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An Introduction to de Branges Spaces of Entire Functions with Applications to Differential Equations of the Sturm–Liouville Type*

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NOTATION

Typically we use the Greek letters, α, β, γ to denote complex variables writing often, for example, $\gamma = a + ib$ or $\beta = c + id$, with a, b, c, and d real. However, when γ appears as the variable of integration, as in $\int(\gamma - \beta)^{-1}f(\gamma)d\gamma$, it is, unless explicitly indicated otherwise, taken to be real. Moreover, when, as in this example, the limits of integration are not indicated explicitly they should be taken as $-\infty$ and $+\infty$. The differential will also be suppressed from time to time when this does not lead to ambiguity. R^1 shall denote the real line, R^2 the complex plane, and $R^{2+}[R^{2-}]$ the open upper [lower] half-plane. $\bar{\gamma}$ denotes the complex conjugate of γ and $f^*(\gamma) \equiv \bar{f}(\bar{\gamma})$. We also write $\log|f| = \log^+|f| - \log^-|f|$, where $\log^+|f| = \max\{\log|f|, 0\}$. If ν is a measure, or nondecreasing function, and I is a subinterval of R^1, then

$$L^2(d\nu : I) = \left\{\text{measurable functions } f : \int_I |f|^2\, d\nu < \infty\right\}.$$

If the measure is not indicated explicitly, as in $L^2(R^1)$, then it is Lebesgue, while if the interval is not indicated explicitly, as in $L^2(d\nu)$, then it is all of R^1.

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1. Introduction

Hilbert spaces of entire functions have been studied extensively by de Branges ([3–9]; for additional references see also the bibliography of [9]). Unfortunately, however, it requires considerable effort to extract many of the beautiful and far reaching results discovered by de Branges from his published work. The objective of the present paper is to make some aspects of the theory more accessible and at the same time illustrate its utility in developing spectral representation formulas.

In particular we shall utilize solutions of the system of equations

$$\begin{aligned} A_t(\gamma) - A_0(\gamma) &= -\gamma \int_0^t B_s(\gamma)\, dQ^-(s), \\ B_t(\gamma) - B_0(\gamma) &= \gamma \int_0^t A_s(\gamma)\, dQ^+(s), \end{aligned} \tag{1.1}$$

where Q^+ and Q^- are continuous nondecreasing functions on $[0, \infty)$, to generate a sequence of Hilbert spaces of entire functions of the kind studied by de Branges. We assume throughout that $Q^+(0) = Q^-(0) = 0$ and from time to time that Q^+ and Q^- are strictly increasing on $[0, \infty)$. Guided by Feller [15] it is easy under this last assumption to introduce the operator $D^+[D^-]$ of differentiation with respect to $Q^+[Q^-]$ and to check first that

$$A_t^- \equiv D^-A_t \equiv \lim_{h\to 0} \frac{A_{t+h} - A_t}{Q^-(t+h) - Q^-(t)} = -\gamma B_t, \tag{1.2a}$$

$$B_t^+ \equiv D^+B_t \equiv \lim_{h\to 0} \frac{B_{t+h} - B_t}{Q^+(t+h) - Q^+(t)} = \gamma A_t, \tag{1.2b}$$

and then to conclude that

$$D^+D^-A_t = -\gamma^2 A_t, \tag{1.3a}$$

$$D^-D^+B_t = -\gamma^2 B_t. \tag{1.3b}$$

If Q^+ and Q^- are differentiable in the ordinary sense with

$$\frac{dQ^+}{dt} = r(t) > 0, \qquad \frac{dQ^-}{dt} = \frac{1}{p(t)} > 0,$$

then Eqs. (1.3) reduce to classical differential equations of the Sturm–Liouville type,

$$\frac{1}{r}(pA_t')' = -\gamma^2 A_t, \tag{1.4a}$$

$$p\left(\frac{1}{r}B_t'\right)' = -\gamma^2 B_t. \tag{1.4b}$$

Note that $Q^+[Q^-]$ is absolutely continuous with respect to

$$Q = Q^+ + Q^-.$$

Hence, by a classical theorem of Lebesgue (Rudin, Ref. [21], Theorem 8.18), there exist a pair of locally Q summable functions q^+ and q^- such that

$$Q^+(t) = \int_0^t q^+(s)\, dQ(s)$$

and

$$Q^-(t) = \int_0^t q^-(s)\, dQ(s).$$

We shall be especially interested in the function $E_t = A_t - iB_t$ when A_t and B_t are the unique solutions of (1.1) subject to initial conditions A_0 and B_0, *which are allowed to depend upon* γ. Specifically we suppose that

(1.5a) $A_0[B_0]$ is an even [odd] entire function of minimal expo nential type (see Comment 1.1 for definition) in the variable γ;

(1.5b) $A_0(0) = 1$;

(1.5c) $A_0^{\#}(\gamma) = A_0(\gamma)$ and $B_0^{\#}(\gamma) = B_0(\gamma)$;

and that either $E_0 = A_0 \quad iB_0$ satisfies the inequalities

(1.5d) $|E_0(\gamma)| > |E_0^{\#}(\gamma)|$ if $\gamma \in R^{2+}$;

(1.5e) $\int (1+\gamma^2)^{-1} |E_0(\gamma)|^{-2}\, d\gamma < \infty$;

or

(1.5d′) $E_0 \equiv 1$ and $Q^+(t) > Q^+(0)$ for every $t > 0$.

Assumption (1.5d′) is not strictly an initial condition but rather incorporates the qualification under which the initial condition $E_0 = 1$

will be considered. It is introduced, as the reader will discover subsequently, to insure that (1.5d) and (1.5e) will be satisfied with E_t in place of E_0 for every choice of $t > 0$. It should further be noted that (1.5a) and (1.5c) together imply that $E_0{}^{\#}(\gamma) = E_0(-\gamma)$ while (1.5d) and (1.5e) imply that E_0 is root-free on the entire closed upper half-plane. This is, of course, the case also if $E_0 = 1$.

Comment 1.1. Recall that an entire function f is said to be of exponential type if there exists a finite constant M such that

$$|f(\gamma)| \leqslant e^{|\gamma|M}$$

for every choice of γ. More precisely f is said to be of exponential type T if

$$\limsup_{R\uparrow\infty} R^{-1} \log\{\max_{0\leqslant\theta<2\pi} |f(Re^{i\theta})|\} = T.$$

It follows of necessity that $T \geqslant 0$; if $T = 0$ f is said to be of minimal type.

Comment 1.2. In reading this paper it is well to think of A_t as the analog of $\cos\gamma t$ and B_t as the analog of $\sin\gamma t$. Indeed if $E_0 = 1$ and $Q^+(t) = Q^-(t) = t$ it is easily checked that $E_t = e^{-i\gamma t}$, $A_t = \cos\gamma t$, and $B_t = \sin\gamma t$. These circumstances will be referred to from time to time throughout the paper as the *classical case*.

The paper is organized as follows: In Section 2 we shall study the solutions of (1.1) under the initial conditions (1.5), showing that E_t is an entire function of exponential type

$$\tau(t) = \int_0^t [q^+(s)q^-(s)]^{1/2}\, dQ(s), \tag{1.6}$$

and that aside from this modification (1.5a)–(1.5e) propogate. That is to say, all the requisite statements hold with A_t, B_t, and E_t put in place of A_0, B_0, and E_0. The "t version" of (1.5d) ($|E_t| > |E_t{}^{\#}|$ on R^{2+}) enables us, as we show in Section 3, to define a Hilbert space of entire functions, $\mathscr{B}(E_t)$, with inner product

$$(f, g)_t = \int f(\gamma)\, \bar{g}(\gamma) |E_t(\gamma)|^{-2}\, d\gamma,$$

norm

$$\|f\|_t = \{(f, f)_t\}^{1/2},$$

and reproducing kernel

$$J_\beta{}^t(\gamma) = \frac{\bar{E}_t(\beta)E_t(\gamma) - \bar{E}_t{}^\#(\beta)E_t{}^\#(\gamma)}{-2\pi i(\gamma - \bar{\beta})}. \tag{1.7}$$

That is to say, $J_\beta{}^t \in \mathscr{B}(E_t)$ for each complex number β, and every $f \in \mathscr{B}(E_t)$ satisfies

$$f(\beta) = (f, J_\beta{}^t)_t. \tag{1.8}$$

The "t version" of (1.5e) helps to insure that $\mathscr{B}(E_t)$ is closed under the mapping $f \to f_\beta = (f - (f_\beta))/(\gamma - \beta)$. This is also noted in Section 3, which is devoted to a general discussion of the kind of Hilbert spaces studied by de Branges, hereafter to be referred to as de Branges spaces. Some of the less evident properties of these spaces require some results from the theory of functions of a complex variable. These are developed in Section 4. In Section 5 we study the family of spaces $\{\mathscr{B}(E_s)\}$, $s > 0$, generated by the solutions to (1.1) under initial conditions (1.5). A principal conclusion is that if Q^+ and Q^- are strictly increasing (as well as continuous[1]) functions on $[0, \infty)$, then the family of spaces $\{\mathscr{B}(E_s)\}$ is ordered by isometric inclusion. Moreover there exists a nondecreasing function μ on R^1 such that $\int(1 + \gamma^2)^{-1}\, d\mu(\gamma) < \infty$ and $L^2(d\mu; R^1) = L^2(d\mu)$ contains isometrically every space $\mathscr{B}(E_s)$ as a closed subspace. In other words, if $0 \leqslant s < t$ then

$$\mathscr{B}(E_s) \subset \mathscr{B}(E_t) \subset L^2(d\mu)$$

and

$$\int |(f/E_s)(\gamma)|^2\, d\gamma = \int |(f/E_t)(\gamma)|^2\, d\gamma = \int |f(\gamma)|^2\, d\mu(\gamma),$$

for every $f \in \mathscr{B}(E_s)$. If $\tau(\infty) = \infty$ then μ is essentially unique and $\bigcup_{s \geqslant 0} \mathscr{B}(E_s)$ is dense in $L^2(d\mu)$. Thus, in this case, the family of spaces $\{\mathscr{B}(E_s)\}$ provide a partial spectral decomposition of $L^2(d\mu)$ running from $\mathscr{B}(E_0)$ to $L^2(d\mu)$.

In Section 6 we go on to study the relationship between the spaces $L^2(dQ^\pm; [0, t])$ and $L^2(d\mu)$. It is perhaps well to pause at this point to recall, for the sake of comparison, the more familiar results from the classical theory of Sturm–Liouville expansions. To this end fix $t > 0$,

[1] This will be a permanent assumption.

assume that p and r are both strictly positive on $[0, t]$, and let A_s denote a solution of (1.4a) for $0 < s < t$ which satisfies the boundary conditions

$$A_0^- = 0, \tag{1.9a}$$

$$A_t = 0. \tag{1.9b}$$

The theory (Jorgens, Ref. [17], Chapter 4, Coddington and Levinson, Ref. [10], Chapter 7) then guarantees the existence of a countable sequence of real constants (eigenvalues)

$$0 < \gamma_1^2 < \gamma_2^2 < \cdots$$

for which the problem (1.4a), (1.9a), (1.9b) possesses solutions (eigenfunctions) $A_s(\gamma_n)$ and that these are orthogonal and complete in $L^2(dQ^+; [0, t])$. Thus if $f \in L^2(dQ^+; [0, t])$ then

$$\pi^{-1} \int_0^t |f(s)|^2 \, dQ^+(s) = \sum_{n=1}^{\infty} |(T_e f)(\gamma_n)|^2 \left\{ \pi^{-1} \int_0^t A_s^2(\gamma_n) \, dQ^+(s) \right\}^{-1}, \tag{1.10}$$

where $T_e f$ denotes the even transform,

$$(T_e f)(\gamma) = \pi^{-1} \int_0^t f(s) A_s(\gamma) \, dQ^+(s). \tag{1.11a}$$

For ease of future reference we define also at this point the odd transform

$$(T_o f)(\gamma) = \pi^{-1} \int_0^t f(s) B_s(\gamma) \, dQ^-(s) \tag{1.11b}$$

for functions $f \in L^2(dQ^-; [0, t])$.

Since $T_e f$ is an even function (of γ) we may, upon introducing the step function ρ_t with jumps of height $1/2\{\pi^{-1} \int_0^t A_s(\gamma_n)^2 \, dQ^+(s)\}^{-1/2}$ at $\pm\gamma_n$, replace the sum in (1.10) by a Stieltjes integral to get

$$\pi^{-1} \int_0^t |f(s)|^2 \, dQ^+(s) = \int_{-\infty}^{\infty} |(T_e f)(\gamma)|^2 \, d\rho_t(\gamma). \tag{1.12}$$

Moreover, if $\tau(t)$ tends to ∞ as $t \uparrow \infty$ [which means that ∞ is not a regular boundary point in Feller's classification (McKean, Ref. [19], p. 522)] and the step functions ρ_t are normalized, say, to be right continuous and to vanish at the origin, then a subsequence of the functions ρ_t converges weakly to a nondecreasing function ρ on R^1 and

(1.11a) and (1.12) now hold with t replaced by ∞ and ρ_t by ρ. The function ρ is the so-called spectral function; it is essentially unique (i.e., up to normalization) and there is a one-to-one correspondence between $L^2(dQ^+; [0, \infty))$ and the even functions in $L^2(d\rho; R^1) = L^2(d\rho)$. In much the same way it follows (upon replacing A_s by B_s, (1.4a) by (1.4b), Q^+ by Q^-, and $T_e f$ by $T_o f$ in the preceding discussion) that there is a one-to-one correspondence between $L^2(dQ^-; [0, \infty))$ and the odd functions in $L^2(d\rho)$.

These conclusions will also be arrived at as one end product of the work in Section 6. Of greater interest, however, is the conclusion that whenever A_s and B_s are solutions of (1.1) subject to (1.5) and $Q^\pm$ are strictly increasing, then the mappings defined in (1.11) act as follows:

$$T_e : f \in L^2(dQ^+; [0, t]) \text{ onto } \{\mathscr{B}(E_t) \ominus \mathscr{B}(E_0)\}_{\text{even}}, \tag{1.13a}$$

$$T_o : f \in L^2(dQ^-; [0, t]) \text{ onto } \{\mathscr{B}(E_t) \ominus \mathscr{B}(E_0)\}_{\text{odd}}, \tag{1.13b}$$

where, for example,

$$\{\mathscr{B}(E_t) \ominus \mathscr{B}(E_0)\}_{\text{even}}$$
$$= \{f \in \mathscr{B}(E_t) : f \text{ is orthogonal to } \mathscr{B}(E_0) \text{ in } L^2(|\, E_t(\gamma)|^{-2}\, d\gamma) \text{ and } f \text{ is even}\}.$$

Since $\mathscr{B}(E_t)$ may also be characterized (Lemma 3.5) as the set of entire functions of exponential type $\leqslant \tau(t)$ for which $\int |(f/E_t)(\gamma)|^2\, d\gamma < \infty$, the correspondence indicated in (1.13) is seen to be a generalization of the Paley–Wiener theorem. Indeed if $E_0 = 1$, then, as we shall see presently, $\mathscr{B}(E_0) \equiv 0$, and if in addition $Q^+(t) = Q^-(t) = t$, so that $E_t = e^{-i\gamma t}$ and $\tau(t) = t$, then the correspondence indicated in (1.13) reduces to the classical Paley–Wiener theorem.

As $t \uparrow \infty$ the mappings T_e and T_o extend naturally to

$$T_e : f \in L^2(dQ^+; [0, \infty)) \text{ into } \{L^2(d\mu) \ominus \mathscr{B}(E_0)\}_{\text{even}},$$

$$T_o : f \in L^2(dQ^-; [0, \infty)) \text{ into} \{L^2(d\mu) \ominus \mathscr{B}(E_0)\}_{\text{odd}},$$

the into becoming onto if $\tau(\infty) = \infty$. In this latter event the function μ is essentially unique and so it must coincide with the spectral function ρ mentioned earlier. We shall accordingly refer to μ as the spectral function also.

In Section 7 the theory developed in Section 6 is used to derive formulas for the spectral function μ in terms of certain solutions to (1.1). The principal results are summarized in Theorems 7.2 and 7.3, and then applied to some examples in Section 8. Under appropriate restrictions the spectral function has a density (derivative) Δ, given by (7.22), which reduces to a form discussed by Levinson [18]. This is elaborated on in Comment 8.1.

We now, in closing this section, list a number of comments in order to single out some points we feel to be of interest.

Comment 1.3. From Lemma 6.1 it follows that if Q^+ and Q^- are strictly increasing then

$$\int (\gamma^{-1}B_I)(\gamma^{-1}B_K)\, d\mu = \pi \int_{I\cap K} dQ^+$$

and

$$\int (\gamma^{-1}A_I)(\gamma^{-1}A_K)\, d\mu = \pi \int_{I\cap K} dQ^-,$$

where I and K are bounded subintervals of $[0, \infty)$ and, for example, A_I denotes $A_t - A_s$ if $I = [s, t]$. In particular it should be noted that these relations imply that Q^+ and Q^- can be recovered from knowledge of A_t, B_t, and μ. This might lead one to suspect that if one starts with an ordered (by isometric inclusion) one-parameter family of de Branges subspaces, of $L^2(d\mu)$, where μ is some nondecreasing function on R^1 which meets the condition $\int(1 + \gamma^2)^{-1}\, d\mu(\gamma) < \infty$, then these spaces must be generated by the solutions of a system of equations akin to (1.1). Subject to some technical qualifications this turns out to be the case. For a concrete realization with applications to the prediction problem of stationary Gaussian processes the reader is referred to Dym and McKean [12]. (A survey paper, Ref. [13], including an outline of the work done along these lines by the Russian mathematician M. G. Kreĭn is currently in preparation.)

Comment 1.4. It is easy to check, with the help of formulas (1.7) and (1.8), that if β_n and β_m are distinct roots of B then

$$(J^t_{\beta_n}, J^t_{\beta_m})_t = J^t_{\beta_n}(\beta_m) = 0.$$

In other words the functions $(\gamma - \beta_n)^{-1}B_t$, where β_n, $n = 1, 2,\ldots$, denote the roots of B_t (which turn out to be real and simple), are logical candidates for an orthogonal basis for $\mathscr{B}(E_t)$. Indeed it may be checked that these functions span $\mathscr{B}(E_t)$ if B_t does not belong to $\mathscr{B}(E_t)$. Assuming this to be the case it follows that we may develop each function in $\mathscr{B}(E_t)$ in an expansion of the form

$$f(\gamma) = \sum \frac{f(\beta_n)}{B_t'(\beta_n)} \frac{B_t(\gamma)}{\gamma - \beta_n}$$

and

$$\pi^{-1} \| f \|_t^2 = \sum \frac{| f(\beta_n) |^2}{B_t'(\beta_n) A_t(\beta_n)},$$

where the summations are carried out over the roots β_n of B_t and $'$ here denotes differentiation with respect to the variable γ. For further details and generalizations the reader is referred to de Branges, Ref. [9], Theorem 22.

Comment 1.5. A portion of the results arrived at in Section 6 may be summarized under the general heading of spectral representation formulas (or perhaps integral transforms) for problems of the Sturm–Liouville type when the boundary conditions are allowed to depend on the parameter. This reflects the fact that we allow the initial conditions A_0 and B_0 to depend upon γ. The simplest example which illustrates the effect of this is Example 8.1. Comment 8.2 touches upon a related example.

2. On the Nature of the Solutions to (1.1)

We consider initially the question of the existence of solutions to the system of equations (1.1) subject to the initial constraints (1.5) when Q^+ and Q^- are continuous nondecreasing functions on $[0, \infty)$. First some notation. For $t \geqslant s \geqslant 0$ let $[F, G]_m(s; t)$ denote the m-fold integral

$$[F, G]_m(s; t) = \int_s^t dF(s_m) \int_s^{s_m} dG(s_{m-1}) \int_s^{s_{m-1}} dF(s_{m-2}) \cdots \int_s^{s_2} d?(s_1), \qquad m \geqslant 2,$$

where

$$\begin{aligned} ?(s_1) &= F(s_1) \qquad \text{if} \quad m \geqslant 3 \text{ is odd} \\ &= G(s_1) \qquad \text{if} \quad m \geqslant 2 \text{ is even,} \end{aligned}$$

and, for the sake of notational unity, let

$$[F, G]_1(s; t) = \int_s^t dF(s_1) = F(t) - F(s)$$

and

$$[F, G]_0(s; t) = 1.$$

In the cases of interest to us F and G will be continuous nondecreasing functions on $[0, \infty)$ so that all these integrals will be well-defined as ordinary Stieltjes integrals (Widder, Ref. [25], Chapter I). Setting

$$Q(s; t) = \int_s^t \{dQ^+ + dQ^-\} = Q^+(t) - Q^+(s) + Q^-(t) - Q^-(s) = Q(t) - Q(s), \quad (2.1)$$

it is easily checked by induction that

$$| [-Q^-, Q^+]_m(s; t)| \leqslant Q^m(s; t)/m! \quad (2.2)$$

and

$$| [Q^+, -Q^-]_m(s; t)| \leqslant Q^m(s; t)/m!, \qquad m = 0, 1, \ldots . \quad (2.3)$$

Now suppose that for each fixed choice of γ there exist a pair of continuous functions (of the index t) A_t and B_t which are solutions of the system (1.1). Then upon successively substituting the solution of each equation of (1.1) into the integrand of the other, one is lead to conjecture that the solution, written in vector form with

$$\mathscr{E}_t(\gamma) = \begin{pmatrix} A_t(\gamma) \\ B_t(\gamma) \end{pmatrix},$$

is given by the matrix power series expansion

$$\mathscr{E}_t = \sum_{m=0}^{\infty} \gamma^m \begin{pmatrix} [-Q^-, Q^+]_m(0; t) & 0 \\ 0 & [Q^+, -Q^-]_m(0; t) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^m \mathscr{E}_0 . \quad (2.4)$$

Invoking the estimates (2.2) and (2.3) it is easily checked that the right side of (2.4) converges both absolutely and uniformly if t and γ are bounded. Thus $\mathscr{E}_t$ is continuous in t, analytic in γ, and is the unique solution of (1.1) which tends to $\mathscr{E}_0$ as $t \to 0$. In fact

$$\| \mathscr{E}_t(\gamma)\| \leqslant e^{|\gamma| Q(0;t)} \| \mathscr{E}_0(\gamma)\|, \quad (2.5)$$

where

$$\| \mathscr{E}_t(\gamma)\| = \{| A_t(\gamma)|^2 + | B_t(\gamma)|^2\}^{1/2}$$

denotes the norm of the vector $\mathscr{E}_t(\gamma)$. Hence A_t, B_t, and $E_t = A_t - iB_t$ are all of exponential type.

We proceed now in a series of lemmas and corollaries to establish results which will give us more information on the character of E_t. The principal conclusions are summarized at the end of the section in Theorem 2.1.

