His hypothesis has not been proved until this day (despite the fact that modern computers can verify that the first 10^{12} zeros are on the critical line), but considerable effort has been put in order to understand the different objects in the theory of the Riemann zeta-function assuming its validity.

For instance, J. E. Littlewood in 1924 [26] showed that under the Riemann hypothesis (RH) we have the following estimate:

$$\log\left|\zeta\left(\tfrac{1}{2}+it\right)\right| \leq \left(C+o(1)\right)\frac{\log t}{\log\log t},$$

for sufficiently large t. This estimate was never improved in its order of magnitude, and the advances have rather focused on diminishing the value of the admissible constant C. In [34] Ramachandra and Sankaranarayanan obtained $C = 0.466$, while in [38] Soundararajan improved this bound, obtaining $C = 0.373$. Recently, Chandee and Soundararajan in [10, Theorem 1] obtained another improvement, currently the best bound, as shown below.

Theorem 3.1 (Upper bound for $\zeta(s)$ in the critical line)**.** *Assume RH. For large real numbers t, we have*

$$\log\left|\zeta\left(\tfrac{1}{2}+it\right)\right| \leq \frac{\log 2}{2}\frac{\log t}{\log\log t} + O\left(\frac{\log t \log\log\log t}{(\log\log t)^2}\right).$$

A generalization of this result to the critical strip $0 < \Re(s) < 1$ was later obtained by Carneiro and Chandee in [3, Theorem 1].

Another object of interest is the argument function defined by (here $t > 0$)

$$S(t) = \tfrac{1}{\pi}\arg\zeta\left(\tfrac{1}{2}+it\right),$$

where the argument is defined by a continuous variation along the line segments joining the points $2, 2+it$ and $\frac{1}{2}+it$, taking $\arg\zeta(2) = 0$, if t is not an ordinate of a zero of $\zeta(s)$. If t is an ordinate of a zero we set

$$S(t) = \tfrac{1}{2}\lim_{\epsilon\to 0}\left\{S(t+\epsilon)+S(t-\epsilon)\right\}.$$

This function appears for instance when counting the number of zeros $N(t)$ of $\zeta(s)$ with imaginary ordinate in the interval $[0,t]$

$$N(t) = \frac{t}{2\pi}\log\frac{t}{2\pi} - \frac{t}{2\pi} + \frac{7}{8} + S(t) + O\left(\frac{1}{t}\right).$$

In the work [26] Littlewood also showed that under RH we have

$$|S(t)| \leq \left(C+o(1)\right)\frac{\log t}{\log\log t},$$

and, as in the case of the size of $\zeta\left(\frac{1}{2}+it\right)$, this estimate has not been improved in its order of magnitude over the years. Efforts to bring down the value of the admissible constant C were carried out by Ramachandra and Sankaranarayanan [34] who proved that $C = 1.119$ is admissible, and later by Fujii [13] who obtained the result for $C = 0.67$.

The application of certain extremal functions of exponential type, that majorize and/or minorize the characteristic functions of intervals, to problems related to the theory of the Riemann zeta-function dates back to the works of Montgomery [31] and Gallagher [14], on the pair correlation of zeros of $\zeta(s)$. In [16] Goldston and Gonek were the first to realize a distinct connection between the Riemann hypothesis and these extremal functions, via the so called Guinand-Weil explicit formula (the method we shall be presenting here). Using this connection they obtained the following bound [16, Theorem 2] for the argument function

$$|S(t)| \leq \left(\frac{1}{2} + o(1)\right) \frac{\log t}{\log\log t}.$$

We shall present here a sharper version of this bound, recently obtained by Carneiro, Chandee and Milinovich in [4, Theorem 2].

Theorem 3.2 (Bound for $S(t)$)**.** *Assume RH. For t sufficiently large we have*

$$|S(t)| \leq \frac{1}{4}\frac{\log t}{\log\log t} + O\left(\frac{\log t \log\log\log t}{(\log\log t)^2}\right).$$

Finally, another important function in the theory of the Riemann zeta-function is the antiderivative of $S(t)$ defined by

$$S_1(t) = \int_0^t S(u)\, \mathrm{d}u.$$

There has been earlier work on establishing explicit bounds for $S_1(t)$. Littlewood [26] was the first to prove that $S_1(t) \ll \log t/(\log\log t)^2$ under the assumption of the Riemann hypothesis. More recently, Karatsuba and Korolëv [23] obtained that

$$\big|S_1(t)\big| \;\leq\; (40+o(1))\, \frac{\log t}{(\log\log t)^2},$$

and Fujii [13] obtained that

$$-\,(0.51+o(1))\, \frac{\log t}{(\log\log t)^2} \;\leq\; S_1(t) \;\leq\; (0.32+o(1))\, \frac{\log t}{(\log\log t)^2}.$$

We present here the following improvement obtained in [4, Theorem 1].

Theorem 3.3 (Bounds for $S_1(t)$)**.** *Assume RH. For t sufficiently large we have*

$$-\left(\frac{\pi}{24}+o(1)\right) \frac{\log t}{(\log\log t)^2} \;\leq\; S_1(t) \;\leq\; \left(\frac{\pi}{48}+o(1)\right) \frac{\log t}{(\log\log t)^2},$$

where the terms $o(1)$ in the above inequalities are $O(\log\log\log t/\log\log t)$.

