ON THE STRUCTURE AND COMPLEX ANALYSIS OF DIRICHLET SERIES

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Preface

> "There is a subject in mathematics (it's a perfectly good and valid subject and it's perfectly good mathematics) which is misleadingly called Analytic Number Theory. In a sense, it was born with Riemann who was definitely not a number-theorist; it was carried on, among others, by Hadamard, and later by Hardy, who were also not number-theorists (I knew Hadamard well); and to the best of my understanding, analytic number theory is not number-theory."
> – Andre Weil in "Two Lectures on Number Theory, Past and Present" [99]

This quote, wholly unfair but still remarkable to modern sensibilities, points to the curious interplay between the analytic and algebraic sides of number theory. A number of results that began as curiosities or seemingly random tricks became greatly formalized, becoming a foundation of much of modern number theory.

In the first chapter we provide a general overview of some of the theory and history of Dirichlet series and L-functions, and their relations to other mathematical objects.

The second chapter deals with those results that come from pure complex analysis, neither fundamentally requiring either a Dirichlet series nor a functional equation but using only the regularity of behaviour that these provide. I also discuss the limits of such an approach. The main theorems (1) and (2) restrict the behaviour of an analytic function in two parts of its domain, based on how large the domain itself is. Applied to $\log\zeta(s)$ this provides another proof of Littlewood's Theorem that the spacing of zeta zeros at height T is unconditionally $O(1/\log_3(T))$. While Littlewood's proof is restricted to finding rectangular strips as zero-free regions, this approach explicitly gives a bound given any candidate shape for a zero-free region. The approach also gives more refined results provided information on growth rates in the critical strip. Finally a careful analysis of gamma factor asymptotics is done in (4) to give a bound on low-lying zeros of L-functions, uniform in all of the parameters.

The third chapter studies Dirichlet series that analytically continue with some given control on the growth of the analytic continuation (resembling the growth of degree 1 L-functions). Theorem 7 shows these to exactly correspond to power series that analytically continue slightly outside their disc of convergence. As a corollary this shows that growth rate strictly between degree 0 and 1 is impossible, even not assuming a functional equation. Section 3.2.1 then uses this to show that all Dirichlet series with growth rate bounded by that of the gamma function are continuous linear combinations of Hurwitz zeta functions, and discusses the consequence for the Lindelof hypothesis. Attempts to bring in the Euler product relate to a conjecture of Schwarz from 1978. It is also shown in Theorem 10 how the Dirichlet series having a functional equation yields an analytic continuation of the power series to almost the entire plane and as Corollary 2 this provides a new proof of the classification of degree 1 elements of the extended Selberg class.

The fourth chapter expands on theorem 10 from chapter 3 and in Theorem 12 a degree d functional equation is shown to be equivalent to a generalized Fourier series having analytic continuation and some given singularities. Unlike in the degree 1 case this does not yield a complete classification but in Section 4.2 partial results are presented for degree 2. A method for recovering an eigenbasis is presented, and applied to the space of cusp forms on $\Gamma_0(N)$ via the Eichler-Selberg trace formula.

The fifth chapter discusses Selberg zeta functions, which also have analytic continuation and are known to satisfy a Riemann Hypothesis, but are higher order functions so their complex analysis is very different. Theorem uses spectral bounds to bound the Selberg zeta function on the critical line, similarly to the results of chapter two.

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I also would not have travelled so far on this path without the support of family, nor for the excellent instruction and support of Tom Griffiths while I was in middle school and high school. Finally, I was in the past at least a bit confused by the acknowledgements of significant others, but now I truly understand and appreciate how important it is to have such a patient, supportive and loving girlfriend, Phan Ha.

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Chapter 1

Introduction

Dirichlet series $L(s) := \sum_{n=1}^{\infty} a_n n^{-s}$ are important as generating functions in number theory, similarly to how standard power series $\sum_{n=1}^{\infty} a_n z^n$ are important in combinatorics [100]. However, analytic properties of Dirichlet series are still much more mysterious than the case of power series. Power series have a well-defined radius of convergence. For Dirichlet series past the abscissa of absolute convergence, it is difficult to tell how far the function can extend analytically. It is difficult to recognize if a Dirichlet series defines an entire function, although some results will be shown when there is a specific growth rate in the region of analytic continuation. The space of functions defined by power series on a given domain defined by power series consists of all analytic functions on the domain, and the zero set may be an arbitrary discrete subset of the domain. For Dirichlet series it is unknown whether there even can be a half-plane with a finite number of zeros. It is unclear what possible growth-rates on vertical lines are, what restrictions on the locations of the zeros this places, and how a Dirichlet series with its zeros on a line can behave to the right of that line. Many typical Dirichlet series studied have extra structure in the form of a functional equation, or an Euler product, and often come from integral transforms of certain special functions on $GL(n)$. This brings up the further questions of understanding what types of functional equations are possible, understanding how to characterize functions with a certain functional equation, and the question of how much of the original $GL(n)$ structure can be canonically recovered.

1.1 Historical Background and Some Motivating Problems

The prime number theorem states that the counting function of the primes $\pi(x) := \sum_{p\leq x} 1$ is asymptotically $\frac{x}{\log x}$.

If we define the counting function $\Psi(x) := \sum^*_{p^k\leq x} \log p$ of the prime powers with weight $\log p$ the corresponding result is that $\Psi(x)$ is asymptotically x.

Riemann in 1859 outlined how to prove the prime number theorem based on his zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p \text{ prime}} (1-p^{-s})^{-1}.$$

This series is defined for $\Re(s) > 1$ but the function has analytic continuation to all $s \neq 1$, and satisfies the functional equation

$$\Lambda(s) := \zeta(s)\Gamma(s/2)\pi^{-s/2} = \Lambda(1-s).$$

This functional equation is equivalent to the Poisson Summation Formula for arbitrary test function f,

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \hat{f}(m),$$

in the sense that each can prove the other directly.

