In fact, by (35), (38) and Theorem 2.11

$$u = c \lim_{r \to 0} r^{n-2} (M^r Q(\Box)u)(o) = c \int_{C_R} (Q(\Box)u)(z)\, d\mu(z)$$

and this is (39).

Remark 5.2. Formula (39) shows that each factor

$$\Box_k = \Box - \kappa(n-k)(k-1) \quad k = 3, 5, \ldots, n-1 \tag{40}$$

in $Q(\Box)$ has fundamental solution supported on the *retrograde conical surface* $\overline{C}_R$. This is known to be the equivalent to the validity of Huygens' principle for the Cauchy problem for the equation $\Box_k u = 0$ (see Gűnther [1991] and [1988], Ch. IV, Cor. 1.13). For a recent survey on Huygens' principle see Berest [1998].

Bibliographical Notes

§1. The construction of the constant curvature spaces (Theorems 1.3 and 1.5) was given by the author ([1959], [1961]). The proof of Lemma 1.4 on the connectivity is adapted from Boerner [1955]. For more information on isotropic manifolds (there is more than one definition) see Tits [1955], p. 183 and Wolf [1967].

§§2-4. This material is based on Ch. IV in Helgason [1959]. Corollary 5.1 with a different proof and the subsequent remark were shown to me by Schlichtkrull. See Schimming and Schlichtkrull [1994] (in particular Lemma 6.2) where it is also shown that the constants $c_k = -\kappa(n-k)(k-1)$ in (40) are *the only ones* for which $\Box + c_k$ satisfies Huygens' principle. Here it is of interest to recall that in the flat Lorentzian case $\mathbf{R}^{2m}$, $\Box + c$ satisfies Huygens' principle only for $c = 0$. Theorem 4.1 was extended to pseudo-Riemannian manifolds of constant curvature by Orloff [1985], [1987]. For recent representative work on orbital integrals see e.g. Bouaziz [1995], Flicker [1996], Harinck [1998], Renard [1997].

CHAPTER V

FOURIER TRANSFORMS AND DISTRIBUTIONS. A RAPID COURSE

§1 The Topology of the Spaces $\mathcal{D}(\mathbf{R}^n)$, $\mathcal{E}(\mathbf{R}^n)$ and $\mathcal{S}(\mathbf{R}^n)$

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Let $\mathbf{R}^n = \{x = (x_1, \dots, x_n) : x_i \in \mathbf{R}\}$ and let ∂_i denote $\partial/\partial x_i$. If $(\alpha_1, \dots, \alpha_n)$ is an n-tuple of integers $\alpha_i \geq 0$ we put $\alpha! = \alpha_1! \cdots \alpha_n!$,

$$D^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}\,,\ x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}\,,\ |\alpha| = \alpha_1 + \cdots + \alpha_n\,.$$

For a complex number c, $\operatorname{Re} c$ and $\operatorname{Im} c$ denote respectively, the real part and the imaginary part of c. For a given compact set $K \subset \mathbf{R}^n$ let

$$\mathcal{D}_K = \mathcal{D}_K(\mathbf{R}^n) = \{f \in \mathcal{D}(\mathbf{R}^n) : \operatorname{supp}(f) \subset K\}\,,$$

where supp stands for support. The space $\mathcal{D}_K$ is topologized by the seminorms

$$\|f\|_{K,m} = \sum_{|\alpha| \leq m} \sup_{x \in K} |(D^\alpha f)(x))|, \quad m \in \mathbb{Z}^+\,. \tag{1}$$

The topology of $\mathcal{D} = \mathcal{D}(\mathbf{R}^n)$ is defined as the largest locally convex topology for which all the embedding maps $\mathcal{D}_K \to \mathcal{D}$ are continuous. This is the so-called *inductive limit* topology. More explicitly, this topology is characterized as follows:

A *convex* set $C \subset \mathcal{D}$ is a neighborhood of 0 in $\mathcal{D}$ if and only if for each compact set $K \subset \mathbf{R}^n$, $C \cap \mathcal{D}_K$ is a neighborhood of 0 in $\mathcal{D}_K$.

A fundamental system of neighborhoods in $\mathcal{D}$ can be characterized by the following theorem. If B_R denotes the ball $|x| < R$ in $\mathbf{R}^n$ then

$$\mathcal{D} = \cup_{j=0}^\infty \mathcal{D}_{\overline{B}_j}\,. \tag{2}$$

Theorem 1.1. *Given two monotone sequences*

$$\begin{aligned} &\{\epsilon\} = \epsilon_0, \epsilon_1, \epsilon_2, \dots, && \epsilon_i \to 0 \\ &\{N\} = N_0, N_1, N_2, \dots, && N_i \to \infty \quad N_i \in \mathbb{Z}^+ \end{aligned}$$

let $V(\{\epsilon\}, \{N\})$ denote the set of functions $\varphi \in \mathcal{D}$ satisfying for each j the conditions

$$|(D^\alpha \varphi)(x)| \leq \epsilon_j \quad \textit{for } |\alpha| \leq N_j, \quad x \notin B_j\,. \tag{3}$$

Then the sets $V(\{\epsilon\}, \{N\})$ form a fundamental system of neighborhoods of 0 in $\mathcal{D}$.

Proof. It is obvious that each $V(\{\epsilon\},\{N\})$ intersects each $\mathcal{D}_K$ in a neighborhood of 0 in $\mathcal{D}_K$. Conversely, let W be a *convex* subset of $\mathcal{D}$ intersecting each $\mathcal{D}_K$ in a neighborhood of 0. For each $j \in \mathbb{Z}^+$, $\exists N_j \in \mathbb{Z}^+$ and $\eta_j > 0$ such that each $\varphi \in \mathcal{D}$ satisfying

$$|D^\alpha \varphi(x)| \leq \eta_j \text{ for } |\alpha| \leq N_j \quad \operatorname{supp}(\varphi) \subset \overline{B}_{j+2}$$

belongs to W. Fix a sequence (β_j) with

$$\beta_j \in \mathcal{D}, \beta_j \geq 0\,,\ \Sigma\beta_j = 1\,,\ \operatorname{supp}(\beta_j) \subset \overline{B}_{j+2} - B_j$$

and write for $\varphi \in \mathcal{D}$,

$$\varphi = \sum_j \frac{1}{2^{j+1}}(2^{j+1}\beta_j\varphi)\,.$$

Then by the convexity of W, $\varphi \in W$ if each function $2^{j+1}\beta_j\varphi$ belongs to W. However, $D^\alpha(\beta_j\varphi)$ is a finite linear combination of derivatives $D^\beta\beta_j$ and $D^\gamma\varphi$, $(|\beta|, |\gamma| \leq |\alpha|)$. Since (β_j) is fixed and only values of φ in $\overline{B}_{j+2} - B_j$ enter, $\exists$ constant k_j such that the condition

$$|(D^\alpha\varphi)(x)| \leq \epsilon_j \text{ for } |x| \geq j \text{ and } |\alpha| \leq N_j$$

implies

$$|2^{j+1}D^\alpha(\beta_j\varphi)(x)| \leq k_j\epsilon_j \quad \text{for } |\alpha| \leq N_j, \quad \text{all } x\,.$$

Choosing the sequence $\{\epsilon\}$ such that $k_j\epsilon_j \leq \eta_j$ for all j we deduce for each j

$$\varphi \in V(\{\epsilon\},\{N\}) \Rightarrow 2^{j+1}\beta_j\varphi \in W\,,$$

whence $\varphi \in W$.

The space $\mathcal{E} = \mathcal{E}(\mathbf{R}^n)$ is topologized by the seminorms (1) for the varying K. Thus the sets

$$V_{j,k,\ell} = \{\varphi \in \mathcal{E}(\mathbf{R}^n) : \|\varphi\|_{\overline{B}_j,k} < 1/\ell \quad j,k,\ell \in \mathbb{Z}^+$$

form a fundamental system of neighborhoods of 0 in $\mathcal{E}(\mathbf{R}^n)$. This system being countable the topology of $\mathcal{E}(\mathbf{R}^n)$ is defined by sequences: A point $\varphi \in \mathcal{E}(\mathbf{R}^n)$ belongs to the closure of a subset $A \subset \mathcal{E}(\mathbf{R}^n)$ if and only if φ is the limit of a sequence in A. It is important to realize that this fails for the topology of $\mathcal{D}(\mathbf{R}^n)$ since the family of sets $V(\{\epsilon\},\{N\})$ is uncountable.

The space $\mathcal{S} = \mathcal{S}(\mathbf{R}^n)$ of rapidly decreasing functions on $\mathbf{R}^n$ is topologized by the seminorms (6), Ch. I. We can restrict the P in (6), Ch. I to polynomials with rational coefficients.

In contrast to the space $\mathcal{D}$ the spaces $\mathcal{D}_K$, $\mathcal{E}$ and $\mathcal{S}$ are Fréchet spaces, that is their topologies are given by a countable family of seminorms.

The spaces $\mathcal{D}_K(M)$, $\mathcal{D}(M)$ and $\mathcal{E}(M)$ can be topologized similarly if M is a manifold.

§2 Distributions

A *distribution* by definition is a member of the dual space $\mathcal{D}'(\mathbf{R}^n)$ of $\mathcal{D}(\mathbf{R}^n)$. By the definition of the topology of $\mathcal{D}$, $T \in \mathcal{D}'$ if and only if the restriction $T|\mathcal{D}_K$ is continuous for each compact set $K \subset \mathbf{R}^n$. Each locally integrable function F on $\mathbf{R}^n$ gives rise to a distribution $\varphi \to \int \varphi(x)F(x)\,dx$. A measure on $\mathbf{R}^n$ is also a distribution.

