Take the two sectors S and S' where $|y| < \frac{1}{2}|x|$; what is important here is that S and S' *have opening* $< 90°$. *Outside* both S and S', $|x| \leqslant 2|y|$, so $|z| \leqslant 3|y|$, and (††) gives

$$(\S) \qquad \log|P(z)| \leqslant M_\varepsilon + 9\varepsilon(1 + |\Im z|).$$

This also holds on the *boundaries* of S and S', where it can be rewritten thus:

$$\log|P(z)| \leqslant M_\varepsilon + 9\varepsilon + \tfrac{9}{2}\varepsilon|\Re z|.$$

Let us consider the sector S. Inside S, $\log|P(z)| - \frac{9}{2}\varepsilon\Re z$ is *subharmonic*, and $\leqslant \text{const.}|z|^2$ for large $|z|$ by (††). On the *boundary* of S, $\log|P(z)| - \frac{9}{2}\varepsilon\Re z \leqslant M_\varepsilon + 9\varepsilon$ as we have just seen. So, since the opening of S is $< 90°$, *this last inequality must in fact hold throughout* S, by the *second* Phragmén–Lindelöf theorem of §C, Chapter III. We thus get

$$\log|P(z)| \leqslant M_\varepsilon + 9\varepsilon + \tfrac{9}{2}\varepsilon|\Re z|$$

in S.

The same reasoning applies to S'. Referring to (§), which holds *outside* S and S', and contenting ourselves with a result *slightly worse* than what we *actually have*, we see that

$$\log|P(z)| \leqslant M_\varepsilon + 9\varepsilon(1 + |z|)$$

throughout $\mathbb{C}$, whenever $\|P\|_W \leqslant 1$.

If, now, $\Phi(z)$ is *any* u.c.c. *limit* of polynomials P with $\|P\|_W \leqslant 1$, we must *also* have

$$\log|\Phi(z)| \leqslant M_\varepsilon + 9\varepsilon(1 + |z|)$$

for all complex z. Since $\varepsilon > 0$ is *arbitrary* (with M_ε depending on ε through (†)), we see that the entire function $\Phi(z)$ must be of *exponential type zero*. Any $\varphi \in \mathscr{C}_W(\mathbb{R})$ with $\|\varphi\|_W < 1$ which is the $\|\ \ \|_W$-limit of polynomials must, *on the set of x where $W(x) < \infty$*, *coincide* with *such an entire function* $\Phi(z)$, as we saw at the end of the preceding subsection.

We are done.

Remark. Let $\varphi \in \mathscr{C}_W(\mathbb{R})$ be such that there exist polynomials P_n with $\|\varphi - P_n\|_W \xrightarrow[n]{} 0$. Then, as the above theorem shows, if

$$\int_{-\infty}^{\infty} \frac{\log W_*(t)}{1+t^2}\,dt \; < \; \infty,$$

$\varphi(x)$ must, on $\{x\colon W(x) < \infty\}$, *coincide* with an entire function $\Phi(z)$ of *exponential type zero*. Sometimes it is useful to know that $\Phi(z)$ satisfies the more precise condition

$$\log|\Phi(z)| \leqslant \log\|\Phi\|_W + M_\varepsilon + 9\varepsilon(1+|z|), \quad z\in\mathbb{C}.$$

Here, $\varepsilon > 0$ is *arbitrary and* M_ε *depends only on* ε (through (†)), *and not on* Φ. This fact follows immediately from the proof just given – we need only note that $\|\varphi\|_W = \|\Phi\|_W$, so that $\|K^{-1}\varphi\|_W < 1$ for every $K > \|\Phi\|_W$.

Remark. Given that $\int_{-\infty}^{\infty}(\log W_*(t)/(1+t^2))\,dt < \infty$, is it true that for *every* entire function $\Psi(z)$ of *exponential type zero* whose restriction, $\Psi(x)$, to $\mathbb{R}$ belongs to $\mathscr{C}_W(\mathbb{R})$, we *do have* a sequence of polynomials P_n with $\|\Psi - P_n\|_W \xrightarrow[n]{} 0$? As we shall see later on, the *answer to this question is no* for some weights $W(x)$ with seemingly rather regular behaviour.

Theorem. *Suppose that* $W(x_k) < \infty$ *for a sequence of points* x_k *going to* ∞, *and, if* $n(t)$ *denotes the number of the points* x_k *in* $[0,t]$, *suppose that*

$$\limsup_{t\to\infty}\frac{n(t)}{t} > 0.$$

Then, if $\int_{-\infty}^{\infty}(\log W_*(t)/(1+t^2))\,dt < \infty$, *the polynomials are not dense in* $\mathscr{C}_W(\mathbb{R})$.

Proof. Take any function $\varphi\in\mathscr{C}_W(\mathbb{R})$ such that $\varphi(x_0) = 1$ but $\varphi(x_k) = 0$ for $k \geqslant 1$. Then φ *cannot* be $\|\quad\|_W$-approximated by polynomials.

If, indeed, it *could* be so approximated, the preceding theorem would furnish an entire function $\Phi(z)$ of exponential type *zero* with $\Phi(x_k) = \varphi(x_k)$ for $k \geqslant 0$. Then in particular $\Phi(x_0) = 1$, so $\Phi(z) \not\equiv 0$. At the same time $\Phi(x_k) = 0$ for $k \geqslant 1$, so, if $N(r)$ denotes the number of zeros of $\Phi(z)$ with modulus $\leqslant r$, $N(r) \geqslant n(r) - 1$, and $\limsup_{r\to\infty}(N(r)/r)$ would have to be > 0 by hypothesis.

This, however, is *impossible*. For, $\Phi(z)$ being of *exponential type zero* and $\not\equiv 0$, we must have $N(r) = o(r)$ for $r \to \infty$ by an easy application of Jensen's formula (see problem 1(a) in Chapter 1!).

The theorem is proved.

Remark. We shall soon see that the condition $\limsup_{t\to\infty}(n(t)/t) > 0$ in this theorem cannot be relaxed much.

3. Strengthened version of Akhiezer's criterion. Pollard's theorem

The Bernstein approximation problem was also studied by the American mathematician Pollard, whose work was largely independent of Akhiezer's and Mergelian's. Pollard published one of the first solutions, I think in fact *before* the appearance of the other two mathematicians' articles. *In one direction*, the criterion given by him strengthens that furnished by the Akhiezer theorem of article 1. The way this happens is shown by the following

Theorem (due, essentially, to Pollard). *If*

$$\sup\left\{\int_{-\infty}^{\infty}\frac{\log|P(x)|}{1+x^2}\,dx : P \text{ a polynomial and } \|P\|_W \leqslant 1\right\}$$

is finite, then $\int_{-\infty}^{\infty}(\log W_*(x)/(1+x^2))\,dx < \infty$.

Proof. As $x \to \pm\infty$, $W(x) \to \infty$ *faster than any power of* x. So, if we take a suitable constant C,

$$\tilde{W}(x) \;=\; C\frac{W(x)}{|x-i|}$$

is $\geqslant 1$ on $\mathbb{R}$. $\tilde{W}(x)$ obviously grows faster than any power of x as $x \to \pm\infty$, and we may consider *weighted polynomial approximation with the weight* $\tilde{W}$. To this situation we apply the Mergelian theorems in §§A.1 and A.3.

Put, for $z \in \mathbb{C}$,

$$\tilde{\Omega}(z) = \sup\left\{|P(z)|: P \text{ a polynomial and } \left|\frac{P(t)}{(t-i)\tilde{W}(t)}\right| \leqslant 1 \text{ on } \mathbb{R}\right\};$$

note that $\tilde{\Omega}(z)$ is just $CW_*(z)$. According to the theorem of §A.1, $\tilde{\Omega}(i) < \infty$ implies that polynomials are *not* $\|\ \ \|_{\tilde{W}}$-dense in $\mathscr{C}_{\tilde{W}}(\mathbb{R})$, and, by §A.3, the latter fact makes

$$\int_{-\infty}^{\infty}\frac{\log\tilde{\Omega}(t)}{1+t^2}\,dt \;<\; \infty,$$

i.e.,

$$\int_{-\infty}^{\infty}\frac{\log W_*(t)}{1+t^2}\,dt \;<\; \infty.$$

In order to show this last relation, it is therefore enough to verify that $\tilde{\Omega}(i) < \infty$, or, what comes to the same thing, that $W_*(i) < \infty$.

Take a sequence of polynomials $\{P_n(z)\}$ with $\|P_n\|_W \leqslant 1$ and $|P_n(i)| \xrightarrow[n]{} W_*(i)$; by §G.2 of Chapter III,

$$\log|P_n(i)| \;\leqslant\; \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\log|P_n(t)|}{1+t^2}\,dt.$$

Under the hypothesis, however, the integrals on the right are *bounded above*. Therefore $W_*(i) < \infty$, which is what we needed. We are done.

In the course of the argument just given, we established a subsidiary result, important in its own right. We state it as a

Corollary. *If* $W_*(\mathrm{i}) < \infty$, *then* $\int_{-\infty}^{\infty} (\log W_*(t)/(1+t^2))\,\mathrm{d}t < \infty$.

Remark. Sometimes it is easier to get an upper bound for

$$\int_{-\infty}^{\infty} \frac{\log|P(t)|}{1+t^2}\,\mathrm{d}t$$

when P is a polynomial with $\|P\|_W \leqslant 1$ than to try to directly obtain good estimates on $W_*(x)$. If we can show that the upper bound is finite, the description of functions $\|\ \ \|_W$-approximable by polynomials given in article 2 is available, and hence the consequences of that description. For this reason, the result proved here is quite useful.

C. Mergelian's criterion really more general in scope than Akhiezer's. Example

Given a weight $W(x) \geqslant 1$ on $\mathbb{R}$ which goes to ∞ faster than any power of x as $x \to \pm\infty$, we can form the two functions

$$\Omega(z) = \sup\left\{|P(z)|\colon P \text{ a polynomial and } \left|\frac{P(t)}{(t-\mathrm{i})W(t)}\right| \leqslant 1 \text{ on } \mathbb{R}\right\},$$

and

$$W_*(z) = \sup\left\{|P(z)|\colon P \text{ a polynomial and } \left|\frac{P(t)}{W(t)}\right| \leqslant 1 \text{ on } \mathbb{R}\right\}.$$

Mergelian's second theorem (§A.3) says that the condition

$$\int_{-\infty}^{\infty} \frac{\log \Omega(t)}{1+t^2}\,\mathrm{d}t \;=\; \infty$$

is necessary and sufficient for polynomials to be $\|\ \ \|_W$-dense in $\mathscr{C}_W(\mathbb{R})$. Akhiezer's theorem (§B.1) says that the condition

$$\int_{-\infty}^{\infty} \frac{\log W_*(t)}{1+t^2}\,\mathrm{d}t \;=\; \infty$$

is *always sufficient* for the $\|\ \ \|_W$-density of polynomials in $\mathscr{C}_W(\mathbb{R})$, and *necessary* for that density to hold *provided that* $W(x)$ *has a certain regularity*. As we saw in §B.1, *continuity* of W is *enough* here; it suffices in fact that $W(x)$ be finite on an infinite closed set with a finite point of accumulation. The work of §B.2 shows that the *set on which* $W(x)$ *is finite need not even have a finite point of accumulation*; it is enough that the set be *infinite and not too sparse*. As long as

$$\limsup_{r\to\infty} \frac{\text{number of points in the set and in } [-r, r]}{r}$$

is *positive*, the necessity of Akhiezer's criterion (involving $\log W_*(t)$) holds good.

These successive relaxations in the regularity required of $W(x)$ for Akhiezer's criterion to hold make us hope that perhaps *all* restrictions on W's regularity may be dispensed with. Maybe the lower polynomial regularization $W_*(x)$ of $W(x)$ is all we need for the study of Bernstein's problem, no matter what the behaviour of the latter function is, and we can *forget* about $\Omega(x)$ altogether. Do we *really need* $\Omega(x)$ at all in order to have a completely general test for $\| \; \|_W$-density of polynomials in $\mathscr{C}_W(\mathbb{R})$?

We do. Here is an *example of a weight* $W(x)$ *such that*

$$\int_{-\infty}^{\infty} \frac{\log W_*(t)}{1+t^2} dt \;<\; \infty,$$

but nevertheless $\Omega(\mathrm{i}) = \infty$, *making the polynomials* $\| \; \|_W$*-dense in* $\mathscr{C}_W(\mathbb{R})$.

Our construction is based on use of the entire function

$$S(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{4^n}\right)$$

of *exponential type zero.* The weight $W(x)$ will be *identically infinite* outside the set of points

$$x_k = \operatorname{sgn} k \cdot 2^{|k|}; \quad k = \pm 1, \pm 2, \ldots;$$

these are just the *zeros* of $S(z)$. *On that set* we take

$$W(x_k) = C\sqrt{|x_k| \cdot |S'(x_k)|}$$

with a constant C chosen so as to make $W(x_k) \geqslant 1$ for all k.

We start with the asymptotic evaluation of $S'(x_k)$ for large $|k|$; on account of symmetry we need only consider positive values of k. For $k \geqslant 1$, then,

$$S'(2^k) \;=\; -\frac{2}{2^k} \prod_{1 \leqslant n < k} \left(1 - \frac{4^k}{4^n}\right) \cdot \prod_{n > k} \left(1 - \frac{4^k}{4^n}\right).$$

Here is a *trick* which can be used to good effect in many calculations of this kind. *Factor* each ratio $4^k/4^n$ with $k > n$ *out of the first product.* One finds that

$$\begin{aligned} S'(2^k) \;&=\; -\frac{2}{2^k}(-1)^{k-1} \prod_{1 \leqslant n < k} 4^{k-n} \cdot \prod_{1 \leqslant n < k} \left(1 - \frac{4^n}{4^k}\right) \prod_{n > k} \left(1 - \frac{4^k}{4^n}\right) \\ &=\; \frac{2(-1)^k}{2^k} \cdot 2^{(k-1)k} \prod_{l=1}^{k-1} \left(1 - \frac{1}{4^l}\right) \prod_{m=1}^{\infty} \left(1 - \frac{1}{4^m}\right). \end{aligned}$$

For large k, this is

$$\sim \; (-1)^k 2^{(k-1)^2} (S(1))^2,$$

and we see that $|S'(x_k)|$ behaves like a constant multiple of $|x_k|^{(|k|-2)}$ for $k \to \pm\infty$. This evidently tends to ∞ faster than any power of x_k as $k \to \pm\infty$; the same is then true of $W(x_k)$.

We need also to consider the partial products

$$S_N(z) \;=\; \prod_{n=1}^{N}\left(1-\frac{z^2}{4^n}\right)$$

of $S(z)$. For $1 \leqslant k \leqslant N$, we have

$$S'_N(2^k) \;=\; -\frac{2}{2^k}\prod_{1\leqslant n<k}\left(1-\frac{4^k}{4^n}\right)\prod_{k<n\leqslant N}\left(1-\frac{4^k}{4^n}\right);$$

comparison of this with the *first* of the above formulas for $S'(2^k)$ shows that

$$|S'_N(x_k)| \geqslant |S'(x_k)| \quad \text{for } 1 \leqslant k \leqslant N.$$

On account of this fact and of the *growth* of $|S'(x_k)|$ for $k \to \pm\infty$, we have, for any *polynomial* P,

$$\sum_{-N}^{N}{}' \frac{P(x_k)}{S'_N(x_k)(z-x_k)} \;\xrightarrow[N]{}\; \sum_{-\infty}^{\infty}{}' \frac{P(x_k)}{S'(x_k)(z-x_k)}$$

as long as z is different from all the x_k.

▶ N.B. *A prime next to a summation sign means that there is no term corresponding to the value zero of the summation index.* (This convention is fairly widespread, by the way.)

Let us fix any polynomial P. As soon as $2N >$ degree of P, we have, by Lagrange's interpolation formula,

$$P(z) \;=\; S_N(z)\sum_{-N}^{N}{}'\frac{P(x_k)}{S'_N(x_k)(z-x_k)},$$

provided that z is different from all the x_k. Fixing such a z, and making $N \to \infty$, we get, by virtue of the previous relation,

$$(*) \qquad P(z) \;=\; S(z)\sum_{-\infty}^{\infty}{}'\frac{P(x_k)}{S'(x_k)(z-x_k)}.$$

This formula is valid, then, for any polynomial P and any z different from all the x_k.

We estimate $W_*(\mathrm{i})$. Take any polynomial P with $\|P\|_W \leqslant 1$, i.e., with

$$|P(x_k)| \leqslant C\sqrt{|x_k|}\cdot|S'(x_k)|.$$

Substituting into $(*)$, we find that

$$|P(\mathrm{i})| \;\leqslant\; CS(i)\sum_{-\infty}^{\infty}{}'\frac{\sqrt{|x_k|}}{|\mathrm{i}-x_k|} \;\leqslant\; 2CS(\mathrm{i})\sum_{1}^{\infty}2^{-k/2} \;=\; \frac{2CS(\mathrm{i})}{\sqrt{2}-1}.$$

Taking the supremum of $|P(\mathrm{i})|$ over all such P, we see that

$$W_*(\mathrm{i}) \leqslant \frac{2CS(\mathrm{i})}{\sqrt{2}-1} < \infty.$$

According to the corollary in §*B*.3, *this implies that*

$$\int_{-\infty}^{\infty} \frac{\log W_*(t)}{1+t^2}\,\mathrm{d}t < \infty.$$

It is now claimed that $\Omega(\mathrm{i}) = \infty$. To see this, consider the polynomials

$$P_N(x) = \sqrt{x_N}\cdot S_N(x);$$

since $P_N(\mathrm{i})/\sqrt{x_N} \underset{N}{\longrightarrow} S(\mathrm{i}) > 0$, it is clear that $P_N(\mathrm{i}) \underset{N}{\longrightarrow} \infty$. It is therefore enough to show that

$$\left|\frac{P_N(x)}{(x-\mathrm{i})W(x)}\right| \leqslant \frac{1}{2CS(1)}$$

on $\mathbb{R}$ in order to conclude that $\Omega(\mathrm{i}) = \infty$. We have, in other words, to verify that

$$|P_N(x_k)| \leqslant \frac{|x_k - \mathrm{i}|\,W(x_k)}{2CS(1)}$$

for $k = \pm 1, \pm 2, \ldots$. This is true for $1 \leqslant |k| \leqslant N$ because then $P_N(x_k) = 0$. Suppose, therefore, that $k > N$. Then

$$\begin{aligned}
|P_N(x_k)| &= \sqrt{x_N}\cdot \prod_{1\leqslant n\leqslant N}\left|\frac{4^k}{4^n} - 1\right| \leqslant \sqrt{x_N}\cdot \prod_{1\leqslant n<k}\left|\frac{4^k}{4^n} - 1\right| \\
&= \frac{\sqrt{x_N}\cdot x_k |S'(x_k)|}{2\prod_{n>k}\left(1 - \dfrac{4^k}{4^n}\right)} \\
&\leqslant \frac{\sqrt{x_N}\cdot x_k |S'(x_k)|}{2S(1)}.
\end{aligned}$$

Taking symmetry into account, we see that, for $|k| > N$,

$$\begin{aligned}
|P_N(x_k)| &\leqslant \frac{|x_k|}{2S(1)}\sqrt{x_N}\cdot|S'(x_k)| \\
&\leqslant \frac{1}{2S(1)}|x_k - \mathrm{i}|\sqrt{|x_k|}\,|S'(x_k)| = \frac{|x_k - \mathrm{i}|\,W(x_k)}{2CS(1)}.
\end{aligned}$$

We thus have $|P_N(x_k)| \leqslant |x_k - \mathrm{i}|\,W(x_k)/2CS(1)$ for all k and every N, and this, as we have seen, ensures that $\Omega(\mathrm{i}) = \infty$.

Because $\Omega(\mathrm{i}) = \infty$, *polynomials are* $\| \quad \|_W$*-dense in* $\mathscr{C}_W(\mathbb{R})$ *by the first Mergelian theorem* (§A.1). *However, as we already have shown,*

$$\int_{-\infty}^{\infty} \frac{\log W_*(t)}{1+t^2} \mathrm{d}t \ < \ \infty.$$

Application of Akhiezer's criterion to $\mathscr{C}_W(\mathbb{R})$ with the weight W considered here would therefore lead to a false result.

D. Some partial results involving the weight W explicitly

Let us see how much information about the $\| \quad \|_W$-density of polynomials in $\mathscr{C}_W(\mathbb{R})$ can be obtained by direct examination of the weight $W(x)$ itself. Here, first of all, is an easy negative result.

Theorem (T. Hall). *If* $\int_{-\infty}^{\infty}(\log W(x)/(1+x^2))\mathrm{d}x < \infty$, *the polynomials are not* $\| \quad \|_W$*-dense in* $\mathscr{C}_W(\mathbb{R})$.

Proof. Trivially, $\Omega(x) \leqslant |x-\mathrm{i}|\,W(x)$ for $x \in \mathbb{R}$, so, if the above integral with $\log W(x)$ converges, so is

$$\int_{-\infty}^{\infty} \frac{\log \Omega(x)}{1+x^2} \mathrm{d}x \ < \ \infty.$$

The desired result now follows from Mergelian's second theorem, §A.3.

One is, naturally, very interested in finding simple conditions on W which will guarantee $\| \quad \|_W$*-density* of polynomials in $\mathscr{C}_W(\mathbb{R})$. In this direction, we begin with a very old result.

Theorem (S. Bernstein). *Let* $W(x) = \sum_{n=0}^{\infty} A_n x^{2n}$ *where the* $A_n \geqslant 0$ *and the series is everywhere convergent. If*

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2} \mathrm{d}x \ = \ \infty,$$

polynomials are $\| \quad \|_W$*-dense in* $\mathscr{C}_W(\mathbb{R})$.

Proof. The polynomials

$$P_N(x) \ = \ \sum_{n=0}^{N} A_n x^{2n}$$

satisfy $\|P_N\|_W \leqslant 1$, because all the A_n are $\geqslant 0$. Clearly $P_N(x) \underset{N}{\longrightarrow} W(x)$ for each x, so *here* $W_*(x) = W(x)$. Our result now follows from Akhiezer's theorem (§B.1).

Corollary. *The polynomials are* $\| \quad \|_W$*-dense in* $\mathscr{C}_W(\mathbb{R})$ *for* $W(x) = \mathrm{e}^{x^2}$.

Corollary. *The polynomials are* $\| \ \|_W$*-dense in* $\mathscr{C}_W(\mathbb{R})$ *for* $W(x) = e^{|x|}$.

Proof. Use the fact that $\frac{1}{2}e^{|x|} \leqslant \cosh x \leqslant e^{|x|}$ and work with the weight

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots.$$

Theorem. *Let* $W(x) \geqslant 1$ *be even, and suppose that, for* $x > 0$, $\log W(x)$ *is a convex function of* $\log x$. *Then, if*

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2}\,dx \;=\; \infty,$$

polynomials are $\| \ \|_W$*-dense in* $\mathscr{C}_W(\mathbb{R})$.

Proof. Starts out like that of the corollary in §C.2, Chapter V, with the use of some material from convex logarithmic regularization. Write

$$S_n \;=\; \sup_{r>0} \frac{r^n}{W(r)} \qquad \text{for } n = 0, 1, 2, \ldots,$$

and then put, for $r > 0$,

$$T(r) \;=\; \sup_{n \geqslant 0} \frac{r^{2n}}{S_{2n}}.$$

Since $\log W(r)$ is a convex function of $\log r$, we have, by the *proof* of the second lemma in Chapter IV, §D, that

$$(*) \qquad \frac{W(r)}{r^2} \;\leqslant\; T(r) \;\leqslant\; W(r)$$

whenever r is sufficiently large. (It's $W(r)/r^2$ on the left and not $W(r)/r$ because we use the *even* powers of r in forming $T(r)$.)

Take any fixed number λ between 0 and 1, and form the function

$$S(x) \;=\; (1-\lambda^2) \sum_{n=0}^{\infty} \lambda^{2n} \frac{x^{2n}}{S_{2n}}.$$

We see from the definition of $T(r)$ and $(*)$ that $S(x) \leqslant W(x)$ for $|x|$ sufficiently large, since W is even. Therefore, as in the proof of the above theorem of Bernstein (the numbers S_{2n} are all positive!), we at least have

$$W_*(x) \geqslant CS(x)$$

for some $C > 0$ chosen so as to make $CS(x) \leqslant W(x)$ for *all* x.

Referring again to $(*)$, we see, however, that

$$S(x) \;\geqslant\; (1-\lambda^2)T(\lambda|x|) \;\geqslant\; (1-\lambda^2)\frac{W(\lambda|x|)}{\lambda^2 x^2}$$

for $|x|$ sufficiently large. Hence, taking a big enough,

$$\int_a^\infty \frac{\log W_*(x)}{x^2}\mathrm{d}x \;\geqslant\; \frac{\log C}{a} + \int_a^\infty \frac{\log S(x)}{x^2}\mathrm{d}x$$

$$\geqslant\; \frac{\log C}{a} + \frac{1}{a}\log\left(\frac{1-\lambda^2}{\lambda^2}\right) - 2\int_a^\infty \frac{\log x}{x^2}\mathrm{d}x + \int_a^\infty \frac{\log W(\lambda x)}{x^2}\mathrm{d}x.$$

The last integral on the right equals $\lambda\int_{\lambda a}^\infty (\log W(\xi)/\xi^2)\mathrm{d}\xi$ which is clearly infinite if $\int_{-\infty}^\infty (\log W(x)/(1+x^2))\mathrm{d}x = \infty$.
So $\int_{-\infty}^\infty (\log W_*(x)/(1+x^2))\mathrm{d}x = \infty$, and the result follows by Akhiezer's theorem.

Remark. Is the theorem still true if the even function $W(x)$ is merely required to be *increasing* for $x > 0$? An example to be given in Chapter VII shows that the answer to this question is *no*.

E. Weighted approximation by sums of imaginary exponentials with exponents from a finite interval

Let $W(x) \geqslant 1$ be a weight which is now merely assumed to tend to ∞ as $x \longrightarrow \pm\infty$. We fix some $A > 0$ and ask whether the collection of *finite sums* of the form

$$\sum_{-A\leqslant\lambda\leqslant A} C_\lambda e^{i\lambda x}$$

is $\|\ \|_W$-dense in $\mathscr{C}_W(\mathbb{R})$. It turns out that the theorems of Mergelian and Akhiezer given in §§A and B above have *complete analogues* in the present situation. We will be able to see this in the present § without having to repeat most of the details from the preceding discussion.

1. Equivalence with weighted approximation by certain entire functions of exponential type. The collection $\mathscr{E}_A$

If $\sigma(t)$ is a finite sum of the form $\sum_{-A\leqslant\lambda\leqslant A} C_\lambda e^{i\lambda t}$ and z_0 is a complex number, the ratio $(\sigma(t) - \sigma(z_0))/(t - z_0)$ is no longer expressible as such a sum. Therefore the useful Markov–Riesz–Pollard trick applied, for instance, in the proof of the second Mergelian theorem (§A.3) is not available for such sums. For this reason, the following result is very important.

Lemma. *If $W(x) \geqslant 1$ and $W(x) \to \infty$ for $x \to \pm\infty$, every entire function of exponential type $\leqslant A$, bounded on the real axis, can be $\|\ \|_W$-approximated by finite sums of the form*

$$\sum_{-A\leqslant\lambda\leqslant A} C_\lambda e^{i\lambda x}.$$

Proof. Take any entire function $f(z)$ of exponential type $\leqslant A$, bounded on $\mathbb{R}$. Since $W(x) \to \infty$ for $x \to \pm\infty$,

$$\sup_{x\in\mathbb{R}} \left| \frac{f(x) - f(\rho x)}{W(x)} \right| \longrightarrow 0$$

for $\rho \to 1$ by *continuity* of f. Given $\varepsilon > 0$, fix a $\rho < 1$ such that the above supremum is $< \varepsilon$. If $h > 0$ is small enough, we will also have

$$\sup_{x\in\mathbb{R}} \left| \frac{1}{W(x)} \left(f(\rho x) - f(\rho x) \frac{\sin hx}{hx} \right) \right| \;<\; \varepsilon;$$

we take such an h so small that

$$g(z) = f(\rho z) \frac{\sin hz}{hz}$$

is *of exponential type* $\leqslant A$, and fix it.

We thus have $\|f - g\|_W < 2\varepsilon$. However, g, besides being of exponential type $\leqslant A$, is *also* in L_2 on the *real axis*. We can therefore apply the Paley–Wiener theorem (Chapter III, §D) to g, obtaining

$$g(x) \;=\; \int_{-A}^{A} e^{i\lambda x} G(\lambda) d\lambda$$

with some $G \in L_2(-A, A)$. This property of G also makes $G \in L_1(-A, A)$, by Schwarz' inequality.

For large integers N, put

$$g_N(x) \;=\; \sum_{k=0}^{N-1} e^{i(-A+(2k/N)A)x} \int_{-A+(2k/N)A}^{-A+((2k+2)/N)A} G(\lambda) d\lambda.$$

For each N, $|g_N(x)| \leqslant \|G\|_1$ for $x \in \mathbb{R}$, and we clearly have $g_N(x) \xrightarrow[N]{} g(x)$ u.c.c. on $\mathbb{R}$. Therefore $\|g - g_N\|_W \xrightarrow[N]{} 0$, so, taking N large enough, we get

$$\|f - g_N\|_W \;\leqslant\; \|f - g\|_W + \|g - g_N\|_W \;<\; 3\varepsilon.$$

Since $g_N(x)$ is a finite sum of the form

$$\sum_{-A\leqslant\lambda\leqslant A} C_\lambda e^{i\lambda x},$$

we are done.

Definition. $\mathscr{E}_A$ denotes the set of entire functions of exponential type $\leqslant A$, bounded on the real axis.

Since every finite sum $\sum_{-A\leqslant\lambda\leqslant A} C_\lambda e^{i\lambda x}$ certainly *belongs* to $\mathscr{E}_A$, the above lemma has the obvious

Corollary. *Let $\varphi \in \mathscr{C}_W(\mathbb{R})$. There are finite sums $\sigma(x)$ of the form*

$$\sum_{-A \leqslant \lambda \leqslant A} C_\lambda e^{i\lambda x}$$

making $\|\varphi - \sigma\|_W$ arbitrarily small if and only if there are $f_n \in \mathscr{E}_A$ with $\|\varphi - f_n\|_W \xrightarrow[n]{} 0$.

