From the theorem of the preceding article we have $x\Omega_{\mathscr{D}}(x) \leqslant Y_{\mathscr{D}}(0)$, so this is in turn

$$\leqslant \; Y_{\mathscr{D}}(0)\int_{-\infty}^{\infty} G_{\mathscr{D}}(x,0)\,\mathrm{d}x,$$

which, however equals $\pi(Y_{\mathscr{D}}(0))^2$ by the second theorem of §A.2.

We are done.

This theorem will be used in establishing the remaining results of the present §. For that work it will be convenient to have at hand an *alternative notation for the energy*

$$E(\mathrm{d}\rho(t),\, \mathrm{d}\rho(t)).$$

*Suppose that we have a real Green potential*

$$u(x) \;=\; \int_0^{\infty} \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t).$$

*If the double integral*

$$\int_0^{\infty}\int_0^{\infty} \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t)\,\mathrm{d}\rho(x)$$

*used to define* $E(\mathrm{d}\rho(t),\, \mathrm{d}\rho(t))$ *is absolutely convergent, we write*

$$\boxed{\|u\|_E^2 \;=\; E(\mathrm{d}\rho(t),\, \mathrm{d}\rho(t)).}$$

*If we have another such Green potential*

$$v(x) \;=\; \int_0^{\infty} \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\sigma(t),$$

*we similarly write*

$$\boxed{\langle u, v\rangle_E \;=\; E(\mathrm{d}\rho(t),\, \mathrm{d}\sigma(t)).}$$

$\langle\ ,\ \rangle_E$ is a bilinear form on the collection of real Green potentials of this kind; according to the remark at the end of §B.5 it is *positive definite*. The reader may wonder whether our use of the symbol $\|u\|_E$ to denote $\sqrt{(E(\mathrm{d}\rho(t), \mathrm{d}\rho(t)))}$ is *legitimate*; could not *the same* function $u(x)$ be the Green potential of *two different measures*? That this cannot occur is easily seen, and boils down to showing that if $\rho(x)$ is *not constant*, the Green potential

$$u(x) \;=\; \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t)$$

*cannot be* $\equiv 0$ on $[0, \infty)$ (provided, of course, that the double integral used to define $E(\mathrm{d}\rho(t), \mathrm{d}\rho(t))$ is absolutely convergent). Here, we have $E(\mathrm{d}\rho(t), \mathrm{d}\rho(t)) \;=\; \int_0^\infty u(x)\mathrm{d}\rho(x)$. Hence, if $u(x) \equiv 0$, the left-hand side is also zero. Then, however, $\rho(x)$ is constant by the second lemma of §B.5.

## 4. Harmonic estimation in $\mathscr{D}$

We are now able to give a fairly general result of the kind envisioned at the beginning of this §. Suppose we have an *even* majorant $M(t) \geqslant 0$ with $M(0) = 0$. In the case where $M(x)/x$ is a *Green potential*

$$\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t)$$

with the double integral defining $E(\mathrm{d}\rho(t), \mathrm{d}\rho(t)) \;=\; \|M(x)/x\|_E^2$ *absolutely convergent*, the following is true:

***Theorem.*** *Let $M(t)$ be a majorant of the kind just described. Given one of our domains $\mathscr{D}$ containing 0, suppose we have a function $v(z)$, subharmonic in $\mathscr{D}$ and continuous up to $\partial\mathscr{D}$, with*

$$v(z) \;\leqslant\; A|\Im z| + \mathrm{O}(1)$$

*for some real* (sic!) *A, and*

$$v(t) \;\leqslant\; M(t) \qquad \textit{for } t \in \partial\mathscr{D}.$$

*Then*

$$\boxed{v(0) \;\leqslant\; Y_{\mathscr{D}}(0)\left\{A \;+\; \int_0^\infty \frac{M(t)}{t^2}\mathrm{d}t \;+\; \sqrt{\pi}\left\|\frac{M(x)}{x}\right\|_E\right\}.}$$

***Remark.*** The assumptions on $v(z)$'s behaviour can be lightened by means of standard Phragmén–Lindelöf arguments (see footnote near beginning of §A.2, after problem 16). Such extensions are left to the reader; what we have here is general enough for the applications in this book.

***Proof of theorem.*** The difference

$$v(z) - AY_{\mathscr{D}}(z)$$

is (by the definition of $Y_{\mathscr{D}}(z)$ in §A.2) *subharmonic* and *bounded above* in $\mathscr{D}$, and continuous up to $\partial\mathscr{D}$, where it coincides with $v(z)$. Hence, by harmonic majoration (Chapter VII, §B.1),

$$v(z) - AY_{\mathscr{D}}(z) \leqslant \int_{\partial\mathscr{D}} v(t)\mathrm{d}\omega_{\mathscr{D}}(t,z) \leqslant \int_{\partial\mathscr{D}} M(t)\mathrm{d}\omega_{\mathscr{D}}(t,z) \qquad \text{for} \quad z\in\mathscr{D}.$$

Taking $z = 0$, we see that we have to estimate $\int_{\partial\mathscr{D}} M(t)\mathrm{d}\omega_{\mathscr{D}}(t,0)$, which, in view of the definition of $\Omega_{\mathscr{D}}(t)$, equals $-\int_0^\infty M(t)\mathrm{d}\Omega_{\mathscr{D}}(t)$, $M(t)$ being *even.*

The *trick* here is to write

$$-\int_0^\infty M(t)\,\mathrm{d}\Omega_{\mathscr{D}}(t) = \int_0^\infty \frac{M(t)}{t}\Omega_{\mathscr{D}}(t)\,\mathrm{d}t - \int_0^\infty \frac{M(t)}{t}\mathrm{d}(t\Omega_{\mathscr{D}}(t)).$$

Since $M(t) \geqslant 0$, the *first* integral on the right is

$$\leqslant Y_{\mathscr{D}}(0)\int_0^\infty \frac{M(t)}{t^2}\mathrm{d}t$$

by the theorem of article 2. In view of our assumption on $M(t)$, the *second* right-hand integral can be rewritten

$$-\int_0^\infty\int_0^\infty \log\left|\frac{t+x}{t-x}\right|\mathrm{d}\rho(x)\,\mathrm{d}(t\Omega_{\mathscr{D}}(t)) = -E(\mathrm{d}\rho(t),\ \mathrm{d}(t\Omega_{\mathscr{D}}(t))).$$

Using Schwarz' inequality on the *positive definite* bilinear form $E(\ ,\ )$ (see remark, end of §B.5), we see that the last expression is in modulus

$$\leqslant \sqrt{(E(\mathrm{d}\rho(t),\ \mathrm{d}\rho(t)))}\cdot\sqrt{(E(\mathrm{d}(t\Omega_{\mathscr{D}}(t)),\ \mathrm{d}(t\Omega_{\mathscr{D}}(t))))}$$

which, by the result of the preceding article, is $\leqslant \|M(x)/x\|_E\sqrt{\pi}\,Y_{\mathscr{D}}(0)$. Putting our two estimates together, we get

$$\boxed{\int_{\partial\mathscr{D}} M(t)\mathrm{d}\omega_{\mathscr{D}}(t,0) \leqslant Y_{\mathscr{D}}(0)\left\{\int_0^\infty \frac{M(t)}{t^2}\mathrm{d}t + \sqrt{\pi}\left\|\frac{M(x)}{x}\right\|_E\right\}.}$$

As we have seen $v(0) - AY_{\mathscr{D}}(0)$ is $\leqslant$ the left-hand integral. The theorem is thus proved.

***Remark.*** This result shows that for special majorants $M(t)$ of the kind described, the *entire dependence* of our bound for $v(0)$ *on the domain* $\mathscr{D}$ is *expressed through the quantity* $Y_{\mathscr{D}}(0)$, $Y_{\mathscr{D}}$ being the Phragmén–Lindelöf function for $\mathscr{D}$.

## 5. When majorant is the logarithm of an entire function of exponential type

The result in the preceding article can be extended so as to apply to certain even majorants $M(x)$ of the form

$$x\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t)$$

for which the iterated integral

$$\int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t)\,\mathrm{d}\rho(x)$$

is *not* absolutely convergent. This can, in particular, be done in the important special case where

$$M(x) = \log|G(x)|$$

with an entire function $G$ of exponential type, 1 at 0, having *even* modulus $\geqslant 1$ on $\mathbb{R}$, and such that

$$\int_{-\infty}^\infty \frac{\log|G(x)|}{1+x^2}\mathrm{d}x < \infty.$$

Then the right side of the boxed formula at the end of the previous article can be simplified so as to involve only $Y_{\mathscr{D}}(0)$, $\int_0^\infty (M(t)/t^2)\mathrm{d}t$, and the type of $G$.

The treatment of *any* majorant $M(x)$, *even or not*, of the form $\log^+|F(x)|$ with $F$ entire, of exponential type, and such that

$$\int_{-\infty}^\infty \frac{\log^+|F(x)|}{1+x^2}\mathrm{d}x < \infty,$$

can be reduced to that of one of the kind just described. Indeed, to any such $M(x)$ corresponds another, $M_1(x) = \log|G(x)|$ with $G$ entire and of exponential type, such that

$$M_1(x) \geqslant M(x) \quad \text{for } |x|\geqslant 1,$$
$$M_1(x) = M_1(-x) \geqslant 0,$$
$$M_1(0) = 0,$$

and

$$\int_0^\infty \frac{M_1(x)}{x^2}\mathrm{d}x < \infty.$$

To see this, put first of all

$$\Phi(z) \;=\; 1 + z^2[F(z)\overline{F(\bar{z})} + F(-z)\overline{F(-\bar{z})}];$$

$\Phi(z)$ is then entire and of exponential type, even, and $\geqslant 1$ on $\mathbb{R}$ with $\Phi(0) = 1$. Clearly

$$\int_{-\infty}^{\infty} \frac{\log \Phi(x)}{1+x^2}\,\mathrm{d}x \;<\; \infty$$

in view of the similar property of $F$, and

$$\Phi(x) \;\geqslant\; |F(x)|^2 \qquad \text{for } |x| \geqslant 1.$$

By the Riesz–Fejer theorem (the *third* one in §G.3 of Chapter III), there is an entire function $G(z)$ of exponential type, *having all its zeros in* $\Im z < 0$ (since here $\Phi(x) \geqslant 1$), such that

$$\Phi(z) \;=\; G(z)\overline{G(\bar{z})}.$$

The majorant $M_1(x) = \log|G(x)|$ then has the required properties.

