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The logarithmic integral II

PAUL KOOSIS

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THE LOGARITHMIC INTEGRAL
II

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$$\int_{-\infty}^{\infty} \frac{\log M(t)}{1+t^2} \mathrm{d}t$$

The logarithmic integral

II

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CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo, Delhi

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521102544

First published 1992
This digitally printed version 2009

A catalogue record for this publication is available from the British Library

Library of Congress Cataloguing in Publication data
(Revised for volume 2)

Koosis, Paul
The logarithmic integral.
(Cambridge studies in advanced mathematics; 21)
Includes bibliographies and indexes.
1. Analytic functions. 2. Harmonic analysis.
3. Integrals, Logarithmic. I. Title. II. Series.
QA331.K7393 1988 515.4 85-28018

ISBN 978-0-521-30907-3 hardback
ISBN 978-0-521-10254-4 paperback

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Contents

Foreword to volume II, *with an example for the end of volume* I

Art is long and life is short. More than four years elapsed between completion of the MS for volume I and its publication; a good deal of that time was taken up with the many tasks, often tedious, called for by the production of any decently printed book on mathematics.

An attempt has been made to speed up the process for volume II. Three quarters of it has been set directly from handwritten MS, with omission of the intermediate preparation of typed copy, so useful for bringing to light mistakes of all kinds. I have tried to detect such deficiencies on the galleys and corrected all the ones I could find there; I hope the result is satisfactory.

Some mistakes did remain in volume I in spite of my efforts to remove them; others crept in during the successive proof revisions. Those that have come to my attention are reported in the *errata* immediately following this foreword.

In volume I the theorem on simultaneous polynomial approximation was incorrectly ascribed to Volberg; it is almost certainly due to T. Kriete, who published it some three years earlier. L. de Branges' name should have been mentioned in connection with the theorem on p. 215, for he gave (with different proof) an essentially equivalent result in 1959. The developments in §§A and C of Chapter VIII have been influenced by earlier work of Akhiezer and Levin. A beautiful paper of theirs made a strong impression on me many years ago. For exact references, see the bibliography at the end of this volume.

I thank Jal Choksi, my friend and colleague, for having frequently helped me to extricate myself from entanglements with the English language while I was writing and revising both volumes.

Suzanne Gervais, maker of animated films, became my friend at a bad time in my life and has constantly encouraged me in my work on this book,

from the time I first decided I would write it early in 1983. Although she had visual work enough of her own to think about, she was always willing to examine my drawings of the figures and give me practical advice on how to do them. For that help and for her friendship which I am fortunate to enjoy, I thank her affectionately.

One point raised at the very end of volume I had there to be left unsettled. This concerned the likelihood that Brennan's improvement of Volberg's theorem, presented in article 1 of the addendum, was essentially best possible. An argument to support that claim was made on pp. 578–83; it depended, however, on an example which had been reported, but not described, by Borichev and Volberg. No description was available before Volume I went to press, so the claim about Brennan's improvement could not be fully substantiated.

Now we are able to complete verification of the claim by providing the missing example. Its description is found at the end of a paper by Borichev and Volberg appearing in the very first issue of the new Leningrad periodical *Algebra i analiz*. We continue using the notation of the addendum to volume I.

Two functions have to be constructed. The first, $h(\xi)$, should be decreasing for $0 < \xi < \infty$ and satisfy $\xi h(\xi) \geqslant 1$, together with the relation

$$\int_0^1 \log h(\xi)\,d\xi \quad = \quad \infty.$$

The second, $F(z)$, is to be continuous on the closed unit disk and $\mathscr{C}_\infty$ in its interior, with

$$\left|\frac{\partial F(z)}{\partial \bar{z}}\right| \quad \leqslant \quad \exp(-h(\log(1/|z|))), \qquad |z| < 1,$$

$$|F(e^{i\vartheta})| \quad > \quad 0 \qquad \text{a.e.},$$

and

$$\int_{-\pi}^{\pi} \log|F(e^{i\vartheta})|\,d\vartheta \quad = \quad -\infty.$$

The function $F(z)$ we obtain will in fact be analytic in most of the unit disk Δ, ceasing to be so only in the neighborhood of some very small segments on the positive radius, accumulating at 1. The function $h(\log(1/x))$ will be very much larger than $1/\log(1/x)$ for most of the

$x \in (0, 1)$ contiguous to those segments.

Three simple ideas form the basis for the entire construction:

(1) In a domain $\mathscr{E}$ with piecewise analytic boundary having a 90° corner (internal measure) at ζ, say, we have

$$\omega_{\mathscr{E}}(I, z) \leqslant K_z|I|^2, \qquad z \in \mathscr{E},$$

for arcs I on $\partial\mathscr{E}$ containing ζ (see volume I, pp 260–1);

(2) The use of a Blaschke product involving factors affected with fractional exponents to 'correct', in an infinitely connected subdomain of Δ, a function analytic there and multiple-valued, but with single valued modulus;

(3) The use of a smoothing operation inside Δ, scaled according to that disk's hyperbolic geometry.

We start by looking at harmonic measure in domains $\mathscr{E} = \Delta \sim [a, 1]$, where $0 < a < 1$. According to (1), if $\eta > 0$ is small (and $< 1-a$), we have

$$\omega_{\mathscr{E}}(E_\eta, 0) \leqslant O(\eta^2)$$

for the sets $E_\eta = [1-\eta, 1] \cup I_\eta$, where I_η is the arc of length η on the unit circle, centered at 1. This is so because $\partial\mathscr{E}$ has two (internal) square corners at 1 that contribute separately to harmonic measure. (The slit $[a, 1]$ can be opened up by making a conformal mapping of $\mathscr{E}$ given by $z \longrightarrow \sqrt{(a-z)}$; when this is done the two corners at 1 are separated and they remain square.) Suppose that, for some given $a \in (0, 1)$, we fix an $\eta > 0$ small enough to make $\omega_{\mathscr{E}}(E_\eta, 0)/\eta$ less than some preassigned amount. Then, if we put $\mathscr{G} = \Delta \sim [a, 1-\eta]$, we will have, by simple comparison of $\omega_{\mathscr{G}}(I_\eta, z)$ and $\omega_{\mathscr{E}}(E_\eta, z)$ in $\mathscr{E}$,

$$\omega_{\mathscr{G}}(I_\eta, 0)/\eta \leqslant \omega_{\mathscr{E}}(E_\eta, 0)/\eta.$$

This relation is taken as the base of an inductive process. Beginning with an $a_1 > 2/(2+\sqrt{3})$ and < 1 (we shall see presently why the first condition is needed), we take a b_1, $a_1 < b_1 < 1$, so close to 1 as to make

$$\frac{\omega_{\mathscr{G}_1}(I_1, 0)}{|I_1|} < \frac{1}{2}$$

for $\mathscr{G}_1 = \Delta \sim [a_1, b_1]$ and the arc I_1 of length $1-b_1$ on $\partial\Delta$ centered at 1. One next chooses a_2, $b_1 < a_2 < 1$, in a way to be specified later on (a_2 will in fact be much *closer* to 1 than b_1), and then takes

b_2, $a_2 < b_2 < 1$, near enough to 1 to have

$$\frac{\omega_{\mathscr{G}_2}(I_2,\ 0)}{|I_2|} \quad < \quad \frac{1}{4}$$

and

$$|I_2| \quad < \quad \tfrac{1}{2}|I_1|$$

for $\mathscr{G}_2 = \Delta \sim [a_2,\ b_2]$ and the arc I_2 of length $1 - b_2$ on $\partial\Delta$ centered at 1. Continuing this procedure indefinitely, we get a sequence of segments

$$J_n \quad = \quad [a_n,\ b_n],$$

where $b_n < a_{n+1} < b_{n+1} < 1$, and nested arcs I_n of length $1 - b_n$ on $\partial\Delta$, each centered at 1, with

$$\frac{\omega_{\mathscr{G}_n}(I_n,\ 0)}{|I_n|} \quad < \quad \frac{1}{2^n}$$

for the corresponding domains $\mathscr{G}_n = \Delta \sim J_n$, and

$$|I_n| \quad < \quad \tfrac{1}{2}|I_{n-1}|.$$

Take now

$$\mathscr{D} \quad = \quad \Delta \sim J_1 \sim J_2 \sim J_3 \sim \cdot \quad \cdot \quad \cdot\ ;$$

then, since $\mathscr{D}$ is contained in each $\mathscr{G}_n$, the principle of extension of domain tells us that

$$\frac{\omega_{\mathscr{D}}(I_n,\ 0)}{|I_n|} \quad \leqslant \quad \frac{\omega_{\mathscr{G}_n}(I_n,\ 0)}{|I_n|} \quad < \quad \frac{1}{2^n}.$$

Our first ingredient in the formation of the desired $F(z)$ is a function $u(z)$ positive and harmonic in $\mathscr{D}$. Let $T_n(\vartheta)$ be periodic of period 2π, with

$$T_n(\vartheta) \quad = \quad \frac{1}{|I_n|}\left(1 \quad - \quad \frac{2|\vartheta|}{|I_n|}\right)^+ \qquad \text{for } -\pi \leqslant \vartheta \leqslant \pi.$$

The graph of $T_n(\vartheta)$ for $|\vartheta| \leqslant \pi$ is an isosceles triangle of height $1/|I_n|$ with its base on the segment $\{|\vartheta| \leqslant |I_n|/2\}$ corresponding to the arc I_n. We have

$$\int_{-\pi}^{\pi} T_n(\vartheta)\,\mathrm{d}\vartheta \quad = \quad \frac{1}{2}$$

while

$$\int_{-\pi}^{\pi} T_n(\vartheta)\,d\omega_{\mathscr{D}}(e^{i\vartheta}, 0) \quad \leqslant \quad \frac{\omega_{\mathscr{D}}(I_n, 0)}{|I_n|} \quad < \quad \frac{1}{2^n},$$

so

$$\int_{-\pi}^{\pi} \sum_{n=1}^{\infty} T_n(\vartheta)\; d\omega_{\mathscr{D}}(e^{i\vartheta}, 0) \quad < \quad \infty.$$

although

$$\int_{-\pi}^{\pi} \sum_{n=1}^{\infty} T_n(\vartheta)\; d\vartheta \quad = \quad \infty$$

For $z \in \mathscr{D}$, we put

$$u(z) \quad = \quad \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} T_n(\vartheta)\; d\omega_{\mathscr{D}}(e^{i\vartheta}, z)\ ;$$

the integral on the right is certainly *finite* by the third of the preceding four relations and Harnack's inequality, so $u(z)$ is harmonic in $\mathscr{D}$ and

$$u(z) \ > \ 0$$

there. For $0 < |\vartheta| \leqslant \pi$, $\sum_{n=1}^{\infty} T_n(\vartheta)$ is continuous (and *even* locally Lip 1 !), so at these values of ϑ,

$$u(z) \quad \longrightarrow \quad \sum_{n=1}^{\infty} T_n(\vartheta) \qquad \text{as } z \longrightarrow e^{i\vartheta}$$

from within $\mathscr{D}$. (It is practically obvious that the corresponding points $e^{i\vartheta}$ are regular for the Dirichlet problem in $\mathscr{D}$ – in fact, *all* points of $\partial\mathscr{D}$ are regular.) Taking $u(e^{i\vartheta})$ *equal* to $\sum_{n=1}^{\infty} T_n(\vartheta)$, we thus get a function $u(z)$ continuous in $\bar{\Delta} \sim \{1\}$, and we have

$$\int_{-\pi}^{\pi} u(e^{i\vartheta})\,d\vartheta \quad = \quad \infty.$$

The function $u(z)$ has, *locally*, a harmonic conjugate $\tilde{u}(z)$ in $\mathscr{D}$. The latter, of course, need not be *single-valued* in the infinitely connected domain $\mathscr{D}$; we nevertheless put

$$f(z) \quad = \quad e^{-(u(z) + i\tilde{u}(z))}$$

for $z \in \mathscr{D}$, obtaining a function *analytic and multiple-valued* in $\mathscr{D}$ whose *modulus*, $e^{-u(z)}$, is *single-valued* there. If $e^{i\vartheta} \neq 1$, *any given branch* of $\tilde{u}(z)$ is *continuous up to* $e^{i\vartheta}$, because $u(e^{it}) \ = \ \sum_{n=1}^{\infty} T_n(t)$ is Lip 1 for t near ϑ.

(To verify this, it suffices to look at $u(z)$ and $\tilde{u}(z)$ in the intersection of $\mathscr{D}$ with a small disk about $e^{i\vartheta}$ avoiding the J_n; if there is still any doubt, map that intersection conformally onto Δ.) It therefore makes sense to talk about the *multiple-valued*, but locally continuous boundary value $f(e^{i\vartheta})$ when $e^{i\vartheta} \neq 1$; the modulus $|f(e^{i\vartheta})|$ is again single-valued, being equal to $\exp(-u(e^{i\vartheta}))$. By the previous relation, we have

$$\int_{-\pi}^{\pi} \log|f(e^{i\vartheta})|d\vartheta \quad = \quad -\infty.$$

It is now necessary to cure the multiple-valuedness of $f(z)$; that is where the second of our ideas comes in. In constructing the $J_n = [a_n, b_n]$ and the arcs I_n, there is nothing to prevent our choosing the a_n so as to have

$$\sum_{n=1}^{\infty} (1-a_n) \quad < \quad \infty;$$

we henceforth assume that this has been done. (A much *faster* convergence of a_n to 1 will indeed be required later on.) Our condition on the a_n guarantees that the sum

$$\sum_{n=1}^{\infty} \mu_n \log\left|\frac{z-a_n}{1-a_nz}\right|$$

converges uniformly in the interior of $\Delta \sim \bigcup_n\{a_n\} \supseteq \mathscr{D}$ whenever the coefficients μ_n are bounded. If $0 \leqslant \mu_n \leqslant 1$, that sum is then equal to a function $v(z)$, *harmonic and* $\leqslant 0$ in $\mathscr{D}$. For the latter, there is a multiple-valued harmonic conjugate $\tilde{v}(z)$ defined in $\mathscr{D}$, and we have finally a function

$$b(z) \quad = \quad e^{v(z)+i\tilde{v}(z)} \quad = \quad \prod_{n=1}^{\infty}\left(\frac{a_n-z}{1-a_nz}\right)^{\mu_n},$$

analytic but multiple-valued in $\mathscr{D}$. The *modulus* $|b(z)| = e^{v(z)}$ is single-valued in $\mathscr{D}$.

The points a_n accumulate only at 1, so any branch of $b(z)$ is continuous up to points $e^{i\vartheta} \neq 1$ of the unit circle. For such points, $|b(e^{i\vartheta})| = 1$, and of course $|b(z)| \leqslant 1$ in $\mathscr{D}$, since $v(z) \leqslant 0$ there.

By proper adjustment of the exponents μ_n we can make the *product* $b(z)f(z)$ *single-valued* in $\mathscr{D}$, and hence analytic there in the ordinary sense. Consider what happens when z describes a simple closed path in the counterclockwise sense about just one of the slits $J_n = [a_n, b_n]$. Each given branch of the harmonic conjugate $\tilde{u}(z)$ will then increase by a certain (real) amount λ_n, independent of the branch. At the same time, every branch of $\tilde{v}(z) = \arg b(z)$ will increase by $2\pi\mu_n$. We take μ_n between 0 and 1 so as to make

$$2\pi\mu_n - \lambda_n$$

an integral multiple of 2π; this is clearly possible, and, once it is done, every branch of $\arg(b(z)f(z)) = \tilde{v}(z) - \tilde{u}(z)$ *increases by that amount* when z goes around a path of the kind just mentioned. Then the *product* $b(z)f(z)$ just comes back to its original value! Choosing in this way a value of μ_n, $0 \leqslant \mu_n \leqslant 1$, for *every n*, we ensure that $b(z)f(z)$ is single-valued in $\mathscr{D}$. Note that we have

$$|b(z)f(z)| \leqslant e^{-u(z)} \leqslant 1, \qquad z \in \mathscr{D},$$

and, since $|b(e^{i\vartheta})| = 1$ for $e^{i\vartheta} \neq 1$,

$$|b(e^{i\vartheta})f(e^{i\vartheta})| = |f(e^{i\vartheta})| > 0, \qquad e^{i\vartheta} \neq 1.$$

Because the product $b(z)f(z)$ is analytic in $\mathscr{D}$, we have there

$$\frac{\partial}{\partial \bar{z}} b(z)f(z) = 0;$$

the expression on the left may therefore be looked on as a *distribution* in Δ, supported on the slits J_n of $\Delta \sim \mathscr{D}$. In order to obtain a $\mathscr{C}_\infty$ function defined in Δ, we *smooth out* $b(z)f(z)$; that is our *third idea*. The smoothing is scaled according to the *square* of the gauge for the hyperbolic metric in Δ, i.e., like $1/(1-|z|)^2$.

Taking a $\mathscr{C}_\infty$ function $\psi(\rho) \geqslant 0$ supported on the interval $[1/4, 1/2]$ of the real axis, with

$$\int_0^{1/2} \psi(\rho)\rho \, d\rho = \frac{1}{2\pi},$$

we put, for $z \in \Delta$,

$$G(z) = \iint_\Delta \psi\left(\frac{|z-\zeta|}{(1-|z|)^2}\right) \frac{b(\zeta)f(\zeta)}{(1-|z|)^4} \, d\xi \, d\eta$$

(writing, as usual, $\zeta = \xi + i\eta$).

The first thing to observe here is that the expression on the right *makes sense*. Although $b(\zeta)f(\zeta)$ is defined merely in $\mathscr{D}$, the slits J_n making up $\Delta \sim \mathscr{D}$ are of *planar Lebesgue measure zero*, so we *only need* the values of the product in $\mathscr{D}$ in order to do the integral. The *second* observation is that $G(z)$ is $\mathscr{C}_\infty$ in $\mathscr{D}$. As a function of ζ, $\psi(|z-\zeta|/(1-|z|)^2)$ vanishes outside the disk $|\zeta - z| \leqslant \frac{1}{2}(1-|z|)^2$ which, however, lies well within Δ

for $z \in \Delta$, since then $|z| + \frac{1}{2}(1-|z|)^2 < 1$. We may therefore differentiate under the integral sign with respect to z or $\bar{z}$ as often as we wish, $\psi(\rho)$ being $\mathscr{C}_\infty$ (its identical vanishing for ρ near 0 helps here), and $|b(\zeta)f(\zeta)|$ being < 1 in $\mathscr{D}$. In this way we verify that $G(z)$ is $\mathscr{C}_\infty$ in Δ, and get (practically 'by inspection') the crude estimate

$$\left|\frac{\partial G(z)}{\partial \bar{z}}\right| \leqslant \frac{\text{const.}}{(1-|z|)^2}, \qquad |z| < 1.$$

As for $G(z)$, just *an average* of the function $b(\zeta)f(\zeta)$, we have

$$|G(z)| < 1, \qquad |z| < 1.$$

The *third* thing to observe is that $G(z)$ is actually *analytic* in a fairly large subset of Δ. Because $\psi(\rho)$ vanishes for $\rho \geqslant 1/2$, the integration in the above formula for $G(z)$ is really over the disk

$$\bar{\Delta}_z = \{\zeta: \ |\zeta - z| \leqslant \tfrac{1}{2}(1-|z|)^2\}$$

which, as we have just seen, *lies in* Δ when $|z| < 1$. Suppose that $\bar{\Delta}_z$ touches *none* of the slits J_n. Then $\bar{\Delta}_z \subseteq \mathscr{D}$ where $b(\zeta)f(\zeta)$ is *analytic* and, writing $\zeta = z + re^{i\vartheta}$, we have

$$G(z) = \int_0^{(1-|z|)^2/2} \int_{-\pi}^{\pi} b(z+re^{i\vartheta})\, f(z+re^{i\vartheta})\, \psi(r/(1-|z|)^2)\, \frac{r\,d\vartheta\,dr}{(1-|z|)^4}.$$

Using Cauchy's theorem to perform the first integration with respect to ϑ and then making the change of variable $r/(1-|z|)^2 = \rho$, we obtain the value $2\pi b(z)f(z)\int_0^{1/2}\psi(\rho)\rho\,d\rho = b(z)f(z)$, i.e.,

$$G(z) = b(z)f(z) \qquad \text{if} \quad \bar{\Delta}_z \subseteq \mathscr{D}.$$

When $\bar{\Delta}_z \subseteq \mathscr{D}$, the disks $\bar{\Delta}_{z'}$ also lie in $\mathscr{D}$ for the z' belonging to some *neighborhood* of z; we thus have $G(z') = b(z')f(z')$ in that neighborhood, and $G(z')$ (like $b(z')f(z')$) is then *analytic at* z. For the z in Δ such that $\bar{\Delta}_z \subseteq \mathscr{D}$, we therefore have

$$\frac{\partial G(z)}{\partial \bar{z}} = 0$$

although, for the *remaining* z in the unit disk, only the above estimate on $\partial G(z)/\partial \bar{z}$ is available. It is necessary to examine the set of those remaining z.

They are precisely the ones for which $\bar{\Delta}_z$ intersects with some J_n. We proceed to describe the set

$$B_n = \{z \in \Delta: \ \bar{\Delta}_z \cap J_n \neq \varnothing\}.$$

Write for the moment $J_n = [a, b]$, dropping the subscripts on a_n and b_n. If $\bar{\Delta}_z$ is to intersect with $[a, b]$, we must have $|z| > 2-\sqrt{3}$. Indeed, a, as one of the a_n, is $\geqslant a_1$ which we initially took $> 2/(2+\sqrt{3})$, while $\bar{\Delta}_z$ lies in the disk $\{|\zeta| \leqslant |z| + \frac{1}{2}(1-|z|)^2\}$ whose radius increases with $|z|$. For $|z| = 2-\sqrt{3}$, that radius works out to $2/(2+\sqrt{3})$, so if $|z| \leqslant 2-\sqrt{3}$, $[a, b]$ would lie *outside* the disk containing $\bar{\Delta}_z$; $|z|$ is thus $> 2-\sqrt{3}$ for $z \in B_n$. Now when $2-\sqrt{3} < |z| < 1$, $|z| - \frac{1}{2}(1-|z|)^2 > 0$, so $\bar{\Delta}_z$ is in fact contained in the *ring*

$$|z| - \tfrac{1}{2}(1-|z|)^2 \leqslant |\zeta| \leqslant |z| + \tfrac{1}{2}(1-|z|)^2$$

(*that's* why a_1 was chosen $> 2/(2+\sqrt{3})$!). Therefore, if $\bar{\Delta}_z$ intersects with $[a, b]$, we must have

$$|z| - \tfrac{1}{2}(1-|z|)^2 \leqslant b,$$
$$|z| + \tfrac{1}{2}(1-|z|)^2 \geqslant a.$$

Both left sides are increasing functions of $|z|$ (for $z \in \Delta$), so these relations are equivalent to

$$a' \leqslant |z| \leqslant b',$$

where

$$a' + \tfrac{1}{2}(1-a')^2 = a,$$
$$b' - \tfrac{1}{2}(1-b')^2 = b.$$

In $(0, 1)$ these equations have the solutions

$$a' = \sqrt{(2a-1)},$$
$$b' = 2-\sqrt{(3-2b)};$$

for the first we need $a > 1/2$ but have in fact $a > 2/(2+\sqrt{3})$. Using differentiation, one readily verifies that $a' < a$ and $b < b' < 1$.

We see that B_n (the set of $z \in \Delta$ for which $\bar{\Delta}_z$ intersects with $[a, b]$) is an oval-shaped region including $[a, b]$ and contained in the ring $a' \leqslant |z| \leqslant b'$; its boundary crosses the x-axis at the points a' and b'. When a is close to 1, B_n is quite thin in the vertical direction because, if $\bar{\Delta}_z$ touches the x-axis at all, we must have $|\Im z| \leqslant \frac{1}{2}(1-|z|)^2$.

One can specify the a_n and b_n so as to ensure *disjointness* of the oval regions B_n. The preceding description shows that this will be the case if the *rings* $a_n' \leqslant |z| \leqslant b_n'$ are disjoint, where (restoring the subscript n)

$$a_n' = \sqrt{(2a_n-1)},$$
$$b_n' = 2 - \sqrt{(3-2b_n)};$$

i.e., if $b'_n < a'_{n+1}$ for $n = 1, 2, 3, \ldots$. It is easy to arrange this in making the successive choices of the a_n and b_n; all we need is to have

$$a_{n+1} = a'_{n+1} + \tfrac{1}{2}(1 - a'_{n+1})^2 > b'_n + \tfrac{1}{2}(1 - b'_n)^2.$$

Here it is certainly true that $b_n < b'_n < 1$ when $0 < b_n < 1$; then, however, the extreme right-hand member of the relation is still < 1, and numbers $a_{n+1} < 1$ satisfying it *are available*. There is obviously no obstacle to our making the a_n increase as rapidly as we like towards 1; we can, in particular, have

$$\sum_{n=1}^{\infty} (1 - a_n) < \infty.$$

We henceforth assume that the last precaution has been heeded in the selection of the a_n. The B_n will then lie in their respective *disjoint* rings $a'_n \leqslant |z| \leqslant b'_n$ besides being all included in the cusp-shaped region $|\Im z| \leqslant \frac{1}{2}(1 - |z|)^2$ and, of course, in the right half plane. According to what we have already seen, $G(z)$ is equal to the analytic function $b(z)f(z)$ for $z \in \Delta$ *outside* all of the B_n, so then $\partial G(z)/\partial \bar{z} = 0$. *Within* any of the B_n, we have only the estimate $|\partial G(z)/\partial \bar{z}| \leqslant \text{const.}/(1 - |z|)^2$.