The *derivative* $\partial_i T$ of a distribution T is by definition the distribution $\varphi \to -T(\partial_i\varphi)$. If $F \in C^1(\mathbf{R}^n)$ then the distributions $T_{\partial_i F}$ and $\partial_i(T_F)$ coincide (integration by parts).

A *tempered distribution* by definition is a member of the dual space $\mathcal{S}'(\mathbf{R}^n)$. Since the imbedding $\mathcal{D} \to \mathcal{S}$ is continuous the restriction of a $T \in \mathcal{S}'$ to $\mathcal{D}$ is a distribution; since $\mathcal{D}$ is dense in $\mathcal{S}$ two tempered distributions coincide if they coincide on $\mathcal{D}$. In this sense we have $\mathcal{S}' \subset \mathcal{D}'$.

Since distributions generalize measures it is sometimes convenient to write

$$T(\varphi) = \int \varphi(x)\,dT(x)$$

for the value of a distribution on the function φ. A distribution T is said to be 0 on an open set $U \subset \mathbf{R}^n$ if $T(\varphi) = 0$ for each $\varphi \in \mathcal{D}$ with support contained in U. Let U be the union of all open sets $U_\alpha \subset \mathbf{R}^n$ on which T is 0. Then $T = 0$ on U. In fact, if $f \in \mathcal{D}(U)$, $\operatorname{supp}(f)$ can be covered by finitely many U_α, say $U_1, \ldots, U_r$. Then $U_1, \ldots, U_r, \mathbf{R}^n - \operatorname{supp}(f)$ is a covering of $\mathbf{R}^n$. If $1 = \sum_1^{r+1} \varphi_i$ is a corresponding partition of unity we have $f = \sum_1^r \varphi_i f$ so $T(f) = 0$. The complement $\mathbf{R}^n - U$ is called the *support of* T, denoted $\operatorname{supp}(T)$.

A distribution T of compact support extends to a unique element of $\mathcal{E}'(\mathbf{R}^n)$ by putting

$$T(\varphi) = T(\varphi\varphi_0)\,, \quad \varphi \in \mathcal{E}(\mathbf{R}^n)$$

if φ_0 is any function in $\mathcal{D}$ which is identically 1 on a neighborhood of $\operatorname{supp}(T)$. Since $\mathcal{D}$ is dense in $\mathcal{E}$, this extension is unique. On the other hand let $\tau \in \mathcal{E}'(\mathbf{R}^n)$, T its restriction to $\mathcal{D}$. Then $\operatorname{supp}(T)$ is compact. Otherwise we could for each j find $\varphi_j \in \mathcal{E}$ such that $\varphi_j \equiv 0$ on $\overline{B}_j$ but $T(\varphi_j) = 1$. Then $\varphi_j \to 0$ in $\mathcal{E}$, yet $\tau(\varphi_j) = 1$ which is a contradiction.

This identifies $\mathcal{E}'(\mathbf{R}^n)$ with the space of distributions of compact support and we have the following canonical inclusions:

$$\begin{array}{ccc} \mathcal{D}(\mathbf{R}^n) & \subset \mathcal{S}(\mathbf{R}^n) & \subset \mathcal{E}(\mathbf{R}^n) \\ \cap & \cap & \cap \\ \mathcal{E}'(\mathbf{R}^n) & \subset \mathcal{S}'(\mathbf{R}^n) & \subset \mathcal{D}'(\mathbf{R}^n)\,. \end{array}$$

If S and T are two distributions, at least one of compact support, their *convolution* is the distribution $S * T$ defined by

$$\varphi \to \int \varphi(x+y)\, dS(x)\, dT(y), \quad \varphi \in \mathcal{D}(\mathbf{R}^n)\,. \tag{4}$$

If $f \in \mathcal{D}$ the distribution $T_f * T$ has the form T_g where

$$g(x) = \int f(x-y)\, dT(y)$$

so we write for simplicity $g = f * T$. Note that $g(x) = 0$ unless $x - y \in \operatorname{supp}(f)$ for some $y \in \operatorname{supp}(T)$. Thus $\operatorname{supp}(g) \subset \operatorname{supp}(f) + \operatorname{supp} T$. More generally,

$$\operatorname{supp}(S * T) \subset \operatorname{supp}(S) + \operatorname{supp} T$$

as one sees from the special case $S = T_g$ by approximating S by functions $S * \varphi_\epsilon$ with $\operatorname{supp}(\varphi_\epsilon) \subset B_\epsilon(0)$.

The convolution can be defined for more general S and T, for example if $S \in \mathcal{S}$, $T \in \mathcal{S}'$ then $S * T \in \mathcal{S}'$. Also $S \in \mathcal{E}'$, $T \in \mathcal{S}'$ implies $S * T \in \mathcal{S}'$.

§3 The Fourier Transform

For $f \in L^1(\mathbf{R}^n)$ the *Fourier transform* is defined by

$$\widetilde{f}(\xi) = \int_{\mathbf{R}^n} f(x) e^{-i\langle x, \xi\rangle}\, dx, \quad \xi \in \mathbf{R}^n\,. \tag{5}$$

If f has compact support we can take $\xi \in \mathbf{C}^n$. For $f \in \mathcal{S}(\mathbf{R}^n)$ one proves quickly

$$i^{|\alpha|+|\beta|} \xi^\beta (D^\alpha \widetilde{f})(\xi) = \int_{\mathbf{R}^n} D^\beta(x^\alpha f(x)) e^{-i\langle x, \xi\rangle}\, dx \tag{6}$$

and this implies easily the following result.

Theorem 3.1. *The Fourier transform is a linear homeomorphism of $\mathcal{S}$ onto $\mathcal{S}$.*

The function $\psi(x) = e^{-x^2/2}$ on $\mathbf{R}$ satisfies $\psi'(x) + x\psi = 0$. It follows from (6) that $\widetilde{\psi}$ satisfies the same differential equation and thus is a constant multiple of $e^{-\xi^2/2}$. Since $\widetilde{\psi}(0) = \int e^{-\frac{x^2}{2}}\, dx = (2\pi)^{1/2}$ we deduce $\widetilde{\psi}(\xi) = (2\pi)^{1/2} e^{-\xi^2/2}$. More generally, if $\psi(x) = e^{-|x|^2/2}$, $(x \in \mathbf{R}^n)$ then by product integration

$$\widetilde{\psi}(\xi) = (2\pi)^{n/2} e^{-|\xi|^2/2}\,. \tag{7}$$

Theorem 3.2. *The Fourier transform has the following properties.*

(i) $\quad f(x) = (2\pi)^{-n} \int \widetilde{f}(\xi) e^{i\langle x,\xi\rangle}\, d\xi \quad$ *for* $f \in \mathcal{S}$.

(ii) $\quad f \to \widetilde{f}$ *extends to a bijection of* $L^2(\mathbf{R}^n)$ *onto itself and*

$$\int_{\mathbf{R}^n} |f(x)|^2 = (2\pi)^{-n} \int_{\mathbf{R}^n} |\widetilde{f}(\xi)|^2 \, d\xi\,.$$

(iii) $\;(f_1 * f_2)^{\sim} = \widetilde{f}_1 \widetilde{f}_2 \quad$ *for* $\; f_1, f_2 \in \mathcal{S}$.

(iv) $\;(f_1 f_2)^{\sim} = (2\pi)^{-n} \widetilde{f}_1 * \widetilde{f}_2 \quad$ *for* $f_1, f_2 \in \mathcal{S}$.

Proof. (i) The integral on the right equals

$$\int e^{i\langle x,\xi\rangle} \left(\int f(y) e^{-i\langle y,\xi\rangle}\, dy \right) d\xi$$

but here we cannot exchange the integrations. Instead we consider for $g \in \mathcal{S}$ the integral

$$\int e^{i\langle x,\xi\rangle} g(\xi) \left(\int f(y) e^{-i\langle y,\xi\rangle}\, dy \right) d\xi\,,$$

which equals the expressions

$$\int \widetilde{f}(\xi) g(\xi) e^{i\langle x,\xi\rangle}\, d\xi = \int f(y) \widetilde{g}(y-x)\, dy = \int f(x+y) \widetilde{g}(y)\, dy\,. \tag{8}$$

Replace $g(\xi)$ by $g(\epsilon\xi)$ whose Fourier transform is $\epsilon^{-n}\widetilde{g}(y/\epsilon)$. Then we obtain

$$\int \widetilde{f}(\xi) g(\epsilon\xi) e^{i\langle x,\xi\rangle}\, d\xi = \int \widetilde{g}(y) f(x+\epsilon y)\, dy\,,$$

which upon letting $\epsilon \to 0$ gives

$$g(0) \int \widetilde{f}(\xi) e^{i\langle x,\xi\rangle}\, d\xi = f(x) \int \widetilde{g}(y)\, dy\,.$$

Taking $g(\xi)$ as $e^{-|\xi|^2/2}$ and using (7) Part (i) follows. The identity in (ii) follows from (8) (for $x = 0$) and (i). It implies that the image $L^2(\mathbf{R}^n)^{\sim}$ is closed in $L^2(\mathbf{R}^n)$. Since it contains the dense subspace $\mathcal{S}(\mathbf{R}^n)$ (ii) follows. Formula (iii) is an elementary computation and now (iv) follows taking (i) into account.

