Expanding the integrand from (‡) in powers of re^{it}, we obtain for that expression the value

$$-\sum_{n=1}^{\infty}\left(\int_0^1 r^{n+2}w(r)\,dr\right)\left(\frac{1}{2\pi}\int_0^{2\pi} e^{i(n-1)t}\Omega(e^{it})\,dt\right)e^{-in\vartheta}.$$

We see that we need to have

$$\Omega(e^{it}) \sim \sum_{-\infty}^{\infty} b_n e^{int}$$

where, for $n > 1$,

$$(\dagger) \qquad b_{1-n} = -\frac{a_{-n}}{\int_0^1 r^{n+2}w(r)\,dr}.$$

We may choose the b_n with *positive* index in any manner compatible with the continuous differentiability of $\Omega(e^{it})$; let us simply *put them all equal to zero.*

By the lemma, the *right side* of (†) is in modulus

$$\leqslant \text{const.}|na_{-n}|e^{M(n)}$$

for $n \geqslant 1$. The b_m given by (†) therefore satisfy

$$\sum_{-\infty}^{0} |mb_m| < \infty$$

according to the hypothesis of our theorem. This means that there *is* a function $\Omega(e^{it})$ satisfying our requirements whose *differentiated Fourier series is absolutely convergent.* Such a function is surely continuously differentiable; that is what was needed.

The theorem is proved.

Remark 1. We are going to use the extension of F to $\{|z| < 1\}$ furnished by Dynkin's theorem in conjunction with the *corollary at the end of article* 1. That corollary involves the integral

$$\frac{1}{2\pi}\iint_{\mathscr{D}} \frac{F_{\bar\zeta}(\zeta)}{F(\zeta)}\frac{d\xi\,d\eta}{(\zeta - z)}$$

where, on the open set $\mathscr{D}$, $|F(\zeta)| > 0$ and $|F_{\bar\zeta}(\zeta)| \leqslant \text{const.}|F(\zeta)|$. The theorem of article 1 is used to take $\partial/\partial\bar z$ of this integral, and the *hypothesis of the corollary* requires that $F(\zeta)$ be $\mathscr{C}_2$ in $\mathscr{D}$ in order to guarantee the legitimacy of that theorem's application. *Our extension $F(z)$ furnished by Dynkin's theorem is, however, only ensured to be $\mathscr{C}_1$* for $|z| < 1$. Are we not in *trouble*?

Not to worry. All that the corollary really *uses* is continuous differentiability of the *quotient* $F_{\bar\zeta}(\zeta)/F(\zeta)$ in $\mathscr{D}$. Our F *is*, however, $\mathscr{C}_1$ in $\mathscr{D}$, where it is also $\neq 0$. And, *from the proof of Dynkin's theorem,* $F_{\bar{\zeta}}(\zeta) = -|\zeta|^2 w(|\zeta|)\Omega(e^{it})$ for $\zeta = |\zeta|e^{it}$ with $|\zeta| < 1$. *The expression on the right is, however,* $\mathscr{C}_1$ for $|\zeta| < 1$; we indeed checked that it *had* that property during the proof (as we *had* to do in order to justify using the theorem of article 1 to show that $F_{\bar{\zeta}}(\zeta)$ was equal to it!). We are all right.

Remark 2. Under the conditions of Volberg's theorem, there is no essential distinction between the functions $M(v)$ and $2M(v)$, and $e^{M(v)}$ goes to infinity much faster than any power of v as $v \to \infty$. (We will need to require that $M(v) \geqslant \text{const.}v^{\alpha}$ for some $\alpha > \frac{1}{2}$ as has already been remarked in article 2.) In the application of Dynkin's theorem to be made below, we will therefore be able to replace the condition

$$\sum_1^{\infty} |n^2 a_{-n}| e^{M(n)} \;<\; \infty$$

figuring in its hypothesis by

$$|a_{-n}| \;\leqslant\; \text{const.}e^{-2M(n)}, \qquad n \geqslant 1,$$

or even (after a suitable unessential modification in the description of $w(r)$) by

$$|a_{-n}| \;\leqslant\; \text{const.}e^{-M(n)}, \qquad n \geqslant 1.$$

4. Material about weighted planar approximation by polynomials

Lemma. *Let* $w(r) \geqslant 0$ *for* $0 \leqslant r < 1$, *with*

$$\int_0^1 w(r) r \, dr \;<\; \infty.$$

If $F(z)$ *is any function analytic in* $\{|z| < 1\}$ *such that*

$$\iint_{|z|<1} |F(z)|^2 w(|z|) \, dx \, dy \;<\; \infty,$$

there are polynomials $Q(z)$ *making*

$$\iint_{|z|<1} |F(z) - Q(z)|^2 w(|z|) \, dx \, dy$$

arbitrarily small.

Proof. The basic idea is that $w(|z|)$ depends only on the modulus of z.

Given $\varepsilon > 0$, take $\rho < 1$ so close to 1 that

$$\iint_{\rho < |z| < 1} |F(z)|^2 w(|z|)\,dx\,dy \;<\; \varepsilon.$$

Note that if $0 < \lambda < 1$ and $0 < r < 1$,

$$\int_0^{2\pi} |F(\lambda r e^{i\vartheta})|^2\,d\vartheta \;\leqslant\; \int_0^{2\pi} |F(re^{i\vartheta})|^2\,d\vartheta.$$

Therefore,

$$\iint_{\rho < |z| < 1} |F(\lambda z)|^2 w(|z|)\,dx\,dy \;=\; \int_\rho^1 \int_0^{2\pi} |F(\lambda r e^{i\vartheta})|^2 w(r) r\,d\vartheta\,dr$$

$$\leqslant\; \int_\rho^1 \int_0^{2\pi} |F(re^{i\vartheta})|^2 w(r) r\,d\vartheta\,dr \;<\; \varepsilon.$$

Once $\rho < 1$ has been fixed, $F(z)$ is *uniformly continuous* for $|z| \leqslant \rho$, so $\iint_{|z|<\rho} |F(z) - F(\lambda z)|^2 w(|z|)\,dx\,dy \longrightarrow 0$ as $\lambda \to 1$ (we use the integrability of $rw(r)$ on $[0, 1)$ here). In view of the preceding calculation, we can thus find (and fix) a $\lambda < 1$ such that

$$\iint_{|z|<1} |F(z) - F(\lambda z)|^2 w(|z|)\,dx\,dy \;<\; 5\varepsilon.$$

The Taylor series for $F(\lambda z)$ converges *uniformly* for $|z| \leqslant 1$. We may therefore take a suitable *partial sum* $Q(z)$ of that Taylor series so as to make

$$\iint_{|z|<1} |F(\lambda z) - Q(z)|^2 w(|z|)\,dx\,dy \;<\; \varepsilon.$$

Then

$$\iint_{|z|<1} |F(z) - Q(z)|^2 w(|z|)\,dx\,dy \;<\; 16\varepsilon.$$

That does it.

At this point, we begin to make systematic use of a corollary to the theorem of Levinson given in §A.5. Oddly enough, Beurling's stronger results from §B are never called for in Volberg's work.

Theorem on simultaneous polynominal approximation (Kriete, Volberg). *Let, for* $0 < r < 1$,

$$w(r) \;=\; \exp\left(-H\left(\log\frac{1}{r}\right)\right)$$

where $H(\xi)$ is decreasing and bounded below on $(0, \infty)$, *and suppose that*

$$\int_0^a \log H(\xi)\,\mathrm{d}\xi \;=\; \infty$$

for all sufficiently small $a > 0$.

Let E be any proper closed subset of the unit circumference, let $p(\mathrm{e}^{\mathrm{i}\vartheta}) \in L_2(E)$, and suppose that $f(z)$ is analytic in $\{|z| < 1\}$, and such that

$$\iint_{|z|<1} |f(z)|^2 w(|z|)\,\mathrm{d}x\,\mathrm{d}y \;<\; \infty.$$

Then there is a sequence of polynomials $P_n(z)$ with

$$\int_E |p(\mathrm{e}^{\mathrm{i}\vartheta}) - P_n(\mathrm{e}^{\mathrm{i}\vartheta})|^2\,\mathrm{d}\vartheta + \iint_{|z|<1} |f(z) - P_n(z)|^2 w(|z|)\,\mathrm{d}x\,\mathrm{d}y \;\xrightarrow[n]{}\; 0.$$

Proof. We use the fact that the collection of functions $F(z)$ analytic in $\{|z| < 1\}$ with $\iint_{|z|<1} |F(z)|^2 w(|z|)\,\mathrm{d}x\,\mathrm{d}y < \infty$ forms a Hilbert space if we bring in the inner product

$$\langle F_1, F_2\rangle_w \;=\; \iint_{|z|<1} F_1(z)\overline{F_2(z)}\,w(|z|)\,\mathrm{d}x\,\mathrm{d}y.$$

This is evident, except perhaps for the completeness property. To verify the latter, it is clearly enough to show that, for any L, the functions $F(z)$ analytic in $\{|z| < 1\}$ and satisfying

$$\iint_{|z|<1} |F(z)|^2 w(|z|)\,\mathrm{d}x\,\mathrm{d}y \;\leqslant\; L$$

form a *normal family* in the open unit disk. However, the weight $w(r)$ we are using is *strictly positive* and *decreasing* on $(0, 1)$. Hence, for any $r < 1$, the previous relation makes

$$\iint_{|z|<r} |F(z)|^2\,\mathrm{d}x\,\mathrm{d}y \;\leqslant\; \frac{L}{w(r)}.$$

It is well known that such functions $F(z)$ form a normal family in $\{|z| < r\}$. Here, $r < 1$ is arbitrary.

Let us turn to the proof of the theorem, reasoning by duality in the Hilbert space $L_2(E) \oplus \mathscr{H}$ where $\mathscr{H}$ is the Hilbert space just described.* Suppose, then, that there is a $p(\mathrm{e}^{\mathrm{i}\vartheta}) \in L_2(E)$ and an $f(z) \in \mathscr{H}$ for which the *conclusion of the theorem fails to hold.* There must then be a *non-zero* element (q, F) of $L_2(E) \oplus \mathscr{H}$ *orthogonal* in that space to *all* the elements of the form $(P(\mathrm{e}^{\mathrm{i}\vartheta}),\ P(z))$ with *polynomials* P. We are going to obtain a *contradiction* by showing that *in fact* $q = 0$ and $F = 0$.

* We are dealing here with the *direct sum* of $L_2(E)$ and $\mathscr{H}$.