The result to be obtained in this article regarding even majorants $\log|G(x)|$ of the abovementioned kind can thus be used in studying problems involving the more general ones of the form $\log^+|F(x)|$.

For entire functions $G(z)$ of exponential type with $G(0) = 1$, $|G(x)| = |G(-x)| \geqslant 1$, and

$$\int_{-\infty}^{\infty} \frac{\log|G(x)|}{1+x^2}\,\mathrm{d}x \;<\; \infty,$$

$\log|G(x)|$ has a simple representation as a Stieltjes integral. When dealing only with the *modulus* of $G$ on $\mathbb{R}$, we may, by the *second* theorem of §G.3, Chapter III, *assume that* $G(z)$ *has all its zeros in the lower half plane*. Forming, for the moment, the entire function $\Phi(z) = G(z)\overline{G(\bar{z})}$, we see that $\Phi(x) = \Phi(-x)$ on $\mathbb{R}$ so that $\Phi(z) = \Phi(-z)$, and every zero of $\Phi(z)$ is also one of $\Phi(-z)$. The zeros of $\Phi(z)$ are just those of $G(z)$ together with their *complex conjugates*, so, since all the former lie in $\Im z < 0$, we have $G(-\bar{\lambda}) = 0$ whenever $G(\lambda) = 0$. The zeros of $G(z)$ thus fall into three groups: those on the *negative imaginary axis*, those in the *open fourth quadrant*, and the *reflections* of *these latter ones* in the *imaginary axis*. The Hadamard factorization (Chapter III, §A) of $G(z)$ can therefore be written

$$G(z) = \mathrm{e}^{\alpha z}\prod_k\left(1+\frac{z}{\mathrm{i}\mu_k}\right)\mathrm{e}^{\mathrm{i}z/\mu_k} \cdot \prod_n\left(1-\frac{z}{\bar{\lambda}_n}\right)\mathrm{e}^{z/\bar{\lambda}_n}\left(1+\frac{z}{\lambda_n}\right)\mathrm{e}^{-z/\lambda_n},$$

where the $\mu_k > 0$, $\Re\lambda_n > 0$ and $\Im\lambda_n > 0$. One (or even both!) of the two products occurring on the right may of course be empty.

Since $|G(x)| = |G(-x)|$, $\alpha$ is pure imaginary. We also know, by the *first* theorem of §G.3, Chapter III, that

$$\sum_k \frac{1}{\mu_k} \quad \text{and} \quad \sum_n \frac{\Im\lambda_n}{|\lambda_n|^2}$$

both converge. The exponential factors figuring in the above product may therefore be grouped together and multiplied out separately, after which the expression takes the form

$$e^{ibz}\prod_k\left(1+\frac{z}{i\mu_k}\right)\cdot\prod_n\left(1-\frac{z}{\bar{\lambda}_n}\right)\left(1+\frac{z}{\lambda_n}\right),$$

with $b$ real. Here, we are only concerned with the *modulus* $|G(x)|$, $x\in\mathbb{R}$; *we may hence take* $b = 0$. *This we do throughout the remainder of this article, working exclusively with entire functions of exponential type of the form*

$$\boxed{G(z) = \prod_k\left(1+\frac{z}{i\mu_k}\right)\cdot\prod_n\left(1-\frac{z}{\bar{\lambda}_n}\right)\left(1+\frac{z}{\lambda_n}\right),}$$

*where the* $\mu_k > 0$, $\Re\lambda_n > 0$ *and* $\Im\lambda_n > 0$. The products on the right are of course assumed to be *convergent*. Our Stieltjes integral representation for such functions is provided by the

***Lemma.*** *Let* $G(z)$, *of exponential type, be of the form just described. Then, for* $\Im z > 0$,

$$\log|G(z)| = \int_0^\infty \log\left|1-\frac{z^2}{t^2}\right| d\nu(t)$$

*with an increasing function* $\nu(t)$ *given by*

$$\frac{d\nu(t)}{dt} = \frac{1}{\pi}\left(\sum_k \frac{\mu_k}{\mu_k^2+t^2} + \sum_n\left(\frac{\Im\lambda_n}{|\lambda_n - t|^2} + \frac{\Im\lambda_n}{|\lambda_n + t|^2}\right)\right).$$

***Proof.*** *Fix* $z$, $\Im z > 0$. Then $\log|1 + z/\lambda|$ is a *harmonic function of* $\lambda$ in $\{\Im\lambda > 0\}$, *bounded* therein for $\lambda$ away from 0, and *continuous* up to $\mathbb{R}$ save at $\lambda = 0$ where it has a logarithmic singularity. We can therefore apply Poisson's formula, getting

$$\log\left|1+\frac{z}{\lambda}\right| = \frac{1}{\pi}\int_{-\infty}^{\infty}\log\left|1-\frac{z}{t}\right|\cdot\frac{\Im\lambda}{|\lambda+t|^2}dt$$

for $\Im\lambda > 0$, from which

$$\log\left|1+\frac{z}{\lambda}\right| + \log\left|1-\frac{z}{\bar\lambda}\right| = \frac{1}{\pi}\int_0^\infty \log\left|1-\frac{z^2}{t^2}\right|\left(\frac{\Im\lambda}{|\lambda+t|^2}+\frac{\Im\lambda}{|\lambda-t|^2}\right)\mathrm{d}t.$$

Similarly, for $\mu > 0$,

$$\log\left|1+\frac{z}{\mathrm{i}\mu}\right| = \frac{1}{\pi}\int_0^\infty \log\left|1-\frac{z^2}{t^2}\right| \frac{\mu}{\mu^2+t^2}\mathrm{d}t.$$

We have

$$\log|G(z)| = \sum_k \log\left|1+\frac{z}{\mathrm{i}\mu_k}\right| + \sum_n\left(\log\left|1+\frac{z}{\lambda_n}\right| + \log\left|1-\frac{z}{\bar\lambda_n}\right|\right).$$

When $\Im z > 0$, we can rewrite each of the terms on the right using the formulas just given, obtaining a certain *sum of integrals*. If $|\Re z| < \Im z$, the *order* of summation and integration in that sum can be *reversed*, for then

$$\log\left|1-\frac{z^2}{t^2}\right| \geqslant 0, \qquad t\in\mathbb{R}.$$

This gives

$$\log|G(z)| = \int_0^\infty \log\left|1-\frac{z^2}{t^2}\right|\mathrm{d}\nu(t),$$

at least for $|\Re z| < \Im z$, with $\nu'(t)$ as in the statement of the lemma.

Both sides of the relation just found are, however, *harmonic in $z$* for $\Im z > 0$; the *left* one by our assumption on $G(z)$ and the *right* one because $\int_0^\infty \log|1+y^2/t^2|\mathrm{d}\nu(t)$, being just equal to $\log|G(\mathrm{i}y)|$ for $y>0$, is *convergent* for every such $y$. (To show that this implies u.c.c. convergence, and hence harmonicity, of the integral involving $z$ for $\Im z > 0$, one may argue as at the beginning of the proof of the second theorem in §A, Chapter III.) The two sides of our relation, equal for $|\Re z| < \Im z$, must therefore coincide for $\Im z > 0$ and finally for $\Im z \geqslant 0$ by a continuity argument.

***Remark.*** Since $G(z)$ has no zeros for $\Im z \geqslant 0$, a branch of $\log G(z)$, and hence of $\arg G(z)$, is defined there. By logarithmic differentiation of the above boxed product formula for $G(z)$, it is easy to check that

$$\frac{\mathrm{d}\arg G(t)}{\mathrm{d}t} = -\pi\nu'(t)$$

with the $\nu$ of the lemma. From this it is clear that $\nu'(t)$ is certainly *continuous* (and even $\mathscr{C}_\infty$) on $\mathbb{R}$.

*In what follows, we will take* $\nu(0) = 0$, $\nu(t)$ being the increasing function in the lemma. Since $\nu'(t)$ is *clearly even*, $\nu(t)$ *is then odd*. With $\nu(t)$ thus specified, we have the easy

***Lemma.*** *If* $G(z)$, *given by the above boxed formula, is of exponential type, the function* $\nu(t)$ *corresponding to it is* $\leqslant$ const.$t$ *for* $t \geqslant 0$.

***Proof.*** By the preceding lemma,

$$\int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| \mathrm{d}\nu(t) \;=\; \log|G(z)|$$

for $\Im z \geqslant 0$, the right side being $\leqslant K|z|$ by hypothesis, since $G(0) = 1$. Calling the left-hand integral $U(z)$, we have, however, $U(z) = U(\bar{z})$, so

$$U(z) \;\leqslant\; K|z|$$

for all $z$.

Reasoning as in the proof of Jensen's formula, Chapter I (what we are dealing with here is indeed nothing but a version of that formula for the subharmonic function $U(z)$), we see, for $t \neq 0$, that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|1 - \frac{re^{i\vartheta}}{t}\right| \mathrm{d}\vartheta \;=\; \begin{cases} \log\dfrac{r}{|t|}, & |t| < r, \\ 0, & |t| \geqslant r. \end{cases}$$

Thence, by Fubini's theorem,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} U(re^{i\vartheta})\,\mathrm{d}\vartheta \;=\; \int_{-r}^{r} \log\frac{r}{|t|}\,\mathrm{d}\nu(t).$$

Integrating the right side by parts, we get the value $2\int_0^r (\nu(t)/t)\,\mathrm{d}t$, $\nu(t)$ being odd and $\nu'(0)$ finite. In view of the above inequality on $U(z)$, we thus have

$$\int_0^r \frac{\nu(t)}{t}\,\mathrm{d}t \;\leqslant\; \tfrac{1}{2}Kr.$$

From this relation we easily deduce that $\nu(r) \;\leqslant\; \frac{1}{2}eKr$ as in problem 1, Chapter I. Done.