Remark. What is important here is that, if $f \in \mathscr{E}_A$ and $z_0 \in \mathbb{C}$, the ratio $(f(t) - f(z_0))/(t - z_0)$ *also* belongs to $\mathscr{E}_A$.

2. The functions $\Omega_A(z)$ and $W_A(z)$. Analogues of Mergelian's and Akhiezer's theorems

In analogy with the definition of $\Omega(z)$ (beginning of §A), we put

$$\Omega_A(z) = \sup\left\{ |f(z)|\colon f \in \mathscr{E}_A \text{ and } \left|\frac{f(t)}{(t-i)W(t)}\right| \leqslant 1 \text{ on } \mathbb{R} \right\}.$$

Remark. A slight extension of the argument used to prove the lemma in the preceding subsection shows that $\Omega_A(z)$ is already obtained if we use only finite sums $f(t)$ of the form $\sum_{-A \leqslant \lambda \leqslant A} C_\lambda e^{i\lambda t}$ in taking the supremum on the right. Verification of this fact is left to the reader.

Observe that, if $f \in \mathscr{E}_A$, so is the function f^* defined by the formula $f^*(z) = \overline{f(\bar{z})}$. This makes $\Omega_A(z) = \Omega_A(\bar{z})$. As we have already noted, when $f(t) \in \mathscr{E}_A$, the quotient $(f(t) - f(z_0))/(t - z_0)$ also belongs to $\mathscr{E}_A$. So, by the way, does

$$\frac{f(t) - f(z_0)}{t - z_0}(t - z_1)$$

belong to $\mathscr{E}_A$ then. These evident facts make it possible for us to virtually *copy* the proof of the first Mergelian theorem as given in §A.1, replacing the collection of *polynomials* by $\mathscr{E}_A$. Keeping the lemma from the previous subsection in mind, we obtain, in this way, the

Theorem. *If $\Omega_A(z) = \infty$ for one non-real z, the collection of finite sums of the form*

$$\sum_{-A \leqslant \lambda \leqslant A} C_\lambda e^{i\lambda x}$$

is $\| \ \|_W$-dense in $\mathscr{C}_W(\mathbb{R})$. Conversely, if such sums are $\| \ \|_W$-dense in $\mathscr{C}_W(\mathbb{R})$, $\Omega_A(z) = \infty$ for all non-real z.

The second theorem in §G.2 of Chapter III applies to the functions $f \in \mathscr{E}_A$, and we have, for them,

$$(*) \qquad \log|f(z)| \;\leqslant\; A|\Im z| + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|\log|f(t)|}{|z-t|^2}\,dt.$$

Using this relation we can copy the proof of Mergelian's *second* theorem (§A.3), to get

Theorem. *The finite sums of the form* $\sum_{-A\leqslant\lambda\leqslant A} C_\lambda e^{i\lambda x}$ *are* $\| \;\|_W$*-dense in* $\mathscr{C}_W(\mathbb{R})$ *if and only if*

$$\int_{-\infty}^{\infty} \frac{\log \Omega_A(x)}{1+x^2}\,dx \;=\; \infty.$$

To obtain analogues of Akhiezer's theorems (§§B.1 and B.2), we write

$$W_A(z) = \sup\{|f(z)|: f\in\mathscr{E}_A \text{ and } \|f\|_W \leqslant 1\}.$$

As in the formation of $\Omega_A(z)$, we can limit the set of functions f occurring on the right to the ones expressible as finite sums $\sum_{-A\leqslant\lambda\leqslant A} C_\lambda e^{i\lambda x}$. $W_A(x)$ is thus a *lower regularization* of $W(x)$ by such finite sums.

Arguing as in §§B.1, B.2, with use of $(*)$ in the appropriate places, we find:

Theorem. *For continuous* W, *finite sums of the form*

$$\sum_{-A\leqslant\lambda\leqslant A} C_\lambda e^{i\lambda x}$$

are $\| \;\|_W$*-dense in* $\mathscr{C}_W(\mathbb{R})$ *if and only if*

$$\int_{-\infty}^{\infty} \frac{\log W_A(x)}{1+x^2}\,dx \;=\; \infty.$$

Theorem. *If* $\int_{-\infty}^{\infty}(\log W_A(x)/(1+x^2))\,dx < \infty$, *any function in* $\mathscr{C}_W(\mathbb{R})$ *which can be* $\| \;\|_W$*-approximated by finite sums of the form* $\sum_{-A\leqslant\lambda\leqslant A} C_\lambda e^{i\lambda x}$ *coincides, on the set of points where* $W(x) < \infty$, *with an entire function* $\Phi(z)$ *satisfying, for each* $\varepsilon > 0$, *an inequality of the form*

$$|\Phi(z)| \;\leqslant\; \|\Phi\|_W M_\varepsilon \exp(A|\Im z| + \varepsilon|z|).$$

Here, M_ε *depends only on* ε, *and is independent of the particular function* Φ *arising in this manner.*

Corollary. *Let* $\int_{-\infty}^{\infty}(\log W_A(x)/(1+x^2))\,dx \;<\; \infty$, *and denote by E the set of points on* $\mathbb{R}$ *where* $W(x) < \infty$. *If either*

$$\limsup_{r\to\infty} \frac{\text{number of points in } E\cap[0,r]}{r} \;>\; \frac{A}{\pi}$$

or

$$\limsup_{r\to\infty} \frac{\text{number of points in } E\cap[-r,0]}{r} > \frac{A}{\pi},$$

then $\mathscr{E}_A$ *cannot be* $\|\quad\|_W$*-dense in* $\mathscr{C}_W(\mathbb{R})$.

Proof. Is based on a result much deeper than the one needed for the corresponding proposition about weighted polynomial approximation (end of §B.2).

Suppose, wlog, that $W(x_k) < \infty$ where $0 \leqslant x_0 < x_1 < x_2 < \cdots$, and that $\limsup_{n\to\infty} n/x_n > A/\pi$. (If the set E has a *finite* limit point, one can give a much simpler argument.) Take any continuous bounded φ (belonging thus to $\mathscr{C}_W(\mathbb{R})$) with $\varphi(x_0) = 1$ and $\varphi(x_k) = 0$ for $k \geqslant 1$; it is claimed that such a function φ cannot be $\|\quad\|_W$-approximated by functions in $\mathscr{E}_A$.

If it could, we would, by the theorem, get an entire function $\Phi(z)$ with $\Phi(x_k) = \varphi(x_k)$, $k \geqslant 0$ (hence $\Phi(x_0) = 1$ so that $\Phi \not\equiv 0$) satisfying, for every $\varepsilon > 0$, an inequality of the form

$$|\Phi(z)| \leqslant C_\varepsilon \exp(A|\Im z| + \varepsilon|z|).$$

This certainly makes Φ of exponential type $\leqslant A$. We also have

$$(\dagger) \qquad \int_{-\infty}^{\infty} \frac{\log^+|\Phi(x)|}{1+x^2}\,dx < \infty.$$

Indeed, there is a sequence of functions $f_n \in \mathscr{E}_A$ with

$$\|\varphi - f_n\|_W \xrightarrow[n]{} 0$$

(hence, wlog, $\|f_n\|_W \leqslant 1$), and $f_n(z) \xrightarrow[n]{} \Phi(z)$ u.c.c. (That's how one shows there *is* such a function Φ – see §§B.1 and B.2!) Since $\|f_n\|_W \leqslant 1$ we have *by definition* $|f_n(z)| \leqslant W_A(z)$, and thus finally $|\Phi(z)| \leqslant W_A(z)$. We are, however, *assuming* that $\int_{-\infty}^{\infty} (\log W_A(x)/(1+x^2))dx < \infty$, and, in the last integral, we may replace log by $\log^+$, because $W_A(z) \geqslant 1$. (Note that $1 \in \mathscr{E}_A$!) Therefore $(\dagger)$ holds.

The *hypothesis of Levinson's theorem,* from §H.3 of Chapter III *is thus satisfied.* If $n_+(r)$ denotes the number of zeros of $\Phi(z)$ in the right half plane having modulus $\leqslant r$, that theorem says that $\lim_{r\to\infty} n_+(r)/r$ *exists,* and *here* has a value $\leqslant A/\pi$. However, $\Phi(x_k) = \varphi(x_k) = 0$ for $k \geqslant 1$, so certainly $n_+(x_k) \geqslant k$. Our assumption that $\limsup_{k\to\infty} k/x_k > A/\pi$ therefore leads to a contradiction. The corollary is proved.

3. Scholium. Pólya's maximum density

We have not really used the full strength of Levinson's theorem in proving the corollary at the end of the preceding article. One can in fact replace the assumption that

$$\limsup_{r\to\infty} \frac{\text{number of points in } E\cap[0,r]}{r} > \frac{A}{\pi}$$

by a *weaker* one, and the *corollary's conclusion will still apply.*

Suppose we have any increasing sequence of points $x_k \geqslant 0$, some of which may be repeated. For $r>0$, denote the number of those points on $[0,r]$ (counting repetitions) by $N(r)$, and, for each positive $\lambda<1$, put

$$D_\lambda = \limsup_{r\to\infty} \frac{N(r)-N(\lambda r)}{(1-\lambda)r}$$

Note that if $\limsup_{r\to\infty} N(r)/r = \bar{D}$ is *finite*, we certainly have $\bar{D} \leqslant D_\lambda$ for each $\lambda<1$, as simple verification shows.

Lemma. $\lim_{\lambda\to 1} D_\lambda$ *exists (it may be infinite).*

Proof. Let $0<\lambda<\lambda'<1$, Writing $\lambda/\lambda' = \mu$, we have the identity

$$\frac{N(r)-N(\lambda r)}{(1-\lambda)r} = \frac{1-\lambda'}{1-\lambda}\frac{N(r)-N(\lambda' r)}{(1-\lambda')r} + \frac{\lambda'-\lambda}{1-\lambda}\frac{N(\lambda' r)-N(\mu\lambda' r)}{(1-\mu)\lambda' r},$$

whence

$$(*) \qquad D_\lambda \leqslant \frac{1-\lambda'}{1-\lambda}D_{\lambda'} + \frac{\lambda'-\lambda}{1-\lambda}D_{\lambda/\lambda'}.$$

Since $N(r)$ is increasing we also have, for $0<\lambda<\lambda'<1$,

$$\frac{N(r)-N(\lambda r)}{(1-\lambda)r} \geqslant \frac{1-\lambda'}{1-\lambda}\frac{N(r)-N(\lambda' r)}{(1-\lambda')r},$$

so

$$D_\lambda \geqslant \frac{1-\lambda'}{1-\lambda}D_{\lambda'}.$$

Suppose first of all that $\limsup_{\lambda\to 1} D_\lambda = \infty$. Then, if we have $D_{\lambda_0} \geqslant M$, say, for some λ_0, $0<\lambda_0<1$, the previous relation shows that $D_\lambda \geqslant (1/(1+\lambda_0))D_{\lambda_0} > \frac{1}{2}M$ for $\lambda_0^2 \leqslant \lambda \leqslant \lambda_0$. However, substituting $\lambda = \lambda_0$ and $\lambda' = \sqrt{\lambda_0}$ in (*), we get $D_{\lambda_0} \leqslant D_{\sqrt{\lambda_0}}$, so also $D_{\sqrt{\lambda_0}} \geqslant M$. Then, by the reasoning just given, $D_\lambda \geqslant M/2$ for $\lambda_0 \leqslant \lambda \leqslant \sqrt{\lambda_0}$. This same argument can evidently be repeated indefinitely, getting $D_{\lambda_0^{1/4}} \geqslant M$, $D_\lambda \geqslant M/2$ for $\sqrt{\lambda_0} \leqslant \lambda \leqslant \lambda_0^{1/4}$, and so forth. Hence $D_\lambda \geqslant M/2$ for $\lambda_0^2 \leqslant \lambda < 1$, so, since M was arbitrary, $D_\lambda \to \infty$ as $\lambda \to 1$.

Consider now the case where $\limsup_{\lambda\to\infty} D_\lambda = L$ is finite, and, picking any $\varepsilon>0$, take any λ_0, $0<\lambda_0<1$, such that $D_{\lambda_0} > L-\varepsilon$ but $D_\lambda < L+\varepsilon$

for $\lambda_0 \leqslant \lambda < 1$. Putting $\lambda = \lambda_0$ in (*), we find, for $\lambda_0 \leqslant \lambda' < 1$,

$$L - \varepsilon \;\leqslant\; \frac{1-\lambda'}{1-\lambda_0} D_{\lambda'} \;+\; \frac{\lambda'-\lambda_0}{1-\lambda_0}(L+\varepsilon),$$

that is,

$$\frac{1-\lambda'}{1-\lambda_0} L \;-\; \varepsilon \;-\; \frac{\lambda'-\lambda_0}{1-\lambda_0}\varepsilon \;\leqslant\; \frac{1-\lambda'}{1-\lambda_0} D_{\lambda'},$$

and

$$D_{\lambda'} \;\geqslant\; L - \frac{1+\lambda'-2\lambda_0}{1-\lambda'}\varepsilon.$$

For $\lambda_0 \leqslant \lambda' \leqslant \sqrt{\lambda_0}$, the right-hand side is $\geqslant\; L - (1+2\sqrt{\lambda_0})\varepsilon \;>\; L - 3\varepsilon$, so we see that in fact

$$L - 3\varepsilon \;<\; D_{\lambda'} \;<\; L + \varepsilon \quad \text{for } \lambda_0 \leqslant \lambda' \leqslant \sqrt{\lambda_0}.$$

As we already saw, $D_{\sqrt{\lambda_0}} \geqslant D_{\lambda_0}$. Therefore $D_{\sqrt{\lambda_0}} > L - \varepsilon$, and we may repeat the last argument with $\sqrt{\lambda_0}$ instead of λ_0 to conclude that

$$L - 3\varepsilon \;<\; D_{\lambda'} \;<\; L + \varepsilon \quad \text{for } \sqrt{\lambda_0} \leqslant \lambda' \leqslant \lambda_0^{1/4}.$$

Continuing in this way, we see that

$$L - 3\varepsilon \;<\; D_{\lambda'} \;<\; L + \varepsilon \quad \text{for } \lambda_0 \leqslant \lambda' < 1,$$

so, since $\varepsilon > 0$ was arbitrary, $D_\lambda \to L$ for $\lambda \to 1$. The lemma is proved.

Definition. $D^* = \lim_{\lambda \to 1} D_\lambda = \lim_{\lambda \to 1-} (\limsup_{r \to \infty} (N(r) - N(\lambda r))/(1-\lambda)r)$ is called the *maximum density* of the sequence $\{x_k\}$.

Since, as we have already remarked,

$$D_\lambda \;\geqslant\; \bar{D} \;=\; \limsup_{r \to \infty} \frac{N(r)}{r},$$

we *certainly* have $\bar{D} \leqslant D^*$. Simple examples (furnished by sequences with *large gaps*) show that D^* may be *much larger* than $\bar{D}$. Therefore, if, in any theorem whose hypothesis requires $\bar{D}$ to *exceed* some value, we can replace $\bar{D}$ by D^*, we obtain a stronger result thereby. This observation applies to the corollary at the end of the preceding article.

Theorem. *Given a weight $W(x) \geqslant 1$ tending to ∞ as $x \longrightarrow \pm\infty$ and a number $A > 0$, suppose that*

$$\int_{-\infty}^{\infty} \frac{\log W_A(x)}{1+x^2} dx \;<\; \infty$$

and that $W(x_k) < \infty$ on a strictly increasing sequence of points $x_k \geqslant 0$. If the maximum density D^* of the sequence $\{x_k\}$ is $> A/\pi$, then $\mathscr{E}_A$ is not *$\| \ \|_W$-dense in $\mathscr{C}_W(\mathbb{R})$.*

Proof. Taking the index k of the sequence $\{x_k\}$ to start from the value $k = 0$, we begin as in the proof of the corollary by choosing a $\varphi \in \mathscr{C}_W(\mathbb{R})$ with $\varphi(x_0) = 1$ and $\varphi(x_k) = 0$ for $k \geqslant 1$, and argue that, if φ *could* be $\| \ \|_W$-approximated by functions in $\mathscr{C}_W(\mathbb{R})$, there would be an entire function $\Phi(z)$ of exponential type $\leqslant A$ with

$$\int_{-\infty}^{\infty} \frac{\log^+ |\Phi(x)|}{1 + x^2} dx \ < \ \infty$$

and $\Phi(x_k) = \varphi(x_k)$, $k \geqslant 0$. Letting $n_+(r)$ be the number of zeros of $\Phi(z)$ with real part $\geqslant 0$ having modulus $\leqslant r$, we have, by Levinson's theorem (§H.3, Chapter III), that

$$\frac{n_+(r)}{r} \longrightarrow \text{some } D \leqslant \frac{A}{\pi}$$

as $r \to \infty$.

The x_k with $k \geqslant 1$ are zeros of $\Phi(z)$; therefore, if $N(r)$ denotes the he number of such x_k in $[0, r]$, we have, for each $\lambda < 1$, $N(r) - N(\lambda r) \leqslant n_+(r) - n_+(\lambda r)$. In view of the limit relation just written, the quantity on the right equals $(1 - \lambda)Dr + o(r)$ for large r, so we get

$$D_\lambda \leqslant D$$

for each $\lambda < 1$. Therefore $D^* = \lim_{\lambda \to 1} D_\lambda$ is *also* $\leqslant D \leqslant A/\pi$, contradicting our assumption that $D^* > A/\pi$. We are done.

Remark. Towards the end of Chapter IX, we will see that the theorem remains true when we replace the *maximum density* D^* of the sequence $\{x_k\}$ by a *still larger* density associated with that sequence.

The maximum density D^* associated with an increasing sequence of positive numbers $\{x_k\}$ has an elegant geometric interpretation.

Definition. Let $\{\xi_n\}$ be an increasing sequence of positive numbers, some of which may be repeated, and let $\nu(r)$ denote the number of points ξ_n in the interval $[0, r]$, counting repeated ξ_n according to their multiplicities as usual. The sequence $\{\xi_n\}$ is called *measurable* if $\lim_{r \to \infty} \nu(r)/r$ *exists and is finite*. The *value* of that *limit* is called the *density* of $\{\xi_n\}$.

We have then the

Theorem (Pólya). *Let the maximum density D^* of the increasing sequence of positive numbers x_k be finite. Then any measurable sequence of positive* *numbers containing all the* x_k *has density* $\geqslant D^*$, *and there* is *such a measurable sequence whose density is* exactly D^*.

If $D^* = \infty$, *there is* no *measurable sequence* (*of finite density*) *containing all the* x_k.

Proof. If $\{x_k\}$ is contained in an increasing sequence of numbers $\xi_n \geqslant 0$, and if, with $\nu(r)$ denoting the number of points ξ_n in $[0, r]$, $\nu(r)/r \to D$ for $r \to \infty$, we see, just as in the proof of the preceding theorem, that $D^* \leqslant D$. The *first* and *last* statements of the present theorem are therefore true. To complete the proof we must, when $D^* < \infty$, show how to *construct* a measurable sequence of density D^* containing the points x_k. The idea here is transparent enough, but the details are a bit fussy.

Call $N(r)$ the number of points x_k (counting repetitions) in $[0, r]$, and write $\lambda_n = 2^{-1/2^n}$. We have $\lambda_n \underset{n}{\uparrow} 1$, so, if we put

$$\varepsilon_n = \sup\{|D^* - D_\lambda|:\ \lambda_n \leqslant \lambda < 1\},$$

ε_n *decreases monotonically* to zero as $n \to \infty$, since, *according to the lemma* at the beginning of this subsection, $D_\lambda \to D^*$ for $\lambda \to 1$. Referring to the definition of D_λ and taking $\lambda = \lambda_n$, we see that for each n there is an r_n with

$$(\dagger) \qquad \frac{N(r) - N(\lambda_n r)}{(1-\lambda_n)r} < D^* + 2\varepsilon_n \quad \text{for} \quad r \geqslant r_n.$$

It is convenient in what follows to write $\Lambda_n = 1/\lambda_n = 2^{1/2^n}$. For each n, take an *integral power* R_n of Λ_{n-1} (*sic*!) which is $> r_n$ and large enough so that

$$(*) \qquad (\Lambda_n - 1)\varepsilon_n R_n > 1.$$

We require also that $R_n > R_{n-1}$ if $n > 1$ so as to make the sequence $\{R_n\}$ *increasing*.

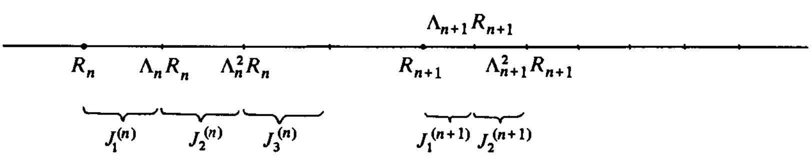

Figure 37

Using the numbers R_n and Λ_n we construct certain intervals, in the following manner. Given n, we have, from R_n up to R_{n+1}, the intervals $(R_n, \Lambda_n R_n]$, $(\Lambda_n R_n, \Lambda_n^2 R_n], \ldots, (\Lambda_n^{K-1} R_n, \Lambda_n^K R_n]$, say, with $\Lambda_n^K R_n = R_{n+1}$. From R_{n+1} onwards, each of the intervals $(\Lambda_n^l R_n, \Lambda_n^{l+1} R_n]$ splits into *two*, both of the form $(\Lambda_{n+1}^m R_{n+1}, \Lambda_{n+1}^{m+1} R_{n+1}]$. After R_{n+2}, each of *those* splits further into two, and so forth. *We denote the intervals of the form* $(\Lambda_n^{p-1}R_n, \Lambda_n^p R_n]$ *lying between* R_n *and* R_{n+1} *by* $J_p^{(n)}$.

Consider any of the intervals $J_p^{(n)}$. Since $\Lambda_n = 1/\lambda_n$, we have, by (†) and the choice of R_n, the inequality

$$\frac{N(\Lambda_n^p R_n) - N(\Lambda_n^{p-1}R_n)}{\Lambda_n^p R_n - \Lambda_n^{p-1}R_n} < D^* + 2\varepsilon_n$$

for the number $N(\Lambda_n^p R_n) - N(\Lambda_n^{p-1}R_n)$ of points x_k in $J_p^{(n)}$. *If the ratio on the left is* $< D^*$, *let us throw new points into* $J_p^{(n)}$ *until we arrive at a total number of such points (including the* x_k *already* $\in J_p^{(n)}$*) lying between*

$$(\Lambda_n^p R_n - \Lambda_n^{p-1}R_n)D^* \quad \textit{and} \quad (\Lambda_n^p R_n - \Lambda_n^{p-1}R_n)(D^* + 2\varepsilon_n).$$

This we can do, thanks to (⁑).

In this manner we adjoin points to the sequence $\{x_k\}$ in each of the intervals $J_p^{(n)}$ lying between R_n and R_{n+1}, to the extent necessary. We do that for every n. When finished, we have a *new sequence of points containing all the original* x_k. It is claimed that this new sequence is *measurable*, and of density D^*.

For $r > 0$, call $\nu(r)$ the number of points *of our new sequence* in $[0, r]$. Suppose that $R > R_n$; then R lies in one of the intervals $J_p^{(m)}$ with $m \geqslant n$, and, since the ε_l *decrease monotonically*,

$$\begin{aligned}(R - R_n)D^* - (D^* + 2\varepsilon_n)|J_p^{(m)}| &\leqslant \nu(R) - \nu(R_m)\\ &\leqslant (R - R_n)(D^* + 2\varepsilon_n) + (D^* + 2\varepsilon_n)|J_p^{(m)}|,\end{aligned}$$

as is evident from our construction. Because $\Lambda_m \underset{m}{\longrightarrow} 1$ and $|J_p^{(m)}| \leqslant (\Lambda_m - 1)R\Lambda_m \leqslant (\Lambda_m - 1)\Lambda_n R$, the last relation shows that

$$D^* - \varepsilon_n \leqslant \frac{\nu(R) - \nu(R_n)}{R - R_n} \leqslant D^* + 3\varepsilon_n$$

as soon as R is sufficiently large $> R_n$.* This means that we have

$$D^* - 2\varepsilon_n \leqslant \frac{\nu(R)}{R} \leqslant D^* + 4\varepsilon_n$$

for R large enough, so, since ε_n can be taken as small as we like,

$$\frac{\nu(R)}{R} \longrightarrow D^* \quad \text{for} \quad R \to \infty.$$

Our new sequence is thus measurable and of density D^*, which is what was needed. We are done.

4. The analogue of Pollard's theorem

Returning from the above digression to the main subject of the

* The upper index m of the interval $J_p^{(m)}$ containing R tends to ∞ with R.

present §, let us complete our exposition of the parallel between weighted approximation by linear combinations of the $e^{i\lambda x}$, $-A \leqslant \lambda \leqslant A$, and that by polynomials. To do this, we need the analogue of the Pollard theorem in §C.3.

In the present situation, we cannot just *copy* the proof given for $W_*(x)$ in §C.3. That's because we now suppose merely that $W(x) \to \infty$ for $x \to \pm\infty$, and *no longer* assume the growth of $W(x)$ to be more rapid than that of any power of x as $x \to \pm\infty$. This means that we no longer necessarily have $W(x)/|x-i| \longrightarrow \infty$ for $x \to \pm\infty$, or even $W(x)/|x-i| \geqslant \text{const.} > 0$ on $\mathbb{R}$.

The *method* of the proof in §B.3 can, however, be *adapted* to the treatment of the present case.

Theorem. *Let $W(x) \geqslant 1$ and $W(x) \to \infty$ for $x \longrightarrow \pm\infty$, and suppose $A > 0$. If $W_A(i) < \infty$, then*

$$\int_{-\infty}^{\infty} \frac{\log W_A(t)}{1+t^2}\,dt \;<\; \infty.$$

Proof. As in §C.3, put $\tilde{W}(x) = W(x)/|x-i|$. For each fixed $z_0 \notin \mathbb{R}$, the ratio $1/(t-z_0)\tilde{W}(t)$ is bounded above on $\mathbb{R}$ and $\to 0$ as $t \to \pm\infty$.

Let us define $\mathscr{C}_{\tilde{W}}(\mathbb{R})$ as the set of functions φ continuous on $\mathbb{R}$ for which $|\varphi(x)/\tilde{W}(x)|$ is bounded and tends to zero as $x \to \pm\infty$ (just as in the situation where $\tilde{W}(x) \geqslant 1$), and put

$$\|\varphi\|_{\tilde{W}} \;=\; \sup_{x\in\mathbb{R}} \left|\frac{\varphi(x)}{\tilde{W}(x)}\right|$$

for such φ. As we have just seen, all the functions $1/(x-z_0)$, $z_0 \notin \mathbb{R}$, *do* belong to $\mathscr{C}_{\tilde{W}}(\mathbb{R})$.

Denote by $\tilde{\mathscr{E}}_A$ the set of functions $f(t)$ in $\mathscr{E}_A$ such that $tf(t)$ *also* belongs to $\mathscr{E}_A$; $\tilde{\mathscr{E}}_A$ is just the set of entire functions f of exponential type $\leqslant A$ with $f(t)$ and $tf(t)$ both bounded on $\mathbb{R}$. There are plenty of such functions; $\sin A(t-z_0)/(t-z_0)$ is one for each complex z_0.

We have $\tilde{\mathscr{E}}_A \subseteq \mathscr{C}_{\tilde{W}}(\mathbb{R})$. It is claimed that, if $W_A(i) < \infty$, $\tilde{\mathscr{E}}_A$ is *not* $\|\;\;\|_{\tilde{W}}$-dense in $\mathscr{C}_{\tilde{W}}(\mathbb{R})$. To see this, it is enough to verify that the function $1/(t-i)$ (which belongs to $\mathscr{C}_{\tilde{W}}(\mathbb{R})$) is not the $\|\;\;\|_{\tilde{W}}$-limit of functions in $\tilde{\mathscr{E}}_A$.

Suppose, for $\eta > 0$, that we *had* an $f \in \tilde{\mathscr{E}}_A$ with

$$\left\|\frac{1}{t-i} - f(t)\right\|_{\tilde{W}} \;\leqslant\; \eta.$$

Then

$$\left|\frac{1-(t-i)f(t)}{W(t)}\right| \;=\; \left|\frac{1-(t-i)f(t)}{(t-i)\tilde{W}(t)}\right| \;\leqslant\; \eta$$

for $t \in \mathbb{R}$, so, putting $G(t) = (1-(t-\mathrm{i})f(t))/\eta$, we would have a $G \in \mathscr{E}_A$ (because $f \in \tilde{\mathscr{E}}_A$) with $\|G\|_W \leqslant 1$ and $G(\mathrm{i}) = 1/\eta$. This means that $1/\eta$ would have to be $\leqslant W_A(\mathrm{i})$, so η cannot be smaller than $1/W_A(\mathrm{i})$, and our assertion holds.

Assuming henceforth that $W_A(\mathrm{i}) < \infty$, we see by the Hahn–Banach theorem (same application as in §A.3) that there is a linear form L on the functions of the form $\varphi/\tilde{W}$, $\varphi \in \mathscr{C}_{\tilde{W}}(\mathbb{R})$, with

$$\left| L\left(\frac{\varphi}{\tilde{W}}\right) \right| \;\leqslant\; \|\varphi\|_{\tilde{W}}, \quad \varphi \in \mathscr{C}_{\tilde{W}}(\mathbb{R}),$$

and $L(f/\tilde{W}) = 0$ for all $f \in \tilde{\mathscr{E}}_A$, whilst $L(1/(t-\mathrm{i})\tilde{W}(t)) \neq 0$.

Let now $G \in \mathscr{E}_A$ (*sic*!), and take any fixed $z_0 \notin \mathbb{R}$. Since the function

$$\frac{G(t) - G(z_0)}{t - z_0}$$

belongs to $\tilde{\mathscr{E}}_A$ (!), we have

$$L\left(\frac{G(t) - G(z_0)}{(t-z_0)\tilde{W}(t)}\right) \;=\; 0,$$

whence (the Markov–Riesz–Pollard trick again!),

$$(\S) \qquad G(z_0) \;=\; L\left(\frac{G(t)}{(t-z_0)\tilde{W}(t)}\right) \Big/ L\left(\frac{1}{(t-z_0)\tilde{W}(t)}\right),$$

provided that the denominator on the right is different from zero.

