Lemma 8.3. *Let v, u be two functions in $\mathcal{S}(\mathbb{R})$. One has for every square-free integer N the identity*

$$\left(v \mid \Psi\left(\mathfrak{T}_N\right) u\right) = \sum_{T|N} \mu(T) \sum_{j,k \in \mathbb{Z}} \overline{v}\left(Tj + \frac{k}{T}\right) u\left(Tj - \frac{k}{T}\right). \tag{8.7}$$

Proof. Together with the operator $2i\pi\mathcal{E}$, let us introduce the operator $2i\pi\mathcal{E}^{\natural} = r\,\frac{\partial}{\partial r} - s\,\frac{\partial}{\partial s}$ when the coordinates (r, s) are used on $\mathbb{R}^2$. One has if $\mathcal{F}_2^{-1}$ denotes the inverse Fourier transformation with respect to the second variable $\mathcal{F}_2^{-1}\left[(2i\pi\mathcal{E})\,\mathfrak{S}\right] = \left(2i\pi\mathcal{E}^{\natural}\right)\mathcal{F}_2^{-1}\mathfrak{S}$ for every tempered distribution $\mathfrak{S}$. From the relation (4.11) between $\mathfrak{T}_N$ and the Dirac comb, and Poisson's formula, one obtains

$$\mathcal{F}_2^{-1}\mathfrak{T}_N = \prod_{p|N}\left(1 - p^{-2i\pi\mathcal{E}^{\natural}}\right)\mathcal{F}_2^{-1}\mathcal{D}ir = \sum_{T|N} \mu(T)\, T^{-2i\pi\mathcal{E}^{\natural}}\mathcal{D}ir, \tag{8.8}$$

explicitly

$$\left(\mathcal{F}_2^{-1}\mathfrak{T}_N\right)(r,\, s) = \sum_{T|N} \mu(T) \sum_{j,k\in\mathbb{Z}} \delta\left(\frac{r}{T} - j\right)\delta(Ts - k). \tag{8.9}$$

The integral kernel of the operator $\Psi\left(\mathfrak{T}_N\right)$ is (2.2)

$$\begin{aligned} K(x,\, y) &= \frac{1}{2}\left(\mathcal{F}_2^{-1}\mathfrak{T}_N\right)\left(\frac{x+y}{2},\, \frac{x-y}{2}\right) \\ &= \sum_{T|N} \mu(T) \sum_{j,k\in\mathbb{Z}} \delta\left(x - Tj - \frac{k}{T}\right)\delta\left(y - Tj + \frac{k}{T}\right). \end{aligned} \tag{8.10}$$

The equation (8.7) follows. □

We now take advantage of (8.2) and Lemma 8.2, after the proof of which we have seen that one could choose $\widetilde{u}(y) = u(y(1 - 2R^2) + 2aN^2)$, with any $a \in \mathbb{Z}$, to obtain a quick verification of the main feature of (7.6). As a particular case of the conditions in Theorem 7.2, we assume that $0 < x^2 - y^2 < 8$ when $v(x)u(y) \neq 0$, say that v is supported in $[2, \sqrt{8}]$ and u in $[0, 1]$, and that N is large enough. From (8.7), one obtains

$$\begin{aligned} &\left(v \mid \Psi\left(Q^{2i\pi\mathcal{E}}\mathfrak{T}_N\right) u\right) = \mu(Q) \sum_{T|N} \mu(T) \sum_{j,k\in\mathbb{Z}} \overline{v}\left(Tj + \frac{k}{T}\right)\widetilde{u}\left(Tj - \frac{k}{T}\right) \\ &= \mu(Q) \sum_{T|N} \mu(T) \sum_{j,k\in\mathbb{Z}} \overline{v}\left(Tj + \frac{k}{T}\right) u\left(\left(Tj - \frac{k}{T}\right)(1 - 2R^2) + 2aN^2\right). \end{aligned} \tag{8.11}$$

Let x and y be the arguments of $\overline{v}$ and u in the last expression. Note that that of u is truly defined mod $2N^2$, while no such proviso is made about v. One has $u(y) \neq 0$ for at most one value of a, which we choose so as to have $x^2 - y^2 = 4$, not only $x^2 - y^2 \equiv 4 \bmod 2N^2$. Then,

$$\begin{aligned} 1 = \frac{x^2-y^2}{4} = \frac{x-y}{2} \times \frac{x+y}{2} \\ &= \frac{1}{2}\left[Tj + \frac{k}{T} - \left(Tj - \frac{k}{T}\right)(1-2R^2) - 2aN^2\right] \\ &\times \frac{1}{2}\left[Tj + \frac{k}{T} + \left(Tj - \frac{k}{T}\right)(1-2R^2) + 2aN^2\right] \\ = \left[\frac{k}{T} + R^2\left(Tj - \frac{k}{T}\right) - aN^2\right]\left[Tj - R^2\left(Tj - \frac{k}{T}\right) + aN^2\right]. \end{aligned} \tag{8.12}$$

With $\alpha = \frac{x-y}{2}$, $\beta = \frac{x+y}{2}$, one has, as a congruence mod $2N^2$,

$$Q\,\beta \equiv Q\left[(1-R^2)Tj + \frac{R^2k}{T}\right] \equiv (1-R^2)QT_j + \frac{NRk}{T}, \tag{8.13}$$

an integer since $T|N$. Writing $R^2 = 1 + 2\lambda Q^2$, so that

$$\alpha \equiv \frac{(1-R^2)k}{T} + R^2Tj = -\frac{2\lambda Q^2 k}{T} + R^2Tj, \tag{8.14}$$

one sees in the same way that $R\alpha \in \mathbb{Z}$.

Since $\alpha\beta \equiv 1$, the numbers $m = Q\beta$ and $n = \frac{R}{\beta}$ are integers. Setting $m = m_1m_2$ and $n = n_1n_2$ with $m_1, n_1|R$ and $m_2, n_2|Q$, the equation $mn = QR$ yields $m_1n_1 = \pm R$, $m_2n_2 = \pm Q$ and $\beta = \frac{R}{n} = \pm\frac{m_1n_1}{n_1n_2} - \pm\frac{m_1}{n_2}$. In other words, $\beta = \frac{R_1}{Q_2}$ with $R_1|R$ and $Q_2|Q$. This is an equality, not just a congruence, since adding to R_1 a multiple of $2N^2$ would put β out of the interval where it must lie to contribute a nonzero term.

This is only a quick verification of Theorem 7.2 (under the condition $R \equiv 1 \bmod 2Q^2$), not a complete second proof since recovering the integers T, j, k from the set $\{R, Q, R_1, Q_2\}$ looks like a complicated task. In particular, we have not verified, here, that the coefficient of $\overline{v}\left(\frac{R_1}{Q_2} + \frac{Q_2}{R_1}\right) u\left(\frac{R_1}{Q_2} - \frac{Q_2}{R_1}\right)$ is $\mu(R_1Q_1) = \mu(Q)\mu(R_1Q_2)$. Theorem 7.2, based on (6.4), gives the coefficients in full.

9. A series $F_\varepsilon(s)$ and its integral version

Definition 9.1. Let $\varepsilon > 0$ be fixed. We associate to any pair v, u of functions in $\mathcal{S}(\mathbb{R})$ the function

$$F_\varepsilon(s) \colon = \sum_{Q \in \mathrm{Sq}^{\mathrm{odd}}} Q^{-s} \left(v \,\Big|\, \Psi\left(Q^{2i\pi\mathcal{E}} \mathfrak{T}_{\frac{\infty}{2}} \right) u_Q \right), \tag{9.1}$$

with $u_Q(y) = Q^{\frac{\varepsilon}{2}}\, u(Q^\varepsilon y)$ (we might have denoted it as u_{Q^ε}).

Theorem 9.2. *Let $v, u \in C^\infty(\mathbb{R})$, with v supported in $[2, \sqrt{8}]$ and u supported in $[0,1]$. Then, the function $F_\varepsilon(s)$ is entire. For* $\mathrm{Re}\, s > 2$*, $F_\varepsilon(s)$ converges as $\varepsilon \to 0$ towards the function $F_0(s)$ introduced in* (5.2).

Proof. When $v(x)u(y) \neq 0$ one has $0 < x^2 - y^2 < 8$, which will make it possible, in the proof of Proposition 9.3, to use the simple integral expression (3.18) of $(v \,|\, \Psi(\mathfrak{E}_{-\nu})\, u)$.

We use Theorem 7.2. Under the given support assumptions, one can rewrite (7.6) as

$$\left(v \,\Big|\, \Psi(Q^{2i\pi\mathcal{E}} \mathfrak{T}_{\frac{\infty}{2}})\, u \right) = \sum_{Q_1 Q_2 = Q} \mu(Q_1) \sum_{\substack{R_1 \in \mathrm{Sq}^{\mathrm{odd}} \\ (R_1, Q) = 1}} \mu(R_1)\, \overline{v}\left(\frac{R_1}{Q_2} + \frac{Q_2}{R_1} \right) u\left(\frac{R_1}{Q_2} - \frac{Q_2}{R_1} \right). \tag{9.2}$$

Indeed, given Q, $\left(v \,\Big|\, \Psi(Q^{2i\pi\mathcal{E}} \mathfrak{T}_{\frac{\infty}{2}})\, u \right)$ coincides according to (4.18) with $\left(v \,\big|\, \Psi(Q^{2i\pi\mathcal{E}} \mathfrak{T}_N)\, u \right)$ for N odd divisible, for some $\beta > 0$ depending on the supports of v and u, by all odd primes $< \beta Q$: we shall take $\beta = \frac{1+\sqrt{8}}{2}$ and fix N accordingly. Now, given R_1 squarefree odd, the condition that R_1 divides some number R with the property that $N = RQ$ is squarefree odd implies the condition $(R_1, Q) = 1$. In the reverse direction, the condition $0 < \frac{R_1}{Q_2} - \frac{Q_2}{R_1} < 1$ implies $R_1 < \frac{1+\sqrt{5}}{2}\, Q$. This implies, since R_1 is squarefree, that $R_1 | N$ or, if $(R_1, Q) = 1$, that $R_1 | R$.