LEMMA 2.1. *The function* $J_\beta{}^t(\gamma)$ *defined in* (1.7) *may also be written in the form*

$$J_\beta{}^t(\gamma) = \pi^{-1} \int_0^t \bar{A}_s(\beta)A_s(\gamma)\, dQ^+(s) + \pi^{-1} \int_0^t \bar{B}_s(\beta)B_s(\gamma)\, dQ^-(s) + J_\beta{}^0(\gamma). \tag{2.6}$$

Proof. It follows readily from equations (1.1) that, locally, A_t and B_t are functions of bounded variation in t and that

$$\gamma \int_0^t \bar{A}_s(\beta)A_s(\gamma)\, dQ^+(s) = \int_0^t \bar{A}_s(\beta)\, dB_s(\gamma).$$

Integrating the right side by parts it is found to equal

$$\bar{A}_s(\beta)B_s(\gamma)\Big|_0^t - \int_0^t B_s(\gamma)\, d\bar{A}_s(\beta).$$

That is to say,

$$\gamma \int_0^t \bar{A}_s(\beta)A_s(\gamma)\, dQ^+(s) - \bar{\beta} \int_0^t \bar{B}_s(\beta)B_s(\gamma)\, dQ^-(s) = \bar{A}_s(\beta)B_s(\gamma)\Big|_0^t.$$

In much the same way it follows that

$$\gamma \int_0^t \bar{B}_s(\beta)B_s(\gamma)\, dQ^-(s) - \bar{\beta} \int_0^t \bar{A}_s(\beta)A_s(\gamma)\, dQ^+(s) = -\bar{B}_s(\beta)A_s(\gamma)\Big|_0^t.$$

The desired formula (2.6) now follows easily upon adding these two equations and noting that in terms of A_t and B_t (1.7) becomes

$$J_\beta{}^t(\gamma) = \frac{\bar{A}_t(\beta)B_t(\gamma) - \bar{B}_t(\beta)A_t(\gamma)}{\pi(\gamma - \bar{\beta})}. \tag{2.7}$$

COROLLARY 1. *Suppose either*

(a) $|E_0(\beta)| > |E_0{}^{\#}(\beta)|$ *for every* $\beta \in R^{2+}$,

or

(b) $E_0(\beta) \equiv 1$ *and* $Q^+(t) > Q^+(0)$ *for every* $t > 0$.

Then

$$|E_t(\beta)| > |E_t^{\#}(\beta)| \quad \textit{for every} \quad \beta \in R^{2+} \tag{2.8}$$

and every $t > 0$.

Proof. Because of (1.7) it is enough to show that $J_\beta^t(\beta) > 0$ for every $\beta \in R^{2+}$. But (2.6) implies that

$$J_\beta^t(\beta) = J_\beta^0(\beta) + \pi^{-1} \int_0^t |A_s(\beta)|^2 \, dQ^+(s) + \pi^{-1} \int_0^t |B_s(\beta)|^2 \, dQ^-(s).$$

In case (a) $J_\beta^0(\beta) > 0$. In case (b) $J_\beta^0(\beta) = 0$ but, as $A_0(\beta) \equiv 1$ and A_s is continuous, $\int_0^t |A_s(\beta)|^2 \, dQ^+(s) > 0$.

COROLLARY 2. $J_\beta^t(\beta) \geqslant J_\beta^s(\beta)$ *for every* $t \geqslant s \geqslant 0$ *and every* β.

We now establish a stronger result on the growth of $J_\beta^t(\beta)$ than Corollary 2. Recall first that

$$\tau(t) = \int_0^t [q^+(s)q^-(s)]^{1/2} \, dQ(s), \tag{2.9}$$

where $q^+[q^-]$ is the derivative of $Q^+[Q^-]$ with respect to $Q = Q^+ + Q^-$.

LEMMA 2.2. *If* $\beta = a + ib$ *and* $t \geqslant s \geqslant 0$ *then*

$$J_\beta^t(\beta) \geqslant e^{2|b|[\tau(t)-\tau(s)]} \, J_\beta^s(\beta).$$

Proof. In view of Corollary 2 above we need only consider the case $b \neq 0$. Now for each such choice of β, $e^{-2|b|\tau(u)} J_\beta^u(\beta)$ is absolutely continuous with respect to $Q = Q^+ + Q^-$. It is thus differentiable a.e. $[Q]$ with locally integrable derivative

$$\begin{aligned} &e^{-2|b|\tau(u)}\{-2|b|[q^+(u)q^-(u)]^{1/2}[\bar{A}_u(\beta)B_u(\beta) - \bar{B}_u(\beta)A_u(\beta)]/(\pi 2ib) \\ &\qquad + \pi^{-1}|A_u(\beta)|^2 q^+(u) + \pi^{-1}|B_u(\beta)|^2 q^-(u)\} \\ &\quad = \pi^{-1} e^{-2|b|\tau(u)} \, |[q^+(u)]^{1/2} A_u(\beta) + i|b|b^{-1}[q^-(u)]^{1/2} B_u(\beta)|^2. \end{aligned}$$

Hence

$$\begin{aligned} &e^{-2|b|\tau(t)} J_\beta^t(\beta) - e^{-2|b|\tau(s)} J_\beta^s(\beta) \\ &\quad = \pi^{-1} \int_s^t e^{-2|b|\tau(u)} \, |[q^+(u)]^{1/2} A_u(\beta) + i|b|b^{-1}[q^-(u)]^{1/2} B_u(\beta)|^2 \, dQ(u) \\ &\quad \geqslant 0. \end{aligned}$$

This completes the proof.

We add for the sake of completeness, though we shall not need it explicitly,

LEMMA 2.3. *E_t is an entire function of exponential type $\tau(t)$.*

Proof. It follows readily from Lemma 2.2 and Eq. (1.7) that

$$| E_t(iR)|^2 \geqslant e^{2R[\tau(t)-\tau(s)]}\, 4\pi R J^s_{iR}(iR)$$

for every $R \geqslant 0$ and every $t > s > 0$. The restriction $s > 0$ is made to insure that $J^s_\gamma(\gamma) > 0$. The corollary to Lemma 4.1 then further implies that $J^s_{iR}(iR) \geqslant J_i^s(i) > 0$ for all $R \geqslant 1$ and hence that

$$\limsup_{R\uparrow\infty} R^{-1} \log | E_t(iR)| \geqslant \tau(t) - \tau(s)$$

for every $s > 0$, which is to say that the exponential type of E_t, which we henceforth denote by type (E_t), is $\geqslant \tau(t)$.

To establish the opposite inequality we let $c > 0$ be a fixed constant,

$$\mathscr{C} = \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix},$$

and rewrite (1.1) in vector form as

$$\mathscr{C}\mathscr{E}_t = \mathscr{C}\mathscr{E}_s + \gamma \int_s^t \mathscr{C} \begin{pmatrix} 0 & -q^-(u) \\ q^+(u) & 0 \end{pmatrix} \mathscr{C}^{-1}\mathscr{C}\mathscr{E}_u \, dQ(u).$$

Upon iterating this expression and computing the norms of the vectors and matrices involved in the iteration we find

$$\| \mathscr{C}\mathscr{E}_t \| \leqslant \exp \left\{ | \gamma | \int_s^t \left\| \mathscr{C} \begin{pmatrix} 0 & -q^-(u) \\ q^+(u) & 0 \end{pmatrix} \mathscr{C}^{-1} \right\| dQ(u) \right\} \| \mathscr{C}\mathscr{E}_s \|. \tag{2.10}$$

Now

$$\begin{aligned} \| \mathscr{C}\mathscr{E}_t(\gamma)\|^2 &= c^2 | A_t(\gamma)|^2 + c^{-2} | B_t(\gamma)|^2 \\ &\geqslant \min(c^2, c^{-2})(| A_t(\gamma)|^2 + | B_t(\gamma)|^2) \\ &\geqslant \min(c^2, c^{-2}) | E_t(\gamma)|^2 (1/2). \end{aligned}$$

On the other hand,

$$\begin{aligned}\|\mathscr{C}\mathscr{E}_t(\gamma)\|^2 &\leqslant (c^2 + c^{-2})(|A_t(\gamma)|^2 + |B_t(\gamma)|^2)\\ &= (c^2 + c^{-2})(|E_t(\gamma)|^2 + |E_t^{\#}(\gamma)|^2)(1/2)\\ &\leqslant (c^2 + c^{-2})|E_t(\gamma)|^2, \qquad \text{if} \quad \gamma \in R^{2+},\end{aligned}$$

because of (2.8). In addition (2.8) implies that the exponential type of E_t,

$$\text{type}\,(E_t) = \limsup_{R\uparrow\infty} R^{-1} \log \{ \max_{0\leqslant\theta\leqslant\pi} |E_t(Re^{i\theta})| \} \tag{2.11}$$

(the point being that θ can be restricted to $[0, \pi]$), which, together with the preceding two inequalities, implies that

$$\text{type}\,(E_t) = \limsup_{R\uparrow\infty} R^{-1} \log\{ \max_{0\leqslant\theta\leqslant\pi} \|\mathscr{C}\mathscr{E}_t(Re^{i0})\|\}.$$

Combining this with (2.10) we see that

$$\text{type}\,(E_t) - \text{type}\,(E_s) \leqslant \int_s^t \left\| \begin{pmatrix} 0 & -c^{-2}q^-(u) \\ c^2q^+(u) & 0 \end{pmatrix} \right\| dQ(u)$$

for every $t \geqslant s \geqslant 0$ and every choice of $c > 0$. But this implies that type (E_t) is an absolutely continuous function with respect to Q and that a.e. $[Q]$

$$\frac{d}{dQ}\,\text{type}\,(E_t) \leqslant \left\| \begin{pmatrix} 0 & -c^{-2}q^-(t) \\ c^2q^+(t) & 0 \end{pmatrix} \right\|, \tag{2.12}$$

where again $c > 0$ is arbitrary. Hence upon setting (as we may by a passage to the limit) $c = 0$ if $q^-(t) = 0$, $c^{-1} = 0$ if $q^+(t) = 0$, and $q^-(t) \neq 0$, and $c^2 = [q^-(t)/q^+(t)]^{1/2}$ if $q^-(t)q^+(t) > 0$, it follows that the right side of (2.12) is equal to $[q^+(t)q^-(t)]^{1/2}$. That is to say, we have shown that

$$\text{type}\,(E_t) - \text{type}\,(E_s) \leqslant \int_s^t [q^+(u)q^-(u)]^{1/2}\, dQ(u),$$

which, upon letting $s \downarrow 0$ establishes the inequality needed to complete the proof, since we have assumed type $(E_0) = 0$.

COROLLARY 1. *A_t and B_t have exponential type $\tau(t)$.*

Proof. It is clear from (2.4) and assumption (1.5c) that $A_t^\# = A_t$ and $B_t^\# = B_t$. The inequality

$$|A_t(\gamma)|^2 + |B_t(\gamma)|^2 \leqslant |E_t(\gamma)|^2 \qquad \text{if} \quad \gamma \in R^{2+}$$

thus clearly implies that

$$\text{type}\,(A_t) \leqslant \tau(t), \qquad \text{and type}\;(B_t) \leqslant \tau(t).$$

On the other hand, since A_t is even and B_t is odd (as functions of γ),

$$\begin{aligned}\pi R J_{iR}^t(iR) &= -iA_t(iR)B_t(iR)\\ &\geqslant e^{2R[\tau(t)-\tau(s)]}\, J_{iR}^s(iR),\end{aligned}$$

from which it follows readily that

$$\limsup_{R\uparrow\infty} R^{-1}\log|A_t(iR)| + \limsup_{R\uparrow\infty} R^{-1}\log|B_t(iR)| \geqslant 2\tau(t),$$

which is to say that

$$\text{type}\,(A_t) + \text{type}\,(B_t) \geqslant 2\tau(t).$$

But this, in conjunction with the previous type estimate, leads immediately to the desired result.

COROLLARY 2. *If $F_t = E_t$ or A_t or B_t, then*

$$\limsup_{R\uparrow\infty} R^{-1}\log|F_t(iR)| = \tau(t).$$

Comment 2.1. The proof of Lemma 2.3 was adapted, with modifications, from de Branges, Ref. [6], Theorem 10. A variant of this adaptation appears in Dym and McKean [12].

We now summarize the principal properties of the solutions to (1.1) under initial conditions (1.5) in

THEOREM 2.1. *If Q^+ and Q^- are continuous nondecreasing functions on* $[0, \infty)$, *then for each choice of the complex constant γ, the system of equations* (1.1), *under initial constraints* (1.5), *possesses a unique pair of solutions $A_t = A_t(\gamma)$ and $B_t = B_t(\gamma)$ which are continuous functions of t, and*

(2.13a) *$A_t[B_t]$ is an even [odd] entire function of exponential type $\tau(t)$ (in the variable γ);*

(2.13b) $A_t(0) = 1$, $B_t(0) = 0$;

(2.13c) $A_t^{\#}(\gamma) = A_t(\gamma)$, $B_t^{\#}(\gamma) = B_t(\gamma)$;

and for every $t > 0$, $E_t = A_t - iB_t$ *satisfies the inequalities*

(2.13d) $|E_t(\gamma)| > |E_t^{\#}(\gamma)|$ *for* $\gamma \in R^{2+}$;

(2.13e) $\int (1 + \gamma^2)^{-1} |E_t(\gamma)|^{-2}\, d\gamma < \infty$;

(2.13f) $\int (1 + \gamma^2)^{-1} \log^+ |E_t(\gamma)|\, d\gamma < \infty$.

Proof. It remains but to check assertions (2.13e) and (2.13f). The former follows from inequality (5.5) in case $E_0 = 1$, and from Step 1 in the proof of Theorem 5.3 otherwise. Inequality (2.13f) then follows from Lemma 4.3.

Comment 2.2. Inequality (2.13e) in conjunction with Lemma 4.3 allows us (as noted in Comment 4.4) to claim equality in (4.6e) with $g = \{(\gamma + i)E_t\}^{-1}$. This in turn yields the representation formula

$$\log |E_t(a + ib)| = \pi^{-1} b \int \{(c - a)^2 + b^2\}^{-1} \log |E_t(c)|\, dc + \tau(t)b,$$

which is valid for every $b > 0$. This formula can also be deduced directly from Theorem 6.5.4 in Boas, Ref. [2], once (2.13f) is known.

Comment 2.3. Note that parts (a) and (c) of (2.13) together imply that $E_t^{\#}(\gamma) = E_t(-\gamma)$, while (d) and (e) imply that E_t is root-free on the entire closed upper half-plane. The fact that E_t has no real roots can also be proved without recourse to (2.13e) as follows: Note first that under initial conditions (1.5) E_0 has no real roots and next that the matrix in (2.4) which links $\mathscr{E}_t$ to $\mathscr{E}_0$ has determinant 1. This last point can be checked by showing that the derivative of the determinant with respect to Q is equal to zero, and then evaluating the determinant when $t = 0$.

Comment 2.4. If Q^+ and Q^- are differentiable in the ordinary sense with derivatives r and $1/p$, respectively, then

$$\tau(t) = \int_0^t [r(s)/p(s)]^{1/2}\, ds.$$

3. de Branges Spaces

Let E be an entire function which satisfies the basic inequality

$$|E| > |E^{\#}|, \text{ on } R^{2+} \text{ (the open upper half-plane)}, \tag{3.1}$$

and let $\mathscr{B}(E)$ denote the collection of all entire functions f such that

(3.2a) $\|f\|^2 \equiv \int |(f/E)(\gamma)|^2 \, d\gamma < \infty$;

(3.2b) $|(f/E)(Re^{i\theta})| \leqslant M(R \sin \theta)^{-1/2}$ for $0 < \theta < \pi$ and $R > R_0$,

(3.2c) $|(f/E^{\#})(Re^{i\theta})| \leqslant M(R|\sin\theta|)^{-1/2}$ for $\pi < \theta < 2\pi$ and $R > R_0$;

where R_0 and M are arbitrary positive constants which may depend upon f.

We shall term $\mathscr{B}(E)$ the de Branges space generated by E.

THEOREM 3.1. *$\mathscr{B}(E)$ is a Hilbert space with reproducing kernel*

$$J_\beta = J_\beta(\gamma) = \frac{\bar{E}(\beta)E(\gamma) - \bar{E}^{\#}(\beta)E^{\#}(\gamma)}{-2\pi i(\gamma - \bar{\beta})}. \tag{3.3}$$

That is to say, for each fixed $\beta \in R^2$, $J_\beta \in \mathscr{B}(E)$ and

$$(f, J_\beta) \equiv \int f(\gamma)\bar{J}_\beta(\gamma)|E(\gamma)|^{-2} \, d\gamma = f(\beta). \tag{3.4}$$

Proof. (3.1), which guarantees that E is root-free in R^{2+}, and (3.2a) together imply that f/E is analytic in the *closed* upper half-plane while (3.2b) implies that for fixed β and sufficiently large R

$$\left| \int_0^\pi \frac{f}{E}(Re^{i\theta}) \frac{iRe^{i\theta}}{Re^{i\theta} - \beta} \, d\theta \right| \leqslant \int_0^\pi \frac{\text{constant}}{(R\sin\theta)^{1/2}} \frac{R}{R - |\beta|} \, d\theta,$$

which tends to zero as $R \uparrow \infty$. Hence Cauchy's formula may be invoked to yield

$$\begin{aligned} \frac{1}{2\pi i} \int \frac{f}{E}(\gamma) \frac{1}{\gamma - \beta} \, d\gamma &= \frac{f}{E}(\beta) \qquad \text{if} \quad \beta \in R^{2+}, \\ &= 0 \qquad \text{if} \quad \beta \in R^{2-}. \end{aligned}$$

A similar argument over the lower half-plane with the help of (3.2c) shows that

$$\begin{aligned} \frac{1}{2\pi i} \int \frac{f}{E^{\#}}(\gamma) \frac{1}{\gamma - \beta} \, d\gamma &= 0 \qquad \text{if} \quad \beta \in R^{2+}, \\ &= -\frac{f}{E^{\#}}(\beta) \qquad \text{if} \quad \beta \in R^{2-}. \end{aligned}$$

Combining these two results it is a simple matter to check that

$$\int f(\gamma) \left\{ \frac{E(\beta)}{E(\gamma)} - \frac{E^{\#}(\beta)}{E^{\#}(\gamma)} \right\} \frac{1}{2\pi i(\gamma - \beta)} \, d\gamma = f(\beta)$$

for *every* value of $\beta \in R^2$, and hence that (3.4) holds. Moreover, as J_β itself belongs to $\mathscr{B}(E)$ we must have

$$(J_\beta, J_\alpha) = J_\beta(\alpha) \tag{3.5}$$

and hence, by Schwarz's inequality, it follows that for every $f \in \mathscr{B}(E)$

$$\begin{aligned} |f(\beta)|^2 &= |(f, J_\beta)|^2 \\ &\leqslant \|f\|^2 (J_\beta, J_\beta) \\ &= \|f\|^2 J_\beta(\beta). \end{aligned}$$

That is to say, if $\beta = a + ib \notin R^1$

$$|f(\beta)|^2 \leqslant \|f\|^2 (|E(\beta)|^2 - |E^{\#}(\beta)|^2)/(4\pi b). \tag{3.6}$$

It is easily checked that *inequality* (3.6) *may be used in place of* (3.2b) *and* (3.2c) *in the definition of* $\mathscr{B}(E)$. Finally note that if $\{f_n\}$ is a Cauchy sequence in (the norm of) $\mathscr{B}(E)$ then (3.6) implies that $\{f_n\}$ is a normal family (Rudin, Ref. [21], p. 271), and so a subsequence of the $\{f_n\}$ converges pointwise to an entire function f. It is easy to check that f must itself sit in $\mathscr{B}(E)$, and hence, upon reapplying (3.6), the full sequence must tend to f pointwise as well as in norm.

This covers the essential points in the proof that $\mathscr{B}(E)$ is a Hilbert space with reproducing kernel J_β.

In the remainder of this section we discuss a number of properties of de Branges spaces assuming, in addition to (3.1), that

$$E^{\#}(\gamma) = E(-\gamma). \tag{3.7}$$

The theory may be developed without this restriction. It is adopted in order to simplify the exposition. An important consequence of this assumption is that $f(-\gamma) \in \mathscr{B}(E)$ whenever $f \in \mathscr{B}(E)$.

It proves useful to introduce the functions

$$A = (E^{\#} + E)/2 \qquad \text{and} \qquad B = (E^{\#} - E)/2i$$

and to note, as is obvious from the definition, that $A = A^{\#}$, $B = B^{\#}$; A and B are real-valued on R^1; $E = A - iB$ and $E^{\#} = A + iB$. Note also that (3.7) is equivalent to assuming A even and B odd.

LEMMA 3.1. *If $f \in \mathscr{B}(E)$ then*

$$\|f\|^2 \equiv \int (A^2 + B^2)^{-1}|f|^2 = \int (k^2A^2 + k^{-2}B^2)^{-1}|f|^2 \tag{3.8}$$

for every real nonzero constant k.

Proof. Writing the reproducing kernel (3.3) in terms of A and B we see that

$$J_\beta(\gamma) = \frac{\bar{A}(\beta)B(\gamma) - \bar{B}(\beta)A(\gamma)}{\pi(\gamma - \bar{\beta})} \tag{3.9}$$

does not change if A is replaced by $A_1 = kA$ and B is replaced by $B_1 = k^{-1}B$. Moreover, as $E_1 = A_1 - iB_1$ satisfies inequality (3.1) there exists a corresponding de Branges space $\mathscr{B}(E_1)$ with, by the foregoing observation, the same reproducing kernel J_β. Thus as the set of functions $\{J_\beta : \beta \in R^2\}$ is dense in $\mathscr{B}(E)$ and

$$\int |J_\beta/E|^2 = J_\beta(\beta) = \int |J_\beta/E_1|^2,$$

we conclude that

$$\int |f/E|^2 = \int |f/E_1|^2$$

for every $f \in \mathscr{B}(E) = \mathscr{B}(E_1)$. This is equivalent to (3.8).

Comment 3.1. In the proof of Lemma 3.1 it was shown that if $A_1 = kA$ and $B_1 = k^{-1}B$ for some nonzero real constant k then $\mathscr{B}(E_1) = \mathscr{B}(E)$.

The converse of this statement is true also. For, if $\mathscr{B}(E) = \mathscr{B}(E_1)$ then the corresponding reproducing kernels must match. Thatis to say, we must have

$$\bar{A}_1(\beta)B_1(\gamma) - \bar{B}_1(\beta)A_1(\gamma) = \bar{A}(\beta)B(\gamma) - \bar{B}(\beta)A(\gamma).$$

The assertion follows upon matching even and odd parts in γ and recalling that $A = A^{\#}$ is even while $B = B^{\#}$ is odd. For a more general result of this type see de Branges, Ref. [5], Theorem 1.

LEMMA 3.2. *If neither A nor B belong to $\mathscr{B}(E)$ then the domain of multiplication by γ,*

$$\mathscr{M}(E) = \{f \in \mathscr{B}(E) : \gamma f \in \mathscr{B}(E)\},$$

is dense in $\mathscr{B}(E)$.

Proof. We first note, as is easy to check, that if $f \in \mathscr{B}(E)$ and $f(\beta) = 0$ for some complex constant $\beta \notin R^1$ then $(\gamma - \beta)^{-1} f \in \mathscr{B}(E)$ and $(\gamma - \bar{\beta})(\gamma - \beta)^{-1} f \in \mathscr{B}(E)$. Now suppose $g \in \mathscr{B}(E)$ is orthogonal to every element f of $\mathscr{M}(E)$. Then for each nonreal choice of β,

$$G_\beta(\gamma) = J_\beta(\beta)\, g(\gamma) - g(\beta) J_\beta(\gamma) - [J_\beta(\bar{\beta})\, g(\gamma) - g(\bar{\beta}) J_\beta(\gamma)](\gamma - \beta)/(\gamma - \bar{\beta})$$

belongs to $\mathscr{B}(E)$ and

$$((\gamma - \beta)f, G_\beta) = 0. \tag{3.10}$$

Moreover, as $G_\beta(\beta) = 0$ it follows that $(\gamma - \beta)^{-1} G_\beta \in \mathscr{M}(E)$ and so may be substituted for f in (3.10). But this implies that $G_\beta \equiv 0$ and hence that g must be a linear combination (with complex coefficients c_1, c_2) of A and $B : g = c_1 A + c_2 B$. However, because of (3.7), $\mathscr{B}(E)$ contains $g(-\gamma)$ as well as g and hence, also $2c_1 A = g(\gamma) + g(-\gamma)$ and $2c_2 B = g(\gamma) - g(-\gamma)$. $c_1 = c_2 = 0$ follows, and the proof is complete.

