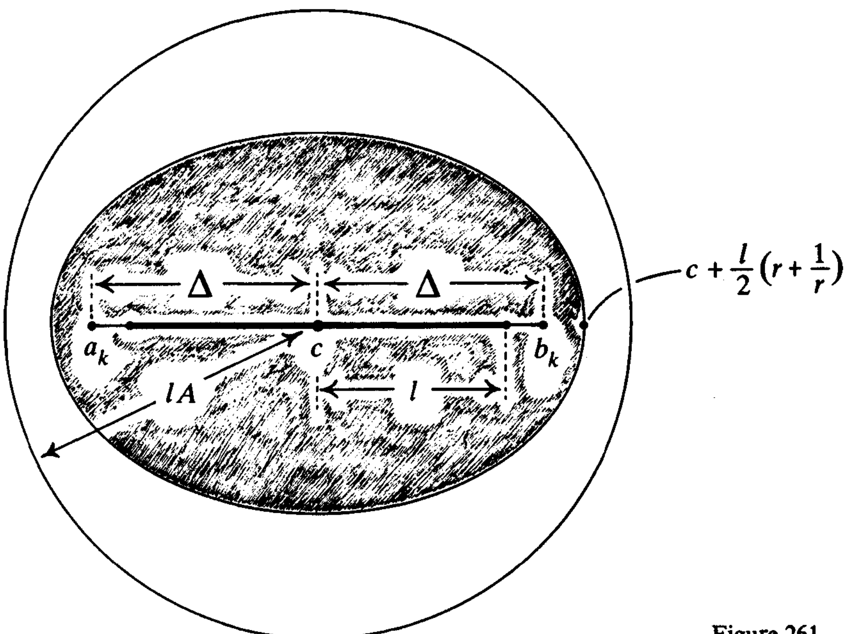

Figure 261

Because

$$U(w) = \int_0^\infty \log\left|1 - \frac{w^2}{t^2}\right| \mathrm{d}s(t),$$

we can, after making the change of variable $t = l\tau + c$, write

$$F(z) = U(lz+c) = \int_{-A}^{A} \log|z-\tau|\,\mathrm{d}s(l\tau+c) + h(z),$$

with $h(z)$ a certain function *harmonic for* $|z| < A$. If, then, $r > 1$ and $\frac{1}{2}(r + 1/r) < A$, the *closed region* of the z-plane *bounded by the ellipse*

$$z = \frac{1}{2}\left(re^{i\vartheta} + \frac{e^{-i\vartheta}}{r}\right)$$

(whose image in the w-plane is shown in the above figure) *has mass* $M(r)$ *equal to*

$$\int_{-\frac{1}{2}(r+1/r)}^{\frac{1}{2}(r+1/r)} \mathrm{d}s(l\tau+c) = s\left(c + \frac{l}{2}\left(r+\frac{1}{r}\right)\right) - s\left(c - \frac{l}{2}\left(r+\frac{1}{r}\right)\right)$$

assigned to it by the measure associated with the Riesz representation for $F(z)$ just given. By the fifth lemma of the last article we thus have

$$\int_1^R \frac{s(c + \frac{1}{2}l(r+r^{-1})) - s(c - \frac{1}{2}l(r+r^{-1}))}{r}\,\mathrm{d}r = \int_1^R \frac{M(r)}{r}\mathrm{d}r$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} U\left(c + \frac{l}{2}\left(Re^{i\vartheta}+\frac{e^{-i\vartheta}}{R}\right)\right)\mathrm{d}\vartheta - \frac{1}{\pi}\int_{-1}^{1} \frac{U(lx+c)}{\sqrt{(1-x^2)}}\,\mathrm{d}x.$$

As in §D.1 of Chapter IX, it is convenient to now write

$$R = e^{\gamma}$$

(thus making γ a certain fixed quantity > 0), and to take a number $\varepsilon > 0$, considerably *smaller* than γ (corresponding to the quantity denoted by η in the passage referred to). If the parameter l is actually

$$\geqslant \frac{\Delta}{\cosh\varepsilon},$$

we will have

$$s\left(c + \frac{l}{2}\left(r+\frac{1}{r}\right)\right) - s\left(c - \frac{l}{2}\left(r+\frac{1}{r}\right)\right) \geqslant s(b_k) - s(a_k)$$
$$\text{for } r \geqslant e^{\varepsilon}$$

