

FORBIDDEN CONDUCTORS AND SEQUENCES OF ± 1 S

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ABSTRACT. We study “forbidden” conductors, i.e. numbers $q > 0$ satisfying algebraic criteria introduced by J. Kaczorowski, A. Perelli and M. Radziejewski, that cannot be conductors of L -functions of degree 2 from the extended Selberg class. We show that the set of forbidden q is dense in the interval $(0, 4)$, solving a problem posed in [6]. We also find positive points of accumulation of rational forbidden q .

The Selberg class \mathcal{S} and the extended Selberg class $\mathcal{S}^\#$ are axiomatically defined classes of L -functions, cf., e.g., [5]. Each $F \in \mathcal{S}^\#$ has a degree $d_F \geq 0$ and a conductor $q_F > 0$. It was shown by J. Kaczorowski and A. Perelli [2, 3], cf. also [1], that if $d_F < 2$, then d_F and q_F are integers. Moreover, if $d \in \{0, 1\}$ and q is a positive integer, then there exists an L -function $F \in \mathcal{S}^\#$ with $d_F = d$ and $q_F = q$ [2, Theorems 1 and 2]. As pointed out by V. Blomer, Theorem 3 of [2] implies that there is no $F \in \mathcal{S}$ with $d_F = 1$ and $q_F = 2$. The structure of

$$\mathcal{S}_2^\# = \{F \in \mathcal{S}^\# : d_F = 2\}$$

is a subject of current research [4]. J. Kaczorowski, A. Perelli and M. Radziejewski [6] have shown the existence of $F \in \mathcal{S}_2^\#$ with $q_F = q$ for every real $q \geq 4$ and for $q = 4 \cos^2(\pi/m)$ with integer $m \geq 3$ [6, Lemma 6]. They also considered fractions

$$(1) \quad c(q, \mathbf{m}) = m_k + \frac{1}{qm_{k-1} + \frac{q}{qm_{k-2} + \frac{q}{\ddots + \frac{q}{qm_0}}}}$$

where $q > 0$ and $\mathbf{m} = (m_0, \dots, m_k) \in \mathbf{Z}^{k+1}$. For a given q the sequence \mathbf{m} is called a path if (1) is well defined, i.e. there are no zeros in denominators. It is called a loop if $c(q, \mathbf{m}) = 0$. A sequence \mathbf{m} is therefore

2020 *Mathematics Subject Classification.* 11M41, 11A55.

Key words and phrases. Selberg class; forbidden conductors; continued fractions.

a path if and only if none of $(m_0), (m_0, m_1), \dots, (m_0, \dots, m_{k-1})$ is a loop. The weight of \mathbf{m} is then defined as

$$(2) \quad w(q, \mathbf{m}) = q^{k/2} \prod_{j=0}^{k-1} |c(q, \mathbf{m}_j)|,$$

where $\mathbf{m}_j = (m_0, \dots, m_j)$ for $0 \leq j \leq k$. We also write $c(q, m_0, \dots, m_n)$ and $w(q, m_0, \dots, m_n)$ where appropriate. The set of loops for a given q is denoted as $L(q)$. In particular, $\mathbf{m} = (0) \in L(q)$ and $w(q, \mathbf{m}) = 1$. It was shown in [6, Theorem 1] that if $q = q_F$ is a conductor of an L -function $F \in \mathcal{S}_2^\#$, then

$$(3) \quad w(q, \mathbf{m}) = 1 \text{ for all } \mathbf{m} \in L(q),$$

and that (3) implies that $w(q, \mathbf{m})$ is a function of $c(q, \mathbf{m})$, i.e. $w(q, \mathbf{m}) = w(q, \mathbf{m}')$ for all paths \mathbf{m}, \mathbf{m}' satisfying $c(q, \mathbf{m}) = c(q, \mathbf{m}')$. In the present paper, for brevity, we call q forbidden if (3) is false.

