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THE THEORY OF FUNCTIONS

BY

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SECOND EDITION

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PREFACE TO SECOND EDITION

A CONSIDERABLE number of minor corrections and improvements have been made in this reprint of the first edition. I have to thank a large number of colleagues who have helped me in the revision. The only major change is that I have inserted in Chapter VIII a short introduction to the theory of meromorphic functions. This has been made possible by compressing some comparatively unimportant sections, and transferring the theory of the gamma-function to Chapter IV, where it now includes a more complete discussion of Stirling's formula.

E. C. T.

PREFACE TO FIRST EDITION

THIS volume is a development of the notes from which I have lectured in recent years to students at University College, London, and Liverpool University. It consists of some rather disconnected introductions to various branches of the theory of functions, both real and complex. I think the average student finds the existing literature on these subjects rather formidable, and I hope that these chapters will do something to bridge the gap between the elementary text-books and the systematic treatises on the theory of functions.

A knowledge of elementary analysis is assumed. By elementary analysis we mean, roughly, what is contained in Hardy's *Course of Pure Mathematics*. Apart from this the work is self-contained. The order in which the chapters occur is to a certain extent arbitrary. The last four chapters might well come after Chapter I. Apart from occasional references forward, the earlier part of the book is independent of these chapters; but what they contain is part of the necessary equipment of the analyst of to-day, just as much as the older theory of analytic functions.

A number of miscellaneous examples are given at the ends of the chapters. Some of them are more or less immediate applications of the book-work. Others are more difficult theorems

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which have not found a place in the text; these are accompanied by indications of the solution, and references to the sources.

When I first proposed to put my notes into the form of a book, Professor Hardy very generously offered to work through them in connexion with his lectures at Oxford, and they have been revised with the help of the notes which he made during this process. I have adopted a very large number of improvements from Professor Hardy's notes, and I wish to express my very deep gratitude for the assistance which he has given.

I have also to thank Mr. U. S. Haslam-Jones and Dr. B. M. Wilson, who have read the proofs and made a large number of useful suggestions.

E. C. T.

REFERENCES TO HARDY'S *PURE MATHEMATICS*

This book refers to the sixth edition of the above but the corresponding references in the seventh edition may be found from the following table:

Sixth Edition	*Seventh Edition*	*Sixth Edition*	*Seventh Edition*	*Sixth Edition*	*Seventh Edition*
99	100	160, 161	165, 166	193–4	200–1
101–2	102–3	167, 168	173, 175	206	213
105–6	106–7	175	181	208	215
125	126	177–8	184–5	213–14	220–1
146	149	180–1	187–8	222	229
153, 154	157, 159	184–5	191–2	224	231
156–64	161–9	189–90	196–7	233	240

The following alterations to references should also be noted:

Page 10, footnote, *for* Ex. 27 *read* Ex. 16
19, line 20, *for* ex. 32 *read* ex. 36
20, line 2 from foot, *for* § 184 *read* § 203
31, line 3, *for* ex. (xv) *read* ex. (xvii)
43, lines 2–3, *for* § 181, exs. LXXVI, 9–10
read § 188, exs. LXXVI, 8–9
51, footnote †, *for* ex. 20 *read* ex. 15
305, footnote *, *for* ex. 42 *read* ex. 44

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CHAPTER I

INFINITE SERIES, PRODUCTS, AND INTEGRALS

1. Introduction. In this opening chapter we supplement the knowledge of elementary analysis which the reader is supposed to have at his disposal. We deal particularly with series, each term of which is a function of a variable; with integrals involving variable parameters; and with a variety of those double-limit problems which are so common in all branches of analysis. As we have explained in the preface, we take Hardy's *Pure Mathematics* (to which we refer as *P.M.*) as a starting-point, and refer to it whenever possible.

We shall use the following notation. In any argument, a number independent of the main variables is called a constant. A number not depending on any variable is called an absolute constant. We use A to denote an absolute positive constant, not necessarily the same one each time it occurs. The reader may find statements such as '$f(x) < A$, hence $2f(x) < A$' a little disconcerting at first, but he will soon get used to them. A constant depending on one or more parameters is usually denoted by K.

By $f(x) = O\{\phi(x)\}$ we mean generally that $|f(x)| < A\phi(x)$ if x is sufficiently near to some given limit. In particular, $O(1)$ means a bounded function. Thus

$$\sin x = O(|x|), \qquad (x+1)^2 = O(1)$$

as $x \to 0$; and

$$\sin x = O(1), \qquad (x+1)^2 = O(x^2)$$

as $x \to \infty$.

Sometimes, however, $f(x) = O\{\phi(x)\}$ is used to mean

$$|f(x)| < K\phi(x),$$

but it is usually sufficiently obvious what parameters are involved.

By $f(x) = o\{\phi(x)\}$ we mean that $f(x)/\phi(x) \to 0$ as x tends to a given limit. Thus

$$\sin x = o(x^2), \qquad (x+1)^2 = o(x^3)$$

as $x \to \infty$. In particular, $o(1)$ means a function which tends to zero.

By $f(x) \sim \phi(x)$ we mean that $f(x)/\phi(x) \to 1$ as x tends to a given limit.

We use ϵ to denote a variable which is to be given arbitrarily small values, and so may be thought of as small.

By $\max(a, b, \ldots)$ we mean the greatest of $a, b, \ldots$, and by $\min(a, b, \ldots)$ the least.

1.1. Uniform convergence. The reader should be familiar with the idea of a *convergent series*.* Our standard notation for an infinite series is

$$u_1 + u_2 + u_3 + \ldots = \sum_{n=1}^{\infty} u_n = \sum u_n,$$

the limits of summation being $(1, \infty)$, unless other limits are definitely assigned. The nth partial sum of the series is

$$s_n = u_1 + u_2 + \ldots + u_n.$$

We begin by recalling the definition of convergence. The series is said to be convergent to the sum s if, given any positive number ϵ, however small, we can find a number n_0, depending on ϵ, such that

$$|s - s_n| < \epsilon \qquad (n > n_0).$$

In other words, s_n tends to the limit s as n tends to infinity.

Suppose now that each term of the series is a function of a real variable x. This variable is usually supposed to range over a closed interval, $a \leqslant x \leqslant b$, say; but the range of variation may equally well be an open interval, $a < x < b$; or indeed any set of points. We now write the series

$$u_1(x) + u_2(x) + \ldots = \sum u_n(x),$$

and its nth partial sum is $s_n(x)$. The series may, of course, be convergent for some values of x and divergent for others. If it is convergent for all the values of x considered, its sum is a function of x, defined for these values of x. We denote it by $s(x)$.

DEFINITION. *The series $\sum u_n(x)$ is said to be uniformly convergent over the interval (a, b) if, given any positive number ϵ, however small, we can find a number n_0, depending on ϵ* **but not on** *x, such that*

$$|s(x) - s_n(x)| < \epsilon$$

for $n > n_0$, and for every value of x in the interval (a, b).

It is clear that uniform convergence implies convergence for every value of x in the interval; but a series may (as we shall

* *P.M.* § 76.

show by examples) be convergent for every value of x in an interval without being uniformly convergent. It may be true that to every pair of values of x and ϵ corresponds a number n_0 such that $|s(x)-s_n(x)|<\epsilon$ for $n>n_0$; but at the same time it may happen that, as x approaches some point of the interval, the number n_0 may become indefinitely large. The series would then not be uniformly convergent.

Notice that uniform convergence is a property associated with *an interval* (or set of points), not with a single point.

1.11. Tests for uniform convergence. Just as there are tests for the convergence of a series of constants, so there are tests for the uniform convergence of a series of functions. The simplest and most useful test, due to Weierstrass, is as follows:

The series $\sum u_n(x)$ is uniformly convergent over the interval (a, b) if there is a convergent series of positive constant terms, $\sum a_n$ say, such that

$$|u_n(x)| \leqslant a_n$$

for all values of n and x.

In the first place, the series $\sum u_n(x)$ is convergent for every value of x, by the ordinary comparison theorem (*P.M.* §§ 167, 184). It therefore has a sum $s(x)$ for every value of x. Also

$$|s(x)-s_n(x)| = |u_{n+1}(x)+u_{n+2}(x)+\ldots| \leqslant a_{n+1}+a_{n+2}+\ldots,$$

which can be made less than any given ϵ by taking n greater than a certain number n_0. Since the a_n series is independent of x, the number n_0 is independent of x. This proves the theorem.

Notice that the result still holds if $|u_n(x)| \leqslant a_n$, not necessarily for *all* values of n, but for all sufficiently large values of n.

A more general test of the same type, which is sometimes useful, is that $\sum u_n(x)$ is uniformly convergent if $|u_n(x)| \leqslant v_n(x)$, and $\sum v_n(x)$ is uniformly convergent. We leave the proof of this to the reader.

Examples. (i) The power series $\sum_{n=0}^{\infty} x^n$ is uniformly convergent for $a \leqslant x \leqslant b$, if $-1<a<b<1$. [Take $a_n = |a|^n$ or $|b|^n$, whichever is the greater.]

(ii) The trigonometrical series

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$

is uniformly convergent over any interval.

(iii) The Dirichlet series $\sum_{n=1}^{\infty} n^{-s}$ is uniformly convergent for $a \leqslant s \leqslant b$, if $1 < a < b$.

[Take $a_n = n^{-a}$; see *P.M.* § 175. The sum of this important series is denoted by $\zeta(s)$.]

(iv) The series

$$\{1-(1-x^2)\}^{\frac{1}{2}} = \sum_{n=0}^{\infty} a_n(1-x^2)^n$$

is uniformly convergent for $-1 \leqslant x \leqslant 1$.

(v) A similar definition of uniform convergence may be framed for series such as

$$\sum_{n=0}^{\infty} r^n \cos n\theta,$$

where the general term is a function of two (or more) variables, here r and θ. This series is uniformly convergent for $0 \leqslant r \leqslant b < 1$ and any range of values of θ.

1.12. Other tests. In a general way, any test for convergence becomes a test for uniform convergence if its conditions are satisfied independently of x. For example (*P.M.* § 168), $\sum u_n(x)$ is convergent for a particular value of x if there is a number r, less than 1, such that

$$\left|\frac{u_{n+1}(x)}{u_n(x)}\right| \leqslant r$$

for all values of n. In general, the value of r for which this is true will depend on x. Suppose, however, that we can find a number r such that the condition is satisfied for all values of x with this same value of r. Then the series is uniformly convergent, provided that $u_1(x)$ is bounded. For repeated application of the above inequality gives

$$|u_n(x)| \leqslant r^{n-1}|u_1(x)| \leqslant Mr^{n-1},$$

if $|u_1(x)| \leqslant M$, and the result follows from the comparison test.

Other tests for convergence may be extended in the same way. Take, for example, Dirichlet's test (*P.M.* § 189). The analogous test for uniform convergence is as follows:

If ϕ_n is a positive function of n which tends steadily to zero as $n \to \infty$, and if there is a constant A such that

$$\left|\sum_{n=1}^{N} u_n(x)\right| \leqslant A$$

for all values of N and x, then the series

$$\sum \phi_n u_n(x)$$

is uniformly convergent.

The reader should have no difficulty in formulating the rigorous proof.*

Examples. (i) If the numbers a_n are positive and decrease steadily to zero, the series

$$\sum a_n \sin nx$$

is uniformly convergent in any closed interval not including a multiple of 2π. [Compare *P.M.* ex. LXXIX, 2. Use the identity

$$\sin x + \sin 2x + \ldots + \sin nx = \frac{\cos \frac{1}{2}x - \cos(n+\frac{1}{2})x}{2 \sin \frac{1}{2}x}.\Big]$$

(ii) Under the same conditions, the series

$$\sum a_n x \sin nx$$

is uniformly convergent in an interval including $x = 0$.

1.13. A necessary and sufficient condition for uniform convergence. *The series $\sum u_n(x)$ is uniformly convergent if and only if the following condition is satisfied. Given any positive number ϵ, we can find n_0, depending on ϵ but not on x, such that*

$$|s_m(x) - s_n(x)| < \epsilon$$

for all values of m and n greater than n_0.

This corresponds to the 'general principle of convergence' for ordinary series (*P.M.* §§ 83, 84).

As in the case of ordinary series, the condition is easily seen to be necessary; for

$$|s_m(x) - s_n(x)| \leqslant |s(x) - s_m(x)| + |s(x) - s_n(x)|,$$

so that, if the series is uniformly convergent, the condition is satisfied. In the case of ordinary series, the proof of sufficiency is more difficult. But, once the difficulty has been overcome in the 'ordinary' case, there is no further difficulty in the 'variable' case. For suppose that the condition is satisfied. Then, by the theorem for ordinary series, the series $\sum u_n(x)$ is convergent for every x. Let its sum be $s(x)$. Given ϵ, choose n_0 so that

$$|s_m(x) - s_n(x)| < \epsilon \qquad (m > n_0,\ n > n_0).$$

Keeping m fixed, make $n \to \infty$. Then, since $s_n(x) \to s(x)$,

$$|s_m(x) - s(x)| \leqslant \epsilon$$

provided only that $m > n_0$. Hence the convergence is uniform.

1.131. The following theorem† on a class of trigonometrical series is an excellent example of the above principle.

* See Bromwich's *Infinite Series*, ed. 2, § 44.

† Chaundy and Jolliffe (1).

If the numbers b_n are positive and steadily decreasing, a necessary and sufficient condition that the series

$$\sum b_n \sin nx$$

should be uniformly convergent throughout any interval is that $nb_n \to 0$.

To show that the condition is necessary, observe that, if $x = \pi/(2p)$, and* $n = [\frac{1}{2}p+1]$,

$$b_n \sin nx + b_{n+1}\sin(n+1)x + \ldots + b_p \sin px$$
$$> b_p(\sin nx + \ldots + \sin px) > b_p(\tfrac{1}{2}p-1)\sin \tfrac{1}{4}\pi,$$

since there are at least $\frac{1}{2}p-1$ terms in the bracket, in each of which $mx > \frac{1}{4}\pi$. Since the given series is uniformly convergent in an interval including the origin, the left-hand side of the above inequality tends to zero as $p \to \infty$. Hence $pb_p \to 0$.

In proving the sufficiency of the condition, we require the following result, known as Abel's lemma:

If $$b_1 \geqslant b_2 \geqslant \ldots \geqslant b_n \geqslant 0,$$

and if $$m \leqslant a_1 + a_2 + \ldots + a_n \leqslant M$$

for all values of n, then

$$b_1 m \leqslant a_1 b_1 + a_2 b_2 + \ldots + a_n b_n \leqslant b_1 M$$

for all values of n.

Let $s_n = a_1 + \ldots + a_n$. Then†

$$a_1 b_1 + \ldots + a_n b_n = b_1 s_1 + b_2(s_2 - s_1) + \ldots + b_n(s_n - s_{n-1})$$
$$= s_1(b_1 - b_2) + s_2(b_2 - b_3) + \ldots + s_{n-1}(b_{n-1} - b_n) + s_n b_n.$$

Since each bracket is positive or zero, the sum is not decreased if each s_m be replaced by M; and this gives

$$M(b_1 - b_2) + M(b_2 - b_3) + \ldots + Mb_n = Mb_1,$$

the required upper bound. Similarly we obtain the required lower bound. This proves the lemma.

In the series in question, it is sufficient to consider the interval $0 \leqslant x \leqslant \pi$, since each term is odd and has the period 2π. Consider the sum

$$s_{n,p} = b_n \sin nx + \ldots + b_p \sin px,$$

where now n and p are unconnected. Let $\mu_n = \max\limits_{m \geqslant n} (mb_m)$, so

* $[x]$ means the integral part of x.

† Compare *P.M.* § 189.

that $\mu_n \to 0$. If $x \geqslant \pi/n$, we apply Abel's lemma. We have

$$|\sin nx + \ldots + \sin rx| = \left|\frac{\cos(n-\frac{1}{2})x - \cos(r+\frac{1}{2})x}{2\sin\frac{1}{2}x}\right| \leqslant \frac{1}{\sin\frac{1}{2}x}$$

for all values of n and r, and, since $\sin\theta/\theta$ is steadily decreasing for $0 < \theta < \frac{1}{2}\pi$,

$$\frac{1}{\sin\frac{1}{2}x} \leqslant \frac{\pi}{x},$$

and we deduce that

$$|s_{n,p}| \leqslant \frac{b_n\pi}{x} \leqslant nb_n \leqslant \mu_n.$$

If $x \leqslant \pi/p$, we have, since $\sin\theta < \theta$,

$$|s_{n,p}| \leqslant b_n nx + \ldots + b_p px \leqslant p\mu_n x \leqslant \pi\mu_n.$$

If $\pi/p < x < \pi/n$, we combine the two arguments. We have

$$|s_{n,p}| \leqslant |s_{n,k}| + |s_{k+1,p}|,$$

and, applying Abel's lemma to the second part, and the other method to the first part, obtain

$$\begin{aligned} |s_{n,p}| &\leqslant k\mu_n x + b_{k+1}\pi/x \\ &\leqslant \mu_n[kx + \pi/\{(k+1)x\}]. \end{aligned}$$

Taking $k = [\pi/x]$, we have

$$|s_{n,p}| \leqslant \mu_n(\pi+1).$$

Hence in any case $\qquad |s_{n,p}| < A\mu_n,$

and, since $\mu_n \to 0$, the result follows.

1.14. Uniform convergence and continuity. So far, of course, we have not suggested any reason for considering uniformly convergent series at all. They are important for many reasons, not all of which can be explained in this chapter. The first reason is the following theorem:

The sum of a uniformly convergent series of continuous functions is a continuous function.

We use the same notation as before, and write

$$s(x) = s_n(x) + r_n(x),$$

so that $r_n(x)$ is the remainder after n terms of the series. Then, if x and $x+h$ are any two points of the interval considered,

$$\begin{aligned} |s(x+h) - s(x)| &= |s_n(x+h) - s_n(x) + r_n(x+h) - r_n(x)| \\ &\leqslant |s_n(x+h) - s_n(x)| + |r_n(x+h)| + |r_n(x)|. \end{aligned}$$

Having given ϵ, we can choose n_0 so that

$$|r_n(x+h)| < \epsilon, \qquad |r_n(x)| < \epsilon \qquad (n > n_0),$$

for all values of h. We now fix on a definite value of n which satisfies this condition. Having fixed n, $s_n(x)$ is a continuous function of x, since it is the sum of n continuous functions. We can therefore choose δ so small that

$$|s_n(x+h)-s_n(x)|<\epsilon \qquad (|h|<\delta).$$

Hence, combining the above inequalities,

$$|s(x+h)-s(x)|<3\epsilon \qquad (|h|<\delta),$$

which proves that $s(x)$ is continuous.

Notice that the result is true if the functions $s_n(x)$ are merely continuous at the single point x considered; for all we have used is that $s_n(x+h)\to s_n(x)$ as $h\to 0$, x being fixed. We can therefore state the result as follows:

The limit of the sum of a uniformly convergent series of functions, each of which tends to a limit, is the sum of the limits of the separate functions.

1.2. Series of complex terms.* The theory of uniform convergence may be extended to series of the form

$$u_1(z)+u_2(z)+\ldots,$$

in which the general term $u_n(z)$ is a function of the complex variable z. Instead of uniform convergence in an interval, we shall now have uniform convergence throughout some region of the z-plane, such as the interior of a circle or a square. The reader should have no difficulty in extending the definitions and tests to this case. It should also be noticed that the theorem on the continuity of the sum of a uniformly convergent series can be extended at once to series of complex functions.

Example. The series $\sum_{n=1}^{\infty} n^{-s}$, where s is a complex variable, is uniformly convergent throughout any finite region in which $\mathbf{R}(s)\geqslant a>1$.

The function $\zeta(s)$, defined as the sum of the series, is continuous at all points of the region $\mathbf{R}(s)>1$.

[Compare ex. (iii), § 1.11.]

1.21. Power series. One of the simplest cases of uniform convergence of a series of complex terms is that of a power series. We know (*P.M.* § 193) that a power series

$$\sum_{n=0}^{\infty} a_n z^n$$

* *P.M.* § 190.

has a *radius of convergence* R (which may be zero or infinite), such that the series is convergent for $|z| < R$, and divergent for $|z| > R$.

The series is uniformly convergent for $|z| \leqslant R'$, where R' is any positive number less than R.

For let ρ be a number between R' and R. Since the series is convergent for $z = \rho$, there is a number K, independent of n, such that $|a_n \rho^n| < K$ for all values of n. Hence, for $|z| \leqslant R'$,

$$|a_n z^n| = \left| a_n \rho^n \cdot \left(\frac{z}{\rho}\right)^n \right| < K\left(\frac{R'}{\rho}\right)^n,$$

and the last term is independent of z, and is the general term of a convergent geometrical progression. Hence (by the analogue for complex functions of the test of § 1.11) the series is uniformly convergent.

We have thus shown that any circle interior to the circle of convergence is a region of uniform convergence. The circle of convergence itself is not necessarily a region of uniform convergence; in fact on the circle the series does not necessarily converge at all.

Example. For the series $\sum z^n/n^2$, the circle of convergence is a region of uniform convergence.

1.22. Abel's theorem. There is one interesting possibility which the above discussion so far leaves open. Suppose, to take the simplest case, that we have a real power series

$$\sum_{n=0}^{\infty} a_n x^n \tag{1}$$

with radius of convergence 1. Suppose further that the series

$$\sum_{n=0}^{\infty} a_n \tag{2}$$

is convergent. Does the interval of uniform convergence, in this case, extend right up to the point $x = 1$? The answer is in the affirmative.

If the series

$$\sum_{n=0}^{\infty} a_n$$

is convergent, and has the sum s, then the series

$$\sum_{n=0}^{\infty} a_n x^n$$

is uniformly convergent for $0 \leqslant x \leqslant 1$, *and*

$$\lim_{x \to 1} \sum_{0}^{\infty} a_n x^n = s.$$

The proof is an immediate consequence of Abel's lemma (see § 1.131). Let

$$s_{n,p} = a_n + a_{n+1} + \ldots + a_p.$$

Then, given ϵ, we can choose n_0 so large that $|s_{n,p}| \leqslant \epsilon$ $(n_0 \leqslant n < p)$. Since the numbers x^n are non-increasing if $x \leqslant 1$, Abel's lemma gives, for $n_0 < n < p$,

$$|a_n x^n + \ldots + a_p x^p| \leqslant \epsilon x^n \leqslant \epsilon \qquad (0 \leqslant x \leqslant 1),$$

and this is the condition for uniform convergence.

The second part of the theorem now follows from the continuity theorem of § 1.14.