Furthermore there is the Von Mangoldt explicit formula

$$\Psi(x) = x - \log(2\pi) - \log(1-x^{-2})/2 - \sum_{\rho} \frac{x^\rho}{\rho}.$$

The sum is over the non-trivial zeros $\rho = \sigma + iT$ of the Riemann Zeta function, and is really a Fourier decomposition with frequencies T and amplitudes x^σ/ρ.

Big open problems, and the limits of the conjectures

There are three particularly long-standing and difficult questions about the Riemann Zeta function.

Definition 1. *(Riemann Hypothesis) All non-trivial zeros of the zeta function have real part* $1/2$*. This is equivalent to having an error bound of* $x^{1/2+\epsilon}$ *in the prime number theorem.*

Definition 2. *(Lindelof Hypothesis)* $\zeta(1/2+iT) \ll |T|^{\epsilon}$*. This is implied by the Riemann Hypothesis.*

Definition 3. *(GUE Conjecture) The (suitably rescaled) zeros of the Riemann Zeta Function are distributed like the eigenvalues of large random matrices from the Gaussian Unitary Ensemble.*

Based on the GUE Conjecture we have very fine conjectures for the zeta zeros. If, however, there were infinitely many Siegel zeros (particular contradictions to the General Riemann Hypothesis) then we would be in the following very different situation.

Definition 4. *(The Alternative Hypothesis) Instead of there being a nice smooth probability measure for the normalized adjacent zero spacings, the limiting distribution (normalized to have mean 1) is supported on integers or half-integers.*

The Alternative Hypothesis is generally not expected to be true, but it demonstrates the limitations of current techniques.

It is expected that all L-functions in the Selberg Class satisfy a Riemann Hypothesis. It would be interesting to determine if there is any analytic function with similar growth rates satisfying a Riemann Hypothesis

Question 1. *Is there an entire function* $f : \mathcal{C} \to \mathcal{C}$ *such that for some* $\delta > 0$ *the following are satisfied.*

1. $f(z)$ *is bounded when* $\Re z > 1 + \delta$*.*
2. $f(z)$ *is unbounded when* $\Re z = 1$*.*
3. $f(z)$ *grows polynomially in vertical strips: for all* σ *there is* $C_\sigma > 0$ *so that* $|f(\sigma + it)| \ll |t|^{C_\sigma}$*.*

4. *(weak RH) $f(z)$ does not vanish when $\Re z > \frac{1}{2}$.*

5. *(strong RH) $f(z)$ vanishes only when $\Re(z) = \frac{1}{2}$ or for $\Im(z)$ in a finite set.*

The road to the Lindelof Hypothesis is often seen to pass through the Riemann Hypothesis it is conceivable that every suitable Dirichlet Series satisfies the Lindelof Hypothesis, which motivates this investigation into the general theory and structure of Dirichlet series.

Question 2. *If $L(s) = \sum a_n n^{-s}$ extends to an entire function of order 1 and the coefficients satisfy $a_n = O(n^\epsilon)$ for every $\epsilon > 0$ is it true that $L(1/2 + iT) = O(T^\epsilon)$ for all $\epsilon > 0$?*

Dirichlet L-functions

In 1837, predating Riemann's work on the Zeta function, Dirichlet introduced his series

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p (1 - \chi(p)p^{-s})^{-1},$$

where χ is a Dirichlet character modulo k, a character of the multiplicative group $\mathbb{Z}/N\mathbb{Z}$ extended to the natural numbers by periodicity.

These share many properties with the Zeta function, and satisfy a functional equation for either $a = 0$ or $a = 1$, depending on parity,

$$\Lambda(s,\chi) := \left(\frac{\pi}{k}\right)^{-(s+a)/2} \Gamma\left(\frac{s+a}{2}\right) L(s,\chi) = \Lambda(1-s,\overline{\chi})\tau(\chi)k^{-1/2}i^{-a}.$$

Here $\tau(\chi) = \sum_{n=1}^{k} \chi(n)\exp(2\pi i n/k)$ and $|\tau(\chi)| = \sqrt{k}$.

These also conjecturally satisfy a Riemann Hypothesis and Lindelof Hypothesis and were most prominently used by Dirichlet to show that there are infinitely many primes congruent to a mod k, as long as a and k are relatively prime. Dirichlet did not study these as analytic functions, so the analytic continuation and functional equation would come later, and fit very nicely into the modern framework of Tate's thesis [93].

As another example of their application, the Miller-Rabin primality test is known to run in polynomial time only if these $L(s,\chi)$ satisfy the Riemann Hypothesis, although the more recent AKS algorithm [2] has unconditional polynomial running time.

Dedekind Zeta Functions

Let K be an algebraic number of field of degree $n = r_1 + 2r_2$ with r_1 real embeddings and $2r_2$ complex embeddings. The Dedekind Zeta function for $K/\mathbb{Q}$ is defined as

$$\zeta_K(s) = \sum F(m)m^{-s} = \sum_{\mathfrak{a}\neq 0}(N\mathfrak{a})^{-s} = \prod_{\mathfrak{p}}(1-(N\mathfrak{p})^{-s})^{-1},$$

where $F(m)$ denotes the number of nonzero integral ideals of norm m in K.

Defining the completed zeta function to be $\Phi(s) := B^s\Gamma(\frac{1}{2}s)^{r_1}\Gamma(s)^{r_2}\zeta_K(s)$, there is the functional equation $\Phi(s) = \Phi(1-s)$.

The zeta functions for imaginary quadratic fields are the same as for modular forms and for elliptic curves. The zeta functions for real quadratic fields give the same functions as one gets from Maass forms.