Comment 3.2. This lemma and its proof are adapted from de Branges, Ref. [9], Theorem 29. It is easy to check that if either A or B belongs to $\mathscr{B}(E)$ (they cannot both belong simultaneously) then it is orthogonal to $\mathscr{M}(E)$. For example, if $B \in \mathscr{B}(E)$ and $f \in \mathscr{M}(E)$ then

$$(f, B) = (\pi\gamma f, J_0) = (\pi\gamma f)(0) = 0.$$

We list more de Branges' space facts although the proofs, which are also given below, depend upon results which are to be established in the next section, wherein some of the deeper implications of inequality (3.1) on the structure of E are discussed.

LEMMA 3.3. *$\mathscr{B}(E)$ is closed under the mapping $f \to f_\beta = (f - f(\beta))/(\gamma - \beta)$ for every complex constant β if and only if*

(3.11) *E is of exponential type, and*

(3.12) $\int (1 + \gamma^2)^{-1} |E|^{-2} < \infty.$

Proof. Suppose $\mathscr{B}(E)$ is closed under the indicated map. Then there must exist a function $f \in \mathscr{B}(E)$ with $f(-i) \neq 0$. Hence, as

$$|f(-i)|^2 |\gamma + i|^{-2} \leqslant 2|f_{-i}(\gamma)|^2 + 2|f(\gamma)|^2 |\gamma + i|^{-2}$$
$$\leqslant 2|f_{-i}(\gamma)|^2 + 2|f(\gamma)|^2$$

for all γ in the closed upper half-plane, it follows that the function $h = (\gamma + i)^{-1}$, which is itself analytic in the closed upper half-plane, satisfies inequalities (3.2a) and (3.2b). This establishes (3.12) directly. (3.11) then follows from Lemma 4.3.

To prove sufficiency we apply the inequality

$$|f_\beta(\gamma)|^2 \leqslant \frac{2|f(\gamma)|^2}{|\gamma - \beta|^2} + 2|f(\beta)|^2 \left| \frac{\gamma - i}{\gamma - \beta} \right|^2 \frac{1}{|\gamma - i|^2}, \qquad \gamma \neq \beta,$$

first to the case $\gamma \in R^1$ to conclude from (3.12) that $\| f_\beta \| < \infty$, and then to the case $\gamma = Re^{i\theta}$, $0 < \theta < \pi$, and $R \uparrow \infty$ to conclude from Lemma 4.3 that f_β satisfies inequality (3.2b) (since f and h do). It remains but to check that f_β satisfies inequality (3.2c). But this follows from the lower half-plane version of Lemma 4.3 which implies in particular that $h^\#$ satisfies (3.2c).

Comment 3.3. This lemma was motivated by de Branges, Ref. [5] Theorem 3. Therein it is shown, albeit very sketchily, that $\mathscr{B}(E)$ is closed under the mapping $f \to f_\beta$ if and only if E satisfies (3.11), (3.12), and (4.2). This last condition is actually superfluous, as follows from Lemma 4.3.

LEMMA 3.4. *Under* (3.11) *and* (3.12) (*and of course* (3.1) *and* (3.7)) *the functions* $(E - E(\beta))/(\gamma - \beta)$, $(A - A(\beta))/(\gamma - \beta)$, *and* $(B - B(\beta))/(\gamma - \beta)$ *belong to* $\mathscr{B}(E)$ *for every choice of the complex constant* β.

Proof. The proof is carried out in much the some fashion as the sufficiency proof of Lemma 3.3. Thus, for example, focusing attention on the functions $(A - A(\beta))/(\gamma - \beta)$, it is easy to check, with the help of (3.12), that (3.2a) is satisfied (even if $\beta \in R^1$). (3.2b) then follows from Lemma 4.3 and the observation (see (2.7)) that

$$|A|^2 \leqslant |A|^2 + |B|^2 + i(\bar{B}A - B\bar{A}) = |E|^2 \qquad \text{on} \quad R^{2+},$$

while (3.2c) follows from the lower half-plane analog of Lemma 4.3 and the fact that

$$|A|^2 \leqslant |E^\#|^2 \qquad \text{on} \quad R^{2-}.$$

LEMMA 3.5. *If E is an entire function of exponential type T which satisfies* (3.1), (3.7), *and* (3.12), *then $\mathscr{B}(E)$ may be characterized as the set of entire functions, f, which satisfy* (3.2a) *and are of exponential type less than or equal to T.*

Proof. Lemma 4.3 implies that $\int(1+\gamma^2)^{-1}\log^+|E|<\infty$, and so if f satisfies (3.2a) it must also obey the inequality

$$\begin{aligned}\int(1+\gamma^2)^{-1}\log^+|f| &\leqslant 1/2\int(1+\gamma^2)^{-1}\log^+|f/E|^2+\int(1+\gamma^2)^{-1}\log^+|E|\\ &\leqslant (1/2)\int|f/E|^2+\int(1+\gamma^2)^{-1}\log^+|E|\\ &<\infty.\end{aligned}$$

Hence, Nevanlinna's representation formula (Boas, Ref. [2], Theorem 6.5.4) is applicable and implies that

$$\log|f(a+ib)|\leqslant\frac{b}{\pi}\int\frac{\log|f(c)|\,dc}{(c-a)^2+b^2}+kb,\qquad b>0,\tag{3.13}$$

where

$$k=\limsup_{R\uparrow\infty}(2/\pi)R^{-1}\int_0^\pi\log|f(Re^{i\theta})|\sin\theta\,d\theta.\tag{3.14}$$

Recalling the definition of the indicator function for f,

$$h(\theta)=\limsup_{R\uparrow\infty}R^{-1}\log|f(Re^{i\theta})|,\tag{3.15}$$

it follows easily from (3.13) that

$$h(\theta)\leqslant k\sin\theta\qquad\text{if}\quad 0<\theta<\pi.$$

Upon replacing (3.13) and (3.14) by their lower half-plane versions it follows further that there exists a constant m such that

$$h(-\theta)\leqslant m\sin\theta\qquad\text{if}\quad 0<\theta<\pi.$$

Thus, as h is continuous (Boas, Ref. [2], Theorem 5.1.4), we must have $h(0)=h(\pi)=0$. Now as $h(\pi/2)\leqslant T$ it follows readily upon applying the Phragmén–Lindelöf theorem (Titchmarsh, Ref. [23], p. 177) to the sector $0\leqslant\theta\leqslant\pi/2$ that there exists a constant M such that for every $\epsilon > 0$ and every $R \geqslant 0$

$$|f(Re^{i\theta})| \leqslant Me^{R[(T+\epsilon)\sin\theta+\epsilon\cos\theta]} \qquad \text{if} \quad 0 \leqslant \theta \leqslant \pi/2.$$

But this implies that

$$|f(Re^{i\theta})| \leqslant Me^{R(T\sin\theta+2\epsilon)}$$

for $0 \leqslant \theta \leqslant \pi/2$ and, in fact, the argument extends to show that aside from a possible adjustment in the value of M, this inequality must hold for $0 \leqslant \theta \leqslant \pi$. Thus upon putting this bound into (3.14) and recalling that ϵ is arbitrary, we see that $k \leqslant T$. Yet on the other hand, applying the Nevanlinna formula to E, which is root-free in the closed upper half-plane, we get

$$\log|E(a+ib)| = \frac{b}{\pi}\int \frac{\log|E(c)|}{(c-a)^2+b^2}\,dc + Tb, \qquad b > 0. \tag{3.16}$$

(The coefficient of b on the right side must equal T since $b^{-1}\log|E(b)| \to T$ as $b \uparrow \infty$.)
Thus

$$\log|(f/E)(a+ib)| \leqslant \frac{b}{\pi}\int \frac{\log|(f/E)(c)|}{(c-a)^2+b^2}\,dc,$$

from which it follows, via Theorem 4.1, that f satisfies (3.2b). Similar estimates for $f/E^{\#}$ carried out over the lower half-plane show that f satisfies (3.2c) also and so we conclude that $f \in \mathscr{B}(E)$.

To complete the proof it remains to show that if E has exponential type T and $f \in \mathscr{B}(E)$ then type $(f) \leqslant T$. But this follows from (3.6) and the fact that

$$J_{Re^{i\theta}}(Re^{i\theta}) \leqslant \text{constant} \cdot M(R) \cdot M(R+1), \tag{3.17}$$

where

$$M(R) = \max_{0\leqslant\theta\leqslant 2\pi} \{|E(Re^{i\theta})|\}.$$

To verify (3.17) we consider two cases. First $|b| > 1/4$, in which case the inequality is apparent from the definition, (3.3), and secondly

$|b| \leqslant 1/4$. In checking the latter it suffices, as $J_\beta(\bar{\beta}) = J_\beta(\beta)$, to consider the case $0 \leqslant b \leqslant 1/4$. But then, since J_β is analytic, we have

$$
\begin{aligned}
| J_\beta(\beta)| &= \left| (2\pi)^{-1} \int_0^{2\pi} J_\beta(\beta + e^{i\theta})\, d\theta \right| \\
&= \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{\bar{E}(\beta)E(\beta + e^{i\theta}) - \bar{E}^{\#}(\beta)E^{\#}(\beta + e^{i\theta})}{-2\pi i(\beta + e^{i\theta} - \bar{\beta})}\, d\theta \right| \\
&\leqslant 2M(R)M(R+1)(2\pi)^{-2} \int_0^{2\pi} \{(2b + \sin\theta)^2 + (\cos\theta)^2\}^{-1/2}\, d\theta \\
&\leqslant 2M(R)M(R+1)(2\pi)^{-2} \left[\pi + \int_\pi^{5\pi/4} |\cos\theta|^{-1}\, d\theta \right. \\
&\qquad \left. + \int_{5\pi/4}^{5\pi/4} (|\sin\theta| - 1/2)^{-1}\, d\theta + \int_{7\pi/4}^{2\pi} \cos\theta\, d\theta \right] \\
&= \text{constant} \cdot M(R)M(R+1),
\end{aligned}
$$

as asserted.

COROLLARY. *If E is an entire function of exponential type T which satisfies* (3.1), (3.7), *and* (3.12), *then*

$$
\limsup_{R\uparrow\infty} R^{-1} \log|F(Re^{i\theta})| = T\sin\theta, \qquad 0 \leqslant \theta \leqslant \pi, \tag{3.18}
$$

if $F = A$ or B or E.

Proof. If $F = E$ then (3.18) follows easily from (3.16) if $0 < \theta < \pi$. That is to say, the indicator function for E, $h_E(\theta) = T\sin\theta$ if $0 < \theta < \pi$. This establishes (3.18) for $F = E$ since h_E is continuous (Boas, Ref. [2], Theorem 5.1.4). A similar argument disposes of the cases $F = A$ and $F = B$ (as (3.16) is applicable in both instances).

Comment 3.4. It is interesting to note that any Hilbert space H of entire functions for which the following hold:

(a) $f(\gamma)(\gamma - \bar{\beta})/(\gamma - \beta) \in H$ whenever $f \in H$ and $f(\beta) = 0$;

(b) Point evaluation is a bounded linear functional;

(c) $f^{\#} \in H$ and has the same norm as f whenever $f \in H$;

is a de Branges space relative to some entire function E which obeys (3.1). A proof is furnished in de Branges, Ref. [4], and Ref. [9], Theorem 23.

4. Some Function Theory

The basic inequality (3.1) has far reaching implications on the behavior of E, especially if E is of exponential type (see, for example, Boas, Ref. [2], Section 7.8). We pause now to explore some of these implications and to consolidate some applicable results from function theory. The casual reader is invited to postpone the proofs and go on to Section 5.

Lemma 4.1. *If E is an entire function of exponential type which satisfies* (3.1) *then*

$$\frac{\partial}{\partial b} \mid E(a+ib)\mid \geqslant 0 \qquad \text{if} \quad b \geqslant 0.$$

Proof. Let $\gamma_n = a_n - ib_n$, $n = 1, 2,\ldots,$ denote the nonzero roots of E. Since $b_n \geqslant 0$, Lindelöf's theorem (Boas, Ref. [2], p. 27) implies that

$$\sum_{n=1}^{\infty} b_n/\mid \gamma_n \mid^2 < \infty. \tag{4.1}$$

Now as E is assumed to be of exponential type its Handamard product representation takes the form

$$E = C\gamma^m e^{\eta\gamma} \prod_{n=1}^{\infty} (1 - \gamma/\gamma_n)\, e^{\gamma/\gamma_n},$$

where C and η are constants. Substituting this into (3.1) we find that

$$1 > \left| \exp \left\{[(\bar{\eta} - \eta) - \sum 2ib_n \mid \gamma_n \mid^{-2}]\gamma\right\} \right| \mid P(\gamma)\mid, \qquad \gamma \in R^{2+},$$

where the convergence of the Blaschke product

$$P(\gamma) = \prod \left(1 - \frac{\gamma}{\bar{\gamma}_n}\right)\Big/\left(1 - \frac{\gamma}{\gamma_n}\right)$$

in R^{2+} is assured by (4.1). Now by the Ahlfors–Heine theorem (Boas, Ref. [2], p. 115)

$$\lim_{R\uparrow\infty} \frac{\log| P(Re^{i\theta})|}{R} = 0$$

for a dense set of θ between 0 and π. Thus

$$0 \geqslant i(\bar{\eta} - \eta) + 2 \sum b_n | \gamma_n |^{-2}.$$

Finally, a routine computation shows that

$$\frac{\partial}{\partial b} \log| E(\gamma)| = bm| \gamma |^{-2} + i(\eta - \bar{\eta})/2 - \sum b_n| \gamma_n |^{-2} + \sum (b_n + b)| \gamma_n - \gamma |^{-2}$$
$$> 0 \qquad \text{for} \quad \gamma \in R^{2+},$$

and the assertion follows.

Comment 4.1. We have not actually utilized the condition $E^{\#}(\gamma) = E(-\gamma)$ in the above proof. Because of it the roots of E must fall symmetrically about the negative imaginary axis, and η in the Hadamard product must be purely imaginary. In fact we must have $i\eta \geqslant 0$ and, as follows from the Hadamard product representation, $| E(Re^{i\theta})| \leqslant | E(iR)|$ for every $R \geqslant 0$.

COROLLARY. *If E is an entire function of exponential type which satisfies* (3.1), *then*

$$\frac{\partial}{\partial b} J_\beta(\beta) \geqslant 0 \qquad \textit{if} \quad \beta = a + ib \in R^{2+}.$$

Proof. Fix $\beta \in R^{2+}$. Then J_β is an entire function of exponential type with roots in R^{2-} only. Hence $J_\beta{}^{\#}/J_\beta$ is analytic on the closed upper half-plane, and, by virtue of (3.1), it is bounded in modulus there by $| \bar{\beta} - \gamma | | \beta - \gamma |^{-1}$. Since this bound approaches one as $| \gamma | \uparrow \infty$ the maximum modulus theorem implies that $| J_\beta{}^{\#}/J_\beta | \leqslant 1$ over the closed upper half-plane. It now follows, upon copying the proof of the lemma, that

$$| J_\beta(\beta + id)| \geqslant J_\beta(\beta) \qquad \text{for} \quad d \geqslant 0 \quad \text{and} \quad \beta \in R^{2+},$$

which, when coupled with the inequality (obtained by applying Schwarz's inequality to (3.5))

$$J_\beta(\beta)J_{\beta+id}(\beta+id) \geqslant |J_\beta(\beta+id)|^2,$$

further implies that

$$J_{\beta+id}(\beta+id) \geqslant J_\beta(\beta) \qquad \text{if} \quad d \geqslant 0 \quad \text{and} \quad \beta \in R^{2+},$$

as asserted.

Lemma 4.2. *If in addition to* (3.1) *E satisfies the inequalities*

$$\int (1+\gamma^2)^{-1} \log^+ |E(\gamma)|\, d\gamma < \infty \tag{4.2}$$

and

$$\log|E(a+ib)| \leqslant (b/\pi) \int \{(c-a)^2 + b^2\}^{-1} \log|E(c)|\, dc + kb \tag{4.3}$$

for every $b > 0$ and some constant k, then E is of exponential type.

Proof. Because of (3.1) it suffices to check that E is of exponential type in the closed upper half-plane. Moreover, in carrying out the proof it is no loss of generality to assume that E satisfies the inequality

$$\log|E(a+ib)| \leqslant (b/\pi) \int \{(c-a)^2 + b^2\}^{-1} \log|E(c)|\, dc \tag{4.3$'$}$$

in place of (4.3). For $e^{i\gamma k}E$ satisfies (4.3′) whenever E satisfies (4.3) and clearly E is of exponential type if and only if $e^{i\gamma k}E$ is. Our first step is to split the integral appearing on the right side of (4.3′) into two pieces,

$$\mathrm{I} + \mathrm{II} = \int\limits_{\{c\,:\,c^2 \geqslant 4a^2\}} + \int\limits_{\{c\,:\,c^2 < 4a^2\}},$$

each to be bounded separately. If $c^2 \geqslant 4a^2$ then

$$(c-a)^2 + b^2 \geqslant c^2/2 - a^2 + b^2 \geqslant (1/4)(c^2 + 4b^2).$$

Hence, as

$$\begin{aligned}(1/4)(c^2+4b^2) &\geqslant (1/4)(c^2+1) &&\text{if} \quad 4b^2 \geqslant 1,\\ &\geqslant b^2(c^2+1) &&\text{if} \quad 4b^2 < 1,\end{aligned}$$

it follows that

$$\mathrm{I} \leqslant \frac{b}{\pi} \int \frac{\log^+ | E(c)|}{(1/4)(c^2 + 4b^2)} \, dc$$

$$\leqslant \{4b + (1/b)\}\pi^{-1} \int (1 + c^2)^{-1} \log^+ | E(c)| \, dc.$$

On the other hand, if $c^2 < 4a^2$ then

$$(c^2 + 1) \leqslant (4a^2 + 1) \leqslant (4a^2 + 1)\{(c - a)^2 + b^2\}b^{-2},$$

which in turn implies that

$$\mathrm{II} \leqslant \frac{b}{\pi} \frac{(4a^2 + 1)}{b^2} \int \frac{\log^+ | E(c)|}{(c^2 + 1)} \, dc.$$

Upon combining these inequalities with (4.3′) it follows easily that

$$R^{-1} \log| E(Re^{i\theta})| \leqslant \left(\frac{4R^2 + 2}{R^2 \sin\theta} \right) \pi^{-1} \int \frac{\log^+ | E(c)|}{c^2 + 1} \, dc.$$

This shows that E is of exponential type in any sector centered about the b axis with aperture less than π.

To estimate E in the vicinity of the axis of reals we first integrate Jensen's inequality (Titchmarsh, Ref. [23], p. 129)

$$\log| E(\beta)| \leqslant \frac{1}{2\pi} \int_0^{2\pi} \log| E(\beta + \rho e^{i\theta})| \, d\theta$$

over ρ to get

$$\log| E(\beta)| \leqslant \frac{1}{\pi\delta^2} \int_{D_\beta} \log^+ | E(u + iv)| \, du \, dv \tag{4.4}$$

in which D_β stands for the disc of radius δ centered at β.
Now upon choosing $\pi/6 > \psi > 0$, $\sin\psi > \delta > 0$, and $R \geqslant 1$, it follows that every disc D_β centered at $\beta = Re^{i\theta}$ with $|\theta| \leqslant \psi$ lies inside the region $\rho e^{i\alpha} : R - \delta \leqslant \rho \leqslant R + \delta$ and $|\alpha| \leqslant 2\psi$. Thus for $|\theta| \leqslant \psi$ and $R \geqslant 1$, (4.4) may be replaced by the uniform bound

$$\log| E(Re^{i\theta})| \leqslant \frac{1}{\pi\delta^2} \int_{-2\psi}^{2\psi} \int_{R-\delta}^{R+\delta} \log^+ | E(\rho e^{i\alpha})| \, \rho \, d\rho \, d\alpha,$$

which, because of (3.1), is

$$\leqslant \frac{2}{\pi\delta^2} \int_0^{2\psi} \int_{R-\delta}^{R+\delta} \log^+ | E(\rho e^{i\alpha})| \, \rho \, d\rho \, d\alpha.$$

Hence, invoking (4.3′) we see that

$$\log| E(Re^{i\theta})| \leqslant \frac{2}{\pi\delta^2} \int_{-\infty}^{\infty} \log^+ | E(c)| F(c) \, dc, \tag{4.5}$$

where

$$F(c) = \frac{1}{\pi} \int_0^{2\psi} \int_{R-\delta}^{R+\delta} \frac{\rho \sin \alpha}{(\rho \cos \alpha - c)^2 + \rho^2 \sin^2 \alpha} \, \rho \, d\rho \, d\alpha.$$

But upon transforming to rectangular coordinates, and recalling that $\int[(u - c)^2 + v^2]^{-1} v \, du = \pi$ for $v > 0$, $F(c)$ is plainly seen to satisfy the bound

$$F(c) \leqslant (R + \delta) \sin 2\psi < (R + \delta).$$

This estimate will be adequate for $0 \leqslant c \leqslant 4R$. Outside this range it is easily checked that

$$\begin{aligned} F(c) &\leqslant \text{constant} \int_0^{2\psi} \int_{R-\delta}^{R+\delta} (c^2 + 1)^{-1} \rho^2 \, d\rho \, d\theta \\ &\leqslant \text{constant} \cdot R^2 \cdot (c^2 + 1)^{-1}. \end{aligned}$$

Substituting these last two estimates into (4.5) yields

$$\begin{aligned} \log| E(Re^{i\theta})| \leqslant \text{constant} \cdot \Big\{ \Big[\int_{-\infty}^{0} + \int_{4R}^{\infty} \Big] \frac{\log^+ | E(c)|}{1 + c^2} \, dc \cdot R^2 \\ + \int_0^{4R} \log^+ | E(c)| \, dc \cdot R \Big\}. \end{aligned}$$

Thus as

$$\int_0^{4R} \log^+ | E(c)| \, dc \leqslant \int_0^{4R} \frac{\log^+ | E(c)| \, dc}{1 + c^2} (1 + 4R^2)$$

it follows readily that

$$\log| E(Re^{i\theta})| \leqslant \text{constant} \cdot R^3$$

for all $R \geqslant 1$ and all $|\theta| < \psi$. In other words, letting c_1, c_2,..., etc. denote positive constants we have

$$|E(Re^{i\theta})| \leqslant c_1 e^{c_2 R^3} \qquad \text{if} \quad |\theta| \leqslant \psi,$$

and, since E is of exponential type "away from" the real axis,

$$|E(Re^{\pm i\psi})| \leqslant c_3 e^{c_4 R} \qquad (= |c_3 e^{(c_4/\cos\psi)\gamma}| \text{ if } \gamma = Re^{\pm i\psi}).$$

Thus, as ψ has been selected smaller than $\pi/6$, the Phragmén–Lindelöf theorem (Titchmarsh, Ref. [23], p. 177) may be invoked to conclude that E is of exponential type on the entire right half-plane and then, by similar arguments, over the left half-plane.