Example. From the expansion (*P.M.* § 213)

$$\log(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \ldots \qquad (|x| < 1),$$

deduce that $$\log 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \ldots.$$

1.23. Tauber's theorem. The direct converse of the 'continuity' part of Abel's theorem would be that if

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \to s$$

as $x \to 1$, then $\sum_{0}^{\infty} a_n$ converges to the sum s. That this is false is shown by the simple example

$$f(x) = \sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1+x},$$

in which $f(x) \to \frac{1}{2}$, but $\sum_{0}^{\infty} a_n$ is not convergent.

If, however, we impose on the coefficients a_n a restriction as to their order of magnitude, it is possible to prove a converse theorem.

If $a_n = o(1/n)$, *and* $f(x) \to s$ *as* $x \to 1$, *then* $\sum_{0}^{\infty} a_n$ *converges to the sum* s.

We first prove the following simple lemma:*

LEMMA. *If* $b_n \to 0$ *as* $n \to \infty$, *then*

$$\frac{b_0 + b_1 + \ldots + b_n}{n+1} \to 0.$$

* *P.M.* Ch. IV, Misc. Ex. 27.

For, if $|b_n| < K$ for all values of n, and $|b_n| < \epsilon$ for $n > n_0$, then

$$\left|\frac{b_0+b_1+\ldots+b_n}{n+1}\right| \leqslant \left|\frac{b_0+\ldots+b_{n_0}}{n+1}\right| + \left|\frac{b_{n_0+1}+\ldots+b_n}{n+1}\right|$$
$$\leqslant \frac{(n_0+1)K}{n+1} + \frac{(n-n_0)\epsilon}{n+1} < 2\epsilon$$

if $n > (n_0+1)K/\epsilon$. This proves the lemma.

To prove Tauber's theorem, it is sufficient to prove that

$$\sum_0^\infty a_n x^n - \sum_0^N a_n \to 0$$

as $x \to 1$, where $N = [1/(1-x)]$. That is, we have to show that

$$\sum_{N+1}^\infty a_n x^n - \sum_0^N a_n(1-x^n) \to 0.$$

Call these two sums S_1 and S_2. Given ϵ, choose N so large that $|na_n| < \epsilon$ $(n > N)$. Then

$$|S_1| = \left|\sum_{N+1}^\infty na_n \cdot \frac{x^n}{n}\right| < \frac{\epsilon}{N+1}\sum_{N+1}^\infty x^n < \frac{\epsilon}{(N+1)(1-x)} < \epsilon.$$

Also $\qquad 1-x^n = (1-x)(1+x+\ldots+x^{n-1}) < n(1-x),$

and so $\qquad |S_2| < (1-x)\sum_0^N n|a_n| \leqslant \frac{1}{N}\sum_0^N n|a_n|,$

which tends to zero, by the lemma. Hence $|S_2| < \epsilon$ if N is large enough, and so $|S_1+S_2| < 2\epsilon$. This proves the theorem.

1.3. Series which are not uniformly convergent. Up to this point, the reader may still suspect that convergence throughout an interval is the same thing as uniform convergence. We shall show by means of examples that this is not so.

Examples. (i) We can construct a series for which

$$s_n(x) = \frac{1}{1+nx} \qquad (0 \leqslant x \leqslant 1)$$

by taking $u_1(x) = 1/(1+x)$, and

$$u_n(x) = \frac{1}{1+nx} - \frac{1}{1+(n-1)x} = \frac{-x}{(1+nx)\{1+(n-1)x\}} \qquad (n > 1).$$

This function $s_n(x)$ is *a continuous function which tends to a discontinuous limit.* For, if $x > 0$, $s_n(x)$ obviously tends to zero as $n \to \infty$. But if $x = 0$, $s_n(x) = 1$ for all values of n, and so its limit is 1. The sum of the series is therefore discontinuous. Hence the series cannot be uniformly convergent.

(ii) Consider the series

$$\sum_{n=0}^\infty xe^{-nx}.$$

Here $s_n(0) = 0$, so that $s(0) = 0$. When $x > 0$,

$$s_n(x) = x\frac{1-e^{-nx}}{1-e^{-x}}, \qquad s(x) = \frac{x}{1-e^{-x}}.$$

As $x \to 0$, $s(x) \to 1$. Hence $s(x)$ is discontinuous, and, as before, the series is not uniformly convergent in any interval ending at $x = 0$.

In fact, if $x = 1/n$,

$$s\left(\frac{1}{n}\right) - s_n\left(\frac{1}{n}\right) = \frac{xe^{-1}}{1-e^{-x}} \to e^{-1},$$

so that $|s(x)-s_n(x)|$ is not 'uniformly small' near $x = 0$.

(iii) Consider similarly

$$\sum_{n=0}^{\infty} x^n(1-x).$$

(iv) As in example (i), we can construct a series for which

$$s_n(x) = nx(1-x)^n \qquad (0 \leqslant x \leqslant 1).$$

Obviously $s(0) = 0$. Also, if $x > 0$, $n(1-x)^n \to 0$ as $n \to \infty$ (*P.M.* § 206). Hence $s(x) = 0$ for all values of x. In this case, therefore, the sum of the series is continuous. But the series is not uniformly convergent. It is a simple exercise in differential calculus to find the maximum of $s_n(x)$; it is

$$\left(\frac{n}{1+n}\right)^{1+n},$$

and thus tends to the limit e^{-1} as $n \to \infty$ (*P.M.* §§ 73, 208). Hence, however large n may be, the function $s_n(x)-s(x)$ takes values nearly as large as e^{-1}. Thus the convergence is not uniform.

The reader should draw the graph of $s_n(x)$. It has a wave which approaches the origin, and diminishes indefinitely in breadth, but not in height.

Notice that uniformity of convergence may be altered by multiplying by a factor independent of n. For example, if

$$s_n(x) = \frac{x}{1+nx},$$

then $|s_n(x)| \leqslant 1/n$ $(0 \leqslant x \leqslant 1)$, so that the series converges uniformly to zero. But the series obtained by multiplying by $1/x$ is not uniformly convergent (ex. (i)).

On the other hand, if we multiply a uniformly convergent series by a *bounded* factor independent of n, the resulting series is also uniformly convergent. This is easily seen from the definition.

1.31. Uniform convergence of series of positive terms. It is clear from the above examples that uniform convergence is not a necessary condition for continuity, though it is a suffi-

cient condition. There is, however, one interesting case in which uniform convergence and continuity are equivalent.*

If $\sum u_n(x) = s(x)$ is a series of continuous positive terms in a given closed interval, a necessary and sufficient condition that $s(x)$ should be continuous is that the series should be uniformly convergent over the interval.

We have to prove that the condition is necessary, i.e. that, if $s(x)$ is continuous, the series is uniformly convergent.

Employing our usual notation, the function $s(x)-s_n(x)$ is continuous, and so (*P.M.* § 102, Th. 2) has an upper bound, ϵ_n say, which is attained at some point x_n of the interval. It is sufficient to prove that $\epsilon_n \to 0$; for, since the terms are positive,

$$|s(x)-s_n(x)| \leqslant |s(x)-s_N(x)| \leqslant \epsilon_N$$

for $n \geqslant N$ and all x; and this implies uniform convergence if $\epsilon_N \to 0$ as $N \to \infty$.

Suppose on the contrary that ϵ_n does not tend to zero. Then, since it is steadily decreasing (because the terms are positive), it has a positive lower bound, δ say. Also the numbers x_n have a limit-point, ξ say, in the interval (*P.M.* § 19). Choose N so large that $s(\xi)-s_N(\xi) < \delta$. Then, if ξ is an interior point of the interval, there is an interval $(\xi-h, \xi+h)$ throughout which $s(x)-s_N(x) < \delta$ (*P.M.* § 101, Th. 1). If ξ is an end-point, the same is true of $(\xi, \xi+h)$ or $(\xi-h, \xi)$. Hence $\epsilon_n < \delta$ for those values of n for which $|x_n-\xi| < h$. This gives a contradiction, and the theorem is proved.

1.4. Infinite products. An infinite product is an expression of the form

$$(1+a_1)(1+a_2)(1+a_3)\ldots \tag{1}$$

containing an infinity of factors. We denote it by

$$\prod_{n=1}^{\infty} (1+a_n).$$

We suppose that no a_n is equal to -1.

Writing p_n for the partial product

$$p_n = \prod_{m=1}^{n} (1+a_m),$$

we say that *the infinite product is convergent if, as $n \to \infty$, p_n tends to a limit other than zero.* We might, of course, admit the

* See Hardy (11) for a detailed discussion.

limit zero as well; but we shall see later that this would often be inconvenient.

If the product is not convergent, it is said to be divergent. If $p_n \to 0$, it is said to diverge to zero.

Examples. (i) The product

$$(1-\tfrac{1}{2})(1+\tfrac{1}{3})(1-\tfrac{1}{4})(1+\tfrac{1}{5})\ldots$$

is convergent.

(ii) If the product (1) is convergent, $a_n \to 0$.

1.41. We begin by considering two simple cases.

If $a_n \geqslant 0$ for all values of n, the product $\prod(1+a_n)$ and the series $\sum a_n$ converge or diverge together.

Since, in this case, p_n is a non-decreasing function of n, it either converges or tends to positive infinity. Now

$$a_1+a_2+\ldots+a_n \leqslant (1+a_1)\ldots(1+a_n) \leqslant e^{a_1+a_2+\ldots+a_n}.$$

The left-hand inequality is obvious on multiplying out the product; and the right-hand inequality follows from the fact that $1+a \leqslant e^a$ for every positive a. The two inequalities show that p_n and $a_1+\ldots+a_n$ are bounded or unbounded together, and this gives the result.

If $a_n \leqslant 0$ for all values of n, write $a_n = -b_n$, and consider the product

$$\prod_{n=1}^{\infty}(1-b_n).$$

If $b_n \geqslant 0$, $b_n \neq 1$, for all values of n, and $\sum b_n$ is convergent, then $\prod(1-b_n)$ is convergent.

Since $\sum b_n$ is convergent, we can choose N so large that

$$b_N+b_{N+1}+\ldots < \tfrac{1}{2},$$

and, in particular, $b_n < 1$ $(n \geqslant N)$. Then

$$(1-b_N)(1-b_{N+1}) \geqslant 1-b_N-b_{N+1},$$

$$\begin{aligned}(1-b_N)(1-b_{N+1})(1-b_{N+2}) &\geqslant (1-b_N-b_{N+1})(1-b_{N+2})\\ &\geqslant 1-b_N-b_{N+1}-b_{N+2},\end{aligned}$$

and so generally

$$(1-b_N)(1-b_{N+1})\ldots(1-b_n) \geqslant 1-b_N-\ldots-b_n > \tfrac{1}{2}.$$

Hence p_n/p_{N-1} is steadily decreasing for $n \geqslant N$, and has a positive lower bound. Hence it tends to a positive limit. Since p_{N-1} is not zero, the result follows.

If $0 \leqslant b_n < 1$ for every n, but $\sum b_n$ diverges, then $\prod(1-b_n)$ diverges to zero.

For $1-b \leqslant e^{-b}$ if $0 \leqslant b < 1$, so that

$$(1-b_1)(1-b_2)\ldots(1-b_n) \leqslant e^{-b_1-b_2\ldots-b_n}.$$

The right-hand side tends to zero, and the result follows.

In particular, *if* $0 \leqslant b_n < 1$, *the product* $\prod(1-b_n)$ *and the series* $\sum b_n$ *converge or diverge together.*

1.42. The general case. Now let the numbers a_n be any numbers, real or complex, other than -1.

DEFINITION. *The product* $\prod(1+a_n)$ *is said to be absolutely convergent if the product* $\prod(1+|a_n|)$ *is convergent.*

It is clear from the first result of § 1.41 that *a necessary and sufficient condition that the product should be absolutely convergent is that* $\sum|a_n|$ *should be convergent.*

We next show that *an absolutely convergent product is convergent.*

To prove this, let p_n denote the same partial product as before, and let

$$P_n = \prod_{m=1}^{n}(1+|a_m|).$$

Then

$$p_n - p_{n-1} = (1+a_1)\ldots(1+a_{n-1})a_n,$$
$$P_n - P_{n-1} = (1+|a_1|)\ldots(1+|a_{n-1}|)|a_n|,$$

and it is plain that

$$|p_n - p_{n-1}| \leqslant P_n - P_{n-1}.$$

Now, if $\prod(1+|a_n|)$ is convergent, P_n tends to a limit, and so $\sum(P_n - P_{n-1})$ is convergent. Hence, by the comparison theorem, $\sum(p_n - p_{n-1})$ is convergent, i.e. p_n tends to a limit.

This limit cannot be zero. For, since $\sum|a_n|$ is convergent and $1+a_n \to 1$, the series

$$\sum\left|\frac{a_n}{1+a_n}\right|$$

is also convergent. Hence, by what we have just proved, the product

$$\prod_{m=1}^{n}\left(1-\frac{a_m}{1+a_m}\right)$$

tends to a limit. But this product is equal to $1/p_n$. Hence the limit of p_n is not zero.

Example. The factors of an absolutely convergent product may be taken in any order, without altering the value of the product. (Compare *P.M.* § 185.)

1.43. The logarithm of an infinite product. If

$$\prod_{n=1}^{\infty}(1+a_n)=p,$$

is it necessarily true that

$$\sum_{n=1}^{\infty}\log(1+a_n)=\log p\ ?$$

Here $\log z$ denotes the principal value of the logarithm of z, i.e. the value whose imaginary part lies between $-\pi$ and π (*P.M.* § 224).

The result is obviously true if all the numbers a_n are real and positive, for then all the logarithms have their ordinary arithmetical value. But, in the general case, the formula requires modification.

Let p_n denote the nth partial product, and let $p_n=\rho_n e^{i\phi_n}$, so that p_n and ρ_n tend to limits, and so does ϕ_n if its values are suitably chosen. Let $1+a_n=r_n e^{i\theta_n}$, where $-\pi<\theta_n\leqslant\pi$; then, since $a_n\to 0$, $\theta_n\to 0$ as $n\to\infty$.

Let
$$s_n=\sum_{m=1}^{n}\log(1+a_m).$$

Then
$$s_n=\log p_n+2k_n i\pi, \tag{1}$$

where k_n is an integer. Now

$$2k_n\pi=\theta_1+\ldots+\theta_n-\phi_n,$$

so that
$$2\pi(k_{n+1}-k_n)=\theta_{n+1}-(\phi_{n+1}-\phi_n),$$

and the right-hand side tends to zero. Hence, if n is sufficiently large,

$$|2\pi(k_{n+1}-k_n)|<2\pi,$$

and so $k_{n+1}=k_n$, since k_n is always an integer. Thus k_n has a constant value, k say, if n is sufficiently large; i.e.

$$s_n=\log p_n+2ki\pi \qquad (n>n_0),$$

and, making $n\to\infty$,

$$\sum\log(1+a_n)=\log p+2ki\pi.$$

The sum of the series is therefore a value, but not necessarily the principal value, of the logarithm of the product.

Notice also that it follows from the proof that

$$\sum_{N+1}^{\infty}\log(1+a_n)=\log p-\log p_N$$

for all sufficiently large values of N.

If we start with the series of logarithms, and assume that

$$\sum_{n=1}^{\infty} \log(1+a_n) = s,$$

we have, on taking the exponential of (1),

$$e^{s_n} = p_n,$$

and so

$$p_n \to p = e^s,$$

i.e. the product converges to the exponential of the sum.

Examples. (i) If $\sum a_n$ and $\sum |a_n|^2$ are convergent, then $\prod (1+a_n)$ is convergent. [Use the equation $\log(1+a_n) = a_n + O(|a_n|^2)$.]

(ii) If $\sum a_n$, $\sum a_n^2, \ldots, \sum a_n^{k-1}$, $\sum |a_n|^k$, are all convergent, then $\prod (1+a_n)$ is convergent.

(iii) If a_n is real, and $\sum a_n$ is convergent, the product $\prod (1+a_n)$ converges, or diverges to zero, according as $\sum a_n^2$ converges or diverges.

(iv) The product

$$\left(1-\frac{1}{\sqrt{2}}\right)\left(1+\frac{1}{\sqrt{3}}\right)\left(1-\frac{1}{\sqrt{4}}\right)\ldots$$

is divergent.

(v) Show that, if

$$a_{2n-1} = -\frac{1}{\sqrt{(n+1)}}, \quad a_{2n} = \frac{1}{\sqrt{(n+1)}} + \frac{1}{n+1} + \frac{1}{(n+1)\sqrt{(n+1)}},$$

the product $\prod (1+a_n)$ converges, though both $\sum a_n$ and $\sum a_n^2$ diverge.

(vi) The product $\prod\left(1+\frac{i}{n}\right)$ is divergent, but $\prod\left|1+\frac{i}{n}\right|$ is convergent.

(vii) If $\sum |u_n^2|$ is convergent, so is $\prod (1-u_n)e^{u_n}$; and if $\sum |u_n|^3$ is convergent, so is $\prod (1-u_n)e^{u_n+\frac{1}{2}u_n^2}$. [As $u \to 0$, $(1-u)e^u = 1+O(u^{-2})$ and $(1-u)e^{u+\frac{1}{2}u^2} = 1+O(u^{-3})$; or we may consider the series of logarithms, as in (i). Products of this type are of great importance in Chapter VIII, and are discussed fully there.]

1.44. Uniform convergence of infinite products. The infinite product

$$\prod_{n=1}^{\infty} \{1+u_n(z)\},$$

where the factors are functions of a variable z, real or complex, is said to be uniformly convergent if the partial product

$$p_n(z) = \prod_{m=1}^{n} \{1+u_m(z)\}$$

converges uniformly in a certain region of values of z to a limit which is never zero.

The simplest test for the uniform convergence of a product is as follows:

The product $\prod\{1+u_n(z)\}$ *is uniformly convergent in any region where the series* $\sum|u_n(z)|$ *converges uniformly to a bounded sum.*

The proof consists of a re-examination of the convergence-argument of § 1.42 from the point of view of uniformity. Let M be the upper bound of the sum $\sum|u_n(z)|$ in the region considered. Then

$$\{1+|u_1(z)|\}\ldots\{1+|u_n(z)|\}<e^{|u_1(z)|+\ldots}\leqslant e^M.$$

Let
$$P_n(z)=\prod_{m=1}^{n}\{1+|u_m(z)|\}.$$

Then

$$P_n(z)-P_{n-1}(z)=\{1+|u_1(z)|\}\ldots\{1+|u_{n-1}(z)|\}|u_n(z)|<e^M|u_n(z)|.$$

Hence $\sum\{P_n(z)-P_{n-1}(z)\}$ is uniformly convergent, and the result follows as in § 1.42.

Examples. (i) The product

$$\prod\left(1-\frac{1}{\varpi^s}\right),$$

where ϖ runs through the prime numbers 2, 3, 5,..., is uniformly convergent in any finite region throughout which $\mathbf{R}(s)\geqslant a>1$; for the same thing is true of the series $\sum|\varpi^{-s}|$, which consists of some of the terms of the series $\sum|n^{-s}|$ (§ 1.2, example).

The value of the product is $1/\zeta(s)$. For

$$(1-2^{-s})\zeta(s)=1+3^{-s}+5^{-s}+\ldots,$$

all terms containing the factor 2 being omitted on the right. Next

$$(1-2^{-s})(1-3^{-s})\zeta(s)=1+5^{-s}+7^{-s}+11^{-s}+\ldots,$$

all terms containing the factors 2 or 3 being omitted. So generally, if ϖ_n is the nth prime,

$$(1-2^{-s})\ldots(1-\varpi_n^{-s})\zeta(s)=1+l^{-s}+\ldots$$

where all numbers containing the factors 2, 3,..., ϖ_n are omitted. Since all the numbers up to ϖ_n are of this form,

$$|(1-2^{-s})\ldots(1-\varpi_n^{-s})\zeta(s)-1|\leqslant|(\varpi_n+1)^{-s}|+|(\varpi_n+2)^{-s}|+\ldots,$$

which tends to 0 as $\varpi_n\to\infty$. Hence

$$\lim_{n\to\infty}(1-2^{-s})\ldots(1-\varpi_n^{-s})\zeta(s)=1,$$

the result stated.

(ii) If $\mathbf{R}(s)\geqslant 2$

$$\log\zeta(s)=-\sum_{\varpi}\log(1-\varpi^{-s}),$$

all the logarithms having their principal values.

[We deduce from the above example and § 1.43 that

$$\log\zeta(s)=-\sum\log(1-\varpi^{-s})+2ki\pi,$$

where k is an integer, which depends *prima facie* on s. If s is real, k is obviously 0. Also, as long as $\mathbf{R}(s)>1$, the real part of $1-\varpi^{-s}$ remains

positive, and so its amplitude remains between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. Each term $\log(1-\varpi^{-s})$ is therefore continuous for $\mathbf{R}(s) > 1$. Hence the sum of the series is continuous.

Similarly $\log \zeta(s)$ is continuous, provided that $\mathbf{R}\{\zeta(s)\} > 0$. This is certainly true if $\mathbf{R}(s) \geqslant 2$, since, if $\mathbf{R}(s) = \sigma \geqslant 2$,

$$\mathbf{R}\{\zeta(s)\} \geqslant 1-2^{-\sigma}-3^{-\sigma}-\ldots \geqslant 1-2^{-2}-3^{-2}-\ldots$$
$$> 1-\frac{1}{1.2}-\frac{1}{2.3}-\ldots = 0.$$

It follows that k is continuous for $\mathbf{R}(s) \geqslant 2$, and so zero throughout this region.]

(iii) The convergence of the product $\prod(1+a_n)$ does not imply that of $\prod(1+a_n x)$, except for $x = 0$ and $x = 1$.*

(iv) The convergence of $\prod(1+a_n)$ does not imply that

$$\lim_{x\to 1} \prod(1+a_n x^n) = \prod(1+a_n).$$

[In fact, Hardy (5) gives an example in which

$$\lim \prod(1+a_n x^n) = 2\prod(1+a_n).$$

The result is, of course, in striking contrast to Abel's theorem on the continuity of power series.]

(v) The product $$\prod\left(1+\frac{e^{in\theta}}{\log n}\right)$$

is not convergent for any rational value of θ/π, but is convergent if θ/π is an algebraic number (*P.M.* Ch. I, ex. 32) which is not rational.

[The problem of the behaviour of this product, suggested by Hardy (5), was solved by Littlewood (2).]