The orthogonality in question is equivalent to the relations

$$\int_E \overline{q(e^{i\vartheta})}e^{in\vartheta}d\vartheta + \iint_{|z|<1} \overline{F(z)}z^n w(|z|)\,dx\,dy = 0, \qquad n = 0, 1, 2, 3, \ldots .$$

Define $q(e^{i\vartheta})$ for *all* of $\{|z|=1\}$ by making it *zero* for $e^{i\vartheta}\notin E$. Then $q(e^{i\vartheta})\in L_2(-\pi,\pi)$, and if we write

$$(*) \qquad q(e^{i\vartheta}) \sim \sum_{-\infty}^{\infty} \alpha_n e^{in\vartheta}$$

we find from the previous relation that

$$\bar{\alpha}_n = -\frac{1}{2\pi}\iint_{|z|<1} \overline{F(z)}z^n w(|z|)\,dx\,dy, \qquad n \geqslant 0.$$

Since

$$\iint_{|z|<1} |F(z)|^2 w(|z|)\,dx\,dy < \infty,$$

the integral on the right is in modulus

$$\leqslant \text{ const.}\sqrt{\int_0^1 r^{2n}w(r)r\,dr}$$

by Schwarz' inequality. However, in terms of $\xi = \log(1/r)$ and the function $H(\xi)$,

$$r^{2n}w(r) = e^{-(H(\xi)+2n\xi)}.$$

Denoting $\inf_{\xi>0}(H(\xi)+\xi v)$ by $M(v)$ as in the last theorem of article 2, we see that the right side is $\leqslant e^{-M(2n)}$. The preceding expression is therefore $\leqslant$ const.$e^{-M(2n)/2}$, i.e.

$$(\ast\!\ast) \qquad |\alpha_n| \leqslant \text{const.}e^{-M(2n)/2}, \qquad n \geqslant 1.$$

Since E is a *proper closed subset of* $\{|z|=1\}$, *its complement on the unit circumference contains an arc J of positive length.* The function $q(e^{i\vartheta})$ *vanishes* outside E, hence on J, and certainly belongs to $L_1(-\pi,\pi)$. Also, *by the last theorem in article 2,*

$$\int_1^\infty \frac{M(v)}{v^2}dv = \infty$$

on account of our hypothesis on $H(\xi)$. Therefore

$$\sum_1^\infty \frac{M(2n)}{2n^2} = \infty,$$

$M(v)$ being increasing, so, *by virtue of the corollary at the end of* §A.5, $(*)$ and $(\substack{*\\*})$ imply that $q(e^{i\vartheta}) = 0$ a.e.

We see that $\alpha_n = 0$ for all n, which means that

$$\iint_{|z|<1} \overline{F(z)}\, z^n w(|z|)\, dx\, dy \;=\; 0$$

for $n = 0, 1, 2, \ldots$. Since $H(\xi)$ is bounded below on $(0, \infty)$, $w(r)$ is *bounded above* for $0 < r < 1$, and we can invoke the lemma, concluding that *polynomials are dense in the Hilbert space* $\mathscr{H}$. The previous relation therefore implies that $F(z) \equiv 0$.

We have thus reached a contradiction by showing that $q = 0$ and $F = 0$. The theorem is proved.

Remark. Some applications involve a weight

$$w(r) \;=\; \exp\left(-h\left(\log\frac{1}{r}\right)\right)$$

where, for $\xi > 0$,

$$h(\xi) \;=\; \sup_{v>0}(M(v) - v\xi),$$

the function $M(v)$ being *merely* supposed *increasing*, and such that $M(0) > -\infty$.

In this situation, we can, from the condition

$$\sum_1^\infty \frac{M(n)}{n^2} \;=\; \infty,$$

conclude that *the rest of the above theorem's statement is valid.*

This can be seen without appealing to the last theorems of article 2. We have here, with $\xi = \log(1/r)$,

$$r^{2n} w(r) \;=\; e^{-(h(\xi) + 2n\xi)},$$

and, since, for any $\xi > 0$,

$$h(\xi) \;\geqslant\; M(v) - v\xi$$

for each $v > 0$, $h(\xi) + 2n\xi \geqslant M(2n)$. We now arrive at $(\substack{*\\*})$ in the same way as above, so, since $M(v)$ is increasing,

$$\sum_1^\infty \frac{M(n)}{n^2} \;=\; \infty \quad \text{makes} \quad \sum_1^\infty \frac{M(2n)}{2n^2} \;=\; \infty$$

and we can conclude by direct application of the corollary from §A.5. (Here, boundedness of $w(r)$ is ensured by the condition $M(0) > -\infty$.)

Remark on a certain change of cariable

If the weight $w(r) = \exp(-H(\log(1/r)))$ *satisfies the hypothesis of the theorem on simultaneous polynomial approximation, so does the weight* $w(r^L) = \exp(-H(L\log(1/r)))$ *for any positive constant* L. That's simply because

$$\int_0^a H(L\xi)\,\mathrm{d}\xi \;=\; \frac{1}{L}\int_0^{aL} H(\xi)\,\mathrm{d}\xi \;!$$

That theorem therefore remains valid if we replace the weight $w(r)$ *figuring in its statement by* $w(r^L)$, L *being any positive constant.*

We will use this fact several times in what follows.

5. Volberg's theorem on harmonic measures

The result to be proved here plays an important role in the establishment of the main theorem of this §. It is also of interest in its own right.

Definition. Let $\mathcal{O}$ be an open subset of $\{|z| < 1\}$, and J any open *arc* of $\{|z| = 1\}$. We say that $\mathcal{O}$ *abuts* on J if, for each $\zeta \in J$, there is a neighborhood V_ζ of ζ with

$$V_\zeta \cap \{|z| < 1\} \;\subseteq\; \mathcal{O}.$$

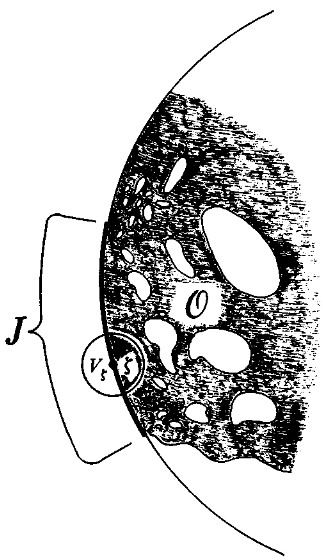

Figure 96

Now we come to the

Theorem on harmonic measures (Volberg). *Let, for* $0 < r < 1$, $w(r) = \exp(-H(\log(1/r)))$, *where* $H(\xi)$ *is* decreasing *and* bounded *below on* $(0, \infty)$, *and tends to* ∞ *sufficiently rapidly as* $\xi \to 0$ *to make* $w(r) = O((1-r)^2)$ *for* $r \to 1$. *(In the situation of Volberg's theorem, we have* $H(\xi) \geqslant \text{const.}\xi^{-c}$ *with* $c > 0$, *so this will certainly be the case.)*

Assume furthermore that

$$\int_0^a \log H(\xi)\, d\xi \;=\; \infty$$

for all sufficiently small $a > 0$.

Let $\mathcal{O}$ *be any connected open set in* $\{|z| < 1\}$ *whose boundary is regular enough to permit the solution of Dirichlet's problem for* $\mathcal{O}$. *Suppose that there are two open arcs* I *and* J *of positive length on* $\{|z| = 1\}$ *such that*:

(i) $\partial\mathcal{O} \cap J$ *is* empty;

(ii) $\mathcal{O}$ abuts *on* I.

Then, if $\omega_{\mathcal{O}}(\ ,z)$ *denotes* harmonic measure *for* $\mathcal{O}$ *(as seen from* $z \in \mathcal{O}$), *we have*

$$\int_{\{|\zeta|<1\}\cap\partial\mathcal{O}} \log\left(\frac{1}{w(|\zeta|)}\right) d\omega_{\mathcal{O}}(\zeta, z_0) \;=\; \infty$$

for each $z_0 \in \mathcal{O}$.

Remark 1. The integral is taken over the part of $\partial\mathcal{O}$ lying *inside* $\{|z| < 1\}$.

Remark 2. The assumption that $\mathcal{O}$ *abuts* on an *arc* I can be relaxed. But the proof uses the full strength of the assumption that $\partial\mathcal{O}$ *avoids* J.

Proof of theorem. We work with the weight $w_1(r) = w(r^3)$. By the *theorem on simultaneous polynomial approximation* and *remark on a change of variable* (previous article), there are polynomials $P_n(z)$ with

$$\int_{\{|\zeta|=1\}\sim J} |P_n(e^{i\vartheta})|^2\, d\vartheta \;\xrightarrow[n]{}\; 0$$

and at the same time

$$\iint_{|z|<1} |P_n(z) - 1|^2 w_1(|z|)\, dx\, dy \;\xrightarrow[n]{}\; 0.$$

The second relation certainly implies that

$$\iint_{|z|<1} |P_n(z)|^2 w_1(|z|)\, dx\, dy \;\leqslant\; C$$

for some $C < \infty$, and all n.

Take any z_0, $|z_0| < 1$; we use the last inequality to get a uniform upper estimate for the values $|P_n(z_0)|$. Put $\rho = \frac{1}{2}(1 - |z_0|)$.

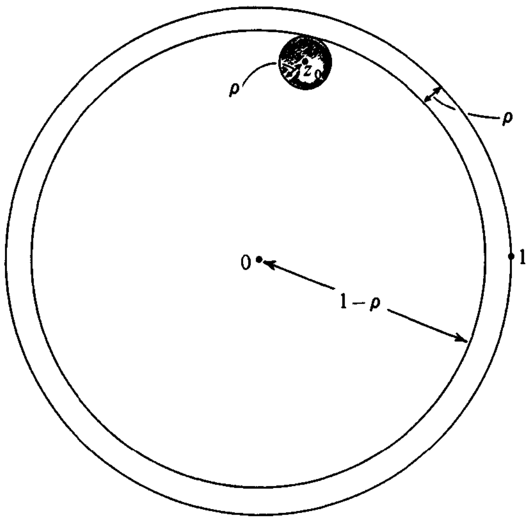

Figure 97

We have

$$|P_n(z_0)|^2 \leqslant \frac{1}{\pi\rho^2}\iint_{|z-z_0|<\rho} |P_n(z)|^2 \, dx \, dy.$$

$w_1(r)$ *decreases*, so the right side is

$$\leqslant \frac{1}{\pi\rho^2 w_1(1-\rho)}\iint_{|z-z_0|<\rho} |P_n(z)|^2 w_1(|z|) \, dx \, dy$$

which, in turn, is

$$\leqslant \frac{C}{\pi\rho^2 w_1(1-\rho)}$$

by the above inequality.