If $T \in \mathcal{S}'(\mathbf{R}^n)$ its Fourier transform is the linear form $\widetilde{T}$ on $\mathcal{S}(\mathbf{R}^n)$ defined by

$$\widetilde{T}(\varphi) = T(\widetilde{\varphi})\,. \tag{9}$$

Then by Theorem 3.1, $\widetilde{T} \in \mathcal{S}'$. Note that

$$\int \varphi(\xi) \widetilde{f}(\xi)\, d\xi = \int \widetilde{\varphi}(x) f(x)\, dx \tag{10}$$

for all $f \in L^1(\mathbf{R}^n)$, $\varphi \in \mathcal{S}(\mathbf{R}^n)$. Consequently

$$(T_f)^\sim = T_{\widetilde{f}} \quad \text{for } f \in L^1(\mathbf{R}^n) \tag{11}$$

so the definition (9) extends the old one (5). If $S_1, S_2 \in \mathcal{E}'(\mathbf{R}^n)$ then $\widetilde{S}_1$ and $\widetilde{S}_2$ have the form T_{s_1} and T_{s_2} where $s_1, s_2 \in \mathcal{E}(\mathbf{R}^n)$ and in addition $(S_1 * S_2)^\sim = T_{s_1 s_2}$. We express this in the form

$$(S_1 * S_2)^\sim = \widetilde{S}_1 \widetilde{S}_2 \,. \tag{12}$$

This formula holds also in the cases

$$S_1 \in \mathcal{S}(\mathbf{R}^n), \quad S_2 \in \mathcal{S}'(\mathbf{R}^n)\,,$$
$$S_1 \in \mathcal{E}'(\mathbf{R}^n), \quad S_2 \in \mathcal{S}'(\mathbf{R}^n)$$

and $S_1 * S_2 \in \mathcal{S}'(\mathbf{R}^n)$ (cf. Schwartz [1966], p. 268).

The classical Paley-Wiener theorem gave an intrinsic description of $L^2(0, 2\pi)^\sim$. We now prove an extension to a characterization of $\mathcal{D}(\mathbf{R}^n)^\sim$ and $\mathcal{E}'(\mathbf{R}^n)^\sim$.

Theorem 3.3. *(i) A holomorphic function $F(\zeta)$ on $\mathbf{C}^n$ is the Fourier transform of a distribution with support in $\overline{B}_R$ if and only if for some constants C and $N \geq 0$ we have*

$$|F(\zeta)| \leq C(1 + |\zeta|^N) e^{R|\operatorname{Im}\zeta|} \,. \tag{13}$$

(ii) $F(\zeta)$ is the Fourier transform of a function in $\mathcal{D}_{\bar{B}_R}(\mathbf{R}^n)$ if and only if for each $N \in \mathbb{Z}^+$ there exists a constant C_N such that

$$|F(\zeta)| \leq C_N (1 + |\zeta|)^{-N} e^{R|\operatorname{Im}\zeta|} \,. \tag{14}$$

Proof. First we prove that (13) is necessary. Let $T \in \mathcal{E}'$ have support in $\overline{B}_R$ and let $\chi \in \mathcal{D}$ have support in $\overline{B_{R+1}}$ and be identically 1 in a neighborhood of $\overline{B}_R$. Since $\mathcal{E}(\mathbf{R}^n)$ is topologized by the semi-norms (1) for varying K and m we have for some $C_0 \geq 0$ and $N \in \mathbb{Z}^+$

$$|T(\varphi)| = |T(\chi\varphi)| \leq C_0 \sum_{|\alpha| \leq N} \sup_{x \in \overline{B_{R+1}}} |(D^\alpha(\chi\varphi))(x)| \,.$$

Computing $D^\alpha(\chi\varphi)$ we see that for another constant C_1

$$|T(\varphi)| \leq C_1 \sum_{|\alpha| \leq N} \sup_{x \in \mathbf{R}^n} |D^\alpha \varphi(x)|, \quad \varphi \in \mathcal{E}(\mathbf{R}^n). \tag{15}$$

Let $\psi \in \mathcal{E}(\mathbf{R})$ such that $\psi \equiv 1$ on $(-\infty, \frac{1}{2})$, and $\equiv 0$ on $(1, \infty)$. Then if $\zeta \neq 0$ the function

$$\varphi_\zeta(x) = e^{-i\langle x, \zeta\rangle} \psi(|\zeta|(|x| - R))$$

belongs to $\mathcal{D}$ and equals $e^{-i\langle x,\zeta\rangle}$ in a neighborhood of $\overline{B}_R$. Hence

$$|\widetilde{T}(\zeta)| = |T(\varphi_\zeta)| \leq C_1 \sum_{|\alpha|\leq N} \sup |D^\alpha \varphi_\zeta| \,. \tag{16}$$

Now $\operatorname{supp}(\varphi_\zeta) \subset \overline{B}_{R+|\zeta|^{-1}}$ and on this ball

$$|e^{-i\langle x,\zeta\rangle}| \leq e^{|x|\,|\operatorname{Im}\zeta|} \leq e^{(R+|\zeta|^{-1})|\operatorname{Im}\zeta|} \leq e^{R|\operatorname{Im}\zeta|+1} \,.$$

Estimating $D^\alpha \varphi_\zeta$ similarly we see that by (16), $\widetilde{T}(\zeta)$ satisfies (13).

The necessity of (14) is an easy consequence of (6).

Next we prove the sufficiency of (14). Let

$$f(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} F(\xi) e^{i\langle x,\xi\rangle}\, d\xi \,. \tag{17}$$

Because of (14) we can shift the integration in (17) to the complex domain so that for any fixed $\eta \in \mathbf{R}^n$,

$$f(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} F(\xi + i\eta) e^{i\langle x,\xi+i\eta\rangle}\, d\xi \,.$$

We use (14) for $N = n+1$ to estimate this integral and this gives

$$|f(x)| \leq C_N e^{R|\eta| - \langle x,\eta\rangle} (2\pi)^{-n} \int_{\mathbf{R}^n} (1+|\xi|)^{-(n+1)}\, d\xi \,.$$

Taking now $\eta = tx$ and letting $t \to +\infty$ we deduce $f(x) = 0$ for $|x| > R$.

For the sufficiency of (13) we note first that F as a distribution on $\mathbf{R}^n$ is tempered. Thus $F = \widetilde{f}$ for some $f \in \mathcal{S}'(\mathbf{R}^n)$. Convolving f with a $\varphi \in \mathcal{D}_{\overline{B}_\epsilon}$ we see that $f * \varphi$ satisfies estimates (14) with R replaced by $R+\epsilon$. Thus $\operatorname{supp}(f * \varphi_\epsilon) \subset \overline{B_{R+\epsilon}}$. Letting $\epsilon \to 0$ we deduce $\operatorname{supp}(f) \subset \overline{B_R}$, concluding the proof.

We shall now prove a refinement of Theorem 3.3 in that the topology of $\mathcal{D}$ is described in terms of $\widetilde{\mathcal{D}}$. This has important applications to differential equations as we shall see in the next section.

Theorem 3.4. *A convex set $V \subset \mathcal{D}$ is a neighborhood of 0 in $\mathcal{D}$ if and only if there exist positive sequences*

$$M_0, M_1, \ldots,\ \delta_0, \delta_1, \ldots$$

such that V contains all $u \in \mathcal{D}$ satisfying

$$|\widetilde{u}(\zeta)| \leq \sum_{k=0}^{\infty} \delta_k \frac{1}{(1+|\zeta|)^{M_k}} e^{k|\operatorname{Im}\zeta|}, \quad \zeta \in \mathbf{C}^n \,. \tag{18}$$

The proof is an elaboration of that of Theorem 3.3. Instead of the contour shift $\mathbf{R}^n \to \mathbf{R}^n + i\eta$ used there one now shifts $\mathbf{R}^n$ to a contour on which the two factors on the right in (14) are comparable.

Let $W(\{\delta\},\{M\})$ denote the set of $u \in \mathcal{D}$ satisfying (18). Given k the set

$$W_k = \{u \in \mathcal{D}_{\overline{B}_k} : |\widetilde{u}(\zeta)| \leq \delta_k (1+|\zeta|)^{-M_k} e^{k|\operatorname{Im}\zeta|}\}$$

is contained in $W(\{\delta\},\{M\})$. Thus if V is a convex set containing $W(\{\delta\},\{M\})$ then $V \cap \mathcal{D}_{\overline{B}_k}$ contains W_k which is a neighborhood of 0 in $\mathcal{D}_{\overline{B}_k}$ (because the bounds on $\widetilde{u}$ correspond to the bounds on the $\|u\|_{\overline{B}_k, M_k}$). Thus V is a neighborhood of 0 in $\mathcal{D}$.