The objective of this chapter is to prove the three theorems above. The general strategy for their proofs is essentially the same. It consists of three main steps: (i) expressing the considered object as a certain sum over the zeros of $\zeta(s)$; (ii) making use of suitable extremal majorants/minorants of exponential type; (iii) applying an appropriate explicit formula to evaluate the sums by taking advantage of the compactly supported Fourier transforms. We shall see in the next section how the functions $f(x)$, $g(x)$ and $h(x)$ are naturally related to Theorems 3.1, 3.2 and 3.3, respectively.

3.2 Representation lemmas and the explicit formula

In this section we let

$$\xi(s) = \frac{1}{2} s(1-s)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

be Riemann's ξ-function. This function is an entire function of order 1 and satisfies the functional equation

$$\xi(s) = \xi(1-s).$$

Hadamard's factorization formula (cf. [12, Chapter 12]) gives us

$$\xi(s) = e^{A+Bs}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho},$$

where $\rho = \frac{1}{2} + i\gamma$ runs over the non-trivial zeros of $\zeta(s)$. We have $B = -\sum_{\rho}\mathrm{Re}(1/\rho)$, with this sum being absolutely convergent. Under RH, γ is real. For $\frac{1}{2} \leq \alpha \leq \frac{3}{2}$ define

$$f_\alpha(x) = \log\left(\frac{x^2+1}{x^2+\left(\alpha-\frac{1}{2}\right)^2}\right). \tag{3.4}$$

Note that our $f(x)$ initially defined in (3.1) is the same $f_{1/2}(x)$ defined above.

Lemma 3.4 (Representation for $\log|\zeta(\alpha+it)|$)**.** *Assume RH and let $f_\alpha(x)$ be defined by* (3.4)*, where $\frac{1}{2} \leq \alpha \leq \frac{3}{2}$. For large t we have*

$$\log|\zeta(\alpha+it)| = \left(\frac{3}{4}-\frac{\alpha}{2}\right)\log t - \frac{1}{2}\sum_{\gamma} f_\alpha(t-\gamma) + O(1), \tag{3.5}$$

uniformly on α, where the sum runs over the non-trivial zeros $\rho = \frac{1}{2}+i\gamma$ of $\zeta(s)$.

Proof. We apply Hadamard's factorization formula at the points $s = \alpha + it$ and $s = \frac{3}{2} + it$ and divide. The absolute convergence of the product allows us to divide term by term to find

$$\left|\frac{\xi(\alpha + it)}{\xi\big(\frac{3}{2} + it\big)}\right| = \prod_{\rho = 1/2 + i\gamma} \left(\frac{(\alpha - \frac{1}{2})^2 + (t-\gamma)^2}{1 + (t-\gamma)^2}\right)^{1/2},$$

and therefore

$$\log|\xi(\alpha + it)| = \log\left|\xi\big(\tfrac{3}{2} + it\big)\right| + \frac{1}{2}\sum_{\gamma} \log\left(\frac{(\alpha - \frac{1}{2})^2 + (t-\gamma)^2}{1 + (t-\gamma)^2}\right). \tag{3.6}$$

Recall Stirling's formula for the Gamma function [12, Chapter 10]

$$\log\Gamma(z) = \tfrac{1}{2}\log 2\pi - z + \big(z - \tfrac{1}{2}\big)\log z + O\big(|z|^{-1}\big),$$

for large $|z|$. Using Stirling's formula and the fact that $\left|\zeta\big(\frac{3}{2} + it\big)\right| \asymp 1$ in (3.6), we obtain (3.5). ☐

Similar representations hold for the argument function $S(t)$ and for the function $S_1(t)$, as reported in [4].

Lemma 3.5 (Representation for $S(t)$)**.** *Assume RH and let $g(x)$ be defined by* (3.2)*. Then, for large t not coinciding with an ordinate of a zero of $\zeta(s)$, we have*

$$S(t) = \frac{1}{\pi}\sum_{\gamma} g(t-\gamma) + O(1), \tag{3.7}$$

where the sum runs over the non-trivial zeros $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$.

Proof. For t not coinciding with an ordinate of a zero of $\zeta(s)$, we have

$$S(t) = -\frac{1}{\pi}\int_{\frac{1}{2}}^{\infty} \Im\frac{\zeta'}{\zeta}(\sigma + it)\,\mathrm{d}\sigma = \frac{1}{\pi}\int_{\frac{3}{2}}^{\frac{1}{2}} \Im\frac{\zeta'}{\zeta}(\sigma + it)\,\mathrm{d}\sigma + O(1).$$

We now replace the integrand on the right-hand side of the above expression by a sum over the non-trivial zeros of $\zeta(s)$. Let $s = \sigma + it$. If s is not a zero of $\zeta(s)$, then the partial fraction decomposition for $\zeta'(s)/\zeta(s)$ (cf. [12, Chapter 12]) and Stirling's formula

$$\frac{\Gamma'(z)}{\Gamma(z)} = \log z + O\big(|z|^{-1}\big), \tag{3.8}$$

valid for large $|z|$ with $\Re(z) > 0$, imply that

$$\begin{aligned} \frac{\zeta'}{\zeta}(s) &= \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2} + 1\right) + O(1) \\ &= \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) - \frac{1}{2}\log t + O(1) \end{aligned} \tag{3.9}$$

uniformly for $\frac{1}{2} \le \sigma \le \frac{3}{2}$ and $t \ge 2$, where the sum runs over the non-trivial zeros ρ of $\zeta(s)$. From (3.9) and the Riemann hypothesis, it follows that

