we have used the fact that $\|H_N g\| = \|g\|$ for $g \in \mathscr{A}$. In the extreme right-hand member of the chain of inequalities just written, the factors $\|q_0\|$ and $\|e^{-2i\varphi_+}\|$ are finite and *only involve* $\varphi = \varphi_\delta$; therefore they depend *only on* δ. The *second term* of $T^{(L)}_{N,\delta} f$ is handled in exactly the same fashion, and, putting together the estimates obtained for *both* terms, we arrive at property 2.

Verification of property (3) remains. This is somewhat long-winded. It is really nothing but an elaborate version of the argument presented in good elementary calculus courses to show that the limit of a product equals the product of the limits. In order not to lose sight of the main idea, let's just compare the *first terms* of $T_{N,\delta} f$ and $T^{(L)}_{N,\delta} f$. The *difference* of these first terms has norm equal to

$$
\begin{aligned}
&\|(S_L f \cdot H_N S_L q_0)_+ (H_N S_L e^{-2i\varphi_+}) \cdot e^{iN\vartheta} \;-\; (f \cdot H_N q_0)_+ (H_N e^{-2i\varphi_+}) e^{iN\vartheta}\| \\
&\quad\leqslant\; \|(S_L f \cdot H_N S_L q_0)_+ - (f H_N q_0)_+\| \, \|H_N S_L e^{-2i\varphi_+}\| \\
&\qquad\qquad + \; \|(f H_N q_0)_+\| \, \|H_N e^{-2i\varphi_+} - H_N S_L e^{-2i\varphi_+}\| \\
&\quad\leqslant\; \|e^{-2i\varphi_+}\| \, \|(S_L f - f) H_N q_0 + (S_L f)(H_N S_L q_0 - H_N q_0)\| \\
&\qquad\qquad + \; \|f\| \, \|q_0\| \, \|e^{-2i\varphi_+} - S_L e^{-2i\varphi_+}\| \\
&\quad\leqslant\; \|e^{-2i\varphi_+}\| \{ \|S_L f - f\| \, \|q_0\| + \|f\| \, \|S_L q_0 - q_0\| \} \\
&\qquad\qquad + \; \|f\| \, \|q_0\| \, \|e^{-2i\varphi_+} - S_L e^{-2i\varphi_+}\|
\end{aligned}
$$

This last expression *does not involve N at all*, and, for fixed $f \in \mathscr{A}$, tends to *zero* as $L \to \infty$. (It depends on δ through the functions φ and $q_0 = e^{i(\varphi_+ - \varphi_-)}$.)

The difference of the *second terms* of $T^{(L)}_{N,\delta} f$ and $T_{N,\delta} f$ is treated in the same way, and we see that property (3) holds. The lemma is proved, and we are done.

2. The example

Theorem (Kargaev). *Let $\Lambda \subseteq \mathbb{Z}$. Suppose that for each positive integer L there is some positive integer N_L with*

$$\Lambda \;\supseteq\; \mathscr{M}(N_L, L) \cap \mathbb{Z},$$

where the sets $\mathscr{M}(N, L)$ are those defined in the previous article. Then, given $\varepsilon > 0$ and $g \in \mathscr{A}$ we can find a $g_\varepsilon \in \mathscr{A}$ such that

(i) $\|g_\varepsilon\| \leqslant K_\varepsilon \|g\|$, *where K_ε depends* only *on* ε;
(ii) $\hat{g}_\varepsilon(n) = 0$ *for* $n \notin \Lambda$;
(iii) $g_\varepsilon(\vartheta) = g(\vartheta)$ *for* $\vartheta \in [0, 2\pi) \sim \Delta$, *where* $|\Delta| < 2\pi\varepsilon$.

Proof. Taking $\varepsilon > 0$, we put $\delta_n = \varepsilon/2^n$ and $\varepsilon_n = 1/2^n C(\delta_{n+1})$ with $C(\delta)$ from property (2) of the *second* lemma in the previous article. There is no harm in supposing that $C(\delta) > 1$; this we *do* in the following construction.

The function g_ε is obtained from a given $g \in \mathscr{A}$ by a process of successive approximations, using the operators $T_{N,\delta}$ and $T_{N,\delta}^{(L)}$ from the two lemmas of the preceding article.

According to the *second* of those lemmas, we can choose an L_1 such that

$$(*) \qquad \| T_{N,\delta_1}^{(L_1)} g - T_{N,\delta_1} g \| \leqslant \varepsilon_1 \| g \|$$

for all values of N simultaneously. If we take any positive integer N, the Fourier coefficients $\hat{h}(n)$ of $h = T_{N,\delta_1}^{(L_1)} g$ all vanish for $n \notin \mathscr{M}(N, L_1)$ by that second lemma. The hypothesis now furnishes a value of N such that

$$\mathscr{M}(N, L_1) \cap \mathbb{Z} \subseteq \Lambda.$$

Fix such a value of N, calling it N_1. Then, if we put $h_1 = T_{N_1,\delta_1}^{(L_1)} g$, we have $\hat{h}_1(n) = 0$ for $n \notin \Lambda$. Let us also write $r_1 = T_{N_1,\delta_1} g - h_1$. Then $(*)$ says that $\| r_1 \| \leqslant \varepsilon_1 \| g \|$, and, by the *first* lemma of the preceding article,

$$g(\vartheta) - h_1(\vartheta) - r_1(\vartheta) = g(\vartheta) - (T_{N_1,\delta_1} g)(\vartheta) = 0$$

for $\vartheta \in E_{N_1,\delta_1}$, a certain subset of $[0, 2\pi)$ with $|E_{N_1,\delta_1}| = 2\pi(1 - \delta_1)$.

We proceed, treating r_1 the way our given function g was just handled. First use the second lemma to get an L_2 such that

$$\| T_{N,\delta_2}^{(L_2)} r_1 - T_{N,\delta_2} r_1 \| \leqslant \varepsilon_2 \| g \|$$

for all positive N simultaneously, then choose (and fix) a value N_2 of N for which $\mathscr{M}(N_2, L_2) \cap \mathbb{Z} \subseteq \Lambda$, such choice being *possible according to the hypothesis.* Writing

$$h_2 = T_{N_2,\delta_2}^{(L_2)} r_1$$

and

$$r_2 = T_{N_2,\delta_2} r_1 - h_2,$$

we will have $\hat{h}_2(n) = 0$ for $n \notin \mathscr{M}(N_2, L_2)$ by the second lemma, hence, *a fortiori*, $\hat{h}_2(n) = 0$ for $n \notin \Lambda$. Our choice of L_2 makes

$$\| r_2 \| \leqslant \varepsilon_2 \| g \|,$$

and, by the first lemma, we have $r_1(\vartheta) = (T_{N_2,\delta_2} r_1)(\vartheta)$, i.e., $r_1(\vartheta) = h_2(\vartheta) + r_2(\vartheta)$ for $\vartheta \in E_{N_2,\delta_2}$, a subset of $[0, 2\pi)$ with $|E_{N_2,\delta_2}| = 2\pi(1 - \delta_2)$. *According to the preceding step,* we then have

$$g(\vartheta) = h_1(\vartheta) + h_2(\vartheta) + r_2(\vartheta) \qquad \text{for } \vartheta \in E_{N_1,\delta_1} \cap E_{N_2,\delta_2}.$$

And $\hat{h}_1(n) + \hat{h}_2(n) = 0$ for $n \notin \Lambda$.

Suppose that functions $h_1, h_2, \ldots, h_{k-1}$ and r_{k-1} (in $\mathscr{A}$) and positive integers $N_1, N_2, \ldots, N_{k-1}$ have been determined with $\|r_{k-1}\| \leqslant \varepsilon_{k-1}\|g\|$, $\hat{h}_j(n) = 0$ for $n \notin \Lambda$, $j = 1, 2, \ldots, k-1$, and $g = h_1 + h_2 + \cdots + h_{k-1} + r_{k-1}$ on the intersection $\bigcap_{j=1}^{k-1} E_{N_j,\delta_j}$. Then choose L_k in such a way that

$$\| T_{N,\delta_k}^{(L_k)} r_{k-1} \; - \; T_{N,\delta_k} r_{k-1} \| \;\leqslant\; \varepsilon_k \|g\|$$

simultaneously for all N (second lemma), and afterwards pick an N_k with $\mathscr{M}(N_k, L_k) \cap \mathbb{Z} \;\subseteq\; \Lambda$ (hypothesis). Putting

$$h_k \;=\; T_{N_k,\delta_k}^{(L_k)} r_{k-1}$$

and

$$r_k \;=\; T_{N_k,\delta_k} r_{k-1} - h_k,$$

we see that $\hat{h}_k(n) = 0$ for $n \notin \Lambda$, that $\|r_k\| \leqslant \varepsilon_k \|g\|$, and that $g = h_1 + h_2 + \cdots + h_{k-1} + h_k + r_k$ on $\bigcap_{j=1}^{k} E_{N_j,\delta_j}$ (first lemma).

Observe now that, by the second lemma, we *also* have

$$\begin{aligned}\|h_k\| \;=\; \|T_{N_k,\delta_k}^{(L_k)} r_{k-1}\| \;&\leqslant\; C(\delta_k)\|r_{k-1}\| \;\leqslant\; C(\delta_k)\varepsilon_{k-1}\|g\| \\ &\leqslant\; \|g\|/2^{k-1}\end{aligned}$$

for $k \geqslant 2$ on account of the way the numbers ε_k were rigged at the beginning of this proof. The series $h_1 + h_2 + h_3 + \cdots$ therefore *converges in the space* $\mathscr{A}$ (hence *uniformly* on $[0, 2\pi]$). Putting

$$g_\varepsilon(\vartheta) \;=\; \sum_{k=1}^{\infty} h_k(\vartheta),$$

we have

$$\|g_\varepsilon\| \;\leqslant\; (C(\delta_1) + \tfrac{1}{2} + \tfrac{1}{4} + \cdots)\|g\| \;=\; (1 + C(\varepsilon/2))\|g\|,$$

and $\hat{g}_\varepsilon(n) = 0$ for $n \notin \Lambda$ since, for such n, we have $\hat{h}_k(n) = 0$ for every k. Finally, since

$$|r_k(\vartheta)| \;\leqslant\; \|r_k\| \;\leqslant\; \varepsilon_k \|g\| \;\xrightarrow[k]{}\; 0,$$

we have

$$\sum_{j=1}^{k} h_j(\vartheta) \;+\; r_k(\vartheta) \;\xrightarrow[k]{}\; g_\varepsilon(\vartheta)$$

uniformly for $0 \leqslant \vartheta \leqslant 2\pi$, so $g_\varepsilon(\vartheta) = g(\vartheta)$ on the intersection

$$E \;=\; \bigcap_{j=1}^{\infty} E_{N_j,\delta_j}.$$

Here, since $|E_{N_j,\delta_j}| = 2\pi(1 - \delta_j)$ and the sets E_{N_j,δ_j} all lie in $[0, 2\pi)$, we have

$$\begin{aligned}|E| \;\geqslant\; 2\pi(1 - \delta_1 - \delta_2 - \delta_3 - \cdots) \;&=\; 2\pi\left(1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{4} - \cdots\right) \\ &=\; 2\pi(1 - \varepsilon).\end{aligned}$$

The theorem is proved.

