where $s = \sigma + it$ and where $N$ is any integer larger than $t$. If $N$ is the first integer larger than $t$, then the first three terms on the right have modulus less than a constant times $t^{-\sigma}$ as $t \longrightarrow \infty$. Backlund's method of estimating the remaining term shows easily that it is less than a constant times $t^{1-\sigma}$, but this method ignores the cancellations in the integral due to the rapid oscillation of $x^{it}$ for large $t$ and the following more refined estimate shows that the fourth term, like the first three, is less than a constant times $t^{-\sigma}$ so that the same is true of $R(\sigma, t)$ itself.

Hardy and Littlewood estimate the integral $\int_N^\infty \bar{B}_1(x)x^{-s-1}\,dx$ by observing that the Fourier series of $\bar{B}_1(x)$, namely, $\bar{B}_1(x) = -\sum_{n=1}^\infty (\sin 2\pi nx)/n\pi$ is boundedly convergent† so that the termwise integration

$$\int_N^\infty \bar{B}_1(x)x^{-s-1}\,dx = -\sum_{n=1}^\infty \frac{1}{n\pi}\int_N^\infty (\sin 2\pi nx)x^{-s-1}\,dx$$

is justified by the Lebesgue dominated convergence theorem.‡ Now

$$\begin{aligned}
&\frac{1}{n\pi}\int_N^\infty (\sin 2\pi nx)x^{-s-1}\,dx \\
&\quad = \frac{1}{2in\pi}\int_N^\infty (e^{2\pi inx} - e^{-2\pi inx})x^{-\sigma-1-it}\,dx \\
&\quad = \frac{1}{2\pi in}\int_N^\infty e^{iw(x)}x^{-\sigma-1}\,dx - \frac{1}{2\pi in}\int_N^\infty e^{-iv(x)}x^{-\sigma-1}\,dx,
\end{aligned}$$

†Let $S_N(x)$ denote the sum of the first $N$ terms. Then for $0 < x < \frac{1}{2}$ the difference $\bar{B}_1(x) - S_N(x)$ can be written in the form

$$\begin{aligned}
&x - \tfrac{1}{2} + \sum_{n=1}^N (\pi n)^{-1}\sin 2\pi nx \\
&\quad = \int_0^x \Big(1 + \sum_{n=1}^N 2\cos 2\pi nt\Big)\,dt - \frac{1}{2} \\
&\quad = \int_0^x \Big(\sum_{n=-N}^N e^{2\pi int}\Big)\,dt - \frac{1}{2} \\
&\quad = \int_0^x \frac{e^{2\pi iNt+\pi it} - e^{-2\pi iNt-\pi it}}{e^{\pi it} - e^{-\pi it}}\,dt - \frac{1}{2} \\
&\quad = \int_0^x \frac{\sin(2N+1)\pi t}{\sin \pi t}\,dt - \frac{1}{2} \\
&\quad = \int_0^x \frac{\sin(2N+1)\pi t}{\pi t}\,dt + \int_0^x \Big(\frac{1}{\sin \pi t} - \frac{1}{\pi t}\Big)\sin(2N+1)\pi t\,dt - \frac{1}{2} \\
&\quad = \frac{1}{\pi}\int_0^{(2N+1)\pi x} \frac{\sin u}{u}\,du + \int_0^x f(t)\sin(2N+1)\pi t\,dt - \frac{1}{2}
\end{aligned}$$

where $f(t) = (\sin \pi t)^{-1} - (\pi t)^{-1} = (\pi t - \sin \pi t)/\pi t \sin \pi t$ is analytic for $|t| < 1$ and therefore bounded for $|t| \le \frac{1}{2}$, say by $K$. Then since $\int_0^h \sin u/u\,du$ is bounded (its maximum occurs at $h = \pi$), it follows that $\bar{B}_1(x) - S_N(x)$ has modulus less than $\pi^{-1}\int_0^\pi u^{-1}\sin u\,du + \frac{1}{2}K + \frac{1}{2}$ for $0 < x \le \frac{1}{2}$ and for *all* $N$. Since $S_N(0) \equiv 0$, $S_N(-x) \equiv -S_N(x)$, the same bound holds for $-\frac{1}{2} \le x \le \frac{1}{2}$ and, since $\bar{B}_1(s)$ is bounded, this implies the desired result.

‡See, for example, Edwards [E1, p. 437].

where $w(x) = 2\pi nx - t \log x$ and $v(x) = 2\pi nx + t \log x$. Since

$$\frac{dw}{dx} = 2\pi n - \frac{t}{x} > 2\pi n - \frac{N}{N}$$

is positive on $\{N \le x < \infty\}$, $x$ can be considered as a function of $w$ and the first integral above can be expressed in terms of $w$

$$\lim_{K\to\infty} \frac{1}{2\pi i n} \int_N^K e^{iw(x)} x^{-\sigma-1}\,dx$$
$$= \lim_{K\to\infty} \frac{1}{2\pi n} \int_{w(N)}^{w(K)} (\sin w - i \cos w) x^{-\sigma-1} \frac{dw}{2\pi n - (t/x)}.$$

The real and imaginary parts of this integral are of the form $\int_A^B F(w) \sin w\,dw$ and $\int_A^B F(w) \cos w\,dw$, respectively, where $F$ is a monotone decreasing positive function. Therefore they can be estimated using the following elementary lemma.

**Lemma** If $F(x)$ is positive and monotone nonincreasing on $\{A \le x \le B\}$, then $\int_A^B F(x) \sin x\,dx$ is at most $2F(A)$ in absolute value. The same estimate applies to $\int_A^B F(x) \cos x\,dx$.

***Proof*** Let $A'$ be an even multiple of $\pi$ less than $A$ and let $B'$ be an odd multiple of $\pi$ greater than $B$, say $2\mu\pi = A' \le A < B \le B' < 2\nu\pi + \pi$. Extend $F$ to $\{A' \le x \le B'\}$ by defining it to be constantly $F(A)$ between $A'$ and $A$ and constantly $F(B)$ between $B$ and $B'$. Then

$$\begin{aligned}\int_{A'}^{B'} F(x)\sin x\,dx &= F(A)\int_{A'}^{A} \sin x\,dx + \int_A^B F(x)\sin x\,dx \\ &\quad + F(B)\int_B^{B'} \sin x\,dx \\ &= F(A)(-\cos A + 1) + \int_A^B F(x)\sin x\,dx \\ &\quad + F(B)(1 + \cos B) \\ &\ge \int_A^B F(x)\sin x\,dx.\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{A'}^{B'} F(x)\sin x\,dx &= \int_{2\mu\pi}^{2\mu\pi+\pi} F(x)\sin x\,dx + \int_{2\mu\pi+\pi}^{2\mu\pi+2\pi} F(x)\sin x\,dx + \cdots \\ &\quad + \int_{2\nu\pi}^{2\nu\pi+\pi} F(x)\sin x\,dx \\ &\le F(2\mu\pi)\int_{2\mu\pi}^{2\mu\pi+\pi} \sin x\,dx + F(2\mu\pi + 2\pi)\int_{2\mu\pi+\pi}^{2\mu\pi+2\pi} \sin x\,dx \\ &\quad + F(2\mu\pi + 2\pi)\int_{2\mu\pi+2\pi}^{2\mu\pi+3\pi} \sin x\,dx + \cdots \\ &\quad + F(2\nu\pi)\int_{2\nu\pi}^{2\nu\pi+\pi} \sin x\,dx \\ &= F(2\mu\pi)\int_{2\mu\pi}^{2\mu\pi+\pi} \sin x\,dx = 2F(A)\end{aligned}$$

which proves $\int_A^B F(x) \sin x\, dx \leq 2F(A)$. The proof of the lemma is now easily completed by using analogous arguments to find a lower estimate of $\int_A^B F(x) \sin x\, dx$ and upper and lower estimates of $\int_A^B F(x) \cos x\, dx$.

Applying the lemma to the integrals above (the real and imaginary parts of the integral involving $v$ as well as the integral involving $w$) shows easily that $|(n\pi)^{-1} \int_N^\infty (\sin 2\pi nx) x^{-s-1}\, dx|$ is less than a constant times $n^{-2} N^{-\sigma-1}$. Summing these estimates over $n$ shows that $|\int_N^\infty \bar{B}_1(x) x^{-s-1}\, dx|$ is less than a constant times $N^{-\sigma-1} \sim t^{-\sigma-1}$. Since $|s| \sim t$, this completes the proof that $R(\sigma, t)$ is less than a constant times $t^{-\sigma}$ as $t \longrightarrow \infty$, and shows, moreover, that the estimate is uniform in $\sigma$.

With this estimate of $R(\sigma, t)$ the evaluation of the average of $|\zeta(\sigma + it)|^2$ is not too difficult. In the first place the average value of $|\sum_{n<t} n^{-\sigma-it}|^2$ is

$$\begin{aligned}
&\frac{1}{T-1}\int_1^T \left(\sum_{m<t}\sum_{n<t} m^{-\sigma-it} n^{-\sigma+it}\right) dt \\
&\quad = \frac{1}{T-1}\int_1^T \sum_{n<t} n^{-2\sigma}\, dt + \frac{1}{T-1}\int_1^T \sum_{n<t}\sum_{m<n} m^{-\sigma} n^{-\sigma}\left(\frac{n}{m}\right)^{it} dt \\
&\quad\quad + \frac{1}{T-1}\int_1^T \sum_{m<t}\sum_{n<m} m^{-\sigma} n^{-\sigma}\left(\frac{n}{m}\right)^{it} dt \\
&\quad = \sum_{n<T} \frac{1}{T-1}\int_n^T n^{-2\sigma}\, dt + \sum_{n<T}\sum_{m<n} \frac{1}{T-1}\int_n^T m^{-\sigma} n^{-\sigma}\left(\frac{n}{m}\right)^{it} dt \\
&\quad\quad + \sum_{m<T}\sum_{n<m} \frac{1}{T-1}\int_m^T m^{-\sigma} n^{-\sigma}\left(\frac{n}{m}\right)^{it} dt.
\end{aligned}$$

This first term is $\sum_{n<T} (T-n)/(T-1)n^{-2\sigma}$ which is essentially the Cesaro sum of the convergent series $\sum n^{-2\sigma} = \zeta(2\sigma)$ and which therefore approaches $\zeta(2\sigma)$ as $T \longrightarrow \infty$. Since $\sum n^{-2\sigma}$ converges uniformly for $\sigma \geq \sigma_0$, it is elementary to show that this limit is approached uniformly in $\sigma$. Thus it is to be shown that the other terms in the average of $|\zeta(\sigma + it)|^2$ approach zero uniformly in $\sigma$. Each of the two remaining terms above has modulus at most

$$\sum_{n<T}\sum_{m<n} \frac{m^{-\sigma} n^{-\sigma}}{T-1}\left|\int_n^T e^{it\log(n/m)}\, dt\right| \leq \sum_{n<T}\sum_{m<n} \frac{2m^{-\sigma} n^{-\sigma}}{(T-1)\log(n/m)}.$$

This sum is monotone decreasing as $\sigma$ increases, so in order to show that it is uniformly small for $\sigma \geq \sigma_0 > \frac{1}{2}$ when $T$ is large, it suffices to show that it is small for any fixed value of $\sigma > \frac{1}{2}$ which, for convenience, can be assumed to be less than 1. Now the above sum can be split into two parts:

$$\sum_{n<T}\sum_{m<n/2} \frac{2m^{-\sigma} n^{-\sigma}}{(T-1)\log(n/m)} + \sum_{n/2\leq m<n<T}\sum \frac{2m^{-\sigma} n^{-\sigma}}{(T-1)\log(n/m)}$$

which with $r = n - m$, $\log(n/m) = -\log[1 - (r/n)] > r/n$ in the second part shows it is less than

$$\begin{aligned}
&\frac{2}{(T-1)\log 2}\sum_{n<T}\sum_{m<T} m^{-\sigma}n^{-\sigma} + \sum_{n<T}\sum_{r\le n/2}\frac{2(n-r)^{-\sigma}n^{-\sigma}}{(T-1)(r/n)} \\
&\le \frac{2}{(T-1)\log 2}\Big(\sum_{n<T} n^{-\sigma}\Big)^2 + \frac{2}{T-1}\sum_{n<T} n^{1-2\sigma}\sum_{r\le n/2}\frac{[1-(r/n)]^{-\sigma}}{r} \\
&\le \frac{2}{(T-1)\log 2}\Big(\sum_{n<T} n^{-\sigma}\Big)^2 + \frac{2\cdot 2^{\sigma}}{T-1}\sum_{n<T} n^{1-2\sigma}\sum_{r\le n/2} r^{-1} \\
&\sim \frac{2}{(T-1)\log 2}\Big(\int_1^T t^{-\sigma}\,dt\Big)^2 + \frac{2\cdot 2^{\sigma}}{T-1}\int_1^T t^{1-2\sigma}\log\frac{t}{2}\,dt \\
&< \text{const}\,\frac{T^{2-2\sigma}}{T-1} + \text{const}\,\frac{T^{2-2\sigma+\epsilon}}{T-1}
\end{aligned}$$

which, because $\sigma > \frac{1}{2}$, approaches zero as $T \to \infty$. Finally $(T-1)^{-1} \times \int_1^T |\zeta(\sigma + it)|^2\,dt$ differs from $(T-1)^{-1}\int_1^T |\sum_{n<t} n^{-\sigma-it}|^2\,dt$ by an amount whose modulus is at most

$$\begin{aligned}
&\frac{1}{T-1}\left|2\,\mathrm{Re}\int_1^T \Big(\sum_{n<t} n^{-\sigma-it}\Big)\overline{R(\sigma,t)}\,dt\right| + \int_1^T |R(\sigma,t)|^2\,dt \\
&\quad\le \frac{2}{T-1}\left(\int_1^T \left|\sum_{n<t} n^{-\sigma-it}\right|^2 dt\right)^{1/2}\left(\int_1^T |R(\sigma,t)|^2\,dt\right)^{1/2} \\
&\qquad + \frac{1}{T-1}\int_1^T |R(\sigma,t)|^2\,dt
\end{aligned}$$

by the Schwarz inequality. Since the average value of $|R(\sigma, t)|^2$ approaches zero as $T \to \infty$, this approaches $2\cdot[\zeta(2\sigma)]^{1/2}\cdot\sqrt{0} + 0 = 0$ as $T \to \infty$ and the proof is complete. Specifically what has been shown is that *given* $\sigma_0 > \frac{1}{2}$ *and given* $\epsilon > 0$, *there is a* $T_0$ *such that*

$$\left|\zeta(2\sigma) - \frac{1}{T-1}\int_1^T |\zeta(\sigma+it)|^2\,dt\right| < \epsilon$$

*whenever* $\sigma \ge \sigma_0$, $T \ge T_0$. This of course implies the statement needed for the Bohr–Landau theorem, namely, that $(T-1)^{-1}\int_1^T |\zeta(\sigma + it)|^2\,dt$ is uniformly bounded for $\sigma \ge \sigma_0 > \frac{1}{2}$, $T \ge T_0$.