Comment 4.2. The assumption $E^{\#}(\gamma) = E(-\gamma)$ was not used here.

The subject matter of the next theorem belongs properly to the theory of Hardy functions of class two over the upper half-plane (H^{2+} for short). These results are collected together here, with a sketch of the proof, for the convenience of the reader. These results are relevant because $(f/E) \in H^{2+}$ whenever $f \in \mathscr{B}(E)$. That is to say, the theorem is applicable to (f/E) whenever $f \in \mathscr{B}(E)$.

THEOREM 4.1. *If $g \in L^2(R^1)$ is analytic in the closed upper half-plane, then the following conditions are equivalent*:

(4.6a) $|g(Re^{i\theta})| \leqslant$ *a constant* $\times (R \sin\theta)^{-1/2}$ *as* $R \uparrow \infty$ *for* $0 < \theta < \pi$;

(4.6b) $g(\beta) = \int g(\gamma)\{2\pi i(\gamma - \beta)\}^{-1}\, d\gamma$ *for* $\beta \in R^{2+}$;

(4.6c) $\hat{g}(\lambda) = (2\pi)^{-1/2} \int e^{-i\lambda c} g(c)\, dc = 0$ *for* $\lambda < 0$;

(4.6d) $\int |g(a + ib)|^2\, da \leqslant \int |g(a)|^2\, da$ *for every* $b \geqslant 0$;

(4.6e) $\log|g(a + ib)| \leqslant (b/\pi) \int \{(c - a)^2 + b^2\}^{-1} \log|g(c)|\, dc - kb$, *for some constant* $k \geqslant 0$ *and every* $b > 0$.

Comment 4.3. The function $\hat{g}$ introduced in (4.6c) is of course the Fourier transform of g. Since $g \in L^2(R^1)$ we must have

$$\int (1 + c^2)^{-1} \log|g(c)|\, dc \leqslant \int (1 + c^2)^{-1} \log^+ |g(c)|\, dc < \infty. \qquad (4.7)$$

(4.6e) then further implies that

$$\int (1 + c^2)^{-1} \log|g(c)|\, dc > -\infty,$$

in accord with the famed Theorem XII of Paley and Wiener, Ref. [20].

Proof, (a) $\Rightarrow$ (b). The proof is identical to the one used to establish (3.3), only here g plays the role of (f/E).

Proof, (b) $\Rightarrow$ (a). This follows from Schwarz's inequality and the fact that $\int |\gamma - \beta|^{-2}\, d\gamma = \pi/b$ if $\beta = a + ib$ and $b > 0$.

Proof, (b) $\Rightarrow$ (c). Let $f(\gamma) = \{-2\pi i(\gamma - \bar{\beta})\}^{-1}$ for fixed $\beta \in R^{2+}$. Then $f \in L^2(R^1)$ and

$$\hat{f}(\lambda) = (2\pi)^{-1/2} \int e^{-i\lambda c}\{-2\pi i(c - \bar{\beta})\}^{-1} dc = (2\pi)^{-1/2} e^{-i\lambda\bar{\beta}} \qquad \text{if} \quad \lambda > 0,$$
$$= 0 \qquad \text{if} \quad \lambda < 0.$$

Thus, invoking the Plancherel formula, we see that

$$\begin{aligned} g(a + ib) &= \int g(c) \bar{f}(c)\, dc \\ &= \int \hat{g}(\lambda) \bar{\hat{f}}(\lambda)\, d\lambda \\ &= (2\pi)^{-1/2} \int_0^\infty (\hat{g}(\lambda)\, e^{-\lambda b})\, e^{i\lambda a}\, d\lambda, \end{aligned}$$

that is to say, the transform of $g(a + ib)$ (viewed as a function of a),

$$(2\pi)^{-1/2} \int g(a + ib)\, e^{-i\lambda a}\, da = \hat{g}(\lambda)\, e^{-\lambda b} \qquad \text{if} \quad \lambda > 0,$$
$$= 0 \qquad \text{if} \quad \lambda < 0.$$

But as this must tend to the transform of $g(a)$ as $b \downarrow 0$ we must have $\hat{g}(\lambda) = 0$ if $\lambda < 0$.

Proof, (c) $\Rightarrow$ (d). If $\hat{g}(\lambda) = 0$ for $\lambda < 0$ then

$$g(a + ib) = (2\pi)^{-1/2} \int_0^\infty (\hat{g}(\lambda)\, e^{-\lambda b})\, e^{i\lambda a}\, d\lambda, \qquad b > 0$$

and so, again applying the Plancherel formula,

$$\begin{aligned} \int_{-\infty}^{\infty} | g(a + ib)|^2\, da &= \int_0^\infty | \hat{g}(\lambda)|^2\, e^{-2\lambda b}\, d\lambda \\ &\leqslant \int_0^\infty | \hat{g}(\lambda)|^2\, d\lambda \\ &= \int_{-\infty}^{\infty} | g(a)|^2\, da. \end{aligned}$$

Proof, (d) $\Rightarrow$ (b). Suppose β lies inside the rectangle with corners $-R$, R, $R + iD$, $-R + iD$, where R and D are both positive numbers. Then by Cauchy's formula we have

$$\begin{aligned} 2\pi i g(\beta) &= \int_{-R}^{R} \frac{g(a)\, da}{a - \beta} + i \int_0^D \frac{g(R + ib)}{R + ib - \beta}\, db \\ &\quad - \int_{-R}^{R} \frac{g(a + iD)\, da}{a + iD - \beta} - i \int_0^D \frac{g(-R + ib)}{-R + ib - \beta}\, db \\ &= I_1(R) + I_2(R) + I_3(R) + I_4(R). \end{aligned}$$

We now integrate both sides of this equation with respect to R from R_1 to $2R_1$. Clearly

$$2\pi i g(\beta) = (1/R_1) \int_{R_1}^{2R_1} 2\pi i g(\beta)\, dR$$

for any choice of $R_1 > 0$, and in particular as $R_1 \uparrow \infty$. On the other hand we claim that

$$\begin{aligned} &\lim_{R_1 \uparrow \infty} \frac{1}{R_1} \int_{R_1}^{2R_1} (I_1 + I_2 + I_3 + I_4)\, dR \\ &\quad = \int_{-\infty}^{\infty} \frac{g(a)}{a - \beta}\, da + 0 - \int_{-\infty}^{\infty} \frac{g(a + iD)}{a + iD - \beta}\, da + 0, \end{aligned} \tag{4.8}$$

from which the desired result follows upon letting $D \uparrow \infty$, since

$$\begin{aligned} \left| \int_{-\infty}^{\infty} \frac{g(a + iD)}{a + iD - \beta}\, da \right|^2 &\leqslant \int_{-\infty}^{\infty} | g(a + iD)|^2\, da \int_{-\infty}^{\infty} | a + iD - \beta |^{-2}\, da \\ &\leqslant \int_{-\infty}^{\infty} | g(a)|^2\, da \cdot \pi \cdot (D - \operatorname{Im}[\beta])^{-1}, \end{aligned}$$

which tends to zero as $D \uparrow \infty$.

It remains for us to check (4.8). First we note that

$$\begin{aligned}
&\left| (1/R_1) \int_{R_1}^{2R_1} [I_1(R) - I_1(R_1)]\, dR \right| \\
&\quad \leqslant (1/R_1) \int_{R_1}^{2R_1} \left[\int_{-\infty}^{-R_1} + \int_{R_1}^{\infty}\right] | g(a)/(a - \beta)|\, da\, dR \\
&\quad = \left[\int_{-\infty}^{-R_1} + \int_{R_1}^{\infty}\right] | g(a)/(a - \beta)|\, da \\
&\quad \leqslant \left\{\int_{-\infty}^{-R_1} | g(a)|^2\, da \int_{-\infty}^{-R_1} | a - \beta |^{-2}\, da\right\}^{1/2} \\
&\qquad + \left\{\int_{R_1}^{\infty} | g(a)|^2\, da \int_{R_1}^{\infty} | a - \beta |^{-2}\, da\right\}^{1/2},
\end{aligned}$$

which tends to 0 as $R_1 \uparrow \infty$. A similar argument shows that

$$(1/R_1) \int_{R_1}^{2R_1} I_3(R)\, dR \to - \int_{-\infty}^{\infty} g(a + iD)(a + iD - \beta)^{-1}\, da$$

as $R_1 \uparrow \infty$. Consider next $I_2(R)$, assuming as we may that $| R + ib - \beta | > R_1/2$ if $R > R_1$. Then

$$\begin{aligned}
\left| (1/R_1) \int_{R_1}^{2R_1} I_2(R)\, dR \right| &\leqslant \int_0^D (1/R_1) \int_{R_1}^{2R_1} | g(R + ib)/(R + ib - \beta)|\, dR\, db \\
&\leqslant \int_0^D 2/R_1^2 \int_{R_1}^{2R_1} | g(R + ib)|\, dR\, db \\
&\leqslant \int_0^D (2/R_1^2) \left\{\int_{R_1}^{2R_1} | g(R + ib)|^2\, dR \cdot R_1\right\}^{1/2} db \\
&\leqslant (2D/R_1^{3/2}) \left(\int_{-\infty}^{\infty} | g(R)|^2\, dR\right)^{1/2},
\end{aligned}$$

which clearly tends to 0 as $R_1 \uparrow \infty$. A similar estimate disposes of $(1/R_1) \int_{R_1}^{2R_1} I_4(R)\, dR$.

Proof, (a) $\Rightarrow$ (e). Carleman's theorem (Boas, Ref. [2], Theorem 1.2.2) together with (4.7) implies that

$$-I_1(R) = -(2/\pi R) \int_0^\pi \log|\,g(Re^{i\theta})|\,\sin\theta\,d\theta$$

is bounded above. Hence, upon writing

$$\log|\,g\,| = \log^+|\,g\,| - \log^-|\,g\,|$$

and using (a) it follows that

$$\begin{aligned}
&(2/\pi R) \int_0^\pi \log^-|\,g(Re^{i\theta})|\,\sin\theta\,d\theta \\
&\qquad \leqslant \text{constant} + (2/\pi R) \int_0^\pi \log^+|\,g(Re^{i\theta})|\,\sin\theta\,d\theta \\
&\qquad \leqslant \text{constant} + (2/\pi R) \int_0^\pi \text{constant} \cdot (R\sin\theta)^{-1/2}\sin\theta\,d\theta \\
&\qquad \leqslant \text{constant},
\end{aligned}$$

and so especially that

$$\begin{aligned}
|\,I_1(R)| &\leqslant (2/\pi R) \int_0^\pi |\,\log|\,g(Re^{i\theta})|\,|\,\sin\theta\,d\theta \\
&\leqslant \text{constant}.
\end{aligned}$$

Now the Poisson formula for a semicircle in the upper half-plane (Boas, Ref. [2], Theorem 1.2.3) states that

$$\log|\,g(a+ib)| - \frac{b}{\pi} \int_{-R}^{R} \frac{\log|\,g(c)|}{(c-a)^2+b^2}\,dc + I_2(R) \leqslant I_3(R), \tag{4.9}$$

with equality if g is root-free, where

$$I_2(R) = \frac{b}{\pi} \int_{-R}^{R} \frac{R^2}{(R^2-ac)^2+(bc)^2} \log|\,g(c)|\,dc$$

and

$$I_3(R) = \frac{2Rb}{\pi} \int_0^\pi \frac{(R^2-|\,\gamma\,|^2)\log|\,g(Re^{i\theta})|\,\sin\theta}{|\,R^2e^{2i\theta}-2Rae^{i\theta}+|\,\gamma\,|^2\,|^2}\,d\theta.$$

The desired result follows upon showing that $I_2(R)$ and $I_3(R) - bI_1(R)$ tend to 0 as $R \uparrow \infty$. For then, as $R \uparrow \infty$, the left side of (4.9) tends to a limit which is less than or equal to $-bk = b \cdot \limsup_{R\uparrow\infty} I_1(R) \leqslant 0$. In case g is root-free equality prevails in (4.9) and so $I_1(R)$ must actually converge to $-k$ as $R \uparrow \infty$. We first estimate $I_2(R)$. Since $\gamma = a + ib$ is fixed we may assume that $R > 2|\gamma|$. Then writing

$$I_2(R) = \int_{-R}^{-M} + \int_{-M}^{M} + \int_{M}^{R}$$

for some "large" fixed choice of $M < R$ it is readily seen that the center term, $\int_{-M}^{M}$, tends to 0 as $R \uparrow \infty$ and that

$$\begin{aligned}\int_M^R &\leqslant \frac{b}{\pi} \int_M^R \frac{R^2(1+R^2)}{(R^2-ac)^2+(bc)^2} \cdot \frac{\log^+|g(c)|}{1+c^2}\,dc \\ &\leqslant \frac{b}{\pi}\,\frac{R^2(1+R^2)}{R^4/4} \int_M^\infty \frac{\log^+|g(c)|}{1+c^2}\,dc,\end{aligned}$$

since $(R^2 - ac)^2 \geqslant R^4/4$ for $M \leqslant c \leqslant R$ and $|a| \leqslant R/2$. But, by (4.7), this last term clearly can be made arbitrarily small by choosing M sufficiently large. A similar argument disposes of $\int_{-R}^{-M}$. To handle $I_3(R) - bI_1(R)$ it is convenient to assume $R > |6\gamma|$, for then

$$|R^2e^{2i\theta} - 2Rae^{i\theta} + |\gamma|^2|^2 \geqslant R^2(R - 6|\gamma|)^2,$$

and consequently

$$\begin{aligned}|I_3(R) - bI_1(R)| &\leqslant \frac{2b}{\pi}\int_0^\pi \left\{\frac{R(R^2-|\gamma|^2)}{R^2(R-6|\gamma|)^2} - \frac{1}{R}\right\} |\log|g(Re^{i\theta})||\sin\theta\,d\theta \\ &\leqslant (\text{constant}/R)\cdot(2/\pi R)\int_0^\pi |\log|g(Re^{i\theta})||\sin\theta\,d\theta,\end{aligned}$$

which tends to zero by an earlier estimate. This completes the proof.

Proof, (e) $\Rightarrow$ (a). It follows from the convexity of the logarithm and Jensen's inequality (Rudin, Ref. [21], p. 61) that

$$\begin{aligned}|g(a+ib)|^2 &\leqslant \exp[(b/\pi)\int \{(c-a)^2+b^2\}^{-1}\log|g(c)|^2\,dc] \\ &\leqslant (b/\pi)\int \{(c-a)^2+b^2\}^{-1}|g(c)|^2\,dc \\ &\leqslant (1/\pi b)\int |g(c)|^2\,dc.\end{aligned}$$

Comment 4.4. The presented proof of (a) $\Rightarrow$ (e) is a modified version of an argument used by Boas to establish a representation formula (Ref. [2], Theorem 6.5.4) akin to (4.6e) under different hypotheses. Note that in case g is root-free in the upper half-plane, equality prevails in (4.6e) and the constant, $-k$, appearing there must equal

$$\lim_{b\to\infty} b^{-1} \log| g(ib)|$$

as well as (in the notation of the proof) $\lim_{R\to\infty} I_1(R)$.

LEMMA 4.3. *If* $h = (\gamma + i)^{-1}$ *satisfies inequality* (3.2a) *then the following conditions are equivalent*:

(a) *h satisfies inequality* (3.2b);

(b) *E satisfies* (4.2) *and* (4.3);

(c) *E is of exponential type.*

Proof. If (a) holds then Theorem 4.1 is applicable to $g = h/E$. Moreover, as g is root-free on the closed upper half-plane equality prevails in (4.6e). (b) now follows easily from the fact that

$$\log| h(a + ib)| = b/\pi \int \{(c - a)^2 + b^2\}^{-1} \log| h(c)|\, dc.$$

If (b) holds then (c) follows by Lemma 4.2. Finally if (c) holds then Lemma 4.1 implies that $| E(a + ib)| \geqslant | E(a)|$ for every $b \geqslant 0$, and so as $| h(a + ib)| \leqslant | h(a)|$ for every $b \geqslant 0$ we must have

$$\int |(h/E)(a + ib)|^2\, da \leqslant \int |(h/E)(a)|^2\, da < \infty$$

for every $b \geqslant 0$. Hence, as an application of Theorem 4.1 to $g = h/E$ clearly shows, h must satisfy inequality (3.2b). That is to say, (c) implies (a), which completes the proof of the Lemma.

5. THE SPACES $\mathscr{B}(E_s)$ GENERATED BY THE SOLUTIONS OF (1.1)

We now consider the specific de Branges spaces $\mathscr{B}(E_s)$, $0 \leqslant s < \infty$, which are generated by the solutions of (1.1). The principal result is that these spaces increase in size with s and, at least if Q^+ and Q^- are strictly increasing, the inclusions are isometric. Moreover, in this case the whole family $\bigcup_{s\geqslant 0}\mathscr{B}(E_s)$ sits (isometrically) inside a space $L^2(d\mu; R^1)$ where μ is a nondecreasing function on R^1 relative to which $(1+\gamma^2)^{-1}$ is summable. That is to say, the spaces $\mathscr{B}(E_s)$ form a monotone increasing family of closed subspaces of $L^2(d\mu; R^1)$. If $\tau(\infty)=\infty$ then the function μ is essentially unique and $\bigcup_{s\geqslant 0}\mathscr{B}(E_s)$ is dense in $L^2(d\mu; R^1)$.

We shall use the notation

$$(f,g) = \int f(\gamma)\,\bar{g}(\gamma)|\,E_s(\gamma)|^{-2}, \qquad (f,g)_\mu = \int f(\gamma)\,\bar{g}(\gamma)\,d\mu(\gamma),$$

and correspondingly,

$$\|f\|_s^2 = (f,f)_s \qquad \text{and} \qquad \|f\|_\mu^2 = (f,f)_\mu\,.$$

THEOREM 5.1. *If $s\leqslant t$ then $\mathscr{B}(E_s)\subset\mathscr{B}(E_t)$ and*

$$\|f\|_t \leqslant \|f\|_s \qquad \textit{for every} \quad f\in\mathscr{B}(E_s),$$

with equality if f belongs to

$$\mathscr{M}(E_s) = \{f\in\mathscr{B}(E_s) : \gamma f\in\mathscr{B}(E_s)\}.$$

Proof. We first note that if $f\in\mathscr{B}(E_s)$, then by (3.6) and Corollary 2 of Lemma 2.1,

$$|f(\gamma)|^2 \leqslant \|f\|_s^2 J_\gamma^{\,s}(\gamma) \leqslant \|f\|_s^2 J_\gamma^{\,t}(\gamma),$$

which is to say that f satisfies (3.2b) and (3.2c) (relative to E_t). It thus follows that $\mathscr{B}(E_s)\subset\mathscr{B}(E_t)$ if and only if $\|f\|_t<\infty$ for every $f\in\mathscr{B}(E_s)$. The rest of the proof proceeds in steps.

STEP 1. *$\|f\|_s=\|f\|_t$ for every $f\in\mathscr{B}(E_s)$ if Q^+ and Q^- increase linearly from s to t with positive slopes q^+ and q^-, respectively.*

Proof of Step 1. In this case it is easily checked that the solutions of (1.1) may be written explicitly in the form

$$A_t = A_s\cos\gamma\eta - B_s(q^-/q^+)^{1/2}\sin\gamma\eta,$$
$$B_t = A_s(q^+/q^-)^{1/2}\sin\gamma\eta + B_s\cos\gamma\eta,$$

where $\eta = (q^+q^-)^{1/2}(t-s)$. (Indeed the same solutions prevail whenever the ratio of q^+ to q^- is constant between s and t, even if the individual slopes themselves are not.) Thus setting $c^2 = q^-/q^+$ we see that

$$c^{-1}A_t^2 + cB_t^2 = c^{-1}A_s^2 + cB_s^2 \qquad \text{if} \quad \gamma \in R^1,$$

and further, invoking Lemma 3.1, we conclude that

$$\int \frac{|f|^2}{A_s^2 + B_s^2} = \int \frac{|f|^2}{(1/c)A_s^2 + cB_s^2} = \int \frac{|f|^2}{(1/c)A_t^2 + cB_t^2} \tag{5.1}$$

for every $f \in \mathscr{B}(E_s)$. But this equality also implies (at least when E_s is of the special form treated herein) that $\| f \|_t < \infty$ for every $f \in \mathscr{B}(E_s)$, and hence that $\mathscr{B}(E_s) \subset \mathscr{B}(E_t)$. Thus Lemma 3.1 can be applied again to conclude that the right side of (5.1) is equal to $\| f \|^2$ for every $f \in \mathscr{B}(E_s)$, which establishes the desired isometry.

STEP 2. *Fix $\epsilon > 0$ and choose $Q_\epsilon^+[Q_\epsilon^-]$ to be a piecewise linear continuous approximation to $Q^+[Q^-]$ on the interval* $[s, t]$ *such that*

$$| Q^+ - Q_\epsilon^+ | \leqslant \epsilon \, [| Q^- - Q_\epsilon^- | < \epsilon]$$

and the slope of each line segment of $Q_\epsilon^+[Q_\epsilon^-]$ is strictly positive, and let

$$L_\epsilon(s; t) = Q^+(t) + Q^-(t) - Q^+(s) - Q^-(s) + Q_\epsilon^+(t) + Q_\epsilon^-(t) - Q_\epsilon^+(s) - Q_\epsilon^-(s).$$

Then for every $m \geqslant 1$, we have, in the notation of Section 2,

$$| [Q^-, Q^+]_m(s; t) - [Q_\epsilon^-, Q_\epsilon^+]_m(s; t)| \leqslant 2\epsilon\{2L_\epsilon(s; t)\}^{m-1}/(m-1)! \tag{5.2}$$

and

$$| [Q^+, Q^-]_m(s; t) - [Q_\epsilon^+, Q_\epsilon^-]_m(s; t)| \leqslant 2\epsilon\{2L_\epsilon(s; t)\}^{m-1}/(m-1)!. \tag{5.3}$$

Proof of Step 2. Both inequalities are clearly valid when $m = 1$. Proceeding by induction we assume them both to hold for all $m \leqslant n - 1$ where $n > 1$. Then, for example,

$$| [Q^-, Q^+]_n(s; t) - [Q_\epsilon^-, Q_\epsilon^+]_n(s; t)| \leqslant \mathrm{I} + \mathrm{II},$$

where

$$\mathrm{I} = \int_s^t | [Q^+, Q^-]_{n-1}(s; u) - [Q_\epsilon^+, Q_\epsilon^-]_{n-1}(s; u)| \, dQ^-(u)$$

and

$$\mathrm{II} = \left| \int_s^t [Q_\epsilon^+, Q_\epsilon^-]_{n-1}(s; u)\{dQ^-(u) - dQ_\epsilon^-(u)\right.$$

By the induction hypothesis

$$\begin{aligned} \mathrm{I} &\leqslant 2\epsilon \int_s^t \{2L_\epsilon(s; u)\}^{n-2}\, dQ^-(u)/(n-2)! \\ &\leqslant 2\epsilon \int_s^t \{2L_\epsilon(s; u)\}^{n-2}\, dL_\epsilon/(n-2)! \\ &= \epsilon\{2L_\epsilon(s; t)\}^{n-1}/(n-1)!. \end{aligned}$$

On the other hand, integrating once by parts, we see that

$$\begin{aligned} \mathrm{II} &= \Big| \{Q^-(t) - Q_\epsilon^-(t)\}[Q_\epsilon^+, Q_\epsilon^-]_{n-1}(s; t) \\ &\qquad - \int_s^t \{Q^-(u) - Q_\epsilon^-(u)\}\, d[Q_\epsilon^+, Q_\epsilon^-]_{n-1}(s; u) \Big| \\ &\leqslant 2\epsilon[Q_\epsilon^+, Q_\epsilon^-]_{n-1}(s, t) \\ &\leqslant 2\epsilon\{L_\epsilon(s; t)\}^{n-1}/(n-1)!. \end{aligned}$$

Hence,

$$\mathrm{I} + \mathrm{II} \leqslant 2\epsilon\{2L_\epsilon(s; t)\}^{n-1}/(n-1)!.$$

This establishes (5.2); (5.3) is proved in much the same way.