$$\begin{aligned}
S(t) &= \frac{1}{\pi}\int_{\frac{3}{2}}^{\frac{1}{2}} \Im\frac{\zeta'}{\zeta}(\sigma+it)\,\mathrm{d}\sigma + O(1)\\
&= \frac{1}{\pi}\int_{\frac{3}{2}}^{\frac{1}{2}} \Im\left(\frac{\zeta'}{\zeta}(\sigma+it) - \frac{\zeta'}{\zeta}(\tfrac{3}{2}+it)\right)\mathrm{d}\sigma + O(1)\\
&= \frac{1}{\pi}\int_{\frac{1}{2}}^{\frac{3}{2}} \sum_{\gamma}\left\{\frac{(t-\gamma)}{(\sigma-\frac{1}{2})^2+(t-\gamma)^2} - \frac{(t-\gamma)}{1+(t-\gamma)^2}\right\}\mathrm{d}\sigma + O(1)\\
&= \frac{1}{\pi}\sum_{\gamma}\int_{\frac{1}{2}}^{\frac{3}{2}} \left\{\frac{(t-\gamma)}{(\sigma-\frac{1}{2})^2+(t-\gamma)^2} - \frac{(t-\gamma)}{1+(t-\gamma)^2}\right\}\mathrm{d}\sigma + O(1)\\
&= \frac{1}{\pi}\sum_{\gamma}\left\{\arctan\left(\frac{1}{(t-\gamma)}\right) - \frac{(t-\gamma)}{1+(t-\gamma)^2}\right\} + O(1)\\
&= \frac{1}{\pi}\sum_{\gamma} g(t-\gamma) + O(1),
\end{aligned}$$

where the interchange of the integral and the sum is justified by dominated convergence since $g(x) = O(x^{-3})$. This proves the lemma. ☐

Lemma 3.6 (Representation for $S_1(t)$)**.** *Assume RH and let $h(x)$ be defined by* (3.3)*. For large t we have*

$$S_1(t) = \frac{1}{4\pi}\log t - \frac{1}{\pi}\sum_{\gamma} h(t-\gamma) + O(1),$$

where the sum runs over the non-trivial zeros $\rho = \frac{1}{2}+i\gamma$ of $\zeta(s)$.

Proof. From [42, Theorem 9.9] we have

$$S_1(t) = \frac{1}{\pi}\int_{1/2}^{3/2} \log\left|\zeta(\alpha+it)\right|\mathrm{d}\alpha + O(1).$$

We replace the integrand by the absolutely convergent sum over the zeros of $\zeta(s)$ given by Lemma 3.4 and integrate term-by-term to obtain

$$S_1(t) = \frac{1}{4\pi}\log t - \frac{1}{\pi}\sum_{\rho} h(t-\gamma) + O(1),$$

where the interchange between integration and sum is justified since all terms are non-negative. Notice that we have used the fact that

$$h(x) = 1 - x\arctan\left(\frac{1}{x}\right) = \frac{1}{2}\int_{1/2}^{3/2}\log\left(\frac{x^2+1}{x^2+\left(\alpha-\frac{1}{2}\right)^2}\right)\mathrm{d}\alpha. \tag{3.10}$$

This completes the proof of the lemma. ☐

We note the similarity of the representations obtained on Lemmas 3.4, 3.5 and 3.6. We were able to write each of our objects (initially a function of t) as a simple function of t plus a sum over the zeros of $\zeta(s)$ plus a small error term. Naturally the hard part to be analyzed is the sum over the zeros of $\zeta(s)$, but fortunately for this matter we can invoke the following version of the Guinand-Weil explicit formula [22, Theorem 5.12] which connects sums over the zeros of $\zeta(s)$ to sums of the Fourier transforms evaluated at the prime powers.

Lemma 3.7 (Guinand-Weil explicit formula). *Let $\Phi(s)$ be analytic in the strip $|\Im(s)| \leq 1/2+\varepsilon$ for some $\varepsilon > 0$, and assume that $|\Phi(s)| \ll (1+|s|)^{-(1+\delta)}$ for some $\delta > 0$ when $|\Re(s)| \to \infty$. Let $\Phi(x)$ be real-valued for real x, and set $\widehat{\Phi}(\xi) = \int_{-\infty}^{\infty} \Phi(x)\, e^{-2\pi i x\xi}\, \mathrm{d}x$. Then*

$$\begin{aligned}\sum_{\rho} \Phi(\gamma) = \Phi\left(\frac{1}{2i}\right) + \Phi\left(-\frac{1}{2i}\right) \\ - \frac{1}{2\pi}\widehat{\Phi}(0)\log\pi + \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(u)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{iu}{2}\right)\mathrm{d}u \\ - \frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\left(\widehat{\Phi}\left(\frac{\log n}{2\pi}\right) + \widehat{\Phi}\left(\frac{-\log n}{2\pi}\right)\right).\end{aligned}$$

where Γ'/Γ is the logarithmic derivative of the Gamma function, and $\Lambda(n)$ is the von Mangoldt function defined as

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m,\ p \text{ prime},\ m \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Observe however that we cannot apply the explicit formula to evaluate the sum of our particular functions f, g and h over the zeros of ζ, since f has singularities on the strip $|\Im(s)| \leq 1/2$, g is not continuous and h is not differentiable at the origin. To overcome this difficulty we adopt the following strategy:

(i) We want to replace each of our functions f, g and h by an appropriate majorant or minorant (to create an inequality), that satisfies the hypothesis of the explicit formula (a real entire function, integrable on $\mathbb{R}$).

Now that we believe we will be able to use the explicit formula, we might want to choose which of its expressions we would like to “keep" or “simplify". For this we will focus on two of its terms.

(ii) We will ask that the term $\widehat{\Phi}(0)$ for these majorants be as close as possible to the original $\widehat{f}(0)$, which is the same as saying that $\int_{\mathbb{R}}\{\Phi - f\}\,\mathrm{d}x$ should be minimal.