If $\|G\|_W \leqslant 1$, we have, since

$$\frac{G(t)}{(t-z_0)\tilde{W}(t)} \;=\; \frac{|t-\mathrm{i}|}{t-z_0} \cdot \frac{G(t)}{W(t)},$$

that

$$\left\| \frac{G(t)}{t-z_0} \right\|_{\tilde{W}} \;\leqslant\; \sup_{t \in \mathbb{R}} \left| \frac{t-\mathrm{i}}{t-z_0} \right|.$$

The quantity on the right was worked out in §A.2 and seen there (in the corollary) to be $\leqslant\; (1+|z_0|)/|\Im z_0|$. Therefore, if $\|G\|_W \leqslant 1$,

$$\left| L\left(\frac{G(t)}{(t-z_0)\tilde{W}(t)}\right) \right| \;\leqslant\; \left\| \frac{G(t)}{t-z_0} \right\|_{\tilde{W}} \;\leqslant\; \frac{1+|z_0|}{|\Im z_0|}.$$

Calling

$$\Phi(z) \;=\; L\left(\frac{1}{(t-z)\tilde{W}(t)}\right) \quad \text{for} \quad z \notin \mathbb{R}$$

we see from this last relation and from (§) that, for $G \in \mathscr{E}_A$,

$$(\S\S) \qquad |G(z)| \;\leqslant\; \frac{1+|z|}{|\Im z|\,|\Phi(z)|} \quad \text{if} \quad \|G\|_W \leqslant 1.$$

We know that $\Phi(\mathrm{i}) \neq 0$. Also, $\Phi(z)$ is analytic for $\Im z > 0$. That's because

$$|t-\mathrm{i}|\left\{\frac{(t-z-\Delta z)^{-1}-(t-z)^{-1}}{\Delta z} - \frac{1}{(t-z)^2}\right\} \;=\; \frac{|t-\mathrm{i}|\,\Delta z}{(t-z)^2(t-z-\Delta z)}$$

tends to zero *uniformly* for $-\infty < t < \infty$ as $\Delta z \to 0$, provided that $z \notin \mathbb{R}$. Since $|t-\mathrm{i}|\,\tilde{W}(t) = W(t)$ *is* $\geqslant 1$ on $\mathbb{R}$, this implies that

$$\frac{\Phi(z+\Delta z)-\Phi(z)}{\Delta z} \;\longrightarrow\; L\left(\frac{1}{(t-z)^2\tilde{W}(t)}\right)$$

when $\Delta z \to 0$ as long as $z \notin \mathbb{R}$, and thus establishes analyticity of $\Phi(z)$ in the upper and lower half planes.

The function $\Phi(z)$, analytic and not $\equiv 0$ for $\Im z > 0$, is not quite bounded in $\{\Im z \geqslant 1\}$; it is, however, not far from being bounded in the latter region. (*Here*, by the way, lies the main difference between our present situation and the one discussed in §A.3.) We have, for $t \in \mathbb{R}$,

$$\left|\frac{1}{(t-z)\tilde{W}(t)}\right| \;=\; \left|\frac{t-\mathrm{i}}{(t-z)W(t)}\right| \;\leqslant\; \left|\frac{t-\mathrm{i}}{t-z}\right|$$

since $W(t) \geqslant 1$, whence, by §A.2, $\|1/(t-z)\|_{\tilde{W}} \leqslant (1+|z|)/|\Im z|$, so

$$|\Phi(z)| \;=\; \left|L\left(\frac{1}{(t-z)\tilde{W}(t)}\right)\right| \;\leqslant\; \frac{1+|z|}{|\Im z|}.$$

The function $\Phi(z)/(z+\mathrm{i})$ is thus analytic and bounded in $\{\Im z > 1\}$ and continuous in the closure of that half plane; it is certainly not identically zero there because $\Phi(\mathrm{i}) \neq 0$. Therefore, by §G.2 of Chapter III,

$$(\dagger\dagger) \qquad \int_{-\infty}^{\infty} \frac{1}{1+x^2} \log^- \left|\frac{\Phi(x+\mathrm{i})}{x+\mathrm{i}}\right| \mathrm{d}x \;<\; \infty.$$

By the definition of W_A and (§§) we now obtain

$$\begin{aligned} W_A(x+\mathrm{i}) &= \sup\{|G(x+\mathrm{i})|:\ G \in \mathscr{E}_A \text{ and } \|G\|_W \leqslant 1\} \\ &\leqslant \frac{1+|x+\mathrm{i}|}{|\Phi(x+\mathrm{i})|} \;\leqslant\; 2\frac{|x+\mathrm{i}|}{|\Phi(x+\mathrm{i})|}, \end{aligned}$$

and

$$\log W_A(x+\mathrm{i}) \;\leqslant\; \log 2 + \log^- \left|\frac{\Phi(x+\mathrm{i})}{x+\mathrm{i}}\right|.$$

Now we are in the hall of mirrors again! Take any $G \in \mathscr{E}_A$ with $\|G\|_W \leqslant 1$. Then, on the one hand,

$$\log|G(x+\mathrm{i})| \leqslant \log W_A(x+\mathrm{i}),$$

while, on the other,

$$\log|G(\xi)| \leqslant A + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log|G(x+\mathrm{i})|}{(\xi-x)^2+1}\mathrm{d}x$$

for $\xi \in \mathbb{R}$, according to the second theorem of §G.2, Chapter III, applied in the half plane $\Im z \leqslant 1$. Substituting into this last inequality the two preceding it, we find

$$\log|G(\xi)| \leqslant A + \log 2 + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log^-|\Phi(x+\mathrm{i})/(x+\mathrm{i})|}{(\xi-x)^2+1}\mathrm{d}x,$$

and, since $\log W_A(\xi)$ is the supremum of $\log|G(\xi)|$ for such G, we see that

$$\log W_A(\xi) \leqslant A + \log 2 + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log^-|\Phi(x+\mathrm{i})/(x+\mathrm{i})|}{(\xi-x)^2+1}\mathrm{d}x.$$

Using Fubini's theorem in the usual way together with this relation, we finally get

$$\int_{-\infty}^{\infty} \frac{\log W_A(\xi)}{\xi^2+1}\mathrm{d}\xi$$
$$\leqslant \pi(A+\log 2) + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{2}{x^2+4}\log^-\left|\frac{\Phi(x+\mathrm{i})}{x+\mathrm{i}}\right|\mathrm{d}x.$$

The integral on the right is, however, *finite* by (††). The theorem is thus proved, and we are done.

Remark. Thus, if $W_A(\mathrm{i}) < \infty$, the fourth theorem of article 2 (Akhiezer's description) and the discussion in article 3 related to it apply.

F. L. de Branges' description of extremal unit measures orthogonal to the $e^{\mathrm{i}\lambda x}/W(x)$, $-A \leqslant \lambda \leqslant A$, when $\mathscr{E}_A$ is not dense in $\mathscr{C}_W(\mathbb{R})$.

We have now about finished with the individual treatment of uniform weighted approximation by polynomials or by functions in $\mathscr{E}_A$. There remains one thing, however.

In his study of the situation where linear combinations of the $\mathrm{e}^{\mathrm{i}\lambda x}$, $-A \leqslant \lambda \leqslant A$, (or polynomials) are *not* $\|\ \ \|_W$-dense in $\mathscr{C}_W(\mathbb{R})$, Louis de Branges obtained a beautiful description (valid for weights which are not too irregular) of the *extremal unit measures orthogonal to the functions*

$e^{i\lambda x}/W(x)$, $-A \leqslant \lambda \leqslant A$ (or to the polynomials divided by W). We should not end the present discussion without giving it.

For the treatment of de Branges' description, we assume that $W(x)$ is continuous (and finite) on a certain closed unbounded subset E of $\mathbb{R}$, and that $W(x) \equiv \infty$ on $\mathbb{R} \sim E$. As always, $W(x) \geqslant 1$.

In this circumstance, the ratios $\varphi(x)/W(x)$ with $\varphi \in \mathscr{C}_W(\mathbb{R})$ are *continuous* on the *locally compact set* E, and tend to *zero* whenever $x \to \pm\infty$ in E. We can write

$$\|\varphi\|_W = \sup_{t\in E}\left|\frac{\varphi(t)}{W(t)}\right| \quad \text{for} \quad \varphi \in \mathscr{C}_W(\mathbb{R}),$$

and we see that the correspondence $\varphi \leftrightarrow \varphi/W$ is an isometric isomorphism between $\mathscr{C}_W(\mathbb{R})$ and $\mathscr{C}_0(E)$, the usual Banach space of functions continuous and zero at ∞ on the locally compact Hausdorff space E. The bounded linear functionals on $\mathscr{C}_0(E)$ are given by the Riesz representation theorem. Therefore the $\|\ \ \|_W$-bounded linear functionals on $\mathscr{C}_W(\mathbb{R})$ are all of the form

$$\int_E \frac{\varphi(t)}{W(t)}\,\mathrm{d}\mu(t) = \int_{-\infty}^{\infty} \frac{\varphi(t)}{W(t)}\,\mathrm{d}\mu(t)$$

with *totally finite complex Radon measures* μ *supported on* E.

We consider in the following discussion the case where linear combinations of the $e^{i\lambda x}$, $-A \leqslant \lambda \leqslant A$, are *not* $\|\ \ \|_W$-dense in $\mathscr{C}_W(\mathbb{R})$. We could also treat the situation where *polynomials* are not $\|\ \ \|_W$-dense in $\mathscr{C}_W(\mathbb{R})$ and obtain a result analogous to the one to be found for approximation by exponentials; here, of course, one needs to make the supplementary assumption that $x^n/W(x) \longrightarrow 0$ for every $n \geqslant 0$ as $x \to \pm\infty$ in E.

Granted, then, that linear combinations of the $e^{i\lambda x}$, $-A \leqslant \lambda \leqslant A$, are not $\|\ \ \|_W$-dense in $\mathscr{C}_W(\mathbb{R})$, there must, by the Hahn–Banach theorem, be some *non-zero* $\|\ \ \|_W$-bounded linear functional on $\mathscr{C}_W(\mathbb{R})$ which is *orthogonal to* (i.e., *annihilates*) all the $e^{i\lambda x}$, $-A \leqslant \lambda \leqslant A$, or, what comes to the same thing (lemma of §E.1), to all the functions $f \in \mathscr{E}_A$. According to the above description of such linear functionals, there is thus a *non-zero totally finite Radon measure* μ on E with

$$(*) \qquad \int_E \frac{f(x)}{W(x)}\,\mathrm{d}\mu(x) = \int_{-\infty}^{\infty} \frac{f(x)}{W(x)}\,\mathrm{d}\mu(x) = 0 \quad \text{for} \quad f \in \mathscr{E}_A.$$

The idea now is to try to obtain a description of the non-zero measures μ on E satisfying $(*)$.

In the first place, if a complex measure μ satisfies $(*)$, *so do its real and* *imaginary parts.* That's because $u = \frac{1}{2}(f + f^*)$ and $v = (1/2i)(f - f^*)$ *both belong to* $\mathscr{E}_A$ *if* f *does* (and conversely). However, $u(x)$ and $v(x)$ are both *real-valued* on $\mathbb{R}$ (recall that $f^*(z) = \overline{f(\bar{z})}$), so, if μ is any *complex* measure on E satisfying $(*)$, we have, given $f \in \mathscr{E}_A$,

$$\int_E \frac{u(x)}{W(x)} d\mu(x) = \int_E \frac{v(x)}{W(x)} d\mu(x) = 0,$$

whence (taking real and imaginary parts)

$$\int_E \frac{u(x)}{W(x)} d\Re\mu(x) = \int_E \frac{u(x)}{W(x)} d\Im\mu(x) = 0,$$

$$\int_E \frac{v(x)}{W(x)} d\Re\mu(x) = \int_E \frac{v(x)}{W(x)} d\Im\mu(x) = 0,$$

and thus

$$\int_E \frac{f(x)}{W(x)} d\Re\mu(x) = \int_E \frac{f(x)}{W(x)} d\Im\mu(x) = 0.$$

This means that a description of the *real-valued* (signed) μ on E satisfying $(*)$ provides us, at the same time, with one for all such *complex* measures μ. Our investigation thus reduces to the study of the *real signed measures* μ *on* E satisfying $(*)$.

Notation. Call Σ the set of *finite real-valued Radon measures* μ *on* E *satisfying* $(*)$ *and such that*

$$\|\mu\| = \int_E |d\mu(x)| \leqslant 1.$$

The set Σ is *convex* and *w*-compact* (over $\mathscr{C}_0(E)$). We can therefore apply to it the celebrated *Krein–Millman theorem*, which says that Σ *is the w*-closed convex hull of its extreme points.* (Recall: an *extreme point* μ of Σ is a member thereof which *cannot be written* as $\lambda\mu_1 + (1-\lambda)\mu_2$ with $0 < \lambda < 1$ and measures μ_1 and μ_2 in Σ *different* from μ.) More explicitly, we can, given any $\mu \in \Sigma$, find a sequence of *finite convex combinations*

$$\mu_N = \sum_k \lambda_k(N) \nu_k(N)$$

of *extreme points* $\nu_k(N)$ of Σ (the $\lambda_k(N) > 0$ and $\sum_k \lambda_k(N) = 1$ for each N) with

$$d\mu_N(x) \longrightarrow d\mu(x) \quad w^* \quad \text{as} \quad N \to \infty.$$

We can, in fact, even do *better* – *Choquet's theorem* furnishes a representation for μ as a kind of *integral over the set of extreme points of* Σ. (The book of Phelps is an excellent introduction to these matters. In the present situation, Σ is *metrizable*, so a particularly simple and elegant form of Choquet's theorem applies to it.) The point here is that a good description of the *extreme points* of Σ will already tell us a *great deal* about all the members of Σ and thus, in turn, about the *complex*-valued μ satisfying $(*)$. Knowledge of those extreme points therefore takes us a long way towards a complete description of the measures μ which satisfy $(*)$.

What Louis de Branges found is an *explicit description of the extreme points* of Σ. We now set out to explain his work.

1. Three lemmas

The main idea behind the following development is contained in

De Branges' lemma. *Let μ be an extreme point of Σ and h a* bounded *real-valued Borel function. Suppose that*

$$\int_{-\infty}^{\infty} \frac{f(x)}{W(x)} h(x)\,d\mu(x) \;=\; 0$$

for all $f \in \mathscr{E}_A$. Then $h(x)$ is a.e. ($|d\mu|$) equal to a constant.

Remark. It is enough to assume that h is *essentially* ($|d\mu|$) *bounded*, as will be clear in the proof.

Proof. We start out by *observing once and for all that an extreme point μ of Σ is never the zero measure*. That's because

$$0 = \tfrac{1}{2}\nu + \tfrac{1}{2}(-\nu)$$

where, for ν, we can take *any non-zero member of* Σ. In the situation of this §, Σ *has* non-zero elements.

In view of this fact, we must have $\int_{-\infty}^{\infty} |d\mu(t)| = 1$ for any extreme point μ of Σ. Otherwise we could write

$$\mu \;=\; (1 - \|\mu\|)\cdot 0 + \|\mu\| \left(\frac{\mu}{\|\mu\|} \right),$$

with $0 < \|\mu\| < 1$ and the measures 0 and $\mu/\|\mu\|$ both belonging to Σ.

Now we take a function h as in the hypothesis, and an extreme point μ of Σ. The relation $(*)$ holds, therefore

$$\int_E \frac{f(x)}{W(x)} (h(x) + C)\,d\mu(x) \;=\; 0$$

for every constant C. Since h is bounded, we will have $h(x) + C \geqslant 0$ for suitable C; *we may therefore just as well assume that* $h(x)$ *is positive to begin with,* since *otherwise* we would *only need to replace* h *by* $h+C$ in the following argument.

We have, then, a *positive* bounded h satisfying the hypothesis. Unless $h(x) \equiv 0$ a.e. ($|d\mu|$) (in which case the lemma is already proved), we have

$$\int_E h(t)|d\mu(t)| \;>\; 0.$$

Multiplication of h by a positive constant will then give us a new positive function like the h in the hypothesis, *which we henceforth also denote by* h, fulfilling the condition

$$\int_E h(t)|d\mu(t)| \;=\; 1.$$

Since h is bounded, there is a λ, $0<\lambda<1$, with $0 \leqslant \lambda h(x) \leqslant 1$. Picking such a λ, we have

$$\begin{aligned}\int_E \left|\frac{1-\lambda h(t)}{1-\lambda}\right| |d\mu(t)| \;&=\; \int_E \frac{1-\lambda h(t)}{1-\lambda}|d\mu(t)| \\ &=\; \frac{1}{1-\lambda}\int_E |d\mu(t)| - \frac{\lambda}{1-\lambda}\int_E h(t)|d\mu(t)| \;=\; 1,\end{aligned}$$

since $\int_E |d\mu(t)| = 1$.

Also,

$$\int_E \frac{f(x)}{W(x)}\frac{1-\lambda h(x)}{1-\lambda}d\mu(x) \;=\; 0$$

for all $f \in \mathscr{E}_A$ by the hypothesis and the property (*). In view of the previous relation, we see that the measure μ_2 on E such that

$$d\mu_2(t) \;=\; \frac{1-\lambda h(t)}{1-\lambda}d\mu(t)$$

belongs to Σ.

The same is true for the measure μ_1 on E with

$$d\mu_1(t) = h(t)d\mu(t).$$

However,

$$d\mu(t) = \lambda d\mu_1(t) + (1-\lambda)d\mu_2(t),$$

and we assumed that μ *was an extreme point of* Σ. Since $0<\lambda<1$, we therefore must have $d\mu_1(t) = d\mu_2(t)$, i.e., $(1-\lambda h(t))/(1-\lambda) = h(t)$ a.e. ($|d\mu|$), and finally $h(t) \equiv 1$ a.e. ($|d\mu|$). The lemma is proved.

Lemma. *Let μ be an extreme point of Σ, let $F \in L_1(|\mathrm{d}\mu|)$, and suppose that $\int_E F(t)\,\mathrm{d}\mu(t) = 0$. Then there are $f_n \in \mathscr{E}_A$ with*

$$\int_E \left| \frac{f_n(t)}{W(t)} - F(t) \right| |\mathrm{d}\mu(t)| \underset{n}{\longrightarrow} 0.$$

Proof. By duality. The usual application of the Hahn–Banach theorem shows that the *infimum*, for f ranging over $\mathscr{E}_A$, of

$$\int_E \left| F(t) - \frac{f(t)}{W(t)} \right| |\mathrm{d}\mu(t)|$$

is equal to the *supremum* of

$$\left| \int_E F(t)h(t)\,\mathrm{d}\mu(t) \right|$$

for Borel functions h such that $|h(t)| \leqslant 1$ a.e. ($|\mathrm{d}\mu|$) and

$$(\dagger) \qquad \int_E \frac{f(t)}{W(t)} h(t)\,\mathrm{d}\mu(t) \;=\; 0$$

whenever $f \in \mathscr{E}_A$.

Since μ is a real measure and any $f \in \mathscr{E}_A$ can be written as $u + iv$ with u and v in $\mathscr{E}_A$ and *real* on $\mathbb{R}$, we see that, if h satisfies (†) for all $f \in \mathscr{E}_A$, so do $\Re h$ and $\Im h$. De Branges' lemma now shows that these latter functions are *constant* a.e. ($|\mathrm{d}\mu|$) if h is bounded; in other words, the functions h over which the above mentioned supremum is taken *are all constant* a.e. ($|\mathrm{d}\mu|$).

But then $\int_E h(t)F(t)\,\mathrm{d}\mu(t) = 0$ for such functions h, according to our assumption on F. So the supremum in question is zero, and the infimum is also zero. Done.

Lemma. *Let $\mu \neq 0$ belong to Σ. The functions $f(z)$ in $\mathscr{E}_A$ with*

$$\int_E \left| \frac{f(t)}{W(t)} \right| |\mathrm{d}\mu(t)| \;\leqslant\; 1$$

form a normal family *in the complex plane. The limit $F(z)$ of any u.c.c. convergent sequence of such functions f is an entire function of exponential type $\leqslant A$ with*

$$\int_{-\infty}^{\infty} \frac{\log^+ |F(t)|}{1+t^2}\,\mathrm{d}t \;<\; \infty.$$

Proof. Since μ is real, the function $\Phi(z) = \int_E (\mathrm{d}\mu(t)/(t-z)W(t))$, which is analytic in both half planes $\Im z > 0$, $\Im z < 0$, *cannot vanish identically in either* (otherwise μ would be 0). $\Phi(z)$ is *bounded* for $\Im z \geqslant 1$ and for $\Im z \leqslant -1$.

If now $f \in \mathscr{E}_A$, the function of t, $(f(t) - f(z))/(t-z)$, *also* belongs to $\mathscr{E}_A$, making $\int_E ((f(t) - f(z))/(t-z))(\mathrm{d}\mu(t)/W(t)) = 0$. Therefore, if $z \notin \mathbb{R}$,

$$f(z) \;=\; \frac{1}{\Phi(z)} \int_E \frac{f(t)\mathrm{d}\mu(t)}{(t-z)W(t)}$$

(the Markov–Riesz–Pollard trick again!). When $\int_E |f(t)/W(t)||\mathrm{d}\mu(t)| \leqslant 1$, this yields, for $z = x \pm \mathrm{i}$, $|f(x \pm \mathrm{i})| \leqslant |1/\Phi(x \pm \mathrm{i})|$, and, by §E of Chapter III,

$$\begin{aligned}\log|f(\zeta)| \;&\leqslant\; A|\Im\zeta \mp 1| + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im\zeta \mp 1|\log^+|f(x \pm \mathrm{i})|}{|\zeta - x \mp \mathrm{i}|^2}\mathrm{d}x \\ &\leqslant\; A|\Im\zeta \mp 1| + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im\zeta \mp 1|\log^-|\Phi(x \pm \mathrm{i})|}{|\zeta - x \mp \mathrm{i}|^2}\mathrm{d}x.\end{aligned}$$

Since $\Phi(z)$ is $\not\equiv 0$ both in $\Im z > 0$ and in $\Im z < 0$, we have

$$\int_{-\infty}^{\infty} (1/(1+x^2))\log^-|\Phi(x+\mathrm{i})|\,\mathrm{d}x \;<\; \infty$$

and

$$\int_{-\infty}^{\infty} (1/(1+x^2))\log^-|\Phi(x-\mathrm{i})|\mathrm{d}x \;<\; \infty.$$

From here on, the proof is like that of Akhiezer's second theorem (§B.2 – see also §E.2, especially the proof of the corollary at the end of that article).

2. De Branges' theorem

Lemma. *Let μ be an extreme point of Σ. Then μ is supported on a countably infinite subset of $\mathbb{R}$ without finite limit point.*

Proof. As we saw in proving de Branges' lemma (previous article), an extreme point μ of Σ *cannot be the zero measure.*

Such a measure μ *cannot have compact support.* Suppose, indeed, that μ were supported on the compact set $K \subseteq E$; then we would have

$$\int_K \frac{\mathrm{e}^{\mathrm{i}\lambda x}}{W(x)}\mathrm{d}\mu(x) \;=\; 0, \quad -A \leqslant \lambda \leqslant A.$$

Here, since K is compact, we can differentiate with respect to λ under the integral sign as many times as we wish, obtaining (for $\lambda = 0$)

$$\int_K \frac{x^n}{W(x)}\mathrm{d}\mu(x) \;=\; 0, \quad n = 0, 1, 2, \ldots.$$

From this, Weierstrass' theorem would give us

$$\int_K \frac{g(x)}{W(x)} \mathrm{d}\mu(x) = 0$$

for all continuous functions g on K. Then, however, μ would have to be *zero*, since $K \subseteq E$, and $W(x)$ is continuous (and $< \infty$) on E.

The fact that μ does *not* have compact support implies the existence of a *finite interval J containing two disjoint open intervals*, I_1 and I_2, with

$$\int_{I_1} |\mathrm{d}\mu(t)| > 0 \quad \text{and} \quad \int_{I_2} |\mathrm{d}\mu(t)| > 0.$$

This means that we can find a Borel function φ, *identically zero outside* $I_1 \cup I_2$ (hence identically zero outside J) with $|\varphi(t)|$ *equal to non-zero constants* on *each* of the intervals I_1, I_2, such that

$$\int_{-\infty}^{\infty} \varphi(t) \mathrm{d}\mu(t) = 0.$$

On account of this relation we have, by the *second* lemma of the preceding article, a sequence of functions $f_n \in \mathscr{E}_A$ with

$$\int_{-\infty}^{\infty} \left| \frac{f_n(t)}{W(t)} - \varphi(t) \right| |\mathrm{d}\mu(t)| \underset{n}{\longrightarrow} 0.$$

The *third* lemma of the above article now shows that a subsequence of the $f_n(z)$ converges u.c.c. in $\mathbb{C}$ to some entire function $F(z)$ of exponential type $\leqslant A$, and we see by Fatou's lemma that

$$\int_{-\infty}^{\infty} \left| \frac{F(t)}{W(t)} - \varphi(t) \right| |\mathrm{d}\mu(t)| = 0,$$

i.e.,

$$\frac{F(t)}{W(t)} = \varphi(t) \text{ a.e. } (|\mathrm{d}\mu|).$$

By its construction, φ is *not a.e.* ($|\mathrm{d}\mu|$) *equal to zero*, hence $F(z) \not\equiv 0$. The function φ *does*, however, vanish identically outside the finite interval J. Therefore

$$\frac{F(t)}{W(t)} \equiv 0 \text{ a.e. } (|\mathrm{d}\mu|),\ t \notin J.$$

Since $F(z) \not\equiv 0$ is entire, F *can only* vanish on a certain countable set without finite limit point. We see that μ, *outside* J, must be supported on this countable set, consisting of zeros of F.

Because the support of μ is not compact, there is a *finite interval* J', *disjoint from* J, and containing two disjoint open intervals I'_1 and I'_2 with

$$\int_{I'_1} |d\mu(t)| > 0, \quad \int_{I'_2} |d\mu(t)| > 0.$$

Repetition of the argument just made, with J' playing the rôle of J, shows now that μ, outside J' (and hence in particular *in* J!) is also supported on a countable set without finite limit point. Therefore the *whole support* of μ in E must be such a set, which is what we had to prove.

Remark. The support of μ must *really be infinite*. Otherwise it would be compact, and this, as we have seen, is impossible.

Now we are ready to establish the

Theorem (Louis de Branges). *Let $W(x) \geqslant 1$ be a weight having the properties stated at the beginning of this §, and let E be the associated closed set on which $W(x)$ is finite.*

Suppose that $\mathscr{E}_A$ is not *$\|\ \|_W$-dense in $\mathscr{C}_W(\mathbb{R})$, and let μ be an* extreme point *of the set Σ of real signed measures ν on E such that*

$$\int_E |d\nu(t)| \leqslant 1$$

and

$$\int_E \frac{f(t)}{W(t)} d\nu(t) = 0 \quad \textit{for all } f \in \mathscr{E}_A.$$

Then μ is supported on an infinite sequence $\{x_n\}$ without finite limit point, lying in E. There is an entire function $S(z)$ of exponential type A having a simple zero *at each point x_n and* no other zeros, *with*

$$\boxed{\mu(\{x_n\}) = \frac{W(x_n)}{S'(x_n)}.}$$

Moreover,

$$\int_{-\infty}^{\infty} \frac{\log^+|S(x)|}{1+x^2} dx < \infty$$

and

$$\lim_{y\to\infty} \frac{\log|S(iy)|}{y} = \lim_{y\to-\infty} \frac{\log|S(iy)|}{|y|} = A.$$

Proof. Let us begin by first establishing an auxiliary proposition.

Given an extreme point μ of Σ, suppose that we have a sequence of functions $f_n \in \mathscr{E}_A$ such that

$$\int_{-\infty}^{\infty} |f_n(x) - f_m(x)| \frac{|d\mu(x)|}{W(x)} \underset{n,m}{\longrightarrow} 0.$$

We know from the third lemma of the preceding article that a subsequence of the f_n tends u.c.c. in $\mathbb{C}$ to some entire function F of exponential type $\leqslant A$, and here it is clear by Fatou's lemma that

$$(*) \qquad \int_{-\infty}^{\infty} |f_n(x) - F(x)| \frac{|d\mu(x)|}{W(x)} \underset{n}{\longrightarrow} 0,$$

whence surely

$$\int_{-\infty}^{\infty} \frac{F(x)}{W(x)} d\mu(x) = 0$$

since $\int_{-\infty}^{\infty} (f_n(x)/W(x)) d\mu(x) = 0$ for all n.

Our auxiliary proposition says that the relations

$$\int_{-\infty}^{\infty} \frac{F(x) - F(a)}{x-a} \frac{d\mu(x)}{W(x)} = 0, \qquad \int_{-\infty}^{\infty} (x-b)\frac{F(x) - F(a)}{x-a} \frac{d\mu(x)}{W(x)} = 0$$

also hold for the limit function F; here, a and b are arbitrary complex numbers. Both of these formulas are proved in the same way, and it is enough to deal here only with the *second* one.

Wlog, $f_n(z) \underset{n}{\longrightarrow} F(z)$ u.c.c., whence $f_n(a) \longrightarrow F(a)$ and thence, by $(*)$,

$$\int_{-\infty}^{\infty} \frac{|f_n(x) - f_n(a) - (F(x) - F(a))|}{W(x)} |d\mu(x)| \underset{n}{\longrightarrow} 0,$$

since $W(x) \geqslant 1$. From this is clear that

$$\int_{|x-a|\geqslant 1} \left| \frac{x-b}{x-a} \left(f_n(x) - f_n(a) - F(x) + F(a) \right) \right| \frac{|d\mu(x)|}{W(x)} \underset{n}{\longrightarrow} 0.$$

Also, the u.c.c. convergence of $f_n(z)$ to $F(z)$ makes

$$\frac{f_n(z) - f_n(a)}{z-a}(z-b) \underset{n}{\longrightarrow} \frac{F(z) - F(a)}{z-a}(z-b)$$

u.c.c., by the elementary theory of analytic functions (Cauchy's formula!). Therefore we also have

$$\int_{|x-a|\leqslant 1} \left| (x-b) \left(\frac{f_n(x) - f_n(a)}{x-a} - \frac{F(x) - F(a)}{x-a} \right) \right| \frac{|d\mu(x)|}{W(x)} \underset{n}{\longrightarrow} 0.$$

and finally

$$\int_{-\infty}^{\infty}\left|\frac{f_n(x)-f_n(a)}{x-a}(x-b)-\frac{F(x)-F(a)}{x-a}(x-b)\right|\frac{|d\mu(x)|}{W(x)} \xrightarrow[n]{} 0.$$

Since, however, the $f_n \in \mathscr{E}_A$, we have $(x-b)(f_n(x)-f_n(a))/(x-a) \in \mathscr{E}_A$, whence

$$\int_{-\infty}^{\infty}\frac{f_n(x)-f_n(a)}{x-a}(x-b)\frac{d\mu(x)}{W(x)} = 0$$

for each n. Referring to the previous relation, we see that

$$\int_{-\infty}^{\infty}\frac{F(x)-F(a)}{x-a}(x-b)\frac{d\mu(x)}{W(x)} = 0,$$

as we set out to show.