## 9.8 FURTHER RESULTS. LANDAU'S NOTATION *o*, *O*

This section is devoted to the statements, without proof, of various refinements and extensions of the theorems proved above. For fuller accounts, with proofs and references to the primary sources, see either of the books [T3] or [T8] of Titchmarsh, which were the source of much of the material of this chapter.

In describing these theorems it will be useful to introduce the following notation of Landau [L3, p.61]: The notation "$f(x) = O(g(x))$ as $x \to \infty$" means that there is a constant $K$ and a value $x_0$ of $x$ such that $|f(x)| < Kg(x)$ whenever $x \geq x_0$. In words, "the modulus of $f$ grows no faster than a constant times $g$ as $x \to \infty$." Here $f$ may be complex valued, but $g$ is real and positive. The notation "$f(x) = o(g(x))$ as $x \to \infty$" means that for every $\epsilon > 0$ there is a value $x_0$ of $x$ such that $|f(x)| < \epsilon g(x)$ whenever $x \geq x_0$. In words, "the modulus of $f$ grows more slowly than $g$ as $x \to \infty$." There are various obvious extensions of this notation such as "$f(x) = O(g(x))$ as $x \to 0$," which means there exist $K$, $x_0$ such that $|f(x)| < Kg(x)$ whenever $0 < |x| < x_0$ or "$f(x) = F(x) + O(g(x))$ as $x \to \infty$," which means "$f(x) - F(x) = O(g(x))$ as $x \to \infty$," etc., which will also be used. In this notation the prime number theorem can be stated $\pi(x) = \mathrm{Li}(x) + o(\mathrm{Li}(x)) = \mathrm{Li}(x) + o(x/\log x)$, de la Vallée Poussin's estimate of the error can be written "$\pi(x) = \mathrm{Li}(x) + o(x \exp[-(c \log x)^{1/2}])$ for some $c > 0$," and the Lindelöf hypothesis can be written "$\zeta(\frac{1}{2} + it) = o(t^{\epsilon})$ for every $\epsilon > 0$," etc.

It was proved in Section 9.2 that $\zeta(1 + it) = O(\log t)$. Weyl [W4] in 1921 proved that, in fact, $\zeta(1 + it) = O(\log t/\log\log t)$ and thus, in particular, $\zeta(1 + it) = o(\log t)$. Weyl's proof was based on a new method of evaluating the "exponential sums" $\sum_{a \leq n \leq b} n^{-1-it} = \sum_{a \leq n \leq b} \exp[-(1 + it)\log n]$ which occur in the Euler–Maclaurin formula for $\zeta(1 + it)$. Weyl's method was improved upon by Vinogradoff who proved $\zeta(1 + it) = O([\log t \log\log t]^{3/4})$. These investigations of $\zeta(1 + it)$ led to improvements in de la Vallée Poussin's estimate (Section 5.2) of the amount by which the real part $\beta$ of a root $\rho = \beta + i\gamma$ must be less than one. Littlewood in 1922 proved that $\beta < 1 - (K \log\log \gamma/\log \gamma)$ and Vinogradov and Korolov in 1958 proved that $\beta < 1 - K(\log t)^{-a}$ for any $a > \frac{2}{3}$. This was improved slightly by Richert (see notes to Walfisz' book [W1]) who proved $\beta < 1 - K(\log t)^{-2/3} (\log\log t)^{-1/3}$. Each of these improvements gives a corresponding improvement of de la Vallée Poussin's estimate of the error in the prime number theorem; for example, Richert's estimate above gives $\pi(x) = \mathrm{Li}(x) + O(x \exp[-c(\log x)^{3/5}(\log\log x)^{-1/5}])$ for some $c > 0$ (see Walfisz [W1]). As is probably clear from the mere statement of the results, the methods used in proving these facts are very demanding. Nonetheless, the improvement over de la Vallée Poussin's result is very slight in comparison with the estimate $\pi(x) = \mathrm{Li}(x) + O(x^{1/2} \log x)$ which would follow from the Riemann hypothesis (see Section 5.5).

It was also proved in Section 9.2 that $\zeta(\frac{1}{2} + it) = O(t^{1/4} \log t)$. Actually the Riemann–Siegel formula (with error estimate) shows that

$$\begin{aligned} |\zeta(\tfrac{1}{2} + it)| = |Z(t)| &= \Big|2 \sum_{2\pi n^2 < t} n^{-1/2} \cos[\vartheta(t) - t \log n]\Big| + O(t^{-1/4}) \\ &= O\Big(\int_1^{(t/2\pi)^{1/2}} t^{-1/2}\, dt\Big) + O(t^{-1/4}) = O(t^{1/4}). \end{aligned}$$

Hardy and Littlewood applied Weyl's method to the estimation of the exponential sum which is the main term of the Riemann–Siegel formula† to prove that this estimate can be improved to $\zeta(\frac{1}{2}+it) = O\big(t^{1/6}(\log t)^{3/2}\big)$. This has been improved to $\zeta(\frac{1}{2}+it) = o(t^{1/6})$ and slightly beyond, for example, to $\zeta(\frac{1}{2}+it) = O(t^{19/116})$. Note that this implies $\mu(\frac{1}{2}) < \frac{1}{6}$ which, by the convexity of Lindelöf's function $\mu$, implies improvements on the bound $\mu(\sigma) \le \frac{1}{2} - \frac{1}{2}\sigma$ throughout $\{0 < \sigma < 1\}$.

Since the average value of $|\zeta(s)|^2$ on $\operatorname{Re} s = \sigma$ is $\zeta(2\sigma)$ for $\sigma > \frac{1}{2}$ and since $\zeta(2\sigma) \to \infty$ as $\sigma \downarrow \frac{1}{2}$, it is to be expected that the average value of $|\zeta(s)|^2$ on $\operatorname{Re} s = \frac{1}{2}$ is infinite. Hardy and Littlewood proved not only that this is true but that

$$\frac{1}{2T}\int_{-T}^{T}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt \sim \log T$$

in the sense that the relative error approaches zero as $T \to \infty$. Similar results hold for the average values of $|\zeta(s)|^4$ on $\operatorname{Re} s = \sigma$. For $\sigma > \frac{1}{2}$ this average is $[\zeta(2\sigma)]^4/\zeta(4\sigma)$ and for $\sigma = \frac{1}{2}$

$$\frac{1}{2T}\int_{-T}^{T}\left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt \sim \frac{1}{2\pi^2}(\log T)^4.$$

The averages of $|\zeta(s)|^{2k}$ for integers $k > 2$ are much more difficult and very little is known about them.

If the Lindelöf hypothesis is true, then of course many of the estimates can be improved. For example, Cramér showed that the Lindelöf hypothesis implies $S(t) = o(\log t)$. This in turn implies that the number of roots $\rho$ with imaginary parts between $T$ and $T+1$ is approximately $(1/2\pi)\log T$ and that the relative error in this approximation approaches zero as $T \to \infty$. Littlewood showed that the Lindelöf hypothesis implies that $\int_0^T S(t)\,dt = o(\log T)$. As for averages of $|\zeta(s)|^{2k}$, Hardy and Littlewood showed that the Lindelöf hypothesis is *equivalent* to the statement that for all positive integers $k$

$$\frac{1}{2T}\int_{-T}^{T}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt = o(T^{\epsilon}) \qquad \text{for all} \quad \epsilon > 0.$$

If the Riemann hypothesis is true, then $\zeta(1+it) = O(\log\log t)$ and $\zeta(\frac{1}{2}+it) = O\big(\exp(c\log t/\log\log t)\big)$ for some $c$. (Note that the last estimate implies the Lindelöf hypothesis.) If the Riemann hypothesis is true, then the estimates of $S(t)$ can be strengthened to $S(t) = O(\log t/\log\log t)$ and $\int_0^T S(t)\,dt = O(\log T/\log\log T)$. On the other hand, the Riemann hypothesis also implies a *lower* bound on the rate of growth of $S(t)$. Namely, it was shown by Bohr and Landau that if the Riemann hypothesis is true, then for every $\epsilon > 0$ the inequalities $S(t) < (\log t)^{(1/2)-\epsilon}$ and $S(t) < -(\log t)^{(1/2)-\epsilon}$ have arbitrarily

†This occurred before the publication of the Riemann–Siegel formula. Hardy and Littlewood derived the main term of this formula independently as a special case of their "approximate functional equation."

large solutions $t$ so that, in particular, $S(t)$ is not bounded either above or below. It is not known whether $S(t)$ is in fact unbounded.* This theorem of Bohr–Landau does not assume the full Riemann hypothesis but only that the number of roots off the line is *finite*, in which form it is the basic tool in Titchmarsh's proof that there must be infinitely many exceptions to Gram's law and in the proof of Section 8.4 that there must be infinitely many exceptions to Rosser's rule.

Since the proof of this theorem is not readily accessible in the literature and since it plays such an important part in Section 8.4, it seems worthwhile to include a proof here. The following proof of the weaker theorem that *if there are only a finite number of exceptions to the Riemann hypothesis, then* $S(t)$ *cannot be bounded below* is taken from a 1911 paper of Landau [L3a]. It uses the 1910 theorem of Bohr [B6a] which states that $\zeta(s)$ is unbounded on $\{\operatorname{Re} s > 1, \operatorname{Im} s \geq 1\}$.

Suppose there is a $q$ such that there are no exceptions to the Riemann hypothesis in the halfplane $\{\operatorname{Im} s \geq q\}$ and suppose that $S(t)$ is bounded below. Then a contradiction to the known fact that $\zeta(s)$ is unbounded on $\{\operatorname{Re} s > 1, \operatorname{Im} s \geq 1\}$ can be derived as follows. Consider $\operatorname{Im} \log \zeta(s)$ on the half-strip $D = \{\frac{1}{2} \leq \operatorname{Re} s \leq 1 + \delta, \operatorname{Im} s \geq q\}$. Since $S(t) = \pi^{-1} \operatorname{Im} \log \zeta(\frac{1}{2} + it)$ at all points where $\zeta(\frac{1}{2} + it) \neq 0$, the assumption on $S(t)$ implies that $\operatorname{Im} \log \zeta(\frac{1}{2} + it)$ is bounded below on the left side of $D$ except at the points where it is not defined. Let $D_0$ be the domain obtained from $D$ by deleting very small semicircular neighborhoods of the roots $\rho$ on its boundary. Since $\zeta(s) = (s - \rho)^n g(s)$, $\operatorname{Im} \log \zeta(s) = n \operatorname{Im} \log (s - \rho) + \operatorname{Im} \log g(s)$ where $\operatorname{Im} \log g(s)$ is continuous and bounded in a neighborhood of $\rho$, and since $\operatorname{Im} \log (s - \rho)$ along any semicircle in $\{\operatorname{Re} s \geq \frac{1}{2}\}$ assumes its smallest value on $\operatorname{Re} s = \frac{1}{2}$ where $S(t)$ is bounded below, $D_0$ can be determined in such a way that $\operatorname{Im} \log \zeta(s)$ is bounded below on the left side of $D_0$, the bound being independent of the size of the deleted semicircles. On the bottom side $\operatorname{Im} s = q$ of $D_0$, $\operatorname{Im} \log \zeta(s)$ is continuous and therefore bounded. On the right side $\operatorname{Re} s = 1 + \delta$ of $D_0$, the inequality $\operatorname{Im} \log \zeta(s) \leq |\log \zeta(s)| \leq \log \zeta(1 + \delta)$ shows that $\operatorname{Im} \log \zeta(s)$ is bounded below. Finally, the estimates of Section 6.7 show that $\operatorname{Im} \log \zeta(\sigma + it) \leq \text{const} \log M \leq \text{const} \log t$ for large $t$ when $\frac{1}{2} \leq \sigma \leq 1 + \delta$ ($\frac{1}{2} + it$ not a root $\rho$). Thus there is a constant $K$ such that $\operatorname{Im} \log \zeta(s) \geq -K$ for all $s$ on the boundary of $D_0$, and for any $\epsilon > 0$ the inequality $\operatorname{Im} \log \zeta(s) + \epsilon t \geq -K$ holds for all $s = \sigma + it$ on the boundary of the domain $\{s \in D_0: \operatorname{Im} s \leq Q\}$ for all sufficiently large $Q$. Since $\operatorname{Im} \log \zeta(s) + \epsilon t$ is harmonic on this domain, it follows that $\operatorname{Im} \log \zeta(s) + \epsilon t \geq -K$ throughout $D_0$. Since $\epsilon$ was arbitrary, $\operatorname{Im} \log \zeta(s) \geq -K$ throughout $D_0$. Since the deleted circles were arbitrary, $\operatorname{Im} \log \zeta(s) \geq -K$ in the interior of $D$. Consider now the smaller strip $D_1 = \{s: (1/2) + (\delta/2) \leq \operatorname{Re} s \leq 1 + (\delta/2), \operatorname{Im} s \geq q + 1\}$. The lemma of Section 2.7 for deducing a bound on $|f|$ from a one-sided bound on its real or imaginary part then implies that $\log \zeta(s)$ is bounded on $D_1$. But then $|\zeta(s)| = e^{\operatorname{Re} \log \zeta(s)}$ would be bounded on $D_1$, which it is not. This contradiction proves the theorem.