(iii) Finally, in order to simplify the sum of the Fourier transforms of over prime powers, we will consider the instances in which this sum is finite, i.e. $\widehat{\Phi}$ has compact support.

With this framework we are essentially asking for the solution of the Beurling-Selberg problem for each of the functions f, g and h.

3.3 Extremal functions revisited

In this section we revisit the extremal function theory to state and prove the main facts concerning the majorants/minorants of the functions $f(x)$, $g(x)$ and $h(x)$ in the precise format we need. Let us start with the minorants for the function $f(x)$, contemplated in the Gaussian subordination framework for even functions in Chapter 2. Observe that there can be no discussion about real entire majorants for this function because of its singularity at the origin.

Lemma 3.8 (Extremal minorants for f)**.** *Let $1 \leq \Delta$ and f be defined by* (3.1). *Then there is a unique real entire function $m_\Delta^- : \mathbb{C} \to \mathbb{C}$ satisfying the following properties:*

(i) For all real x we have

$$\frac{-C}{1+x^2} \leq m_\Delta^-(x) \leq f(x)$$

for some positive constant C. For any complex number $x+iy$ we have

$$\big|m_\Delta^-(x+iy)\big| \ll \frac{\Delta^2}{1+\Delta|x+iy|} e^{2\pi\Delta|y|}.$$

(ii) The Fourier transform of m_Δ^-, namely

$$\widehat{m}_\Delta^-(\xi) = \int_{-\infty}^{\infty} m_\Delta^-(x)\, e^{-2\pi i x\xi}\, \mathrm{d}x,$$

is a continuous real-valued function supported on the interval $[-\Delta, \Delta]$ and satisfies

$$\big|\widehat{m}_\Delta^-(\xi)\big| \ll 1$$

for each $\xi \in [-\Delta, \Delta]$.

(iii) The L^1-distance to f is given by

$$\int_{-\infty}^{\infty} \big\{f(x) - m_\Delta^-(x)\big\}\, \mathrm{d}x = \frac{2}{\Delta}\Big\{ \log 2 - \log\big(1+e^{-2\pi\Delta}\big)\Big\}.$$

Proof. (i) Observe that the desired function $m_\Delta^-(x)$ we seek is the extremal minorant of exponential type $2\pi\Delta$ of $f(x)$. To adjust to our work in Chapter 2 (done for exponential type 2π) we simply consider the function

$$F_\Delta(x) := f\left(\frac{x}{\Delta}\right) = \log\left(\frac{x^2+\Delta^2}{x^2}\right).$$

We do know (by Section 2.3.5 - Example 3) that such F_Δ admits an minorant of type 2π, that we shall call $L_\Delta(z)$, given by

$$L_\Delta(z) = \Big(\frac{\cos \pi z}{\pi}\Big)^2 \sum_{n=-\infty}^{\infty} \left\{ \frac{F_\Delta\big(n-\frac{1}{2}\big)}{\big(z-n+\frac{1}{2}\big)^2} + \frac{F_\Delta'\big(n-\frac{1}{2}\big)}{\big(z-n+\frac{1}{2}\big)} \right\},$$
$$= \sum_{n=-\infty}^{\infty} \left(\frac{\sin \pi\big(z-n+\frac{1}{2}\big)}{\pi\big(z-n+\frac{1}{2}\big)}\right)^2 \left\{ f\left(\frac{n-\frac{1}{2}}{\Delta}\right) + \frac{\big(z-n+\frac{1}{2}\big)}{\Delta} f'\left(\frac{n-\frac{1}{2}}{\Delta}\right) \right\}. \tag{3.11}$$

For any complex number ξ we have

$$\left(\frac{\sin(\pi\xi)}{\pi\xi}\right)^2 \ll \frac{e^{2\pi|\Im(\xi)|}}{1+|\xi|^2},$$

and we also have $f(x) \leq 1/x^2$ and $|f'(x)| \leq 2/(|x|(x^2+1))$ for real x. From (3.11) we conclude that

$$\big|L_\Delta(x+iy)\big| \leq \frac{\Delta^2}{1+|x+iy|}\, e^{2\pi|y|}. \tag{3.12}$$

Since $f(x) \geq 0$ and $f'(-x) = -f'(x)$, by pairing the terms $n \geq 1$ with the terms $1-n \leq 0$ in the sum (3.11) we find, for $x \in \mathbb{R}$, that

$$L_\Delta(x) \geq \sum_{n=1}^{\infty} \left(\frac{\sin \pi\big(x-n+\frac{1}{2}\big)}{\pi\big(x^2-\big(n-\frac{1}{2}\big)\big)^2}\right)^2 \frac{2\left(n-\frac{1}{2}\right)}{\Delta} f'\left(\frac{n-\frac{1}{2}}{\Delta}\right), \tag{3.13}$$

and from this we can deduce that there is a constant C such that

$$-\,C\frac{\Delta^2}{\Delta^2+x^2} \leq L_\Delta(x) \leq F_\Delta(x). \tag{3.14}$$

We now consider $m_\Delta^-(z) = L_\Delta(\Delta z)$. Part (i) of the lemma plainly follows from (3.12) and (3.14).

(ii) We know that m_Δ^- is an (even) entire function of exponential type $2\pi\Delta$ that is uniformly integrable on $\mathbb{R}$ (with integral independent of the parameter $\Delta \geq 1$, by part (i)). From the Paley-Wiener theorem we have that $\widehat{m}_\Delta^-$ is a continuous real-valued function supported on the interval $[-\Delta, \Delta]$ and satisfies

$$\big|\widehat{m}_\Delta^-(\xi)\big| \ll 1$$

for each $\xi \in [-\Delta, \Delta]$.