1.5. Convergence of infinite integrals. We assume that the reader is familiar with the elementary properties of the Riemann integral of a continuous function (*P.M.* §§ 156–64). If $f(x)$ is continuous over a finite closed interval (a, b), the Riemann integral

$$\int_a^b f(x)\,dx$$

exists. Similarly the indefinite integral

$$F(x) = \int_a^x f(t)\,dt$$

exists for $a \leqslant x \leqslant b$; and $F(x)$ is continuous, and has a differential coefficient equal to $f(x)$. We assume a knowledge of the usual rules of integration by parts, integration by substitution, etc., and of the mean-value theorems (*P.M.* §§ 160–1).

We next extend the definition to a class of discontinuous

functions. Suppose that the interval (a, b) can be divided up into a finite number of parts (a, x_1), (x_1, x_2),..., (x_n, b), such that $f(x)$ is continuous except at x_1,..., x_n, and such that the limits $f(x_1-0)$, $f(x_1+0)$,..., etc., exist (*P.M.* § 99). Then the integral of $f(x)$ over each partial interval exists, and the integral over the whole interval is defined as being the sum of the integrals over the partial intervals; i.e.

$$\int_a^b f(x)\,dx = \int_a^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \ldots + \int_{x_n}^{b} f(x)\,dx.$$

An infinite integral is defined in *P.M.* § 177. If $f(t)$ is integrable over (a, x) for all values of x, and

$$\lim_{x\to\infty} \int_a^x f(t)\,dt = l,$$

then we say that the infinite integral

$$\int_a^\infty f(t)\,dt$$

is convergent, and has the value l.

Similarly, if $f(t)$ tends to infinity, or oscillates, as $x \to c$, but

$$\lim_{x\to c} \int_a^x f(t)\,dt = l,$$

then we define the integral of $f(t)$ over (a, c) to be equal to l (*P.M.* § 180).

There is no difficulty in extending the rules for integration by parts and substitution to these cases.

A number of tests for convergence, such as the comparison test, for the case where $f(x)$ is positive, are given in *P.M.* § 178.

Suppose now that $f(x)$ is not necessarily positive. If $f(x)$ and $|f(x)|$ are both integrable in one of the senses already explained, and if the integral

$$\int_a^\infty |f(t)|\,dt$$

is convergent, then the integral

$$\int_a^\infty f(t)\,dt$$

is said to be *absolutely convergent* (cf. *P.M.* § 184).

An absolutely convergent integral is convergent. For, if the

integral of $|f(t)|$ is convergent, so are the integrals of

$$\phi(t) = |f(t)|+f(t), \qquad \psi(t) = |f(t)|-f(t),$$

by the comparison test, ϕ and ψ both being positive. Hence the integral of $\frac{1}{2}\{\phi(t)-\psi(t)\} = f(t)$ is convergent.

The result may be extended to the case where $f(t)$ is a continuous complex function, by considering separately its real and imaginary parts.

An integral which is convergent, but not absolutely convergent, is said to be *conditionally convergent.*

The most important tests for conditionally convergent integrals are the analogues of Dirichlet's and Abel's tests for series (*P.M.* § 189).

Analogue of Dirichlet's test. *If $\phi(x)$ has a continuous derivative, and decreases steadily to zero as $x \to \infty$, and*

$$F(x) = \int_a^x f(t)\,dt$$

is bounded, then the integral

$$\int_a^\infty \phi(x)f(x)\,dx$$

is convergent.

We integrate by parts, this being the process for integrals analogous to the 'partial summation' by which Dirichlet's test is proved. We have

$$\int_a^X \phi(x)f(x)\,dx = \phi(X)F(X)+\int_a^X \{-\phi'(x)\}F(x)\,dx.$$

The integrated term tends to zero as $X \to \infty$; and the last integral is absolutely convergent by the comparison test; for $|F(x)|$ is bounded and $-\phi'(x)$ is positive, and

$$\int_a^X \{-\phi'(x)\}\,dx = \phi(a)-\phi(X) \to \phi(a).$$

This proves the theorem.

Examples. (i) The integrals

$$\int_0^\infty \frac{\sin x}{x}\,dx, \qquad \int_1^\infty \frac{\cos x}{x}\,dx$$

are conditionally convergent.

(ii) State and prove the analogue for integrals of Abel's test.

We note finally a *necessary and sufficient condition* for the convergence of the integral

$$\int_a^\infty f(x)\,dx;$$

it is that, given ϵ, we can find X_0 such that

$$\left|\int_X^{X'} f(x)\,dx\right| < \epsilon$$

for $X' > X \geqslant X_0$. This may be proved in the same way as (or deduced from) the corresponding theorem for series (*P.M.* §§ 83–4).

Examples. (i) Use this principle to prove that an absolutely convergent integral is convergent.

(ii) Prove Dirichlet's test for convergence by means of this principle and the second mean-value theorem (*P.M.* § 161, exs. 11–12).

1.51 Uniform convergence of infinite integrals. We can now extend the idea of uniform convergence to infinite integrals. Let $f(x, y)$ be an integrable function of x over the interval $a \leqslant x \leqslant b$, for $\alpha \leqslant y \leqslant \beta$, and for all values of b. Suppose that the integral

$$\phi(y) = \int_a^\infty f(x, y)\,dx$$

is convergent for all values of y in the interval (α, β). *Then the integral is said to be uniformly convergent if, given ϵ, we can find a number X_0, depending on ϵ but not on y, such that*

$$\left|\phi(y) - \int_a^X f(x, y)\,dx\right| < \epsilon, \qquad (X \geqslant X_0).$$

A similar definition may be framed for integrals which are infinite by reason of the integrand becoming infinite in the range of integration.

The simplest test for uniform convergence is the analogue of the series-test of § 1.11. *The above integral is uniformly convergent if there is a positive function $g(x)$, independent of y, such that $|f(x, y)| \leqslant g(x)$ for all values of x and y, and such that the integral*

$$\int_a^\infty g(x)\,dx$$

is convergent.

This may be proved in the same way as the corresponding result for series.

Other tests may be extended in a similar way. For example, in Dirichlet's test, if f and ϕ are functions of x and y, we assume that $\partial\phi/\partial x$ is continuous, and $\phi(x)$ tends to zero steadily and uniformly with respect to y, and that $|F|$ is less than a constant independent of x and y. The integral of ϕf is then uniformly convergent.

Examples. (i) Consider the convergence of the integral

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt.$$

[Suppose first that x is real. The integral is convergent at the upper limit for all values of x, since $t^{x+1}e^{-t}$ is bounded for all x, and we can compare the integral with that of $1/t^2$; but for convergence at the lower limit we must have $x > 0$ (*P.M.* § 180).

The integral is uniformly convergent over any finite x-interval (a, b), where $a > 0$. To prove this, we divide it into integrals over $(0, 1)$ and $(1, \infty)$, and compare the two parts with

$$\int_0^1 t^{a-1}\,dt, \qquad \int_1^\infty t^{b-1}e^{-t}\,dt,$$

which are convergent and independent of x.

Similarly, if x is complex, the integral is uniformly convergent over any finite region throughout which $\mathbf{R}(x) \geqslant a > 0$; for if $x = \xi + i\eta$, then $|t^{x-1}| = t^{\xi-1}$, and the result can now be proved as before.]

(ii) The integral

$$\int_0^\infty \frac{\sin xy}{x^s}\,dx$$

is absolutely convergent for $1 < s < 2$, and any y. For a fixed s in this range, it is uniformly convergent for $0 < \alpha \leqslant y \leqslant \beta$, for any β.

It is conditionally convergent if $0 < s \leqslant 1$, $y > 0$, and uniformly convergent for s in this range and $0 < \alpha \leqslant y \leqslant \beta$.

For fixed $y > 0$, it is absolutely and uniformly convergent in $1 < s_1 \leqslant s \leqslant s_2 < 2$, and uniformly, but not absolutely, convergent in $0 < s_1 \leqslant s \leqslant 1$.

1.52. The continuity theorem. In this section we shall prove the analogue for integrals of the theorem that the sum of a uniformly convergent series of continuous functions is continuous.

We first require the following theorem on continuous functions

of two variables, similar to the theorem for one variable proved in *P.M.* § 106.

Let $f(x, y)$ be a continuous function of x and y throughout the rectangle $a \leqslant x \leqslant b$, $\alpha \leqslant y \leqslant \beta$. Then, given ϵ, we can divide up the given rectangle into a finite number of sub-rectangles $x_\mu \leqslant x \leqslant x_{\mu+1}$, $y_\nu \leqslant y \leqslant y_{\nu+1}$, in such a way that

$$|f(x, y) - f(\xi, \eta)| < \epsilon$$

provided that (x, y) and (ξ, η) belong to the same sub-rectangle.

We prove this by the method of subdivision. Suppose that the given rectangle has not the required property. Then, if we divide it into quarters by the lines $x = \frac{1}{2}(a+b)$, $y = \frac{1}{2}(\alpha+\beta)$, at least one of the four quarter-rectangles has not the required property. Choose that one which has not; or, if more than one have not, choose one of them—to give a definite rule, choose one on the left-hand side if possible, and then, having fixed the side, choose the lower of two on the same side.

We next subdivide the chosen rectangle into quarters; and so the process of subdivision proceeds indefinitely, there being always at least one quarter which has not the required property. The left-hand sides of the chosen rectangles form an increasing sequence, and the right-hand sides form a decreasing sequence, and so each sequence has a limit; and the limits are the same, since the length of the side tends to zero. Call the limit X. Similarly the upper and lower sides tend to a limit Y.

We now use the fact that the function is continuous at (X, Y). Given ϵ, we can find δ so that

$$|f(x, y) - f(X, Y)| < \tfrac{1}{2}\epsilon \qquad (|x - X| < \delta,\ |y - Y| < \delta),$$

and so

$$|f(x, y) - f(\xi, \eta)| < \epsilon$$

if (x, y) and (ξ, η) both lie in the square with centre (X, Y) and side 2δ. Thus the rectangles chosen in the construction have the required property when they lie in this square, as they ultimately do. We have thus obtained a contradiction, and the theorem is proved.

We also deduce the following result: *Given ϵ, we can find δ such that*

$$|f(x, y) - f(\xi, \eta)| < \epsilon$$

provided that $|x - \xi| < \delta$ and $|y - \eta| < \delta$, δ depending on ϵ only, and not on x, y, ξ, or η.

For divide up the rectangle so that $|f(x,y)-f(\xi,\eta)|<\frac{1}{2}\epsilon$ if (x,y) and (ξ,η) belong to the same sub-rectangle. Let δ be the minimum of the sides of sub-rectangles. Then δ is the required number. For if $|x-\xi|<\delta$ and $|y-\eta|<\delta$, (x,y) and (ξ,η) belong to the same or to adjacent rectangles, and in either case the theorem follows.

The result may be expressed by saying that *a function of two variables which is continuous in a rectangle (boundary included) is uniformly continuous in the rectangle.*

We can now proceed with the properties of integrals.

If $f(x,y)$ is continuous in the rectangle $a\leqslant x\leqslant b$, $\alpha\leqslant y\leqslant\beta$, then

$$\phi(y)=\int_a^b f(x,y)\,dx$$

is a continuous function of y in (α,β).

For

$$\phi(y+k)-\phi(y)=\int_a^b \{f(x,y+k)-f(x,y)\}\,dx,$$

and, given ϵ, we can choose k_0 so that

$$|f(x,y+k)-f(x,y)|\leqslant\epsilon \qquad (|k|<k_0),$$

for all values of x and y. Hence

$$|\phi(y+k)-\phi(y)|\leqslant\epsilon(b-a) \qquad (|k|<k_0),$$

the required result.

If $f(x,y)$ is continuous in the rectangle $a\leqslant x\leqslant b$, $\alpha\leqslant y\leqslant\beta$, for all values of b, and the integral

$$\phi(y)=\int_a^\infty f(x,y)\,dx$$

converges uniformly with respect to y in the interval (α,β), then $\phi(y)$ is a continuous function of y in this interval.

We have

$$|\phi(y+k)-\phi(y)|=\left|\int_a^\infty \{f(x,y+k)-f(x,y)\}\,dx\right|$$

$$\leqslant\left|\int_a^X \{f(x,y+k)-f(x,y)\}\,dx\right|+\left|\int_X^\infty f(x,y+k)\,dx\right|+\left|\int_X^\infty f(x,y)\,dx\right|.$$

Given ϵ, we can choose X_0 so that each of the last two terms is less than ϵ for $X>X_0$, for all values of k. Having fixed X, the first term tends to zero with k, by the previous theorem. The result now follows.

In the above theorems, the continuity of $\phi(y)$ at the end-points α and β is one-sided, e.g. $\phi(y) \to \phi(\alpha)$ as $y \to \alpha$ by values greater than α.

Examples. (i) If the integral

$$\int_0^\infty f(x)\,dx$$

is convergent, then

$$\int_0^\infty e^{-xy} f(x)\,dx$$

is uniformly convergent in $0 \leqslant y \leqslant \beta$, and so continuous at $y = 0$.

[This is the analogue for integrals of Abel's theorem on power series. It may be proved as follows. Let

$$F(x) = \int_x^\infty f(t)\,dt,$$

so that $F(x) \to 0$ as $x \to \infty$. Suppose that $|F(x)| < \epsilon$ for $x > X_0$. Then

$$\left|\int_X^{X'} f(x)e^{-xy}\,dx\right| = \left|F(X)e^{-Xy} - F(X')e^{-X'y} - y\int_X^{X'} F(x)e^{-xy}\,dx\right|$$

$$\leqslant \epsilon + \epsilon + y\epsilon \int_X^{X'} e^{-xy}\,dx < 3\epsilon$$

for $X' > X > X_0$ and all $y \geqslant 0$; and the result follows.]

(ii) The integral

$$\int_0^\infty \frac{\sin xy}{x}\,dx$$

is convergent for every y; but it is not uniformly convergent in the neighbourhood of $y = 0$, since it is discontinuous at this point.

[To prove this, observe that it is (a) constant for $y > 0$ (put $x = u/y$), (b) positive for $y = 1$ (express it as

$$\sum \int_{(n-1)\pi}^{n\pi} \frac{\sin x}{x}\,dx,$$

i.e. as a series of decreasing terms of alternate signs), and (c) an odd function of y.

We may prove directly that it is not uniformly convergent by considering the 'remainder'

$$\int_X^\infty \frac{\sin xy}{x}\,dx = \int_{Xy}^\infty \frac{\sin u}{u}\,du,$$

and putting, e.g., $X = \pi/y$.

The value of the integral will be obtained later (§ 1.76).]

1.6. Double series. A double series consists of a double array of terms

$$\sum\sum a_{m,n},$$

where each of the suffixes m and n runs from 1 to infinity. There is no single method of summing the series, such as the '$\lim s_n$' method for single series, which obviously claims our attention. We can form partial sums of the series in a great variety of different ways, and each way gives rise to a method of summing the series. We may, for example, consider 'rectangular' sums

$$\sum_{m=1}^{M}\sum_{n=1}^{N} a_{m,n},$$

and then make M and N tend to infinity in various ways. Or we may consider sums such as

$$\sum_{m+n\leqslant N} a_{m,n},$$

taken over triangular regions. Or, finally, we may convert the double series into a 'repeated' series, first evaluating the sums

$$\sum_{n=1}^{\infty} a_{m,n},$$

and then finding the sum of their sums. We write this repeated series as

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n},$$

the inner sum being found first. We call this the 'sum by rows'. If we proceed in the opposite order, we obtain another repeated series

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{m,n}.$$

We call this the 'sum by columns'.

1.61. Double series of positive terms. *If all the terms $a_{m,n}$ of the series are positive, all methods of summation are equivalent. Either we obtain a finite limit, the same in all cases; or, however we sum, the series diverges to positive infinity.*

To prove this we consider in turn the various possibilities.

We call a set of pairs of numbers (m,n) a region. Let Δ_p $(p=1,2,\ldots)$ denote a sequence of finite regions, each of which includes the one before, and such that, however large N is, Δ_p includes the square $m\leqslant N$, $n\leqslant N$, if p is large enough.

There are now two possible cases. Suppose first that the finite sums

$$a_{m_1,n_1}+a_{m_2,n_2}+\ldots+a_{m_k,n_k},$$

selected in any manner from the series, have an upper bound G. Then plainly

$$\sum_{\Delta_p} a_{m,n} \leqq G$$

for all values of p. On the other hand, given ϵ, we can find one of these finite sums greater than $G-\epsilon$. But Δ_p includes every term of this finite sum, if p is large enough, and then

$$\sum_{\Delta_p} a_{m,n} > G-\epsilon.$$

Hence, since $\sum_{\Delta_p}$ is non-decreasing,

$$\lim_{p\to\infty} \sum_{\Delta_p} a_{m,n} = G;$$

that is to say, the series is convergent when summed in this particular way, and its sum is G. In this case the series is said to be convergent, it being unnecessary to specify the particular sequence of regions taken.

Suppose secondly that there is no such upper bound G. Then, having given any positive number H, there is a finite sum

$$a_{m_1,n_1}+\ldots+a_{m_k,n_k} > H.$$

Since we can find a number p such that Δ_p includes this sum, we have

$$\sum_{\Delta_p} a_{m,n} > H$$

for this value of p. Hence

$$\sum_{\Delta_p} a_{m,n} \to \infty.$$

In this case the series is said to be divergent.

These two cases are the only possibilities; and, since the results are independent of the particular regions Δ_p considered, we have proved the theorem, so far as finite partial sums are concerned.

Repeated series do not, so far, come under our analysis. To include them we have to replace our finite regions Δ_p by infinite regions.

Suppose first that the double series is convergent. Let D be any region, finite or infinite. Let $b_{m,n} = a_{m,n}$ if (m, n) is a point of D, and otherwise $b_{m,n} = 0$. Then clearly $\sum b_{m,n}$ converges if $\sum a_{m,n}$ does. We write

$$\sum_D a_{m,n} = \sum b_{m,n},$$

this being the definition of the left-hand side. It is clear that

$$\sum_{D} a_{m,n} \leqslant G.$$

Now let D_p $(p = 1, 2, \ldots)$ be a sequence of regions, finite or infinite, having the property characteristic of Δ_p. Then we have

$$\sum_{D_p} a_{m,n} \leqslant G,$$

and we can prove precisely as before that

$$\sum_{D_p} a_{m,n} > G - \epsilon \qquad (p \geqslant p_0).$$

Hence

$$\lim_{p \to \infty} \sum_{D_p} a_{m,n} = G.$$

In particular, if we take D_p to be the infinite region defined by $m \leqslant p$, we find that the sum by rows is equal to G. Similarly the sum by columns is G.

Secondly, suppose that the double series is divergent. Then it may happen that the series

$$\sum_{D_p} a_{m,n}$$

is divergent for a definite value of p. In this case the process comes to an end at this point. On the other hand, if $\sum_{D_p}$ is convergent for every p, we can, as before, show that

$$\sum_{D_p} a_{m,n} > H$$

for every H and $p \geqslant p_0(H)$. Hence

$$\sum_{D_p} a_{m,n} \to \infty.$$

In particular, if the double series is divergent, either some column is divergent, or every column is convergent, but their sums form a divergent series. The same thing is true of rows.

1.62. As the case of repeated series is particularly interesting, we give an alternative proof for this case.

If $a_{m,n} \geqslant \theta$, *then*

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{m,n}, \tag{1}$$

in the sense that, if either side converges, then so does the other, and to the same sum.

Suppose, for example, that the left-hand side is convergent. This means that all the series

$$\sum_{n=1}^{\infty} a_{m,n}$$

are convergent, to sums A_m, say, and that

$$\sum_{m=1}^{\infty} A_m$$

is convergent, to sum S, say.

Since $a_{m,n} \leqslant A_m$ for all values of m and n, it follows from the comparison test that all the series

$$\sum_{m=1}^{\infty} a_{m,n}$$

are convergent. Let their sums be $A^{(n)}$. Then

$$\sum_{n=1}^{N} A^{(n)} = \sum_{n=1}^{N} \sum_{m=1}^{\infty} a_{m,n} = \sum_{m=1}^{\infty} \sum_{n=1}^{N} a_{m,n} \leqslant \sum_{m=1}^{\infty} A_m = S.$$

Hence the series $\sum A^{(n)}$ is convergent, and, if its sum is S', then $S' \leqslant S$. But we can now reverse the whole argument, and, starting with the convergence of the right-hand side, prove that $S \leqslant S'$. Hence $S = S'$.

1.621. Still another method of proof is as follows. Suppose that the left-hand side of 1.62 (1) is convergent. Then

$$\sum_{n=1}^{N} \sum_{m=1}^{\infty} a_{m,n} = \sum_{m=1}^{\infty} \sum_{n=1}^{N} a_{m,n}, \tag{1}$$

this being merely the addition of a finite number of convergent series, whose convergence follows from that of $\sum A_m$. It is now sufficient to prove that, as $N \to \infty$,

$$\sum_{m=1}^{\infty} \sum_{n=N+1}^{\infty} a_{m,n} \to 0; \tag{2}$$

for this expression, together with the right-hand side of (1), is S; and it will then follow that the left-hand side of (1) tends to S, which is what is required.

This, however, follows from the uniform convergence theorem of § 1.14; for the series (2) is of the form

$$\sum_{m=1}^{\infty} u_m(N).$$

It converges uniformly with respect to N, since

$$|u_m(N)| \leqslant A_m,$$

and $\sum A_m$ is convergent; and, for each value of m, $u_m(N) \to 0$. This proves the theorem.

This method is of interest for the following reason. In less simple cases, where the numbers $a_{m,n}$ are not all positive, we

can still start from (1), and so reduce the problem to the proof of (2). This can then be proved by some special method—see, for example, § 1.66, ex. (xvii).

1.63. The comparison test. Series of positive and negative terms. We first note the comparison test for the convergence of a double series of positive terms: *if* $a_{m,n} \leqslant b_{m,n}$, *and* $\sum b_{m,n}$ *is convergent, then* $\sum a_{m,n}$ *is convergent.* We leave the proof of this to the reader.

Suppose now that some of the numbers $a_{m,n}$ are positive, and some are negative. Then *the series* $\sum a_{m,n}$ *is said to be absolutely convergent if the series* $\sum |a_{m,n}|$ *is convergent.*

Let $\alpha_{m,n} = a_{m,n}$ if $a_{m,n} \geqslant 0$, and otherwise let $\alpha_{m,n} = 0$; let $\beta_{m,n} = -a_{m,n}$ if $a_{m,n} \leqslant 0$, and otherwise $\beta_{m,n} = 0$. Then $0 \leqslant \alpha_{m,n} \leqslant |a_{m,n}|$, $0 \leqslant \beta_{m,n} \leqslant |a_{m,n}|$. Hence, by the comparison test, the series

$$\sum \alpha_{m,n}, \qquad \sum \beta_{m,n}$$

are convergent if $\sum |a_{m,n}|$ is convergent. Let the sums of these series be α and β. Then, with our previous notation,

$$\sum_{\Delta_p} a_{m,n} = \sum_{\Delta_p} \alpha_{m,n} - \sum_{\Delta_p} \beta_{m,n} \to \alpha - \beta.$$

The same thing is true if the finite region Δ_p is replaced by an infinite region D_p, but now the above equation is taken as the definition of the left-hand side.