* *Note added in second printing:* This statement is an error. Selberg proved that $S(t)$ is unbounded (Contributions to the theory of the Riemann zeta-function, *Arch. for Math. og Naturv.*, B, **48** (1946), no. 5).

Chapter 10

# *Fourier Analysis*

## 10.1 INVARIANT OPERATORS ON $R^+$ AND THEIR TRANSFORMS

One of the basic ideas in Riemann's original paper is, as Chapter 1 shows, the idea of Fourier analysis. This chapter is devoted to the formulation of a more modern approach to Fourier analysis which I believe sheds some light on the meaning of the zeta function and on its relation to the distribution of primes.

The approach to Fourier analysis I have in mind is that in which the fundamental object of study is the algebra of invariant operators on the functions on a group. The group in this instance is the multiplicative group of positive real numbers, which will be denoted $R^+$. Consider the vector space $V$ of all complex-valued functions on $R^+$ and consider linear operators $V \longrightarrow V$. The simplest such operators are the translation operators, which are defined as follows. For each positive real number $u$ the translation operator $T_u$ is the operator which carries a function $f$ in $V$ to the function whose value at $x$ is the value of $f$ at $ux$, in symbols

$$T_u\colon f(x) \mapsto f(ux),$$

where $f(x)$ is a generic element of $V$ and where the "barred arrow" $\mapsto$ is used to denote the effect of the operator in question on a generic element of its domain. Translation operators $V \longrightarrow V$ are defined for all of $V$. However, many of the most interesting operators are defined only on subspaces of $V$—for example, only for smooth functions or functions which vanish at $\infty$—and the term "operator" should not be taken to imply that the domain is necessarily all of $V$.

An operator $V \longrightarrow V$ is said to be *invariant* if it commutes with all translation operators. More precisely, a linear operator $L$: $V \longrightarrow V$ is said to be invariant if its domain is invariant under the action of the translation operators $T_u$ and if $T_uLf = LT_uf$ for all $u \in R^+$ and all $f$ in the domain of $L$. The simplest examples of invariant operators are the translation operators themselves, which are invariant by virtue of the fact that the group $R^+$ is commutative. Then superpositions of translation operators are also invariant, for example, the summation operator $f(x) \mapsto \sum_{n=1}^{\infty} f(nx)$ or integral operators $f(x) \mapsto \int_0^\infty f(ux)F(u)\,du$. (The exact domain of definition of these operators will not be important in what follows since functions will merely be assumed to satisfy whatever conditions are needed. For the summation operator the domain might be taken to be the set of functions $f \in V$ which satisfy $\lim_{x\to\infty} x^{1+\epsilon}|f(x)| = 0$ for some $\epsilon > 0$. For an integral operator in which $F(u)$ is continuous and $|F(u)|$ is bounded, the domain might be taken to be those continuous functions $f \in V$ which satisfy $\lim_{x\to\infty} x^{1+\epsilon}|f(x)| = 0$ and $\lim_{x\to 0} x^{1-\epsilon}|f(x)| = 0$ for some $\epsilon > 0$, etc.) A different sort of invariant operator is the differential operator $\lim_{h\to 1}(T_h - T_1)/(h-1)$ which carries $f(x)$ to $xf'(x)$ provided $f$ is differentiable.

For any complex number $s$ the one-dimensional subspace of $V$ generated by the function† $f(x) = x^{-s}$ has the property of being invariant under all invariant operators; otherwise stated, an invariant operator carries $f(x) = x^{-s}$ to a multiple of itself. This can be proved as follows. For fixed $u$, $T_u$ multiplies $f$ by $u^{-s}$. Hence, since $L$ is linear, $LT_uf$ is $u^{-s}$ times $Lf$. On the other hand, $LT_uf = T_uLf$, so $T_u$ multiplies $Lf$ by $u^{-s}$ for every $u$. But this means $Lf(u) = (T_uLf)(1) = u^{-s}(Lf)(1)$, so $Lf$ is $(Lf)(1)$ times $f$ as was to be shown. For example, the summation operator $f(x) \mapsto \sum_1^\infty f(nx)$ carries $f(x) = x^{-s}$ to $\zeta(s)f$ (provided Re $s > 1$), the integral operator $f(x) \mapsto \int_0^\infty f(ux)F(u)\,du$ carries it to $\int_0^\infty u^{-s}F(u)\,du$ times $f$ (provided $F$ and $s$ are such that this integral converges—that is, $f$ is in the domain of the operator), and the differential operator $f(x) \mapsto xf'(x)$ carries it to $(-s)$ times $f$. The fundamental idea of Fourier analysis is to *analyze* invariant operators—literally to take them apart—and to study them in terms of their actions on these one-dimensional invariant subspaces.

The *transform* of an invariant operator is the function whose domain is the set of complex numbers $s$ such that the function $f(x) = x^{-s}$ lies in the domain of the operator and whose value for such an $s$ is the factor by which the operator multiplies $f(x) = x^{-s}$. Thus the entire function $s \mapsto u^{-s}$ is the transform of the translation operator $T_u$, the zeta function $s \mapsto \zeta(s)$ for Re

†The reason for taking $x^{-s}$ as the basic function rather than $x^s$ or $x^{is}$ is to put the equations in the most familiar form and, in particular, to make Re $s = \frac{1}{2}$ the line of symmetry in the functional equation.

$s > 1$ is the transform of the summation operator $f(x) \mapsto \sum_{n=1}^{\infty} f(nx)$, and the function $s \mapsto -s$ is the transform of the differential operator $f(x) \mapsto xf'(x)$. The basic formula $\log \zeta(s) = \int_0^\infty x^{-s}\, dJ(x)$ for $\operatorname{Re} s > 1$ of Riemann's paper [(3) of Section 1.11] is the statement that $\log \zeta(s)$ for $\operatorname{Re} s > 1$ is the transform of the invariant operator $f(x) \mapsto \int_0^\infty f(ux)\, dJ(u)$. Similarly, the formula $-\zeta'(s)/\zeta(s) = \int_0^\infty x^{-s}\, d\psi(x)$ which was the starting point of the derivation of von Mangoldt's formula for $\psi(x)$ (see Section 3.1) is the statement that $-\zeta'(s)/\zeta(s)$ is the transform of $f(x) \mapsto \int_0^\infty f(ux)\, d\psi(u)$. The technique by which Riemann derived his formula for $J(x)$ and von Mangoldt his formula for $\psi(x)$ is a technique of *inversion*, of going from the transform to the operator, which might well be called Fourier *synthesis*—putting the operator back together again when its effect on the invariant subspaces is known.

## 10.2 ADJOINTS AND THEIR TRANSFORMS

In order to define the adjoint of an invariant operator (on the vector space $V$ of complex-valued functions on the multiplicative group $R^+$ of positive real numbers), it is necessary to define an inner product on the vector space $V$. The natural definition would be $\langle f, g\rangle = \int_0^\infty f(x)\overline{g(x)}\, d\mu(x)$, where $d\mu$ is the invariant measure on the group $R^+$, namely, $d\mu(x) = d \log x = x^{-1}\, dx$. However, the functional equation $\xi(s) = \xi(1 - s)$ involves an inner product on $V$ which is natural with respect to the additive structure of $R^+$ rather than the multiplicative structure, namely, the inner product

$$\langle f, g\rangle = \int_0^\infty f(x)\overline{g(x)}\, dx.$$

This inner product is defined on a rather small subset of $V$—for example, the inner product of $f_1(x) = x^{-s_1}$ and $f_2(x) = x^{-s_2}$ is undefined for any pair of complex numbers $s_1, s_2$—but it suffices for the definition of a *formal adjoint* $L^*$ of an operator $L$ as an operator such that $\langle Lf, g\rangle = \langle f, L^*g\rangle$ whenever both sides are defined. Again, touchy points regarding the domains of definition of $L$ and $L^*$ can be avoided by restricting consideration to particular cases.

For example, the adjoint of the translation operator $T_u$ is easily found by a change of variable in the integral

$$\langle T_u f, g\rangle = \int_0^\infty f(ux)\overline{g(x)}\, dx = \int_0^\infty f(y)\overline{g(y/u)}u^{-1}\, dy$$

to be the operator $g(x) \mapsto u^{-1}g(u^{-1}x)$, that is, the operator $u^{-1}T_{u^{-1}}$. Since the adjoint of a sum is the sum of the adjoints, this implies the adjoint of the summation operator $f(x) \mapsto \sum_{n=1}^{\infty} f(nx)$ is the operator $f(x) \mapsto \sum_{n=1}^{\infty} n^{-1}f(n^{-1}x)$

and the adjoint of the integral operator $f(x) \mapsto \int_0^\infty f(ux)F(u)\,du$ is

$$f(x) \mapsto \int_0^\infty u^{-1}f(u^{-1}x)\overline{F(u)}\,du = \int_0^\infty f(vx)v^{-1}\overline{F(v^{-1})}\,dv.$$

The adjoint of the differential operator $f(x) \mapsto xf'(x)$ is found by integration by parts,

$$\int_0^\infty xf'(x)\overline{g(x)}\,dx = xf(x)\overline{g(x)}\Big|_0^\infty - \int_0^\infty f(x)\frac{d}{dx}[x\overline{g(x)}]\,dx,$$

to be the operator $g(x) \mapsto -d[xg(x)]/dx$.

Now in terms of transforms this operation is, from the above examples, clearly related to the substitution $s \mapsto 1 - s$ since they show that an operator whose transform is $u^{-s}$ has an adjoint whose transform is $u^{-(1-s)}$, that an operator whose transform is $\zeta(s)$ (Re $s > 1$) has an adjoint whose transform is $\zeta(1 - s)$ (Re $s < 0$), that an operator whose transform is $\int_0^\infty u^{-s}F(u)\,du$ has adjoint whose transform is $\int_0^\infty u^{-(1-s)}\overline{F(u)}\,du$, and that an operator whose transform is $-s$ has an adjoint whose transform is $-(1 - s)$. The general rule is that an operator whose transform is $\phi(s)$ has an adjoint whose transform is $\overline{\phi(1 - \bar{s})}$. This can be thought of as "conjugate transpose," where the "transpose" operation is $\phi(s) \mapsto \phi(1 - \bar{s})$. If $\phi$ is analytic and real on the real axis, then by the reflection principle $\phi(\bar{s}) = \overline{\phi(s)}$, so in this case $\overline{\phi(1 - \bar{s})}$ can be written simply $\phi(1 - s)$.

In this way the functional equation $\xi(s) = \xi(1 - s)$ seems to be saying that some operator is self-adjoint. A specific sense in which this is true is described in the following section.

## 10.3 A SELF-ADJOINT OPERATOR WITH TRANSFORM $\xi(s)$

Riemann's second proof of the functional equation (see Section 1.7) depends on the functional equation $1 + 2\psi(x) = x^{-1/2}[1 + 2\psi(x^{-1})]$ from the theory of theta functions. With $x = u^2$ and $G(u) = 1 + 2\psi(u^2) = \sum_{-\infty}^{\infty} \exp(-\pi n^2u^2)$, this equation takes the simple form $G(u) = u^{-1}G(u^{-1})$, which implies, by Section 10.2, that the invariant operator

(1) $$f(x) \mapsto \int_0^\infty f(ux)G(u)\,du$$

is formally self-adjoint. This operator has no transform at all; that is, the integral

(2) $$\int_0^\infty u^{-s}G(u)\,du$$

does not converge for any $s$. However, as will be shown in this section, it can be modified in such a way as to be made convergent, and when this is done the formal self-adjointness of this operator is in essence equivalent to the functional equation of the zeta function.

The integral (2) would converge at $\infty$ if the constant term of $G(u) = 1 + 2\sum_{n=1}^{\infty} \exp(-\pi n^2 x^2)$ were absent. The constant term can be eliminated by differentiating $G$ or, better, by applying the invariant operator $G(u) \mapsto uG'(u)$. By the definition of "adjoint" this is formally the same as applying the adjoint $f(u) \mapsto -d[uf(u)]/du$ to the first factor in the integrand of (2), which multiplies the first factor by $s - 1$. To preserve self-adjointness, it is natural then to apply $G(u) \mapsto -d[uG(u)]/du$ to the second factor or $f(u) \mapsto uf'(u)$ to the first, which multiplies the first factor by $-s$. Thus, formally,

$$(3) \qquad \int_0^\infty u^{-s}\left[\left(-\frac{d}{du}u^2\frac{d}{du}\right)G(u)\right]du = \int_0^\infty \left[\left(-u\frac{d^2}{du^2}u\right)u^{-s}\right]G(u)\,du$$
$$= s(1-s)\int_0^\infty u^{-s}G(u)\,du.$$

The right side is $s(1 - s)$ times the (formal) transform of a real self-adjoint operator and therefore is in some sense invariant under the substitution $s \to 1 - s$. However, the left side is in fact a well-defined analytic function of $s$ for all $s$. To see this, let

$$H(u) = \frac{d}{du}\left[u^2\frac{d}{du}G(u)\right]$$

so that the integral in question is $-\int_0^\infty u^{-s}H(u)\,du$. Then

$$H(u) = \frac{d}{du}\left[u^2\frac{d}{du}\{G(u) - 1\}\right]$$

goes to zero faster than any power of $u$ as $u \to \infty$, and the integral $-\int_0^\infty u^{-s}H(u)\,du$ therefore converges at $\infty$. On the other hand, applying $(d/du)u^2(d/du)$ to both sides of the functional equation $G(u) = (1/u)G(1/u)$ gives two expressions,

$$(4) \qquad 2uG'(u) + u^2G''(u) = \frac{2}{u^2}G'\left(\frac{1}{u}\right) + \frac{1}{u^3}G''\left(\frac{1}{u}\right),$$

for $H(u)$ and hence gives $H(u) = (1/u)H(1/u)$; that is, $H$ satisfies the same functional equation as $G$. This implies that $H(u)$ goes to zero faster than any power of $u$ as $u \downarrow 0$ and hence that the integral $-\int_0^\infty u^{-s}H(u)\,du$ converges at 0. Therefore the left side of (3) is an entire function of $s$ which, because of (3), would be expected to be invariant under $s \to 1 - s$. That this is the case follows immediately from $H(x) = x^{-1}H(x^{-1})$ which gives

$$-\int_0^\infty u^{-s}H(u)\,du = \int_\infty^0 u^{1-s}H(u)\,d\log u$$
$$= -\int_0^\infty v^{s-1}H\left(\frac{1}{v}\right)d\log v = -\int_0^\infty u^{s-1}H(u)\,du.$$

Since, as will now be shown, $\int_0^\infty u^{-s}H(u)\,du = 2\xi(1-s)$, this proves the functional equation $\xi(s) = \xi(1-s)$.