(iii) Recall from Section 2.3.5 - Example 3 that

$$\int_{-\infty}^{\infty} \{F_\Delta(x) - L_\Delta(x)\}\, \mathrm{d}x = 2\Big\{ \log 2 - \log\big(1+e^{-2\pi\Delta}\big)\Big\}.$$

A simple change of variables $x \mapsto \Delta x$ gives the desired result. $\square$

The proofs of next two lemmas are similar to the previous one and are omitted here. The interested reader can check the details in [4, Lemmas 6 and 8].

Lemma 3.9 (Extremal functions for g)**.** *Let $1 \leq \Delta$ and g be defined by (3.2). Then there are unique real entire functions $m_\Delta^+ : \mathbb{C} \to \mathbb{C}$ and $m_\Delta^- : \mathbb{C} \to \mathbb{C}$ satisfying the following properties:*

(i) For all real x we have

$$\frac{-C}{1+x^2} \leq m_\Delta^-(x) \leq g(x) \leq m_\Delta^+(x) \leq \frac{C}{1+x^2},$$

for some positive constant C. For any complex number $x+iy$ we have

$$\left|m_\Delta^\pm(x+iy)\right| \ll \frac{\Delta^2}{1+\Delta|x+iy|} e^{2\pi\Delta|y|}.$$

(ii) The Fourier transforms of $m_\Delta^\pm$ are continuous functions supported on the interval $[-\Delta, \Delta]$ and satisfy

$$\left|\widehat{m}_\Delta^\pm(\xi)\right| \ll 1$$

for each $\xi \in [-\Delta, \Delta]$.

(iii) The L^1-distances to g are given by

$$\int_{-\infty}^{\infty} \left\{m_\Delta^+(x) - g(x)\right\} \,\mathrm{d}x = \int_{-\infty}^{\infty} \left\{g(x) - m_\Delta^-(x)\right\} \,\mathrm{d}x = \frac{\pi}{2\Delta}.$$

Lemma 3.10 (Extremal functions for h)**.** *Let $1 \leq \Delta$ and h be defined by (3.3). Then there are unique real entire functions $m_\Delta^+ : \mathbb{C} \to \mathbb{C}$ and $m_\Delta^- : \mathbb{C} \to \mathbb{C}$ satisfying the following properties:*

(i) For all real x we have

$$\frac{-C}{1+x^2} \leq m_\Delta^-(x) \leq h(x) \leq m_\Delta^+(x) \leq \frac{C}{1+x^2},$$

for some positive constant C. For any complex number $x+iy$ we have

$$\left|m_\Delta^\pm(x+iy)\right| \ll \frac{\Delta^2}{1+\Delta|x+iy|} e^{2\pi\Delta|y|}.$$

(ii) The Fourier transforms of $m_\Delta^\pm$ are continuous real-valued functions supported on the interval $[-\Delta, \Delta]$ and satisfy

$$\left|\widehat{m}_\Delta^\pm(\xi)\right| \ll 1$$

for each $\xi \in [-\Delta, \Delta]$.

(iii) The L^1-distances to h are given by

$$\int_{-\infty}^{\infty} \left\{h(x) - m_\Delta^-(x)\right\} \mathrm{d}x$$
$$= \int_{1/2}^{3/2} \frac{1}{\Delta} \left\{\log\left(1 + e^{-(2\sigma-1)\pi\Delta}\right) - \log\left(1 + e^{-2\pi\Delta}\right)\right\} \mathrm{d}\sigma,$$

and

$$\int_{-\infty}^{\infty} \left\{m_\Delta^+(x) - h(x)\right\} \mathrm{d}x$$
$$= \int_{1/2}^{3/2} \frac{1}{\Delta} \left\{\log\left(1 - e^{-2\pi\Delta}\right) - \log\left(1 - e^{-(2\sigma-1)\pi\Delta}\right)\right\} \mathrm{d}\sigma.$$

3.4 Proofs of the main theorems

We now make use of the extremal functions described on the last section, together with the representation formulas to provide the proofs of the main theorems.

3.4.1 Proof of Theorem 3.1

With f defined by (3.1) and m_Δ^- defined as in Lemma 3.8, we can use Lemma 3.4 to obtain

$$\begin{aligned} \log\left|\zeta\left(\tfrac{1}{2} + it\right)\right| &= \frac{1}{2}\log t - \frac{1}{2}\sum_\gamma f(t-\gamma) + O(1) \\ &\leq \frac{1}{2}\log t - \frac{1}{2}\sum_\gamma m_\Delta^-(t-\gamma) + O(1). \end{aligned} \tag{3.15}$$

We now apply the explicit formula (Lemma 3.7) with $\Phi(z) = m_\Delta^-(t-z)$. In this context we have $\widehat{\Phi}(\xi) = \widehat{m}_\Delta^-(-\xi)e^{-2\pi i\xi t}$ and therefore

$$\begin{aligned} \sum_\rho m_\Delta^-(t-\gamma) = &\left\{m_\Delta^-\left(t - \tfrac{1}{2i}\right) + m_\Delta^-\left(t + \tfrac{1}{2i}\right)\right\} - \frac{1}{2\pi}\widehat{m}_\Delta^-(0)\log\pi \\ &+ \frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta^-(t-u)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{iu}{2}\right) \mathrm{d}u \\ &- \frac{1}{2\pi}\sum_{n=2}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \left\{n^{-it}\,\widehat{m}_\Delta^-\left(-\frac{\log n}{2\pi}\right) + n^{it}\,\widehat{m}_\Delta^-\left(\frac{\log n}{2\pi}\right)\right\}. \end{aligned} \tag{3.16}$$

Let us split this sum into four terms and quote each of these separately.