Here, $w_1(1-\rho) = w(1 - 3\rho + 3\rho^2 - \rho^3)$ is $\geqslant$ $w(1-2\rho) = w(|z_0|)$ when $3\rho^2 - \rho^3 \leqslant \rho$, i.e., for $\rho \leqslant (3-\sqrt{5})/2$. Also, $w(r)$ is bounded above ($H(\xi)$ being bounded below), so, for $(3-\sqrt{5})/2 < \rho < \frac{1}{2}$, $w_1(1-\rho) \geqslant w(\sqrt{5}-2) \geqslant \text{const.} w(|z_0|)$. The result just found therefore reduces to

$$|P_n(z_0)|^2 \leqslant \frac{\text{const.}}{(1-|z_0|)^2 w(|z_0|)},$$

with the right-hand side in turn

$$\leqslant \frac{\text{const.}}{(w(z_0))^2}$$

according to the hypothesis. Thus, since z_0 was arbitrary,

$$(*) \qquad \log|P_n(z)| \leqslant \text{const.} + \log\left(\frac{1}{w(|z|)}\right), \quad |z|<1.$$

A similar (and simpler) argument, applied to $P_n(z)-1$, shows that

$$(\ast\ast) \qquad P_n(z) \underset{n}{\longrightarrow} 1, \quad |z|<1.$$

Let us now fix our attention on $\partial\mathcal{O}\cap\{|\zeta|=1\}$, *which we henceforth denote by* S, in order to simplify the notation. The *open unit disk* Δ includes $\mathcal{O}$, therefore, by the *principle of extension of domain* (see §B.1), for $z_0\in\mathcal{O}$,

$$\mathrm{d}\omega_{\mathcal{O}}(\zeta, z_0) \leqslant \mathrm{d}\omega_{\Delta}(\zeta, z_0)$$

for ζ varying on S. In other words,

$$(\dagger) \qquad \mathrm{d}\omega_{\mathcal{O}}(\zeta, z_0) \leqslant K(z_0)|\mathrm{d}\zeta|$$

for ζ on S with a number $K(z_0)$ depending only on z_0.

Since $\mathcal{O}$ *abuts* on the arc I with $|I|>0$, *we surely have*

$$\omega_{\mathcal{O}}(S, z_0) > 0$$

for each $z_0\in\mathcal{O}$ by Harnack's theorem. Since $S \subseteq \{|\zeta|=1\}\sim J$ ((i) of the hypothesis), we have, by (†) and the relation between arithmetic and geometric means,

$$\int_S \log|P_n(e^{i\vartheta})|\,\mathrm{d}\omega_{\mathcal{O}}(e^{i\vartheta}, z_0)$$

$$\leqslant \tfrac{1}{2}\omega_{\mathcal{O}}(S, z_0)\log\left(\frac{1}{\omega_{\mathcal{O}}(S, z_0)}\int_S |P_n(e^{i\vartheta})|^2\,\mathrm{d}\omega_{\mathcal{O}}(e^{i\vartheta}, z_0)\right)$$

$$\leqslant \tfrac{1}{2}\omega_{\mathcal{O}}(S, z_0)\log\left(\frac{K(z_0)}{\omega_{\mathcal{O}}(S, z_0)}\int_{\{|\zeta|=1\}\sim J} |P_n(e^{i\vartheta})|^2\,\mathrm{d}\vartheta\right)$$

for $z_0\in\mathcal{O}$. *This last expression, however, tends to* $-\infty$ *as* $n\to\infty$ because

$$\int_{\{|\zeta|=1\}\sim J} |P_n(e^{i\vartheta})|^2\,\mathrm{d}\vartheta \underset{n}{\longrightarrow} 0$$

At the same time, for any $z_0\in\mathcal{O}$, $\log|P_n(z_0)| \underset{n}{\longrightarrow} 0$ by ($\ast\ast$). Therefore, by the *theorem on harmonic estimation* in §B.1 (whose extension to possibly infinitely connected domains $\mathcal{O}$ of the kind considered here presents no difficulty, at least for *polynomials* $P_n(z)$), we see that

$$\int_S \log|P_n(e^{i\vartheta})|\,d\omega_{\mathcal{O}}(e^{i\vartheta}, z_0) + \int_{\partial\mathcal{O}\cap\Delta} \log|P_n(\zeta)|\,d\omega_{\mathcal{O}}(\zeta, z_0)$$
$$= \int_{\partial\mathcal{O}} \log|P_n(\zeta)|\,d\omega_{\mathcal{O}}(\zeta, z_0) \geqslant \log|P_n(z_0)| \underset{n}{\longrightarrow} 0.$$

As we have just shown, the *first* of the two integrals in the left-hand member *tends to* $-\infty$ as $n \to \infty$. *Hence the second must tend to* ∞ *as* $n \to \infty$ (!). However, by (*),

$$\log|P_n(\zeta)| \leqslant \text{const.} + \log\left(\frac{1}{w(|\zeta|)}\right)$$

for $\zeta \in \partial\mathcal{O} \cap \Delta$. So we must have

$$\int_{\partial\mathcal{O}\cap\Delta} \log\left(\frac{1}{w(|\zeta|)}\right) d\omega_{\mathcal{O}}(\zeta, z_0) = \infty \quad \text{for} \quad z_0 \in \mathcal{O}.$$

The theorem is proved.

Remark. The result just established holds in particular for weights $w(r) = \exp(-h(\log(1/r)))$ with $h(\xi) = \sup_{v>0}(M(v) - v\xi)$ for $\xi > 0$, $M(v)$ being increasing, provided that $M(0) > -\infty$, that $\sum_1^\infty (M(n)/n^2) = \infty$, and that $M(v) \to \infty$ as $v \to \infty$ fast enough to make $w(r) = O((1-r)^2)$ for $r \to 1$. See remark following the theorem on simultaneous polynomial approximation (previous article).

Corollary. *Let the connected open set* $\mathcal{O}$ *and the weight* $w(r)$ *be as in the theorem (or the last remark). Let* $G(z)$ *be* analytic *in* $\mathcal{O}$ *and continuous up to* $\partial\mathcal{O}$, *and suppose that, for some* ρ, $0 < \rho < 1$, *we have*

$$|G(\zeta)| \leqslant w(|\zeta|) \quad \textit{for } \zeta \in \partial\mathcal{O} \textit{ with } 1 - \rho < |\zeta| < 1 \ (\textit{sic!}).$$

Then $G(z) \equiv 0$ *in* $\mathcal{O}$.

Proof. Take any $z_0 \in \mathcal{O}$. Since $G(z)$ is continuous on $\bar{\mathcal{O}}$, it is bounded there, so, since $w(r)$ *decreases*, we surely have $G(z) \leqslant \text{const.}w(|z|)$ for $z \in \bar{\mathcal{O}}$ and $|z| \leqslant 1 - \rho$. Therefore our hypothesis in fact implies that

$$|G(\zeta)| \leqslant Cw(|\zeta|) \quad \text{for} \quad \zeta \in \partial\mathcal{O} \cap \{|\zeta| < 1\}$$

with a certain constant C.

Write $\partial\mathcal{O} \cap \{|\zeta| < 1\} = \gamma$, and denote the intersection $\partial\mathcal{O} \sim \gamma$ of $\partial\mathcal{O}$ with the unit circumference by S as in the proof of the theorem. By the *theorem on harmonic estimation* (§B.1), we have

$$\log|G(z_0)| \leqslant \int_S \log|G(\zeta)|\,d\omega_{\mathcal{O}}(\zeta, z_0) + \int_\gamma \log|G(\zeta)|\,d\omega_{\mathcal{O}}(\zeta, z_0).$$

If M is a bound for $G(z)$ on $\bar{\mathcal{O}}$, the *first* integral on the right is $\leqslant \log M$. The *second* is

$$\leqslant \log C + \int_\gamma \log w(|\zeta|)\, d\omega_{\mathcal{O}}(\zeta, z_0)$$

by the above inequality. The integral just written is, however, equal to $-\infty$ by the theorem on harmonic measures. Hence $G(z_0) = 0$, as required.

Scholium. L. Carleson observed that the result furnished by the theorem on harmonic measures cannot be essentially improved. By this he meant the following:

If $w(r) = \exp(-h(\log(1/r)))$ *with* $h(\xi)$ *strictly decreasing, convex, and bounded below on* $(0, \infty)$, *and if*

$$\int_0^a \log h(\xi)\, d\xi \;<\; \infty$$

for all sufficiently small $a > 0$, *there is a simply connected open set* $\mathcal{O}$ *in* $\Delta = \{|z| < 1\}$ *fulfilling the conditions of the above theorem for which*

$$\int_{\partial\mathcal{O}\cap\Delta} \log\left(\frac{1}{w(|\zeta|)}\right) d\omega_{\mathcal{O}}(\zeta, z_0) \;<\; \infty, \qquad z_0 \in \mathcal{O}.$$

To see this, observe that the convergence of $\int_0^a \log h(\xi)\, d\xi$ for all sufficiently small $a > 0$ implies that

$$(\S) \qquad \int_0^a \log|h'(\xi)|\, d\xi \;<\; \infty$$

for such a. (See the proof of the *second* theorem in article 2.) We use (§) in order to construct a domain $\mathcal{O}$ like this

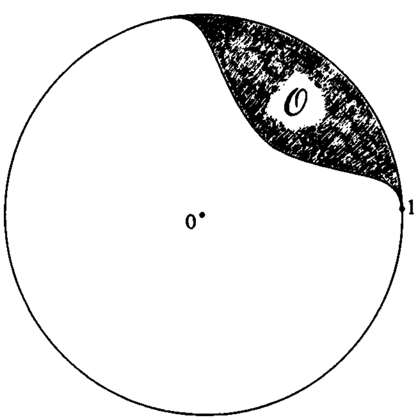

Figure 98

for which

$$\int_{\partial\mathcal{O}\cap\Delta} \log\left(\frac{1}{w(|\zeta|)}\right) d\omega_{\mathcal{O}}(\zeta, z_0) \ < \ \infty, \qquad z_0 \in \mathcal{O},$$

with $w(r) = \exp(-h(\log(1/r)))$. It is convenient to map our (as yet undetermined) region $\mathcal{O}$ *conformally* onto another one, $\mathcal{D}$, by taking $z = re^{i\vartheta}$ to $\varphi = i\log(1/z) = \vartheta + i\log(1/r) = \vartheta + i\xi$. Here ξ has its usual significance.