Our example is now furnished by the following

Corollary. *There exists a non-zero measure* μ *having gaps* (a_n, b_n) *in its support, with*

$$0 < a_1 < b_1 < a_2 < b_2 < a_3 < \cdots$$

and

$$\sum_1^\infty \left(\frac{b_l - a_l}{a_l} \right)^2 = \infty$$

(*and the ratios* $(b_l - a_l)/a_l$ *even tending to* ∞ *as rapidly as we want!*), *while* $\hat{\mu}(\lambda) = 0$ *on a set of positive measure.*

Proof. For $l = 1, 2, 3, \ldots$, take the sets

$$\mathscr{M}_l = \bigcup_{k=-2l-1}^{2l+1}{}' \; [N_l k - l, \; N_l k + l]$$

(term with $k = 0$ omitted), with the positive integers N_l so chosen that $N_l > 2l$ and that N_{l+1} is *much larger* than $(2l+1)(N_l + 1)$. *There is no obstacle to our taking* N_{l+1} *as large as we wish in relation to* $(2l+1)(N_l+1)$ *for each* l.

Put

$$\Lambda = \bigcup_{l=1}^{\infty} (\mathscr{M}_l \cap \mathbb{Z});$$

it is clear that Λ satisfies the hypothesis of the theorem.

Choose any $g \in \mathscr{A}$ such that $g(\vartheta) > 0$ for $\pi/2 < \vartheta < 3\pi/2$ and $g(\vartheta) = 0$ for $0 \leqslant \vartheta \leqslant \pi/2$ and for $3\pi/2 \leqslant \vartheta \leqslant 2\pi$.

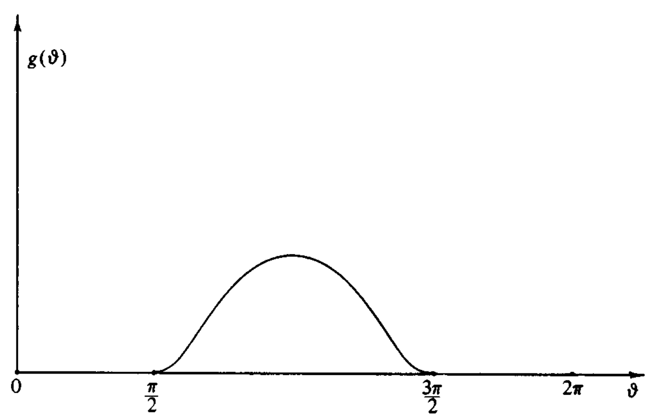

Figure 82

There are plenty of such functions g; we fix one of them.

Apply the theorem with $\varepsilon = \frac{1}{4}$, getting a function g_ε in $\mathscr{A}$ with $\hat{g}_\varepsilon(n) = 0$ for $n \notin \Lambda$ and $g_\varepsilon(\vartheta) = g(\vartheta)$ for all $\vartheta \in [0, 2\pi)$ outside a set of measure $\leqslant \pi/2$. Then *certainly* $g_\varepsilon(\vartheta)$ must be > 0 on a set in $[0, 2\pi)$ of measure $\geqslant \pi/2$ (*hence, in particular*, $g_\varepsilon \not\equiv 0$), while *at the same time* $g_\varepsilon(\vartheta) = 0$ *on a set of measure* $\geqslant \pi/2$ lying *in* $[0, 2\pi)$.

We have

$$g_\varepsilon(\vartheta) \;=\; \sum_{n\in\Lambda} \hat{g}_\varepsilon(n) e^{in\vartheta}$$

with

$$\sum_{n\in\Lambda} |\hat{g}_\varepsilon(n)| \;<\; \infty,$$

so, if we define a measure μ *supported on* $\Lambda \subseteq \mathbb{Z}$ by putting $\mu(E) = \sum_{n\in E} \hat{g}_\varepsilon(n)$, we have $\mu \not\equiv 0$, but $\hat{\mu}(\vartheta) = g_\varepsilon(\vartheta)$ *vanishes on a set of positive measure.*

The support ($\subseteq \Lambda$) of μ has the *gaps* $((2l+1)N_l + l,\ N_{l+1} - l - 1)$ *in it.* By choosing N_{l+1} *sufficiently large in relation to* $(2l+1)(N_l+1)$ for each l, *we can make the ratios*

$$\frac{(N_{l+1} - l - 1) - ((2l+1)N_l + l)}{(2l+1)N_l + l}$$

go to ∞ *as rapidly as we please for* $l \longrightarrow \infty$.

We are done.

D. Volberg's work

Let $f(\vartheta) \in L_1(-\pi, \pi)$; say

$$f(\vartheta) \;\sim\; \sum_{-\infty}^{\infty} a_n e^{in\vartheta}.$$

Suppose that the Fourier coefficients a_n with *negative* indices n are *small enough* to satisfy the relation

$$(*) \qquad \sum_{-\infty}^{-1} \frac{1}{n^2} \log\left(\frac{1}{\sum_{-\infty}^{n} |a_k|}\right) \;=\; \infty.$$

According to a corollary to Levinson's theorem (§ A.5), $f(\vartheta)$ then *cannot vanish on an interval of positive length* unless $f \equiv 0$. If we also assume (for instance) that $\sum_k |a_k| < \infty$, Beurling's improvement of Levinson's theorem (§B.2) shows that $f(\vartheta)$ *cannot even vanish on a set of positive measure* without being identically zero when $(*)$ holds.

It is therefore natural to ask *how small* $|f(\vartheta)|$ *can actually be* for a *non-*

zero f whose Fourier coefficients a_n satisfy $(*)$, or something like it. Suppose for instance, that

$$|a_n| \leqslant e^{-M(|n|)}, \qquad n < 0,$$

with a regularly increasing $M(m)$ for which

$$\sum_1^\infty \frac{M(m)}{m^2} = \infty.$$

Volberg's surprising result is that if the behaviour of $M(m)$ is *regular enough*, then we *must have*

$$\int_{-\pi}^{\pi} \log|f(\vartheta)|\,d\vartheta > -\infty$$

unless $f \equiv 0$. *Very* loosely speaking, this amounts to saying that if $f \not\equiv 0$ and

$$\sum_{-\infty}^{-1} \frac{1}{n^2} \log\left|\frac{1}{\hat{f}(n)}\right| = \infty,$$

then

$$\int_{-\pi}^{\pi} \log|f(\vartheta)|\,d\vartheta > -\infty,$$

at least when the *decrease* of $|\hat{f}(n)|$ for $n \longrightarrow -\infty$ is *sufficiently regular. If one logarithmic integral* (the *sum*) *diverges, the other must converge*!

One could improve this result *only* by finding a way to relax the regularity conditions imposed on $M(m)$.

Indeed, if $p(\vartheta) \geqslant 0$ is *any* function in $L_1(-\pi,\pi)$ with

$$\int_{-\pi}^{\pi} \log p(\vartheta)\,d\vartheta > -\infty,$$

we can *get* a function

$$f(\vartheta) \sim \sum_0^\infty a_n e^{in\vartheta}$$

such that $|f(\vartheta)| = p(\vartheta)$ a.e. by putting

$$f(\vartheta) = \lim_{\substack{z \not\to e^{i\vartheta} \\ |z|<1}} \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\log p(t)\,dt\right\}$$

(see Chapter II, § A). Here, the Fourier coefficients of *negative* index *are all zero*, i.e., for $n < 0$,

$$|a_n| \leqslant e^{-M(|n|)}$$

with $M(|n|) \equiv \infty$. This means that from the condition

$$|a_n| \leqslant e^{-M(|n|)}, \qquad n < 0,$$

with

$$\sum_1^\infty \frac{M(m)}{m^2} = \infty$$

one can *never hope to deduce a more stringent restriction* on the *smallness* of $|f(\vartheta)|$ than

$$\int_{-\pi}^{\pi} \log|f(\vartheta)|\,d\vartheta > -\infty.$$

Also, if $M(m)$ is *increasing*, from a *less* stringent condition than

$$\sum_1^\infty \frac{M(m)}{m^2} = \infty$$

one can *never hope to deduce any limitation on the smallness of* $|f(\vartheta)|$ for functions $f \not\equiv 0$ with $|a_n| \leqslant e^{-M(|n|)}$, $n < 0$. That is the content of

Problem 12

Let $M(m) > 0$ be increasing for $m > 0$, and such that

$$\sum_1^\infty \frac{M(m)}{m^2} < \infty.$$

Given h, $0 < h < \pi$, show that there is a function $f(\vartheta)$, continuous and of period 2π, with $f(\vartheta) = 0$ for $h \leqslant |\vartheta| \leqslant \pi$ but $f \not\equiv 0$, such that

$$|a_n| \leqslant e^{-M(|n|)}, \quad n \neq 0\,(sic!),$$

for the Fourier coefficients a_n of $f(\vartheta)$. (Hint: Use the theorems of Chapter IV, § D and Chapter III, § D. Take a suitable convolution.)