Because of the configuration of the B_n, $G(z)$ is continuous up to the points of $\partial\Delta \sim \{1\}$. Indeed, when $z \in \Delta$ tends to $e^{i\vartheta} \neq 1$, it must eventually *leave* the region $\{\Re z > 0, \ |\Im z| \leqslant \frac{1}{2}(1 - |z|)^2\}$ in which all the B_n lie, and then $G(z)$ becomes equal to $b(z)f(z)$ which has the continuous limit $b(e^{i\vartheta})f(e^{i\vartheta})$ away from 1 on the unit circumference.

If $z \in \Delta$ tends to 1 from *outside* any sector with vertex at 1 of the form $|\arg(1 - z)| \leqslant \alpha$, $0 < \alpha < \pi/2$, we have

$$G(z) \longrightarrow 0.$$

To see this, we argue that such z must leave the region $|\Im z| \leqslant \frac{1}{2}(1 - |z|)^2$, making $G(z) = b(z)f(z)$. Then, however,

$$\log|b(z)f(z)| \leqslant -u(z) = -\int_{-\pi}^{\pi} \sum_{n=1}^{\infty} T_n(\vartheta)\, d\omega_{\mathscr{D}}(e^{i\vartheta}, z),$$

and it suffices to show that the expression on the right tends to $-\infty$ whenever $z \longrightarrow 1$ from outside any of the sectors just mentioned. This is so due to the fact that $\sum_{n=1}^{\infty} T_n(\vartheta) \longrightarrow \infty$ for $\vartheta \longrightarrow 0$, as may be verified by taking the region

$$\mathscr{E} = \Delta \sim [1/2, 1] \subseteq \mathscr{D}$$

and comparing harmonic measure for $\mathscr{D}$ with that for $\mathscr{E}$. By the principle of extension of domain, $d\omega_{\mathscr{E}}(e^{i\vartheta}, z) \leqslant d\omega_{\mathscr{D}}(e^{i\vartheta}, z)$ for $z \in \mathscr{E}$, so we need only check that

$$\int_{-\pi}^{\pi} \sum_{n=1}^{\infty} T_n(\vartheta) \, d\omega_{\mathscr{E}}(e^{i\vartheta}, z) \longrightarrow \infty$$

as $z \longrightarrow 1$ from outside any of the sectors in question. That, however, should be *clear*. Let the reader imagine that $\mathscr{E}$ has been mapped conformally onto the upper half plane so as to take the vertices of its two corners at 1 to -2 and 2, say, and then think about how the *ordinary* Poisson integral corresponding to the last expression must behave as one moves towards -2 or 2 from the upper half plane.

We put finally

$$F(z) = c \exp\left(-K \frac{1+z}{1-z} \right) G(z)$$

for $z \in \Delta$, with c a *small* constant > 0 and K a *large* one. The exponential serves two purposes. It is, in the first place, < 1 in modulus in Δ and continuous up to $\partial\Delta \sim \{1\}$ where it has boundary values of modulus 1. When $z \longrightarrow 1$ from *within* any sector $|\arg(1-z)| \leqslant \alpha$, $0 < \alpha < \pi/2$, the exponential tends to *zero*, making $F(z) \longrightarrow 0$, since $|G(z)| < 1$ in Δ. This, however, is also true when $z \longrightarrow 1$ from *outside* such a sector because then $G(z) \longrightarrow 0$ as we have just seen. Thus,

$$F(z) \longrightarrow 0 \quad \text{for } z \longrightarrow 1, \ z \in \Delta.$$

We have already remarked that $G(z)$ is continuous up to $\partial\Delta \sim \{1\}$, where it coincides with $b(z)f(z)$, so we have

$$F(z) \longrightarrow c\, e^{-Ki\cot(\vartheta/2)} b(e^{i\vartheta}) f(e^{i\vartheta})$$

when $z \in \Delta$ tends to $e^{i\vartheta} \neq 1$. Denoting the boundary value on the right by $F(e^{i\vartheta})$, we have $|F(e^{i\vartheta})| = c|f(e^{i\vartheta})| = c\exp(-u(e^{i\vartheta}))$, and this tends to *zero* as $\vartheta \longrightarrow 0$ since $u(e^{i\vartheta}) = \sum_{n=1}^{\infty} T_n(\vartheta)$ then tends to ∞. *The function $F(z)$ thus extends continuously up to the unit circumference* thanks to the factor $\exp(-K(1+z)/(1-z))$. We have $|F(e^{i\vartheta})| = c|f(e^{i\vartheta})| > 0$ for $e^{i\vartheta} \neq 1$, and

$$\int_{-\pi}^{\pi} \log|F(e^{i\vartheta})| \, d\vartheta = 2\pi \log c + \int_{-\pi}^{\pi} \log|f(e^{i\vartheta})| \, d\vartheta = -\infty.$$

Since $G(z)$ is $\mathscr{C}_\infty$ inside Δ, so is $F(z)$. The second service rendered by the factor $\exp(-K(1+z)/(1-z))$ is to make $\partial F(z)/\partial\bar{z}$ *small* near $\partial\Delta$.

Outside the B_n, $F(z)$ (like $G(z)$) is *analytic*, so $\partial F(z)/\partial \bar{z} = 0$. *Within* any of the B_n, we use the formula

$$\frac{\partial F(z)}{\partial \bar{z}} = c\exp\left(-K\frac{1+z}{1-z}\right)\frac{\partial G(z)}{\partial \bar{z}},$$

which holds because the exponential is analytic in Δ. The B_n all lie in the right half plane, and in them,

$$|\Im z| \leqslant \tfrac{1}{2}(1-|z|)^2 < \tfrac{1}{2}(1-|z|),$$

whence

$$\Re\frac{1+z}{1-z} \geqslant \frac{\text{const.}}{1-|z|}.$$

This makes

$$\left|\frac{\partial F(z)}{\partial \bar{z}}\right| \leqslant c\exp\left(-K\frac{\text{const.}}{1-|z|}\right)\left|\frac{\partial G(z)}{\partial \bar{z}}\right|$$

for z belonging to any of the B_n. As we have seen, the last factor on the right is $\leqslant \text{const.}/(1-|z|)^2$ which, for $|z| < 1$ near 1, is *greatly outweighed* by the exponential. Bearing in mind that $\log(1/|z|) \sim 1-|z|$ for $|z| \longrightarrow 1$, we see that the constants c and K can be adjusted so as to have

$$\left|\frac{\partial F(z)}{\partial \bar{z}}\right| \leqslant \exp\left(-\frac{1}{\log(1/|z|)}\right),$$

within the B_n at least. But then this holds *outside* them as well (in Δ, including in the neighborhood of 0), because $\partial F(z)/\partial \bar{z} = 0$ there.

$F(z)$ has now been shown to enjoy all the properties enumerated at the beginning of this exposition except the one involving the function $h(\xi)$, not yet constructed. That construction comes almost as an afterthought. Since the sets B_n lie inside the disjoint rings $a'_n \leqslant |z| \leqslant b'_n$, we start by putting $h(\log(1/|z|)) = 1/\log(1/|z|)$ on each of the latter; in view of the preceding relation, this *already implies* that

$$\left|\frac{\partial F(z)}{\partial \bar{z}}\right| \leqslant \exp(-h(\log(1/|z|)))$$

throughout Δ, no matter *how* $h(\log(1/|z|))$ is defined for the remaining $z \in \Delta$, because the left side is *zero* outside the B_n. To complete the definition of $h(\xi)$ for $0 < \xi < \infty$, we continue to use $h(\log(1/|z|)) = 1/\log(1/|z|)$ on the range $0 < |z| \leqslant a'_1$ and then take $h(\log(1/|z|))$ to be *linear in*

$|z|$ on each of the *complementary* rings

$$b'_n \leqslant |z| \leqslant a'_{n+1}, \qquad n = 1, 2, 3, \ldots .$$

The function $h(\xi)$ we obtain in this fashion is certainly *decreasing* (in ξ); $h(\log(1/|z|))$ is also > 1 for $|z| \geqslant b'_1$, because $b'_1 > a_1 > 2/(2+\sqrt{3}) > 1/\mathrm{e}$. $h(\log(1/|z|))$ is moreover $\geqslant 1/\log(1/|z|)$ on the complementary rings, for $1/\log(1/x)$ is a *convex* function of x for $1/\mathrm{e}^2 < x < 1$, and $b'_1 > 1/\mathrm{e}^2$. In terms of the variable $\xi = \log(1/|z|)$ we therefore have

$$\xi h(\xi) \geqslant 1, \qquad 0 < \xi < \infty.$$

The trick in arranging to have

$$\int_0^1 \log h(\xi)\,\mathrm{d}\xi = \infty$$

is to use *linearity* of $h(\log(1/x))$ in x on each interval $b'_n \leqslant x \leqslant a'_{n+1}$ to get lower bounds on the integrals

$$\int_{\log(1/a'_{n+1})}^{\log(1/b'_n)} \log h(\xi)\,\mathrm{d}\xi.$$

We have indeed $h(\xi) > 1$ for $\xi \leqslant \log(1/b'_n) \leqslant \log(1/b'_1)$ and $h(\log(1/a'_{n+1})) = 1/\log(1/a'_{n+1})$, so the linearity just mentioned makes $h(\xi) \geqslant 1/2\log(1/a'_{n+1})$ for $(b'_n + a'_{n+1})/2 \leqslant \mathrm{e}^{-\xi} \leqslant a'_{n+1}$, i.e., for $\log(1/a'_{n+1}) \leqslant \xi \leqslant \log(2/(b'_n + a'_{n+1}))$. The preceding integral is therefore

$$\geqslant \log\left(\frac{2a'_{n+1}}{a'_{n+1} + b'_n}\right) \cdot \log^+\left(\frac{1}{2\log(1/a'_{n+1})}\right),$$

since $\log h(\xi)$ is > 0 on the whole range of integration. For any *given* value of b'_n, $0 < b'_n < 1$, *the last expression tends to* ∞ *as* $a'_{n+1} \longrightarrow 1$! We can therefore make it $\geqslant 1$ by taking $a'_{n+1} > b'_n$ *close enough to* 1, and that can in turn be achieved by choosing $a_{n+1} = a'_{n+1} + \frac{1}{2}(1 - a'_{n+1})^2$ sufficiently near 1. We therefore *select the successive* a_n *in accordance with this requirement* in carrying out the inductive procedure followed at the beginning of our construction. That will certainly guarantee that $b'_n < a'_{n+1}$ (which we needed), and may obviously be done so as to have $\sum_{n=1}^{\infty}(1 - a_n) < \infty$ (by making the a_n tend more rapidly towards 1 we can only improve matters).

Once the a_n have been specified in this way, we will have

$$\int_{\log(1/a'_{n+1})}^{\log(1/b'_n)} \log h(\xi)\,\mathrm{d}\xi \geqslant 1$$

for each n, and therefore

$$\int_0^1 \log h(\xi)\,d\xi \quad = \quad \infty.$$

Our construction of the functions $F(z)$ and $h(\xi)$ with the desired properties is thus complete, and the gap in the second half of article 2 in the addendum to volume I filled in. This means, in particular, that in the hypothesis of Brennan's result (top of p. 574, volume I), *the condition that $M(v)/v^{1/2}$ be increasing cannot be replaced by the weaker one that* $M(v)/v^{1/2} \geqslant 2$.

January 26, 1990
Outremont, Québec.

Errata for volume I

Location	Correction
page 66	At end of the theorem's statement, words in roman should be in italic, and words in italic in roman.
pages 85, 87	In running title, delete bar under second M_n but keep it under first one.
page 102	In heading to §E, delete bar under M_n in first and third $\mathscr{C}_R(\{M_n\})$ but keep it in second one.
page 126, line 8	In statement of theorem, change *determinant* to *determinate.*
page 135, line 11	In displayed formula, change w^* to w^*.
page 136, line 4 from bottom	In displayed formula $\|P(x_0)\|^2 v(\{x_0\})$ should stand on the right.
page 177, line 11 from bottom	The sentence beginning 'Since, as we already' should start on a new line, separated by a horizontal space from the preceding one
page 190	In last displayed formula, change x'' to x^n
page 212 and following even numbered pages up to page 232 inclusive	Add to running title: *Comparison of* $\mathscr{C}_W(0)$ *to* $\mathscr{C}_W(0+)$
page 230	In the last two displayed formulas replace $(1-\alpha^2)$ throughout by $\|1-\alpha^2\|$.
page 241, line 3	Change b_b^2 in denominator of right-hand expression to b_n^2.
page 270, line 10	Change $F(z)$ to $F(Z)$.
page 287	In figure 69, B_1 and B_2 should designate the lower and upper sides of $\mathscr{D}_0$, not $\mathscr{D}$.

page 379, line 8 from bottom	Change comma after 'theorem' to a full stop, and capitalize 'if'.
page 394, line 3	Change y_1 to y_l.
page 466, last line	Delete full stop.
page 563, line 9	Change 'potential' to 'potentials'.
page 574, line 9 from bottom	Delete full stop after 'following'.
page 604	In running title, '*volume*' should not be capitalized.
page 605	In titles of §§C.1 and C.4 change 'Chapter 8' to 'Chapter VIII'.

IX

Jensen's formula again

The derivations of the two main results in this chapter – Pólya's gap theorem and a *lower* bound for the completeness radius of a set of imaginary exponentials – are both based on the same simple idea: application of Jensen's formula with a circle of varying radius and *moving* centre. I learned about this device from a letter that J.-P. Kahane sent me in 1958 or 1959, where it was used to prove the first of the results just mentioned. Let us begin our discussion with an exposition of that proof.

A. Pólya's gap theorem

Consider a Taylor series expansion

$$f(w) = \sum_0^\infty a_n w^n$$

with radius of convergence equal to 1. The function $f(w)$ must have at least one singularity on the circle $|w| = 1$. It was observed by Hadamard that *if many of the coefficients a_n are zero*, i.e., if, as we say, *the Taylor series has many gaps*, $f(w)$ must have *lots of singularities on the series' circle of convergence*. In a certain sense, *the more gaps the power series has, the more numerous must be the singularities associated thereto on its circle of convergence.*

This phenomenon was studied by Hadamard and by Fabry; the best result was given by Pólya. In order to formulate it, Pólya invented the maximum density bearing his name which has already appeared in Chapter VI.

In this §, it will be convenient to denote by $\mathbb{N}$ *the set of integers* $\geqslant 0$ (and *not just the ones* $\geqslant 1$ as is usually done, and as we will do in §B!). If $\Sigma \subseteq \mathbb{N}$, *we denote by* $n_\Sigma(t)$ *the number of elements of* Σ *in* $[0, t]$, $t \geqslant 0$. The *Pólya* *maximum density* of Σ, studied in §E.3 of Chapter VI, is the quantity

$$D_\Sigma^* = \lim_{\lambda\to 1-}\left(\limsup_{r\to\infty}\frac{n_\Sigma(r)-n_\Sigma(\lambda r)}{(1-\lambda)r}\right).$$

We have shown in the article referred to that the outer limit really does exist for any Σ, and that D_Σ^* is the *minimum of the densities of the measurable sequences containing* Σ. In this §, we use a property of D_Σ^* furnished by the following

Lemma. *Given $\varepsilon > 0$, we have, for $\rho \geqslant \varepsilon r$,*

$$\frac{n_\Sigma(r+\rho)-n_\Sigma(r)}{\rho} \leqslant D_\Sigma^* + \varepsilon$$

when r is large enough (depending on ε).

Proof. According to the above formula, if N is large enough and

$$\lambda = (1+\varepsilon)^{-1/N},$$

we will have

$$\frac{n_\Sigma(r)-n_\Sigma(\lambda r)}{(1-\lambda)r} < D_\Sigma^* + \frac{\varepsilon}{2}$$

for $r \geqslant R$, say. *Fix* such an N.

When $r \geqslant R$, we certainly have

$$\frac{n_\Sigma(\lambda^{-k-1}r)-n_\Sigma(\lambda^{-k}r)}{(\lambda^{-k-1}-\lambda^{-k})r} < D_\Sigma^* + \frac{\varepsilon}{2}$$

for $k = 0, 1, 2, \ldots$, so

$$\frac{n_\Sigma(\lambda^{-k}r)-n_\Sigma(r)}{(\lambda^{-k}-1)r} < D_\Sigma^* + \frac{\varepsilon}{2}$$

for $k = 1, 2, 3, \ldots$. Let

$$\rho \geqslant \varepsilon r = (\lambda^{-N}-1)r.$$

Then, if k is the least integer such that $(\lambda^{-k}-1)r \geqslant \rho$, we have $k \geqslant N$, so, $n_\Sigma(t)$ being increasing,

$$\frac{n_\Sigma(r+\rho)-n_\Sigma(r)}{\rho} \leqslant \frac{n_\Sigma(\lambda^{-k}r)-n_\Sigma(r)}{(\lambda^{-k}-1)r}\cdot\frac{(\lambda^{-k}-1)r}{\rho}$$

$$< \left(D_\Sigma^* + \frac{\varepsilon}{2}\right)\frac{\lambda^{-k}-1}{\lambda^{-k+1}-1}$$

$$\leqslant \quad \left(D_\Sigma^* + \frac{\varepsilon}{2}\right)\frac{\lambda^{-N} - 1}{\lambda^{-N+1} - 1} \quad = \quad \frac{\varepsilon}{(1+\varepsilon)^{(N-1)/N} - 1}\left(D_\Sigma^* + \frac{\varepsilon}{2}\right)$$

when $r \geqslant R$. If N is chosen large enough to begin with, the last number is $\leqslant D_\Sigma^* + \varepsilon$. This does it.

Theorem (Pólya). *Let the power series*

$$f(w) \quad = \quad \sum_{n \in \Sigma} a_n w^n$$

have radius of convergence 1. *Then, on every arc of* $\{|w| = 1\}$ *with length* $> 2\pi D_\Sigma^*$, $f(w)$ *has at least one singularity.*

Proof (Kahane). Assume that $f(w)$ can be continued analytically through an arc on the unit circle of length $> 2\pi D$, which we may wlog take to be symmetric about -1. We then have to prove that $D \leqslant D_\Sigma^*$. We may of course take $D > 0$. There is also no loss of generality in assuming $D < 1$, for here the power series' circle of convergence, which does include at least one singularity of $f(w)$, has length 2π.

Pick any $\delta > 0$. In the formula

$$a_n \quad = \quad \frac{1}{2\pi i}\int_{|w| = e^{-\delta}} f(w) w^{-n-1}\, dw$$

(we are, of course, taking a_n as *zero* for $n \notin \Sigma$, $n \geqslant 0$) one may, thanks to the analyticity of $f(w)$, *deform* the path of integration $\{|w| = e^{-\delta}\}$ to the contour Γ_δ shown here:

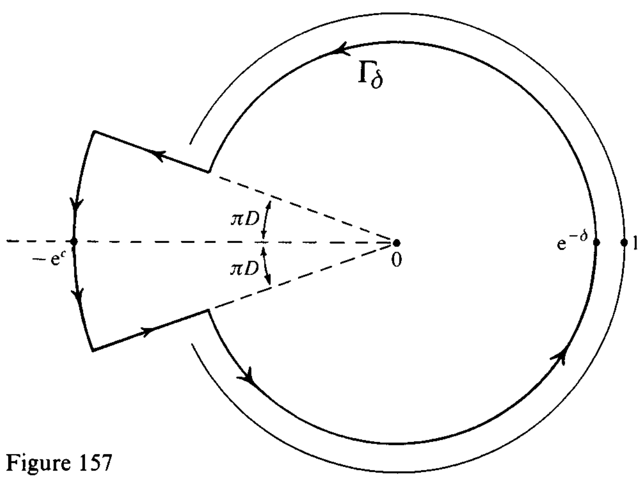


Figure 157

The quantity $c > 0$ is fixed once D is given, and independent of δ.

In the integral around Γ_δ, make the change of variable $w = \mathrm{e}^{-s}$, where $s = \sigma + \mathrm{i}\tau$ with τ ranging from $-\pi$ to π. Our expression then goes over into

$$\frac{1}{2\pi\mathrm{i}}\int_{\gamma_\delta} f(\mathrm{e}^{-s})\mathrm{e}^{ns}\,\mathrm{d}s \quad = \quad a_n$$

with this path γ_δ:

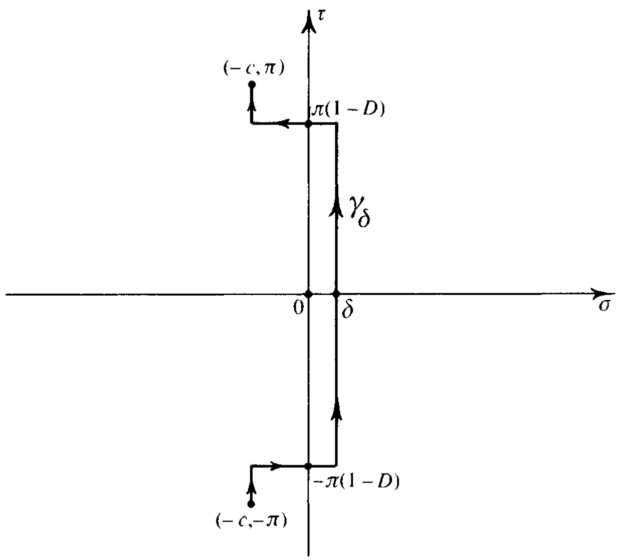


Figure 158

Write

$$F(z) \quad = \quad \frac{1}{2\pi\mathrm{i}}\int_{\gamma_\delta} f(\mathrm{e}^{-s})\mathrm{e}^{zs}\,\mathrm{d}s$$

so that $F(n) \; = \; a_n$ for $n \in \mathbb{N}$ (and is hence *zero* for $n \in \mathbb{N} \sim \Sigma$); $F(z)$ is of course entire and of exponential type. We break up the integral along γ_δ into three pieces, I, II and III, coming from the *front vertical*, *horizontal* and *rear vertical* parts of γ_δ respectively.

On the *front vertical* part of γ_δ, $|f(\mathrm{e}^{-s})| \;\leqslant\; M_\delta$ and $|\mathrm{e}^{sz}| \;\leqslant\; \mathrm{e}^{\delta x + \pi(1-D)|y|}$ (writing as usual $z = x + \mathrm{i}y$); hence

$$|\mathrm{I}| \;\leqslant\; M_\delta \mathrm{e}^{\delta x + \pi(1-D)|y|}$$

On the *horizontal* parts of γ_δ, $|f(\mathrm{e}^{-s})| \leqslant C$ (a number independent of δ, by the way), and $|\mathrm{e}^{sz}| \;\leqslant\; \mathrm{e}^{\delta x + \pi(1-D)|y|}$ for $x > 0$, whence

$$|\mathrm{II}| \;\leqslant\; C\mathrm{e}^{\delta x + \pi(1-D)|y|}, \quad x > 0.$$

Finally, on the *rear vertical* parts of γ_δ, $|f(\mathrm{e}^{-s})| \leqslant C$ and $|\mathrm{e}^{sz}| \leqslant \mathrm{e}^{-cx+\pi|y|}$ for $x > 0$, making

$$|\mathrm{III}| \leqslant Ce^{-cx+\pi|y|}, \quad x > 0.$$

Adding these three estimates, we get

$$|F(z)| \leqslant (M_\delta + C)\,\mathrm{e}^{\delta x + \pi(1-D)|y|} + C\mathrm{e}^{-cx+\pi|y|}$$

for $x > 0$. Since $c > 0$, the *second* term on the right will be $\leqslant$ the *first* in the sector

$$S = \left\{ z\colon \ |\Im z| \leqslant \frac{c}{\pi D}\Re z \right\}$$

with opening *independent of* δ. We thus have

$$|F(z)| \leqslant K_\delta \mathrm{e}^{\delta x + \pi(1-D)|y|} \quad \text{for } z \in S,$$

K_δ being a constant depending on δ. The idea here is that *the availability, for* $f(w)$, *of an analytic continuation through the arc* $\{\mathrm{e}^{\mathrm{i}\vartheta}\colon |\vartheta - \pi| \leqslant \pi D\}$ *has made it possible for us to diminish the term* $\pi|y|$, *which would normally occur in the exponent on the right, to* $\pi(1-D)|y|$, thanks to the term $-cx$ figuring in the previous expression.