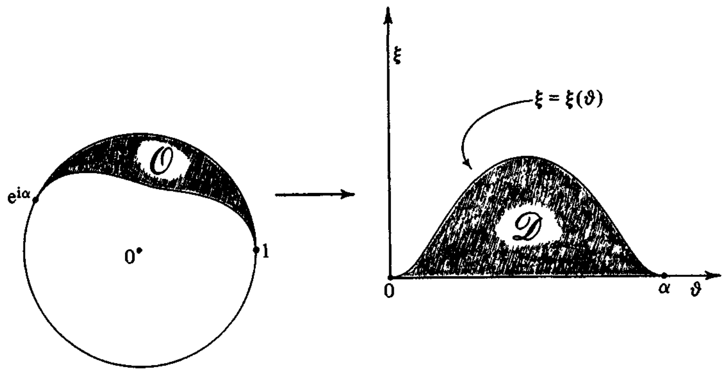

Figure 99

If, in this mapping, the point $z_0 \in \vartheta$ goes over to $p \in \mathcal{D}$, we have, clearly,

$$\int_{\partial\mathcal{O}\cap\Delta} \log\left(\frac{1}{w(|\zeta|)}\right) d\omega_{\mathcal{O}}(\zeta, z_0) \ = \int_{\partial\mathcal{D}\cap\{\xi>0\}} h(\xi)\, d\omega_{\mathcal{D}}(\varphi, p).$$

We see that it is enough to determine the equation $\xi = \xi(\vartheta)$ of the *upper bounding curve* of $\mathcal{D}$ (see picture) in such fashion as to have

$$\int_0^\alpha h(\xi(\vartheta))\, d\omega_{\mathcal{D}}(\vartheta + i\xi(\vartheta), p) \ < \ \infty$$

when $p \in \mathcal{D}$. The easiest way to proceed is to construct a function $\xi(\vartheta) = \xi(\alpha - \vartheta)$, making the upper bounding curve *symmetric about the vertical line through its midpoint*. Then we need only determine an increasing function $\xi(\vartheta)$ on the range $0 \leqslant \vartheta \leqslant \alpha/2$ in such a way that

$$\int_0^{\theta_0} h(\xi(\vartheta))\, d\omega_{\mathcal{D}}(\vartheta + i\xi(\vartheta), p) \ < \ \infty$$

for some $\theta_0 > 0$ and *some* $p \in \mathcal{D}$. From this, the same inequality will follow for every $p \in \mathcal{D}$ by Harnack's theorem, and we can arrive at the full result by adding two such integrals.

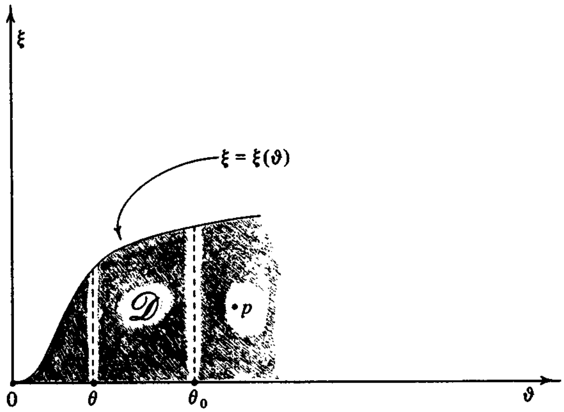

Figure 100

For $0 < \theta < \theta_0$, we have, integrating by parts,

$$\int_\theta^{\theta_0} h(\xi(\vartheta))\mathrm{d}\omega_{\mathscr{D}}(\vartheta + \mathrm{i}\xi(\vartheta), p) \;=\; h(\xi(\theta_0))\omega(\theta_0) - h(\xi(\theta))\omega(\theta)$$

$$- \int_\theta^{\theta_0} \omega(\vartheta)\mathrm{d}h(\xi(\vartheta)),$$

where $\omega(\vartheta)$ denotes the *harmonic measure* (at p) of the segment of the *upper bounding curve* having abscissae between 0 and ϑ, viz.,

$$\omega(\vartheta) \;=\; \int_0^\vartheta \mathrm{d}\omega_{\mathscr{D}}(\tau + \mathrm{i}\xi(\tau), p).$$

Making $\theta \to 0$ and remembering that $h(\xi)$ *decreases*, we see that what we *want* is

(§§) $$\int_0^{\theta_0} \omega(\vartheta)|h'(\xi(\vartheta))|\mathrm{d}\xi(\vartheta) \;<\; \infty.$$

For $\omega(\vartheta)$ we may use the *Carleman–Ahlfors estimate for harmonic measure in curvilinear strips*, to be derived in Chapter IX.* According to that, if p is *fixed* and to the *right* of θ_0,

$$\omega(\vartheta) \;\leqslant\; \text{const.}\exp\left(-\pi\int_\vartheta^{\theta_0}\frac{\mathrm{d}\tau}{\xi(\tau)}\right)$$

for $0 < \vartheta < \theta_0$. The most simple-minded way of ensuring (§§) is then to cook

* See Remark 1 following the third theorem of §E.1 in that chapter. The upper bound arrived at by the method explained there applies in fact to a harmonic measure *larger* than $\omega(\vartheta)$.

the positive increasing function $\xi(\tau)$ so as to have

$$\log|h'(\xi(\vartheta))| \; - \; \pi\int_{\vartheta}^{\theta_0}\frac{d\tau}{\xi(\tau)} \; = \; \text{const.}$$

It is the relation (§) which makes it possible for us to do this.

In order to avoid being fussy, *let us at this point make the additional* (and not really restrictive) *assumption that $h'(\xi)$ is continuously differentiable.* Then we can differentiate the previous equation with respect to ϑ, getting

$$\frac{d\log|h'(\xi)|}{d\xi}\frac{d\xi}{d\vartheta} \; = \; -\frac{\pi}{\xi}$$

for our unknown increasing function $\xi = \xi(\vartheta)$. Calling $\vartheta(\xi)$ the *inverse* to the function $\xi(\vartheta)$, we have

$$\frac{d\vartheta(\xi)}{d\xi} \; = \; -\frac{1}{\pi}\xi\frac{d\log|h'(\xi)|}{d\xi}.$$

In view of (§), this has the solution

$$\vartheta(\xi) \; = \; \frac{1}{\pi}\int_0^{\xi}(\log|h'(t)| - \log|h'(\xi)|)dt$$

with $\vartheta(0) = 0$. Since $h'(t)$ is < 0 and *increasing* (i.e., $\log|h'(t)|$ *decreases*), we see that the function $\vartheta(\xi)$ given by this formula *is* strictly increasing, and therefore *has* an increasing inverse $\xi(\vartheta)$ for which (§§) holds.

This completes our construction, and Carleson's observation is verified.

Let us remark that one can, by the same method, establish a version of the Levinson loglog theorem which we will give at the end of this § (accompanied, however, by a proof based on a different idea). V.P. Gurarii showed me this simple argument (Levinson's original proof of the loglog theorem, found in his book, is quite hard) at the 1966 International Congress in Moscow.

6. Volberg's theorem on the logarithmic integral

We are finally in a position to undertake the proof of the main result of this §. This is what we will establish:*

Theorem. *Let $M(v)$ be increasing for $v \geqslant 0$.*

Suppose that

$$M(v)/v \text{ is decreasing,}$$

that

$$M(v) \; \geqslant \; \text{const.}v^{\alpha}$$

* A refinement of the following result due to Brennan is given in the *Addendum* at the end of the present volume.

with some $\alpha > \frac{1}{2}$ for all large ν, and that

$$\sum_1^\infty \frac{M(n)}{n^2} = \infty.$$

Let

$$F(e^{i\vartheta}) \sim \sum_{-\infty}^{\infty} a_n e^{in\vartheta}$$

be continuous and not identically zero.

Then, if

$$|a_{-n}| \leqslant e^{-M(n)} \quad \textit{for} \quad n \geqslant 1,$$

we have

$$\int_{-\pi}^{\pi} \log|F(e^{i\vartheta})| d\vartheta > -\infty.$$

Remark. Volberg states this theorem for functions $F(e^{i\vartheta}) \in L_1(-\pi, \pi)$.* He replaces our second displayed condition on $M(\nu)$ by a weaker one, requiring only that

$$\nu^{-\frac{1}{2}} M(\nu) \longrightarrow \infty$$

for $\nu \to \infty$, but includes an additional restrictive one, to the effect that

$$\nu^{1/2} M(\nu^{1/2}) \leqslant \text{const.} M(\nu)$$

for large ν. This extra requirement serves to ensure that the function

$$h(\xi) = \sup_{\nu > 0} (M(\nu) - \nu\xi)$$

satisfies the relation $h(K\xi) \leqslant (h(\xi))^{1-c}$ with some $K > 1$ and $c > 0$ for small $\xi > 0$; here we have entirely dispensed with it.

Proof of theorem (essentially Volberg's). This will be quite long.

We start by making some simple reductions. First of all, we assume that $M(\nu)/\nu \longrightarrow 0$ for $\nu \to \infty$, since, in the contrary situation, the theorem is easily verified directly (see article 2).

According to the first theorem of article 2, our condition that $M(\nu)/\nu$ decrease implies that the *smallest concave majorant* $M^*(\nu)$ of $M(\nu)$ is $\leqslant 2M(\nu)$; this means that the hypothesis of the theorem is satisfied if, in it, we replace $M(\nu)$ by the *concave increasing function* $M^*(\nu)/2$.

There is thus no loss of generality in supposing to begin with that $M(\nu)$ is *also concave.* We may also assume that $M(0) \geqslant 3$. To see this, suppose that $M(0) < 3$; in that case we may draw a straight line $\mathscr{L}$ from $(0,3)$ *tangent to the graph* of $M(\nu)$ vs. ν:

* See the addendum for such an extension of Brennan's result.