Using the two results just proved in conjunction with the *first* lemma of §B.4, we now obtain, without further ado, the

***Theorem.*** *Let the entire function* $G(z)$ *of exponential type be given by the above boxed formula, and let* $\nu(t)$ *be the increasing function associated to* $G$ *in the way described above. Then, for* $x > 0$,

$$\log|G(x)| \;=\; -x\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\left(\frac{\nu(t)}{t}\right).$$

For our functions $G(z)$, $(\log|G(x)|)/x$ is thus a *Green potential* on $(0, \infty)$. *This makes it possible for us to apply the result of the preceding article to majorants*

$$M(t) = \log|G(t)|.$$

With that in mind, let us give a more quantitative version of the second of the above lemmas.

***Lemma.*** *If $G(z)$, given by the above boxed formula, is $\geqslant 1$ in modulus on $\mathbb{R}$ and of exponential type $a$, the increasing function $\nu(t)$ associated to it satisfies*

$$\frac{\nu(t)}{t} \leqslant \frac{\mathrm{e}}{2}a + \frac{\mathrm{e}}{\pi}\int_{-\infty}^{\infty} \frac{\log|G(x)|}{x^2}\mathrm{d}x. \qquad t \geqslant 0.$$

***Remark.*** We are not striving for a best possible inequality here.

***Proof of lemma.*** The function $U(z)$ used in proving the previous lemma is *subharmonic* and $\leqslant K|z|$. Assuming that

$$\int_{-\infty}^{\infty} \frac{\log|G(t)|}{t^2}\mathrm{d}t < \infty$$

(the only situation we need consider), let us find an explicit estimate for $K$.

Under our assumption, we have, for $\Im z > 0$,

$$\log|G(z)| \leqslant a\Im z + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log|G(t)|\mathrm{d}t$$

by §E of Chapter III. When $-y \leqslant x \leqslant y$, we have, however, for $z = x + \mathrm{i}y$,

$$|z-t|^2 = t^2 - 2xt + x^2 + y^2 \geqslant \frac{t^2}{2} + \frac{t^2}{2} - 2xt + 2x^2$$

$$\geqslant \frac{t^2}{2}, \qquad t \in \mathbb{R},$$

whence, $\log|G(t)|$ being $\geqslant 0$,

$$\log|G(z)| \leqslant ay + \frac{2y}{\pi}\int_{-\infty}^{\infty} \frac{\log|G(t)|}{t^2}\mathrm{d}t.$$

Thus, since $U(z) = U(\bar{z}) = \log|G(z)|$ for $\Im z \geqslant 0$,

$$U(z) \leqslant \left(a + \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\log|G(t)|}{t^2}\mathrm{d}t\right)|\Im z|$$

in both of the sectors $|\Re z| \leqslant |\Im z|$.

Because $U(z) \leqslant \text{const.}|z|$ we can apply the second Phragmén–Lindelöf theorem of §C, Chapter III, to the difference

$$U(z) \; - \; \left(a + \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\log|G(t)|}{t^2}\,\mathrm{d}t\right)\Re z$$

in the 90° sector $|\Im z| \leqslant \Re z$, and find that it is $\leqslant 0$ in that sector. One proceeds similarly in $\Re z \leqslant -|\Im z|$, and we have

$$U(z) \; \leqslant \; \left(a + \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\log|G(t)|}{t^2}\,\mathrm{d}t\right)|\Re z|$$

for $|\Im z| \leqslant |\Re z|$.

Combining the two estimates for $U(z)$ just found, we get

$$U(z) \; \leqslant \; K|z|$$

with

$$K \; = \; a + \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\log|G(t)|}{t^2}\,\mathrm{d}t.$$

This value of $K$ may now be plugged into the *proof* of the previous lemma. That yields the desired result.

**Problem 27**

Let $\Phi(z)$ be entire and of exponential type, with $\Phi(0) = 1$. *Suppose that* $\Phi(z)$ *has all its zeros in* $\Im z < 0$ *and that* $|\Phi(x)| \geqslant 1$ *on* $\mathbb{R}$. Show that *then*

$$\int_{-\infty}^{\infty} \frac{\log|\Phi(x)|}{x^2}\,\mathrm{d}x \; < \; \infty.$$

(Hint: First use Lindelöf's theorem from Chapter III, §B, to show that the Hadamard factorization for $\Phi(z)$ can be cast in the form

$$\Phi(z) \; = \; e^{cz}\prod_n \left(1 - \frac{z}{\lambda_n}\right)e^{z\Re(1/\lambda_n)},$$

where the $\Im\lambda_n < 0$. Taking $\Psi(z) = \Phi(z)\exp(-iz\Im c)$, show that $\partial\log|\Psi(z)|/\partial y \geqslant 0$ for $y \geqslant 0$, and then look at $1/\Psi(z)$. )

Suppose now that we have an entire function $G(z)$ given by the above boxed representation, of exponential type $a$ and $\geqslant 1$ in modulus on $\mathbb{R}$. *If the double integral*

$$\int_0^{\infty}\int_0^{\infty} \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\left(\frac{\nu(t)}{t}\right)\mathrm{d}\left(\frac{\nu(x)}{x}\right)$$

*is absolutely convergent*, we may, as in the previous two articles, speak of the *energy*

$$E\left(\mathrm{d}\left(\frac{\nu(t)}{t}\right),\ \mathrm{d}\left(\frac{\nu(t)}{t}\right)\right);$$

in terms of the Green potential

$$\frac{\log|G(x)|}{x} = -\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\left(\frac{\nu(t)}{t}\right),$$

this is just $\|(\log|G(x)|)/x\|_E^2$ according to the notation introduced at the end of article 3.

To Beurling and Malliavin is due the important observation that $\|(\log|G(x)|)/x\|_E$ can be expressed in terms of $a$ and $\int_0^\infty(\log|G(x)|/x^2)\mathrm{d}x$ under the present circumstances. Since $\log|G(t)| \geqslant 0$ and $\nu(t)$ increases, we have indeed

$$\begin{aligned}\|(\log|G(x)|)/x\|_E^2 &= -\int_0^\infty \frac{\log|G(x)|}{x}\mathrm{d}\left(\frac{\nu(x)}{x}\right)\\ &= \int_0^\infty \frac{\log|G(x)|}{x^2}\left(\frac{\nu(x)}{x}\mathrm{d}x - \mathrm{d}\nu(x)\right)\\ &\leqslant \left(\sup_{x>0}\frac{\nu(x)}{x}\right)\cdot\int_0^\infty \frac{\log|G(x)|}{x^2}\mathrm{d}x.\end{aligned}$$

Using the preceding lemma and remembering that $|G(x)|$ is even, we find that

$$\boxed{\left\|\frac{\log|G(x)|}{x}\right\|_E^2 \leqslant \left(\frac{ea}{2} + \frac{2e}{\pi}\int_0^\infty \frac{\log|G(x)|}{x^2}\mathrm{d}x\right)\cdot\int_0^\infty \frac{\log|G(x)|}{x^2}\mathrm{d}x.}$$

Take now an even majorant $M(t) \geqslant 0$ equal to $\log|G(t)|$, and consider one of our domains $\mathscr{D}$ with $0\in\mathscr{D}$. From the result just obtained and the boxed formula near the end of the previous article, we get

$$\int_{\partial\mathscr{D}} M(t)\,\mathrm{d}\omega_{\mathscr{D}}(t,0) \leqslant Y_{\mathscr{D}}(0)\left\{J + \sqrt{\left(2eJ\left(J+\frac{\pi a}{4}\right)\right)}\right\},$$

with

$$J = \int_0^\infty \frac{\log|G(t)|}{t^2}\mathrm{d}t = \int_0^\infty \frac{M(t)}{t^2}\mathrm{d}t,$$

*at least in the case where*

$$\int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\left(\frac{\nu(t)}{t}\right) \mathrm{d}\left(\frac{\nu(x)}{x}\right)$$

*is absolutely convergent.* On the right side of this relation, the coefficient $Y_{\mathscr{D}}(0)$ is multiplied by a factor *involving only a, the type of G, and the integral* $\int_0^\infty (M(t)/t^2)\,\mathrm{d}t$ (essentially, the one this book is about!).

*It is very important that the requirement of absolute convergence on the above double integral can be lifted, and the preceding relation still remains true.* This will be shown by bringing in the *completion*, for the norm $\|\ \|_E$, of the collection of real Green potentials associated with absolutely convergent energy integrals – that completion is a real Hilbert space, since $\|\ \|_E$ comes from a positive definite bilinear form. The details of the argument take up the remainder of this article.

Starting with our entire function $G(z)$ of exponential type and the increasing function $\nu(t)$ associated to it, put

$$Q(x) = \frac{\log|G(x)|}{x} = -\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\left(\frac{\nu(t)}{t}\right),$$

and, for $n = 1, 2, 3, \ldots$,

$$Q_n(x) = \frac{1}{x}\int_0^n \log\left|1 - \frac{x^2}{t^2}\right| \mathrm{d}\nu(t).$$

In terms of

$$\nu_n(t) = \begin{cases} \nu(t), & 0 \leqslant t \leqslant n, \\ \nu(n), & t > n, \end{cases}$$

we have

$$Q_n(x) = -\int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\left(\frac{\nu_n(t)}{t}\right)$$

by the first lemma of §B.4; evidently, $Q_n(x) \longrightarrow Q(x)$ u.c.c. in $[0, \infty)$ as $n \to \infty$.

*Each of the integrals*

$$\int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\left(\frac{\nu_n(t)}{t}\right) \mathrm{d}\left(\frac{\nu_n(x)}{x}\right)$$

*is absolutely convergent.* This is easily verified using the facts that

$$\mathrm{d}\left(\frac{\nu_n(t)}{t}\right) \sim \tfrac{1}{2}\nu''(0)\,\mathrm{d}t$$

near 0 ($\nu(t)$ being $\mathscr{C}_\infty$ by a previous remark), and that

$$d\left(\frac{\nu_n(t)}{t}\right) = -\frac{\nu(n)}{t^2}dt \qquad \text{for } t > n.$$

***Lemma.*** *If $|G(x)| \geqslant 1$ on $\mathbb{R}$, the functions $Q_n(x)$ are $\geqslant 0$ for $x > 0$, and*

$$\|Q_n\|_E \leqslant \frac{\pi}{2}\nu'(0).$$

***Proof.*** For $t > 0$, $\log|1 - x^2/t^2| \geqslant 0$ when $x \geqslant \sqrt{2}t$, so

$$xQ_n(x) = \int_0^n \log\left|1 - \frac{x^2}{t^2}\right| d\nu(t)$$

is $\geqslant 0$ for $x \geqslant \sqrt{2}n$. Again, for $0 \leqslant x \leqslant \sqrt{2}t$, $\log|1 - x^2/t^2| \leqslant 0$, so, for $0 \leqslant x \leqslant \sqrt{2}n$,

$$\int_n^\infty \log\left|1 - \frac{x^2}{t^2}\right| d\nu(t) \leqslant 0,$$

and finally $xQ_n(x)$, equal to $\log|G(x)|$ *minus* this integral, is $\geqslant 0$ since $|G(x)| \geqslant 1$.