Now we turn to the theorem itself. According to the lemma at the beginning of this article, our measure μ is supported on a countable set $\{x_n\} \subseteq E$ without finite limit point, and, by the remark following that lemma, $\mu(\{x_n\}) \neq 0$ *for infinitely many* of the points x_n. There is thus no loss of generality in supposing that $\mu(\{x_n\}) \neq 0$ *for each n*.

Take any two points from among the x_n, say x_0 and x_1, and put

$$\varphi(x_0) = \frac{1}{\mu(\{x_0\})(x_0-x_1)},$$

$$\varphi(x_1) = \frac{1}{\mu(\{x_1\})(x_1-x_0)},$$

and $\varphi(x) = 0$ for $x \neq x_0$ or x_1. Then

$$\int_{-\infty}^{\infty}\varphi(x)d\mu(x) = 0,$$

so, as in the proof of the preceding lemma, there is a sequence of $f_n \in \mathscr{E}_A$ with

$$\int_{-\infty}^{\infty}\left|\frac{f_n(x)}{W(x)} - \varphi(x)\right| |d\mu(x)| \xrightarrow[n]{} 0$$

and, wlog, $f_n(z) \xrightarrow[n]{} F(z)$ u.c.c., F being some entire function of exponential type $\leqslant A$. We see that

$$(*) \qquad \frac{F(x)}{W(x)} = \varphi(x) \text{ a.e. } (|d\mu|),$$

whence

$$(\dagger) \qquad \int_{-\infty}^{\infty} \frac{|f_n(x) - F(x)|}{W(x)} |d\mu(x)| \xrightarrow[n]{} 0.$$

From ($\ast\!\ast$) and the definition of φ, we have $F(x_0) \neq 0$, $F(x_1) \neq 0$, so $F(z) \not\equiv 0$. For the same reasons, however, *$F(x)$ vanishes at all the other points x_n, $n \neq 0, 1$.*

Put

$$S(z) = F(z)(z - x_0)(z - x_1).$$

Then S, like F, is an entire function of exponential type $\leqslant A$. $S(z)$ vanishes at each of the points x_n in the support of μ. Finally,

$$\int_{-\infty}^{\infty} \frac{\log^+ |S(x)|}{1 + x^2} dx \ < \ \infty,$$

since, by the third lemma of the preceding article, the function F has this property.

Let us compute the quantities $S'(x_n)$. We already know that

$$S'(x_0) \ = \ F(x_0)(x_0 - x_1) \ = \ W(x_0)\varphi(x_0)(x_0 - x_1) \ = \ \frac{W(x_0)}{\mu(\{x_0\})},$$

and similarly $S'(x_1) = W(x_1)/\mu(\{x_1\})$. Take any other point x_n, $n \neq 0, 1$, and form the function

$$\frac{S(x)}{(x - x_0)(x - x_n)} \ = \ \frac{F(x)}{x - x_n}(x - x_1).$$

Since $F(x_n) = 0$, ($\dagger$) implies, *by our auxiliary proposition*, that

$$\int_{-\infty}^{\infty} \frac{F(x)}{x - x_n}(x - x_1)\frac{d\mu(x)}{W(x)} \ = \ 0.$$

The function $S(x)/(x - x_0)(x - x_n)$ vanishes at all the x_k, save x_0 and x_n. The previous relation therefore reduces to

$$\frac{S'(x_0)\mu(\{x_0\})}{(x_0 - x_n)W(x_0)} \ + \ \frac{S'(x_n)\mu(\{x_n\})}{(x_n - x_0)W(x_n)} \ = \ 0,$$

i.e.,

$$\frac{S'(x_n)}{W(x_n)}\mu(\{x_n\}) \ = \ 1,$$

and finally $S'(x_n) = W(x_n)/\mu(\{x_n\})$.

The function $S(z)$ can have no zeros apart from the x_n. Suppose, indeed, that $S(a) = 0$ with a different from all the x_n; then we would also have $F(a) = 0$, so, in the identity

$$\int_{-\infty}^{\infty} \frac{S(x)}{(x-x_0)(x-a)} \frac{d\mu(x)}{W(x)} = \int_{-\infty}^{\infty} \frac{F(x)}{x-a}(x-x_1)\frac{d\mu(x)}{W(x)},$$

the *right-hand integral would have to vanish* by our auxiliary proposition. The quantity $(F(x)/(x-a))(x-x_1)$ is, however, *different from* 0 *at only one of the points* x_n *in* μ's *support, namely, at* x_0, where

$$\frac{F(x_0)}{x_0 - a}(x_0 - x_1) = \frac{S'(x_0)}{x_0 - a}.$$

We would thus get

$$\frac{S'(x_0)}{x_0 - a} \cdot \frac{\mu(\{x_0\})}{W(x_0)} = 0,$$

i.e., in view of the computation made in the previous paragraph,

$$\frac{1}{x_0 - a} = 0,$$

which is absurd.

The function $S(z)$ thus *vanishes once at each* x_n, and *only at those points.* As we have already seen, $\mu(\{x_n\}) = W(x_n)/S'(x_n)$, $S(z)$ is of exponential type $\leqslant A$, and

$$\int_{-\infty}^{\infty} \frac{\log^+ |S(x)|}{1+x^2} dx < \infty.$$

To complete our proof, we have to show that $\log|S(iy)|/|y| \longrightarrow A$ for $y \to \pm\infty$.

In order to do this, let us first derive the partial fraction decomposition

$$\frac{1}{S(z)} = \sum_n \frac{1}{(z-x_n)S'(x_n)}.$$

Note that $\sum_n |1/S'(x_n)|$ is *surely convergent* because $\mu(\{x_n\}) = W(x_n)/S'(x_n)$, μ is a *finite* measure, and $W(x) \geqslant 1$. Take the function $G(t) = F(t)(t-x_1) = S(t)/(t-x_0)$, and, for *fixed* z, observe that

$$\frac{G(t)-G(z)}{t-z} = \frac{F(t)(t-x_1) - F(z)(z-x_1)}{t-z}$$

$$= \frac{F(t)-F(z)}{t-z}(t-x_1) + F(z).$$

By our auxiliary proposition,

$$\int_{-\infty}^{\infty} \frac{F(t)-F(z)}{t-z}(t-x_1)\frac{d\mu(t)}{W(t)} = 0,$$

and of course

$$\int_{-\infty}^{\infty} F(z)\cdot\frac{d\mu(t)}{W(t)} = 0$$

since $1 \in \mathscr{E}_A$. Therefore

$$\int_{-\infty}^{\infty} \frac{G(t)-G(z)}{t-z}\frac{d\mu(t)}{W(t)} = 0,$$

or, since $G(t)$ vanishes at *all the* x_n *save* x_0,

$$\frac{G(x_0)}{x_0-z}\frac{\mu(\{x_0\})}{W(x_0)} - G(z)\sum_n \frac{1}{(x_n-z)S'(x_n)} = 0.$$

This is the same as

$$\frac{S'(x_0)}{x_0-z}\cdot\frac{1}{S'(x_0)} = \frac{S(z)}{z-x_0}\sum_n \frac{1}{(x_n-z)S'(x_n)},$$

or

$$S(z)\sum_n \frac{1}{(z-x_n)S'(x_n)} = 1,$$

the desired relation.

From the result just found we derive a more general *interpolation formula*. Let $-A \leqslant \lambda \leqslant A$. Then $(e^{i\lambda t}-e^{i\lambda z})/(t-z)$ belongs, as a function of t, to $\mathscr{E}_A$, so

$$\int_{-\infty}^{\infty} \frac{e^{i\lambda t}-e^{i\lambda z}}{t-z}\frac{d\mu(t)}{W(t)} = 0.$$

In other words,

$$\sum_n \frac{e^{i\lambda x_n}}{(x_n-z)S'(x_n)} = e^{i\lambda z}\sum_n \frac{1}{(x_n-z)S'(x_n)}.$$

According to our previous result, the right-hand side is just $-e^{i\lambda z}/S(z)$. Therefore

$$\boxed{\frac{e^{i\lambda z}}{S(z)} = \sum_n \frac{e^{i\lambda x_n}}{(z-x_n)S'(x_n)} \quad \text{for} \quad -A \leqslant \lambda \leqslant A.}$$

(An analogous formula with $e^{i\lambda t}$ replaced by any $f(t) \in \mathscr{E}_A$ also holds, by the way – the proof is the same.)

In the boxed relation, put $\lambda = -A$ and take $z = iy$, $y > 0$. We get

$$\frac{e^{Ay}}{S(iy)} = \sum_n \frac{e^{-iAx_n}}{(iy - x_n)S'(x_n)}.$$

Since $\sum_n |1/S'(x_n)| < \infty$ and the x_n are real, the right side tends to 0 for $y \to \infty$. Thus,

$$\liminf_{y \to \infty} \frac{\log|S(iy)|}{y} \geqslant A.$$

But $\limsup_{y\to\infty} \log|S(iy)|/y \leqslant A$ since S is of exponential type $\leqslant A$. Therefore $\log|S(iy)|/y \to A$ for $y \to \infty$.

On taking $\lambda = A$ in the above boxed formula and making $y \to -\infty$, we see in like manner that

$$\frac{\log|S(iy)|}{|y|} \longrightarrow A \quad \text{for} \quad y \to -\infty.$$

De Branges' theorem is now completely proved. We are done.

3. Discussion of the theorem

De Branges' description of the extreme points of Σ is a most beautiful result; I still do not understand the full meaning of it.

Since $\int_{-\infty}^{\infty} (\log^+|S(x)|/(1+x^2))\,dx < \infty$ and $S(z)$ is of exponential type, the set of zeros $\{x_n\}$ of S, on which the extremal measure μ corresponding to S is supported, has a distribution governed by *Levinson's theorem* (Chapter III; here the version in §H.2 suffices). Because

$$\frac{\log|S(iy)|}{|y|} \longrightarrow A \quad \text{for} \quad y \longrightarrow \pm\infty,$$

we see by that theorem that

$$\frac{\text{number of } x_n \text{ in } [0,t]}{t} \longrightarrow \frac{A}{\pi} \quad \text{as} \quad t \to \infty$$

and

$$\frac{\text{number of } x_n \text{ in } [-t,0]}{t} \longrightarrow \frac{A}{\pi} \quad \text{as} \quad t \to \infty.$$

The zeros of $S(z)$ *are distributed roughly* (very *roughly*!) *like the points*

$$\frac{\pi}{A}n, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots.$$

(We shall see towards the end of Chapter IX that a certain refinement of this description is possible; we cannot, however obtain much more information about the actual *position* of the points x_n.)

De Branges' result is an *existence theorem*. It says that, if W is a weight of the kind considered in this § such that the $e^{i\lambda x}$, $-A \leqslant \lambda \leqslant A$, are *not* $\| \quad \|_W$-dense in $\mathscr{C}_W(\mathbb{R})$, *then* there *exists* an entire function $\Phi(z)$ of exponential type A with

$$\frac{\log|\Phi(iy)|}{|y|} \longrightarrow A \quad \text{for} \quad y \to \pm\infty, \qquad \int_{-\infty}^{\infty} \frac{\log^+|\Phi(x)|}{1+x^2}\,dx < \infty,$$

and $|\Phi(x_n)| \geqslant W(x_n)$ on a set of points x_n with $x_n \sim (\pi/A)n$ for $n \to \pm\infty$.

It suffices to take $\Phi(x) = S'(x)$ with one of the functions $S(z)$ furnished by the theorem. (There *will be* such a function S because here Σ is not reduced to $\{0\}$, and *will have* extreme points by the Krein–Millman theorem!) If $\{x_n\}$ is the set of zeros of S, we have

$$\sum_n \frac{W(x_n)}{|S'(x_n)|} = \int_{-\infty}^{\infty} |d\mu(x)| = 1,$$

so $|S'(x_n)| \geqslant W(x_n)$. Let us verify that

$$\int_{-\infty}^{\infty} \frac{\log^+|S'(x)|}{1+x^2}\,dx < \infty.$$

Our function $S(z)$ is of exponential type; *therefore, so is* $S'(z)$. The desired relation will hence follow in now familiar fashion via Fubini's theorem and §E of Chapter III from the inequality

$$\int_{-\infty}^{\infty} \frac{\log^+|S'(x+i)|}{1+x^2}\,dx < \infty,$$

which we proceed to establish (cf. the hall of mirrors argument at the end of §E.4).

Since $S(z)$ is free of zeros in $\Im z > 0$, we have there, by §G.1 of Chapter III,

$$\begin{aligned}\log|S(z)| &= A\Im z + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log|S(t)|}{|z-t|^2}\,dt \\ &= A\Im z + \frac{1}{\pi}\int_{-\infty}^{\infty} \Im\left(\frac{1}{t-z}\right)\log|S(t)|\,dt.\end{aligned}$$

For the same reason one can define an analytic function $\log S(z)$ in $\Im z > 0$. Using the previous relation together with the Cauchy–Riemann equations we thus find that

$$\frac{S'(z)}{S(z)} = \frac{\mathrm{d}\log S(z)}{\mathrm{d}z} = \left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right)\log|S(z)|$$

$$= -\mathrm{i}A - \frac{\mathrm{i}}{\pi}\int_{-\infty}^{\infty}\frac{\log|S(t)|}{(z-t)^2}\mathrm{d}t, \quad \Im z > 0,$$

whence, taking $z = x + \mathrm{i}$,

$$\left|\frac{S'(x+\mathrm{i})}{S(x+\mathrm{i})}\right| \leqslant A + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|\log|S(t)||}{(x-t)^2+1}\mathrm{d}t.$$

Here, since

$$\int_{-\infty}^{\infty}\frac{\log^+|S(t)|}{t^2+1}\mathrm{d}t < \infty,$$

we of course have

$$\int_{-\infty}^{\infty}\frac{\log^-|S(t)|}{t^2+1}\mathrm{d}t < \infty,$$

(Chapter III, §G.2) so

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|\log|S(t)||}{1+t^2}\mathrm{d}t = \text{say } C,$$

a finite quantity. By §B.2 we also have

$$\left|\frac{t-\mathrm{i}}{t-\mathrm{i}-x}\right|^2 \leqslant (|x|+2)^2, \quad t \in \mathbb{R},$$

so

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|\log|S(t)||}{(t-x)^2+1}\mathrm{d}t \leqslant C(|x|+2)^2$$

and thence, by the previous relation,

$$\frac{|S'(x+\mathrm{i})|}{|S(x+\mathrm{i})|} \leqslant A + C(|x|+2)^2.$$

This means, however, that

$$\log|S'(x+\mathrm{i})| \leqslant \log(A + C(|x|+2)^2) + \log|S(x+\mathrm{i})|,$$

from which

$$\int_{-\infty}^{\infty}\frac{\log^+|S'(x+\mathrm{i})|}{x^2+1}\mathrm{d}t \leqslant \int_{-\infty}^{\infty}\frac{\log^+|S(x+\mathrm{i})|}{1+x^2}\mathrm{d}x$$
$$+ \int_{-\infty}^{\infty}\frac{\log^+(A + C(|x|+2)^2)}{x^2+1}\mathrm{d}x.$$

Both integrals on the right are *finite*, however, the *first* because

$$\int_{-\infty}^{\infty} \frac{\log^+ |S(x)|}{1+x^2}\,dx \;<\; \infty,$$

and the *second* by inspection. Therefore

$$\int_{-\infty}^{\infty} \frac{\log^+ |S'(x+i)|}{1+x^2}\,dx \;<\; \infty,$$

which is what we needed to show.

We still have to check that

$$\frac{\log|S'(iy)|}{|y|} \longrightarrow A \quad \text{for} \quad y \longrightarrow \pm\infty.$$

There are several ways of doing this; one goes as follows. Since the limit relation in question is *true* for S, we have, for each $\varepsilon > 0$,

$$|S(z)| \;\leqslant\; M_\varepsilon \exp(A|\Im z| + \varepsilon|z|)$$

(see discussion at end of §B.2). Using Cauchy's formula (for the derivative) with circles of radius 1 centered on the imaginary axis, we see from this relation that

$$|S'(iy)| \;\leqslant\; \text{const.}e^{(A+\varepsilon)|y|},$$

so, since $\varepsilon > 0$ is arbitrary,

$$\limsup_{y\to\pm\infty} \frac{\log|S'(iy)|}{|y|} \;\leqslant\; A.$$

However, $S(iy) = S(0) + i\int_0^y S'(i\eta)\,d\eta$. Therefore the above limit superior along either direction of the imaginary axis must *be* A, otherwise $\log|S(iy)|/|y|$ could not tend to A as $y \to \pm\infty$. By a remark at the end of §G.1, Chapter III, it will follow from this fact that the ratio $\log|S'(iy)|/|y|$ actually *tends* to A as $y \to \pm\infty$, if we can verify that $S'(z)$ *has only real zeros.*

To see this, write the Hadamard factorization (Chapter III, §A) for S:

$$S(z) \;=\; Ae^{cz}\prod_n\left(1 - \frac{z}{x_n}\right)e^{z/x_n}.$$

(We are assuming that none of the zeros x_n of S is equal to 0; if one of them is, a slight modification in this formula is necessary.) Here, as we know, all the x_n are real, therefore

$$\left|\frac{S(iy)}{S(-iy)}\right| \;=\; e^{-2y\Im c}.$$

Since $\log|S(\mathrm{i}y)|/y$ and $\log|S(-\mathrm{i}y)|/y$ both tend to the same limit, A, as $y \to \infty$, we must have $\Im c = 0$, i.e., c is *real*. Logarithmic differentiation of the above Hadamard product now yields

$$\frac{S'(z)}{S(z)} = c + \sum_n \left(\frac{1}{z - x_n} + \frac{1}{x_n} \right),$$

whence

$$\Im\left(\frac{S'(z)}{S(z)} \right) = -\sum_n \frac{\Im z}{|z - x_n|^2}.$$

The expression on the right is < 0 for $\Im z > 0$ and > 0 for $\Im z < 0$; $S'(z)$ can hence *not vanish* in *either of those half planes*. This argument (which goes back to Gauss, by the way), shows that *all the zeros of* $S'(z)$ *must be real*, as required.

We have now finished showing that the function $\Phi(z) = S'(z)$ has all the properties claimed for it. As an observation of general interest, let us just mention one more fact: *the zeros of* $S'(z)$ *are simple and lie between the zeros* x_n *of* $S(z)$. To see that, differentiate the above formula for $S'(z)/S(z)$ one more time, getting

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{S'(z)}{S(z)} \right) = -\sum_n \frac{1}{(z - x_n)^2}.$$

From this it is clear that $S'(x)/S(x)$ decreases strictly from ∞ to $-\infty$ on each open interval with endpoints at two successive points x_n, and hence *vanishes precisely once therein*. $S'(z)$ therefore has exactly *one* zero in each such interval, and, since all its zeros are *real*, *no others*.

This property implies that the (real) zeros of $S'(z)$ *have the same asymptotic distributions as the* x_n. From that it is easy to obtain another proof of the limit relation

$$\frac{\log|S'(\mathrm{i}y)|}{|y|} \longrightarrow A, \quad y \longrightarrow \pm\infty.$$

Just use the Hadamard factorization of $S'(z)$ to write $\log|S'(\mathrm{i}y)|$ as a *Stieltjes integral*, then perform an integration by parts in the latter. The desired result follows without difficulty (see a similar computation in §H.3, Chapter III).

Let us summarize. If, for a weight $W(x)$, the $\mathrm{e}^{\mathrm{i}\lambda x}$, $-A \leqslant \lambda \leqslant A$, are *not* $\|\ \|_W$-dense in $\mathscr{C}_W(\mathbb{R})$, Louis de Branges' theorem furnishes entire functions of *precise exponential type* A having *convergent* $\log^+$ *integrals* which are at the same time *large* ($\geqslant W$ in absolute value) *fairly often*, namely on a set of points x_n with $x_n \sim (\pi/A)n$ for $n \to \pm\infty$. These points

x_n are of course located in the set E where $W(x) < \infty$; the theorem, unfortunately, does not provide much more information about their position, even though some refinement in the description of their asymptotic distribution is possible (Chapter IX). One would like to know more about the location of the x_n.

4. Scholium. Krein's functions

Entire functions whose reciprocals have partial fraction decompositions like the one for $1/S(z)$ figuring in the proof of de Branges' theorem arise in the study of various questions. They were investigated by M.G. Krein, in connection, I believe, with the inverse Sturm–Liouville problem. We give some results about such functions here, limiting the discussion to those with *real zeros*. More material on Krein's work (he allowed complex zeros) can be found, together with references, in Levin's book.

Theorem. *Let $S(z)$ be entire, of exponential type, and have only the real simple zeros $\{x_n\}$. Suppose that $S(z) \to \infty$ as $z \to \infty$ along each of four rays*

$$\arg z = \alpha_k,$$

with

$$0 < \alpha_1 < \frac{\pi}{2} < \alpha_2 < \pi < \alpha_3 < \frac{3\pi}{2} < \alpha_4 < 2\pi,$$

and that also

$$\sum_n |1/S'(x_n)| < \infty.$$

Then

$$\frac{1}{S(z)} = \sum_n \frac{1}{(z - x_n)S'(x_n)}.$$

Proof. The function

$$L(z) = \sum_n \frac{S(z)}{(z - x_n)S'(x_n)}$$

is entire, since $S(x_n) = 0$ for each n and $\sum_n |1/S'(x_n)| < \infty$.

I claim that $L(z)$ *is of exponential type*. Clearly,

$$|L(z)| \leqslant \text{const.} \frac{|S(z)|}{|\Im z|},$$

so the growth of $L(z)$ is dominated by that of $S(z)$ outside the strip $|\Im z| \leqslant 1$. For $|\Im z| \leqslant 1$, one may use the following trick. The function $\sqrt{|L(z)|}$ is *subharmonic*, therefore

$$\sqrt{|L(z)|} \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \sqrt{|L(z+2e^{i\theta})|}\,d\theta.$$

Substituting the preceding inequality into the integral on the right, we obtain for it a bound of the form const.$e^{K|z|/2}\int_{-\pi}^{\pi} d\theta/\sqrt{|\Im z + 2\sin\theta|}$, and this is clearly $\leqslant$ const.$e^{K|z|/2}$ for $|\Im z| \leqslant 1$. We see in this way that $|L(z)| \leqslant Ce^{K|z|}$ for all z.

We have $L(x_k) = 1$ at each x_k. Therefore

$$\frac{L(z)-1}{S(z)} = \sum_n \frac{1}{(z-x_n)S'(x_n)} - \frac{1}{S(z)}$$

is *entire*; *as the ratio of two entire functions of exponential type it is also of exponential type by Lindelöf's theorem.* (See *third* theorem of §B, Chapter III.)

Since $\sum_n |1/S'(x_n)| < \infty$,

$$\sum_n \frac{1}{(z-x_n)S'(x_n)} \longrightarrow 0$$

as $z \to \infty$ along each of the rays $\arg z = \alpha_k$, $k = 1,2,3$ and 4, and by hypothesis $1/S(z) \to 0$ for $z \to \infty$ along each of those rays. Therefore $(L(z)-1)/S(z)$ is certainly *bounded* on each of those rays, so, since it is *entire and of exponential type*, it is *bounded in each of the four sectors separated by them* (and having opening $< 180°$) according to the *second Phragmén–Lindelöf theorem* of §C, Chapter III. The entire function $(L(z)-1)/S(z)$ is thus *bounded in* $\mathbb{C}$, hence *equal to a constant*, by Liouville's theorem. Since, as we have seen, it tends to *zero* for z tending to ∞ along certain rays, *the constant must be zero*. Hence

$$\sum_n \frac{1}{(z-x_n)S'(x_n)} - \frac{1}{S(z)} \equiv 0, \qquad \text{Q.E.D.}$$

Remark. The hypothesis of the theorem just proved is very ungainly, and one would *like* to be able to affirm the following more general result:

Let $S(z)$, *of exponential type, have only the real simple zeros* x_n, *let* $\sum_n |1/S'(x_n)| < \infty$, *and suppose that* $S(iy) \to \infty$ *for* $y \to \pm\infty$. *Then*

$$\frac{1}{S(z)} = \sum_n \frac{1}{(z-x_n)S'(x_n)}.$$

One can waste much time attempting to prove this statement, all in vain, because it is false! In order to lay this ghost for good, here is a *counter example.*

Take

$$S(z) \;=\; \prod_1^\infty \left(1 - \frac{z}{2^n}\right) e^{z/2^n};$$

since $\sum_1^\infty 2^{-n} < \infty$, $S(z)$ is of exponential type. One readily computes $|S'(2^n)|$ by the method used in §C, and finds that

$$|S'(2^n)| \;\sim\; \frac{1}{e} 2^{(n(n-3)/2)} e^{2^n} (S(1))^2$$

for $n \to \infty$; we thus *certainly have*

$$\sum_1^\infty |1/S'(2^n)| < \infty.$$

It is also true that

$$|S(iy)|^2 \;=\; \prod_1^\infty \left(1 + \frac{y^2}{4^n}\right) \;\longrightarrow\; \infty$$

for $y \to \pm\infty$. However, $\prod_1^\infty (1 + |z|/2^n) \leqslant e^{o(|z|)}$ for $z \to \infty$, again by convergence of $\sum_1^\infty 2^{-n}$ (see calculations in §A, Chapter III!). So, for x *real and negative,*

$$S(x) \;=\; e^x \prod_1^\infty \left(1 + \frac{|x|}{2^n}\right) \;\leqslant\; e^{-|x| + o(|x|)},$$

and, for $x \to -\infty$, $1/S(x)$ tends to ∞ like an exponential (!). Therefore $1/S(x)$ certainly *cannot* equal

$$\sum_1^\infty \frac{1}{(x - 2^n)S'(2^n)}$$

which *tends to zero* as $x \to -\infty$.

Theorem (Krein). *Let an entire function $S(z)$ have only the real simple zeros x_n; suppose that $\sum_n |1/S'(x_n)| < \infty$ and that*

$$\frac{1}{S(z)} \;=\; \sum_n \frac{1}{(z - x_n)S'(x_n)}.$$

Then $S(z)$ is of exponential type, *and*

$$\int_{-\infty}^{\infty} \frac{\log^+ |S(x)|}{1 + x^2} dx \;<\; \infty.$$

Remark. In particular, for functions $S(z)$ satisfying the hypothesis of the *previous* theorem, we have

$$\int_{-\infty}^{\infty} \frac{\log^+ |S(x)|}{1+x^2} dx \ < \ \infty.$$

The reader who wants only *this* result may skip *all but the last paragraph* of the following demonstration.

Proof of theorem. Without loss of generality, $\sum_n |1/S'(x_n)| = 1$, whence, by the assumed representation for $1/S(z)$,

$$\left|\frac{1}{S(z)}\right| \ \leqslant \ \frac{1}{|\Im z|}.$$

Given any $h > 0$, the reciprocal $1/S(z)$ is thus *bounded and non-zero* in each of the half-planes $\{\Im z \geqslant h\}$, $\{\Im z \leqslant -h\}$, as well as being *analytic* in slightly larger open half planes containing them. The representation of §G.1, Chapter III, therefore applies in each of those half planes, and we find that in fact

$$\int_{-\infty}^{\infty} \frac{\log^- |1/S(t+ih)|}{1+t^2} dt \ < \ \infty,$$

and that

$$\log\left|\frac{1}{S(z)}\right| \ = \ -A_h(\Im z - h) + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(\Im z - h)\log|1/S(t+ih)|}{|z-t-ih|^2} dt$$

for $\Im z > h$, while $\int_{-\infty}^{\infty} (\log^- |1/S(t-ih)|/(1+t^2))\, dt < \infty$ and

$$\log\left|\frac{1}{S(z)}\right| \ = \ -B_h(|\Im z| - h) + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(|\Im z| - h)\log|1/S(t-ih)|}{|z-t+ih|^2} dt$$

when $\Im z < -h$.

Here, A_h and B_h are constants which, *a priori*, depend on h. *In fact, they do not*, because, by the remark at the end of §G.1, Chapter III, $\lim_{y\to\infty}(1/y)\log|1/S(iy)|$ *exists* and equals $-A_h$, with a similar relation involving B_h for $y \to -\infty$. All the numbers $-A_h$ for $h > 0$ are thus equal to the limit just mentioned, say to $-A$, and all the B_h are similarly equal to some number B.

For each $h > 0$ we thus have, for $\Im z > h$,

$$\begin{aligned} \log|S(z)| \ &= \ A(\Im z - h) + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(\Im z - h)\log|S(t+ih)|}{|z-t-ih|^2} dt \\ &\leqslant \ A(\Im z - h) + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(\Im z - h)\log^+ |S(t+ih)|}{|z-t-ih|^2} dt. \end{aligned}$$

Similar relations involving B, which we do not bother to write down, hold for $\Im z < -h$. Let us fix some value of h, say $h = 1$. We have

$$\int_{-\infty}^{\infty} \frac{\log^+ |S(t \pm \mathrm{i})|}{1+t^2} \mathrm{d}t = \int_{-\infty}^{\infty} \frac{\log^- |1/S(t \pm \mathrm{i})|}{1+t^2} \mathrm{d}t,$$

both of which are *finite*. Knowing this we can, by using the two inequalities for $\log|S(z)|$ involving integrals with $\log^+$, in $\{\Im z > 1\}$ and in $\{\Im z < -1\}$, verify immediately that $S(z)$ is *of exponential growth at most* in *each* of the two sectors $\delta < \arg z < \pi - \delta$, $\pi + \delta < \arg z < 2\pi - \delta$, $\delta > 0$ being arbitrary. This verification proceeds in the same way as the corresponding one made while proving Akhiezer's *second* theorem, §B.2.