It is important to note that Volberg's theorem *relates specifically to the unit circle; its analogue for the real line is false*. Take, namely, $F(x) = e^{-x^2}$, so that

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} \log F(x)\,dx = -\infty.$$

Here,

$$\hat{F}(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda x} e^{-x^2}\,dx = \sqrt{\left(\frac{\pi}{2}\right)} e^{-\lambda^2/4},$$

so

$$\int_{-\infty}^{0} \frac{1}{1+\lambda^2} \log\left(\frac{1}{\hat{F}(\lambda)}\right) \mathrm{d}\lambda \;=\; \infty.$$

and even

$$\int_{-\infty}^{0} \frac{1}{1+\lambda^2} \log\left(\frac{1}{\int_{-\infty}^{\lambda} \hat{F}(t)\,\mathrm{d}t}\right) \mathrm{d}\lambda \;=\; \infty.$$

This example shows that a *function and its Fourier transform can both get very small on* $\mathbb{R}$ (in terms of the logarithmic integral).

1. The planar Cauchy transform

Notation. If $G(z)$ is differentiable as a function of x and y we write

$$\frac{\partial G(z)}{\partial z} \;=\; G_z(z) \;=\; \frac{\partial G(z)}{\partial x} - \mathrm{i}\frac{\partial G(z)}{\partial y}$$

and

$$\frac{\partial G(z)}{\partial \bar{z}} \;=\; G_{\bar{z}}(z) \;=\; \frac{\partial G(z)}{\partial x} + \mathrm{i}\frac{\partial G(z)}{\partial y}.$$

Nota bene. Nowadays, most people take $\partial G/\partial z$ and $\partial G/\partial \bar{z}$ as *one-half* of the respective right-hand quantities.

Remark. If $G = U + \mathrm{i}V$ with *real* functions U and V, the equation $G_{\bar{z}} = 0$ reduces to

$$\begin{cases} U_x = V_y, \\ U_y = -V_x, \end{cases}$$

i.e., the *Cauchy–Riemann equations* for U and V. The condition that $G_{\bar{z}} \equiv 0$ in a domain $\mathscr{D}$ is thus *equivalent to analyticity of* $G(z)$ *in* $\mathscr{D}$.

Theorem. *Let $F(z)$ be bounded and $\mathscr{C}_1$ in a bounded domain $\mathscr{D}$, and put*

$$G(z) \;=\; \frac{1}{2\pi}\iint_{\mathscr{D}} \frac{F(\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta}{z-\zeta},$$

where, as usual, $\zeta = \xi + \mathrm{i}\eta$. Then $G(z)$ is $\mathscr{C}_1$ in $\mathscr{D}$ and

$$\frac{\partial G(z)}{\partial \bar{z}} \;=\; F(z), \qquad z \in \mathscr{D}.$$

Remark. The integral in question *converges absolutely* for each z, as is seen by going over to the polar coordinates (ρ, ψ) with

$$\zeta - z = \rho e^{i\psi}.$$

$G(z)$ is called the *planar Cauchy transform of* $F(z)$.

Proof of theorem. We first establish the differentiability of $G(z)$ in $\mathscr{D}$.

Let $z_0 \in \mathscr{D}$ with $\operatorname{dist}(z_0, \partial\mathscr{D}) = 3\rho$, say. Take any infinitely differentiable function $\varphi(\zeta)$ of ζ with $0 \leqslant \varphi(\zeta) \leqslant 1$ and

$$\varphi(\zeta) \;=\; \begin{cases} 1, & |\zeta - z_0| \leqslant \rho, \\ 0, & |\zeta - z_0| \geqslant 2\rho. \end{cases}$$

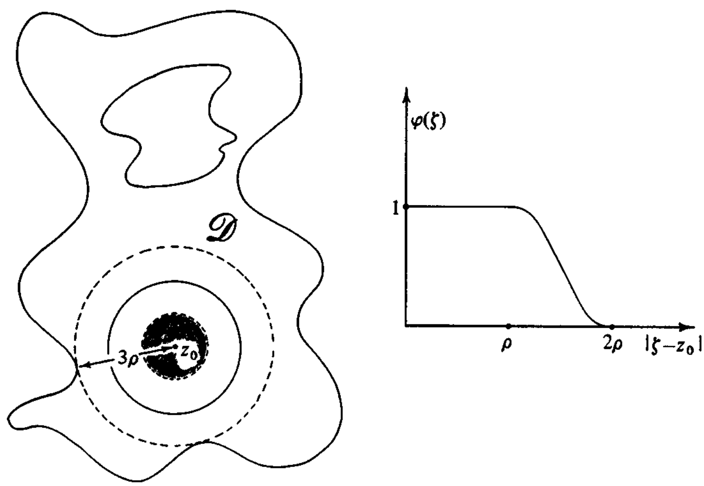

Figure 83

We can write

$$G(z) \;=\; \frac{1}{2\pi}\iint_{|\zeta - z_0| \leqslant 2\rho} \frac{\varphi(\zeta)F(\zeta)}{z - \zeta}\, d\xi\, d\eta$$
$$+ \frac{1}{2\pi}\iint_{\substack{|\zeta - z_0| > \rho \\ \zeta \in \mathscr{D}}} \frac{(1 - \varphi(\zeta))F(\zeta)}{z - \zeta}\, d\xi\, d\eta.$$

The *second* integral on the right is obviously a $\mathscr{C}_\infty$ function of z for $|z - z_0| < \rho$; it remains to consider the *first* one. After a change of variable, the *latter can be rewritten as*

$$\frac{1}{2\pi}\iint_{\mathbb{C}} \frac{F_1(z - w)}{w}\, du\, dv$$

(where $w = u + iv$, as usual) with $F_1(\zeta) = \varphi(\zeta)F(\zeta)$. Here, $F_1(\zeta)$ is *of compact support*, and has as much differentiability as $F(\zeta)$. Hence, since

$$\iint_{|w|<R} \frac{du\,dv}{|w|} \;<\; \infty$$

for any finite R, we can differentiate $(1/2\pi)\iint_{\mathbb{C}}(F_1(z-w)/w)\,du\,dv$ with respect to x and y *under the integral sign*, and thus see that that expression is $\mathscr{C}_1$ in those variables.

We have shown that $G(z)$ is $\mathscr{C}_1$ in the neighborhood of any $z_0 \in \mathscr{D}$; there remains the evaluation of $G_{\bar{z}}(z_0)$ in terms of F. This turns out to be surprisingly difficult if we try to do it directly, and we resort to the following dodge.

Let $r > 0$ be small, and $z_0 \in \mathscr{D}$. By the differentiability of $G(z)$ at z_0,

$$\begin{aligned} G(z_0 + re^{i\vartheta}) &= G(z_0) + G_x(z_0)r\cos\vartheta + G_y(z_0)r\sin\vartheta + o(r) \\ &= G(z_0) + \tfrac{1}{2}G_z(z_0)re^{i\vartheta} + \tfrac{1}{2}G_{\bar{z}}(z_0)re^{-i\vartheta} + o(r). \end{aligned}$$

Multiplying the last expression by $e^{i\vartheta}\,d\vartheta$ and integrating ϑ from 0 to 2π, we find the value $\pi r G_{\bar{z}}(z_0) + o(r)$; therefore

$$G_{\bar{z}}(z_0) \;=\; \lim_{r\to 0} \frac{1}{\pi r}\int_0^{2\pi} G(z_0 + re^{i\vartheta})e^{i\vartheta}\,d\vartheta.$$

Plugging in the expression for G in terms of F and changing the order of integration, this becomes

$$G_{\bar{z}}(z_0) \;=\; \lim_{r\to 0} \frac{1}{2\pi^2 r}\iint_{\mathscr{D}}\int_0^{2\pi} \frac{F(\zeta)e^{i\vartheta}}{z_0 + re^{i\vartheta} - \zeta}\,d\vartheta\,d\xi\,d\eta.$$

However,

$$\int_0^{2\pi} \frac{ire^{i\vartheta}\,d\vartheta}{re^{i\vartheta} - (\zeta - z_0)} \;=\; \begin{cases} 2\pi i, & |\zeta - z_0| < r, \\ 0, & |\zeta - z_0| > r. \end{cases}$$

Therefore, by the previous relation, we have

$$G_{\bar{z}}(z_0) \;=\; \lim_{r\to 0}\frac{1}{\pi r^2}\iint_{|\zeta - z_0|<r} F(\zeta)\,d\xi\,d\eta \;=\; F(z_0),$$

F having been assumed to be $\mathscr{C}_1$ in $\mathscr{D}$. We are done.

Corollary. *Let $\mathscr{D}$ be a bounded domain. Suppose that $F(z)$ is $\mathscr{C}_2$ in $\mathscr{D}$, that $|F(z)| > 0$ there, and that there is a constant C such that*

$$\left|\frac{\partial F(z)}{\partial \bar{z}}\right| \;\leqslant\; C|F(z)|, \qquad z \in \mathscr{D}.$$

Then

$$\Phi(z) \;=\; F(z)\exp\left\{\frac{1}{2\pi}\int\!\!\int_{\mathscr{D}}\frac{F_{\bar\zeta}(\zeta)\,d\xi\,d\eta}{F(\zeta)(\zeta - z)}\right\}$$

is analytic in $\mathscr{D}$, and $|\Phi(z)|$ lies between two constant multiples of $|F(z)|$ therein.

Proof. $F_{\bar z}(z)/F(z)$ is $\mathscr{C}_1$ in $\mathscr{D}$ and bounded there by hypothesis, so we can apply the theorem, which tells us first of all that $\Phi(z)$ is differentiable in $\mathscr{D}$, and secondly that

$$\frac{\partial\Phi(z)}{\partial\bar z} \;=\; \left(F_{\bar z}(z) - F(z)\frac{F_{\bar z}(z)}{F(z)}\right)\exp\left(\frac{1}{2\pi}\int\!\!\int_{\mathscr{D}}\frac{F_{\bar\zeta}(\zeta)\,d\xi\,d\eta}{F(\zeta)(\zeta - z)}\right) \;=\; 0$$

there. The *Cauchy–Riemann equations for $\Re\Phi(z)$ and $\Im\Phi(z)$ are thus satisfied* (see remark at the beginning of this article), so $\Phi(z)$ is *analytic* in $\mathscr{D}$.