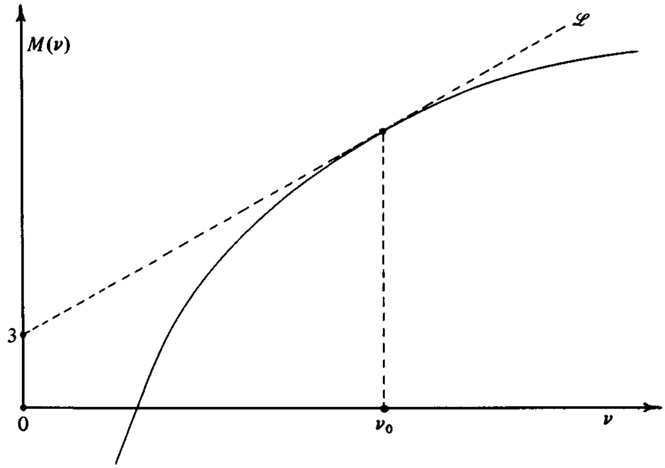

Figure 101

If the point of tangency is at $(\nu_0, M(\nu_0))$, we may then take the new increasing concave function $M_0(\nu)$ equal to $M(\nu)$ for $\nu \geqslant \nu_0$ and to the height of $\mathscr{L}$ at the abscissa ν for $0 \leqslant \nu < \nu_0$. Our Fourier coefficients a_n will satisfy

$$|a_{-n}| \leqslant \text{const.}e^{-M_0(n)}$$

for $n \geqslant 1$, and the rest of the hypothesis will hold with $M_0(\nu)$ in place of $M(\nu)$.

We may now use the simple constructions of $M_1(\nu)$ and $M_2(\nu)$ given in article 2 to obtain an infinitely differentiable, increasing and *strictly* concave function $M_2(\nu)$ (with $M_2''(\nu) < 0$) which is *uniformly close* (within $\frac{1}{2}$ unit, say) to $M_0(\nu)$ on $[0, \infty)$. (Here uniformly close on *all* of $[0, \infty)$ because our present function $M_0(\nu)$ has a *bounded* first derivative on $(0, \infty)$.) We will then still have

$$|a_{-n}| \leqslant \text{const.}e^{-M_2(n)}$$

for $n \geqslant 1$, and the rest of the hypothesis will hold with $M_2(\nu)$ in place of $M(\nu)$.

Since $M(\nu) \geqslant \text{const.}\nu^\alpha$ for large ν, where $\alpha > \frac{1}{2}$, we certainly (and by far!) have

$$n^4 \exp(-M_2(n)/2) \longrightarrow 0, \quad n \to \infty.$$

Therefore

$$\sum_{n=1}^{\infty} |n^2 a_{-n}| e^{M_2(n)/2} < \infty.$$

So, putting $\bar{M}(\nu) = M_2(\nu)/2$, we have

$$\sum_1^\infty |n^2 a_{-n}| e^{\bar{M}(n)} < \infty$$

with a function $\bar{M}(\nu)$ which is *increasing, strictly concave, and infinitely differentiable on* $(0, \infty)$, *having* $\bar{M}''(\nu) < 0$ *there.* The hypothesis of the theorem *holds* with $\bar{M}(\nu)$ standing in place of $M(\nu)$. Besides, $\bar{M}(0)$ $(= \lim_{\nu \to 0} \bar{M}(\nu))$ is $\geqslant 1$ since $M_0(0) \geqslant 3$. Later on, this property will be helpful technically.

▶ *Let us henceforth simply write* $M(\nu)$ *instead of* $\bar{M}(\nu)$. Our *new* function $M(\nu)$ thus *satisfies the hypothesis of Dynkin's theorem* (article 3). We put, as usual,

$$h(\xi) = \sup_{\nu > 0} (M(\nu) - \nu\xi) \quad \text{for} \quad \xi > 0,$$

and then form the weight

$$w(r) = \exp\left(-h\left(\log\frac{1}{r}\right)\right), \qquad 0 < r < 1.$$

Because $M(0) \geqslant 1$, we have $h(\xi) \geqslant 1$ for $\xi > 0$, so $w(r) \leqslant 1/\mathrm{e}$. *Applying* Dynkin's theorem, we obtain a *continuous extension* $F(z)$ of our given function $F(\mathrm{e}^{\mathrm{i}\vartheta})$ to $\{|z| \leqslant 1\}$ with $F(z)$ *continuously differentiable in the open unit disk* and

$$\left|\frac{\partial F(z)}{\partial \bar{z}}\right| \leqslant \text{const.} w(|z|), \qquad |z| < 1.$$

We note here that the properties of $M(\nu)$ assumed in the hypothesis certainly make $w(r) \to 0$ (indeed rather rapidly) as $r \to 1$; see article 2.

Let

$$B_0 = \{z\colon\ |z| < 1 \text{ and } |F(z)| \leqslant w(|z|)\}.$$

We cover each of the closed sets

$$B_0 \cap \left\{1 - \frac{1}{n} \leqslant |z| \leqslant 1 - \frac{1}{n+1}\right\}, \qquad n = 1, 2, 3, \ldots,$$

by a finite number of open disks lying in $\{|z| < 1\}$ on which

$$|F(z)| < 2w(|z|).$$

This gives us altogether a countable collection of open disks lying in the unit circle and covering B_0; *the closure of the union of those disks is denoted by* B. We then have

$$|F(z)| \leqslant 2w(|z|), \qquad z \in B,^*$$

* *including* for $z \in B$ of modulus 1, as long as we take $w(1) = 0$! See argument for *Step 1*, p. 361

and

$$|F(z)| \;>\; w(|z|) \quad \text{for } z\notin B \text{ and } |z|<1.$$

Put

$$\mathcal{O} \;=\; \{|z|<1\}\cap(\sim B);$$

$\mathcal{O}$ is an *open* subset of the unit disk and $|F(z)| > w(|z|)$ therein, as we have just seen. We will see presently that $\mathcal{O}$ fills much of the unit disk. Let us at this point simply observe that *$\mathcal{O}$ is certainly not empty*. If, indeed, it *were* empty, B would fill the unit disk and we would have $|F(z)| \leqslant 2w(|z|)$ for $|z|<1$. The fact that $w(r)\to 0$ for $r\to 1$ would then make $F(e^{i\vartheta})\equiv 0$, *contrary to our hypothesis*, by virtue of the continuity of $F(z)$ on $\{|z|\leqslant 1\}$.

Although the open set $\mathcal{O}$ may have an exceedingly complicated structure, *the Dirichlet problem for it can be solved.* This will follow from a well-known result in elementary potential theory (for a proof of which see, for instance, pp. 35–6 of Gamelin's book, the latter part of the one by Kellog, or any other work on potential theory – I cannot, after all, prove *everything*!), if we show that the *Poincaré cone condition for $\mathcal{O}$ is satisfied at every point ζ of $\partial\mathcal{O}$*, i.e., if, for each such ζ, *there is a small triangle with vertex at ζ lying outside $\mathcal{O}$.* To check this, observe that the small disks used to build up B can accumulate only at points of the unit circumference. Hence, if $|\zeta|<1$ and $\zeta\in\partial\mathcal{O}$, *$\zeta$ is on the boundary of one of those small disks, inside* of which a triangle with vertex at ζ may be drawn. If, however, $|\zeta| = 1$, we may take a triangle lying *outside the unit disk* with vertex at ζ.

Since $|F_{\bar z}(z)| \leqslant Cw(|z|)$ in $\{|z|<1\}$ while $|F(z)| > w(|z|)$ in $\mathcal{O}$, we have

$$\left|\frac{1}{F(z)}\,\frac{\partial F(z)}{\partial \bar z}\right| \;\leqslant\; C, \qquad z\in\mathcal{O}.$$

Volberg's idea is to take advantage of this relation and use the function

$$\Phi(z) \;=\; F(z)\exp\left\{\frac{1}{2\pi}\int_{\mathcal{O}}\!\!\int \frac{F_{\bar\zeta}(\zeta)}{F(\zeta)}\,\frac{d\xi\, d\eta}{(\zeta - z)}\right\}$$

on $\{|z|\leqslant 1\}$. (N.B. *Again we are writing $\zeta = \xi + i\eta$, in conflict with the notation $\xi = \log(1/r)$ used in discussing $h(\xi)$.*) According to *Remark* 1 to Dynkin's theorem (end of article 3), $F(z)$ has enough differentiability in $\{|z|<1\}$ for us to be able to use the *corollary from the end of article* 1. By that corollary, *$\Phi(z)$ is analytic in $\mathcal{O}$*, and *$|\Phi(z)|$ lies between two constant multiples of $|F(z)|$ there* (*and, actually, on* $\{|z|\leqslant 1\}$ as the last part of that corollary's proof shows). We thus certainly have $|\Phi(z)| > 0$ in $\mathcal{O}$ since $|F(z)| > w(|z|)$ there.

Volberg now applies the *theorem on harmonic estimation* (§B.1) to

$\Phi(z)$ and the open set $\mathcal{O}$ in order to eventually get at

$$\int_{-\pi}^{\pi} \log|F(e^{i\vartheta})|\,d\vartheta.$$

His procedure is to show that the whole unit circumference is contained in $\partial\mathcal{O}$, and that the part of $\partial\mathcal{O}$ lying *inside* the unit disk is so unimportant as to make *harmonic measure for* $\mathcal{O}$ act like *ordinary Lebesgue measure* on the unit circumference. We will carry out this program *in 5 steps*. Before going to step 1, we should, however, acknowledge that here the theorem on harmonic estimation will be used under conditions somewhat more general than those allowed for in §B.1. The open set $\mathcal{O}$ may not even be connected, and its components may be infinitely connected! Nevertheless, extension of the theorem in question to the present situation involves no real difficulty – $\Phi(z)$ *is* continuous on $\{|z| \leqslant 1\}$ and the Dirichlet problem for $\mathcal{O}$ *is* solvable.

We proceed.

Step 1. *$B \cap \{|\zeta| = 1\}$ contains no arc of positive length.*

Assume the contrary. By construction of B, each of its points on the unit circumference is a *limit* of a sequence of z_n having modulus < 1 for which $|F(z_n)| < 2w(|z_n|)$. Since $w(r) \to 0$ for $r \to 1$ we thus have $F(e^{i\vartheta}) = 0$ for every point of the form $e^{i\vartheta}$ in B.

If now this happens for each point *belonging to an arc J of positive length* on the unit circumference, we will have $F(e^{i\vartheta}) \equiv 0$ on J. At the same time, the *Fourier coefficients* a_n of $F(e^{i\vartheta})$ satisfy $|a_{-n}| \leqslant \text{const.}e^{-M(n)}$ for $n \geqslant 1$ by hypothesis. *Therefore $F(e^{i\vartheta})$ vanishes identically by the corollary to Levinson's theorem at the end of* §A.5. This, however, is *contrary to our hypothesis.*

Before going on, we note that the properties of $M(v)$ came into play in the preceding argument *only* when we looked at the Fourier coefficients of $F(e^{i\vartheta})$ and *not* when we brought in $w(r) = \exp(-h(\log(1/r)))$, even though $h(\xi)$ *is* related to $M(v)$ in the usual way. *Any other* weight $w(r) \geqslant 0$ tending to zero as $r \to 1$ would have worked just as well.