STEP 3. *If $f \in \mathscr{B}(E_s)$ and $s \leqslant t$, then $\|f\|_t \leqslant \|f\|_s$ and so $\mathscr{B}(E_s) \subset \mathscr{B}(E_t)$, too.*

Proof of Step 3. Let

$$\mathscr{E}_t^\epsilon = \sum_{m=0}^{\infty} \gamma^m \begin{pmatrix} [-Q_\epsilon^-, Q_\epsilon^+]_m(s; t) & 0 \\ 0 & [Q_\epsilon^+, -Q_\epsilon^-]_m(s; t) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathscr{E}_s$$

denote the solution of (1.1) when Q^+ and Q^- are replaced by piecewise linear continuous approximations Q_ϵ^+ and Q_ϵ^- over the interval $[s, t]$. That is to say, we assume that $Q_\epsilon^+[Q_\epsilon^-]$ agrees with $Q^+[Q^-]$ on $[0, s]$. Then (5.2) and (5.3) imply that

$$\begin{aligned} \|\mathscr{E}_t(\gamma) - \mathscr{E}_t^\epsilon(\gamma)\| &\leqslant \sum_{m=1}^{\infty} 2\epsilon|\gamma|^m\{2L_\epsilon(s; t)\}^{m-1}\|\mathscr{E}_s(\gamma)\|/(m-1)! \\ &= 2\epsilon|\gamma|\, e^{|\gamma| 2L_\epsilon(s;t)}\, \|\mathscr{E}_s(\gamma)\|, \end{aligned}$$

and hence that $\mathscr{E}_t^\epsilon(\gamma)$ tends to $\mathscr{E}_t(\gamma)$ locally uniformly as ϵ tends to zero. Thus setting $\epsilon = 1/n$ and correspondingly denoting the components of $\mathscr{E}_t^\epsilon(\gamma)$ by $A_t^n(\gamma)$ and $B_t^n(\gamma)$ we see also that $A_t^n - iB_t^n = E_t^n$ tends to E_t locally uniformly (in γ) as $n \to \infty$. Moreover, by Step 1,

$$\int |f/E_s|^2 = \int |f/E_t^n|^2$$

for every $f \in \mathscr{B}(E_s)$. Thus, by Fatou's lemma,

$$\begin{aligned}\int |f/E_t|^2 &= \int \lim_{n\to\infty} |f/E_t^n|^2 \\ &\leqslant \lim_{n\to\infty} \int |f/E_t^n|^2 \\ &= \int |f/E_s|^2,\end{aligned}$$

which establishes the desired result.

STEP 4. *If both f and $\gamma f \in \mathscr{B}(E_s)$, then*

$$\int |f/E_s|^2 = \int |f/E_t|^2, \qquad s \leqslant t.$$

Proof. We must show that

$$\lim_n \int |f/E_t^n|^2 = \int |f/E_t|^2.$$

Clearly equality prevails if the domain of integration is restricted to be finite, for E_t^n tends to E_t locally uniformly as $n \to \infty$ and, by Comment 2.3, E_t has no real zeros. It thus remains to show that the tails of the integral of $|f/E_t^n|^2$ are uniformly small in n. But if $\gamma f \in \mathscr{B}(E_s)$ we then have, for example,

$$\begin{aligned}\int_R^\infty |f/E_t^n|^2 &\leqslant R^{-2} \int_R^\infty |\gamma f/E_t^n|^2 \\ &\leqslant R^{-2} \int_{-\infty}^\infty |\gamma f/E_t^n|^2 \\ &= R^{-2} \int_{-\infty}^\infty |\gamma f/E_s|^2.\end{aligned}$$

The rest is clear.

DEFINITION 5.1. We shall say that $t \in (0, \infty)$ is a *growth point* if

$$[Q^+(t) - Q^+(s)][Q^-(t) - Q^-(s)] > 0$$

for every $s < t$.

DEFINITION 5.2. We shall say that $s \geqslant 0$ is a *regular point* if it is either a growth point or an accumulation point of growth points.

LEMMA 5.1. *The functions* $\gamma^{-1}(A_t - A_s)$ *and* $\gamma^{-1}(B_t - B_s)$ *belong to* $\mathscr{B}(E_t)$ *for every choice of* $t \geqslant s \geqslant 0$ *and satisfy the following inequalities*:

(5.4a) $\| \gamma^{-1}(B_t - B_s)\|_t^2 \leqslant \pi[Q^+(t) - Q^+(s)]$;

(5.4b) $\| \gamma^{-1}(A_t - A_s)\|_t^2 \leqslant \pi[Q^-(t) - Q^-(s)]$.

If s is a regular point then equality prevails in both (5.4a) *and* (5.4b).

Proof. Clearly $\gamma^{-1}B_s = \pi J_0{}^s$ belongs to $\mathscr{B}(E_s) \subset \mathscr{B}(E_t)$ for $s \leqslant t$, and

$$\begin{aligned}\int (B_t/\gamma)(B_t/\gamma - 2B_s/\gamma)|\, E_t\,|^{-2} &= \pi^2[J_0{}^t(0) - 2J_0{}^s(0)]\\ &= \pi[Q^+(t) - 2Q^+(s) - \pi J_0{}^0(0)].\end{aligned}$$

In addition, by Theorem 5.1, we have

$$\begin{aligned}\int (B_s/\gamma)^2|\, E_t\,|^{-2} &\leqslant \int (B_s/\gamma)^2|\, E_s\,|^{-2}\\ &= \pi^2 J_0{}^s(0)\\ &= \pi Q^+(s) + \pi^2 J_0{}^0(0).\end{aligned}$$

Add this inequality to the above equality to get (5.4a). The asserted equality when s is a regular point is easily verified with the help of Lemma 5.3.

Now suppose $s > 0$. Then $\mathscr{B}(E_s) \not\equiv 0$, since $Q^+(s) > 0$, and so contains an element f with $f(0) = 1$ [e.g., $\{J_0{}^s(0)\}^{-1} J_0{}^s$]. This allows us to write

$$(A_t - A_s)/\gamma = (A_t - f)/\gamma + (f - A_s)/\gamma$$

and to conclude via routine estimates that the first term on the right belongs to $\mathscr{B}(E_t)$ while the second belongs to $\mathscr{B}(E_s) \subset \mathscr{B}(E_t)$. This shows that $\gamma^{-1}(A_t - A_s)$ belongs to $\mathscr{B}(E_t)$ for every choice of $t \geqslant s > 0$ and so too for $s = 0$ as will follow from an application of Fatou's lemma to inequality (5.4b). The proof of (5.4b) is given in Step 3 of the proof of Lemma 5.2.

LEMMA 5.2. *If $A_t[B_t]$ belongs to $\mathscr{B}(E_t)$ for some point $t > 0$ then there exists an $\epsilon > 0$ such that $Q^-[Q^+]$ is constant on the interval $[t - \epsilon, t]$.*

Proof. The proof proceeds in steps.

STEP 1. *$A_t[B_t]$ is perpendicular to $\mathscr{M}(E_s)$ in $L^2(|E_t|^{-2})$.*

Proof of Step 1. Let $f \in \mathscr{M}(E_t)$ be even. Then

$$\begin{aligned}
\int A_t f |E_t|^{-2} &= \lim_{R\to\infty} \int_{-R}^{R} (A_t + iB_t) f (E_t E_t^{\#})^{-1} \\
&= \lim_{R\to\infty} \int_{-R}^{R} f/E_t \\
&= \lim_{R\to\infty} - 2\pi i \int_0^{\pi} (f/E_t)(Re^{i\theta}) iRe^{i\theta}\, d\theta \\
&= 0,
\end{aligned}$$

by an application of inequality (3.2b) to γf, which belongs to $\mathscr{B}(E_t)$ by assumption. A similar argument goes through with B_t in place of A_t.

STEP 2. *If $A_t \in \mathscr{B}(E_t)$ for some point $t > 0$ then Q^- is constant on a left-hand neighborhood of t.*

Proof of Step 2. The idea behind our proof is to suppose the assertion false, which gives $Q^-(s) < Q^-(t)$ for every $s < t$, and then reach a contradiction by writing

$$\begin{aligned}
0 &= \lim_{s\uparrow t} \int A_t(A_t - A_s)\gamma^{-2}[Q^-(t) - Q^-(s)]^{-1} |E_t|^{-2} \\
&= -\int A_t B_t \gamma^{-1} |E_t|^{-2} \\
&= -\pi(A_t, J_0{}^t)_t \\
&= -\pi.
\end{aligned}$$

It remains to justify lines 1 and 2. The rest should be clear. Line 1 is a direct consequence of Step 1 since $(A_t - A_s)\gamma^{-2} \in \mathscr{M}(E_t)$. To get the second line it is enough to check that

$$\gamma^{-1}B_t + [Q^-(t) - Q^-(s)]^{-1}(A_t - A_s)\gamma^{-2}$$
$$= [Q^-(t) - Q^-(s)]^{-1} \int_s^t \gamma^{-1}(B_t - B_u)\, dQ^-(u)$$

tends to zero in $\mathscr{B}(E_t)$ as s increases to t. But as

$$\left\| \int_s^t \gamma^{-1}(B_t - B_u)\, dQ^-(u) \right\|_t^2 \leqslant \left\{ \int_s^t \| \gamma^{-1}(B_t - B_u)\|_t \, dQ^-(u) \right\}^2$$
$$\leqslant \left\{ \int_s^t \{\pi[Q^+(t) - Q^+(u)]\}^{1/2}\, dQ^-(u) \right\}^2,$$

by inequality (5.4a), this must clearly be the case.

STEP 3. *We now verify inequality* (5.4b).

Proof of Step 3. Suppose first that $Q^-(s) > 0$. Then there exists a function $h \in \mathscr{M}(E_s)$ which is real valued on R^1 and is such that $h(0) = 1$ [e.g., one can set $h = -\gamma^{-2}(A_s - A_0)\pi^{-1}\{\int_0^s J_0{}^u(0)\, dQ^-(u)\}^{-1}$], and because of Step 1 we can write

$$\|\gamma^{-1}(A_t - A_s)\|_t^2 = -\int A_s(A_t - A_s)\gamma^{-2}|\, E_t\,|^{-2}$$
$$= \int (A_t - A_s)(h - A_s)\gamma^{-2}|\, E_t\,|^{-2} - \int h(A_t - A_s)\gamma^{-2}|\, E_t\,|^{-2}.$$

But

$$\int (A_t - A_s)(h - A_s)\gamma^{-2}|\, E_t\,|^{-2}$$
$$= -\int A_s(h - A_s)\gamma^{-2}|\, E_t\,|^{-2}$$
$$= \int (h - A_s)^2\gamma^{-2}|\, E_t\,|^{-2} - \int h(h - A_s)\gamma^{-2}|\, E_t\,|^{-2}$$
$$\leqslant \int (h - A_s)^2\gamma^{-2}|\, E_s\,|^{-2} - \int h(h - A_s)\gamma^{-2}|\, E_t\,|^{-2},$$

by Theorem 5.1. Moreover, as both h and $(h - A_s)\gamma^{-2}$ belong to $\mathscr{M}(E_s)$

we can replace $| E_t |^{-2}$ by $| E_s |^{-2}$ in the second term on the right in the last line. Doing this and then combining terms it follows that

$$\int (A_t - A_s)(h - A_s)\gamma^{-2} | E_t |^{-2} \leqslant -\int A_s(h - A_s)\gamma^{-2} | E_s |^{-2},$$

which equals zero by Step 1. To this point therefore we have

$$\begin{aligned} \| \gamma^{-1}(A_t - A_s) \|_t^2 &\leqslant -\int h(A_t - A_s)\gamma^{-2} | E_t |^{-2} \\ &= \pi \int_{-\infty}^{\infty} h(\gamma) \int_s^t J_0{}^u(\gamma)\, dQ^-(u) | E_t(\gamma)|^{-2}\, d\gamma \\ &= \pi \int_s^t \left\{ \int_{-\infty}^{\infty} h(\gamma) J_0{}^u(\gamma) | E_t(\gamma)|^{-2}\, d\gamma \right\} dQ^-(u). \end{aligned}$$

In evaluating this last integral notice that we might as well assume that Q^- is strictly increasing on $[s, t]$. But this means that $A_u \notin \mathscr{B}(E_u)$ and hence that either $\gamma^{-1}B_u \in \mathscr{M}(E_u)$ or else $\mathscr{M}(E_u)$ is dense in $\mathscr{B}(E_u)$. In either event Theorem 5.1 guarantees that

$$\begin{aligned} \int hJ_0{}^u | E_t |^{-2} &= \int hJ_0{}^u | E^u |^{-2} \\ &= 1, \end{aligned}$$

and inequality (5.4b) follows, at least in case $Q^-(s) > 0$. However, simple limiting arguments show that the inequality is valid even if $Q^-(s) = 0$. In addition a review of the proof with Lemma 5.3 in hand readily leads one to the conclusion that equality prevails in (5.4b) when s is a regular point.

STEP 4. *If $B_t \in \mathscr{B}(E_t)$ for some point $t > 0$, then Q^+ is constant on a left-hand neighborhood of t.*

Proof of Step 4. The proof is similar in spirit to the one furnished in Step 2. Again one supposes the assertion false. This gives $Q^+(s) < Q^+(t)$ for all $s < t$ and leads to the following contradiction:

$$\begin{aligned} \pi &= \lim_{s \uparrow t} \int (B_t/\gamma)(B_t/\gamma - B_s/\gamma)[Q^+(t) - Q^+(s)]^{-1} | E_t |^{-2} \\ &= \int (B_t/\gamma) A_t | E_t |^{-2} \\ &= 0. \end{aligned}$$

To get the last line notice that $\gamma^{-1}B_t \in \mathcal{M}(E_t)$, since B_t is assumed to belong to $\mathcal{B}(E_t)$, and apply Step1. This completes the proof.

COROLLARY 1. *For every $t \geqslant 0$ we have*

$$\int (A_0)^2(1+\gamma^2)^{-1}|E_t|^{-2} \leqslant 2\pi[Q^-(t) - Q^-(0) + 1]. \tag{5.5}$$

Proof. Apply Lemma 5.1 after first noting that

$$\begin{aligned}\int (A_0)^2(1+\gamma^2)^{-1}|E_t|^{-2} &\leqslant 2\int \{(A_t - A_0)^2 + A_t^2\}(1+\gamma^2)^{-1}|E_t|^{-2}\\ &\leqslant 2\|\gamma^{-1}(A_t - A_0)\|_t^2 + 2\int (1+\gamma^2)^{-1}\,d\gamma.\end{aligned}$$

Comment 5.1. It is possible to have strict inequality in (5.4) as the example $E_0 = 1$, $Q^-(t) = 0$ indicates, for a routine compution shows that $A_s = 1$ and $B_s = \gamma Q^+(s)$ for $s \leqslant t$ and

$$\|\gamma^{-1}(B_t - B_s)\|_t^2 = \pi[Q^+(t) - Q^+(s)]^2/Q^+(t).$$

LEMMA 5.3. *If $s \geqslant 0$ is a regular point then $\|f\|_s = \|f\|_t$ for every $f \in \mathcal{B}(E_s)$ and every $t \geqslant s$.*

Proof. Let $f \in \mathcal{B}(E_s)$ and fix $t > s$. If s is a growth point then neither A_s nor B_s can belong to $\mathcal{B}(E_s)$ because of Lemma 5.2. This means that $\mathcal{M}(E_s)$ is dense in $\mathcal{B}(E_s)$ [Lemma 3.2] and hence, by Theorem 5.1, $\|f\|_s = \|f\|_t$ for every $t \geqslant s$. Suppose next that s is the limit of a decreasing sequence of growth points $s_1 > s_2 > \cdots > s$. Then by Theorem 5.1 and Fatou's lemma we have

$$\begin{aligned}\|f\|_t^2 &\leqslant \|f\|_s^2\\ &= \int \lim_{n\to\infty} |f|^2|E_{s_n}|^{-2}\\ &\leqslant \lim_{n\to\infty}\inf\|f\|_{s_n}^2\\ &= \|f\|_t^2.\end{aligned}$$

Thus we have isometry in this case also. Finally, if s is the limit of an increasing sequence of growth points it is itself a growth point, and so this case is covered by the argument furnished earlier. This completes the proof.

We now summarize the implications of our work to this point in

THEOREM 5.2. *If $s \geqslant 0$ is a regular point then $\mathscr{B}(E_s)$ is contained isometrically in $\mathscr{B}(E_t)$ for every $t \geqslant s$. If Q^+ and Q^- are strictly increasing on $[0, \infty)$ then every point $s \geqslant 0$ is a regular point.*

Comment 5.2. The conclusions of Theorem 5.2 are essentially contained in de Branges, Ref. [6], Theorem IV. To illustrate the possibilities when s is not a growth point suppose that there exists a point $r \in [0, s)$ such that $Q^+(r) = Q^+(s)$. Then for every $u \in [r, s]$ the solutions to (1.1) take the form

$$A_u = A_r - [Q^-(u) - Q^-(r)]\gamma B_r, \qquad B_u = B_r,$$

and

$$J_\beta{}^u(\gamma) = J_\beta{}^r(\gamma) + [Q^-(u) - Q^-(r)]\pi^{-1}\bar{B}_r(\beta)B_r(\gamma).$$

The spaces $\mathscr{B}(E_u)$, $u \in (r, s)$, are thus seen to be at most one-dimensional extensions of $\mathscr{B}(E_r)$. Especially this is the case if B_r does not belong to $\mathscr{B}(E_r)$ and $Q^-(u) > Q^-(r)$ for every $u \in (r, s]$. If $B_r \in \mathscr{B}(E_r)$ then all the spaces $\mathscr{B}(E_u)$, $u \in [r, s]$, contain the same elements and $\| f \|_u = \| f \|_r$ if $f \in \mathscr{M}(E_r)$, while

$$\| B_r \|_u^2 = \| B_r \|_r^2 - \pi^{-1}[Q^-(u) - Q^-(r)] \| B_r \|_u^2 \| B_r \|_r^2 .$$

The last formula follows, at least formally, upon writing (for $f \in \mathscr{B}(E_r)$)

$$\begin{aligned} \| f \|_u^2 &= \int f(\gamma) | E_u(\gamma)|^{-2} \int \bar{f}(\beta) J_\gamma{}^r(\beta) | E_r(\beta)|^{-2} \\ &= \int f(\gamma) | E_u(\gamma)|^{-2} \int \bar{f}(\beta) \{ J_\gamma{}^u(\beta) - \pi^{-1}[Q^-(u) - Q^-(r)] B_r(\beta) \bar{B}_r(\gamma) \} | E_r(\beta)|^{-2} \end{aligned}$$

and then setting $f = B_r$ and interchanging the order of integration. (In doing this recall that $J_\gamma{}^u(\beta) = \bar{J}_\beta{}^u(\gamma)$.) This argument can be made legitimate by simply replacing f by f_I, the restriction of f to the finite interval I, and noting that as I tends to R^1

$$\int f_I(\beta) \bar{J}_\gamma{}^u(\beta) | E_u(\beta)|^{-2} = \text{projection of } f_I \text{ onto } \mathscr{B}(E_u)$$

tends to f in both $\mathscr{B}(E_r)$ and $\mathscr{B}(E_u)$, and that $\int f_I(\beta)\ \bar{J}_\gamma{}^r(\beta) | E_r(\beta)|^{-2}$ tends to f in both $\mathscr{B}(E_r)$ and $\mathscr{B}(E_u)$. (In checking these statements it is helpful to recall Theorem 5.1 and that in the case at hand $\mathscr{B}(E_r)$ and $\mathscr{B}(E_u)$ contain the same elements.)

THEOREM 5.3. *There exists a nondecreasing function μ on R^1 such that*

$$\int (1+\gamma^2)^{-1}\, d\mu(\gamma) < \infty$$

and

$$\|f\|_\mu \leqslant \|f\|_s$$

for every $f \in \mathscr{B}(E_s)$ and every $s \in [0, \infty)$. The two norms agree if $f \in \mathscr{M}(E_s)$ and also for every $f \in \mathscr{B}(E_s)$ if s is a regular point.

Proof. The proof proceeds in steps.

STEP 1. *There exists a nondecreasing function μ.*

Proof of Step 1. In view of the initial conditions (1.5) and inequality (5.5) there exists a point $r \geqslant 0$ such that $\mathscr{B}(E_r) \not\equiv 0$ and

$$\int (1+\gamma^2)^{-1} |E_r(\gamma)|^{-2} < \infty.$$

Hence there exists a function $f \in \mathscr{B}(E_r)$ such that $f(i) \neq 0$ and, because of Lemma 3.3, $f_i = (\gamma - i)^{-1}(f - f(i)) \in \mathscr{B}(E_r)$ also. Thus, by Theorem 5.1, both f and f_i belong to $\mathscr{B}(E_t)$ for every $t \geqslant r$ and

$$\begin{aligned} \int |f(i)|^2(\gamma^2+1)^{-1} |E_t(\gamma)|^{-2} &\leqslant 2\|f_i\|_t^2 + 2\|f\|_t^2 \\ &\leqslant 2\|f_i\|_r^2 + 2\|f\|_r^2 . \end{aligned}$$

That is to say, there exists a finite constant M such that the functions

$$\sigma_t(\lambda) = \int_{-\infty}^{\lambda} (1+\gamma^2)^{-1} |E_t(\gamma)|^{-2}\, d\gamma \leqslant M$$

for all $t \geqslant r$ and all $\lambda \in R^1$. Hence (Widder, Ref. [25], pp. 26–30) there exists a nondecreasing function σ, bounded by M, and a sequence of points $t_1 < t_2 < \cdots$ tending to $+\infty$ such that σ_{t_i} converges weakly to σ as $i \to \infty$ (i.e., $\sigma_{t_i}(\lambda) \to \sigma(\lambda)$ at each continuity point λ of σ). Let $d\mu = (1+\gamma^2)\, d\sigma$. Clearly $\int (1+\gamma^2)^{-1}\, d\mu \leqslant M$.

STEP 2. $\|f\|_\mu \leqslant \|f\|_s$ *for every* $f \in \mathscr{B}(E_s)$ *and every* $s \in [0, \infty)$.

Proof of Step 2. Let c and d be continuity points of μ. It then follows from the Helly–Bray Theorem (Widder, Ref. [25], p. 31), that for every continuous function f,

$$\begin{aligned}\int_c^d |f/E_{t_i}|^2 &= \int_c^d |f(\gamma)|^2(1+\gamma^2)\, d\sigma_{t_i}(\gamma)\\ &\to \int_c^d |f(\gamma)|^2(1+\gamma^2)\, d\sigma(\gamma)\\ &= \int_c^d |f(\gamma)|^2\, d\mu(\gamma)\end{aligned}$$

as $t_i \to \infty$. But Theorem 5.1 implies that for every $s \in [0, \infty)$ and every $f \in \mathscr{B}(E_s)$

$$\begin{aligned}\int_c^d |f/E_{t_i}|^2 &\leqslant \|f\|_{t_i}^2\\ &\leqslant \|f\|_s^2\end{aligned}$$

if $s \leqslant t_i$. It follows easily that

$$\int_c^d |f|^2\, d\mu \leqslant \|f\|_s^2$$

and so too that

$$\|f\|_\mu \leqslant \|f\|_s,$$

as asserted.