Let $s$ be a negative real number. Then

$$\int_0^\infty u^{-s}H(u)\,du = \int_0^\infty u^{-s}\left(\frac{d}{du}u^2\frac{d}{du}G(u)\right)du$$
$$= \int_0^\infty u^{-s}\left(\frac{d}{du}u^2\frac{d}{du}[G(u)-1]\right)du$$
$$= -\int_0^\infty \left(\frac{d}{du}u^{-s}\right)\left(u^2\frac{d}{du}[G(u)-1]\right)du$$

(because $G(u) - 1$ and all its derivatives vanish more rapidly than any power of $u$ as $u \longrightarrow \infty$ and because, therefore, $G(u) - u^{-1} = u^{-1}[G(u^{-1}) - 1]$ and all its derivatives vanish more rapidly than any power of $u$ as $u \downarrow 0$, which implies that $u^2\,d[G(u)-1]/du = u^2\,d[G(u)-u^{-1}]/du + u^2\,d[u^{-1}-1]/du$ is bounded as $u \downarrow 0$)

$$= s\int_0^\infty u^{1-s}\frac{d}{du}[G(u)-1]\,du$$
$$= -s\int_0^\infty \left(\frac{d}{du}u^{1-s}\right)[G(u)-1]\,du$$

[because $G(u) - 1$ vanishes more rapidly than any power of $u$ as $u \longrightarrow \infty$, while $G(u) - 1 = G(u) - u^{-1} + u^{-1} - 1$ grows no more rapidly than $u^{-1}$ as $u \downarrow 0$]

$$= s(s-1)\int_0^\infty u^{-s}[G(u)-1]\,du$$
$$= s(s-1)\int_0^\infty u^{-s}2\sum_{n=1}^\infty e^{-\pi n^2u^2}\,du$$
$$= s(s-1)\sum_{n=1}^\infty \int_0^\infty u^{1-s}e^{-\pi n^2u^2}2\,d\log u$$

(the interchange being valid by absolute convergence)

$$= s(s-1)\sum_{n=1}^\infty \int_0^\infty \left(\frac{v}{\pi n^2}\right)^{(1-s)/2}e^{-v}\,d\log v$$
$$= s(s-1)\pi^{(s-1)/2}\sum_{n=1}^\infty \frac{1}{n^{1-s}}\int_0^\infty e^{-v}v^{(1-s)/2}v^{-1}\,dv$$
$$= s(s-1)\pi^{(s-1)/2}\zeta(1-s)\,\Pi\left(\frac{1-s}{2}-1\right)$$
$$= (-s)2\Pi\left(\frac{1-s}{2}\right)\pi^{-(1-s)/2}\zeta(1-s) = 2\xi(1-s).$$

Therefore, by analytic continuation, the same equation holds for all $s$. Note that (3) then states that formally

$$\int_0^\infty u^{-u}G(u)\,du = \frac{2\xi(s)}{s(s-1)};$$

that is, the function $2\xi(s)/s(s-1)$ is formally the transform of the operator (1), but, since this operator has no transform, some "convergence factor" such as the replacement of $G$ by $H$ is necessary. For a different method of giving meaning to the idea that $2\xi(s)/s(s-1)$ is the transform of the self-adjoint operator $f(x) \mapsto \int_0^\infty f(ux)G(u)\,du$, see Section 10.5.

In summary, the functional equation $\xi(s) = \xi(1-s)$ can be deduced from the functional equation $G(u) = u^{-1}G(u^{-1})$ as follows. First show that the function $H(u) = (d/du)u^2(d/du)G(u)$ satisfies the same functional equation as $G$ does, that is, $H(u) = u^{-1}H(u^{-1})$. This is immediate from (4). Then show that $\int_0^\infty u^{-s}H(u)\,du$ converges for all $s$, and that for negative real $s$ it is $2\xi(1-s)$. This shows that $\xi(1-s)$ is an entire function and, because $H(u) = u^{-1}H(u^{-1})$, that it is invariant under $s \longrightarrow 1-s$.

## 10.4 THE FUNCTIONAL EQUATION

It was shown in the preceding section that the functional equation $\xi(s) = \xi(1-s)$ can be deduced simply and naturally from the fact that the function $G(u) = \sum_{-\infty}^{\infty} \exp(-\pi n^2u^2)$ satisfies the functional equation $G(u) = u^{-1}G(u^{-1})$ or, what is the same, from the fact that the operator $f(x) \mapsto \int_0^\infty f(ux)G(u)\,du$ is formally self-adjoint in the sense of Section 10.2. Thus in order to understand the functional equation of $\xi$, it is natural to study the functional equation of $G$.

The functional equation of $G$ results immediately from two formulas—the Poisson summation formula and the formula

(1) $$e^{-\pi u^2} = \int_{-\infty}^{\infty} e^{-\pi x^2}e^{2\pi ixu}\,dx.$$

Consider Poisson summation first.

Let the *Fourier transform* of a complex-valued function $f(x)$ on the real line be defined by

$$\hat{f}(u) = \int_{-\infty}^{\infty} f(x)e^{2\pi ixu}\,dx,$$

a definition which differs from the usual definition by a factor of $2\pi$ in the exponential. Then the *Poisson summation formula* states that under suitable conditions on $f$ (involving its smoothness and its vanishing at $\infty$), the sum of the Fourier transform is the sum of the function; that is,

(2) $$\sum_{n=-\infty}^{\infty} \hat{f}(n) = \sum_{n=-\infty}^{\infty} f(n).$$

This fact follows very easily from the theory of Fourier series when one considers the "periodified" function $F(x) = \sum_{n=-\infty}^{\infty} f(x+n)$ (assuming $f$ vanishes rapidly enough at infinity for this sum to converge for all $x$ between 0 and 1). This function $F$ is periodic with period 1 and therefore (again under suitable assumptions about $f$) by the theory of Fourier series can be expanded as a series $\sum_{-\infty}^{\infty} a_n e^{2\pi i x n}$ in which the coefficients are given by

$$\begin{aligned} a_n &= \int_0^1 F(x)e^{-2\pi i n x}\,dx = \int_0^1 \sum_{m=-\infty}^{\infty} f(x+m)e^{-2\pi i n x}\,dx \\ &= \sum_{m=-\infty}^{\infty} \int_m^{m+1} f(y)e^{-2\pi i n(y-m)}\,dy = \int_{-\infty}^{\infty} f(y)e^{-2\pi i n y}\,dy = \hat{f}(-n). \end{aligned}$$

Therefore, setting $x = 0$ in $F(x) = \sum a_n e^{2\pi i n x}$ gives the Poisson summation formula (2). Since the above operations are all valid in the case of the function $f(x) = \exp(-\pi x^2 u^{-2})$ for any positive $u$, this together with (1) gives

$$\begin{aligned} G(u) &= \sum_{n=-\infty}^{\infty} e^{-\pi n^2 u^2} = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x n u}\,dx \\ &= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\pi (y/u)^2} e^{2\pi i n y} u^{-1}\,dy \\ &= u^{-1} \sum_{n=-\infty}^{\infty} \hat{f}(n) = u^{-1} \sum_{n=-\infty}^{\infty} e^{-\pi n^2 u^{-2}} = u^{-1} G(u^{-1}) \end{aligned}$$

as was to be shown.

Finally, consider the proof of (1). In the special case $u = 0$ this is the formula

$$1 = \int_{-\infty}^{\infty} e^{-\pi x^2}\,dx \tag{3}$$

which is one of the basic formulas of calculus, being essentially equivalent to the formula $\Pi(-\frac{1}{2}) = \pi^{1/2}$, to the fact that the constant in Stirling's formula is $\frac{1}{2}\log 2\pi$ (see Section 6.3), or to Wallis' product for $\pi$. It can be proved simply by

$$\begin{aligned} \left(\int_{-\infty}^{\infty} e^{-\pi x^2}\,dx\right)^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-\pi y^2}\,dx\,dy \\ &= \int_0^{2\pi}\int_0^{\infty} e^{-\pi r^2} r\,dr\,d\theta = \int_0^{2\pi} \frac{1}{2\pi}\,d\theta = 1 \end{aligned}$$

which, since the integral must be positive, proves (3). But then the change of variable $x = y - iu$ and Cauchy's theorem gives

$$1 = \int_{-\infty+iu}^{\infty+iu} e^{-\pi(y-iu)^2}\,dy = \int_{-\infty}^{\infty} e^{-\pi(y-iu)^2}\,dy$$

which is the desired formula (1).

This completes the proof that $G(u) = u^{-1}G(u^{-1})$. The structure of this proof can be interpreted in the following way. Let $W$ be a complex vector space with inner product and let $M$ be a linear transformation of $W$. A linear transformation $A\colon W \longrightarrow W$ is said to be *self-reciprocal relative to* $M$ if $A^*M$

$= A$, where $A^*$ denotes the adjoint of $A$. *If $A$ and $B$ are both self-reciprocal relative to $M$ and if $A$ and $B$ commute, then $A^*B$ is self-adjoint* because $A^*B = (A^*M)^*B = M^*AB = M^*BA = (B^*M)^*A = B^*A = (A^*B)^*$. This is roughly the situation in the above proof, with $W$ equal to the vector space of complex-valued functions on the entire real line $R$, with the inner product equal to $\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx$, with $M$ equal to the operator $f(x) \mapsto \int_{-\infty}^{\infty} f(ux) e^{2\pi i u}\,du$ resembling Fourier transform, with $A$ equal to the summation operator $f(x) \mapsto \sum_{n=-\infty}^{\infty} f(nx)$, and with $B$ equal to the operator $f(x) \mapsto \int_{-\infty}^{\infty} f(ux) \exp(-\pi u^2)\,du$. Then $A$ and $B$ are formally self-reciprocal relative to $M$ and $A^*B$ is the operator $f(x) \mapsto \int_{-\infty}^{\infty} f(ux)G(u)\,du$; so the above theorem states that $\int_{-\infty}^{\infty} f(ux)G(u)\,du$ is self-adjoint. If $V$ is identified with the subspace of $W$ consisting of even functions (doubling the inner product on $V$), this implies $G(u) = u^{-1}G(u^{-1})$ as desired. Of course this is only roughly true and certain problems about domains of definition have to be ignored; however, since a rigorous proof of $G(u) = u^{-1}G(u^{-1})$ has already been given and since the only issue here is to understand the structure of the theorem, these problems will be passed over.

Consider first the statement that $A$ is self-reciprocal relative to $M$, that is, $A^*M = A$. Now $A$ is invariant relative to the multiplicative structure of $R$ [it is a superposition of multiplicative "translation "operators $f(x) \mapsto f(nx)$]; hence, formally, so is $A^*$ invariant [this is the part of the proof which is only formally correct because the adjoint of the "translation" $f(x) \mapsto f(0x)$ is not defined because it involves division by 0]. Now an important characteristic of invariant operators is that if they are applied to $f(ux)$ considered as a function of $x$ for fixed $u$, the result is the same as if they are applied to $f(ux)$ considered as a function of $u$ for fixed $x$ because both of these are equal to the operator applied to $f(y)$ as a function of $y$ evaluated at $y = ux$; this follows from $L[f(ux)]$ (as a function of $x$) $= (L \circ T_u)f = T_u Lf$ (by invariance) $=(Lf)(ux)$. Therefore, to apply $A^*$ to

$$(Mf)(x) = \int_{-\infty}^{\infty} f(ux) e^{2\pi i u}\,du$$

is the same as to apply $A^*$ to $f(ux)$ in this integral considered as a function of $u$ for fixed $x$. But by the definition of $A^*$ (because $A$ is real), this is the same as to apply $A$ to the second factor $e^{2\pi i u}$ of the integrand. Thus

$$(A^*Mf)(x) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(ux) e^{2\pi i n u}\,du.$$

But with $F(u) = f(ux)$ the Poisson summation formula shows that this is (formally) equal to

$$\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} F(u) e^{2\pi i n u}\,du = \sum_{-\infty}^{\infty} \hat{F}(n) = \sum_{-\infty}^{\infty} F(n) = \sum_{-\infty}^{\infty} f(nx)$$
$$= (Af)(x).$$

Hence $A^*M = A$ as was to be shown. Similarly $B^*M$ is formally equal to the result of applying $B$ to the second factor of $Mf$; that is,

$$(B^*Mf)(x) = \int_{-\infty}^{\infty} f(ux)\left[\int_{-\infty}^{\infty} e^{2\pi i v u} e^{-\pi v^2}\, dv\right] du$$
$$= \int_{-\infty}^{\infty} f(ux)e^{-\pi u^2}\, du = (Bf)(x)$$

by (1); hence $B^*M = B$. Finally, by the same argument, $A^*B$ is formally equal to the result of applying $A$ to the second factor in the integral $Bf$; that is,

$$(A^*Bf)(x) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(ux)e^{-\pi n^2 u^2}\, du = \int_{-\infty}^{\infty} f(ux)G(u)\, du$$

as was to be shown.