Then, for $\mathrm{Re}\, s > 2$,

$$F_\varepsilon(s):\ = \sum_{Q \in \mathrm{Sq}^{\mathrm{odd}}} Q^{-s} \left(v \,\middle|\, \Psi\left(Q^{2i\pi\mathcal{E}} \mathfrak{T}_{\frac{\infty}{2}} \right) u_Q \right) = \sum_{Q \in \mathrm{Sq}^{\mathrm{odd}}} Q^{-s+\frac{\varepsilon}{2}} \sum_{Q_1 Q_2 = Q} \mu(Q_1)$$
$$\sum_{\substack{R_1 \in \mathrm{Sq}^{\mathrm{odd}} \\ (R_1, Q) = 1}} \mu(R_1)\, \overline{v}\left(\frac{R_1}{Q_2} + \frac{Q_2}{R_1} \right) u\left(Q^\varepsilon \left(\frac{R_1}{Q_2} - \frac{Q_2}{R_1} \right) \right). \quad (9.3)$$

For all nonzero terms of this series, one has for some absolute constant C

$$\left| \frac{R_1}{Q_2} - \frac{Q_2}{R_1} \right| \leq Q^{-\varepsilon}, \quad \left| \frac{R_1}{Q_2} - 1 \right| \leq C\, Q^{-\varepsilon}, \quad |R_1 - Q_2| \leq C\, Q_2 Q^{-\varepsilon}. \quad (9.4)$$

The number of available R_1 is at most $C\, Q_2 Q^{-\varepsilon}$. On the other hand,

$$\left| \frac{R_1}{Q_2} + \frac{Q_2}{R_1} - 2 \right| \leq \left| \frac{R_1}{Q_2} - 1 \right| + \left| \frac{Q_2}{R_1} - 1 \right| \leq C\, Q^{-\varepsilon}. \quad (9.5)$$

Since $v(x)$ is flat at $x = 2$, one has for every $M \geq 1$ and some constant C_M

$$\left| \left(v \,\middle|\, \Psi\left(Q^{2i\pi\mathcal{E}} \mathfrak{T}_N \right) u_Q \right) \right| \leq C C_M\, Q^{\frac{\varepsilon}{2}} \sum_{Q_1 Q_2 = Q} (Q_2 Q^{-\varepsilon})\, Q^{-M\varepsilon}$$
$$\leq C C_M C_2\, Q^{\frac{\varepsilon}{2} + 1 - (M+1)\varepsilon + \varepsilon'} = C C_M C_2\, Q^{1 - (M + \frac{1}{2})\varepsilon + \varepsilon'}, \quad (9.6)$$

where $C_2\, Q^{\varepsilon'}$ is a bound (ε' can be taken arbitrarily small) for the number of divisors of Q. The theorem follows.

□

We wish now to compute $F_\varepsilon(s)$ by analytic methods. The novelty, in comparison to the analysis made in Section 5, is that we have replaced u by u_Q: in just the same way as the distribution $\mathfrak{T}_{\frac{\infty}{2}}$ was decomposed into Eisenstein distributions, it is u that we must now decompose into homogeneous components. Let us recall (Mellin transformation, or Fourier transformation up to a change of variable) that functions on the line decompose into generalized eigenfunctions of the self-adjoint operator $i\left(\frac{1}{2} + y\frac{d}{dy}\right)$, according to the decomposition (analogous to (3.10))

$$u = \frac{1}{i} \int_{\mathrm{Re}\,\mu = 0} u^\mu \, d\mu, \quad (9.7)$$

with

$$u^\mu(y) = \frac{1}{2\pi} \int_0^\infty \theta^{\mu - \frac{1}{2}} u(\theta y)\, d\theta. \quad (9.8)$$

The function u^μ is homogeneous of degree $-\frac{1}{2}-\mu$. Note that μ is in the superscript position, in order not to confuse u^μ with a case of u_Q. A justification of (9.7), (9.8) is as follows.

Given $u = u(y)$ in $\mathcal{S}(\mathbb{R})$ and $y \neq 0$, consider the function $f(t) = e^{\pi t} u\left(e^{2\pi t} y\right)$. The functions f and $\widehat{f}$ are integrable. Define

$$u^{i\lambda}(y) \colon = \widehat{f}(-\lambda) = \frac{1}{2\pi} \int_0^\infty \theta^{i\lambda - \frac{1}{2}}\, u(\theta y)\, d\theta. \tag{9.9}$$

Then, $u(y) = f(0) = \int_{-\infty}^{\infty} \widehat{f}(-\lambda)\, d\lambda = \frac{1}{i} \int_{\mathrm{Re}\,\mu=0} u^\mu\, d\mu$.

Proposition 9.3. *Let $v,\, u \in C^\infty(\mathbb{R})$ satisfy the support conditions in Theorem 9.2. For every $\nu \in \mathbb{C}$, the function*

$$\mu \mapsto \Phi(v,\, u;\, \nu,\, \mu) \colon = \int_0^\infty t^{\nu-1}\, \overline{v}(t+t^{-1})\, u^\mu(t-t^{-1})\, dt, \tag{9.10}$$

initially defined for $\mathrm{Re}\,\mu > -\frac{1}{2}$, extends as an entire function, rapidly decreasing in vertical strips. One has

$$(v \,|\, \Psi(\mathfrak{E}_{-\nu})\, u) = \frac{1}{i} \int_{\mathrm{Re}\,\mu=0} \Phi(v,\, u;\, \nu,\, \mu)\, d\mu. \tag{9.11}$$

Proof. When $\overline{v}(t+t^{-1}) \neq 0$, t is bounded and bounded away from zero. The last factor of the integrand of (9.10) has a singularity at $t=1$, taken care of by the fact that $(t-t^{-1})^2 \geq C^{-1}\,(t+t^{-1}-2)$ for $t^{\pm 1}$ bounded, while $v = v(x)$ is flat at $x=2$. Powers of $(1+|\mu|)^{-1}$ can be gained with the help of repeated integrations by parts associated to the identity

$$(-\frac{1}{2}-\mu)\, u^\mu(t-t^{-1}) = (t+t^{-1})^{-1}\, t\frac{d}{dt}\left(u^\mu(t-t^{-1})\right). \tag{9.12}$$

Let us prove (9.11). Using (9.7), one has

$$\frac{1}{i} \int_{\mathrm{Re}\,\mu=0} \Phi(v,\, u;\, \nu,\, \mu)\, d\mu = \int_0^\infty t^{\nu-1}\, \overline{v}(t+t^{-1})\, u(t-t^{-1})\, dt. \tag{9.13}$$

The right-hand side is the same as $(v \,|\, \Psi(\mathfrak{E}_{-\nu})\, u)$ according to Theorem 3.4.

The equations (9.7) and (9.11) do not imply, however, that $\Phi(v,\, u;\, \nu,\, \mu)$ is the same as $(v \,|\, \Psi(\mathfrak{E}_{-\nu})\, u^\mu)$, because the pair v, u^μ does not satisfy the support conditions in the second part of Theorem 3.4. Actually, as seen from the proof of Theorem 3.4,

$$\Phi(v,\, u;\, \nu,\, \mu) = (v \,\big|\, \Psi(\mathfrak{S}_1)\, u^{\mu}), \tag{9.14}$$

where $\mathfrak{S}_1(x,\, \xi) \;=\; |x|^{\nu-1} \exp\left(-\frac{2i\pi\xi}{x}\right)$ is the term corresponding to the choice $r = 1$ in the Fourier expansion (3.18) of $\mathfrak{E}_{-\nu}$. Such a reduction has been made possible by the demands made on the supports of v, u

□

We shall also need to prove that, with a loss tempered by a power of $|\mu|$, the function $\nu \mapsto \Phi(v,\, u;\, \nu,\, \mu)$ is integrable on lines $\mathrm{Re}\,\nu = c$ with $c > 1$.

Proposition 9.4. *Let $v = v(x)$ and $u = u(y)$ be two functions satisfying the conditions in Theorem 9.2. Defining the operators, to be applied to v, such that*

$$D^{\mu}_{-1} v = v'', \quad D^{\mu}_{0} = -2\mu\left(xv' + \frac{v}{2}\right), \qquad D^{\mu}_{1} v = \left(\frac{1}{2} + \overline{\mu}\right)\left(\frac{3}{2} + \overline{\mu}\right) x^2 v, \tag{9.15}$$

one has if $\mathrm{Re}\,\mu < \frac{1}{2}$

$$\left(\frac{1}{2} + \nu^2\right) \Phi(v,\, u;\, \nu,\, \mu) = \sum_{j=-1,0,1} \Phi\left(D^{\mu}_{j} v,\; y^{-2j} u;\; \nu,\, \mu + 2j\right). \tag{9.16}$$

As a function of ν on any line $\mathrm{Re}\,\nu = c$ with $c > 1$, the function $\Phi(v,\, u;\, \nu,\, \mu)$ is a $\mathrm{O}((1+|\mathrm{Im}\,\nu|)^{-2})$, with a loss of uniformity relative to μ bounded by $|\mu|^2$.

Proof. The equation (9.10) and the identity $\nu^2\, t^{\nu} \;=\; (t\frac{d}{dt})^2 t^{\nu}$ give (noting that the operator $t\frac{d}{dt}$ is the negative of its transpose if one uses the measure $\frac{dt}{t}$)

$$\nu^2\, \Phi(v,\, u;\, \nu,\, \mu) \colon = \int_0^{\infty} t^{\nu-1}\, (t\frac{d}{dt})^2\, \left(\overline{v}(t + t^{-1})\, u^{\mu}(t - t^{-1})\right)\, dt. \tag{9.17}$$

To facilitate the calculations which follow, observe, setting $|s|_1^{\alpha} = |s|^{\alpha}\mathrm{sign} s$, that $\frac{d}{ds}\, |s|^{\alpha} \;=\; \alpha\, |s|_1^{\alpha-1}$ and $\frac{d}{ds}\, |s|_1^{\alpha} \;=\; \alpha\, |s|^{\alpha-1}$, finally $t\frac{d}{dt}\, v(t + t^{-1}) \;=\; (t - t^{-1})\, v'(t + t^{-1})$ and $t\frac{d}{dt}\, u(t - t^{-1}) = (t + t^{-1})\, u'(t - t^{-1})$.

One has

$$t\frac{d}{dt}\left[\overline{v}(t+t^{-1})\,|t-t^{-1}|^{-\mu-\frac{1}{2}}\right]$$
$$=\overline{v'}(t+t^{-1})\,|t-t^{-1}|_1^{-\mu+\frac{1}{2}}-(\mu+\frac{1}{2})\,\overline{v}(t+t^{-1})\,(t+t^{-1})\,|t-t^{-1}|_1^{-\mu-\frac{3}{2}}. \tag{9.18}$$

Next,

$$(t\frac{d}{dt})^2\left[\overline{v}(t+t^{-1})\,|t-t^{-1}|^{-\mu-\frac{1}{2}}\right]$$
$$=\overline{v''}(t+t^{-1})\,|t-t^{-1}|^{-\mu+\frac{3}{2}}-2\mu\,(t+t^{-1})\,\overline{v'}(t+t^{-1})\,|t-t^{-1}|^{-\mu-\frac{1}{2}}$$
$$-(\mu+\frac{1}{2})\,\overline{v}(t+t^{-1})\,|t-t^{-1}|^{-\mu-\frac{1}{2}}$$
$$+(\mu+\frac{1}{2})(\mu+\frac{3}{2})\,(t+t^{-1})^2\,\overline{v}(t+t^{-1})\,|t-t^{-1}|^{-\mu-\frac{5}{2}}. \tag{9.19}$$

Now, one has $y^2u^\mu=\left(y^2u\right)^{\mu-2}$, $y^{-2}u^\mu=\left(y^{-2}u\right)^{\mu+2}$. If u is even, so that $u^\mu(y)$ is a multiple of $|y|^{-\mu-\frac{1}{2}}$, using these identities with $y=t-t^{-1}$ leads to the identity (9.16). Just exchanging the signed and unsigned versions of the power function gives the same result if u is odd. The loss by a factor $(1+|\mu|)^2$ is insignificant in view of Proposition 9.3.

