where for $\xi \in \mathbf{P}^n$, $\xi - y$ denotes the translate of the hyperplane ξ by $-y$. Then

$$(f \times \varphi)^{\vee} = f * \check{\varphi}\,.$$

In fact, if ξ_0 is any hyperplane through 0,

$$\begin{aligned}(f \times \varphi)^{\vee}(x) &= \int_K dk \int_{\mathbf{R}^n} f(y)\varphi(x + k\cdot\xi_0 - y)\,dy \\ &= \int_K dk \int_{\mathbf{R}^n} f(x-y)\varphi(y + k\cdot\xi_0)\,dy = (f * \check{\varphi})(x)\,.\end{aligned}$$

By the definition of $\widehat{T}$, the support assumption on $\widehat{T}$ is equivalent to

$$T(\check{\varphi}) = 0$$

for all $\varphi \in \mathcal{D}(\mathbf{P}^n)$ with support in $\mathbf{P}^n - C\ell(\beta_A(0))$. Let $\epsilon > 0$, let $f \in \mathcal{D}(\mathbf{R}^n)$ be a symmetric function with support in $C\ell(B_\epsilon(0))$ and let $\varphi = \mathcal{D}(\mathbf{P}^n)$ have support contained in $\mathbf{P}^n - C\ell(\beta_{A+\epsilon}(0))$. Since $d(0, \xi - y) \leq d(0,\xi) + |y|$ it follows that $f \times \varphi$ has support in $\mathbf{P}^n - C\ell(\beta_A(0))$; thus by the formulas above, and the symmetry of f,

$$(f * T)(\check{\varphi}) = T(f * \check{\varphi}) = T((f \times \varphi)^{\vee}) = 0\,.$$

But then

$$(f * T)\widehat{\ }(\varphi) = (f * T)(\check{\varphi}) = 0\,,$$

which means that $(f*T)\widehat{\ }$ has support in $C\ell(\beta_{A+\epsilon}(0))$. But now Theorem 2.6 implies that $f * T$ has support in $C\ell(B_{A+\epsilon})(0)$. Letting $\epsilon \to 0$ we obtain the desired conclusion, $\mathrm{supp}(T) \subset C\ell(B_A(0))$.

We can now extend the inversion formulas for the Radon transform to distributions. First we observe that the Hilbert transform $\mathcal{H}$ can be extended to distributions T on $\mathbf{R}$ of compact support. It suffices to put

$$\mathcal{H}(T)(F) = T(-\mathcal{H}F)\,, \quad F \in \mathcal{D}(\mathbf{R})\,.$$

In fact, as remarked at the end of §3, the mapping $F \longrightarrow \mathcal{H}F$ is a continuous mapping of $\mathcal{D}(\mathbf{R})$ into $\mathcal{E}(\mathbf{R})$. In particular $\mathcal{H}(T) \in \mathcal{D}'(\mathbf{R})$.

Theorem 5.5. *The Radon transform* $S \longrightarrow \widehat{S}$ $\quad (S \in \mathcal{E}'(\mathbf{R}^n))$ *is inverted by the following formula*

$$cS = (\Lambda\widehat{S})^{\vee}\,, \quad S \in \mathcal{E}'(\mathbf{R}^n)\,,$$

where the constant $c = (-4\pi)^{(n-1)/2}\Gamma(n/2)/\Gamma(1/2)$.

In the case when n *is odd we have also*

$$c\,S = L^{(n-1)/2}((\widehat{S})^{\vee})\,.$$

Remark 5.6. Since $\widehat{S}$ has compact support and since Λ is defined by means of the Hilbert transform the remarks above show that $\Lambda\widehat{S} \in \mathcal{D}'(\mathbf{P}^n)$ so the right hand side is well defined.

Proof. Using Theorem 3.6 we have

$$(\Lambda\widehat{S})^\vee(f) = (\Lambda\widehat{S})(\widehat{f}) = \widehat{S}(\Lambda\widehat{f}) = S((\Lambda\widehat{f})^\vee) = cS(f)\,.$$

The other inversion formula then follows, using the lemma.

In analogy with β_A we define the "sphere" σ_A in $\mathbf{P}^n$ as

$$\sigma_A = \{\xi \in \mathbf{P}^n : d(0,\xi) = A\}\,.$$

From Theorem 5.5 we can then deduce the following complement to Theorem 5.4.

Corollary 5.7. *Suppose n is odd. Then if $S \in \mathcal{E}'(\mathbf{R}^n)$,*

$$\operatorname{supp}(\widehat{S}) \in \sigma_R \Rightarrow \operatorname{supp}(S) \subset S_R(0)\,.$$

To see this let $\epsilon > 0$ and let $f \in \mathcal{D}(\mathbf{R}^n)$ have $\operatorname{supp}(f) \subset B_{R-\epsilon}(0)$. Then $\operatorname{supp}\widehat{f} \in \beta_{R-\epsilon}$ and since Λ is a differential operator, $\operatorname{supp}(\Lambda\widehat{f}) \subset \beta_{R-\epsilon}$. Hence

$$cS(f) = S((\Lambda\widehat{f})^\vee) = \widehat{S}(\Lambda\widehat{f}) = 0$$

so $\operatorname{supp}(S) \cap B_{R-\epsilon}(0) = \emptyset$. Since $\epsilon > 0$ is arbitrary,

$$\operatorname{supp}(S) \cap B_R(0) = \emptyset\,.$$

On the other hand by Theorem 5.4, $\operatorname{supp}(S) \subset \overline{B_R(0)}$. This proves the corollary.

Let M be a manifold and $d\mu$ a measure such that on each local coordinate patch with coordinates $(t_1, \ldots, t_n)$ the Lebesque measure $dt_1, \ldots, dt_n$ and $d\mu$ are absolutely continuous with respect to each other. If h is a function on M locally integrable with respect to $d\mu$ the distribution $\varphi \to \int \varphi h\, d\mu$ will be denoted T_h.

Proposition 5.8. *(a) Let $f \in L^1(\mathbf{R}^n)$ vanish outside a compact set. Then the distribution T_f has Radon transform given by*

$$\widehat{T_f} = T_{\widehat{f}}\,. \tag{47}$$

(b) Let φ be a locally integrable function on $\mathbf{P}^n$. Then

$$(T_\varphi)^\vee = T_{\check{\varphi}}\,. \tag{48}$$

Proof. The existence and local integrability of $\widehat{f}$ and $\check{\varphi}$ was established during the proof of Lemma 5.1. The two formulas now follow directly from Lemma 5.1.

As a result of this proposition the smoothness assumption can be dropped in the inversion formula. In particular, we can state the following result.

Corollary 5.9. *(n odd.) The inversion formula*

$$cf = L^{(n-1)/2}((\widehat{f})^{\vee}),$$

$c = (-4\pi)^{(n-1)/2}\Gamma(n/2)/\Gamma(1/2)$, *holds for all* $f \in L^1(\mathbf{R}^n)$ *vanishing outside a compact set, the derivative interpreted in the sense of distributions.*

Examples. If μ is a measure (or a distribution) on a closed submanifold S of a manifold M the distribution on M given by $\varphi \to \mu(\varphi|S)$ will also be denoted by μ.

(a) Let δ_0 be the delta distribution $f \to f(0)$ on $\mathbf{R}^n$. Then

$$\widehat{\delta_0}(\varphi) = \delta_0(\check{\varphi}) = \Omega_n^{-1} \int_{S^{n-1}} \varphi(\omega, 0)\, d\omega$$

so

$$\widehat{\delta_0} = \Omega_n^{-1} m_{\mathbf{S}^{n-1}} \tag{49}$$

the normalized measure on $\mathbf{S}^{n-1}$ considered as a distribution on $\mathbf{S}^{n-1} \times \mathbf{R}$.

(b) Let ξ_0 denote the hyperplane $x_n = 0$ in $\mathbf{R}^n$, and δ_{ξ_0} the delta distribution $\varphi \to \varphi(\xi_0)$ on $\mathbf{P}^n$. Then

$$(\delta_{\xi_0})^{\vee}(f) = \int_{\xi_0} f(x)\, dm(x)$$

so

$$(\delta_{\xi_0})^{\vee} = m_{\xi_0}, \tag{50}$$

the Euclidean measure of ξ_0.

(c) Let χ_B be the characteristic function of the unit ball $B \subset \mathbf{R}^n$. Then by (47),

$$\widehat{\chi}_B(\omega, p) = \begin{cases} \frac{\Omega_{n-1}}{n-1}(1-p^2)^{(n-1)/2} & , |p| \le 1 \\ 0 & , |p| > 1 \end{cases}.$$

(d) Let Ω be a bounded convex region in $\mathbf{R}^n$ whose boundary is a smooth surface. We shall obtain a formula for the volume of Ω in terms of the areas of its hyperplane sections. For simplicity we assume n odd. The characteristic function χ_Ω is a distribution of compact support and $(\chi_\Omega)\widehat{\ }$ is thus well defined. Approximating χ_Ω in the L^2-norm by a sequence $(\psi_n) \subset \mathcal{D}(\Omega)$ we see from Theorem 4.1 that $\partial_p^{(n-1)/2}\widehat{\psi}_n(\omega,p)$ converges in the L^2-norm on $\mathbf{P}^n$. Since

$$\int \widehat{\psi}(\xi)\varphi(\xi)\,d\xi = \int \psi(x)\check{\varphi}(x)\,dx$$

it follows from Schwarz' inequality that $\widehat{\psi}_n \longrightarrow (\chi_\Omega)\widehat{\ }$ in the sense of distributions and accordingly $\partial^{(n-1)/2}\widehat{\psi}_n$ converges as a distribution to $\partial^{(n-1)/2}((\chi_\Omega)\widehat{\ })$. Since the L^2 limit is also a limit in the sense of distributions this last function equals the L^2 limit of the sequence $\partial^{(n-1)/2}\widehat{\psi}_n$. From Theorem 4.1 we can thus conclude the following result:

Theorem 5.10. *Let $\Omega \subset \mathbf{R}^n$ (n odd) be a convex region as above and $V(\Omega)$ its volume. Let $A(\omega,p)$ denote the $(n-1)$-dimensional area of the intersection of Ω with the hyperplane $\langle x,\omega\rangle = p$. Then*

$$V(\Omega) = \tfrac{1}{2}(2\pi)^{1-n}\int_{\mathbf{S}^{n-1}}\int_{\mathbf{R}}\left|\frac{\partial^{(n-1)/2}A(\omega,p)}{\partial p^{(n-1)/2}}\right|^2 dp\,d\omega\,. \tag{51}$$

§6 Integration over d-planes. X-ray Transforms. The Range of the d-plane Transform

Let d be a fixed integer in the range $0 < d < n$. We define the *d-dimensional Radon transform* $f \to \widehat{f}$ by

$$\widehat{f}(\xi) = \int_\xi f(x)\,dm(x) \qquad \xi \quad \text{a } d\text{-plane}\,. \tag{52}$$

Because of the applications to radiology indicated in § 7,b) the 1-dimensional Radon transform is often called the *X-ray transform*. Since a hyperplane can be viewed as a disjoint union of parallel d-planes parameterized by $\mathbf{R}^{n-1-d}$ it is obvious from (4) that the transform $f \to \widehat{f}$ is injective. Similarly we deduce the following consequence of Theorem 2.6.