In principle, to show that a fixed $q \in (0, 4)$ is forbidden, it is enough to find a loop $\mathbf{m} \in L(q)$ with $w(q, \mathbf{m}) \neq 1$. For rational $q = a/b$ this process bears some similarity to Euclid's Algorithm, but in general we have no deterministic algorithm to find such a loop. The search for loops of non-unit weight becomes harder for large a and, in particular, when q gets closer to 4. It was shown in [6] that the following qs are forbidden:

$$(4) \quad q = \frac{4}{n} \cos^2(\pi \ell / (2k + 1))$$

with integers k, ℓ, n such that $k \geq 1$, $1 \leq \ell < 2k + 1$, $(\ell, 2k + 1) = 1$, $n \geq 2$. The qs satisfying (4) are dense in the interval $0 < q < 2$. In addition, some rational qs were also determined to be forbidden:

- infinitely many with $0 < q < 1$ (with a point of accumulation in 0),
- 16271 qs with $1 < q < 2$,
- 3865 qs with $2 < q < 3$,
- 293 qs with $3 < q < 4$,

cf. the online table of computation results accompanying [6]. In the present paper we solve one of the problems posed in [6] and construct a set of forbidden q that is dense in $(0, 4)$. We also show that the set of forbidden rational qs has positive points of accumulation, including $\frac{3 \pm \sqrt{5}}{2}$.

Theorem 1. *The set of forbidden qs is dense in the interval $(0, 4)$.*

Theorem 2. *The set of accumulation points of forbidden rational qs contains $\frac{3 - \sqrt{5}}{2} \cong 0.381966$ and $\frac{3 + \sqrt{5}}{2} \cong 2.618034$.*

We start by listing further definitions and notation to be used in the paper. Next we show that, when q is close to 4, a loop $\mathbf{m} \in L(q)$ must contain a long chain of alternating ± 1 s, except for some trivial loops, that always have a unit weight. This fact will not be used directly in our main result, but it serves to set the scene, explaining why we consider loops based on such chains.

In Section 3 we prove our main result, Theorem 1. The proof is based on the study of loops of the form

$$(5) \quad (m_0, \dots, m_n) = (1, -1, \dots, (-1)^{n-1}, (-1)^n + c).$$

If $c = 0$ or $c = (-1)^{n+1}$, such loops must have unit weight, however, for other values of c this is not so, cf. Lemma 9. Using the fact that $c(q, \mathbf{m})$ is a rational function of q and the Darboux property we construct a dense set of forbidden qs .

In Section 4 we show Theorem 2 by finding rational qs for which (5) with $n = 4$ is a loop. The problem comes down to finding integer values of a given rational function at rational arguments a/b . The denominator of our rational function can be interpreted as the norm of an element $a - b\omega$ in an algebraic number field $\mathbf{Q}(\omega)$. Hence we only need to find enough units of the form $a - b\omega$, as for these the value of the function will be an integer. This approach strictly depends on loop length being equal to 4, as in that case the field $\mathbf{Q}(\omega)$ is quadratic, so the general form of units coincides with the form of factors in the decomposition of a homogeneous polynomial in a and b .

The problem of finding integer values of a rational function seems to be of independent interest.

Problem. Given a rational function $f \in \mathbf{Q}(x)$ determine the set of $x \in \mathbf{Q}$ such that $f(x) \in \mathbf{Z}$.

1. PRELIMINARIES

We use the notation $e(x) = e^{2\pi i x}$ and write $\lfloor x \rfloor$ for the largest integer $\leq x$. Given $q > 0$ we call a loop $\mathbf{m} = (m_0, \dots, m_n)$ proper if $n = 0$ or $m_j \neq 0$ for all j . It was shown in [6] that the study of loops, and of condition (3) in particular, can be reduced to proper loops, through an equivalence reminiscent of homotopy. In fact proper loops have a group structure and weight is a multiplicative homomorphism on proper loops.

