CHAPTER 10

Composition operators

DEFINITION 10.1. Let $\mathcal{K}$ be a Hilbert space of analytic functions on X with reproducing kernel k and let $\varphi : X \to X$ be an analytic function. To φ we associate a *composition operator* C_φ given by $C_\varphi(f) := f \circ \varphi$.

The study of such operators was originally inspired by the following result.

THEOREM 10.2. **(Littlewood's subordination principle)** *For any analytic $\varphi : \mathbb{D} \to \mathbb{D}$, the operator C_φ is bounded on $H^2(\mathbb{D})$.*

J. Shapiro proved in 1987 [**Sha87**] that C_φ is compact on $H^2(\mathbb{D})$, if and only if "φ does not get too close to $\partial\mathbb{D}$ too often."

An interesting property of composition operators is that their adjoints permute the kernel functions. Indeed,

$$\begin{aligned} \langle f, C_\varphi^* k_\zeta \rangle &= \langle C_\varphi f, k_\zeta \rangle \\ &= \langle f \circ \varphi, k_\zeta \rangle \\ &= f(\varphi(\zeta)) \\ &= \langle f \circ \varphi, k_\zeta \rangle \\ &= \langle f, k_{\varphi(\zeta)} \rangle, \end{aligned}$$

so $C_\varphi^* k_\zeta = k_{\varphi(\zeta)}$.

Recently, various properties of C_φ were studied in terms of properties of φ on the Hardy space, the Dirichlet space and the Bergman space.

We now gather some results about composition operators on $\mathcal{H}^2$. Let $\Phi : \Omega_{1/2} \to \Omega_{1/2}$ be an analytic function. Note that $C_\Phi : f \mapsto f \circ \Phi$ might not map Dirichlet series to Dirichlet series. Indeed, if $f \sim \sum_{n=1}^\infty a_n n^{-s}$, then $(f \circ \Phi) \sim \sum_n a_n n^{-\Phi(s)}$. The next two theorems are due to J. Gordon and H. Hedenmalm [**GH99**].

THEOREM 10.3. *An analytic function $\Phi : \Omega_{1/2} \to \Omega_{1/2}$ gives rise to a composition operator $C_\Phi : \mathcal{H}^2 \to \mathcal{D}$, if and only if $\Phi(s) = c_0 s + \varphi(s)$, where $c_0 \in \mathbb{N}$ and $\varphi \in \mathcal{D}$.*

THEOREM 10.4. **(Gordon, Hedenmalm)** *An analytic function* $\Phi : \Omega_{1/2} \to \Omega_{1/2}$ *gives rise to a bounded composition operator* $C_\Phi : \mathcal{H}^2 \to \mathcal{H}^2$*, if and only if* $\Phi(s) = c_0 s + \varphi(s)$*, where* $c_0 \in \mathbb{N}$*,* $\varphi \in \mathcal{D}$*, and* Φ *has an analytic extension to* Ω_0 *such that* $\Phi(\Omega_0) \subset \Omega_0$*, if* $c_0 > 0$ *and* $\Phi(\Omega_0) \subset \Omega_{1/2}$*, if* $c_0 = 0$.

They also proved that C_Φ is a contraction (i.e., $\|C_\Phi\| \leq 1$), if and only if $c_0 > 0$ in the above theorem. Furthermore, the same theorem holds for $\mathcal{H}^p$ with $2 \leq p < \infty$ and the conditions are necessary for $1 < p < 2$.

Compactness of composition operators was studied by F. Bayart. He proved the following theorem [**Bay03**]:

THEOREM 10.5. **(Bayart)** *The composition operator* C_Φ *is compact on* $\mathrm{Mult}\,(\mathcal{H}^2_w)$*, if and only if* $\Phi(\Omega_0) \subset \Omega_\varepsilon$*, for some* $\varepsilon > 0$.

He also proved that if C_Φ is a compostion operator on $\mathcal{H}^2$, then $\mathcal{Q} C_\Phi \mathcal{Q}^{-1}$ is a composition operator on $H^2(\mathbb{T}^\infty)$, i.e., there exists $\psi : \mathbb{D}^\infty \cap \ell^2 \to \mathbb{D}^\infty \cap \ell^2$ such that $C_\psi = \mathcal{Q} C_\Phi \mathcal{Q}^{-1}$. This allows one to construct compact composition operators on $\mathcal{H}^2$ that are not Hilbert-Schmidt.