Corollary 6.1. *Let $f, g \in C(\mathbf{R}^n)$ satisfy the rapid decrease condition: For each $m > 0$, $|x|^m f(x)$ and $|x|^m g(x)$ are bounded on $\mathbf{R}^n$. Assume for the d-dimensional Radon transforms*

$$\widehat{f}(\xi) = \widehat{g}(\xi)$$

whenever the d-plane ξ lies outside the unit ball. Then

$$f(x) = g(x) \quad \textit{for} \quad |x| > 1\,.$$

We shall now generalize the inversion formula in Theorem 3.1. If φ is a continuous function on the space of d-planes in $\mathbf{R}^n$ we denote by $\check{\varphi}$ the point function

$$\check{\varphi}(x) = \int_{x \in \xi} \varphi(\xi)\, d\mu(\xi)\,,$$

where μ is the unique measure on the (compact) space of d-planes passing through x, invariant under all rotations around x and with total measure 1. If σ is a fixed d-plane through the origin we have in analogy with (16),

$$\check{\varphi}(x) = \int_K \varphi(x + k \cdot \sigma)\, dk\,. \tag{53}$$

Theorem 6.2. *The d-dimensional Radon transform in $\mathbf{R}^n$ is inverted by the formula*

$$cf = (-L)^{d/2}((\widehat{f})^{\vee})\,, \tag{54}$$

where $c = (4\pi)^{d/2}\Gamma(n/2)/\Gamma((n-d)/2)$. Here it is assumed that $f(x) = 0(|x|^{-N})$ for some $N > n$.

Proof. We have in analogy with (34)

$$\begin{aligned} (\widehat{f})^{\vee}(x) &= \int_K \left(\int_\sigma f(x + k \cdot y)\, dm(y) \right) dk \\ &= \int_\sigma dm(y) \int_K f(x + k \cdot y)\, dk = \int_\sigma (M^{|y|} f)(x)\, dm(y)\,. \end{aligned}$$

Hence

$$(\widehat{f})^{\vee}(x) = \Omega_d \int_0^\infty (M^r f)(x) r^{d-1}\, dr$$

so using polar coordinates around x,

$$(\widehat{f})^{\vee}(x) = \frac{\Omega_d}{\Omega_n} \int_{\mathbf{R}^n} |x - y|^{d-n} f(y)\, dy\,. \tag{55}$$

The theorem now follows from Proposition 5.7 in Chapter V.

As a consequence of Theorem 2.10 we now obtain a generalization, characterizing the image of the space $\mathcal{D}(\mathbf{R}^n)$ under the d-dimensional Radon transform.

The set $\mathbf{G}(d, n)$ of d-planes in $\mathbf{R}^n$ is a manifold, in fact a homogeneous space of the group $\mathbf{M}(n)$ of all isometries of $\mathbf{R}^n$. Let $\mathbf{G}_{d,n}$ denote the manifold of all d-dimensional subspaces (d-planes through 0) of $\mathbf{R}^n$. The parallel translation of a d-plane to one through 0 gives a mapping π of $\mathbf{G}(d, n)$ onto $\mathbf{G}_{d,n}$. The inverse image $\pi^{-1}(\sigma)$ of a member $\sigma \in \mathbf{G}_{d,n}$ is naturally identified with the orthogonal complement $\sigma^\perp$. Let us write

$$\xi = (\sigma, x'') = x'' + \sigma \text{ if } \sigma = \pi(\xi) \text{ and } x'' = \sigma^\perp \cap \xi\,.$$

(See Fig. I.6.) Then (52) can be written

(56)

$$\widehat{f}(x''+\sigma) = \int_\sigma f(x'+x'')\,dx' \, .$$

For $k \in \mathbb{Z}^+$ we consider the polynomial

(57)

$$P_k(u) = \int_{\mathbf{R}^n} f(x)\langle x,u\rangle^k\,dx \, .$$

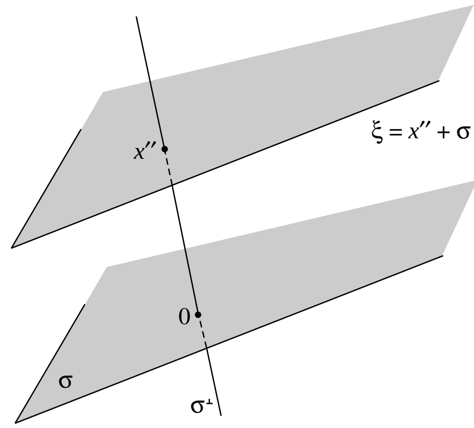

FIGURE I.6.

If $u = u'' \in \sigma^\perp$ this can be written

$$\int_{\mathbf{R}^n} f(x)\langle x,u''\rangle^k\,dx = \int_{\sigma^\perp}\int_\sigma f(x'+x'')\langle x'',u''\rangle^k\,dx'\,dx''$$

so the polynomial

$$P_{\sigma,k}(u'') = \int_{\sigma^\perp} \widehat{f}(x''+\sigma)\langle x'',u''\rangle^k\,dx''$$

is the restriction to $\sigma^\perp$ of the polynomial P_k.

In analogy with the space $\mathcal{D}_H(\mathbf{P}^n)$ in No. 2 we define the space $\mathcal{D}_H(\mathbf{G}(d,n))$ as the set of $\mathcal{C}^\infty$ functions

$$\varphi(\xi) = \varphi_\sigma(x'') = \varphi(x''+\sigma) \quad (\text{if } \xi = (\sigma, x''))$$

on $\mathbf{G}(d,n)$ of compact support satisfying the following condition.

(H) : *For each $k \in \mathbb{Z}^+$ there exists a homogeneous k^{th} degree polynomial P_k on $\mathbf{R}^n$ such that for each $\sigma \in \mathbf{G}_{d,n}$ the polynomial*

$$P_{\sigma,k}(u'') = \int_{\sigma^\perp} \varphi(x''+\sigma)\langle x'',u''\rangle^k\,dx'' \, , \quad u'' \in \sigma^\perp \, ,$$

coincides with the restriction $P_k|\sigma^\perp$.

Theorem 6.3. *The d-dimensional Radon transform is a bijection of $\mathcal{D}(\mathbf{R}^n)$ onto $\mathcal{D}_H(\mathbf{G}(d,n))$.*

Proof. For $d = n-1$ this is Theorem 2.10. We shall now reduce the case of general $d \le n-2$ to the case $d = n-1$. It remains just to prove the surjectivity in Theorem 6.3.

We shall actually prove a stronger statement.

Theorem 6.4. *Let $\varphi \in \mathcal{D}(\mathbf{G}(d,n))$ have the property: For each pair $\sigma, \tau \in \mathbf{G}_{d,n}$ and each $k \in \mathbb{Z}^+$ the polynomials*

$$\begin{aligned} P_{\sigma,k}(u) &= \int_{\sigma^\perp} \varphi(x'' + \sigma)\langle x'', u\rangle^k \, dx'' \qquad u \in \mathbf{R}^n \\ P_{\tau,k}(u) &= \int_{\tau^\perp} \varphi(y'' + \tau)\langle y'', u\rangle^k \, dy'' \qquad u \in \mathbf{R}^n \end{aligned}$$

agree for $u \in \sigma^\perp \cap \tau^\perp$. Then $\varphi = \widehat{f}$ for some $f \in \mathcal{D}(\mathbf{R}^n)$.

Proof. Let $\varphi = \mathcal{D}(\mathbf{G}(d,n))$ have the property above. Let $\omega \in \mathbf{R}^n$ be a unit vector. Let $\sigma, \tau \in \mathbf{G}_{d,n}$ be perpendicular to ω. Consider the $(n-d-1)$-dimensional integral

$$\Psi_\sigma(\omega, p) = \int_{\langle \omega, x''\rangle = p,\, x'' \in \sigma^\perp} \varphi(x'' + \sigma) d_{n-d-1}(x''), \quad p \in \mathbf{R}. \tag{58}$$

We claim that

$$\Psi_\sigma(\omega, p) = \Psi_\tau(\omega, p).$$

To see this consider the moment

$$\begin{aligned} &\int_{\mathbf{R}} \Psi_\sigma(\omega, p) p^k \, dp \\ &= \int_{\mathbf{R}} p^k \left(\int \varphi(x'' + \sigma) d_{n-d-1}(x'') \right) dp = \int_{\sigma^\perp} \varphi(x'' + \sigma)\langle x'', \omega\rangle^k \, dx'' \\ &= \int_{\tau^\perp} \varphi(y'' + \tau)\langle y'', \omega\rangle^k \, dy'' = \int_{\mathbf{R}} \Psi_\tau(\omega, p) p^k \, dp. \end{aligned}$$

Thus $\Psi_\sigma(\omega, p) - \Psi_\tau(\omega, p)$ is perpendicular to all polynomials in p; having compact support it would be identically 0. We therefore put $\Psi(\omega, p) = \Psi_\sigma(\omega, p)$. Observe that Ψ is smooth; in fact for ω in a neighborhood of a fixed ω_0 we can let σ depend smoothly on ω so by (58), $\Psi_\sigma(\omega, p)$ is smooth.

Writing

$$\langle x'', \omega\rangle^k = \sum_{|\alpha|=k} p_\alpha(x'')\omega^\alpha, \qquad \omega^\alpha = \omega_1^{\alpha_1} \dots \omega_n^{\alpha_n}$$

we have

$$\int_{\mathbf{R}} \Psi(\omega, p) p^k \, dp = \sum_{|\alpha|=k} A_\alpha \omega^\alpha,$$

where

$$A_\alpha = \int_{\sigma^\perp} \varphi(x'' + \sigma) p_\alpha(x'') \, dx''.$$

Here A_α is independent of σ if $\omega \in \sigma^\perp$; in other words, viewed as a function of ω, A_α has for each σ a constant value as ω varies in $\sigma^\perp \cap S_1(0)$. To see that this value is the same as the value on $\tau^\perp \cap S_1(0)$ we observe that there exists a $\rho \in \mathbf{G}_{d,n}$ such that $\rho^\perp \cap \sigma^\perp \neq 0$ and $\rho^\perp \cap \tau^\perp \neq 0$. (Extend the 2-plane spanned by a vector in $\sigma^\perp$ and a vector in $\tau^\perp$ to an $(n-d)$-plane.) This shows that A_α is constant on $S_1(0)$ so $\Psi \in \mathcal{D}_H(\mathbf{P}^n)$. Thus by Theorem 2.10,

$$\Psi(\omega, p) = \int_{\langle x,\omega\rangle = p} f(x)\, dm(x) \tag{59}$$

for some $f \in \mathcal{D}(\mathbf{R}^n)$. It remains to prove that

$$\varphi(x'' + \sigma) = \int_\sigma f(x' + x'')\, dx' \,. \tag{60}$$

But as x'' runs through an arbitrary hyperplane in $\sigma^\perp$ it follows from (58) and (59) that both sides of (60) have the same integral. By the injectivity of the $(n-d-1)$-dimensional Radon transform on $\sigma^\perp$ equation (60) follows. This proves Theorem 6.4.