Thanks to what we found in step 1, $\{|\zeta| = 1\} \cap (\sim B)$ is *non-empty* (and even *dense* on the unit circumference). *It is open, hence equal to a countable union of disjoint open arcs I_k on $\{|\zeta| = 1\}$.* (B, remember, is *closed*.) $\mathcal{O} = \{|z| < 1\} \cap (\sim B)$ *clearly abuts on each of the I_k.* (See definition, beginning of article 5.)

Take any ρ_0, $0 < \rho_0 < 1$, and denote by $\Omega_k(\rho_0)$ (or just by Ω_k, if it is not necessary to keep the value of ρ_0 in mind) the *connected component of $\mathcal{O} \cap \{\rho_0 < |z| < 1\}$ abutting on I_k.*

Step 2. *All the $\Omega_k(\rho_0)$ are the same. In other words, $\bigcup_k \Omega_k(\rho_0)$ is connected.*

Assume the contrary. Then we must have two different arcs I_k – call them I_1 and I_2 – for which the corresponding components Ω_1 and Ω_2 are *disjoint*.

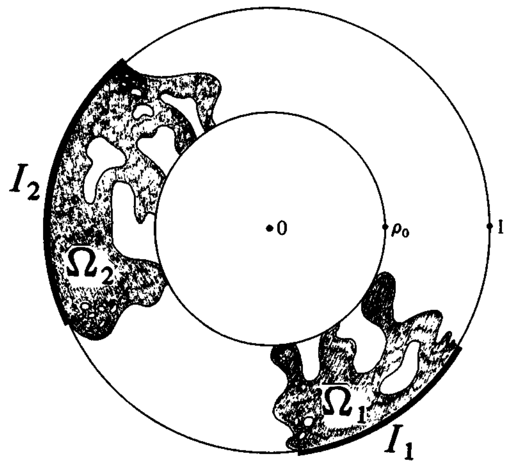

Figure 102

The function $\Phi(z)$ is analytic in Ω_1 and continuous on its closure. As stated above, it's *continuous on* $\{|z| \leqslant 1\}$ – that's because $F(z)$ is, and because the ratio $F_{\bar{\zeta}}(\zeta)/F(\zeta)$ appearing in the formula for $\Phi(z)$ is *bounded* on the region $\mathcal{O}$ over which the double integral figuring therein is taken (see beginning of the proof of the theorem in article 1).

The points ζ on $\partial\Omega_1$ with $\rho_0 < |\zeta| < 1$ (*sic*!) must belong to B, therefore $|F(\zeta)| \leqslant 2w(|\zeta|)$ for them. So, since $A|F(z)| \leqslant |\Phi(z)| \leqslant A'|F(z)|$ for $|z| \leqslant 1$, we have

$$|\Phi(\zeta)| \;\leqslant\; \text{const.}w(|\zeta|) \qquad \text{for } \zeta \in \partial\Omega_1 \text{ and } \rho_0 < |\zeta| < 1.$$

We have $w(r) = \exp(-h(\log(1/r)))$ with $h(\zeta)$ *decreasing* and *bounded below* for $\xi > 0$. In the present case, where $h(\xi) = \sup_{v>0}(M(v) - v\xi)$ and $\int_1^\infty (M(v)/v^2)\mathrm{d}v = \infty$, $\int_0^a \log h(\xi)\mathrm{d}\xi = \infty$ for all sufficiently small $a > 0$ (next to last theorem of article 2 – in the present circumstances we could even bypass that theorem as in the remark following the one of article 4). Finally, the condition (given!) that $M(v) \;\geqslant\; \text{const.}v^\alpha$ for large v, with $\alpha > \frac{1}{2}$, makes $h(\xi) \;\geqslant\; \xi^{-\lambda}$ for small $\xi > 0$, where $\lambda > 1$, by a lemma of article 2. Therefore (and by far!) $w(r) = O((1-r)^2)$ as $r \to 1$.

In our present situation, Ω_1 *abuts* on I_1 and $\partial\Omega_1$ *avoids* I_2. Here, *all the conditions* of *Volberg's theorem on harmonic measures* (previous article) *are fulfilled*. Therefore, by the corollary to that theorem, $\Phi(z) \equiv 0$ in Ω_1. *This, however, is impossible since $\Omega_1 \subseteq \mathcal{O}$ on which $|\Phi(z)| > 0$.*

As we have just seen, the union $\bigcup_k \Omega_k(\rho_0)$ is *connected. We denote that union by* $\Omega(\rho_0)$, or sometimes just by Ω. $\Omega(\rho_0)$ is an open subset of $\mathcal{O}$ lying in the ring $\rho_0 < |z| < 1$ and *abutting on each arc of* $\{|\zeta| = 1\}$ *contiguous to* $\{|\zeta| = 1\} \cap B$.

Step 3. *If* $|\zeta| = 1$, *there are values of* $r < 1$ *arbitrarily close to* 1 *with* $r\zeta \in \Omega(\rho_0)$, *and hence, in particular, with*

$$|F(r\zeta)| \; > \; w(r).$$

Take, wlog, $\zeta = 1$, and assume that for some a, $\rho_0 < a < 1$, *the whole segment* $[a, 1]$ *fails to intersect* Ω. The function $\sqrt{(z-a)/(1-az)}$ can then be defined so as to be analytic and single valued in Ω, and, if we introduce the new variable

$$s \; = \; \sqrt{\frac{z-a}{1-az}},$$

the mapping $z \to s$ takes Ω conformally onto a new domain – call it $\Omega_{\surd}$ – lying in $\{|s| < 1\}$:

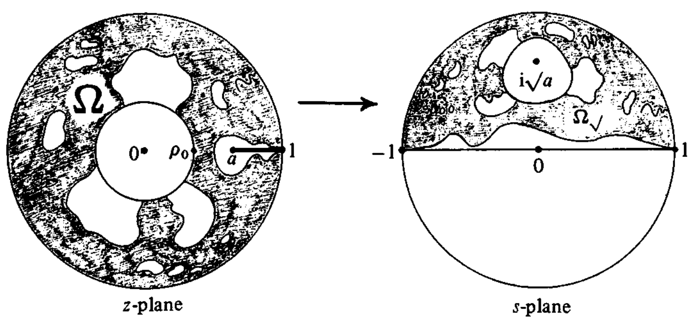

Figure 103

In terms of the variable s, write

$$\Phi(z) = \Psi(s), \qquad z \in \Omega;$$

$\Psi(s)$ is obviously analytic in $\Omega_{\surd}$ and continuous on its closure. If $s \in \Omega_{\surd}$ has $|s| \; > \; \sqrt{a}$, we have, since

$$z \; = \; \frac{s^2 + a}{1 + as^2},$$

that $1 - |z| \; \leqslant \; ((1+a)/(1-a))(1 - |s|^2)$; proof of this inequality is an elementary exercise in the geometry of linear fractional transformations which the reader should do. Hence, for $s \in \Omega_{\surd}$ with $|s| \; > \; \sqrt{a}$,

$$|z| \; \geqslant \; 1 - \frac{1+a}{1-a}(1 - |s|^2),$$

and, if $|s|$ is *close* to 1, this last expression is $\geqslant |s|^L$, where we can take for L a number $> 2(1+a)/(1-a)$ (depending on the closeness of $|s|$ to 1). The same relation between $|s|$ and $|z|$ holds for $s \in \partial\Omega_{\surd}$ with $|s|$ close to 1.

Suppose $|s| < 1$ is *close* to 1 and $s \in \partial\Omega_{\surd}$. The corresponding z then lies on $\partial\mathcal{O}$ with $|z| < 1$, so $|\Phi(z)| \leqslant \text{const.}w(|z|)$; therefore $|\Psi(s)| = |\Phi(z)| \leqslant \text{const.}w(|s|^L)$ by the relation just found, $w(r)$ being *decreasing*. The open set $\Omega_{\surd}$ *certainly abuts* on some arcs of $\{|s| = 1\}$ having positive length, since Ω abuts on the I_k. And $\partial\Omega_{\surd}$ *does not intersect the arc* $\pi < \arg s < 2\pi$ on $\{|s| = 1\}$ – that's why we *did* the conformal mapping $z \to s$! Here, the weight $w(r^L)$ is just as good (or just as bad) as $w(r)$ – see the *remark on a certain change of variable* at the end of article 4. We can therefore apply the *corollary* of the *theorem on harmonic measures* (end of article 5) to $\Psi(s)$ and the domain $\Omega_{\surd}$, and conclude that $\Psi(s) \equiv 0$ in $\Omega_{\surd}$. This, however, would make $\Phi(z) \equiv 0$ in Ω *which is impossible*, since $\Omega \subseteq \mathcal{O}$ where $|\Phi(z)| > 0$. *Step* 3's assertion must therefore hold.

The result just proved certainly implies that $\partial\Omega(\rho_0)$ *includes the unit circumference*. Since the Dirichlet problem can be solved for $\mathcal{O}$, it can be solved for Ω. It therefore makes sense to speak of the *harmonic measure* $\omega_\Omega(E, z)$ of an arbitrary closed subset E of $\{|\zeta| = 1\}$ ($\subseteq \partial\Omega$) relative to Ω, as seen from $z \in \Omega$. As was said above, our *aim* is to show that $\omega_\Omega(E, z_0) \geqslant k(z_0)|E|$ for such sets E; the analyticity of $\Phi(z)$ in Ω together with the fact that $|\Phi(z)|$ is > 0 and lies between two constant multiples of $|F(z)|$ there will *then* make

$$\int_{-\pi}^{\pi} \log|F(e^{i\vartheta})|\,d\vartheta \;>\; -\infty$$

by the *theorem on harmonic estimation* (§B.1), $|F(z)|$ being in any case *bounded above* in the closed unit disk. According to Harnack's theorem, the desired inequality for $\omega_\Omega(E, z_0)$ will follow from a *local version* of it, which is thus all that we need establish.

At this point the condition, assumed in the hypothesis, that $M(v) \geqslant \text{const.}v^\alpha$ with some $\alpha > \frac{1}{2}$ for large v, begins to play a more important rôle in our construction. We have already made *some* use of that property; it has not yet, however, been used *essentially*.