□

Proposition 9.5. *Let v,u satisfy the conditions of Theorem 9.2. Assuming $c>1$ and $\mathrm{Re}\,s$ large, one has the identity*

$$F_\varepsilon(s)=\frac{1}{2i\pi}\int_{\mathrm{Re}\,\nu=c}\frac{(1-2^{-\nu})^{-1}}{\zeta(\nu)}\;H_\varepsilon(s,\,\nu)\,d\nu, \tag{9.20}$$

with

$$H_\varepsilon(s,\,\nu)=\frac{1}{i}\int_{\mathrm{Re}\,\mu=0}f(s-\nu+\varepsilon\mu)\,\Phi(v,\,u;\,\nu,\,\mu)\,d\mu \tag{9.21}$$

and $f(s)=(1+2^{-s})^{-1}\frac{\zeta(s)}{\zeta(2s)}$.

Proof. Using (9.1) and (4.17), one has for $c>1$

$$F_\varepsilon(s)=\frac{1}{2i\pi}\sum_{Q\in\mathrm{Sq}^{\mathrm{odd}}}Q^{-s}\int_{\mathrm{Re}\,\nu=c}\frac{(1-2^{-\nu})^{-1}}{\zeta(\nu)}\,Q^\nu\,\left(v\,\middle|\,\Psi\left(\mathfrak{E}_{-\nu}\right)u_Q\right)\,d\nu. \tag{9.22}$$

One has $(u_Q)^\mu = (u^\mu)_Q = Q^{-\varepsilon\mu} u^\mu$, and, according to (9.10) and (9.11),

$$\big(v \,\big|\, \Psi\left(\mathfrak{E}_{-\nu}\right) u_Q\big) = \frac{1}{i} \int_{\mathrm{Re}\,\mu=0} Q^{-\varepsilon\mu} \Phi(v,\, u;\, \nu,\, \mu)\, d\mu. \tag{9.23}$$

Recall (Proposition 9.3) that $\Phi(v,\, u;\, \nu,\, \mu)$ is a rapidly decreasing function of μ in vertical strips.

To insert this equation into (9.22), we use Proposition 9.4, which provides the $d\nu$-summability, at the price of losing at most the factor $(1+|\mu|)^2$. We obtain if $c > 1$ and $\mathrm{Re}\, s$ is large enough

$$F_\varepsilon(s) = -\frac{1}{2\pi} \sum_{Q \in \mathrm{Sq}^{\mathrm{odd}}} \int_{\mathrm{Re}\,\nu=c} Q^{-s+\nu}\, \frac{(1-2^{-\nu})^{-1}}{\zeta(\nu)}\, d\nu \int_{\mathrm{Re}\,\mu=0} Q^{-\varepsilon\mu}\, \Phi(v,\, u;\, \nu,\, \mu)\, d\mu \tag{9.24}$$

or, using (5.4),

$$F_\varepsilon(s) = -\frac{1}{2\pi} \int_{\mathrm{Re}\,\nu=c} \frac{(1-2^{-\nu})^{-1}}{\zeta(\nu)}\, d\nu \int_{\mathrm{Re}\,\mu=0} f(s-\nu+\varepsilon\mu)\, \Phi(v,\, u;\, \nu,\, \mu)\, d\mu, \tag{9.25}$$

which is the announced proposition.

□

A resolvent of the self-adjoint operator $i\left(y\frac{d}{dy}+\frac{1}{2}\right)$ in $L^2(\mathbb{R})$ is given by the equations

$$\left[\left(y\frac{d}{dy}+\frac{1}{2}+\mu\right)^{-1} u\right](y) = \begin{cases} \int_0^1 \theta^{\mu-\frac{1}{2}}\, u(\theta y)\, d\theta & \text{if } \mathrm{Re}\,\mu > 0, \\ -\int_1^\infty \theta^{\mu-\frac{1}{2}}\, u(\theta y)\, d\theta & \text{if } \mathrm{Re}\,\mu < 0. \end{cases} \tag{9.26}$$

Let $u \in \mathcal{S}(\mathbb{R})$ be flat at 0 and let $\mu \in \mathbb{C}$. Starting from (9.8), setting $\delta = y\frac{d}{dy}$ and using the integration by parts associated to the identity $\theta\frac{d}{d\theta}\theta^{\mu+\frac{1}{2}} = (\mu+\frac{1}{2})\,\theta^{\mu+\frac{1}{2}}$, one obtains for $N = 0, 1, \dots$ and $y \neq 0$ the identity

$$u^\mu(y) = \frac{(-1)^N}{(\mu+\frac{1}{2})^N} \cdot \frac{1}{2\pi} \int_0^\infty \theta^{\mu-\frac{1}{2}}\, (\delta^N u)(\theta y)\, d\theta. \tag{9.27}$$

We define

$$u_+^{\mu,N}(y) = \frac{(-1)^N}{(\mu+\frac{1}{2})^N} \cdot \frac{1}{2\pi} \int_0^1 \theta^{\mu-\frac{1}{2}} \, (\delta^N u)(\theta y)\, d\theta,$$
$$u_-^{\mu,N}(y) = \frac{(-1)^N}{(\mu+\frac{1}{2})^N} \cdot \frac{1}{2\pi} \int_1^\infty \theta^{\mu-\frac{1}{2}} \, (\delta^N u)(\theta y)\, d\theta, \qquad y \neq 0. \tag{9.28}$$

One has if $\operatorname{Re}\mu > -\frac{1}{2}$

$$\left| u_+^{\mu,N}(y) \right| \leq \frac{1}{2\pi} \, |\mu + \frac{1}{2}|^{-N-1} \sup |\delta^N u| \tag{9.29}$$

and, under the assumption that $\operatorname{Re}\mu < -\frac{1}{2}$, the same inequality holds after one has substituted $u_-^{\mu,N}$ for $u_+^{\mu,N}$.

We decompose accordingly the function in (9.10), setting

$$\Phi_N^{\pm}(v,\, u;\, \nu,\, \mu) = \int_0^\infty \overline{v}(t+t^{-1})\, t^{\nu-1}\, u_\pm^{\mu,N}(t-t^{-1})\, dt. \tag{9.30}$$

The sum $\Phi(v,\, u;\, \nu,\, \mu) = \Phi_N^+(v,\, u;\, \nu,\, \mu) + \Phi_N^-(v,\, u;\, \nu,\, \mu)$ does not depend on N.

Proposition 9.6. *Let v, u satisfy the assumptions of Theorem* 9.2. *For every pair C_1, C_2 of non-negative integers, the function*

$$(1+|\operatorname{Im}\nu|)^{C_2} \; (1+|\mu|)^{C_1} \; \Phi_N^+(v,\, u;\, \nu,\, \mu) \tag{9.31}$$

is bounded if $N+1 \geq C_1 + C_2$, in a uniform way relative to $\operatorname{Re}\mu \geq \alpha > -\frac{1}{2}$ *if $|\operatorname{Re}\nu|$ is bounded. The function*

$$(1+|\operatorname{Im}\nu|)^{C_2} \; (1+|\mu|)^{C_1} \; \Phi_N^-(v,\, u;\, \nu,\, \mu) \tag{9.32}$$

is bounded under the same condition, in a uniform way relative to $\operatorname{Re}\mu \leq \beta < -\frac{1}{2}$ *if $|\operatorname{Re}\nu|$ is bounded.*

Proof. The way $\Phi_N^{\pm}(v,\, u;\, \nu,\, \mu)$ behaves in terms of μ as $|\operatorname{Im}\mu| \to \infty$ is a consequence of the similar estimates relative to $u_\pm^{\mu,N}(t-t^{-1})$. Following the proof of Proposition 9.4, one can improve the bounds by powers of $(1+|\operatorname{Im}\nu|)^{-1}$, losing the corresponding power of $1+|\mu|$, and the claimed uniformity is preserved.

$\square$

Remark 9.1. Anticipating on the application in the next section, we could dispense with factors of the kind $(1+|\mathrm{Im}\,\nu|)^{C_2}$, relying instead on Proposition 5.5 to improve the $d\nu$-convergence at infinity. Indeed, after the contour change in Theorem 10.2 has produced the residue in (10.8), an extra factor such as ν^{-M} in (10.4) becomes $(s-1+\varepsilon\mu)^{-M}$, just what is needed, as shown in Lemma 10.1, to take care of the bound at infinity of $(\zeta(s-1+\varepsilon\mu))^{-1}$. Taking $N=1$ would then suffice to ensure $d\mu$-summability.

10. A refutation of the Riemann hypothesis

This section is entirely based on Cauchy-type analysis. Note the following benefit – not the only one – of using the approximation $F_\varepsilon(s)$ of $F_0(s)$. As soon as Theorem 10.3 has been established, the whole task consists in establishing bounds, uniform with respect to ε, for integrals the convergence of which is not in question. On the other hand, with one exception in the proof of Theorem 10.4, all changes of contour are made under the assumption that $\mathrm{Re}\,s$ is large: it is only, as explained in Remark 5.2, the results of the changes that must be analyzed for s in the domains of interest.

Lemma 10.1. *If $d\in]\frac{1}{2},1[$ is such that, for some $\delta>0$, zeta has no zero in the strip $\{s\colon d-\delta<\mathrm{Re}\,s<d+\delta\}$, one has for some pair C,M the estimate $|\zeta(d+i\tau)|^{-1}\leq C\,(1+|\tau|)^M$ for every $\tau\in\mathbb{R}$.*

Proof. Recall first the Borel-Caratheodory lemma: if a function $f(w)$ is holomorphic and satisfies the condition $\mathrm{Re}\,f(w)\leq A$ for $|w|<\delta$, finally if $f(0)=0$, one has for $|w|\leq\frac{\delta}{2}$ the estimate $|f(w)|\leq 4A$. With $s_0=d+i\tau$, let $f(w)=\log\frac{\zeta(d+i\tau+w)}{\zeta(d+i\tau)}$. The Lindelöf convexity inequality [7, p.201], considerably more precise than the estimate $|\zeta(d+i\tau)|=\mathrm{O}(|\tau|)$ for $|\tau|\to\infty$, sufficient for our purpose, gives for $|\tau|$ large

$$\mathrm{Re}\,f(w)=\log|\zeta(d+i\tau+w)|-\log|\zeta(d+i\tau)|\leq C\,\log|\tau|,\qquad(10.1)$$

for some $C>0$, and it follows [7, p.225] from the Borel-Caratheodory lemma that one has if $|\tau|\geq 2$ the estimate

$$|\zeta(d+i\tau)|^{-1}\leq\,|\tau|^{4C}\,|\zeta(d+i\tau+w)|^{-1}.\qquad(10.2)$$

Then, we use the following result, due to Valiron [13], which provides a sequence of heights at which crossing the critical strip is reasonably safe.