Theorem 6.4 raises the following elementary question: If a function f on $\mathbf{R}^n$ is a polynomial on each k-dimensional subspace, is f itself a polynomial? The answer is no for $k = 1$ but yes if $k > 1$. See Proposition 6.13 below, kindly communicated by Schlichtkrull.

We shall now prove another characterization of the range of $\mathcal{D}(\mathbf{R}^n)$ under the d-plane transform (for $d \leq n-2$). The proof will be based on Theorem 6.4.

Given any $d+1$ points $(x_0, \ldots, x_d)$ in general position let $\xi(x_0, \ldots, x_d)$ denote the d-plane passing through them. If $\varphi \in \mathcal{E}(\mathbf{G}(d,n))$ we shall write $\varphi(x_0, \ldots, x_d)$ for the value $\varphi(\xi(x_0, \ldots, x_d))$. We also write $V(\{x_i - x_0\}_{i=1,d})$ for the volume of the parallelepiped spanned by vectors $(x_i - x_0), (1 \leq i \leq d)$. The mapping

$$(\lambda_1, \ldots, \lambda_d) \to x_0 + \Sigma_{i=1}^d \lambda_i (x_i - x_0)$$

is a bijection of $\mathbf{R}^d$ onto $\xi(x_0, \ldots, x_d)$ and

$$\widehat{f}(x_0, \ldots, x_d) = V(\{x_i - x_0\}_{i=1,d}) \int_{\mathbf{R}^d} f(x_0 + \Sigma_i \lambda_i (x_i - x_0))\, d\lambda \,. \tag{61}$$

The range $\mathcal{D}(\mathbf{R}^n)$ can now be described by the following alternative to Theorem 6.4. Let x_i^k denote the k^{th} coordinate of x_i.

Theorem 6.5. *If $f \in \mathcal{D}(\mathbf{R}^n)$ then $\varphi = \widehat{f}$ satisfies the system*

$$(\partial_{i,k}\partial_{j,\ell} - \partial_{j,k}\partial_{i,\ell})\big(\varphi(x_0, \ldots, x_d)/V(\{x_i - x_0\}_{i=1,d})\big) = 0 \,, \tag{62}$$

where

$$0 \leq i, j \leq d \,,\ 1 \leq k, \ell \leq n \,, \partial_{i,k} = \partial/\partial x_i^k \,.$$

Conversely, if $\varphi \in \mathcal{D}(\mathbf{G}(d,n))$ satisfies (62) then $\varphi = \widehat{f}$ for some $f \in \mathcal{D}(\mathbf{R}^n)$.

The validity of (62) for $\varphi = \widehat{f}$ is obvious from (61) just by differentiation under the integral sign. For the converse we first prove a simple lemma.

Lemma 6.6. *Let $\varphi \in \mathcal{E}(\mathbf{G}(d,n))$ and $A \in \mathbf{O}(n)$. Let $\psi = \varphi \circ A$. Then if $\varphi(x_0, \dots, x_d)$ satisfies (62) so does the function*

$$\psi(x_0, \dots, x_d) = \varphi(Ax_0, \dots, Ax_d)\,.$$

Proof. Let $y_i = Ax_i$ so $y_i^\ell = \Sigma_p a_{\ell p} x_i^p$. Then, if $D_{i,k} = \partial/\partial y_i^k$,

$$(\partial_{i,k}\partial_{j,\ell} - \partial_{j,k}\partial_{i,\ell}) = \Sigma_{p,q=1}^n a_{pk} a_{q\ell} (D_{i,p}D_{j,q} - D_{i,q}D_{j,p})\,. \tag{63}$$

Since A preserves volumes, the lemma follows.

Suppose now φ satisfies (62). We write $\sigma = (\sigma_1, \dots, \sigma_d)$ if (σ_j) is an orthonormal basis of σ. If $x'' \in \sigma^\perp$, the $(d+1)$-tuple

$$(x'', x'' + \sigma_1, \dots, x'' + \sigma_d)$$

represents the d-plane $x'' + \sigma$ and the polynomial

$$\begin{aligned} (64)\ P_{\sigma,k}(u'') &= \int_{\sigma^\perp} \varphi(x'' + \sigma) \langle x'', u''\rangle^k \, dx'' \\ &= \int_{\sigma^\perp} \varphi(x'', x'' + \sigma_1, \dots, x'' + \sigma_d) \langle x'', u''\rangle^k \, dx''\,, \quad u'' \in \sigma^\perp\,, \end{aligned}$$

depends only on σ. In particular, it is invariant under orthogonal transformations of $(\sigma_1, \dots, \sigma_d)$. In order to use Theorem 6.4 we must show that for any $\sigma, \tau \in \mathbf{G}_{d,n}$ and any $k \in \mathbb{Z}^+$,

$$P_{\sigma,k}(u) = P_{\tau,k}(u) \ \text{ for } \ u \in \sigma^\perp \cap \tau^\perp\,, \quad |u| = 1\,. \tag{65}$$

The following lemma is a basic step towards (65).

Lemma 6.7. *Assume $\varphi \in \mathbf{G}(d,n)$ satisfies (62). Let*

$$\sigma = (\sigma_1, \dots, \sigma_d), \tau = (\tau_1, \dots, \tau_d)$$

be two members of $\mathbf{G}_{d,n}$. Assume

$$\sigma_j = \tau_j \quad \textit{for } 2 \le j \le d\,.$$

Then

$$P_{\sigma,k}(u) = P_{\tau,k}(u) \quad \textit{for } u \in \sigma^\perp \cap \tau^\perp\,, \quad |u| = 1\,.$$

Proof. Let $e_i (1 \le i \le n)$ be the natural basis of $\mathbf{R}^n$ and $\epsilon = (e_1, \dots, e_d)$. Select $A \in \mathbf{O}(n)$ such that

$$\sigma = A\epsilon\,, \quad u = Ae_n\,.$$

Let

$$\eta = A^{-1}\tau = (A^{-1}\tau_1, \dots, A^{-1}\tau_d) = (A^{-1}\tau_1, e_2, \dots, e_d)\,.$$

The vector $E = A^{-1}\tau_1$ is perpendicular to e_j $(2 \le j \le d)$ and to e_n (since $u \in \tau^\perp$). Thus

$$E = a_1 e_1 + \sum_{d+1}^{n-1} a_i e_i \qquad (a_1^2 + \sum_i a_i^2 = 1)\,.$$

In (64) we write $P^\varphi_{\sigma,k}$ for $P_{\sigma,k}$. Putting $x'' = Ay$ and $\psi = \varphi \circ A$ we have

$$P^\varphi_{\sigma,k}(u) = \int_{\epsilon^\perp} \varphi(Ay, A(y+e_1), \dots, A(y+e_d))\langle y, e_n\rangle^k \; dy = P^\psi_{\epsilon,k}(e_n)$$

and similarly

$$P^\varphi_{\tau,k}(u) = P^\psi_{\eta,k}(e_n)\,.$$

Thus, taking Lemma 6.6 into account, we have to prove the statement:

$$P_{\epsilon,k}(e_n) = P_{\eta,k}(e_n)\,, \tag{66}$$

where $\epsilon = (e_1, \dots, e_d), \eta = (E, e_2, \dots, e_d)$, E being any unit vector perpendicular to e_j $(2 \le j \le d)$ and to e_n. First we take

$$E = E_t = \sin t\, e_1 + \cos t\, e_i \quad (d < i < n)$$

and put $\epsilon_t = (E_t, e_2, \dots, e_d)$. We shall prove

$$P_{\epsilon_t,k}(e_n) = P_{\epsilon,k}(e_n)\,. \tag{67}$$

Without restriction of generality we can take $i = d+1$. The space $\epsilon_t^\perp$ consists of the vectors

$$x_t = (-\cos t\, e_1 + \sin t\, e_{d+1})\lambda_{d+1} + \sum_{i=d+2}^{n} \lambda_i e_i\,, \quad \lambda_i \in \mathbf{R}\,. \tag{68}$$

Putting $P(t) = P_{\epsilon_t,k}(e_n)$ we have

$$P(t) = \int_{\mathbf{R}^{n-d}} \varphi(x_t, x_t + E_t, x_t + e_2, \dots, x_t + e_d)\lambda_n^k \, d\lambda_n \dots d\lambda_{d+1}\,. \tag{69}$$

In order to use (62) we replace φ by the function

$$\psi(x_0, \ldots, x_d) = \varphi(x_0, \ldots, x_d)/V(\{x_i - x_0\}_{i=1,d})\,.$$

Since the vectors in (69) span volume 1 replacing φ by ψ in (69) does not change $P(t)$. Applying $\partial/\partial t$ we get (with $d\lambda = d\lambda_n \ldots d\lambda_{d+1}$),

$$\begin{aligned} P'(t) &= \int_{\mathbf{R}^{n-d}} \Big[\sum_{j=0}^{d} \lambda_{d+1}(\sin t\, \partial_{j,1}\psi + \cos t\, \partial_{j,d+1}\psi) \\ &+ \cos t\, \partial_{1,1}\psi - \sin t\, \partial_{1,d+1}\psi \Big] \lambda_n^k \, d\lambda\,. \end{aligned} \tag{70}$$

Now φ is a function on $\mathbf{G}(d,n)$. Thus for each $i \neq j$ it is invariant under the substitution

$$y_k = x_k \; (k \neq i),\; y_i = s x_i + (1-s)x_j = x_j + s(x_i - x_j)\,, \quad s > 0$$

whereas the volume changes by the factor s. Thus

$$\psi(y_0, \ldots, y_d) = s^{-1}\psi(x_0, \ldots, x_d)\,.$$

Taking $\partial/\partial s$ at $s = 1$ we obtain

$$\psi(x_0, \ldots, x_d) + \sum_{k=1}^{n} (x_i^k - x_j^k)(\partial_{i,k}\psi)(x_0, \ldots, x_d) = 0\,. \tag{71}$$

Note that in (70) the derivatives are evaluated at

$$(x_0, \ldots, x_d) = (x_t, x_t + E_t, x_t + e_2, \ldots, x_t + e_d)\,. \tag{72}$$

Using (71) for $(i,j) = (1,0)$ and $(i,j) = (0,1)$ and adding we obtain

$$\sin t\, (\partial_{0,1}\psi + \partial_{1,1}\psi) + \cos t\, (\partial_{0,d+1}\psi + \partial_{1,d+1}\psi) = 0\,. \tag{73}$$