It remains to show that $S(z)$ is of at most exponential growth in each of the two sectors $|\arg z| < \delta$, $|\arg z - \pi| < \delta$. This can be done by choosing $\delta < \pi/4$ and then following the Phragmén–Lindelöf procedure used at the end of the proof of Akhiezer's second theorem, *provided* that we know that $|S(z)| \leqslant \exp(\mathrm{O}(|z|^2))$ for large $|z|$ in *each* of those two sectors. This property we now proceed to establish.

The method followed here is like that used to discuss $L(z)$ in the proof of the previous theorem. For $h > 0$, we have $|S(t + \mathrm{i}h)| \geqslant h$, so

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log^- |S(t+\mathrm{i}h)|}{1+t^2} \mathrm{d}t \leqslant \log^+ \frac{1}{h}.$$

At the same time,

$$A + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log |S(t+\mathrm{i}h)|}{t^2+1} \mathrm{d}t = \log|S(\mathrm{i}+\mathrm{i}h)|,$$

and, since $\mathrm{i} + \mathrm{i}h$ lies on the positive imaginary axis, we *already know* that $\log|S(\mathrm{i}+\mathrm{i}h)| \leqslant C(h+1)$ for $h > 0$, for the positive imaginary axis lies in the sector $\delta < \arg z < \pi - \delta$ where (at most) exponential growth of $S(z)$ *is* clear. Because $\log^+|S(t+\mathrm{i}h)| = \log|S(t+\mathrm{i}h)| + \log^-|S(t+\mathrm{i}h)|$, the above relations yield, for $h > 0$.

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log^+ |S(t+\mathrm{i}h)|}{t^2+1} \mathrm{d}t \leqslant -A + C(h+1) + \log^+ \frac{1}{h}.$$

In like manner,

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log^+ |S(t-\mathrm{i}h)|}{t^2+1} \mathrm{d}t \leqslant -B + C'(h+1) + \log^+ \frac{1}{h}$$

when $h > 0$.

From these two inequalities we now find, for large (!) R, that

$$\frac{1}{\pi}\int_{-R}^{R}\int_{-\infty}^{\infty} \frac{\log^+ |S(t+\mathrm{i}y)|}{1+t^2} \mathrm{d}t\,\mathrm{d}y \leqslant \text{const.} R^2$$

(note that $\int_{-R}^{R} \log^+(1/|y|)\,dy \leqslant \text{const}!$). This inequality yields, in turn

$$\int_{-R}^{R}\int_{-R}^{R} \log^+|S(z)|\,dx\,dy \;\leqslant\; \text{const.}R^4$$

for large R.

Let z_0 be given. Since $\log^+|S(z)|$ is *subharmonic*,

$$\log^+|S(z_0)| \;\leqslant\; \frac{1}{\pi|z_0|^2}\iint_{|z-z_0|\leqslant|z_0|} \log^+|S(z)|\,dx\,dy$$

$$\leqslant\; \frac{4}{\pi R^2}\int_{-R}^{R}\int_{-R}^{R} \log^+|S(z)|\,dx\,dy,$$

where $R = 2|z_0|$. By what we have just seen, the expression on the *right* is $\leqslant (1/R^2)O(R^4) = O(|z_0|^2)$ for large values of $|z_0|$, i.e., $|S(z_0)| \leqslant \exp(O(|z_0|^2))$ when $|z_0|$ is large. This is what we wanted to show; as explained above, it implies that $S(z)$ is actually of exponential growth in the two sectors $|\arg z| < \delta$, $|\arg z - \pi| < \delta$, and hence, finally, that the entire function $S(z)$ *is of exponential type.*

We still have to show that

$$\int_{-\infty}^{\infty} \frac{\log^+|S(x)|}{1+x^2}\,dx \;<\; \infty.$$

That is, however, immediate. In the course of the argument just completed, we had (taking, for instance, $h = 1$) the relation

$$\int_{-\infty}^{\infty} \frac{\log^+|S(t+i)|}{1+t^2}\,dt \;<\; \infty.$$

Because S is of *exponential type*, the desired inequality follows from this one by §E of Chapter III (applied in the half plane $\Im z < 1$) and Fubini's theorem, in the usual fashion (hall of mirrors). We are done.

Remark. We remind the reader that, since the functions $S(z)$ considered here have no zeros either in $\Im z > 0$ or in $\Im z < 0$, the representation of §G.1, Chapter III holds for them in each of those half planes. That is,

$$\log|S(z)| \;=\; A\Im z + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z \log|S(t)|}{|z-t|^2}\,dt \quad \text{for} \quad \Im z > 0,$$

and

$$\log|S(z)| \;=\; B|\Im z| + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\Im z|\log|S(t)|}{|z-t|^2}\,dt \quad \text{for} \quad \Im z < 0.$$

Problem 9

Let $x_{-n} = -x_n$, let $\sum_1^\infty 1/x_n^2 < \infty$, and suppose that $\sum'^{\infty}_{-\infty} |1/S'(x_n)| < \infty$, where

$$S(z) = \prod_1^\infty \left(1 - \frac{z^2}{x_n^2}\right).$$

The x_n are assumed to be *real*.

Show that

$$\frac{1}{S(z)} = \sum_{-\infty}^{\infty}{}' \frac{1}{(z - x_n)S'(x_n)},$$

and hence that $S(z)$ is of *exponential type*, and that

$$\int_{-\infty}^{\infty} \frac{\log^+ |S(x)|}{1 + x^2} \mathrm{d}x < \infty.$$

(Hint: First put $S_R(z) = \prod_{0 < x_n \leqslant R}(1 - z^2/x_n^2)$, and show that one can make $R \to \infty$ in the Lagrange formula

$$1 = \sum_{|x_n| \leqslant R} \frac{S_R(z)}{(z - x_n)S_R'(x_n)}$$

so as to obtain

$$1 = \sum_{-\infty}^{\infty}{}' \frac{S(z)}{(z - x_n)S'(x_n)}.$$

At this point, one may either invoke Krein's theorem, or else look at the Poisson representation of the (negative) harmonic function $\log|1/S(z)|$ in a suitable half-plane $\{\Im z > H\}$, noting that here $|S(z)| \leqslant S(\mathrm{i}|z|)$.)

Problem 10

Let $S(z)$ be entire, of exponential type, and satisfy the rest of the hypothesis of the *first* theorem of this article. That is, S has only the real simple zeros x_n, $\sum_n |1/S'(x_n)| < \infty$, and $S(z) \to \infty$ for z tending to ∞ along *four rays*, *one* in the *interior* of *each* of the *four quadrants*. Suppose also that the *two limits* (which *exist* by the above discussion) of $\log|S(\mathrm{i}y)|/|y|$, for $y \to \infty$ and for $y \to -\infty$, *are equal*, say to $A > 0$. The purpose of this problem is to prove that

$$\sum_n \frac{\mathrm{e}^{\mathrm{i}\lambda x_n}}{S'(x_n)} = 0 \quad \text{for} \quad -A \leqslant \lambda \leqslant A.$$

(a) If

$$F(z) = \sum_n \frac{S(z)\mathrm{e}^{\mathrm{i}\lambda x_n}}{(z - x_n)S'(x_n)},$$

show that $F(z)$ is entire and of exponential type A, and that

$$\int_{-\infty}^{\infty} \frac{\log^+|F(x)|}{1+x^2}\,dx < \infty.$$

(Hint: Refer to the trick with $L(z)$, pp. 203–4.)

(b) Show that $Q(z) = (F(z) - e^{i\lambda z})/S(z)$ is *entire and of exponential type*, and that

$$\int_{-\infty}^{\infty} \frac{\log^+|Q(x)|}{1+x^2}\,dx < \infty.$$

(c) If $-A < \lambda < A$ and $Q(z)$ is the function constructed in (b), show that $Q(z) \equiv 0$. (Hint: First show that $Q(iy) \to 0$ for $y \to \pm\infty$ when $-A < \lambda < A$. Use this fact and the result proved in (b) to show that

$$\limsup_{R\to\infty} \frac{\log|Q(Re^{i\varphi})|}{R} \leqslant 0$$

if $\varphi \neq 0$ or π. Then use boundedness of Q on the imaginary axis and apply the Poisson representation for $\log|Q(z)|$ (or else Phragmén–Lindelöf) in the *right* and *left* half-planes.)

(d) $\sum_{-\infty}^{\infty}(e^{i\lambda x_n}/S'(x_n)) = 0$ for $-A \leqslant \lambda \leqslant A$. (Hint: Show this for $-A < \lambda < A$ and argue by continuity. Here, one may observe that

$$\sum_{-\infty}^{\infty} \frac{e^{i\lambda x_n}}{S'(x_n)} = \lim_{y\to\infty}\left(iy\cdot\sum_{-\infty}^{\infty}\frac{e^{i\lambda x_n}}{(iy-x_n)S'(x_n)}\right),$$

and use the result of (c).)

G. Weighted approximation with L_p norms

The results established in §§A–E apply to *uniform* weighted approximation, i.e., to approximation using the norm

$$\|\varphi\|_W = \sup_{t\in\mathbb{R}}\left|\frac{\varphi(t)}{W(t)}\right|.$$

One may ask what happens if, instead of *this* norm, we use a weighted L_p one, viz.

$$\|\varphi\|_{W,p} = \sqrt[p]{\int_{-\infty}^{\infty}\left|\frac{\varphi(x)}{W(x)}\right|^p dx};$$

here, p is some number $\geqslant 1$. The answer is that *all the results except for de Branges' theorem* (§F) *carry over with hardly any change, not even in the proofs.* Here, we of course have to assume that, for $x \to \pm\infty$, $W(x) \to \infty$ rapidly enough to make

$$\int_{-\infty}^{\infty}\left(\frac{1}{W(x)}\right)^p dx < \infty.$$

(some weakening of this restriction is possible; compare with the discussion in §E.4.)

It is enough to merely peruse the proofs of Mergelian's and Akhiezer's theorems, whether for approximation by polynomials or by functions in $\mathscr{E}_A$, to see that they are applicable *as is* with the norms $\| \ \|_{W,p}$. Here the functions $\Omega(z)$, $\Omega_A(z)$, $W_*(z)$ and $W_A(z)$ have evidently to be defined using the appropriate norm $\| \ \|_{W,p}$ instead of $\| \ \|_W$. And it is no longer necessarily true that $W_*(x) \leqslant W(x)$.

Verification of all this is left to the reader. In general, in the *kind* of approximation problem considered here (that of the *density* of a certain simple class of functions in the whole space), *it makes very little difference which L_p norm is chosen.* If the proofs vary in difficulty, they are hardest for the L_1 norm or for the uniform one. Here, the *continuous functions* (with the uniform norm) play the rôle of '$\lim_{p\to\infty} L_p$', and *not* L_∞, which is not even separable.

H. Comparison of weighted approximation by polynomials and by functions in $\mathscr{E}_A$

We now turn to the examination of the *relations* between the $\| \ \|_W$-closed subspaces of $\mathscr{C}_W(\mathbb{R})$ generated by the *polynomials* and by the *linear combinations of the* $e^{i\lambda x}$, $-A \leqslant \lambda \leqslant A$, for $A > 0$.

▶ *In order to consider the former subspace, it is of course necessary to assume that*

$$x^n/W(x) \longrightarrow 0 \quad \textit{for } x \longrightarrow \pm\infty$$

when $n \geqslant 0$. This we do throughout the present §.

We also use systematically the following

Notation. $\mathscr{C}_W(0)$ is the $\| \ \|_W$-closure of the set of polynomials in $\mathscr{C}_W(\mathbb{R})$. For $A > 0$, $\mathscr{C}_W(A)$ is the $\| \ \|_W$-closure of the set of finite linear combinations of the $e^{i\lambda x}$; $-A \leqslant \lambda \leqslant A$. (*Equivalently, $\mathscr{C}_W(A)$ is the $\| \ \|_W$-closure of $\mathscr{E}_A$*; see §E.1.)

It also turns out to be useful to introduce some intersections:

Definition. For $A \geqslant 0$ (*sic*!),

$$\mathscr{C}_W(A+) = \bigcap_{A' > A} \mathscr{C}_W(A').$$

In this §, we shall be especially interested in $\mathscr{C}_W(0+)$, the set of functions in $\mathscr{C}_W(\mathbb{R})$ *which can be* $\|\ \|_W$*-approximated by entire functions of arbitrarily small exponential type.*

We clearly have $\mathscr{C}_W(A) \subseteq \mathscr{C}_W(A+)$ for $A > 0$. But also:

Lemma. $\mathscr{C}_W(0) \subseteq \mathscr{C}_W(0+)$.

Proof. We have to show that $\mathscr{C}_W(0) \subseteq \mathscr{C}_W(A)$ for every $A > 0$. *Fix* any such A.

We have $x/W(x) \longrightarrow 0$ for $x \to \pm\infty$. Therefore, for the functions

$$f_h(x) = \frac{e^{i(\lambda+h)x} - e^{i\lambda x}}{h}, \quad h > 0,$$

we have $\|f_h\|_W \leqslant \text{const.}$, $h > 0$, and $f_h(x)/W(x) \longrightarrow 0$ *uniformly* for $h > 0$ as $x \to \pm\infty$. Since $f_h(x) \longrightarrow xe^{i\lambda x}$ u.c.c. in x for $h \to 0$, we thus have $\|f_h(x) - xe^{i\lambda x}\|_W \longrightarrow 0$ as $h \to 0$, and $xe^{i\lambda x} \in \mathscr{C}_W(A)$ if $-A < \lambda < A$.

By iterating this procedure, we find that $x^n e^{i\lambda x} \in \mathscr{C}_W(A)$ for $n = 0, 1, 2, 3, \ldots$ if $-A < \lambda < A$. In particular, then, all the *powers* x^n, $n = 0, 1, 2, \ldots$, belong to $\mathscr{C}_W(A)$, so $\mathscr{C}_W(0) \subseteq \mathscr{C}_W(A)$, as required.

Remark. This *justifies* the notation $\mathscr{C}_W(0)$ for the $\|\ \|_W$-closure of *polynomials* in $\mathscr{C}_W(\mathbb{R})$.

Once we know that $\mathscr{C}_W(0) \subseteq \mathscr{C}_W(0+)$, it is natural to ask whether $\mathscr{C}_W(0) = \mathscr{C}_W(0+)$ for the weights considered in this §, and, if the equality does not hold for *all* such weights, for *which ones* it *is* true. In other words, if a given function can be $\|\ \|_W$-approximated by *entire functions of arbitrarily small exponential type*, can it be $\|\ \|_W$-approximated by *polynomials*? This question, which interested some probabilists around 1960, was studied by Levinson and McKean who used the quadratic norm $\|\ \|_{W,2}$ (§G) instead of $\|\ \|_W$, and, simultaneously and independently, by me, in terms of the uniform norm $\|\ \|_W$. I learned later, around 1967, that *I.O. Khachatrian* had done some of the same work that I had a couple of years before me, in a somewhat different way. He has a paper in the Kharkov University Mathematics and Mechanics Faculty's *Uchonye Zapiski* for 1964, and a short note in the (more accessible) 1962 *Doklady* (vol. 145).

The remainder of this § is concerned with the question of equality of the subspaces $\mathscr{C}_W(0)$ and $\mathscr{C}_W(0+)$. It turns out that *in general* they are *not equal*, but that they *are* equal when the weight $W(x)$ enjoys a certain *regularity*.

1. Characterization of the functions in $\mathscr{C}_W(A+)$

Akhiezer's second theorem (§§B.2 and E.2) generally furnishes only a partial description of the functions in $\mathscr{C}_W(A)$ when that subspace does not coincide with $\mathscr{C}_W(\mathbb{R})$. One important reason for introducing the intersections $\mathscr{C}_W(A+)$ is that we can give a *complete* description of the functions belonging to any one of them which is properly contained in $\mathscr{C}_W(\mathbb{R})$.

Lemma. *Suppose that $f(z)$ is an entire function of exponential type with*

$$|f(z)| \leqslant C_\varepsilon \exp(A|\Im z| + \varepsilon|z|)$$

for each $\varepsilon > 0$. Then, if $\delta > 0$, the Fourier transform

$$F_\delta(\lambda) = \int_{-\infty}^{\infty} e^{-\delta|x|} e^{i\lambda x} f(x)\, dx$$

belongs to $L_1(\mathbb{R})$, and, if $A' > A$,

$$(*) \qquad \int_{|\lambda| > A'} |F_\delta(\lambda)|\, d\lambda \longrightarrow 0 \quad \textit{for} \quad \delta \to 0.$$

Proof. For each $\delta > 0$, $e^{-\delta|x|}f(x)$ is in $L_1(\mathbb{R})$ (choose $\varepsilon < \delta$ in the given condition on $f(x)$), so $F_\delta(\lambda)$ is *continuous* and *therefore* integrable on $[-A', A']$. The whole lemma will thus follow as soon as we prove $(*)$.

Fix $A' > A$, and suppose for the moment that $\delta > 0$ is also fixed. Take an $\varepsilon > 0$ *less than* both $\delta/2$ and $(A' - A)/2$. If $\lambda \geqslant A'$, we then have, for $y = \Im z \geqslant 0$,

$$|e^{-\delta z} e^{i\lambda z} f(z)| \leqslant C_\varepsilon e^{(A - A')y - \delta x + \varepsilon|z|},$$

and, for $x = \Re z \geqslant 0$, this is in turn $< C_\varepsilon e^{-\varepsilon x - \varepsilon y}$.

Let us now apply Cauchy's theorem using the following contour Γ_R:

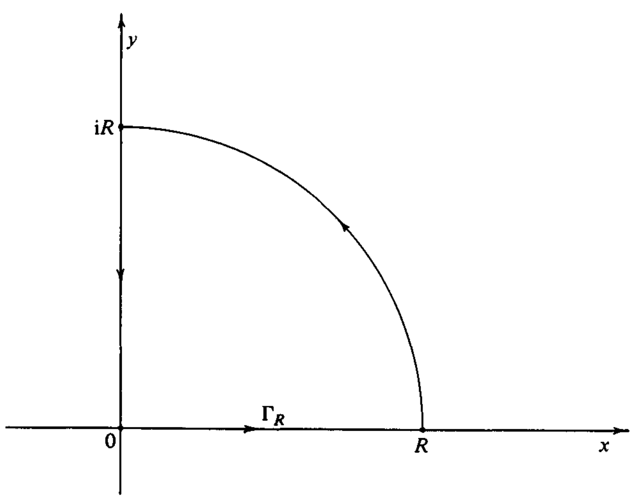

Figure 38

We have $\int_{\Gamma_R} e^{-\delta z} e^{i\lambda z} f(z) dx = 0$. For large R, $|e^{-\delta z} e^{i\lambda z} f(z)|$ is, by the preceding inequality, $< C_\varepsilon e^{-\varepsilon R/\sqrt{2}}$ on the *circular* part of Γ_R. Therefore the portion of our integral taken along this circular part tends to *zero* as $R \to \infty$, and we see that

$$\int_0^\infty e^{-\delta x} e^{i\lambda x} f(x) dx \;=\; i \int_0^\infty e^{-i\delta y} e^{-\lambda y} f(iy) dy.$$

This formula is valid whenever $\lambda \geqslant A' > A$ and $\delta > 0$. By integrating around the following contour

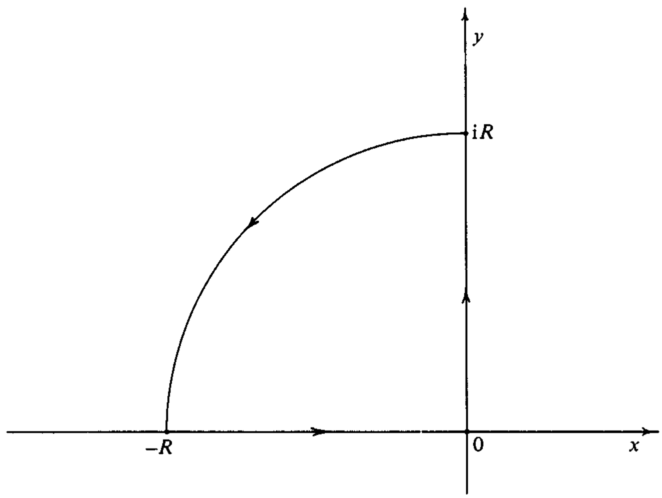

Figure 39

we see in like manner, on making $R \to \infty$, that

$$\int_{-\infty}^0 e^{\delta x} e^{i\lambda x} f(x) dx \;=\; -i \int_0^\infty e^{i\delta y} e^{-\lambda y} f(iy) dy,$$

whenever $\delta > 0$ and $\lambda \geqslant A' > A$. Combining this with the previous formula we get

$$F_\delta(\lambda) \;=\; \int_{-\infty}^\infty e^{-\delta|x|} e^{i\lambda x} f(x) dx \;=\; 2\int_0^\infty e^{-\lambda y} \sin \delta y f(iy) dy;$$

this holds whenever $\lambda \geqslant A' > A$ and $\delta > 0$.

Take now any η, $0 < \eta < (A' - A)/2$, and *fix* it for the following computation. Since, for $y \geqslant 0$, $|f(iy)| \leqslant C_\eta e^{(A+\eta)y}$, the formula just derived yields, for $\lambda \geqslant A'$,

$$|F_\delta(\lambda)| \;\leqslant\; 2\int_0^\infty C_\eta \mathrm{e}^{(A+\eta-\lambda)y}|\sin\delta y|\,\mathrm{d}y.$$

By Schwarz' inequality, the right-hand side is in turn

$$\leqslant\; 2C_\eta\sqrt{\left(\int_0^\infty \mathrm{e}^{-(\lambda-A-\eta)y}\,\mathrm{d}y\int_0^\infty \mathrm{e}^{-(\lambda-A-\eta)y}\sin^2\delta y\,\mathrm{d}y\right)}$$

For the second integral under the radical we have

$$\int_0^\infty \mathrm{e}^{-(\lambda-A-\eta)y}\sin^2\delta y\,\mathrm{d}y \;=\; \tfrac{1}{2}\Re\int_0^\infty \mathrm{e}^{-(\lambda-A-\eta)y}(1-\mathrm{e}^{2\mathrm{i}\delta y})\mathrm{d}y$$

$$= \frac{2\delta^2}{(\lambda-A-\eta)|\lambda-A-\eta-2\mathrm{i}\delta|^2}.$$

Therefore, for $\lambda \geqslant A'$,

$$|F_\delta(\lambda)| \;\leqslant\; \frac{2\sqrt{2}C_\eta\delta}{(\lambda-A-\eta)|\lambda-A-\eta-2\mathrm{i}\delta|} \;\leqslant\; \frac{2\sqrt{2}C_\eta\delta}{(\lambda-A-\eta)^2}.$$

And

$$\int_{A'}^\infty |F_\delta(\lambda)|\,\mathrm{d}\lambda \;\leqslant\; \frac{2\sqrt{2}C_\eta\delta}{A'-A-\eta} \;\leqslant\; \frac{2\sqrt{2}C_\eta\delta}{\eta}.$$

We see now that

$$\int_{A'}^\infty |F_\delta(\lambda)|\,\mathrm{d}\lambda \longrightarrow 0 \quad \text{for} \quad \delta \to 0.$$

Working with contours in the *lower* half plane, we see in the same way that

$$\int_{-\infty}^{-A'} |F_\delta(\lambda)|\,\mathrm{d}\lambda \longrightarrow 0 \quad \text{as} \quad \delta \to 0.$$

We have proved $(*)$, and are done.

Theorem (de Branges). *Let $f(z)$ be entire, and suppose that*

$$|f(z)| \;\leqslant\; C_\varepsilon \mathrm{e}^{A|\Im z|+\varepsilon|z|}$$

for each $\varepsilon > 0$. Then,

$$\text{if} \quad f\in\mathscr{C}_W(\mathbb{R}), \quad f\in\mathscr{C}_W(A+).$$

Proof. We have to show that, if $f\in\mathscr{C}_W(\mathbb{R})$, then in fact $f\in\mathscr{C}_W(A')$ for each $A' > A$; this we do by duality.

Fix any $A' > A$. According to the Hahn–Banach theorem it is enough to show that if L is any bounded linear functional on functions of the form $\varphi(t)/W(t)$ with $\varphi\in\mathscr{C}_W(\mathbb{R})$, and if

$$L\left(\frac{e^{i\lambda t}}{W(t)}\right) = 0 \quad \text{for} \quad -A'\leqslant\lambda\leqslant A',$$

then

$$L\left(\frac{f(t)}{W(t)}\right) = 0.$$

To see this, observe in the first place that $\|f(t)-e^{-\delta|t|}f(t)\|_W \longrightarrow 0$ for $\delta\to 0$, so surely

$$L\left(\frac{f(t)}{W(t)}\right) = \lim_{\delta\to 0} L\left(\frac{e^{-\delta|t|}f(t)}{W(t)}\right).$$

Our task thus reduces to showing that the limit on the right is zero; this we do with the help of the above lemma.

Writing, as in the lemma,

$$F_\delta(\lambda) = \int_{-\infty}^{\infty} e^{-\delta|x|}e^{i\lambda x}f(x)\,dx,$$

we have $F_\delta\in L_1(\mathbb{R})$ as we have seen. Hence, by the Fourier inversion formula,

$$e^{-\delta|t|}f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda t}F_\delta(\lambda)\,d\lambda.$$

In order to bring the functional L into play, we approximate the integral on the right by *finite sums.*

Put

$$S_N(t) = \frac{1}{2\pi}\sum_{k=-N^2}^{N^2-1} e^{-i(k/N)t}\int_{k/N}^{(k+1)/N} F_\delta(\lambda)\,d\lambda;$$

since $F_\delta\in L_1(\mathbb{R})$,

$$S_N(t) \longrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda t}F_\delta(\lambda)\,d\lambda = e^{-\delta|t|}f(t)$$

u.c.c. in t as $N\to\infty$, and, at the same time, $|S_N(t)|\leqslant\|F_\delta\|_1$ on $\mathbb{R}$ for all N. Therefore, since $W(t)\to\infty$ for $t\to\pm\infty$,

$$\|e^{-\delta|t|}f(t)-S_N(t)\|_W \underset{N}{\longrightarrow} 0,$$

so, by the *boundedness* of L,

$$L\left(\frac{e^{-\delta|t|}f(t)}{W(t)}\right) = \lim_{N\to\infty} L\left(\frac{S_N(t)}{W(t)}\right).$$

However, $\|e^{-i\lambda t} - e^{-i\lambda' t}\|_W \longrightarrow 0$ when $|\lambda - \lambda'| \to 0$, so $L(e^{-i\lambda t}/W(t))$ is a *continuous function* of λ on $\mathbb{R}$ as well as being *bounded* there (note that $|e^{i\lambda t}| = 1$!). Hence, since $F_\delta(\lambda)\in L_1(\mathbb{R})$, we have

$$L\left(\frac{S_N(t)}{W(t)}\right) = \frac{1}{2\pi}\sum_{k=-N^2}^{N^2-1} L\left(\frac{e^{-i(k/N)t}}{W(t)}\right)\int_{k/N}^{(k+1)/N} F_\delta(\lambda)\,d\lambda$$

$$\longrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} L\left(\frac{e^{-i\lambda t}}{W(t)}\right)F_\delta(\lambda)\,d\lambda$$

for $N\to\infty$. In view of the previous relation, we thus get

$$L\left(\frac{e^{-\delta|t|}f(t)}{W(t)}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} L\left(\frac{e^{-i\lambda t}}{W(t)}\right)F_\delta(\lambda)\,d\lambda.$$

We are assuming that

$$L\left(\frac{e^{-i\lambda t}}{W(t)}\right) = 0 \quad \text{for} \quad -A' \leqslant \lambda \leqslant A'.$$

The integral on the right thus reduces to

$$\frac{1}{2\pi}\int_{|\lambda|\geqslant A'} L\left(\frac{e^{-i\lambda t}}{W(t)}\right)F_\delta(\lambda)\,d\lambda.$$

Here, as already noted,

$$|L(e^{-i\lambda t}/W(t))| \leqslant \text{const.}, \quad \lambda\in\mathbb{R},$$

so the last integral is bounded in absolute value by

$$\text{const.}\int_{|\lambda|\geqslant A'} |F_\delta(\lambda)|\,d\lambda.$$

This, however, tends to 0 by the lemma as $\delta\to 0$. We see that

$$L\left(\frac{e^{-\delta|t|}f(t)}{W(t)}\right) \longrightarrow 0$$

for $\delta\to 0$, which is what was needed. The theorem is proved.

Remark. Since we are not supposing anything about *continuity* of $W(t)$, we are *not* in general permitted to write

$$L\left(\frac{f(t)}{W(t)}\right) \quad \text{as} \quad \int_{-\infty}^{\infty} \frac{f(t)}{W(t)}\,d\mu(t)$$

with a finite (complex-valued) Radon *measure* on $\mathbb{R}$. This makes the above proof *appear* a little more involved than in the case where use of such a measure is allowed. The difference is only in the *appearance*, however. The argument with a measure *is the same*, and only *looks* simpler.

Thanks to the above result, we can strengthen Akhiezer's second theorem so as to arrive at the following *characterization of the subspaces* $\mathscr{C}_W(A+)$. Recall the definition (§E.2):

$$W_A(z) = \sup\{|f(z)|: \ f \in \mathscr{E}_A \text{ and } \|f\|_W \leqslant 1\}.$$

Then we have the

Theorem. *Let $A \geqslant 0$. Either*

$$\int_{-\infty}^{\infty} \frac{\log W_{A'}(x)}{1+x^2}\,dx \ = \ \infty$$

for every $A' > A$, in which case $\mathscr{C}_W(A+)$ is equal to $\mathscr{C}_W(\mathbb{R})$, or else $\mathscr{C}_W(A+)$ consists precisely *of all the entire functions f such that $f(x)/W(x) \longrightarrow 0$ for $x \to \pm\infty$ and*

$$|f(z)| \ \leqslant \ C_\varepsilon e^{A|\Im z| + \varepsilon|z|}$$

for each $\varepsilon > 0$.