The 'sum' $\alpha - \beta$ is independent of the region Δ_p or D_p. We state the result simply by saying that *the series* $\sum a_{m,n}$ *is convergent; that is, an absolutely convergent double series is convergent.*

The comparison test can now be extended to series of this type.

1.64. Series of complex terms. Suppose now that $a_{m,n}$ is a complex number, say $a_{m,n} = b_{m,n} + ic_{m,n}$. Then the series $\sum a_{m,n}$ is said to be absolutely convergent if $\sum |a_{m,n}|$ is convergent. Since

$$|b_{m,n}| \leqslant |a_{m,n}|, \qquad |c_{m,n}| \leqslant |a_{m,n}|,$$

this involves the absolute convergence, and so the convergence, of $\sum b_{m,n}$ and $\sum c_{m,n}$. If b and c are the sums of these series,

$$\sum_{\Delta_p \text{ (or } D_p)} a_{m,n} = \sum_{\Delta_p} b_{m,n} + i \sum_{\Delta_p} c_{m,n} \to b + ic,$$

and the series is said to be convergent; that is, *an absolutely convergent series of complex terms is convergent.*

All our conclusions about series of positive terms can now be extended to absolutely convergent series.

1.65. Multiplication of series. Let

$$a_0+a_1+a_2+\ldots, \qquad b_0+b_1+b_2+\ldots$$

be two absolutely convergent series. The series obtained by multiplying these series by Cauchy's rule is

$$c_0+c_1+c_2+\ldots$$

where $$c_n = a_0b_n+a_1b_{n-1}+\ldots+a_nb_0.$$

The rule has its origin in the case of power series, where $a_n = \alpha_n x^n$, $b_n = \beta_n x^n$, and where, in the multiplied series, we collect together terms involving the same power of x.

If the series $\sum_0^\infty a_n$, $\sum_0^\infty b_n$, are absolutely convergent, and their sums are a and b, then $\sum_0^\infty c_n$ is absolutely convergent, and its sum is ab.

This follows at once from the above theorems on double series. For the double series $\sum a_m b_n$ is absolutely convergent. Its sum by rows or columns is ab, and

$$\sum_{n=0}^{N} c_n = \sum_{m+n\leqslant N} a_m b_n,$$

which also tends to the limit ab.

If the series $\sum_0^\infty a_n$, $\sum_0^\infty b_n$, $\sum_0^\infty c_n$, are all convergent, to sums a, b, and c, then $c = ab$.

We apply the above theorem to the series $\sum_0^\infty a_n x^n$, $\sum_0^\infty b_n x^n$, $\sum_0^\infty c_n x^n$, which are absolutely convergent if $0 < x < 1$, and then make $x \to 1$ and use Abel's theorem (§ 1.22).

1.66. Miscellaneous examples on double and repeated series.

(i) If $|a_{m,n}| < Am^\alpha n^\beta$, where A, α, β are constants, and $|x| < 1$, $|y| < 1$, then $\sum a_{m,n}x^m y^n$ is absolutely convergent.

(ii) If $\sum a_{m,n}x_0^m y_0^n$ is absolutely convergent, then $\sum a_{m,n}x^m y^n$ is absolutely convergent for $|x| \leqslant |x_0|$, $|y| \leqslant |y_0|$.

(iii) The series $\sum m^{-\alpha}n^{-\beta}$ is convergent if $\alpha > 1$, $\beta > 1$.

(iv) The series $\sum (m^2+n^2)^{-\alpha}$ is convergent if $\alpha > 1$.

[Compare the terms for which $m \leqslant n$ with $\sum_{n=1}^{\infty} \sum_{m=1}^{n} n^{-2\alpha}$.]

The same result holds for the corresponding doubly infinite series, in which each variable ranges from $-\infty$ to ∞, omitting $m = n = 0$.

(v) The series $\sum (am^2+2bmn+cn^2)^{-\alpha}$ is convergent if $a > 0$, $b^2 < ac$, and $\alpha > 1$. [For

$$\frac{am^2+2bmn+cn^2}{m^2+n^2}$$

has a positive minimum value.]

(vi) If the ratio z/z' is complex, and a is not equal to any of the numbers $-mz-nz'$, then $\sum |a+mz+nz'|^{-\alpha}$ is convergent if $\alpha > 2$.

(vii) By expanding the function

$$\log(1-2x\cos\theta+x^2) = \log(1-xe^{i\theta})+\log(1-xe^{-i\theta})$$

in two different ways, show that

$$\cos n\theta = 2^{n-1}\cos^n\theta - \frac{n}{1!}2^{n-3}\cos^{n-2}\theta + \frac{n(n-3)}{2!}2^{n-5}\cos^{n-4}\theta - \dots,$$

and also obtain a series of ascending powers of $\cos\theta$.

[The rearrangement may be justified by the double series theorem; we also use the theorem on the uniqueness of a power series, *P.M.* § 194.]

(viii) If $|a| < 1$ and $|x| < 1$,

$$\sum_{n=0}^{\infty} \frac{1}{n!}\frac{a^n}{1+x^2a^{2n}} = e^a - x^2e^{a^3} + x^4e^{a^5} - \dots.$$

(ix) If x is not a negative integer,

$$\sum_{n=0}^{\infty} \frac{1}{x(x+1)\dots(x+n)} = e\left\{\frac{1}{x} - \frac{1}{1!}\frac{1}{(x+1)} + \frac{1}{2!}\frac{1}{(x+2)} - \dots\right\}.$$

(x) If $d(n)$ denotes the number of divisors of n, and $|x| < 1$,

$$\sum_{n=1}^{\infty} \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} d(n)x^n.$$

(xi) **Dirichlet multiplication.** If $\sum a_n n^{-s}$, $\sum b_n n^{-s}$ are absolutely convergent, and $c_n = \sum_{pq=n} a_p b_q$, then

$$\sum a_n n^{-s} \cdot \sum b_n n^{-s} = \sum c_n n^{-s}.$$

In particular $$\{\zeta(s)\}^2 = \sum d(n) n^{-s}.$$

(xii) If $a_{m,n} = 1$ $(m = n+1,\ n = 1, 2, \dots)$, $a_{m,n} = -1$ $(m = n-1,\ n = 2, 3, \dots)$, and otherwise $a_{m,n} = 0$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n} \neq \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{m,n}.$$

(xiii) Prove a similar result if

$$a_{m,n} = \frac{1}{m^2-n^2} \ (m \neq n), \qquad a_{m,n} = 0 \ (m = n).$$

[Here (terms $m = n$ being omitted)

$$\sum_{m=1}^{\infty}{}' \frac{1}{m^2-n^2} = \lim_{N\to\infty} \frac{1}{2n} \sum_{m=1}^{N}{}' \left(\frac{1}{m-n} - \frac{1}{m+n}\right)$$

$$= \lim_{N\to\infty} \frac{1}{2n}\left(-\sum_{\nu=1}^{n-1} \frac{1}{\nu} + \sum_{\nu=1}^{N-n} \frac{1}{\nu} - \sum_{\nu=n+1}^{N+n} \frac{1}{\nu} + \frac{1}{2n}\right)$$

$$= \frac{3}{4n^2},$$

and the sums by columns and rows are $\pi^2/8$ and $-\pi^2/8$. See Hardy (3).]

(xiv) If $\sum |u_n|$ is convergent, then $\prod (1+u_n z)$ is absolutely and uniformly convergent in any finite region, and it may be rearranged as a power series in z,

$$\prod (1+u_n z) = 1+z \sum u_n + z^2 \sum_{m\neq n}\sum u_m u_n + \dots .$$

[The first part has been proved already (§ 1.44), and it is simply a question of justifying the rearrangement. Let z be fixed. Let

$$P_N = \prod_1^N (1+|u_n||z|) = 1 + C_1^{(N)}|z| + \dots + C_N^{(N)}|z|^N.$$

Then $P_N \to P$, and, for each m, $C_m^{(N)}$ is non-decreasing and $\leqslant P|z|^{-m}$. Hence $C_m^{(N)} \to C_m$, say. Plainly

$$1+\sum_{m=1}^{k} C_m^{(N)}|z|^m \leqslant P_N \leqslant 1+\sum_{m=1}^{N} C_m|z|^m \qquad (k \leqslant N).$$

Making $N \to \infty$, then $k \to \infty$, we deduce that

$$P = 1+\sum_{m=1}^{\infty} C_m|z|^m.$$

Let
$$p_N = \prod_1^N (1+u_n z) = 1+c_1^{(N)}z+\dots+c_N^{(N)}z^N.$$

By an obvious extension of the theorem of absolute convergence to multiple series, $c_m^{(N)} \to c_m$, and plainly $|c_m - c_m^{(N)}| \leqslant C_m - C_m^{(N)} \leqslant C_m$. Hence if $k \leqslant N$,

$$\left|\sum_{m=1}^{\infty} c_m z^m - \sum_{m=1}^{N} c_m^{(N)} z^m\right| \leqslant \sum_{m=1}^{k} (C_m - C_m^{(N)})|z|^m + \sum_{k+1}^{\infty} C_m|z|^m,$$

which tends to 0 by choosing first k and then N. Hence the result.

(xv) Assuming that

$$\frac{\sin x}{x} = \prod_1^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right),$$

deduce that
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

(xvi) Let $s_{m,n} \to s$ mean that $|s_{m,n} - s| < \epsilon$ if m and n are both $> n_0 = n_0(\epsilon)$.

Show that, if $s_{m,n} \to s$, and $\lim\limits_{n\to\infty} s_{m,n}$ exists for each m, then

$$\lim_{m\to\infty}\left(\lim_{n\to\infty} s_{m,n}\right) = s.$$

It has been proved by Pringsheim that, if the double series $\sum\sum a_{m,n}$ converges to sum s in the sense that

$$\sum_{\mu=1}^{m}\sum_{\nu=1}^{n} a_{\mu,\nu} \to s,$$

and the single series $\quad \sum_{m=1}^{\infty} a_{m,n}, \qquad \sum_{n=1}^{\infty} a_{m,n}$

all converge, then the sum by rows and the sum by columns are also both equal to s. See Bromwich, *Infinite Series*, § 30.

(xvii) From the formula

$$1-\frac{1}{3}+\frac{1}{5}-\ldots=\frac{\pi}{4}$$

(*P.M.* § 214), deduce that

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=\frac{\pi^2}{8}.$$

[The result follows from the formula

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{(m+\frac{1}{2})(m+n+\frac{1}{2})}=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{(-1)^n}{(m+\frac{1}{2})(m+n+\frac{1}{2})}.$$

For the left-hand side (putting $n = r-m$) is equal to

$$\sum_{m=-\infty}^{\infty}\frac{(-1)^{-m}}{m+\frac{1}{2}}\sum_{r=-\infty}^{\infty}\frac{(-1)^r}{r+\frac{1}{2}}=\left(\frac{2}{\frac{1}{2}}-\frac{2}{\frac{3}{2}}+\ldots\right)^2=\pi^2.$$

Also, if $n \neq 0$,

$$\sum_{m=-\infty}^{\infty}\frac{1}{(m+\frac{1}{2})(m+n+\frac{1}{2})}=\frac{1}{n}\sum_{m=-\infty}^{\infty}\left(\frac{1}{m+\frac{1}{2}}-\frac{1}{m+n+\frac{1}{2}}\right)=0,$$

so that the right-hand side is equal to

$$\sum_{m=-\infty}^{\infty}\frac{1}{(m+\frac{1}{2})^2}=8\left(\frac{1}{1^2}+\frac{1}{3^2}+\ldots\right).$$

We have therefore to justify the inversion of the repeated series. The associated double series is clearly not absolutely convergent, so that a special method is required.

We have, for every N,

$$\sum_{m=-\infty}^{\infty}\sum_{n=-N}^{N}=\sum_{n=-N}^{N}\sum_{m=-\infty}^{\infty},$$

and hence (see § 1.621) it is sufficient to prove that, as $N\to\infty$,

$$\sum_{m=-\infty}^{\infty}\sum_{n=N+1}^{\infty}\to 0, \qquad \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{-N-1}\to 0.$$

Consider the former sum. If $m \geqslant -N-1$, we have

$$\left|\sum_{n=N+1}^{\infty} \frac{(-1)^n}{m+n+\frac{1}{2}}\right| < \frac{1}{m+N+\frac{3}{2}}.$$

This part of the sum is therefore less in absolute value than

$$\sum_{m=-N-1}^{\infty} \frac{1}{|m+\frac{1}{2}|(m+N+\frac{3}{2})} \leqslant \sum_{-N-1\leqslant m<-\frac{1}{2}N-\frac{1}{2}} \frac{1}{\frac{1}{2}N(m+N+\frac{3}{2})} +$$

$$+ \sum_{-\frac{1}{2}N-\frac{1}{2}\leqslant m\leqslant N} \frac{1}{|m+\frac{1}{2}|\frac{1}{2}N} + \sum_{N+1}^{\infty} \frac{1}{(m+\frac{1}{2})^2},$$

which plainly tends to zero. In the rest of the sum, we write

$$\sum_{n=N+1}^{\infty} \frac{(-1)^n}{m+n+\frac{1}{2}} = (-1)^m\pi - \sum_{n=-\infty}^{N} \frac{(-1)^n}{m+n+\frac{1}{2}}.$$

The sum formed from the terms involving π plainly tends to zero, and the last term is similar to the one already considered, and so also gives a sum which tends to zero.

Finally, the other sum can be dealt with in the same way, and the result follows.]

1.7. Integration of series. Having completed our discussion of repeated summations, we now turn to a similar set of problems, in which one of the summations is replaced by an integration. Since a finite integral is itself a limit, whereas a finite sum is not, this makes everything one degree more complicated.

We first consider the term-by-term integration of a series over a finite range.

1.71. *A uniformly convergent series of continuous functions may be integrated term by term; that is, if $u_1(x)$, $u_2(x)$,... are continuous, and*

$$u_1(x)+u_2(x)+\ldots = s(x)$$

converges uniformly over (a, b), then

$$\int_a^b u_1(x)\,dx + \int_a^b u_2(x)\,dx + \ldots = \int_a^b s(x)\,dx.$$

Since $s(x)$ is continuous (§ 1.14), it has a Riemann integral. Also the sum of the first n terms of the integrated series is (with our usual notation)

$$\int_a^b s_n(x)\,dx.$$

We have therefore to prove that

$$\int_a^b s_n(x)\,dx \to \int_a^b s(x)\,dx,$$

or that

$$\int_a^b \{s(x)-s_n(x)\}\,dx \to 0.$$

But, given ϵ, we can find n_0 such that

$$|s(x)-s_n(x)| < \epsilon$$

for $n > n_0$ and all values of x. Hence, by *P.M.* § 160, (7),

$$\left|\int_a^b \{s(x)-s_n(x)\}\,dx\right| < \epsilon(b-a),$$

and the result follows.

Examples. (i) If $0 < x < 1$,

$$\log\frac{1}{1-x} = \int_0^x \frac{dt}{1-t} = \int_0^x (1+t+t^2+\ldots)\,dt = x+\tfrac{1}{2}x^2+\tfrac{1}{3}x^2+\ldots.$$

(ii) Similarly, $$\arctan x = x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots.$$

(iii) Prove that $$\int_0^1 \log\frac{1+x}{1-x}\,\frac{dx}{x} = \frac{\pi^2}{4}.$$

[Use Abel's continuity theorem.]

(iv) Show that, if $r < 1$, and n is a positive integer,

$$\int_0^\pi \frac{1-r^2}{1-2r\cos\theta+r^2}\cos n\theta\,d\theta = \pi r^n.$$

1.72. *A series may be differentiated term by term if the differentiated series is a uniformly convergent series of continuous functions; that is, if*

$$u_1(x)+u_2(x)+\ldots = s(x),$$

and the functions $u_1(x)$, $u_2(x)$,... *have continuous derivatives* $u_1'(x)$,... *such that the series*

$$u_1'(x)+u_2'(x)+\ldots$$

converges uniformly to $f(x)$ *in* (a, b), *then* $f(x) = s'(x)$ *for* $a < x < b$.

By the previous theorem, the second series may be integrated term by term over (a, x), so that

$$\{u_1(x)-u_1(a)\}+\{u_2(x)-u_2(a)\}+\ldots = \int_a^x f(t)\,dt.$$

But the left-hand side is also equal to $s(x)-s(a)$. Hence

$$s(x)-s(a)=\int_a^x f(t)\,dt,$$

and, differentiating, the result follows, since $f(x)$ is continuous.

Examples. (i) If $|x|<1$,

$$\sum_{n=k}^{\infty} n(n-1)\ldots(n-k+1)x^{n-k}=\frac{k!}{(1-x)^{k+1}}.$$

(ii) If $s>1$,

$$\zeta'(s)=-\sum_{n=2}^{\infty} n^{-s}\log n.$$

1.73. *A real power series may be integrated or differentiated any number of times within the interval of convergence.* That is to say, the result of any number of formal term-by-term integrations or differentiations is true, provided that we are inside the interval of convergence.

Let the power series be

$$f(x)=\sum_0^{\infty} a_n x^n \qquad (|x|<R).$$

We can integrate once, by uniform convergence (§ 1.21), and obtain

$$\int_0^x f(t)\,dt=\sum_{n=0}^{\infty}\frac{a_n}{n+1}x^{n+1} \qquad (|x|<R).$$

The interval of convergence, and so of uniform convergence, is plainly at least as wide for this series as for the previous one, and so the process may be repeated.

Term-by-term differentiation gives

$$f'(x)=\sum_1^{\infty} na_n x^{n-1}.$$

This series also is convergent for $|x|<R$. For, if $0<\rho<R$, $|a_n\rho^n|<K$. Hence

$$|na_n x^{n-1}|=n|a_n\rho^n|\cdot\left|\frac{x^{n-1}}{\rho^n}\right|<\frac{K}{\rho}n\left(\frac{|x|}{\rho}\right)^{n-1},$$

and hence the differentiated series is convergent for $|x|<\rho$, by comparison with the convergent series

$$\sum_1^{\infty} n\left(\frac{|x|}{\rho}\right)^{n-1}=\frac{1}{(1-|x|/\rho)^2}.$$

Hence the differentiated series is uniformly convergent over any interval included in $(-R,R)$, and so term-by-term dif-

ferentiation is justified. The process can, of course, now be repeated.

It follows in particular that *a function represented by a power series has derivatives of all orders.*

It is also clear, since neither integration nor differentiation can decrease the interval of convergence, and the two processes are reciprocal, that neither can increase the interval.

Example. The Maclaurin expansion of $f(x) = \sum_{n=0}^{\infty} a_n x^n$ in powers of x is the original series.

1.74. *If x is real, and*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \qquad (|x| < R),$$

then $f(x+h)$ may be expanded by Taylor's theorem in powers of h, provided that $|x| < R$ and $|h| < R - |x|$.

The formal expansion is

$$f(x+h) = \sum_{m=0}^{\infty} \frac{h^m}{m!} f^{(m)}(x),$$

where, by the previous theorem,

$$f^{(m)}(x) = \sum_{n=m}^{\infty} n(n-1)\ldots(n-m+1) a_n x^{n-m}.$$

To prove that this actually holds, we write

$$f(x+h) = \sum_{n=0}^{\infty} a_n (x+h)^n = \sum_{n=0}^{\infty} a_n \sum_{m=0}^{n} \frac{n(n-1)\ldots(n-m+1)}{m!} x^{n-m} h^m$$

$$= \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} a_n \frac{n(n-1)\ldots(n-m+1)}{m!} x^{n-m} h^m = \sum_{m=0}^{\infty} \frac{h^m}{m!} f^{(m)}(x).$$

We have to justify the inversion; and it is justified by absolute convergence if

$$\sum_{n=0}^{\infty} |a_n| \sum_{m=0}^{n} \frac{n(n-1)\ldots(n-m+1)}{m!} |x|^{n-m} |h|^m = \sum_{n=0}^{\infty} |a_n| (|x| + |h|)^n$$

is convergent. This is true if $|x| + |h| < R$, which proves the theorem.

Notice that the interval of convergence obtained for the new series extends just up to one end of the interval of convergence of the original series. The actual interval of convergence may

be no larger, e.g. in the case of

$$f(x) = \frac{1}{1-x^2} = 1+x^2+x^4+\dots.$$

But it may in some cases extend further; e.g. if

$$f(x) = \frac{1}{1+x} = 1-x+x^2-\dots$$

we have
$$f^{(m)}(x) = \frac{(-1)^m m!}{(1+x)^{m+1}},$$

and
$$f(\tfrac{1}{2}+h) = \sum_{m=0}^{\infty} (-1)^m (\tfrac{2}{3})^{m+1} h^m,$$

which is convergent for $|h| < \frac{3}{2}$.

It is impossible to give a satisfactory account of this phenomenon so long as we consider *real* power series only, and we must postpone further discussion until we have considered functions of a complex variable.

1.75. Series which cannot be integrated term by term. A simple example of such a series is obtained by putting

$$s_n(x) = n^2 x(1-x)^n \cdot \quad (0 \leqslant x \leqslant 1).$$

Then $s(x) = 0$ for all values of x, so that

$$\int_0^1 s(x)\,dx = 0.$$

But
$$\int_0^1 s_n(x)\,dx = \frac{n^2}{(n+1)(n+2)} \to 1,$$

so that term-by-term integration gives an incorrect result. The series is, of course, not uniformly convergent.

On the other hand, uniform convergence is not a necessary condition for term-by-term integration. Some of the non-uniformly convergent series of § 1.3, e.g. those for which

$$s_n(x) = \frac{1}{1+nx}, \qquad s_n(x) = nx(1-x)^n,$$

can be integrated term by term.

This leads us to consider more general classes of series which can be integrated term by term.

1.76. Boundedly convergent series. *A series*

$$u_1(x)+u_2(x)+\dots$$

is said to be boundedly convergent in an interval (a, b), *if it con-*

verges for every value of x in the interval, and if there is a constant M such that $|s_n(x)| \leqslant M$ for all values of n and x $(a \leqslant x \leqslant b)$.