For $i \geq 2$ we have

$$x_i - x_0 = e_i,\; x_i - x_1 = -\sin t\, e_1 - \cos t\, e_{d+1} + e_i\,,$$

and this gives the relations (for $j = 0$ and $j = 1$)

$$\psi(x_0, \ldots, x_d) + (\partial_{i,i}\psi)(x_0, \ldots, x_d) = 0\,, \tag{74}$$

$$\psi - \sin t\, (\partial_{i,1}\psi) - \cos t\, (\partial_{i,d+1}\psi) + \partial_{i,i}\psi = 0\,. \tag{75}$$

Thus by (73)–(75) formula (70) simplifies to

$$P'(t) = \int_{\mathbf{R}^{n-d}} \left[\cos t\, (\partial_{1,1}\psi) - \sin t\, (\partial_{1,d+1}\psi)\right] \lambda_n^k \, d\lambda\,.$$

In order to bring in 2nd derivatives of ψ we integrate by parts in λ_n,

$$(76)\quad (k+1)P'(t) = \int_{\mathbf{R}^{n-d}} -\frac{\partial}{\partial\lambda_n}\left[\cos t\ (\partial_{1,1}\psi) - \sin t\ (\partial_{1,d+1}\psi)\right]\lambda_n^{k+1}\,d\lambda\,.$$

Since the derivatives $\partial_{j,k}\psi$ are evaluated at the point (72) we have in (76)

$$(77)\qquad \frac{\partial}{\partial\lambda_n}(\partial_{j,k}\psi) = \sum_{i=0}^{d}\partial_{i,n}(\partial_{j,k}\psi)$$

and also, by (68) and (72),

$$(78)\qquad \frac{\partial}{\partial\lambda_{d+1}}(\partial_{j,k}\psi) = -\cos t\sum_{0}^{d}\partial_{i,1}(\partial_{j,k}\psi) + \sin t\sum_{0}^{d}\partial_{i,d+1}(\partial_{j,k}\psi)\,.$$

We now plug (77) into (76) and then invoke equations (62) for ψ which give

$$(79)\quad \sum_{0}^{d}\partial_{i,n}\partial_{1,1}\psi = \partial_{1,n}\sum_{0}^{d}\partial_{i,1}\psi\,,\qquad \sum_{0}^{d}\partial_{i,n}\partial_{1,d+1}\psi = \partial_{1,n}\sum_{0}^{d}\partial_{i,d+1}\psi\,.$$

Using (77) and (79) we see that (76) becomes

$$-(k+1)P'(t) =$$
$$\int_{\mathbf{R}^{n-d}}\left[\partial_{1,n}(\cos t\,\Sigma_i\partial_{i,1}\psi - \sin t\,\Sigma_i\partial_{i,d+1}\psi)\right](x_t, x_t+E_t,\ldots,x_t+e_d)\lambda_n^{k+1}\,d\lambda$$

so by (78)

$$(k+1)P'(t) = \int_{\mathbf{R}^{n-d}}\frac{\partial}{\partial\lambda_{d+1}}(\partial_{1,n}\psi)\lambda_n^{k+1}\,d\lambda\,.$$

Since $d+1<n$, the integration in λ_{d+1} shows that $P'(t)=0$, proving (67).

This shows that without changing $P_{\epsilon,k}(e_n)$ we can pass from $\epsilon = (e_1,\ldots,e_d)$ to

$$\epsilon_t = (\sin\ t\,e_1 + \cos\ t\,e_{d+1}, e_2,\ldots,e_d)\,.$$

By iteration we can replace e_1 by

$$\sin t_{n-d-1}\ldots\sin t_1 e_1 + \sin t_{n-d-1}\ldots\sin t_2\cos t_1 e_{d+1} + \cdots + \cos t_{n-d-1}e_{n-1},$$

but keeping $e_2,\ldots,e_d$ unchanged. This will reach an arbitrary E so (66) is proved.

We shall now prove (65) in general. We write σ and τ in orthonormal bases, $\sigma = (\sigma_1, \ldots, \sigma_d), \tau = (\tau_1, \ldots, \tau_d)$. Using Lemma 6.7 we shall pass from σ to τ without changing $P_{\sigma,k}(u)$, u being fixed.

Consider τ_1. If two members of σ, say σ_j and σ_k, are both not orthogonal to τ_1 that is ($\langle \sigma_j, \tau_1 \rangle \neq 0, \langle \sigma_k, \tau_1 \rangle \neq 0$) we rotate them in the (σ_j, σ_k)-plane so that one of them becomes orthogonal to τ_1. As remarked after (63) this has no effect on $P_{\sigma,k}(u)$. We iterate this process (with the same τ_1) and end up with an orthogonal frame $(\sigma_1^*, \ldots, \sigma_d^*)$ of σ in which at most one entry σ_i^* is not orthogonal to τ_1. In this frame we replace this σ_i^* by τ_1. By Lemma 6.7 this change of σ does not alter $P_{\sigma,k}(u)$.

We now repeat this process with $\tau_2, \tau_3 \ldots$, etc. Each step leaves $P_{\sigma,k}(u)$ unchanged (and u remains fixed) so this proves (65) and the theorem.

We consider now the case $d = 1, n = 3$ in more detail. Here $f \to \widehat{f}$ is the X-ray transform in $\mathbf{R}^3$. We also change the notation and write ξ for x_0, η for x_1 so $V(\{x_1 - x_0\})$ equals $|\xi - \eta|$. Then Theorem 6.5 reads as follows.

Theorem 6.8. *The X-ray transform $f \to \widehat{f}$ in $\mathbf{R}^3$ is a bijection of $\mathcal{D}(\mathbf{R}^3)$ onto the space of $\varphi \in \mathcal{D}(\mathbf{G}(1,3))$ satisfying*

$$\left(\frac{\partial}{\partial \xi_k}\frac{\partial}{\partial \eta_\ell} - \frac{\partial}{\partial \xi_\ell}\frac{\partial}{\partial \eta_k}\right)\left(\frac{\varphi(\xi,\eta)}{|\xi - \eta|}\right) = 0\,, \quad 1 \leq k, \ell \leq 3\,. \tag{80}$$

Now let $\mathbf{G}'(1,3) \subset \mathbf{G}(1,3)$ denote the open subset consisting of the *non-horizontal* lines. We shall now show that for $\varphi \in \mathcal{D}(\mathbf{G}(1,n))$ (and even for $\varphi \in \mathcal{E}(\mathbf{G}'(1,n))$) the validity of (80) for $(k,\ell) = (1,2)$ implies (80) for general (k,ℓ). Note that (71) (which is also valid for $\varphi \in \mathcal{E}(\mathbf{G}'(1,n))$) implies

$$\frac{\varphi(\xi,\eta)}{|\xi-\eta|} + \sum_1^3 (\xi_i - \eta_i)\frac{\partial}{\partial \xi_i}\left(\frac{\varphi(\xi,\eta)}{|\xi-\eta|}\right) = 0\,.$$

Here we apply $\partial/\partial\eta_k$ and obtain

$$\left(\sum_{i=1}^3 (\xi_i - \eta_i)\frac{\partial^2}{\partial \xi_i \partial \eta_k} - \frac{\partial}{\partial \xi_k} + \frac{\partial}{\partial \eta_k}\right)\left(\frac{\varphi(\xi,\eta)}{|\xi-\eta|}\right) = 0\,.$$

Exchanging ξ and η and adding we derive

$$\sum_{i=1}^3 (\xi_i - \eta_i)\left(\frac{\partial^2}{\partial \xi_i \partial \eta_k} - \frac{\partial^2}{\partial \xi_k \partial \eta_i}\right)\left(\frac{\varphi(\xi,\eta)}{|\xi-\eta|}\right) = 0 \tag{81}$$

for $k = 1, 2, 3$. Now assume (80) for $(k,\ell) = (1,2)$. Taking $k = 1$ in (81) we derive (80) for $(k,\ell) = (1,3)$. Then taking $k = 3$ in (81) we deduce (80) for $(k,\ell) = (3,2)$. This verifies the claim above.

We can now put this in a simpler form. Let $\ell(\xi,\eta)$ denote the line through the points $\xi \neq \eta$. Then the mapping

$$(\xi_1, \xi_2, \eta_1, \eta_2) \to \ell((\xi_1, \xi_2, 0), (\eta_1, \eta_2, -1))$$

is a bijection of $\mathbf{R}^4$ onto $\mathbf{G}'(1,3)$. The operator

$$\Lambda = \frac{\partial^2}{\partial\xi_1\partial\eta_2} - \frac{\partial^2}{\partial\xi_2\partial\eta_1} \tag{82}$$

is a well defined differential operator on the dense open set $\mathbf{G}'(1,3)$. If $\varphi \in \mathcal{E}(\mathbf{G}(1,3))$ we denote by ψ the restriction of the function $(\xi,\eta) \to \varphi(\xi,\eta)/|\xi-\eta|$ to $\mathbf{G}'(1,3)$. Then we have proved the following result.

Theorem 6.9. *The X-ray transform $f \to \widehat{f}$ is a bijection of $\mathcal{D}(\mathbf{R}^3)$ onto the space*

$$\{\varphi \in \mathcal{D}(\mathbf{G}(1,3)) : \Lambda\psi = 0\}\,. \tag{83}$$

We shall now rewrite the differential equation (83) in Plücker coordinates. The line joining ξ and η has Plücker coordinates $(p_1,p_2,p_3,q_1,q_2,q_3)$ given by

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \xi_1 & \xi_2 & \xi_3 \\ \eta_1 & \eta_2 & \eta_3 \end{vmatrix} = p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}\,, \qquad q_i = \begin{vmatrix} \xi_i & 1 \\ \eta_i & 1 \end{vmatrix}$$

which satisfy

$$p_1q_1 + p_2q_2 + p_3q_3 = 0\,. \tag{84}$$

Conversely, each ratio $(p_1 : p_2 : p_3 : q_1 : q_2 : q_3)$ determines uniquely a line provided (84) is satisfied. The set $\mathbf{G}'(1,3)$ is determined by $q_3 \neq 0$. Since the common factor can be chosen freely we fix q_3 as 1. Then we have a bijection $\tau : \mathbf{G}'(1,3) \to \mathbf{R}^4$ given by

$$x_1 = p_2 + q_2,\ x_2 = -p_1 - q_1,\ x_3 = p_2 - q_2,\ x_4 = -p_1 + q_1 \tag{85}$$

with inverse

$$(p_1,p_2,p_3,q_1,q_2) = \Big(\tfrac{1}{2}(-x_2-x_4), \tfrac{1}{2}(x_1+x_3), \tfrac{1}{4}(-x_1^2-x_2^2+x_3^2+x_4^2), \tfrac{1}{2}(-x_2+x_4), \tfrac{1}{2}(x_1-x_3)\Big).$$

Theorem 6.10. *If $\varphi \in \mathcal{D}(\mathbf{G}(1,3))$ satisfies (83) then the restriction $\varphi|\mathbf{G}'(1,3)$ (with $q_3 = 1$) has the form*

$$\varphi(\xi,\eta) = |\xi-\eta|\ u(p_2+q_2, -p_1-q_1, p_2-q_2, -p_1+q_1) \tag{86}$$

where u satisfies

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2} - \frac{\partial^2 u}{\partial x_4^2} = 0\,. \tag{87}$$

On the other hand, if u satisfies (87) then (86) defines a function φ on $\mathbf{G}'(1,3)$ which satisfies (80).