The second lemma of §B.4 is applicable to the functions $\nu_n(t)$. Using it and the positivity of $Q_n(x)$, already established, we get

$$\begin{aligned}
\|Q_n\|_E^2 &= \int_0^\infty \int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\left(\frac{\nu_n(t)}{t}\right) d\left(\frac{\nu_n(x)}{x}\right) \\
&= \int_0^\infty Q_n(x) \left\{\frac{\nu_n(x)}{x^2}dx - \frac{d\nu_n(x)}{x}\right\} \\
&\leqslant \int_0^\infty Q_n(x)\frac{\nu_n(x)}{x^2}dx = \frac{\pi^2}{4}(\nu_n'(0))^2 = \frac{\pi^2}{4}(\nu'(0))^2.
\end{aligned}$$

We are done.

***Theorem.*** *Let $G(z)$ be an entire function of exponential type $a$, 1 at 0, with $|G(x)|$ even and $\geqslant 1$ on $\mathbb{R}$, and such that*

$$\int_{-\infty}^\infty \frac{\log|G(x)|}{1+x^2}dx < \infty.$$

*If $\mathscr{D}$ is one of our domains containing 0, we have*

$$\boxed{\begin{array}{l} \displaystyle\int_{\partial\mathscr{D}} \log|G(t)|\,d\omega_{\mathscr{D}}(t,0) \;\leqslant\; Y_{\mathscr{D}}(0)\left\{J + \sqrt{\left(2eJ\left(J+\frac{\pi a}{4}\right)\right)}\right\} \\ \textit{where} \\ \displaystyle J \;=\; \int_0^\infty \frac{\log|G(t)|}{t^2}\,dt. \end{array}}$$

***Proof.*** According to the discussion at the beginning of this article we may, without loss of generality,* assume that $G(z)$ has the above boxed product representation.

Beginning as in the proof of the theorem from the preceding article, we have

$$\int_{\partial\mathscr{D}} \log|G(x)|\,d\omega_{\mathscr{D}}(x,0) \;=\; \int_0^\infty \frac{\log|G(x)|}{x}\,\Omega_{\mathscr{D}}(x)\,dx \;-\; \int_0^\infty \frac{\log|G(x)|}{x}\,d(x\Omega_{\mathscr{D}}(x)).$$

The *first* term on the right is of course

$$\leqslant\; Y_{\mathscr{D}}(0)\int_0^\infty \frac{\log|G(x)|}{x^2}\,dx$$

by the theorem of article 2, $\log|G(x)|$ being positive. The *second*, equal to

$$-\int_0^\infty Q(x)\,d(x\Omega_{\mathscr{D}}(x)),$$

can be looked at in two different ways.

In the first place, for $x > 0$,

$$Q(x) \;=\; \lim_{n\to\infty} Q_n(x)$$

with the functions $Q_n(x)$ introduced above. Also, for each $n$,

* Dropping the factor $\exp(ibz)$ from the second displayed expression on p. 557 can only *diminish* the overall exponential type, for, if $G(z)$ is given by the *boxed formula* on that page, the limsups of $\log|G(iy)|/|y|$ for $y$ tending to $\infty$ and to $-\infty$ are *equal.* To see that, observe that the limsup for $y\to\infty$ is actually a *limit* (see remark, p. 49), and that $\overline{G(\bar{z})}/G(z) = B(z)$ is a Blaschke product like the one figuring in the remark on p. 58. The argument of pp. 57–8 shows, however, that then the limsup of $\log|B(iy)|/y$ for $y\to\infty$ is *zero.*

$$Q_n(x) \leqslant \frac{1}{x}\int_0^n \log\left|1+\frac{x^2}{t^2}\right| \mathrm{d}\nu(t)$$

$$\leqslant \frac{1}{x}\int_0^\infty \log\left|1+\frac{x^2}{t^2}\right| \mathrm{d}\nu(t), \qquad x>0.$$

Since $\nu(t) \leqslant Kt$, the right-hand member comes out $\leqslant \pi K$ on integrating by parts. This, together with the preceding lemma, shows that

$$0 \leqslant Q_n(x) \leqslant \pi K \qquad \text{for } x>0.$$

However, for *large* $x$,

$$\mathrm{d}(x\Omega_{\mathscr{D}}(x)) = \left(\frac{\text{const.}}{x^3} + O\left(\frac{1}{x^5}\right)\right)\mathrm{d}x$$

(see just before the theorem of article 3). Therefore

$$\int_0^\infty Q(x)\mathrm{d}(x\Omega_{\mathscr{D}}(x)) = \lim_{n\to\infty}\int_0^\infty Q_n(x)\mathrm{d}(x\Omega_{\mathscr{D}}(x))$$

by dominated convergence.

The right-hand limit can also be expressed as an inner product in a certain real Hilbert space. The latter – call it $\mathfrak{H}$ – is the *completion with respect to the norm* $\|\ \|_E$ of the collection of real Green potentials

$$u(x) = \int_0^\infty \log\left|\frac{x+t}{x-t}\right| \mathrm{d}\rho(t)$$

such that

$$\int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right| |\mathrm{d}\rho(t)||\mathrm{d}\rho(x)| < \infty;$$

the positive definite bilinear form $\langle\ ,\ \rangle_E$ extends by continuity to $\mathfrak{H}$ for which it serves as inner product. For each $n$, we have

$$\int_0^\infty Q_n(x)\mathrm{d}(x\Omega_{\mathscr{D}}(x)) = E\left(\mathrm{d}\left(\frac{\nu_n(t)}{t}\right),\ \mathrm{d}(t\Omega_{\mathscr{D}}(t))\right) = \langle Q_n, P\rangle_E,$$

where

$$P(x) = x(G_{\mathscr{D}}(x,0)+G_{\mathscr{D}}(-x,0))\ ;$$

here only Green potentials associated with *absolutely convergent* energy integrals are involved. By the lemma, however,

$$\|Q_n\|_E \leqslant \frac{\pi}{2}\nu'(0),$$

so a *subsequence* of $\{Q_n\}$, which we may as well *also* denote by $\{Q_n\}$, *converges weakly in* $\mathfrak{H}$ *to some element* $q$ *of that space.* (Here, we do *not need* to 'identify' $q$ with the function $Q(x)$, although that can easily be done.) In view of the previous limit relation, we see that

$$\int_0^{\infty} Q(x)\mathrm{d}(x\Omega_{\mathscr{D}}(x)) \;=\; \lim_{n\to\infty}\langle Q_n, P\rangle_E \;=\; \langle q, P\rangle_E.$$

Thence, by Schwarz' inequality and the result of article 3,

$$\left|\int_0^{\infty} Q(x)\mathrm{d}(x\Omega_{\mathscr{D}}(x))\right| \;\leqslant\; \|q\|_E\|P\|_E$$
$$=\; \|q\|_E\sqrt{(E(\mathrm{d}(t\Omega_{\mathscr{D}}(t)),\,\mathrm{d}(t\Omega_{\mathscr{D}}(t))))} \;\leqslant\; \sqrt{\pi Y_{\mathscr{D}}(0)}\,\|q\|_E.$$

Returning to the beginning of this proof, we see that

$$\int_{\partial\mathscr{D}} \log|G(x)|\mathrm{d}\omega_{\mathscr{D}}(x,0) \;\leqslant\; Y_{\mathscr{D}}(0)\int_0^{\infty}\frac{\log|G(x)|}{x^2}\mathrm{d}x$$
$$+\; \sqrt{\pi Y_{\mathscr{D}}(0)}\,\|q\|_E,$$

and thus need an estimate for $\|q\|_E$. The obvious one, $\|q\|_E \leqslant \liminf_{n\to\infty}\|Q_n\|_E \leqslant \pi\nu'(0)/2$, is not good enough to give us what we want here, so we argue as follows.

The weak convergence of $Q_n$ to $q$ in $\mathfrak{H}$ implies first of all that

$$\|q\|_E^2 \;=\; \lim_{n\to\infty}\langle q, Q_n\rangle_E.$$

*Fix* any $n$; then, by weak convergence again,

$$\langle q, Q_n\rangle_E \;=\; \lim_{k\to\infty}\langle Q_k, Q_n\rangle_E \;=\; -\lim_{k\to\infty}\int_0^{\infty} Q_k(x)\,\mathrm{d}\left(\frac{\nu_n(x)}{x}\right).$$

Here, $\mathrm{d}(\nu_n(x)/x)$ is just $-(\nu(n)/x^2)\mathrm{d}x$ for $x > n$, so, since $0 \leqslant Q_k(x) \leqslant \pi K$, we have, by dominated convergence,

$$-\lim_{k\to\infty}\int_0^{\infty} Q_k(x)\,\mathrm{d}\left(\frac{\nu_n(x)}{x}\right) \;=\; -\int_0^{\infty} Q(x)\,\mathrm{d}\left(\frac{\nu_n(x)}{x}\right)$$

which, $Q(x)$ being positive, is

$$\leqslant\; \int_0^{\infty} Q(x)\frac{\nu_n(x)}{x^2}\mathrm{d}x.$$

Again, $\nu_n(x) \leqslant \nu(x)$ for $x \geqslant 0$, so finally

$$\langle q, Q_n\rangle_E \;\leqslant\; \int_0^{\infty} Q(x)\frac{\nu(x)}{x^2}\mathrm{d}x \;=\; \int_0^{\infty}\frac{\log|G(x)|}{x^2}\frac{\nu(x)}{x}\mathrm{d}x$$

for each fixed $n$. The right-hand integral was already estimated above, before the preceding lemma, and found to be

$$\leqslant \frac{2e}{\pi}\left(\frac{\pi a}{4} + \int_0^\infty \frac{\log|G(x)|}{x^2}dx\right)\int_0^\infty \frac{\log|G(x)|}{x^2}dx.$$

This quantity is thus $\geqslant \lim_{n\to\infty}\langle q, Q_n\rangle_E = \|q\|_E^2$, giving us an upper bound on $\|q\|_E$.