Because $\sum_{n\in\Sigma} a_n w^n$ has *radius of convergence* 1, there is a subsequence Σ' of Σ with

$$\frac{\log|a_n|}{n} \longrightarrow 0 \quad \text{for } n \longrightarrow \infty \text{ in } \Sigma'.$$

Let 2α be the opening (*independent* of δ) of our sector S, i.e.,

$$\alpha = \arctan\frac{c}{\pi D}$$

With $n \in \Sigma'$, write Jensen's formula for $F(z)$ and the circle of radius $n \sin\alpha$ about n (this is Kahane's idea). That is just

$$\int_0^{n\sin\alpha} \frac{N(\rho, n)}{\rho}\,\mathrm{d}\rho = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|F(n + n\sin\alpha\,\mathrm{e}^{\mathrm{i}\vartheta})|\,\mathrm{d}\vartheta - \log|a_n|,$$

where $N(\rho, n)$ denotes the number of zeros of $F(z)$ in the disk $\{|z - n| \leqslant \rho\}$.

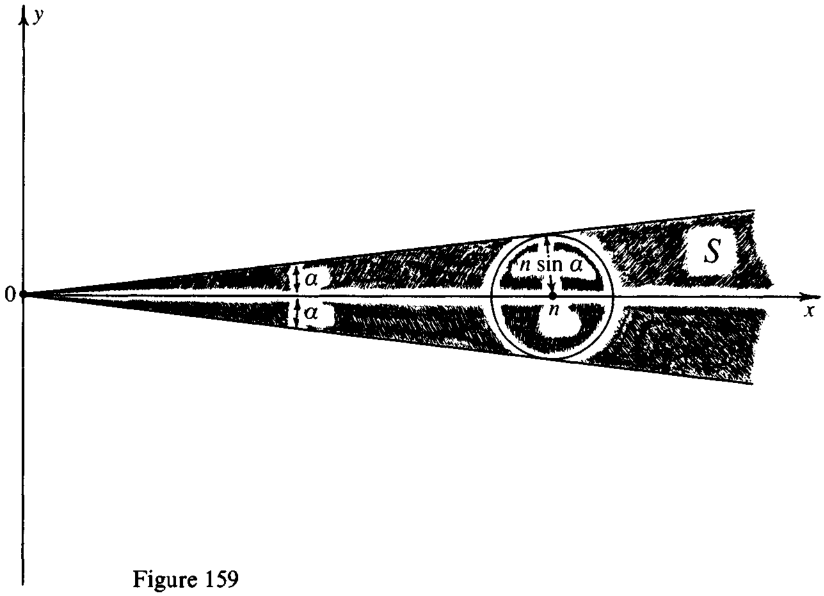


Figure 159

By the above estimate on $|F(z)|$ for $z \in S$, the *right side* of the relation just written is

$$\leqslant \log K_\delta + \frac{1}{2\pi}\int_{-\pi}^{\pi} (\delta n + \delta n \sin\alpha\cos\vartheta + n\sin\alpha\cdot\pi(1-D)|\sin\vartheta|)\,d\vartheta - \log|a_n|$$

$$= \log K_\delta + \delta n + 2(1-D)n\sin\alpha - \log|a_n|.$$

The *left side* we estimate *from below*, using the lemma. Since $F(m) = a_m = 0$ for $m \in \mathbb{N} \sim \Sigma$, we have, for $0 < \rho \leqslant n$,

$$\begin{aligned} N(\rho, n) \;&\geqslant\; \text{number of integers in } [n-\rho,\ n+\rho] \;-\; \\ &\quad \text{number of elements of } \Sigma \text{ in } [n-\rho,\ n+\rho] \\ &\geqslant\; 2\rho - (n_\Sigma(n+\rho) - n_\Sigma(n-\rho)) - 2. \end{aligned}$$

Fix any ε, $0 < \varepsilon < \sin\alpha$. According to the lemma, for n sufficiently large,

$$n_\Sigma(n+\rho) - n_\Sigma(n-\rho) \;\leqslant\; 2\rho(D^*_\Sigma + \varepsilon)$$

when $\varepsilon(1-\sin\alpha)n \;\leqslant\; 2\rho \;\leqslant\; 2n\sin\alpha$, so, for such ρ (and large n),

$$N(\rho, n) \;\geqslant\; 2(1 - D^*_\Sigma - \varepsilon)\rho - 2.$$

Hence, since $n_\Sigma(t)$ increases,

$$\int_0^{n\sin\alpha} \frac{N(\rho, n)}{\rho}\,d\rho \;\geqslant\; 2(1 - D_\Sigma^* - \varepsilon)\left(\sin\alpha - \frac{\varepsilon(1-\sin\alpha)}{2}\right)n$$
$$- \;\; 2\log\frac{2\sin\alpha}{\varepsilon(1-\sin\alpha)}.$$

Use this inequality together with the preceding estimate for the right side of the above Jensen formula. After dividing by $2n\sin\alpha$, one finds that

$$(1 - D_\Sigma^* - \varepsilon)\left(1 - \frac{\varepsilon(1-\sin\alpha)}{2\sin\alpha}\right) \;\leqslant\; \frac{\delta}{2\sin\alpha} + 1 - D$$
$$- \;\frac{\log|a_n|}{2n\sin\alpha} + O\left(\frac{1}{n}\right)$$

for large n, whence, making $n \longrightarrow \infty$ in Σ',

$$(1 - D_\Sigma^* - \varepsilon)\left(1 - \frac{\varepsilon(1-\sin\alpha)}{2\sin\alpha}\right) \;\leqslant\; 1 - D + \frac{\delta}{2\sin\alpha},$$

on account of the behaviour of $\log|a_n|$ for $n \in \Sigma'$.

The quantity ε, $0 < \varepsilon < \sin\alpha$, is arbitrary, and so is $\delta > 0$ with, as we have remarked, the opening 2α of S independent of δ. We thence deduce from the previous relation that $1 - D_\Sigma^* \leqslant 1 - D$, i.e., that

$$D \;\leqslant\; D_\Sigma^*.$$

This, however, is what we had to prove. We are done.

Remark. We see from the proof that it is really the presence in the Taylor series of *many gaps 'near' those $n \in \Sigma$ for which $|a_n|$ is 'big'* (the $n \in \Sigma'$) that gives rise to *large numbers of singularites* on the circle of convergence. The reader is invited to formulate a precise statement of this observation, obtaining a theorem in which the behaviour of the a_n and that of Σ both figure.

Pólya's gap theorem has various generalizations to Dirichlet series. For these, the reader should first look in the last chapter of Boas' book, after which the one by Levinson may be consulted. The most useful work on this subject is, however, the somewhat older one of V. Bernstein. Two of Mandelbrojt's books – the one published in 1952 and an earlier Rice Institute pamphlet on Dirichlet series – also contain interesting material, as does J.-P. Kahane's thesis, beginning with part II. There is, in addition, a recent monograph by Leontiev.

B. Scholium. A converse to Pólya's gap theorem

The quantity D_Σ^* figuring in the result of the preceding § is a kind of *upper density* for sequences Σ of positive integers. Before continuing with the main material of this chapter, it is natural to ask whether D_Σ^* is *the right kind of density measure to use* for a sequence Σ when investigating the distribution of the singularities associated with

$$\sum_{n\in\Sigma} a_n w^n$$

on that series' circle of convergence. Maybe there is always a singularity on each arc of that circle having opening greater than $2\pi d_\Sigma$, with d_Σ a quantity $\leqslant D_\Sigma^*$ associated to Σ which is really $< D_\Sigma^*$ for some sequences Σ. It turns out that *this is not the case*; D_Σ^* is *always* the critical parameter associated with the sequence Σ insofar as distribution of singularities on the circle of convergence is concerned.

This fact, which shows Pólya's gap theorem to be *definitive*, is not well known in spite of its clear scientific importance. It is the content of the following

Converse to Polya's gap theorem *Given any sequence Σ of positive integers with Pólya maximum density $D_\Sigma^* > 0$, there is, for any δ, $0 < \delta < D_\Sigma^*$, a Taylor series*

$$\sum_{n\in\Sigma} a_n w^n$$

with radius of convergence 1, *equal, for $|w| < 1$, to a function which can be continued analytically through the arc*

$$\{e^{i\vartheta}:\ |\vartheta| < \pi(D_\Sigma^* - \delta)\}.$$

The present § is devoted to the establishment of this result in its full generality.

1. Special case. Σ measurable and of density $D > 0$.

If $\lim_{t\to\infty} n_\Sigma(t)/t$ exists and equals a number $D > 0$ ($n_\Sigma(t)$ denoting the number of elements of Σ in $[0,\ t]$), the converse* to Pólya's theorem is easy – I think it is due to Pólya himself. The contour integration technique used to study this case goes back to Lindelöf; it was extensive-

* in a strengthened version, with analytic continuation through the arc $|\vartheta| < \pi D_\Sigma^* = \pi D$

ly used by V. Bernstein in his work on Dirichlet series, and later on by L. Schwartz in his thesis on sums of exponentials.

Restricting our attention to sequences Σ of strictly positive integers clearly involves no loss in generality; we do so throughout the present § because that makes certain formulas somewhat simpler. Denote by $\mathbb{N}$ *the set of integers* > 0 (N.B. *this is different from the notation of* §A, *where* $\mathbb{N}$ *also included* 0), and by Λ the sequence of *positive integers complementary to* Σ, i.e.,

$$\Lambda = \mathbb{N} \sim \Sigma.$$

For $t \geqslant 0$, we simply write $n(t)$ for the *number of elements of* Λ (N.B.!) in $[0, t]$. Put*

$$C(z) = \prod_{n \in \Lambda} \left(1 - \frac{z^2}{n^2}\right);$$

in the present situation

$$\frac{n(t)}{t} \longrightarrow 1 - D \quad \text{for} \quad t \longrightarrow \infty$$

and on account of this, $C(z)$ turns out to be an entire function of exponential type with quite regular behaviour.

Problem 29

(a) By writing $|\log C(z)|$ as a Stieltjes integral and integrating by parts, show that

$$\frac{\log|C(iy)|}{|y|} \longrightarrow \pi(1 - D)$$

for $y \longrightarrow \pm\infty$

(b) Show that for $x > 0$,

$$\log|C(x)| = 2\int_0^1 \left(\frac{n(x\tau)}{\tau} - \tau n\left(\frac{x}{\tau}\right)\right)\frac{d\tau}{1 - \tau^2}.$$

(Hint: First write the left side as a Stieltjes integral, then integrate by parts. Make appropriate changes of variable in the resulting expression.)

(c) Hence show that for $x > 0$,

$$\log|C(x)| \leqslant 2n(x)\log\frac{1}{\gamma} + 2\int_0^\gamma \left(\frac{n(x\tau)}{\tau} - \tau n\left(\frac{x}{\tau}\right)\right)\frac{d\tau}{1 - \tau^2},$$

with γ any number between 0 and 1.

* When $D = 1$, the complementary sequence Λ has density zero and may even be empty. In the last circumstance we take $C(z) \equiv 1$; the function $f(w)$ figuring in the construction given below then reduces simply to $w/\pi(1 + w)$.

(d) By making an appropriate choice of the number γ in (c), show that $\log|C(x)| \leqslant \varepsilon x$ for large enough x, $\varepsilon > 0$ being arbitrary.

(e) Use an appropriate Phragmén–Lindelöf argument to deduce from (a) and (d) that

$$\limsup_{r\to\infty} \frac{\log|C(re^{i\vartheta})|}{r} \leqslant \pi(1-D)|\sin\vartheta|.$$

(f) Show that in fact

$$\frac{\log|C(n)|}{n} \longrightarrow 0 \quad \text{for} \quad n \longrightarrow \infty \text{ in } \Sigma,$$

and that we have *equality* in the result of (e).
(Hint: Form the function

$$K(z) = \prod_{n\in\Sigma}\left(1-\frac{z^2}{n^2}\right);$$

then, as in (e),

$$\limsup_{r\to\infty} \frac{\log|K(re^{i\vartheta})|}{r} \leqslant \pi D|\sin\vartheta|.$$

Show that the same result holds if $K(re^{i\vartheta})$ is replaced by $K'(re^{i\vartheta})$. Observe that

$$\pi z K(z)C(z) = \sin\pi z.$$

Look at the derivative of the left-hand side at points $n\in\Sigma$.)

We are going to use the function $C(z)$ to construct a power series

$$\sum_{n\in\Sigma} a_n w^n$$

having *radius of convergence* 1, and representing a function which can be *analytically continued into the whole sector* $|\arg w| < \pi D$.

Start by putting

$$f(w) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{C(\zeta)}{\sin\pi\zeta} w^{\zeta}\, d\zeta$$

for $|\arg w| < \pi D$. Given any $\varepsilon > 0$, we see, by part (e) of the above problem, that

$$\left|\frac{C(\frac{1}{2}+i\eta)}{\sin\pi(\frac{1}{2}+i\eta)}\right| \leqslant \frac{\text{const.}}{\cosh\pi\eta} e^{(\pi(1-D)+\varepsilon)|\eta|}$$

for real η, where the constant on the right depends on ε. At the same time,

$$|w^{\frac{1}{2}+i\eta}| = |w|^{1/2}e^{-\eta \arg w},$$

so the above integral *converges absolutely and uniformly* for w ranging over any bounded part of the sector

$$|\arg w| \leqslant \pi D - 2\varepsilon.$$

The function $f(w)$ is hence analytic in the interior of that sector, and thus finally for

$$|\arg w| < \pi D,$$

since $\varepsilon > 0$ was arbitrary.

We proceed now to obtain a series expansion *in powers of* w for $f(w)$, valid for w of small modulus with $|\arg w| < \pi D$. For this purpose the *method of residues* is used. Taking a large *integer* R, let us consider the integral

$$\frac{1}{2\pi i}\int_{\Gamma_R} \frac{C(\zeta)}{\sin \pi\zeta} w^{\zeta}\, d\zeta$$

around the following contour Γ_R:

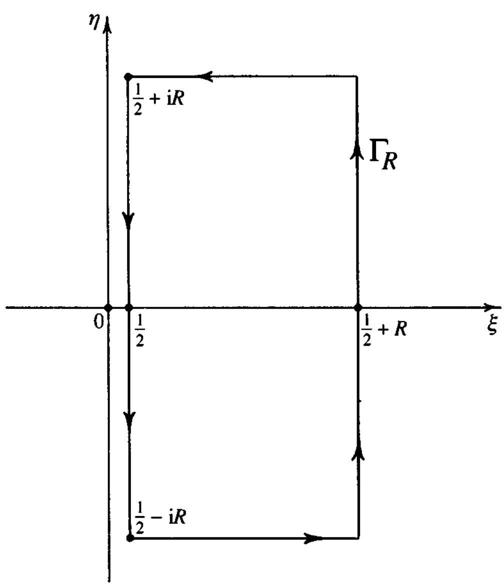


Figure 160

On the *top horizontal* side of Γ_R,

$$\left|\frac{C(\zeta)}{\sin \pi\zeta}\right| \quad \leqslant \quad \text{const.} \frac{e^{(\pi(1-D)+\varepsilon)R}}{e^{\pi R}} \quad = \quad \text{const.}\, e^{(\varepsilon - \pi D)R}$$

by part (e) of our problem. Here, $\varepsilon > 0$ is arbitrary, and the constant depends on it. The same estimate holds on the *lower horizontal* side of Γ_R. Also, if $|w| < 1$ and

$$|\arg w| \quad < \quad \pi D - 2\varepsilon,$$

we have

$$|w^\zeta| \quad \leqslant \quad e^{(\pi D - 2\varepsilon)R}$$

for ζ on the *horizontal sides of* Γ_R; in this circumstance the contribution of the horizontal sides to the contour integral is thus

$$\leqslant \quad \text{const.}\, R e^{-\varepsilon R}$$

in absolute value, and *that tends to zero as* $R \longrightarrow \infty$.

Along the *right vertical* side of Γ_R, by part (e) of the problem,

$$\left|\frac{C(\zeta)}{\sin \pi\zeta}\right| \quad \leqslant \quad \text{const.} \frac{e^{\pi(1-D)|\eta| + \varepsilon R}}{\cosh \pi\eta}$$

with $\varepsilon > 0$ arbitrary as before (we write as usual $\zeta = \xi + i\eta$). For $|\arg w| \; < \; \pi D - 2\varepsilon$ and ζ on that side,

$$|w^\zeta| \quad \leqslant \quad |w|^R e^{(\pi D - 2\varepsilon)|\eta|},$$

so, if also $|w| < e^{-2\varepsilon}$, the contribution of the *right vertical* side of Γ_R to the contour integral is in absolute value

$$\leqslant \quad \text{const.}\, e^{-\varepsilon R} \int_{-\infty}^{\infty} e^{-2\varepsilon|\eta|}\, d\eta,$$

and *this tends to zero as* $R \longrightarrow \infty$.

Putting together the two results just found, we see that

$$\frac{1}{2\pi i} \int_{\Gamma_R} \frac{C(\zeta)}{\sin \pi\zeta} w^\zeta d\zeta \quad \longrightarrow \quad -\frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{C(\zeta)}{\sin \pi\zeta} w^\zeta d\zeta \quad = \quad -f(w)$$

as $R \longrightarrow \infty$ for $|w| < 1$ and $|\arg w| < \pi D$, since $\varepsilon > 0$ is arbitrary.

By the residue theorem we have, however (taking $|\arg w| \leqslant \pi$, say),

$$\frac{1}{2\pi i} \int_{\Gamma_R} \frac{C(\zeta)}{\sin \pi\zeta} w^\zeta d\zeta \quad = \quad \frac{1}{\pi} \sum_{n=1}^{R} (-1)^n C(n) w^n$$

Here, $C(n) = 0$ for $n \in \Lambda = \mathbb{N} \sim \Sigma$, and, *by part* (f) *of the above problem*, the power series

$$\frac{1}{\pi}\sum_{n\in\Sigma}(-1)^n C(n) w^n$$

has *radius of convergence* 1. We thus see that

$$\lim_{R\to\infty}\frac{1}{2\pi i}\int_{\Gamma_R}\frac{C(\zeta)}{\sin\pi\zeta}w^{\zeta}\,d\zeta$$

equals the sum of that power series – call it $g(w)$ – for $|w| < 1$ and $|\arg w| < \pi D$. In that region, $g(w)$ must then coincide with $-f(w)$ by the calculation of the preceding limit just made. For $|w| < 1$ and $|\arg w| < \pi D$, we therefore have

$$f(w) \;=\; -\frac{1}{\pi}\sum_{n\in\Sigma}(-1)^n C(n) w^n$$

This relation furnishes an *analytic continuation* of $-g(w)$, analytic in $|w| < 1$, *to the whole sector* $|\arg w| < \pi D$ where, as we have seen, $f(w)$ is analytic. The power series on the right has radius of convergence 1.

For our measurable sequence Σ, D_Σ^* and D coincide. Hence *Pólya's gap theorem cannot be improved* in the case of such Σ.

2. General case; Σ not measurable. Beginning of Fuchs' construction

As stated at the beginning of this §, the converse to Pólya's gap theorem holds for *any* sequence Σ of positive integers, and for non-measurable Σ, the critical size for singularity-free arcs on the circle of convergence is $2\pi D_\Sigma^*$ radians, where D_Σ^* is the *maximum density* of Σ. This remarkable extension of the preceding article's result is not generally known. Malliavin makes passing mention of it in his 1957 *Illinois Journal* paper (one exceedingly difficult to read, by the way), but it really goes back to a publication of W. Fuchs in the 1954 *Proceedings* of the Edinburgh mathematical society, being entirely dependent on the beautiful construction given there. Fuchs, however, does not mention this (almost immediate) application of his construction in that paper.

The treatment for the general case involves a contour integral like the one used in the preceding article. Now, however, we cannot make do with just an entire function of exponential type like $C(z)$, but need another more complicated one besides. The latter, analytic and of exponential type in the *right half plane* (but *not* entire), is obtained by means of Fuchs' construction.

We start with a *non-measurable* sequence Σ of *strictly positive* integers (this last being no real restriction), and assume, throughout the remaining articles of the present §, that

$$D_\Sigma^* = \lim_{\lambda \to 1-} \left(\limsup_{r \to \infty} \frac{n_\Sigma(r) - n_\Sigma(\lambda r)}{(1-\lambda)r} \right)$$

is > 0. By the second theorem of §E.3, Chapter III, we know that Σ is included in a *measurable* sequence Σ^* of positive numbers with *density* D_Σ^*. *In the present case, we may take* $\Sigma^* \subseteq \mathbb{N}$. Indeed, since $\Sigma \subseteq \mathbb{N}$, D_Σ^* is certainly $\leqslant 1$. If $D_\Sigma^* = 1$, we can just put $\Sigma^* = \mathbb{N}$. A glance at the construction used in proving the theorem referred to shows that *the choice of new elements to be adjoined to* Σ *so as to make up* Σ^* *is fairly arbitrary, and that when* $D_\Sigma^* < 1$ *we may always take them to be distinct positive integers*. Here, this will yield a sequence $\Sigma^* \subseteq \mathbb{N}$ when $D_\Sigma^* < 1$.

Having obtained $\Sigma^* \subseteq \mathbb{N}$, we take the *complement*

$$\Lambda_1 = \mathbb{N} \sim \Sigma^*;$$

since Σ^* is *measurable*, so is Λ_1, and Λ_1 has *density* $1 - D_\Sigma^*$. The complement of Σ in $\mathbb{N}$ consists of Λ_1 together with another sequence

$$\Lambda_0 = \Sigma^* \sim \Sigma$$

*distinct** from Λ_1; *most of the work in the rest of this § will be with* Λ_0. For $t \geqslant 0$ we *denote by* $n(t)$ *the number of elements of* Λ_0 *in* $[0, t]$. (We write $n(t)$ instead of $n_{\Lambda_0}(t)$ in order to simplify the notation.) If $n_{\Sigma^*}(t)$ denotes the number of points of Σ^* in $[0, t]$, we have

$$n_{\Sigma^*}(t) = n(t) + n_\Sigma(t),$$

so, since

$$\frac{n_{\Sigma^*}(t)}{t} \longrightarrow D_\Sigma^* \quad \text{for} \quad t \longrightarrow \infty,$$

the relation

$$\boxed{\lim_{\lambda \to 1-} \left(\liminf_{r \to \infty} \frac{n(r) - n(\lambda r)}{(1-\lambda)r} \right) = 0}$$

* The sequence Λ_0 is certainly non-void and indeed infinite since Σ is *non-measurable*, as we are *assuming* throughout this and the next 6 articles. But Λ_1, of density $1 - D_\Sigma^*$, may even be *empty* when $D_\Sigma^* = 1$ (if we then take $\Sigma^* = \mathbb{N}$).

must hold, in view of the above formula for D_Σ^*. We may say that the sequence $\Lambda_0 \subseteq \mathbb{N}$ has *minimum density zero.*

Lemma. *Given* $\varepsilon > 0$, *there is an increasing sequence of numbers* X_j *tending to* ∞ *and an* $\alpha > 0$, *both depending on* ε, *such that*

$$n(x) - n(X_j) \leqslant \frac{\varepsilon}{2}(x - X_j) \quad \textit{for} \quad X_j \leqslant x \leqslant (1+\alpha)X_j.$$

Proof. For a certain fixed $c > 0$ we have, with $\lambda = 1/(1+c)$,

$$\liminf_{r\to\infty} \frac{n(r) - n(\lambda r)}{(1-\lambda)r} < \frac{\varepsilon}{4}$$

by the above boxed relation. There are hence *arbitrarily large* numbers R such that

$$\frac{n((1+c)R) - n(R)}{cR} < \frac{\varepsilon}{4}.$$

Take such a number R. It is claimed that if the integer M is *large* enough (*independently* of R) and we put

$$1 + \alpha = (1+c)^{1/M},$$

there exists an X,

$$R \leqslant X \leqslant (1+\alpha)^{M-1}R,$$

such that

$$n(x) - n(X) \leqslant \frac{\varepsilon}{2}(x - X) \quad \text{for } X \leqslant x \leqslant (1+\alpha)X.$$

This assertion, once verified, *will establish the lemma*, for we can then take a sequence of numbers R tending to ∞ and choose a corresponding sequence $\{X_j\}$ of numbers X.