Proving the converse amounts to proving that given $V(\{\epsilon\},\{N\})$ in Theorem 1.1 there exist $\{\delta\},\{M\}$ such that

$$W(\{\delta\},\{M\}) \subset V(\{\epsilon\},\{N\})\,.$$

For this we shift the contour in (17) to others where the two factors in (14) are comparable. Let

$$\begin{aligned} x &= (x_1,\dots,x_n), & x' &= (x_1,\dots,x_{n-1}) \\ \zeta &= (\zeta_1,\dots,\zeta_n) & \zeta' &= (\zeta_1,\dots,\zeta_{n-1}) \\ \zeta &= \xi + i\eta, & \xi,\eta &\in \mathbf{R}^n\,. \end{aligned}$$

Then

$$\int_{\mathbf{R}^n} \widetilde{u}(\xi) e^{i\langle x,\xi\rangle}\, d\xi = \int_{\mathbf{R}^{n-1}} e^{i\langle x',\xi'\rangle}\, d\xi' \int_{\mathbf{R}} e^{ix_n\xi_n} \widetilde{u}(\xi',\xi_n)\, d\xi_n\,. \tag{19}$$

In the last integral we shift from $\mathbf{R}$ to the contour in $\mathbf{C}$ given by

$$\gamma_m : \zeta_n = \xi_n + im\log(1 + (|\xi'|^2 + \xi_n^2)^{1/2}) \tag{20}$$

$m \in \mathbb{Z}^+$ being fixed.

We claim that (cf. Fig. V.1)

$$\int_{\mathbf{R}} e^{ix_n\xi_n} \widetilde{u}(\xi',\xi_n)\, d\xi_n = \int_{\gamma_m} e^{ix_n\zeta_n} \widetilde{u}(\xi',\zeta_n)\, d\zeta_n\,. \tag{21}$$

Since (14) holds for each N, $\widetilde{u}$ decays between ξ_n−axis and γ_m faster than any $|\zeta_n|^{-M}$. Also

$$\left|\frac{d\zeta_n}{d\xi_n}\right| = \left|1 + i\,m\,\frac{1}{1+|\xi|} \cdot \frac{\partial(|\xi|)}{\partial\xi_n}\right| \leq 1 + m\,.$$

Thus (21) follows from Cauchy's theorem in *one* variable. Putting

$$\Gamma_m = \{\zeta \in \mathbf{C}^n : \zeta' \in \mathbf{R}^{n-1}, \zeta_n \in \gamma_m\}$$

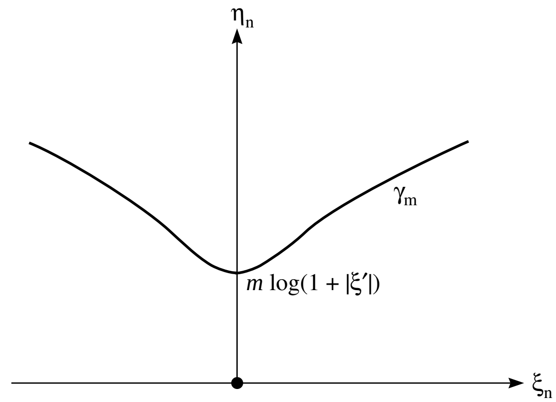

FIGURE V.1.

we thus have with $d\zeta = d\xi_1 \dots d\xi_{n-1}\, d\zeta_n$,

$$u(x) = (2\pi)^{-n} \int_{\Gamma_m} \widetilde{u}(\zeta) e^{i\langle x,\zeta\rangle}\, d\zeta\,. \tag{22}$$

Now suppose the sequences $\{\epsilon\}$, $\{N\}$ and $V(\{\epsilon\}, \{N\})$ are given as in Theorem 1.1. We have to construct sequences $\{\delta\}$ $\{M\}$ such that (18) implies (3). By rotational invariance we may assume $x = (0, \dots, 0, x_n)$ with $x_n > 0$. For each n-tuple α we have

$$(D^\alpha u)(x) = (2\pi)^{-n} \int_{\Gamma_m} \widetilde{u}(\zeta)(i\zeta)^\alpha e^{i\langle x,\zeta\rangle}\, d\zeta\,.$$

Starting with positive sequences $\{\delta\}$, $\{M\}$ we shall modify them successively such that (18) $\Rightarrow$ (3). Note that for $\zeta \in \Gamma_m$

$$e^{k|\operatorname{Im}\zeta|} \leq (1 + |\xi|)^{km}$$

$$|\zeta^\alpha| \leq |\zeta|^{|\alpha|} \leq ([|\xi|^2 + m^2(\log(1+|\xi|))^2]^{1/2})^{|\alpha|}\,.$$

For (3) with $j = 0$ we take $x_n = |x| \geq 0$, $|\alpha| \leq N_0$ so

$$|e^{i\langle x,\zeta\rangle}| = e^{-\langle x, \operatorname{Im}\zeta|} \leq 1 \quad \text{for } \zeta \in \Gamma_m\,.$$

Thus if u satisfies (18) we have by the above estimates

$$\begin{aligned} &|(D^\alpha u)(x)| \\ &\leq \sum_0^\infty \delta_k \int_{\mathbf{R}^n} (1 + [|\xi|^2 + m^2(\log(1+|\xi|))^2]^{1/2})^{N_0 - M_k} (1+|\xi|)^{km}(1+m)\, d\xi\,. \end{aligned} \tag{23}$$

We can choose sequences $\{\delta\}$, $\{M\}$ (all δ_k, $M_k > 0$) such that this expression is $\leq \epsilon_0$. This then verifies (3) for $j = 0$. We now fix δ_0 and M_0. Next we want to prove (3) for $j = 1$ by shrinking the terms in $\delta_1, \delta_2, \dots$ and increasing the terms in $M_1, M_2, \dots$ (δ_0, M_0 having been fixed).

Now we have $x_n = |x| \geq 1$ so

$$|e^{i\langle x,\zeta\rangle}| = e^{-\langle x, \operatorname{Im}\zeta\rangle} \leq (1+|\xi|)^{-m} \text{ for } \zeta \in \Gamma_m \tag{24}$$

so in the integrals in (23) the factor $(1+|\xi|)^{km}$ is replaced by $(1+|\xi|)^{(k-1)m}$.

In the sum we separate out the term with $k = 0$. Here M_0 has been fixed but now we have the factor $(1+|\xi|)^{-m}$ which assures that this $k = 0$ term is $< \frac{\epsilon_1}{2}$ for a sufficiently large m which we now fix. In the remaining terms in (23) (for $k > 0$) we can now increase $1/\delta_k$ and M_k such that the sum is $< \epsilon_1/2$. Thus (3) holds for $j = 1$ and it will remain valid for $j = 0$. We now fix this choice of δ_1 and M_1.

Now the inductive process is clear. We assume $\delta_0, \delta_1, \dots, \delta_{j-1}$ and $M_0, M_1, \dots, M_{j-1}$ having been fixed by this shrinking of the δ_i and enlarging of the M_i.

We wish to prove (3) for this j by increasing $1/\delta_k$, M_k for $k \geq j$. Now we have $x_n = |x| \geq j$ and (24) is replaced by

$$|e^{i\langle x,\zeta\rangle}| = e^{-\langle x, \operatorname{Im}\zeta\rangle} \leq 1 + |\xi|^{-jm}$$

and since $|\alpha| \leq N_j$, (23) is replaced by

$$\begin{aligned} &|(D^\alpha f)(x)| \\ &\leq \sum_{k=0}^{j-1} \delta_k \int_{\mathbf{R}^n} (1 + [|\xi|^2 + m^2(\log(1+|\xi|))^2]^{1/2})^{N_j - M_k} (1+|\xi|)^{(k-j)m} (1+m)\, d\xi \\ &+ \sum_{k \geq j} \delta_k \int_{\mathbf{R}^n} (1 + [|\xi|^2 + m^2(\log(1+|\xi|))^2]^{1/2})^{N_j - M_k} (1+|\xi|)^{(k-j)m} (1+m)\, d\xi \,. \end{aligned}$$

In the first sum the M_k have been fixed but the factor $(1+|\xi|)^{(k-j)m}$ decays exponentially. Thus we can fix m such that the first sum is $< \frac{\epsilon_j}{2}$.

In the latter sum the $1/\delta_k$ and the M_k can be increased so that the total sum is $< \frac{\epsilon_j}{2}$. This implies the validity of (3) for this particular j and it remains valid for $0, 1, \dots j-1$. Now we fix δ_j and M_j.

This completes the induction. With this construction of $\{\delta\}$, $\{M\}$ we have proved that $W(\{\delta\}, \{M\}) \subset V(\{\epsilon\}, \{N\})$. This proves Theorem 3.4.

§4 Differential Operators with Constant Coefficients

The description of the topology of $\mathcal{D}$ in terms of the range $\widetilde{\mathcal{D}}$ given in Theorem 3.4 has important consequences for solvability of differential equations on $\mathbf{R}^n$ with constant coefficients.

Theorem 4.1. *Let $D \neq 0$ be a differential operator on $\mathbf{R}^n$ with constant coefficients. Then the mapping $f \to \mathcal{D}f$ is a homeomorphism of $\mathcal{D}$ onto $D\mathcal{D}$.*

Proof. It is clear from Theorem 3.3 that the mapping $f \to Df$ is injective on $\mathcal{D}$. The continuity is also obvious.

For the continuity of the inverse we need the following simple lemma.

Lemma 4.2. *Let $P \neq 0$ be a polynomial of degree m, F an entire function on $\mathbf{C}^n$ and $G = PF$. Then*

$$|F(\zeta)| \leq C \sup_{|z|\leq 1} |G(z+\zeta)|, \quad \zeta \in \mathbf{C}^n\,,$$

where C is a constant.