It is clear that the sum of a boundedly convergent series is bounded. So far we have no method of integrating bounded functions in general, so boundedness by itself does not enable us (at this stage) to integrate term by term. We have to combine it with another condition.

We say that a series *is uniformly convergent over (a, b), except in the neighbourhood of the point c*, if it is uniformly convergent over the intervals $(a, c-\delta)$, $(c+\delta, b)$, however small δ may be. We can then justify term-by-term integration under the following conditions:

If the series is uniformly convergent over (a, b), except in the neighbourhood of a finite number of points, and also boundedly convergent over the whole interval, then it may be integrated term by term over (a, b).

To prove this, it is sufficient to suppose that there is one exceptional point, c. Suppose that $|s_n(x)| \leqslant M$. Then $|s(x)| \leqslant M$ also. The integral of $s(x)$ exists in the sense of § 1.5, and

$$\left|\int_a^b \{s(x)-s_n(x)\}\,dx\right| \leqslant \left|\int_a^{c-\delta} \{s(x)-s_n(x)\}\,dx\right| + \left|\int_{c+\delta}^b \{s(x)-s_n(x)\}\,dx\right| +$$

$$+ \left|\int_{c-\delta}^{c+\delta} s(x)\,dx\right| + \left|\int_{c-\delta}^{c+\delta} s_n(x)\,dx\right|$$

$$\leqslant \left|\int_a^{c-\delta}\right| + \left|\int_{c+\delta}^b\right| + 4\delta M.$$

We can choose δ so small that the last term is less than a given ϵ, for all n. Then, having fixed δ, the other terms tend to zero, by uniform convergence. This proves the theorem.

Various extensions of the theorem are possible. It is not necessary that the series should be uniformly convergent over $(a, c-\delta)$ and $(c+\delta, b)$, if term-by-term integration over these intervals can be justified in some other way. A more important observation is that the theorem remains true if we insert in the integral a factor $\phi(x)$, which is integrable, but not necessarily bounded. Suppose, for example, that $\phi(x)$ is continuous over (a, b) except at $x = a$, in the neighbourhood of which it is

unbounded; and that

$$\int_a^b |\phi(x)|dx$$

exists as an infinite integral. Then we may multiply the series by $\phi(x)$ and integrate term by term. For

$$\left|\int_a^{a+\delta} \{s(x)-s_n(x)\}\phi(x)\,dx\right| \leqslant 2M \int_a^{a+\delta} |\phi(x)|\,dx,$$

which can be made less than any given ϵ, by choice of δ, for all values of n; and the integral over $(a+\delta, b)$ may be dealt with as before.

We observe finally that later, when we have developed the theory of the Lebesgue integral, we can put all these theorems into a much more satisfactory form. All the restrictions involving continuity and uniform convergence are only necessary because of the limitations of the Riemann integral, and disappear in the final form of the theorem.

Examples. (i) The series for which

$$s_n(x) = \frac{1}{1+nx}, \qquad s_n(x) = nx(1-x)^n \quad (0 \leqslant x \leqslant 1),$$

are boundedly convergent.

(ii) Consider the series

$$\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \dots.$$

It follows from one of the general tests (see § 1.12, ex. (i)) that this series is uniformly convergent except in the neighbourhood of the points $x = 0, \pm 2\pi, \pm 4\pi, \dots$. To show that it is boundedly convergent, and to sum it, we use a more special method.

Since each term has the period 2π, it is sufficient to consider the interval $0 \leqslant x < 2\pi$. Here we write

$$s_n(x) = \int_0^x (\cos t + \cos 2t + \dots + \cos nt)\,dt = \int_0^x \frac{\sin(n+\frac{1}{2})t - \sin\frac{1}{2}t}{2\sin\frac{1}{2}t}\,dt$$

$$= \int_0^x \frac{\sin(n+\frac{1}{2})t}{t}\,dt + \int_0^x \left(\frac{1}{2\sin\frac{1}{2}t} - \frac{1}{t}\right)\sin(n+\tfrac{1}{2})t\,dt - \tfrac{1}{2}x$$

$$= \int_0^{(n+\frac{1}{2})x} \frac{\sin u}{u}\,du + \int_0^x \left(\frac{1}{2\sin\frac{1}{2}t} - \frac{1}{t}\right)\sin(n+\tfrac{1}{2})t\,dt - \tfrac{1}{2}x.$$

Now
$$\int_0^h \frac{\sin u}{u}\,du$$
is always positive, and has an absolute maximum at $h = \pi$ (*P.M.* § 181, exs. LXXVI, 9–10). Hence
$$|s_n(x)| \leqslant \int_0^\pi \frac{\sin u}{u}\,du + \int_0^\pi \left(\frac{1}{2\sin\frac{1}{2}t} - \frac{1}{t}\right) dt + \tfrac{1}{2}\pi$$
for $0 \leqslant x \leqslant \pi$, i.e. the series is boundedly convergent in this interval. Since each term is odd, it is boundedly convergent in $(-\pi, 0)$, and so, by periodicity, in any interval.

To sum the series, let x be fixed, $0 < x < 2\pi$, and make $n \to \infty$. Then
$$\lim_{n\to\infty} \int_0^{(n+\frac{1}{2})x} \frac{\sin u}{u}\,du = \int_0^\infty \frac{\sin u}{u}\,du$$
exists (§ 1.5). Let us denote the value of this integral by I. Also
$$\int_0^x \left(\frac{1}{2\sin\frac{1}{2}t} - \frac{1}{t}\right) \sin(n+\tfrac{1}{2})t\,dt$$
$$= -\left(\frac{1}{2\sin\frac{1}{2}x} - \frac{1}{x}\right)\frac{\cos(n+\frac{1}{2})x}{n+\frac{1}{2}} + \frac{1}{n+\frac{1}{2}}\int_0^x \frac{d}{dt}\left(\frac{1}{2\sin\frac{1}{2}t} - \frac{1}{t}\right)\cos(n+\tfrac{1}{2})t\,dt,$$
which tends to zero on account of the factor $n+\frac{1}{2}$ in the denominator (the other factors being bounded). Hence, if $s(x)$ is the sum,
$$s(x) = I - \tfrac{1}{2}x \qquad (0 < x < 2\pi).$$
But plainly $s(\pi) = 0$, so that $I = \frac{1}{2}\pi$. This gives at the same time the sum of the series and the value of the infinite integral.

The reader should draw the graph of the sum of the series, noticing its discontinuities at the points $x = 0, \pm 2\pi, \pm 4\pi, \ldots$. [See also Ch. XIII, ex. 11.]

(iii) Prove, without using integrals, that the above series is boundedly convergent, using a method similar to that used in § 1.131.

(iv) Sum the series
$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$
by integrating the above series over $(0, \pi)$.

(v) Prove that
$$\sum_{n=1}^\infty \frac{\cos nx}{n^2} = \frac{\pi^2}{6} - \frac{\pi x}{2} + \frac{x^2}{4} \qquad (0 < x < \pi).$$

1.77. Term-by-term integration when the integrals are infinite. We now pass to the general case of term-by-term integration over an infinite range, or of functions which become

infinite in the range of integration. In each case the results are similar to those already obtained for repeated series. For convenience we state them as a single theorem.

Suppose that $u_n(x) \geqslant 0$ for all values of n and x, and that

$$\int_a^c \{\sum u_n(x)\}\, dx = \sum \int_a^c u_n(x)\, dx \tag{1}$$

for all values of c less than b (or for all finite values of c). Then

$$\int_a^b \{\sum u_n(x)\}\, dx = \sum \int_a^b u_n(x)\, dx \tag{2}$$

(*or*

$$\int_a^\infty \{\sum u_n(x)\}\, dx = \sum \int_a^\infty u_n(x)\, dx \tag{2a}$$

in the second case), provided that either side of the resulting equation is convergent.

The proof is the same in the two cases. Let us take the case of a finite interval (a, b).

Suppose that the series on the right of (2) is convergent, say

$$\sum \int_a^b u_n(x)\, dx = S.$$

Then, since $u_n(x) \geqslant 0$,

$$\int_a^c \{\sum u_n(x)\}\, dx = \sum \int_a^c u_n(x)\, dx \leqslant S$$

for all values of c less than b. Hence (see *P.M.*, 7th edition, § 185) the integral on the left of (2) exists, as an infinite integral at b; and, if its value is I, then $I \leqslant S$. On the other hand,

$$\sum_{n=1}^N \int_a^b u_n(x)\, dx = \int_a^b \left\{\sum_1^N u_n(x)\right\} dx \leqslant \int_a^b \left\{\sum_1^\infty u_n(x)\right\} dx = I,$$

and, making $N \to \infty$, we see that $S \leqslant I$. Hence in fact $S = I$.

A similar method may be used if the left-hand side of (2) or (2a) is assumed to exist. The reader should write it out in detail.

As in the case of series, we can omit our 'positive' condition if we assume instead that one of the sides of (2) or (2a) is 'absolutely convergent', i.e. remains convergent if $u_n(x)$ is replaced by its modulus.

The above results remain true for functions $u_n(x)$ which may have either sign, or be complex, provided that either of

$$\int_a^b \{\sum |u_n(x)|\}\, dx, \qquad \sum \int_a^b |u_n(x)|\, dx,$$

$$(\textit{or} \qquad \int_a^\infty \{\sum |u_n(x)|\}\, dx, \qquad \sum \int_a^\infty |u_n(x)|\, dx$$

in the second case), is convergent.

It follows from the theorem already proved that the two conditions are equivalent. We then obtain the result precisely as in the case of double series. If $u_n(x)$ is real, we consider the functions $|u_n(x)| \pm u_n(x)$, each of which is positive. If $u_n(x)$ is complex, say $u_n(x) = \alpha_n(x) + i\beta_n(x)$, we consider the four functions $|u_n(x)| \pm \alpha_n(x)$, $|u_n(x)| \pm \beta_n(x)$, each of which is positive.

1.78. Miscellaneous examples on term-by-term integration.

(i) Prove that
$$\int_0^1 \log \frac{1}{1-x}\, dx = 1$$

by expanding in powers of x and integrating term by term. [Notice that the series is not uniformly or even boundedly convergent in the neighbourhood of $x = 1$.]

(ii) We have

$$\int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx = \int_0^\infty \{\sum x^{s-1} e^{-nx}\}\, dx = \sum \int_0^\infty x^{s-1} e^{-nx}\, dx$$

$$= \sum n^{-s} \int_0^\infty y^{s-1} e^{-y}\, dy = \sum n^{-s} \Gamma(s) = \Gamma(s)\zeta(s).$$

Justify this process (a) for $s > 1$, (b) for s complex, $\mathbf{R}(s) > 1$.

(iii) Prove that
$$\int_0^\infty e^{-ax} \cos bx\, dx = \frac{a}{a^2 + b^2}$$

by expanding $\cos bx$ in powers of x. [The process is justified by absolute convergence if $\mathbf{R}(a) > |b|$, though the result holds in a wider range than this.]

(iv) Prove that, if $p > 0$,

$$\int_0^1 \frac{x^{p-1}}{1-x} \log \frac{1}{x}\, dx = \frac{1}{p^2} + \frac{1}{(p+1)^2} + \dots.$$

(v) Prove that, if $p > 0$,

$$\int_0^1 \frac{x^{p-1}}{1+x}\,dx = \frac{1}{p} - \frac{1}{p+1} + \dots .$$

[Here the absolute convergence test fails. Integrate over $(0, \xi)$, where $0 < \xi < 1$, and use Abel's theorem.]

(vi) Prove that $$\int_0^\infty \frac{\sinh ax}{\sinh bx}\,dx = \frac{\pi}{2b}\tan\frac{a\pi}{2b} \qquad (0 < a < b).$$

[On expanding in powers of e^{-bx}, we obtain the series

$$\sum \frac{2a}{(2n-1)^2b^2-a^2}.$$

For the summation of this we must refer to Chapter III.]

(vii) Prove that $$\int_0^\infty \frac{\cosh ax}{\cosh bx}\,dx = \frac{\pi}{2b}\sec\frac{a\pi}{2b} \qquad (0 < a < b).$$

[The general test fails, but the integral can be evaluated by means of (v) above.]

(viii) If $u_n(x) = ae^{-nax} - be^{-nbx}$ $(0 < a < b)$, show that

$$\sum \int_0^\infty u_n(x)\,dx \neq \int_0^\infty \{\sum u_n(x)\}\,dx.$$

[We have $$\sum \int_0^\infty u_n(x)\,dx = \sum \left(\frac{1}{n} - \frac{1}{n}\right) = 0,$$

but $$\int_0^\infty \{\sum u_n(x)\}\,dx = \int_0^\infty \left(\frac{a}{e^{ax}-1} - \frac{b}{e^{bx}-1}\right) dx > 0,$$

since the integrand is positive for all values of x, $u/(e^u-1)$ being a steadily decreasing function of u.

It is easy to prove directly that

$$\sum \int_0^\infty |u_n(x)|\,dx$$

is divergent.]

(ix) Consider the integral

$$\int_0^\infty e^{-x^{\frac{1}{2}}} \sin x^{\frac{1}{2}} \cos ax\,dx.$$

Here we shall anticipate for a moment some of the results of Chapter III.

If we expand $\cos ax$ in powers of x and integrate term by term, we obtain

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^{2n}}{(2n)!} \int_0^{\infty} e^{-x^{\frac{1}{2}}} \sin x^{\frac{1}{2}} x^{2n}\, dx,$$

every term of which is zero (§ 3.125). But the given integral is not identically zero (see § 3.13).

The test of § 1.77 fails; for the integral obtained by replacing $u_n(x)$ by $|u_n(x)|$ is

$$\int_0^{\infty} e^{-x^{\frac{1}{2}}} |\sin x^{\frac{1}{2}}| \cosh ax\, dx,$$

which is divergent.

1.79. As a final example of term-by-term integration under special conditions, we prove the following theorem:

The power series $\sum_{n=0}^{\infty} a_n x^n$, *supposed convergent for* $x > 0$, *may be multiplied by* e^{-x} *and integrated term by term over* $(0, \infty)$, *provided only that the resulting series is convergent.**

The formula is

$$\int_0^{\infty} e^{-x} \left\{ \sum_{n=0}^{\infty} a_n x^n \right\} dx = \sum_{n=0}^{\infty} a_n \int_0^{\infty} e^{-x} x^n\, dx = \sum_{n=0}^{\infty} a_n n!,$$

and we have to justify the inversion on the assumption that $\sum a_n n!$ is convergent.

Put $a_n n! = b_n$, so that $\sum b_n$ is convergent. Then b_n is bounded, $|b_n| < B$, say, and

$$\left| \frac{b_n x^n}{n!} \right| < B \frac{X^n}{n!} \qquad (0 \leqslant x \leqslant X).$$

Hence $\sum b_n x^n / n!$ is uniformly convergent over $(0, X)$, and we may multiply by e^{-x} and integrate term by term over this range. Thus

$$\int_0^{X} e^{-x} \left\{ \sum_{n=0}^{\infty} \frac{b_n x^n}{n!} \right\} dx = \sum_{n=0}^{\infty} \frac{b_n}{n!} \int_0^{X} e^{-x} x^n\, dx. \tag{1}$$

We are given that

$$\sum_{n=0}^{\infty} \frac{b_n}{n!} \int_0^{\infty} e^{-x} x^n\, dx = \sum_{n=0}^{\infty} b_n$$

* Hardy, (1), (6).

is convergent. Hence (1) may be written

$$\int_0^X e^{-x}\left\{\sum_{n=0}^{\infty}\frac{b_n x^n}{n!}\right\}dx = \sum_{n=0}^{\infty} b_n - \sum_{n=0}^{\infty}\frac{b_n}{n!}\int_X^{\infty} e^{-x}x^n\,dx, \tag{2}$$

and it remains to be proved that the last term tends to zero as $X \to \infty$. Now

$$\int_X^{\infty} e^{-x}x^n\,dx = e^{-X}\{X^n + nX^{n-1} + \ldots + n!\},$$

so that the series in question is

$$e^{-X}\sum_{n=0}^{\infty} b_n \sum_{m=0}^{n}\frac{X^m}{m!}. \tag{3}$$

Let $r_n = \sum_{\nu=n}^{\infty} b_\nu$, so that $r_n \to 0$ as $n \to \infty$. Then $b_n = r_n - r_{n+1}$, and

$$\sum_{n=0}^{N} b_n \sum_{m=0}^{n}\frac{X^m}{m!} = \sum_{n=0}^{N}(r_n - r_{n+1})\sum_{m=0}^{n}\frac{X^m}{m!}$$

$$= \sum_{n=0}^{N} r_n\frac{X^n}{n!} - r_{N+1}\left(1 + X + \ldots + \frac{X^N}{N!}\right).$$

The last term tends to zero as $N \to \infty$, for any X, so that, making $N \to \infty$, (3) takes the form

$$e^{-X}\sum_{n=0}^{\infty} r_n\frac{X^n}{n!}.$$

The result now easily follows. Given any positive ϵ, we can find N so that $|r_n| < \epsilon$ for $n > N$. Then, since $|r_n| < A$ for all values of n,

$$\left|e^{-X}\sum_{n=0}^{\infty} r_n\frac{X^n}{n!}\right| < Ae^{-X}\sum_{0}^{N}\frac{X^n}{n!} + \epsilon e^{-X}\sum_{N+1}^{\infty}\frac{X^n}{n!}$$

$$< Ae^{-X}\sum_{0}^{N}\frac{X^n}{n!} + \epsilon,$$

and, having fixed N, we can choose X so large that the first term also is less than ϵ. Hence the result.

1.8. Repeated integrals. A repeated integral is one degree more complicated than a series of integrals. Even if the limits

of each integration are finite, a repeated limit is involved, and its inversion requires justification. If the limits of each integral are infinite, four successive limiting operations are involved.

1.81. We consider first continuous functions and finite limits.

If $f(x,y)$ is a continuous function of x and y in the rectangle $a \leqslant x \leqslant b$, $\alpha \leqslant y \leqslant \beta$, then

$$\int_a^b dx \int_\alpha^\beta f(x,y)\,dy = \int_\alpha^\beta dy \int_a^b f(x,y)\,dx.$$

Since $f(x,y)$ is continuous, the integral

$$F(x) = \int_\alpha^\beta f(x,y)\,dy$$

is a continuous function of x (§ 1.52). We can therefore integrate it over (a,b). The result is the left-hand side of the equation. Similarly the right-hand side has a meaning.

To prove that the two sides are equal, divide up the ranges of integration by points x_μ and y_ν ($a = x_0$, $b = x_m$, $\alpha = y_0$, $\beta = y_n$), such that $x_{\mu+1} - x_\mu < \delta$, $y_{\nu+1} - y_\nu < \delta$, for all values of μ and ν. Let $m_{\mu,\nu}$, $M_{\mu,\nu}$ be the lower and upper bounds of $f(x,y)$ in the rectangle $(x_\mu, x_{\mu+1}; y_\nu, y_{\nu+1})$. Then for $y_\nu \leqslant y \leqslant y_{\nu+1}$

$$m_{\mu,\nu}(x_{\mu+1} - x_\mu) \leqslant \int_{x_\mu}^{x_{\mu+1}} f(x,y)\,dx \leqslant M_{\mu,\nu}(x_{\mu+1} - x_\mu),$$

and hence, integrating with respect to y,

$$m_{\mu,\nu}(x_{\mu+1} - x_\mu)(y_{\nu+1} - y_\nu) \leqslant \int_{y_\nu}^{y_{\nu+1}} dy \int_{x_\mu}^{x_{\mu+1}} f(x,y)\,dx$$
$$\leqslant M_{\mu,\nu}(x_{\mu+1} - x_\mu)(y_{\nu+1} - y_\nu).$$

Summing with respect to μ and ν, we have

$$\sum\sum m_{\mu,\nu}(x_{\mu+1} - x_\mu)(y_{\nu+1} - y_\nu) \leqslant \int_\alpha^\beta dy \int_a^b f(x,y)\,dx$$
$$\leqslant \sum\sum M_{\mu,\nu}(x_{\mu+1} - x_\mu)(y_{\nu+1} - y_\nu).$$

The same inequalities are also satisfied by the other repeated integral. Also, when $\delta \to 0$, the difference between the extreme terms of the inequality tends to zero; for we can choose δ so

small that the maximum value of $M_{\mu,\nu}-m_{\mu,\nu}$ is less than any given ϵ; and then

$$\sum\sum(M_{\mu,\nu}-m_{\mu,\nu})(x_{\mu+1}-x_\mu)(y_{\nu+1}-y_\nu)$$
$$\leqslant \epsilon\sum\sum(x_{\mu+1}-x_\mu)(y_{\nu+1}-y_\nu)=\epsilon(b-a)(\beta-\alpha).$$

Hence the two repeated integrals are equal.

1.82. Extension to discontinuous functions. Suppose next that the rectangle is crossed by a continuous monotonic curve $y=\phi(x)$, or $x=\psi(y)$, from $x=a$ to $x=c$; and let $f(x,y)$ be bounded, and continuous except on this curve. Then the repeated integrals still exist and are equal.

In the first place, the function $F(x)$ is still continuous. For if $a<x<c$,

$$F(x)=\int_\alpha^{\phi(x)} f(x,y)\,dy+\int_{\phi(x)}^{\beta} f(x,y)\,dy=F_1(x)+F_2(x),$$

say; and

$$F_1(x+h)-F_1(x)=\int_\alpha^{\phi(x)}\{f(x+h,y)-f(x,y)\}\,dy+\int_{\phi(x)}^{\phi(x+h)} f(x+h,y)\,dy,$$

which plainly tends to zero with h. Hence $F_1(x)$ is continuous, and similarly $F_2(x)$ is continuous. Hence the first repeated integral exists, and, similarly, so does the second one.

To prove that they are equal, consider the strip

$$\phi(x)-\eta<y<\phi(x)+\eta,$$

and suppose for simplicity that $\phi(x)$ is steadily increasing. Construct rectangles $(x_\mu, x_{\mu+1}; y_\nu, y_{\nu+1})$ with sides less than δ, as before. Then the area of those rectangles between x_μ and $x_{\mu+1}$ which contain any point of the strip is less than

$$(x_{\mu+1}-x_\mu)[\{\phi(x_{\mu+1})+\eta\}-\{\phi(x_\mu)-\eta\}+2\delta]$$
$$<\delta\{\phi(x_{\mu+1})-\phi(x_\mu)\}+(x_{\mu+1}-x_\mu)(2\eta+2\delta);$$

the total area of such rectangles is therefore less than

$$\delta(\beta-\alpha)+(b-a)(2\eta+2\delta).$$

Hence, if $\sum_1$ denotes a summation over these rectangles, and $|f(x,y)|\leqslant M$,

$$\sum_1(M_{\mu,\nu}-m_{\mu,\nu})(x_{\mu+1}-x_\mu)(y_{\nu+1}-y_\nu)$$
$$<2M\{\delta(\beta-\alpha)+(2\eta+2\delta)(b-a)\},$$

which can be made arbitrarily small by choice of η and δ.