First term. From Lemma 3.8 (i) we see that

$$\left|m_\Delta^-\big(t - \tfrac{1}{2i}\big) + m_\Delta^-\big(t + \tfrac{1}{2i}\big)\right| \ll \frac{\Delta^2}{1+\Delta t} e^{\pi\Delta}. \tag{3.17}$$

Second term. From Lemma 3.8 (ii) we have

$$\left|\widehat{m}_\Delta^-(0)\right| \ll 1. \tag{3.18}$$

Third term. Using Stirling's formula (3.8), Lemma 3.8 (i) and (iii), and the fact that $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi$ we have

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta^-(t-u)\, \Re\left[\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{iu}{2}\right)\right]\,\mathrm{d}u &= \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta^-(u)\,\big(\log t + O(\log(2+|u|))\big)\,\mathrm{d}u \\ &= \log t - \frac{\log t}{\pi\Delta}\log\left(\frac{2}{1+e^{-2\pi\Delta}}\right) + O(1).\end{aligned} \tag{3.19}$$

Fourth term. Finally, we use the fact that the Fourier transform of m_Δ^- is compactly supported on the interval $[-\Delta, \Delta]$, as given in Lemma 3.8 (ii), to bound the sum over the prime powers

$$\begin{aligned}&\left|\frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\left\{n^{-it}\widehat{m}_\Delta^-\left(-\frac{\log n}{2\pi}\right) + n^{it}\widehat{m}_\Delta^-\left(\frac{\log n}{2\pi}\right)\right\}\right| \\ &\le \frac{1}{2\pi}\sum_{n=2}^{e^{2\pi\Delta}}\frac{\Lambda(n)}{\sqrt{n}}\left\{\left|\widehat{m}_\Delta^-\left(-\frac{\log n}{2\pi}\right)\right| + \left|\widehat{m}_\Delta^-\left(\frac{\log n}{2\pi}\right)\right|\right\} \\ &\ll \sum_{n=2}^{e^{2\pi\Delta}}\frac{\Lambda(n)}{\sqrt{n}} \ll e^{\pi\Delta},\end{aligned} \tag{3.20}$$

where the last expression was evaluated via summation by parts.

Conclusion. Combining expressions (3.15)-(3.20) we arrive at

$$\log\left|\zeta\big(\tfrac{1}{2}+it\big)\right| \le \frac{\log t}{2\pi\Delta}\log\left(\frac{2}{1+e^{-2\pi\Delta}}\right) + O\left(\frac{\Delta^2 e^{\pi\Delta}}{(1+\Delta t)} + e^{\pi\Delta} + 1\right). \tag{3.21}$$

Until now we did all of our estimates without prescribing any particular value for Δ. It turns out that the choice

$$\pi\Delta = \log\log t - 3\log\log\log t$$

in (3.21) concludes the proof of Theorem 3.1.

3.4.2 Proof of Theorem 3.2

This follows by a very similar argument. With g defined as in (3.2), and $m_\Delta^\pm$ defined as in Lemma 3.9, we can use Lemma 3.5 to obtain

$$\begin{aligned}&\frac{1}{\pi}\sum_\gamma m_\Delta^-(t-\gamma)+O(1)\\&\qquad\le S(t)=\frac{1}{\pi}\sum_\gamma g(t-\gamma)+O(1)\le\frac{1}{\pi}\sum_\gamma m_\Delta^+(t-\gamma)+O(1).\end{aligned}$$

We then use the explicit formula with $m_\Delta^\pm$ and bound the first, second and fourth terms as done in the proof of Theorem 3.1, now using Lemma 3.9. For the third term we use Stirling's formula (3.8), Lemma 3.9 (i) and (iii), and the fact that $\int_{-\infty}^{\infty} g(x)\,\mathrm{d}x=0$ to get

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta^\pm(t-u)\,\Re\left[\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{iu}{2}\right)\right]\,\mathrm{d}u=\\&\qquad\qquad=\frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta^\pm(u)\,\big(\log t+O(\log(2+|u|)\big)\,\mathrm{d}u\\&\qquad\qquad=\pm\frac{\log t}{4\Delta}+O(1).\end{aligned}$$

We thus arrive at

$$|S(t)|\le\frac{\log t}{4\pi\Delta}+O\left(\frac{\Delta^2e^{\pi\Delta}}{(1+\Delta t)}+e^{\pi\Delta}+1\right).$$

and again it is just a matter of choosing $\pi\Delta=\log\log t-3\log\log\log t$ to conclude the proof of Theorem 3.2.

3.4.3 Proof of Theorem 3.3

Let h be defined as in (3.3), and $m_\Delta^\pm$ defined as in Lemma 3.10. From Lemma 3.6 we have

$$\begin{aligned}&\frac{1}{4\pi}\log t-\frac{1}{\pi}\sum_\gamma m_\Delta^+(t-\gamma)+O(1)\\&\qquad\le S_1(t)=\frac{1}{4\pi}\log t-\frac{1}{\pi}\sum_\gamma h(t-\gamma)+O(1)\\&\qquad\qquad\le\frac{1}{4\pi}\log t-\frac{1}{\pi}\sum_\gamma m_\Delta^-(t-\gamma)+O(1).\end{aligned}$$

Once more we apply the explicit formula with $m_\Delta^\pm$ and bound the first, second and fourth terms as done in the proof of Theorem 3.1, now using Lemma 3.10.