CHAPTER 11

Appendix

11.1. Multi-index Notation

When dealing with power series in several variables, it is easy to become overwhelmed with subscripts. Multi-index notation is a way to make formulas easier to read.

We fix the number of variables, d say, and assume that is understood. We write

$$\alpha \ = \ (\alpha_1, \alpha_2, \dots, \alpha_d)$$

for a multi-index, where α is in $\mathbb{N}^d$ or $\mathbb{Z}^d$. Then

$$\sum c_\alpha z^\alpha$$

stands for

$$\sum c_{\alpha_1, \alpha_2, \dots, \alpha_d} z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_d^{\alpha_d}.$$

We define

$$\begin{aligned} |\alpha| &= \sum_{r=1}^{d} |\alpha_r| \\ \alpha! &= \alpha_1! \alpha_2! \cdots \alpha_d! \end{aligned}$$

11.2. Schwarz-Pick lemma on the polydisk

Schwarz's lemma on the disk has a non-infinitesimal version, called the Schwarz-Pick lemma. Both these lemmata generalize to the polydisk.

Lemma 11.1. **(Schwarz-Pick)** *If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, then*

$$\left| \frac{f(w) - f(z)}{1 - \overline{f(w)} f(z)} \right| \ \leq \ \left| \frac{w - z}{1 - \overline{w} z} \right|,$$

for all $z, w \in \mathbb{D}$.

Proof: For $\xi \in \mathbb{D}$, let ψ_ξ be the automorphism of the disk that exchanges 0 and ξ, that is, $\psi_\xi(z) = \frac{\xi - z}{1 - \bar{\xi} z}$. Consider the function g :

$\mathbb{D} \to \mathbb{D}$ given by $g = \psi_{f(w)} \circ f \circ \psi_w$. Choose $\zeta = \psi_w(z)$ so that

$$|g(\zeta)| = |(\psi_{f(w)} \circ f \circ \psi_w)(\psi_w(z))| = |\psi_{f(w)}(f(z))| = \left|\frac{f(w) - f(z)}{1 - \overline{f(w)}f(z)}\right|$$

and

$$|\zeta| = |\psi_w(z)| = \left|\frac{w - z}{1 - \overline{w}z}\right|.$$

Also, $g(0) = 0$ so that $|g(\zeta)| \leq |\zeta|$ by the classical Schwarz lemma. □

LEMMA 11.2. **Schwarz's lemma on the polydisk** *Let $f \in H^\infty(\mathbb{D}^N)$ satisfies $\|f\|_\infty \leq 1$ and $f(0) = 0$. Then*

$$|f(w_1, \ldots, w_N)| \leq \max_{1 \leq i \leq N} |w_i|.$$

Proof: Let

$$r = \max_{i=1,\ldots,N} |w_i|.$$

Define $g \in H^\infty(\mathbb{D})$ by

$$g(z) := f(\frac{z}{r}(w_1, \ldots, w_N)).$$

Then $\|g\|_\infty \leq 1$, and $g(0) = 0$. Apply Schwarz's lemma to g to conclude $|g(r)| \leq r$. □

LEMMA 11.3. *Let $f \in H^\infty(\mathbb{D})$ satisfies $\|f\|_\infty \leq K$ and $f(0) = 1$. Then $f \neq 0$ on $\frac{1}{K}\mathbb{D}$.*

Proof: We may assume that g is non-constant. Consider $g(z) = \frac{f(z)}{K}$, then $g(0) = 1/K$ and $g : \mathbb{D} \to \mathbb{D}$. If $f(z) = 0$, then, by the Schwarz-Pick lemma applied to g and $w = 0$

$$\frac{1}{K} = \left|\frac{\frac{1}{K}f(0) - 0}{1 - \overline{f(0)/K} \cdot 0}\right| = \left|\frac{g(0) - g(z)}{1 - \overline{g(0)}g(w)}\right| \leq \left|\frac{0 - z}{1 - 0 \cdot z}\right| = |z|.$$