Suppose, for some large integer M and for α related to it by the above formula, that *there is no such* X. There must then be a number x_1, $R \leqslant x_1 \leqslant (1+\alpha)R$, with

$$n(x_1) - n(R) > \frac{\varepsilon}{2}(x_1 - R).$$

This certainly makes $n(x_1) \geqslant n(R) + 1$ since $n(t)$ increases by 1 at each of its jumps. By the same token, there is an x_2, $x_1 \leqslant x_2 \leqslant (1+\alpha)x_1$, with

$$n(x_2) - n(x_1) \;>\; \frac{\varepsilon}{2}(x_2 - x_1)$$

(so in particular $n(x_2) \geqslant n(x_1) + 1$). The process continues, yielding x_3, $x_2 \leqslant x_3 \leqslant (1+\alpha)x_2$, x_4, and so forth, with

$$n(x_{k+1}) - n(x_k) \;>\; \frac{\varepsilon}{2}(x_{k+1} - x_k),$$

as long as the number x_k already obtained is $\leqslant (1+\alpha)^{M-1}R$. Since $n(x_{k+1}) - n(x_k) \geqslant 1$, *$x_k$ cannot remain* $\leqslant (1+\alpha)^{M-1}R$ *indefinitely* (we must eventually have $n(x_k) > n((1+\alpha)^{M-1}R))$). Let x_l be the *last* x_k which is $\leqslant (1+\alpha)^{M-1}R$; then we can still get an x_{l+1} between $(1+\alpha)^{M-1}R$ and $(1+\alpha)^M R$, such that

$$n(x_{l+1}) - n(x_l) \;>\; \frac{\varepsilon}{2}(x_{l+1} - x_l).$$

Adding to this the corresponding inequalities already obtained, we get

$$n(x_{l+1}) - n(R) \;>\; \frac{\varepsilon}{2}(x_{l+1} - R).$$

Since $(1+\alpha)^M = 1 + c$, $x_{l+1} \leqslant (1+c)R$, so

$$n((1+c)R) \;\geqslant\; n(x_{l+1}).$$

And

$$x_{l+1} - R \;\geqslant\; (1+\alpha)^{M-1}R - R \;=\; \frac{(1+c)^{(M-1)/M} - 1}{c}\cdot cR.$$

The relation just found therefore implies that

$$\frac{n((1+c)R) - n(R)}{cR} \;>\; \frac{(1+c)^{(M-1)/M} - 1}{c}\cdot\frac{\varepsilon}{2}.$$

However, if M is *large enough* (depending *only* on c and not on R !), we have

$$\frac{(1+c)^{(M-1)/M} - 1}{c} \;>\; \frac{1}{2}.$$

This would make the *left-hand side* of the previous relation $> \varepsilon/4$, *in contradiction with our choice of the number R*. For such large M, then, a number X with the properties specified above *must exist*. This establishes our claim, and proves the lemma.

Lemma. *Given $\varepsilon > 0$, let $\alpha > 0$ and $X = X_j$ be as in the statement of the previous lemma. There is then a β, $\alpha/3 \leqslant \beta \leqslant \alpha$, such that*

$$n((1+\beta)X) - n(x) \leqslant 2\varepsilon((1+\beta)X - x) \quad \text{for} \quad X \leqslant x \leqslant (1+\beta)X.$$

Proof. By the argument used to prove the lemma about Bernstein intervals near the beginning of §B.2, Chapter VIII.

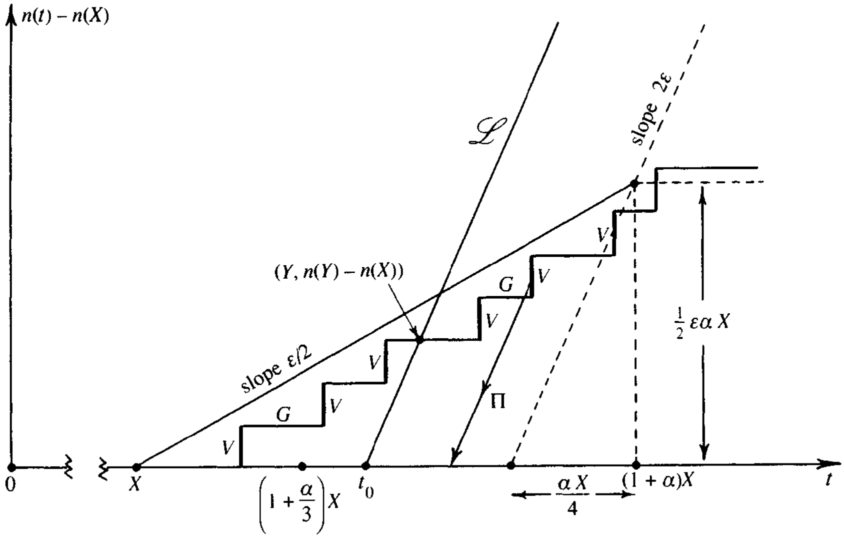


Figure 161

Denote by G the *graph* of $n(t) - n(X)$ vs. t for $X \leqslant t \leqslant (1+\alpha)X$, and by V the union of the *vertical portions* of G (corresponding to the *jumps* of $n(t)$). Let Π be the operation of *downward projection, along a line of slope* 2ε, *onto the t-axis*. Then, since $2\varepsilon > \varepsilon/2$,

$$\Pi(V) \subseteq [X, (1+\alpha)X],$$

and we see from the figure that

$$|\Pi(V)| \leqslant \frac{1}{2\varepsilon}\cdot\frac{\varepsilon}{2}\alpha X = \frac{\alpha X}{4}.$$

Therefore $\Pi(V)$ *cannot cover the segment* $[(1+\alpha/3)X, (1+3\alpha/4)X]$ *of length* $(5/12)\alpha X$, *so there is a* t_0 *in that segment not belonging to* $\Pi(V)$.

The line $\mathscr{L}$ of slope 2ε through $(t_0, 0)$ must, from the figure, cut G, say at a point $(Y, n(Y) - n(X))$, with $(1+\alpha/3)X \leqslant Y \leqslant (1+\alpha)X$. Since $\mathscr{L}$, passing through that point, does not touch any part of V (t_0

being $\notin \Pi(V)$), we have

$$n(Y) - n(t) \leqslant 2\varepsilon(Y - t) \quad \text{for} \quad X \leqslant t \leqslant Y.$$

Calling $Y/X = 1 + \beta$, we have the lemma.

Let us combine the two results just proved. We see that, given $\varepsilon > 0$, there are *two sequences* $\{X_j\}$ and $\{Y_j\}$ tending to ∞ and an $\alpha > 0$ depending on ε, such that

$$1 + \frac{\alpha}{3} \leqslant \frac{Y_j}{X_j} \leqslant 1 + \alpha$$

and that the *simultaneous relations*

$$\begin{cases} n(x) - n(X_j) \leqslant 2\varepsilon(x - X_j), \\ n(Y_j) - n(x) \leqslant 2\varepsilon(Y_j - x) \end{cases}$$

hold on each of the intervals $[X_j, Y_j]$. (Of course, the first of these relations can be replaced by an even better one!).

We henceforth assume that $\varepsilon \leqslant 1/6$. *That* being granted, we can, at the cost of ending with slightly worse inequalities, *modify* the above constructions *so as to make the* X_j *and* Y_j *half-odd integers*. To see this, we again use an idea from §B.2 of Chapter VIII.

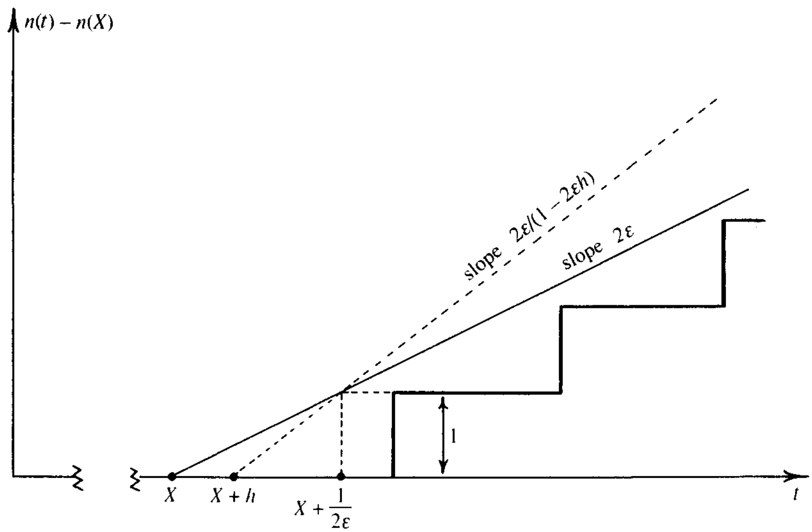


Figure 162

Choosing one of the intervals $[X_j, Y_j]$ described above, we drop the index j, writing simply X for X_j and Y for Y_j. *The function* $n(t)$ *increases* *by* 1 *at each of its jumps.* Therefore, since

$$n(t) - n(X) \leqslant 2\varepsilon(t - X)$$

for $X \leqslant t \leqslant Y$, we must have

$$n(t) = n(X) \quad \text{for } X \leqslant t < X + \frac{1}{2\varepsilon}.$$

If $0 \leqslant h < 1/2\varepsilon$, the line of slope $2\varepsilon/(1 - 2\varepsilon h)$ through $(X + h,\ 0)$ must then *lie entirely above* the graph of $n(t) - n(X)$ vs. t for $X + h \leqslant t \leqslant Y$, as the above figure shows. Choosing h, $0 \leqslant h < 1$, *so as to make*

$$X' = X + h$$

a half-odd integer, we thus have

$$n(t) - n(X') \leqslant 3\varepsilon(t - X') \quad \text{for } X' \leqslant t \leqslant Y$$

because for such h,

$$\frac{2\varepsilon}{1 - 2\varepsilon h} \leqslant 3\varepsilon,$$

ε being $< 1/6$.

The same kind of reasoning shows that if we take a *half-odd integer* Y' with $Y - 1 < Y' \leqslant Y$, we will still have

$$n(Y') - n(t) \leqslant 3\varepsilon(Y' - t) \quad \text{for} \quad X' \leqslant t \leqslant Y'.$$

Since

$$1 + \frac{\alpha}{3} \leqslant \frac{Y}{X} \leqslant 1 + \alpha,$$

we have

$$\frac{Y'}{X'} \leqslant 1 + \alpha$$

and also

$$\frac{Y'}{X'} \geqslant \frac{Y - 1}{X + 1} \geqslant 1 + \frac{\alpha}{4}$$

as long as X *is large.*

From now on, we work with the intervals $[X', Y']$, *and write* $X = X_j$ *instead of* X' *and* $Y = Y_j$ *instead of* Y'. Also, since ε, $0 < \varepsilon < 1/6$, is *arbitrary, we may just as well write* ε *instead of* 3ε. By the above considerations we have then proved the following

Theorem. *Given* ε, $0<\varepsilon<1/2$, *there are sequences* $\{X_j\}$ *and* $\{Y_j\}$ *of half-odd integers tending to* ∞ *and an* $\alpha>0$ *such that*

$$\left(1+\frac{\alpha}{4}\right)X_j \;\leqslant\; Y_j \;\leqslant\; (1+\alpha)X_j$$

and that

$$\left.\begin{aligned} n(t)-n(X_j) &\leqslant \varepsilon(t-X_j) \\ n(Y_j)-n(t) &\leqslant \varepsilon(Y_j-t) \end{aligned}\right\} \textit{ for } X_j\leqslant t\leqslant Y_j.$$

When X_j is *large*, $Y_j - X_j \geqslant (\alpha/4)X_j$ is *also* large, so the segment $[X_j, Y_j]$ *contains lots of integers.* Recalling the meaning of $n(t)$, we see by the theorem that if the intervals $[X_j, Y_j]$ are constructed for a *small* value of ε, *most of the integers in them will not belong to our sequence* Λ_0.

The purpose of Fuchs' construction is to obtain a function $\Phi(z)$, analytic in $\Re z>0$ and of *small exponential type there*, such that, for large $n\in\Lambda_0$, $|\Phi(n)|^{1/n}$ is *at most* $e^{-\delta}$ times the limsup of $|\Phi(m)|^{1/m}$ *for m tending to* ∞ in Σ, δ being some constant >0. The function $\Phi(z)$ is constructed so as to *vanish at the points of* Λ_0 *belonging to a sparse sequence of the intervals* $[X_j, Y_j]$, and so as to make

$$|\Phi(x)| \;\leqslant\; \text{const.}e^{(k-\delta)x}$$

for $x>0$ *outside* of those intervals, while

$$|\Phi(m)| \;\geqslant\; \text{const.}e^{km}$$

for most of the integers *m inside* them *that don't belong to* Λ_0. As we have just observed, there will be plenty of the latter.

3. Bringing in the gamma function

For obtaining the function $\Phi(z)$ mentioned at the end of the preceding article, procedures yielding entire functions will not work.* Fuchs' idea is to construct $\Phi(z)$ by using products of the form

$$\prod\left(\frac{1-z/n}{1+z/n}\right)e^{2z/n}$$

taken over *certain* sets of positive integers *n*; these are analytic in the *right* half plane, but have poles in the left half plane. The exponential factors ensure convergence.

* at least, so it seems

The prototype of such a product is

$$\prod_{n=1}^{\infty}\left(\frac{1-z/n}{1+z/n}\right)e^{2z/n};$$

this can be expressed in terms of the gamma function.

$\Gamma(z)$ is the reciprocal of an entire function of order 1 defined by means of a certain infinite product designed to make

$$\Gamma(x+1) \;=\; \int_0^{\infty} t^x e^{-t}\,dt$$

for real $x > -1$. Starting from this formula, successive integrations by parts yield

$$\Gamma(x+1) \;=\; \frac{1}{x+1}\int_0^{\infty} t^{x+1}e^{-t}\,dt \;=\; \frac{1}{(x+1)(x+2)}\int_0^{\infty} t^{x+2}e^{-t}\,dt$$

$$= \;\cdot\;\cdot\;\cdot\; = \;\frac{\int_0^{\infty} t^{x+m}e^{-t}\,dt}{(1+x)(2+x)\cdots(m+x)}.$$

The last expression can be rewritten

$$\frac{\exp\left(-\sum_{k=1}^{m}(x/k)\right)}{\prod_{k=1}^{m}\left(1+\frac{x}{k}\right)e^{-x/k}}\cdot\frac{\int_0^{\infty} t^{x+m}e^{-t}\,dt}{m!}$$

One has, of course, $m! = \int_0^{\infty} t^m e^{-t}\,dt$, and, for real x tending to ∞, Stirling's formula,

$$\int_0^{\infty} t^x e^{-t}\,dt \;\sim\; \sqrt{(2\pi x)}\cdot\left(\frac{x}{e}\right)^x$$

is valid. (The latter may be proved by applying Laplace's method to the integral on the left.) Using these relations to simplify the expression just written, we find that

$$\Gamma(x+1) \;=\; \lim_{m\to\infty}\frac{m^x\exp\left(-\sum_{k=1}^{m}(x/k)\right)}{\prod_{k=1}^{m}\left(1+\frac{x}{k}\right)e^{-x/k}}$$

We have

$$\exp\left(x\sum_{k=1}^{m}(1/k)\right)m^{-x} \;=\; \exp\left\{x\left(\frac{1}{m} + \sum_{k=1}^{m-1}\frac{1}{k} - \log m\right)\right\}.$$

By drawing a picture, one sees that as $m \to \infty$,

$$\sum_{k=1}^{m-1} \frac{1}{k} \quad - \quad \log m$$

increases steadily to a certain finite limit C (called Euler's constant). Therefore, by the preceding formula,

$$\Gamma(x+1) \quad = \quad 1 \Big/ \exp(Cx) \prod_{m=1}^{\infty} \left(1 + \frac{x}{k}\right) e^{-x/k}.$$

For general complex z, one just *defines*

$$\boxed{\Gamma(z+1) \quad = \quad 1 \Big/ \exp(Cz) \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}.}$$

By a slight adaptation of the work in Chapter III, §§A, B one easily shows using this formula that

$$|1/\Gamma(z+1)| \quad \leqslant \quad K_\varepsilon \exp(|z|^{1+\varepsilon})$$

for each $\varepsilon > 0$. ($1/\Gamma(z+1)$ is NOT, by the way, *of exponential type*, on account of *Lindelöf's theorem* if for no other reason!)

Since

$$\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right),$$

we have

$$\prod_{n=1}^{\infty} \left(\frac{1 - z/n}{1 + z/n}\right) e^{2z/n} \quad = \quad e^{2Cz} \frac{\sin \pi z}{\pi z} (\Gamma(z+1))^2.$$

Use of this relation together with Stirling's formula for complex z enables us to get a good grip on the behaviour of the right-hand product.

Problem 30

To extend Stirling's formula to complex values of z in the right half plane. Write

$$g(z) \quad = \quad \sqrt{(2\pi z)} \cdot \left(\frac{z}{e}\right)^z \Big/ \Gamma(z+1).$$

(a) Show that $g(z)$ is of order 1 – i.e., that

$$|g(z)| \quad \leqslant \quad M_\varepsilon \exp(|z|^{1+\varepsilon})$$

for each $\varepsilon > 0$ – in any open sector of the form $|\arg z| \leqslant \pi - \delta$, $\delta > 0$, and is continuous up to the boundary of such a sector.

(b) Show that for real y,

$$|g(iy)| \;=\; \sqrt{2}\cdot e^{-(\pi|y|/2)}\sqrt{(\sinh \pi|y|)}.$$

(c) Hence show that $g(z)$ is bounded for $\Re z \geqslant 0$. (Hint: Use Stirling's formula to estimate $g(x)$ for $x > 0$. Then use Phragmén–Lindelöf in the *first* and *fourth* quadrants.)

(d) Hence show that $g(z) \longrightarrow 1$ uniformly for z tending to ∞ in any sector $|\arg z| \leqslant \pi/2 - \delta$, $\delta > 0$. (Hint: $g(x) \longrightarrow 1$ for $x \longrightarrow \infty$ by Stirling's formula. In view of (c), a theorem of Lindelöf may be applied.)

(e) Show that $g(re^{\pm(2\pi i/3)}) \longrightarrow 1$ as $r \longrightarrow \infty$. (Hint: First show that $\Gamma(1+z)\Gamma(1-z) = \sin \pi z/\pi z$. Use this in conjunction with the result from (d), noting that $e^{\pm(2\pi i/3)} = -e^{\mp(\pi i/3)}$.)

(f) Hence show that $g(z) \longrightarrow 1$ uniformly for z tending to ∞ in any sector of the form

$$|\arg z| \;\leqslant\; \tfrac{2}{3}\pi - \delta, \quad \delta > 0.$$

(Hint: Use Phragmén–Lindelöf and the theorem of Lindelöf referred to in the hint to part (d) again.)

From part (f) of this problem we have in particular

$$\Gamma(z+1) \;\sim\; \sqrt{(2\pi z)}\cdot\left(\frac{z}{e}\right)^{z}$$

for $|\arg z| \leqslant \pi/2$ and $|z|$ large. This means that

$$\boxed{z^{-2z}e^{2(1-C)z}\prod_{n=1}^{\infty}\left(\frac{1-z/n}{1+z/n}\right)e^{2z/n} \;\sim\; 2\sin \pi z}$$

for $\Re z \geqslant 0$ when $|z|$ is large. The expression on the left is thus certainly of exponential type π in the right half plane.

Fuchs takes* the intervals $[X_j, Y_j]$ constructed in the previous article,

* His construction is, of course, needed by us only for the case of *non-measurable* Σ, when the sequence Λ_0 is certainly *available*, and indeed *infinite*.

corresponding to a *small* value of $\varepsilon > 0$. He then fixes a *large* integer L with, however, $L < 1/\varepsilon$, and forms the function

$$F(z/L) \quad = \quad \left(\frac{z}{L}\right)^{-2z/L} e^{2(1-C)z/L} \prod_{n=1}^{\infty} \left(\frac{1 - z/nL}{1 + z/nL}\right) e^{2z/nL}.$$

According to the above boxed formula, this has very regular behaviour in the right half plane, and is of exponential type π/L there.

Fuchs' idea is now to *modify* the product on the right side of this last relation by *throwing away* the factors

$$\left(\frac{1 - z/nL}{1 + z/nL}\right) e^{2z/nL}$$

corresponding to the n for which nL belongs to *certain* of the intervals $[X_j, Y_j]$. *Those* factors are *replaced* by *others* of the form

$$\left(\frac{1 - z/\lambda}{1 + z/\lambda}\right) e^{2z/\lambda}$$

corresponding to the $\lambda \in \Lambda_0$ belonging to the *same* intervals $[X_j, Y_j]$. This alteration of $F(z/L)$ produces a new function, vanishing at the points of Λ_0 lying in certain of the $[X_j, Y_j]$. We have to see how much the behaviour of the latter differs from that of the former.

4. Formation of the group products $R_j(z)$

We want, then, to *remove* from the product

$$\prod_{n=1}^{\infty} \left(\frac{1 - z/nL}{1 + z/nL}\right) e^{2z/nL}$$

the group of factors

$$\prod_{nL \in [X_j, Y_j]} \left(\frac{1 - z/nL}{1 + z/nL}\right) e^{2z/nL}$$

and to *insert*

$$\prod_{\lambda \in \Lambda_0 \cap [X_j, Y_j]} \left(\frac{1 - z/\lambda}{1 + z/\lambda}\right) e^{2z/\lambda}$$

in their place, doing this for infinitely many of the intervals $[X_j, Y_j]$ constructed in article 2, corresponding to some fixed small $\varepsilon > 0$. This amounts to multiplying our original product by expressions of the form

$$\prod_{nL \in [X_j, Y_j]} \left(\frac{nL + z}{nL - z}\right) e^{-2z/nL} \prod_{\lambda \in \Lambda_0 \cap [X_j, Y_j]} \left(\frac{\lambda - z}{\lambda + z}\right) e^{2z/\lambda}.$$

As we said at the end of the preceding article, Fuchs takes the integer $L < 1/\varepsilon$. Therefore, since

$$n(Y_j) - n(X_j) \leqslant \varepsilon(Y_j - X_j)$$

by the theorem of article 2, there are *fewer* $\lambda \in \Lambda_0$ than integral multiples of L in $[X_j, Y_j]$, and the exponential factors in the above expression do not multiply out to 1. Their presence would cause difficulties later on, and we would like to get rid of them.

For this reason, Fuchs brings in a small multiple a_j of X_j, chosen so as to make

$$\sum_{nL\in[X_j,Y_j]} \frac{1}{nL} - \sum_{\lambda\in\Lambda_0\cap[X_j,Y_j]} \frac{1}{\lambda} = \frac{q_j}{a_j}$$

with a *positive integer* q_j, and then *adjoins* to the previous expression an *additional dummy factor* of the form

$$\left(\frac{a_j - z}{a_j + z}\right)^{q_j} e^{2q_j z/a_j}$$

Once this is done, the exponential factors $e^{2q_j z/a_j}$, $e^{-2z/nL}$ and $e^{2z/\lambda}$ figuring in the resulting product cancel each other out.

The details in this step involve some easy estimates. *In this and the succeeding articles, when concentrating on any particular interval* $[X_j, Y_j]$, *we will simplify the notation by dropping the subscript j, writing just X for* X_j, *Y for* Y_j, *a for* a_j, *and so forth.*

By the theorem of article 2, $n(t) - n(X) \leqslant \varepsilon(t - X)$ for $X \leqslant t \leqslant Y$, and X and Y are *half-odd*, while the $\lambda \in \Lambda_0$ are *integers*. Hence,

$$\begin{aligned}\sum_{\lambda\in\Lambda_0\cap[X,Y]} \frac{1}{\lambda} &= \int_X^Y \frac{dn(t)}{t} = \frac{n(Y) - n(X)}{Y} + \int_X^Y \frac{n(t) - n(X)}{t^2}\,dt \\ &\leqslant \frac{\varepsilon(Y - X)}{Y} + \varepsilon\int_X^Y \frac{t - X}{t^2}\,dt = \varepsilon\log\left(\frac{Y}{X}\right).\end{aligned}$$

In the theorem just mentioned, $Y - X \geqslant (\alpha/4)X$ with a constant $\alpha > 0$ depending on ε. Therefore, if X is a *very large* X_j (*which we always assume henceforth*), there *will be* numbers $nL \in [X, Y]$, $n \in \mathbb{N}$, and then, as a simple picture shows,

$$\sum_{nL\in[X,Y]} \frac{1}{nL} \geqslant \frac{1}{L}\log\frac{Y}{X + L} \geqslant \frac{1}{L}\log\frac{Y}{X} - \frac{1}{X}.$$

Thus, when X is large,

$$\sum_{nL\in[X,Y]} \frac{1}{nL} - \sum_{\lambda\in\Lambda_0\cap[X,Y]} \frac{1}{\lambda} \geqslant \left(\frac{1}{L} - \varepsilon\right)\log\frac{Y}{X} - \frac{1}{X}.$$

Here, $1/L > \varepsilon$ and $Y \geqslant (1+\alpha/4)X$, so the right side is bounded below by a constant > 0 depending on L, ε and α for X large enough (again depending on those parameters).