For the third term we use Stirling's formula (3.8), Lemma 3.10 (i) and (iii), and the fact that

$$\int_{-\infty}^{\infty} h(x)\,\mathrm{d}x = \frac{\pi}{2}$$

to get

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta^-(t-u)\,\Re\left[\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{iu}{2}\right)\right]\mathrm{d}u\\
&\quad= \frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta^-(u)\Big(\log t + O\big(\log(2+|u|)\big)\Big)\,\mathrm{d}u\\
&\quad= \frac{1}{4}\log t - \frac{\log t}{2\pi\Delta}\int_{1/2}^{3/2}\left(\log\left(1+e^{-(2\sigma-1)\pi\Delta}\right)-\log\left(1+e^{-2\pi\Delta}\right)\right)\mathrm{d}\sigma + O(1)\\
&\quad\geq \frac{1}{4}\log t - \frac{\log t}{2\pi\Delta}\int_{1/2}^{\infty}\log\left(1+e^{-(2\sigma-1)\pi\Delta}\right)\mathrm{d}\sigma + O(1)\\
&\quad= \frac{1}{4}\log t - \frac{\log t}{2\pi^2\Delta^2}\int_{0}^{\infty}\log\left(1+e^{-2\alpha}\right)\mathrm{d}\alpha + O(1).
\end{aligned}$$

Now observe that (cf. [17, §4.291])

$$\int_0^\infty \log\left(1+e^{-2\alpha}\right)\mathrm{d}\alpha = \frac{1}{2}\int_0^1 \frac{\log(1+u)}{u}\,\mathrm{d}u = \frac{\pi^2}{24}.$$

Therefore, by combining these estimates, we arrive at

$$S_1(t) \leq \frac{\log t}{48\pi\Delta^2} + O\left(\frac{\Delta^2 e^{\pi\Delta}}{(1+\Delta t)} + e^{\pi\Delta} + 1\right).$$

Choosing $\pi\Delta = \log\log t - 3\log\log\log t$ in the inequality above gives us

$$S_1(t) \leq \frac{\pi}{48}\frac{\log t}{(\log\log t)^2} + O\left(\frac{\log t\log\log\log t}{(\log\log t)^3}\right),$$

which is the upper bound for $S_1(t)$ stated in Theorem 3.3. To prove the lower bound we proceed similarly by observing that

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta^+(t-u)\,\Re\left[\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{iu}{2}\right)\right]\mathrm{d}u\\
&\quad= \frac{1}{2\pi}\int_{-\infty}^{\infty} m_\Delta^+(u)\Big(\log t + O\big(\log(2+|u|)\big)\Big)\,\mathrm{d}u\\
&\quad= \frac{1}{4}\log t - \frac{\log t}{2\pi\Delta}\int_{1/2}^{3/2}\left(\log\left(1-e^{-(2\sigma-1)\pi\Delta}\right)-\log\left(1-e^{-2\pi\Delta}\right)\right)\mathrm{d}\sigma + O(1)\\
&\quad\leq \frac{1}{4}\log t - \frac{\log t}{2\pi\Delta}\int_{1/2}^{\infty}\log\left(1-e^{-(2\sigma-1)\pi\Delta}\right)\mathrm{d}\sigma + O(1)\\
&\quad= \frac{1}{4}\log t - \frac{\log t}{2\pi^2\Delta^2}\int_{0}^{\infty}\log\left(1-e^{-2\alpha}\right)\mathrm{d}\alpha + O(1).
\end{aligned}$$

We now invoke the identity (cf. [17, §4.291])

$$\int_0^\infty \log\left(1-e^{-2\alpha}\right)\,\mathrm{d}\alpha = \frac{1}{2}\int_0^1 \frac{\log(1-u)}{u}\,\mathrm{d}u = -\frac{\pi^2}{12}$$

to arrive at

$$S_1(t) \geq -\frac{\log t}{24\pi\Delta^2} + O\left(\frac{\Delta^2 e^{\pi\Delta}}{(1+\Delta t)} + e^{\pi\Delta} + 1\right).$$

Finally, choosing $\pi\Delta = \log\log t - 3\log\log\log t$ in the inequality above gives us

$$S_1(t) \geq -\frac{\pi}{24}\frac{\log t}{(\log\log t)^2} + O\left(\frac{\log t \log\log\log t}{(\log\log t)^3}\right),$$

and this completes the proof of Theorem 3.3.

Bibliography

[1] J. T. Barton, H. L. Montgomery, and J. D. Vaaler, Note on a Diophantine inequality in several variables, Proc. Amer. Math. Soc. 129 (2001), 337–345.

[2] E. Carneiro, Sharp approximations to the Bernoulli periodic functions by trigonometric polynomials, J. Approx. Theory 154 (2008), 90–104.

[3] E. Carneiro and V. Chandee, Bounding $\zeta(s)$ in the critical strip, J. Number Theory 131 (2011), 363–384.

[4] E. Carneiro, V. Chandee and M. Milinovich, Bounding $S(t)$ and $S_1(t)$ under the Riemann hypothesis, to appear in Math. Ann.

[5] E. Carneiro and F. Littmann, Bandlimited approximations to the truncated Gaussian and applications, to appear in Const. Approx.

[6] E. Carneiro and F. Littmann, Entire approximations for a class of truncated and odd functions, preprint.