## 10.5 $2\xi(s)/s(s-1)$ AS A TRANSFORM

The statement that $2\xi(s)/s(s-1)$ is formally the transform of $f(x) \mapsto \int_0^\infty f(ux)G(u)\, du$ can be given substance as follows. A continuous analog of Euler's ludicrous formula

$$\sum_{-\infty}^{\infty} x^n = (1 + x + x^2 + \cdots) + (x^{-1} + x^{-2} + \cdots)$$
$$= \frac{1}{1-x} + \frac{1}{x-1} = 0$$

is

(1) $$\int_0^\infty x^{-s}\, dx = \int_0^1 x^{-s}\, dx + \int_1^\infty x^{-s}\, dx$$
$$= \frac{1}{1-s} - \frac{1}{1-s} = 0.$$

This is of course nonsense because the values of $s$ for which the above integrals converge are mutually exclusive—the first integral being convergent for $\operatorname{Re} s < 1$ and the second being convergent for $\operatorname{Re} s > 1$—but it does suggest that the formal transform of $f(x) \mapsto \int_0^\infty f(ux)\, dx$ is zero and hence that the formal transform of $f(x) \mapsto \int_0^\infty f(ux)[G(u) - 1]\, du$ ought to be $2\xi(s)/s(s-1)$. However, since this operator actually has a transform for $\operatorname{Re} s < 0$, this suggests the correct formula

$$\int_0^\infty u^{-s}[G(u) - 1]\, du = \frac{2\xi(s)}{s(s-1)} \qquad (\operatorname{Re} s < 0)$$

which was proved in Section 10.3. Setting $u = v^{-1}$ in this formula gives

$$\int_0^\infty v^{s-1}\left[G(v) - \frac{1}{v}\right] dv = \frac{2\xi(s)}{s(s-1)} \qquad (\operatorname{Re} s < 0)$$

and, hence,

$$\int_0^\infty u^{-s}\left[G(u) - \frac{1}{u}\right] du = \frac{2\xi(s)}{s(s-1)} \qquad (\operatorname{Re} s > 1)$$

[using $\xi(s) = \xi(1-s)$]. This too can be interpreted as saying that $2\xi(s)/s(s-1)$ is formally the transform of $f(x) \mapsto \int_0^\infty f(ux)G(u)\,du$ because the formal transform of $f(x) \mapsto \int_0^\infty f(ux)u^{-1}\,du$ is zero by (1). Now since

$$\int_0^\infty u^{-s}\left[G(u) - 1 - \frac{1}{u}\right] du$$

is convergent for $s$ in the critical strip $\{0 < \operatorname{Re} s < 1\}$, the same considerations lead one to expect that the value of this integral in the strip where it converges will also be $2\xi(s)/(s-1)$. That this is actually the case can be proved by considering the function

$$\int_1^\infty u^{-s}[G(u) - 1]\,du + \int_0^1 u^{-s}\left[G(u) - 1 - \frac{1}{u}\right] du - \frac{1}{s}$$

which is defined and analytic throughout the halfplane $\{\operatorname{Re} s < 1\}$ except for the pole at $s = 0$. Since this function agrees with

$$\int_0^\infty u^{-s}[G(u) - 1]\,du \qquad \text{on} \quad \{\operatorname{Re} s < 0\}$$

and with

$$\int_0^\infty u^{-s}\left[G(u) - 1 - \frac{1}{u}\right] du \qquad \text{on} \quad \{0 < \operatorname{Re} s < 1\}$$

it follows that these two functions are analytic continuations of one another. Thus

$$\frac{2\xi(s)}{s(s-1)} = \begin{cases} \displaystyle\int_0^\infty u^{-s}[G(u) - 1]\,du & (\operatorname{Re} s < 0), \\ \displaystyle\int_0^\infty u^{-s}\left[G(u) - 1 - \frac{1}{u}\right] du & (0 < \operatorname{Re} s < 1), \\ \displaystyle\int_0^\infty u^{-s}\left[G(u) - \frac{1}{u}\right] du & (1 < \operatorname{Re} s), \end{cases}$$

are all three literally true in the stipulated ranges of $s$ and all three say that formally $2\xi(s)/(s-1)$ is the transform of $f(x) \mapsto \int_0^\infty f(ux)G(u)\,du$.

## 10.6 FOURIER INVERSION

The problem of Fourier inversion or Fourier synthesis is the problem of finding an invariant operator when its transform—that is, its effect on the one-dimensional invariant subspaces—is known. Riemann's technique of accomplishing this by changes of variable in Fourier's theorem (see Section

1.12) gives more generally

$$(1) \qquad \Phi(s) = \int_0^\infty u^{-s} \frac{\phi(u)}{u} \, du$$

$$\Longleftrightarrow \phi(u) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \Phi(s) u^s \, ds$$

provided $\phi$ and/or $\Phi$ satisfy suitable conditions. That is, under suitable conditions, an operator with the given transform $\Phi(s)$ can be found by defining

$$\phi(u) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \Phi(s) u^s \, du$$

and taking the operator to be $f(x) \mapsto \int_0^\infty f(ux)\phi(u) \, d \log u$. In actual practice this formula is perhaps best understood as a heuristic one because the problems of proving convergence and of proving the applicability of Fourier's theorem are substantial.

In this context the proof of von Mangoldt's formula for $\psi$ in Chapter 3 can be described as follows. If $\phi(u)$ is defined to be 0 for $u < 1$ and 1 for $u > 1$, then $\Phi(s)$ is $1/s$ for $\operatorname{Re} s > 0$ and the basic integral formula of Section 3.3 is the statement that the inversion formula (1) is valid in this case provided $a > 0$. This result can be generalized by changing $\phi$ to

$$\phi(u) = \begin{cases} 0 & \text{for} \quad u < y, \\ 1 & \text{for} \quad u > y, \end{cases}$$

which changes $\Phi(s)$ to $y^{-s}/s$. Then the inversion formula (1) is still valid. On the other hand $\Phi(s)$ can be changed from $1/s$ to $1/(s - \alpha)$ which changes $\phi$ to

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{u^s \, ds}{s - \alpha} = \frac{1}{2\pi i} \int_{a-i\infty-\alpha}^{a+i\infty+\alpha} \frac{u^{z+\alpha} \, dz}{z}$$

which, when $a > \alpha$, gives $\phi(u) = u^\alpha$ for $u > 1$ and zero for $u < 1$, and the inversion (1) is valid provided $a > \operatorname{Re} \alpha$. (If $\alpha$ is real, this is immediate. If $\alpha$ is not real, then the final estimate of Section 3.3 must be applied to the evaluation of the conditionally convergent integral from $a - i\infty - \alpha$ to $a + i\infty - \alpha$ in order to show it is the same as the integral from $a - \operatorname{Re} \alpha - i\infty$ to $a - \operatorname{Re} \alpha + i\infty$.) Now

$$-\frac{\zeta'(s)}{\zeta(s)} = \int_0^\infty u^{-s} \, d\psi(u) = -\int_0^\infty \psi(u) \, d(u^{-s}) = s \int_0^\infty u^{-s} \frac{\psi(u)}{u} \, du,$$

so when $\phi(u) = \psi(u)$, the transform $\Phi(s)$ is $-\zeta'(s)/s\zeta(s)$. But this function can be written in two different ways as superpositions of functions with known inverse transforms, namely, as

$$-\frac{\zeta'(s)}{s\zeta(s)} = \frac{1}{s} \int_0^\infty u^{-s} \, d\psi(u) = \sum_{n=1}^\infty \Lambda(n) \frac{n^{-s}}{s}$$

and as

$$-\frac{\zeta'(s)}{s\zeta(s)} = \frac{1}{s-1} - \sum_\rho \frac{1}{\rho(s-\rho)} + \sum_n \frac{1}{2n(s+2n)} - \frac{c}{s},$$

where $c = \zeta'(0)/\zeta(0)$ [see (7) of Section 3.2]. Hence termwise application of the inversion (1) gives a $\phi(u)$ which, on the one hand, is equal to

$$\sum_{n=1}^{\infty} \Lambda(n)\phi_n(u) = \psi(u),$$

where $\phi_n(u)$ is the function which is 1 for $u < n$ and 0 for $u > n$, and, on the other hand, is equal to

$$u - \sum \frac{u^\rho}{\rho} + \sum \frac{u^{-2n}}{2n} - c$$

for $u > 1$ and 0 for $u < 1$. The proof of von Mangoldt's formula is simply the proof that these termwise inversions are valid.

## 10.7 PARSEVAL'S EQUATION

One of the basic theorems of Fourier analysis is Parseval's theorem, which states that under suitable conditions Fourier transform is a unitary transformation; that is, if $\hat{f}$ is the Fourier transform of $f$

$$\hat{f}(u) = \int_{-\infty}^{\infty} f(x)e^{2\pi i x u}\,dx,$$

then

$$\text{(1)} \qquad \int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |\hat{f}(x)|^2\,dx.$$

This statement of the theorem deals of course with Fourier transforms relative to the additive structure of the real numbers, but it can easily be translated into a theorem concerning Fourier transforms on the multiplicative group $R^+$ as follows.

If $\Phi(s)$ is the transform of the invariant operator $f(x) \mapsto \int_0^\infty f(ux)\phi(u)\,du$, that is, if

$$\Phi(s) = \int_0^\infty u^{-s}\phi(u)\,du,$$

then

$$\begin{aligned}\Phi(a+it) &= \int_0^\infty u^{-a-it}\phi(u)\,du \\ &= \int_0^\infty e^{-it\log u}u^{1-a}\phi(u)\,d\log u \\ &= \int_{-\infty}^{\infty} e^{-itv}e^{(1-a)v}\phi(e^v)\,dv,\end{aligned}$$

$$\Phi(a-2\pi i x) = \int_{-\infty}^{\infty} e^{(1-a)v}\phi(e^v)e^{2\pi i x v}\,dv;$$

so (1) with $f(x) = e^{(1-a)x}\phi(e^x)$ and consequently with $\hat{f}(u) = \Phi(a - 2\pi i u)$ gives

$$\int_{-\infty}^{\infty} |\Phi(a - 2\pi i x)|^2 \, dx = \int_{-\infty}^{\infty} e^{2(1-a)x} |\phi(e^x)|^2 \, dx,$$

$$\frac{-1}{2\pi i} \int_{a+2\pi i\infty}^{a-2\pi i\infty} |\Phi(s)|^2 \, ds = \int_0^{\infty} e^{2(1-a)\log u} |\phi(e^{\log u})|^2 \, d\log u,$$

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} |\Phi(s)|^2 \, ds = \int_0^{\infty} u^{1-2a} |\phi(u)|^2 \, du,$$

and in particular

$$\frac{1}{2\pi i} \int_{(1/2)-i\infty}^{(1/2)+i\infty} |\Phi(s)|^2 \, dx = \int_0^{\infty} |\phi(u)|^2 \, du.$$

This theorem is used in Chapter 11 in the study of zeros of $\xi(s)$ on the line $\operatorname{Re} s = \frac{1}{2}$. For a proof of the theorem in the needed cases see Bochner [B5] or Titchmarsh [T7].

## 10.8 THE VALUES OF $\zeta(-n)$

The zeta function can be evaluated at negative integers as follows. Consider the operator whose transform is $(1 - s)\zeta(s)$, namely, the composition of $f(x) \mapsto d[xf(x)]/dx$ with the operator $f(x) \mapsto \sum_{n=1}^{\infty} f(nx)$. Since $(1 - s)\zeta(s)$ is an entire function, this operator is defined, at least formally, for all functions of the form $f(x) = x^{-s}$. Consider the effect of this operator on $f(x) = e^{-x}$. This can be found in two different ways as follows.

On the one hand $f(x) = \sum_{n=0}^{\infty} (-1)^n x^n/n!$, so if the operator whose transform is $(1 - s)\zeta(s)$ is applied termwise to this series, one finds the function

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (1 + n)\zeta(-n) x^n$$

as the resulting function because the operator multiplies $x^{-(-n)}$ by $[1 - (-n)]$ $\zeta(-n)$. On the other hand the summation operator carries $f(x) = e^{-x}$ to

$$e^{-x} + e^{-2x} + e^{-3x} + \cdots = \frac{1}{e^x - 1},$$

and its composition with $f(x) \mapsto d[xf(x)]/dx$ carries it to

$$\frac{d}{dx} \frac{x}{e^x - 1} = \frac{d}{dx} \sum_{n=0}^{\infty} \frac{B_n x^n}{n!} = \sum_{n=1}^{\infty} \frac{n B_n x^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{B_{n+1} x^n}{n!}$$

by the definition of the Bernoulli numbers [see (1) of Section 1.5]. Thus, equating the coefficients of $x^n$ in the two expressions, one finds

$$(-1)^n (n + 1)\zeta(-n) = B_{n+1},$$

$$\zeta(-n) = (-1)^n [B_{n+1}/(n + 1)] \qquad (n = 0, 1, 2, \ldots)$$

which agrees with the value found in Section 1.5.

However, it must be admitted that this argument is very far from being a rigorous proof. For one thing, the series expansion of $x(e^x - 1)^{-1}$ in terms of Bernoulli numbers is valid only for $|x| < 2\pi$. Actually the evaluation of $\zeta(-n)$ using Riemann's integral as in Section 1.5 can be regarded as a method of making mathematical sense out of the above nonsense.

## 10.9 MÖBIUS INVERSION

The Möbius inversion formula is simply the inverse transform of the Euler product formula

$$\zeta(s) \prod_p \left(1 - \frac{1}{p^s}\right) = 1$$

in that it states that the summation operator $f(x) \mapsto \sum f(nx)$ with transform $\zeta(s)$ can be inverted by composing it with the operators $f(x) \mapsto f(x) - f(px)$ with transform $1 - p^{-s}$, where $p$ ranges over all primes. If $\sum f(nx)$ converges absolutely (and in particular if $f$ is zero for all sufficiently large $x$), this follows easily from the fact that after a finite number of steps the above operations reduce $\sum f(nx)$ to $\sum f(kx)$, where $k$ ranges over all integers not divisible by any of the primes that have been used; since this means that the first $k$ past $k = 1$ is very large, it implies that $\sum f(kx)$ approaches $f(x)$ as more and more primes are used (see note, Section 1.17).