Proof. Suppose first $n = 1$ and that $P(z) = \sum_0^m a_k z^k (a_m \neq 0)$. Let $Q(z) = z^m \sum_0^m \overline{a}_k z^{-k}$. Then, by the maximum principle,

$$|a_m F(0)| = |Q(0)F(0) \leq \max_{|z|=1} |Q(z)F(z)| = \max_{|z|=1} |P(z)F(z)|\,. \tag{25}$$

For general n let A be an $n \times n$ complex matrix, mapping the ball $|\zeta| < 1$ in $\mathbf{C}^n$ into itself and such that

$$P(A\zeta) = a\zeta_1^m + \sum_0^{m-1} P_k(\zeta_2, \ldots, \zeta_n)\zeta_1^k, \quad a \neq 0\,.$$

Let

$$F_1(\zeta) = F(A\zeta), \quad G_1(\zeta) = G(A\zeta), \quad P_1(\zeta) = P(A\zeta)\,.$$

Then

$$G_1(\zeta_1 + z, \zeta_2, \ldots, \zeta_n) = F_1(\zeta_1 + z, \zeta_2, \ldots, \zeta_n) P_1(\zeta + z, \zeta_2, \ldots, \zeta_n)$$

and the polynomial

$$z \to P_1(\zeta_1 + z, \ldots, \zeta_n)$$

has leading coefficient a. Thus by (25)

$$|aF_1(\zeta)| \leq \max_{|z|=1} |G_1(\zeta_1 + z, \zeta_2, \ldots, \zeta_n)| \leq \max_{\substack{z \in \mathbf{C}^n \\ |z|\leq 1}} |G_1(\zeta + z)|\,.$$

Hence by the choice of A

$$|aF(\zeta)| \leq \sup_{\substack{z \in \mathbf{C}^n \\ |z|\leq 1}} |G(\zeta + z)|$$

proving the lemma.

For Theorem 4.1 it remains to prove that if V is a convex neighborhood of 0 in $\mathcal{D}$ then there exists a convex neighborhood W of 0 in $\mathcal{D}$ such that

$$f \in \mathcal{D},\, Df \in W \Rightarrow f \in V\,. \tag{26}$$

We take V as the neighborhood $W(\{\delta\},\{M\})$. We shall show that if $W = W(\{\epsilon\},\{M\})$ (same $\{M\}$) then (26) holds provided the ϵ_j in $\{\epsilon\}$ are small enough. We write $u = Df$ so $\widetilde{u}(\zeta) = P(\zeta)\widetilde{f}(\zeta)$ where P is a polynomial. By Lemma 4.2

$$|\widetilde{f}(\zeta)| \leq C \sup_{|z|\leq 1} |\widetilde{u}(\zeta + z)|\,. \tag{27}$$

But $|z| \leq 1$ implies

$$(1 + |z+\zeta|)^{-M_j} \leq 2^{M_j}(1+|\zeta|)^{-M_j}\,, \quad |\mathrm{Im}\,(z+\zeta)| \leq |\mathrm{Im}\,\zeta| + 1\,,$$

so if $C2^{M_j}e^j\epsilon_j \leq \delta_j$ then (26) holds.

Q.e.d.

Corollary 4.3. *Let $D \neq 0$ be a differential operator on $\mathbf{R}^n$ with constant (complex) coefficients. Then*

$$D\mathcal{D}' = \mathcal{D}'\,. \tag{28}$$

In particular, there exists a distribution T on $\mathbf{R}^n$ such that

$$DT = \delta\,. \tag{29}$$

Definition. A distribution T satisfying (29) is called a *fundamental solution* for D.

To verify (28) let $L \in \mathcal{D}'$ and consider the functional $D^*u \to L(u)$ on $D^*\mathcal{D}$ ($*$ denoting adjoint). Because of Theorem 3.3 this functional is well defined and by Theorem 4.1 it is continuous. By the Hahn-Banach theorem it extends to a distribution $S \in \mathcal{D}'$. Thus $S(D^*u) = Lu$ so $DS = L$, as claimed.

Corollary 4.4. *Given $f \in \mathcal{D}$ there exists a smooth function u on $\mathbf{R}^n$ such that*

$$Du = f\,.$$

In fact, if T is a fundamental solution one can put $u = f * T$.

We conclude this section with the mean value theorem of Ásgeirsson which entered into the range characterization of the X-ray transform in Chapter I. For another application see Theorem 5.9 below

Theorem 4.5. *Let u be a C^2 function on $B_R \times B_R \subset \mathbf{R}^n \times \mathbf{R}^n$ satisfying*

$$L_x u = L_y u\,. \tag{30}$$

Then

$$\int_{|y|=r} u(0,y)\,dw(y) = \int_{|x|=r} u(x,0)\,dw(x) \quad r < R\,. \tag{31}$$

Conversely, if u is of class C^2 near $(0,0) \subset \mathbf{R}^n \times \mathbf{R}^n$ and if (31) holds for all r sufficiently small then

$$(L_x u)(0,0) = (L_y u)(0,0)\,. \tag{32}$$

Remark 4.6. Integrating Taylor's formula it is easy to see that on the space of analytic functions the mean value operator M^r (Ch. I, §2) is a power series in the Laplacian L. (See (44) below for the explicit expansion.) Thus (30) implies (31) for analytic functions.

For u of class C^2 we give another proof.

We consider the mean value operator on each factor in the product $\mathbf{R}^n \times \mathbf{R}^n$ and put

$$U(r,s) = (M_1^r M_2^s u)(x,y)$$

where the subscript indicates first and second variable, respectively. If u satisfies (30) then we see from the Darboux equation (Ch. I, Lemma 3.2) that

$$\frac{\partial^2 U}{\partial r^2} + \frac{n-1}{r}\frac{\partial U}{\partial r} = \frac{\partial^2 U}{\partial s^2} + \frac{n-1}{s}\frac{\partial U}{\partial s}\,.$$

Putting $F(r,s) = U(r,s) - U(s,r)$ we have

$$\frac{\partial^2 F}{\partial r^2} + \frac{n-1}{r}\frac{\partial F}{\partial r} - \frac{\partial^2 F}{\partial s^2} - \frac{n-1}{s}\frac{\partial F}{\partial s} = 0\,, \tag{33}$$

$$F(r,s) = -F(s,r)\,. \tag{34}$$

After multiplication of (33) by $r^{n-1}\frac{\partial F}{\partial s}$ and some manipulation we get

$$-r^{n-1}\frac{\partial}{\partial s}\left[\left(\frac{\partial F}{\partial r}\right)^2 + \left(\frac{\partial F}{\partial s}\right)^2\right] + 2\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial F}{\partial r}\frac{\partial F}{\partial s}\right) - 2r^{n-1}\frac{n-1}{s}\left(\frac{\partial F}{\partial s}\right)^2 = 0\,.$$

Consider the line MN with equation $r+s=$ const. in the (r,s)-plane and integrate the last expression over the triangle OMN (see Fig. V.2).

Using the divergence theorem (Ch. I, (26)) we then obtain, if $\mathbf{n}$ denotes the outgoing unit normal, $d\ell$ the element of arc length, and $\cdot$ the inner product,

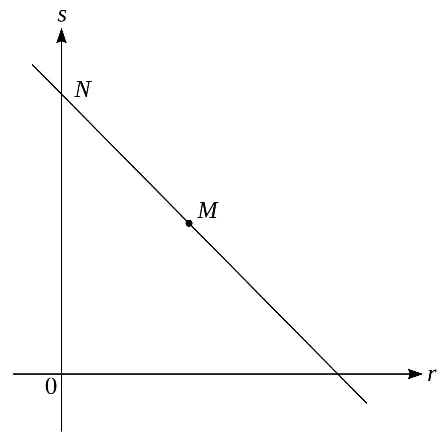

FIGURE V.2.

$$(35)\quad \int_{OMN} \left(2r^{n-1}\frac{\partial F}{\partial r}\frac{\partial F}{\partial s}\,, -r^{n-1}\left[\left(\frac{\partial F}{\partial r}\right)^2 + \left(\frac{\partial F}{\partial s}\right)^2\right]\right)\cdot \mathbf{n}\, d\ell$$
$$= 2\iint_{OMN} r^{n-1}\frac{n-1}{s}\left(\frac{\partial F}{\partial s}\right)^2 dr\, ds\,.$$

On $OM : \overline{n} = (2^{-1/2}, -2^{-1/2})$, $\quad F(r,r)=0$ so $\frac{\partial F}{\partial r}+\frac{\partial F}{\partial s}=0$.
On $MN : \overline{n} = (2^{-1/2}, 2^{-1/2})$.

Taking this into account, (35) becomes

$$2^{-\frac{1}{2}}\int_{MN} r^{n-1}\left(\frac{\partial F}{\partial r}-\frac{\partial F}{\partial s}\right)^2 d\ell + 2\iint_{OMN} r^{n-1}\frac{n-1}{s}\left(\frac{\partial F}{\partial s}\right)^2 dr\, ds = 0\,.$$

This implies F constant so by (34) $F\equiv 0$. In particular, $U(r,0)=U(0,r)$ which is the desired relation (31).

For the converse we observe that the mean value $(M^r f)(0)$ satisfies (by Taylor's formula)

$$(M^r f)(0) = f(0) + c_n r^2 (Lf)(0) + o(r^2)$$

where $c_n \neq 0$ is a constant. Thus

$$r^{-1}\frac{dM^r f(0)}{dr} \to 2c_n (Lf)(0) \ \text{ as } \ r\to 0\,.$$

Thus (31) implies (32) as claimed.