These can also be placed in a broader context as follows. The field K, as a vector space over $\mathbb{Q}$, is a n-dimensional representation of $\mathrm{Gal}(K/\mathbb{Q})$ under the standard action. The Artin L-function for this representation is the same as ζ_K. In particular if $\mathrm{Gal}(K/\mathbb{Q})$ is abelian then the irreducible representations are 1 dimensional, and the decomposition of the standard representation into irreducibles gives a decomposition $\zeta_K(s) = \zeta(s)\prod L(s,\chi)$, for some collection of $L(s,\chi)$.

Ramanujan τ function

Consider the power series defined by the infinite product,

$$\sum_{n=1}^{\infty}\tau(n)q^n = q\prod_{n\geq 1}(1-q^n)^{24} = \Delta(z).$$

Ramanujan conjectured in 1916 that the coefficients $\tau(n)$ satisfy the following.

- (Multiplicativity) $\tau(mn) = \tau(m)\tau(n)$ if $\gcd(m,n) = 1$.
- $\tau(p^{r+1}) = \tau(p)\tau(p^r) - p^{11}\tau(p^{r-1})$ for p prime and $r > 0$.
- (Ramanujan Hypothesis) $|\tau(p)| \leq 2p^{11/2}$ for all primes p.

The first two conjectures are a consequence of $\Delta(z)$ being the unique (up to a constant multiple) holomorphic cusp form of weight 12 and level 1 and were proved by Mordell in 1917. The third was proved by Deligne in 1974 as a consequence of his proof of the Weil conjectures.

The corresponding Dirichlet series

$$L(s) := \sum_{n=1}^{\infty} \frac{\tau(n)}{n^{s+11/2}}$$

has a degree 2 functional equation

$$\Lambda(s) := L(s)(2\pi)^{11/2-s}\Gamma(s+11/2) = \Lambda(1-s).$$

Relation to summation formulae

The functional equation for $\zeta(s)$ is equivalent to the Poisson summation formula for the Fourier transform, which can be written as

$$\sum_{n\in\mathbb{Z}} \delta_n \stackrel{\mathcal{F}}{\longmapsto} \sum_{n\in\mathbb{Z}} \delta_n.$$

For a primitive Dirichlet character χ mod q, the functional equation for $L(s,\chi)$ is equivalent to the Twisted Poisson summation formula

$$\sum_{n\in\mathbb{Z}} \chi(n)\delta_{n/q} \stackrel{\mathcal{F}}{\longmapsto} \tau\chi \sum_{n\in\mathbb{Z}} \overline{\chi}(-n)\delta_n.$$

The functional Equation for $\zeta(s)^2$ is equivalent to the Voronoi summation formula for the Fourier-Bessel transform, written as

$$\sum_{n\in\mathbb{N}} d(n)\delta_{\sqrt{n}} \overset{\mathcal{FB}}{\longmapsto} \log + 2\gamma + \sum_{n\in\mathbb{N}} d(n)\delta_{\sqrt{n}}.$$

Since it is possible to construct infinitely many degree d Dirichlet series $L(s)$ so that $L(s/d)$ has the same functional equation as the Zeta function [20], the relation gives a twist of Poisson summation formula by coefficients of $L(s)$. Namely, for every d there are complex $a_{n^{1/d}}, b_{n^{1/d}}$ and positive Q so that

$$\sum_{n\in\pm\mathbb{N}^{1/d}} a_{|n|}\delta_n \overset{\mathcal{F}}{\longmapsto} \sum_{n\in\pm\mathbb{N}^{1/d}} b_{|n|}\delta_{Qn}.$$

The degree conjecture then claims that d must be a natural number, which potentially says something interesting about resonances at non-integers. Such sums are sometimes called aperiodic Dirac combs and 1-dimensional quasi-crystals, but are typically studied for support having bounded lower density.

1.2 Axiomatization: the Selberg Class

The class $\mathcal{S}_d$ of Dirichlet Series of degree d in the Selberg class consists of those Dirichlet series $L(s) = \sum a_n n^{-s}$ satisfying the following properties.

1. (Analyticity) $(s-1)^m L(s)$ is an entire function of finite order for some non-negative integer m.

2. (Ramanujan Hypothesis) $a_n \ll n^\epsilon$ for any fixed $\epsilon > 0$.

3. (Functional Equation) There must be a function $\gamma(s)$ of the form $\gamma(s) = \epsilon Q^s \prod_{i=1}^r \Gamma(\lambda_i s + \mu_i)$ where $|\epsilon| = 1$, $Q > 0$, $\lambda_i > 0$, $2\sum_{i=1}^r \lambda_i = d$, and $\Re\mu_i \geq 0$ such that
$$\Lambda(s) := \gamma(s)L(s) = \overline{\Lambda}(1-s).$$

4. (Euler product) $a_1 = 1$ and

$$\log F(s) = \sum_{n=p^k}^{\infty} \frac{b_n}{n^s},$$

where the sum ranges over prime powers and $b_n \ll n^{\theta}$ for some $\theta < 1/2$.

If L and L' are two different elements of the Selberg class that cannot be expressed as products of simpler elements, Selberg made a number of conjectures [87]. There is conjectured to be regularity, $\sum_{p \le x} |a_p|^2/p \sim \log\log x + O(1)$, and orthonormality, $\sum_{p \le x} \frac{a_p a'_p}{p} = O(1)$, of the coefficients at primes. It is conjectured that the Riemann Hypothesis holds for all such L. It is also conjectured that the degree d must be an integer, which is currently known for $d \le 2$ [57]. The first two conjectures imply both the Dedekind conjecture and the Artin conjecture. It is also expected that everything in the Selberg class comes from an automorphic L-function on $GL(n, \mathbb{R})$.