If R is the *diameter* of $\mathscr{D}$, we easily check that

$$e^{-CR}|F(z)| \;\leqslant\; |\Phi(z)| \;\leqslant\; e^{CR}|F(z)|$$

for $z \in \mathscr{D}$. This does it.

The corollary has been extensively used by Lipman Bers and by Vekua in the study of partial differential equations. Volberg also uses it so as to bring analytic functions into his treatment.

Problem 13

Show that the condition that $|F(z)| > 0$ in $\mathscr{D}$ can be *dropped* from the hypothesis of the corollary, provided that we *maintain* the assumption that $|F_{\bar z}(z)| \leqslant C|F(z)|$, $z \in \mathscr{D}$, and define the ratio $F_{\bar z}(z)/F(z)$ in a satisfactory way on the set where $F(z) = 0$. Hence show that a function F satisfying the inequality $|F_{\bar z}(z)| \leqslant C|F(z)|$ *can have only isolated zeros in $\mathscr{D}$*, unless $F \equiv 0$ there. (Hint. On $E = \{z \in \mathscr{D}: F(z) = 0\}$, assign any constant value to the ratio $F_{\bar z}(z)/F(z)$. The function $\Phi(z)$ defined in the statement of the corollary is surely *analytic* in $\mathscr{D} \sim E$; it is also analytic in E° (if that set is non-empty) because it vanishes identically there. To check *existence* of

$$\Phi'(z_0) \;=\; \lim_{z\to z_0}\frac{\Phi(z) - \Phi(z_0)}{z - z_0}$$

at a point $z_0 \in \partial E \cap \mathscr{D}$, note that both $F(z_0)$ and $F_{\bar z}(z_0)$ must vanish, so, *near* z_0,

$$F(z) \;=\; \tfrac{1}{2}F_z(z_0)(z - z_0) + o(|z - z_0|).$$

If $F_z(z_0) = 0$, $\Phi'(z_0)$ *exists* and equals *zero*. If $F_z(z_0) \neq 0$, $|F(z)| > 0$ in some punctured neighborhood $0 < |z - z_0| < \eta$ of z_0, so such a punctured neighborhood is included in $\mathscr{D} \sim E$.)

2. The function $M(\nu)$ and its Legendre transform $h(\xi)$

As explained at the beginning of this chapter, Volberg's work deals with functions

$$f(\vartheta) \sim \sum_{-\infty}^{\infty} a_n e^{in\vartheta}$$

for which the a_n with negative index are very small; more precisely,

$$|a_{-n}| \leqslant e^{-M(n)}, \quad n > 0,$$

where $M(n)$ is *increasing* and such that

$$\sum_1^{\infty} \frac{M(n)}{n^2} = \infty.$$

► It will be convenient to assume throughout this § that $M(\nu)$ *is defined for all real values of* $\nu \geqslant 0$ *and not just the integral ones, and is increasing on* $[0, \infty)$. We do not, to begin with, exclude the possibility that $M(0) = -\infty$. Whether this happens or not will turn out to make no difference as far as our final result is concerned.

Volberg's treatment makes essential use of a *weight* $w(r) > 0$ defined for $0 < r < 1$ by means of the formula

$$\log\left(\frac{1}{w(r)}\right) = \sup_{\nu > 0}\left(M(\nu) - \nu \log\frac{1}{r}\right).$$

It is therefore necessary to make a study of the relation between $M(\nu)$ and the function

$$\boxed{h(\xi) = \sup_{\nu > 0}(M(\nu) - \nu\xi),}$$

defined for $\xi > 0$, and to find out how various properties of $M(\nu)$ are connected to others of $h(\xi)$. We take up these matters in the present article.

The formula for the function $h(\xi)$ (sometimes called the *Legendre transform* of $M(\nu)$) is reminiscent of material discussed extensively in Chapter IV, beginning with § A.2 therein. It is perhaps a good idea to start by showing how the situation now under consideration is related to that of Chapter IV, and especially how it *differs* from the latter.

Our present function $M(\nu)$ can be interpreted as $\log T(\nu)$, where $T(r)$ is the Ostrowski function used in Chapter IV. ($M(n)$ is *not*, as the similarity in letters might lead one to believe, a version of the $\{M_n\}$ – or of $\log M_n$ –

from Chapter IV!) Suppose indeed that

$$f(\vartheta) \sim \sum_{-\infty}^{\infty} a_n e^{in\vartheta}$$

is infinitely differentiable and in the class $\mathscr{C}(\{M_n\})$ considered in Chapter IV – in order to simplify matters, let us say that

$$|f^{(n)}(\vartheta)| \leqslant M_n, \quad n \geqslant 0.$$

We have

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\vartheta} f(\vartheta)\, d\vartheta,$$

and the right side, after k integrations by parts, becomes

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} (in)^{-k} f^{(k)}(\vartheta)\, d\vartheta$$

when $n \neq 0$. Using the above inequality on the derivatives of $f(\vartheta)$ in this integral, we see that

$$|a_n| \leqslant \inf_{k \geqslant 0} \frac{M_k}{|n|^k} = \frac{1}{T(|n|)}$$

where, as in Chapter IV,

$$T(r) = \sup_{k \geqslant 0} \frac{r^k}{M_k} \quad \text{for} \quad r > 0.$$

On putting $T(v) = e^{M(v)}$, we get

$$|a_n| \leqslant e^{-M(|n|)}.$$

This connection makes it possible to apply the final result of the present § to certain classes $\mathscr{C}(\{M_n\})$ of periodic functions, of period 2π. But that application does not show its real scope. The inequality for the a_n obtained by assuming that $f \in \mathscr{C}(\{M_n\})$ is a *two-sided* one; it shows that the a_n go to zero rapidly as $n \longrightarrow \pm\infty$. The *hypothesis* for the theorem on the logarithmic integral is, however, *one-sided*; it is only necessary to assume that

$$|a_{-n}| \leqslant e^{-M(n)}$$

for $n > 0$ in order to reach the desired conclusion.

There is another essential difference between our present situation and that of Chapter IV. *Here* we look at the function

$$h(\xi) = \sup_{v > 0} (M(v) - v\xi),$$

i.e., in terms of $T(v)$,

$$h(\xi) \;=\; \sup_{v>0}(\log T(v) - v\xi).$$

There we used the convex logarithmic regularisation $\{\underline{M}_n\}$ given by

$$\log \underline{M}_n = \sup_{v>0}((\log v)n - \log T(v)).$$

There is, first of all, a *change in sign.* Besides this, the former expression involves terms $v\xi$, *linear in the parameter* v, where the latter has terms *linear in* $\log v$. On account of these differences it usually turns out that the function $h(\xi)$ considered here *tends to* ∞ *for* $\xi \to 0$, whereas $\log \underline{M}_n$ *usually tended to* ∞ *for* $n \to \infty$.

Let us begin our examination of $h(\xi)$ by verifying the statement just made about its behaviour for $\xi \to 0$.

Lemma. *If* $M(v) \to \infty$ *for* $v \to \infty$, $h(\xi) \to \infty$ *for* $\xi \to 0$.

Proof. Take any v_0. Then, if $0 < \xi < \frac{1}{2}M(v_0)/v_0$,

$$h(\xi) \;\geqslant\; M(v_0) - v_0\xi \;>\; \tfrac{1}{2}M(v_0). \qquad \text{Q.E.D.}$$

The function

$$h(\xi) = \sup_{v>0}(M(v) - v\xi),$$

as the supremum of *decreasing* functions of ξ, is *decreasing*. As the *supremum* of *linear functions* of ξ, *it is convex.* The *upper supporting line of slope* ξ *to the graph of* $M(v)$ vs v *has ordinate intercept equal to* $h(\xi)$:

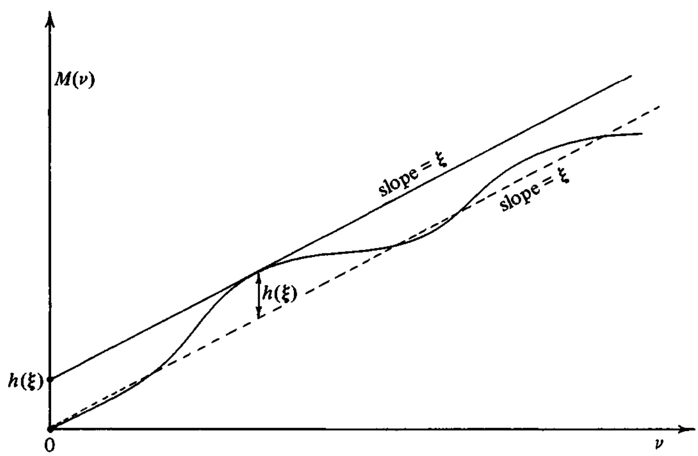

Figure 84

From this picture, we see immediately that

$$M^*(\nu) \;=\; \inf_{\xi>0}\,(h(\xi)+\xi\nu)$$

is the smallest concave increasing function which is $\geqslant M(\nu)$. Therefore, *if* $M(\nu)$ is *also concave,* $M^*(\nu)=M(\nu)$. We will come back to this relation later on.

Here is a graph dual to the one just drawn:

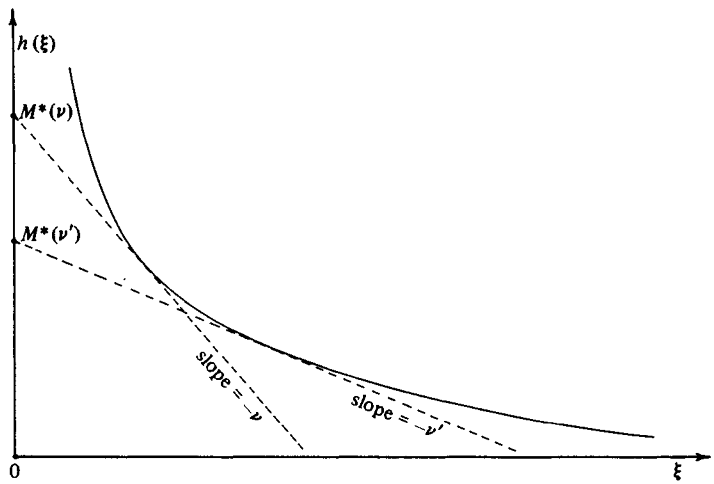

Figure 85

We see that $M^*(\nu)$ is the *ordinate intercept of the* (lower) *supporting line to the convex graph of* $h(\xi)$ having slope $-\nu$.