Proof. First assume $\varphi \in \mathcal{D}(\mathbf{G}(1,3))$ satisfies (83) and define $u \in \mathcal{E}(\mathbf{R}^4)$ by

$$u(\tau(\ell)) = \varphi(\ell)(1+q_1^2+q_2^2)^{-\frac{1}{2}}\,, \tag{88}$$

where $\ell \in \mathbf{G}'(1,3)$ has Plücker coordinates $(p_1,p_2,p_3,q_1,q_2,1)$. On the line ℓ consider the points ξ, η for which $\xi_3 = 0, \eta_3 = -1$ (so $q_3 = 1$). Then since

$$p_1 = -\xi_2,\ p_2 = \xi_1,\ q_1 = \xi_1 - \eta_1,\ q_2 = \xi_2 - \eta_2$$

we have

$$\frac{\varphi(\xi,\eta)}{|\xi-\eta|} = u(\xi_1+\xi_2-\eta_2,\ -\xi_1+\xi_2+\eta_1,\ \xi_1-\xi_2+\eta_2,\ \xi_1+\xi_2-\eta_1)\,. \tag{89}$$

Now (83) implies (87) by use of the chain rule.

On the other hand, suppose $u \in \mathcal{E}(\mathbf{R}^4)$ satisfies (87). Define φ by (88). Then $\varphi \in \mathcal{E}(\mathbf{G}'(1,3))$ and by (89),

$$\Lambda\left(\frac{\varphi(\xi,\eta)}{|\xi-\eta|}\right) = 0\,.$$

As shown before the proof of Theorem 6.9 this implies that the whole system (80) is verified.

We shall now see what implications Ásgeirsson's mean-value theorem (Theorem 4.5, Chapter V) has for the range of the X-ray transform. We have from (85),

$$\int_0^{2\pi} u(r\cos\varphi, r\sin\varphi, 0, 0)\,d\varphi = \int_0^{2\pi} u(0,0,r\cos\varphi, r\sin\varphi)\,d\varphi\,. \tag{90}$$

The first points $(r\cos\varphi, r\sin\varphi, 0, 0)$ correspond via (85) to the lines with

$$(p_1,p_2,p_3,q_1,q_2,q_3) = (-\tfrac{r}{2}\sin\varphi, \tfrac{r}{2}\cos\varphi, -\tfrac{r^2}{4}, -\tfrac{r}{2}\sin\varphi, \tfrac{r}{2}\cos\varphi, 1)$$

containing the points

$$\begin{aligned}(\xi_1,\xi_2,\xi_3) &= (\tfrac{r}{2}\cos\varphi, \tfrac{r}{2}\sin\varphi, 0)\\ (\eta_1,\eta_2,\eta_3) &= (\tfrac{r}{2}(\sin\varphi+\cos\varphi), +\tfrac{r}{2}(\sin\varphi-\cos\varphi), -1)\end{aligned}$$

with $|\xi-\eta|^2 = 1+\frac{r^2}{4}$. The points $(0,0,r\cos\varphi, r\sin\varphi)$ correspond via (85) to the lines with

$$(p_1,p_2,p_3,q_1,q_2,q_3) = (-\tfrac{r}{2}\sin\varphi, \tfrac{r}{2}\cos\varphi, \tfrac{r^2}{4}, \tfrac{r}{2}\sin\varphi, -\tfrac{r}{2}\cos\varphi, 1)$$

containing the points

$$\begin{aligned}(\xi_1,\xi_2,\xi_3) &= (\tfrac{r}{2}\cos\varphi, \tfrac{r}{2}\sin\varphi, 0)\\ (\eta_1,\eta_2,\eta_3) &= (\tfrac{r}{2}(\cos\varphi-\sin\varphi), \tfrac{r}{2}(\cos\varphi+\sin\varphi), -1)\end{aligned}$$

with $|\xi-\eta|^2 = 1+\frac{r^2}{4}$. Thus (90) takes the form

$$(91)\quad \int_0^{2\pi} \varphi(\tfrac{r}{2}\cos\theta, \tfrac{r}{2}\sin\theta, 0, \tfrac{r}{2}(\sin\theta+\cos\theta), \tfrac{r}{2}(\sin\theta-\cos\theta), -1)\, d\theta$$

$$= \int_0^{2\pi} \varphi(\tfrac{r}{2}\cos\theta, \tfrac{r}{2}\sin\theta, 0, \tfrac{r}{2}(\cos\theta-\sin\theta), \tfrac{r}{2}(\cos\theta+\sin\theta), -1)\, d\theta\,.$$

The lines forming the arguments of φ in these integrals are the two families of generating lines for the hyperboloid (see Fig. I.7)

$$x^2+y^2 = \tfrac{r^2}{4}(z^2+1)\,.$$

Definition. A function $\varphi \in \mathcal{E}(\mathbf{G}'(1,3))$ is said to be a *harmonic line function* if

$$\Lambda\left(\frac{\varphi(\xi,\eta)}{|\xi-\eta|}\right) = 0\,.$$

Theorem 6.11. *A function $\varphi \in \mathcal{E}(\mathbf{G}'(1,3))$ is a harmonic line function if and only if for each hyperboloid of revolution H of one sheet and vertical axis the mean values of φ over the two families of generating lines of H are equal. (The variable of integration is the polar angle in the equatorial plane of H.).*

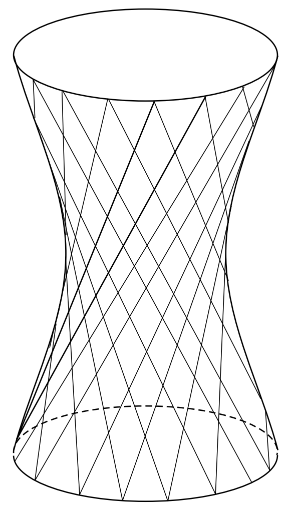

FIGURE I.7.

The proof of (91) shows that φ harmonic implies the mean value property for φ. The converse follows since (90) (with $(0,0)$ replaced by an arbitrary point in $\mathbf{R}^2$) is equivalent to (87) (Chapter V, Theorem 4.5).

Corollary 6.12. *Let $\varphi \in \mathcal{D}(\mathbf{G}(1,3))$. Then φ is in the range of the X-ray transform if and only if φ has the mean value property for arbitrary hyperboloid of revolution of one sheet (and arbitrary axis).*

We conclude this section with the following result due to Schlichtkrull mentioned in connection with Theorem 6.4.

Proposition 6.13. *Let f be a function on $\mathbf{R}^n$ and $k \in \mathbb{Z}^+$, $1 < k < n$. Assume that for each k-dimensional subspace $E_k \subset \mathbf{R}^n$ the restriction $f|E_k$ is a polynomial on E_k. Then f is a polynomial on $\mathbf{R}^n$.*

For $k=1$ the result is false as the example $f(x,y) = xy^2/(x^2+y^2)$, $f(0,0)=0$ shows. We recall now the Lagrange interpolation formula. Let $a_0,\ldots,a_m$ be distinct numbers in $\mathbf{C}$. Then each polynomial $P(x)$ $(x \in \mathbf{R})$

of degree $\leq m$ can be written

$$P(x) = P(a_0)Q_0(x) + \cdots + P(a_m)Q_m(x)\,,$$

where

$$Q_i(x) = \prod_{j=0}^{m}(x-a_j)/(x-a_i)\prod_{j\neq i}(a_i-a_j)\,.$$

In fact, the two sides agree at $m+1$ distinct points. This implies the following result.

Lemma 6.14. *Let $f(x_1,\ldots,x_n)$ be a function on $\mathbf{R}^n$ such that for each i with $x_j (j\neq i)$ fixed the function $x_i \to f(x_1,\ldots,x_n)$ is a polynomial. Then f is a polynomial.*

For this we use Lagrange's formula on the polynomial $x_1 \longrightarrow f(x_1,x_2,\ldots,x_n)$ and get

$$f(x_1,\ldots,x_n) = \sum_{j=0}^{m} f(a_j,x_2,\ldots,x_m)Q_j(x_1)\,.$$

The lemma follows by iteration.

For the proposition we observe that the assumption implies that f restricted to each 2-plane E_2 is a polynomial on E_2. For a fixed $(x_2,\ldots,x_n)$ the point $(x_1,\ldots,x_n)$ is in the span of $(1,0,\ldots,0)$ and $(0,x_2,\ldots,x_n)$ so $f(x_1,\ldots,x_n)$ is a polynomial in x_1. Now the lemma implies the result.

§7 Applications

A. Partial differential equations.

The inversion formula in Theorem 3.1 is very well suited for applications to partial differential equations. To explain the underlying principle we write the inversion formula in the form

$$f(x) = \gamma\, L_x^{\frac{n-1}{2}}\left(\int_{\mathbf{S}^{n-1}} \widehat{f}(\omega,\langle x,\omega\rangle)\,d\omega\right). \tag{92}$$

where the constant γ equals $\frac{1}{2}(2\pi i)^{1-n}$. Note that the function $f_\omega(x) = \widehat{f}(\omega,\langle x,\omega\rangle)$ is a *plane wave with normal* ω, that is, it is constant on each hyperplane perpendicular to ω.

Consider now a differential operator

$$D = \sum_{(k)} a_{k_1\ldots k_n}\partial_1^{k_1}\ldots\partial_n^{k_n}$$

with constant coefficients $a_{k_1,\dots,k_n}$, and suppose we want to solve the differential equation

$$Du = f \tag{93}$$

where f is a given function in $\mathcal{S}(\mathbf{R}^n)$. To simplify the use of (92) we assume n to be odd. We begin by considering the differential equation

$$Dv = f_\omega\,, \tag{94}$$

where f_ω is the plane wave defined above and we look for a solution v which is also a plane wave with normal ω. But a plane wave with normal ω is just a function of one variable; also if v is a plane wave with normal ω so is the function Dv. The differential equation (94) (with v a plane wave) is therefore an *ordinary* differential equation with constant coefficients. Suppose $v = u_\omega$ is a solution and assume that this choice can be made smoothly in ω. Then the function

$$u = \gamma\, L^{\frac{n-1}{2}} \int_{\mathbf{S}^{n-1}} u_\omega \, d\omega \tag{95}$$

is a solution to the differential equation (93). In fact, since D and $L^{\frac{n-1}{2}}$ commute we have

$$Du = \gamma L^{\frac{n-1}{2}} \int_{\mathbf{S}^{n-1}} Du_\omega \, d\omega = \gamma L^{\frac{n-1}{2}} \int_{\mathbf{S}^{n-1}} f_\omega \, d\omega = f\,.$$

This method only assumes that the plane wave solution u_ω to the ordinary differential equation $Dv = f_\omega$ exists and can be chosen so as to depend smoothly on ω. This cannot always be done because D might annihilate all plane waves with normal ω. (For example, take $D = \partial^2/\partial x_1 \partial x_2$ and $\omega = (1,0)$.) However, if this restriction to plane waves is never 0 it follows from a theorem of Trèves [1963] that the solution u_ω can be chosen depending smoothly on ω. Thus we can state

Theorem 7.1. *Assuming the restriction D_ω of D to the space of plane waves with normal ω is $\neq 0$ for each ω formula (95) gives a solution to the differential equation $Du = f \quad (f \in \mathcal{S}(\mathbf{R}^n))$.*

The method of plane waves can also be used to solve the Cauchy problem for hyperbolic differential equations with constant coefficients. We illustrate this method by means of the wave equation $\mathbf{R}^n$,

$$Lu = \frac{\partial^2 u}{\partial t^2},\ u(x,0) = f_0(x),\ u_t(x,0) = f_1(x)\,, \tag{96}$$

f_0, f_1 being given functions in $\mathcal{D}(\mathbf{R}^n)$.