1.3 Recovering greater structure

Hecke's Correspondence Theorem

Let $\{a_n\}$ and $\{b_n\}$ be complex sequences $O(n^c)$ for some $c > 0$. Let $\lambda > 0$, $k \in \mathbb{R}$, $\gamma \in \mathbb{C}$. For $\sigma > c + 1$ define the functions

$$L_a(s) = \sum a_n n^{-s} \quad \text{and} \quad L_b(s) = \sum b_n n^{-s},$$

and the completed functions

$$\Lambda_a(s) = (2\pi/\lambda)^{-s}\Gamma(s)L_a(s) \quad \text{and} \quad \Lambda_b(s) = (2\pi/\lambda)^{-s}\Gamma(s)L_b(s).$$

For $\tau \in \mathcal{H}$, define the corresponding Fourier series,

$$f_a(\tau) = \sum_{n=1}^{\infty} a_n e(n\tau/\lambda) \quad \text{and} \quad f_b(\tau) = \sum_{n=1}^{\infty} b_n e(n\tau/\lambda).$$

Then the following two assertions are equivalent [7].

1. f_a and f_b are involutions of each other: $f_a(\tau) = \gamma(\tau/i)^{-k} f_b(-1/\tau)$.

2. $L_a(s)$ is entire and bounded in vertical strips. Moreover, the completed functions are related by
$$\Lambda_a(s) = \gamma \Lambda_b(k-s).$$

Furthermore, define $M_0(\lambda, k, \gamma)$ to be the set of such self-symmetric functions, $f(\tau) = \sum a_n e(n\tau/\lambda)$ such that

1. $f(-1/\tau) = \gamma(\tau/i)^k f(\tau)$ where $k > 0$ and $\gamma = \pm 1$.

2. $a_n = O(n^c)$ for some c as $n \to \infty$.

This is another way of saying that f is an entire automorphic form of weight k and multiplier γ with respect to $G(\lambda)$, the group of linear transforms generated by $T : \tau \mapsto -1/\tau$ and $S_\lambda : \tau \mapsto \tau + \lambda$.

Hecke [7] is able to fully classify and construct these spaces $M_0(\lambda, k, \gamma)$. First, there is a fundamental domain for the action of $G(\lambda)$ only if $\lambda \geq 2$ or $\lambda = 2\cos(\pi/q)$ with $q \geq 3$ and integer, since otherwise the action is not discrete. If $\lambda > 2$ then $\dim M_0(\lambda, k, \gamma) = \infty$ for every $k > 0$ and $\gamma = \pm 1$. If $\lambda = 2\cos(\pi/q)$ then k must be of the form $\frac{4m}{q-2} + 1 - \gamma$ for $m \in \mathbb{N}$ and

$$\dim M(\lambda, k, \gamma) = 1 + \left[\frac{m + (\gamma - 1)/2}{q} \right]$$

Weil Converse Theorem

In addition to a functional equation for $L(s) := \sum a_n n^{-s}$ suppose that for given N there is an appropriate functional equation for all character twists $L(s, \chi) := \sum a_n \chi(n) n^{-s}$,

$$\Lambda(s, \chi) := (2\pi)^{-s} \Gamma(s) L(s, \chi) = i^k \chi(N) \frac{\tau(\chi)^2}{D} (D^2 N)^{-s+k/2} \overline{\lambda}(k - s, \overline{\chi}),$$

where χ is a character modulo m and m is relatively prime to N. Suppose also that all of this twisted Dirichlet series extend to entire functions for bounded on vertical lines. Then $f(\tau)$ is a modular form of weight k with respect to $\Gamma_0(N)$.

Recovering a Maass form (sketch)

Hecke's Correspondence deals with the case of a gamma factor $\Gamma(s)$, which is what you have for Dedekind Zeta functions of imaginary quadratic fields, and for elliptic curve L-functions. Maass developed the theory of non-holomorphic Maass forms to deal with the situation of the Dedekind Zeta function of a real quadratic field, where the gamma factor is instead $\Gamma(s/2)^2$.

The function $\sum a_n e(nx)$ is periodic but, unlike in the Hecke case, it has no functional equation under the transform $x \mapsto \lambda/x$. The function $\sum a_n K_0(ny)$, on the other hand, has a functional equation under the transform $y \mapsto \lambda/y$ but is not periodic.

Combine these together to form the function (appropriately made to be either even or odd)

$$f(x, y) = \sum_{n \in \mathbb{Z}} a_n e(nx) K_0(2\pi ny)$$

This function is an eigenfunction of the hyperbolic Laplacian.

The involution $(0, y) \mapsto (0, \lambda/y)$ is the restriction to $x = 0$ of the hyperbolic involution $(x, y) \mapsto \left(\frac{-\lambda x}{x^2+y^2}, \frac{\lambda y}{x^2+y^2}\right)$. The involution leaves the hyperbolic Laplacian invariant, so sends $f(x, y)$ to a similarly even or odd function that agrees with $f(x, y)$ on $x = 0$ and satisfies the same second order differential equation, so must be the same function.

In effect we have reconstructed a function on the higher dimensional space $\Gamma \backslash SL_2(\mathbb{R})$.

$\mathbf{SL}_3(\mathbb{Z})$ Converse Theorem

(Jacquet, Piatetski-Shapiro, and Shalika) Consider a Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ with $a_1 = 1$. Define $A(n,m)$ so that $A(n,1) = a_n$ and these coefficients satisfy the multiplicativity relations[36] that result in

$$\sum_{m,n=1}^{\infty} A(m,n) m^{-s} n^{-t} = \frac{L(s)\overline{L}(s)}{\zeta(s+t)}.$$

Suppose that all of the Dirichlet twists

$$\sum_{n=1}^{\infty} A(n,1)\chi(n) n^{-s}$$

satisfy a suitable degree 3 functional equation.

Then

$$\begin{aligned} f(z) \quad = \quad & \sum_{\gamma \in U_2(\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{Z})} \sum_{m_1=1}^{\infty} \sum_{m_2 \neq 0} \frac{A(m_1,m_2)}{m_1|m_2|} \\ & \times W_{\text{Jacquet}} \left(\begin{pmatrix} m1|m_2| & & \\ & m_2 & \\ & & 1 \end{pmatrix} \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z, \nu, \Psi_{1,\frac{m_2}{|m_2|}} \right), \end{aligned}$$

is a Maass form for $\mathrm{SL}(3,\mathbb{Z})$. In other words it is a function on $\mathrm{SL}_3(\mathbb{Z})\backslash \mathrm{SL}_3(\mathbb{R})/O_3$ that is an eigenfunction of the center of the universal enveloping algebra.