The inverse transform of the expanded product

$$\prod_p \left(1 - \frac{1}{p^s}\right) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

states that the composition of the operators $f(x) \mapsto f(x) - f(px)$ over all primes $p$ can also be written in the form $f(x) \mapsto \sum_{n=1}^{\infty} \mu(n)f(nx)$. [Here $\mu(n)$ is zero unless $n$ is a product of distinct prime factors, is 1 if $n$ is a product of an even number of distinct prime factors, and is $-1$ if $n$ is a product of an odd number of distinct prime factors.] This too is very easily proved in the case where $\sum f(nx)$ is absolutely convergent. Thus the Möbius inversion formula can be written in the form

$$g(x) = \sum_{n=1}^{\infty} f(nx) \Longleftrightarrow f(x) = \sum_{n=1}^{\infty} \mu(n)g(nx)$$

provided $\sum f(nx)$ and $\sum g(nx)$ both converge absolutely. Yet another statement of it is

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \mu(n)f(mnx)$$

under suitable conditions on $f$. For example, if $\sum \sum f(mnx)$ converges absolutely, then the double series can be rearranged,

$$f(x) = \sum_{N=1}^{\infty} \left[\sum_{m|N} \mu(m)\right] f(Nx), \tag{1}$$

and the inversion formula is equivalent to the identity

$$\sum_{m|N} \mu(m) = \begin{cases} 1 & \text{if } N = 1, \\ 0 & \text{otherwise,} \end{cases}$$

which can be proved by applying Möbius inversion in the form (1) to the function $f(x)$ which is 1 for $x \le 1$ and 0 for $f(x) > 1$, and then setting $x = 1$, $x = \frac{1}{2}$, $x = \frac{1}{3}$, etc. In Chapter 12 the slightly different statement

$$g(x) = \sum_{n=1}^{\infty} f\left(\frac{x}{n}\right) \Longleftrightarrow f(x) = \sum_{n=1}^{\infty} \mu(n) g\left(\frac{x}{n}\right)$$

of Möbius inversion will be needed.

## 10.10 RAMANUJAN'S FORMULA

Hardy, in his book [H4] on Ramanujan, states that Ramanujan "was especially fond and made continual use" of the formula

$$\int_0^\infty x^{s-1}[\phi(0) - x\phi(1) + x^2\phi(2) - \cdots]\, dx = \frac{\pi}{\sin \pi s}\phi(-s).$$

In the discussion that follows it will be convenient to recast this in the equivalent form

$$(1) \qquad \int_0^\infty x^{-s}\left[\phi(1) - x\phi(2) + \frac{x^2}{2!}\phi(3) - \cdots + (-1)^n \frac{x^n}{n!}\phi(n+1) + \cdots\right] dx = \Pi(-s)\phi(s)$$

in which $s$ has been replaced by $1 - s$, $\phi(s)$ by $\phi(s + 1)/\Pi(s)$, and $\pi(\sin \pi s)^{-1}$ by $\Pi(-s)\Pi(s - 1)$ [see (6) of Section 1.3]. In this form Ramanujan's formula can be deduced from

$$(2) \qquad \int_0^\infty x^{-s}e^{-x}\, dx = \Pi(-s)$$

by observing that application of an operator with transform $\phi(s)$ to the first factor $x^{-s}$ of the integrand on the one hand multiplies the integral by $\phi(s)$ but on the other hand is the same as application of an operator with transform $\phi(1 - \bar{s})$ (the conjugate of the adjoint) to the second factor; since $e^{-x} = \sum_{n=0}^{\infty} (-1)^n x^n/n!$ and since an operator with transform $\phi(1 - \bar{s})$ multiplies $x^n = x^{-(-n)}$ by $\phi(1 - (-n)) = \phi(n + 1)$, this gives formula (1).

This heuristic argument is of course not a proof, but it does show how the formula can be proved for certain functions $\phi(s)$. For example, if $\phi(s) = a^{-s}$ for $a > 0$, then the integral on the left side of (1) is simply $\int_0^\infty x^{-s}e^{-x/a}a^{-1}\, dx = \int_0^\infty (ay)^{-s}e^{-y}\, dy = \phi(s)\Pi(-s)$ as was to be shown. Note that the integral is convergent only for Re $s < 1$ and that the right side of (1) gives an analytic continuation of the value of the integral past the pole at $s = 1$.

As a second example consider the binomial coefficient function $\phi(s) = \binom{-s}{n}$ for a fixed integer $n \geq 0$. This is the transform of the operator $f(x) \mapsto x^n/n!\, d^n f(x)/dx^n$ and the right side of (1) can be expressed as an integral $1/n! \int_0^\infty [x^n\, d^n x^{-s}/dx^n] e^{-x}\, dx$ for Re $s < 1$. Integration by parts is valid for Re $s < 1$ and puts this integral in the form $(-1)^n/n! \int_0^\infty x^{-s}\, (d^n[x^n e^{-x}]/dx^n)\, dx$. Since by simple termwise operations on power series

$$\begin{aligned}
&\frac{(-1)^n}{n!}\frac{d^n}{dx^n}[x^n e^{-x}] \\
&\quad= \frac{(-1)^n}{n!}\frac{d^n}{dx^n}\sum_{m=0}^{\infty}\frac{(-1)^m x^{m+n}}{m!} \\
&\quad= \sum_{m=0}^{\infty}\frac{(-1)^n(m+n)(m+n-1)\cdots(m+1)}{n!}\cdot\frac{(-1)^m x^m}{m!} \\
&\quad= \sum_{m=0}^{\infty}\binom{-m-1}{n}\frac{(-1)^m x^m}{m!},
\end{aligned}$$

this puts the integral in the desired form and shows that Ramanujan's formula (1) is valid for this $\phi$. Note that the integral again converges only for Re $s < 1$ and that the right side of (1) gives the analytic continuation of the value of the integral past the pole at $s = 1$. Since it is linear in $\phi$, Ramanujan's formula is true in the same sense for any linear combination of the polynomials $\phi(s) = \binom{-s}{n}$ and hence for any polynomial $\phi(s)$.

As a third example consider the case $\phi(s) = \Pi(s - 1)$. In this case the series in the integral is $\sum_{n=0}^{\infty} \phi(n+1)(-x)^n/n! = \sum_{n=0}^{\infty} (-x)^n = (1+x)^{-1}$, so the integral is $\int_0^\infty x^{-s}(1+x)^{-1}\, dx$, which is convergent for $0 < \text{Re } s < 1$. Ramanujan's formula says that the value of this integral should be $\Pi(-s)\,\Pi(s-1)$, a fact which is easily proved by applying the operator $f(x) \mapsto \int_0^\infty f(ux)e^{-u}\, du$ with transform $\Pi(-s)$ to the second factor of the integrand in $\Pi(-s) = \int_0^\infty x^{-s}e^{-x}\, dx$ and evaluating the result in two ways to find

$$\begin{aligned}
\int_0^\infty x^{-s}\left(\int_0^\infty e^{-ux}e^{-u}\, du\right) dx &= \int_0^\infty x^{-s}\,\frac{e^{-u(x+1)}}{-(x+1)}\bigg|_{u=0}^{u=\infty} dx \\
&= \int_0^\infty x^{-s}(x+1)^{-1}\, dx
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^\infty x^{-s}\left(\int_0^\infty e^{-ux}e^{-u}\, du\right) dx \\
&\quad= \int_0^\infty e^{-u}\left(\int_0^\infty x^{1-s}e^{-ux}\, d\log x\right) du \\
&\quad= \int_0^\infty e^{-u}\left(u^{s-1}\int_0^\infty y^{1-s}e^{-y}\, d\log y\right) du \\
&\quad= \Pi(s-1)\Pi(-s)
\end{aligned}$$

which proves the desired equation.

Ramanujan's formula fails in the case of the function $\phi(s) = \sin \pi s$ because in this case the integral is identically zero, but $\Pi(-s)\phi(s)$ is not. However, if the formula is regarded as a method of extending a given function $\phi(1), \phi(2), \phi(3), \ldots$ defined at positive integers to an analytic function $\phi(s)$ defined for all $s$ (or as many $s$ as possible), then the formula works very well, extending the function $0 = \phi(1) = \phi(2) = \phi(3) = \cdots$ to the function $\phi(s) \equiv 0$ rather than to the more complicated function $\phi(s) = \sin \pi s$.

Now if $\phi(s)$ is an analytic function, then $\Pi(-s)\phi(s)$ has poles at positive integers $s = n$, and the residues of these poles are

$$\begin{aligned}\lim_{s\to n}(s-n)\Pi(-s)\phi(s) &= \lim_{s\to n}(s-n)\frac{\Pi(n-s)\phi(s)}{(n-s)(n-1-s)\cdots(1-s)} \\ &= -1\frac{\Pi(0)\phi(n)}{(-1)(-2)\cdots(1-n)} \\ &= (-1)^n\frac{\phi(n)}{(n-1)!}\end{aligned}$$

and, conversely, if $F(s)$ is a function with simple poles of residue $(-1)^n[\phi(n)/(n-1)!]$ at positive integers $n$, then $F(s)/\Pi(-s) = \phi(s)$ defines a function with values $\phi(n)$ at positive integers. Thus Ramanujan's formula can be regarded as the statement that the analytic function defined by

$$\int_0^\infty x^{-s}\left[\phi(1) - \phi(2)x + \frac{\phi(3)x^2}{2!} - \cdots\right]dx$$

has an analytic continuation [if $\phi(1) \neq 0$, this integral does not converge for $\operatorname{Re} s > 1$] with poles at $s = 1, 2, 3, \ldots$ with residues $-\phi(1), \phi(2), -\phi(3)/2!, \ldots$. In this way Ramanujan's formula becomes—provided the series $\sum_{n=0}^\infty \phi(n+1)(-x)^n/n!$ has a sum which is $O(x^{-\alpha})$ for some positive $\alpha$ as $x \to \infty$—a special case of the following theorem.

**Theorem** Let $\Phi(x)$ be a continuous function on the positive real axis which satisfies $\Phi(x) = O(x^{-\alpha})$ for some $\alpha > 0$ as $x \to \infty$ and which has an asymptotic expansion $\Phi(x) \sim \sum_{n=0}^\infty a_n x^n$ as $x \downarrow 0$. Then the analytic function $F(s)$ defined in the strip $\{1 - \alpha < \operatorname{Re} s < 1\}$ by the integral $F(s) = \int_0^\infty x^{-s}\Phi(x)\,dx$ has an analytic continuation to the entire halfplane $\{1 - \alpha < \operatorname{Re} s\}$ with no singularities other than simple poles at positive integers $n$, and the residue of the pole at $n$ is $-a_{n-1}$.

***Proof*** The integral

$$F_n(s) = \int_0^\infty x^{-s}[\Phi(x) - a_0 - a_1 x - \cdots - a_{n-1}x^{n-1}]\,dx$$

is convergent at $\infty$ if $\operatorname{Re} s > n$ and convergent at 0 if $\operatorname{Re} s < n + 1$ (because the integrand is like $x^{-s}a_n x^n$ as $x \longrightarrow 0$), so it defines an analytic function $F_n(s)$ in the strip $\{n < \text{Re } s < n + 1\}$. In the strip $\{n - 1 < \text{Re } s < n + 1\}$ the function

$$\int_0^1 x^{-s}[\Phi(x) - a_0 - \cdots - a_{n-1}x^{n-1}]\,dx + \frac{a_{n-1}}{n - s}$$
$$+ \int_1^\infty x^{-s}[\Phi(x) - a_0 - \cdots - a_{n-2}x^{n-2}]\,dx$$

is defined and analytic, except for a simple pole with residue $-a_{n-1}$ at $s = n$, and agrees with $F_{n-1}(s)$ and $F_n(s)$ in their respective strips of definition. Thus the functions $F_n(s)$ are all analytic continuations of each other and in the same way $F_1(s)$ is an analytic continuation of $F(s)$. Since the analytic function they all define has the stated properties, this proves the theorem. [If $\Phi$ is analytic on a neighborhood of the positive real axis and analytic at 0, then the theorem can also be proved quite easily using Riemann's method of Section 1.4 of considering the integral $\int_{+\infty}^{+\infty} (-x)^{-s}\Phi(x)\,dx$.]

Applying a trivial modification of this theorem to Abel's formula

$$\Pi(-s)\zeta(1 - s) = \int_0^\infty x^{-s}\left(\frac{1}{e^x - 1}\right) dx \qquad (\text{Re } s < 0) \tag{3}$$

[(1) of Section 1.4] shows that since

$$(e^x - 1)^{-1} = \frac{1}{x} - \frac{1}{2} + \frac{B_2 x}{2!} + \frac{B_4 x^3}{4!} - \cdots,$$

the function $\Pi(-s)\zeta(1 - s)$ has an analytic continuation to the entire complex plane with simple poles at $0, 1, 2, \ldots$ having residues $-1, \frac{1}{2}, -B_2/2, 0, -B_4/4!, \ldots, -B_n/n!, \ldots$. Thus $\zeta(1 - s)$ has an analytic continuation which has a simple pole with residue $-1$ at $s = 0$ but which is analytic at all positive integers $n$ and has the value

$$-\frac{B_n}{n!} \cdot \frac{(n-1)!}{(-1)^n} = (-1)^{n-1}\frac{B_n}{n}$$

at $n$. Thus the theorem very easily gives the analytic continuation of $\zeta$ and its values at $0, -1, -2, \ldots$.