Thus f cannot vanish on $\frac{1}{K}\mathbb{D}$. □

LEMMA 11.4. *Let $f \in H^\infty(\mathbb{D}^N)$ satisfies $\|f\|_\infty \leq K$ and $f(0) = 1$. Then $f \neq 0$ on $\frac{1}{K}\mathbb{D}^N$.*

Proof: Fix $w = (w_1, \ldots, w_N) \in \mathbb{D}^N$, and define $|w|_\infty = \max_{i=1,\ldots,N} |w_i|$. Define $g \in H^\infty(\mathbb{D})$ by $g(z) := f(\frac{zw}{|w|_\infty})$, then $\|g\|_\infty \leq K$. If $f(w) = 0$, then $g(|w|_\infty) = 0$. Thus, by the preceding lemma, $|w|_\infty \geq 1/K$. □

11.3. Reproducing kernel Hilbert spaces

Let $\mathcal{H}$ be a Hilbert space of functions on a set X such that evaluation at each point of X is continuous. (Note: when we speak of a Hilbert space of functions on X, we assume that any function that is identically zero on X is zero in the Hilbert space). Then by the Riesz representation theorem, for each $w \in X$, there must be some function $k_w \in \mathcal{H}$ such that

$$f(w) \;=\; \langle f, k_w \rangle.$$

One can think of k_w as a function in its own right, $k_w(z)$ say. We call the function $k(z,w) = k_w(z)$ the kernel function for $\mathcal{H}$, and we call k_w the reproducing kernel at w.

Proposition 11.5. *Let $\mathcal{H}$ be a Hilbert function space on X, and let $\{e_i\}_{i \in \mathcal{I}}$ be any orthonormal basis for $\mathcal{H}$. Then*

$$k(z,w) \;=\; \sum_{i \in \mathcal{I}} \overline{e_i(w)} e_i(z). \tag{11.6}$$

Proof: This is just Parseval's identity:

$$\begin{aligned} k(z,w) \;&=\; \langle k_w, k_z \rangle \\ &= \sum_{i \in \mathcal{I}} \langle k_w, e_i \rangle \, wa e_i, k_\zeta \rangle \\ &= \sum_{i \in \mathcal{I}} \overline{e_i(w)} e_i(z). \end{aligned}$$ □

It follows from (11.6) that $k(z,w) = \overline{k(w,z)}$.

Proposition 11.7. *Let $\mathcal{H}$ be a Hilbert space of analytic functions on a topological space X such that the function $\kappa : X \to \mathcal{H}$ given by $\kappa(w) := k_w$ is continuous. Let $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$ be a bounded sequence. Then, the following are equivalent*

(1) *$\langle f_n, g \rangle \to \langle f, g \rangle$ for all g in some set $S \subset \mathcal{H}$, whose span is dense in $\mathcal{H}$,*
(2) *$f_n \to f$ weakly in $\mathcal{H}$,*
(3) *$f_n \to f$ uniformly on compact subsets of X,*
(4) *$f_n \to f$ pointwise in X.*

Proof: (1) $\implies$ (2) : By linearity, $\langle f_n, g \rangle \to \langle f, g \rangle$ for all $g \in$ span S. Now choose an arbitrary $g \in \mathcal{H}$, fix $\varepsilon > 0$ and find $g_0 \in$ span S such that $\|g - g_0\| < \varepsilon$. Then

$$\begin{aligned} \lim_{n \to \infty} |\langle f_n - f, g \rangle| \quad &\le \quad \lim_{n \to \infty} |\langle f_n - f, g - g_0 \rangle| \;+\; \lim_{n \to \infty} |\langle f_n - f, g_0 \rangle| \\ &\le \quad \lim_{n \to \infty} \|f_n - f\| \cdot \|g - g_0\| \;+\; 0 \end{aligned}$$

$$\leq \quad M\varepsilon,$$

where $M = \sup_{n\in\mathbb{N}} \|f_n\|$. Since ε was arbitrary, we conclude that $f_n \to f$ weakly.