► ***Remark.*** *In the second case, $\mathscr{C}_W(A+)$ may* still *coincide with $\mathscr{C}_W(\mathbb{R})$. (If, for example, the set of points x where $W(x) < \infty$ is sufficiently sparse. See* §C *and end of* §E.2)

Proof. For the Mergelian function $\Omega_A(z)$ defined in §E.2, we have $\Omega_A(z) \geqslant W_A(z)$, so, if the *first alternative holds,*

$$\int_{-\infty}^{\infty} \frac{\log \Omega_{A'}(x)}{1+x^2}\,dx \ = \ \infty$$

for every $A' > A$. Then, by Mergelian's second theorem, $\mathscr{C}_W(A') = \mathscr{C}_W(\mathbb{R})$ for each $A' > A$, so $\mathscr{C}_W(A+) = \mathscr{C}_W(\mathbb{R})$.

The supremum $W_{A'}(z)$ is an *increasing* function of A' for each fixed z by virtue of the obvious inclusion of $\mathscr{E}_{A'}$ in $\mathscr{E}_{A''}$ when $A' \leqslant A''$. Therefore, if the *second alternative holds*, we have

$$(\dagger) \qquad \int_{-\infty}^{\infty} \frac{\log W_{A'}(x)}{1+x^2}\,dx \ < \ \infty$$

for each $A' \leqslant A_0$, some number *larger* than A.

Let $\varepsilon > 0$ be given, wlog $\varepsilon < A_0 - A$, and put $\delta = \varepsilon/2$. Then, if $f \in \mathscr{C}_W(A+)$, surely $f \in \mathscr{C}_W(A')$, where $A' = A + \delta$. For *this* A', (†) holds, so, by Akhiezer's *second theorem* (§E.2), we have

$$|f(z)| \leqslant K_\delta e^{A'|\Im z| + \delta|z|}.$$

Therefore

$$|f(z)| \leqslant K_{\varepsilon/2} e^{A|\Im z| + \varepsilon|z|}.$$

Saying that $f(x)/W(x) \longrightarrow 0$ for $x \to \pm\infty$ is simply another way of expressing the fact that $f \in \mathscr{C}_W(\mathbb{R})$. Thus, in the event of the *second alternative, all* the functions f in $\mathscr{C}_W(A+)$ *have* the two asserted properties.

However, *any* entire function f with those two properties *does* belong to $\mathscr{C}_W(A+)$. For such a function will be in $\mathscr{C}_W(\mathbb{R})$, and then *must* belong to $\mathscr{C}_W(A+)$ by the preceding theorem. The subspace $\mathscr{C}_W(A+)$ thus consists *precisely* of the functions having the two properties in question (and no others) when the second alternative holds. We are done.

Corollary. *For the intersections* $\mathscr{C}_W(A+)$ *the following alternative holds: Either* $\mathscr{C}_W(A+) = \mathscr{C}_W(\mathbb{R})$, *or, if* $\mathscr{C}_W(A+) \neq \mathscr{C}_W(\mathbb{R})$, *the former space consists* precisely *of the entire functions* $f(z)$ *belonging to* $\mathscr{C}_W(\mathbb{R})$ *with*

$$|f(z)| \leqslant C_\varepsilon e^{A|\Im z| + \varepsilon|z|}$$

for each $\varepsilon > 0$.

Remark. Even when $\mathscr{C}_W(A+) = \mathscr{C}_W(\mathbb{R})$, *all* the functions in $\mathscr{C}_W(A+)$ *may* have the form described in the *second* clause of this statement. That happens when $\mathscr{C}_W(\mathbb{R})$ consists entirely of the restrictions of such functions to the set of real x where $W(x) < \infty$. See remark following the statement of the preceding theorem.

2. Sufficient conditions for equality of $\mathscr{C}_W(0)$ and $\mathscr{C}_W(0+)$

Lemma. *Let* $w(z) = c\prod_1^N (z - a_k)$, *where the* a_k *are distinct, with* $\Im a_k < 0$. *Let* $g(z)$ *be an entire function of exponential type* $\leqslant A$ *with* $|(x+i)g(x)|$ *bounded for real* x. *Then, for* $x \in \mathbb{R}$,

$$\left| \frac{e^{-iAx} g(x)}{w(x)} - \sum_{k=1}^N \frac{g(a_k) e^{-iAa_k}}{w'(a_k)(x - a_k)} \right|$$
$$\leqslant \frac{e}{\pi} \int_{-\infty}^{\infty} \left| \frac{g(t + (i/A))}{w(t + (i/A))} \right| \left| \frac{\sin A(x-t)}{x - t - (i/A)} \right| dt.$$

Proof. $(z + i)g(z)$ is of exponential type $\leqslant A$ and is bounded on $\mathbb{R}$, hence has modulus $\leqslant \text{const.}e^{A|\Im z|}$ by the third Phragmén–Lindelöf theorem of §C, Chapter III. Hence

$$(*) \qquad |g(z)| \leq \text{const.}\frac{e^{A|\Im z|}}{|z+i|}.$$

We are going to use $(*)$ together with some contour integration.

Fix $b > 0$, take any large R, and let Γ_R be the following contour:

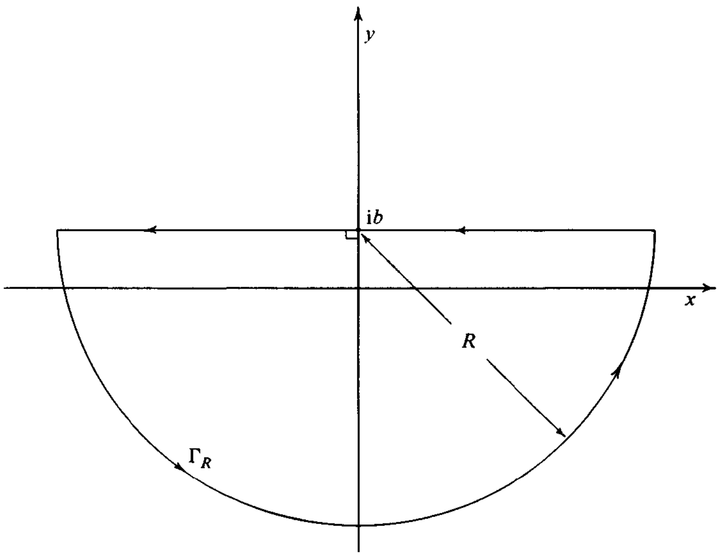

Figure 40

If R is large enough for Γ_R to encircle all the a_k and the real point x, the calculus of residues gives

$$\frac{1}{2\pi i}\int_{\Gamma_R} \frac{g(\zeta)e^{iA(x-\zeta)}}{w(\zeta)(x-\zeta)}d\zeta = \sum_{k=1}^{N} \frac{g(a_k)e^{iA(x-a_k)}}{w'(a_k)(x-a_k)} - \frac{g(x)}{w(x)}.$$

By $(*)$, $|g(\zeta)e^{-iA\zeta}|$ is $O(1/(|\zeta|-1)) = O(1/R)$ on the semi-circular part of Γ_R, so, as $R \to \infty$, the portion of the integral taken along that part of the contour tends to zero. Therefore

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{g(t+ib)e^{iA(x-t-ib)}}{w(t+ib)(x-t-ib)}dt = \frac{g(x)}{w(x)} - \sum_k \frac{g(a_k)e^{iA(x-a_k)}}{w'(a_k)(x-a_k)}.$$

We rewrite this relation as follows:

$$(\ast\ast) \qquad \frac{e^{Ab}}{2\pi i}\int_{-\infty}^{\infty} \frac{g(t+ib)e^{iA(x-t)}}{w(t+ib)(x-t-ib)}dt = \frac{g(x)}{w(x)} - \sum_k \frac{g(a_k)e^{iA(x-a_k)}}{w'(a_k)(x-a_k)}.$$

Let now Γ'_R be the contour obtained by reflecting Γ_R in the line $\Im z = b$:

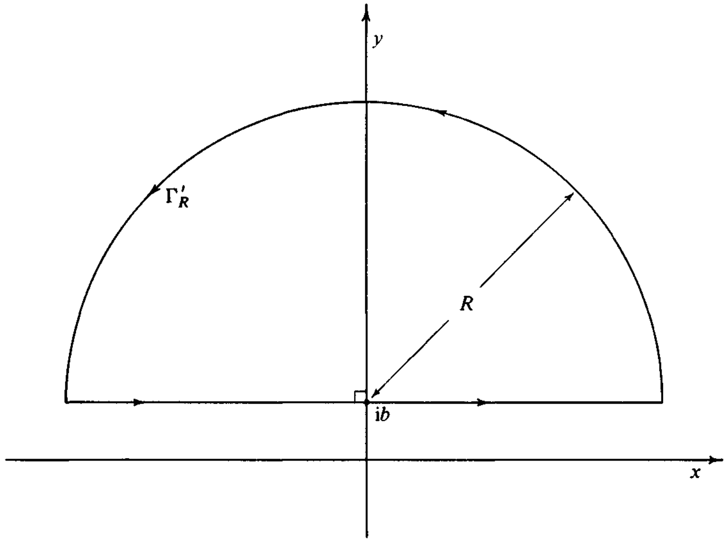

Figure 41

We have

$$\int_{\Gamma'_R} \frac{g(\zeta)e^{-iA(x-\zeta)}}{w(\zeta)(x-\zeta)} d\zeta \;=\; 0.$$

Here, $|g(\zeta)e^{iA\zeta}| = O(1/R)$ on the semi-circular part of Γ'_R, so, making $R \to \infty$, we get

$$\int_{-\infty}^{\infty} \frac{g(t+ib)e^{-iA(x-t-ib)}}{w(t+ib)(x-t-ib)} dt \;=\; 0,$$

that is,

$$\int_{-\infty}^{\infty} \frac{g(t+ib)e^{-iA(x-t)}}{w(t+ib)(x-t-ib)} dt \;=\; 0.$$

Multiplying the last relation by $e^{Ab}/2\pi i$ and subtracting the result from the left side of ($\ast\atop\ast$), we find

$$\frac{e^{Ab}}{\pi}\int_{-\infty}^{\infty} \frac{g(t+ib)\sin A(x-t)}{w(t+ib)(x-t-ib)} dt \;=\; \frac{g(x)}{w(x)} \;-\; \sum_k \frac{g(a_k)e^{iA(x-a_k)}}{w'(a_k)(x-a_k)}.$$

Now put $b = 1/A$ and multiply what has just been written by e^{-iAx}. After taking absolute values, we see that

$$\left|\frac{g(x)e^{-iAx}}{w(x)} - \sum_k \frac{g(a_k)e^{-iAa_k}}{w'(a_k)(x-a_k)}\right|$$
$$\leqslant \frac{e}{\pi}\int_{-\infty}^{\infty}\left|\frac{g(t+(i/A))}{w(t+(i/A))}\right|\left|\frac{\sin A(x-t)}{x-t-(i/A)}\right| dt,$$

Q.E.D.

Corollary. *Let $w(z)$ be as in the lemma, and suppose that $f(z)$ is entire, of exponential type $\leqslant A/2$, and bounded on $\mathbb{R}$. Then there is a polynomial $P(z)$ of degree less than that of $w(z)$, such that*

$$\left|\frac{P(x) - e^{-iAx}\dfrac{\sin(Ax/2)}{(Ax/2)}f(x)}{w(x)}\right| \leqslant \frac{Ke}{\pi}\sup_{t\in\mathbb{R}}\left|\frac{f(t+(i/A))}{w(t+(i/A))}\right| \quad \text{for } x\in\mathbb{R}.$$

Here K is an absolute numerical constant, whose value we do not bother to calculate.

Proof. Put $g(z) = (\sin(Az/2)/(Az/2))f(z)$; then $g(z)$ satisfies the hypothesis of the previous lemma, so, with the polynomial

$$P(x) = \sum_{k=1}^{N}\frac{w(x)g(a_k)e^{-iAa_k}}{w'(a_k)(x-a_k)},$$

we get, for $x\in\mathbb{R}$,

$$(\dagger)\qquad \left|\frac{P(x)-e^{-iAx}g(x)}{w(x)}\right| \leqslant \frac{e}{\pi}\sup_{t\in\mathbb{R}}\left|\frac{f(t+(i/A))}{w(t+(i/A))}\right|$$
$$\times\int_{-\infty}^{\infty}\left|\frac{\sin\dfrac{A}{2}\left(t+\dfrac{i}{A}\right)}{\dfrac{A}{2}\left(t+\dfrac{i}{A}\right)}\right|\left|\frac{\sin A(x-t)}{x-t-(i/A)}\right| dt.$$

In the integral on the right, make the substitutions $At/2 = \tau$, $Ax/2 = \xi$. That integral then becomes

$$2\int_{-\infty}^{\infty}\left|\frac{\sin(\tau+(i/2))}{\tau+(i/2)}\right|\left|\frac{\sin 2(\xi-\tau)}{2(\xi-\tau)-i}\right| d\tau.$$

By Schwarz, this last is

$$\leqslant 2\sqrt{\left(\int_{-\infty}^{\infty}\left|\frac{\sin(\tau+(i/2))}{\tau+(i/2)}\right|^2 d\tau\cdot\int_{-\infty}^{\infty}\frac{\sin^2 2(\tau-\xi)}{4(\tau-\xi)^2}d\tau\right)},$$

a *finite quantity* – call it K – *independent of* ξ, hence (clearly) *independent* of A and x.

The right-hand side of (†) is thus bounded above by

$$K\cdot\frac{\mathrm{e}}{\pi}\sup_{t\in\mathbb{R}}\left|\frac{f(t+(\mathrm{i}/A))}{w(t+(\mathrm{i}/A))}\right|,$$

and the corollary is established.

Theorem. *Let* $W(x)=\sum_0^\infty a_{2k}x^{2k}$, where the a_{2k} are all $\geqslant 0$, *with* $a_0\geqslant 1$ *and* $a_{2k}>0$ *for infinitely many values of k. Then* $\mathscr{C}_W(0)=\mathscr{C}_W(0+)$.

Remark. We require $a_0\geqslant 1$ because our weights $W(x)$ are supposed to be $\geqslant 1$. We require $a_{2k}>0$ for infinitely many k because $W(x)$ is supposed to go to ∞ faster than any polynomial as $x\to\pm\infty$.

Proof of theorem. Let $\varphi\in\mathscr{C}_W(0+)$. Then there are finite sums

$$f_n(x) \;=\; \sum_{-1/2n\leqslant\lambda\leqslant 1/2n} a_n(\lambda)\mathrm{e}^{\mathrm{i}\lambda x}$$

with

$$\|f_n-\varphi\|_W \;\underset{n}{\longrightarrow}\; 0.$$

We put $g_n(x)=(\sin(x/2n)/(x/2n))f_n(x)$, and set out to apply the above corollary with $f=f_n$ and suitable polynomials w. Note that f_n is entire, of exponential type $1/2n$, and bounded on the real axis. Since $\|f_n-\varphi\|_W \underset{n}{\longrightarrow} 0$, we also have

$$\|\mathrm{e}^{-\mathrm{i}x/n}g_n(x)-\varphi(x)\|_W \;\underset{n}{\longrightarrow}\; 0,$$

in view of the fact that $W(x)\to\infty$ for $x\to\pm\infty$.*

Choose any $\varepsilon>0$. The norms $\|f_n\|_W$ must be bounded; wlog $|f_n(x)/W(x)|\leqslant 1$, say, for $x\in\mathbb{R}$ and every n. For the function $\varphi\in\mathscr{C}_W(\mathbb{R})$ we of course have $\varphi(x)/W(x)\underset{n}{\longrightarrow}0$ for $x\to\pm\infty$, so, since $\|f_n-\varphi\|_W\underset{n}{\longrightarrow}0$, *there must be an* L (depending on ε) such that

$$\left|\frac{f_n(x)}{W(x)}\right| \;\leqslant\; \varepsilon \quad\text{for}\quad |x|\geqslant L$$

whenever n is sufficiently large. Take such an L and *fix* it.

Pick any n large enough for the previous relation to be true, and *fix* it *for the moment.* Our individual function $f_n(x)$ is *bounded* on $\mathbb{R}$ (true, with perhaps an enormous bound!), so, for some N_0, we will surely have

$$\left|f_n(x)\Big/\sum_0^{N_0}a_{2k}x^{2k}\right| \;<\; \varepsilon \quad\text{for}\quad |x|>A,\ \text{say},$$

* Note that $\|\mathrm{e}^{-\mathrm{i}x/n}(\sin(x/2n)/(x/2n))\varphi(x) \;-\; \varphi(x)\|_W \;\underset{n}{\longrightarrow}\; 0$ for any $\varphi\in\mathscr{C}_W(\mathbb{R})$.

where, wlog, $A > L$, the number chosen above. Also,

$$1 \leqslant \sum_0^N a_{2k}x^{2k} \xrightarrow[N]{} W(x),$$

the sums on the left being monotone increasing with N $(a_{2k} \geqslant 0!)$. Therefore,

$$\left| f_n(x) \middle/ \sum_0^N a_{2k}x^{2k} \right| \xrightarrow[N]{} |f_n(x)/W(x)|$$

uniformly for $-A \leqslant x \leqslant A$, and, if $N \geqslant N_0$ is large enough, we have, in view of the previous inequality,

$$(\S) \qquad \left| f_n(x) \middle/ \sum_0^N a_{2k}x^{2k} \right| \leqslant 2 \quad \text{for} \quad x \in \mathbb{R}$$

(since $\| f_n \|_W \leqslant 1$), and also

$$(\dagger\dagger) \qquad \left| f_n(x) \middle/ \sum_0^N a_{2k}x^{2k} \right| \leqslant 2\varepsilon \quad \text{for} \quad |x| \geqslant L.$$

Fix such an N for the moment (it depends of course on n which we have already fixed!), and call

$$V(x) = \sum_0^N a_{2k}x^{2k}.$$

Because $V(x) \geqslant 1$ on $\mathbb{R}$, we can find another polynomial $w(x)$, *with all its zeros in* $\Im z < 0$, such that $|w(x)| = V(x)$, $x \in \mathbb{R}$.

There is no loss of generality in supposing that the zeros of w are *distinct*. There are, in any case, a finite number ($2N$) of them, lying in the open lower half plane. Separating each *multiple* zero (if there are any*) into a cluster of *simple* ones, *very close together*, will change $w(x)$ to a polynomial $\tilde{w}(x)$ having the new zeros, and such that

$$(1-\delta)|w(x)| \leqslant |\tilde{w}(x)| \leqslant (1+\delta)|w(x)|$$

on $\mathbb{R}$, with $\delta > 0$ *as small as we like*. One may then run through the following argument with $\tilde{w}$ in place of w; the effect of this will merely be to render the final inequality worse by a harmless factor of $(1+\delta)/(1-\delta)$.

Let us proceed, then, assuming that the zeros of w *are* simple. Desiring, as we do, to use the above corollary, we need an estimate for

$$\sup_{t \in \mathbb{R}} \left| \frac{f_n(t+in)}{w(t+in)} \right|.$$

The function $e^{iz/2n} f_n(z)/w(z)$ is *analytic and bounded* for $\Im z > 0$, and continuous up to $\mathbb{R}$. Therefore we can use Poisson's formula (lemma of

* and there *are*! – *all* zeros of $w(z)$ are of *even order*!

§H.1, Chapter III), getting

$$\frac{e^{i(t+in)/2n}f_n(t+in)}{w(t+in)} = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{n}{(t-x)^2+n^2}\cdot\frac{e^{ix/2n}f_n(x)}{w(x)}dx.$$

Since $|w(x)| = V(x)$, we see, by (§) and (††), that the integral on the right is in absolute value

$$\leqslant \frac{2}{\pi}\int_{-L}^{L}\frac{n}{(t-x)^2+n^2}dx + \frac{2\varepsilon}{\pi}\int_{|x|\geqslant L}\frac{n}{(t-x)^2+n^2}dx.$$

This is in turn

$$\leqslant \frac{2}{\pi}\left(\arctan\frac{t+L}{n} - \arctan\frac{t-L}{n}\right) + 2\varepsilon \leqslant \frac{4}{\pi}\frac{L}{n} + 2\varepsilon,$$

so we have

$$\left|\frac{e^{-1/2}f_n(t+in)}{w(t+in)}\right| \leqslant \frac{4}{\pi}\frac{L}{n} + 2\varepsilon, \quad t\in\mathbb{R}.$$

Apply now the corollary with $f = f_n$ and $A = 1/n$. According to it and to the inequality just proved, there is a *polynomial* $P_n(x)$ (depending, of course, partly on our $w(x)$ whose choice *also* depended on the n we have taken!) such that, for $x\in\mathbb{R}$,

$$\left|\frac{P_n(x) - e^{-ix/n}g_n(x)}{w(x)}\right| \leqslant \frac{Ke}{\pi}\cdot e^{1/2}\left(\frac{4}{\pi}\frac{L}{n} + 2\varepsilon\right),$$

where (as we recall),

$$g_n(x) = \frac{\sin(x/2n)}{(x/2n)}f_n(x).$$

Therefore, since $|w(x)| = V(x) \leqslant W(x)$ (!),

$$\left|\frac{P_n(x) - e^{-ix/n}g_n(x)}{W(x)}\right| \leqslant \frac{Ke^{3/2}}{\pi}\left(\frac{4}{\pi}\frac{L}{n} + 2\varepsilon\right), \quad x\in\mathbb{R}.$$

Our number L depended *only* on ε, and the intermediate partial sum

$$V(x) = \sum_{0}^{N}a_{2k}x^{2k}$$

(which depended on n) is now *gone. The only restriction on n* (which was kept fixed during the above argument) was that it be *sufficiently large* (*how* large depended on L). For *fixed* L, then, there is, for *each* sufficiently large n, a polynomial $P_n(x)$ satisfying the above relation. If such an n is also $> 2L/\pi\varepsilon$, we will thus certainly have

$$\left|\frac{P_n(x) - e^{-ix/n}g_n(x)}{W(x)}\right| \leqslant 4\varepsilon\frac{Ke^{3/2}}{\pi}$$

for $x \in \mathbb{R}$.

Let us return to our function $\varphi \in \mathscr{C}_W(0+)$. For each *sufficiently large n*, we have a *polynomial* $P_n(x)$ with

$$\|P_n - \varphi\|_W \leqslant \|\varphi(x) - e^{-ix/n}g_n(x)\|_W + \|e^{-ix/n}g_n(x) - P_n(x)\|_W,$$

which, according to the inequality we have finally established, is

$$\leqslant 4\varepsilon\frac{Ke^{3/2}}{\pi} + \|e^{-ix/n}g_n(x) - \varphi(x)\|_W.$$

However, $\|e^{-ix/n}g_n(x) - \varphi(x)\|_W \underset{n}{\longrightarrow} 0$ by choice of our functions f_n. Therefore $\|P_n - \varphi\|_W \leqslant 8\varepsilon(Ke^{3/2}/\pi)$ for all sufficiently large n. Since $\varepsilon > 0$ was arbitrary, we have, then, $\|P_n - \varphi\|_W \underset{n}{\longrightarrow} 0$, and $\varphi \in \mathscr{C}_W(0)$.

This proves that $\mathscr{C}_W(0+) \subseteq \mathscr{C}_W(0)$. Since the reverse inclusion is always true, we are done.

Remark. An analogous result holds for approximation in the norms $\| \quad \|_{W,p}$, $1 < p < \infty$. There, a much *easier* proof can be given, based on duality and the fact that the Hilbert transform is a bounded operator on $L_p(\mathbb{R})$ for $1 < p < \infty$. The reader is encouraged to try to work out such a proof.

We can apply the technique of convex logarithmic regularisation developed in Chapter IV together with the theorem just proved so as to obtain another result in which a *regularity condition* on $W(x)$ replaces the explicit representation for it figuring above.

Theorem. *Let* $W(x) \geqslant 1$ *be* even, *with* $\log W(x)$ *a* convex *function of* $\log x$ *for* $x > 0$. *Suppose that for* each $\Lambda > 1$ *there is a constant* C_Λ *such that*

$$x^2 W(x) \leqslant C_\Lambda W(\Lambda x), \quad x \in \mathbb{R}.$$

Then $\mathscr{C}_W(0) = \mathscr{C}_W(0+)$.

Remark. Speaking, as we are, of $\mathscr{C}_W(0)$, we of course require that $x^n/W(x) \longrightarrow 0$ for $x \to \pm\infty$ and all $n \geqslant 0$, so $W(x)$ must tend to ∞ fairly rapidly as $x \to \pm\infty$. But one *cannot derive* the condition involving numbers $\Lambda > 1$ from this fact and the convexity of $\log W(x)$ in $\log|x|$. Nor have I been able to *dispense* with that ungainly condition.

Proof of theorem. Let us first show that, if $\varphi \in \mathscr{C}_W(\mathbb{R})$ and we write $\varphi_\lambda(x) = \varphi(\lambda^2 x)$ for $\lambda < 1$, then $\|\varphi - \varphi_\lambda\|_W \longrightarrow 0$ as $\lambda \to 1$.

We know that $\log W(x)$ tends to ∞ as $x \to \pm\infty$. Hence, since that function is *convex* in $\log x$ for $x > 0$, it must be *increasing* in x for all *sufficiently large* x. Take any $\varphi \in \mathscr{C}_W(\mathbb{R})$; since φ is continuous on $\mathbb{R}$ we certainly have $|\varphi(x) - \varphi_\lambda(x)| \longrightarrow 0$ uniformly on *any* interval $[-M, M]$ as $\lambda \to 1$. Also, $|\varphi(x)/W(x)| < \varepsilon$ for $|x|$ sufficiently large. Choose M big enough so that this inequality holds for $|x| \geqslant M/4$ and also $W(x)$ increases for $x \geqslant M/4$. Then, if $\frac{1}{2} < \lambda < 1$ and $|x| \geqslant M$,

$$\left|\frac{\varphi_\lambda(x)}{W(x)}\right| = \left|\frac{\varphi(\lambda^2 x)}{W(x)}\right| \leqslant \left|\frac{\varphi(\lambda^2 x)}{W(\lambda^2 x)}\right| < \varepsilon,$$

as well as $|\varphi(x)/W(x)| < \varepsilon$, so

$$\left|\frac{\varphi(x) - \varphi_\lambda(x)}{W(x)}\right| < 2\varepsilon$$

for $|x| \geqslant M$ and $\frac{1}{2} < \lambda < 1$. Making λ close enough to 1, we get the quantity on the left $< 2\varepsilon$ for $-M \leqslant x \leqslant M$ also, so $\|\varphi - \varphi_\lambda\|_W < 2\varepsilon$.

Take now any $\varphi \in \mathscr{C}_W(0+)$. We have to show that φ also belongs to $\mathscr{C}_W(0)$, and, by what we have just proved, this will follow if we establish that $\varphi_\lambda \in \mathscr{C}_W(0)$ for *each* $\lambda < 1$. We proceed to verify that fact.

We may, wlog, assume that $W(x) \equiv 1$ for $|x| \leqslant 1$ and *increases* for $x \geqslant 1$. For $n = 0, 1, 2, \ldots$, put

$$S_n = \sup_{r>0} \frac{r^n}{W(r)}$$

and, then, for $r > 0$, write

$$T(r) = \sup_{n \geqslant 0} \frac{r^{2n}}{S_{2n}}.$$

Since $\log W(r)$ increases for $r \geqslant 1$, the *proof* of the *second* lemma from §D of Chapter IV shows that

$$\frac{W(r)}{r^2} \leqslant T(r) \leqslant W(r) \quad \text{for} \quad r \geqslant 1$$

(cf. proof of second theorem in §D, this chapter). Take now

$$(\S\S) \qquad S(x) = 1 + \sum_{n=0}^{\infty} \frac{x^{2n+2}}{S_{2n}}.$$

Then, by the preceding inequalities, for $|x| \geqslant 1$,

$$S(x) \geqslant x^2 T(|x|) \geqslant W(x)$$

whilst, for any λ, $0<\lambda<1$,

$$S(x) \;=\; 1+x^2\sum_0^\infty \lambda^{2n}\frac{(x/\lambda)^{2n}}{S_{2n}} \;\leqslant\; 1+\frac{x^2}{1-\lambda^2}T\left(\frac{|x|}{\lambda}\right)$$
$$\leqslant\; 1+\frac{x^2}{1-\lambda^2}W\left(\frac{x}{\lambda}\right).$$

The first of these relations* clearly also holds for $|x|\leqslant 1$, because $W(x)\equiv 1$ there. *So does the second.* For, the inequality between its last two members is true for $|x|\geqslant\lambda$, while $T(|x|/\lambda)$ is, by its definition, *increasing* when $0<|x|<\lambda$, and $W(x/\lambda)$ *constant* for such x. We thus have

$$W(x) \;\leqslant\; S(x) \;\leqslant\; 1+\frac{x^2}{1-\lambda^2}W\left(\frac{x}{\lambda}\right)$$

for all x.

According to the hypothesis, there is a constant K_λ for each $\lambda<1$ with

$$\frac{x^2}{1-\lambda^2}W\left(\frac{x}{\lambda}\right) \;\leqslant\; K_\lambda W\left(\frac{x}{\lambda^2}\right).$$

We may of course take $K_\lambda\geqslant 1$, and thus get finally

$$(\ddagger) \qquad W(x) \;\leqslant\; S(x) \;\leqslant\; 2K_\lambda W\left(\frac{x}{\lambda^2}\right), \quad x\in\mathbb{R}.$$

Given our function $\varphi\in\mathscr{C}_W(0+)$, we have a sequence of functions f_n, $f_n\in\mathscr{E}_{1/n}$, with

$$\|\varphi-f_n\|_W \;\underset{n}{\longrightarrow}\; 0.$$

Thence, by ($\ddagger$), *a fortiori*,

$$\|\varphi-f_n\|_S \;\underset{n}{\longrightarrow}\; 0,$$

so $\varphi\in\mathscr{C}_S(0+)$ as well. Now, however, $S(x)$ has the form (§§), so *we may apply the previous theorem, getting* $\varphi\in\mathscr{C}_S(0)$. There is thus a sequence of *polynomials* $P_n(x)$ with

$$\|\varphi-P_n\|_S \;\underset{n}{\longrightarrow}\; 0.$$

From this we see, by ($\ddagger$) again, that

$$\sup_{x\in\mathbb{R}}\left|\frac{\varphi(x)-P_n(x)}{W(x/\lambda^2)}\right| \;\underset{n}{\longrightarrow}\; 0,$$

* i.e., that between $S(x)$ and $W(x)$

i.e.,

$$\sup_{x\in\mathbb{R}} \left| \frac{\varphi(\lambda^2 x) - P_n(\lambda^2 x)}{W(x)} \right| \xrightarrow[n]{} 0$$

for each λ, $0 < \lambda < 1$.