Finally, since $f(x,y)$ is continuous in each of the remaining regions, we can choose δ so small that

$$\max(M_{\mu,\nu}-m_{\mu,\nu})<\epsilon$$

in the remaining rectangles. The result now follows as before.

We can, of course, now extend the result to functions which have any finite number of discontinuities of the above type. In particular, the result holds for an integral taken over a non-rectangular region bounded by curves of the above type; for this can be considered as an integral over a rectangle, the function being continuous in a limited part of it and zero elsewhere.

Notice finally the following inequality. Suppose that $f(x,y)$ is continuous, and $|f(x,y)|\leqslant M$, in a region of the above type, and that $f(x,y)=0$ elsewhere. Let $F(x,y)=M$ in the region, and $F(x,y)=0$ elsewhere. Then

$$\left|\int_\alpha^\beta dy\int_a^b f(x,y)\,dx\right|\leqslant\int_\alpha^\beta dy\int_a^b F(x,y)\,dx.$$

The reader should have no difficulty in deducing this from the above analysis.

1.83. Change of variables in a repeated integral. The formula by which the variables in a repeated integral are changed may be obtained as follows. Consider the integral

$$\int dy\int f(x,y)\,dx$$

over a given region, and let

$$x=\phi(u,v),\qquad y=\psi(u,v).$$

Suppose that these functions are such that, if y is constant, x is a monotonic differentiable function of u. If we transform the integral with respect to x into one with respect to u, we obtain

$$\int f(x,y)\,dx=\int f\frac{dx}{du}\,du.$$

But* $$\frac{dx}{du}=\frac{\partial\phi}{\partial u}+\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial u},\qquad 0=\frac{\partial\psi}{\partial u}+\frac{\partial\psi}{\partial v}\frac{\partial v}{\partial u},$$

so that† $$\frac{dx}{du}=\left(\frac{\partial\phi}{\partial u}\frac{\partial\psi}{\partial v}-\frac{\partial\phi}{\partial v}\frac{\partial\psi}{\partial u}\right)\Big/\frac{\partial\psi}{\partial v}=\frac{\partial(\phi,\psi)}{\partial(u,v)}\Big/\frac{\partial\psi}{\partial v}.$$

* See *P.M.* § 153.

† For the Jacobian notation see *P.M.* ch. VII, ex. 20.

Hence the repeated integral becomes

$$\int dy \int f\frac{\partial(\phi,\psi)}{\partial(u,v)}\Big/\frac{\partial\psi}{\partial v}\,du = \int du \int f\frac{\partial(\phi,\psi)}{\partial(u,v)}\Big/\frac{\partial\psi}{\partial v}\,dy$$

if the order of integration may be inverted. Finally, expressing y as a function of v (for a fixed u), and assuming that it is monotonic, we have

$$\frac{dy}{dv} = \frac{\partial\psi}{\partial v},$$

and the integral becomes

$$\int du \int F(u,v)\frac{\partial(\phi,\psi)}{\partial(u,v)}\,dv,$$

where $F(u,v) = f\{\phi(u,v), \psi(u,v)\}$.

The process is valid if, for example, the integrand at every stage is a continuous function, and the region is bounded by monotonic curves as in § 1.82. Some care is needed in verifying this in particular cases. Consider, for example, the integral

$$I = \int_0^a dy \int_0^{\sqrt{(a^2-y^2)}} f(x,y)\,dx,$$

where $f(x,y)$ is continuous, and transform to polar coordinates (r,θ) given by $x = r\cos\theta$, $y = r\sin\theta$. Transforming first to (r,y), we have $x = \sqrt{(r^2-y^2)}$, and

$$\frac{dx}{dr} = \frac{r}{\sqrt{(r^2-y^2)}},$$

which becomes infinite at $r = y$. To avoid this difficulty, consider instead the integral

$$I_\delta = \int_0^{\sqrt{(a^2-\delta^2)}} dy \int_\delta^{\sqrt{(a^2-y^2)}} f(x,y)\,dx,$$

where $0 < \delta < a$. Transforming first to (r,y), then to (r,θ), in the above manner, we obtain

$$I_\delta = \int_\delta^a r\,dr \int_0^{\arccos(\delta/r)} f(r\cos\theta, r\sin\theta)\,d\theta.$$

Now let $\delta \to 0$. Then $I_\delta \to I$, and the last integral tends to

$$\int_0^a r\,dr \int_0^{\frac{1}{2}\pi} f(r\cos\theta, r\sin\theta)\,d\theta.$$

Each of these statements is readily proved by means of the inequality noticed at the end of § 1.82.

For the general theory of these transformations see Goursat's *Cours d'Analyse*, t. 1, ch. 6.

1.84. Repeated integrals, one range being infinite. The most important theorem here is the analogue of the theorem on term-by-term integration of a uniformly convergent series.

Suppose that

$$\int_a^b dx \int_\alpha^\beta f(x,y)\,dy = \int_\alpha^\beta dy \int_a^b f(x,y)\,dx$$

for all values of b greater than a, and that

$$\int_a^\infty f(x,y)\,dx$$

is uniformly convergent in the range $\alpha \leqslant y \leqslant \beta$. Then

$$\int_a^\infty dx \int_\alpha^\beta f(x,y)\,dy = \int_\alpha^\beta dy \int_a^\infty f(x,y)\,dx.$$

For
$$\int_a^n f(x,y)\,dx = s_n(y) \to s(y)$$

uniformly in (α, β). Hence, using the result for sequences,

$$\int_a^n dx \int_\alpha^\beta f(x,y)\,dy = \int_\alpha^\beta dy \int_a^n f(x,y)\,dx$$

$$= \int_\alpha^\beta s_n(y)\,dy \to \int_\alpha^\beta s(y)\,dy = \int_\alpha^\beta dy \int_a^\infty f(x,y)\,dx,$$

the required result.

A similar theorem holds for infinite integrals of the second kind.

The same results also hold if the integral is not uniformly convergent in the neighbourhood of certain points, but is boundedly convergent. In fact the same proof holds under these conditions.

1.85. Repeated infinite integrals. The following theorem is the analogue for repeated integrals of the theorems of § 1.62 and § 1.77 for double series and series of integrals.

Suppose that $f(x, y)$ is positive, and that

$$\int_a^c dx \int_\alpha^\beta f(x, y)\, dy = \int_\alpha^\beta dy \int_a^c f(x, y)\, dx \tag{1}$$

for all values of c less than b, and that

$$\int_a^b dx \int_\alpha^\gamma f(x, y)\, dy = \int_\alpha^\gamma dy \int_a^b f(x, y)\, dx \tag{2}$$

for all values of γ less than β. Then

$$\int_a^b dx \int_\alpha^\beta f(x, y)\, dy = \int_\alpha^\beta dy \int_a^b f(x, y)\, dx, \tag{3}$$

provided that either side of this equation is convergent.

The theorem is still true if one or both of b and β is replaced by infinity.

Take, for example, the case of two finite intervals, and suppose that the left-hand side of (3) exists. Since $f(x, y) \geqslant 0$

$$\int_\alpha^\gamma f(x, y)\, dy \leqslant \int_\alpha^\beta f(x, y)\, dy \qquad (\alpha < \gamma < \beta),$$

and hence

$$\int_\alpha^\gamma dy \int_a^b f(x, y)\, dx = \int_a^b dx \int_\alpha^\gamma f(x, y)\, dy \leqslant \int_a^b dx \int_\alpha^\beta f(x, y)\, dy.$$

Making $\gamma \to \beta$, we see that the right-hand side of (3) exists, and that

$$\int_\alpha^\beta dy \int_a^b f(x, y)\, dx \leqslant \int_a^b dx \int_\alpha^\beta f(x, y)\, dy.$$

The same process may now be reversed, and it yields the reversed inequality. Hence the two sides are equal.

The same proof holds if b, or β, or both of them, are infinite.

Though the actual proof is simple, we have, of course, made rather far-reaching assumptions; and, in applying the theorem, we have to justify (1) and (2) on other grounds, for example, by uniform convergence. This is made necessary by the limitations of the Riemann integral; for from the mere fact that

$$\int_\alpha^\beta dy \int_a^c f(x, y)\, dx$$

is bounded as $c \to b$, we cannot deduce that

$$\int_a^b f(x,y)\,dx$$

is integrable in Riemann's sense. When we come to the Lebesgue integral, we shall see that difficulties of this kind disappear.

The theorem remains true for functions $f(x,y)$ which may have either sign, or be complex, provided that either of the integrals

$$\int_a^b dx \int_\alpha^\beta |f(x,y)|\,dy, \qquad \int_\alpha^\beta dy \int_a^b |f(x,y)|\,dx$$

is convergent.

The extension is made as in the case of series. If $f(x,y)$ is real, we consider the functions $|f(x,y)| \pm f(x,y)$, and if $f(x,y)$ is complex, we consider $|f(x,y)| \pm \mathbf{R}f(x,y)$, $|f(x,y)| \pm \mathbf{I}f(x,y)$.

1.86. The Gamma-function. The function

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt \tag{1}$$

is, as we have already observed (§§ 1.51, 1.52), continuous for $\mathbf{R}(x) > 0$. We are now in a position to investigate its properties more fully. We shall suppose throughout this section and the following one that x and y are real, leaving it to the reader to consider how far the results are true for complex values of the variables.

If $x > 1$, we may integrate by parts, and obtain

$$\Gamma(x) = \left[-t^{x-1}e^{-t}\right]_0^\infty + (x-1)\int_0^\infty t^{x-2}e^{-t}\,dt.$$

The integrated terms vanish, and we have

$$\Gamma(x) = (x-1)\Gamma(x-1) \qquad (x > 1). \tag{2}$$

Since $\Gamma(1) = \int_0^\infty e^{-t}\,dt = 1$, repeated application of (2) gives

$$\Gamma(n) = (n-1)! \tag{3}$$

if n is a positive integer. Thus $\Gamma(x)$ may be regarded as a generalization of a factorial.

Now consider the product

$$\Gamma(x)\Gamma(y) = \int_0^\infty t^{x-1}e^{-t}\,dt \int_0^\infty u^{y-1}e^{-u}\,du \quad (x>0,\, y>0).$$

We may regard this as a repeated integral. Putting $u = tv$, and inverting the order of integration, we have, formally,

$$\begin{aligned}\Gamma(x)\Gamma(y) &= \int_0^\infty t^{x-1}e^{-t}\,dt \int_0^\infty t^y v^{y-1}e^{-tv}\,dv \\ &= \int_0^\infty v^{y-1}\,dv \int_0^\infty t^{x+y-1}e^{-t(1+v)}\,dt \\ &= \int_0^\infty v^{y-1}\,dv \int_0^\infty \frac{w^{x+y-1}e^{-w}\,dw}{(1+v)^{x+y}} \\ &= \Gamma(x+y)\int_0^\infty \frac{v^{y-1}}{(1+v)^{x+y}}\,dv.\end{aligned}$$

Hence
$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \phi(x,y) \qquad (x>0,\, y>0), \tag{4}$$
where
$$\begin{aligned}\phi(x,y) = \int_0^\infty \frac{v^{y-1}}{(1+v)^{x+y}}\,dv &= 2\int_0^{\frac{1}{2}\pi} (\cos\theta)^{2x-1}(\sin\theta)^{2y-1}\,d\theta \\ &= \int_0^1 \lambda^{x-1}(1-\lambda)^{y-1}\,d\lambda.\end{aligned}$$

The difficulty of the proof lies in the inversion of the repeated integral

$$\int_0^\infty dt \int_0^\infty t^{x+y-1}v^{y-1}e^{-t(1+v)}\,dv.$$

Since each integral is infinite at each limit if the indices of t and v are negative, this requires several applications of the theorem of § 1.85. It is easily seen that the t- and v-integrals are both uniformly convergent over any finite range which excludes the origin. Hence the integrals over $(0, T; v_0, V)$ and $(t_0, T; 0, V)$ may be inverted if $t_0 > 0$, $v_0 > 0$. Hence, the integrand being positive, the integral over $(0, T; 0, V)$ may be inverted. Since also the integral over $(t_0, T; 0, \infty)$ may be inverted by uniform convergence, it follows that the integral over $(0, T; 0, \infty)$ may

be inverted. Similarly the integral over $(0, \infty; 0, V)$ may be inverted. Hence, finally the whole integral may be inverted.

Again, putting $x = y = \frac{1}{2}$ in (4), we have

$$\{\Gamma(\tfrac{1}{2})\}^2 = 2\Gamma(1) \int_0^{\frac{1}{2}\pi} d\theta = \pi,$$

or, since $\Gamma(\frac{1}{2})$ is plainly positive,

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}. \tag{5}$$

Next, putting $y = x$ in (4),

$$\frac{\{\Gamma(x)\}^2}{\Gamma(2x)} = \int_0^1 \lambda^{x-1}(1-\lambda)^{x-1}\, d\lambda = 2\int_0^{\frac{1}{2}} \lambda^{x-1}(1-\lambda)^{x-1}\, d\lambda.$$

Putting $\lambda = \frac{1}{2} - \frac{1}{2}\sqrt{\mu}$, so that $\lambda(1-\lambda) = \frac{1}{4} - \frac{1}{4}\mu$, this gives

$$\tfrac{1}{2}\int_0^1 (\tfrac{1}{4} - \tfrac{1}{4}\mu)^{x-1}\mu^{-\frac{1}{2}}\, d\mu = 2^{1-2x}\int_0^1 (1-\mu)^{x-1}\mu^{-\frac{1}{2}}\, d\mu = 2^{1-2x}\frac{\Gamma(x)\Gamma(\frac{1}{2})}{\Gamma(x+\frac{1}{2})}.$$

Hence we obtain the 'duplication formula'

$$\Gamma(2x)\Gamma(\tfrac{1}{2}) = 2^{2x-1}\Gamma(x)\Gamma(x+\tfrac{1}{2}). \tag{6}$$

1.87. Asymptotic behaviour of $\Gamma(x)$ as $x \to \infty$. Consider first the case where x is an integer, n say, so that $\Gamma(x) = (n-1)!$. We use the well-known method of comparing a sum $\sum \phi(n)$ with the corresponding integral $\int \phi(t)\, dt$. We have

$$\log\{(n-1)!\} = \sum_{\nu=1}^{n-1} \log \nu.$$

Now
$$\int_{\nu-\frac{1}{2}}^{\nu+\frac{1}{2}} \log t\, dt = \int_0^{\frac{1}{2}} \{\log(\nu+t) + \log(\nu-t)\}\, dt$$

$$= \int_0^{\frac{1}{2}} \left\{\log \nu^2 + \log\left(1 - \frac{t^2}{\nu^2}\right)\right\} dt = \log \nu + C_\nu,$$

say, where clearly $C_\nu = O(1/\nu^2)$. Hence

$$\log \Gamma(n) = \log\{(n-1)!\} = \int_{\frac{1}{2}}^{n-\frac{1}{2}} \log t\, dt - \sum_{\nu=1}^{n-1} C_\nu$$

$$= (n-\tfrac{1}{2})\log(n-\tfrac{1}{2}) - (n-\tfrac{1}{2}) - \tfrac{1}{2}\log\tfrac{1}{2} + \tfrac{1}{2} - \sum_{\nu=1}^{\infty} C_\nu + o(1)$$

$$= (n-\tfrac{1}{2})\log n - n + C + o(1), \tag{1}$$

where C is a constant.

We can extend this result to non-integral values of x by means of the following lemma.

LEMMA. *If a is constant, as $x \to \infty$,*

$$\frac{\Gamma(x)}{\Gamma(x+a)} \sim x^{-a}. \tag{2}$$

Suppose that $a > 1$. Then

$$\frac{\Gamma(x)\Gamma(a)}{\Gamma(x+a)} = \int_0^1 (1-\lambda)^{a-1}\lambda^{x-1}\,d\lambda = \int_0^\infty (1-e^{-t})^{a-1}e^{-xt}\,dt$$

$$= \int_0^\infty t^{a-1}e^{-xt}\,dt - \int_0^\infty \{t^{a-1}-(1-e^{-t})^{a-1}\}e^{-xt}\,dt.$$

The first integral is $\Gamma(a)x^{-a}$, and the second is $O(x^{-a-1})$. For

$$1-e^{-t} < t \quad (t > 0), \qquad 1-e^{-t} > t-\tfrac{1}{2}t^2 \quad (0 < t < 1).$$

Hence the second integral is positive, and less than

$$\int_0^1 \{1-(1-\tfrac{1}{2}t)^{a-1}\}t^{a-1}e^{-xt}\,dt + \int_1^\infty t^{a-1}e^{-xt}\,dt$$

$$< K\int_0^1 t^a e^{-xt}\,dt + \int_1^\infty t^a e^{-xt}\,dt < Kx^{-a-1},$$

where K depends on a only. This proves the lemma for $a > 1$. The result for other values of a then follows from 1.86 (2).

If now x is not an integer, let $x = n+a$, where n is an integer and $0 < a < 1$. Then

$$\begin{aligned}\log\Gamma(x) &= \log\Gamma(n+a) = \log\Gamma(n) + a\log n + o(1)\\ &= (n-\tfrac{1}{2})\log n - n + C + a\log n + o(1)\\ &= (x-a-\tfrac{1}{2})\log(x-a) - x + a + C + a\log(x-a) + o(1)\\ &= (x-\tfrac{1}{2})\log x - x + C + o(1),\end{aligned} \tag{3}$$

the required result.

To find the value of C, we use the duplication formula 1.86 (6). Taking logarithms and using (3), we have

$$(2x-\tfrac{1}{2})\log 2x - 2x + C + \log\sqrt{\pi} + o(1)$$
$$= (2x-1)\log 2 + (x-\tfrac{1}{2})\log x + x\log(x+\tfrac{1}{2}) - 2x - \tfrac{1}{2} + 2C + o(1),$$

and equating the constant terms, we obtain

$$C = \log\sqrt{(2\pi)}.$$

Hence finally

$$\Gamma(x) = x^{x-\frac{1}{2}}e^{-x}\sqrt{(2\pi)}\{1+o(1)\}. \tag{4}$$

This result is known as Stirling's theorem.

1.88. Differentiation of integrals. The two following theorems cover most of the cases that are ordinarily met with.

If $f(x,y)$ *and* $\dfrac{\partial f}{\partial y}$ *are continuous in the rectangle* $a \leqslant x \leqslant b$, $y_0-\eta \leqslant y \leqslant y_0+\eta$ $(\eta > 0)$, *then*

$$\frac{d}{dy}\int_a^b f(x,y)\,dx = \int_a^b \frac{\partial f}{\partial y}\,dx \tag{1}$$

for $y = y_0$.

Let $$\phi(y) = \int_a^b f(x,y)\,dx, \qquad g(x,y) = \frac{\partial f}{\partial y}.$$

Then $$\frac{\phi(y_0+k)-\phi(y_0)}{k} = \frac{1}{k}\int_a^b \{f(x,y_0+k)-f(x,y_0)\}\,dx$$
$$= \int_a^b g(x,y_0+\theta k)\,dx,$$

where $0 < \theta < 1$. Since $g(x,y)$ is uniformly continuous, we have, as in § 1.52,

$$\lim_{k\to 0}\int_a^b g(x,y_0+\theta k)\,dx = \int_a^b g(x,y_0)\,dx,$$

the required result.

If the equation (1) *is true for all values of* b *greater than* a, *and if the integral*

$$\int_a^\infty f\,dx$$

is convergent, and

$$\int_a^\infty \frac{\partial f}{\partial y}\,dx$$

is uniformly convergent in the interval $(y_0-\eta, y_0+\eta)$, *then*

$$\frac{d}{dy}\int_a^\infty f(x,y)\,dx = \int_a^\infty \frac{\partial f}{\partial y}\,dx.$$

We can deduce this from the corresponding theorem for series (§ 1.72). For let

$$\int_{a+n-1}^{a+n} f(x,y)\,dx = u_n(y).$$

Then
$$\int_a^\infty f(x,y)\,dx = \sum u_n(y),$$
and, by the previous theorem,
$$\int_a^\infty \frac{\partial f}{\partial y}\,dx = \sum \int_{a+n-1}^{a+n} \frac{\partial f}{\partial y}\,dx = \sum \frac{d}{dy}\int_{a+n-1}^{a+n} f(x,y)\,dx = \sum \frac{d}{dy}\,u_n(y).$$
The result now follows at once from the theorem for series.

MISCELLANEOUS EXAMPLES

1. Consider the uniform convergence of the series
$$\sum_{n=1}^\infty \frac{1}{x^2+n^2}, \qquad \sum_{n=1}^\infty \frac{1}{x^2n^2}, \qquad \sum_{n=1}^\infty \frac{1}{x^2-n^2},$$
and
$$\sum_{n=1}^\infty \frac{\sin nx}{n^\alpha} \quad (\alpha > 0).$$

2. Discuss with reference to uniformity the convergence of the series
$$\sum_{n=1}^\infty \frac{1}{n^2}\,\frac{z^n}{1+z^n}, \qquad \sum_{n=1}^\infty \frac{(-1)^n}{n}\,\frac{z^n}{1+z^n}$$
(i) for z real and positive, (ii) for general complex values of z.

3. Consider the uniform convergence of the integrals
$$\int_0^\infty \frac{dx}{x^2+y^2}, \qquad \int_0^\infty e^{-xy}\,dx, \qquad \int_0^\infty \cosh xy\, e^{-x^2}\,dx.$$

4. Consider the uniform convergence of the integral
$$\int_0^\infty \frac{\sin^2 xy}{x^2}\,dx.$$
Evaluate the integral by differentiating with respect to y.