Lemma 7.2. *Let $h \in C^2(\mathbf{R})$ and $\omega \in \mathbf{S}^{n-1}$. Then the function*

$$v(x,t) = h(\langle x, \omega\rangle + t)$$

satisfies $Lv = (\partial^2/\partial t^2)v$.

The proof is obvious. It is now easy, on the basis of Theorem 3.6, to write down the unique solution of the Cauchy problem (96).

Theorem 7.3. *The solution to (96) is given by*

$$u(x,t) = \int_{\mathbf{S}^{n-1}} (Sf)(\omega, \langle x, \omega\rangle + t)\, d\omega \tag{97}$$

where

$$Sf = \begin{cases} c(\partial^{n-1}\widehat{f}_0 + \partial^{n-2}\widehat{f}_1)\,, & n \text{ odd} \\ c\,\mathcal{H}(\partial^{n-1}\widehat{f}_0 + \partial^{n-2}\widehat{f}_1)\,, & n \text{ even}\,. \end{cases}$$

Here $\partial = \partial/\partial p$ and the constant c equals

$$c = \tfrac{1}{2}(2\pi i)^{1-n}\,.$$

Lemma 7.2 shows that (97) is annihilated by the operator $L - \partial^2/\partial t^2$ so we just have to check the initial conditions in (96).

(a) If $n > 1$ is odd then $\omega \to (\partial^{n-1}\widehat{f}_0)(\omega, \langle x, \omega\rangle)$ is an even function on $\mathbf{S}^{n-1}$ but the other term in Sf, that is the function $\omega \to (\partial^{n-2}\widehat{f}_1)(\omega, \langle x, \omega\rangle)$, is odd. Thus by Theorem 3.6, $u(x,0) = f_0(x)$. Applying $\partial/\partial t$ to (97) and putting $t = 0$ gives $u_t(x,0) = f_1(x)$, this time because the function $\omega \to (\partial^n \widehat{f}_0)(\omega, \langle x, \omega\rangle)$ is odd and the function $\omega \to (\partial^{n-1}\widehat{f}_1)(\omega, \langle x, \omega\rangle)$ is even.

(b) If n is even the same proof works if we take into account the fact that $\mathcal{H}$ interchanges odd and even functions on $\mathbf{R}$.

Definition. For the pair $f = \{f_0, f_1\}$ we refer to the function Sf in (97) as the *source*.

In the terminology of Lax-Philips [1967] the wave $u(x,t)$ is said to be

(a) *outgoing* if $u(x,t) = 0$ in the *forward cone* $|x| < t$;

(b) *incoming* if $u(x,t) = 0$ in the *backward cone* $|x| < -t$.

The notation is suggestive because "outgoing" means that the function $x \to u(x,t)$ vanishes in larger balls around the origin as t increases.

Corollary 7.4. *The solution $u(x,t)$ to (96) is*

(i) outgoing if and only if $(Sf)(\omega, s) = 0$ for $s > 0$, all ω.

(ii) incoming if and only if $(Sf)(\omega, s) = 0$ for $s < 0$, all ω.

Proof. For (i) suppose $(Sf)(\omega, s) = 0$ for $s > 0$. For $|x| < t$ we have $\langle x, \omega\rangle + t \geq -|x| + t > 0$ so by (97) $u(x,t) = 0$ so u is outgoing. Conversely, suppose $u(x,t) = 0$ for $|x| < t$. Let $t_0 > 0$ be arbitrary and let $\varphi(t)$ be a smooth function with compact support contained in (t_0, ∞).

Then if $|x| < t_0$ we have

$$\begin{aligned} 0 &= \int_{\mathbf{R}} u(x,t)\varphi(t)\,dt = \int_{\mathbf{S}^{n-1}} d\omega \int_{\mathbf{R}} (Sf)(\omega, \langle x, \omega\rangle + t)\varphi(t)\,dt \\ &= \int_{\mathbf{S}^{n-1}} d\omega \int_{\mathbf{R}} (Sf)(\omega, p)\varphi(p - \langle x, \omega\rangle)\,dp\,. \end{aligned}$$

Taking arbitrary derivative $\partial^k/\partial x_{i_1} \ldots \partial x_{i_k}$ at $x = 0$ we deduce

$$\int_{\mathbf{R}} \left(\int_{\mathbf{S}^{n-1}} (Sf)(\omega, p)\omega_{i_1} \ldots \omega_{i_k}\,d\omega \right) (\partial^k \varphi)(p)\,dp = 0$$

for each k and each $\varphi \in \mathcal{D}(t_0, \infty)$. Integrating by parts in the p variable we conclude that the function

$$p \to \int_{\mathbf{S}^{n-1}} (Sf)(\omega, p)\omega_{i_1} \ldots \omega_{i_k}\,d\omega\,, \quad p \in \mathbf{R} \tag{98}$$

has its k^{th} derivative $\equiv 0$ for $p > t_0$. Thus it equals a polynomial for $p > t_0$. However, if n is odd the function (98) has compact support so it must vanish identically for $p > t_0$.

On the other hand, if n is even and $F \in \mathcal{D}(\mathbf{R})$ then as remarked at the end of §3, $\lim_{|t|\to\infty}(\mathcal{H}F)(t) = 0$. Thus we conclude again that expression (98) vanishes identically for $p > t_0$.

Thus in both cases, if $p > t_0$, the function $\omega \to (Sf)(\omega, p)$ is orthogonal to all polynomials on $\mathbf{S}^{n-1}$, hence must vanish identically.

One can also solve (96) by means of the *Fourier transform*

$$\widetilde{f}(\zeta) = \int_{\mathbf{R}^n} f(x)e^{-i\langle x, \zeta\rangle}\,dx\,.$$

Assuming the function $x \to u(x,t)$ in $\mathcal{S}(\mathbf{R}^n)$ for a given t we obtain

$$\widetilde{u}_{tt}(\zeta, t) + \langle \zeta, \zeta\rangle \widetilde{u}(\zeta, t) = 0\,.$$

Solving this ordinary differential equation with initial data given in (96) we get

$$\widetilde{u}(\zeta, t) = \widetilde{f}_0(\zeta)\cos(|\zeta|t) + \widetilde{f}_1(\zeta)\frac{\sin(|\zeta|t)}{|\zeta|}\,. \tag{99}$$

The function $\zeta \to \sin(|\zeta|t)/|\zeta|$ is entire of exponential type $|t|$ on $\mathbf{C}^n$ (of at most polynomial growth on $\mathbf{R}^n$). In fact, if $\varphi(\lambda)$ is even, holomorphic on $\mathbf{C}$ and satisfies the exponential type estimate (13) in Theorem 3.3, Ch. V, then the same holds for the function Φ on $\mathbf{C}^n$ given by $\Phi(\zeta) = \Phi(\zeta_1, \dots, \zeta_n) = \varphi(\lambda)$ where $\lambda^2 = \zeta_1^2 + \cdots + \zeta_n^2$. To see this put

$$\lambda = \mu + iv\,, \quad \zeta = \xi + i\eta \qquad \mu, \nu \in \mathbf{R}\,, \quad \xi\,, \eta \in \mathbf{R}^n\,.$$

Then

$$\mu^2 - \nu^2 = |\xi|^2 - |\eta|^2\,, \quad \mu^2\nu^2 = (\xi \cdot \eta)^2\,,$$

so

$$|\lambda|^4 = (|\xi|^2 \,-\, |\eta|^2)^2 + 4(\xi \cdot \eta)^2$$

and

$$2|\operatorname{Im}\lambda|^2 = |\eta|^2 - |\xi|^2 + \left[(|\xi|^2 - |\eta|^2)^2 + 4(\xi \cdot \eta)^2\right]^{1/2}\,.$$

Since $|(\xi \cdot \eta)| \leq |\xi|\,|\eta|$ this implies $|\operatorname{Im}\lambda| \leq |\eta|$ so the estimate (13) follows for Φ. Thus by Theorem 3.3, Chapter V there exists a $T_t \in \mathcal{E}'(\mathbf{R}^n)$ with support in $\overline{B_{|t|}(0)}$ such that

$$\frac{\sin(|\zeta|t)}{|\zeta|} = \int_{\mathbf{R}^n} e^{-i\langle\zeta, x\rangle}\, dT_t(x)\,.$$

Theorem 7.5. *Given $f_0, f_1 \in \mathcal{E}(\mathbf{R}^n)$ the function*

$$u(x,t) = (f_0 * T_t')(x) + (f_1 * T_t)(x) \tag{100}$$

satisfies (96). Here T_t' stands for $\partial_t(T_t)$.

Note that (96) implies (100) if f_0 and f_1 have compact support. The converse holds without this support condition.

Corollary 7.6. *If f_0 and f_1 have support in $B_R(0)$ then u has support in the region*

$$|x| \leq |t| + R\,.$$

In fact, by (100) and support property of convolutions (Ch. V, §2), the function $x \to u(x,t)$ has support in $B_{R+|t|}(0)^-$. While Corollary 7.6 implies that for $f_0, f_1 \in \mathcal{D}(\mathbf{R}^n)$ u has support in a suitable solid cone we shall now see that Theorem 7.3 implies that if n is odd u has support in a conical shell (see Fig. I.8).

Corollary 7.7. *Let n be odd. Assume f_0 and f_1 have support in the ball $B_R(0)$.*

(i) Huygens' Principle. The solution u to (96) has support in the conical shell

$$|t| - R \leq |x| \leq |t| + R\,, \tag{101}$$

which is the union for $|y| \leq R$ of the light cones,

$$C_y = \{(x,t) : |x - y| = |t|\}\,.$$

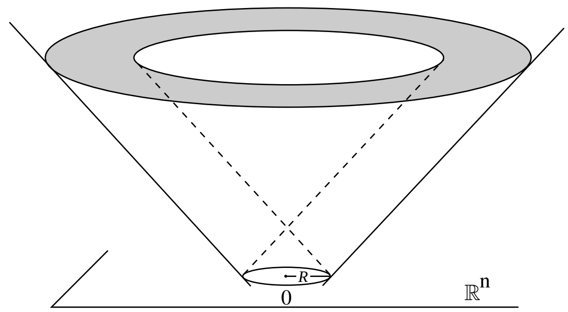

FIGURE I.8.