[7] E. Carneiro, F. Littmann, and J. D. Vaaler, Gaussian subordination for the Beurling-Selberg extremal problem, to appear in Trans. Amer. Math. Soc.

[8] E. Carneiro and J. D. Vaaler, Some extremal functions in Fourier analysis, II, Trans. Amer. Math. Soc. 362 (2010), 5803–5843.

[9] E. Carneiro and J. D. Vaaler, Some extremal functions in Fourier analysis, III, Constr. Approx. 31, No. 2 (2010), 259–288.

[10] V. Chandee and K. Soundararajan, Bounding $|\zeta(\frac{1}{2}+it)|$ on the Riemann Hypothesis, Bull. London Math. Soc. 43(2) (2011), 243–250.

[11] K. Chandrasekharan, *Elliptic Functions*, Springer-Verlag, Berlin, 1985.

[12] H. Davenport, *Multiplicative number theory*, Third edition, Graduate Texts in Mathematics 74, Springer-Verlag, New York, 2000.

[13] G. B. Folland, *Real Analysis: Modern Techniques and Their Applications*, John Wiley and Sons Inc., 1999.

[14] P. X. Gallagher, Pair correlation of zeros of the zeta-function, J. Reine Angew. Math. 362 (1985), 72–86.

[15] M. I. Ganzburg and D. S. Lubinsky, Best approximating entire functions to $|x|^\alpha$ in L^2, Complex analysis and dynamical systems III, 93–107, Contemp. Math. 455, Amer. Math. Soc., Providence, RI, 2008.

[16] D. A. Goldston and S. M. Gonek, A note on $S(t)$ and the zeros of the Riemann zeta-function, Bull. Lond. Math. Soc. 39 (2007), no. 3, 482–486.

[17] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, Translated from Russian. Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger. Seventh edition. Elsevier/Academic Press, Amsterdam, 2007.

[18] L. Grafakos, *Classical and Modern Fourier Analysis*, Pearson, Upper Saddle River, New Jersey, 2004.

[19] S. W. Graham and J. D. Vaaler, A class of Extremal Functions for the Fourier Transform, Tran. Amer. Math. Soc. 265, (1981), 283–302.

[20] J. Holt and J. D. Vaaler, The Beurling-Selberg extremal functions for a ball in the Euclidean space, Duke Math. Journal, 83, (1996), 203–247.

[21] L. Hörmander, *Linear Partial Differential Operators*, Springer-Verlag, 1964.

[22] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, American Mathematical Society Colloquium Publications, vol. 53, 2004.

[23] A. A. Karatsuba and M. A. Korolëv, The argument of the Riemann zeta-function, Russian Math. Surveys 60 (2005) no. 3, 433–488 (in English), Uspekhi Mat. Nauk 60 (2005), no. 3, 41–96 (in Russian).

[24] M. G. Krein, On the best approximation of continuous differentiable functions on the whole real axis, Dokl. Akad. Nauk SSSR 18 (1938), 615-624. (Russian).

[25] X. J. Li and J. D. Vaaler, Some trigonometric extremal functions and the Erdös-Turán type inequalities, Indiana Univ. Math. J. 48 (1999), no. 1, 183–236.

[26] J. E. Littlewood, On the zeros of the Riemann zeta-function, Proc. Camb. Philos. Soc. 22 (1924), 295–318.

[27] F. Littmann, One-sided approximation by entire functions, J. Approx. Theory 141 (2006), no. 1, 1–7.

[28] F. Littmann, Entire majorants via Euler-Maclaurin summation, Trans. Amer. Math. Soc. 358 (2006), no. 7, 2821–2836.

[29] F. Littmann, Entire approximations to the truncated powers, Constr. Approx. 22 (2005), no. 2, 273–295.

[30] F. Littmann, Quadrature and extremal bandlimited functions, preprint.

[31] H. L. Montgomery, The pair correlation of zeros of the zeta-function, Proc. Symp. Pure Math. 24, Providence (1973), 181–193.

[32] H. L. Montgomery and R. C. Vaughan, Hilbert's Inequality, J. London Math. Soc. (2), 8, (1974), 73–81.

[33] M. Plancherel and G. Polya, Fonctions entiéres et intégrales de Fourier multiples, (Seconde partie) Comment. Math. Helv. 10, (1938), 110–163.

[34] K. Ramachandra and A. Sankaranarayanan, On some theorems of Littlewood and Selberg. I., J. Number Theory 44 (1993), 281–291.

[35] I. J. Schoenberg, Metric spaces and completely monotone functions, Ann. of Math. (2) 39 (1938), 811–841.

[36] L. Schwartz, *Théorie des Distributions*, Vol. I et II, Hermann 1950, 1951.

[37] A. Selberg, Lectures on Sieves, *Atle Selberg: Collected Papers*, Vol. II, Springer-Verlag, Berlin, 1991, pp. 65–247.

[38] K. Soundararajan, Moments of the Riemann zeta-function, Ann. of Math. (2) 170 (2009), no. 2, 981–993.

[39] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

[40] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean spaces*, Princeton University Press, 1973.

[41] B. Sz.-Nagy, Über gewisse Extremalfragen bei transformierten trigonometrischen Entwicklungen II, Ber. Math.-Phys. Kl. Sächs. Akad. Wiss. Leipzig 91, 1939.

[42] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, 2nd edition (The Clarendon Press/Oxford University Press, New York, 1986).

[43] J. D. Vaaler, Some extremal functions in Fourier analysis, Bull. Amer. Math. Soc. 12 (1985), 183–215.

[44] A. Zygmund, *Trigonometric Series*, Vol II, Cambridge University Press, 1959.