STEP 3. $\|f\|_s = \|f\|_u$ *for every* $f \in \mathscr{M}(E_s)$ *and extends to* $f \in \mathscr{B}(E_s)$ *if* s *is a regular point.*

Proof of Step 3. Let $f \in \mathscr{M}(E_s)$. Then for every $t \geqslant s$ it follows, from Theorem 5.1, that

$$\begin{aligned}&\left| \int_c^d |f/E_s|^2 - \int_c^d |f/E_t|^2 \right|\\ &\qquad \leqslant 2\int_d^\infty \{|f/E_s|^2 + |f/E_t|^2\} + 2\int_{-\infty}^c \{|f/E_s|^2 + |f/E_t|^2\},\end{aligned}$$

which can be made uniformly small for all $t \geqslant s$ by choosing $-c$ and d sufficiently large, since, for example,

$$\int_d^\infty |f/E_t|^2 \leqslant d^{-2} \int_d^\infty |\gamma f/E_t|^2$$

$$\leqslant d^{-2} \| \gamma f \|_t^2$$

$$\leqslant d^{-2} \| \gamma f \|_s^2 .$$

Combining this observation with the work of the preceding step it follows readily that if $f \in \mathscr{M}(E_s)$ then $\|f\|_s = \|f\|_\mu$. If s is a growth point, then $\mathscr{M}(E_s)$ is dense in $\mathscr{B}(E_s)$ and so $\|f\|_s = \|f\|_\mu$ for all $f \in \mathscr{B}(E_s)$. The same conclusion holds if s is a regular point.

Comment 5.3. The reader may find it of interest to compare the conclusions of this theorem and the next with Theorem VIII and Theorem XI of de Branges in Ref. [6] and Theorem 42 of de Branges in Ref. [9]. The latter gives somewhat better sufficiency conditions for the uniqueness of μ than Theorem 5.4 below.

COROLLARY. *If Q^+ and Q^- are strictly increasing on $[0, \infty)$ then $\|f\|_s = \|f\|_\mu$ for every $f \in \mathscr{B}(E_s)$ and every $s \geqslant 0$.*

LEMMA 5.4. *If $s \leqslant t$ and $f \in \mathscr{B}(E_s)$ then $e^{i\gamma\delta} f \in \mathscr{B}(E_t)$ for every real number δ which satisfies the inequality $|\delta| \leqslant \tau(t) - \tau(s)$.*

Proof. Theorem 5.1 guarantees that

$$\| e^{i\gamma\delta} f \|_t = \|f\|_t \leqslant \|f\|_s < \infty,$$

while on the other hand, Lemma 2.2 insures that

$$| e^{i\gamma\delta} f(\gamma)|^2 \leqslant e^{2|b|(\tau(t)-\tau(s))} \|f\|_s^2 J_\gamma{}^s(\gamma)$$

$$\leqslant \text{constant} \cdot J_\gamma{}^t(\gamma)$$

for every choice of $\gamma = a + ib$.

THEOREM 5.4. *$\bigcup \mathscr{B}(E_s)$, taken over all regular points s, is a dense subset of $L^2(d\mu)$ if and only if $\tau(\infty) = \infty$. If $\tau(\infty) = \infty$ then the function μ is essentially unique.*

Proof. Suppose first that $\tau(\infty) = +\infty$. This guarantees that the collection of growth points is unbounded. Thus if $f \in \bigcup \mathscr{B}(E_s)$ [here and henceforth it is to be understood that the union is taken over all regular points] then, by Lemma 5.4, $e^{i\gamma t}f \in \bigcup \mathscr{B}(E_s)$ for every real choice of t. Hence, if there exists a function $g \in L^2(d\mu)$ which is orthogonal to $\bigcup \mathscr{B}(E_s)$ then $\int e^{i\gamma t} f\bar{g}\, d\mu = 0$ for every real choice of t and this, by a standard theorem of Fourier analysis (Sz.-Nagy, Ref. [22], p. 316), implies that $f\bar{g} = 0$ a.e. $[d\mu]$. Now choosing a point β for which $f(\beta) \neq 0$ and repeating the argument with $f_\beta = (\gamma - \beta)^{-1}(f - f(\beta))$ in place of f, or for that matter with any function in $\bigcup \mathscr{B}(E_s)$ whose zeros are distinct from those of f, it follows that $\bar{g} = 0$, a.e. $[d\mu]$. But this means that $\bigcup \mathscr{B}(E_s)$ is dense in $L^2(d\mu)$, as asserted. The argument for essential uniqueness follows in much the same spirit, for if ν is a nondecreasing function which enjoys the same relationship to the spaces $\mathscr{B}(E_s)$ as μ then we must have

$$\int e^{i\gamma t} | f |^2 \, d\mu = \int e^{i\gamma t} | f |^2 \, d\nu$$

and

$$\int e^{i\gamma t} | f_\beta |^2 \, d\mu = \int e^{i\gamma t} | f_\beta |^2 \, d\nu$$

for every real choice of t. We thus conclude that $\mu(d) - \mu(c) = \nu(d) - \nu(c)$ whenever c and d are continuity points of μ.

Next let us suppose that $\tau(\infty) = T < \infty$. Then, as follows from the estimates in the proof of Lemma 3.5,

$$J_\beta^{\,t}(\beta) \leqslant J_\beta^{\infty}(\beta) \leqslant M e^{(2T+\epsilon)|\beta|}$$

for every $\beta \in R^2$, where M is a constant depending only upon the choice of $\epsilon > 0$. Now let f be the limit in $L^2(d\mu)$ of a sequence of functions $\{f_n\}$ drawn from $\bigcup \mathscr{B}(E_s)$. Then, as the inequality

$$| f_n(\gamma) - f_m(\gamma)|^2 \leqslant \| f_n - f_m \|_\mu^2 \, M e^{(2T+\epsilon)|\gamma|}$$

implies that $\{f_n\}$ is a normal family (Rudin, Ref. [21], p. 271), we may assume that $f_n \to f$ locally uniformly and that f is an entire function. Moreover, as

$$\begin{aligned} | f(\gamma)| &\leqslant | f(\gamma) - f_n(\gamma)| + | f_n(\gamma)| \\ &\leqslant | f(\gamma) - f_n(\gamma)| + \| f_n \|_\mu \, M^{1/2} \, e^{(T+\epsilon/2)|\gamma|} \\ &\leqslant | f(\gamma) - f_n(\gamma)| + M_1 \, e^{(T+\epsilon/2)|\gamma|}, \end{aligned}$$

where M_1 is independent of n, it follows, upon letting $n \uparrow \infty$, that f is of exponential type $\leqslant T$. Thus to complete the proof of the assertion that $\bigcup \mathscr{B}(E_s)$ is not dense in $L^2(d\mu)$ it is enough to exhibit an entire function in $L^2(d\mu)$ whose type exceeds T. The function $\gamma^{-1}(E_1 - 1)e^{-i\gamma 2T}$ serves.

6. On Mapping $L^2(dQ^{\pm}; [0, t])$ into $\mathscr{B}(E_t)$

Let $\mathscr{L}^{\pm}(t)$ denote the span in $L^2(dQ^{\pm}; [0, t])$ of finite linear combinations of indicator functions χ_I of subintervals I of $[0, t]$ with regular end points. The basic result of this section is

Theorem 6.1. *Let* 0 *and* $t > 0$ *be regular points. Then the solutions* A_s *and* B_s *of the system* (1.1) *give rise to a pair of transforms*

(6.1a) $T_e : f \in \mathscr{L}^{+}(t)$
$$\to \pi^{-1} \int_0^t f(s)\, A_s(\gamma)\, dQ^{+}(s) \in \{\mathscr{B}(E_t) \ominus \mathscr{B}(E_0)\}_{\text{even}},$$

(6.1b) $T_o : f \in \mathscr{L}^{-}(t)$
$$\to \pi^{-1} \int_0^t f(s)\, B_s(\gamma)\, dQ^{-}(s) \in \{\mathscr{B}(E_t) \ominus \mathscr{B}(E_0)\}_{\text{odd}},$$

where, for example,

$$\{\mathscr{B}(E_t) \ominus \mathscr{B}(E_0)\}_{\text{even}}$$
$$= \{f \in \mathscr{B}(E_t)\colon f \text{ is orthogonal to } \mathscr{B}(E_0) \text{ in } L^2(|\, E_t\,|^{-2}) \text{ and } f \text{ is even}\}.$$

Both of these mappings are one-to-one, onto, and, apart from a factor of π, *are norm-preserving,*

(6.2a) $\int_0^t |f(s)|^2\, dQ^{+}(s) = \pi \|\, T_e f\,\|_t^2,$

(6.2b) $\int_0^t |f(s)|^2\, dQ^{-}(s) = \pi \|\, T_o f\,\|_t^2.$

The inverse mappings are given by

(6.3a) $T_e^{-1} : f \in \mathscr{B}(E_t) \to \lim_n \int_{-n}^{n} f(\gamma)\, A_s(\gamma)\, |\, E_t(\gamma)|^{-2}\, d\gamma,$

(6.3b) $T_o^{-1} : f \in \mathscr{B}(E_t) \to \lim_n \int_{-n}^{n} f(\gamma)\, B_s(\gamma)\, |\, E_t(\gamma)|^{-2}\, d\gamma,$

where the limits in question are to be taken in $L^2(dQ^{+})$ *for* (6.3a) *and* $L^2(dQ^{-})$ *for* (6.3b).

Comment 6.1. This theorem should be compared with de Branges in Ref. [7], Theorem 3, and Ref. [9], Theorem 44, pp. 152–156. In the classical case $[dQ^{\pm}(s) = ds, E_t = e^{-i\gamma t}, \mathscr{B}(E_0) = 0]$ Theorem 6.1 reduces to the Paley–Wiener strengthened version of the Plancherel theorem. That is to say, for example,

$$\begin{aligned}\{\mathscr{B}(E_t) \ominus \mathscr{B}(E_0)\}_{\text{even}} &= \{\mathscr{B}(E_t)\}_{\text{even}} \\ &= \{f \in L^2(R^1)\colon f \text{ is an even entire function of exponential type} \leqslant t\}.\end{aligned}$$

Theorem 6.1 then implies that the cosine transform of such a function f, $(T_{\mathrm{e}}^{-1}f)(s)$, vanishes for $s > t$.

Comment 6.2. Formulas (6.1) and (6.3) are perhaps best motivated by recalling Eq. (2.6), which exhibits the reproducing kernel in diagonal form. For upon allowing a formal interchange in the order of integration, (2.6) leads to the following representation formula for $f \in \mathscr{B}(E_t)$:

$$\begin{aligned} f(\beta) &= (f, J_\beta{}^t)_t \\ &= \pi^{-1} \int_0^t A_s(\beta) \left\{ \int_{-\infty}^{\infty} f(\gamma) A_s(\gamma) | E_t(\gamma)|^{-2} \, d\gamma \right\} dQ^+(s) \\ &\quad + \pi^{-1} \int_0^t B_s(\beta) \left\{ \int_{-\infty}^{\infty} f(s) B_s(\gamma) | E_t(\gamma)|^{-2} \, d\gamma \right\} dQ^-(s) + (f, J_\beta{}^0)_t \,. \end{aligned}$$

In the classical case, of course, the last term does not appear, since $J_\beta{}^0 \equiv 0$.

The proof of Theorem 6.1 rests on Lemma 6.1, which is accordingly treated first.

LEMMA 6.1. *Let K and I denote bounded subintervals of* $[0, t]$ *with regular end points. Then*

(6.4a) $\int (\gamma^{-1} B_I)(\gamma^{-1} B_K) \mid E_t \mid^{-2} = \pi Q^+(I \cap K);$

(6.4b) $\int (\gamma^{-1} A_I)(\gamma^{-1} A_K) \mid E_t \mid^{-2} = \pi Q^-(I \cap K);$

where, for example, B_I denotes $B_s - B_r$ if $I = [r, s]$ and $Q^+(I) = \int_I dQ^+$.

Proof. In view of Lemma 5.1 if suffices to show that $(A_s - A_r)/\gamma$ and $(B_s - B_r)/\gamma$ are both orthogonal to $\mathscr{B}(E_p)$ in $L^2(| E_t(\gamma)|^{-2} \, d\gamma)$ whenever $s > r$ are regular points in $[0, t]$ and $p \leqslant r$. We shall work only with

$(A_s - A_r)/\gamma$, as this case displays all the central ideas, and is in fact technically a little harder to deal with than $(B_s - B_r)/\gamma$ since its components A_s/γ and A_r/γ are not analytic.

To get on with the argument, choose $f \in \mathscr{B}(E_p)$. Then,

$$\int [(A_s - A_r)/\gamma] f \,|\, E_t\,|^{-2} = \int [(A_s - A_r)/\gamma] f \,|\, E_s\,|^{-2}$$

by Theorem 5.2. Clearly we can assume that f is odd, which implies that $(f/\gamma) \in \mathscr{M}(E_p)$, and hence that

$$\int A_s(f/\gamma) |\, E_s\,|^{-2} = 0,$$

by Step 1 of Lemma 5.2. Next choose a function $h \in \mathscr{B}(E_r)$ so that $h(0) = 1$. Then $(h - A_r)/\gamma$ belongs to $\mathscr{B}(E_r)$ and so we can invoke Theorem 5.1 to get

$$\begin{aligned} -\int A_r(f/\gamma) |\, E_s\,|^{-2} &= \int [(h - A_r)/\gamma] f \,|\, E_s\,|^{-2} - \int h(f/\gamma) |\, E_s\,|^{-2} \\ &= \int [(h - A_r)/\gamma] f \,|\, E_r\,|^{-2} - \int h(f/\gamma) |\, E_r\,|^{-2} \\ &= -\int A_r(f/\gamma) |\, E_r\,|^{-2} \\ &= 0. \end{aligned}$$

The last line is by Step 1 of Lemma 5.2. This completes the argument.

We turn now to the Proof of Theorem 6.1. The proof comes in three pieces.

Proof of (6.2a). Let χ_I denote the indicator function of a subinterval I of $[0, t]$ with regular end points. Then

$$T_e(\chi_I) = \pi^{-1} \int \chi_I A_s \, dQ^+(s) = (\gamma\pi)^{-1} B_I$$

clearly belongs to $\{\mathscr{B}(E_t) \ominus \mathscr{B}(E_0)\}_{\text{even}}$ and, by (6.4a),

$$\begin{aligned} \| T_e(\chi_I)\|_t^2 &= \pi^{-2} \| \gamma^{-1} B_I \|_t^2 \\ &= \pi^{-1} Q^+(I) \\ &= \pi^{-1} \int_0^t |\, \chi_I(s)|^2 \, dQ^+(s). \end{aligned}$$

In the same way it follows that (6.2a) holds for all finite linear combinations of such indicator functions and hence for all of $\mathscr{L}^+(t)$.

Proof that the mapping (6.1a) *is onto* $\{\mathscr{B}(E_t) \ominus \mathscr{B}(E_0)\}_{\text{even}}$. We show first that $\mathscr{L}^+(t)$ contains (a version of) $A_s(\beta)$ for every choice of the complex number β. This amounts to checking that the increment $A_I(\beta)$ is equivalent to zero in $L^2(dQ^+; [0, t])$ for every subinterval I of $[0, t]$ for which $Q^+(I)Q^-(I) = 0$. But this is clearly the case. For if $Q^-(I) = 0$ then

$$A_I(\beta) = -\beta \int_I B_s(\beta)\, dQ^-(s) = 0,$$

while, on the other hand, if $Q^+(I) = 0$ then

$$\int_I | A_s(\beta)|^2\, dQ^+ = 0.$$

Now as the reproducing kernel for $\{\mathscr{B}(E_t) \ominus \mathscr{B}(E_0)\}_{\text{even}}$ coincides with the transform of $\bar{A}_s(\beta)$,

$$\pi^{-1} \int_0^t \bar{A}_s(\beta) A_s(\gamma)\, dQ^+(s) = (1/2)\{J_\beta{}^t(\gamma) + J_\beta{}^t(-\gamma) - J_\beta{}^0(\gamma) - J_\beta{}^0(-\gamma)\},$$

it follows easily that the mapping is onto. For if there exists a function $g \in \{\mathscr{B}(E_t) \ominus \mathscr{B}(E_0)\}_{\text{even}}$ which is orthogonal to $J_\beta{}^t - J_\beta{}^0$ in $L^2(| E_t |^{-2})$ for every β then

$$g(\beta) = (g, J_\beta{}^t - J_\beta{}^0)_t = 0.$$

Proof of (6.3a). To verify the inversion formula it is enough to check that

$$\int_0^t | \chi_I - \int_{-n}^n (\pi\gamma)^{-1} B_I(\gamma) A_s(\gamma) | E_t(\gamma)|^{-2}\, d\gamma\, |^2\, dQ^+(s) \to 0 \qquad \text{as} \quad n \to \infty$$

for every subinterval I of $[0, t]$ with regular end points. But letting f_n denote $(\pi\gamma)^{-1} B_I(\gamma) \chi_{-nn}$ we find, upon expanding, that this term equals

$$\begin{aligned} Q^+(I) &- 2 \int_I \int_{-n}^n f_n(\gamma) A_s(\gamma) | E_t(\gamma)|^{-2}\, d\gamma\, dQ^+(s) \\ &+ \int_0^t \left\{ \int_{-n}^n f_n(\gamma) A_s(\gamma) | E_t(\gamma)|^{-2}\, d\gamma \right\}^2 dQ^+(s) \\ = Q^+(I) &- 2\pi \| f_n \|_t^2 \\ &+ \int_n^n f_n(\gamma) | E_t(\gamma)|^{-2}\, d\gamma \int_{-n}^n f_n(\beta) \int_0^t A_s(\gamma) A_s(\beta)\, dQ^+(s) | E_t(\beta)|^{-2}\, d\beta, \end{aligned}$$

which tends to zero as $n \to \infty$, as follows from (6.4a) and the fact that the last term is equal to

$$\pi \int_{-\infty}^{\infty} f_n(\gamma) |E_t(\gamma)|^{-2}\, d\gamma \int_{-\infty}^{\infty} f_n(\beta)\{J_\gamma^t(\beta) - J_\gamma^0(\beta)\}| E_t(\beta)|^{-2}\, d\beta$$

$$= \pi \int_{-\infty}^{\infty} f_n(\gamma) \cdot \{\text{projection of } f_n \text{ onto } \mathscr{B}(E_t) \ominus \mathscr{B}(E_0)\}| E_t(\gamma)|^{-2}\, d\gamma$$

$$= \pi \int_{-\infty}^{\infty} |\{\text{projection of } f_n \text{ onto } \mathscr{B}(E_t) \ominus \mathscr{B}(E_0)\}|^2 \,| E_t |^{-2}$$

$$\leqslant \pi \int_{-\infty}^{\infty} | f_n |^2 \,| E_t |^{-2}.$$

The proof that T_0, defined in (6.1b), is a one-to-one mapping onto the indicated range and that (6.2b) and (6.3b) hold may be carried out in much the same way to complete the proof of the theorem.

Comment 6.3. The mapping T_e is clearly well-defined on all of $L^2(dQ^+;\ [0, t])$. It is restricted to the subspace $\mathscr{L}^+(t)$ in order to insure that isometry prevails. The example furnished in Comment 5.1 shows that this is not a moot point. If Q^+ and Q^- are strictly increasing on $[0, t]$ then such refinements are not necessary since the two spaces coincide. Similar remarks hold for the odd transform T_o. It is perhaps of interest to note that $\mathscr{L}^+(t)$ can be characterized as the closure in $L^2(dQ^+;\ [0, t])$ of finite linear combinations of the form $\sum c_j A_s(\beta_j)$.

Our next objective is to extend the mappings T_e and T_o to the interval $[0, \infty)$. We assume that $\tau(\infty) = \infty$ in order to insure the existence of a sequence of regular points $t_1 < t_2 < \cdots$ tending to $+\infty$, and let $\mathscr{L}^\pm(\infty)$ denote the closure in $L^2(dQ^\pm;\ [0, \infty))$ of the increasing family of spaces $\mathscr{L}^\pm(t_n)$. In this situation there exists, by Theorems 5.3 and 5.4, an essentially unique monotone function μ such that the spaces $\mathscr{B}(E_{t_n})$ are contained isometrically in $L^2(d\mu)$ and their union is dense in $L^2(d\mu)$. Moreover, it is clear from (6.2a) and Theorem 5.3 that if $f \in \mathscr{L}^+(\infty)$ then $T_e(\chi_{[0,t_n]}f)$ converges in $L^2(d\mu)$ as $n \to \infty$. It is thus natural to set

$$T_e f = \lim_{n\to\infty} T_e(\chi_{[0,t_n]}f) \qquad \text{for} \quad f \in \mathscr{L}^+(\infty), \tag{6.5a}$$

and correspondingly to set

$$T_o f = \lim_{n\to\infty} T_o(\chi_{[0,t_n]}f) \qquad \text{for} \quad f \in \mathscr{L}^-(\infty). \tag{6.5b}$$

In both cases the limit, which is taken in $L^2(d\mu)$, is clearly independent of the particular sequence of regular points chosen. Combining the conclusions of Theorem 5.4 with those of Theorem 6.1 we then have the following result:

THEOREM 6.2. *If $\tau(\infty) = \infty$ and 0 is a regular point, then the mappings defined in* (6.5),

$$T_e : f \in \mathscr{L}^+(\infty) \quad \textit{onto} \quad \{L^2(d\mu) \ominus \mathscr{B}(E_0)\}_{\text{even}}$$

and

$$T_0 : f \in \mathscr{L}^-(\infty) \quad \textit{onto} \quad \{L^2(d\mu) \ominus \mathscr{B}(E_0)\}_{\text{odd}} .$$

Both these mappings are, apart from a factor of π, norm-preserving and invertible.

For ease of future reference we list the Plancherel formula as a

COROLLARY. *If f and $g \in \mathscr{L}^+(\infty)$, then*

$$\int_0^\infty f(s)\, \bar{g}(s)\, dQ^+(s) = \pi(T_e f, T_e g)_\mu . \tag{6.6a}$$

There is, of course, a corresponding (6.6b) result for odd functions which we do not list.

Comment 6.4. If Q^+ and Q^- are strictly increasing on $[0, \infty)$ then every point $t \geqslant 0$ is a regular point and it is permissible to replace $\mathscr{L}^\pm(\infty)$ by $L^2(dQ^\pm; [0, \infty))$ in Theorem 6.2 and formula (6.6), since these spaces are then the same.

Finally we note that Theorem 6.1 and 6.2 lead readily to a spectral representation formula (of the type noted formally in Comment 6.2) for functions $f \in L^2(d\mu)$.

We record this as

THEOREM 6.3. *If $\tau(\infty) = \infty$ and 0 is a regular point then every function $f \in L^2(d\mu)$ may be written*

$$f(\beta) = (f, J_\beta{}^0)_\mu + \pi^{-1} \int_0^\infty A_s(\beta) \left\{ \int_{-\infty}^\infty f(\gamma) A_s(\gamma)\, d\mu(\gamma) \right\} dQ^+(s)$$

$$+ \pi^{-1} \int_0^\infty B_s(\beta) \left\{ \int_{-\infty}^\infty f(\gamma) B_s(\gamma)\, d\mu(\gamma) \right\} dQ^-(s), \tag{6.7}$$

where, as usual, each integral is to be interpreted in the appropriate L^2 sense.