(ii) The solution to (96) is outgoing if and only if

$$\widehat{f}_0(\omega, p) = \int_p^\infty \widehat{f}_1(\omega, s)\, ds\,, \quad p > 0\,,\ \textit{all}\ \omega \tag{102}$$

and incoming if and only if

$$\widehat{f}_0(\omega, p) = -\int_{-\infty}^p \widehat{f}_1(\omega, s)\, ds\,, \quad p < 0\,,\ \textit{all}\ \omega\,.$$

Note that Part (ii) can also be stated: The solution is outgoing (incoming) if and only if

$$\int_\pi f_0 = \int_{H_\pi} f_1 \qquad \left(\int_\pi f_0 = -\int_{H_\pi} f_1\right)$$

for an arbitrary hyperplane $\pi (0 \notin \pi)$ H_π being the halfspace with boundary π which does not contain 0.

To verify (i) note that since n is odd, Theorem 7.3 implies

$$u(0, t) = 0 \quad \text{for} \quad |t| \geq R\,. \tag{103}$$

If $z \in \mathbf{R}^n$, $F \in \mathcal{E}(\mathbf{R}^n)$ we denote by F^z the translated function $y \to F(y + z)$. Then u^z satisfies (96) with initial data f_0^z, f_1^z which have support contained in $B_{R+|z|}(0)$. Hence by (103)

$$u(z, t) = 0 \quad \text{for} \quad |t| > R + |z|\,. \tag{104}$$

The other inequality in (101) follows from Corollary 7.6.

For the final statement in (i) we note that if $|y| \leq R$ and $(x, t) \in C_y$ then $|x - y| = t$ so $|x| \leq |x - y| + |y| \leq |t| + R$ and $|t| = |x - y| \leq |x| + R$ proving (101). Conversely, if (x, t) satisfies (101) then $(x, t) \in C_y$ with $y = x - |t|\frac{x}{|x|} = \frac{x}{|x|}(|x| - t)$ which has norm $\leq R$.

For (ii) we just observe that since $\widehat{f}_i(\omega, p)$ has compact support in p, (102) is equivalent to (i) in Corollary 7.4.

Thus (102) implies that for $t > 0$, $u(x, t)$ has support in the thinner shell $|t| \leq |x| \leq |t| + R$.

B. X-ray Reconstruction.

The classical interpretation of an X-ray picture is an attempt at reconstructing properties of a 3-dimensional body by means of the X-ray projection on a plane.

In modern X-ray technology the picture is given a more refined mathematical interpretation. Let $B \subset \mathbf{R}^3$ be a body (for example a part of a human body) and let $f(x)$ denote its density at a point x. Let ξ be a line in $\mathbf{R}^3$ and suppose a thin beam of X-rays is directed at B along ξ. Let I_0 and I respectively, denote the intensity of the beam before entering B and after leaving B (see Fig. I.9). As the X-ray traverses the distance Δx along ξ it will undergo the relative intensity loss $\Delta I/I = f(x)\,\Delta x$. Thus $dI/I = -f(x)\,dx$ whence

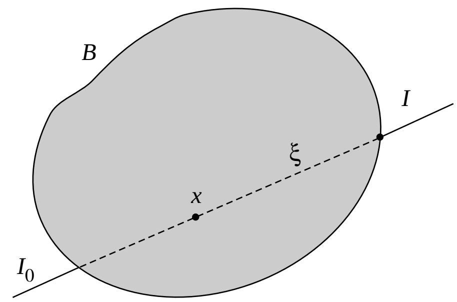

FIGURE I.9.

$$\log(I_0/I) = \int_\xi f(x)\,dx\,, \tag{105}$$

the integral $\widehat{f}(\xi)$ of f along ξ. Since the left hand side is determined by the X-ray picture, the *X-ray reconstruction problem* amounts to the determination of the function f by means of its line integrals $\widehat{f}(\xi)$. *The inversion formula in Theorem 3.1 gives an explicit solution of this problem.*

If $B_0 \subset B$ is a convex subset (for example the heart) it may be of interest to determine the density of f outside B_0 using only X-rays which do not intersect B_0. *The support theorem (Theorem 2.6, Cor. 2.8 and Cor. 6.1) implies that f is determined outside B_0 on the basis of the integrals $\widehat{f}(\xi)$ for which ξ does not intersect B_0.* **Thus the density outside the heart can be determined by means of X-rays which bypass the heart.**

In practice one can of course only determine the integrals $\widehat{f}(\xi)$ in (105) for *finitely* many directions. A compensation for this is the fact that only an approximation to the density f is required. One then encounters the mathematical problem of selecting the directions so as to optimize the approximation.

As before we represent the line ξ as the pair $\xi = (\omega, z)$ where $\omega \in \mathbf{R}^n$ is a unit vector in the direction of ξ and $z = \xi \cap \omega^\perp$ ($\perp$ denoting orthogonal complement). We then write

$$\widehat{f}(\xi) = \widehat{f}(\omega, z) = (P_\omega f)(z)\,. \tag{106}$$

The function $P_\omega f$ is the X-ray picture or the *radiograph* in the direction ω. Here f is a function on $\mathbf{R}^n$ vanishing outside a ball B around the origin and for the sake of Hilbert space methods to be used it is convenient to assume in addition that $f \in L^2(B)$. Then $f \in L^1(\mathbf{R}^n)$ so by the Fubini theorem we have: for each $\omega \in \mathbf{S}^{n-1}$, $P_\omega f(z)$ is defined for almost all $z \in \omega^\perp$. Moreover, we have in analogy with (4),

$$\widetilde{f}(\zeta) = \int_{\omega^\perp} (P_\omega f)(z) e^{-i\langle z,\zeta\rangle}\, dz \quad (\zeta \in \omega^\perp)\,. \tag{107}$$

Proposition 7.8. *An object is determined by any infinite set of radiographs.*

In other words, a compactly supported function f is determined by the functions $P_\omega f$ for any infinite set of ω.

Proof. Since f has compact support $\widetilde{f}$ is an analytic function on $\mathbf{R}^n$. But if $\widetilde{f}(\zeta) = 0$ for $\zeta \in \omega^\perp$ we have $\widetilde{f}(\eta) = \langle \omega, \eta\rangle g(\eta) \quad (\eta \in \mathbf{R}^n)$ where g is also analytic. If $P_{\omega_1} f, \dots, P_{\omega_k} f \dots$ all vanish identically for an infinite set $\omega_1, \dots, \omega_k \dots$ we see that for each k

$$\widetilde{f}(\eta) = \prod_{i=1}^{k} \langle \omega_i, \eta\rangle g_k(\eta)\,,$$

where g_k is analytic. But this would contradict the power series expansion of $\widetilde{f}$ which shows that for a suitable $\omega \in \mathbf{S}^{n-1}$ and integer $r \geq 0$, $\lim_{t\to 0} \widetilde{f}(t\omega) t^{-r} \neq 0$.

If only finitely many radiographs are used we get the opposite result.

Proposition 7.9. *Let $\omega_1, \dots, \omega_k \in \mathbf{S}^{n-1}$ be an arbitrary finite set. Then there exists a function $f \in \mathcal{D}(\mathbf{R}^n), f \not\equiv 0$ such that*

$$P_{\omega_i} f \equiv 0 \quad \textit{for all} \quad 1 \leq i \leq k\,.$$

Proof. We have to find $f \in \mathcal{D}(\mathbf{R}^n), f \not\equiv 0$, such that $\widetilde{f}(\zeta) = 0$ for $\zeta \in \omega_i^\perp (1 \leq i \leq k)$. For this let D be the constant coefficient differential operator such that

$$(Du)\widetilde{\ }(\eta) = \prod_{1}^{k} \langle \omega_i, \eta\rangle \widetilde{u}(\eta) \quad \eta \in \mathbf{R}^n\,.$$

If $u \not\equiv 0$ is any function in $\mathcal{D}(\mathbf{R}^n)$ then $f = Du$ has the desired property.

We next consider the problem of *approximate reconstruction* of the function f from a finite set of radiographs $P_{\omega_1} f, \dots, P_{\omega_k} f$.

Let N_j denote the null space of P_{ω_j} and let P_j the orthogonal projection of $L^2(B)$ on the plane $f + N_j$; in other words

$$P_j g = Q_j(g - f) + f\,, \tag{108}$$

where Q_j is the (linear) projection onto the subspace $N_j \subset L^2(B)$. Put $P = P_k \dots P_1$. Let $g \in L^2(B)$ be arbitrary (the initial guess for f) and form the sequence $P^m g, m = 1, 2, \dots$. Let $N_0 = \cap_1^k N_j$ and let P_0 (resp. Q_0) denote the orthogonal projection of $L^2(B)$ on the plane $f + N_0$ (subspace N_0). We shall prove that the sequence $P^m g$ converges to the projection $P_0 g$. This is natural since by $P_0 g - f \in N_0$, $P_0 g$ and f have the same radiographs in the directions $\omega_1, \dots, \omega_k$.

Theorem 7.10. *With the notations above,*

$$P^m g \longrightarrow P_0 g \quad \textit{as} \quad m \longrightarrow \infty$$

for each $g \in L^2(B)$.

Proof. We have, by iteration of (108)

$$(P_k \dots P_1)g - f = (Q_k \dots Q_1)(g - f)$$

and, putting $Q = Q_k \dots Q_1$ we obtain

$$P^m g - f = Q^m (g - f) \,.$$

We shall now prove that $Q^m g \longrightarrow Q_0 g$ for each g; since

$$P_0 g = Q_0 (g - f) + f$$

this would prove the result. But the statement about Q^m comes from the following general result about abstract Hilbert space.

Theorem 7.11. *Let $\mathcal{H}$ be a Hilbert space and Q_i the projection of $\mathcal{H}$ onto a subspace $N_i \subset \mathcal{H}(1 \le i \le k)$. Let $N_0 = \cap_1^k N_i$ and $Q_0 : \mathcal{H} \longrightarrow N_0$ the projection. Then if $Q = Q_k \dots Q_1$*

$$Q^m g \longrightarrow Q_0 g \quad \textit{for each} \quad g \in \mathcal{H}, .$$

Since Q is a contraction ($\|Q\| \le 1$) we begin by proving a simple lemma about such operators.

Lemma 7.12. *Let $T : \mathcal{H} \longrightarrow \mathcal{H}$ be a linear operator of norm ≤ 1. Then*

$$\mathcal{H} = C\ell((I - T)\mathcal{H}) \oplus \ \textit{Null space}\ (I - T)$$

is an orthogonal decomposition, $C\ell$ denoting closure, and I the identity.