(see once more the preceding figure). By construction of our intervals $(a_k,\ b_k)$, the quantity on the right is equal to

$$\frac{\delta}{\pi}(b_k - a_k) = \frac{2\delta}{\pi}\Delta.$$

The previous relation thus yields

$$\frac{2\delta}{\pi}\Delta\int_{e^\varepsilon}^{e^\gamma}\frac{dr}{r} \;\leqslant\; \int_1^R \frac{M(r)}{r}dr \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} U(c \;+\; l\cosh(\gamma+i\vartheta))d\vartheta$$

$$-\;\frac{1}{\pi}\int_{-1}^{1}\frac{U(lx+c)}{\sqrt{(1-x^2)}}dx \qquad \text{for}\;\; \frac{\Delta}{\cosh\varepsilon} \leqslant l \leqslant \Delta.$$

Using the inequality

$$U(c \;+\; l\cosh(\gamma+i\vartheta)) \;\leqslant\; \beta l\,|\Im\cosh(\gamma+i\vartheta)| \;=\; \beta l\sinh\gamma\,|\sin\vartheta|$$

to estimate the next-to-the last integral on the right and making the change of variable $\xi = lx$ in the last one, one finds, after rearrangement, that

$$\int_{-l}^{l}\frac{U(\xi+c)}{\sqrt{(l^2-\xi^2)}}d\xi \;\leqslant\; 2\beta l\sinh\gamma \;-\; 2\delta(\gamma-\varepsilon)\Delta$$

$$\leqslant\; 2\sinh\gamma\left(\beta \;-\; \delta\frac{\gamma-\varepsilon}{\sinh\gamma}\right)\Delta, \qquad \text{for}\;\; \frac{\Delta}{\cosh\varepsilon} \leqslant l \leqslant \Delta.$$

Recall that in our present construction, we have

$$\beta \;<\; \frac{\delta}{e};$$

it is therefore certainly possible (and by far!) to choose $\gamma > 0$ *small enough* to make

$$\delta\frac{\gamma}{\sinh\gamma} \;>\; \beta$$

(*thus* is the value of $R = e^\gamma$ finally specified!), and then to fix an $\varepsilon > 0$ *yet smaller*, so as to *still have*

$$\delta\,\frac{\gamma-\varepsilon}{\sinh\gamma} \;>\; \beta.$$

These choices having been made, we write

$$a \;=\; \delta\,\frac{\gamma-\varepsilon}{\sinh\gamma} \;-\; \beta,$$

so that $a > 0$, and the above inequality becomes

$$\int_{-l}^{l}\frac{U(\xi+c)}{\sqrt{(l^2-\xi^2)}}d\xi \;\leqslant\; -2a\Delta\sinh\gamma, \qquad \frac{\Delta}{\cosh\varepsilon} \leqslant l \leqslant \Delta.$$

This relation is now multiplied by $l\,\mathrm{d}l$, and both sides integrated over the range

$$\frac{\Delta}{\cosh\varepsilon} \leqslant l \leqslant \Delta.$$

In our circumstances, $U(t) \leqslant 0$ on the real axis, and the computation at this point is practically identical to the one in §D.1 of Chapter IX. It therefore suffices to merely give the result, which reads

$$\int_{a_k}^{b_k} U(t)\,\mathrm{d}t = \int_{-\Delta}^{\Delta} U(\xi + c)\,\mathrm{d}\xi \leqslant -a\Delta^2 \sinh\gamma \tanh\varepsilon$$

$$= -\frac{a}{4}(\sinh\gamma \tanh\varepsilon)(b_k - a_k)^2.$$

The last inequality, involving the quantities a, γ, and ε, now *fixed* (and > 0), holds for *any* of the intervals (a_k, b_k), $k \geqslant 1$. Since the a_k are $\geqslant a_0 > 0$ and $U(t) \leqslant 0$ on $\mathbb{R}$, we see from it that

$$\int_{a_k}^{b_k} \frac{U(t)}{t^2+1}\,\mathrm{d}t \leqslant -\frac{aa_0^2}{4(a_0^2+1)} \sinh\gamma \tanh\varepsilon \left(\frac{b_k - a_k}{b_k}\right)^2.$$