It is the fact that there exists a sequence $(T_k)_{k\geq 1}$, $T_k \in [k, k+1[$, and a pair B, M_1 such that

$$\inf_{0\leq\sigma\leq 1}|\zeta(\sigma+iT_k)| \geq B^{-1}\, k^{-M_1}. \tag{10.3}$$

A modern proof, which I owe to Gerald Tenenbaum, is to be found as [9, Lemma 4.2].

From $d+i\tau$, one can reach a point $d+iT_k$ by adding consecutively a number of the order of $\frac{2}{\delta}$ of increments w with $|w| \leq \frac{\delta}{2}$. The lemma follows. $\square$

Theorem 10.2. *Let $v,\, u \in C^\infty(\mathbb{R})$ satisfy the assumptions of Theorem 9.2, here recalled: v is supported in $[2, \sqrt{8}]$ and u in $[0,1]$. For $c>1$ and $2 < \mathrm{Re}\, s < c+1$, one has $F_\varepsilon(s) = E_\varepsilon(s) - i\, G_\varepsilon(s)$, with*

$$E_\varepsilon(s) = -\frac{1}{2\pi}\int_{\mathrm{Re}\,\mu=0}\int_{\mathrm{Re}\,\nu=c} \frac{(1-2^{-\nu})^{-1}}{\zeta(\nu)}\, f(s-\nu+\varepsilon\mu)\,\Phi(v,\,u;\,\nu,\,\mu)\, d\mu\, d\nu \tag{10.4}$$

and

$$G_\varepsilon(s) = \frac{4}{\pi^2}\int_{\mathrm{Re}\,\mu=0} \frac{\left(1-2^{-s+1-\varepsilon\mu}\right)^{-1}}{\zeta(s-1+\varepsilon\mu)}\,\Phi(v,\,u;\,s-1+\varepsilon\mu,\,\mu)\, d\mu. \tag{10.5}$$

We recall that $f(s) = (1+2^{-s})^{-1}\,\frac{\zeta(s)}{\zeta(2s)}$ and that the function $\Phi(v,\,u;\,\nu,\,\mu)$ was defined in Proposition 9.3.

Set $\sigma_0 = \sup\{\mathrm{Re}\,\rho\colon \zeta(\rho)=0\}$. The function $G_\varepsilon(s)$, as defined by (10.5) *for* $\mathrm{Re}\,(s-1) > \sigma_0$*, extends analytically to the half-plane* $\mathrm{Re}\,(s-1) > \frac{\sigma_0}{2}$*, and the decomposition $F_\varepsilon(s) = E_\varepsilon(s) - i\, G_\varepsilon(s)$ is valid for $c + \frac{\sigma_0}{2} <$* $\mathrm{Re}\, s < c+1$.

Proof. We start from Proposition 9.5 here recalled,

$$F_\varepsilon(s) = -\frac{1}{2\pi}\int_{\mathrm{Re}\,\nu=c}\int_{\mathrm{Re}\,\mu=0} \frac{(1-2^{-\nu})^{-1}}{\zeta(\nu)}\, f(s-\nu+\varepsilon\mu)\,\Phi(v,\,u;\,\nu,\,\mu)\, d\nu\, d\mu, \tag{10.6}$$

an identity valid if $c>1$ and $\mathrm{Re}\, s > c+1$. The double integral is absolutely convergent in view of Propositions 9.3 and 9.4. The same integral converges for $c+\frac{\sigma_0}{2} < \mathrm{Re}\, s < c+1$. Indeed, in that case, the function $f(s-\nu+\varepsilon\mu)$ is still non-singular, since the numerator of its expression just recalled is non-singular and the denominator is nonzero. The bound $\zeta(s-\nu+\varepsilon\mu) = \mathrm{O}(|\mathrm{Im}\,(s-\nu+\varepsilon\mu)|^{\frac{1}{2}})$ since $\mathrm{Re}\,(s-\nu) > 0$ [7, p.201] completes the estimates.

However, the convergence in the cases for which $\mathrm{Re}\, s > c+1$ and $c + \frac{\sigma_0}{2} < \mathrm{Re}\, s < c+1$ holds for incompatible reasons. To make the jump possible, we choose c_1 such that $1 < c_1 < c$ and observe that one can replace in (10.6) the integral sign $\int_{\mathrm{Re}\,\nu=c}$ by $\int_{\mathrm{Re}\,\nu=c_1}$, enlarging the domain of validity of the new identity to the half-plane $\mathrm{Re}\, s > c_1+1$. In the case when $c_1+1 < \mathrm{Re}\, s < c+1$, one has for $\mathrm{Re}\,\mu = 0$

$$\begin{aligned} \frac{1}{2i\pi} \int_{\mathrm{Re}\,\nu=c_1} & \frac{\left(1-2^{-\nu}\right)^{-1}}{\zeta(\nu)}\, f(s-\nu+\varepsilon\mu)\, \Phi(v,\, u;\, \nu,\, \mu)\, d\nu \\ &= \frac{1}{2i\pi} \int_{\mathrm{Re}\,\nu=c} \frac{\left(1-2^{-\nu}\right)^{-1}}{\zeta(\nu)}\, f(s-\nu+\varepsilon\mu)\, \Phi(v,\, u;\, \nu,\, \mu)\, d\nu \\ &\quad - \mathrm{Res}_{\nu=s-1+\varepsilon\mu} \left[\frac{\left(1-2^{-\nu}\right)^{-1}}{\zeta(\nu)}\, f(s-\nu+\varepsilon\mu)\, \Phi(v,\, u;\, \nu,\, \mu) \right]. \end{aligned} \tag{10.7}$$

The $d\mu$-integral of the left-hand side coincides with $F_\varepsilon(s)$ for $\mathrm{Re}\, s > c_1+1$ according to Proposition 9.5, hence extends as an entire function. As already observed, the $d\nu$-integral which is the first term on the right-hand side is analytic in the domain $\{s\colon c + \frac{\sigma_0}{2} < \mathrm{Re}\, s < c+1\}$ and so is its $d\mu$-integral $E_\varepsilon(s)$.

The residue of the function $f(s) = \left(1+2^{-s}\right)^{-1} \frac{\zeta(s)}{\zeta(2s)}$ at $s=1$ is $\frac{4}{\pi^2}$. The second term on the right-hand side of (10.7) is thus

$$\frac{4}{\pi^2}\, \frac{\left(1-2^{-s+1-\varepsilon\mu}\right)^{-1}}{\zeta(s-1+\varepsilon\mu)}\, \Phi(v,\, u;\, s-1+\varepsilon\mu,\, \mu), \tag{10.8}$$

and $G_\varepsilon(s)$ is for $\mathrm{Re}\, s > 1+\sigma_0$ the $d\mu$-integral of this expression.

One has by definition $G_\varepsilon(s) = i(F_\varepsilon(s) - E_\varepsilon(s))$ if $c_1+1 < \mathrm{Re}\, s < c+1$. The right-hand side of this identity is analytic for $c + \frac{\sigma_0}{2} < \mathrm{Re}\, s < c+1$. The function $G_\varepsilon(s)$, in the definition of which $c>1$ is not present, thus extends analytically for $\mathrm{Re}\, s > 1 + \frac{\sigma_0}{2}$, and the identity that defines it extends for $c + \frac{\sigma_0}{2} < \mathrm{Re}\, s < c+1$.

$\square$

Theorem 10.3. *Let $Z = \{\mathrm{Re}\,\rho\colon 0 < \mathrm{Re}\,\rho < 1,\, \zeta(\rho)=0\}$, and let $\sigma_0 = \sup Z$. There is no interval $[a,b]$ with $\frac{\sigma_0}{2} < a < \sigma_0 < b$ such that for every pair v, u satisfying the support conditions in Theorem 10.2 and every compact subset K of the strip $\{s\colon a < \mathrm{Re}\,(s-1) < b\}$, the function $G_\varepsilon(s)$ is* *bounded in K, in a uniform way relative to $\varepsilon > 0$.*

Proof. The condition $a > \frac{\sigma_0}{2}$ ensures that $G_\varepsilon(s)$ is well-defined for $\mathrm{Re}\,(s-1) \geq a$. Let us assume that the interval (a, b) satisfies the conditions the impossibility of which is to be proven. In view of Theorem 5.3 and Corollary 5.4, it suffices to show that, under the given assumption, the function $F_0(s)$ could be continued to the half-plane $\mathrm{Re}\,(s-1) > a$.

By a compactness argument (Montel's theorem), replacing the set $\{\varepsilon\colon \varepsilon < \varepsilon_0\}$ by an appropriate sequence going to zero, one may assume that $G_\varepsilon(s)$ converges in the strip $a < \mathrm{Re}\,(s-1) < b$, as $\varepsilon \to 0$, towards an analytic function $G_0^\sharp(s)$.

On the other hand, starting from (10.5) and using Proposition 9.3 and Lemma 10.1, finally (9.11), one sees that, for $\mathrm{Re}\,(s-1) > \sigma_0$, $G_\varepsilon(s)$ converges as $\varepsilon \to 0$ towards the function

$$\begin{aligned} G_0(s) &= \frac{4}{\pi^2} \int_{\mathrm{Re}\,\mu=0} \frac{\left(1-2^{-s+1}\right)^{-1}}{\zeta(s-1)}\, \Phi_N(v,\, u;\, s-1,\, \mu)\, d\mu \\ &= \frac{4i}{\pi^2} \frac{\left(1-2^{-s+1}\right)^{-1}}{\zeta(s-1)} \left(v \,\middle|\, \Psi(\mathfrak{E}_{1-s})\, u\right), \end{aligned} \tag{10.9}$$

which must coincide with $G_0^\sharp(s)$ for $\sigma_0 < \mathrm{Re}\,(s-1) < b$. We have thus obtained a continuation of the function $G_0(s)$ to the half-plane $\mathrm{Re}\,(s-1) > a$. Next, the function $E_\varepsilon(s)$, which coincides as has been seen in Theorem 10.2 with the sum $F_\varepsilon(s) + i\, G_\varepsilon(s)$ in the strip $c - 1 + \frac{\sigma_0}{2} < \mathrm{Re}\,(s-1) < c$, has there a limit $E_0(s)$, obtained by taking $\varepsilon = 0$ on the right-hand side of (10.4). We choose c such that $1 < c < a + 1 - \frac{\sigma_0}{2}$.

We sum up what precedes. As $\varepsilon \to 0$, the function $G_\varepsilon(s)$ has a locally uniform limit $G_0(s)$ for $\mathrm{Re}\,(s-1) > a$, and $E_\varepsilon(s)$ has a locally uniform limit $E_0(s)$ for $c - 1 + \frac{\sigma_0}{2} < \mathrm{Re}\,(s-1) < c$: one has $F_\varepsilon(s) = E_\varepsilon(s) - i\, G_\varepsilon(s)$ in the second interval so that, since $a > c - 1 + \frac{\sigma_0}{2}$, the function $F_\varepsilon(s)$ has a locally uniform limit for $a < \mathrm{Re}\,(s-1) < c$. This limit is $F_0(s)$ when $\mathrm{Re}\,(s-1) > 1$, as seen for instance from Theorem 9.2, so that it is for $\mathrm{Re}\,(s-1) > a$ a continuation of $F_0(s)$.