Volberg's construction depends in an essential way on a theorem of Dynkin, to be proved in the next article, which requires *concavity* of the function $M(\nu)$. Insofar as inequalities of the form

$$|a_{-n}| \;\leqslant\; e^{-M(n)}$$

are concerned, this concavity is pretty much *equivalent* to the cruder property that $M(\nu)/\nu$ be *decreasing.* It is, first of all, fairly evident that *the concavity of* $M(\nu)$ *makes* $M(\nu)/\nu$ *decreasing* (and even *strictly* decreasing, save in the trivial case where $M(\nu)/\nu \equiv \text{const.}$) *for all sufficiently large* ν. We have, in the other direction, the following

Theorem. *Let* $M(\nu)$ *be* >0 *and increasing for* $\nu>0$, *and denote by* $M^*(\nu)$ *the* smallest concave majorant *of* $M(\nu)$. *If* $M(\nu)/\nu$ *is* decreasing.

$$M^*(\nu) \;<\; 2M(\nu).$$

Problem 14(a)

Prove this result. (Hint: The graph of $M^*(v)$ vs v coincides with that of $M(v)$, save on certain open intervals (a_n, b_n) on each of which $M^*(v)$ is *linear*, with $M^*(a_n) = M(a_n)$ and $M^*(b_n) = M(b_n)$:

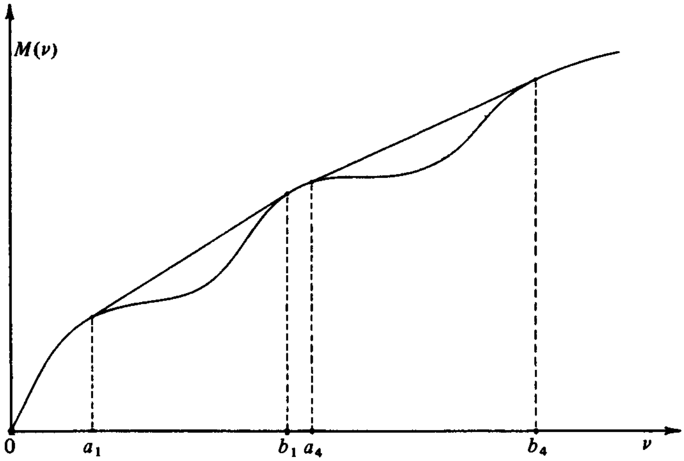

Figure 86

The (a_n, b_n) *may*, of course, be disposed like the contiguous intervals to the Cantor set, for instance. Consider any one of them, say, wlog, (a_1, b_1):

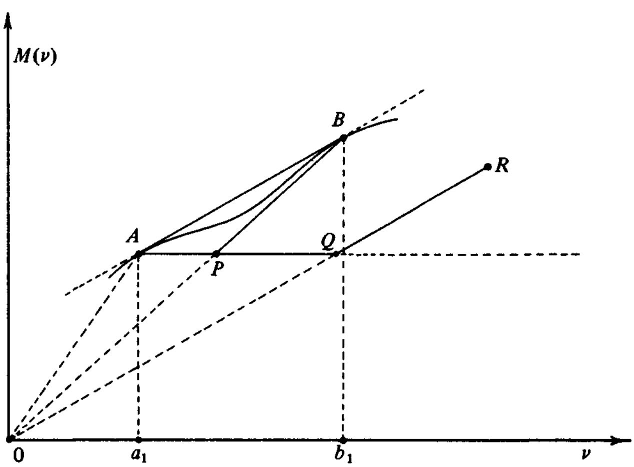

Figure 87

For $a_1 \leqslant \nu \leqslant b_1$, $(\nu, M(\nu))$ must lie *above* the broken line path APB, and $(\nu, M^*(\nu))$ lies on the segment $\overline{AB}$. Work with the broken line path $\overline{AQR}$, where OR is a line through the origin parallel to $\overline{AB}$.)

Because of this fact, the Fourier coefficients a_n of a given function which satisfy an inequality of the form

$$|a_{-n}| \leqslant e^{-M(n)}, \qquad n \geqslant 1,$$

with an *increasing* $M(\nu) > 0$ *such that* $M(\nu)/\nu$ *decreases* also satisfy

$$|a_{-n}| \leqslant e^{-M^*(n)/2}, \qquad n \geqslant 1$$

with the *concave majorant* $M^*(\nu)$ of $M(\nu)$. Clearly, $\sum_1^\infty M^*(n)/n^2 = \infty$ if $\sum_1^\infty M(n)/n^2 = \infty$. This circumstance makes it possible to simplify much of the computational work by *supposing to begin with that* $M(\nu)$ *is concave as well as increasing.*

A further (really, mainly formal) simplification results if we consider only functions $M(\nu)$ for which $M(\nu)/\nu \longrightarrow 0$ as $\nu \to \infty$ (see the next lemma). *As far as Volberg's work is concerned, this entails no restriction.* Since we will be assuming (at least) that $M(\nu)/\nu$ is decreasing, $\lim_{\nu\to\infty}(M(\nu)/\nu)$ *certainly exists. In case that limit is strictly positive,* the inequalities

$$|a_{-n}| \leqslant e^{-M(n)}, \qquad n \geqslant 1,$$

imply that

$$F(z) = \sum_{-\infty}^{\infty} a_n z^n$$

is *analytic in some annulus* $\{\rho < |z| < 1\}$, $\rho < 1$. This makes it possible for us to apply the *theorem on harmonic estimation* (§B.1), at least when $F(z)$ is continuous up to $\{|z| = 1\}$ (which will be the case in our version of Volberg's result). We find in this way that

$$\int_{-\pi}^{\pi} \log|F(e^{i\vartheta})|\,d\vartheta > -\infty$$

unless $F(z) \equiv 0$, using a simple estimate for harmonic measure in an annulus. (If the reader has any trouble working out that estimate, he or she may find it near the *very end* of the proof of Volberg's theorem in article 6 below.) The *conclusion* of Volberg's theorem is thus *verified* in the special case that $\lim_{\nu\to\infty}(M(\nu)/\nu) > 0$.

For this reason, we will mostly only consider functions $M(\nu)$ *for which* $\lim_{\nu\to\infty}(M(\nu)/\nu) = 0$ *in the present* §.

Once we decide to work with *concave* functions $M(v)$, it costs but little to *further restrict our attention* to *strictly concave infinitely differentiable* $M(v)$'s. Given any concave increasing $M(v)$, we may, first of all, add to it a *bounded strictly concave* increasing function (with second derivative <0

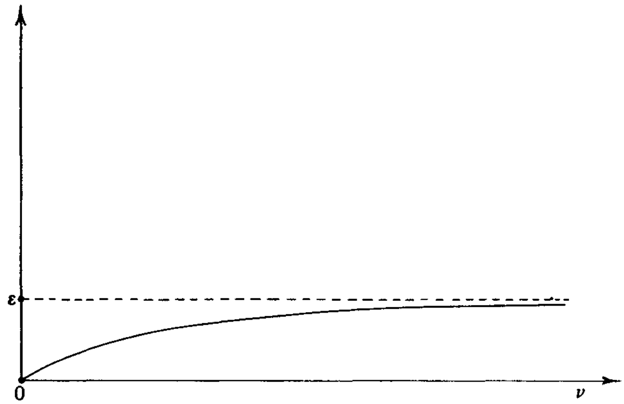

Figure 88

on $(0, \infty)$) whose graph has a *horizontal asymptote* of height ε, and thus obtain a new *strictly concave* increasing function $M_1(v)$, with $M_1''(v) < 0$, differing by at most ε from $M(v)$. We may then take an infinitely differentiable positive function φ supported on $[0, 1]$ and having $\int_0^1 \varphi(t)\,dt = 1$,

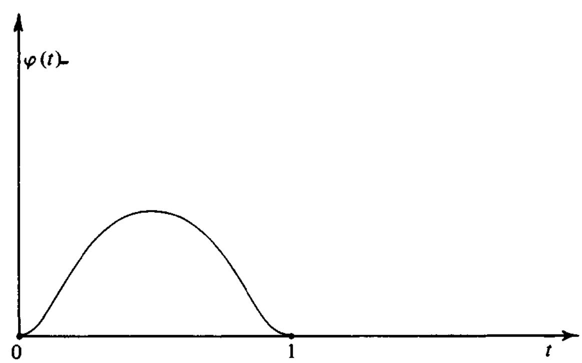

Figure 89

and form the function

$$M_2(v) = \frac{1}{h}\int_0^h M_1(v+\tau)\varphi(\tau/h)\,d\tau,$$

using a small value of $h > 0$. $M_2(v)$ will also be strictly concave with

$M_2''(v) < 0$ on $(0, \infty)$, and increasing, and *infinitely differentiable besides* for $0 < v < \infty$. It will differ by less than ε from $M_1(v)$ for $v \geqslant a$ when a is any given number > 0, if $h > 0$ is small enough (depending on a). That's because $0 \leqslant M_1'(v) \leqslant M_1'(a) < \infty$ for $v \geqslant a$.