§5 Riesz Potentials

We shall now study some examples of distributions in detail. If $\alpha \in \mathbf{C}$ satisfies $\operatorname{Re}\alpha > -1$ the functional

$$x_+^\alpha : \varphi \to \int_0^\infty x^\alpha \varphi(x)\,dx, \quad \varphi \in \mathcal{S}(\mathbf{R})\,, \tag{36}$$

is a well-defined tempered distribution. The mapping $\alpha \to x_+^\alpha$ from the half-plane $\operatorname{Re}\alpha > -1$ to the space $\mathcal{S}'(\mathbf{R})$ of tempered distributions is holomorphic (that is $\alpha \to x_+^\alpha(\varphi)$ is holomorphic for each $\varphi \in \mathcal{S}(\mathbf{R})$). Writing

$$x_+^\alpha(\varphi) = \int_0^1 x^\alpha(\varphi(x) - \varphi(0))\,dx + \frac{\varphi(0)}{\alpha+1} + \int_1^\infty x^\alpha \varphi(x)\,dx$$

the function $\alpha \to x_+^\alpha$ is continued to a holomorphic function in the region $\operatorname{Re}\alpha > -2, \alpha \neq -1$. In fact

$$\varphi(x) - \varphi(0) = x \int_0^\infty \varphi'(tx)\,dt\,,$$

so the first integral above converges for $\operatorname{Re}\alpha > -2$. More generally, $\alpha \to x_+^\alpha$ can be extended to a holomorphic $\mathcal{S}'(\mathbf{R})$-valued mapping in the region

$$\operatorname{Re}\alpha > -n-1, \quad \alpha \neq -1, -2, \dots, -n\,,$$

by means of the formula

$$\begin{aligned} x_+^\alpha(\varphi) = &\int_0^1 x^\alpha \left[\varphi(x) - \varphi(0) - x\varphi'(0) - \cdots - \frac{x^{n-1}}{(n-1)!}\varphi^{(n-1)}(0)\right] dx \\ &+ \int_1^\infty x^\alpha \varphi(x)\,dx + \sum_{k=1}^n \frac{\varphi^{(k-1)}(0)}{(k-1)!(\alpha+k)}\,. \end{aligned} \tag{37}$$

In this manner $\alpha \to x_+^\alpha$ is a meromorphic distribution-valued function on $\mathbf{C}$, with simple poles at $\alpha = -1, -2, \dots$. We note that the residue at $\alpha = -k$ is given by

$$\operatorname*{Res}_{\alpha=-k} x_+^\alpha = \lim_{\alpha\to -k}(\alpha+k)x_+^\alpha = \frac{(-1)^{k-1}}{(k-1)!}\delta^{(k-1)}\,. \tag{38}$$

Here $\delta^{(h)}$ is the h^{th} derivative of the delta distribution δ. We note that x_+^α is always a tempered distribution.

Next we consider for $\operatorname{Re}\alpha > -n$ the distribution r^α on $\mathbf{R}^n$ given by

$$r^\alpha : \varphi \to \int_{\mathbf{R}^n} \varphi(x)|x|^\alpha\,dx, \quad \varphi \in \mathcal{D}(\mathbf{R}^n)\,.$$

Lemma 5.1. *The mapping $\alpha \to r^\alpha$ extends uniquely to a meromorphic mapping from $\mathbf{C}$ to the space $\mathcal{S}'(\mathbf{R}^n)$ of tempered distributions. The poles are the points*

$$\alpha = -n - 2h \quad (h \in \mathbb{Z}^+)$$

and they are all simple.

Proof. We have for $\operatorname{Re}\alpha > -n$

$$r^\alpha(\varphi) = \Omega_n \int_0^\infty (M^t\varphi)(0) t^{\alpha+n-1}\, dt\,. \tag{39}$$

Next we note (say from (15) in §2) that the mean value function $t \to (M^t\varphi)(0)$ extends to an even $\mathcal{C}^\infty$ function on $\mathbf{R}$, and its odd order derivatives at the origin vanish. Each even order derivative is nonzero if φ is suitably chosen. Since by (39)

$$r^\alpha(\varphi) = \Omega_n t_+^{\alpha+n-1}(M^t\varphi)(0) \tag{40}$$

the first statement of the lemma follows. The possible (simple) poles of r^α are by the remarks about x_+^α given by $\alpha + n - 1 = -1, -2, \ldots$. However if $\alpha + n - 1 = -2, -4, \ldots$, formula (38) shows, since $(M^t\varphi(0))^{(h)} = 0$, ($h$ odd) that $r^\alpha(\varphi)$ is holomorphic at the points $a = -n-1, -n-3, \ldots$.

The remark about the even derivatives of $M^t\varphi$ shows on the other hand, that the points $\alpha = -n-2h$ ($h \in \mathbb{Z}^+$) are genuine poles. We note also from (38) and (40) that

$$\operatorname*{Res}_{\alpha=-n} r^\alpha = \lim_{\alpha\to -n} (\alpha+n) r^\alpha = \Omega_n \delta\,. \tag{41}$$

We recall now that the Fourier transform $T \to \widetilde{T}$ of a tempered distribution T on $\mathbf{R}^n$ is defined by

$$\widetilde{T}(\varphi) = T(\widetilde{\varphi}) \qquad \varphi = \mathcal{S}(\mathbf{R}^n)\,.$$

We shall now calculate the Fourier transforms of these tempered distributions r^α.

Lemma 5.2. *We have the following identity*

$$(r^\alpha)^\sim = 2^{n+\alpha}\pi^{\frac{n}{2}} \frac{\Gamma((n+\alpha)/2)}{\Gamma(-\alpha/2)} r^{-\alpha-n}, \quad -\alpha - n \notin 2\mathbb{Z}^+\,. \tag{42}$$

For $\alpha = 2h\,(h \in \mathbb{Z}^+)$ the singularity on the right is removable and (42) takes the form

$$(r^{2h})^\sim = (2\pi)^n (-L)^h \delta, \quad h \in \mathbb{Z}^+\,. \tag{43}$$

Proof. We use the fact that if $\psi(x) = e^{-|x|^2/2}$ then $\widetilde{\psi}(u) = (2\pi)^{\frac{n}{2}} e^{-|u|^2/2}$ so by the formula $\int f\widetilde{g} = \int \widetilde{f}g$ we obtain for $\varphi \in \mathcal{S}(\mathbf{R}^n), t > 0$,

$$\int \widetilde{\varphi}(x) e^{-t|x|^2/2}\,dx = (2\pi)^{n/2} t^{-n/2} \int \varphi(u) e^{-|u|^2/2t}\,du\,.$$

We multiply this equation by $t^{-1-\alpha/2}$ and integrate with respect to t. On the left we obtain the expression

$$\Gamma(-\alpha/2) 2^{-\frac{\alpha}{2}} \int \widetilde{\varphi}(x)|x|^\alpha\,dx\,,$$

using the formula

$$\int_0^\infty e^{-t|x|^2/2} t^{-1-\alpha/2}\,dt = \Gamma(-\tfrac{\alpha}{2}) 2^{-\frac{\alpha}{2}} |x|^\alpha\,,$$

which follows from the definition

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\,dt\,.$$

On the right we similarly obtain

$$(2\pi)^{\frac{n}{2}} \Gamma((n+\alpha)/2)\, 2^{\frac{n+\alpha}{2}} \int \varphi(u)|u|^{-\alpha-n}\,du\,.$$

The interchange of the integrations is valid for α in the strip $-n < \operatorname{Re}\alpha < 0$ so (42) is proved for these α. For the remaining ones it follows by analytic continuation. Finally, (43) is immediate from the definitions and (6).

By the analytic continuation, the right hand sides of (42) and (43) agree for $\alpha = 2h$. Since

$$\operatorname*{Res}_{\alpha=2h} \Gamma(-\alpha/2) = -2(-1)^h/h!$$

and since by (40) and (38),

$$\operatorname*{Res}_{\alpha=2h} r^{-\alpha-n}(\varphi) = -\Omega_n \frac{1}{(2h)!} \left[\left(\frac{d}{dt}\right)^{2h} (M^t\varphi)\right]_{t=0}$$

we deduce the relation

$$\left[\left(\frac{d}{dt}\right)^{2h} (M^t\varphi)\right]_{t=0} = \frac{\Gamma(n/2)}{\Gamma(h+n/2)} \frac{(2h)!}{2^{2h}h!} (L^h\varphi)(0)\,.$$

This gives the expansion

$$M^t = \sum_{h=0}^\infty \frac{\Gamma(n/2)}{\Gamma(h+n/2)} \frac{(t/2)^{2h}}{h!} L^h \tag{44}$$

on the space of analytic functions so M^t is a modified Bessel function of $tL^{1/2}$. This formula can also be proved by integration of Taylor's formula (cf. end of §4).