According to a lemma in article 2, the condition is equivalent to the property that $h(\xi) \geqslant \text{const.}\xi^{-\alpha/(1-\alpha)}$ for small $\xi > 0$. There is, in other words, an η, $0 < \eta < \frac{1}{2}$, with

$$\frac{1}{\xi} \leqslant \text{const.}(h(\xi))^{1-2\eta}$$

for small $\xi > 0$. *We fix such an η and put*

$$\boxed{H(\xi) \;=\; (h(2\xi))^{\eta}}$$

for $\xi > 0$.

Since $h(\xi)$ is *decreasing, so is* $H(\xi)$. Also, the property $h(\xi) \geqslant 1$ (due to the condition $M(0) \geqslant 1$) makes $H(\xi) \leqslant h(2\xi)$, whence, *a fortiori*,

$$H(\xi) \;\leqslant\; h(\xi) \quad \text{for} \quad \xi > 0.$$

We have

$$\int_0^a \log H(\xi)\,\mathrm{d}\xi \;=\; \frac{\eta}{2}\int_0^{2a} \log h(x)\,\mathrm{d}x \;=\; \infty$$

for all sufficiently small $a > 0$ by a theorem in article 2, since, as we are assuming, $\sum_1^\infty M(n)/n^2 \;=\; \infty$.

Using $H(\xi)$, let us form the new weight

$$w_1(r) \;=\; \exp\left(-H\left(\log\frac{1}{r}\right)\right), \quad 0 \leqslant r < 1.$$

Then

$$w_1(r) \;\geqslant\; w(r), \quad 0 \leqslant r < 1;$$

we still have, however,

$$w_1(r) \;=\; \mathrm{O}((1-r)^2) \quad \text{for} \quad r \to 1,$$

although, when r is close to 1, $w_1(r)$ is *much larger than* $w(r)$. Starting with $w_1(r)$, we proceed to construct a new open set $\mathcal{O}_1 \subseteq \mathcal{O}$ on which $|F(z)| > w_1(|z|)$ in much the same fashion as $\mathcal{O}$ was formed by use of $w(r)$.

Take first the set

$$B_0' \;=\; \{z\colon\ |z| < 1 \text{ and } |F(z)| \leqslant w_1(|z|)\}.$$

Since $w_1(r) \geqslant w(r)$, B_0' contains the set B_0 used above in the construction of B. Note that on each of the little open disks used to cover B_0 and form the set B – call those disks Δ_j – we have $|F(z)| < 2w_1(|z|)$ since $|F(z)| < 2w(|z|)$ on them. If the Δ_j also cover B_0', we take $B_1 = B$. Otherwise, we form the difference

$$B_0'' \;=\; B_0' \cap \sim \bigcup_j \Delta_j$$

and cover each closed set

$$B_0'' \cap \left\{1 - \frac{1}{n} \;\leqslant\; |z| \;\leqslant\; 1 - \frac{1}{n+1}\right\}, \qquad n = 1, 2, 3, \ldots,$$

by a *finite number* of open disks $\tilde{\Delta}_k(n)$ lying in $\{|z|<1\}$, on which $|F(z)| < 2w_1(|z|)$. We then take B_1 as the *closure of the union*

$$\left(\bigcup_j \Delta_j\right) \cup \left(\bigcup_{n=1}^{\infty} \bigcup_k \tilde{\Delta}_k(n)\right).$$

For $z \in B_1$, $|F(z)| \leqslant 2w_1(|z|)$. B_1 contains B and has clearly the same general structure as B; B_1 includes all the points z of $\{|z|<1\}$ for which $|F(z)| \leqslant w_1(|z|)$.

We now put

$$\mathcal{O}_1 \;=\; \{|z|<1\} \cap \sim B_1.$$

The set $\mathcal{O}_1$ is open and contained in $\mathcal{O}$ since $B_1 \supseteq B$. For $z \in \mathcal{O}_1$, $|F(z)| > w_1(|z|)$, and on $\partial\mathcal{O}_1 \cap \{|z|<1\}$ we have $|F(z)| \leqslant 2w_1(|z|)$ since the points of the later set must belong to B_1. The function $\Phi(z)$ introduced above is thus *analytic in* $\mathcal{O}_1$ (and continuous on its closure) and, since $A|F(z)| \;\leqslant\; |\Phi(z)| \;\leqslant\; A'|F(z)|$ for $|z|<1$, satisfies

$$|\Phi(z)| \;>\; \text{const.}w_1(|z|), \qquad z \in \mathcal{O}_1,$$

as well as

$$|\Phi(z)| \;\leqslant\; \text{const.}w_1(|z|) \quad \text{for } z \in \partial\mathcal{O}_1 \text{ and } |z|<1 \; (\textit{sic!}).$$

Our new weight $w_1(r)$ and the function $H(\xi)$ to which it is associated *fulfill the conditions for the theorem on harmonic measures* (article 5). *Hence, in view of the above two inequalities satisfied by* $\Phi(z)$, *there is nothing to prevent our going through steps* 1, 2, *and* 3 *again, with* $\mathcal{O}_1$ *in place of* $\mathcal{O}$ *and* $w_1(r)$ *in place of* $w(r)$. *We henceforth consider this done.*

Once step 3 for $\mathcal{O}_1$ and $w_1(r)$ is carried out, we know that *for each* ζ, $|\zeta|=1$, *there are* $r<1$ *arbitrarily close to* 1 *for which* $|F(r\zeta)| > w_1(r)$. The open set $\mathcal{O}_1$ was brought into our discussion in order to obtain this result, which will be used to play off $w_1(r)$ against $w(r)$. Having now served its purpose, $\mathcal{O}_1$ will not appear again.

Given ζ_0, $|\zeta_0|=1$, consider any $\rho<1$ for which

$$(*) \qquad |F(\rho\zeta_0)| \;>\; w_1(\rho)$$

and form the domain $\Omega(\rho^2)$; this is the *connected* (step 2) component of $\mathcal{O} \cap \{\rho^2 < |z| < 1\}$ ($\mathcal{O}$ and *not* $\mathcal{O}_1$ here!) which *abuts* on *each* of the arcs I_k making up $\{|\zeta|=1\} \cap (\sim B)$.

Step 4. *If, for given* ζ_0 *of modulus* 1, (*) *holds with* ρ *close enough to* 1, *we have* $\rho\zeta_0 \in \Omega(\rho^2)$.

Assuming the contrary, we shall obtain a contradiction. Wlog, $\zeta_0 = 1$.

When $\rho \to 1$, the ratio

$$w_1(\rho)/w(\rho) = \exp\left(h\left(\log\frac{1}{\rho}\right) - \left(h\left(2\log\frac{1}{\rho}\right)\right)^\eta\right)$$

tends to ∞ since $h(\log(1/\rho))$ tends to ∞ then, h is decreasing, and $0 < \eta < 1$. Hence, if $|F(\rho)| > w_1(\rho)$ and $\rho > 1$ is close enough to 1, we surely have $|F(\rho)| > 2w(\rho)$, i.e., $\rho \notin B$, so $\rho \in \mathcal{O}$. The point ρ must then belong to *some* component of $\mathcal{O} \cap \{\rho^2 < |z| < 1\}$, so, *if it is not* in the *component* $\Omega(\rho^2)$, abutting on the I_k, of that intersection, *it must be in some other one, which* we may call $\mathcal{D}$. $\mathcal{D}$, being *disjoint* from $\Omega(\rho^2)$, *can thus abut on none of the arcs* I_k *of* $\{|\zeta| = 1\}$ *contiguous to* $\{|\zeta| = 1\} \cap B$.

It is now claimed that

$$\partial\mathcal{D} \cap \{\rho^2 \leqslant |z| \leqslant 1\} \quad (sic!)$$

is contained in B. Let ζ be in that intersection; if $|\zeta| < 1$, $\zeta \in \partial\mathcal{O} \cap \{|z| < 1\} \subseteq B$, so suppose that $|\zeta| = 1$. If $\zeta \notin B$, then ζ must lie in some contiguous arc I_k, say $\zeta \in I_1$. Then, for some open disk V_ζ with centre at ζ, $V_\zeta \cap \{|z| < 1\} \subseteq \mathcal{O}$. The intersection on the left is, however, *connected*, and, since $\zeta \in \partial\mathcal{D}$, it contains some points from the (*connected*!) open set $\mathcal{D}$. Therefore, $V_\zeta \cap \{|z| < 1\}$ must lie entirely in $\mathcal{D}$, and $\mathcal{D}$ *intersects* with the component $\Omega(\rho^2)$ of $\mathcal{O} \cap \{\rho^2 < |z| < 1\}$ abutting on I_1. But it doesn't! This contradiction shows that we must have $\zeta \in B$, as claimed.

Because $\partial\mathcal{D} \cap \{\rho^2 < |z| \leqslant 1\}$ is contained in B, we have

$$|\Phi(\zeta)| \leqslant \text{const.}|F(\zeta)| \leqslant \text{const.}w(|\zeta|)$$

for $\zeta \in \partial\mathcal{D}$ and $\rho^2 < |z| \leqslant 1$ (*sic*!).* The function $\Phi(z)$ is of course *analytic* in $\mathcal{D} \subseteq \mathcal{O}$, and continuous up to $\partial\mathcal{D}$. In order to help the reader follow the argument, let us *try* to draw a picture of $\mathcal{D}$:

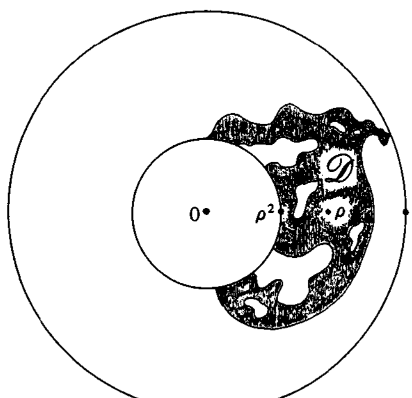

Figure 104

* We are taking $w(1) = 0$. See footnote, p. 359.

(N.B. $\mathscr{D}$ *can't really look like this* as we can see by applying an argument like the one of *step* 3 to $\Omega(\rho^2)$. One of the reasons why the present material is so hard is the *difficulty in drawing correct pictures* of what can happen.)