Our function $M_2(v)$, infinitely differentiable, increasing, and strictly concave, thus differs by less than 2ε from $M(v)$ when v is large. This, however, means that $h_2(\xi) = \sup_{v>0}(M_2(v) - v\xi)$ differs by less than 2ε from

$$h(\xi) \;=\; \sup_{v>0}(M(v) - v\xi)$$

for small values of $\xi > 0$, the suprema in question being attained for *large values* of v if ξ is *small*:

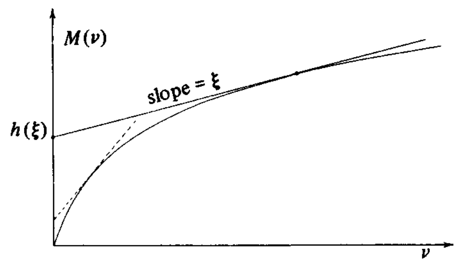

Figure 90

Hence, in studying the *order of magnitude of* $h(\xi)$ *for* ξ *near zero* (which is what we will be mainly concerned with in this §), *we may as well assume to begin with that* $M(v)$ is *strictly concave* and *infinitely differentiable.*

When this restriction holds, one can obtain some useful relations in connection with the duality between $M(v)$ and $h(\xi)$.*

Lemma. *If* $M(v)$ *is strictly concave and increasing with* $M(v)/v \longrightarrow 0$ *for* $v \to \infty$, *there is for each* $\xi > 0$ *a* unique $v = v(\xi)$ *such that*

$$h(\xi) \;=\; M(v) - v\xi.$$

$h(\xi)$ *has a* derivative *for* $\xi > 0$, *and* $h'(\xi) = -\, v(\xi)$.

Proof. Since $M(v)/v \longrightarrow 0$ as $v \to \infty$, the supporting line of slope ξ to the graph of $M(v)$ vs v *does* touch that graph *somewhere* (see preceding diagram), say at $(v_1, M(v_1))$. Thus,

$$h(\xi) \;=\; M(v_1) - v_1\xi.$$

* In the following 3 lemmas, it is tacitly assumed that $\xi > 0$ ranges over some *small* interval with left endpoint at the origin, for they will be used only for such values of ξ. This eliminates our having to worry about the behaviour of $M(v)$ for small v.

Suppose that $v_2 \neq v_1$ and also

$$h(\xi) = M(v_2) - v_2\xi;$$

wlog say that $v_2 > v_1$. Then

$$M(v_2) = M(v_1) + \xi(v_2 - v_1).$$

Therefore, for $v_1 < v < v_2$, by *strict* concavity of $M(v)$,

$$M(v) > M(v_1) + \xi(v - v_1),$$

i.e.,

$$M(v) - v\xi > M(v_1) - v_1\xi = h(\xi).$$

This, however, contradicts the definition of $h(\xi)$, so there can *be* no $v_2 \neq v_1$ with

$$h(\xi) = M(v_2) - v_2\xi.$$

Since $M(v)$ is already concave, it is *equal* to its smallest concave majorant, $M^*(v)$, i.e.,

$$M(v) = \inf_{\xi > 0} (h(\xi) + \xi v).$$

The function $h(\xi)$ is convex, so if it does *not* have a derivative at a point $\xi_0 > 0$, it has a *corner* there, with two *different* supporting lines, of slopes $-v_1$ and $-v_2$, touching the graph of $h(\xi)$ vs ξ at $(\xi_0, h(\xi_0))$:

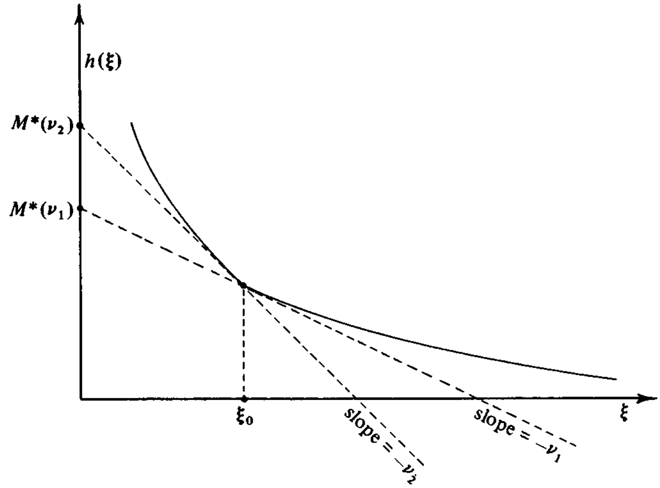

Figure 91

Those two supporting lines have ordinate intercepts equal to $M^*(v_1)$ and $M^*(v_2)$, i.e., to $M(v_1)$ and $M(v_2)$. But then $h(\xi_0) = M(v_1) - v_1\xi_0 = M(v_2) - v_2\xi_0$, which we have already seen to be impossible. $h'(\xi_0)$ must therefore *exist*, and it is now clear that derivative must have the value $-v(\xi_0)$, the slope of the *unique* supporting line to the graph of $h(\xi)$ vs ξ at the point $(\xi_0, h(\xi_0))$.

Lemma. *If $M(v)$ is differentiable and strictly concave and $M(v)/v \longrightarrow 0$ for $v \to \infty$,*

$$\frac{\mathrm{d}M(v)}{\mathrm{d}v} = \xi \quad \textit{for} \quad v = v(\xi).$$

Proof. $v(\xi)$ is the abscissa at which the supporting line of slope ξ to the graph of $M(v)$ vs v touches that graph.

Recall that, for the strictly concave functions $M(v)$ we are dealing with here, *we actually have* $M''(v) < 0$ on $(0, \infty)$ – refer to the above construction of $M_1(v)$ and $M_2(v)$ from $M(v)$.

Lemma. *If $M(v)$ is twice continuously differentiable and $M''(v) < 0$ on $(0, \infty)$, and if $M(v)/v \longrightarrow 0$ for $v \to \infty$, $h''(\xi)$ exists for $\xi > 0$.*

Proof. $M(v)$ is certainly strictly concave, so, by the preceding two lemmas, $h'(\xi)$ exists and we have the implicit relation

$$M'(-h'(\xi)) = \xi.$$

Since $M''(v)$ exists, is continuous, and is < 0, we can apply the *implicit function theorem* to conclude that $h''(\xi)$ exists and equals $-1/M''(-h'(\xi))$.

Volberg's construction, besides depending (through Dynkin's theorem) on the concavity of $M(v)$, makes essential use of one *additional special property*, namely, that

$$\boxed{\xi^{-K} \leqslant \mathrm{const}.h(\xi)}$$

for some $K > 1$ as $\xi \to 0$. Let us express this in terms of $M(v)$.

Lemma. *For concave $M(v)$, the preceding boxed relation holds with some $K > 1$ for $\xi \to 0$ iff*

$$M(v) \geqslant \mathrm{const}.v^{K/(K+1)} \quad \textit{for large } v.$$

Proof. Since $M(v)$ is concave, it is equal to $\inf_{\xi>0}(h(\xi) + v\xi)$. If the boxed relation holds and v is large, this expression is $\geqslant \inf_{\xi>0}(\mathrm{const}.\xi^{-K} + v\xi)$

whose value is readily seen to be of the form const.$\nu^{K/(K+1)}$.

To go the other way, compute $\sup_{\nu>0}(\text{const}.\nu^{K/(K+1)} - \nu\xi)$.

Remark. One *might think* that the *concavity of* $M(\nu)$ and the fact that

$$\sum_1^\infty M(n)/n^2 \;=\; \infty$$

together imply that $M(\nu) \geqslant \nu^\rho$ with *some* positive ρ (say $\rho = \frac{1}{2}$) for large ν. That, however, is *not so.* A counter example may easily be constructed by building the graph of $M(\nu)$ vs ν out of exceedingly long straight segments chosen one after the other so as to alternately cut the graph of ν^ρ vs. ν *from below* and *from above.*

Here is one more rather trivial fact which we will have occasion to use.

Lemma. *For* increasing $M(\nu)$,

$$h(\xi) \;\geqslant\; M(0) \quad \textit{for} \quad \xi > 0$$

and hence $\lim_{\xi\to\infty} h(\xi)$ *is* finite *if* $M(0) > -\infty$.

Proof. $h(\xi)$ is *decreasing*, so $\lim_{\xi\to\infty} h(\xi)$ *exists*, but is perhaps equal to $-\infty$. The rest is clear.

The principal result on the connection between $M(\nu)$ and $h(\xi)$ was published independently by Beurling and by Dynkin in 1972. It says that, if $a > 0$ is sufficiently small (so that $\log h(\xi) > 0$ for $0 < \xi \leqslant a$), the convergence of $\int_0^a \log h(\xi)\,d\xi$ is equivalent to that of $\int_1^\infty (M(\nu)/\nu^2)\,d\nu$ (compare with the material in §C of Chapter IV). More precisely:

Theorem. *If* $M(\nu)$ *is* increasing *and* concave, *and*

$$h(\xi) \;=\; \sup_{\nu>0}(M(\nu) - \nu\xi),$$

there is an $a > 0$ *such that*

$$\int_0^a \log h(\xi)\,d\xi \;<\; \infty$$

iff

$$\int_1^\infty \frac{M(\nu)}{\nu^2}\,d\nu \;<\; \infty.$$

Proof. In the first place, if $\lim_{\nu\to\infty} M(\nu)/\nu = c > 0$, the function $h(\xi) = \sup_{\nu>0}(M(\nu) - \nu\xi)$ is *infinite* for $0 < \xi < c$. In this case, the integrals involved in the theorem *both diverge.* For the remainder of the proof we may thus suppose that $M(\nu)/\nu \longrightarrow 0$ as $\nu \to \infty$.

Again, by the *first* lemma of this article, $h(\xi) \longrightarrow \infty$ for $\xi \to 0$ unless $M(\nu)$ is bounded for $v \to \infty$, and in that case both of the integrals in question are obviously finite. There is thus no loss of generality in supposing that $h(\xi) \longrightarrow \infty$ for $\xi \to 0$, and we may take an $a > 0$ with $h(a) \geqslant 2$, say.