Take now a small parameter $\eta > 0$ which is to remain *fixed* throughout all the following constructions – later on we will see how η is to be chosen. *Here*, we observe that if X is *large, there is a number a between $\eta X/2$ and ηX whose product with the left side of the preceding inequality is an integer.* Picking such an a, we call the corresponding integer q, and we have

$$\boxed{\frac{q}{a} = \sum_{nL\in[X,Y]} \frac{1}{nL} - \sum_{\lambda\in\Lambda_0\cap[X,Y]} \frac{1}{\lambda}.}$$

Because $\eta X/2 \leqslant a \leqslant \eta X$, the inequality

$$\sum_{X\leqslant nL\leqslant Y} \frac{1}{nL} \leqslant \frac{1}{X}\left(\frac{Y-X}{L}+1\right)$$

gives us the useful upper estimate

$$\boxed{q \leqslant \frac{\eta}{L}(Y-X)+\eta.}$$

Definition. We write

$$\boxed{R(z) = \left(\frac{a-z}{a+z}\right)^q \prod_{nL\in[X,Y]}\left(\frac{nL+z}{nL-z}\right) \prod_{\lambda\in\Lambda_0\cap[X,Y]}\left(\frac{\lambda-z}{\lambda+z}\right).}$$

When using the subscript j with X, Y, a and q, we also write $R_j(z)$ instead of $R(z)$.

Let us establish some simple properties of the group product $R(z)$. In the first place, *we can put* the aforementioned *exponential factors* (needed for convergence) *back* into $R(z)$ *if we want to*:

$$R(z) \;=\; \left(\frac{1-z/a}{1+z/a}\right)^q e^{2qz/a} \prod_{nL\in[X,Y]} \left(\frac{1+z/nL}{1-z/nL}\right) e^{-2z/nL}$$
$$\times \prod_{\lambda\in\Lambda_0\cap[X,Y]} \left(\frac{1-z/\lambda}{1+z/\lambda}\right) e^{2z/\lambda}.$$

This is so by the above boxed relation involving q/a.

In the second place, we have the

Lemma. *If $\eta > 0$ is chosen small enough (depending only on L and ε) and X is large,*

$$|R(x)| \;\leqslant\; 1 \quad \textit{for } 0 \leqslant x < a.$$

Proof.

$$\log|R(x)| \;=\; q\log\left|\frac{1-x/a}{1+x/a}\right| \;+\; \sum_{nL\in[X,Y]} \log\left|\frac{1+x/nL}{1-x/nL}\right|$$
$$+ \sum_{\lambda\in\Lambda_0\cap[X,Y]} \log\left|\frac{1-x/\lambda}{1+x/\lambda}\right|.$$

Since $a \leqslant \eta X < X$ we can, for $0 \leqslant x < a$, expand the logarithms in powers of x. Collecting terms, we find, thanks to the above boxed formula for q/a, that the *coefficient* of x *vanishes*, and we get

$$\log|R(x)| \;=\; \sum_{\substack{N=3\\ N\text{ odd}}}^{\infty} \left\{ \sum_{nL\in[X,Y]} \frac{1}{(nL)^N} - \sum_{\lambda\in\Lambda_0\cap[X,Y]} \frac{1}{\lambda^N} - \frac{q}{a^N} \right\} \cdot \frac{2x^N}{N}.$$

By a previous inequality for the *right side* of the boxed formula for q/a,

$$\frac{q}{a} \;\geqslant\; \left(\frac{1}{L} - \varepsilon\right) \log\frac{Y}{X} \;-\; \frac{1}{X}.$$

Hence, since $a \leqslant \eta X$,

$$\frac{q}{a^N} \;\geqslant\; \frac{\left(\frac{1}{L} - \varepsilon\right) X^{-(N-1)}}{\eta^{N-1}} \log\frac{Y}{X} \;-\; \frac{1}{\eta^{N-1} X^N}$$

for $N > 1$.

At the same time, we already have

$$\sum_{nL\in[X,Y]} \frac{1}{(nL)^N} \;\leqslant\; \frac{1}{X^N}\left(\frac{Y-X}{L}+1\right).$$

Thus, since $(1+\alpha/4)X \;\leqslant\; Y \;\leqslant\; (1+\alpha)X$,

$$\sum_{nL\in[X,Y]} \frac{1}{(nL)^N} \;\leqslant\; \left(\frac{\alpha}{L}+\frac{1}{X}\right)\cdot\frac{1}{X^{N-1}},$$

while, by the preceding calculation,

$$\frac{q}{a^N} \;\geqslant\; \left[\left(\frac{1}{L}-\varepsilon\right)\log\left(1+\frac{\alpha}{4}\right) \;-\; \frac{1}{X}\right]\cdot\frac{1}{(\eta X)^{N-1}}.$$

Since $1/L-\varepsilon \;>\; 0$, *for sufficiently small values of* η, *the second of these quantities exceeds the first for every* $N\geqslant 2$, as long as X is large enough. The required smallness of η is determined here by α, L and ε, and therefore really by the *last two* of these quantities, for α itself depends on ε.

Under the circumstances just described, *all the coefficients in the above power series expansion of* $\log|R(x)|$ *will be negative.* This makes $\log|R(x)|\leqslant 0$ for $0\leqslant x<a$,

Q.E.D.

Another result goes in the opposite direction.

Lemma. *If* $\eta>0$ *is taken* small enough (*depending* only *on* L *and* ε) *and* X *is large,*

$$|R(X/2)| \;>\; 1.$$

Proof.

$$\log|R(X/2)| \;=\; q\log\left|\frac{\frac{1}{2}X-a}{\frac{1}{2}X+a}\right| \;+\; \sum_{nL\in[X,Y]}\log\left|\frac{nL+\frac{1}{2}X}{nL-\frac{1}{2}X}\right| \;+\; \sum_{\lambda\in\Lambda_0\cap[X,Y]}\log\left|\frac{\lambda-\frac{1}{2}X}{\lambda+\frac{1}{2}X}\right|.$$

Since $a \;\leqslant\; \eta X$, the first term on the right is

$$\geqslant\; \eta\left(\frac{Y-X}{L}+1\right)\log\frac{1-2\eta}{1+2\eta}$$

by the previous boxed estimate on q.

Recall that in Fuchs' construction, L is *an integer*, Λ_0 consists of integers, and (by the theorem of article 2) X and Y are *half-odd integers.* The sum of the second and third terms on the right in the previous relation can therefore be written as

$$\int_X^Y \log\left|\frac{t+\frac{1}{2}X}{t-\frac{1}{2}X}\right| \mathrm{d}([t/L]-n(t))$$
$$= \left\{[Y/L]-[X/L]-(n(Y)-n(X))\right\}\log\left(\frac{2Y+X}{2Y-X}\right)$$
$$+ \int_X^Y \frac{4X}{4t^2-X^2}([t/L]-[X/L]-(n(t)-n(X)))\mathrm{d}t$$

(we are using the symbol $[p]$ to denote the *biggest integer* $\leqslant p$). We have

$$[t/L]-[X/L] \geqslant [(t-X)/L].$$

Also, $n(t)-n(X) \leqslant \varepsilon(t-X)$ for $X \leqslant t \leqslant Y$, *with the left side integer-valued*, so *in fact*

$$n(t)-n(X) \leqslant [\varepsilon(t-X)], \quad X \leqslant t \leqslant X.$$

Here $1/L > \varepsilon$, so surely $[(t-X)/L] \geqslant [\varepsilon(t-X)]$, and the right-hand integral in the last formula is positive. Since $(1+\alpha/4)X \leqslant Y \leqslant (1+\alpha)X$, we thus have

$$\int_X^Y \log\left|\frac{t+\frac{1}{2}X}{t-\frac{1}{2}X}\right| \mathrm{d}([t/L]-n(t))$$
$$\geqslant \left(\left(\frac{1}{L}-\varepsilon\right)(Y-X)-2\right)\log\frac{3+2\alpha}{1+2\alpha}.$$

Combining this estimate with the one previously obtained, we see that

$$\log|R(X/2)| \geqslant (Y-X)\left\{\left(\frac{1}{L}-\varepsilon\right)\log\frac{3+2\alpha}{1+2\alpha} - \frac{\eta}{L}\log\frac{1+2\eta}{1-2\eta}\right\}$$
$$- 2\log\frac{3+2\alpha}{1+2\alpha} - \eta\log\frac{1+2\eta}{1-2\eta}.$$

Because $Y-X \geqslant (\alpha/4)X$ and $1/L-\varepsilon > 0$, the right side will be *positive for all large X provided that* $\eta > 0$ *is sufficiently small* (depending on L, ε and α, hence on L and ε). This does it.

5. Behaviour of $(1/x)\log|(x-\lambda)/(x+\lambda)|$.

We are going to have to study $(1/x)\log|R_j(x)|$ for the products $R_j(z)$ constructed in the preceding article. For this purpose, frequent use will be made of the

Lemma. *If* $\lambda > 0$,

$$\frac{\partial}{\partial x}\left(\frac{1}{x}\log\left|\frac{x-\lambda}{x+\lambda}\right|\right) \quad \textit{is} \ < \ 0 \textit{ for } 0<x<\lambda \textit{ and } > \ 0 \textit{ for } x>\lambda.$$

Also,

$$\frac{\partial^2}{\partial\lambda\partial x}\left(\frac{1}{x}\log\left|\frac{x-\lambda}{x+\lambda}\right|\right) > 0 \quad \textit{for } x>0 \textit{ different from } \lambda.$$

Proof.

$$\frac{\partial}{\partial x}\left(\frac{1}{x}\log\left|\frac{x-\lambda}{x+\lambda}\right|\right) = \frac{1}{x^2}\log\left|\frac{x+\lambda}{x-\lambda}\right| + \frac{2\lambda}{x(x^2-\lambda^2)}.$$

The right side is $>$ 0 for $x > \lambda$. When $0 < x < \lambda$ we rewrite the right side as

$$\frac{1}{x^2}\left\{\log\left(\frac{1+\xi}{1-\xi}\right) - \frac{2\xi}{1-\xi^2}\right\}$$

with $\xi = x/\lambda$, and then expand the quantity in curly brackets in powers of ξ. This yields

$$\frac{2}{x^2}\left(\xi + \frac{\xi^3}{3} + \frac{\xi^5}{5} + \cdots - \xi - \xi^3 - \xi^5 - \cdots\right),$$

which is $<$ 0 since $0 < \xi < 1$.

Finally,

$$\frac{\partial^2}{\partial\lambda\partial x}\left(\frac{1}{x}\log\left|\frac{x-\lambda}{x+\lambda}\right|\right) = \frac{4x}{(\lambda^2-x^2)^2} > 0$$

for $x > 0$, $x \neq \lambda$.

We are done.

Corollary. *If* $0 < \lambda < \lambda'$,

$$\frac{1}{x}\log\left|\frac{x-\lambda}{x+\lambda}\right| - \frac{1}{x}\log\left|\frac{x-\lambda'}{x+\lambda'}\right|$$

is a decreasing function *of* x for $0 < x < \lambda$ *and for* $x > \lambda'$.

Proof. By the second derivative inequality from the lemma.

In like manner, we have the

Corollary. *If* $0 < x < x'$,

$$\frac{1}{x'}\log\left|\frac{x'-\lambda}{x'+\lambda}\right| - \frac{1}{x}\log\left|\frac{x-\lambda}{x+\lambda}\right|$$

is an increasing function *of* λ *for* $\lambda > x'$ *and for* $0 < \lambda < x$.

6. Behaviour of $(1/x)\log|R_j(x)|$ outside the interval $[X_j, Y_j]$

Turning now to the group products $R_j(z)$ constructed in article 4, we have the

Lemma. *If the parameter* $\eta > 0$ *is taken* sufficiently small *(depending only on* L *and* ε*),* $(1/x)\log|R_j(x)|$ *is decreasing for* $x \geqslant Y_j$ *provided that* X_j *is large enough.*

Proof. Dropping the subscript j, we have, for $x \geqslant Y$,

$$\frac{1}{x}\log|R(x)| = \frac{q}{x}\log\left(\frac{x-a}{x+a}\right) + \sum_{\lambda\in\Lambda_0\cap[X,Y]}\frac{1}{x}\log\left(\frac{x-\lambda}{x+\lambda}\right) - \sum_{nL\in[X,Y]}\frac{1}{x}\log\left(\frac{x-nL}{x+nL}\right).$$

We are going to make essential use of the property

$$n(Y) - n(t) \leqslant \varepsilon(Y-t), \quad X \leqslant t \leqslant Y$$

(see theorem of article 2). Since $1/L > \varepsilon$, we have a picture like the following:

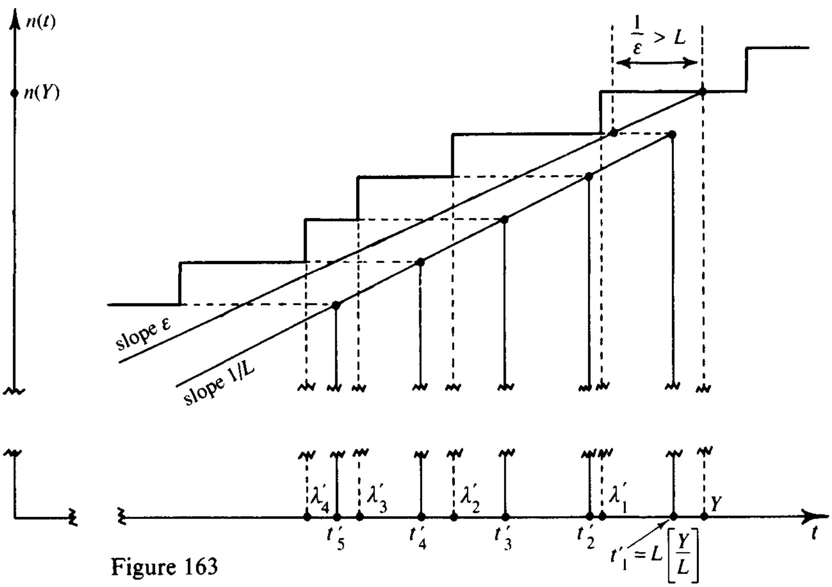


Figure 163

Number the members of Λ_0 in $[X, Y]$ *downwards*, calling the *largest* of those λ'_1, the *next largest* λ'_2, and so forth. We also denote by $t'_1 = L[Y/L]$ the *largest integral multiple* of L in $[X, Y]$, by t'_2 the *next largest one*, and so on. Since L and the members of Λ_0 are *integers* while Y is *half-odd* (theorem, article 2), we in fact have $\lambda'_1 < Y$ and $t'_1 < Y$.

By the above property of $n(t)$,

$$\lambda'_1 \leqslant Y - 1/\varepsilon,$$
$$\lambda'_2 \leqslant Y - 2/\varepsilon,$$

etc. Since, however, $L < 1/\varepsilon$, it is also true that

$$t'_1 > Y - 1/\varepsilon,$$
$$t'_2 > Y - 2/\varepsilon,$$

and so on.

We can *pair off* each of the λ'_k, $X \leqslant \lambda'_k \leqslant Y$, with t'_k, and *still have some t'_ks left over after all the λ'_k are taken care of* in this way. Indeed, there are *at most* $n(Y) - n(X) \leqslant \varepsilon(Y - X)$ of the $\lambda \in \Lambda_0$ in $[X, Y]$, X being half-odd – in fact there are at most $[\varepsilon(Y - X)]$ of them, since $n(Y) - n(X)$ is integer-valued. At the same time, there are at least $[(Y - X)/L]$ integral multiples of L in that interval. Since $1/L > \varepsilon$, there are *more* of the latter than the former, and, after pairing off each λ'_k with t'_k, there will *still be* at least

$$[(Y - X)/L] - [\varepsilon(Y - X)] \geqslant \left[\left(\frac{1}{L} - \varepsilon\right)(Y - X)\right]$$

integral multiples of L *left over* in $[X, Y]$.

Now from article 4,

$$q \leqslant \frac{\eta}{L}(Y - X) + \eta.$$

Since $Y - X \geqslant (\alpha/4)X$, we see that if $\eta/L < 1/L - \varepsilon$, $[(1/L - \varepsilon)(Y - X)]$ is *larger than* q, provided only that X is big enough. Under these circumstances, *there will be more than enough of the points t'_k left over to pair off with the q-fold point a, after each λ'_k has been paired with the t'_k* corresponding to it.

Write $n(Y) - n(X) = N$. Then, after the pairings just described, the above formula for $(1/x)\log|R(x)|$ can be rewritten thus:

$$\frac{1}{x}\log|R(x)| = \sum_{k=1}^{N}\left(\frac{1}{x}\log\left(\frac{x - \lambda'_k}{x + \lambda'_k}\right) - \frac{1}{x}\log\left(\frac{x - t'_k}{x + t'_k}\right)\right)$$
$$+ \sum_{k=N+1}^{N+q}\left(\frac{1}{x}\log\left(\frac{x - a}{x + a}\right) - \frac{1}{x}\log\left(\frac{x - t'_k}{x + t'_k}\right)\right) + \sum_{k>N+q}\frac{1}{x}\log\left(\frac{x + t'_k}{x - t'_k}\right).$$

As we have seen, for $1 \leqslant k \leqslant N$,

$$\lambda_k' \quad \leqslant \quad Y - \frac{k}{\varepsilon} \quad < \quad t_k'.$$

Therefore, according to the *first corollary* from the preceding article, each of the terms in the *first right-hand sum* is a *decreasing function of* x *for* $x \geqslant Y$. The same holds good for the terms of the *second right-hand sum*, because $a < X < t_k'$. The terms of the *third* sum are decreasing for $x \geqslant Y$ by the lemma of article 5, since $t_k' < Y$.

All in all, then, $(1/x)\log|R(x)|$ decreases for $x \geqslant Y$. That is what we had to prove.

Lemma. *Under the hypothesis of the preceding lemma,* $(1/x)\log|R_j(x)|$ *is* increasing *for* $a_j < x \leqslant X_j$.

Proof. We drop the subscript j and argue as in the last proof.

Here, we number the elements of Λ_0 and the whole multiples of L belonging to $[X, Y]$ in *increasing* order, calling the *smallest* of the former λ_1, the *next smallest* λ_2, and so on, and similarly denoting by t_1 the *smallest* integral multiple of L in $[X, Y]$, by t_2 the *next smallest* one, etc. In the present situation, the property

$$n(t) - n(X) \;\leqslant\; \varepsilon(t - X), \quad X \leqslant t \leqslant Y,$$

ensured by the theorem of article 2, is the relevant one for us. By its help we see, remembering that $1/L > \varepsilon$, that

$$t_k \quad < \quad X + \frac{k}{\varepsilon} \quad \leqslant \quad \lambda_k.$$

(The reader may wish to make a diagram like the one accompanying the proof of the preceding lemma.)

There are of course *fewer* λ_ks than t_ks, just as in the proof referred to. Denoting by N the number of the *former*, we can write

$$\frac{1}{x}\log|R(x)| \quad = \quad \frac{q}{x}\log\frac{x-a}{x+a} + \sum_{k=1}^{N}\left(\frac{1}{x}\log\left(\frac{\lambda_k - x}{\lambda_k + x}\right) - \frac{1}{x}\log\left(\frac{t_k - x}{t_k + x}\right)\right)$$
$$+ \sum_{k>N}\frac{1}{x}\log\left(\frac{t_k + x}{t_k - x}\right)$$

for $a < x \leqslant X$. Using the lemma from the preceding article and its first corollary, we readily see that the right-hand expression is an increasing function of x for $a < x \leqslant X$. Done.

Combining the above two lemmas with those from article 4 we now get the

Theorem. *If $\eta > 0$ is taken small enough (depending only on the values of L and ε), and if X_j is* large, *the* maximum value *of* $(1/x)\log|R_j(x)|$ *for* $x \geqslant 0$ outside of (X_j, Y_j) *is attained for* $x = X_j$ *or* $x = Y_j$, *and it is* positive.

Proof. Under the hypothesis, we have, by the results in article 4,

$$\frac{1}{x}\log|R(x)| \;\leqslant\; 0, \quad 0 \leqslant x < a,$$

whereas

$$\frac{2}{X}\log|R(X/2)| \;>\; 0$$

(as usual, we suppress the subscript j). The maximum in question is hence $>\;0$, and it is attained for $a < x \leqslant X$ or for $x \geqslant Y$. Now use the preceding two lemmas. The theorem follows.

7. Behaviour of $(1/x)\log|R_j(x)|$ inside $[X_j, Y_j]$

The essential step in Fuchs' construction consists in showing that $(1/x)\log|R_j(x)|$ *really gets larger inside* $[X_j, Y_j]$ than *outside* that interval.

Lemma. *If $\kappa > 0$ is sufficiently small (depending* only *on L and ε) and ξ, $(1 + (\kappa/5))X_j \leqslant \xi \leqslant (1 + (4\kappa/5))X_j$, is an* integer not in Λ_0, *we have*

$$\frac{1}{\xi}\log|R_j(\xi)| \;\geqslant\; \frac{1}{X_j}\log|R_j(X_j)| \;+\; \frac{1}{2}\left(\frac{1}{L} - \varepsilon\right)\sigma\log\frac{1}{\sigma},$$

where

$$1 + \sigma \;=\; \frac{\xi}{X_j},$$

provided that X_j is large enough *(depending on κ, L and ε).*

Proof. As usual, we suppress the subscript j. For $X < \xi < Y$, we have

$$\frac{1}{\xi}\log|R(\xi)| - \frac{1}{X}\log|R(X)| \;=\; \frac{q}{\xi}\log\left(\frac{\xi - a}{\xi + a}\right) - \frac{q}{X}\log\left(\frac{X - a}{X + a}\right)$$
$$+ \sum_{\lambda\in\Lambda_0\cap[X,Y]}\left(\frac{1}{\xi}\log\left|\frac{\xi - \lambda}{\xi + \lambda}\right| - \frac{1}{X}\log\left|\frac{X - \lambda}{X + \lambda}\right|\right)$$
$$+ \sum_{nL\in[X,Y]}\left(\frac{1}{\xi}\log\left|\frac{\xi + nL}{\xi - nL}\right| - \frac{1}{X}\log\left|\frac{X + nL}{X - nL}\right|\right).$$

Let us start by looking at the *third* term on the right – the summation over λ.

Taking a $\kappa > 0$ which is fairly small in relation to α (the number > 0 depending on ε such that $(1 + (\alpha/4))X \leqslant Y \leqslant (1 + \alpha)X$), we break up the sum in question as

$$\sum_{\substack{X \leqslant \lambda < (1+\kappa)X \\ \lambda \in \Lambda_0}} \quad + \quad \sum_{\substack{(1+\kappa)X \leqslant \lambda \leqslant Y \\ \lambda \in \Lambda_0}},$$

and look initially at the *second* of these terms, which, in fact, gives the main contribution.

Number the elements of Λ_0 belonging to $[X, Y]$ in *increasing* order, calling them $\lambda_1 (> X)$, $\lambda_2 > \lambda_1$, $\lambda_3 > \lambda_2$ and so on. As in the proof of the second lemma from the preceding article, we have

$$\lambda_k \geqslant X + \frac{k}{\varepsilon}.$$

It is, in the first place, possible that some of the λ_k with $1 \leqslant k < \varepsilon\kappa X + 1$ are $\geqslant (1 + \kappa)X$. Denoting by S the set of indices k for which this occurs (if there are any), we can write

$$\sum_{\substack{(1+\kappa)X \leqslant \lambda \leqslant Y \\ \lambda \in \Lambda_0}} = \sum_{\lambda_k, k \in S} + \sum_{\lambda_k, k \geqslant \varepsilon\kappa X + 1}.$$

The first sum on the right is

$$\sum_{k \in S} \left(\frac{1}{\xi} \log \left| \frac{\xi - \lambda_k}{\xi + \lambda_k} \right| - \frac{1}{X} \log \left| \frac{X - \lambda_k}{X + \lambda_k} \right| \right).$$

For $X \leqslant \xi < (1 + \kappa)X$, each of the differences in this expression is *negative* by the lemma of article 5, and each is made *smaller* (*more* negative) when the corresponding λ_k is *moved downwards* to $(1 + \kappa)X$, by the *second* corollary to that lemma. Since it involves at most $\varepsilon\kappa X + 1$ of the λ_k, the sum just written is therefore

$$\geqslant (\varepsilon\kappa X + 1) \left\{ \frac{1}{\xi} \log\left(\frac{(1+\kappa)X - \xi}{(1+\kappa)X + \xi} \right) - \frac{1}{X} \log\left(\frac{(1+\kappa)X - X}{(1+\kappa)X + X} \right) \right\}$$

Writing $\xi = (1 + \sigma)X$, this last works out to

$$\frac{\varepsilon\kappa + (1/X)}{1 + \sigma} \left\{ \log\left(\frac{\kappa - \sigma}{\kappa} \right) + \log\left(\frac{2 + \kappa}{2 + \kappa + \sigma} \right) - \sigma \log\left(\frac{\kappa}{2 + \kappa} \right) \right\},$$

and this is $\geqslant \varepsilon O(\kappa)$ for $\kappa/5 \leqslant \sigma \leqslant 4\kappa/5$ and $X \geqslant 1/\varepsilon\kappa$ when κ is small. The $O(\kappa)$ factor involved here depends *only* on κ and *not* on any of the *other* parameters.