□

Let us consider the factors of the integral (10.5) for $G_\varepsilon(s)$. In view of Lemma 10.1, for $\mathrm{Re}\,(s-1)$ lying in a closed subinterval of an open interval in which there are no real parts of zero, the factor $(\zeta(s-1+\varepsilon\mu))^{-1}$ is bounded by $C(1+|\mathrm{Im}\,(\varepsilon\mu)|)^M$ as long as $\mathrm{Im}\,(s-1)$ is bounded. On the other hand, the Φ-function is rapidly decreasing as a function of $\mathrm{Im}\,\mu$, and this will remain true when the line $\mathrm{Re}\,\mu=0$ has been changed to another one. However, there is no uniformity in the latter set of estimates with respect to the real part of μ, as will be necessary presently when letting ε go to 0: to depend on it, we shall have to use Proposition 9.6, which demands cutting the Φ-function into two pieces.

Theorem 10.4. *There cannot exist any pair a,η with $\frac{\sigma_0}{2}<a<a+\eta<\sigma_0<a+2\eta$ such that the function zeta has no zero with a real part in the interval $[a,a+\eta]$.*

Proof. Let v,u be a pair of functions satisfying the support conditions in Theorem 10.2. Recall from Theorem 10.2 that $G_\varepsilon(s)$, as defined by (10.5) for $\mathrm{Re}\,(s-1)>\sigma_0$, extends analytically to the half-plane $\mathrm{Re}\,(s-1)>\frac{\sigma_0}{2}$. Our first task will consist in building an integral representation of $G_\varepsilon(s)$ in the half-plane $\mathrm{Re}\,(s-1)>a$. We use the decompositions (9.30) $\Phi(v,\,u;\,\nu,\,\mu)=\Phi_N^+(v,\,u;\,\nu,\,\mu)+\Phi_N^-(v,\,u;\,\nu,\,\mu)$, and $G_{\varepsilon,N}^+(s)+G_{\varepsilon,N}^-(s)$, with

$$G_{\varepsilon,N}^\pm(s)=\frac{4}{\pi^2}\int_{\mathrm{Re}\,\mu=0}\frac{\left(1-2^{-s+1-\varepsilon\mu}\right)^{-1}}{\zeta(s-1+\varepsilon\mu)}\,\Phi_N^\pm(v,\,u;\,s-1+\varepsilon\mu,\,\mu)\,d\mu \quad (10.10)$$

if $\mathrm{Re}\,s>1+\sigma_0$. Given a pair contradicting the claim of the theorem, we shall show that either term of the decomposition (10.10) can be continued to the half-plane $\{s\colon \mathrm{Re}\,s>a\}$, proving at the same time the required bounds independent of $\varepsilon>0$ for $a<\mathrm{Re}\,(s-1)<a+2\eta$. The sum of the continuations will of course coincide with the restriction of $G_\varepsilon(s)$ to this half-plane and, as $\frac{\sigma_0}{2}<a<\sigma_0<a+2\eta$, the contradiction will be a consequence of Theorem 10.3. We analyze the two terms separately.

Choose β such that $\frac{1}{2\varepsilon}<\beta=\mathrm{O}(\frac{1}{\varepsilon})$. For $\mathrm{Re}\,(s-1)>\frac{\sigma_0}{2}$, even the more so for $\mathrm{Re}\,(s-1)>a$, one has $\mathrm{Re}\,(s-1+\varepsilon\beta)>\mathrm{Re}\,(s-\frac{1}{2})>\frac{1+\sigma_0}{2}\geq\sigma_0$. We may thus write for $\mathrm{Re}\,(s-1)>\frac{\sigma_0}{2}$, after a contour change,

$$G_{\varepsilon,N}^+(s)=\frac{4}{\pi^2}\int_{\mathrm{Re}\,\mu=\beta}\frac{\left(1-2^{-s+1-\varepsilon\mu}\right)^{-1}}{\zeta(s-1+\varepsilon\mu)}\,\Phi_N^+(v,\,u;\,s-1+\varepsilon\mu,\,\mu)\,d\mu, \quad (10.11)$$

benefitting from Proposition 9.3 to ensure convergence. On one hand, the real part of $\nu=s-1+\varepsilon\mu$ is bounded as $\varepsilon\to 0$, in opposition to that of μ. On the other hand, even though $\beta \to \infty$ as $\varepsilon \to 0$, the estimate of $\Phi_N^+(v,\, u;\, \nu,\, \mu)$, for any given C, by a $\mathrm{O}\left((1+|\mathrm{Im}\,\mu|)^{-C}\right)$ is uniform in view of (9.31). The part of the analysis concerning $G_{\varepsilon,N}^+(s)$ follows. Recall that the change of contour is made under the assumption that $\mathrm{Re}\, s$ is large, in this case $\mathrm{Re}\,(s-1) > \sigma_0$ (cf. Remark 5.2): after the change has been made, we use (10.11) to define the sought-after continuation of $G_{\varepsilon,N}^+(s)$.

Some more care is needed when dealing with $G_{\varepsilon,N}^-(s)$. Besides the function

$$G_{\varepsilon,N}^-(s) = \frac{4}{\pi^2} \int_{\mathrm{Re}\,\mu=0} \frac{\left(1-2^{-s+1-\varepsilon\mu}\right)^{-1}}{\zeta(s-1+\varepsilon\mu)}\, \Phi_N^-(v,\, u;\, s-1+\varepsilon\mu,\, \mu)\, d\mu, \tag{10.12}$$

as defined for $\mathrm{Re}\,(s-1) > \sigma_0$, we introduce the function $K_{\varepsilon,N}^-(s)$ seemingly defined by exactly the same integral, but regarded in the domain $\{s\colon a < \mathrm{Re}\,(s-1) < a+\eta\}$, where the integral converges in view of Lemma 10.1. As $a+\eta < \sigma_0$, the function $K_{\varepsilon,N}^-$ may not (not yet, actually) be regarded as a continuation of the function $G_{\varepsilon,N}^-$: an intermediary will be required.

Choose η' such that $0 < \eta' < \eta$ and $a+\eta+\eta' > \sigma_0$, and consider with $\alpha = -\frac{\eta'}{\varepsilon}$ the integral

$$H_{\varepsilon,N}^-(s)\colon = \frac{4}{\pi^2} \int_{\mathrm{Re}\,\mu=\alpha} \frac{\left(1-2^{-s+1-\varepsilon\mu}\right)^{-1}}{\zeta(s-1+\varepsilon\mu)}\, \Phi_N^-(v,\, u;\, s-1+\varepsilon\mu,\, \mu)\, d\mu. \tag{10.13}$$

It is convergent if $a < \mathrm{Re}\,(s-1-\eta') < a+\eta$, i.e., $a+\eta' < \mathrm{Re}\,(s-1) < a+\eta+\eta'$. In the non-void domain $\{s\colon a+\eta' < \mathrm{Re}\,(s-1) < a+\eta\}$, the integrals for $K_{\varepsilon,N}^-(s)$ and $H_{\varepsilon,N}^-(s)$ are both convergent, and the second one is the result of the application to the first one of a change of the μ-contour. Beware that in this instance, one cannot rely on Remark 5.2 since there is an upper limit on $\mathrm{Re}\,(s-1)$: but, during the move, $t = \mathrm{Re}\,(\varepsilon\mu)$ stays between $-\eta'$ and 0, so that $a < \mathrm{Re}\,(s-1+t) < a+\eta$ if $a+\eta' < \mathrm{Re}\,(s-1) < a+\eta$, which justifies the change of contour. Hence, $K_{\varepsilon,N}^-(s) = H_{\varepsilon,N}^-(s)$ when $a+\eta' < \mathrm{Re}\,(s-1) < a+\eta$ and one obtains a continuation of this function to the union $\{s\colon a < \mathrm{Re}\,(s-1) < a+\eta+\eta'\} = \{s\colon a < \mathrm{Re}\,(s-1) < a+\eta\} \cup \{s\colon a+\eta' < \mathrm{Re}\,(s-1) < a+\eta+\eta'\}$ when piecing the two definitions.

On the other side, the identity (10.13) is valid for $a+\eta' < \mathrm{Re}\,(s-1) < a+\eta+\eta'$. As $a+\eta' < a+\eta < \sigma_0 < a+\eta+\eta'$, (10.13) is valid for

$\sigma_0 < \mathrm{Re}\,(s-1) < a+\eta+\eta'$, a subdomain of the half-plane $\mathrm{Re}\,(s-1) > \sigma_0$ in which (10.12) is valid: we have obtained, using $H^-_{\varepsilon,N}(s)$ as an intermediary, a continuation of $G^-_{\varepsilon,N}(s)$ to the half-plane $\{s\colon \mathrm{Re}\,(s-1) > a\}$, given when $a < \mathrm{Re}\,(s-1) < a+\eta$ as $K^-_{\varepsilon,N}(s)$ and when $a+\eta' < \mathrm{Re}\,(s-1) < a+\eta+\eta'$ as $H^-_{\varepsilon,N}(s)$.

As zeta has no zero with a real part in $[a, a+\eta]$, it follows from the integral representation of $K^-_{\varepsilon,N}(s)$ that this function is bounded in a locally uniform way in the strip $\{s\colon a < \mathrm{Re}\,(s-1) < a+\eta\}$, in a uniform way relative to $\varepsilon > 0$. Again, the estimate of $\Phi^-_N(v,\, u;\, s-1+\varepsilon\mu,\, \mu)$ by a rapidly decreasing function of $\mathrm{Im}\,\mu$ is uniform relative to ε as $\mathrm{Re}\,\mu \to -\infty$, and it follows from (10.13) that $H^-_{\varepsilon,N}(s)$ is bounded in a locally uniform way in the strip $\{s\colon a+\eta' < \mathrm{Re}\,(s-1) < a+\eta+\eta'\}$, in a uniform way relative to $\varepsilon > 0$.

This concludes the proof of Theorem 10.4.

$\square$

Remarks 10.1 (i) In Theorem 10.4, the condition $a + 2\eta > \sigma_0$ is exactly what is needed, in the sense that it could not be replaced, for any $\lambda > 1$, by $a + (1+\lambda)\,\eta > \sigma_0$. Indeed, the integral (10.13) converges in the domain $\{s\colon a - \varepsilon\alpha < \mathrm{Re}\,(s-1) < a+\eta-\varepsilon\alpha\,\}$. But, under the new condition, this is useful only if $a+\eta-\varepsilon\alpha \geq a+(1+\lambda)\eta$, in other words $\alpha \leq -\frac{\lambda\eta}{\varepsilon}$. Then, $a-\varepsilon\alpha \geq a+\lambda\eta > a+\eta$, the two intervals $[a, a+\eta]$ and $[a-\varepsilon\alpha,\, a+\eta-\varepsilon\alpha]$ are disjoint and Theorem 10.3 is unapplicable.