(2) $\implies$ (3) : Let $K \subset X$ be compact, then by continuity of κ, the set $\tilde{K} := \{k_w;\ w \in K\}$ is also compact. Fix $\varepsilon > 0$ and find a finite ε-net $\{k_{w_1}, \ldots, k_{w_m}\}$ in $\tilde{K}$. Find $N \in \mathbb{N}$ such that for all $n > N$ $\langle f_n - f, k_{w_j}\rangle < \varepsilon$ holds for $j = 1, \ldots, m$. Then for any $w \in K$ and $n > N$:

$$\begin{aligned} |f_n(w) - f(w)| &= |\langle f_n - f, k_w\rangle| \\ &\leq |\langle f_n - f, k_{w_i}\rangle| + |\langle f_n - f, k_w - k_{w_i}\rangle| \\ &\leq \varepsilon + \|f_n - f\| \cdot \|k_w - k_{w_i}\| \\ &\leq \varepsilon + 2M\varepsilon \\ &= (2M+1)\varepsilon, \end{aligned}$$

for a suitable i (such i exists since $\{k_{w_1}, \ldots, k_{w_m}\}$ is an ε-net). Since $\varepsilon > 0$ was arbitrary, we conclude that $f_n \to f$ uniformly in K.

(3) $\implies$ (4) : Obvious.

(4) $\implies$ (1) : Follow immediately, since (4) means that (1) holds with $S = \{k_w\}_{w\in X}$ □

Corollary 11.8. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a bounded sequence with $\mathcal{H}$ as in Proposition 11.7. Then there exists a subsequence that satisfies all the equivalent conditions of Proposition 11.7.*

Proof: Since any bounded set in a Hilbert space weakly sequentially compact, there exists a subsequence that converges weakly. By Proposition 11.7, it satisfies all four conditions. □

11.4. Multiplier Algebras

If $\mathcal{H}$ is a Hilbert space of functions on X, we let $\mathrm{Mult}(\mathcal{H})$ denote the multiplier algebra, *i.e.* the set

$$\mathrm{Mult}(\mathcal{H}) \ = \ \{\phi : \phi f \in \mathcal{H}\ \forall\ f \in \mathcal{H}\}.$$

It follows from the closed graph theorem that if ϕ is in $\mathrm{Mult}(\mathcal{H})$, then the operator M_ϕ of multiplication by ϕ is bounded. The adjoint M_ϕ^* has all the kernel functions as eigenvectors.

Proposition 11.9. *Let $\mathcal{H}$ be a Hilbert function space on X, and let ϕ be in* $\mathrm{Mult}(\mathcal{H})$*. Then*

$$M_\phi^* k_w \ = \ \overline{\phi(w)} k_w, \quad \forall\ w \in X. \tag{11.10}$$

$$\|M_\phi\| \ \geq \ \sup_X |\phi|. \tag{11.11}$$

If the norm on $\mathcal{H}$ is an L^2-norm on X, then (11.11) *becomes an equality.*

PROOF: Let f be an arbitrary function in $\mathcal{H}$. Then

$$\begin{aligned} \langle f, M_\phi^* k_w \rangle &= \langle \phi f, k_w \rangle \\ &= \phi(w) f(w) \\ &= \langle f, \overline{\phi(w)} k_w \rangle. \end{aligned}$$

This proves (11.10).

As

$$\begin{aligned} \|M_\phi^*\| &\geq \sup_{w \in X} \|M_\phi^* k_w\| / \|k_w\| \\ &= \sup_{w \in X} |\phi(w)|, \end{aligned}$$

we get (11.11).

Finally, if the norm on $\mathcal{H}$ is the $L^2(\mu)$-norm, then the inequality

$$\int_X |\phi f|^2 d\mu \;\leq\; \|\phi\|_\infty^2 \int_X |f|^2 d\mu$$

means $\|M_\phi\| \leq \|\phi\|_\infty$. □

PROPOSITION 11.12. *Let $\mathcal{H}$ be a Hilbert function space on X, and assume* Mult$(\mathcal{H})$ *separates the points of X. Then* Mult$(\mathcal{H})$ *equals its commutant in the bounded linear operators on $\mathcal{H}$.*

PROOF: Suppose T is in the commutant of Mult$(\mathcal{H})$. Then T^* has each kernel function k_w as an eigenvector, since Mult$(\mathcal{H})$ separates the points of X. Therefore

$$T^* k_w \;=\; \overline{\phi(w)} k_w,$$

for some function ϕ. Therefore $T = M_\phi$, and since T is bounded, this means ϕ is a multipler. □

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