Now we define polynomials in variables λ, m_0, m_1, \dots . They are related to the numerator and denominator of (1) when $q = \lambda^2$. Let \mathcal{I} denote the set of finite subsets of $I \subset \mathbf{N}_0$ such that

$$i \equiv |[0, i) \cap I| \pmod{2} \quad \text{for all } i \in I.$$

Let

$$(6) \quad f_n = f_n(\lambda, m_0, \dots, m_{n-1}) = \sum_{\substack{0 \leq k \leq n \\ k \equiv n \pmod{2}}} \left(\sum_{\substack{I \in \mathcal{I} \\ I \subseteq [0, n) \\ |I| = k}} \prod_{i \in I} m_i \right) \lambda^k, \quad n \geq 0.$$

The polynomials P_n and Q_n in [6, Proof of Theorem 2] are related to f_n by $P_n(\lambda^2, \mathbf{m}) = \lambda^n f_{n+1}(\lambda, \mathbf{m})$ and $Q_n(\lambda^2, \mathbf{m}) = \lambda^{n+1} f_n(\lambda, \mathbf{m}_{n-1})$.

Fact 3. f_n has degree n with respect to λ and the leading coefficient is $\prod_{i=0}^{n-1} m_i$.

Lemma 4. We have

$$\begin{aligned} f_0 &= 1, \\ f_1 &= m_0 \lambda, \end{aligned}$$

and

$$(7) \quad f_{n+1} = m_n \lambda f_n + f_{n-1}, \quad n \geq 1.$$

Proof. For $n \in I \subset [0, n+1)$, $I \in \mathcal{I}$ and $|I| \equiv n+1 \pmod{2}$, we have $I = \{n\} \cup I'$ with $I' \subset [0, n)$ and $|I'| \equiv n \pmod{2}$. If $n \notin I \subset [0, n+1)$, $I \in \mathcal{I}$ and $|I| \equiv n+1 \pmod{2}$, then also $n-1 \notin I$, as otherwise we would get

$$n \equiv |I \setminus \{n-1\}| = |I \cap [0, n-1)| \equiv n-1 \pmod{2},$$

a contradiction. This implies (7). \square

Lemma 5. If $\lambda > 0$ and $\mathbf{m} = (m_0, \dots, m_n) \in \mathbf{Z}^{n+1}$ is a path for $q = \lambda^2$, then

$$(8) \quad c(\lambda^2, \mathbf{m}) = \frac{f_{n+1}(\lambda, \mathbf{m})}{\lambda f_n(\lambda, \mathbf{m}_{n-1})}$$

and

$$(9) \quad w(\lambda^2, \mathbf{m}) = |f_n(\lambda, \mathbf{m}_{n-1})|.$$

Proof. By (7) we have

$$(10) \quad \begin{aligned} \frac{f_{n+1}}{\lambda f_n} &= \frac{m_n \lambda f_n + f_{n-1}}{\lambda f_n} \\ &= m_n + \frac{1}{\lambda^2 \frac{f_n}{f_{n-1}}}, \quad n \geq 1 \end{aligned}$$

and (8) follows. Then, by (2) and (8) we obtain (9). \square

2. A NECESSARY CONDITION FOR LOOPS

Suppose $2 < q < 4$. Let $x_0 = +\infty$, $x_1 = 1$ and

$$x_{n+1} = \begin{cases} 1 - \frac{1}{qx_n}, & x_n \geq \frac{1}{q}, \\ 0 & \text{otherwise,} \end{cases}$$

for $n \geq 1$. The sequence (x_n) is weakly decreasing and bounded, so it has a limit point. The map $x \mapsto 1 - \frac{1}{qx}$ has no fixed point in $(0, +\infty)$, therefore $\lim_{n \rightarrow \infty} x_n = 0$. Let $C(q)$ denote the largest integer such that $x_{C(q)} \geq \frac{1}{q}$.