Consider now the application of Ramanujan's formula to guess the values of $\zeta$ on the basis of the values

$$\zeta(2n) = \frac{(2\pi)^{2n}(-1)^{n+1}B_{2n}}{2(2n)!} \qquad (n = 1, 2, \ldots)$$

found by Euler [(2) of Section 1.5]. Setting $\phi(s) = \zeta(2s)$ in Ramanujan's formula leads to a series which cannot be summed in any obvious way, but setting $\phi(s) = \Pi(s-1)\zeta(2s)$ leads to the series

$$\sum_{n=1}^{\infty} \phi(n)\frac{(-1)^{n-1}x^{n-1}}{(n-1)!} = \frac{1}{x}\sum_{n=1}^{\infty} \frac{(2\pi)^{2n}(-1)^{n+1}B_{2n}}{2(2n)!}(-1)^{n-1}x^n$$
$$= \frac{1}{2x}\,{}'\!\sum_{m=2}^{\infty} \frac{(2\pi x^{1/2})^m B_m}{m!}$$

(because $B_3 = B_5 = \cdots = 0$) which has the sum

$$\frac{1}{2x}\left(\frac{2\pi x^{1/2}}{\exp(2\pi x^{1/2}) - 1} - 1 + \frac{2\pi x^{1/2}}{2}\right).$$

Thus Ramanujan's formula would give

(4) $$\int_0^\infty x^{-s}\frac{1}{2x}\left(\frac{2\pi x^{1/2}}{\exp(2\pi x^{1/2}) - 1} - 1 + \frac{2\pi x^{1/2}}{2}\right)dx$$
$$= \Pi(-s)\Pi(s-1)\zeta(2s)$$

if it were true in this case. The theorem proved above shows that this formula gives the correct values of $\zeta(2n)$ $(n = 1, 2, 3, \ldots)$ but does not show that it gives the correct value of $\zeta(s)$ for other values of $s$. However, the integral on the left can be rewritten

$$\int_0^\infty \left(\frac{y}{2\pi}\right)^{-2s}\left(\frac{y}{e^y - 1} - 1 + \frac{y}{2}\right) d\log y$$
$$= (2\pi)^{2s}\int_0^\infty y^{-2s}\left(\frac{1}{e^y - 1} - \frac{1}{y} + \frac{1}{2}\right) dy$$

which shows—by the principle of Section 10.5—that it has the analytic continuation

(5) $$(2\pi)^{2s}\int_0^\infty y^{-2s}\left(\frac{1}{e^y - 1}\right) dy$$

and therefore, by Abel's formula (3), that it is $(2\pi)^{2s}\Pi(-2s)\zeta(1-2s)$. The method of Section 10.5 and of the proof of the theorem above can be used to prove very easily that (5) is indeed an analytic continuation of the integral in (4) and therefore to prove that *Ramanujan's formula in the case* (4) *is equivalent to*

$$(2\pi)^{2s}\Pi(-2s)\zeta(1-2s) = \Pi(-s)\Pi(s-1)\zeta(2s)$$

*which is the functional equation of the zeta function* [see (4) of Section 1.6]. Thus Ramanujan's formula does hold in this case even though the theorem above does not suffice to prove it.

As a final example, consider the case $\phi(s) = \Pi(s)\zeta(1-s)$. In this case the series is $\sum_{n=0}^{\infty}(n+1)!\zeta(-n)(-1)^n x^n/n! = \sum_{n=0}^{\infty} B_{n+1}x^n$ and Ramanujan's formula takes the form

(6) $$\int_0^\infty x^{-s}(-\tfrac{1}{2} + B_2 x + B_4 x^3 + \cdots)\,dx = \Pi(-s)\Pi(s)\zeta(1-s).$$

This formula is meaningless as it stands because the power series is divergent for all $x \neq 0$. However, it is an asymptotic expansion for small $x$ of a function which can be identified and the integral can be made meaningful as follows. By Stirling's formula

$$\log \Pi(x) \sim \left(x + \frac{1}{2}\right) \log x - x + \frac{1}{2} \log 2\pi + \frac{B_2}{2x} + \frac{B_4}{4\cdot 3x^3} + \cdots,$$

$$\frac{\Pi'(x)}{\Pi(x)} \sim \log x + \frac{1}{2x} - \frac{B_2}{2x^2} - \frac{B_4}{4x^4} - \frac{B_6}{6x^6} - \cdots,$$

$$\frac{d}{dx}\left[\frac{\Pi'(x)}{\Pi(x)} - \log x\right] \sim -\frac{1}{2x^2} + \frac{B_2}{x^3} + \frac{B_4}{x^5} + \frac{B_6}{x^7} + \cdots.$$

On the other hand the change of variable $y = x^{-1}$ in the integral of (6) puts it in the form

$$\int_0^\infty y^{s-1}\left(-\frac{1}{2} + B_2 y^{-1} + B_4 y^{-3} + \cdots\right) d\log y = \int_0^\infty y^s\left(-\frac{1}{2y^2} + \frac{B_2}{y^3} + \frac{B_4}{y^5} + \cdots\right) dy$$

so (6) suggests the equation

$$\int_0^\infty y^s \frac{d}{dy}\left(\frac{\Pi'(y)}{\Pi(y)} - \log y\right) dy = \Pi(-s)\Pi(s)\zeta(1-s)$$

which by integration by parts [the function $\Pi'(y)/\Pi(y) - \log y$ is asymptotic to $(2y)^{-1}$ as $y \to \infty$ and asymptotic to $-\log y$ as $y \to 0$, so the integral is convergent for $0 < \operatorname{Re} s < 1$] is equivalent to

$$-s \int_0^\infty y^{s-1} \left(\frac{\Pi'(y)}{\Pi(y)} - \log y\right) dy = \Pi(-s)\Pi(s)\zeta(1-s)$$

or, with $s \to 1 - s$, equivalent to

$$\int_0^\infty y^{-s}\left(\log y - \frac{\Pi'(y)}{\Pi(y)}\right) dy = \frac{\pi}{\sin \pi s}\zeta(s), \tag{7}$$

a formula which is in fact true for all $s$ in the strip $\{0 < \operatorname{Re} s < 1\}$ where the integral converges; see Titchmarsh [T8, formula (2.9.2)].

A stronger version of Ramanujan's formula (1) than the one embodied in the theorem above can be proved using Fourier inversion. Since this involves the behavior of $\phi(s)$ on lines $\operatorname{Re} s = \text{const}$ in the complex plane, it necessarily involves considering $\phi$ as a function of a complex variable and therefore, as Hardy observes, it lay outside the range of Ramanujan's ideas and techniques in a very essential way. For example, Fourier inversion can be used to prove the following theorem.

**Theorem** Let $F(s)$ be analytic in a halfplane $\{\text{Re } s > 1 - \alpha\}$ (for $\alpha > 0$) except for simple poles at $s = 1, 2, 3, \ldots$ with residues $-a_0, -a_1, -a_2, \ldots,$ respectively. Suppose, moreover, that the growth of $F(s)$ in the complex plane satisfies suitable conditions and in particular that $F(s) \longrightarrow 0$ very rapidly as $\text{Im } s \longrightarrow \pm\infty$ along lines $\text{Re } s = \text{const}$. Then $F(s)$ in the strip $\{1 - \alpha < \text{Re } s < 1\}$ can be represented in the form $F(s) = \int_0^\infty x^{-s}\Phi(x)\, dx$, where $\Phi(x)$ is analytic for $x > 0$, where $\Phi(x) = O(x^{-\alpha+\epsilon})$ as $x \longrightarrow \infty$ (for every $\epsilon > 0$), and where $\Phi(x) \sim \sum_{n=0}^\infty a_n x^n$ is an asymptotic expansion of $\Phi(x)$ as $x \downarrow 0$. If, moreover, $F(s)$ does not grow too rapidly as $\text{Re } s \longrightarrow \infty$, then $\Phi(x)$ is analytic at 0 and, consequently, has $\sum a_n x^n$ as its power series expansion near 0.

***Proof*** The idea of the proof is simply the Fourier inversion formula

$$F(s) = \int_0^\infty x^{-s}\Phi(x)\, dx \Longleftrightarrow \Phi(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)x^{s-1}\, ds$$

[in (1) of Section 10.6 set $\phi(u) = u\Phi(u)$ and set $\Phi(s) = F(s)$]. Let $c$ lie between $1 - \alpha$ and 1, and let the formula on the right define $\Phi$. The assumption that $F(s) \longrightarrow 0$ rapidly as $s \longrightarrow c \pm i\infty$ guarantees that $\Phi(x)$ is then defined and analytic for all $x$ in the slit plane and that $\Phi(x)$ is independent of the choice of $c$. Fourier inversion shows that $F(s)$ has the desired representation, and it remains only to show that $\Phi(x)$ has the stated properties as $x \longrightarrow \infty$ and as $x \downarrow 0$. Since $\Phi(x)$ is a superposition of functions $x^{s-1}$ ($\text{Re } s = c$) all of which are $O(x^{c-1+\epsilon})$ as $x \longrightarrow \infty$, the same is true—by passage to the limit under the integral sign—of $\Phi(x)$. Since $c$ is any number greater than $1 - \alpha$, this gives $\Phi(x) = O(x^{-\alpha+\epsilon})$ as $x \longrightarrow \infty$. The integral of $(1/2\pi i)F(s)x^{s-1}\, ds$ over the boundary of the strip $\{c \leq \text{Re } s \leq n + \frac{1}{2}\}$ is on the one hand the sum of the residues in the strip—which is $-a_0 - a_1 x - \cdots - a_{n-1}x^{n-1}$—and is on the other hand $1/2\pi i \int_{n+1/2-i\infty}^{n+1/2+i\infty} F(s)x^{s-1}\, ds - \Phi(x)$. This shows that $\Phi(x) - a_0 - a_1 x - \cdots - a_{n-1}x^{n-1}$ is a superposition of functions which are $O(x^{n-(1/2)})$ as $x \longrightarrow 0$, hence by passage to the limit under the integral sign that $\Phi(x) \sim a_0 + a_1 x + a_2 x^2 + \cdots$ is an asymptotic expansion. To prove that it actually converges to $\Phi(x)$ for small $x$, it suffices to prove that $\lim_{n\to\infty} \int_{n+1/2-i\infty}^{n+1/2+i\infty} F(s) e^{s \log x}\, ds$ is zero; since $e^{s \log x} \longrightarrow 0$ very rapidly for small $x$, this will be true if the decrease of $F(s)$ for $\text{Im } s \longrightarrow \pm\infty$ is uniform as $\text{Re } s \longrightarrow \infty$ and if $|F(s)|$ does not grow rapidly as $\text{Re } s \longrightarrow \infty$.

In the case $F(s) = \pi/\sin \pi s$ the elementary estimate $F(c + it) \leq \text{const}\; e^{-\pi|t|}$ as $t \longrightarrow \infty$ (uniform in $c$) shows that the theorem applies and gives $\pi/\sin \pi s = \int_0^\infty x^{-s}[1 - x + x^2 - x^3 + \cdots]\, dx = \int_0^\infty x^{-s}(1 + x)^{-1}\, dx = \Pi(-s)\Pi(s - 1)$ (see above), that is, the theorem gives the product formula for the sine (6) of Section 1.3. In the case $F(s) = \Pi(-s)\Pi(s - 1)\zeta(2s) = (\pi/\sin \pi s)\zeta(2s)$, the fact that $\zeta(2s)$ is bounded for $\text{Re } s > \frac{3}{4}$ shows that the theorem applies to give formula (4) and hence the functional equation of the zeta function [given the formula for the values of $\zeta(2), \zeta(4), \zeta(6), \ldots$]. Formula (7) does not quite come under the theorem as stated because $(\pi/\sin \pi s)$ $\zeta(s)$ has a double pole at $s = 1$. However, the same methods can be applied to prove that

$$\frac{\pi}{\sin \pi s}\zeta(s) = \int_0^\infty x^{-s}\left(\log x + \gamma + \sum_{n=1}^\infty \zeta(n+1)(-x)^n\right) dx,$$

so (7) is equivalent to the elementary formula $-\Pi'(x)/\Pi(x) = \gamma + \sum_{n=1}^\infty$ $\zeta(n+1)(-x)^n$; this formula can be proved by taking the logarithmic derivative of Euler's formula (4) of Section 1.3 and using $(x+n)^{-1} = n^{-1} \sum_{m=0}^\infty$ $(-x/n)^m$.

Chapter 11

# Zeros on the Line

## 11.1 HARDY'S THEOREM

In 1914 Hardy [H3] proved that *there are infinitely many roots $\rho$ of $\xi(s) = 0$ on the line* $\operatorname{Re} s = \frac{1}{2}$. Except for the numerical work of Gram and Backlund, this was the first concrete result concerning zeros on the line. As was stated in Section 1.9, Hardy and Littlewood [H6] later proved—in 1921—that *the number of roots on the line segment from $\frac{1}{2}$ to $\frac{1}{2} + iT$ is at least $KT$* for some positive constant $K$ and all sufficiently large $T$, and still later—in 1942—Selberg [S1] proved that the *number of such roots is at least $KT$* $\log T$ for some positive constant $K$ and all sufficiently large $T$. This chapter is devoted to the proofs of these three theorems. Note that each of them supersedes the preceding one so that logically it would suffice to prove just Selberg's estimate. However, each of the three proofs is essentially an elaboration of the preceding one, so it is natural—both logically and historically—to prove all three. The proofs given here follow those of Titchmarsh's book [T8]. Although the basic ideas of these proofs are essentially the same as in the originals, Titchmarsh has simplified and clarified them considerably.