We now turn to the *second* right-hand sum in the above decomposition, namely

$$\sum_{k\geqslant\varepsilon\kappa X+1}\left(\frac{1}{\xi}\log\left(\frac{\lambda_k-\xi}{\lambda_k+\xi}\right)-\frac{1}{X}\log\left(\frac{\lambda_k-X}{\lambda_k+X}\right)\right).$$

Since $\lambda_k \geqslant X + k/\varepsilon$, the summand involving λ_k is

$$\geqslant \varepsilon\int_{X+(k-1)/\varepsilon}^{X+k/\varepsilon}\left(\frac{1}{\xi}\log\left(\frac{t-\xi}{t+\xi}\right)-\frac{1}{X}\log\left(\frac{t-X}{t+X}\right)\right)\mathrm{d}t,$$

because it is $\geqslant$ *the integrand* here for *each* $t \in [X+(k-1)/\varepsilon,\ X+k/\varepsilon]$ according to the *second* corollary of article 5. In our present *sum*, the index k ranges from the smallest integer $\geqslant \varepsilon\kappa X + 1$ up to $n(Y) - n(X)$ (which we assume is not less than the former quantity; otherwise the sum is just *zero*). That sum is thus

$$\geqslant \varepsilon\int_{X+\kappa X}^{X+(n(Y)-n(X))/\varepsilon}\left\{\frac{1}{\xi}\log\left(\frac{t-\xi}{t+\xi}\right)-\frac{1}{X}\log\left(\frac{t-X}{t+X}\right)\right\}\mathrm{d}t$$

where, by the lemma of article 5, *the integrand is negative* (for $X \leqslant \xi < X + \kappa X$). We have $n(Y) - n(X) \leqslant \varepsilon(Y - X)$. The last expression is therefore

$$\geqslant \varepsilon\int_{(1+\kappa)X}^{Y}\left\{\frac{1}{\xi}\log\left(\frac{t-\xi}{t+\xi}\right)-\frac{1}{X}\log\left(\frac{t-X}{t+X}\right)\right\}\mathrm{d}t.$$

This is worked out by partial integration. It is convenient to write $Y = (1+\beta)X$ (so that $\alpha/4 \leqslant \beta \leqslant \alpha$) and make the substitution $t = X + \tau X$. For $x = (1+s)X$ with $0 \leqslant s < \kappa$, we find that

$$\frac{1}{x}\int_{(1+\kappa)X}^{Y}\log\left(\frac{t-x}{t+x}\right)\mathrm{d}t \;=\; \frac{1}{1+s}\int_{\kappa}^{\beta}\log\left(\frac{\tau-s}{2+\tau+s}\right)\mathrm{d}\tau$$

$$= \frac{1}{1+s}\Big\{(\beta-s)\log(\beta-s) \;-\; (\beta+s+2)\log(\beta+s+2)$$

$$-\;(\kappa-s)\log(\kappa-s)\;+\;(\kappa+s+2)\log(\kappa+s+2)\Big\}.$$

Putting first $s = \sigma$, then $s = 0$ and subtracting, and afterwards multiplying the result by ε, we get for the previous integral the value

$$\varepsilon\left\{\frac{\beta-\sigma}{1+\sigma}\log(\beta-\sigma) \;-\; \beta\log\beta \;-\; \frac{\beta+\sigma+2}{1+\sigma}\log(\beta+\sigma+2)\;+\right.$$

$$+ \quad (\beta+2)\log(\beta+2) \quad + \quad \frac{\kappa+\sigma+2}{1+\sigma}\log(\kappa+\sigma+2)$$
$$\left. - \quad (\kappa+2)\log(\kappa+2) \quad - \quad \frac{\kappa-\sigma}{1+\sigma}\log(\kappa-\sigma) \quad + \quad \kappa\log\kappa \right\}.$$

Here, $\alpha/4 \leqslant \beta \leqslant \alpha$, so if $\kappa/5 \leqslant \sigma \leqslant 4\kappa/5$ with κ quite small in relation to α, the *first two terms* in curly brackets are readily seen to amount to $O(\sigma)$, taken together. The same is true for the *third* and *fourth* terms. The *fifth* and *sixth* terms again come to $O(\sigma)$. (The first two quantities $O(\sigma)$ obtained in this manner depend on α, which depends on ε.) There remain the *last two* terms. Those work out to $\sigma\log\sigma + O(\sigma)$. The whole expression thus equals $\varepsilon(\sigma\log\sigma + O(\sigma))$; *this*, then, is the value of the above integral, which, in turn, is a *lower bound* for

$$\sum_{k\geqslant\varepsilon\kappa X+1}\left(\frac{1}{\xi}\log\left(\frac{\lambda_k-\xi}{\lambda_k+\xi}\right)-\frac{1}{X}\log\left(\frac{\lambda_k-X}{\lambda_k+X}\right)\right).$$

That sum is thus

$$\geqslant \quad -\varepsilon\left(\sigma\log\frac{1}{\sigma}+O(\sigma)\right).$$

We combine this estimate with the one for the sum in which k ranges over S, obtained previously. That yields, for $X \geqslant 1/\varepsilon\kappa$,

$$\sum_{\substack{(1+\kappa)X\leqslant\lambda\leqslant Y\\ \lambda\in\Lambda_0}}\left(\frac{1}{\xi}\log\left(\frac{\lambda-\xi}{\lambda+\xi}\right)-\frac{1}{X}\log\left(\frac{\lambda-X}{\lambda+X}\right)\right)$$
$$\geqslant \quad -\varepsilon\sigma\log\frac{1}{\sigma}-\varepsilon O(\sigma)-\varepsilon O(\kappa) \quad = \quad -\varepsilon\left(\sigma\log\frac{1}{\sigma}+O(\sigma)\right),$$

where the $O(\sigma)$ term depends on α (and thus on ε).

We lay the sum

$$\sum_{\substack{X\leqslant\lambda<(1+\kappa)X\\ \lambda\in\Lambda_0}}$$

aside for the moment, and proceed to the examination of

$$\sum_{X\leqslant nL\leqslant Y}\left(\frac{1}{\xi}\log\left|\frac{\xi+nL}{\xi-nL}\right|-\frac{1}{X}\log\left|\frac{X+nL}{X-nL}\right|\right),$$

which we break up as

$$\sum_{X\leqslant nL\leqslant(1+\kappa)X} \qquad + \qquad \sum_{(1+\kappa)X<nL\leqslant Y}.$$

The *second* of these sums can be handled just as the one over λ_k with $k \geqslant \varepsilon\kappa X + 1$ was treated above. Since the numbers nL are *already equally spaced*, that procedure furnishes an *approximate equality* in the present situation, namely*

$$\sum_{(1+\kappa)X < nL \leqslant Y} \left(\frac{1}{\xi} \log\left(\frac{nL+\xi}{nL-\xi}\right) - \frac{1}{X}\log\left(\frac{nL+X}{nL-X}\right)\right)$$
$$= \quad \frac{1}{L}\left(\sigma \log\frac{1}{\sigma} + \mathrm{O}(\sigma)\right) \;+\; \mathrm{O}\left(\frac{1}{X}\right).$$

For the *first* sum,

$$\sum_{X \leqslant nL \leqslant (1+\kappa)X} \left(\frac{1}{\xi}\log\left|\frac{\xi+nL}{\xi-nL}\right| - \frac{1}{X}\log\left|\frac{X+nL}{X-nL}\right|\right),$$

we proceed to find a *lower bound, assuming that ξ is an integer between* $(1+(\kappa/5))X$ *and* $(1+(4\kappa/5))X$. (Up to now, we have *not used* the fact that $\xi \in \mathbb{N}$, which is part of the hypothesis.) In case ξ is *divisible by the integer* L, the expression in question is *infinite*, for X is half-odd. We therefore need do the computation only when $\xi \in \mathbb{N}$ is *not divisible by L*. In that case, we rewrite the sum as

$$\int_X^{(1+\kappa)X} \left(\frac{1}{\xi}\log\left|\frac{\xi+t}{\xi-t}\right| - \frac{1}{X}\log\left|\frac{X+t}{X-t}\right|\right) \mathrm{d}[t/L],$$

and begin by working out

$$\int_X^{(1+\kappa)X} \log\left|\frac{1}{\xi-t}\right| \mathrm{d}[t/L].$$

(As usual, $[t/L]$ denotes the largest integer $\leqslant t/L$.)

Since $\xi \in \mathbb{N}$ is *not divisible by L*, the last integral is equal to

$$\left(\int_X^{\xi-\gamma} + \int_{\xi+\gamma}^{(1+\kappa)X}\right)\log\left|\frac{1}{t-\xi}\right| \mathrm{d}[t/L]$$

for all sufficiently small $\gamma > 0$. Integrate by parts, using $[t/L] - [\xi/L]$ as

* Thanks to article 5's second corollary, the sum in question can differ from a corresponding integral by at most *twice* the value of $(1/\xi)\log((nL+\xi)/(nL-\xi)) - (1/X)\log((nL+X)/(nL-X))$ for $n = [(1+\kappa)X/L]$. This is $\mathrm{O}(1/X)$ for large X when $\kappa/5 \leqslant \sigma \leqslant 4\kappa/5$.

a primitive for $\mathrm{d}[t/L]$. After making $\gamma \longrightarrow 0$, we obtain the value

$$\Big([\xi/L] - [X/L]\Big)\log\frac{1}{\xi - X} + \Big([(1+\kappa)X/L] - [\xi/L]\Big)\log\frac{1}{(1+\kappa)X - \xi}$$

$$+ \int_X^{(1+\kappa)X} \frac{[t/L] - [\xi/L]}{t - \xi}\,\mathrm{d}t.$$

The *integral* in this expression is *positive*, and

$$\log\frac{1}{\xi - X} = \log\frac{1}{X} + \log\frac{1}{\sigma} = \log\frac{1}{X} + \log\frac{1}{\kappa} + \mathrm{O}(1),$$

$$\log\frac{1}{(1+\kappa)X - \xi} = \log\frac{1}{X} + \log\frac{1}{\kappa - \sigma} = \log\frac{1}{X} + \log\frac{1}{\kappa} + \mathrm{O}(1),$$

because $\kappa/5 \leqslant \sigma \leqslant 4\kappa/5$. Therefore,

$$\int_X^{(1+\kappa)X} \log\left|\frac{1}{\xi - t}\right| \mathrm{d}[t/L]$$

$$\geqslant \quad ([(1+\kappa)X/L] - [X/L])\left(\log\frac{1}{X} + \log\frac{1}{\kappa} + \mathrm{O}(1)\right).$$

On the other hand, since $X < \xi < (1+\kappa)X$, we clearly have

$$\int_X^{(1+\kappa)X} \log(t + \xi)\,\mathrm{d}[t/L]$$

$$= \quad ([(1+\kappa)X/L] - [X/L])(\log X + \mathrm{O}(1)).$$

This, together with the calculation just made, gives

$$\frac{1}{\xi}\int_X^{(1+\kappa)X} \log\left|\frac{t+\xi}{t-\xi}\right| \mathrm{d}[t/L]$$

$$\geqslant \quad \frac{[(1+\kappa)X/L] - [X/L]}{(1+\sigma)X}\left(\log\frac{1}{\kappa} + \mathrm{O}(1)\right).$$

Turning to the similar integral involving X in place of ξ, we first get

$$\int_X^{(1+\kappa)X} \log\left(\frac{1}{t - X}\right)\mathrm{d}[t/L] = \Big([(1+\kappa)X/L] - [X/L]\Big)\log\frac{1}{\kappa X}$$

$$+ \int_X^{(1+\kappa)X} \frac{[t/L] - [X/L]}{t - X}\,\mathrm{d}t \quad \leqslant$$

$$\leqslant \left([(1+\kappa)X/L]-[X/L]\right)\left(\log\frac{1}{X}+\log\frac{1}{\kappa}\right) + \frac{\kappa X}{L} + \mathrm{O}(\log X)$$

after taking account of the fact that $[t/L]-[X/L] \leqslant (t-X)/L + 1$ actually vanishes for $0 \leqslant t-X < 1/2$, X being half-odd and L an integer. Thence, it is readily seen that

$$\frac{1}{X}\int_X^{(1+\kappa)X} \log\left(\frac{t+X}{t-X}\right)\mathrm{d}[t/L]$$
$$\leqslant \frac{[(1+\kappa)X/L]-[X/L]}{X}\left(\log\frac{1}{\kappa}+\mathrm{O}(1)\right) + \frac{\kappa}{L} + \mathrm{O}\left(\frac{\log X}{X}\right).$$

Combining this with the estimate already obtained for the similar integral involving ξ, we find that

$$\sum_{X\leqslant nL\leqslant(1+\kappa)X}\left(\frac{1}{\xi}\log\left|\frac{\xi+nL}{\xi-nL}\right| - \frac{1}{X}\log\left|\frac{X+nL}{X-nL}\right|\right)$$
$$\geqslant \frac{[(1+\kappa)X/L]-[X/L]}{X}\left\{\left(\frac{1}{1+\sigma}-1\right)\log\frac{1}{\kappa}+\mathrm{O}(1)\right\}$$
$$- \frac{\kappa}{L} - \mathrm{O}\left(\frac{\log X}{X}\right)$$

for integers $\xi = (1+\sigma)X$ between $(1+(\kappa/5))X$ and $(1+(4\kappa/5))X$. When $X/\log X$ is *large in relation to* L/κ, the right side of this inequality reduces to

$$-\frac{\kappa\sigma}{L}\log\frac{1}{\kappa} - \mathrm{O}\left(\frac{\kappa}{L}\right) = \frac{\mathrm{O}(\sigma)}{L}$$

for small κ. Referring to the estimate for

$$\sum_{(1+\kappa)X<nL\leqslant Y}$$

given above, we thus obtain

$$\boxed{\sum_{X\leqslant nL\leqslant Y}\left(\frac{1}{\xi}\log\left|\frac{\xi+nL}{\xi-nL}\right| - \frac{1}{X}\log\left|\frac{X+nL}{X-nL}\right|\right) \geqslant \frac{1}{L}\sigma\log\frac{1}{\sigma} - \frac{\mathrm{O}(\sigma)}{L},}$$

valid for *small* values of $\kappa > 0$ with *integers* $\xi = (1+\sigma)X$ such that $\kappa/5 \leqslant \sigma \leqslant 4\kappa/5$, and for *large* values of X (depending on L and κ).

Let us return to the sum

$$\sum_{\substack{X \leqslant \lambda < (1+\kappa)X \\ \lambda \in \Lambda_0}} \left(\frac{1}{\xi} \log\left|\frac{\xi - \lambda}{\xi + \lambda}\right| - \frac{1}{X} \log\left|\frac{X - \lambda}{X + \lambda}\right| \right)$$

which was set aside earlier; *here we assume that* ξ *is an integer not belonging to* Λ_0, with $(1 + (\kappa/5))X \leqslant \xi \leqslant (1 + (4\kappa/5))X$. The calculation is like the one just made for the similar sum over nL with $X \leqslant nL \leqslant (1 + \kappa)X$. In order to keep our notation simple, we assume that $(1 + \kappa)X \notin \Lambda_0$ – actually, whether this is true or not makes no real difference.

Consider the integral

$$\int_X^{(1+\kappa)X} \log|t - \xi| \, \mathrm{d}n(t).$$

Following the procedure used above for the similar integral involving $\mathrm{d}[t/L]$ instead of $\mathrm{d}n(t)$, this works out to

$$(n((1 + \kappa)X) - n(X))(\log X + \log \kappa + O(1))$$
$$- \int_X^{(1+\kappa)X} \frac{n(t) - n(\xi)}{t - \xi} \mathrm{d}t.$$

We have $\Lambda_0 \subseteq \mathbb{N}$, so, if $\xi \in \mathbb{N}$ does *not* belong to Λ_0, $|n(t) - n(\xi)| \leqslant |t - \xi|$. The last expression therefore *differs* from

$$(n((1 + \kappa)X) - n(X))(\log X + \log \kappa + O(1))$$

by at most κX in absolute value.

From here on, the work goes just as that done above for the integrals involving $\mathrm{d}[t/L]$. One finds without trouble that when κ is small,

$$\sum_{\substack{X \leqslant \lambda < (1+\kappa)X \\ \lambda \in \Lambda_0}} \left(\frac{1}{\xi} \log\left|\frac{\xi - \lambda}{\xi + \lambda}\right| - \frac{1}{X} \log\left|\frac{X - \lambda}{X + \lambda}\right| \right) = O(\sigma)$$

for *integers* $\xi = (1 + \sigma)X \notin \Lambda_0$, $\kappa/5 \leqslant \sigma \leqslant 4\kappa/5$, and *sufficiently large* X (depending on κ). (This calculation *really gives* just $O(\sigma)$ and *not* $\varepsilon O(\sigma)$, because the relation $n(t) - n(\xi) \leqslant |t - \xi|$ was all we had available for our choice of ξ.)

The result just obtained is now combined with the earlier one for the sum

$$\sum_{\substack{(1+\kappa)X \leqslant \lambda \leqslant Y \\ \lambda \in \Lambda_0}} .$$

We get

$$\boxed{\sum_{\substack{X\leqslant\lambda\leqslant Y\\ \lambda\in\Lambda_0}}\left(\frac{1}{\xi}\log\left|\frac{\xi-\lambda}{\xi+\lambda}\right|-\frac{1}{X}\log\left|\frac{X-\lambda}{X-\lambda}\right|\right) \;\geqslant\; -\varepsilon\sigma\log\frac{1}{\sigma} \;-\; \mathrm{O}(\sigma),}$$

valid for ξ of the kind described above, provided that X is *large enough* (depending on κ).

We still have to look at the difference

$$\frac{q}{\xi}\log\left(\frac{\xi-a}{\xi+a}\right) \;-\; \frac{q}{X}\log\left(\frac{X-a}{X+a}\right);$$

it, however, is clearly $\geqslant 0$ by the lemma of article 5, since $a < X < \xi$!*

The relation last noted and two boxed estimates above are finally plugged into the formula from the very beginning of this proof. That gives, *for* $\kappa > 0$ *small in relation to* α,

$$\frac{1}{\xi}\log|R(\xi)| \;-\; \frac{1}{X}\log|R(X)| \;\geqslant\; \left(\frac{1}{L}-\varepsilon\right)\sigma\log\frac{1}{\sigma} \;-\; \mathrm{O}(\sigma),$$

provided that the *integer* $\xi = (1+\sigma)X$ *with* $\kappa/5 \leqslant \sigma \leqslant 4\kappa/5$ *is* $\notin \Lambda_0$ and that X is *sufficiently large* (depending on κ, L and ε). *Formally*, the $\mathrm{O}(\sigma)$ term on the right depends on L, ε and α; *in fact*, however, since ε is *small* and L *large*, it is *essentially* dependent *on* α *alone* (that is, on ε), as one sees by looking again at how the individual $\mathrm{O}(\sigma)$ terms arise in the above computations.

We are using $L < 1/\varepsilon$ *in the present construction*. Therefore, if $\kappa > \sigma > 0$ is *small enough* (depending on L and ε), the term $(1/L-\varepsilon)\sigma\log 1/\sigma$ in the right-hand side of the preceding inequality *will greatly outweigh* the $\mathrm{O}(\sigma)$ term appearing there. Then, for *sufficiently large* X (depending on κ, L and ε) we will have

$$\frac{1}{\xi}\log|R(\xi)| \;-\; \frac{1}{X}\log|R(X)| \;\geqslant\; \frac{1}{2}\left(\frac{1}{L}-\varepsilon\right)\sigma\log\frac{1}{\sigma}$$

when the integer $\xi = (1+\sigma)X$ with $\kappa/5 \leqslant \sigma \leqslant 4\kappa/5$ lies *outside* Λ_0.

The lemma is proved.

* Using the relation $a \leqslant \eta X$ and the boxed estimate on q from article 4, the difference in question is readily worked out to be $\leqslant$ const. $\sigma\eta^2((\alpha/L)+(1/X))$.

Remark. Since $n(t)-n(X_j) \leqslant \varepsilon(t-X_j)$ for $X_j \leqslant t \leqslant Y_j$, there are *at most* $\frac{4}{5}\kappa\varepsilon X_j$ members of Λ_0 in the interval $[(1+(\kappa/5))X_j,\ (1+(4\kappa/5))X_j]$ which, however, contains *at least* $[(3\kappa/5)X_j]$ *integers*. In our construction, $\varepsilon > 0$ is *small*. Hence, given $\kappa > 0$ there are *at least* $(1-\frac{4}{3}\varepsilon)\cdot(3\kappa/5)X_j - 1$ *integers* ξ *in* $[(1+(\kappa/5))X_j,\ (1+(4\kappa/5))X_j]$ *to which the lemma applies* when X_j is large.

It is important that the ratio $(1/x)\log|R_j(x)|$ has behaviour similar to that described by the preceding lemma when x is near the *right* endpoint Y_j of $[X_j, Y_j]$. Arguing very much as in the long proof just given, but using the inequality

$$n(Y_j)-n(t) \leqslant \varepsilon(Y_j-t), \quad X_j \leqslant t \leqslant Y_j,$$

instead of

$$n(t)-n(X_j) \leqslant \varepsilon(t-X_j), \quad X_j \leqslant t \leqslant Y_j,$$

one establishes the

Lemma. *If* $\kappa > 0$ *is* sufficiently small (*depending on* L *and* ε) *and* ξ, $(1-(4\kappa/5))Y_j \leqslant \xi \leqslant (1-(\kappa/5))Y_j$, *is an* integer not in Λ_0, *we have*

$$\frac{1}{\xi}\log|R_j(\xi)| \geqslant \frac{1}{Y_j}\log|R_j(Y_j)| + \frac{1}{2}\left(\frac{1}{L}-\varepsilon\right)\sigma\log\frac{1}{\sigma}$$

with $1-\sigma = \xi/Y_j$, *provided that* X_j *is* large enough (*depending on* κ, L *and* ε).

It is recommended that the reader *think through* how the steps in the proof of the previous result can be adapted to that of the present one, without actually writing out the details.

8. Formation of Fuchs' function $\Phi(z)$. Discussion

$\Phi(z)$, which actually involves the parameters ε, L and η, is constructed as follows. One starts by taking a *small* $\varepsilon > 0$ and then getting a sequence of intervals $[X_j, Y_j]$ with $X_j \xrightarrow[j]{} \infty$, corresponding to ε in the manner described by the theorem at the end of article 2. One then picks a *large integer* $L < 1/\varepsilon$ and finally, choosing a small value > 0 for η, takes for each j a number a_j,

$$\tfrac{1}{2}\eta X_j \leqslant a_j \leqslant \eta X_j,$$

and an integer q_j, according to the procedure of article 4. The parameter

$\eta > 0$ is chosen *small enough* (depending on L and ε) for the results of articles 4 and 6 to apply.

We next take an *exceedingly sparse* sequence of the intervals $[X_j, Y_j]$ – by this we mean that the ratios X_{j+1}/X_j are to *increase very rapidly*, in a way to be determined presently. *The rest of the construction uses only the* $[X_j, Y_j]$ *from this sparse sequence.* In terms of *these*, we write

$$\Omega \quad = \quad (0, \infty) \sim \bigcup_j [X_j, Y_j],$$

and finally put

$$\Phi(z) \quad = \quad \left(\frac{z}{L}\right)^{-2z/L} e^{(2-2C)z/L} \prod_{\substack{nL\in\Omega \\ n\in\mathbb{N}}} \left(\frac{1 - z/nL}{1 + z/nL}\right) e^{2z/nL}$$

$$\times \prod_j \left\{ \left(\frac{1 - z/a_j}{1 + z/a_j}\right)^{q_j} e^{2q_j z/a_j} \cdot \prod_{\lambda\in\Lambda_0\cap[X_j, Y_j]} \left(\frac{1 - z/\lambda}{1 + z/\lambda}\right) e^{2z/\lambda} \right\}.$$

Here, C is Euler's constant (see article 3).

The function $\Phi(z)$ is analytic in $\Re z > 0$ and vanishes at the positive integral multiples of L *outside* our sparse sequence of intervals $[X_j, Y_j]$, as well as at *the points of* Λ_0 lying *within the latter*. It also has a q_j-fold zero at each a_j. Using the group products defined in article 4 and studied in the previous two articles, we can write

$$\Phi(z) \quad = \quad F(z/L) \prod_j R_j(z),$$

where

$$F(z) \quad = \quad z^{-2z} e^{(2-2C)z} \prod_{n=1}^{\infty} \left(\frac{1 - z/n}{1 + z/n}\right) e^{2z/n}$$

$$= \quad z^{-2z} e^{2z} \frac{\sin \pi z}{\pi z} (\Gamma(z+1))^2,$$

a function already looked at in article 3. (In this last formula for $\Phi(z)$, it is of course the product of the $R_j(z)$ *corresponding to our sparse sequence of intervals* $[X_j, Y_j]$ that is understood.)

Lemma. *If* $Y_j \; < \; \frac{1}{2}\eta X_{j+1}$ *we have*

$$|\Phi(z)| \quad \leqslant \quad \text{const.} \exp\left(\frac{\pi}{L}|\Im z| + A\Re z\right)$$

for $\Re z \geqslant 0$, *A being a constant depending on* L, ε and η.