(ii) Another look at the equations (9.28) and (9.30) shows that, in Proposition 9.6, one could, when dealing with Φ^+_N, replace the claim "is bounded if $N+1 \geq C_1 + C_2$ in a uniform way ..." by "goes to zero as $\mathrm{Re}\,\mu \to \infty$". Something similar goes when Φ^-_N is considered. This implies that $G^+_{\varepsilon,N}(s)$ goes to zero as $\varepsilon \to 0$ for $a < \mathrm{Re}\,(s-1) < a+2\eta$ and, using (10.13), that $G^-_{\varepsilon,N}(s)$ goes to zero as $\varepsilon \to 0$ for $a+\eta' < \mathrm{Re}\,(s-1) < a+\eta'+\eta$. Using the Montel argument in the proof of Theorem 10.3, one sees that, for some sequence (ε_j) going to zero, G_{ε_j} has a (locally uniform) limit for $a < \mathrm{Re}\,(s-1) < a+2\eta$: of necessity, this limit is zero since it is analytic and zero in the subdomain $\{s\colon a+\eta' < \mathrm{Re}\,(s-1) < a+\eta+\eta'\}$. This is of no consequence as, in order that the function $G_0^\sharp(s) = \lim_{j\to\infty} G_{\varepsilon_j}(s)$ (with the notation in the proof of Theorem 10.3) should agree with $G_0(s)$ in some non-void domain, we would have to assume that $a+2\eta > a+\eta+\eta' > \sigma_0$. But it is precisely the result of Theorem 10.4 that this is to be excluded.

As seen in the proof of Theorem 10.3, the function $G_0^\sharp$ has no significance unless its domain of analyticity intersects that of G_0.

Theorem 10.5. *One has $\sigma_0 = \sup Z \geq \frac{4}{7}$. In particular, the Riemann hypothesis does not hold. The set Z of real parts of non-trivial zeros is infinite.*

Proof. By definition, zeta has no zero with a real part in $]\sigma_0, 1]$; as a consequence of the functional equation, it does not have any zero with a real part in $[0, 1-\sigma_0[$. Applying Theorem 10.4, one obtains a contradiction if one can find a pair a, η such that

$$\frac{\sigma_0}{2} < a < a + \eta < 1 - \sigma_0, \qquad a + 2\eta > \sigma_0. \tag{10.14}$$

This is indeed the case if $\sigma_0 < \frac{4}{7}$. For, then, one can find η such that $2\sigma_0 - 1 < \eta < \frac{\sigma_0}{4}$. Choosing a such that $\sigma_0 - 2\eta < a < 1 - \sigma_0 - \eta$, as is possible since $\eta > 2\sigma_0 - 1$, we are done: all that remains to be done is noting that $a > \frac{\sigma_0}{2}$, or $\sigma_0 - 2\eta > \frac{\sigma_0}{2}$, a consequence of $\eta < \frac{\sigma_0}{4}$.

Now, assume that the set Z is finite, let σ_1 be the point of Z immediately inferior to the largest point σ_0. Choose η such that $\frac{\sigma_0 - \sigma_1}{2} < \eta < \sigma_0 - \sigma_1$, then a such that $\sigma_1 < a < \sigma_0 - \eta$. Then, $a + \eta < \sigma_0$ and $a + 2\eta > \sigma_0$: this contradicts the result of Theorem 10.4.

□

In [1], Bombieri (almost) proved that if the Riemann hypothesis does not hold, the set of zeros of zeta is infinite. Concentrating on the real parts of zeros has proved decisive.

Remark 10.2. The inequality $\sigma_0 \geq \frac{4}{7}$ can be improved to $\sigma_0 \geq \frac{2}{3}$ in the following way. In the definition (10.4) of $E_\varepsilon(s)$, replace the condition $c > 1$ by $c > \sigma_0$. Then, in Theorem 10.4, one can replace the condition $a > \frac{\sigma_0}{2}$ by $a > \frac{\sigma_0}{2} + c - 1$, or $a > \frac{3\sigma_0}{2} - 1$. The set of conditions (10.14) leads to the desired improvement.

Actually, this improvement has an only temporary role, since we shall prove that $\sigma_0 = 1$. To obtain this, we shall drop in the main series (5.2) or (9.1) the assumption that the summation variable Q is squarefree.

11. Zeros of zeta accumulate on the line $\operatorname{Re} s = 1$

In Theorem 10.5 and Remark 10.2, we have reached, with the inequality $\sigma_0 \geq \frac{2}{3}$, the limit of the method. If one replaces in the series (5.2) the sum over all squarefree odd integers by the sum over all odd integers, one can reach, as we shall see, the correct value $\sigma_0 = 1$.

We must first answer the natural question "why did we not do this in the first place ?". The answer is that when, years ago, we embarked on this program, it was under the conviction that the Riemann hypothesis did hold. In a version of the criterion (4.19), in which R.H. appears as a consequence of an estimate regarding $\left(v \,\middle|\, \Psi(Q^{2i\pi\mathcal{E}}\mathfrak{T}_{\frac{\infty}{2}})\, u\right)$, putting more constraints on Q led to a situation which we regarded as more favorable. As it turns out, R.H. is disproved in the present paper, and a more favorable situation is obtained, especially in Theorem 9.2, if one lowers the constraint on the summation variable Q in the series there.

Yes, a slightly shorter proof would have been obtained if we had started in the right direction from the beginning. But we do not regret this very small détour, one point of which is being satisfied that the present proof is an in-depth analysis of the Riemann hypothesis, not just the consequence of a favorable accident. Also, in this direction, Remark 11.1 below will relate our understanding of the Riemann hypothesis to an arithmetic question.

First, we generalize Theorem 7.2, recalling (paragraph following (4.2)) that $Q_\bullet$ is the squarefree version of Q .

Theorem 11.1. *Let $N = RQ$, where we assume that $(R, Q) = 1$, that R is squarefree and that both factors are odd. Let $v, u \in C^\infty(\mathbb{R})$, compactly supported, satisfying the conditions that $x > 0$ and $0 < x^2 - y^2 < 8$ when $v(x)u(y) \neq 0$. Then, if N is large enough,*

$$\left(v \,\middle|\, \Psi(Q^{2i\pi\mathcal{E}}\mathfrak{T}_N)\, u\right) = \sum_{\substack{R_1R_2 = R \\ Q_1Q_2 = Q_\bullet}} \mu(R_1Q_1)$$
$$\overline{v}\left(\frac{R_1}{Q_2}\frac{Q_\bullet}{Q} + \frac{Q_2}{R_1}\frac{Q}{Q_\bullet}\right) u\left(\frac{R_1}{Q_2}\frac{Q_\bullet}{Q} - \frac{Q_2}{R_1}\frac{Q}{Q_\bullet}\right). \tag{11.1}$$

Proof. Theorem 6.1 remains valid if one drops the assumption that N is squarefree: the only point that matters here is that R and Q should be relatively prime. We thus compute, with $b(j,\,k) = a((j,k,N)) = 1 - p\,\mathrm{char}(j \equiv k \equiv N \equiv 0 \bmod p)$, the sum $f_N(j,\,s)$ defined in (6.3). The Eulerian reduction (7.4) is still valid if $(N_1, N_2) = 1$ and one can reduce the computation to the case when $N = Q = p^\gamma$ for some prime p: there is no difference with the earlier situation so far as the R-factor is concerned. One has

$$\begin{aligned} f_N(j,s) &= \frac{1}{p^\gamma} \sum_{k \bmod p^\gamma} [1 - p\ \mathrm{char}(j \equiv k \equiv 0 \bmod p)] \exp\left(\frac{2i\pi ks}{p^\gamma}\right) \\ &= \frac{1}{p^\gamma} \sum_{k \bmod p^\gamma} \exp\left(\frac{2i\pi ks}{p^\gamma}\right) - \mathrm{char}(j \equiv 0 \bmod p)\, \frac{1}{p^{\gamma-1}} \sum_{k_1 \bmod p^{\gamma-1}} \exp\left(\frac{2i\pi k_1 s}{p^{\gamma-1}}\right) \\ &= \mathrm{char}(s \equiv 0 \bmod p^\gamma) - \mathrm{char}(j \equiv 0 \bmod p)\,\mathrm{char}(s \equiv 0 \bmod p^{\gamma-1}) \\ &= \mathrm{char}(s \equiv 0 \bmod p^{\gamma-1}) \times \left[\mathrm{char}\left(\frac{s}{p^{\gamma-1}} \equiv 0 \bmod p\right) - \mathrm{char}(j \equiv 0 \bmod p)\right]. \end{aligned} \tag{11.2}$$

If $Q = \prod_p p^{\gamma_p}$, one has $\prod_p p^{\gamma_p - 1} = \frac{Q}{Q_\bullet}$. Piecing together the equations (11.2) for all values of p dividing N, one obtains

$$\begin{aligned} f_N(j,,s) = \mathrm{char}\left(s \equiv 0 \bmod \frac{Q}{Q_\bullet}\right) \times \sum_{M_1 M_2 = N_\bullet} \mu(M_1) \\ \mathrm{char}\left(\frac{s}{Q/Q_\bullet} \equiv 0 \bmod M_2\right) \mathrm{char}(j \equiv 0 \bmod M_1). \end{aligned} \tag{11.3}$$

The transformation θ_N is defined as in (6.5). Just as in the proof of Theorem 7.2, one has $(\theta_N v)(m) = v\left(\frac{m}{N}\right)$ and $(\theta_N u)(n) = u\left(\frac{n}{N}\right)$ if N is large. Applying the recipe in Theorem 6.1 and setting $M_2 = R_2 Q_2$, $M_1 =$

R_1Q_1, one finds

$$\left(v \,\middle|\, \Psi(Q^{2i\pi\mathcal{E}}\mathfrak{T}_N)\, u\right) = \sum_{\substack{R_1R_2=R\\ Q_1Q_2=Q_\bullet}} \mu(R_1Q_1)\, \overline{v}\left(\frac{m}{N}\right)\, u\left(\frac{n}{N}\right)$$

$$\text{char}\left(m-n \equiv 0 \bmod 2Q\,(\frac{Q}{Q_\bullet})\,(R_2Q_2)\right)\,\text{char}(m+n\equiv 0 \bmod 2RR_1Q_1). \tag{11.4}$$

Setting

$$m+n = 2R(R_1Q_1)\,a, \qquad m-n = 2Q\left(\frac{Q}{Q_\bullet}\right)(R_2Q_2)\,b \tag{11.5}$$

with $a, b \in \mathbb{Z}$ and $a > 0$, one has

$$\frac{m^2-n^2}{N^2} = \frac{R(R_1Q_1)\,\times\, Q\left(\frac{Q}{Q_\bullet}\right)R_2Q_2}{N^2}\,\times\, 4ab = 4ab, \tag{11.6}$$

so that $a = b = 1$. One has

$$\frac{RR_1Q_1}{N} = \frac{R_1Q_1}{Q} = \frac{R_1}{Q_2}\times\frac{Q_\bullet}{Q}, \qquad \frac{Q}{N}\left(\frac{Q}{Q_\bullet}\right)R_2Q_2 = \frac{1}{R}\,\frac{QR_2Q_2}{Q_\bullet} = \frac{Q_2}{R_1}\,\frac{Q}{Q_\bullet} \tag{11.7}$$

and, finally,

$$\frac{m}{N} = \frac{R_1}{Q_2}\,\frac{Q_\bullet}{Q} + \frac{Q_2}{R_1}\,\frac{Q}{Q_\bullet}, \qquad \frac{n}{N} = \frac{R_1}{Q_2}\,\frac{Q_\bullet}{Q} - \frac{Q_2}{R_1}\,\frac{Q}{Q_\bullet}. \tag{11.8}$$

The equation (11.1) follows.