But this means that $\varphi_\lambda \in \mathscr{C}_W(0)$ for each such λ, the fact we had to verify. The theorem is proved.

Remark. *Some* regularity in $W(x)$ is *necessary* for the equality of $\mathscr{C}_W(0)$ and $\mathscr{C}_W(0+)$; exactly *what kind* is not yet known. In the next article we give an example showing that the behaviour of $W(x)$ *cannot depart too much* from that required in the above two theorems if we are to have $\mathscr{C}_W(0) = \mathscr{C}_W(0+)$. In the first part of the next chapter we will give another example, of an even weight $W(x)$, *increasing* for $x > 0$, such that $\mathscr{C}_W(0) \neq \mathscr{C}_W(0+) = \mathscr{C}_W(\mathbb{R})$.

3. Example of a weight W with $\mathscr{C}_W(0) \neq \mathscr{C}_W(0+) \neq \mathscr{C}_W(\mathbb{R})$

The idea for this example comes from a letter J.-P. Kahane sent me in 1963.

Take

$$S(z) = \prod_1^\infty \left(1 - \frac{z^2}{4^n}\right),$$

pick any fixed number λ_1, $1 < \lambda_1 < 2$, and write

$$C(z) = \left(1 - \frac{z^2}{\lambda_1^2}\right) \prod_2^\infty \left(1 - \frac{z^2}{4^n}\right).$$

This function $S(z)$ is the same as the one used in §C, and $C(z)$ differs from it *only* in that the two zeros, -2 and 2, of $S(z)$ *closest to the origin* have been *moved* slightly, the first towards -1 and the second towards 1.

Let us write $\lambda_{-1} = -\lambda_1$, and, for $|n| \geqslant 2$, $\lambda_n = (\operatorname{sgn} n) 2^{|n|}$. Then

$$C(z) = \prod_1^\infty \left(1 - \frac{z^2}{\lambda_n^2}\right),$$

and

$$\sum_{-\infty}^{\infty}{}' \frac{\lambda_n S(\lambda_n)}{C'(\lambda_n)} = 2\frac{\lambda_1 S(\lambda_1)}{C'(\lambda_1)} < 0.$$

For large n, we clearly have

$$C'(\lambda_n) \sim \frac{4}{\lambda_1^2} S'(\lambda_n),$$

where $S'(\lambda_n)$ was studied in §C. There we found that

$$|S'(\lambda_n)| \;=\; |S'(2^n)| \;\sim\; \text{const.}2^{(n-1)^2}$$

for large n, so surely (in view of the evenness of $C(z)$),

$$\sum_{-\infty}^{\infty}{}' \frac{|\lambda_n|^p}{|C'(\lambda_n)|} \;<\; \infty \quad \text{for} \quad p = 0, 1, 2, 3, \ldots$$

Use of the Lagrange interpolation formula now shows, as in §C (where an analogous result was proved with $S'(\lambda_n)$ in place of $C'(\lambda_n)$), that

$$P(z) \;=\; C(z) \sum_{-\infty}^{\infty}{}' \frac{P(\lambda_n)}{(z-\lambda_n)C'(\lambda_n)}$$

for any *polynomial* P. Taking $P(z) = z^{p+1}$ and then putting $z = 0$ gives us

$$\sum_{-\infty}^{\infty}{}' \frac{\lambda_n^p}{C'(\lambda_n)} \;=\; 0, \quad p = 0, 1, 2, \ldots .$$

We are ready to construct our weight W. Taking a large constant K (chosen so as to make $W(x)$ come out $\geqslant 1$), put $W(x) = KS(1)$ for $|x| \leqslant 1$. For $|x| \geqslant 1$, make $W(x) = Kx^2|S(x)|$ when x lies *outside all the intervals*

$$[2^n(1-2^{-4n}), \quad 2^n(1+2^{-4n})].$$

Finally, if $2^n(1-2^{-4n}) \;\leqslant\; |x| \;\leqslant\; 2^n(1+2^{-4n})$ for some $n \geqslant 1$, define $W(x)$ as

$$\sup\{K\xi^2|S(\xi)|: \quad |\xi - 2^n| \leqslant 2^{-3n}\}.$$

We see first of all that $xS(x)/W(x) \longrightarrow 0$ for $x \to \pm\infty$, so $xS(x) \in \mathscr{C}_W(\mathbb{R})$. Hence, since $S(z)$, and therefore $zS(z)$, is of exponential type *zero* we have $xS(x) \in \mathscr{C}_W(0+)$ by the *first* theorem of article 1.

We need some information about the asymptotic behaviour of $S(x)$ for $x \to \infty$. This may be obtained by the method followed in §C. Suppose that $x = 2^n\alpha$ with $1/\sqrt{2} \;\leqslant\; \alpha \;\leqslant\; \sqrt{2}$. Then we have

$$|S(x)| \;=\; \prod_{k=1}^{n-1}\left|1 - \frac{4^n\alpha^2}{4^k}\right| \times |1-\alpha^2| \times \prod_{k=n+1}^{\infty}\left|1 - \frac{4^n\alpha^2}{4^k}\right|$$

$$= \; |1-\alpha^2| \prod_{k=1}^{n-1}\left(\frac{4^n\alpha^2}{4^k}\right) \prod_{l=1}^{n-1}\left(1 - \frac{1}{4^l\alpha^2}\right) \prod_{l=1}^{\infty}\left(1 - \frac{\alpha^2}{4^l}\right),$$

and this last is

$$\sim \; |1-\alpha^2| \frac{(4^n\alpha^2)^{n-1}}{2^{n(n-1)}} S\left(\frac{1}{\alpha}\right) S(\alpha) \;=\; |1-\alpha^2|(2^n\alpha^2)^{n-1} S\left(\frac{1}{\alpha}\right) S(\alpha).$$

Thence, for large n,

$$\sup\{|S(\xi)|: \ |\xi - 2^n| \leqslant 2^{-3n}\} \ \sim \ \text{const.}2^{n^2-5n},$$

so, for $2^n(1-2^{-4n}) \ \leqslant \ |x| \ \leqslant \ 2^n(1+2^{-4n})$,

$$W(x) \ \sim \ \text{const.}2^{n^2-3n},$$

and, in particular,

$$W(\lambda_n) \ \sim \ \text{const.}2^{n^2-3n}.$$

Comparing this with the relation $|C'(\lambda_n)| \sim \text{const.}2^{(n-1)^2}$, valid for large n, which we already know, we see that

$$\sum_{-\infty}^{\infty}{}' \frac{W(\lambda_n)}{|C'(\lambda_n)|} \ < \ \infty.$$

This permits us to define a *finite* signed Radon measure μ on the set of points λ_n, $n = \pm 1, \pm 2, \ldots$, by putting

$$\mu(\{\lambda_n\}) \ = \ \frac{W(\lambda_n)}{C'(\lambda_n)}.$$

Then, for $p = 0, 1, 2, \ldots$,

$$\int_{-\infty}^{\infty} \frac{x^p}{W(x)} \mathrm{d}\mu(x) \ = \ \sum_{-\infty}^{\infty}{}' \frac{\lambda_n^p}{C'(\lambda_n)},$$

which is *zero*, as we have seen, whilst

$$\int_{-\infty}^{\infty} \frac{xS(x)}{W(x)} \mathrm{d}\mu(x) \ = \ \sum_{-\infty}^{\infty}{}' \frac{\lambda_n S(\lambda_n)}{C'(\lambda_n)} \ < \ 0.$$

So $xS(x) \in \mathscr{C}_W(0+)$ *can't* be in $\mathscr{C}_W(0)$, and $\mathscr{C}_W(0) \neq \mathscr{C}_W(0+)$.

In the present example, $\mathscr{C}_W(0+)$ is a *proper subspace* of $\mathscr{C}_W(\mathbb{R})$. Indeed, this is almost immediate. By the above asymptotic computation of $S(x)$ we clearly have

$$W(x) \ = \ \text{const.}|x|^{\log_2|x| + \theta(x)}$$

with a quantity $\theta(x)$ *varying between two constants*. Therefore,

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2} \mathrm{d}x \ < \ \infty,$$

so we surely have $\mathscr{C}_W(A) \neq \mathscr{C}_W(\mathbb{R})$ for each A by an obvious extension of

T. Hall's theorem (p. 169). Hence $\mathscr{C}_W(0+) \neq \mathscr{C}_W(\mathbb{R})$. The construction of our example is finished.*

Remark. Let $\Omega(x) = \prod_1^\infty (1 + x^2/4^n)$. The asymptotic evaluation of $\Omega(x)$ for $x \to \infty$ can be made in fashion similar to that for $S(x)$, and is, in fact, *easier* than the latter. As is clear after a moment's thought, here one also obtains

$$\Omega(x) \sim \text{const.}|x|^{\log_2|x| + \varphi(x)} \quad \text{for} \quad |x| \to \infty$$

with a certain $\varphi(x)$ varying between two constants. Thus,

$$\frac{\Omega(x)}{W(x)} = \text{const.}|x|^{\psi(x)}, \quad x \in \mathbb{R},$$

where $A \leqslant \psi(x) \leqslant B$, say.

However, $\mathscr{C}_\Omega(0) = \mathscr{C}_\Omega(0+)$. This follows from the first theorem of the previous article, in view of the evident fact that $\Omega(x) = 1 + a_2x^2 + a_4x^4 + \cdots$ with $a_{2k} > 0$.

The difference in behaviour of $\Omega(x)$ and $W(x)$ is small in comparison to their size, and yet $\mathscr{C}_\Omega(0) = \mathscr{C}_\Omega(0+)$ although $\mathscr{C}_W(0) \neq \mathscr{C}_W(0+)$.

The question of *how* a weight W's *local behaviour* is related to the *equality* of $\mathscr{C}_W(0)$ and $\mathscr{C}_W(0+)$ *merits further study.*

* $W(x)$ has jump discontinuities among the points $\pm(2^n \pm 2^{-3n})$, $n \geqslant 1$, but a continuous weight with the same properties as W is furnished by an evident elaboration of the procedure in the text.

VII

How small can the Fourier transform of a rapidly decreasing non-zero function be?

Let us consider functions $F(x) \in L_1(\mathbb{R})$ whose modulus goes to zero *rapidly* as $x \to \infty$, in such fashion that

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} \log^{-}\left(\int_x^{\infty} |F(t)|\,\mathrm{d}t \right) \mathrm{d}x \;=\; \infty .$$

The general theme of this chapter is that, for such a function F, the Fourier transform

$$\hat{F}(\lambda) \;=\; \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\lambda x} F(x)\,\mathrm{d}x$$

cannot be too small anywhere unless F vanishes identically.

The first result of this kind (obtained by Levinson) said that if (for such an F) *$\hat{F}$ vanishes throughout an interval of positive length*, then $F \equiv 0$. This was refined by Beurling, who proved that $\hat{F}(\lambda)$ *cannot even vanish on a set of positive measure unless* $F \equiv 0$. Analogues of these theorems hold for measures as well as functions F; they, and the methods used to establish them, have various important consequences, some of which apply to material already taken up in the present book.

These things have been known for more than 20 years. Until recently, the only developments since the sixties in the subject matter of this chapter had to do mainly with aspects of its presentation. That state of affairs was changed in 1982 by the appearance of a remarkable result, due to A.L. Volberg, which says that if

$$f(\vartheta) \;=\; \sum_{-\infty}^{\infty} \hat{f}(n)^{\mathrm{i}n\vartheta}$$

has

$$|\hat{f}(n)| \;\leqslant\; \mathrm{e}^{-M(n)} \quad \text{for} \quad n > 0$$

with $M(n)$ sufficiently regular and increasing, and if

$$\sum_1^\infty \frac{M(n)}{n^2} = \infty,$$

then

$$\int_{-\pi}^{\pi} \log|f(\vartheta)|\mathrm{d}\vartheta > -\infty$$

unless $f(\vartheta) \equiv 0$. The proof of this uses new ideas (coming from the study of weighted *planar* approximation by polynomials) and is very long; its inclusion has necessitated a considerable extension of the present chapter. I still do not completely understand the result's meaning; it applies to the *unit circle* and seems to *not have* a natural analogue for the *real line* which would generalize Levinson's and Beurling's theorems.

There are not too many easily accessible references for this chapter. The earliest results are in Levinson's book; material relating to them can also be found in the book by de Branges (some of it being set as problems). The main source for the first two §§ of this chapter consists, however, of the famous mimeographed notes for Beurling's Standford lectures prepared by P. Duren; those notes came out around 1961. Volberg published his theorem in a 6-page (!) *Doklady* note at the beginning of 1982. That paper is quite difficult to get through on account of its being so condensed.

A. The Fourier transform vanishes on an interval. Levinson's result

Levinson originally proved his theorem by means of a complicated argument, involving contour integration, which figured later on as one of the main ingredients in Beurling's proof of his deeper result. Beurling observed that Levinson's theorem (and others related to it) could be obtained more easily by the use of test functions, and then de Branges simplified that treatment by bringing Akhiezer's first theorem from §E.2 of Chapter VI into it. I follow this procedure in the present §. The particularly convenient and elegant test function used here (which has several other applications, by the way) was suggested to me by my reading of a paper of H. Widom.

1. Some shop math

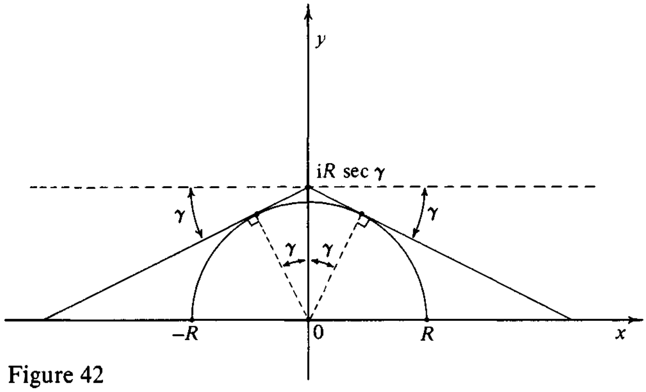

Figure 42

The circle of radius R about 0 lies *under* the two straight lines of slopes $\pm\ \tan\gamma$ passing through the point $\mathrm{i}R\sec\gamma$. Therefore, if $A>0$,

$$A\sqrt{(R^2-x^2)} \;\leqslant\; AR\sec\gamma-(A\tan\gamma)|x|, \quad -R\leqslant x\leqslant R.$$

Consider any function $\omega(x)\geqslant 0$ such that

$$|\omega(x)-\omega(x')| \;\leqslant\; (A\tan\gamma)|x-x'|.$$

If we adjust R so as to make $AR\sec\gamma=\omega(0)$, we have, for $-R\leqslant x\leqslant R$,

$$\omega(x) \;\geqslant\; \omega(0)-(A\tan\gamma)|x|,$$

which, by the above, is $\geqslant A\sqrt{(R^2-x^2)}$. The function $\cos(A\sqrt{(x^2-R^2)})$ is, however, in modulus $\leqslant 1$ for $|x|>R$, and for $-R\leqslant x\leqslant R$ it equals $\cosh(A\sqrt{(R^2-x^2)}) \;\leqslant\; \exp(A\sqrt{(R^2-x^2)})$. Therefore, for $x\in\mathbb{R}$,

$$\omega(x) \;\geqslant\; \log|\cos(A\sqrt{(x^2-R^2)})|$$

when

$$R \;=\; \frac{\omega(0)}{A\sec\gamma}.$$

Let us apply these considerations to a function $W(x)\geqslant 1$ defined on $\mathbb{R}$ and satisfying

$$|\log W(x)-\log W(x')| \;\leqslant\; C|x-x'|$$

there. Taking any fixed $A>0$, we determine an acute angle γ such that $A\tan\gamma=C$. Suppose $x_0\in\mathbb{R}$ is given. Then we translate x_0 to the origin, using the above calculation with

$$\omega(x-x_0) \;=\; \log W(x).$$

We see that

$$|\cos(A\sqrt{((x-x_0)^2-R^2)})| \;\leqslant\; W(x) \quad \text{for} \quad x\in\mathbb{R},$$

where

$$R = \frac{\log W(x_0)}{A\sec\gamma} = \frac{\log W(x_0)}{\sqrt{(A^2+C^2)}}.$$

Here, $\cos(A\sqrt{((z-x_0)^2-R^2)})$ *is an entire function of* z because the Taylor development of $\cos w$ about the origin contains only *even powers* of w. It is clearly *of exponential type* A, and, for $z = x_0$, has the value

$$\cosh AR \geqslant \tfrac{1}{2}e^{AR} = \tfrac{1}{2}(W(x_0))^{A/\sqrt{(A^2+C^2)}}.$$

Recall now the definition of the Akhiezer function $W_A(x)$ given in Chapter VI, §E.2, namely

$$W_A(x) = \sup\{|f(x)|: f \text{ entire of exponential type} \leqslant A, \text{ bounded on } \mathbb{R} \text{ and } |f(t)/W(t)| \leqslant 1 \text{ on } \mathbb{R}\}.$$

In terms of W_A, we have, by the computation just made, the

Theorem. *Let* $W(x) \geqslant 1$ *on* $\mathbb{R}$, *with*

$$|\log W(x) - \log W(x')| \leqslant C|x-x'|$$

for x *and* $x' \in \mathbb{R}$. *Then, if* $A > 0$,

$$W_A(x) \geqslant \tfrac{1}{2}(W(x))^{A/\sqrt{(A^2+C^2)}}, \quad x \in \mathbb{R}.$$

Corollary. *Let* $W(x) \geqslant 1$, *with* $\log W(x)$ *uniformly* Lip 1 *on* $\mathbb{R}$. *Then, if*

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2}\,dx = \infty,$$

we have

$$\int_{-\infty}^{\infty} \frac{\log W_A(x)}{1+x^2}\,dx = \infty$$

for each $A > 0$.

According to Akhiezer's first theorem (Chapter VI, §E.2), this in turn implies the

Theorem. *Let* $W(x) \geqslant 1$, *with* $\log W(x)$ *uniformly* Lip 1 *on* $\mathbb{R}$, *and* $W(x)$ *tending to* ∞ *as* $x \longrightarrow \pm\infty$. *If*

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2}\,dx = \infty,$$

linear combinations of $e^{i\lambda x}$, $-A \leqslant \lambda \leqslant A$, *are, for each* $A > 0$, $\|\ \|_W$*-dense in* $\mathscr{C}_W(\mathbb{R})$.

2. Beurling's gap theorem

As a first application of the above fairly easy result, let us prove the following beautiful proposition of Beurling:

Theorem. *Let μ be a totally finite complex Radon measure on $\mathbb{R}$ with $|d\mu(t)| = 0$ on each of the disjoint intervals (a_n, b_n), $0 < a_1 < b_1 < a_2 < b_2 < \cdots$, and suppose that*

$$(*) \qquad \sum_1^\infty \left(\frac{b_n - a_n}{a_n}\right)^2 = \infty.$$

If $\hat{\mu}(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda x} d\mu(x)$ vanishes identically on some real interval of positive length, then $\mu \equiv 0$.

Remark. This is not the only time we shall encounter the condition $(*)$ in the present book.

Proof of theorem (de Branges). We start by taking an *even* function $T(x) \geqslant 1$ whose logarithm is uniformly Lip 1 on $\mathbb{R}$, and which *increases to* ∞ *so slowly as* $|x| \to \infty$, that

$$\int_{-\infty}^{\infty} T(x)|d\mu(x)| < \infty.$$

(Construction of such a function T is in terms of the given measure μ, and is left to the reader as an easy exercise.)

For each n, let b'_n be *the lesser* of b_n and $2a_n$. Then, given that $(*)$ holds, we also have

$$\sum_n \left(\frac{b'_n - a_n}{a_n}\right)^2 = \infty.$$

Indeed, this sum certainly diverges if the one in $(*)$ does, when $(b'_n - a_n)/a_n$ differs from $(b_n - a_n)/a_n$ for only *finitely many* n. But the sum in question *also* diverges when *infinitely many* of its terms differ from the corresponding ones in $(*)$, since $(b'_n - a_n)/a_n = 1$ when $b'_n = 2a_n$.

Let $\omega(x)$ be *zero* outside the intervals (a_n, b'_n), and *on each one of those intervals let the graph of* $\omega(x)$ *vs* x *be a* 45° *triangle with base on* (a_n, b'_n).

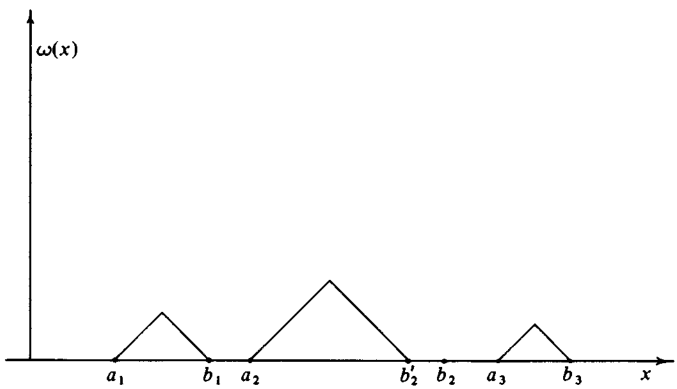

Figure 43

The function $\omega(x)$ is clearly uniformly Lip 1 on $\mathbb{R}$.

Put $W(x) = e^{\omega(x)}T(x)$. Then $W(x) \geqslant 1$ and $\log W(x)$ is uniformly Lip 1 on $\mathbb{R}$; also, $W(x) \to \infty$ for $x \to \pm\infty$. Since $|d\mu(x)| = 0$ throughout each interval (a_n, b_n') and $\omega(x)$ is *zero* outside those intervals,

$$\int_{-\infty}^{\infty} W(x)|d\mu(x)| \;=\; \int_{-\infty}^{\infty} T(x)|d\mu(x)| \;<\; \infty.$$

The complex Radon measure ν with

$$d\nu(x) = W(x)d\mu(x)$$

is therefore *totally finite.*

Suppose now that $\hat{\mu}(\lambda)$ vanishes on some *interval*; say, wlog, that

$$\int_{-\infty}^{\infty} e^{i\lambda x} d\mu(x) \;=\; 0 \quad \text{for} \quad -A \leqslant \lambda \leqslant A.$$

This can be rewritten as

$$(*) \qquad \int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{W(x)} d\nu(x) \;=\; 0, \quad -A \leqslant \lambda \leqslant A.$$

However, $\log W(x) \;=\; \omega(x) + \log T(x) \;\geqslant\; \omega(x)$, and

$$\int_{a_1}^{\infty} \frac{\omega(x)}{x^2} dx \;\geqslant\; \sum_1^{\infty} \left(\frac{1}{b_n'}\right)^2 \left(\frac{b_n' - a_n}{2}\right)^2 \;\geqslant\; \frac{1}{4}\sum_1^{\infty}\left(\frac{b_n' - a_n}{2a_n}\right)^2,$$

which is *infinite*, as we saw above. Therefore

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2} dx \;=\; \infty,$$

and *linear combinations* of the $e^{i\lambda x}$, $-A \leqslant \lambda \leqslant A$, are $\| \;\|_W$*-dense* in $\mathscr{C}_W(\mathbb{R})$ by the *second* theorem of the preceding article.

Referring to $(*)$, we thence see that $\nu \equiv 0$, i.e., $d\nu(x) = W(x)d\mu(x) \equiv 0$ and $\mu = 0$. Q.E.D.

Problem 11

Let μ be a finite complex measure on $\mathbb{R}$, and put

$$e^{-\sigma(x)} \;=\; \int_{-\infty}^{\infty} e^{-|x-t|}|d\mu(t)|.$$

Suppose that $\int_{-\infty}^{\infty}(\sigma(x)/(1+x^2))dx = \infty$. Then, if $\hat{\mu}(\lambda)$ vanishes identically on any interval, $\mu \equiv 0$ (Beurling). (Hint. Wlog, $\int_{-\infty}^{\infty}|d\mu(t)| \leqslant 1$ so that

$\sigma(x) \geqslant 0$. Assuming that $\hat{\mu}(\lambda) \equiv 0$ for $-A \leqslant \lambda \leqslant A$, write the relation

$$\int_{-\infty}^{\infty} e^{\sigma(x)-|x-t|}|d\mu(t)| \;=\; 1,$$

and use the picture

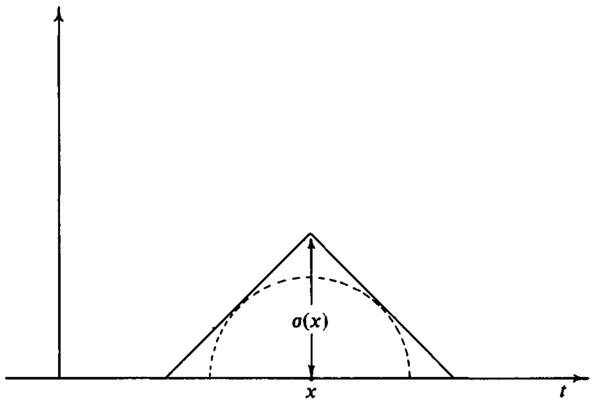

Figure 44

to estimate the supremum of $|f(x)|$ for entire functions f of exponential type $\leqslant A$, bounded on $\mathbb{R}$, and such that

$$\int_{-\infty}^{\infty} |f(t)|\;|d\mu(t)| \;\leqslant\; 1.)$$

Remark. Beurling generalized the result of problem 11 to complex Radon measures μ *which are not necessarily totally finite.* This extension will be taken up in Chapter X.

3. Weights which increase along the positive real axis

Lemma. *Let* $T(x) \geqslant 1$ *be defined and* increasing *for* $x \geqslant 0$, *and denote by* $\mathcal{T}(x)$ *the* largest minorant of $T(x)$ *with the property that* $|\log \mathcal{T}(x) - \log \mathcal{T}(x')| \leqslant |x - x'|$ *for x and* $x' \geqslant 0$. *If* $\int_1^\infty (\log T(x)/x^2)dx = \infty$, *then also* $\int_1^\infty (\log \mathcal{T}(x)/x^2)dx = \infty$.

Proof. The graph of $\log \mathcal{T}(x)$ vs x is obtained from that of $\log T(x)$ by means of the following construction:

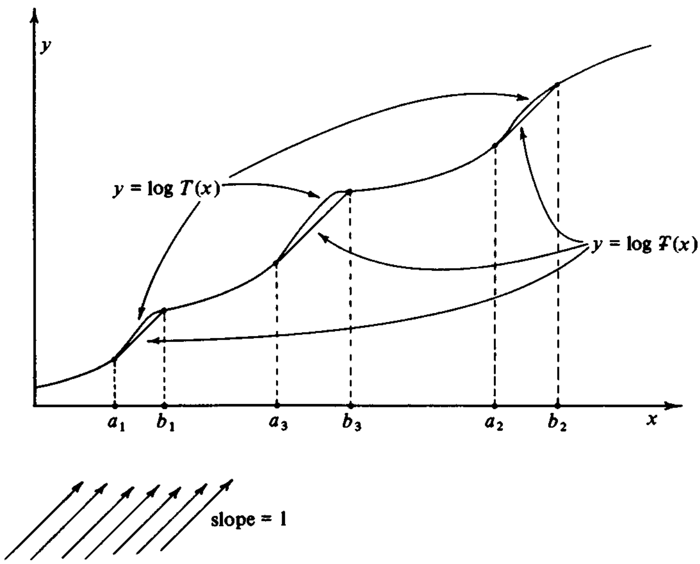

Figure 45

One imagines rays of light of slope 1 shining upwards *underneath* the graph of $\log T(x)$ vs x. The graph of $\log \mathcal{T}(x)$ is made up of the portions of the *former* one which are *illuminated* by those rays of light and some straight segments of slope 1. Those segments lie over certain intervals $[a_n, b_n]$ on the x-axis, of which there are generally countably many, that cannot necessarily be indexed in such fashion that $b_n \leqslant a_{n+1}$ for all n. The *open* intervals (a_n, b_n) are *disjoint*, and on *any one of them* we have

$$\log \mathcal{T}(x) \;=\; \log T(a_n) + (x - a_n).$$

On $[0, \infty) \sim \bigcup_n (a_n, b_n)$, $\mathcal{T}(x)$ and $T(x)$ are *equal*.

In order to prove the lemma, let us *assume* that $\int_1^\infty (\log \mathcal{T}(x)/x^2)\mathrm{d}x < \infty$ and then *show* that $\int_1^\infty (\log T(x)/x^2)\mathrm{d}x < \infty$. If, in the first place, (a_n, b_n) is any of the aforementioned intervals with $1 \leqslant a_n < b_n/2$, we have, since $\log T(a_n) \geqslant 0$,

$$\int_{a_n}^{b_n} \frac{\log \mathcal{T}(x)}{x^2}\mathrm{d}x \;\geqslant\; \int_{a_n}^{b_n} \frac{x - a_n}{x^2}\mathrm{d}x \;>\; \int_1^2 \frac{\xi - 1}{\xi^2}\mathrm{d}\xi$$
$$= \;\log 2 - \tfrac{1}{2}. > \; 0.$$

We can therefore only have *finitely many intervals* (a_n, b_n) with $b_n > 2a_n$ and $a_n \geqslant 1$ if $\int_1^\infty (\log \mathcal{T}(x)/x^2)\mathrm{d}x$ is *finite*.

This being granted, consider any *other* of the intervals (a_n, b_n) with $a_n \geqslant 1$.