Substituting the estimate just obtained into the above inequality for $\int_{\partial\mathscr{D}}\log|G(x)|d\omega_{\mathscr{D}}(x,0)$, we have the theorem. The proof is complete.

***Corollary***. *Let $G(z)$ and the domain $\mathscr{D}$ be as in the hypothesis of the theorem. If $v(z)$, subharmonic in $\mathscr{D}$ and continuous up to $\partial\mathscr{D}$, satisfies*

$$v(t) \leqslant \log|G(t)|, \qquad t\in\partial\mathscr{D},$$

*and*

$$v(z) \leqslant A|\Im z| + O(1)$$

*with some real $A$, we have*

$$v(0) \leqslant Y_{\mathscr{D}}(0)\left\{A + J + \sqrt{\left(2eJ\left(J+\frac{\pi a}{4}\right)\right)}\right\},$$

*where*

$$J = \int_0^\infty \frac{\log|G(x)|}{x^2}dx$$

*and $a$ is the type of $G$.*

This result will be used in proving the Beurling–Malliavin multiplier theorem in Chapter XI.

**Problem 28**

Let $G(z)$, entire and of exponential type, be given by the above boxed product formula and satisfy the hypothesis of the preceding theorem. Suppose also that

$$\frac{\log|G(iy)|}{|y|} \longrightarrow a \quad \text{for} \quad y \longrightarrow \pm\infty.$$

The purpose of this problem is to improve the estimate of $\|(\log|G(x)|)/x\|_E$ obtained above.

(a) Show that $v'(0) = a/\pi + 2J/\pi^2$ and that $v(t)/t \longrightarrow a/\pi$ as $t\to\infty$. Here, $J$ has the same meaning as in the statement of the theorem.
(Hint. For the second relation, one may just indicate how to adapt the argument from §H.2 of Chapter III.)

(b) Show that

$$\int_0^\infty \frac{\log|G(x)|}{x}\frac{\nu(x)}{x^2}\,\mathrm{d}x = \frac{\pi^2}{4}\left((\nu'(0))^2 - \left(\lim_{t\to\infty}\frac{\nu(t)}{t}\right)^2\right).$$

(Hint. Integral on left is the *negative* of

$$\int_0^\infty\int_0^\infty \log\left|\frac{x+t}{x-t}\right|\mathrm{d}\left(\frac{\nu(t)}{t}\right)\frac{\nu(x)}{x^2}\,\mathrm{d}x.$$

Here, direct application of the method used to prove the second lemma of §B.4 is hampered by $(\mathrm{d}/\mathrm{d}t)(\nu(t)/t)$'s lack of regularity for large $t$; however, the following procedure works and is quite general.

For small $\delta > 0$ and large $L$ one can get $\varepsilon$, $0 < \varepsilon < \delta$, and $R > L$ making

$$\int_\delta^L\int_\varepsilon^R \log\left|\frac{x+t}{x-t}\right|\mathrm{d}\left(\frac{\nu(t)}{t}\right)\frac{\nu(x)}{x^2}\,\mathrm{d}x$$

*nearly equal* to the above iterated integral. The order of integration can now be reversed and then the second mean value theorem applied to show that $\int_\varepsilon^\delta\int_\delta^L$ and $\int_L^R\int_\delta^L$ are *both small in magnitude* when $\delta > 0$ is small and $L$ large. Our initial expression is thus closely approximated by

$$\int_\delta^L\int_\delta^L \log\left|\frac{x+t}{x-t}\right|\frac{\nu(x)}{x^2}\,\mathrm{d}x\,\mathrm{d}\left(\frac{\nu(t)}{t}\right).$$

Apply to this a suitable modification of the reasoning in the proof of the aforementioned lemma, and then make $\delta \to 0$, $L \to \infty$.)

(c) Hence show that

$$\int_0^\infty \frac{\log|G(x)|}{x^2}\frac{\nu(x)}{x}\,\mathrm{d}x = \frac{1}{\pi^2}J(J+\pi a)$$

so that

$$\int_0^\infty\left(\int_0^\infty \log\left|\frac{x+t}{x-t}\right|\mathrm{d}\left(\frac{\nu(t)}{t}\right)\right)\mathrm{d}\left(\frac{\nu(x)}{x}\right) \leqslant \frac{1}{\pi^2}J(J+\pi a).$$

# Addendum

## Improvement of Volberg's Theorem on the Logarithmic Integral. Work of Brennan, Borichev, Jöricke and Volberg.

Writing of §D in Chapter VII was completed early in 1984, and some copies of the MS were circulated that spring. At the beginning of 1987 I learned, first from V.P. Havin and then from N.K. Nikolskii, that the persons named in the title had extended the theorem of §D.6. Expositions of their work did not come into my hands until April and May of 1987, when I had finished going through the second proof sheets for this volume.

In these circumstances, time and space cannot allow for inclusion of a thorough presentation of the recent work here. It nevertheless seems important to describe *some* of it because the strengthened version of Volberg's theorem first obtained by Brennan is very likely close to being best possible. I am thankful to Nikolskii, Volberg and Borichev for having made sure that the material got to me in time for me to be able to include the following account.

The development given below is based on the methods worked out in §D of Chapter VII, and familiarity with that § on the part of the reader is assumed. In order to save space and avoid repetition, we will refer to §D frequently and use the symbols employed there whenever possible.

### 1. Brennan's improvement, for $M(\nu)/\nu^{1/2}$ monotone increasing

Let us return to the proof of the theorem in §D.6 of Chapter VII, starting from the place on p. 359 where $h(\xi)$ and the weight $w(r) = \exp(-h(\log(1/r)))$ were brought into play. We take over the notation used in that discussion without explaining it anew.

What is shown by the reasoning of pp. 359–73 is that *unless* $F(e^{i\vartheta})$

*vanishes identically,*

$$\int_{-\pi}^{\pi} \log|F(e^{i\vartheta})|\,d\vartheta \;>\; -\infty$$

provided that

$$h(\xi) \;\geqslant\; \text{const.}\ \xi^{-(1+\delta)}$$

with some $\delta > 0$ as $\xi \longrightarrow 0$, and that

$$\int_0^a \log h(\xi)\,d\xi \;=\; \infty$$

for small $a > 0$. *Brennan's result is that the first condition on h can be replaced by the requirement that $\xi h(\xi)$ be decreasing for small $\xi > 0$.* (The second condition then obviously implies that $\xi h(\xi) \longrightarrow \infty$ as $\xi \longrightarrow 0$.)

Borichev and Volberg made the important observation that Brennan's result is yielded by Volberg's original argument. To see how this comes about, we begin by noting that in §D.6 of Chapter VII, *no real use of the property* $h(\xi) \geqslant$ const. $\xi^{-(1+\delta)}$ *is made until one comes to step 5* on p. 369. Up to then, it is more than enough to have $h(\xi) \geqslant$ const. $\xi^{-c}$ with *some* $c > 0$ together with the integral condition on $\log h(\xi)$. *Step 5* itself, however, is carried out in rather clumsy fashion (see p. 370). The reader was probably aware of this, and especially of the wasteful manner of using that step's conclusion in the subsequent local estimate of $\omega(E, z)$ (pp. 370–2). At the top of p. 372, the smallness of $\int_{\gamma_\rho}(1/(1-|\zeta|))\,d\omega(\zeta,\rho)$ was used where its *smallness in relation to* $1/(1-\rho)$ would have sufficed!

Instead of verifying the conclusion of *step* 5, let us show that *the quantity*

$$(1-\rho)\int_{\gamma_\rho} \frac{d\omega(\zeta,\rho\zeta_0)}{1-|\zeta|}$$

*can be made as small as we please* for $\rho$ sufficiently close to 1 chosen according to the specifications at the bottom of p. 368, *under the assumption* that $\xi h(\xi)$ *decreases*, with the integral of $\log h(\xi)$ *divergent*.

The original argument for *step 5* is *unchanged* up to the point where the relation

$$(\ast) \qquad \int_{\gamma_\rho} h\left(\log\frac{1}{|\zeta|}\right) d\omega(\zeta,\rho) \;\leqslant\; \text{const.} \;+\; (h(\log(1/\rho^2)))^{\eta}$$

is obtained at the top of p. 370; here $\eta$ can be chosen *at pleasure* in the interval $(0, 1)$, the construction following *step 3* (pp. 365–6) and subsequent carrying out of *step 4* being in no way hindered. Write now

$$P(\xi) \;=\; \xi h(\xi);$$

under the present circumstances $P(\xi)$ is *decreasing* for small $\xi > 0$. Since $\gamma_\rho$, recall, lies in the ring $\{\rho^2 \leqslant |\zeta| < 1\}$, we then have, for $\rho$ near 1,

$$\int_{\gamma_\rho} \frac{\mathrm{d}\omega(\zeta,\rho)}{1-|\zeta|} \;\leqslant\; 2\int_{\gamma_\rho} \frac{\mathrm{d}\omega(\zeta,\rho)}{\log(1/|\zeta|)} \;=\; 2\int_{\gamma_\rho} \frac{h(\log(1/|\zeta|))}{P(\log(1/|\zeta|))}\mathrm{d}\omega(\zeta,\rho)$$
$$\leqslant\; \frac{2}{P(2\log(1/\rho))}\int_{\gamma_\rho} h(\log(1/|\zeta|))\mathrm{d}\omega(\zeta,\rho).$$

Referring to ($\ast\atop\ast$), we see that the last expression is

$$\leqslant\; \frac{2}{P(2\log(1/\rho))}\{\text{const.} \;+\; (h(2\log(1/\rho)))^\eta\}.$$

Here, the monotoneity of $P(\xi)$ makes it tend to $\infty$ for $\xi \longrightarrow 0$; otherwise $\int_0^a \log h(\xi)\,\mathrm{d}\xi$ would be *finite* for small $a > 0$ as already remarked. The function $h(\xi)$ also tends to $\infty$ for $\xi \longrightarrow 0$, so, for $\rho$ close to 1 the preceding quantity is

$$\leqslant\; 3\left\{\frac{h(2\log(1/\rho))}{P(2\log(1/\rho))}\right\}^\eta \;=\; \frac{3}{(\log(1/\rho^2))^\eta} \;\leqslant\; \frac{3}{(1-\rho)^\eta}.$$

We thus have

$$\int_{\gamma_\rho} \frac{\mathrm{d}\omega(\zeta,\rho)}{1-|\zeta|} \;\leqslant\; 3(1-\rho)^{-\eta} \;=\; \mathrm{o}(1/(1-\rho))$$

for values of $\rho$ tending to 1 chosen in the way mentioned above, and our substitute for *step 5* is established.