Proposition 6. *Let $2 < q < 4$ and let $\mathbf{m} = (m_0, \dots, m_k) \in L(q)$ be a non-zero proper loop. Let ℓ be the smallest non-negative integer such that $|c(q, \mathbf{m}_j)| \leq 1$ for $\ell \leq j \leq k$. Then we have $k \geq \ell + C(q)$ and*

$$m_j = (-1)^j \varepsilon, \quad \ell \leq j \leq \ell + C(q) - 1,$$

for some $\varepsilon = \pm 1$.

Proof. Suppose for some $i \in \{0, \dots, k-1\}$ we have $|c(q, \mathbf{m}_i)| > \frac{1}{q}$ and $|c(q, \mathbf{m}_{i+1})| \leq 1$. Then we have

$$(11) \quad x_j < |c(q, \mathbf{m}_i)| \leq x_{j-1}$$

for some $1 \leq j \leq h$. We have $|c(q, \mathbf{m}_{i+1})| \leq 1$ and

$$0 < \left| \frac{1}{qc(q, \mathbf{m}_i)} \right| < \frac{1}{qx_j} \leq 1,$$

and the difference

$$m_{i+1} = c(q, \mathbf{m}_{i+1}) - \frac{1}{qc(q, \mathbf{m}_i)}$$

is a non-zero integer. This implies that $m_{i+1} = \pm 1$, moreover $\text{sgn}(m_{i+1}) = \text{sgn}(c(q, \mathbf{m}_{i+1})) = -\text{sgn}(c(q, \mathbf{m}_i))$ and $0 < |c(q, \mathbf{m}_{i+1})| < 1$, in particular $i \leq k-2$. We also have

$$c(q, \mathbf{m}_{i+1}) = \text{sgn}(c(q, \mathbf{m}_i)) \left(-1 + \frac{1}{q |c(q, \mathbf{m}_i)|} \right),$$

therefore

$$x_{j+1} < |c(q, \mathbf{m}_{i+1})| \leq x_j.$$

If $\ell > 0$, then (11) holds for $i = \ell - 1$ and $j = 1$, hence also for $1 \leq j \leq C(q)$ and $i = \ell + j - 2$. Moreover, we have

$$m_{\ell+j-1} = (-1)^j \text{sgn}(c(q, \mathbf{m}_{\ell-1})), \quad j = 1, \dots, C(q),$$

and $k \geq \ell + C(q)$.

If $\ell = 0$, then $m_0 = c(q, \mathbf{m}_0) = \pm 1$, in particular $m_0 = \text{sgn}(m_0)$. Now (11) holds for $j = 2$ and $i = 0$, hence also for $2 \leq j \leq C(q)$ and $i = j - 2$. Moreover, we have

$$m_{j-1} = (-1)^{j-1} m_0, \quad j = 2, \dots, C(q),$$

and $k \geq C(q) = \ell + C(q)$. \square

3. PROOF OF THEOREM 1

Let

$$g_n(\lambda) = f_n(\lambda, 1, -1, \dots, (-1)^{n-1}), \quad n \geq 0.$$

We are going to consider loops of the form (5). It follows from (8) and (10) that such loops correspond to integer values of $\frac{g_{n+1}(\lambda)}{\lambda g_n(\lambda)}$.