Finally (using again the fact that $U(t) \leqslant 0$ on $\mathbb{R}$), we get

$$\sum_{k=1}^{\infty} \left(\frac{b_k - a_k}{b_k}\right)^2 \leqslant -\frac{4(a_0^2+1)}{aa_0^2 \sinh\gamma \tanh\varepsilon} \int_0^{\infty} \frac{U(t)}{t^2+1}\,\mathrm{d}t < \infty,$$

and our claim that $\sum_k ((b_k - a_k)/a_k)^2 < \infty$ is thereby established*.

From this result, *property* (ii) *for the function* $\omega(x)$ readily follows. As we know,

$$\tilde{\omega}'(x) = \pi \sum_1^{\infty} \sigma_n'(x) = \pi \sum_1^{N-1} \sigma_n'(x) + \pi s'(x).$$

It has already been noted that *each one* of the derivatives $\sigma_n'(x)$ is *bounded* for $x > 0$; there is thus a number K such that

$$\pi \sum_1^{N-1} \sigma_n'(x) \leqslant K - \delta, \qquad x > 0.$$

The derivative $\pi s'(x)$ is, at the same time, $\leqslant \delta$ for all the $x > 0$ *outside*

$$\mathcal{O} = \bigcup_{k \geqslant 1} (a_k, b_k).$$

* for the preceding displayed relation implies in particular that the ratios b_k/a_k are *bounded* – cf. discussion, top of p. 81

Therefore

$$\tilde{\omega}'(x) - K = \pi \sum_{1}^{N-1} \sigma_n'(x) - (K-\delta) + \pi s'(x) - \delta$$

is $\leqslant 0$ for $x \in (0, \infty) \sim \mathcal{O}$.

On any of the components (a_k, b_k) of $\mathcal{O}$, we have

$$(\tilde{\omega}'(x) - K)^+ \leqslant \left(\pi \sum_{1}^{N-1} \sigma_n'(x) - (K-\delta)\right)^+ + (\pi s'(x) - \delta)^+$$

$$\leqslant 0 + \pi s'(x)$$

($s(x)$ being increasing!), so

$$\int_{a_k}^{b_k} (\tilde{\omega}'(x) - K)^+ \mathrm{d}x \leqslant \pi \int_{a_k}^{b_k} s'(x)\mathrm{d}x = \pi(s(b_k) - s(a_k)) = \delta(b_k - a_k).$$

Property (ii) *therefore holds*, the sum $\sum_k ((b_k - a_k)/a_k)^2$ being convergent.

The *necessity* of our condition is thus proved.

2° ***The sufficiency.*** Suppose that there *is* an even $\mathscr{C}_\infty$ function

$$\omega(x) \geqslant \log W(x)$$

having the properties enumerated in the theorem's statement; we must show that W admits multipliers. Let, then, $\delta > 0$ be given; corresponding to it we have a K and an open subset $\mathcal{O}$ of $(0, \infty)$ with

$$\tilde{\omega}'(x) \leqslant K \quad \text{for } x \in (0, \infty) \sim \mathcal{O},$$

and, if $\mathcal{O}$ is the union of the disjoint intervals (a_k, b_k),

$$\sum_k \left(\frac{b_k - a_k}{a_k}\right)^2 < \infty,$$

while

$$\int_{a_k}^{b_k} (\tilde{\omega}'(x) - K)^+ \mathrm{d}x \leqslant \delta(b_k - a_k)$$

for each k.