This, as already noted, is all we need for the reasoning at the top of p. 372. The local estimate for $\omega(E,\rho)$ obtained on pp. 370–2 is therefore valid, and proof of the relation

$$\int_{-\pi}^{\pi} \log|F(\mathrm{e}^{\mathrm{i}\vartheta})|\,\mathrm{d}\vartheta \;>\; -\infty$$

is completed as on pp. 372–3.

It may well appear that the argument just made did not make full use of the monotoneity of $\xi h(\xi)$. However that may be, this requirement does not seem capable of further significant relaxation, as we shall see in the next two articles. At present, let us translate our conclusion into a result involving the majorant $M(\nu)$ figuring in Volberg's theorem (p. 356).

In the statement of that theorem, two regularity properties are required of the increasing function $M(\nu)$ in addition to the divergence of $\sum_1^\infty M(n)/n^2$, namely, *that $M(\nu)/\nu$ be decreasing* and that

$$M(\nu) \geqslant \text{const.}\,\nu^\alpha$$

for large $\nu$, where $\alpha > 1/2$. The *first* of these properties is (for us) practically equivalent to *concavity* of $M(\nu)$ by the theorem on p. 326. The concavity is needed for Dynkin's theorem (p. 339) and is not at issue here. Our interest is in replacing the *second* property by a *weaker* one. That being the object, there is no point in trying to gild the lily, and we may as well phrase our result for *concave majorants* $M(\nu)$. Indeed, nothing is really lost by sticking to *infinitely differentiable ones* with $M''(\nu) < 0$ and $M'(\nu) \longrightarrow 0$ for $\nu \longrightarrow \infty$, as long as that simplifies matters. See the theorem, p. 326 and the subsequent discussion on pp. 328–30; see also the beginning of the proof of the theorem in the next article.

With this simplification granted, passage from the result just arrived at to one stated in terms of $M(\nu)$ is provided by the easy

***Lemma.*** Let $M(\nu)$ be infinitely differentiable for $\nu > 0$ with $M''(\nu) < 0$ and $M'(\nu) \longrightarrow 0$ for $\nu \longrightarrow \infty$, and put (as usual)

$$h(\xi) \;=\; \sup_{\nu>0}(M(\nu) - \nu\xi).$$

Then $\xi h(\xi)$ is decreasing for small $\xi > 0$ if and only if $M(\nu)/\nu^{1/2}$ is increasing for large $\nu$.

***Proof.*** Under the given conditions, when $\xi > 0$ is sufficiently small, $h(\xi) = M(\nu) - \nu\xi$ for the *unique* $\nu$ with $M'(\nu) = \xi$ by the lemmas on pp. 330 and 332. Thus,

$$M'(\nu)h(M'(\nu)) \;=\; M(\nu)M'(\nu) - \nu(M'(\nu))^2,$$

so, since $M'(\nu)$ tends monotonically to zero as $\nu \longrightarrow \infty$, $\xi h(\xi)$ is *decreasing* for small $\xi > 0$ if and only if the right side of the last relation is *increasing* for large $\nu$. But

$$\frac{\mathrm{d}}{\mathrm{d}\nu}(M(\nu)M'(\nu) - \nu(M'(\nu))^2) \;=\; M''(\nu)M(\nu) - 2\nu M''(\nu)M'(\nu)$$

$$= \; -2\nu^{3/2}M''(\nu)\frac{\mathrm{d}}{\mathrm{d}\nu}\left(\frac{M(\nu)}{\nu^{1/2}}\right).$$

Since $M''(\nu) < 0$, the lemma is clear.

Referring now to the above result, we get, almost without further ado, the

***Theorem*** (Brennan). Let $M(\nu)$ be infinitely differentiable for $\nu > 0$, with $M''(\nu) < 0$,

$$\frac{M(\nu)}{\nu^{1/2}} \text{ increasing for large } \nu,$$

and

$$\sum_1^\infty M(n)/n^2 \;=\; \infty.$$

Suppose that

$$F(e^{i\vartheta}) \;\sim\; \sum_{-\infty}^{\infty} a_n e^{in\vartheta}$$

is continuous, with

$$|a_n| \;\leqslant\; \text{const.}\,e^{-M(|n|)} \qquad \text{for } n < 0.$$

Then, unless $F(e^{i\vartheta})$ vanishes identically,

$$\int_{-\pi}^{\pi} \log|F(e^{i\vartheta})|\,d\vartheta \;>\; -\infty.$$

Indeed, this follows directly by the lemma unless $\lim_{\nu\to\infty} M'(\nu) > 0$. Then, however, the theorem is true anyway – see p. 328.

## 2. Discussion

Brennan's result *really is* more general than the theorem on p. 356. That's because the hypothesis of the former one is fulfilled for any function $F(e^{i\vartheta})$ satisfying the hypothesis of the latter, thanks to the following

***Theorem.*** Let $M(\nu)$, increasing and with $M(\nu)/\nu$ decreasing, satisfy the condition $\sum_1^\infty M(n)/n^2 \;=\; \infty$ and have $M(\nu) \;\geqslant\; \text{const.}\,\nu^{\frac{1}{2}+\delta}$ for large $\nu$, where $\delta > 0$. Then there is an infinitely differentiable function $M_0(\nu)$, with $M_0''(\nu) < 0$,

$$M_0(\nu) \;\leqslant\; M(\nu) \text{ for large } \nu,$$

$M_0(\nu)/\nu^{1/2}$ increasing, and $\sum_1^\infty \; M_0(n)/n^2 \;=\; \infty.$

***Proof.*** By the theorem on p. 326 we can, wlog, take $M(\nu)$ to be *actually concave*. It is then sufficient to obtain *any concave minorant* $M_*(\nu)$ of $M(\nu)$

with $M_*(\nu)/\nu^{1/2}$ *increasing* and $\int_1^\infty (M_*(\nu)/\nu^2)\mathrm{d}\nu$ *divergent*, for from such a minorant one easily obtains an $M_0(\nu)$ with the additional regularity affirmed by the theorem.

The procedure for doing this is like the one of pp. 229–30. Starting with an $M_*(\nu)$, one first puts $M_1(\nu) = M_*(\nu) + \nu^{1/2}$ and then, using a $\mathscr{C}_\infty$ function $\varphi(\tau)$ having the graph shown on p. 329, takes

$$M_0(\nu) = c\int_0^1 M_1(\nu-\tau)\varphi(\tau)\,\mathrm{d}\tau$$

for $\nu > 1$ with a suitable small constant $c$. This function $M_0(\nu)$ (defined in any convenient fashion for $0 < \nu \leqslant 1$) is readily seen to do the job.

Our main task is thus the construction of an $M_*(\nu)$. For that it is helpful to make a further reduction, arranging for $M(\nu)$ to have a *piecewise linear graph starting out from the origin.* That poses no problem; we simply replace our *given* concave function $M(\nu)$ by *another*, with graph consisting of a straight segment going from the origin to a point on the graph of the original function followed by suitably chosen *successive chords* of that graph. This having been attended to, we let $R(\nu)$ be the *largest increasing minorant* of $M(\nu)/\nu^{1/2}$ and then put

$$M_*(\nu) = \nu^{1/2}R(\nu);$$

this of course makes $M_*(\nu)/\nu^{1/2}$ automatically increasing and $M_*(\nu) \leqslant M(\nu)$.

Thanks to our initial adjustment to the graph of $M(\nu)$, we have $M(\nu)/\nu^{1/2} \longrightarrow 0$ for $\nu \longrightarrow 0$. Hence, since $M(\nu) \geqslant$ const. $\nu^{\frac{1}{2}+\delta}$ for large $\nu$, $R(\nu)$ must tend to $\infty$ for $\nu \longrightarrow \infty$, and *coincides* with $M(\nu)/\nu^{1/2}$ *save on certain disjoint intervals* $(\alpha_k, \beta_k) \subset (0, \infty)$ for which

$$\frac{M(\alpha_k)}{\alpha_k^{1/2}} = R(\nu) = \frac{M(\beta_k)}{\beta_k^{1/2}}, \qquad \alpha_k \leqslant \nu \leqslant \beta_k.$$

*Concavity* of $M_*(\nu)$ follows from that of $M(\nu)$. The graph of $M_*(\nu)$ coincides with that of $M(\nu)$, save over the intervals $(\alpha_k, \beta_k)$, where it has *concave* arcs (along which $M_*(\nu)$ is proportional to $\nu^{1/2}$), lying *below* the corresponding arcs for $M(\nu)$ and *meeting those* at their *endpoints.* The *former* graph is thus clearly concave if the other one is.

Proving that $\sum_1^\infty M_*(n)/n^2 = \infty$ is trickier. There would be no trouble at all here if we could be sure that the ratios $\beta_k/\alpha_k$ were *bounded*, but we cannot assume that and our argument makes strong use of the fact that $\delta > 0$ in the condition $M(\nu) >$ const. $\nu^{\frac{1}{2}+\delta}$.