Lemma 5.3. *The action of the Laplacian is given by*

$$Lr^{\alpha} = \alpha(\alpha+n-2)r^{\alpha-2}, \quad (-\alpha-n+2 \notin 2\mathbb{Z}^+) \tag{45}$$

$$Lr^{2-n} = (2-n)\Omega_n\delta \quad (n \neq 2)\,. \tag{46}$$

For $n = 2$ this 'Poisson equation' is replaced by

$$L(\log r) = 2\pi\delta\,. \tag{47}$$

Proof. For $\operatorname{Re}\alpha$ sufficiently large (45) is obvious by computation. For the remaining ones it follows by analytic continuation. For (46) we use the Fourier transform and the fact that for a tempered distribution S,

$$(-LS)^{\sim} = r^2\widetilde{S}\,.$$

Hence, by (42),

$$(-Lr^{2-n})^{\sim} = 4\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}-1)} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}(n-2)\widetilde{\delta}\,.$$

Finally, we prove (47). If $\varphi \in \mathcal{D}(\mathbf{R}^2)$ we have, putting $F(r) = (M^r\varphi)(0)$,

$$(L(\log r))(\varphi) = \int_{\mathbf{R}^2} \log r(L\varphi)(x)\,dx = \int_0^{\infty} (\log r)2\pi r(M^r L\varphi)(0)\,dr\,.$$

Using Lemma 3.2 in Chapter I this becomes

$$\int_0^{\infty} \log r\;\, 2\pi r(F''(r) + r^{-1}F'(r))\,dr\,,$$

which by integration by parts reduces to

$$\Big[\log r(2\pi r)F'(r)\Big]_0^{\infty} - 2\pi\int_0^{\infty} F'(r)\,dr = 2\pi F(0)\,.$$

This proves (47).

Another method is to write (45) in the form $L(\alpha^{-1}(r^{\alpha}-1)) = \alpha r^{\alpha-2}$. Then (47) follows from (41) by letting $\alpha \to 0$.

We shall now define fractional powers of L, motivated by the formula

$$(-Lf)^{\sim}(u) = |u|^2\widetilde{f}(u)\,,$$

so that formally we should like to have a relation

$$((-L)^p f)^{\sim}(u) = |u|^{2p}\widetilde{f}(u)\,. \tag{48}$$

Since the Fourier transform of a convolution is the product of the Fourier transforms, formula (42) (for $2p = -\alpha - n$) suggests defining

$$(-L)^p f = I^{-2p}(f)\,, \tag{49}$$

where I^γ is the *Riesz potential*

$$(I^\gamma f)(x) = \frac{1}{H_n(\gamma)} \int_{\mathbf{R}^n} f(y)|x-y|^{\gamma-n}\, dy \tag{50}$$

with

$$H_n(\gamma) = 2^\gamma \pi^{\frac{n}{2}} \frac{\Gamma(\frac{\gamma}{2})}{\Gamma(\frac{n-\gamma}{2})}\,. \tag{51}$$

Note that if $-\gamma \in 2\mathbb{Z}^+$ the poles of $\Gamma(\gamma/2)$ cancel against the poles of $r^{\gamma-n}$ because of Lemma 5.1. Thus if $\gamma - n \notin 2\mathbb{Z}^+$ we can write

$$(I^\gamma f)(x) = (f * (H_n(\gamma)^{-1} r^{\gamma-n}))(x), \quad f \in \mathcal{S}(\mathbf{R}^n)\,. \tag{52}$$

By (12) and Lemma 5.2 we then have

$$(I^\gamma f)^\sim(u) = |u|^{-\gamma} \widetilde{f}(u)\,, \qquad \gamma - n \notin 2\mathbb{Z}^+ \tag{53}$$

as tempered distributions. Thus we have the following result.

Lemma 5.4. *If $f \in \mathcal{S}(\mathbf{R}^n)$ then $\gamma \to (I^\gamma f)(x)$ extends to a holomorphic function in the set $\mathbf{C}_n = \{\gamma \in \mathbf{C} : \gamma - n \notin 2\mathbb{Z}^+\}$. Also*

$$I^0 f = \lim_{\gamma \to 0} I^\gamma f = f\,, \tag{54}$$

$$I^\gamma L f = L I^\gamma f = -I^{\gamma-2} f\,. \tag{55}$$

We now prove an important property of the Riesz' potentials. Here it should be observed that $I^\gamma f$ is defined for all f for which (50) is absolutely convergent and $\gamma \in \mathbf{C}_n$.

Proposition 5.5. *The following identity holds:*

$$I^\alpha(I^\beta f) = I^{\alpha+\beta} f \text{ for } f \in \mathcal{S}(\mathbf{R}^n), \quad \operatorname{Re}\alpha, \operatorname{Re}\beta > 0, \quad \operatorname{Re}(\alpha+\beta) < n\,,$$

$I^\alpha(I^\beta f)$ being well defined. The relation is also valid if

$$f(x) = 0(|x|^{-p}) \text{ for some } p > \operatorname{Re}\alpha + \operatorname{Re}\beta\,.$$

Proof. We have

$$\begin{aligned} I^\alpha(I^\beta f)(x) &= \frac{1}{H_n(\alpha)} \int |x-z|^{\alpha-n} \left(\frac{1}{H_n(\beta)} \int f(y)|z-y|^{\beta-n}\, dy \right) dz \\ &= \frac{1}{H_n(\alpha) H_n(\beta)} \int f(y) \left(\int |x-z|^{\alpha-n} |z-y|^{\beta-n}\, dz \right) dy\,. \end{aligned}$$

The substitution $v = (x-z)/|x-y|$ reduces the inner integral to the form

$$|x-y|^{\alpha+\beta-n} \int_{\mathbf{R}^n} |v|^{\alpha-n} |w-v|^{\beta-n}\, dv\,, \tag{56}$$

where w is the unit vector $(x-y)/|x-y|$. Using a rotation around the origin we see that the integral in (56) equals the number

$$c_n(\alpha,\beta)=\int_{\mathbf{R}^n}|v|^{\alpha-n}|e_1-v|^{\beta-n}\,dv\,, \tag{57}$$

where $e_1=(1,0,\ldots,0)$. The assumptions made on α and β insure that this integral converges. By the Fubini theorem the exchange order of integrations above is permissible and

$$I^\alpha(I^\beta f)=\frac{H_n(\alpha+\beta)}{H_n(\alpha)H_n(\beta)}c_n(\alpha,\beta)I^{\alpha+\beta}f\,. \tag{58}$$

It remains to calculate $c_n(\alpha,\beta)$. For this we use the following lemma which was already used in Chapter I, §2. As there, let $\mathcal{S}^*(\mathbf{R}^n)$ denote the set of functions in $\mathcal{S}(\mathbf{R}^n)$ which are orthogonal to all polynomials.

Lemma 5.6. *Each $I^\alpha(\alpha\in\mathbf{C}_n)$ leaves the space $\mathcal{S}^*(\mathbf{R}^n)$ invariant.*

Proof. We recall that (53) holds in the sense of tempered distributions. Suppose now $f\in\mathcal{S}^*(\mathbf{R}^n)$. We consider the sum in the Taylor formula for $\widetilde{f}$ in $|u|\leq 1$ up to order m with $m>|\alpha|$. Since each derivative of $\widetilde{f}$ vanishes at $u=0$ this sum consists of terms

$$(\beta!)^{-1}u^\beta(D^\beta\widetilde{f})(u^*)\,,\qquad |\beta|=m$$

where $|u^*|\leq 1$. Since $|u^\beta|\leq|u|^m$ this shows that

$$\lim_{u\to 0}|u|^{-\alpha}\widetilde{f}(u)=0\,. \tag{59}$$

Iterating this argument with $\partial_i(|u|^{-\alpha}\widetilde{f}(u))$ etc. we conclude that the limit relation (59) holds for each derivative $D^\beta(|u|^{-\alpha}\widetilde{f}(u))$. Because of (59), relation (53) can be written

$$\int_{\mathbf{R}^n}(I^\alpha f)^\sim(u)g(u)\,du=\int_{\mathbf{R}^n}|u|^{-\alpha}\widetilde{f}(u)g(u)\,du\,,\quad g\in\mathcal{S}\,, \tag{60}$$

so (53) holds as an identity for functions $f\in\mathcal{S}^*(\mathbf{R}^n)$. The remark about $D^\beta(|u|^{-\alpha}\widetilde{f}(u))$ thus implies $(I^\alpha f)^\sim\in\mathcal{S}_0$ so $I^\alpha f\in\mathcal{S}^*$ as claimed.

We can now finish the proof of Prop. 5.5. Taking $f_o\in\mathcal{S}^*$ we can put $f=I^\beta f_o$ in (53) and then

$$\begin{aligned}(I^\alpha(I^\beta f_0))^\sim(u)&=(I^\beta f_0)^\sim(u)|u|^{-\alpha}=\widetilde{f}_0(u)|u|^{-\alpha-\beta}\\&=(I^{\alpha+\beta}f_0)^\sim(u)\,.\end{aligned}$$

This shows that the scalar factor in (58) equals 1 so Prop. 5.5 is proved. In the process we have obtained the evaluation

$$\int_{\mathbf{R}^n}|v|^{\alpha-n}|e_1-v|^{\beta-n}\,dv=\frac{H_n(\alpha)H_n(\beta)}{H_n(\alpha+\beta)}\,.$$

We now prove a variation of Prop. 5.5 needed in the theory of the Radon transform.

Proposition 5.7. *Let $0 < k < n$. Then*

$$I^{-k}(I^k f) = f \qquad f \in \mathcal{E}(\mathbf{R}^n)$$

if $f(x) = 0(|x|^{-N})$ for some $N > n$.