Proof. We have

$$\Phi(z) \;=\; \left(\frac{z}{L}\right)^{-2z/L} e^{(2-2C)z/L} \prod_{k=1}^{\infty} \left(\frac{1 - z/\mu_k}{1 + z/\mu_k}\right) e^{2z/\mu_k},$$

where $\{\mu_k\}$ is a certain increasing sequence consisting of the numbers nL in Ω, $n \in \mathbb{N}$, the points of Λ_0 in our intervals $[X_j, Y_j]$, and the q_j-fold repeated points a_j.

If z, $\Re z > 0$, is given, we have, for $\mu_k \geqslant 2|z|$,

$$\begin{aligned}\left(\frac{1 - z/\mu_k}{1 + z/\mu_k}\right) e^{2z/\mu_k} \;&=\; \exp\left(-\frac{2z^3}{3\mu_k^3} - \frac{2z^5}{5\mu_k^5} - \cdots\right) \\ &=\; \exp\left(-\frac{2z^3}{3\mu_k^3}\left(1 + o\left(\frac{z}{\mu_k}\right)\right)\right).\end{aligned}$$

Clearly,

$$\sum_{\mu_k \geqslant 2|z|} \frac{1}{\mu_k^3} \;=\; O\left(\frac{1}{|z|^2}\right).$$

Hence

$$\left|\prod_{\mu_k \geqslant 2|z|} \left(\frac{1 - z/\mu_k}{1 + z/\mu_k}\right) e^{2z/\mu_k}\right| \;\leqslant\; e^{O(|z|)}.$$

For $0 < \mu_k < 2|z|$, since $x = \Re z > 0$,

$$\left|\left(\frac{1 - z/\mu_k}{1 + z/\mu_k}\right) e^{2z/\mu_k}\right| \;\leqslant\; e^{2x/\mu_k} \;(!),$$

whence

$$\left|\prod_{\mu_k < 2|z|} \left(\frac{1 - z/\mu_k}{1 + z/\mu_k}\right) e^{2z/\mu_k}\right| \;\leqslant\; \exp\left(2x \sum_{\mu_k < 2|z|} \frac{1}{\mu_k}\right).$$

Suppose now that $Y_j \leqslant 2|z| < a_{j+1}$ for some j – note that our condition $Y_j < \frac{1}{2}\eta X_{j+1}$ does make $Y_j < a_{j+1}$ for each j. Then, since the a_l and q_l were chosen so as to make

$$\frac{q_l}{a_l} \;+\; \sum_{\lambda \in \Lambda_0 \cap [X_l, Y_l]} \frac{1}{\lambda} \;=\; \sum_{nL \in [X_l, Y_l]} \frac{1}{nL}$$

for each l (see article 4), we have

$$\begin{aligned}\sum_{\mu_k < 2|z|} \frac{1}{\mu_k} \;&=\; \sum_{\substack{nL < 2|z| \\ n \in \mathbb{N}}} \frac{1}{nL} \;=\; \frac{1}{L}\left(\log\left(\frac{2|z|}{L}\right) + O(1)\right) \\ &=\; \frac{1}{L}\log|z| \;-\; \frac{O(\log L)}{L}.\end{aligned}$$

This formula *remains true* when $a_{j+1} \leqslant 2|z| < Y_{j+1}$. For then $\sum_{\mu_k < 2|z|}(1/\mu_k)$ can differ from $\sum_{nL < 2|z|, n\in\mathbb{N}}(1/nL)$ by *at most*

$$\frac{q_{j+1}}{a_{j+1}} \quad + \sum_{\substack{X_{j+1}\leqslant\lambda\leqslant Y_{j+1}\\ \lambda\in\Lambda_0}} \frac{1}{\lambda} \quad + \sum_{\substack{X_{j+1}\leqslant nL\leqslant Y_{j+1}\\ n\in\mathbb{N}}} \frac{1}{nL},$$

and by the above relation this equals

$$2\sum_{X_{j+1}\leqslant nL\leqslant Y_{j+1}} \frac{1}{nL} \quad\leqslant\quad \frac{2}{X_{j+1}} + \frac{2}{L}\log\frac{Y_{j+1}}{X_{j+1}} \quad\leqslant\quad \frac{2}{X_{j+1}} + \frac{2\log(1+\alpha)}{L},$$

since

$$\left(1 + \frac{\alpha}{4}\right)X_{j+1} \quad\leqslant\quad Y_{j+1} \quad\leqslant\quad (1+\alpha)X_{j+1}.$$

There is of course no loss of generality in assuming all the X_l to be $> L$, so, this being granted, we have

$$\sum_{\mu_k < 2|z|} \frac{1}{\mu_k} \quad=\quad \frac{1}{L}\log|z| \;+\; \frac{O(\log L)}{L}$$

in the present case also; the formula is thus true generally.

Using the relation just found with the preceding estimate and then combining the result with the one obtained previously we get, for $\Re z > 0$,

$$\left|\prod_{k=1}^{\infty}\left(\frac{1 - z/\mu_k}{1 + z/\mu_k}\right)e^{2z/\mu_k}\right| \quad\leqslant\quad \exp\left(\frac{2x}{L}\log|z| + O(|z|)\right).$$

Hence, in the right half plane,

$$|\Phi(z)| \quad\leqslant\quad \left|\frac{z}{L}\right|^{-2x/L} e^{(2y/L)\arg z}\, e^{(2-2C)x/L}\, e^{(2x/L)\log|z| + O(|z|)} \quad=\quad e^{O(|z|)},$$

the exponential in $(2x/L)\log|z|$ being cancelled by $|z|^{-2x/L}$. The function $\Phi(z)$ is thus of *exponential type* in the half plane $\{\Re z > 0\}$.

Once this is known, we can use Φ's obvious continuity up to the imaginary axis and apply the *second* Phragmén–Lindelöf theorem from §C of Chapter III. For $z = iy$ *pure imaginary*, the product over the μ_k in the above formula for $\Phi(z)$ *has modulus* 1, and we see that

$$|\Phi(iy)| \quad=\quad e^{(2y/L)\arg(iy)} \quad=\quad e^{\pi|y|/L}$$

On the other hand, $|\Phi(x)| \leqslant \text{const.}\, e^{Ax}$ with a certain constant A, by what has just been shown. The function

$$e^{\pi i z/L}e^{-Az}\,\Phi(z)$$

is thus *bounded* on the *sides* of the *first quadrant*, and *hence within it*, by Phragmén–Lindelöf. Similarly,

$$e^{-\pi iz/L}e^{-Az}\Phi(z)$$

is *bounded* in the *fourth quadrant*. Thus,

$$|\Phi(z)| \quad \leqslant \quad \text{const.}\, e^{(\pi|y|/L)+Ax} \quad \text{for } x \geqslant 0,$$

Q.E.D.

Remark. Paying a little more attention to the computation at the beginning of the proof just given, one sees that the constant A can be taken to be *small* if L is *large*. Our function $\Phi(z)$ is thus of *small exponential type in* $\Re z \geqslant 0$. This fact will not be used in our application.

We now return to our *original non-measurable sequence* $\Sigma \subseteq \mathbb{N}$ with Polya maximum density $D_\Sigma^* > 0$. At the beginning of article 2, the complement $\mathbb{N} \sim \Sigma$ was broken up into two disjoint sequences: Λ_0, infinite and of *minimum density zero*, which has figured in the constructions of articles 2–7, and a *measurable sequence* Λ_1 of *density* $1 - D_\Sigma^*$. *The main purpose of all the above work has been to arrive at the function* $\Phi(z)$, *having properties described by the preceding lemma and by the*

Theorem. *If* $\varepsilon > 0$ *is small enough and the integer* L, $0 < L < 1/\varepsilon$ *is large, if, moreover,*

$$\frac{1}{L} + 2\varepsilon \quad < \quad D_\Sigma^*$$

and the sequence of intervals $[X_j, Y_j]$ *used in the construction of* $\Phi(z)$ *is sparse enough, we have*

$$\boxed{\limsup_{\substack{m\to\infty\\ m\in\Lambda_0}} \frac{\log \Phi(m)}{m} \quad \leqslant \quad \limsup_{\substack{n\to\infty\\ n\in\Sigma}} \frac{\log|\Phi(n)|}{n} \; - \; \delta(L,\varepsilon),}$$

where $\delta(L,\varepsilon)$ *is a quantity* > 0 *depending on* L *and* ε. *The quantity*

$$\limsup_{\substack{n\to\infty\\ n\in\Sigma}} \frac{\log|\Phi(n)|}{n}$$

is finite *and* > 0.

Proof. Using the group products $R_j(z)$ constructed in article 4, we have

$$\Phi(z) \;=\; F(z/L)\prod_j R_j(z)$$

with the function F studied in article 3. For each fixed j, the modulus of

$$R_j(z) \;=\; \left(\frac{a_j - z}{a_j + z}\right)^{q_j} \prod_{nL\in[X_j,Y_j]} \left(\frac{nL+z}{nL-z}\right) \prod_{\lambda\in\Lambda_0\cap[X_j,Y_j]} \left(\frac{\lambda - z}{\lambda + z}\right)$$

tends obviously to 1 when $z\to\infty$. $R_j(z)$ also tends to 1 when $z\to 0$, and *in a manner dependent only on the ratio* $|z|/X_j^{2/3}$, *while otherwise independent of* j. To see this, recall that $a_j \geqslant \frac{1}{2}\eta X_j$ so that, for $|z| \leqslant \frac{1}{4}\eta X_j$, say, we can expand $\log R_j(z)$ in powers of z, as in the proof of the *first* lemma in article 4. As we saw there, the first degree term in z is absent from this expansion, and we can readily deduce from the latter that

$$|R_j(z)-1| \;\leqslant\; \text{const.}|z|^3\left\{\frac{q_j}{a_j^3} \;+\; \sum_{\lambda\in\Lambda_0\cap[X_j,Y_j]}\frac{1}{\lambda^3} \;+\; \sum_{nL\in[X_j,Y_j]}\frac{1}{(nL)^3}\right\}$$

for $|z| \leqslant \frac{1}{4}\eta X_j$. The sum in curly brackets is clearly $\leqslant$ const.$/X_j^2$ so we have

$$|R_j(z)-1| \;\leqslant\; \text{const.}\frac{|z|^3}{X_j^2}, \qquad |z| \;\leqslant\; \tfrac{1}{4}\eta X_j,$$

verifying our claim.

Thanks to this behaviour of the $R_j(z)$, we can select a sequence of the numbers X_j *increasing sufficiently rapidly* so that, for $x > 0$, the product $\prod_j |R_j(x)|$ *will be sensibly equal to* 1 *unless* x *is much nearer to one of the intervals* $[X_j, Y_j]$ – *to* $[X_l, Y_l]$ *say – than to any of the others. In the latter situation,* $\prod_{j\neq l}|R_j(x)|$ *will be practically equal to* 1 *and the whole product* $\prod_j |R_j(x)|$ *essentially equal to* $|R_l(x)|$.

By the asymptotic behaviour of $F(z)$ obtained in article 3 and the formula at the beginning of this proof,

$$\Phi(x) \;\sim\; \left(2\sin\frac{\pi x}{L}\right)\prod_j R_j(x)$$

for large $x > 0$. For $m \in \mathbb{N}$ *not* divisible by L, the asymptotic behaviour of $\log|\Phi(m)|$ is thus governed by that of $\prod_j |R_j(m)|$. And, as we have just seen, the latter is practically 1 unless m is *much closer* to some $[X_l, Y_l]$ than to *any of the other* $[X_j, Y_j]$, in which case the product is nearly equal to $|R_l(m)|$.

Suppose, first of all, that $m \in \Lambda_0$. If also $m \in [X_l, Y_l]$, then $R_l(m) = 0$

by our definition of the $R_j(z)$. If, on the other hand, $m \notin [X_l, Y_l]$ we have, by the theorem at the end of article 6,

$$\frac{\log|R_l(m)|}{m} \leqslant \max\left(\frac{\log|R_l(X_l)|}{X_l}, \frac{\log|R_l(Y_l)|}{Y_l}\right),$$

a strictly positive quantity. Thus, if $m \in \Lambda_0$ is *near* the interval $[X_l, Y_l]$, we have

$$\frac{\log|\Phi(m)|}{m} \leqslant \frac{\text{const.}}{m} + \max\left(\frac{\log|R_l(X_l)|}{X_l}, \frac{\log|R_l(Y_l)|}{Y_l}\right).$$

Fix now a number $\kappa > 0$ small enough to ensure the conclusions of the two lemmas in article 7. By the remark following the *first* of those lemmas, the interval $[(1 + \kappa/5)X_l, (1 + 4\kappa/5)X_l]$ contains *at least*

$$\left(1 - \frac{4}{3}\varepsilon\right)\cdot\frac{3\kappa}{5}X_l - 1$$

integers not belonging to Λ_0, when X_l is *large*. Since Λ_1 is *measurable and of density* $1 - D_\Sigma^*$, *at most*

$$\left(1 - D_\Sigma^* + \frac{2\varepsilon}{3}\right)\cdot\frac{3\kappa}{5}X_l$$

of the integers just mentioned can belong to Λ_1, when X_l is large. And at most

$$\frac{1}{L}\cdot\frac{3\kappa}{5}X_l + 1$$

of them can be divisible by L. We are, however, assuming that $D_\Sigma^* > 1/L + 2\varepsilon$. Hence

$$1 - D_\Sigma^* + \frac{2\varepsilon}{3} + \frac{1}{L} < 1 - \frac{4}{3}\varepsilon,$$

so, if X_l is large, *there are at least*

$$\left\{\left(1 - \frac{4}{3}\varepsilon\right) - \left(1 - D_\Sigma^* + \frac{2\varepsilon}{3} + \frac{1}{L}\right)\right\}\frac{3\kappa}{5}X_l - 2$$
$$= \left(D_\Sigma^* - \frac{1}{L} - 2\varepsilon\right)\cdot\frac{3\kappa}{5}X_l - 2$$

integers ξ in the above interval not divisible by L, and belonging neither to Λ_0 nor to Λ_1. Such ξ are thus in Σ. For them, by the first lemma of article

7, *we have*

$$\frac{\log|R_l(\xi)|}{\xi} \geqslant \frac{\log|R_l(X_l)|}{X_l} + \frac{1}{2}\left(\frac{1}{L}-\varepsilon\right)\sigma\log\frac{1}{\sigma}$$

with $1+\sigma = \xi/X_l$, if X_l is large enough. Here, $\kappa/5 \leqslant \sigma \leqslant 4\kappa/5$, so (wlog $\kappa < 1/e$!)

$$\frac{1}{2}\left(\frac{1}{L}-\varepsilon\right)\sigma\log\frac{1}{\sigma} \geqslant \frac{1}{10}\left(\frac{1}{L}-\varepsilon\right)\kappa\log\frac{5}{\kappa}.$$

The *choice* of our small *fixed* number $\kappa > 0$ depended on L and ε (refer to the first lemma in article 7). The *right side* of the last relation is therefore a certain *strictly positive quantity* $\delta(L, \varepsilon)$ dependent on L and ε. The integers $\xi \in \Sigma$ now under consideration are not divisible by L, so

$$\left|\sin\frac{\pi\xi}{L}\right| \geqslant \sin\frac{\pi}{L},$$

and it thence follows from the above inequality that

$$\frac{\log|\Phi(\xi)|}{\xi} \geqslant \frac{\text{const.}}{\xi} + \frac{\log|R_l(X_l)|}{X_l} + \delta(L, \varepsilon)$$

for them when X_l is large.

The ξ satisfying this relation are in Σ and also in the interval

$$\left[\left(1+\frac{\kappa}{5}\right)X_l,\ \left(1+\frac{4\kappa}{5}\right)X_l\right].$$

An argument just like the one used to get them, but based on the *second* lemma of article 7 instead of the *first*, will similarly give us *other* $\xi \in \Sigma$, this time in

$$\left[\left(1-\frac{4\kappa}{5}\right)Y_l,\ \left(1-\frac{\kappa}{5}\right)Y_l\right],$$

such that

$$\frac{\log|\Phi(\xi)|}{\xi} \geqslant \frac{\text{const.}}{\xi} + \frac{\log|R_l(Y_l)|}{Y_l} + \delta(L, \varepsilon),$$

provided that X_l is sufficiently large.

From this and the preceding inequality we see in the first place that

$$\limsup_{\substack{\xi\to\infty \\ \xi\in\Sigma}} \frac{\log|\Phi(\xi)|}{\xi}$$

is certainly > 0 by the theorem of article 6 – it is, on the other hand, finite by the preceding lemma. The *second statement* of our theorem is thus verified.

For the *first statement*, we confront the two inequalities just obtained with the previous estimate on $(\log|\Phi(m)|)/m$ for $m \in \Lambda_0$ close to $[X_l, Y_l]$. In view of the behaviour of the product $\prod_j |R_j(m)|$ described earlier, we see in that way that

$$\limsup_{\substack{\xi \to \infty \\ \xi \in \Sigma}} \frac{\log|\Phi(\xi)|}{\xi} \geqslant \limsup_{\substack{m \to \infty \\ m \in \Lambda_0}} \frac{\log|\Phi(m)|}{m} + \delta(L, \varepsilon).$$

The theorem is now completely proved. We are done.

Discussion. Let us look back and try to grasp the idea behind this and the preceding 6 articles, taken as a whole. On the sequence Λ_0, $|\Phi(m)|$ is smaller by a factor of roughly $e^{-\delta m}$ than on a certain sequence Σ in the complement $\mathbb{N} \sim \Lambda_0$. It seems at first glance as though we had succeeded in 'controlling' the magnitude of $\Phi(m)$ on Λ_0 by causing it to have zeros at the points of the latter contained in an *extremely sparse* sequence of intervals $[X_j, Y_j]$, that is, by using only an *insignificantly small part* of Λ_0. This is hard to believe. What is going on?

The truth is that we are *not so much controlling* $\Phi(m)$ *on* Λ_0 *as making it large at the points of* $\mathbb{N} \sim \Lambda_0$ *in the intervals* $[X_j, Y_j]$; $|\Phi(m)|$ is of about *the same* order of magnitude *outside* those intervals *whether* $m \in \Lambda_0$ *or not,* as long as L does not divide m. $|\Phi(m)|$ is made large *inside* the $[X_j, Y_j]$ by what amounts to the *removal of some of the zeros that* $F(z/L)$ *has in them.* The latter function vanishes at the points nL, $n \in \mathbb{N}$, and behaves like $2\sin((\pi/L)z)$ on the real axis; $\Phi(z)$ is obtained from it by essentially *replacing* its zeros in each $[X_j, Y_j]$, which are about $(1/L)(Y_j - X_j)$ in number, by the elements of Λ_0 therein, of which there are at most $\varepsilon(Y_j - X_j)$. Since $1/L > \varepsilon$, we are in effect just *throwing away* some of the *zeros that* $F(z/L)$ *has in each interval* $[X_j, Y_j]$ *in order to arrive at* $\Phi(z)$, and the result of this is to make $|\Phi(m)|$ *considerably larger than* $|F(m/L)|$ at the integers $m \notin \Lambda_0$ therein. *Outside* the $[X_j, Y_j]$ (where the modification has taken place), this effect is *less pronounced.* Its evaluation in the two cases (m *inside* one of the intervals or *outside all of them*) depends ultimately on the *behaviour of factorials* – that is the real origin (somewhat disguised by the use of integrals) of the (crucial) terms in $\sigma \log 1/\sigma$ appearing in the lemmas of article 7.

It is the simple monotoneity properties of $(1/x)\log|(x - \lambda)/(x + \lambda)|$ given in article 5 that make the computations work out the way they do; those properties form the basis for Fuchs' construction with the factors

$$\left(\frac{1-z/\lambda}{1+z/\lambda}\right)e^{2z/\lambda}$$

and the resulting appearance of the gamma function. The use of such factors leads of course to *functions analytic in the right half plane rather than to entire functions.* Analogous constructions with entire functions of exponential type would involve the somewhat more complicated monotoneity properties of $(1/x)\log|1-x/\lambda|$ or of $(1/x)\log|1-x^2/\lambda^2|$; for such work one should consult Rubel's 1955 paper and especially the one of Malliavin and Rubel published in 1961.

Malliavin's very difficult 1957 paper is *also* based on use of the factors

$$\left(\frac{1-z/\lambda}{1+z/\lambda}\right)e^{2z/\lambda}$$

(he works mainly with the logarithms of their absolute values), and is thus in part a generalization of Fuchs' work. Keeping this in mind should help anyone who wishes to understand Malliavin's article.

9. Converse of Pólya's gap theorem in general case

Based on Fuchs' construction, we can now establish the

Theorem. *Let $\Sigma \subseteq \mathbb{N}$ have Pólya maximum density $D_\Sigma^* > 0$. Given any $D < D_\Sigma^*$, there is an analytic function*

$$f(w) = \sum_{n\in\Sigma} a_n w^n$$

whose expansion in powers of w has radius of convergence 1, and which can be analytically continued across the arc

$$\{e^{i\vartheta}:\ -\pi D < \vartheta < \pi D\}$$

of the unit circle.

Proof. The method is from the end of Malliavin's 1957 paper. We start by picking a small $\varepsilon > 0$ and a large integer $L < 1/\varepsilon$ with

$$\frac{1}{L} + 2\varepsilon < D_\Sigma^*.$$

Our result has already been established for *measurable* sequences Σ in article 1, so here we may as well assume Σ to be *non-measurable.* Then, as described in article 2, the complement $\mathbb{N} \sim \Sigma$ can be split up into two *disjoint* sequences: a *measurable* one Λ_1 of *density* $1 - D_\Sigma^*$ and *another*, Λ_0, of *minimum density zero*. Using Λ_0 which, in the present circumstances, is really infinite, we form the Fuchs function described in the preceding article, corresponding to the parameters L and ε. From Λ_1 we construct the entire function of exponential type

$$C(z) = \prod_{\lambda \in \Lambda_1} \left(1 - \frac{z^2}{\lambda^2}\right)$$

already considered in article 1.*

According to the theorem of the preceding article, the quantity

$$\gamma = \limsup_{\substack{n \to \infty \\ n \in \Sigma}} \frac{\log|\Phi(n)|}{n}$$

is *finite*. To get our function $f(w)$, we first look at

$$g(w) = \frac{1}{2\pi i} \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} \frac{e^{-\gamma\zeta}\Phi(\zeta)C(\zeta)}{\sin \pi\zeta} w^\zeta \, d\zeta$$

for $|\arg w| < \pi(D_\Sigma^* - 1/L - 2\varepsilon)$; it is claimed that the integral converges *absolutely and uniformly* for w in that sector, making $g(w)$ *analytic* there.

To check this, observe that

$$|C(\zeta)| \leqslant \text{const.}\, e^{\pi((1 - D_\Sigma^*)|\eta| + \varepsilon|\zeta|)}$$

by problem 29 (article 1) – as usual, we are writing $\zeta = \xi + i\eta$. Again by the lemma of article 8,

$$|\Phi(\tfrac{1}{2} + i\eta)| \leqslant \text{const.}\, e^{\pi|\eta|/L}$$

for real η. Hence, for $\zeta = \frac{1}{2} + i\eta$, $\eta \in \mathbb{R}$,

$$\left|\frac{w^\zeta e^{-\gamma\zeta}\Phi(\zeta)C(\zeta)}{\sin \pi\zeta}\right| \leqslant \text{const.} \exp\left\{\left(|\arg w| + \frac{\pi}{L} + \pi\varepsilon - \pi D_\Sigma^*\right)|\eta|\right\},$$

and the asserted convergence is manifest.

We now proceed as in article 1, approximating the above integral by others taken around rectangles. For $\Re\zeta \geqslant 0$, by the lemma of the preceding article,

$$|\Phi(\zeta)| \leqslant \text{const.} \exp\left(\frac{\pi}{L}|\eta| + A\xi\right),$$

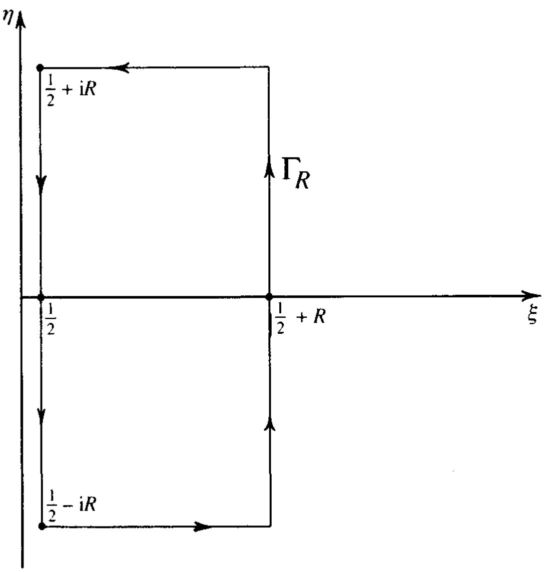
* Should Λ_1 be empty, $C(z)$ is taken equal to 1.