We again appeal to the special structure of $M(\nu)$'s graph to argue that the *local maxima* of $M(\nu)/\nu^{1/2}$, and hence the *intervals* $(\alpha_k, \beta_k)$, *cannot accumulate* at any finite point. Those intervals can therefore be indexed from left to right, and in the event that two adjacent ones should touch at their endpoints, we can consolidate them to form a single larger interval and then relabel. In this fashion, we arrive at a set-up where

$$0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots,$$

with $M_*(v) = M(v)$ *outside* the union of the (perhaps new) $(\alpha_k, \beta_k)$, and

$$M_*(v) = \left(\frac{v}{\alpha_k}\right)^{1/2} M(\alpha_k) = \left(\frac{\beta_k}{v}\right)^{1/2} M(\beta_k) \qquad \text{for } \alpha_k \leqslant v \leqslant \beta_k.$$

It is convenient to fix a $\beta_0$ with $0 < \beta_0 < \alpha_1$. Then, since $M(v)/v$ *decreases*, $M(\alpha_1) \leqslant (\alpha_1/\beta_0)M(\beta_0)$, so, by the preceding relation,

$$M(\beta_1) = \left(\frac{\beta_1}{\alpha_1}\right)^{1/2} M(\alpha_1) \leqslant \left(\frac{\beta_1}{\alpha_1}\right)^{1/2} \frac{\alpha_1}{\beta_0} M(\beta_0).$$

In like manner we find first that $M(\alpha_2) \leqslant (\alpha_2/\beta_1)M(\beta_1)$ and thence that $M(\beta_2) \leqslant (\beta_2/\alpha_2)^{1/2}(\alpha_2/\beta_1)M(\beta_1)$ which, substituted into the previous, yields

$$M(\beta_2) \leqslant \left(\frac{\beta_2}{\alpha_2}\right)^{1/2} \frac{\alpha_2}{\beta_1}\left(\frac{\beta_1}{\alpha_1}\right)^{1/2} \frac{\alpha_1}{\beta_0} M(\beta_0).$$

Continuing in this fashion, we see that

$$M(\beta_n) \leqslant \left(\frac{\beta_n}{\alpha_n}\right)^{1/2} \frac{\alpha_n}{\beta_{n-1}}\left(\frac{\beta_{n-1}}{\alpha_{n-1}}\right)^{1/2} \frac{\alpha_{n-1}}{\beta_{n-2}} \cdots \left(\frac{\beta_1}{\alpha_1}\right)^{1/2} \frac{\alpha_1}{\beta_0} M(\beta_0).$$

Now by hypothesis, $M(\beta_n) \geqslant C\beta_n^{\frac{1}{2}+\delta}$ where, wlog, $C = 1$. Use this with the relation just found and then divide the resulting inequality by $\alpha_n^{\frac{1}{2}+\delta}$, noting that

$$\alpha_n = \frac{\alpha_n}{\beta_{n-1}} \frac{\beta_{n-1}}{\alpha_{n-1}} \frac{\alpha_{n-1}}{\beta_{n-2}} \cdots \frac{\beta_1}{\alpha_1} \frac{\alpha_1}{\beta_0} \beta_0.$$

One gets

$$\left(\frac{\beta_n}{\alpha_n}\right)^{\frac{1}{2}+\delta} \leqslant \left(\frac{\beta_n}{\alpha_n}\right)^{\frac{1}{2}} \frac{\dfrac{\alpha_n}{\beta_{n-1}}\left(\dfrac{\beta_{n-1}}{\alpha_{n-1}}\right)^{1/2} \cdots \left(\dfrac{\beta_1}{\alpha_1}\right)^{1/2} \dfrac{\alpha_1}{\beta_0} M(\beta_0)}{\left\{\dfrac{\alpha_n}{\beta_{n-1}} \dfrac{\beta_{n-1}}{\alpha_{n-1}} \cdots \dfrac{\beta_1}{\alpha_1} \dfrac{\alpha_1}{\beta_0} \beta_0\right\}^{1/2+\delta}}.$$

After cancelling $(\beta_n/\alpha_n)^{1/2}$ from both sides and rearranging, this becomes

$$\left(\frac{\beta_n}{\alpha_n} \frac{\beta_{n-1}}{\alpha_{n-1}} \cdots \frac{\beta_1}{\alpha_1}\right)^{\delta} \leqslant \left(\frac{\alpha_n}{\beta_{n-1}} \frac{\alpha_{n-1}}{\beta_{n-2}} \cdots \frac{\alpha_1}{\beta_0}\right)^{\frac{1}{2}-\delta} \frac{M(\beta_0)}{\beta_0^{\frac{1}{2}+\delta}}.$$

There is of course no loss of generality here in assuming $\delta < 1/2$. The last formula can be rewritten

$$\sum_{k=1}^{n}\log\left(\frac{\beta_k}{\alpha_k}\right) \leqslant c + \frac{1-2\delta}{2\delta}\sum_{k=1}^{n}\log\left(\frac{\alpha_k}{\beta_{k-1}}\right)$$

where $c = (1/\delta)\log(M(\beta_0)/\beta_0^{1/2+\delta})$ is independent of $n$, and this estimate makes it possible for us to compare some integrals of $M(v)/v^2$ over complementary sets.

Since $M(v)/v$ is decreasing, we have

$$\int_{\beta_{n-1}}^{\alpha_n}\frac{M(v)}{v^2}\mathrm{d}v \geqslant \frac{M(\alpha_n)}{\alpha_n}\int_{\beta_{n-1}}^{\alpha_n}\frac{\mathrm{d}v}{v} = \frac{M(\alpha_n)}{\alpha_n}\log\frac{\alpha_n}{\beta_{n-1}},$$

and at the same time,

$$\int_{\alpha_n}^{\beta_n}\frac{M(v)}{v^2}\mathrm{d}v \leqslant \frac{M(\alpha_n)}{\alpha_n}\int_{\alpha_n}^{\beta_n}\frac{\mathrm{d}v}{v} = \frac{M(\alpha_n)}{\alpha_n}\log\frac{\beta_n}{\alpha_n}.$$

From the *second* inequality,

$$\sum_{n=1}^{N}\int_{\alpha_n}^{\beta_n}\frac{M(v)}{v^2}\mathrm{d}v \leqslant \sum_{n=1}^{N}\frac{M(\alpha_n)}{\alpha_n}\log\frac{\beta_n}{\alpha_n},$$

and partial summation converts the right side to

$$\sum_{n=1}^{N-1}\left\{\frac{M(\alpha_n)}{\alpha_n} - \frac{M(\alpha_{n+1})}{\alpha_{n+1}}\right\}\sum_{k=1}^{n}\log\frac{\beta_k}{\alpha_k} + \frac{M(\alpha_N)}{\alpha_N}\sum_{k=1}^{N}\log\frac{\beta_k}{\alpha_k}.$$

The ratios $M(\alpha_n)/\alpha_n$ are, however, decreasing, so we may apply the estimate obtained above to see that the last expression is

$$\leqslant \sum_{n=1}^{N-1}\left\{\frac{M(\alpha_n)}{\alpha_n} - \frac{M(\alpha_{n+1})}{\alpha_{n+1}}\right\}\left\{\frac{1-2\delta}{2\delta}\sum_{k=1}^{n}\log\frac{\alpha_k}{\beta_{k-1}} + c\right\}$$
$$+ \frac{M(\alpha_N)}{\alpha_N}\left\{\frac{1-2\delta}{2\delta}\sum_{k=1}^{N}\log\frac{\alpha_k}{\beta_{k-1}} + c\right\},$$

which, by reverse summation by parts, boils down to

$$\frac{1-2\delta}{2\delta}\sum_{n=1}^{N}\frac{M(\alpha_n)}{\alpha_n}\log\frac{\alpha_n}{\beta_{n-1}} + c\frac{M(\alpha_1)}{\alpha_1}.$$

This in turn is

$$\leqslant \frac{1-2\delta}{2\delta}\sum_{n=1}^{N}\int_{\beta_{n-1}}^{\alpha_n}\frac{M(v)}{v^2}\mathrm{d}v + c\frac{M(\alpha_1)}{\alpha_1}$$

by the *first* of the above inequalities, so, since $M(v) = M_*(v)$ on each of the intervals $[\beta_{n-1}, \alpha_n]$, we have finally

$$\sum_{n=1}^{N} \int_{\alpha_n}^{\beta_n} \frac{M(v)}{v^2} \mathrm{d}v \leqslant \frac{1-2\delta}{2\delta} \sum_{n=1}^{N} \int_{\beta_{n-1}}^{\alpha_n} \frac{M_*(v)}{v^2} \mathrm{d}v + c\frac{M(\alpha_1)}{\alpha_1}.$$

Adding $\sum_{n=1}^{N} \int_{\beta_{n-1}}^{\alpha_n} (M(v)/v^2)\mathrm{d}v = \sum_{n=1}^{N} \int_{\beta_{n-1}}^{\alpha_n} (M_*(v)/v^2)\mathrm{d}v$ to both sides of this relation one gets (*a fortiori*!)

$$\int_{\beta_0}^{\beta_N} \frac{M(v)}{v^2} \mathrm{d}v < c\frac{M(\alpha_1)}{\alpha_1} + \frac{1}{2\delta} \int_{\beta_0}^{\alpha_N} \frac{M_*(v)}{v^2} \mathrm{d}v,$$

and thence

$$\int_{\beta_0}^{\infty} \frac{M(v)}{v^2} \mathrm{d}v \leqslant c\frac{M(\alpha_1)}{\alpha_1} + \frac{1}{2\delta} \int_{\beta_0}^{\infty} \frac{M_*(v)}{v^2} \mathrm{d}v.$$

In the present circumstances, however, divergence of $\sum_1^\infty M(n)/n^2$ is equivalent to that of the left-hand integral and divergence of $\sum_1^\infty M_*(n)/n^2$ equivalent to that of the integral on the right. Our assumptions on $M(v)$ thus make $\sum_1^\infty M_*(n)/n^2 = \infty$, and the proof of the theorem is complete.

The second observation to be made about Brennan's theorem is that its *monotoneity requirement* on $M(v)/v^{1/2}$ is probably *incapable of much further relaxation*. That depends on an example mentioned at the end of Borichev and Volberg's preprint. Unfortunately, they do not describe the construction of the example, so I cannot give it here. *Let us, in the present addendum, assume that their construction is right and show how to deduce from this supposition that Brennan's result is close to being best possible in a sense to be soon made precise.*

The example of Borichev and Volberg, if correct, furnishes a decreasing function $h(\xi)$ with $\xi h(\xi) \geqslant 1$ and $\int_0^1 \log h(\xi)\,\mathrm{d}\xi = \infty$ together with $F(z)$, bounded and $\mathscr{C}_\infty$ in $\{|z| < 1\}$ and having the non-tangential boundary value $F(\mathrm{e}^{\mathrm{i}\vartheta})$ a.e. on $\{|z| = 1\}$, such that

$$\left|\frac{\partial F(z)}{\partial \bar{z}}\right| \leqslant \exp\left(-h\left(\log\frac{1}{|z|}\right)\right) \qquad \text{for } |z| < 1,$$

while

$$\int_{-\pi}^{\pi} \log|F(\mathrm{e}^{\mathrm{i}\vartheta})|\,\mathrm{d}\vartheta = -\infty$$

although $F(\mathrm{e}^{\mathrm{i}\vartheta})$ *is not* a.e. *zero*.