The idea of the proof of Hardy's theorem is to apply Fourier inversion (see Section 10.6) to one of the expressions of $\xi$ as a transform (see Section 10.5), say

$$\frac{2\xi(s)}{s(s-1)} = \int_0^\infty u^{-s}\left[G(u) - 1 - \frac{1}{u}\right] du \qquad (0 < \operatorname{Re} s < 1),$$

to find a formula such as

$$(1) \qquad G(x) - 1 - \frac{1}{x} = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{2\xi(s)}{s(s-1)} x^{s-1}\, ds \qquad (0 < a < 1).$$

The applicability of Fourier inversion in this case follows from the most basic theorems of the theory of Fourier integrals—see, for example, Taylor [T2]. With $a = \frac{1}{2}$ the right side of this equation is an integral involving the function $\xi(\frac{1}{2} + it)$ to be studied and the left side is a function about which a great deal was known in the nineteenth century. (It is essentially the function $\psi$ which occurs in Riemann's second proof of the functional equation—see Section 1.7.) The idea of Hardy's proof is to use information about the function on the left to draw conclusions about the integrand on the right.

The function $G(x) = \sum_{-\infty}^{\infty} \exp(-\pi n^2 x^2)$ is defined whenever $\operatorname{Re} x^2 > 1$, which means that it is defined not only for positive real $x$ as in (1) but also for complex values of $x$ in the wedge $\{-\pi/4 < \operatorname{Im} \log x < \pi/4\}$. In this wedge it is of course an analytic function of the complex variable $x$. However, it very definitely has singularities on the boundary of the wedge, and in fact these singularities of (1) for complex $x$ are what Hardy's proof uses. Specifically, $G(x)$ *and all its derivatives approach zero as $x$ approaches* $i^{1/2}$ $(= e^{i\pi/4})$. This fact about $G$, which originally was discovered in connection with the theory of $\theta$-functions, can easily be proved as follows:

$$\begin{aligned} G(x) &= \sum_{-\infty}^{\infty} e^{-\pi n^2 x^2} = \sum_{-\infty}^{\infty} e^{-\pi n^2 i} e^{-\pi n^2 (x^2 - i)} = \sum_{-\infty}^{\infty} (-1)^n e^{-\pi n^2 (x^2 - i)} \\ &= -G\big((x^2 - i)^{1/2}\big) + 2G\big(2(x^2 - i)^{1/2}\big). \end{aligned}$$

The functional equation $G(x) = x^{-1}G(x^{-1})$ then gives

$$\begin{aligned} G(x) &= -\frac{1}{(x^2 - i)^{1/2}} G\left(\frac{1}{(x^2 - i)^{1/2}}\right) + \frac{2}{2(x^2 - i)^{1/2}} G\left(\frac{1}{2(x^2 - i)^{1/2}}\right) \\ &= \frac{1}{(x^2 - i)^{1/2}} \left[\sum_{-\infty}^{\infty} e^{-\pi n^2 2^{-2}(x^2 - i)^{-1}} - \sum_{-\infty}^{\infty} e^{-\pi n^2 (x^2 - i)^{-1}}\right] \\ &= \frac{1}{(x^2 - i)^{1/2}} \sum_{n = \text{odd}} e^{-\pi n^2 2^{-2} (x^2 - i)^{-1}}. \end{aligned}$$

Since $e^{-1/u}$ approaches zero as $u \downarrow 0$ more rapidly than any power of $u$ does, this shows that $G(x)$ and all its derivatives approach zero as $x$ approaches $i^{1/2}$ from within the wedge $\{-\pi/4 < \operatorname{Im} \log x < \pi/4\}$, say along the circle $|x| = 1$.

Consider now the integral on the right side of (1) for complex values of $x$. It will converge provided $\xi(a + it)$ goes to zero rapidly enough as $t \to \pm\infty$. Now $|\xi(s)| = |\Pi(s/2)\pi^{-s/2}(s - 1)\zeta(s)|$ for $s = a + it$ is easily estimated; $|\zeta(s)|$ grows less rapidly than a constant times $t^2$ as $t \to \pm\infty$ (see Section 6.7), $|s - 1|$ grows like $|t|$, $|\pi^{-s/2}|$ is constant, and $|\Pi(s/2)|$ grows like $e^B$ where

$$\begin{aligned} B &= \operatorname{Re} \log \Pi\left(\frac{a + it}{2}\right) \\ &= \operatorname{Re}\left\{\left(\frac{a + it + 1}{2}\right) \log\left(\frac{a + it}{2}\right) - \frac{a + it}{2} + \cdots\right\} \\ &= \operatorname{Re}\left\{\left(\frac{a + it + 1}{2}\right) \log\left(\frac{it}{2}\right) + \cdots\right\} \end{aligned}$$

$$= \frac{a+1}{2}\log\frac{|t|}{2} - \frac{t}{2}\operatorname{Im}\log\left(\frac{it}{2}\right) + \cdots$$
$$= \frac{a+1}{2}\log|t| - \frac{|t|\pi}{4} + \cdots,$$

where the omitted terms remain bounded as $t \to \pm\infty$. Thus $|\xi(a+it)|$ is less than a power of $|t|$ times $e^{-|t|\pi/4}$ as $t \to \pm\infty$. Since the factor $x^{s-1} = x^{a-1}x^{it} = \text{const } e^{it\log x}$ grows like $\exp(\pm|t||\operatorname{Im}\log x|)$ as $t \to \pm\infty$, this shows that the decrease of $\xi(s)$ overwhelms the increase of $x^{s-1}$ in the integral (1) provided $|\operatorname{Im}\log x| < \pi/4$ and hence that *the integral* (1) *converges throughout the wedge* $\{-\pi/4 < \operatorname{Im}\log x < \pi/4\}$. By analytic continuation, then, formula (1) remains valid throughout this wedge.

Formula (1) takes a simpler form if the operator $x(d^2/dx^2)x$ is applied to both sides to give

$$H(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} 2\xi(s)x^{s-1}\,ds \qquad (0 < a < 1).$$

Clearly $H(x) = x(d^2/dx^2)x\,G(x)$ has, like $G$, the property that it and all its derivatives approach zero as $x \to i^{1/2}$. Moreover, the above estimates of the integrand justify termwise integration to give, when $a = \frac{1}{2}$,

$$H(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \xi\left(\frac{1}{2} + it\right)x^{-1/2}x^{it}\,dt,$$
$$x^{1/2}H(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \xi\left(\frac{1}{2} + it\right)\sum_0^\infty \frac{(it\log x)^n}{n!}\,dt$$
$$= \sum_0^\infty c_n(i\log x)^n,$$

where

$$c_n = \frac{1}{\pi n!}\int_{-\infty}^{\infty} \xi\left(\frac{1}{2} + it\right)t^n\,dt.$$

The integrals $c_n$ are zero for odd $n$ by the symmetry of $\xi$. If Hardy's theorem were *false*, that is, if there were only a finite number of zeros of $\xi(\frac{1}{2} + it)$, then $\xi(\frac{1}{2} + it)$ would have the same sign for all large $t$ and one would expect—because of the high weight it places on large values of $t$—that $c_{2n}$ would have this same sign for all sufficiently large $n$. This can be proved simply by observing that if $\xi(\frac{1}{2} + it)$ is positive for $t \geq T$, then

$$\begin{aligned}
\Pi(2n)\pi c_{2n} &= 2\int_0^\infty \xi(\tfrac{1}{2} + it)t^{2n}\,dt \\
&\geq 2\int_0^{T+2} \xi(\tfrac{1}{2} + it)t^{2n}\,dt \\
&\geq 2\left\{-\int_0^T |\xi(\tfrac{1}{2} + it)|\,T^{2n}\,dt \right. \\
&\qquad \left. + \int_{T+1}^{T+2} \xi(\tfrac{1}{2} + it)(T+1)^{2n}\,dt\right\} \\
&\geq \text{const}\,(T+1)^{2n} - \text{const}\,(T)^{2n}
\end{aligned}$$

is positive for all sufficiently large $n$ and similarly that if $\xi(\frac{1}{2} + it)$ is negative for $t \geq T$, then $c_{2n}$ is negative for all sufficiently large $n$. Thus if the above formula for $x^{1/2}H(x)$ is differentiated sufficiently many times with respect to $i \log x$, then the right side becomes an even power series in which all terms have the same sign. Thus if $x \longrightarrow i^{1/2}$ upward along the circle $|x| = 1$, it follows that $i \log x \downarrow -\pi/4$ through real values and the value of this even power series cannot approach zero. On the other hand it must approach zero because to differentiate with respect to $i \log x$ is the same as to apply $ix(d/dx)$, and doing this any number of times carries $x^{1/2}H(x)$ to a function which approaches zero as $x \longrightarrow i^{1/2}$. This contradiction proves Hardy's theorem.

Another proof of Hardy's theorem which is worthy of mention is that of Titchmarsh [T4]. Titchmarsh showed that in using the Riemann–Siegel formula† at Gram points $g_n$ *on average* only the first term

$$Z(g_n) = 2\cos\vartheta(g_n) + \cdots = (-1)^n \cdot 2 + \cdots$$

counts. More specifically, he proved that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} Z(g_{2n}) = 2, \qquad \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} Z(g_{2n+1}) = -2.$$

This of course proves that $Z$ must change sign infinitely often and hence proves Hardy's theorem. It also proves that on the average Gram's law is true in the strong sense that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \operatorname{Re} \zeta\left(\frac{1}{2} + ig_n\right) = 2$$

since $\zeta(\frac{1}{2} + ig_n) = \operatorname{Re} \zeta(\frac{1}{2} + ig_n) = (-1)^n Z(g_n)$.

## 11.2 THERE ARE AT LEAST *KT* ZEROS ON THE LINE

The proof of the fact that there are positive constants $K$, $T_0$ such that the number of roots $\rho$ on the line segment from $\frac{1}{2}$ to $\frac{1}{2} + iT$ is at least $KT$ whenever $T \geq T_0$ begins, as did the proof of the preceding section, with the formula

$$\frac{2\xi(s)}{s(s-1)} = \int_0^\infty u^{-s}\left[G(u) - 1 - \frac{1}{u}\right] du$$

(valid for $s$ in the so-called critical strip $0 < \operatorname{Re} s < 1$). The proof of the preceding section depended on the fact that the integral

$$\frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \frac{2\xi(s)}{s(s-1)} x^{s-1}\, ds$$

†However, as with the Hardy–Littlewood estimates of $\zeta(\frac{1}{2} + it)$ described in Section 9.8, this work preceded publication of the Riemann–Siegel formula and was based instead on the so-called approximate functional equation.

approaches zero very rapidly as $x$ approaches $i^{1/2}$ from below along the circle $|x| = 1$. The present proof depends on a *local* study of this integral, that is, on a study of the integral over finite intervals of the line $\operatorname{Re} s = \frac{1}{2}$. Let $s$ denote the midpoint of the interval under consideration, let $2k$ denote its length, and let

$$I_{x,k}(s) = \frac{1}{2\pi i}\int_{s-ik}^{s+ik} \frac{2\xi(v)}{v(v-1)} x^{v-1}\, dv. \tag{1}$$

Then $I_{x,k}(s)$ can be rewritten in the form

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{s-ik}^{s+ik}\int_0^\infty \left(\frac{u}{x}\right)^{-v}\left[G(u) - 1 - \frac{1}{u}\right]x^{-1}\, du\, dv \\
&= \frac{1}{2\pi i}\int_{s-ik}^{s+ik}\int_0^{\infty/x} w^{-v}\left[G(xw) - 1 - \frac{1}{xw}\right] dw\, dv \\
&= \frac{1}{2\pi i}\int_{s-ik}^{s+ik}\int_0^{\infty} w^{-v}\left[G(xw) - 1 - \frac{1}{xw}\right] dw\, dv \\
&= \int_0^\infty \left(\frac{1}{2\pi i}\int_{s-ik}^{s+ik} w^{-v}\, dv\right)\left[G(xw) - 1 - \frac{1}{xw}\right] dw \\
&= \frac{1}{\pi}\int_0^\infty \frac{w^{-s}\sin(k\log w)}{\log w}\left[G(xw) - 1 - \frac{1}{xw}\right] dw
\end{aligned}$$

[where use is made of the fact that $G(u) - 1$ approaches zero very rapidly as $u$ goes to infinity along any ray $u = xw$ in the wedge $-(\pi/4) \le \operatorname{Im}\log x \le (\pi/4)$, $w =$ real]. This expresses $I_{x,k}(s)$ as the transform of an operator and shows, by virtue of Parseval's theorem (Section 10.7), that

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{(1/2)-i\infty}^{(1/2)+i\infty} |I_{x,k}(s)|^2\, ds \\
&\qquad = \frac{1}{\pi^2}\int_0^\infty \left|\frac{\sin(k\log w)}{\log w}\right|^2 \left|G(xw) - 1 - \frac{1}{xw}\right|^2 dw.
\end{aligned} \tag{2}$$

The idea of the proof is to use this formula with explicit estimates of the function $G$ to find an upper bound for $\int |I|^2\, ds$ and to show that for suitable choices of $x$, $k$ it is much smaller than it could be if $\xi(\frac{1}{2} + it)$ did not change sign frequently.

The first step, therefore, is to derive an upper estimate of $\int |I|^2\, ds$. Note first that the symmetries of $G$ and $x^{-1} = \bar{x}$ imply that the integral on the right side of (2) is equal to twice the integral from 1 to $\infty$. [The factor $\sin(k\log w)/\log w$ is unchanged under $w \longrightarrow w^{-1}$. The factor $G(xw) - 1 - (xw)^{-1}$ becomes $G(1/\bar{x}w) - 1 - \bar{x}w = \bar{x}w[(\bar{x}w)^{-1}G(1/\bar{x}w) - (\bar{x}w)^{-1} - 1] = \bar{x}w[G(\bar{x}w) - 1 - (\bar{x}w)^{-1}]$ under $w \longrightarrow w^{-1}$, so the square of its modulus is multiplied by $w^2$. Since $dw$ becomes $-dw/w^2$, it follows that the integral from 0 to 1 becomes the integral from 1 to $\infty$.] Now

$$\left|\frac{\sin ky}{y}\right| \le \begin{cases} k & \text{for} \quad 0 \le y < \pi/k, \\ y^{-1} & \text{for} \quad \pi/k \le y < \infty; \end{cases}$$