A being a certain constant. From this and our estimate on $|C(\zeta)|$, we thus have

$$\left|\frac{w^\zeta e^{-\gamma\zeta}\Phi(\zeta)C(\zeta)}{\sin\pi\zeta}\right| \leqslant \text{const.}\exp\left\{\xi\log|w| + |\eta||\arg w| - \gamma\xi + \frac{\pi}{L}|\eta| + A\xi + \pi(1-D_\Sigma^*)|\eta| + \pi\varepsilon|\zeta| - \pi|\eta|\right\}$$

$$\leqslant \text{const.}\, e^{(A-\gamma+\pi\varepsilon+\log|w|)\xi}\, e^{(|\arg w|+\pi/L+\pi\varepsilon-\pi D_\Sigma^*)|\eta|}$$

for $\Re\zeta \geqslant 0$, *as long as the distance between* ζ *and the integers* (*zeros of* $\sin\pi\zeta$) stays bounded away from 0. With the help of this relation we now easily see as in article 1 that if R is a *large integer* and Γ_R the contour



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the contributions from the *right-hand*, *top*, and *bottom* parts of Γ_R to the value of

$$\frac{1}{2\pi i}\int_{\Gamma_R}\frac{e^{-\gamma\zeta}\Phi(\zeta)C(\zeta)}{\sin\pi\zeta}w^\zeta\, d\zeta$$

will be *very small* when

$$|w| < e^{-(A-\gamma+2\pi\varepsilon)}$$

and

$$|\arg w| \;<\; \pi\left(D_\Sigma^* \;-\; \frac{1}{L} \;-\; 2\varepsilon\right).$$

This means that for *such* w,

$$\frac{1}{2\pi i}\int_{\Gamma_R} \frac{e^{-\gamma\zeta}\Phi(\zeta)C(\zeta)}{\sin \pi\zeta} w^\zeta \, d\zeta \quad\longrightarrow\quad -\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{e^{-\gamma\zeta}\Phi(\zeta)C(\zeta)}{\sin \pi\zeta} w^\zeta \, d\zeta$$

as the integer R tends to infinity.

As in article 1, by the residue theorem

$$\frac{1}{2\pi i}\int_{\Gamma_R} \frac{e^{-\gamma\zeta}\Phi(\zeta)C(\zeta)}{\sin \pi\zeta} w^\zeta \, d\zeta \;=\; \frac{1}{\pi}\sum_{n=1}^{R} (-1)^n e^{-\gamma n}\Phi(n)C(n)w^n$$

for integers R. Here,

$$|\Phi(n)C(n)| \;\leqslant\; e^{O(n)}$$

for large n, so the power series

$$\frac{1}{\pi}\sum_{1}^{\infty}(-1)^n e^{-\gamma n}\Phi(n)C(n)w^n$$

certainly has a *positive radius of convergence*. For $|w| > 0$ sufficiently small and

$$|\arg w| \;<\; \pi\left(D_\Sigma^* \;-\; \frac{1}{L} \;-\; 2\varepsilon\right),$$

its *sum* must then be equal to

$$-\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{e^{-\gamma\zeta}\Phi(\zeta)C(\zeta)}{\sin \pi\zeta} w^\zeta \, d\zeta \;=\; -g(w)$$

by the preceding two relations.

Let us look more carefully at the power series just written. The function

$$C(z) \;=\; \prod_{\lambda\in\Lambda_1}\left(1 - \frac{z^2}{\lambda^2}\right)$$

vanishes at the points of Λ_1. Therefore, since $\mathbb{N} \sim \Lambda_1 \;=\; \Sigma \cup \Lambda_0$ our series can be written as

$$\frac{1}{\pi}\left(\sum_{n\in\Sigma} \;+\; \sum_{n\in\Lambda_0}\right)(-1)^n e^{-\gamma n}\Phi(n)C(n)w^n.$$

By our choice of γ,

$$\limsup_{\substack{n\to\infty \\ n\in\Sigma}} |e^{-\gamma n}\Phi(n)|^{1/n} \quad = \quad 1,$$

and $|C(n)|^{1/n} \longrightarrow 1$ as $n \longrightarrow \infty$ with $n \notin \Lambda_1$, according to problem 29, part (f) (article 1). Hence

$$\frac{1}{\pi}\sum_{n\in\Sigma}(-1)^n e^{-\gamma n}\Phi(n)C(n)w^n,$$

the *first* of the two power series into which our original one was split, *has radius of convergence* 1 and is *equal*, in $\{|w| < 1\}$, *to a certain function* $f(w)$, *analytic there.*

It is at this point that we apply the main part of the theorem from the preceding article. According to that theorem, if the sequence of intervals $[X_j, Y_j]$ used in constructing the Fuchs function $\Phi(z)$ is sparse enough,

$$\limsup_{\substack{n\to\infty \\ n\in\Lambda_0}} \frac{\log|\Phi(n)|}{n} \quad \leqslant \quad \limsup_{\substack{m\to\infty \\ m\in\Sigma}} \frac{\log|\Phi(m)|}{m} - \delta \quad = \quad \gamma - \delta$$

with a certain constant $\delta > 0$ depending on L and ε. In view of our previous relation involving $C(n)$ we thus have

$$\limsup_{\substack{n\to\infty \\ n\in\Lambda_0}} |e^{-\gamma n}\Phi(n)C(n)|^{1/n} \quad \leqslant \quad e^{-\delta},$$

and the radius of convergence of

$$\frac{1}{\pi}\sum_{n\in\Lambda_0}(-1)^n e^{-\gamma n}\Phi(n)C(n)w^n$$

(the *second* of the series into which our original one was broken) is $\geqslant e^{\delta} > 1$. There is thus a function $h(w)$, *analytic* for $|w| < e^{\delta}$ and *equal* there to the *sum* of this *second series.*

For $|\arg w| < \pi(D_\Sigma^* - 1/L - 2\varepsilon)$ and $|w| > 0$ small enough,

$$f(w) + h(w) \quad = \quad \frac{1}{\pi}\sum_{1}^{\infty}(-1)^n e^{-\gamma n}\Phi(n)C(n)w^n$$

is, as we have just seen, equal to $-g(w)$, a function *analytic in the whole* sector

$$|\arg w| < \pi\left(D_\Sigma^* - \frac{1}{L} - 2\varepsilon\right).$$

The formula

$$f(w) \;=\; -g(w) \;-\; h(w)$$

thus furnishes *an analytic continuation* of $f(w)$ from the unit disk into the *intersection* of our sector with the disk $\{|w| \;<\; e^{\delta}\}$:

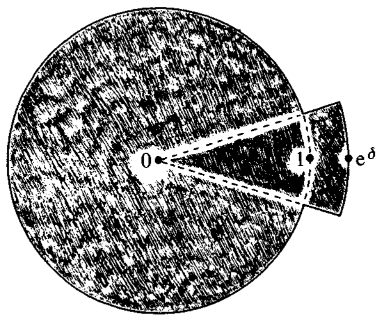

Figure 165

The function given, for $|w| \;<\; 1$, by the power series

$$\frac{1}{\pi}\sum_{n\in\Sigma}(-1)^n e^{-\gamma n}\Phi(n)C(n)w^n,$$

having convergence radius 1 can, in other words, be *continued analytically across the arc*

$$\left\{e^{i\vartheta}\colon\;\; |\vartheta| \;<\; \pi\left(D^*_\Sigma \;-\; \frac{1}{L} \;-\; 2\varepsilon\right)\right\}$$

of the unit circle. Here, $\varepsilon > 0$ can be as small as we like and L is any large integer $<\; 1/\varepsilon$. Hence $D \;=\; D^*_\Sigma \;-\; 1/L \;-\; 2\varepsilon$ can be made as close as we like to D^*_Σ.

Our theorem is proved.

C. A Jensen formula involving confocal ellipses instead of circles

Suppose that we are only interested in the *real zeros* of a function $f(z)$ analytic in some disk $\{|z| < R\}$ with $f(0) \;\neq\; 0$. If we denote by $n(r)$ the number of zeros of f on the segment $[-r,\; r]$, Jensen's formula implies that

$$\int_0^r \frac{n(\rho)}{\rho}\,\mathrm{d}\rho \;\;\leqslant\;\; \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\vartheta})|\,\mathrm{d}\vartheta \;-\; \log|f(0)|$$

for $0 < r < R$. This relation can be used to *estimate* $n(r)$ for certain values of r, and that application has been frequently made in the present book: in Chapter III, for instance, and in §A of this one. Such use of it does, however, involve a drawback – it furnishes a kind of average of $n(\rho)$ for $\rho \leqslant r$ rather than $n(r)$ itself. In order to alleviate this shortcoming, we proceed to derive a similar formula by working with confocal ellipses instead of concentric circles.

The standard Joukowski mapping

$$w \longrightarrow z = \frac{1}{2}\left(w + \frac{1}{w}\right)$$

takes $\{|w| > 1\}$ conformally onto the *complement* (in $\mathbb{C}$) of the real segment $[-1,\ 1]$, and each of the circles $|w| = R > 1$ onto an *ellipse*

$$\frac{4x^2}{(R+R^{-1})^2} + \frac{4y^2}{(R-R^{-1})^2} = 1$$

with foci at 1 and -1:

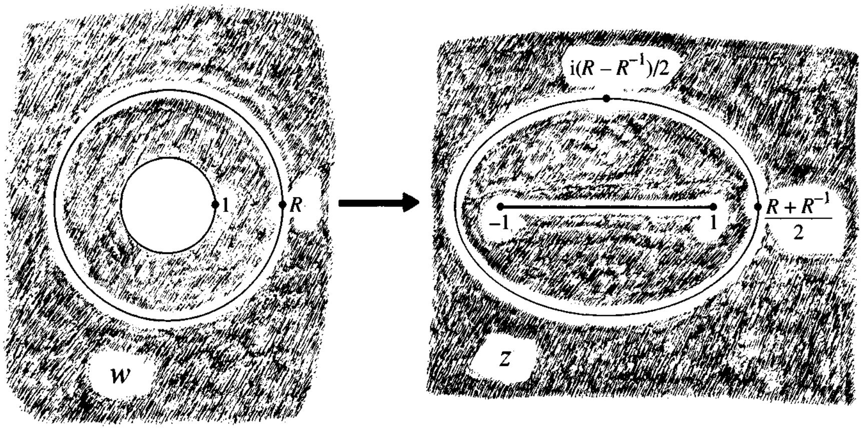

Figure 166

For such an ellipse we have the parametric representation

$$z = \frac{1}{2}\left(Re^{i\vartheta} + \frac{e^{-i\vartheta}}{R}\right),$$

and as R increases, the ellipse gets bigger.

Theorem. *Let $f(z)$ be analytic inside and on the ellipse*

$$z = \frac{1}{2}\left(Re^{i\vartheta} + \frac{e^{-i\vartheta}}{R}\right),$$

$R > 1$, *and, for* $1 < r \leqslant R$, *denote by* $N(r)$ *the* number of zeros *of* f (*counting multiplicities*) inside or on the ellipse

$$z = \frac{1}{2}\left(re^{i\vartheta} + \frac{e^{-i\vartheta}}{r}\right).$$

Then

$$\int_1^R N(r)\frac{dr}{r} = \frac{1}{2\pi}\int_0^{2\pi}\log\left|f\left(\frac{1}{2}\left(Re^{i\vartheta}+\frac{e^{-i\vartheta}}{R}\right)\right)\right|d\vartheta - \frac{1}{\pi}\int_{-1}^{1}\frac{\log|f(x)|}{\sqrt{(1-x^2)}}dx.$$

Proof. Like that of a theorem of Littlewood given at the end of Chapter III in Titchmarsh's *Theory of Functions.* (Our result can in fact be derived from that theorem.)

Suppose that $1 \leqslant r_0 < r_1 \leqslant R$, and that $f(z)$ *has no zeros* inside or on the boundary of the ring-shaped open region bounded by the two ellipses

$$z = \frac{1}{2}\left(r_0e^{i\vartheta} + \frac{e^{-i\vartheta}}{r_0}\right), \qquad z = \frac{1}{2}\left(r_1e^{i\vartheta} + \frac{e^{-i\vartheta}}{r_1}\right).$$

Then $\log f(z)$ can be defined so as to be *analytic and single-valued* in the *simply connected* domain obtained by removing the segment

$$\left(-\tfrac{1}{2}(r_1 + r_1^{-1}),\quad -\tfrac{1}{2}(r_0 + r_0^{-1})\right)$$

from that region:

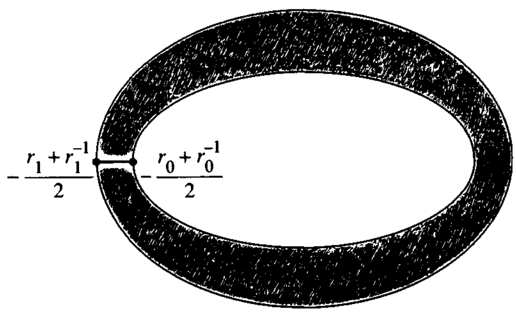

Figure 167

Along the *upper and lower sides* of the removed segment, $\arg f$ will generally have *two different* determinations. Indeed, by the principle of argument,

$$\arg f(x + i0) - \arg f(x - i0) = 2\pi N(r_0)$$

for

$$-\tfrac{1}{2}(r_1 + r_1^{-1}) \leqslant x \leqslant -\tfrac{1}{2}(r_0 + r_0^{-1}).$$

Let us parametrize the *boundary* of our simply connected domain by putting $z = \frac{1}{2}(w + 1/w)$ and then having w go around the path γ shown here:

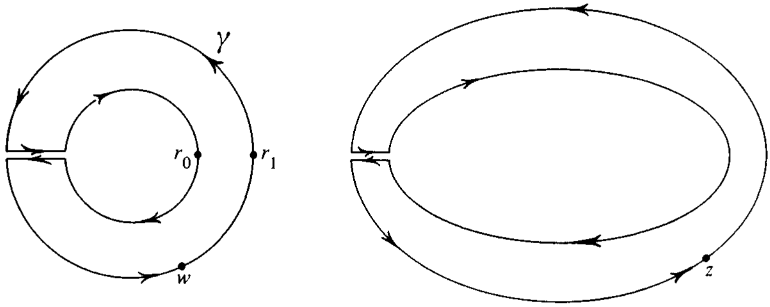

Figure 168

Then, using appropriate determinations of $\log f(\frac{1}{2}(u + 1/u))$ along the upper and lower horizontal stretches of γ, we have by Cauchy's theorem

$$\int_\gamma \log f\left(\frac{1}{2}\left(w+\frac{1}{w}\right)\right)\frac{\mathrm{d}w}{w} \quad = \quad 0.$$

Taking the imaginary part of this relation, we get

$$\int_{-r_1}^{-r_0}\left\{\arg f\left(\frac{u+u^{-1}}{2}+\mathrm{i}0\right) - \arg f\left(\frac{u+u^{-1}}{2}-\mathrm{i}0\right)\right\}\frac{\mathrm{d}u}{u}$$
$$- \int_{-\pi}^{\pi}\log\left|f\left(\frac{1}{2}\left(r_0\mathrm{e}^{\mathrm{i}\vartheta}+\frac{\mathrm{e}^{-\mathrm{i}\vartheta}}{r_0}\right)\right)\right|\mathrm{d}\vartheta$$
$$+ \int_{-\pi}^{\pi}\log\left|f\left(\frac{1}{2}\left(r_1\mathrm{e}^{\mathrm{i}\vartheta}+\frac{\mathrm{e}^{-\mathrm{i}\vartheta}}{r_1}\right)\right)\right|\mathrm{d}\vartheta \quad = \quad 0,$$

i.e.,

$$2\pi N(r_0)\log\frac{r_1}{r_0} \quad = \quad \int_{-\pi}^{\pi}\log\left|f\left(\frac{1}{2}\left(r_1\mathrm{e}^{\mathrm{i}\vartheta}+\frac{\mathrm{e}^{-\mathrm{i}\vartheta}}{r_1}\right)\right)\right|\mathrm{d}\vartheta$$
$$- \int_{-\pi}^{\pi}\log\left|f\left(\frac{1}{2}\left(r_0\mathrm{e}^{\mathrm{i}\vartheta}+\frac{\mathrm{e}^{-\mathrm{i}\vartheta}}{r_0}\right)\right)\right|\mathrm{d}\vartheta,$$

in view of the previous formula.

The integral $\int_{-\pi}^{\pi}\log|f(\frac{1}{2}(r\mathrm{e}^{\mathrm{i}\vartheta} + \mathrm{e}^{-\mathrm{i}\vartheta}/r))|\mathrm{d}\vartheta$ is, however, a *continuous function* of r for $1 \leqslant r \leqslant R$, even at values of r for which the ellipse $z = \frac{1}{2}(r\mathrm{e}^{\mathrm{i}\vartheta} + \mathrm{e}^{-\mathrm{i}\vartheta}/r)$ has zeros of f lying on it – this is immediate by an argument like the one used in Chapter I. The result just found therefore remains valid when f *has* zeros on one or both of the ellipses

$$z = \frac{1}{2}\left(r_0 e^{i\vartheta} + \frac{e^{-i\vartheta}}{r_0}\right), \qquad z = \frac{1}{2}\left(r_1 e^{i\vartheta} + \frac{e^{-i\vartheta}}{r_1}\right),$$

as long as it has *none in the region between them.* Hence, if all the zeros of $f(z)$ inside or on the ellipse $z = \frac{1}{2}(Re^{i\vartheta} + e^{-i\vartheta}/R)$ occur on the ellipses (or segment!)

$$z = \frac{1}{2}\left(\rho_k e^{i\vartheta} + \frac{e^{-i\vartheta}}{\rho_k}\right)$$

with

$$1 \leqslant \rho_1 < \rho_2 < \cdots < \rho_m \leqslant R,$$

we have

$$N(\rho_{k-1})\log\frac{\rho_k}{\rho_{k-1}} = \frac{1}{2\pi}\int_0^{2\pi}\log\left|f\left(\frac{1}{2}\left(\rho_k e^{i\vartheta} + \frac{e^{-i\vartheta}}{\rho_k}\right)\right)\right|d\vartheta$$
$$- \frac{1}{2\pi}\int_0^{2\pi}\log\left|f\left(\frac{1}{2}\left(\rho_{k-1} e^{i\vartheta} + \frac{e^{-i\vartheta}}{\rho_{k-1}}\right)\right)\right|d\vartheta$$

for $k = 2, 3, \ldots, m$. If $\rho_1 > 1$, the same relation holds with $\rho_0 = 1$ and ρ_1, and, if $\rho_m < R$, with ρ_m and $\rho_{m+1} = R$. Since $N(r) = N(\rho_{k-1})$ for $\rho_{k-1} \leqslant r < \rho_k$, addition of all these formulas yields

$$\int_1^R N(r)\frac{dr}{r} = \frac{1}{2\pi}\int_0^{2\pi}\log\left|f\left(\frac{1}{2}\left(Re^{i\vartheta} + \frac{e^{-i\vartheta}}{R}\right)\right)\right|d\vartheta$$
$$- \frac{1}{2\pi}\int_0^{2\pi}\log|f(\cos\vartheta)|\,d\vartheta.$$

Putting $\cos\vartheta = x$ in the second integral on the right now gives us the theorem. Done.

Our desired amelioration of the information obtainable from Jensen's formula is provided by the following

Corollary. *Let a and $\gamma > 0$, and suppose that $F(z)$ is analytic inside and on the ellipse*

$$z = \frac{a}{2}\left(e^{\gamma}e^{i\vartheta} + \frac{e^{-i\vartheta}}{e^{\gamma}}\right) = a\cosh(\gamma + i\vartheta),$$

$0 \leqslant \vartheta \leqslant 2\pi$. *If* $0 \leqslant \eta < \gamma$ *and* $F(z)$ *has at least* N *zeros (counting* *multiplicities) on the segment* $[-a\cosh\eta,\ a\cosh\eta]$ *of the real axis, then*

$$(\gamma-\eta)N \leqslant \frac{1}{2\pi}\int_0^{2\pi}\log|F(a\cosh(\gamma+\mathrm{i}\vartheta))|\,\mathrm{d}\vartheta - \frac{1}{\pi}\int_{-a}^{a}\frac{\log|F(t)|}{\sqrt{(a^2-t^2)}}\mathrm{d}t.$$

Proof. Apply the theorem to $f(z) = F(az)$ with $R = \mathrm{e}^{\gamma}$.

D. A condition for completeness of a collection of imaginary exponentials on a finite interval

Suppose we are given a one-way or two-way strictly increasing sequence of *real* numbers λ_n, with $\lambda_n > 0$ for $n > 0$, $\lambda_n \longrightarrow \infty$ as $n \longrightarrow \infty$, and, if there are λ_n with negative indices, $\lambda_n \leqslant 0$ for $n \leqslant 0$ and $\lambda_n \longrightarrow -\infty$ as $n \longrightarrow -\infty$. Considering the corresponding set of *imaginary exponentials* $\mathrm{e}^{\mathrm{i}\lambda_n t}$, we take any number $L > 0$ and ask whether the finite linear combinations of these exponentials are *uniformly dense* in $\mathscr{C}(-L, L)$. If they *are*, the exponentials $\mathrm{e}^{\mathrm{i}\lambda_n t}$ are said to be *complete* on $[-L, L]$; otherwise, they are *incomplete* on that interval.

If the $\mathrm{e}^{\mathrm{i}\lambda_n t}$ are complete on $[-L, L]$, they are obviously complete on $[-L', L']$ for any L' with $0 < L' < L$. There is thus a certain number A associated with those exponentials, $0 \leqslant A \leqslant \infty$, such that the former are *complete* on $[-L, L]$ if $0 < L < A$ and *incomplete* on $[-L, L]$ if $L > A$. We are, of course, not limited here to consideration of intervals centred at the origin; when $0 < A < \infty$ it is immediate (by translation!) that the $\mathrm{e}^{\mathrm{i}\lambda_n t}$ will in fact be complete on any real interval of length $< 2A$ and incomplete on any one of length $> 2A$. In the extreme case where $A = 0$, the given exponentials are incomplete on any real interval of length > 0, and, when $A = \infty$, they are complete on all finite intervals. Simple examples show that both of these extreme cases are possible.

Regarding completeness of the exponentials on intervals of length *exactly equal* to $2A$, nothing can be said *a priori*. There are examples in which the $\mathrm{e}^{\mathrm{i}\lambda_n t}$ are *complete* on $[-A, A]$ and others where they are *incomplete* thereon. Without going into the matter at all, it seems clear that the outcome in this borderline situation must depend in very delicate and subtle fashion on the sequence of frequencies λ_n. We will not consider that particular question in this book; various fragmentary results concerning it may be found in Levinson's monograph and in Redheffer's expository article.

What interests us is the more basic problem of finding out how the number A – L. Schwartz called it the *completeness radius* associated with the λ_n – actually depends on those frequencies. We would like, if possible, to get a *formula* relating A to the distribution of the λ_n.

This important question was investigated by Paley and Wiener, Levinson, L. Schwartz and others. A complete solution was obtained around 1960 by Beurling and Malliavin, whose work involved two main steps:

(i) The determination of a certain *lower bound* for A,

(ii) Proof that the lower bound found in (i) is also an *upper bound* for A.

The first of these can be presented quite simply using the formula from the preceding §; that is what we will do presently. The second step is much more difficult; its completion required a deep existence theorem established expressly for that purpose by Beurling and Malliavin. That part of the solution will be given in Chapter X, with proof of the existence theorem itself deferred until Chapter XI.

The first step amounts to a *proof of completeness* of the $e^{i\lambda_n t}$ on intervals $[-L, L]$ with L small enough (depending on the λ_n). The idea for this goes back to Szasz and to Paley and Wiener.

Reasoning by contradiction, we take an $L > 0$ and assume *incompleteness* of the $e^{i\lambda_n t}$ on $[-L, L]$. *Duality* (Hahn–Banach theorem) then gives us a *non-zero complex measure* μ on $[-L, L]$ with

$$\int_{-L}^{L} e^{i\lambda_n t} d\mu(t) = 0$$

for each λ_n, i.e., $\hat{\mu}(\lambda_n) = 0$ for the Fourier–Stieltjes transform

$$\hat{\mu}(z) = \int_{-L}^{L} e^{izt} d\mu(t).$$

The function $\hat{\mu}(z)$ is entire, of exponential type $\leqslant L$, and bounded on the real axis. Using a familiar result from Chapter III, §G.2, together with the one from the preceding §, one now shows that for *small enough* $L > 0$, the *zeros* λ_n of $\hat{\mu}(z)$ *cannot* (in some suitable sense) *be too dense without forcing* $\hat{\mu}(z) \equiv 0$, contrary to our choice of μ.

The details of this argument are given in the following article. Before proceeding to it, we should observe how duality can be used to demonstrate one very important fact: *the completeness radius A associated with a sequence $\{\lambda_n\}$ is not really specific to the topology of uniform convergence and the spaces $\mathscr{C}(-L, L)$. If, in place of $\mathscr{C}(-L, L)$, we take any of the spaces $L_p(-L, L)$, $1 \leqslant p < \infty$, the value of A corresponding to a given sequence of frequencies λ_n turns out to be the same.*