Proof. If $f \in L^2(d\mu)$ then

$$f - (f, J_\beta{}^0)_\mu \in \{L^2(d\mu) \ominus \mathscr{B}(E_0)\}.$$

Hence, by Theorems 6.1 and 6.2, formula (6.7) must hold if f is replaced by $f - (f, J_\beta{}^0)_\mu$ in the two integrals on the right side. However, $(f, J_\beta{}^0)_\mu \in \mathscr{B}(E_0)$ and we claim, as follows readily by imitating the proof of formula (6.3a), that for any function $g \in \mathscr{B}(E_0)$

$$\int_{-\infty}^{\infty} g(\gamma) A_s(\gamma)\, d\mu(\gamma) = 0$$

in the sense that

$$\int_0^t \left(\int_{-n}^{n} g(\gamma) A_s(\gamma)\, d\mu(\gamma) \right)^2 dQ^+(s) \to 0 \qquad \text{as} \quad n \uparrow \infty.$$

But this says that the term $(f, J_\beta{}^0)_\mu$ does not contribute anything to the first integral. For similar reasons it does not contribute anything to the second integral and (6.7) follows.

7. A Formula for the Spectral Function μ

The objective of this section is to derive explicit formulas for the function μ, discussed in Theorem 5.3, when $\tau(\infty) = \infty$. We shall assume for the sake of simplicity that Q^+ and Q^- are strictly increasing on $[0, \infty)$. This allows us to view A_t and B_t as solutions of the generalized differential equations introduced in Section 1:

(7.1a) $D^+D^-A_t(\beta) = -\beta^2 A_t(\beta)$;

(7.1b) $D^-D^+B_t(\beta) = -\beta^2 B_t(\beta)$.

Much as in the classical case it can be checked that each of these generalized second-order differential equations has two linearly independent solutions, and following Feller, Ref. [15], p. 105, if $u = u(t, \beta)$ is a second solution of (7.1a) then the Wronskian

$$W(\beta) = A_t^-(\beta) u(t, \beta) - A_t(\beta) u^-(t, \beta) \tag{7.2}$$

is independent of t. Next let K_β denote the integral operator which carries the function h having compact support into

$$(K_\beta h)(t) = u(t, \beta) \int_0^t h(s)A_s(\beta)\, dQ^+(s) + A_t(\beta) \int_t^\infty h(s)u(s, \beta)\, dQ^+(s). \quad (7.3)$$

Then since

$$\left[u(t, \beta) \int_t^{t+\delta} h(s)A_s(\beta)\, dQ^+(s) - A_t(\beta) \int_t^{t+\delta} h(s)u(s, \beta)\, dQ^+(s)\right] [Q^-(t + \delta) - Q^-(t)]^{-1}$$

tends to zero as $\delta \to 0$ it follows easily that

$$(\beta^2 + D^+D^-)(K_\beta h)(t) = -W(\beta)h(t) \quad (7.4)$$

at every continuity point of h. This formula is helpful in computing the finite even transform of $K_\beta h$, for we have

$$\beta^2 \int_0^T (K_\beta h)(s)A_s(\gamma)\, dQ^+(s)$$
$$= -\int_0^T [D^+D^-K_\beta h)(s) + W(\beta)h(s)]A_s(\gamma)\, dQ^+(s). \quad (7.5)$$

Integrating by parts gives

$$\int_0^T (D^+D^-K_\beta h)(s)A_s(\gamma)\, dQ^+(s)$$
$$= (K_\beta h)^-(s)A_s(\gamma) \Big|_0^T - \int_0^T (K_\beta h)^-(s)\, dA_s(\gamma)$$
$$= (K_\beta h)^-(s)A_s(\gamma) \Big|_0^T + \gamma \int_0^T (K_\beta h)^-(s)B_s(\gamma)\, dQ^-(s)$$
$$= [(K_\beta h)^-(s)A_s(\gamma) + \gamma(K_\beta h)(s)B_s(\gamma)] \Big|_0^T - \gamma \int_0^T (K_\beta h)(s)\, dB_s(\gamma)$$
$$= [(K_\beta h)^-(s)A_s(\gamma) - (K_\beta h)(s)A_s^-(\gamma)] \Big|_0^T - \gamma^2 \int_0^T (K_\beta h)(s)A_s(\gamma)\, dQ^+(s).$$

Upon putting this into (7.5) we find

$$(\gamma^2 - \beta^2) \int_0^T (K_\beta h)(s)A_s(\gamma)\, dQ^+(s)$$
$$= W(\beta) \int_0^T h(s)A_s(\gamma)\, dQ^+(s) + [(K_\beta h)^-(s)A_s(\gamma) - (K_\beta h)(s)A_s^-(\gamma)] \Big|_0^T. \quad (7.6)$$

Now let J denote the subinterval $[c, d]$ of $[0, T]$ and set

$$h_J(s) = \begin{cases} 1 & \text{if } s \in J, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$(K_\beta h_J)(t) = \begin{cases} A_t(\beta) \int_J u(s, \beta)\, dQ^+(s) & \text{if } t \leqslant c \\ u(t, \beta) \int_c^t A_s(\beta)\, dQ^+(s) + A_t(\beta) \int_t^d u(s, \beta)\, dQ^+(s) & \text{if } t \in J, \\ u(t, \beta) \int_J A_s(\beta)\, dQ^+(s) & \text{if } t \geqslant d, \end{cases}$$

and so, by (7.6),

$$\begin{aligned}(\gamma^2 - \beta^2) &\int_0^T (K_\beta h_J)(s) A_s(\gamma)\, dQ^+(s) \\ &= W(\beta)\gamma^{-1}B_J(\gamma) + [u^-(T, \beta)A_T(\gamma) - u(T, \beta)A_T^-(\gamma)]\beta^{-1}B_J(\beta) \\ &\quad - [A_0^-(\beta)A_0(\gamma) - A_0(\beta)A_0^-(\gamma)] \int_J u(s, \beta)\, dQ^+(s). \qquad (7.7)\end{aligned}$$

We next let I denote a second subinterval of $[0, T]$ and use the Plancherel formula (6.6a) together with the evaluation (7.7) to get

$$\begin{aligned}\int_0^T &(K_\beta h_J)(s) h_I(s)\, dQ^+(s) \\ &= W(\beta) \int \frac{[B_I(\gamma)/\gamma][B_J(\gamma)/\gamma]}{\pi(\gamma^2 - \beta^2)|\, E_T(\gamma)|^2}\, d\gamma \\ &\quad + \beta^{-1}B_J(\beta) \int \frac{u^-(T, \beta)A_T(\gamma) - u(T, \beta)A_T^-(\gamma)}{\pi(\gamma^2 - \beta^2)|\, E_T(\gamma)|^2} [B_I(\gamma)/\gamma]\, d\gamma. \qquad (7.8)\end{aligned}$$

In doing this computation we have taken advantage of the fact that

$$(\gamma^2 - \beta^2)^{-1}[A_0^-(\beta)A_0(\gamma) - A_0(\beta)A_0^-(\gamma)]$$

belongs to $\mathscr{B}(E_0)$ and so is orthogonal to $\gamma^{-1}B_I$ in $L^2(|\, E_T(\gamma)|^{-2}\, d\gamma)$. Further reductions are possible in (7.8), for

$$\begin{aligned}\int \frac{A_T(\gamma)[B_I(\gamma)/\gamma]}{\pi(\gamma^2 - \beta^2)E_T(\gamma)E_T^{\#}(\gamma)}\, d\gamma &= \int \frac{[B_I(\gamma)/\gamma]}{\pi(\gamma - \beta)(\gamma + \beta)E_T(\gamma)}\, d\gamma \\ &= \frac{iB_I(\beta)}{\beta^2 E_T(\beta)}\end{aligned}$$

for $\beta \in R^{2+}$, by Cauchy's formula, and in the same spirit,

$$\begin{aligned}\int \frac{A_T^-(\gamma)[B_I(\gamma)/\gamma]}{\pi(\gamma^2-\beta^2)E_T(\gamma)E_T^{\#}(\gamma)}\,d\gamma &= -\int \frac{B_T(\gamma)B_I(\gamma)}{\pi(\gamma^2-\beta^2)E_T(\gamma)E_T^{\#}(\gamma)}\,d\gamma \\ &= i\int \frac{B_I(\gamma)\,d\gamma}{\pi(\gamma-\beta)(\gamma+\beta)E_T(\gamma)} \\ &= -\frac{B_I(\beta)}{\beta E_T(\beta)} \qquad (\beta \in R^{2+}).\end{aligned}$$

In both of these evaluations inequality (3.2b) is applied to $\gamma^{-1}B_I$ to show that the integral along the contour $Re^{i\theta}$, $0 \leqslant \theta \leqslant \pi$, tends to zero as $R \to \infty$. Putting this information into (7.8) gives

$$\begin{aligned}&\int_I (K_\beta h_J)(s)\,dQ^+(s) \\ &\quad = W(\beta)\int \frac{[B_I(\gamma)/\gamma][B_J(\gamma)/\gamma]}{\pi(\gamma^2-\beta^2)|\,E_T(\gamma)|^2}\,d\gamma + \beta^{-1}B_J(\beta)\beta^{-2}B_I(\beta)\times \\ &\qquad \times [iu^-(T,\beta)+\beta u(T,\beta)]/E_T(\beta)\end{aligned} \tag{7.9}$$

for $\beta \in R^{2+}$. Dividing both sides by $Q^+(I)Q^+(J)$ and letting I contract to s and J contract to t, with $s \leqslant t$, it is then a routine matter to conclude that

$$\begin{aligned}&A_s(\beta)u(t,\beta) \\ &\quad = W(\beta)\int \frac{A_s(\gamma)A_t(\gamma)}{\pi(\gamma^2-\beta^2)|\,E_T(\gamma)|^2}\,d\gamma + \frac{A_s(\beta)A_t(\beta)}{\beta E_T(\beta)}[iu^-(T,\beta)+\beta u(T,\beta)]\end{aligned} \tag{7.10}$$

for $\beta \in R^{2+}$. We have thus proved

THEOREM 7.1. *Let Q^+ and Q^- be strictly increasing on* $[0, T]$ *and let u denote any solution of* (7.1a). *Then for every choice of $s \leqslant t$ in the interval* $[0, T]$ *and every choice of $\beta \in R^{2+}$ formulas* (7.9) *and* (7.10) *are valid.*

In (7.9) I and J are subintervals of $[0, T]$ about the points s and t, respectively. We are now ready to tackle

THEOREM 7.2. *Let Q^+ and Q^- be strictly increasing on* $[0, \infty)$ *and suppose $\tau(\infty) = \infty$. Then there exists a solution $u = u(t, \beta)$ of* (7.1a) *which is linearly independent of A_t, and analytic in R^{2+}, and such that the formula*

$$A_s(\beta)u(t,\beta) = W(\beta)\int \frac{A_s(\gamma)A_t(\gamma)}{\pi(\gamma^2-\beta^2)}\,d\mu(\gamma) \tag{7.11}$$

holds for every $\beta \in R^{2+}$ *and every choice of* $s \leqslant t$ *in* $[0, \infty)$. *This function* u *is unique up to a multiplicative factor which depends only upon* β. *It may be characterized as the only solution of* (7.1a) (*apart from this multiplicative factor*) *which meets the constraint*

$$|u(t, ib)| + |u^{-}(t, ib)| \leqslant \text{a finite constant independent of } t,$$

for $b > 0$. (7.12)

Proof. Let

$$C_t(\beta) = A_t(\beta) \int_0^t [A_s(\beta)]^{-2} \, dQ^{-}(s). \tag{7.13}$$

Then C_t is a second solution of (7.1a) which is linearly independent of A_t. In fact the Wronskian

$$A_t^{-} C_t - A_t C_t^{-} = -1. \tag{7.14}$$

The rest of the proof proceeds in steps.

STEP 1. *Let* $u = u(T, \beta)$ *be any solution of* (7.1a); *then the limit*

$$U(\beta) = \lim_{T \to \infty} \frac{iu^{-}(T, \beta) + \beta u(T, \beta)}{\beta E_T(\beta)} \tag{7.15}$$

exists, for $\beta \in R^{2+}$, *and*

$$A_s(\beta) u(t, \beta) = W(\beta) \int \frac{A_s(\gamma) A_t(\gamma)}{\pi(\gamma^2 - \beta^2)} \, d\mu(\gamma) + A_s(\beta) A_t(\beta) U(\beta) \tag{7.16}$$

for every $0 \leqslant s \leqslant t$ *and every* $\beta \in R^{2+}$.

Proof of Step 1. From the information furnished in the proof of Theorem 5.3 it is clear that every sequence of real numbers tending to $+\infty$ possesses a subsequence $T_1 < T_2 < \cdots$ tending to $+\infty$ such that the monotone functions (of λ)

$$\int_{-\infty}^{\lambda} (1 + \gamma^2)^{-1} |E_{T_n}(\gamma)|^{-2} \, d\gamma \to \int_{-\infty}^{\lambda} (1 + \gamma^2)^{-1} \, d\mu(\gamma)$$

weakly, and

$$\int \frac{[B_I(\gamma)/\gamma][B_J(\gamma)/\gamma]}{\pi(\gamma^2 - \beta^2) |E_{T_n}(\gamma)|^2} \, d\gamma \to \int \frac{[B_I(\gamma)/\gamma][B_J(\gamma)/\gamma]}{\pi(\gamma^2 - \beta^2)} \, d\mu(\gamma)$$

as $n \to \infty$. This, together with (7.9), guarantees the existence of the limit $U(\beta)$ and shows that

$$\int_I (K_\beta h_J)(s)\, dQ^+(s) = W(\beta) \int \frac{[B_I(\gamma)/\gamma][B_J(\gamma)/\gamma]}{\pi(\gamma^2 - \beta^2)}\, d\mu(\gamma) + \beta^{-1}B_I(\beta)\beta^{-1}B_J(\beta)U(\beta).$$

To get (7.16) one has simply to divide through by $Q^+(I)Q^+(J)$ and let I contract to the point s and J contract to the point t, $s \leqslant t$.

STEP 2. *The integral* $\int_t^\infty [A_s(\beta)]^{-2}\, dQ^-(s)$ *converges for every choice of* $\beta \in R^{2+}$ *and every* $t \geqslant 0$.

Proof of Step 2. Put $u = C_t(\beta)$ in formula (7.15) to conclude that

$$U(\beta) = \lim_{T\to\infty} \left\{ \int_0^T [A_s(\beta)]^{-2}\, dQ^-(s) - [iA_T(\beta)\beta E_T(\beta)]^{-1} \right\} \text{ exists.}$$

The convergence of the integral follows from the fact that the last term on the right tends to zero as $T \to \infty$ when $\beta \in R^{2+}$, since

$$| A_T(\beta)E_T(\beta)| \geqslant \operatorname{Re}\{A_T(\beta)\bar{E}_T(\beta)\} = | A_T(\beta)|^2 + \pi b J_\beta{}^T(\beta),$$

which tends to $+\infty$, by Lemma 2.2.

STEP 3. *There exists a solution* u *of* (7.1a) *for which formula* (7.11) *holds.*

Proof of Step 3. Let

$$u(t, \beta) = A_t(\beta) \int_t^\infty [A_s(\beta)]^{-2}\, dQ^-(s). \tag{7.17}$$

The integral converges for $\beta \in R^{2+}$ by Step 2, and u is a solution of (7.1a) which is linearly independent of A_t. In fact the Wronskian

$$A_t^-(\beta)u(t, \beta) - A_t(\beta)u^-(t, \beta) = 1. \tag{7.18}$$

Now put this solution into formula (7.15). It follows readily that $U \equiv 0$ on R^{2+} and so (7.16) reduces to (7.11).

STEP 4. *If t and b are both positive then*

$$\int_t^\infty [A_s(ib)]^{-2}\, dQ^-(s) \leqslant \{\pi b^2 J_{ib}^t(ib)\}^{-1} \tag{7.19}$$

Proof of Step 4. Note first with the help of (7.14) that

$$D^+[C_t^-(ib)/A_t^-(ib)] = -[iB_t(ib)]^{-2} < 0.$$

This shows that

$$\frac{C_t^-(ib)}{A_t^-(ib)} = \int_0^t [A_s(ib)]^{-2}\, dQ^-(s) + \{\pi b^2 J_{ib}^t(ib)\}^{-1}$$

is a decreasing function of t and hence that

$$\int_t^T [A_s(ib)]^{-2}\, dQ^-(s) \leqslant \frac{1}{\pi b^2}\left\{\frac{1}{J_{ib}^t(ib)} - \frac{1}{J_{ib}^T(ib)}\right\}$$

when $t < T$. The desired inequality (7.19) now follows from Lemma 2.2 by letting $T \to \infty$.

STEP 5. *Let* $b > 0$ *be fixed. Then the solution* $u(t, ib)$ *given in* (7.17) *meets the constraint* (7.12).

Proof of Step 5. Suppose for the sake of definiteness that $A_0(ib) > 0$. Because

$$\pi b^2 J_{ib}^t(ib) = A_t^-(ib) A_t(ib) = -ibB_t(ib)A_t(ib) \geqslant 0$$

this means that $A_0^-(ib) \geqslant 0$, and, as follows easily from the integral equations (1.1), both $A_t(ib)$ and $A_t^-(ib)$ are positive increasing functions of t. [Keep in mind here that $D^+A_t^-(ib) = b^2 A_t(ib) > 0$.] Now $u(t, ib)$ is clearly positive and

$$\begin{aligned} u^-(t, ib) &= A_t^-(ib) \int_t^\infty [A_s(ib)]^{-2}\, dQ^-(s) - [A_t(ib)]^{-1} \\ &= \left\{\pi b^2 J_{ib}^t(ib) \int_t^\infty [A_s(ib)]^{-2}\, dQ^-(s) - 1\right\} [A_t(ib)]^{-1}, \end{aligned}$$

which is less than zero by Step 4. Thus as $D^+u^-(t, ib) = b^2 u(t, ib) > 0$ it follows that $u^-(t, ib)$ is a negative increasing function of t while $u(t, ib)$ is a positive decreasing function of t. This shows that u and u^- are bounded as asserted, and clearly this holds true even if $A_0(ib) < 0$.

STEP 6. *The function* u *defined in* (7.17) *is* (*apart from a multiplicative* *factor which is independent of t) the only solution of* (7.1a) *which meets the constraint* (7.12).

Proof of Step 6. Since u is linearly independent of A_t the most general solution v of (7.1a) can be written in the form

$$v(t, ib) = k_1 A_t(ib) + k_2 u(t, ib),$$

where k_1 and k_2 are constants which depend only on b. But now as

$$\pi b^2 J^t_{ib}(ib) = A_t^-(ib) A_t(ib) \to \infty$$

as $t \to \infty$, when b is positive it is clear that v satisfies (7.12) if and only if $k_1 = 0$.

An application of the ideas underlying the Stieltjes inversion formula to (7.11) leads next to

THEOREM 7.3. *Let u be a solution of* (7.1a) *which meets the constraint* (7.12). *Let*

$$R_t(\beta) = A_t(\beta) u(t, \beta)/W(\beta), \tag{7.20}$$

where $W(\beta)$ is the Wronskian defined in (7.2), *and let $d > c$ be continuity points of μ. Then*

$$\int_c^d A_s^2(\gamma)\, d\mu(\gamma) = \lim_{b \downarrow 0} \int_c^d \mathrm{Im}[(a + ib) R_s(a + ib)]\, da \tag{7.21}$$

for every $s \geqslant 0$. If $\beta R_s(\beta)$ is continuous in a neighborhood of the point $c \in R^1$ then μ is differentiable at c with derivative

$$\Delta(c) = \frac{c[u^-(t, c)\overline{u(t, c)} - u(t, c)\overline{u^-(t, c)}]}{2i |W(c)|^2}. \tag{7.22}$$

Proof. Fix $s = t$ in (7.11) and set $d\nu = A_s^2(\gamma)\, d\mu$ *to get*

$$\pi R_s(\beta) = \int (\gamma^2 - \beta^2)^{-1}\, d\nu(\gamma).$$

Since $\int (\gamma^2 + 1)^{-1}\, d\nu(\gamma) < \infty$ it follows that $\nu(\gamma) = o(\gamma^2)$ as $\gamma^2 \uparrow \infty$, (Widder, Ref. [25], p. 330) and hence, an integration by parts leads to

$$\pi R_s(\beta) = -\int \nu(\gamma) \frac{d}{d\gamma} \left\{ \frac{1}{\gamma^2 - \beta^2} \right\} d\gamma.$$

Thus

$$\pi 2\beta R_s(\beta) = -\int \nu(\gamma) \frac{d}{d\gamma} \left\{ \frac{1}{\gamma - \beta} - \frac{1}{\gamma + \beta} \right\} d\gamma$$
$$= \int \nu(\gamma) \frac{d}{d\beta} \left\{ \frac{1}{\gamma - \beta} + \frac{1}{\gamma + \beta} \right\} d\gamma,$$

which implies that

$$\pi \int_c^d 2(a + ib)R_s(a + ib)\, da = \int \nu(\gamma) \left\{ \left(\frac{1}{\gamma - a - ib} + \frac{1}{\gamma + a + ib} \right) \Big|_c^d \right\} d\gamma$$

and consequently that

$$\pi \int_c^d \{2(a + ib)R_s(a + ib) - 2(a - ib)\bar{R}_s(a + ib)\}\, da$$
$$= \int \nu(\gamma) \left\{ \left(\frac{2ib}{(\gamma - a)^2 + b^2} - \frac{2ib}{(\gamma + a)^2 + b^2} \right) \Big|_c^d \right\} d\gamma.$$

Hence, selecting $d > c$ to be continuity points of the function ν we find, upon letting $b \downarrow 0$, that,

$$\lim_{b \downarrow 0} \pi \int_c^d \operatorname{Im}[(a + ib)R_s(a + ib)]\, da = (\pi/2)[\nu(d) - \nu(c) - \nu(-d) + \nu(-c)]$$
$$= \pi[\nu(d) - \nu(c)].$$

This proves (7.21).

If $\beta R_s(\beta)$ is continuous in a neighborhood of the interval $[c, d]$ then this reduces to

$$[\nu(d) - \nu(c)] = \int_c^d \operatorname{Im}[aR_s(a)]\, da.$$

Thus ν, and so too μ, is seen to be differentiable at c. In fact dividing both sides by $d - c$ and letting $d \downarrow c$ we find

$$c[R_s(c) - \bar{R}_s(c)]/(2i) = A_s^2(c)\, \Delta(c),$$

where $\Delta(c)$ denotes the derivative of μ. The desired formula (7.22) now follows easily upon using (7.20) to eliminate $R_s(c)$.

Comment 7.1. Both the numerator of $\Delta(c)$ and $W(c)$, which appears in the denominator, are Wronskians and as such are independent of t.

As will be discussed in the next section, it is often advantageous in examples to use asymptotic formulas for $u(t, \beta)$ as $t \uparrow \infty$ to evaluate the numerator of Δ, and to evaluate the denominator, or more precisely $W(c)$, by letting $t \downarrow 0$.

8. Examples

In this section we present a few elementary examples to illustrate some of the ideas developed to this point. Particular attention is paid to the computation of the spectral function μ.