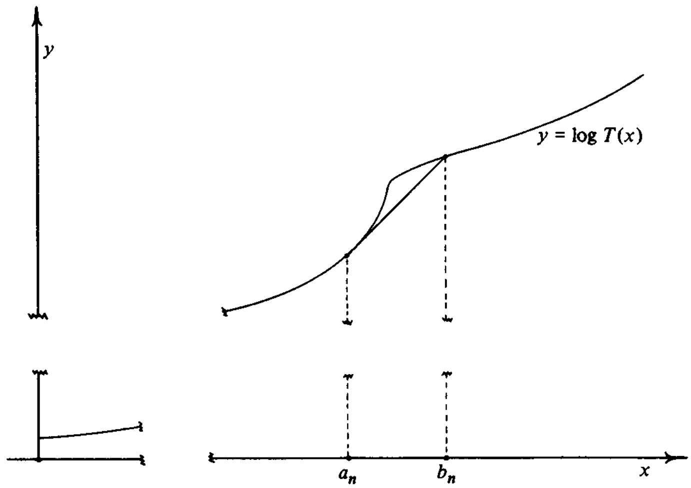

Figure 46

By shop math,

$$\int_{a_n}^{b_n} \frac{1}{x^2} \log \text{Ŧ}(x)\mathrm{d}x \;\geqslant\; \frac{1}{b_n^2}\int_{a_n}^{b_n} \log \text{Ŧ}(x)\mathrm{d}x$$

$$= \frac{1}{b_n^2}\cdot\frac{\log T(a_n) + \log T(b_n)}{2}\cdot(b_n - a_n) \;\geqslant\; \frac{(b_n - a_n)\log T(b_n)}{2b_n^2}.$$

At the same time, since $T(x)$ *increases*,

$$\int_{a_n}^{b_n} \frac{1}{x^2} \log T(x)\mathrm{d}x \;\leqslant\; \frac{(b_n - a_n)\log T(b_n)}{a_n^2} \;\leqslant\; 8\cdot\frac{(b_n - a_n)\log T(b_n)}{2b_n^2}$$

when $b_n \leqslant 2a_n$. Therefore, for all the intervals (a_n, b_n) with $a_n \geqslant 1$ and $b_n \leqslant 2a_n$, hence, certainly, *for all save a finite number of the* (a_n, b_n) *contained in* $[1, \infty)$, we have

$$\int_{a_n}^{b_n} \frac{1}{x^2} \log T(x)\mathrm{d}x \;\leqslant\; 8\int_{a_n}^{b_n} \frac{1}{x^2} \log \text{Ŧ}(x)\mathrm{d}x.$$

The *sum* of the integrals $\int_{a_n}^{b_n}(1/x^2)\log T(x)\mathrm{d}x$ for the *remaining finite number of* (a_n, b_n) in $[1, \infty)$ *is surely finite* – note that *none* of those intervals can have *infinite length,* for such a one would be of the form (a_l, ∞), and in that case we would have

$$\int_{a_l}^{\infty} \frac{1}{x^2} \log \text{Ŧ}(x)\mathrm{d}x \;\geqslant\; \int_{a_l}^{\infty} \frac{x - a_l}{x^2}\mathrm{d}x \;=\; \infty,$$

contrary to our assumption on $\mathcal{T}(x)$. We see that

$$\sum_{a_n \geqslant 1} \int_{a_n}^{b_n} \frac{\log T(x)}{x^2} \mathrm{d}x \; < \; \infty,$$

since

$$\sum_{\substack{a_n \geqslant 1 \\ 2b_n \leqslant a_n}} 8 \int_{a_n}^{b_n} \frac{\log \mathcal{T}(x)}{x^2} \mathrm{d}x$$

is finite.

On the complement

$$E = [1, \infty) \cap \sim \bigcup_n (a_n, b_n),$$

$T(x) = \mathcal{T}(x)$ by our construction. Hence

$$\int_E \frac{\log T(x)}{x^2} \mathrm{d}x \; = \; \int_E \frac{\log \mathcal{T}(x)}{x^2} \mathrm{d}x \; < \; \infty.$$

The whole half line $[1, \infty)$ can differ from the union of E and the (a_n, b_n) with $a_n \geqslant 1$ by *at most* an interval of the form $[1, b_m)$, which happens when there is an m such that $a_m < 1 < b_m$. If there *is* such an m, however, b_m must be *finite* (see above), and then

$$\int_1^{b_m} \frac{\log T(x)}{x^2} \mathrm{d}x \; < \; \infty.$$

Putting everything together, we see that

$$\int_1^{\infty} \frac{\log T(x)}{x^2} \mathrm{d}x \; < \; \infty,$$

which is what we had to show. We are done.

Corollary. *Let* $W(x) \geqslant 1$ *be defined on* $\mathbb{R}$ *and* increasing *for* $x \geqslant 0$. *If*

$$\int_1^{\infty} \frac{\log W(x)}{x^2} \mathrm{d}x \; = \; \infty,$$

we have

$$\int_1^{\infty} \frac{\log W_A(x)}{x^2} \mathrm{d}x \; = \; \infty$$

for each of the Akhiezer functions W_A, $A > 0$ (Chapter VI, §E.2).

Proof. Let, for $x \geqslant 0$, $\mathcal{T}(x)$ be the *largest minorant* of $W(x)$ on $[0, \infty)$ with

$$|\log \mathcal{T}(x) - \log \mathcal{T}(x')| \; \leqslant \; |x - x'|$$

there, and put $\mathcal{T}(x) = \mathcal{T}(0)$ for $x < 0$. By the lemma, $\int_1^\infty (\log W(x)/x^2)\mathrm{d}x = \infty$ implies that $\int_1^\infty (\log \mathcal{T}(x)/x^2)\mathrm{d}x = \infty$. Here, $\log \mathcal{T}(x)$ is certainly uniformly Lip 1 (and $\geqslant 0$) on $\mathbb{R}$, so, by the corollary of article 1, we see that

$$\int_1^\infty \frac{\log \mathcal{T}_A(x)}{x^2}\mathrm{d}x \;=\; \infty$$

for each $A > 0$.

We have $\mathcal{T}(x) \leqslant W(x) + \mathcal{T}(0)$ (the term $\mathcal{T}(0)$ on the right being perhaps needed for negative x). Therefore

$$\mathcal{T}_A(x) \;\leqslant\; (1 + \mathcal{T}(0))W_A(x),$$

and

$$\int_1^\infty \frac{\log W_A(x)}{x^2}\mathrm{d}x \;=\; \infty$$

for each $A > 0$ by the previous relation. Q.E.D.

From this, Akhiezer's first theorem (Chapter VI, §E.2) gives, without further ado, the following

Theorem. *Let $W(x) \geqslant 1$ on $\mathbb{R}$, with $W(x) \to \infty$ for $x \longrightarrow \pm\infty$. Suppose that $W(x)$ is* monotone *on* one *of the two half lines $(-\infty, 0]$, $[0, \infty)$, and that the* integral *of* $\log W(x)/(1 + x^2)$, *taken over whichever of those half lines on which monotoneity holds,* diverges. *Then $\mathscr{C}_W(A) = \mathscr{C}_W(\mathbb{R})$ for* every *$A > 0$, so $\mathscr{C}_W(0+) = \mathscr{C}_W(\mathbb{R})$.*

Remark. The notation is that of §E.2, Chapter VI. This result is due to Levinson. It is remarkable because only the monotoneity of $W(x)$ on a *half line* figures in it.

4. Example on the comparison of weighted approximation by polynomials and that by exponential sums

If $W(x) \geqslant 1$ tends to ∞ as $x \to \pm\infty$, we know that $\mathscr{C}_W(A)$ is *properly contained* in $\mathscr{C}_W(\mathbb{R})$ for each $A > 0$ in the case that

$$\int_{-\infty}^\infty \frac{\log W(x)}{1 + x^2}\mathrm{d}x \;<\; \infty.$$

(See Chapter VI, §E.2 and also the beginning of §D.) The theorem of the previous article shows that *mere monotoneity of $W(x)$ on $[0, \infty)$ without any additional regularity*, when accompanied by the condition

$$\int_0^\infty \frac{\log W(x)}{1 + x^2}\mathrm{d}x \;=\; \infty,$$

already guarantees the equality of $\mathscr{C}_W(A)$ and $\mathscr{C}_W(\mathbb{R})$ for each $A > 0$.

The question arises as to whether this also works for $\mathscr{C}_W(0)$, the $\| \quad \|_W$-closure of the *polynomials* in $\mathscr{C}_W(\mathbb{R})$. (Here, of course we must assume that $x^n/W(x) \longrightarrow 0$ as $x \longrightarrow \pm\infty$ for all $n \geqslant 0$.) *The following example will show that the answer to this question is NO.*

We start with a *very rapidly increasing* sequence of numbers λ_n. It will be sufficient to take

$$\lambda_1 = 2,$$
$$\lambda_2 = e^{\lambda_1},$$

and, in general, $\lambda_n = e^{\lambda_{n-1}}$. Let us check that $\lambda_n > \lambda_{n-1}^2$ for $n > 1$. We have $e^2 > 2^2 = 4$, and $(d/dx)(e^x - x^2) = e^x - 2x$ is >0 for $x = 2$. Also $(d^2/dx^2)(e^x - x^2) = e^x - 2 > 0$ for $x \geqslant 2$, so $e^x - x^2$ continues to increase strictly on $[2, \infty)$. Therefore $e^x > x^2$ for $x \geqslant 2$, so $\lambda_n = e^{\lambda_{n-1}} > \lambda_{n-1}^2$. We note that λ_{n-1}^2 is turn $\geqslant 2\lambda_{n-1}$, since the numbers λ_{n-1} are $\geqslant 2$.

We proceed to the construction of the weight W. For $0 \leqslant x \leqslant \lambda_1$, take $\log W(x) = 0$, and for $2\lambda_{n-1} \leqslant x \leqslant \lambda_n$ with $n > 1$ put $\log W(x) = n\lambda_{n-1}/2$ (by the computation just made we do have $2\lambda_{n-1} < \lambda_n$). We then specify $\log W(x)$ on the segments $[\lambda_{n-1}, 2\lambda_{n-1}]$ by making it *linear on each of them*, and finally define $W(x)$ for negative x by putting $W(-x) = W(x)$.

Here is the picture:

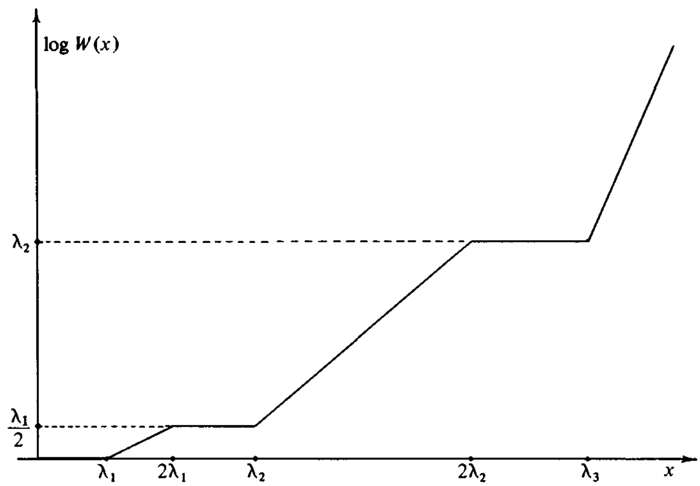

Figure 47

$W(x)$ is $\geqslant 1$ and *increasing* for $x \geqslant 0$, and, for large n,

$$\int_{2\lambda_n}^{\lambda_{n+1}} \frac{\log W(x)}{x^2}\,dx \;=\; \frac{(n+1)\lambda_n}{2}\left(\frac{1}{2\lambda_n} - \frac{1}{\lambda_{n+1}}\right)$$

is $\geqslant (n+1)\lambda_n/8\lambda_n = \frac{1}{8}(n+1)$. Therefore $\int_0^\infty (\log W(x)/(1+x^2))\mathrm{d}x = \infty$, so, *by the theorem of the previous article, $\mathscr{C}_W(A) = \mathscr{C}_W(\mathbb{R})$ for each $A > 0$ and $\mathscr{C}_W(0+) = \mathscr{C}_W(\mathbb{R})$.*

For $2\lambda_{n-1} \leqslant |x| \leqslant 2\lambda_n$,

$$W(x) \;\geqslant\; \mathrm{e}^{n\lambda_{n-1}/2} \;=\; \lambda_n^{n/2} \;\geqslant\; |x/2|^{n/2}.$$

Hence

$$\frac{x^p}{W(x)} \longrightarrow 0 \quad \text{as} \quad x \longrightarrow \pm\infty$$

for every $p \geqslant 0$, and it makes sense to talk about the space $\mathscr{C}_W(0)$. *It is claimed that $\mathscr{C}_W(0) \neq \mathscr{C}_W(\mathbb{R})$.*

To see this, take the entire function

$$C(z) \;=\; \prod_1^\infty \left(1 - \frac{z^2}{\lambda_n^2}\right).$$

Because the λ_n go to ∞ so rapidly, $C(z)$ is of *zero exponential type*. For $n > 1$,

$$|C'(\lambda_n)| \;=\; \frac{2}{\lambda_n}\left(\frac{\lambda_n}{\lambda_1}\right)^2\left(\frac{\lambda_n}{\lambda_2}\right)^2 \cdots \left(\frac{\lambda_n}{\lambda_{n-1}}\right)^2$$

$$\times \prod_{k=1}^{n-1}\left(1 - \frac{\lambda_k^2}{\lambda_n^2}\right) \cdot \prod_{l=n+1}^{\infty}\left(1 - \frac{\lambda_n^2}{\lambda_l^2}\right).$$

Since the ratios λ_{j+1}/λ_j are always > 2 and $\to \infty$ as $j \to \infty$, the two products written with the sign $\prod$ on the right are both *bounded below* by *strictly positive constants* for $n > 1$ and indeed tend to 1 as $n \to \infty$. The product standing before them,

$$2\frac{\lambda_n}{\lambda_1^2}\cdot\frac{\lambda_n}{\lambda_2^2}\cdots\frac{\lambda_n}{\lambda_{n-1}^2}\cdot\lambda_n^{n-2},$$

far exceeds $2\lambda_n^{n-2}$ because $\lambda_j > \lambda_{j-1}^2$. Therefore we surely have

$$|C'(\lambda_n)| \;\geqslant\; \lambda_n^{n-2}$$

for large n.

At the same time, $W(\lambda_n) = \mathrm{e}^{n\lambda_{n-1}/2} = \lambda_n^{n/2}$, whence, for large n,

$$\frac{W(\lambda_n)}{|C'(\lambda_n)|} \;\leqslant\; \frac{\lambda_n^{n/2}}{\lambda_n^{n-2}} \;=\; \frac{1}{\lambda_n^{(n/2)-2}}.$$

Since the sequence $\{\lambda_n\}$ tends to ∞, we thus have

$$\sum_1^\infty \frac{W(\lambda_n)}{|C'(\lambda_n)|} \;<\; \infty.$$

For $n = 1, 2, 3, \ldots$ it is convenient to put $\lambda_{-n} = -\lambda_n$. Let us then define a discrete measure μ supported on the points λ_n, $n = \pm 1, \pm 2, \ldots$, by putting

$$\mu(\{\lambda_n\}) = \frac{W(\lambda_n)}{C'(\lambda_n)}.$$

The functions $W(x)$ and $C(x)$ are even, hence

$$\int_{-\infty}^{\infty} |d\mu(x)| < \infty$$

by the calculation just made.

We can now verify, just as in §H.3 of Chapter VI, that

$$(\dagger) \qquad \int_{-\infty}^{\infty} \frac{x^p}{W(x)} d\mu(x) = 0 \quad \text{for} \quad p = 0, 1, 2, \ldots.$$

The integral on the right is just the (absolutely convergent) sum

$$\sum_{-\infty}^{\infty}{}' \frac{\lambda_n^p}{C'(\lambda_n)},$$

and we have to show that this is zero for $p \geqslant 0$. Taking

$$C_N(z) = \prod_{n=1}^{N} \left(1 - \frac{z^2}{\lambda_n^2}\right)$$

(cf. §C, Chapter VI), we have the Lagrange interpolation formula

$$z^l = \sum_{-N}^{N}{}' \frac{\lambda_n^l C_N(z)}{(z - \lambda_n) C_N'(\lambda_n)},$$

valid for $0 \leqslant l < 2N$. Fix l. Clearly, $|C_N'(\lambda_n)| \geqslant |C'(\lambda_n)|$ for $-N \leqslant n \leqslant N$. Therefore, since $\sum_{-\infty}^{\prime \infty} |\lambda_n^l / C'(\lambda_n)| < \infty$, we can make $N \to \infty$ in the preceding relation and use dominated convergence to obtain

$$z^l = \sum_{-\infty}^{\infty}{}' \frac{\lambda_n^l C(z)}{(z - \lambda_n) C'(\lambda_n)}.$$

Putting $l = p + 1$ and specializing to $z = 0$, the desired result follows, and we have ($\dagger$).

Our measure μ is not zero. The strict inclusion of $\mathscr{C}_W(0)$ in $\mathscr{C}_W(\mathbb{R})$ is thus a consequence of ($\dagger$), and the construction of our example is completed.

Let us summarize what we have. We have found *an even weight $W(x) \geqslant 1$, increasing on $[0, \infty)$ at a rate faster than that of any power of x, such that $\mathscr{C}_W(0) \neq \mathscr{C}_W(\mathbb{R})$ but $\mathscr{C}_W(0+) = \mathscr{C}_W(\mathbb{R})$.* This was promised at the end of §H.2, Chapter VI. In §H.3 of that chapter we constructed an even weight W with $\mathscr{C}_W(0) \neq \mathscr{C}_W(0+)$ and $\mathscr{C}_W(0+) \neq \mathscr{C}_W(\mathbb{R})$.

Scholium

As the work of Chapter VI shows, the condition

$$\int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2} \mathrm{d}x < \infty$$

is sufficient to guarantee proper inclusion in $\mathscr{C}_W(\mathbb{R})$ of each of the spaces $\mathscr{C}_W(0)$ and $\mathscr{C}_W(A)$, $A > 0$ (for $\mathscr{C}_W(0)$ see §D of that chapter). The question is, *how much regularity do we have to impose on $W(x)$ in order that the contrary property*

$$(\dagger\dagger) \qquad \int_{-\infty}^{\infty} \frac{\log W(x)}{1+x^2} \mathrm{d}x = \infty$$

should imply that $\mathscr{C}_W(0) = \mathscr{C}_W(\mathbb{R})$ or that $\mathscr{C}_W(A) = \mathscr{C}_W(\mathbb{R})$ for $A > 0$?

As we saw in the previous article, *monotoneity* of $W(x)$ on $[0, \infty)$ is *enough* for ($\dagger\dagger$) to make $\mathscr{C}_W(A) = \mathscr{C}_W(\mathbb{R})$ when $A > 0$, in the case of *even* weights W. In §D, Chapter VI, it was also shown that ($\dagger\dagger$) implies $\mathscr{C}_W(0) = \mathscr{C}_W(\mathbb{R})$ for *even weights W with* $\log W(x)$ *convex in* $\log|x|$. The example just given shows that *logarithmic convexity cannot be replaced by monotoneity when weighted polynomial approximation is involved*, even though the *later* is *good enough* when we deal with *weighted approximation by exponential sums.*

We have here a *qualitative difference* between weighted polynomial approximation and that by linear combinations of the $e^{i\lambda x}$, $-A \leqslant \lambda \leqslant A$, and in fact the *first real distinction* we have seen between these two kinds of approximation. In Chapter VI, the study of the latter paralleled that of the former in almost every detail.

The reason for this difference is that (for weights W which are finite reasonably often) the $\| \quad \|_W$-density of polynomials in $\mathscr{C}_W(\mathbb{R})$ is governed by the *lower polynomial regularization* $W_*(x)$ of W, whereas that of $\mathscr{E}_A$ is determined by the lower regularization $W_A(x)$ of W based on the use of entire functions of exponential type $\leqslant A$. The latter are better than polynomials for getting at $W(x)$ from underneath. As the example shows, they are qualitatively better.

5. **Levinson's theorem**

There is one other easy application of the material in article 1 which should be mentioned. Although the result obtained in that way has been superseded by a deeper (and more difficult) one of Beurling, to be given in the next §, it is still worthwhile, and serves as a basis for Volberg's very refined work presented in the last § of this chapter.

Theorem (Levinson). *Let μ be a finite Radon measure on $\mathbb{R}$, and suppose that*

$$\int_0^\infty \frac{1}{1+x^2}\log\left(\frac{1}{\int_x^\infty |d\mu(t)|}\right)dx \;=\; \infty.$$

Then the Fourier–Stieltjes transform

$$\hat{\mu}(\lambda) \;=\; \int_{-\infty}^{\infty} e^{i\lambda x}\, d\mu(x)$$

cannot vanish identically over any interval of positive length unless $\mu \equiv 0$.

Remark 1. Of course, the same result holds if

$$\int_{-\infty}^0 \frac{1}{1+x^2}\log\left(\frac{1}{\int_{-\infty}^x |d\mu(t)|}\right)dx \;=\; \infty.$$

Remark 2. Beurling's theorem, to be proved in the next §, says that under the stated condition on $\log(\int_x^\infty |d\mu(t)|)$, $\hat{\mu}(\lambda)$ cannot even vanish on a *set of positive measure* unless $\mu \equiv 0$.

Proof of theorem. It is enough, in the first place, to establish the result for *absolutely continuous* measures μ. Suppose, indeed, that μ is *any* measure satisfying the hypothesis; from it let us form the absolutely continuous measures μ_h, $h > 0$, having the densities

$$\frac{d\mu_h(x)}{dx} \;=\; \frac{1}{h}\int_x^{x+h} d\mu(t).$$

Then

$$\hat{\mu}_h(\lambda) \;=\; \frac{1-e^{-i\lambda h}}{i\lambda h}\,\hat{\mu}(\lambda),$$

so $\hat{\mu}_h(\lambda)$ *vanishes wherever* $\hat{\mu}(\lambda)$ *does.* Also,

$$\int_x^\infty |d\mu_h(t)| \;\leqslant\; \int_x^\infty |d\mu(t)|$$

for $x > 0$, so

$$\int_0^\infty \frac{1}{1+x^2}\log\left(\frac{1}{\int_x^\infty |d\mu_h(t)|}\right)dx \;=\; \infty$$

for each $h > 0$ by the hypothesis. Truth of our theorem for absolutely continuous measures would thus make the μ_h all zero if $\hat{\mu}(\lambda)$ vanishes on an interval of length > 0. But then $\mu \equiv 0$.

We may therefore take μ to be absolutely continuous. Assume, without loss of generality, that

$$\int_{-\infty}^{\infty} |\mathrm{d}\mu(t)| \leqslant 1$$

and that $\hat{\mu}(\lambda) \equiv 0$ for $-A \leqslant \lambda \leqslant A$, $A > 0$.

For $x \geqslant 0$, write $W(x) = (\int_x^\infty |\mathrm{d}\mu(t)|)^{-1/2}$, and, for $x < 0$, put

$$W(x) = \left(\int_{-\infty}^{x} |\mathrm{d}\mu(t)|\right)^{-1/2}$$

The function $W(x)$ (perhaps discontinuous at 0) is $\geqslant 1$ and tends to ∞ as $x \to \pm\infty$. It is monotone on $(-\infty, 0)$ and on $[0, \infty)$, and *continuous* on each of those intervals (in the *extended sense*, as it *may take the value* ∞).

By integral calculus (!), we now find that

$$\int_0^\infty W(x)|\mathrm{d}\mu(x)| = \int_0^\infty \frac{|\mathrm{d}\mu(x)|}{\sqrt{\int_x^\infty |\mathrm{d}\mu(t)|}} = 2\sqrt{\int_0^\infty |\mathrm{d}\mu(t)|} < \infty,$$

and, in like manner,

$$\int_{-\infty}^{0} W(x)|\mathrm{d}\mu(x)| < \infty.$$

The measure ν with $\mathrm{d}\nu(x) = W(x)\mathrm{d}\mu(x)$ is therefore *totally finite* on $\mathbb{R}$. (If $W(x)$ is infinite on any semi-infinite interval J, we of course must have $\mathrm{d}\mu(x) \equiv 0$ on J, so $\mathrm{d}\nu(x)$ is also zero there.) For $-A \leqslant \lambda \leqslant A$,

$$(\S) \qquad \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\lambda x}}{W(x)}\mathrm{d}\nu(x) = \hat{\mu}(\lambda) = 0.$$

However, by hypothesis,

$$\int_0^\infty \frac{\log W(x)}{1+x^2}\mathrm{d}x = \frac{1}{2}\int_0^\infty \frac{1}{1+x^2}\log\left(\frac{1}{\int_x^\infty |\mathrm{d}\mu(t)|}\right)\mathrm{d}x = \infty,$$

so, since $W(x)$ is *increasing* on $[0, \infty)$, $\mathscr{C}_W(A)$ is $\|\ \|_W$-dense in $\mathscr{C}_W(\mathbb{R})$ according to the theorem of article 3. Therefore, by (§),

$$\int_{-\infty}^{\infty} \varphi(x)\mathrm{d}\mu(x) = \int_{-\infty}^{\infty} \frac{\varphi(x)}{W(x)}\mathrm{d}\nu(x) = 0$$

for every continuous φ of compact support. This means that $\mu \equiv 0$. We are done.

The proof of Volberg's theorem uses the following

Corollary. Let $f(\vartheta) \sim \sum_{-\infty}^{\infty} \hat{f}(n)\mathrm{e}^{in\vartheta}$ *belong to* $L_1(-\pi, \pi)$, *and suppose that* $f(\vartheta) = 0$ *a.e. on an interval* J *of positive length. If* $|\hat{f}(n)| \leqslant \mathrm{e}^{-M(n)}$ *for*

$n > 0$ *with* $M(n)$ increasing, *and such that*

$$\sum_1^\infty \frac{M(n)}{n^2} = \infty,$$

then $f(\vartheta) \equiv 0, \ -\pi \leqslant \vartheta \leqslant \pi.$

Proof. Take any small $h > 0$ and form the convolution

$$f_h(\vartheta) = \frac{1}{h}\int_{-h}^{h}\left(1 - \frac{|t|}{h}\right) f(\vartheta - t)\,\mathrm{d}t.$$

If $h < \frac{1}{2}$ (length of J), $f_h(\vartheta)$ *also vanishes identically on an interval of positive length*

From the rudiments of Fourier series, we have

$$f_h(\vartheta) = \sum_{-\infty}^{\infty}\left(\frac{\sin(nh/2)}{nh/2}\right)^2 \hat{f}(n)\mathrm{e}^{\mathrm{i}n\vartheta}.$$

The sum on the right can be rewritten in evident fashion as $\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\vartheta x}\,\mathrm{d}\mu(x)$ with a (discrete) totally finite measure μ. Let $x > 0$ be given. If n is *the next integer* $\geqslant x$ we have, since $M(n)$ *increases*,

$$\int_x^\infty |\mathrm{d}\mu(t)| = \sum_{l \geqslant n}\left(\frac{\sin(lh/2)}{lh/2}\right)^2 |\hat{f}(l)|$$

$$\leqslant \mathrm{e}^{-M(n)}\sum_{l \geqslant n}\frac{4}{h^2 l^2} \leqslant \frac{\text{const.}}{h^2}\mathrm{e}^{-M(n)}.$$

Because $\sum_1^\infty M(n)/n^2 = \infty$, we see that

$$\int_0^\infty \frac{1}{1+x^2}\log^-\left(\int_x^\infty |\mathrm{d}\mu(t)|\right)\mathrm{d}x = \infty,$$

and conclude by the theorem that $f_h \equiv 0$. Making $h \to 0$, we see that $f \equiv 0$,

Q.E.D.

B. The Fourier transform vanishes on a set of positive measure. Beurling's theorems

Beurling was able to extend considerably the theorem of Levinson given at the end of the preceding §. The main imprôvement in technique which made this extension possible involved the use of harmonic measure.

Harmonic measure will play an increasingly important rôle in the remaining chapters of this book. We therefore begin this § with a brief general discussion of what it is and what it does.

1. What is harmonic measure?

Suppose we have a finitely connected bounded domain $\mathscr{D}$ whose boundary, $\partial\mathscr{D}$, consists of several piecewise smooth Jordan curves. The *Dirichlet problem* for $\mathscr{D}$ requires us to find, for any given φ continuous on $\partial\mathscr{D}$, a function $U_\varphi(z)$ *harmonic in* $\mathscr{D}$ *and continuous up to* $\partial\mathscr{D}$ with $U_\varphi(\zeta) = \varphi(\zeta)$ for $\zeta \in \partial\mathscr{D}$. It is well known that the Dirichlet problem can always be solved for domains like those considered here. Many books on complex variable theory or potential theory contain proofs of this fact, which we henceforth take for granted.

Let us, however, tarry long enough to remind the reader of one particularly easy proof, available for *simply connected* domains $\mathscr{D}$. There, the Riemann mapping theorem provides us with a *conformal mapping* F of $\mathscr{D}$ onto the unit disk $\{|w| < 1\}$. Such a function F extends continuously up to $\partial\mathscr{D}$ and maps the latter in one–one fashion onto $\{|\omega| = 1\}$; this is true by a famous theorem of Carathéodory and can also be directly verified in many cases where $\partial\mathscr{D}$ has a simple explicit description (including all the ones to be met with in this book).

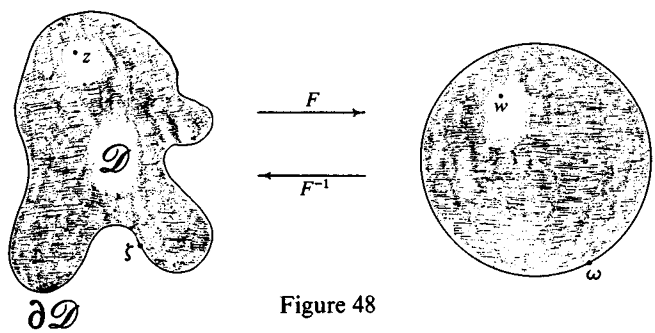

Figure 48

Denote by F^{-1} the inverse mapping to F. The function $\psi(\omega) = \varphi(F^{-1}(\omega))$ is then continuous on $\{|\omega| = 1\}$, and, if U_φ is the harmonic function sought which is to agree with φ on $\partial\mathscr{D}$, $V(w) = U_\varphi(F^{-1}(w))$ must be harmonic in $\{|w| < 1\}$ and continuous up to $\{|\omega| = 1\}$, where $V(\omega)$ must equal $\psi(\omega)$. A function V with these properties (there is only *one* such) can, however, be obtained from ψ by *Poisson's formula*:

$$V(w) \;=\; \frac{1}{2\pi}\int_{|\omega|=1} \frac{1 - |w|^2}{|w - \omega|^2}\,\psi(\omega)\,|d\omega|.$$

Going back to $\mathscr{D}$, and writing $z = F^{-1}(w)$, $\zeta = F^{-1}(\omega)$, we get

$$U_\varphi(z) \;=\; \frac{1}{2\pi}\int_{\partial\mathscr{D}} \frac{1 - |F(z)|^2}{|F(z) - F(\zeta)|^2}\,\varphi(\zeta)\,|dF(\zeta)|$$