1.4 Functional Equations

Since a number of examples of functional equation have been mentioned, it will be good here to introduce the general definitions that will be use throughout this thesis.

Definition: symmetric form

Consider two Dirichlet series $L_a(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ and $L_b(s) = \sum_{n=1}^{\infty} b_n n^{-s}$ absolutely convergent in some right half-plane, where $a_n, b_n \in \mathbb{C}$. Suppose $L_a(s)$ and $L_b(s)$ have analytic continuation to entire function with polynomial growth on vertical lines. Define Gamma-factors

$$
\begin{aligned}
\gamma_a(s) &:= \prod_{i=1}^{r_a} \pi\Gamma(\lambda_{a,i}s + \mu_{a,i}) \\
\gamma_b(s) &:= \prod_{i=1}^{r_b} \Gamma(\lambda_{b,i}s + \mu_{b,i})
\end{aligned}
$$

and suppose that $L_a(s)$ and $L_b(s)$ are related by a functional equation where $\lambda_{a,i}, \lambda_{b,i}, Q_a, Q_b$ all exceed zero.

$$
\begin{aligned}
\Lambda(s) &:= L_a(s)\gamma_a(s)Q_a^s \\
\Lambda(1-s) &= L_b(s)\gamma_b(s)Q_b^{s-1}
\end{aligned}
$$

Definition: non-symmetric form

Equivalently, the functional equation can also be written in the non-symmetric form

$$L_a(s) = L_b(1-s)\gamma(1-s)\theta(1-s)Q^s,$$

where $Q = Q_b/Q_a$, there is the nonsymmetric gamma factor,

$$
\begin{aligned}
\gamma(s) &:= \prod_{i=1}^{r} \Gamma(\lambda_i s + \mu_i) \\
&= \prod_{i=1}^{r_b} \Gamma(\lambda_{b,i}s + \mu_{b,i}) \prod_{i=1}^{r_a} \Gamma(\lambda_{a,i}s + 1 - \lambda_{a,i} - \mu_{a,i}),
\end{aligned}
$$

and a product of sines and cosines that we denote

$$\begin{aligned}\theta(s) &:= \prod_{i=1}^{r_a} \sin\left(\pi\left(s\lambda_i - \mu_i - \lambda_i\right)/2\right) \\ &:= \sum \alpha_j e\left(\omega_j s/4\right),\end{aligned}$$

where $\{\omega_j\} = \{\pm\lambda_1 \pm \lambda_2 \pm \cdots\}$. The degree of the functional equation is defines to be $d := \sum \lambda_i = 2\sum \lambda_{a,i} = 2\sum \lambda_{b,i}$.

Let $q^{-1} = Q\prod_{i=1}^{r} \lambda_i^{\lambda_i}$. Let $\mu = \sum_{i=1}^{r_a}(\mu_i - 1/2)$.

Definition Gamma-like functions

Lemma 6 in Appendix A lead to the useful result for $1 = \sum_{i=1}^{r} \lambda_i$ that

$$\prod_{i=1}^{r} \Gamma(\lambda_i s + \mu_i) = \Gamma(s+\mu) q^s s^{\mu}(A + O(1/s)),$$

where $q = \prod_{i=1}^{r} \lambda_i^{\lambda_i}$, $\mu - 1/2 = \sum_{i=1}^{r}(\mu_i - \frac{1}{2})$, and $A = (2\pi)^{(r-1)/2}\prod \lambda_i^{\mu_i - 1/2}$.

To avoid the technical complication of dealing with these types of products directly, call a meromorphic function $g(s)$ "gamma-like" if $g(s) \sim \Gamma(s)(1 + O(1/s))$ away from poles of both sides.

Definition: Functional equation, fully general

Without loss of generality let $\mu = 0$.

Expanding out $\theta(s)$ and L_b we arrive at the expression

$$\sum a_n n^{-s} = \gamma(1-s)\sum_x c_x x^{1-s},$$

where x ranges over the set Spec $:= ne(\omega_j/4)q$, on the Riemann surface of $\log x$, and when $x = ne(\omega j/4)q$ then $c_x = b_n \alpha_j C$ for some constant C. Analogously to Kaczorowski and Perelli [52] we call this set the Spec of L. The exact form of $\gamma(1-s)$ will not matter and it may be replaced by similarly growing functions that are parametrized by x.

The most general context is convenient since non-linear twists do not retain a standard functional equation but they do retain a functional equation in this more general sense.

Definition 5. *(Most general functional equation) $L(s) = \sum a_n n^{-s}$ for some set Spec, complex b_x bounded by some power of $|x|$, and some uniformly Gamma-like $\gamma_x(s)$ satisfies the equality*

$$L(s) = \sum_{x \in Spec} b_x x^{s-1} \gamma_x(1 - ds + \mu).$$

By translating s we may without loss of generality take $\mu = 0$.

Conjectures

Conjecture 1. *The following is a list of conjectures in strictly increasing order of strength:*

1. *Selberg's degree conjecture that $d \in \mathbb{N}$.*
2. *All ω_j are in $\mathbb{N}$ and have equal parity. Equivalently Spec always lies over the real or imaginary axis, depending on the parity of d.*
3. *All $\lambda_{a,i}$ and $\lambda_{b,i}$ may be taken to be 1/2.*
4. *All λ_i can be taken to be 1.*

Chapter 2

Unconditional zero-spacing results

2.1 Introduction

The analytic theory of the zeta function, $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, was first developed by Riemann [84] who was motivated by its connection to the distribution of prime numbers, and used it to outline a proof of the prime number theorem. This connection is made fully explicit in Weil's Explicit Formula [48] that relates the summation of a test function over the log of prime powers to the summation of its Fourier transform over the zeros of $\zeta(s)$. Thus in theory, if not in practice, perfect knowledge of the zeros gives perfect knowledge of the primes.