5. The Bessel function $J_\nu(z)$ is defined for $\nu > -1$ by the series
$$J_\nu(z) = \sum_{n=0}^\infty \frac{(-1)^n(\frac{1}{2}z)^{\nu+2n}}{n!\,\Gamma(\nu+n+1)}.$$
Prove that, if $\nu > -\frac{1}{2}$,
$$J_\nu(z) = \frac{z^\nu}{2^{\nu-1}\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}\int_0^{\frac{1}{2}\pi} \cos(z\cos\theta)\sin^{2\nu}\theta\,d\theta,$$

and that, if $\mu > -1$, $\nu > -1$,

$$J_{\mu+\nu+1}(z) = \frac{z^{\nu+1}}{2^\nu \Gamma(\nu+1)} \int_0^{\frac{1}{2}\pi} J_\mu(z\sin\theta)\sin^{\mu+1}\theta\cos^{2\nu+1}\theta\, d\theta.$$

6. Prove that

$$\int_0^\infty e^{-ax} J_0(bx)\, dx = \frac{1}{\sqrt{(a^2+b^2)}} \qquad (0 < b < a),$$

and that

$$\int_0^\infty e^{-x} J_0(x)\, dx = \frac{1}{\sqrt{2}}.$$

7. Prove that $$\int_0^\infty J_\nu(at) e^{-t^2} t^{\nu+1}\, dt = \frac{a^\nu}{2^{\nu+1}} e^{-\frac{1}{4}a^2}$$

8. Prove that

$$\sum_{n=1}^\infty \int_0^\infty \frac{\sin nx}{n^2 x}\, dx = \int_0^\infty \left\{ \sum_{n=1}^\infty \frac{\sin nx}{n^2 x} \right\} dx.$$

9. Show that the repeated integrals

$$\int_1^\infty dx \int_1^\infty \frac{x-y}{(x+y)^3}\, dy, \qquad \int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3}\, dy,$$

$$\int_1^\infty dx \int_1^\infty \frac{x^2-y^2}{(x^2+y^2)^2}\, dy, \qquad \int_0^1 dx \int_1^\infty (ae^{-axy} - be^{-bxy})\, dy,$$

are not equal to the integrals obtained by inverting the order of integration.

10. Prove that $$\int_0^\infty e^{-x^2} \cos 2xy\, dx = \tfrac{1}{2}\sqrt{\pi} e^{-y^2}$$

(i) by expanding $\cos 2xy$ in powers of x and integrating term by term, and (ii) by proving that the integral satisfies the differential equation

$$\frac{dI}{dy} = -2yI.$$

11. If $$\phi(y) = \int_0^\infty \frac{\sin xy}{x(a^2+x^2)}\, dx,$$

prove that $$\phi''(y) - a^2\phi(y)$$

is a constant, and hence that

$$\phi(y) = \frac{\pi}{2a^2}(1 - e^{-ay}) \qquad (y > 0).$$

12. Deduce from the previous example the values of the integrals

$$\int_0^\infty \frac{\cos xy}{a^2+x^2}\,dx, \qquad \int_0^\infty \frac{x\sin xy}{a^2+x^2}\,dx.$$

13. If
$$\phi(p, q, a, b) = \Gamma(p)\int_0^\infty \frac{e^{-bx}x^{q-1}}{(a+cx)^p}\,dx,$$

where a, b, p, and q are positive, prove that

$$\phi(p, q, a, b) = \phi(q, p, b, a).$$

14. If
$$\psi(y) = \int_0^\infty e^{-x^2-y^2/x^2}\,dx,$$

prove that
$$\psi(y) = \tfrac{1}{2}\sqrt{\pi}e^{-2y} \qquad (y > 0)$$

(i) by proving that $\psi'(y) = -2\psi(y)$, and (ii) by means of the substitution $w = x-y/x$.

15. Show that the repeated integral

$$\int_0^\infty dx \int_0^\infty \cos 2mx . ye^{-y^2(1+x^2)}\,dy$$

may be inverted; and hence deduce the value of the first integral of example 12 from those of examples 10 and 14.

16. Prove that if $\lambda > 0$, $\mu > 0$,

$$\int_0^\infty e^{-x^2}x^{2\lambda-1}\,dx \int_0^\infty e^{-y^2}y^{2\mu-1}\,dy = \int_0^\infty e^{-r^2}r^{2\lambda+2\mu-1}\,dr \int_0^{\frac{1}{2}\pi} \cos^{2\lambda-1}\theta \sin^{2\mu-1}\theta\,d\theta,$$

and hence obtain another proof of the formulae of § 1.86.

17. Show that, in the repeated integral

$$\int_0^\infty \sin ax\,dx \int_0^\infty f(y)e^{-xy}\,dy,$$

the order of integration may be inverted if the integrals

$$\int_0^1 |f(y)|\,dy, \qquad \int_1^\infty |f(y)|y^{-2}\,dy$$

are convergent. Hence show that

$$\int_0^\infty \frac{\sin ax}{\sqrt{(1+x^2)}}\,dx = a\int_0^\infty \frac{J_0(y)}{a^2+y^2}\,dy.$$

[The integral $\int_\xi^X \sin ax\,dx \int_0^\infty f(y)e^{-xy}\,dy$ may be inverted, since the

y-integral is uniformly convergent for $0 < \xi \leqslant x \leqslant X$. Hence it is sufficient to prove that the integrals

$$\int_0^\infty f(y)\,dy \int_0^\xi \sin ax\, e^{-xy}\,dx, \qquad \int_0^\infty f(y)\,dy \int_X^\infty \sin ax\, e^{-xy}\,dx$$

tend to zero as $\xi \to 0$ and $X \to \infty$. Since $|\sin ax| \leqslant ax$, the modulus of the former does not exceed

$$\int_0^\infty |f(y)|\,dy \int_0^\xi ax\, e^{-xy}\,dx < a\int_0^Y |f(y)|\,dy \int_0^\xi x\,dx + a\int_Y^\infty |f(y)|\,dy \int_0^\infty xe^{-xy}\,dx$$

$$= \tfrac{1}{2}a\xi^2 \int_0^Y |f(y)|\,dy + a\int_Y^\infty |f(y)|y^{-2}\,dy,$$

which can be made arbitrarily small by choosing first Y and then ξ. The modulus of the latter integral is

$$\left|\int_0^\infty f(y)\frac{y\sin aX + a\cos aX}{a^2+y^2}e^{-Xy}\,dy\right| < \frac{1}{a}\int_0^\infty |f(y)|e^{-Xy}\,dy,$$

which tends to zero.]

18. If $\alpha > 0$, $\beta > 0$,

$$\int_y^x (x-t)^{\alpha-1}(t-y)^{\beta-1}\,dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-y)^{\alpha+\beta-1}.$$

19. If $\alpha > 0$, $\beta > 0$, and $\lambda < y$ or $\lambda > x$,

$$\int_y^x \frac{(x-t)^{\alpha-1}(t-y)^{\beta-1}}{|t-\lambda|^{\alpha+\beta}}\,dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,\frac{(x-y)^{\alpha+\beta-1}}{|x-\lambda|^\beta|y-\lambda|^\alpha}.$$

[Consider the general linear transformation of the interval (y, x) into itself, and use the previous example.]

20. Prove that if $c > b > 0$, $c-a-b > 0$,

$$\sum_{n=0}^\infty \frac{a(a+1)\ldots(a+n-1)}{n!}\int_0^1 t^{b+n-1}(1-t)^{c-b-1}\,dt = \int_0^1 t^{b-1}(1-t)^{c-a-b-1}\,dt.$$

Deduce that

$$\sum_{n=0}^\infty \frac{a\ldots(a+n-1)b\ldots(b+n-1)}{n!\,c\ldots(c+n-1)} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

CHAPTER II

ANALYTIC FUNCTIONS

2.1. Functions of a complex variable. It is just as easy to construct a function of a complex variable $z = x+iy$ as it is to construct a function of a real variable x. Any finite or infinite convergent expression involving z gives such a function. For example, z^2, $1/z$, e^z are functions of the complex variable z. The reader of Hardy's *Pure Mathematics* is already familiar with many such functions.

Throughout this chapter and the next all functions are supposed to be one-valued in the region in which they are defined.

Our first task is to give a general definition which will be appropriate to all such functions.

We might say that w is a function of z if to every value of z in a certain region corresponds one or more values of w. This is modelled on the usual definition of a function of a real variable. It is perfectly legitimate, but, as is explained in Hardy's *Pure Mathematics*, it is futile because it is too wide. It makes a function of the complex variable z exactly the same thing as a complex function

$$u(x,y)+iv(x,y)$$

of two real variables x and y. Of course this is not what we meant when we began to speak of functions of a complex variable.

Our method of procedure is to assign various properties to our function which appear to be desirable, and to see whether any such properties distinguish between what we feel to be 'proper' and 'improper' functions of z.

2.11. Continuity. Let $f(z)$ be a function of z defined in the above way. *It is said to be continuous at the point $z = z_0$ if, given any positive number ϵ, we can find a number δ such that*

$$|f(z)-f(z_0)| < \epsilon$$

provided that $|z-z_0| < \delta$.

This is quite satisfactory as far as it goes, but it does not go very far. A continuous function of z is merely a continuous complex function of the two variables x and y. For if

$$f(z) = u(x,y)+iv(x,y),$$

and $z_0 = x_0 + iy_0$, then

$$|u(x,y) - u(x_0,y_0)| \leqslant |f(z) - f(z_0)| < \epsilon$$

if $|z - z_0| < \delta$, which is true if

$$|x - x_0| < \frac{\delta}{\sqrt{2}}, \qquad |y - y_0| < \frac{\delta}{\sqrt{2}}.$$

Hence $u(x,y)$ is continuous, and so is $v(x,y)$. Conversely, if u and v are continuous, so is $f(z)$.

2.12. Differentiability. From the class of continuous complex functions, we next select those which can be differentiated. The meaning of this term for complex functions must now be defined.

Following the suggestion of real differential calculus, we write

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

and we say that $f(z)$ is differentiable if the limit on the right exists. The limit is called the derivative or differential coefficient of $f(z)$. As in the definition of continuity, the approach of z to its limit z_0 can take place in all possible ways. More precisely, we interpret the above formula as meaning that, given any positive number ϵ, we can find a number δ such that

$$\left| f'(z_0) - \frac{f(z) - f(z_0)}{z - z_0} \right| < \epsilon$$

provided that $0 < |z - z_0| < \delta$. Thus we assert that, along whatever path z approaches z_0, the ratio

$$\frac{f(z) - f(z_0)}{z - z_0}$$

always tends to a limit, and that all the limiting values are the same. Our requirements are therefore somewhat exacting.

This property of differentiability is, however, one which belongs to many familiar functions. A constant is differentiable. A positive integral power of z is differentiable; for the familiar proof for x^n applies word for word to the case of z^n. Similarly, the sum or product of two (or any finite number) of differentiable functions is differentiable; and the quotient of two differentiable functions is differentiable provided that the denominator does not vanish. Finally, a differentiable function of a differentiable

function is differentiable. All these theorems are proved in the same way for functions of z as for functions of x.

For example, any rational function of z is differentiable for all values of z, other than zeros of its denominator.

2.13. We now naturally ask whether this property of differentiability corresponds to a simple property of the functions $u(x, y)$ and $v(x, y)$ which are the real and imaginary parts of $f(z)$.

Suppose first that $z-z_0$ is purely real, so that

$$z_0 = x_0 + iy_0, \qquad z = x + iy_0.$$

Then

$$\frac{f(z)-f(z_0)}{z-z_0} = \frac{\{u(x, y_0)+iv(x, y_0)\}-\{u(x_0, y_0)+iv(x_0, y_0)\}}{x-x_0}$$

$$= \frac{u(x, y_0)-u(x_0, y_0)}{x-x_0} + i\frac{v(x, y_0)-v(x_0, y_0)}{x-x_0}.$$

If this tends to a limit as $x \to x_0$, then its real and imaginary parts separately tend to limits. But this means simply that the partial differential coefficients

$$\frac{\partial u}{\partial x}, \quad \frac{\partial v}{\partial x}$$

exist at the point (x_0, y_0). Also

$$f'(z_0) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)_{x=x_0,\, y=y_0} \tag{1}$$

Similarly, if we take $z-z_0$ to be purely imaginary, say

$$z_0 = x_0 + iy_0, \qquad z = x_0 + iy,$$

we obtain

$$\frac{f(z)-f(z_0)}{z-z_0} = \frac{\{u(x_0, y)+iv(x_0, y)\}-\{u(x_0, y_0)+iv(x_0, y_0)\}}{iy-iy_0}$$

$$= \frac{v(x_0, y)-v(x_0, y_0)}{y-y_0} - i\,\frac{u(x_0, y)-u(x_0, y_0)}{y-y_0}.$$

Hence the partial differential coefficients

$$\frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y}$$

exist at the point (x_0, y_0); and

$$f'(z_0) = \left(\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}\right)_{x=x_0,\, y=y_0} \tag{2}$$

Further, on comparing (1) and (2), and equating real and imaginary parts, we see that

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y} \qquad (x=x_0,\ y=y_0). \tag{3}$$

We now see that the results of assuming *differentiability* are much more far-reaching than those of assuming *continuity*. Not only do the functions $u(x,y)$, $v(x,y)$ possess partial differential coefficients of the first order, but they are connected by the differential equations (3). These are called the Cauchy-Riemann equations.

Thus, even if u and v are functions of x and y with partial differential coefficients of the first order, $u+iv$ will not in general be a differentiable function of z.

Examples. (i) Let $f(z)=\mathbf{R}(z)=x$. Then

$$\frac{\partial u}{\partial x}=1, \qquad \frac{\partial u}{\partial y}=0, \qquad \frac{\partial v}{\partial x}=0, \qquad \frac{\partial v}{\partial y}=0.$$

The partial differential coefficients all exist, but the Cauchy-Riemann equations are not satisfied for any value of z.

(ii) Let $f(z)=|z|^2=x^2+y^2$. Then

$$\frac{\partial u}{\partial x}=2x, \qquad \frac{\partial u}{\partial y}=2y, \qquad \frac{\partial v}{\partial x}=0, \qquad \frac{\partial v}{\partial y}=0.$$

The Cauchy-Riemann equations are satisfied at the point $z=0$ only.

2.14. Analytic functions. Since the property which we have been discussing goes far beyond what we ordinarily think of as differentiability, we give it a special name. A function which is differentiable in this sense is said to be *analytic*.

The property of being analytic is in fact the distinguishing property for 'proper' functions of a complex variable for which we have been seeking.

We have seen that the truth of the Cauchy-Riemann equations is a *necessary* condition for the function to be analytic. But it is not a *sufficient* condition. This is perhaps to be expected, since we obtained the equations as particular cases only of the general property of differentiability.

Consider, for example, the function

$$f(z)=\sqrt{|xy|}.$$

This vanishes on both axes, so that at $z=0$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}=0,$$

and the Cauchy-Riemann equations are satisfied. But $f(z)$ is not differentiable at $z=0$. For

$$\frac{f(z)}{z}=\frac{\sqrt{|xy|}}{x+iy},$$

and, if $x=\alpha r, y=\beta r$, where α and β are constants, this tends to

$$\frac{\sqrt{|\alpha\beta|}}{\alpha+i\beta}$$

as $r\to 0$. The limit is therefore not unique, and so the function is not analytic.

This example shows that $f(z)$ may not be analytic if we merely know that

$$\{f(z)-f(z_0)\}/(z-z_0)$$

tends to a limit along two straight lines at right angles. Actually the definition fails if we restrict ourselves to any special class of paths. Consider, for example, the function

$$f(z)=\frac{xy^2(x+iy)}{x^2+y^4} \qquad (z\neq 0), \qquad f(0)=0.$$

Then it is easily seen that

$$\lim\{f(z)-f(0)\}/z=0$$

as $z\to 0$ along any straight line. But, on the curve $x=y^2$,

$$\frac{f(z)-f(0)}{z}=\frac{y^4}{y^4+y^4}=\frac{1}{2}.$$

Hence $f(z)$ is not analytic at $z=0$.

2.15. *Suppose, however, that the four partial derivatives of the first order exist throughout a region, and are continuous at all points of the region. Then the truth of the Cauchy-Riemann equations is a necessary and sufficient condition for $f(z)$ to be analytic throughout the region.*

We have seen already that the condition is necessary. To prove that it is sufficient, we use the mean-value theorem for functions of two variables (*P.M.* § 154). Consider a point (x, y) of the region, and a neighbouring point $(x+\delta x, y+\delta y)$. Then

$$\begin{aligned}\delta u &= u(x+\delta x, y+\delta y)-u(x, y)\\ &= \left(\frac{\partial u}{\partial x}+\epsilon\right)\delta x+\left(\frac{\partial u}{\partial y}+\eta\right)\delta y,\end{aligned}$$

where ϵ and η tend to zero with δx and δy. Similarly

$$\delta v=\left(\frac{\partial v}{\partial x}+\epsilon'\right)\delta x+\left(\frac{\partial v}{\partial y}+\eta'\right)\delta y.$$

Hence, using the Cauchy-Riemann equations,

$$\delta u+i\delta v=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)(\delta x+i\delta y)+\rho,$$

where $$|\rho|\leqslant(|\epsilon|+|\epsilon'|)|\delta x|+(|\eta|+|\eta'|)|\delta y|.$$

Hence

$$\frac{f(z+\delta z)-f(z)}{\delta z}=\frac{\delta u+i\delta v}{\delta x+i\delta y}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}+\frac{\rho}{\delta x+i\delta y},$$

and $$|\rho/(\delta x+i\delta y)|\leqslant|\epsilon|+|\epsilon'|+|\eta|+|\eta'|\to 0.$$

Hence $f(z)$ is analytic.

2.16. *A power series represents an analytic function inside its circle of convergence.*

We shall see later that this is merely a particular case of a general theorem on series which represent analytic functions. But the following direct proof may be inserted at this point. Let

$$f(z)=\sum_{n=0}^{\infty}a_nz^n$$

be convergent for $|z|<R$. Then, if $\rho<R$, $a_n\rho^n$ is bounded, say $|a_n\rho^n|\leqslant K$. Let

$$g(z)=\sum_{n=1}^{\infty}na_nz^{n-1}.$$

Then if $|z|<\rho$ and $|z|+|h|<\rho$,

$$\frac{f(z+h)-f(z)}{h}-g(z)=\sum_{n=0}^{\infty}a_n\left\{\frac{(z+h)^n-z^n}{h}-nz^{n-1}\right\}.$$

Now

$$\left|\frac{(z+h)^n-z^n}{h}-nz^{n-1}\right|=\left|\frac{n(n-1)}{1.2}z^{n-2}h+\ldots+h^{n-1}\right|$$

$$\leqslant\frac{n(n-1)}{1.2}|z|^{n-2}|h|+\ldots+|h|^{n-1}=\frac{(|z|+|h|)^n-|z|^n}{|h|}-n|z|^{n-1}.$$

Hence

$$\left|\frac{f(z+h)-f(z)}{h}-g(z)\right|\leqslant K\sum_{n=0}^{\infty}\frac{1}{\rho^n}\left\{\frac{(|z|+|h|)^n-|z|^n}{|h|}-n|z|^{n-1}\right\}$$

$$=K\left\{\frac{1}{|h|}\left(\frac{\rho}{\rho-|z|-|h|}-\frac{\rho}{\rho-|z|}\right)-\frac{\rho}{(\rho-|z|)^2}\right\}$$

$$=\frac{K\rho|h|}{(\rho-|z|-|h|)(\rho-|z|)^2},$$

which tends to zero with h. Hence $f(z)$ has the derivative $g(z)$.

2.17. Functions analytic in a region. A function is said to be analytic in a region if it is analytic at all points of the region. Henceforth we always consider functions which are analytic *in a region*. No particular interest attaches to the fact that a function (like $|z|^2$) happens to be analytic at certain points, or even on a certain curve. It is when it is analytic in a region that interesting consequences follow.

Examples. (i) The function

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \dots$$

is analytic for $|z| < 1$.

(ii) The functions

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad \cos z = \tfrac{1}{2}(e^{iz}+e^{-iz}), \qquad \sin z = \frac{1}{2i}(e^{iz}-e^{-iz}),$$

are analytic for all finite values of z.

(iii) The function

$$f(z) = e^{-z^{-4}} \quad (z \neq 0), \qquad f(0) = 0$$

is analytic for all finite values of z, except $z = 0$. At the point $z = 0$, the Cauchy-Riemann equations are satisfied; for at $z = 0$

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{e^{-x^{-4}}}{x} = 0, \qquad \frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{0}{x} = 0,$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{e^{-y^{-4}}}{y} = 0, \qquad \frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{0}{y} = 0,$$

so that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

In spite of this, $f(z)$ is not analytic at $z = 0$. For suppose that $z = re^{\frac{1}{4}i\pi}$. Then

$$f(z) = \exp\{-(re^{\frac{1}{4}i\pi})^{-4}\} = e^{r^{-4}},$$

which tends to infinity as $r \to 0$.

2.2. The complex differential calculus. The reader might expect that we should now proceed after the manner of the real differential calculus. There, having distinguished the special class of differentiable functions, we next consider the still more special class of functions which have a second differential coefficient. Some of these functions have differential coefficients of the third order, and so on. Finally, from among functions which have differential coefficients of all orders, we pick out those which can be expanded in a power series by Taylor's theorem.

There is no such process of successive specialization for analytic functions of a complex variable. A function which is analytic in a region has differential coefficients of all orders at all points of the region, and the function can be expanded in a power series, after the manner of Taylor's theorem, about any point of the region.

All these facts follow from the definition of an analytic function by means of its first differential coefficient.

The reader would perhaps expect us to begin by proving that an analytic function has a second differential coefficient. We are unable to do this.

The results have of course been proved, or we should not have been able to announce what they were. But they have never been proved directly. They all depend on the complex integral calculus, and it is to this that we must now turn.

2.3. Complex integration. The reader of Hardy's *Pure Mathematics* should know what a complex integral is (*P.M.* § 222). We shall, however, introduce the subject in a slightly different way.

Let AB be an arc C of a curve defined by the equations

$$x = \phi(t), \qquad y = \psi(t),$$

where ϕ and ψ are functions of t with continuous differential coefficients $\phi'(t)$ and $\psi'(t)$, and suppose that, as t varies from t_A to t_B, the point (x, y) moves along the curve steadily from A to B.