In all the examples presented Q^+ and Q^- are at least C^2 functions on $[0, \infty)$ with strictly positive derivatives $r = p$ and $1/p$, respectively. This means that Q^+ and Q^- are strictly increasing, $\tau(t) = t$, and $\tau(\infty) = \infty$. As p itself is assumed to be differentiable (7.1a) can be written in the form

$$y'' + (p'/p)\, y' + \gamma^2 y = 0. \tag{8.1}$$

We reserve the symbol u for that solution of (7.1a) which meets the constraints (7.12). Such a solution exists by Theorem 7.2, and enters into the formulas for the spectral function μ supplied in Theorem 7.3.

Comment 8.1. If p is twice differentiable it is easy to check that

$$\eta = (p(s))^{1/2} u(s, \gamma) \qquad \text{and} \qquad \varphi = (p(s))^{1/2} A_s(\gamma)$$

are solutions of the differential equation

$$y'' + (\gamma^2 - V)\, y, \tag{8.2}$$

where (the so-called potential)

$$V = p''/(2p) - \{p'/(2p)\}^2,$$

and (7.22) may be rewritten as

$$\Delta(\gamma) = \frac{\gamma[\eta'(s, \gamma)\bar{\eta}(s, \gamma) - \eta(s, \gamma)\bar{\eta}'(s, \gamma)]}{2i\,|\, \varphi(s, \gamma)\eta'(s, \gamma) - \varphi'(s, \gamma)\eta(s, \gamma)|^2}\,.$$

The differential equation (8.2) has been studied by Levinson in Ref. [18] under various summability constraints on V. (Do not overlook the addendum in Ref. [18], pp. 27–29). When these conditions are met Levinson's results imply that there exist functions, f, Γ, and Φ of $\gamma \in R^1$ only, such that

$$\eta(s, \gamma) - e^{is\gamma} f(\gamma) \to 0,$$

$$\eta'(s, \gamma) - i\gamma e^{is\gamma} f(\gamma) \to 0,$$

$$\varphi(s, \gamma) - \Gamma(\gamma) \cos(\gamma s - \Phi(\gamma)) \to 0,$$

$$\varphi'(s, \gamma) + \Gamma(\gamma)\gamma \sin(\gamma s - \Phi(\gamma)) \to 0$$

as $s \uparrow \infty$, and in addition that the differential equation (8.1) subject to suitable initial conditions has a unique differentiable spectral function whose derivative is equal to $|\Gamma(\gamma)|^{-2}$. Note that our formula for Δ reduces to this when η and φ are replaced by the asymptotes recorded above. In other words, it is a general fact, for a large class of potentials V, that μ is differentiable with density Δ and that

$$(p(s))^{1/2} A_s(\gamma) \sim \Delta(\gamma)^{-1/2} \cos(\gamma s - \Phi(\gamma)), \qquad \gamma \in R^1,$$

as $s \uparrow \infty$. A number of the examples considered below exhibit this kind of behavior. It is perhaps also worth noting that conceptually (although the normalizations are not quite the same) the function u (or η) has much in common with the Jost solution used by physicists in scattering theory (De Alfaro and Regge, Ref. [1], Chapter 4).

EXAMPLE 1. $p(s) = 1$.

Let $u(t, \beta) = e^{i\beta t}$. Clearly u meets the constraint (7.12), and, by (7.2), $W(\beta) = -e^{i\beta t} = -i\beta E_0$. Thus

$$\beta R_t(\beta) = \frac{-A_t(\beta)}{iE_t(\beta)}$$

is continuous (in fact analytic) on the closed upper half-plane. It thus follows from Theorem 7.3 that μ is differentiable with density (given by (7.22))

$$\Delta(\gamma) = |E_t(\gamma)|^{-2} = |E_0(\gamma)|^{-2} \qquad (\gamma \in R^1).$$

Furthermore, as is readily checked, we may write

$$\begin{aligned} A_t(\gamma) &= A_0(\gamma)\cos\gamma t + B_0(\gamma)\sin\gamma t \\ &= |\,E_0(\gamma)|\cos\{\gamma t - \Phi(\gamma)\} \\ &= \Delta(\gamma)^{-1/2}\cos\{\gamma t - \Phi(\gamma)\}, \qquad \gamma \in R^1, \end{aligned}$$

where $\cos\Phi = A_0/|\,E_0\,|$ and $\sin\Phi = B_0/|\,E_0\,|$. This is in accord with Comment 8.1.

A particular case of interest occurs if we choose $A_0 = 1$ and $B_0 = \gamma$. Then, as the initial conditions (1.5) are met, the preceding results are applicable and we find

$$\begin{aligned} \Delta(\gamma) &= (1+\gamma^2)^{-1}, \\ A_t(\gamma) &= \cos\gamma t - \gamma\sin\gamma t, \\ B_t(\gamma) &= \sin\gamma t + \gamma\cos\gamma t, \qquad t \geqslant 0. \end{aligned}$$

In terms of these functions we may, by virtue of Theorems 6.1 and 6.2, write

$$\begin{aligned} f(t) = \pi^{-1}\int_{-\infty}^{\infty}\left(\int_0^{\infty} f(s)A_s(\gamma)\,ds\right) A_t(\gamma)(1+\gamma^2)^{-1}\,d\gamma \\ + \pi^{-1}\int_{-\infty}^{\infty}\left(\int_0^{\infty} f(s)B_s(\gamma)\,ds\right) B_t(\gamma)(1+\gamma^2)^{-1}\,d\gamma \end{aligned}$$

for every function $f \in L^2(ds;\,[0,\infty))$ with the customary interpretation of the integrals in question. On the other hand, if we start with $g \in L^2((1+\gamma^2)^{-1}\,d\gamma;\,R^1)$ then by (6.7) we have

$$\begin{aligned} g(\beta) = \int_{-\infty}^{\infty} g(\gamma)(1/\pi)(1+\gamma^2)^{-1}\,d\gamma \\ + \pi^{-1}\int_0^{\infty}\int_{-\infty}^{\infty} g(\gamma)A_s(\gamma)(1+\gamma^2)^{-1}\,d\gamma A_s(\beta)\,ds \\ + \pi^{-1}\int_0^{\infty}\int_{-\infty}^{\infty} g(\gamma)B_s(\gamma)(1+\gamma^2)^{-1}\,d\gamma B_s(\beta)\,ds, \end{aligned}$$

since $J_\beta{}^0 = 1/\pi$. The last equality reflects the fact that the collection of entire functions of minimal exponential type which sit in $L^2((1+\gamma^2)^{-1}\,d\gamma;\,R^1)$, that is to say $\mathscr{B}(E_0)$, consists solely of constants.

Comment 8.2. The reader may find it of interest to compare this last formula with the representation formula derived by Cohen in Ref. [11], (Eq. (17)) for the differential equation

(a) $y'' + \gamma^2 y = 0$

subject to the boundary conditions

(b) $y'(0, \gamma) - \gamma^2 y(0, \gamma) = 0$

and

(c) $|y(s, y)| \to 0$ as $s \to \infty$.

In making the comparison, it should be noted that A_s is a solution of (a) which satisfies the boundary condition $A_0'(\gamma) + \gamma^2 A_0(\gamma) = 0$.

Another particular case along the lines of the first (which we add partially for sentimental reasons as it appeared also in Ref. [12]) occurs if we choose $A_0 = 1 - 3\gamma^2$ and $B_0 = 3\gamma - \gamma^3$. Then E_0 satisfies the requisite initial conditions (1.5) and, as is readily checked,

$$\Delta(\gamma) = (1 + \gamma^2)^{-3},$$

$$A_t(\gamma) = (1 - 3\gamma^2) \cos \gamma t - (3\gamma - \gamma^3) \sin \gamma t,$$

$$B_t(\gamma) = (3\gamma - \gamma^3) \cos \gamma t + (1 - 3\gamma^2) \sin \gamma t.$$

Furthermore, an easy manipulation shows that if $g \in L^2((1 + \gamma^2)^{-3}\, d\gamma; R^1)$ then

$$\int g(\gamma) J_\beta^0(\gamma)(1 + \gamma^2)^{-3}\, d\gamma = \sum_{j=0}^{2} \varphi_j(\beta) \int \varphi_j(\gamma) g(\gamma)(1 + \gamma^2)^{-3}\, d\gamma,$$

where the φ_j are the orthonormal polynomials of degree j which belong to $L^2((1 + \gamma^2)^{-3}\, d\gamma; R^1)$,

$$\varphi_0 = \{8/(9\pi)\}^{1/2}, \qquad \varphi_1 = (8/\pi)^{1/2}\gamma, \qquad \varphi_2 = \{1/(3\pi)\}^{1/2}(1 - 3\gamma^2)\}.$$

That is to say, in this case $\mathscr{B}(E_0)$ is the span of the polynomials φ_0, φ_1, and φ_2. The remaining details in the representation of g follow simply from (6.7) and are left to the reader.

EXAMPLE 2. $p(s) = (s + m)^{2\nu+1}$, $m > 0$, ν real.

For this choice of p the differential equation (8.1) reduces to a modified form of Bessel's equation. The solutions are summarized in convenient form in Hildebrand, Ref. [16], Section 4.10. For our "u solution" we take

$$u(s, \beta) = (s + m)^{-\nu} H_\nu^{(1)}(\beta(s + m)),$$

where $H_\nu^{(1)}$ denotes the Hankel function of the first kind of order ν. Then as

$$u'(s, \beta) = -\beta(s + m)^{-\nu} H_{\nu+1}^{(1)}(\beta(s + m))$$

for every real choice of ν, and

$$H_\nu^{(1)}(\beta(s + m)) = \sqrt{\frac{2}{\pi\beta(s + m)}}\, e^{i[\beta(s+m)-\nu\pi/2-\pi/2]}\,[1 + O(|\,\beta(s + m)|\ \ )] \quad (8.3)$$

for every real choice of ν and every $\beta \in R^{2+}$ as $s \uparrow \infty$ (Erdelyi, Ref. [14], 7.13(1)), it follows readily that u meets the constraint (7.12). We take u to be analytic in the β plane cut along the negative real axis. Since u is linearly independent of the even function $A_s(\beta)$ for every choice of $\beta > 0$ it follows that $W(\beta)$ cannot vanish for $\beta > 0$ and hence that $\beta R_s(\beta) = \beta A_s(\beta) u(s, \beta)/W(\beta)$ is analytic and μ is differentiable when $\beta > 0$. We now (replacing β by γ) check the formula for $\Delta(\gamma)$ keeping $\gamma > 0$. Following the strategy elucidated in Comment 7.1 we first evaluate the numerator by letting $s \uparrow \infty$ and then invoking (8.3). Doing so we find that

$$\gamma p(t)[u'(t, \gamma)\bar{u}(t, \gamma) - u(t, \gamma)\bar{u}'(t, \gamma)] = \gamma 4i/\pi \qquad \text{if} \quad \gamma > 0.$$

That is to say,

$$\Delta(\gamma) = (2\gamma/\pi)|\,W(\gamma)|^{-2} \qquad \text{if} \quad \gamma > 0.$$

Since $\Delta(\gamma)$ is an even function this essentially completes the computation. It remains but to check that the spectral function μ has no jump at the origin. This will be the case if $\nu \geqslant -1/2$, for then

$$\beta R_s(\beta) = \frac{(H_\nu^{(1)}/H_{\nu+1}^{(1)})A_s}{(s + m)^{2\nu+1}A_s - (H_\nu^{(1)}/H_{\nu+1}^{(1)})B_s}$$

stays bounded as $\beta = a + ib$, $b > 0$, tends towards 0 since the ratio of the Hankel functions $H_\nu^{(1)}/H_{\nu+1}^{(1)}$, which are evaluated at $\beta(s + m)$, stays bounded (for $\nu \geqslant -1/2$) and $A_s(0) = 1$ and $B_s(0) = 0$. If

$\nu = l + 1/2$, where l is a nonnegative integer, the formula for Δ can be written more explicitly as

$$\Delta(\gamma) = \{2/(\pi\gamma)\}|\, m^{\nu+1}H_{\nu+1}^{(1)}(m\gamma)A_0(\gamma) - m^{-\nu}H_\nu^{(1)}(m\gamma)B_0(\gamma)|^{-2}$$

$$= \gamma^{2l+2}\left|\, 2^{-(l+1)} \sum_{k=0}^{l+1} (2i\gamma m)^{l+1-k} \frac{(l+1+k)!}{k!\,(l+1-k)!} A_0(\gamma)\right.$$

$$\left. - 2^{-l}m^{-(2l+1)} \sum_{k=0}^{l} (2i\gamma m)^{l-k} \frac{(l+k)!}{k!\,(l-k)!}\gamma B_0(\gamma)\,\right|^{-2}.$$

In particular, if $m = 2$, $l = 0$, $A_0 = 1$, $B_0 = 2\gamma$ (a choice motivated by an example in Ref. [12]), then

$$\Delta(\gamma) = \gamma^2(1 + \gamma^2)^{-2},$$

$$A_s(\gamma) = (s + 2)^{-1}[2 \cos \gamma s - \gamma \sin \gamma s + \gamma^{-1} \sin \gamma s],$$

$$B_s(\gamma) = (2s + 3) \sin \gamma s + (s + 2)\, \gamma \cos \gamma s - \gamma^{-1}s \cos \gamma s + \gamma^{-2} \sin \gamma s,$$

$$\mathscr{B}(E_0) = \text{the constants} \qquad (J_\beta{}^0 = 2/\pi).$$

For this general class of examples it is not hard to check that if we write

$$A_s(\gamma) = (s + m)^{-\nu}\{c(\gamma)J_\nu(\gamma(s + m)) + d(\gamma)Y_\nu(\gamma(s + m))\},$$

then

$$(p(s))^{1/2}A_s(\gamma) \sim \{2[c^2(\gamma) + d^2(\gamma)]/(\pi\gamma)\}^{1/2} \cos[\gamma(s + m) - \Phi(\gamma)]$$
$$= \Delta(\gamma)^{-1/2} \cos[\gamma(s + m) - \Phi(\gamma)] \qquad (\gamma > 0, s \to \infty),$$

in accord with Comment 8.1.

Notice that if $\nu < -1$ then the constant function belongs to $L^2(dQ^+; [0, \infty))$ and is potentially admissible as an eigenfunction of the differential operator corresponding to the eigenvalue 0. In other words, the spectral function μ may have a jump at the origin. For example, if $p(s) = (s+1)^{-2}$, so that $\nu = -3/2$, and the initial conditions $A_0 = 1$ and $B_0 = 0$ are imposed then

$$\beta R_0(\beta) = i(1 + i/\beta)$$

and correspondingly

$$\begin{aligned}\mu(d) - \mu(c) &= d - c &&\text{for}\quad d > c > 0,\\ &= d - c + \pi &&\text{for}\quad d > 0 > c.\end{aligned}$$

Comment 8.3. Notice, in this last example ($\nu = -3/2$), that the jump in the spectral function has no influence on the odd transform.

EXAMPLE 3.

$$p(s) = \frac{\{(s+m)^m - m\}^2}{m^2(s+m)^{m-1}}, \qquad m = 3, 5, \ldots .$$

For this choice of p the transformation discussed in Comment 8.1 seems to carry the differential equation (8.1) into its most tractable form,

$$y'' + (\gamma^2 - V)y = 0,$$

where the "potential term"

$$V(s) = (m^2 - 1)/[4(s+m)^2].$$

This is a modified version of Bessel's equation which may be solved as in Example 2, by following the prescription furnished in Hildebrand, Ref. [16], Section 4.10. For our "u solution" of (8.1) we take

$$u(s, \beta) = [(s+m)/p(s)]^{1/2} H^{(1)}_{m/2}(\beta(s+m)).$$

It again follows readily from (8.3) that u meets the constraint (7.12). In addition, taking u to be analytic in the β plane, cut along the negative real axis, and noting that it is independent of the even function A_s for every choice of $\beta > 0$ it follows that $W(\beta)$ cannot vanish for $\beta > 0$ and hence that

$$\beta R_s(\beta) = A_s(\beta) u(s, \beta)\beta / W(\beta)$$

is continuous (in fact analytic) there. Indeed, as only "spherical" Hankel functions are involved, a more detailed analysis shows that $\beta R_s(\beta)$ is analytic for $\beta \geqslant 0$. It follows that μ is differentiable with even density Δ over R^1. It thus suffices to check the formula for Δ on the half-line $(0, \infty)$. We treat the numerator first, writing it in terms of Hankel functions and then letting $s \uparrow \infty$ in order to invoke (8.3). This shows that

$$p(s)[u'(s,\gamma)\bar{u}(s,\gamma) - u(s,\gamma)\bar{u}'(s,\gamma)] = 4i/\pi,$$

and hence that

$$\Delta(\gamma) = (2\gamma/\pi)|\,W(\gamma)|^{-2} \qquad \text{if} \quad \gamma > 0,$$

where

$$\begin{aligned} -W(\gamma) = {} & m^{m/2}[(1 - m^{1-m})\gamma H_{m/2-1}(m\gamma) - H_{m/2}(m\gamma)]A_0(\gamma) \\ & + m^{m/2}(m^{m-1} - 1)^{-1} H_{m/2}(m\gamma)\gamma B_0(\gamma). \end{aligned}$$

Again, writing

$$A_s(\gamma) = \{(s+m)/p(s)\}^{1/2}\{c(\gamma)J_{m/2}(\gamma(s+m)) + d(\gamma)Y_{m/2}(\gamma(s+m))\}$$

as in example 2, it follows readily that

$$\begin{aligned} (p(s))^{1/2}A_s(\gamma) & \sim \{2[c^2(\gamma) + d^2(\gamma)]/(\pi\gamma)\}^{1/2}\cos[\gamma(s+m) - \Phi(\gamma)] \\ & = \Delta(\gamma)^{-1/2}\cos[\gamma(s+m) - \Phi(\gamma)], \qquad (\gamma \geqslant 0, s \uparrow \infty). \end{aligned}$$

This is in accord with Comment 8.1.

For a more concrete example set $m = 3$, and choose initial conditions $A_0 = 1$ and $B_0 = 8\gamma/3$. It is then easily checked that (1.5) is satisfied, that $\Delta(\gamma) = \gamma^4(1 + \gamma^2)^{-3}$, and that $\mathscr{B}(E_0)$ consists of constants. The functions A_s and B_s are somewhat cumbersome combinations of $\cos \gamma s$ and $\sin \gamma s$ and so are not recorded. The interested reader will find them in Dym and McKean [12].

Example 4. $p(s) = e^{-s^2}$

This example is adapted from Titchmarsh, Ref. [24], Section 4.2. For the indicated choice of p the function

$$\begin{aligned} u(s,\beta) = {} & \int_1^\infty e^{-[1/4\rho^2 + s\rho + (1/2)(\beta^2+2)\log\rho]}\, d\rho\{e^{-i\pi\beta^2} - 1\} \\ & + \int_0^{2\pi} e^{-[(1/4)e^{2i\theta} + se^{i\theta} + (1/2)(\beta^2+2)i\theta]}\, ie^{i\theta}\, d\theta \end{aligned}$$

is a solution of (8.1) which meets the constraint (7.12). A second solution of (8.1) is given by $u(s, -\beta)$. It is linearly independent of $u(s,\beta)$ providing that $\beta^2/2$ is not an integer. Imposing the initial conditions $A_0 = 1$ and $B_0 = 0$ we therefore find that

$$A_s(\beta) = \frac{u(s,\beta) + u(-s,\beta)}{2u(0,\beta)}$$

while

$$\beta R_0(\beta) = \frac{-\beta u(0,\beta)}{u'(0,\beta)}$$
$$= \frac{\beta\Gamma(-\beta^2/4)}{2\Gamma(1/2-\beta^2/4)}\,.$$

Thus $\beta R_0(\beta)$ is seen to have simple poles at the points

$$\beta_0 = 0,\quad \beta_n = 2\sqrt{n},\quad n = 1, 2,\ldots,\quad \text{and}\quad \beta_{-n} = -\beta_n\,,\quad n = 1, 2,\ldots,$$

with real residues

$$r_0 = -2(\pi)^{-1/2} \qquad \text{at}\quad \beta_0 \quad \text{and}$$

$$r_n = -\frac{(\pi)^{-1/2}(2n)!}{(n!)^2 4^n} \qquad \text{at}\quad \beta_n \quad\text{and}\quad \beta_{-n}\,,\quad n = 1, 2,\ldots\,.$$

Correspondingly, as follows from (7.21), the spectral function μ is discrete with jumps of height

$$\mu_0 = -\pi r_0 \qquad \text{at}\quad \beta_0 \quad \text{and}$$

$$\frac{\mu_n}{2} = -\pi r_n \qquad \text{at}\quad \beta_n \quad\text{and}\quad \beta_{-n}\,,\quad n = 1, 2,\ldots\,.$$

The Plancherel formula (6.6a)

$$\int_0^\infty |f(s)|^2 e^{-s^2}\,ds = \pi \int_{-\infty}^\infty \left| \int_0^\infty f(s) A_s(\gamma)\, e^{-s^2} \frac{ds}{\pi} \right|^2 d\mu(\gamma)$$

thus reduces to the formula

$$\int_0^\infty |f(s)|^2 e^{-s^2}\,ds = \pi \sum_{n=0}^\infty \left| \int_0^\infty f(s) A_s(\beta_n)\, e^{-s^2} \frac{ds}{\pi} \right|^2 \mu_n\,.$$

But

$$u(s, 2\sqrt{n}) = \int_0^{2\pi} e^{-[se^{i\theta}+(1/4)e^{i\theta 2}]}\, e^{-i\theta(2n+1)} ie^{i\theta}\, d\theta$$

$$= \frac{2\pi i}{(2n)!} \frac{d^{2n}}{dz^{2n}} \{e^{-(sz+(1/4)z^2)}\}, \quad \text{evaluated at} \quad z = 0,$$

$$= \frac{2\pi i}{(2n)!} 2^{-2n} e^{s^2} \frac{d^{2n}}{ds^{2n}} e^{-s^2}$$

$$= \frac{2\pi i}{(2n)!\, 4^n} \cdot H_{2n}(s),$$

where $H_{2n}(s)$ denotes the Hermite polynomial of degree $2n$. Thus, as

$$A_s(\beta_n) = \frac{H_{2n}(s)}{H_{2n}(0)} = \frac{(-1)^n n!}{(2n)!} H_{2n}(s), \qquad n = 0, 1, \dots,$$

we finally get

$$\int_0^\infty |f(s)|^2 e^{-s^2}\, ds = \sum_{n=0}^\infty \left| \int_0^\infty f(s) H_{2n}(s)\, e^{-s^2}\, ds \right|^2 \left(\frac{2\pi^{-1/2}}{4^n (2n)!} \right).$$

This formula, which is valid for every function $f \in L^2(dQ^+; [0, \infty))$, is of course a version of the classical Hermite expansion formula.

In much the same way the "B version" of (6.6a) leads to the expansion formula

$$\int_0^\infty |f(s)|^2 e^{s^2}\, ds = \pi \sum_{n=1}^\infty \left| \int_0^\infty f(s) B_s(2\sqrt{n})\, e^{s^2} \frac{ds}{\pi} \right|^2 \mu_n,$$

which is valid for each function $f \in L^2(dQ^-; [0, \infty))$, where

$$B_s(2\sqrt{n}) = \frac{-e^{-s^2}}{2\sqrt{n}} A_s'(2\sqrt{n})$$

$$= \frac{-e^{-s^2}}{2\sqrt{n}} \frac{4nH_{2n-1}(s)}{H_{2n}(0)}, \qquad n = 1, 2, \dots .$$

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