Call $\Gamma = \partial\mathscr{D} \cap \{\rho^2 < |z| \leqslant 1\}$, and denote harmonic measure for $\mathscr{D}$ by $\omega_{\mathscr{D}}(\ ,z)$. Since $|\Phi(z)|$ is bounded above on the unit disk and $|\Phi(\zeta)| \leqslant \text{const.}w(|\zeta|)$ on Γ, we have, by the theorem on harmonic estimation (§B.1),

$$\log|\Phi(\rho)| \;\leqslant\; \text{const.} \;+\; \omega_{\mathscr{D}}(\Gamma,\rho)\log w(\rho^2),$$

for $|\zeta| > \rho^2$ on Γ and $w(r)$ decreases. An almost trivial application of the *principle of extension of domain* shows that $\omega_{\mathscr{D}}(\Gamma,\rho)$ is *larger* than the *harmonic measure of the circle* $\{|\zeta| = 1\}$ *for the ring* $\{\rho^2 < |z| < 1\}$, *seen from* ρ. However, *that harmonic measure is* $\frac{1}{2}$! We thus see by the previous inequality that

$$\log|\Phi(\rho)| \;\leqslant\; \text{const.} \;+\; \tfrac{1}{2}\log w(\rho^2),$$

i.e., since

$$|\Phi(\rho)| \;\geqslant\; \text{const.}|F(\rho)| \;\geqslant\; \text{const.}w_1(\rho)$$

by $(*)$, that

$$\frac{1}{2}h\left(\log\frac{1}{\rho^2}\right) \;\leqslant\; \text{const.} \;+\; \left(h\left(\log\frac{1}{\rho^2}\right)\right)^{\eta}$$

in terms of the function h, with a constant independent of ρ.

Now $0 < \eta < 1$ and $h(\xi) \longrightarrow \infty$ for $\xi \to 0$. *This means that the relation just obtained is impossible for values of* ρ *sufficiently close to* 1. Therefore, if $\rho < 1$ is close enough to 1 and $|F(\rho)| > w_1(\rho)$, we *must have* $\rho \in \Omega(\rho^2)$, the conclusion we desired to make. (Part of the idea for the preceding argument is due to Peter Jones.)

Take now any ζ_0, $|\zeta_0| = 1$, and pick a $\rho < 1$ *very close to* 1 such that

$$|F(\rho\zeta_0)| \;>\; w_1(\rho)$$

(which is possible, as we have already observed), and that therefore $\rho\zeta_0 \in \Omega(\rho^2)$ (by step 4, just completed). We are going to show that if E is a closed set on the arc of the unit circumference going from $\zeta_0 e^{i\log\rho}$ to $\zeta_0 e^{-i\log\rho}$, then

$$\omega_{\Omega(\rho^2)}(E,\rho\zeta_0) \;\geqslant\; C(\zeta_0,\rho)|E|$$

with some constant $C(\zeta_0,\rho)$ depending on ζ_0 and on ρ. Here, of course, $\omega_{\Omega(\rho^2)}(\ ,z)$ denotes harmonic measure for the domain $\Omega(\rho^2)$.

In order to do this, we write

$$\gamma_\rho = \partial\Omega(\rho^2) \cap \{\rho^2 < |z| < 1\} \quad (sic!)$$

and carry out

Step 5. *If ρ, chosen according to the above specifications, is close enough to* 1,

$$\int_{\gamma_\rho} \frac{1}{1-|\zeta|} \mathrm{d}\omega_{\Omega(\rho^2)}(\zeta, \rho\zeta_0)$$

is as small as we please.

In order to simplify the notation, let us write

$$\boxed{\omega(\ \ ,z) \quad \text{for} \quad \omega_{\Omega(\rho^2)}(\ \ ,z)}$$

during the *remainder* of the *present discussion.* The proof of our statement uses *almost the full strength* of the property that

$$\frac{1}{\xi} \leqslant \text{const.}(h(\xi))^{1-2\eta}$$

for small $\xi > 0$ (with $0 < \eta < \frac{1}{2}$), equivalent to our condition that

$$M(v) \geqslant \text{const.} v^\alpha$$

(with $\alpha > \frac{1}{2}$) for large v.

Take, wlog, $\zeta_0 = 1$. Then, if $\rho \in \Omega(\rho^2)$, we have, by the theorem on harmonic estimation (§ B.1),

$$\log|\Phi(\rho)| \leqslant \int_{\partial\Omega(\rho^2)} \log|\Phi(\zeta)| \mathrm{d}\omega(\zeta, \rho),$$

$\Phi(z)$ being analytic in $\Omega(\rho^2)$ and continuous up to that set's boundary. The subset γ_ρ of $\partial\Omega(\rho^2)$ is of course contained in B, so, for $\zeta \in \gamma_\rho$,

$$|\Phi(\zeta)| < \text{const.} |F(\zeta)| \leqslant \text{const.} w(|\zeta|);$$

that is,

$$\log|\Phi(\zeta)| \leqslant \text{const.} - h\left(\log\frac{1}{|\zeta|}\right), \qquad \zeta \in \gamma_\rho.$$

The function $|\Phi(z)|$ is in any event bounded on $\{|z| \leqslant 1\}$, so this relation, together with the previous one, yields

$$\log|\Phi(\rho)| \leqslant \text{const.} - \int_{\gamma_\rho} h\left(\log\frac{1}{|\zeta|}\right) \mathrm{d}\omega(\zeta, \rho).$$

If ρ is chosen in such a way that we also have $|F(\rho)| > w_1(\rho)$, this becomes

$$(\ast)\qquad \int_{\gamma_\rho} h\left(\log\frac{1}{|\zeta|}\right) d\omega(\zeta,\rho) \;\leqslant\; \text{const.} \;+\; \left(h\left(\log\frac{1}{\rho^2}\right)\right)^{\eta}.$$

Since $1/\xi \leqslant \text{const.}(h(\xi))^{1-2\eta}$ for small $\xi > 0$, we have (when $\rho < 1$ is close to 1),

$$\int_{\gamma_\rho} \frac{1}{1-|\zeta|} d\omega(\zeta,\rho) \;\leqslant\; \text{const.} \int_{\gamma_\rho} \left(h\left(\log\frac{1}{|\zeta|}\right)\right)^{1-2\eta} d\omega(\zeta,\rho).$$

Rewrite (!) the right-hand integral as

$$\text{const.} \int_{\gamma_\rho} \frac{\left(h\left(\log\frac{1}{|\zeta|}\right)\right)^{1-\eta}}{\left(h\left(\log\frac{1}{|\zeta|}\right)\right)^{\eta}} d\omega(\zeta,\rho).$$

Since γ_ρ lies in the ring $\{\rho^2 < |z| < 1\}$ and $h(\xi)$ decreases, the expression just written is

$$\leqslant \;\; \frac{\text{const.}}{\left(h\left(\log\frac{1}{\rho^2}\right)\right)^{\eta}} \int_{\gamma_\rho} \left(h\left(\log\frac{1}{|\zeta|}\right)\right)^{1-\eta} d\omega(\zeta,\rho),$$

so, since $h(\xi) \longrightarrow \infty$ for $\xi \to 0$, we see that

$$\int_{\gamma_\rho} \frac{1}{1-|\zeta|} d\omega(\zeta,\rho) \;\leqslant\; \frac{\delta_\rho}{\left(h\left(\log\frac{1}{\rho^2}\right)\right)^{\eta}} \int_{\gamma_\rho} h\left(\log\frac{1}{|\zeta|}\right) d\omega(\zeta,\rho)$$

with a quantity δ_ρ going to zero for $\rho \to 1$. Plugging $(\ast)$ into the right-hand side, we obtain finally

$$\int_{\gamma_\rho} \frac{1}{1-|\zeta|} d\omega(\zeta,\rho) \;\leqslant\; \delta_\rho\left(1 \;+\; \frac{\text{const.}}{(h(\log(1/\rho^2)))^{\eta}}\right).$$

Here, the right side tends to *zero* as $\rho \to 1$. *Step* 5 is finished.

Now we can make the *local estimate* of $\omega(E, z)$ for closed E on the unit circumference, promised just before step 5. Taking any fixed ζ_0, $|\zeta_0| = 1$, we pick a $\rho < 1$ very close to 1, in such fashion that the conclusions reached in *steps* 4 and 5 apply. Let E be any closed set on the arc of the unit circumference going from $\zeta_0 e^{i\log\rho}$ to $\zeta_0 e^{-i\log\rho}$.

In order to keep the notation simple, consider, as before, the case where $\zeta_0 = 1$.

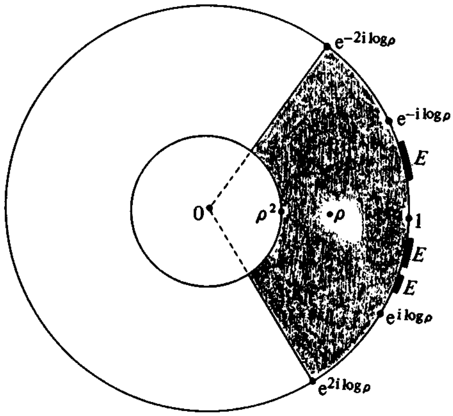

Figure 105

Denote harmonic measure for the *ring* $\{\rho^2 < |z| < 1\}$ by $\tilde{\omega}(\ \ ,z)$. If we *compare* $\tilde{\omega}(E,z)$ with the harmonic measure of E *for the sectorial box shown in the above diagram*, we see immediately that

$$(\dagger) \qquad \tilde{\omega}(E,\rho) \;\geqslant\; C\frac{|E|}{1-\rho}$$

with a numerical constant C independent of $|E|$ and of ρ.

Put, as in *step* 5,

$$\gamma_\rho \;=\; \partial\Omega(\rho^2)\cap\{\rho^2 < |z| < 1\},$$

and continue to denote the harmonic measure for $\Omega(\rho^2)$ by $\omega(\ \ ,z)$. Since

$$\Omega(\rho^2) \;\subseteq\; \{\rho^2 < |z| < 1\}$$

while

$$\partial\Omega(\rho^2) \;\supseteq\; \{|\zeta| = 1\},$$

we can apply a formula established near the end of § B.1, getting

$$\omega(E,z) \;=\; \tilde{\omega}(E,z) - \int_{\gamma_\rho} \tilde{\omega}(E,\zeta)\,\mathrm{d}\omega(\zeta,z)$$

for $z\in\Omega(\rho^2)$.

Comparing $\tilde{\omega}(E,z)$ with harmonic measure of E for the *whole unit disk*, we get the estimate

$$\tilde{\omega}(E,\zeta) \;\leqslant\; \frac{|E|}{\pi(1-|\zeta|)}.$$