Lemma 7. *We have*

$$g_n = \pm \prod_{j=1}^n \left(\lambda - \left(e \left(\frac{j}{2n+2} \right) + e \left(-\frac{j}{2n+2} \right) \right) \right).$$

Proof. A part of the argument in this proof was contained in the proof of Theorem 4 in [6], but we repeat it because of the different generality and setup. We have, by (6),

$$g_n = \sum_{0 \leq \ell \leq n/2} \left(\sum_{\substack{I \in \mathcal{I} \\ I \subseteq [0, n] \\ |I| = n-2\ell}} (-1)^{\lfloor (n-2\ell)/2 \rfloor} \right) \lambda^{n-2\ell}.$$

The number of sets I in the inner sum is equal to the number of $(n-2\ell)$ -element subsets of $\{0, 2, \dots, 2n-2\}$, by the bijective mapping where $I = \{a_0, \dots, a_{n-2\ell-1}\}$ with $a_0 < \dots < a_{n-2\ell-1}$ corresponds to

$$I' = \{a_0, a_1 + 1, \dots, a_{n-2\ell-1} + n - 2\ell - 1\} \subseteq \{0, 2, \dots, 2n - 2\ell - 2\}.$$

Hence

$$g_n = \pm \sum_{0 \leq \ell \leq n/2} (-1)^\ell \binom{n-\ell}{n-2\ell} \lambda^{n-2\ell}.$$

By Fact 3 it is enough to show that

$$g_n \left(e \left(\frac{j}{2n+2} \right) + e \left(-\frac{j}{2n+2} \right) \right) = 0, \quad j = 1, \dots, n.$$

Fix j and $\omega = e\left(\frac{j}{2n+2}\right) + e\left(-\frac{j}{2n+2}\right)$. We have

$$\begin{aligned} g_n(\omega) &= \pm \sum_{0 \leq \ell \leq n/2} (-1)^\ell \binom{n-\ell}{n-2\ell} \sum_{0 \leq k \leq n-2\ell} \binom{n-2\ell}{k} e\left(\frac{(2k-n+2\ell)j}{2n+2}\right) \\ &= \pm \sum_{0 \leq m \leq n} e\left(\frac{(2m-n)j}{2n+2}\right) \sum_{0 \leq \ell \leq n/2} (-1)^\ell \binom{n-\ell}{n-2\ell} \binom{n-2\ell}{m-\ell} \\ &= \pm \sum_{0 \leq m \leq n} e\left(\frac{(2m-n)j}{2n+2}\right) = 0, \end{aligned}$$

where we use the fact that $\binom{n-2\ell}{m-\ell} = 0$ when $m-\ell > n-2\ell$, and the identity

$$\sum_{0 \leq \ell \leq n/2} (-1)^\ell \binom{n-\ell}{n-2\ell} \binom{n-2\ell}{m-\ell} = 1, \quad 0 \leq m \leq n,$$

which follows by induction. \square

Lemma 8. *We have*

$$(12) \quad g_n^2 + g_{n+1}g_{n-1} = 1, \quad n \geq 1.$$

Proof. By Lemma 4 we have

$$g_0 = 1,$$

$$g_1 = \lambda,$$

and

$$g_2 = -\lambda^2 + 1,$$

so (12) holds for $n = 1$. If (12) holds for a certain $n \geq 1$, then, using (7) twice, we obtain

$$\begin{aligned} g_{n+1}^2 + g_{n+2}g_n &= g_{n+1}^2 + ((-1)^{n+1}\lambda g_{n+1} + g_n)g_n \\ &= g_n^2 + (g_{n+1} - (-1)^n\lambda g_n)g_{n+1} \\ &= g_n^2 + g_{n+1}g_{n-1} = 1, \end{aligned}$$

so the assertion follows by induction. \square

Lemma 9. *If $n \geq 1$, $c \in \mathbf{Z} \setminus \{0, (-1)^{n+1}\}$ and $q > 0$ are such that $\mathbf{m} = (1, -1, \dots, (-1)^{n-1}, (-1)^n + c) \in \mathbf{Z}^{n+1}$ satisfies*

$$\mathbf{m} \in L(q),$$

then $w(q, \mathbf{m}) \neq 1$.