□

Theorem 11.2. *Zeros of $\zeta(s)$ accumulate on the line* $\mathrm{Re}\, s = 1$*. For every $\delta < 1$, there exists a zero of zeta with a real part in* $[\delta, \frac{1+\delta}{2}]$.

Proof. First, Theorem 9.2 generalizes as follows. Still defining $u_Q(y) = Q^{\frac{\varepsilon}{2}}\, u(Q^{\varepsilon}y)$, set

$$\widetilde{F}_\varepsilon(s) \colon = \sum_{Q\,\text{odd}} Q^{-s}\left(v \,\middle|\, \Psi\left(Q^{2i\pi\mathcal{E}}\mathfrak{T}_{\frac{\infty}{2}}\right) u_Q\right). \tag{11.9}$$

Assuming that v is supported in $[2, \sqrt{8}]$ and u in $[0, 1]$, the function $\widetilde{F}_\varepsilon(s)$ is entire for every $\varepsilon > 0$; for $\mathrm{Re}\, s > 1 + \sigma_0$, it converges as $\varepsilon \to 0$ towards the function

$$\widetilde{F}_0(s) = \sum_{Q\,\text{odd}} Q^{-s}\left(v \,\middle|\, \Psi\left(Q^{2i\pi\mathcal{E}}\mathfrak{T}_{\frac{\infty}{2}}\right) u\right). \tag{11.10}$$

The proof of this follows the proof of Theorem 9.2, after one has noted that (4.18) (a reduction of $Q^{2i\pi\mathcal{E}}\mathfrak{T}_{\frac{\infty}{2}}$ to $Q^{2i\pi\mathcal{E}}\mathfrak{T}_N$) extends without modification if dropping the assumption that Q is squarefree.

From this point on, the proof follows that of Theorem 10.5. The function $(1+2^{-s})^{-1}\,\frac{\zeta(s)}{\zeta(2s)}$ must be replaced by $\widetilde{f}(s)=\sum_{Q\,\mathrm{odd}} Q^{-s}=(1-2^{-s})\,\zeta(s)$ and $f(s-\nu+\varepsilon\mu)$ by $\widetilde{f}(s-\nu+\varepsilon\mu)$. Half-zeros of zeta do no longer enter the picture, and the condition $a>\frac{\sigma_0}{2}$ in the proof of Theorem 10.5 can be dropped.

That, for every $\delta\in]\frac{1}{2},1[$, there exists a zero of zeta with a real part in $[\delta,\frac{1+\delta}{2}]$, is a consequence of Theorem 10.4. $\square$

Remark 11.1. Just so as to answer a possibly natural question, if one replaces for some $\kappa=1,2,\dots,$ in the definition of $F_0(s)$, the summation over all squarefree odd integers by that over all odd integers in the decompositions of which all primes are taken to powers with exponents $\leq\kappa$, the result obtained by a generalization of the proof of Theorem 10.5 is that $\sigma_0\geq\frac{2}{3+\frac{1}{\kappa+1}}$. It suffices to substitute for the function $f(s)=(1+2^{-s})^{-1}\frac{\zeta(s)}{\zeta(2s)}$ the function $(1+2^{-s}+\cdots+2^{-\kappa s})^{-1}\,\frac{\zeta(s)}{\zeta((\kappa+1)s)}$. Replacing in (10.4) the condition $c>1$ by $c>\sigma_0$, as done in Remark 10.2, one can improve this to $\sigma_0\geq\frac{3}{4+\frac{1}{\kappa+1}}$, or $\sigma_0\geq\frac{3}{4}$ if using all values of κ: this is still less than the result of Theorem 11.2.

Generalizing the result to the case of Dirichlet L-series does not present any new difficulty.

Theorem 11.3. *Let χ be a Dirichlet character mod M ($M=2,3,\dots$), to wit a function χ on $\mathbb{Z}$ such that $\chi(n)=0$ if $(n,M)>1$ while, if $(n,M)=1$, $\chi(n)$ coincides with the value on the class of n of some character of the group $(\mathbb{Z}/M\mathbb{Z})^\times$. We assume that χ is primitive, i.e., not induced by a Dirichlet character mod M', in which M' would be a divisor of M distinct from M. Consider the associated Dirichlet L-series*

$$L(s,\,\chi)=\prod_p\left(1-\frac{\chi(p)}{p^s}\right)^{-1}=\sum_{n\geq 1}\frac{\chi(n)}{n^s},\tag{11.11}$$

and recall that the function $L(s,\,\chi)$ is entire unless the character χ is trivial [7, p.331]*. One has* $\sigma_0\colon = \sup\{\mathrm{Re}\,\rho\colon L(\rho,\,\chi)=0\}=1$.

Proof. It is an adaptation of the proof just given for the Riemann zeta function. Setting, for $r\neq 0$, $a^{\chi}(r)=\prod_{p|r}(1-p\,\chi(p))$, we introduce for every squarefree integer N the distribution

$$\mathfrak{T}_N^{\chi}(x,\xi)=\sum_{j,k\in\mathbb{Z}} a^{\chi}((j,k,N))\,\delta(x-j)\,\delta(\xi-k), \tag{11.12}$$

its truncation $\left(\mathfrak{T}_N^{\chi}\right)^{\times}$ obtained by dropping the term for which $j=k=0$, and the weak limit as $N\nearrow\infty$

$$\mathfrak{T}_{\infty}^{\chi}=\frac{1}{2i\pi}\int_{\mathrm{Re}\,\nu=c}\frac{\mathfrak{E}_{-\nu}}{L(\nu,\,\chi)}\,d\nu,\qquad c\geq 1. \tag{11.13}$$

Discarding the prime 2 in the definition as a product of $L(s,\,\chi)$, we define in a similar way $\mathfrak{T}^{\chi}_{\frac{\infty}{2}}$.

The novelties in the analogue of Theorem 6.1 are the following. We define now, as in (6.3),

$$f_N^{\chi}(j,\,s)=\frac{1}{N}\sum_{k \bmod N} b^{\chi}(j,\,k)\,\exp\left(\frac{2i\pi ks}{N}\right) \tag{11.14}$$

with, this time, $b^{\chi}(j,\,k)=a^{\chi}((j,k,N))=\prod_{p|(j,k,N)}(1-p\,\chi(p))$. Then, $f_N^{\chi}=\otimes_{p|N}f_p^{\chi}$, with (following the proof of (7.4))

$$\begin{aligned} f_p^{\chi}(j,\,s)&=\frac{1}{p}\sum_{k \bmod p}(1-p\,\chi(p)\,\mathrm{char}(j\equiv k\equiv 0 \bmod p))\,\exp\left(\frac{2i\pi ks}{p}\right)\\ &=\mathrm{char}(s\equiv 0 \bmod p)-\chi(p)\,\mathrm{char}(j\equiv 0 \bmod p). \end{aligned} \tag{11.15}$$

It follows that

$$f_N^{\chi}(j,\,s)=\sum_{N_1N_2=N}\mu(N_1)\,\chi(N_1)\,\mathrm{char}(s\equiv 0 \bmod N_2)\,\mathrm{char}(j\equiv 0 \bmod N_1), \tag{11.16}$$

which leads to a generalization of Theorem 6.1. In terms of the reflection map, introducing the distribution $\mathfrak{T}_N^{\overline{\chi}}$, one has

$$c_{R,Q}\left(\mathfrak{T}_N^{\chi},\,m\,n\right)=\mu(Q)\,\chi(Q)\,c_{N,1}\left(\mathfrak{T}_N^{\overline{\chi}};m,\,\overset{\vee}{n}\right). \tag{11.17}$$

One can conclude with the method used in Section 10 or the one used in the present section.

Be sure to use the series $\sum Q^{-s+2i\pi\mathcal{E}}$ as in the case of the Riemann zeta function, certainly not the series $\sum \chi(Q)\, Q^{-s+2i\pi\mathcal{E}}$: the pole of $\zeta(s)$ at $s = 1$ played a crucial role as soon as in the proof of Theorem 5.3. Only, in (10.4), it is the function $\frac{\left(1-\chi(2)\, 2^{-\nu}\right)^{-1}}{L(\nu,\chi)}$ that takes the place of $\frac{\left(1-2^{-\nu}\right)^{-1}}{\zeta(\nu)}$. □

Remark 11.1. It looks very unlikely that one will "ever" find an explicit non-trivial zero of zeta not on the critical line. It is even hard to see how one could find a number $\tau > 0$, in the spirit of Skewes' number (a huge bound on the location of the first sign change in the difference $\pi(n) - \mathrm{li}(n)$), such that one could be assured that there exist non-trivial zeros with a positive imaginary part less than τ. The developments in the present section were conclusive because we concentrated on the real parts, not the imaginary parts of zeros. In this respect, short of finding a spectral interpretation of the real parts of zeros, we shall build in Section 2.1 a family $(ds_{\Sigma}^{(\rho)})$ of automorphic one-dimensional objects in the hyperbolic half-plane with the following property: assuming that ρ is real, $ds_{\Sigma}^{(\rho)}$ misses some Eisenstein series $E^*_{\frac{1-i\lambda}{2}}$ in its spectral decomposition if and only if $\frac{\rho}{2}$ is the real part of a zero of zeta.

12. The role of the Lax-Phillips automorphic scattering theory

The theory of automorphic distributions could have been born from several approaches: the Lax-Phillips automorphic scattering theory [5], representation-theoretic facts such as the way [2] the principal series of representations can be realized by means of functions on the hyperbolic half-plane or by distributions in the plane, or the theory of the Radon transformation [3] in one of its simplest examples. As a matter of fact, it grew out of a cooperation between the Lax-Phillips theory and pseudodifferential operator theory. In the course of the two decades between its introduction [8] and its application ([12] and the present book) to important problems, we were led to stressing the pseudodifferential and representation-theoretic aspects, and we almost forgot the Lax-Phillips origin of the construction. We feel that recovering this aspect of the theory may contribute to a good understanding of the whole rich situation.