The broad distribution of zeros is well understood. Let $N(T)$ count the number of of zeros with imaginary parts in $[0, T]$. Then

$$N(T) = \frac{T}{2\pi} \log\left(\frac{T}{2\pi e}\right) + \frac{7}{8} + S(T) + O(1/T),$$

where $\pi S(T) = \arg(1/2 + iT)$ is obtained by continuous variation on the straight line from $2 + iT$[42]. $S(T)$ is on average 0, meaning that the average gap between zeros is $2\pi/\log T$, and $S(T)$ measures the cumulative discrepancy. The GUE conjecture in Random matrix theory suggests a particularly nice limiting distribution for the normalized zero spacings, but this is very far from what can be proven. Being able to prove that there are infinitely many zero pairs within half of the average spacing would be number theoretically significant but as far as we know the Alternative Hypothesis could be true: the limiting distribution is supported on half-integers [31]. While typical values of $S(T)$ are on the order of $(\log\log T)^{1/2}$[79][86] and extreme values are as large as $\Omega\left(\sqrt{\frac{\log T}{\log\log T}}\right)$[74], $|S(T) < 2$ for $|T| < 6.8 \cdot 10^6$ and the largest value of $|S(T)|$ found computationally is around 3.2 [81]. Given that $S(T)$ jumps by 1 at each zero, this demonstrates how relatively tame $\zeta(s)$ is in the region in which we have been able to compute, particularly given that a counterexample of the Riemann Hypothesis would make $S(T)$ jump by 2.

Unconditionally $S(T) = O(\log T)$ (so the zero spacings are bounded) and conditional on the Riemann Hypothesis Littlewood showed [65] that $S(T) = O(\log T/\log\log T)$ so that the zero spacings are $O(1/\log\log T)$. This bound has subsequently been improved to $|S(T)| \leq (\frac{1}{2} + o(1))\frac{\log T}{\log\log T}$ by Goldston and Gonek[42], and to $|S(T)| \leq (\frac{1}{4} + o(1))\frac{\log T}{\log\log T}$ by Carneiro, Chandee, and Milinovich.

In his 1924 paper, "Two notes on the Riemann Zeta-function", Littlewood [66] proved that the maximal spacing between consecutive zeros at height T is bounded by $\frac{16}{\log\log\log T}$. Remarkably this is purely a result in complex analysis and not specific to the Zeta function, requiring as its only inputs that $\zeta(s)$ is bounded for $\Re(s) > 2$, of known size for $\Re(s) < -1$, not growing exponentially in the critical strip, and providing as its output that the domain of $\log\zeta(s)$ cannot be too large, and hence that zeros cannot be too spread out. Newer proofs are provided in [60] and [95], with [94] being a good reference. All of these proofs are based on a mix of the Three Circles Theorem, Borel-Caratheodory, and either a study of particular conformal maps or some cleverness with piecing together bounds on various discs.

Littlewood's result is outlined in section 1.1.

In Section 2 I give a new proof, essentially a continuous hybrid of the above proofs. Rather than being restricted to rectangular domains, the result constrains the modulus of an analytic function at two arbitrary parts of an arbitrary domain, based on how constrained the path between these is. Applied to a horizontal rectangular strip this gives the same result as above. Conditional on RH we can take a larger domain and determine a maximal spacing of $\ll \frac{1}{(\log\log T)^A}$. Extra information on the modulus of the function can be used as well, for example assuming that

$|\log \zeta(1/2 + iT)| \ll (\log T)^{1-\epsilon}$ the maximal spacing is $\ll \frac{1}{(\log \log T)}$ unconditionally, and $\ll (\log T)^{-C}$ for some C conditional on the Riemann Hypothesis.

In section 3 I apply Littlewood's result to get a uniform bound for low-lying zeros in the Selberg class. While more or less a direct application, previously published bounds have all had implicit constants dependent on the degree or on the shape of the functional equation, so would not have worked. Instead of T the pertinent parameter becomes $q^{1/d}$ where d is the degree and q is essentially the analytic conductor. In particular the lowest lying zero goes to the real axis as long as $q^{1/d} \to \infty$. Without making any further assumptions this is the best possible result, since high powers of an L-function have ever increasing q but have their zeros bounded away from the real axis. It is still plausible that for primitive L-functions the lowest lying zero is $o(1)$ as $q \to \infty$, but this would be far more subtle to prove. For example, if the smallest natural number to not split in a number field is very large then the L-function looks like a power of zeta and the approximate functional equation shows that there are no zeros near the real axis.

2.1.1 Littlewood's Proof for ζ

Following [95] I will first outline the simpler proof that the gaps between zeros are bounded. This result follows directly from the formula for $N(T)$ and so is not interesting in its own right, but it introduces the main techniques and leads directly into the stronger result. Consider a system of four concentric discs C_1, C_2, C_3, C_4 centered at $3 + iT$ and with radii 1,4,5, and 6 respectively. Suppose $\zeta(s)$ does not vanish in or on C_4. Then $\log \zeta(s)$ is regular in C_4. Let M_1, M_2, M_3 be its maximum modulus on C_1, C_2, and C_3 respectively. Now, $\Re \log \zeta(s) \ll \log T$ in C_4. The Borel Caratheodory theorem gives $M_3 \ll \log T$. But $M_1 \ll 1$ so Hadamard's Three-Circles Theorem applied to C_1, C_2, C_3 gives $M_2 \ll \log^{0.9} T$ so in particular $\zeta(-1 + iT) \ll T^{\epsilon}$, but from the functional equation we know $|\zeta(-1 + iT)| \gg T^{3/2}$. This gives a contradiction.