Let $f(z)$ be any complex function of z, continuous along C. Let

$$f(z) = u(x, y) + iv(x, y).$$

Let $z_0, z_1, \ldots, z_n$ be points on C, z_0 being A and z_n being B. Consider the sum

$$\sum_{m=1}^{n} f(\zeta_m)(z_m - z_{m-1}), \tag{1}$$

where ζ_m is a point of the curve between z_{m-1} and z_m. Writing $\zeta_m = \xi_m + i\eta_m$, $u_m = u(\xi_m, \eta_m)$, $v_m = v(\xi_m, \eta_m)$, this is

$$\sum_{m=1}^{n} (u_m + iv_m)(x_m + iy_m - x_{m-1} - iy_{m-1}).$$

Now
$$x_m - x_{m-1} = \phi(t_m) - \phi(t_{m-1}) = \phi'(\tau_m)(t_m - t_{m-1}),$$
$$y_m - y_{m-1} = \psi(t_m) - \psi(t_{m-1}) = \psi'(\tau'_m)(t_m - t_{m-1}),$$

where $t_{m-1} \leqslant \tau_m \leqslant t_m$, $t_{m-1} \leqslant \tau'_m \leqslant t_m$. Hence the sum may be written

$$\sum_{m=1}^{n} (u_m + iv_m)\{\phi'(\tau_m) + i\psi'(\tau'_m)\}(t_m - t_{m-1}). \tag{2}$$

Since all the functions concerned are continuous (and therefore uniformly continuous), we can, given ϵ, find δ so that

$$|u_m \phi'(\tau_m) - u(x_m, y_m)\phi'(t_m)| < \epsilon$$

for every m, provided that each $|t_m - t_{m-1}| < \delta$. Also

$$\sum_{m=1}^{n} \epsilon(t_m - t_{m-1}) = \epsilon(t_B - t_A).$$

It follows that, as ϵ and δ tend to zero,

$$\sum_{m=1}^{n} u_m \phi'(\tau_m)(t_m - t_{m-1})$$

tends to the same limit as

$$\sum_{m=1}^{n} u(x_m, y_m)\phi'(t_m)(t_m - t_{m-1}),$$

viz. to the limit $$\int_{t_A}^{t_B} u\{\phi(t), \psi(t)\}\phi'(t)\, dt.$$

Similarly the other terms of (2) tend to limits, and we find that the whole sum tends to the limit

$$\int_{t_A}^{t_B} (u + iv)\{\phi'(t) + i\psi'(t)\}\, dt, \tag{3}$$

this integral being interpreted in the obvious way as the sum of two real integrals, one of which is multiplied by i.

This limit is taken as the definition of the complex integral of $f(z)$ along C, and it is written

$$\int_C f(z)\, dz. \tag{4}$$

In particular, the above analysis holds for any function $f(z)$ which is analytic throughout a region including C.

Some of the most obvious properties of real integrals extend at once to complex integrals; for example,

$$\int_C \{f(z) + g(z)\}\, dz = \int_C f(z)\, dz + \int_C g(z)\, dz,$$

and, if k is a constant,

$$\int_C kf(z)\, dz = k \int_C f(z)\, dz.$$

Also, if C' denotes the contour C described in the opposite direction,

$$\int_{C'} f(z)\,dz = -\int_C f(z)\,dz.$$

Examples. (i) Let $f(z) = k$, a constant, and let C be any curve joining the points $z = a$ and $z = b$. Then

$$\sum_{m=1}^{n} f(\zeta_m)(z_m - z_{m-1}) = k\sum_{m=1}^{n}(z_m - z_{m-1}) = k(b-a).$$

Hence
$$\int_C k\,dz = k(b-a).$$

Since the result is independent of the particular curve C taken, we may write the result as

$$\int_a^b k\,dz = k(b-a).$$

(ii) Let $f(z) = z$, and let C be any curve joining the points $z = a$ and $z = b$.

First take $\zeta_m = z_m$. Then

$$\sum_{m=1}^{n} f(\zeta_m)(z_m - z_{m-1}) = \sum_{m=1}^{n} z_m(z_m - z_{m-1}).$$

Taking $\zeta_m = z_{m-1}$, the sum is

$$\sum_{m=1}^{n} z_{m-1}(z_m - z_{m-1}).$$

These sums tend to the same limit, and hence so does half their sum, viz.

$$\tfrac{1}{2}\sum(z_m^2 - z_{m-1}^2) = \tfrac{1}{2}(b^2 - a^2).$$

Hence
$$\int_C z\,dz = \tfrac{1}{2}(b^2 - a^2),$$

and the integral is again independent of the path.

(iii) Calculate the integral

$$\int_C \frac{dz}{z},$$

where C is the circle with centre the origin and radius ρ.

[Here we can put

$$x = \rho\cos\theta = \phi(\theta), \qquad y = \rho\sin\theta = \psi(\theta),$$

where θ varies from 0 to 2π. Now

$$\phi'(\theta) + i\psi'(\theta) = -\rho\sin\theta + \rho i\cos\theta = \rho i e^{i\theta}.$$

Hence the integral is

$$\int_0^{2\pi} \frac{1}{\rho e^{i\theta}}\rho i e^{i\theta}\,d\theta = i\int_0^{2\pi} d\theta = 2\pi i.]$$

(iv) Prove similarly that $\displaystyle\int_C z^n\,dz = 0,$

where n is any integer, positive or negative, other than -1.

2.31. An inequality for complex integrals. We may define the length of the curve $x=\phi(t)$, $y=\psi(t)$, where $\phi'(t)$ and $\psi'(t)$ are continuous, to be the integral

$$\int [\{\phi'(t)\}^2+\{\psi'(t)\}^2]^{\frac{1}{2}}\, dt,$$

taken between appropriate limits. For the justification of this, see *P.M.* § 146.

If M is the upper bound of $|f(z)|$ on the curve C, and L is the length of C, then

$$\left|\int f(z)\, dz\right| \leqslant ML.$$

In the first place, if $F(t)$ is any continuous complex function of a real variable t,

$$\left|\int_a^b F(t)\, dt\right| \leqslant \int_a^b |F(t)|\, dt. \tag{1}$$

For $\quad |\sum F(t_m)(t_m-t_{m-1})| \leqslant \sum |F(t_m)|(t_m-t_{m-1}),$

and (1) follows on proceeding to the limit.

Hence, with our previous notation,

$$\begin{aligned}\left|\int_C f(z)\, dz\right| &= \left|\int_{t_A}^{t_B} (u+iv)\{\phi'(t)+i\psi'(t)\}\, dt\right| \\ &\leqslant \int_{t_A}^{t_B} M[\{\phi'(t)\}^2+\{\psi'(t)\}^2]^{\frac{1}{2}}\, dt \\ &= ML.\end{aligned}$$

2.32. Contours. By a contour we mean a continuous curve consisting of a finite number of arcs of the type already considered, that is to say, arcs defined by equations $x=\phi(t)$, $y=\psi(t)$, where $\phi'(t)$ and $\psi'(t)$ are continuous. The contour is closed if the end-point of the last arc is the same as the starting-point of the first.

Let C be a closed contour. Suppose that there is an interval (a,b) such that, if $a<x'<b$, the line $x=x'$ meets C in just two points, say $y_1(x')$ and $y_2(x')$, where $y_1<y_2$; while if $x'<a$ or $x'>b$, the line $x=x'$ does not meet C. Suppose similarly that there is an interval (α,β), such that if $\alpha<y'<\beta$, the line $y=y'$ meets C in just two points, say $x_1(y')$ and $x_2(y')$, where $x_1<x_2$; while if $y'<\alpha$ or $y'>\beta$, the line $y=y'$ does not meet C.

Then the point (x,y) is said to be inside C if $a<x<b$ and

$y_1(x) < y < y_2(x)$. A point not inside C or on C is said to be outside C.

A contour which satisfies these conditions is called a simple closed contour. For example, a circle, square, or ellipse in any position is a simple closed contour. The definition of 'inside' and 'outside' which we have given may strike the reader as unnecessarily elaborate, and the class of curves considered unnecessarily restricted. But the general study of questions of this kind is not quite so easy as might be supposed, and we regard it as outside our scope. It forms the subject known as 'analysis situs', and is dealt with, for example, in Watson's *Complex Integration and Cauchy's Theorem*. On the other hand, the reader who prefers to ignore our explanations and trust to geometrical intuition will find that he gets on perfectly well.

We can extend the class of contours to which our theorems apply by 'addition' and 'subtraction' of simple closed contours. Suppose that C and C' are two simple closed contours having one or more arcs in common, but lying outside each other. We form a new closed contour C'' by deleting the common boundary. The interior of C'' consists of the interiors of C and C', together with points on the deleted boundary. Similarly, if all points inside C' are also inside C, we form a new closed contour C''', the interior of which consists of points inside C but outside C'.

A closed contour of this kind which is often useful is formed by the semi-circles $|z| = \rho$, $|z| = R$ $(0 < \rho < R)$, in the upper half-plane, joined by intervals of the real axis.

Still more complicated contours can be introduced by further additions; for example, add the contour just described to its reflection in the real axis, and delete the common boundary from $z = -R$ to $z = -\rho$. We obtain a closed contour with a definite inside and outside. The outside consists of the regions $|z| < \rho$ and $|z| > R$. In describing the contour, the interval from $z = \rho$ to $z = R$ is described twice in opposite directions.

2.33. Cauchy's theorem. The keystone of the theory of analytic functions is the following theorem of Cauchy:

If a function $f(z)$ is analytic and one-valued inside and on a simple closed contour C, then

$$\int_C f(z)\,dz = 0.$$

To prove this, we divide up the region inside C into a large number of small parts by a network of lines parallel to the real and imaginary axes. Suppose that this divides the inside of C into a number of squares, $C_1,\ldots, C_M$ say, and a number of irregular regions, $D_1,\ldots, D_N$ say, parts of whose boundaries are parts of C. Then

$$\int_C f(z)\,dz = \sum_{m=1}^{M} \int_{C_m} f(z)\,dz + \sum_{n=1}^{N} \int_{D_n} f(z)\,dz, \tag{1}$$

where each contour is described in the positive (anti-clockwise) direction. Consider, for example, two squares $ABCD$ and $DCEF$ with a common side CD. The side CD is described from C to D in the first square, and from D to C in the second. Hence the two integrals along CD cancel. So all the integrals cancel, except those which form part of C itself, since these are described once only. This proves (1).

We now use the fact that $f(z)$ is analytic at every point. This means that, if z_0 is any point inside or on C, then

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| < \epsilon$$

provided that $0 < |z-z_0| < \delta = \delta(z_0)$; i.e. if $|z-z_0| < \delta$,

$$|f(z)-f(z_0)-(z-z_0)f'(z_0)| \leqslant \epsilon\,|z-z_0|. \tag{2}$$

If we consider any particular region C_m or D_n in the above construction, it is evident that we can choose its sides so small that (2) is satisfied if z_0 is a given point of the region, and z any other point. It is not, however, immediately obvious that we can choose the whole network so that the conditions are satisfied in all the partial regions at the same time. We shall prove that this is actually possible.

Having given ϵ, we can choose the network in such a way that, in every partial region C_m or D_n, there is a point z_0 such that (2) *holds for every z in this region.*

This means, substantially, that the function is *uniformly differentiable* throughout the interior of C.

Let us assume for the moment that this is true. Consider one of the squares C_m, of side l_m. Here, by (2),

$$f(z) = f(z_0) + (z-z_0)f'(z_0) + \phi(z),$$

where $$|\phi(z)| \leqslant \epsilon\,|z-z_0|.$$

Hence

$$\int_{C_m} f(z)\,dz = \int_{C_m} \{f(z_0)+(z-z_0)f'(z_0)\}\,dz + \int_{C_m} \phi(z)\,dz.$$

The first integral on the right is zero, by § 2.3, examples (i) and (ii). Also, by § 2.31,

$$\left|\int_{C_m} \phi(z)\,dz\right| < \epsilon\sqrt{2}l_m \cdot 4l_m,$$

since $|z-z_0| \leqslant \sqrt{2}l_m$, and the length of C_m is $4l_m$.

In the case of one of the irregular regions D_n, the length is not greater than $4l_n+s_n$, where s_n is the length of the curved part of the boundary. Hence

$$\left|\int_{D_n} \phi(z)\,dz\right| < \epsilon\sqrt{2}l_n(4l_n+s_n).$$

Adding all the parts, we obtain

$$\left|\int_C f(z)\,dz\right| < 4\sqrt{2}\epsilon \sum (l_m^2+l_n^2)+\epsilon\sqrt{2}l \sum s_n. \tag{3}$$

Now $\sum (l_m^2+l_n^2)$ is the area of a region which just includes C, and is therefore bounded; in fact, if $(a, b; \alpha, \beta)$ is a rectangle including C,

$$\sum (l_m^2+l_n^2) \leqslant (b-a)(\beta-\alpha).$$

Also $\sum s_n$ is the length of the contour C. Hence the right-hand side of (3) is less than a constant multiple of ϵ. But the left-hand side is independent of ϵ. It must therefore be zero.

2.34. We have still to prove the assumption which we have made. This is done by the well-known process of subdivision. Suppose that we start with a network of parallel lines at constant distance l. Some of the squares formed by these lines may contain a point z_0 with the required property. We leave these unchanged. The rest we subdivide by lines midway between the previous ones. If there still remain any parts which have not the required property, we subdivide them again in the same way. There are now *a priori* two possibilities. The process may terminate after a finite number of steps, and then the result is obtained; or it may go on indefinitely. In the second case there is at least one region which we can subdivide indefinitely without obtaining the required result. Call this region (boundary included) R_1. After the first subdivision we obtain a part R_2, contained in R_1, with the same property. So we shall have an infinity of regions $R_1, R_2, \ldots, R_n, \ldots$ each contained in the previous one, and for each of which (2) is impossible.

There is a point z_0 common to all the regions R_n; and, since the dimensions of R_n decrease indefinitely, $|z-z_0|<\delta$ if n is sufficiently large (say $n>n_0$) and z is in R_n. But $f(z)$ is analytic at z_0. Hence (2) holds with this z_0 in R_n, if $n>n_0$. We have thus arrived at a contradiction, and so proved the theorem.

2.35. Cauchy's theorem may obviously be extended at once to a closed contour of any of the types defined in § 2.32. It may also be expressed in slightly different forms. Suppose, for example, that z_0 and z_1 are points connected by two different curves C and C' such that C, and C' reversed, together make up a simple closed contour, or a closed contour of one of the other types described in § 2.32. Let $f(z)$ be a function analytic in the whole region between C and C', and on the curves themselves. Then Cauchy's theorem obviously gives

$$\int_C f(z)\,dz = \int_{C'} f(z)\,dz. \tag{1}$$

Suppose again that C is a simple closed contour, and C' another simple closed contour lying entirely inside C. Let $f(z)$ be analytic and one-valued at all points in the ring-shaped region between C and C'. Then

$$\int_C f(z)\,dz = \int_{C'} f(z)\,dz. \tag{2}$$

For we can join C to C' by a straight line l, parallel, say, to the real axis. Then the region between C and C', cut by l, is the inside of a closed contour Γ, formed by C described positively, C' negatively, and l described twice in opposite directions. Now

$$\int_\Gamma f(z)\,dz = \int_C f(z)\,dz - \int_{C'} f(z)\,dz + \int_l f(z)\,dz - \int_l f(z)\,dz.$$

Since the integral round Γ is zero, the result follows.

A similar result holds if there are any finite number of contours C', C'',... inside C, and $f(z)$ is analytic in the region between them. Then

$$\int_C f(z)\,dz = \int_{C'} f(z)\,dz + \int_{C''} f(z)\,dz + \dots. \tag{3}$$

Another important remark is that, for the truth of Cauchy's theorem, it is not necessary that $f(z)$ should be analytic on C, provided that it is analytic inside it and continuous up to and on C. For if $f(z)$ is continuous, it can be shown that

$$\int_C f(z)\,dz = \lim \int_{C'} f(z)\,dz, \tag{4}$$

where C' is a contour inside C and tending to it. It is perhaps not worth while describing how this is to be done in all cases—if C is a circle, C' is a concentric circle, and so on. Now the right-hand side of (4) is zero for all positions of C' inside C. Hence the left-hand side is zero.

2.36. A complex integral as a function of its upper limit. Let $f(z)$ be a function analytic in a region D. Let

$$F(z) = \int_{z_0}^{z} f(w)\, dw,$$

the path of integration being any contour lying entirely inside D. It follows from Cauchy's theorem that the value of $F(z)$ depends on z only, and not on the particular path of integration taken from z_0 to z. Our notation has, of course, anticipated this.

The function $F(z)$ is analytic in D. For

$$F(z+\delta z)-F(z) = \int_{z}^{z+\delta z} f(w)\, dw,$$

where (by Cauchy's theorem) we may suppose the integral to be taken along the straight line from z to $z+\delta z$. Hence

$$\frac{F(z+\delta z)-F(z)}{\delta z} - f(z) = \frac{1}{\delta z} \int_{z}^{z+\delta z} \{f(w)-f(z)\}\, dw.$$

Since $f(z)$ is continuous,

$$|f(w)-f(z)| < \epsilon \qquad (|w-z| < \delta).$$

Hence, if $0 < |\delta z| < \delta$,

$$\left|\frac{F(z+\delta z)-F(z)}{\delta z} - f(z)\right| < \epsilon.$$

This proves that $F(z)$ is analytic, and that its derivative is $f(z)$.

As in the theory of functions of a real variable, we call $F(z)$ the *indefinite integral* of $f(z)$.

Suppose, on the other hand, that we know an analytic function $G(z)$ such that

$$G'(z) = f(z)$$

throughout D. Then

$$\frac{d}{dz}\{F(z)-G(z)\} = 0.$$

Let

$$F(z)-G(z) = X+iY.$$

Then (as in the proof of the Cauchy-Riemann equations)

$$\frac{\partial X}{\partial x}=0, \qquad \frac{\partial X}{\partial y}=0, \qquad \frac{\partial Y}{\partial x}=0, \qquad \frac{\partial Y}{\partial y}=0.$$

Hence X and Y are constant, i.e. $F-G$ is constant. Hence

$$\int_a^b f(z)\,dz = G(b)-G(a).$$

2.37. Integration and differentiation of complex series. *A uniformly convergent series of analytic functions of a complex variable may be integrated term by term along any path in the region of uniform convergence.*

This may be proved precisely as in the case of real functions (using the inequality of § 2.31).

A series of analytic functions may be differentiated term by term at any point inside a region where the differentiated series is uniformly convergent.

This also may be proved in the same way as for real functions. It will, however, be superseded later (§ 2.8) by a much more useful theorem, which is a characteristic achievement of complex function theory, and which has no analogue in the theorems of Chapter I.

2.4. Cauchy's integral. Let $f(z)$ be a function analytic inside and on a simple closed contour C. Let z be any point inside C. Consider the function of w

$$\frac{f(w)}{w-z}.$$

This function is analytic except at $w=z$, where the denominator vanishes. Hence

$$\int_C \frac{f(w)}{w-z}\,dw = \int_\gamma \frac{f(w)}{w-z}\,dw,$$

where γ is any other closed contour inside C and including $w=z$. Let γ be the circle with centre z and radius ρ. Since $f(w)$ is continuous, we can take ρ so small that

$$|f(w)-f(z)|<\epsilon$$

on γ. Then

$$\int_\gamma \frac{f(w)}{w-z}\,dw = f(z)\int_\gamma \frac{dw}{w-z} + \int_\gamma \frac{f(w)-f(z)}{w-z}\,dw.$$

The first term is equal to $2\pi i f(z)$ (§ 2.3, ex. 3). Also, by § 2.31, the modulus of the second term does not exceed

$$\frac{\epsilon}{\rho} \cdot 2\pi\rho = 2\pi\epsilon.$$

Hence
$$\left| \int_C \frac{f(w)}{w-z}\, dw - 2\pi i\, f(z) \right| < 2\pi\epsilon.$$

But the left-hand side is independent of ϵ. It must therefore be zero. Thus

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z}\, dw.$$

This is Cauchy's integral formula. It expresses the value of $f(z)$ at any point inside C in terms of its values on C.

2.41. The derivatives of an analytic function. Let z be any point inside C, and $z+h$ a neighbouring point, also inside C. Then

$$f(z+h) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z-h}\, dw.$$

Subtracting the previous result from this, and dividing by h,

$$\frac{f(z+h)-f(z)}{h} = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z)(w-z-h)}\, dw. \qquad (1)$$

When $h \to 0$ the integrand tends to the limit $f(w)/(w-z)^2$. To prove that we can proceed to the limit under the integral sign, consider the difference:

$$\int_C \frac{f(w)}{(w-z)(w-z-h)}\, dw - \int_C \frac{f(w)}{(w-z)^2}\, dw$$
$$= h \int_C \frac{f(w)}{(w-z)^2(w-z-h)}\, dw. \qquad (2)$$

Suppose that $|f(w)| \leqslant M$ on C, and that the distance from z to C (i.e. the minimum of $|w-z|$ as w describes C) is δ. Let the length of C be L. Then if $|h| < \delta$,

$$\left| \int_C \frac{f(w)}{(w-z)^2(w-z-h)}\, dw \right| \leqslant \frac{ML}{\delta^2(\delta-|h|)}$$

which is bounded as $|h| \to 0$. Hence the right-hand side of (2) tends to zero with $|h|$. Hence, by (1),

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z)^2}\, dw. \qquad (3)$$

This is Cauchy's formula for $f'(z)$.

The existence of $f'(z)$ was, of course, our original hypothesis. But now that we have obtained this formula for it, we can repeat the above process. We have

$$\frac{f'(z+h)-f'(z)}{h}=\frac{1}{2\pi i}\int_C \frac{2w-2z-h}{(w-z)^2(w-z-h)^2}f(w)\,dw$$

and we prove as before that when $h\to 0$ this tends to the limit

$$\frac{2}{2\pi i}\int_C \frac{f(w)}{(w-z)^3}\,dw.$$

Hence $f''(z)$, the derivative of $f'(z)$, exists, and is given by the formula

$$f''(z)=\frac{2}{2\pi i}\int_C \frac{f(w)}{(w-z)^3}\,dw.$$

The argument can obviously be repeated indefinitely; *hence $f(z)$ has derivatives of all orders, the nth being given by the formula*

$$f^{(n)}(z)=\frac{n!}{2\pi i}\int_C \frac{f(w)}{(w-z)^{n+1}}\,dw.$$

2.42. Morera's theorem. This is a sort of converse of Cauchy's theorem.

If $f(z)$ is a continuous function of z in a region D, and if the integral

$$\int f(z)\,dz$$

taken round any closed contour in D is zero, then $f(z)$ is analytic inside D.

In this theorem the precise sense of the word 'contour' does not matter. The result holds even if we restrict ourselves, say, to convex polygons.

Consider the function

$$F(z)=\int_{z_0}^{z} f(w)\,dw.$$

Its value is independent of the path of integration; and

$$\frac{F(z+h)-F(z)}{h}-f(z)=\frac{1}{h}\int_z^{z+h}\{f(w)-f(z)\}\,dw,$$

where the path of integration may be taken to be the straight line. This tends to zero with h, since $f(w)$ is continuous. Hence $F(z)$ is analytic, and has the derivative $f(z)$. But we have just proved that the derivative of an analytic function is analytic. Hence $f(z)$ is analytic.