Proof. It follows from (10) and (8) that

$$(13) \quad \frac{g_{n+1}(\sqrt{q})}{\sqrt{q}g_n(\sqrt{q})} = \frac{f_{n+1}(\sqrt{q}, \mathbf{m})}{\sqrt{q}f_n(\sqrt{q}, \mathbf{m}_{n-1})} - c = -c.$$

By (12) and (7) we have

$$g_{n+1}^2(\sqrt{q}) + (-1)^{n+1}\sqrt{q}g_{n+1}(\sqrt{q})g_n(\sqrt{q}) + g_n^2(\sqrt{q}) = 1,$$

so (13) implies

$$g_n^2(\sqrt{q}) (c^2q + (-1)^n cq + 1) = 1.$$

Hence, by (9), we have

$$\begin{aligned} w(q, \mathbf{m}) &= |f_n(\sqrt{q}, \mathbf{m}_{n-1})| \\ &= |g_n(\sqrt{q})| \\ &= \frac{1}{\sqrt{|1 + cq(c + (-1)^n)|}}. \end{aligned}$$

□

To complete the proof of Theorem 1 let

$$U_n = \left\{ \left(e \left(\frac{j}{2n+2} \right) + e \left(-\frac{j}{2n+2} \right) \right)^2 : 1 \leq j \leq n, (j, n+1) = 1 \right\},$$

$n \geq 1.$

We show that every element of the set

$$U = \bigcup_{n=1}^{\infty} U_n$$

is an accumulation point of the set of forbidden qs . Since U is dense in the interval $[0, 4)$ and the set of accumulation points on the real line is closed, the assertion follows.

Note that the sets U_n are pairwise disjoint. Fix n and $t_0 \in U_n$. Let $t_1 > t_0$ be such that $t_1 < 4$ and

$$(14) \quad (t_0, t_1) \cap U_k = \emptyset \quad \text{for } k \in \{1, \dots, n-1, n+1\}.$$

Let $\epsilon = \text{sgn}(g_{n+1}(t_1)/g_n(t_1))$. By Lemma 7 and (14) the rational function

$$\frac{g_{n+1}(x)}{\sqrt{x}g_n(x)}$$

has no roots and no singularities in the interval $x \in (t_0, t_1)$. Moreover, by (8) and (14), every sequence of the form (5) is a path for all $q \in (t_0, t_1)$. We have

$$\lim_{x \rightarrow t_0^+} \frac{\epsilon g_{n+1}(x)}{\sqrt{x} g_n(x)} = +\infty,$$

so, by the Darboux property, there exists a sequence $q_k \xrightarrow[k \rightarrow \infty]{} t_0^+$ such that

$$\frac{\epsilon g_{n+1}(q_k)}{\sqrt{x} g_n(q_k)} = c_k \in \mathbf{Z} \cap [3, +\infty),$$

and in fact $c_k \xrightarrow[k \rightarrow \infty]{} +\infty$. It follows from (10) and (8) that for $\mathbf{m} \in \mathbf{Z}^{n+1}$,

$$\mathbf{m} = (1, -1, \dots, (-1)^{n-1}, (-1)^n - \epsilon c_k),$$

we have $\mathbf{m} \in L(q_k)$. Hence q_k is forbidden by Lemma 9. \square

4. PROOF OF THEOREM 2

For $\mathbf{m} = (1, -1, 1, -1, c)$ and rational positive q we have, by (6), (8) and (9),

$$c(q, \mathbf{m}) = c + \frac{2 - q}{1 - 3q + q^2}$$

and

$$w(q, \mathbf{m}) = |1 - 3q + q^2|.$$

We put $q = a/b$, $(a, b) = 1$, and we obtain

$$c(q, \mathbf{m}) = c + \frac{2b^2 - ab}{b^2 - 3ab + a^2}.$$

Existence of $c \in \mathbf{Z}$ such that $(1, -1, 1, -1, c) \in L(q)$ is therefore equivalent to $b^2 - 3ab + a^2 \mid 2b^2 - ab$ which, in turn, reduces to