Taking the existence of a conformal map between rectangles and disks almost immediately shows the spacings to be $o(1)$, and a careful investigation of this map allowed Littlewood to prove the following Lemma.

Lemma 1. *Let Γ_1, Γ_2, Γ_4 be three concentric, similar, and similarly oriented rectangles, the ratio of the sides of any one of them being v, and the dimensions of Γ_2 and Γ_4 being respectively 2 and 4 times those of Γ_1. Suppose now that a function f is regular, and satisfies the inequality $\Re f \leqslant P$ in the interior of Γ_4, and satifies $|f| \leqslant M_1$ on the boundary of Γ_1. Then if $v > 100$,*

$$|f| \;\leqslant\; (M_1 + P)\, e^{v} \left(\frac{M_1}{M_1 + P}\right)^{e^{-v}}$$

on the boundary of Γ_2.

Take the rectangles to be axis-aligned, having center $5+iT$, and top left-corners $2+i\,(T+h)$, $-1+i\,(T+h)$, $-7+i\,(T+h)$ respectively. If ζ does not vanish on Γ_4 then taking f to be $\log\zeta$ we have $M_1 \asymp 1$ and $P \sim 7.5\log T$. Also, $|\log\zeta\,(-1+iT)\,| > \log T$, so in the lemma (with $v = 3/h$):

$$\log T \;\ll\; \log{(T)}\, e^{v} \log{(T)}^{-e^{-v}} \ll e^{v} \log{(T)}^{1-e^{-v}}\,.$$

Hence $e^{v} \gg e^{\log(\log(T))e^{-v}}$, so $v \gg \log\log\log T$. Making the constants effective gives $h \leq \frac{16}{\log\log\log T}$.

2.2 Growth rates of analytic functions

In the following formulae, c_k will refer to a constant, and $c_k(\cdots)$ will refer to a constant depending only on the specified parameters.

Theorem 1. *Let f be an analytic function on some domain $\mathcal{D}$. Suppose that for $x \in \mathcal{D}$, $\mathcal{D}$ contains a ball around x of radius $r_x > 0$. Let M be the maximum of $\log|f|$ on the domain. Let $m(x,r)$ denote the maximum of $\log|f|$ on a disc of radius r around x. Then for $x, y \in \mathcal{D}$ and $0 < \alpha < 1$:*

$$\left|\log\left(\frac{M - m(y,\alpha r_y)}{M - m(x,\alpha r_x)}\right)\right| \leq d_\alpha(x,y),$$

where $d_\alpha(x,y) = \frac{1+\alpha}{\alpha\log(1/\alpha)} \int_x^y \frac{1}{r_t}|dt|$.

Essentially this says that for all x and y, $f(x)$ and $f(y)$ must be equally close to the maximum modulus or equally far from the maximum modulus unless all paths from x to y go through a narrow section of the domain (a pinch-point). Taking $\alpha = 1/3$ makes the constant $\frac{1+\alpha}{\alpha \log(1/\alpha)}$ on the right hand side just under 4.

Proof. Let m_x and m_r denote $\frac{\partial m}{\partial x}$ and $\frac{\partial m}{\partial r}$ respectively. By the maximum principle applied to concentric circles: $m_r > 0$. By the maximum principle applied to tangent circles: $m_r \pm m_x > 0$, so $|m_x| < m_r$, also $\frac{\partial r_x}{\partial x} \leq 1$ [1]. Hadamards Three Circle Theorem states that $m(x,r)$ is convex as a function of $\log r$, namely $\frac{\partial m}{\partial \log r}(x,r) = m_r(x,r) \cdot r$ is a non-decreasing function in r.

By definition of M,

$$M \geq m(x, r_x).$$

By the fundamental theorem of calculus this is

$$M \geq m(x,r) + \int_r^{r_x} m_r(x,t)dt,$$

and then bringing in the Three Circle bound this is

$$M \geq m(x,r) + \int_r^{r_x} m_r(x,r) \cdot \frac{r}{t} dt$$

which equals

$$M \geq m(x,r) + m_r(x,r) r \log(r_x/r).$$

Rearranging, this tells us that

$$m_r(x,r) \leq \frac{M - m(x,r)}{r \log(r_x/r)}$$

and specifically,

$$m_x(x, \alpha r_x) \leq m_r(x, \alpha r_x) \leq \frac{M - m(x, \alpha r_x)}{\alpha r_x \log(1/\alpha)}.$$

[1] Note that the derivative may not exist, so this should be viewed as a bound on the derivates: $\limsup_{h \to 0} \left| \frac{f(x+h,r) - f(x)}{h} \right| < m_r(x,r)$

Comparing this bound to the following derivative we find

$$-\frac{\partial}{\partial x}\log(M - m(x,\alpha r_x) = \frac{m_x(x,\alpha r_x) + m_r(x,\alpha r_x)\alpha\frac{\partial}{\partial x}r_x}{M - m(x,\alpha r_x)} \leq \frac{1+\alpha}{\alpha\log(1/\alpha)r_x}.$$

And the result follows by integrating both sides:

$$\log(M - m(y,\alpha r_y)) - \log(M - m(x,\alpha r_x)) \geq -\frac{1+\alpha}{\alpha\log(1/\alpha)}\int_x^y \frac{1}{r_t}dt.$$

Similarly

$$\log(M - m(y,\alpha r_y)) - \log(M - m(x,\alpha r_x)) \leq \frac{1+\alpha}{\alpha\log(1/\alpha)}\int_x^y \frac{1}{r_t}dt.$$

□

When applying this to $\log\zeta(s)$ it will be convenient to have a form of this theorem that requires a maximum for $\Re f$ on the domain and not a maximum for $|f|$. Rather than going through the Borel-Caratheodory Theorem, a tighter bound can be derived by composing with the Mobius transform between half planes and discs.