Proof. By Prop. 5.5 we have if $f(y) = 0(|y|^{-N})$

$$I^\alpha(I^k f) = I^{\alpha+k} f \quad \text{for } 0 < \operatorname{Re}\alpha < n-k\,. \tag{61}$$

We shall prove that the function $\varphi = I^k f$ satisfies

$$\sup_x |\varphi(x)|\,|x|^{n-k} < \infty\,. \tag{62}$$

For an $N > n$ we have an estimate $|f(y)| \leq C_N(1+|y|)^{-N}$ where C_N is a constant. We then have

$$\begin{aligned}\left(\int_{\mathbf{R}^n} f(y)|x-y|^{k-n}\,dy\right) \leq\; & C_N \int_{|x-y|\leq\frac{1}{2}|x|} (1+|y|)^{-N}|x-y|^{k-n}\,dy \\ & + C_N \int_{|x-y|\geq\frac{1}{2}|x|} (1+|y|)^{-N}|x-y|^{k-n}\,dy\,.\end{aligned}$$

In the second integral, $|x-y|^{k-n} \leq (\frac{|x|}{2})^{k-n}$ so since $N > n$ this second integral satisfies (62). In the first integral we have $|y| \geq \frac{|x|}{2}$ so the integral is bounded by

$$\left(1+\frac{|x|}{2}\right)^{-N} \int_{|x-y|\leq\frac{|x|}{2}} |x-y|^{k-n}\,dy = \left(1+\frac{|x|}{2}\right)^{-N} \int_{|z|\leq\frac{|x|}{2}} |z|^{k-n}\,dz$$

which is $0(|x|^{-N}|x|^k)$. Thus (62) holds also for this first integral. This proves (62) provided

$$f(x) = 0(|x|^{-N}) \text{ for some } N > n\,.$$

Next we observe that $I^\alpha(\varphi) = I^{\alpha+k}(f)$ is holomorphic for $0 < \operatorname{Re}\alpha < n-k$. For this note that by (39)

$$\begin{aligned}(I^{\alpha+k}f)(0) &= \frac{1}{H_n(\alpha+k)} \int_{\mathbf{R}^n} f(y)|y|^{\alpha+k-n}\,dy \\ &= \frac{1}{H_n(\alpha+k)}\Omega_n \int_0^\infty (M^t f)(0)t^{\alpha+k-1}\,dt\,.\end{aligned}$$

Since the integrand is bounded by a constant multiple of $t^{-N}t^{\alpha+k-1}$, and since the factor in front of the integral is harmless for $0 < k + \operatorname{Re}\alpha < n$, the holomorphy statement follows.

We claim now that $I^\alpha(\varphi)(x)$, which as we saw is holomorphic for $0 < \operatorname{Re}\alpha < n-k$, extends to a holomorphic function in the half-plane $\operatorname{Re}\alpha < n-k$. It suffices to prove this for $x = 0$. We decompose $\varphi = \varphi_1 + \varphi_2$ where φ_1 is a smooth function identically 0 in a neighborhood $|x| < \epsilon$ of 0, and $\varphi_2 \in \mathcal{S}(\mathbf{R}^n)$. Since φ_1 satisfies (62) we have for $\operatorname{Re}\alpha < n-k$,

$$\begin{aligned}\left|\int \varphi_1(x)|x|^{\operatorname{Re}\alpha - n}\,dx\right| &\leq C\int_\epsilon^\infty |x|^{k-n}|x|^{\operatorname{Re}\alpha-n}|x|^{n-1}d|x| \\ &= C\int_\epsilon^\infty |x|^{\operatorname{Re}\alpha+k-n-1}d|x| < \infty\end{aligned}$$

so $I^\alpha\varphi_1$ is holomorphic in this half-plane. On the other hand $I^\alpha\varphi_2$ is holomorphic for $\alpha \in \mathbf{C}_n$ which contains this half-plane. Now we can put $\alpha = -k$ in (61). As a result of (39), $f(x) = 0(|x|^{-N})$ implies that $(I^\lambda f)(x)$ is holomorphic near $\lambda = 0$ and $I^0 f = f$. Thus the proposition is proved.

Denoting by C_N the class of continuous functions f on $\mathbf{R}^n$ satisfying $f(x) = 0(|x|^{-N})$ we proved in (62) that if $N > n$, $0 < k < n$, then

$$I^k C_N \subset C_{n-k}\,. \tag{63}$$

More generally, we have the following result.

Proposition 5.8. *If* $N > 0$ *and* $0 < \operatorname{Re}\gamma < N$*, then*

$$I^\gamma C_N \subset C_s$$

where $s = \min(n, N) - \operatorname{Re}\gamma \quad (n \neq N)$.

Proof. Modifying the proof of Prop. 5.7 we divide the integral

$$I = \int (1+|y|)^{-N}|x-y|^{\operatorname{Re}\gamma - n}\,dy$$

into integrals I_1, I_2 and I_3 over the disjoint sets

$$A_1 = \{y : |y-x| \leq \tfrac{1}{2}|x|\}, \qquad A_2 = \{y : |y| < \tfrac{1}{2}|x|\,,$$

and the complement $A_3 = \mathbf{R}^n - A_1 - A_2$. On A_1 we have $|y| \geq \frac{1}{2}|x|$ so

$$I_1 \leq \left(1 + \frac{|x|}{2}\right)^{-N}\int_{A_1}|x-y|^{\operatorname{Re}\gamma-n}\,dy = \left(1+\frac{|x|}{2}\right)^{-N}\int_{|z|\leq|x|/2}|z|^{\operatorname{Re}\gamma-n}\,dz$$

so

$$I_1 = 0(|x|^{-N+\operatorname{Re}\gamma})\,. \tag{64}$$

On A_2 we have $|x| + \frac{1}{2}|x| \geq |x-y| \geq \frac{1}{2}|x|$ so

$$|x-y|^{\operatorname{Re}\gamma-n} \leq C|x|^{\operatorname{Re}\gamma-n}, \quad C = \text{const.}.$$

Thus

$$I_2 \leq C|x|^{\operatorname{Re}\gamma-n}\int_{A_2}(1+|y|)^{-N}\,.$$

If $N > n$ then

$$\int_{A_2}(1+|y|)^{-N}\,dy \leq \int_{\mathbf{R}^n}(1+|y|)^{-N}\,dy < \infty\,.$$

If $N < n$ then

$$\int_{A_2}(1+|y|)^{-N}\,dy \leq C|x|^{n-N}\,.$$

In either case

$$I_2 = 0(|x|^{\operatorname{Re}\gamma-\min(n,N)})\,. \tag{65}$$

On A_3 we have $(1+|y|)^{-N} \leq |y|^{-N}$. The substitution $y = |x|u$ gives (with $e = x/|x|$)

$$I_3 \leq |x|^{\operatorname{Re}\gamma-N}\int_{|u|\geq\frac{1}{2},|e-u|\geq\frac{1}{2}}|u|^{-N}|e-u|^{\operatorname{Re}\gamma-n}\,du = 0(|x|^{\operatorname{Re}\gamma-N})\,. \tag{66}$$

Combining (64)–(66) we get the result.

We conclude with a consequence of Theorem 4.5 observed in John [1935]. Here the Radon transform maps functions $\mathbf{R}^n$ into functions on a space of $(n+1)$ dimensions and the range is the kernel of a single differential operator. This may have served as a motivation for the range characterization of the X-ray transform in John [1938]. As before we denote by $(M^r f)(x)$ the average of f on $S_r(x)$.

Theorem 5.9. *For f on $\mathbf{R}^n$ put*

$$\widehat{f}(x,r) = (M^r f)(x)\,.$$

Then

$$\mathcal{E}(\mathbf{R}^n)\widehat{\ } = \{\varphi \in \mathcal{E}(\mathbf{R}^n \times \mathbf{R}^+) : L_x\varphi = \partial_r^2\varphi + \frac{n-1}{r}\partial_r\varphi\}\,.$$

The inclusion $\subset$ follows from Lemma 3.2, Ch. I. Conversely suppose φ satisfies the Darboux equation. The extension $\Phi(x,y) = \varphi(x,|y|)$ then satisfies $L_x\Phi = L_y\Phi$. Using Theorem 4.5 on the function $(x,y) \to \Phi(x+x_0,y)$ we obtain $\varphi(x_0,r) = (M^r f)(x_0)$ so $\widehat{f} = \varphi$ as claimed.

Bibliographical Notes

§1-2 contain an exposition of the basics of distribution theory following Schwartz [1966]. The range theorems (3.1–3.3) are also from there but we have used the proofs from Hörmander [1963]. Theorem 3.4 describing the topology of $\mathcal{D}$ in terms of $\widetilde{\mathcal{D}}$ is from Hörmander [1983], Vol. II, Ch. XV. The idea of a proof of this nature involving a contour like Γ_m appears already in Ehrenpreis [1956] although not correctly carried out in details. In the proof we specialize Hörmander's convex set K to a ball; it simplifies the proof a bit and requires Cauchy's theorem only in a single variable. The consequence, Theorem 4.1, and its proof were shown to me by Hörmander in 1972. The theorem appears in Ehrenpreis [1956].

Theorem 4.5, with the proof in the text, is from Ásgeirsson [1937]. Another proof, with a refinement in odd dimension, is given in Hörmander [1983], Vol. I. A generalization to Riemannian homogeneous spaces is given by the author in [1959]. The theorem is used in the theory of the X-ray transform in Chapter I.

§5 contains an elementary treatment of the results about Riesz potentials used in the book. The examples x_+^λ are discussed in detail in Gelfand-Shilov [1959]. The potentials I^λ appear there and in Riesz [1949] and Schwartz [1966]. In the proof of Proposition 5.7 we have used a suggestion by R. Seeley and the refinement in Proposition 5.8 was shown to me by Schlichtkrull. A thorough study of the composition formula (Prop. 5.5) was carried out by Ortner [1980] and a treatment of Riesz potentials on L^p-spaces (Hardy-Littlewood-Sobolev inequality) is given in Hörmander [1983], Vol. I, §4.

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