These things being granted, let us, as in the previous discussion, approximate $M(v)$ to within ε on $[A, \infty)$, $A > 0$, by an infinitely differentiable *strictly* concave function $M_\varepsilon(v)$, with $M_\varepsilon''(v) < 0$. If $\varepsilon > 0$ and $A > 0$ are small enough, the corresponding function

$$h_\varepsilon(\xi) \;=\; \sup_{v>0}(M_\varepsilon(v) - v\xi)$$

approximates $h(\xi)$ *to within* 1 *unit* (say) on $(0, a]$. But then

$$\int_0^a \log h(\xi)\,d\xi \quad \text{and} \quad \int_0^a \log h_\varepsilon(\xi)\,d\xi$$

converge simultaneously, and the *same* is true for the integrals

$$\int_1^\infty \frac{M(v)}{v^2}\,dv \quad \text{and} \quad \int_1^\infty \frac{M_\varepsilon(v)}{v^2}\,dv.$$

It is therefore enough to establish the theorem for $M_\varepsilon(v)$ and $h_\varepsilon(\xi)$; in other words, we may, wlog, *assume to begin with that $M(v)$ is infinitely differentiable and strictly concave, with $M''(v) < 0$, and that $M(v)/v \longrightarrow 0$ for $v \to \infty$.*

In these circumstances, we can use the relations furnished by the preceding lemmas. It is convenient to work with $\log|h'(\xi)|$ instead of $\log h(\xi)$, so for this purpose let us first show that

$$\int_0^a \log h(\xi)\,d\xi \quad \text{and} \quad \int_0^a \log|h'(\xi)|\,d\xi$$

converge simultaneously. First of all,

$$h(\xi) \;\leqslant\; h(a) + (a-\xi)|h'(\xi)| \;\leqslant\; h(a) + a|h'(\xi)| \quad \text{for} \quad 0 < \xi < a$$

by the *convexity* of $h(\xi)$, as the following diagram shows:

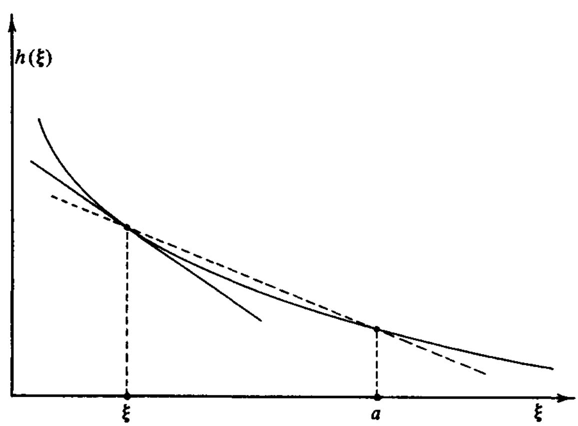

Figure 92

Therefore convergence of the *second* integral implies that of the *first*. Again, for $0 < \xi < a$, $h(\xi) \geqslant 2$, so

$$h\left(\frac{\xi}{2}\right) \;\geqslant\; 2 + \frac{\xi}{2}|h'(\xi)| \;\geqslant\; \frac{\xi}{2}|h'(\xi)|$$

for such ξ:

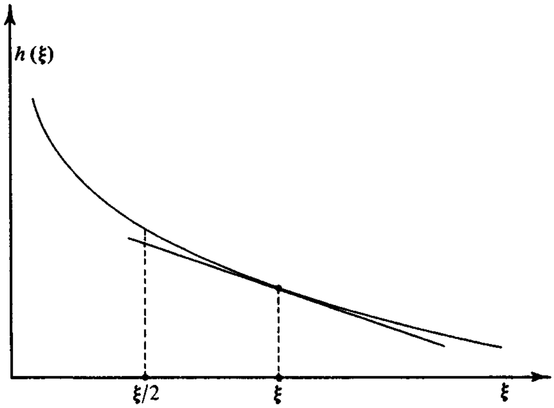

Figure 93

So, since $\int_0^a |\log \xi| \mathrm{d}\xi < \infty$, convergence of the *first* integral implies that of the *second*.

We have

$$h(\xi) \;=\; M(\nu(\xi)) - \xi\nu(\xi)$$

with $\nu(\xi) = -h'(\xi)$, and $M'(\nu(\xi)) = \xi$. Therefore

$$\xi \mathrm{d}\log|h'(\xi)| \;=\; M'(\nu(\xi))\frac{\mathrm{d}\nu(\xi)}{\nu(\xi)}.$$

Taking a number b, $0 < b < a$, and integrating by parts, we find that

$$\int_b^a \xi \mathrm{d}\log|h'(\xi)| \;=\; a\log|h'(a)| - b\log|h'(b)| - \int_b^a \log|h'(\xi)|\mathrm{d}\xi$$

$$= \frac{M(\nu(a))}{\nu(a)} - \frac{M(\nu(b))}{\nu(b)} + \int_{\nu(b)}^{\nu(a)} \frac{M(\nu)}{\nu^2}\mathrm{d}\nu.$$

Here, $\nu(\xi)$ is decreasing, so $\nu(b) \geqslant \nu(a)$. Turning things around, we thus have

$$\int_b^a \log|h'(\xi)|\mathrm{d}\xi + b\log|h'(b)| - a\log|h'(a)|$$

$$= \frac{M(\nu(b))}{\nu(b)} - \frac{M(\nu(a))}{\nu(a)} + \int_{\nu(a)}^{\nu(b)} \frac{M(\nu)}{\nu^2}\mathrm{d}\nu.$$

$M(\nu)/\nu$ is decreasing (concavity of $M(\nu)$!) and, as $b \to 0$, $\nu(b) \to \infty$. We see, then, that

$$\int_0^a \log|h'(\xi)|\,\mathrm{d}\xi \;<\; \infty$$

if

$$\int_{\nu(a)}^{\infty} \frac{M(\nu)}{\nu^2}\,\mathrm{d}\nu \;<\; \infty.$$

Also, $|h'(\xi)|$ *decreases*, so $b\log|h'(b)| \leqslant \int_0^b \log|h'(\xi)|\,\mathrm{d}\xi$. Therefore

$$\int_{\nu(a)}^{\nu(b)} \frac{M(\nu)}{\nu^2}\,\mathrm{d}\nu$$

is bounded above for $b \to 0$ if $\int_0^a \log|h'(\xi)|\,\mathrm{d}\xi < \infty$, i.e., $\int_{\nu(a)}^{\infty}(M(\nu)/\nu^2)\,\mathrm{d}\nu < \infty$. We are done.

Problem 14(b)

Let $H(\xi)$ be decreasing for $\xi > 0$ with $H(\xi) \to \infty$ for $\xi \to 0$, and denote by $h(\xi)$ the *largest convex minorant* of $H(\xi)$. Show that, if, for some small $a > 0$, $\int_0^a \log h(\xi)\,\mathrm{d}\xi < \infty$, then $\int_0^a \log H(\xi)\,\mathrm{d}\xi < \infty$. Hint: Use the following picture:

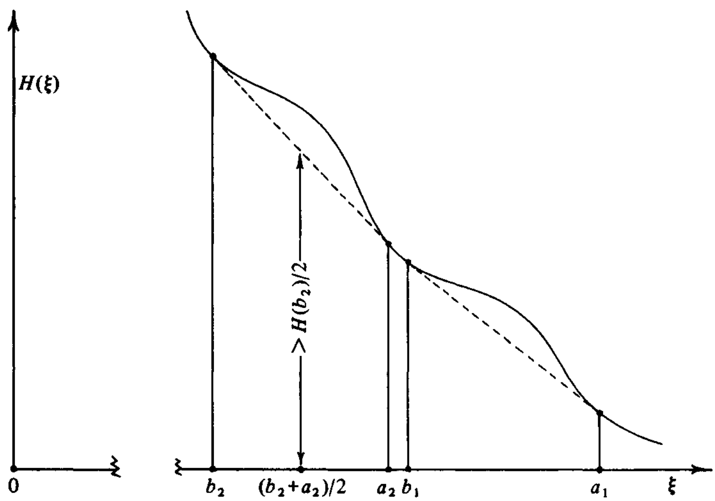

Figure 94

Problem 14(c)

If $M(\nu)$ is increasing, it is in general *false* that $\int_1^{\infty}(M(\nu)/\nu^2)\,\mathrm{d}\nu < \infty$ makes $\int_1^{\infty}(M^*(\nu)/\nu^2)\,\mathrm{d}\nu < \infty$ for the *smallest concave majorant* $M^*(\nu)$ of $M(\nu)$. (Hint: In one counter example, $M^*(\nu)$ has a broken line graph with vertices on the one of $\nu/\log\nu$ (ν large).)

Theorem. *Let $H(\xi)$ be decreasing for $\xi > 0$ and tend to ∞ as $\xi \to 0$. For $v > 0$, put*

$$M(v) \;=\; \inf_{\xi>0}\,(H(\xi) + \xi v).$$

Then

$$\int_0^a \log H(\xi)\,d\xi \;<\; \infty$$

for some (and hence for all) arbitrarily small values of $a > 0$ iff

$$\int_1^\infty \frac{M(v)}{v^2}\,dv \;<\; \infty.$$

Proof. As the infimum of linear functions of v, $M(v)$ *is* concave; it is obviously *increasing*. The function

$$h(\xi) \;=\; \sup_{v>0}\,(M(v) - v\xi)$$

is the largest *convex minorant* of $H(\xi)$ because its height at any abscissa ξ is the supremum of the heights of all the (lower) supporting lines with slopes $-v < 0$ to the graph of H:

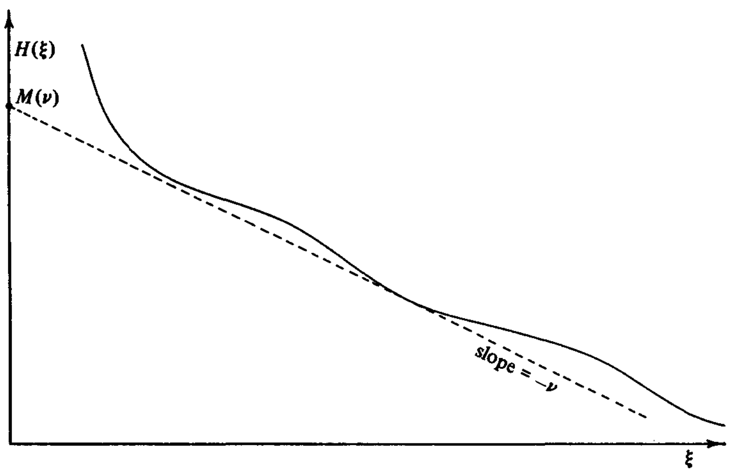

Figure 95

Therefore $\int_0^a \log H(\xi)\,d\xi = \infty$ makes $\int_0^a \log h(\xi)\,d\xi = \infty$ *by problem* 14(b), so in that case $\int_1^\infty (M(v)/v^2)\,dv = \infty$ *by the preceding theorem.* If, on the other hand, $\int_1^\infty (M(v)/v^2)\,dv$ *does diverge*, $\int_0^a \log h(\xi)\,d\xi = \infty$ for each small enough $a > 0$ by that same theorem, so certainly $\int_0^a \log H(\xi)\,d\xi = \infty$ for such a. This does it.