Theorem 2. *Let f be an analytic function on some domain $\mathcal{D}$. Suppose that for $x \in \mathcal{D}$, $\mathcal{D}$ contains a ball around x of radius $r_x > 0$. Let $P > 0$ be the supremum of $\Re f$ on the domain. Let $m(x,r)$ denote the maximum of $\log|f|$ on a disc of radius r around x. Then for $x, y \in \mathcal{D}$:*

$$\Re f(y) \leq P\left(\frac{1}{P}\exp(m(x,\alpha r_x))\right)^{e^{-d}},$$

where $d = d_\alpha(x,y) = \frac{1+\alpha}{\alpha\log(1/\alpha)}\int_x^y \frac{1}{r_t}|dt|$. Thus, as long as $\Re f(y) > 0$, we may extract a bound on d:

$$d \geq \log\left(\frac{\log(P) - m(x,\alpha r_x)}{\log(P/\Re f(y))}\right).$$

Proof. Let $g(z) = f(z)/(2P - f(z))$, chosen so that $|g| \leq 1$ on $\mathcal{D}$. Let $d = \frac{1+\alpha}{\alpha\log(1/\alpha)}\int_x^y \frac{1}{r_t}dt$. Then applying Theorem 1 to g (using m_f and m_g to refer to suprema of f and g respectively) yields

$$\log(-m_g(y,\alpha r_y)) \geq \log(-m_g(x,\alpha r_x)) - d.$$

Exponentiated this gives

$$m_g(y,\alpha r_y) \leq m_g(x,\alpha r_x)e^{-d},$$

so therefore

$$|g(y)| \leq \exp(m_g(y,\alpha r_y)) \leq \exp(m_g(x,\alpha r_x)e^{-d}).$$

Note that $|g(z)| \leq |f(z)|/P$ so $m_g \leq m_f - \log P$ and so

$$|g(y)| \leq \left(\frac{1}{P}\exp(m_f(x,\alpha r_x))\right)^{e^{-d}}.$$

Now, $f(z) = 2P \cdot \frac{g(z)}{g(z)+1}$ so

$$\Re f(y) \leq |f(y)| \quad \leq \quad P|g(y)| \leq P\left(\frac{1}{P}\exp(m_f(x,\alpha r_x))\right)^{e^{-d}}.$$

□

In order to apply this to $\zeta(s)$ we will need the following bounds from chapter 5 of [95], which come from the functional equation.

Lemma 2. *When $\sigma > 1.5$, $\zeta(\sigma+iT) \asymp 1$. When $\sigma < -0.5$, $\zeta(\sigma+iT) \asymp T^{1/2+\sigma}$. When $\sigma \geq -0.5$, $\zeta(\sigma+iT) = O(T)$.*

Theorem 3. *There is a constant C such that every horizontal strip $T-h \leq \Im(s) \leq T+h$ contains a zero of $\zeta(s)$ as long as $h > C/\log\log\log T$.*

More generally, any zero-free region $\mathcal{D}$ of $\zeta(s)$ must satisfy $d_{1/4}(-1+iT, 2+iT) \gg \log\log\log T$.

Proof. For given T let $f(s) = \log\zeta(s+iT)$. Suppose f is analytic on the rectangle defined by $|\Im(s)| < h = o(1/\log T)$ and $|\Re(s)| < 3$. Following the notation of Theorem 2, we bound r_x, P, $\Re f$ and m, in order to apply the theorem.

By definition, $r_x = h$ when $-1 \leq x \leq 2$. For some constants c_1, c_2, c_3, c_4 and for h sufficiently small, lemma 2 implies that $c_3 < m(2, h/4) < c_4$, that $\Re f(-1) \sim c_2 \log T$, and that $P \sim \sum c_1 \log T$.

Theorem 2 then concludes that

$$18/h \geq d_{1/4}(2, -1) \geq \log\left(\frac{\log\log T + C}{c_1 - c_2}\right) \sim \log\log\log(T).$$

$\square$

Assuming the Riemann Hypothesis, instead of a rectangle one may take a bowtie-shaped domain such that $r_{x+1/2}\ h + c|x|$ for some $0 < c < 1$ and all $-1 \leq x + 1/2 \leq 2$. Then

$$\int_{-1}^{2} \frac{1}{r_t} dt = 2 \int_{0}^{1.5} \frac{1}{h + ct} dt = 2 \log(h + ct)/c|_0^{1.5} \sim 2 \log h,$$

so therefore

$$\log\log\log T \ll d_{1/4}(2, -1) \asymp \log h,$$

and we can conclude that for some C

$$h \ll (\log\log T)^{-C}.$$

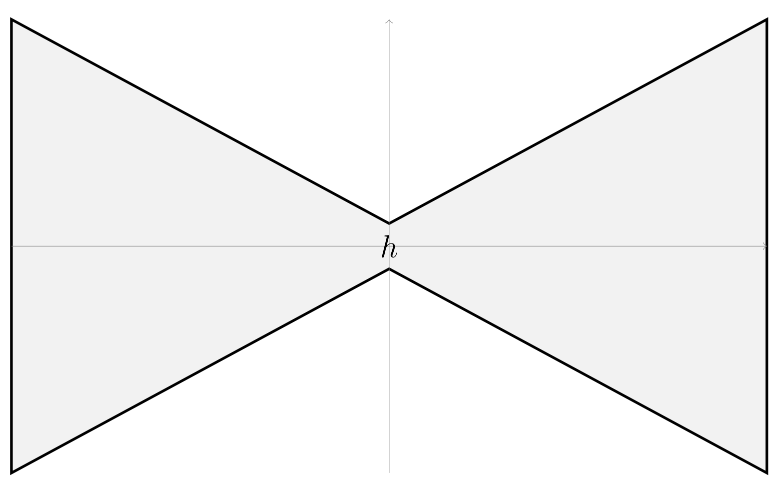

Figure 2.1: A "bowtie" domain

These results can also be strengthened given bounds on $\zeta(s)$ in the critical strip.