$$(15) \quad b^2 - 3ab + a^2 = \pm 1$$

when $(a, b) = 1$. If (15) holds for any integers a, b (not necessarily relatively prime) such that

$$(16) \quad b^2 > 1 \quad \text{and} \quad a/b \neq 1, 2,$$

then

$$\mathbf{m} = (1, -1, 1, -1, -(2b^2 - ab)/(b^2 - 3ab + a^2)),$$

is a path for $q \neq 1, 2, \frac{3 \pm \sqrt{5}}{2}$ by (8) and Lemma 7. Hence $\mathbf{m} \in L(a/b)$. We have $w(a/b, \mathbf{m}) = 1/b^2 \neq 1$, so $q = a/b$ is forbidden.

To find a and b satisfying (15) and (16) we rewrite (15) as

$$(17) \quad (a - \omega_1 b)(a - \omega_2 b) = \pm 1$$

where $\omega_1 = \frac{3+\sqrt{5}}{2}$ and $\omega_2 = \frac{3-\sqrt{5}}{2}$. Let $M > 0$ be large. Since $\{1, \omega_2\}$ is a basis of $\mathbf{Z}\left[\frac{1+\sqrt{5}}{2}\right]$, we can find k large enough and integers a, b such that

$$a - b\omega_2 = \left(\frac{1+\sqrt{5}}{2}\right)^k > M,$$

in particular $b \neq 0$. The norm of a unit is ± 1 , so (17) holds. Therefore

$$\left|\frac{a}{b} - \omega_1\right| = \frac{1}{|b|} |a - b\omega_1| < \frac{1}{M}.$$

In particular $\left|\frac{a}{b} - \omega_1\right| < \frac{1}{3}$ implies $b \neq \pm 1$ and $a/b \neq 1, 2$. Hence $q = a/b$ is forbidden and it is arbitrarily close to ω_1 . The case of ω_2 is analogous. \square

We remark that for general $\mathbf{m} = (m_0, m_1, m_2, m_3, m_4)$ with non-zero terms we would get

$$c(a/b, \mathbf{m}) = m_4 + \frac{(m_0 + m_2)b^2 + m_0m_1m_2ab}{b^2 + (m_0m_1 + m_0m_3 + m_2m_3)ab + m_0m_1m_2m_3a^2},$$

so we would essentially have to solve (17) for

$$\omega_{1,2} = \frac{1}{2} \left(-u - v_1 - v_2 \pm \sqrt{u^2 + v_1^2 + v_2^2 - 2v_1v_2 + 2uv_1 + 2uv_2} \right)$$

where $v_1 = \frac{1}{m_0m_1}$, $v_2 = \frac{1}{m_2m_3}$ and $u = \frac{1}{m_1m_2}$. However, we cannot obtain an accumulation point $\max(\omega_1, \omega_2) > \frac{3+\sqrt{5}}{2}$ in this way, as by Proposition 6, we would need to have $(m_i, m_{i+1}, m_{i+2}) = (\epsilon, -\epsilon, \epsilon)$ for $i = 0$ or 1 and $\epsilon = \pm 1$. In the case $i = 0$ we have $v_1 = u = -1$ and $v_2 = \frac{\epsilon}{m_3}$, so

$$\omega_{1,2} = \frac{1}{2} \left(2 - \frac{\epsilon}{m_3} + \sqrt{4 + \frac{1}{m_3^2}} \right) \leq \frac{3 + \sqrt{5}}{2}.$$

The case $i = 1$ is analogous.

Reaching higher accumulation points would therefore require the study of longer loops. This, in turn, leads to searching for units in extensions $\mathbf{Q}(\omega)$ of a higher degree, and these units would not, in general, be of the form $a + b\omega$ that we require.

5. ACKNOWLEDGMENTS

This research was supported by the grant 2021/41/B/ST1/00241 from the National Science Centre, Poland. The author thanks Jerzy Kaczorowski for his remarks.

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