In the eighties, with the aim of teaching ourselves some basics of harmonic analysis and modular form theory, we developed generalizations of the Weyl pseudodifferential calculus of operators in which domains such as the hyperbolic half-plane $\mathbb{H}$ could serve as so-called phase spaces (the place where symbols live). The space of functions serving as arguments of the operators could be taken as any of the spaces in the holomorphic discrete series of representations of $SL(2,\mathbb{R})$ or its continuation $(\mathcal{D}_{\tau+1})_{\tau>-1}$ (Knapp's notation). While representation-theoretic and Hilbert-type facts were quite satisfactory, the symbolic calculus of operators was not: the explicit formulas for the composition of symbols (the notion corresponding to the composition of operators) were not as easy as the ones of the Weyl calculus. We soon found out [8, theor. 9.9] that, in order to save the situation, we had to use for symbols pairs of functions on $\mathbb{H}$, and let the two parts of the calculus act on a pair of spaces associated to parameters $\tau, \tau+1$: the case when $\tau = -\frac{1}{2}$ is the one that would coincide, after some changes of variables, to the Weyl calculus.

This necessity to use pairs of functions on $\mathbb{H}$ brought our interest to the Lax-Phillips automorphic scattering theory. It starts with the consideration of the cone

$$C = \{\eta = (\eta_0, \eta_1, \eta_2) \in \mathbb{R}^3 \colon \eta_0 > 0,\, \eta_0^2 - \eta_1^2 - \eta_2^2 > 0\}, \tag{12.1}$$

its boundary ∂C (the forward light-cone in $(1+2)$-dimensional spacetime), and the "mass hyperboloid" $\mathcal{H} = \{\eta \colon \eta_0 > 0,\, \eta_0^2 - \eta_1^2 - \eta_2^2 = 1\}$. As is well-known, the hyperbolic half-plane $\mathbb{H}$ and $\mathcal{H}$ are equivalent models of the symmetric space G/K, with $G = SL(2,\mathbb{R})$, $K = SO(2)$. One enriches the correspondence between the two by means of rescalings, so as to fill up C, obtaining the map

$$(t,\, z) \mapsto \begin{pmatrix} \eta_0 + \eta_1 & \eta_2 \\ \eta_2 & \eta_0 - \eta_1 \end{pmatrix} = e^t \begin{pmatrix} \frac{|z|^2}{y} & \frac{x}{y} \\ \frac{x}{y} & \frac{1}{y} \end{pmatrix} \tag{12.2}$$

from $\mathbb{R} \times \mathbb{H}$ to C. In C, one takes interest in the "d'Alembertian" operator

$$\Box = \frac{\partial^2}{\partial \eta_0^2} - \frac{\partial^2}{\partial \eta_1^2} - \frac{\partial^2}{\partial \eta_2^2}. \tag{12.3}$$

On $\mathbb{H}$, one disposes of the Laplacian Δ. Now [5, p.11], under the map (12.2) and the gauge transformation $u \mapsto W = e^{-\frac{t}{2}} u$, the equation $\Box W = 0$ inside C is equivalent to the wave equation

$$\frac{\partial^2 u}{\partial t^2} + (\Delta - \frac{1}{4})\, u = 0. \tag{12.4}$$

On one hand, one considers the problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + (\Delta - \frac{1}{4})\, u = 0 \\ u(0,\, z) = f_0(z) \\ \frac{\partial u}{\partial t}(0,\, z) = f_1(z), \end{cases} \tag{12.5}$$

in which a solution of (12.4) is characterized by a pair of Cauchy data. On the other hand, ∂C is totally characteristic for the operator $\Box$ and, in a strikingly different way, just one datum on ∂C is needed to characterize W.

Indeed, given a solution of the equation $\Box W = 0$ inside C, extend it by zero outside C, obtaining a function $\breve{W}$ in $\mathbb{R}^3$. The function W is characterized by the density (a function on ∂C) of $\Box \breve{W}$ (taken in the distribution sense) with respect to the measure $\frac{d\eta_1 d\eta_2}{\eta_0}$. To make the correspondence from the pair (f_0, f_1) of Cauchy data to this density, we prefer, having pseudodifferential analysis in mind, to substitute even functions h on $\mathbb{R}^2$ for functions on ∂C by means of the map $h \mapsto Qh$ such that

$$h(x,\, \xi) = (Qh)\left(\frac{x^2+\xi^2}{2},\, \frac{x^2-\xi^2}{2},\, x\xi\right). \tag{12.6}$$

Then, define as $\widetilde{Qh}$ the measure on $\mathbb{R}^3$, supported in ∂C, the density of which with respect to the measure $\frac{d\eta_1 d\eta_2}{\eta_0}$ is the function Qh.

Let Z_2 be the fundamental solution of $\Box$ supported in the closure of C, as provided by M.Riesz' theory [6], given as

$$Z_2(\eta) = \frac{1}{2\pi}\,(\eta_0^2 - \eta_1^2 - \eta_2^2)^{-\frac{1}{2}}, \qquad \eta \in C. \tag{12.7}$$

The function $\breve{W} = Z_2 * \widetilde{Qh}$, supported in the closure of C, satisfies the (distribution) equation $\Box(Z_2 * \widetilde{Qh}) = \widetilde{Qh}$ in $\mathbb{R}^3$: in particular $\Box W = 0$ inside C. Just one function (the function h) is thus needed to build a solution W of $\Box W = 0$ inside C.

All solutions can be obtained in this way, under some explicit regularity conditions. To prove this, we must make the correspondence between the pair (f_0, f_1) in the Cauchy problem (12.5) and the function h, when u and W are linked by the Lax-Phillips transformation, explicit. To this effect, we introduce the Radon transformation V from functions in $\mathbb{H}$ to even functions in $\mathbb{R}^2$ which is the adjoint of the map V^*, the operator from functions, or distributions in $\mathbb{R}^2$ to functions in $\mathbb{H}$ defined by the equation, when convergent,

$$(V^* h)(g\,.\,i) = \int_K h((gk)\,.\,\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right))\,dk, \tag{12.8}$$

where $g \in SL(2,\mathbb{R})$ and $K = SO(2)$. This is the simplest case of an extensive theory developed by Helgason [3]. On the other hand, we introduce the operator T on even functions in the plane, the function in the spectral-theoretic sense of $2i\pi\mathcal{E}$ given as [8, (4.10)]

$$T = (\frac{\pi}{2})^{\frac{1}{2}}\,\frac{\Gamma(\frac{1}{2} - i\pi\mathcal{E})}{\Gamma(-1\pi\mathcal{E})} = \pi^{-\frac{1}{2}}\,(-i\pi\mathcal{E})\int_0^\infty t^{-\frac{1}{2}}(1+t)^{-1+i\pi\mathcal{E}}\,dt. \tag{12.9}$$

After some calculations [8, p. 29 and p. 195-197] based on the use of the Radon transformation and the M.Riesz transform, one ends up with the explicit formula

$$h = 2^{\frac{1}{2}}\,(TV\,f_1 + (i\pi\mathcal{E})\,TV\,f_0). \tag{12.10}$$

What is especially important for us in the whole construction is the question of finding and using a square root of the operator $\Delta - \frac{1}{4}$. The answer provided by operators on $L^2(\mathbb{H})$ the integral kernels of which are functions of the point-pair invariant cosh $d(z,w)$ leads to the spectral theory of the Laplacian, in the open space $\mathbb{H}$ or in other situations, including the automorphic case. There is another answer, which consists in replacing the space $L^2(\mathbb{H})$ by its square, and taking the matrix operator $\begin{pmatrix} 0 & I \\ -\Delta + \frac{1}{4} & 0 \end{pmatrix}$. While it may first look as a joke, it is far form being one, and it is the main object of interest in [5]. Under the correspondence from pairs (f_0, f_1) of functions in $\mathbb{H}$ to even functions h in the plane, this transfers to the operator $i\pi\mathcal{E}$ [8, p. 195-197], an operator which we have used consistently in this paper.

The two "square roots" of $\Delta - \frac{1}{4}$ are totally distinct, and have totally distinct applications. The first is the useful one in analysis in $L^2(\mathbb{H})$ and is by essence a non-negative self-adjoint operator. The operator $2i\pi\mathcal{E}$ differs from $-2i\pi\mathcal{E}$ and the two operators give rise to a notion close (but simpler) to that of ingoing and outgoing waves which is the main subject of automorphic scattering theory. The terminology is justified by the equation

$$g^+(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d\lambda}{(1+i\lambda)^2}\int_0^1 \theta^{1+i\lambda}(\delta^2 g)(\theta y, \theta\eta)\,\frac{d\theta}{\theta}, \tag{12.11}$$

and the similar one regarding $g^-(y)$. The splitting $g = g^+ + g^-$ plays an essential role in [12, prop. 8.2], on the way to a proof of the Ramanujan-Petersson conjecture for Maass forms.

A similar role is played, in the present paper, in relation to the one-dimensional operator $i\left(t\frac{d}{dt}+\frac{1}{2}\right)$, by the decomposition (9.28).

References

[1] E.Bombieri, *Remarks on Weil's quadratic functional in the theory of prime numbers.* Att.Accad.Naz.Lincei, Cl.Sci, Rend.Lincei (9), Math.Appl. **11**,3 (2000), 183-233.

[2] I.M.Gelfand, M.I.Graev, I.I.Pyatetskii-Shapiro, *Representation theory and automorphic functions*, W.B.Saunders Co, Philadelphia-London-Toronto, 1969.

[3] S.Helgason, *Geometric Analysis on Symmetric spaces*, Math.Surveys and Monographs **39**, Amer.Math.Soc, Providence, 1984.

[4] A.W.Knapp, *Representation Theory of Semi-Simple Groups*, Princeton Univ.Press, Princeton, 1986.

[5] P.D.Lax, R.S.Phillips, *Scattering Theory for Automorphic Functions*, Ann.Math.Studies **87**, Princeton Univ.Press, 1976.

[6] M.Riesz, *L'intégrale de Riemann-Liouville et le problème de Cauchy*, Acta Math. **81**(1949), 1-223.

[7] G.Tenenbaum, *Introduction à la théorie analytique et probabiliste des nombres*, Dunod, Paris, 2022.

[8] A.Unterberger, *Quantization and non–holomorphic modular forms*, Lecture Notes in Math. **1742**, Springer–Verlag, Berlin–Heidelberg, 2000.

[9] A.Unterberger, *Aspects of the zeta function originating from pseudodifferential analysis*, J.Pseudo-Differ. Operators Appl., (2014), 157-214.

[10] A.Unterberger, *Pseudodifferential operators with automorphic symbols*, Pseudodifferential Operators **11**, Birkhäuser, Basel–Boston–Berlin, 2015.

[11] A.Unterberger, *Pseudodifferential methods in number theory*, Pseudodifferential Operators **13**, Birkhäuser, Basel-Boston-Berlin, 2018.

[12] A.Unterberger, *The Ramanujan-Petersson conjecture for Maass forms*, arXiv:2001.10956 [math.GR], submitted.

[13] G.Valiron, *Thèse* (1916), *in* wikipedia, *Histoire de la fonction zêta de Riemann*

[14] H.Weyl, *Gruppentheorie und Quantenmechanik*, reprint of 2nd edition, Wissenschaftliche Buchgesellschaft, Darmstadt, 1977.

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