3. Dynkin's extension of $F(e^{i\vartheta})$ to $\{|z| \leqslant 1\}$ with control on $|F_{\bar{z}}(z)|$.

As stated near the beginning of the previous article, a very important role in Volberg's construction is played by a weight $w(r) > 0$ defined for $0 < r < 1$ by the formula

$$w(r) \;=\; \exp\left(-h\left(\log\frac{1}{r}\right)\right),$$

where, for $\xi > 0$,

$$h(\xi) \;=\; \sup_{v>0}(M(v) - v\xi).$$

Here $M(v)$ is an increasing (usually concave) function such that $\int_1^\infty (M(v)/v^2)\,dv = \infty$; this makes $h(\xi)$ increase to ∞ rather rapidly as ξ decreases towards 0, so that $w(r)$ *decreases very rapidly towards zero* as $r \to 1$.

A typical example of the kind of functions $M(v)$ figuring in Volberg's theorem is obtained by putting

$$M(v) \;=\; \frac{v}{\log v}$$

for $v \geqslant e^2$, say, and defining $M(v)$ in any convenient fashion for $0 \leqslant v < e^2$ so as to keep it increasing and concave on that range. Here we find without difficulty that

$$h(\xi) \;\sim\; \frac{\xi^2}{e}e^{1/\xi} \quad \text{for} \quad \xi \longrightarrow 0,$$

and $w(r)$ decreases towards zero like

$$\exp\left(-\frac{(1-r)^2}{e}e^{1/(1-r)}\right)$$

as $r \to 1$; this is *really* fast. It is good to keep this example in mind during the following development.

Lemma. *Let $M(v)$ be increasing and strictly concave for $v > 0$ with $M(v)/v \longrightarrow 0$ for $v \to \infty$, put*

$$h(\xi) \;=\; \sup_{v>0}(M(v) - v\xi),$$

and write $w(r) = \exp(-h(\log(1/r)))$ for $0 < r < 1$. Then

$$\int_0^1 r^{n+2}w(r)\,dr \;>\; \frac{\text{const.}}{n}e^{-M(n)}$$

for $n \geqslant 1$.

Proof. In terms of $\xi = \log(1/r)$, $r^n w(r) = \exp(-h(\xi) - \xi n)$. Since $M(v)$ is strictly concave, we have, by the previous article,

$$\inf_{\xi > 0} (h(\xi) + \xi n) \ = \ M(n),$$

the infimum being attained at the value $\xi = \xi_n = M'(n)$. Put $r_n = e^{-\xi_n}$. Then, $r_n^n w(r_n) = e^{-M(n)}$. Because $w(r)$ decreases, we now see that

$$\int_0^1 r^{n+2} w(r)\,dr \ \geqslant \ w(r_n)\int_0^{r_n} r^{n+2}\,dr \ = \ \frac{r_n^3}{n+3}(r_n^n w(r_n))$$
$$= \ \frac{r_n^3}{n+3} e^{-M(n)}.$$

Here,

$$r_n^3 \ = \ e^{-3M'(n)},$$

and this is $\geqslant e^{-3M'(1)}$ since $M'(v)$ decreases, when $n \geqslant 1$. From the previous relation, we thus find that

$$\int_0^1 r^{n+2} w(r)\,dr \ \geqslant \ \frac{e^{-3M'(1)}}{4n} e^{-M(n)}$$

for $n \geqslant 1$, Q.E.D.

Theorem (Dynkin (the younger), 1972). *Let $M(v)$ be increasing on $(0, \infty)$, as well as strictly concave and infinitely differentiable on $(0, \infty)$, with $M''(v) < 0$ there and $M(v)/v \longrightarrow 0$ for $v \to \infty$. Let $M(0) > -\infty$.*

For $0 < r < 1$, put

$$w(r) \ = \ \exp\left(-h\left(\log\frac{1}{r}\right)\right),$$

where $h(\xi)$ is related to $M(v)$ in the usual fashion.

Suppose that

$$F(e^{i\vartheta}) \ \sim \ \sum_{-\infty}^{\infty} a_n e^{in\vartheta}$$

is continuous on the unit circumference, and that

$$\sum_1^{\infty} |n^2 a_{-n}| e^{M(n)} \ < \ \infty.$$

Then F has a continuous extension $F(z)$ onto $\{|z| \leqslant 1\}$ with $F(z)$ continuously differentiable for $|z| < 1$ and $|\partial F(z)/\partial \bar{z}| \leqslant \text{const.}w(|z|)$, $|z| < 1$.

Remark. The sense of Dynkin's theorem is that *rapid growth* of $M(n)$

to ∞ for $n \to \infty$ (which corresponds to *rapid growth* of $h(\xi)$ to ∞ for ξ tending to 0) makes it possible to extend F continuously to $\{|z| \leqslant 1\}$ in such a way as to have $|\partial F(z)/\partial \bar{z}|$ *dropping off to zero very quickly for* $|z| \to 1$.

Proof of theorem. We start by taking a continuously differentiable function $\Omega(e^{it})$, to be determined presently, and putting*

$$(*) \qquad G(z) = \sum_0^\infty a_n z^n + \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 \frac{\Omega(e^{it})r^2 w(r) r\,dr\,dt}{re^{it} - z}$$

for $|z| \leqslant 1$. The reason for using the factor r^2 with $w(r)$ will soon be apparent.

The idea now is to specify $\Omega(e^{it})$ in such fashion as to make $G(e^{i\vartheta})$ have the same Fourier series as $F(e^{i\vartheta})$. If we can do that, the function $G(z)$ will be a continuous extension of $F(e^{i\vartheta})$ to $\{|z| \leqslant 1\}$.

To see this, observe that our hypothesis certainly makes the trigonometric series

$$\sum_{-\infty}^{-1} a_n e^{in\vartheta}$$

absolutely convergent, so, since $F(e^{i\vartheta})$ is continuous,

$$\sum_0^\infty a_n e^{in\vartheta}$$

must also be the Fourier series of some continuous function, and hence the power series on the right in $(*)$ a continuous function of z for $|z| \leqslant 1$. According to a lemma from the previous article, the property $M(0) > -\infty$ makes $h(\xi)$ *bounded below* for $\xi > 0$ and hence $w(r)$ *bounded above* in $(0, 1)$. The right-hand integral in $(*)$ is thus of the form

$$\frac{1}{2\pi}\iint_{|\zeta|<1} \frac{b(\zeta)\,d\xi\,d\eta}{\zeta - z}$$

with a *bounded function* $b(\zeta)$. (Here, we are writing $\zeta = \xi + i\eta$ which *conflicts* with our frequent use of ξ to denote $\log(1/r)$. *No confusion should thereby result.*) It is well known that such an integral gives a *continuous* function of z; that's because it's a *convolution* on $\mathbb{R}^2$, with

$$\iint_{|\zeta|<R} \frac{d\xi\,d\eta}{|\zeta|}$$

finite for each finite R. We see in this way that the function $G(z)$ given by $(*)$ *will* be continuous for $|z| \leqslant 1$. If also the Fourier series of $G(e^{i\vartheta})$ and $F(e^{i\vartheta})$ coincide, those two functions must obviously be equal.

We wish to apply the theorem of article 1 to the right-hand integral in $(*)$. In order to stay honest, we should therefore check *continuous*

* The power series on the right in $(*)$ may not actually be *convergent* for $|z| = 1$, but *does* represent a *continuous function* for $|z| \leqslant 1$, as will be clear in a moment.

differentiability (for $|\zeta| < 1$) of the function $b(\zeta)$ figuring in that double integral, viz.,

$$b(re^{it}) = r^2 w(r)\Omega(e^{it}),$$

because continuous differentiability (at least) is required in the hypothesis of the theorem. It is for this purpose that the factor r^2 has been included; that factor ensures differentiability of $b(\zeta)$ at 0. Thanks to it, the desired property of $b(\zeta)$ follows from the continuous differentiability of $\Omega(e^{it})$ together with the continuity of $rw'(r)$ on $[0, 1)$ which we now verify.

We have $rw'(r) = h'(\log(1/r))w(r)$ for $0 < r < 1$. By a lemma in the previous article, $h''(\xi)$ exists for each $\xi > 0$ since $M''(v) < 0$. This certainly makes $h'(\log(1/r))$ continuous for $0 < r < 1$, so $rw'(r)$ is continuous for such r. When $r \to 0$, $w(r)$ *increases* and tends to a *finite* limit (since $M'(0) > -\infty$), and $h'(\log(1/r))$ *increases* (convexity of $h'(\xi)$), remaining, however, always $\leqslant 0$. Hence $rw'(r)$ *tends to a finite limit* as $r \to 0$, and (with obvious definition of $rw'(r)$ *at* the origin) is thus continuous at 0.

Having justified the application of the theorem from article 1 by this rather fussy argument, we see through its use that

$$\frac{\partial G(z)}{\partial \bar{z}} = -|z|^2 w(|z|)\Omega(e^{i\vartheta})$$

for

$$z = |z|e^{i\vartheta}, \quad |z| < 1,$$

after taking account of the fact that

$$\frac{\partial}{\partial \bar{z}}\left(\sum_0^\infty a_n z^n\right) = 0, \qquad |z| < 1.$$

This relation certainly makes

$$\left|\frac{\partial G(z)}{\partial \bar{z}}\right| \leqslant \text{const.}w(|z|)$$

for $|z| < 1$, so, if $G(z)$ *does* coincide with $F(z)$ for $|z| = 1$, *we will* have the theorem on putting $F(z) = G(z)$ for $|z| < 1$.

Everything thus depends on our being able to determine a continuously differentiable $\Omega(e^{it})$ which will make

$$(*) \qquad \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 \frac{\Omega(e^{it})r^2 w(r) r\,dr\,dt}{re^{it} - e^{i\vartheta}}$$

have the Fourier series

$$\sum_{-\infty}^{-1} a_n e^{in\vartheta}.$$