We start by expressing $\tilde{\omega}(x)$ as the difference of two functions, each continuous and *increasing* on $[0, \infty)$. The given properties of $\tilde{\omega}'(x)$ make it possible for us to define a bounded measurable function $p(x)$ on $[0, \infty)$

with $0 \leqslant p(x) \leqslant \delta$ by taking

$$p(x) = \delta \quad \text{for } x \in [0,\infty) \sim \mathcal{O}$$

and then, on each of the interval components (a_k, b_k) of $\mathcal{O}$, having $p(x)$ assume the constant value needed to make

$$\int_{a_k}^{b_k} \{(\tilde{\omega}'(x) - K)^+ + p(x)\}\,dx = \delta(b_k - a_k).$$

Put now

$$\pi v_1(x) = \int_0^x \{(\tilde{\omega}'(t) - K)^+ + p(t)\}\,dt \qquad \text{for } x \geqslant 0$$

and

$$\pi v_2(x) = \int_0^x \{(K - \tilde{\omega}'(t))^+ + p(t)\}\,dt, \qquad x \geqslant 0.$$

We have $\tilde{\omega}(0) = 0$, for, since $\omega(x)$ is $\mathscr{C}_\infty$ and even, $\tilde{\omega}(x)$ is $\mathscr{C}_\infty$ and odd. Therefore, when $x \geqslant 0$,

$$\tilde{\omega}(x) - Kx = \int_0^x (\tilde{\omega}'(t) - K)\,dt$$

$$= \int_0^x \{(\tilde{\omega}'(t) - K)^+ - (\tilde{\omega}'(t) - K)^-\}\,dt = \pi v_1(x) - \pi v_2(x);$$

i.e.,

$$\pi v_1(x) + Kx - \pi v_2(x) = \tilde{\omega}(x), \qquad x \geqslant 0,$$

with $v_1(x)$ and $v_2(x)$ both *increasing* and *continuous* for $x \geqslant 0$.

The ratio $v_1(x)/x$ is *bounded* for $x > 0$. Indeed, if $x \in [0,\infty) \sim \mathcal{O}$, $\pi v_1(x) = \delta x$ by the definition of our function $p(x)$. And if $a_k < x < b_k$ for some k,

$$\frac{\pi v_1(x)}{x} < \frac{\pi v_1(b_k)}{a_k} = \delta \frac{b_k}{a_k}.$$

The ratios b_k/a_k are, however, *bounded above*, since $\sum_k ((b_k - a_k)/a_k)^2$ is convergent. Hence

$$\frac{v_1(x)}{x} \leqslant \text{const.} \qquad \text{for } x > 0.$$

The *odd* continuous increasing function $\mu(x)$ equal to $\pi\nu_1(x) + Kx$ for $x \geqslant 0$ is thus $O(x)$ on $[0, \infty)$, and $\tilde{\omega}(x) - \mu(x)$, *also odd* and equal, for $x \geqslant 0$, to $-\pi\nu_2(x)$ by the above formula, is *decreasing* on $\mathbb{R}$. The *fourth* lemma of article 1 can therefore be invoked (with $F(x) = -\omega(x)$, $\omega(x)$ being $\mathscr{C}_\infty$, $\geqslant 0$ and enjoying property (i) by hypothesis). This yields

$$|\tilde{\omega}(x)| \leqslant \text{const.}|x| \qquad \text{for } x \in \mathbb{R},$$

which in turn makes

$$\nu_2(x) \leqslant \text{const.}\, x \qquad \text{for } x \geqslant 0$$

in view of the preceding estimate on $\nu_1(x)$ and the formula just referred to.

All the conditions for application of the *third* lemma in article 1 (again with $F(x) = -\omega(x)$) are now verified. By that result we get, for $x \in \mathbb{R}$,

$$\omega(x) = \omega(0) - \frac{1}{\pi}\int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\tilde{\omega}(t)$$

$$= \omega(0) + \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\nu_2(t)$$

$$- \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\nu_1(t) - \frac{K}{\pi}\int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| dt.$$

The very last integral on the right is of course zero, so we have

$$\boxed{\begin{aligned}&\omega(x) + \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\nu_1(t)\\ &= \omega(0) + \int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\nu_2(t), \qquad x \in \mathbb{R}.\end{aligned}}$$

The rest of our work here is based mainly on this formula.

Before looking more closely at the increasing function $\nu_1(t)$ and the expression

$$\int_0^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\nu_1(t)$$

corresponding to it, we should attend to a detail regarding the *location* of the open set $\mathcal{O}$. We can, namely, arrange to *ensure that* $\mathcal{O} \subseteq (1, \infty)$