The procedure we are about to follow comes from the paper of Jöricke and Volberg, and will be used again to investigate the more complicated situation taken up in the next article. In order that the reader may first see its main idea unencumbered by detail, let us *for now* make *an additional assumption that the function* $F(z)$ *supplied by the Borichev–Volberg construction is continuous up to* $|z| = 1$. At the end of the next article we will see that a counter-example to further extension of the $L_1$ version of Brennan's result given there can be obtained *without this continuity*. Assuming it here enables us to just *take over* the constructions of §D.6, Chapter VII.

The present function $F(z)$ is to be subjected to the treatment applied to the one thus denoted in §D.6, beginning on p. 359. We also employ the symbols

$$w(r) \;=\; \exp\left(-h\left(\log\frac{1}{r}\right)\right),$$

$\mathcal{O}$, $B$, $\Phi$, $\Omega$, &c with the meanings adopted there.

Starting with $F(z)$, we construct a *continuous* function $g(e^{i\vartheta})$ on $\{|z| = 1\}$ and a *concave* increasing majorant $M(v)$ having the following properties:

(i) $\quad g(e^{i\vartheta}) \;\not\equiv\; 0,$

(ii) $\quad \displaystyle\int_{-\pi}^{\pi} \log|g(e^{i\vartheta})|\,d\vartheta \;=\; -\infty,$

(iii) $\quad \displaystyle\sum_1^{\infty} M(n)/n^2 \;=\; \infty,$

(iv) $\quad M(v)/v^{1/2} \;\geqslant\; 2,$

(v) $\quad g(e^{i\vartheta}) \;\sim\; \displaystyle\sum_{-\infty}^{\infty} a_n e^{in\vartheta}$ with $|a_n| \;\leqslant\;$ const.$\,e^{-M(|n|)}$ for $n < 0$.

It is clear from this *how close* Brennan's result comes to being *best possible* provided that the above assumptions are granted.

The weight $w(r)$ we are now using is decreasing, and, since $\xi h(\xi) \geqslant 1$, goes to zero rapidly enough for the reasoning followed in *steps 2* and *3* of §D.6, Chapter VII to carry over without change.

But the argument made for *step 1* on p. 361 requires modification. Here, since $F(z)$ is continuous on the closed unit disk and $\neq 0$ on its circumference, there is a non-empty open arc $I$ of that circumference on which $|F(e^{i\vartheta})|$ is *bounded away from zero*. Then, because $w(r) \longrightarrow 0$ for $r \longrightarrow 1$, the open set $\mathcal{O}$ must have a *component* – call it $\mathcal{O}'$ – *abutting* on $I$. If, at the same time, $B$ contained a non-void open arc $J$ of the unit circumference, we would have $\partial\mathcal{O}' \cap J = \emptyset$. In that event one could reason with the analytic function $\Phi(z)$ as at the bottom of p. 362, because $|\Phi(\zeta)| \leqslant \text{const.}\, w(|\zeta|)$ on $\partial\mathcal{O}' \cap \{|\zeta| < 1\}$. In that way, one would find that $\Phi(z) \equiv 0$ in $\mathcal{O}'$, making $F(e^{i\vartheta}) \equiv 0$ on $I$, a contradiction. Hence no such arc as $J$ can exist.

Once *steps 1, 2* and *3* are carried out, we fix any $\rho$, $0 < \rho < 1$ and take the connected set $\Omega = \Omega(\rho) \subseteq \{\rho < |z| < 1\}$ described at the top of p. 363. As pointed out on p. 364, $\partial\Omega$ *includes the whole unit circumference.*

Fix now a $z_0 \in \Omega$. Given an integer $n \geqslant 0$, let us apply *Poisson's formula* to the function $z^n\Phi(z)$, *harmonic* (since analytic!) in $\Omega$ and *continuous* up to $\partial\Omega$. We get

$$z_0^n\Phi(z_0) = \int_{\partial\Omega} \zeta^n\Phi(\zeta)\, d\omega_\Omega(\zeta, z_0)$$

where, as usual, $\omega_\Omega(\ ,\ )$ is harmonic measure for $\Omega$. The boundary $\partial\Omega$ consists of the unit circumference together with

$$\gamma = \partial\Omega \cap \{|z| < 1\},$$

so the last relation can be rewritten

$$\int_{-\pi}^{\pi} e^{in\vartheta}\Phi(e^{i\vartheta})\, d\omega_\Omega(e^{i\vartheta}, z_0) = z_0^n\Phi(z_0) - \int_\gamma \zeta^n\Phi(\zeta) d\omega_\Omega(\zeta, z_0).$$

Let us first examine the *right side* of this formula.

With $\log\dfrac{1}{|z_0|} = \xi_0 > 0$, the *first* term on the right has modulus $|\Phi(z_0)|e^{-n\xi_0}$. Concerning the *second* term, we recall that by the construction of $\mathcal{O}$, $|\Phi(\zeta)| \leqslant \text{const.}\, w(|\zeta|)$ on $\gamma$, including on any arcs thereof lying on $\{|\zeta| = \rho\}$ and in $\mathcal{O}$, as long as the constant is chosen large enough. Therefore, writing

$$M(v) = \inf_{\xi > 0}(h(\xi) + \xi v)$$

we have, since $w(|\zeta|) = \exp(-h(\xi))$ with $\xi = \log(1/|\zeta|)$,

$$|\zeta^n\Phi(\zeta)| \leqslant \text{const.}\, e^{-M(n)}, \qquad \zeta \in \gamma.$$

Harmonic measure of course has total mass 1. Our second term is hence $\leqslant \text{const.}\, e^{-M(n)}$ in magnitude, and we find that altogether, for $n \geqslant 0$,

$$\left|\int_{-\pi}^{\pi} e^{in\vartheta}\Phi(e^{i\vartheta})\, d\omega_\Omega(e^{i\vartheta}, z_0)\right| \leqslant \text{const.}\,(e^{-n\xi_0} + e^{-M(n)}).$$

It will be seen presently that $e^{-M(n)}$ dominates $e^{-n\xi_0}$ for large $n$, so that the latter term can be dropped from this last relation. On account of that, we next turn our attention to $M(v)$. This function is *concave* by its definition, and, since $h(\xi) \geqslant 1/\xi$, easily seen to be $\geqslant 2v^{1/2}$ and thus enjoy property (iv) of the above list. Because $h(\xi)$ is decreasing and $\int_0^1 \log h(\xi)\,d\xi = \infty$, we have $\int_1^\infty (M(v)/v^2)\,dv = \infty$ by the theorem on p. 337. That, however, implies that $\sum_1^\infty M(n)/n^2 = \infty$, which is property (iii).

We look now at the measure $\Phi(e^{i\vartheta})\,d\omega_\Omega(e^{i\vartheta}, z_0)$ appearing *on the left* in the preceding relation. In the first place, $d\omega_\Omega(e^{i\vartheta}, z_0)$ is *absolutely continuous* with respect to $d\vartheta$ on $\{|\zeta| = 1\}$, and indeed $\leqslant C\,d\vartheta$ there, the constant $C$ depending on $z_0$. This follows immediately by comparison of $d\omega_\Omega(e^{i\vartheta}, z_0)$ with harmonic measure for the whole unit disk. We can therefore write

$$\Phi(e^{i\vartheta})\,d\omega_\Omega(e^{i\vartheta}, z_0) = g(e^{i\vartheta})\,d\vartheta$$

with a *bounded* function $g$, and have just the *moduli of* $2\pi g(e^{i\vartheta})$'s *Fourier coefficients* (of negative index) standing on the left in the above relation.

In fact, $d\omega_\Omega(e^{i\vartheta}, z_0)$ has *more regularity* than we have just noted. The *derivative* $d\omega_\Omega(e^{i\vartheta}, z_0)/d\vartheta$ is, for instance, *strictly positive* in the interior of each arc $I_k$ of the unit circumference contiguous to $B$'s intersection therewith. To see this one may, given $I_k$, construct a very shallow sectorial box $\mathscr{S}$ in the unit disk with base on $I_k$ and *slightly shorter* than the latter. A shallow enough $\mathscr{S}$ will have none of $\partial\Omega$ in its interior since $\Omega$ *abuts* on $I_k$. One may therefore compare $d\omega_\Omega(e^{i\vartheta}, z)$ with harmonic measure for $\mathscr{S}$ when $z \in \mathscr{S}$ and $e^{i\vartheta}$ is on that box's base, and an application of Harnack then leads to the desired conclusion.

From this we can already see that $|g(e^{i\vartheta})|$ is *bounded away from zero* inside some of the arcs $I_k$, for instance, on the arc $I$ used at the beginning of this discussion. But there is more – $g(e^{i\vartheta})$ is *continuous* on the unit circumference. That follows immediately from *four* properties: the *continuity* of $\Phi(e^{i\vartheta})$, its *vanishing* for $e^{i\vartheta} \in B$, the *boundedness* of $d\omega_\Omega(e^{i\vartheta}, z_0)/d\vartheta$, and, finally, the *continuity* of this derivative in the interior of each arc $I_k$ contiguous to $B \cap \{|\zeta| = 1\}$. The first three of these we are sure of, so it suffices to verify the fourth.

For that purpose, it is easiest to use the formula

$$\frac{d\omega_\Omega(e^{i\vartheta}, z_0)}{d\vartheta} = \frac{d\omega_\Delta(e^{i\vartheta}, z_0)}{d\vartheta} - \int_\gamma \frac{d\omega_\Delta(e^{i\vartheta}, \zeta)}{d\vartheta}\,d\omega_\Omega(\zeta, z_0),$$

where $\omega_\Delta(\ \ , z_0)$ is ordinary harmonic measure for the unit disk $\Delta$ (cf. p. 371). For $e^{i\vartheta}$ moving along an arc $I_k$,

$$d\omega_\Delta(e^{i\vartheta}, \zeta)/d\vartheta = (1 - |\zeta|^2)/2\pi|\zeta - e^{i\vartheta}|^2$$

varies *continuously*, and *uniformly so*, for $\zeta$ ranging over any subset of $\Delta$