Proof. If $Tg = g$ then since $\|T^*\| = \|T\| \le 1$ we have

$$\|g\|^2 = (g, g) = (Tg, g) = (g, T^*g) \le \|g\| \|T^*g\| \le \|g\|^2$$

so all terms in the inequalities are equal. Hence

$$\|g - T^*g\|^2 = \|g\|^2 - (g, T^*g) - (T^*g, g) + \|T^*g\|^2 = 0$$

so $T^*g = g$. Thus $I - T$ and $I - T^*$ have the same null space. But $(I - T^*)g = 0$ is equivalent to $(g, (I - T)\mathcal{H}) = 0$ so the lemma follows.

Definition. An operator T on a Hilbert space $\mathcal{H}$ is said to have *property* S if

(109) $$\|f_n\| \leq 1, \|Tf_n\| \longrightarrow 1 \text{ implies } \|(I-T)f_n\| \longrightarrow 0\,.$$

Lemma 7.13. *A projection, and more generally a finite product of projections, has property* (S).

Proof. If T is a projection then

$$\|(I-T)f_n\|^2 = \|f_n\|^2 - \|Tf_n\|^2 \leq 1 - \|Tf_n\|^2 \longrightarrow 0$$

whenever

$$\|f_n\| \leq 1 \text{ and } \|Tf_n\| \longrightarrow 1\,.$$

Let T_2 be a projection and suppose T_1 has property (S) and $\|T_1\| \leq 1$. Suppose $f_n \in \mathcal{H}$ and $\|f_n\| \leq 1, \|T_2T_1f_n\| \longrightarrow 1$. The inequality implies $\|T_1f_n\| \leq 1$ and since

$$\|T_1f_n\|^2 = \|T_2T_1f_n\|^2 + \|(I-T_2)(T_1f_n)\|^2$$

we also deduce $\|T_1f_n\| \longrightarrow 1$. Writing

$$(I - T_2T_1)f_n = (I-T_1)f_n + (I-T_2)T_1f_n$$

we conclude that T_2T_1 has property (S). The lemma now follows by induction.

Lemma 7.14. *Suppose T has property (S) and $\|T\| \leq 1$. Then for each $f \in \mathcal{H}$*

$$T^nf \longrightarrow \pi f \quad \textit{as} \quad n \longrightarrow \infty\,,$$

where π is the projection onto the fixed point space of T.

Proof. Let $f \in \mathcal{H}$. Since $\|T\| \leq 1, \|T^nf\|$ decreases monotonically to a limit $\alpha \geq 0$. If $\alpha = 0$ we have $T^nf \longrightarrow 0$. By Lemma 7.12 $\pi T = T\pi$ so $\pi f = T^n\pi f = \pi T^n f$ so $\pi f = 0$ in this case. If $\alpha > 0$ we put $g_n = \|T^nf\|^{-1}(T^nf)$. Then $\|g_n\| = 1$ and $\|Tg_n\| \to 1$. Since T has property (S) we deduce

$$T^n(I-T)f = \|T^nf\|(I-T)g_n \longrightarrow 0,.$$

Thus $T^nh \longrightarrow 0$ for all h in the range of $I-T$. If g is in the closure of this range then given $\epsilon > 0$ there exist $h \in (I-T)\mathcal{H}$ such that $\|g-h\| < \epsilon$. Then

$$\|T^ng\| \leq \|T^n(g-h)\| + T^nh\| < \epsilon + \|T^nh\|$$

whence $T^ng \longrightarrow 0$. On the other hand, if h is in the null space of $I-T$ then $Th = h$ so $T^nh \longrightarrow h$. Now the lemma follows from Lemma 7.12.

In order to deduce Theorem 7.11 from Lemmas 7.13 and 7.14 we just have to verify that N_0 is the fixed point space of Q. But if $Qg = g$ then

$$\|g\| = \|Q_k \ldots Q_1 g\| \leq \|Q_{k-1} \ldots Q_1 g\| \leq \ldots \leq \|Q_1 g\| \leq \|g\|$$

so equality signs hold everywhere. But the Q_i are projections so the norm identities imply

$$g = Q_1 g = Q_2 Q_1 g = \ldots = Q_k \ldots Q_1 g$$

which shows $g \in N_0$. This proves Theorem 7.11.

Bibliographical Notes

§§*1-2.* The inversion formulas

(i) $f(x) = \frac{1}{2}(2\pi i)^{1-n} L_x^{(n-1)/2} \int_{\mathbf{S}^{n-1}} J(\omega, \langle \omega, x \rangle)\, , d\omega$ (n odd)

(ii) $f(x) = \frac{1}{2}(2\pi i)^{-n} L_x^{(n-2)/2} \int_{\mathbf{S}^{n-1}} d\omega \int_{-\infty}^{\infty} \frac{dJ(\omega,p)}{p-(\omega,x)}$ (n even)

for a function $f \in \mathcal{D}(X)$ in terms of its plane integrals $J(\omega, p)$ go back to Radon [1917] and John [1934], [1955]. According to Bockwinkel [1906] the case $n = 3$ had been proved before 1906 by H.A. Lorentz, but fortunately, both for Lorentz and Radon, the transformation $f(x) \to J(\omega, p)$ was not baptized "Lorentz transformation". In John [1955] the proofs are based on the Poisson equation $Lu = f$. Other proofs, using distributions, are given in Gelfand-Shilov [1960]. See also Nievergelt [1986]. The dual transforms, $f \to \widehat{f}, \varphi \to \check{\varphi}$, the unified inversion formula and its dual,

$$cf = L^{(n-1)/2}((\widehat{f})^{\vee})\, , \quad c\varphi = \square^{(n-1)/2}((\check{\varphi})\widehat{)}$$

were given by the author in [1964]. The second proof of Theorem 3.1 is from the author's paper [1959]. It is valid for constant curvature spaces as well. The version in Theorem 3.6 is also proved in Ludwig [1966].

The support theorem, the Paley-Wiener theorem and the Schwartz theorem (Theorems 2.4,2.6, 2.10) are from Helgason [1964], [1965a]. The example in Remark 2.9 was also found by D.J. Newman, cf. Weiss' paper [1967], which gives another proof of the support theorem. See also Droste [1983]. The local result in Corollary 2.12 goes back to John [1935]; our derivation is suggested by the proof of a similar lemma in Flensted-Jensen [1977], p. 81. Another proof is in Ludwig [1966].See Palamodov and Denisjuk [1988] for a related inversion formula.

The simple geometric Lemma 2.7 is from the authors paper [1965a] and is extended to hyperbolic spaces in [1980b]. In the Proceedings containing

[1966a] the author raised the problem (p. 174) to extend Lemma 2.7 to each complete simply connected Riemannian manifold M of negative curvature. If in addition M is analytic this was proved by Quinto [1993b] and Grinberg and Quinto [1998]. This is an example of injectivity and support results obtained by use of the techniques of microlocal analysis and wave front sets. As further samples involving very general Radon transforms we mention Boman [1990], [1992], [1993], Boman and Quinto [1987], [1993], Quinto [1983], [1992], [1993b], [1994a], [1994b], Agranovsky and Quinto [1996], Gelfand, Gindikin and Shapiro [1979].

Corollary 2.8 is derived by Ludwig [1966] in a different way. There he proposes alternative proofs of the Schwartz- and Paley-Wiener theorems by expanding $\widehat{f}(\omega, p)$ in spherical harmonics in ω. However, the principal point—the smoothness of the function F in the proof of Theorem 2.4—is overlooked. Theorem 2.4 is from the author's papers [1964] [1965a].

Since the inversion formula is rather easy to prove for odd n it is natural to try to prove Theorem 2.4 for this case by showing directly that if $\varphi \in \mathcal{S}_H(\mathbf{P}^n)$ then the function $f = cL^{(n-1)/2}(\check{\varphi})$ for n odd belongs to $\mathcal{S}(\mathbf{R}^n)$ (in general $\check{\varphi} \notin \mathcal{S}(\mathbf{R}^n)$). This approach is taken in Gelfand-Graev-Vilenkin [1966], pp. 16-17. However, this method seems to offer some unresolved technical difficulties. For some generalizations see Kuchment and Lvin [1990], Aguilar, Ehrenpreis and Kuchment [1996] and Katsevich [1997]. Cor. 2.5 is stated in Semyanisty [1960].

§*5.* The approach to Radon transforms of distributions adopted in the text is from the author's paper [1966a]. Other methods are proposed in Gelfand-Graev-Vilenkin [1966] and in Ludwig [1966]. See also Ramm [1995].

§*6.* The d-plane transform and Theorem 6.2 are from Helgason [1959], p. 284. Formula (55) was already proved by Fuglede [1958]. The range characterization for the d-plane transform in Theorem 6.3 is from the 1980-edition of this book and was used by Kurusa [1991] to prove Theorem 6.5, which generalizes John's range theorem for the X-ray transform in $\mathbf{R}^3$ [1938]. The geometric range characterization (Corollary 6.12) is also due to John [1938]. Papers devoted to the d-plane range question for $\mathcal{S}(\mathbf{R}^n)$ are Gelfand-Gindikin and Graev [1982], Grinberg [1987], Richter [1986] and Gonzalez [1991]. This last paper gives the range as the kernel of a single 4^{th} order differential operator on the space of d-planes. As shown by Gonzalez, the analog of Theorem 6.3 fails to hold for $\mathcal{S}(\mathbf{R}^n)$. An L^2-version of Theorem 6.3 was given by Solmon [1976], p. 77. Proposition 6.13 was communicated to me by Schlichtkrull.

Some difficulties with the d-plane transform on $L^2(\mathbf{R}^n)$ are pointed out by Smith and Solmon [1975] and Solmon [1976], p. 68. In fact, the function $f(x) = |x|^{-\frac{1}{2}n}(\log|x|)^{-1} \quad (|x| \geq 2)$, 0 otherwise, is square integrable on $\mathbf{R}^n$ but is not integrable over any plane of dimension $\geq \frac{n}{2}$. Nevertheless, see for example Rubin [1998a], Strichartz [1981] for L^p-extensions of the d-plane transform.

§ 7. The applications to partial differential equations go in part back to Herglotz [1931]; see John [1955]. Other applications of the Radon transform to partial differential equations with constant coefficients can be found in Courant-Lax [1955], Gelfand-Shapiro [1955], John [1955], Borovikov [1959], Gårding [1961] and Ludwig [1966]. Our discussion of the wave equation (Theorem 7.3 and Corollary 7.4) is closely related to the treatment in Lax-Phillips [1967], Ch. IV, where however, the dimension is assumed to be odd. Applications to general elliptic equations were given by John [1955].

While the Radon transform on $\mathbf{R}^n$ can be used to "reduce" partial differential equations to ordinary differential equations one can use a Radon type transform on a symmetric space X to "reduce" an invariant differential operator D on X to a partial differential operator with constant coefficients. For an account of these applications see the author's monograph [1994b], Chapter V.

While the applications to differential equations are perhaps the most interesting to mathematicians, the tomographic applications of the X-ray transform have revolutionized medicine. These applications originated with Cormack [1963], [1964] and Hounsfield [1973]. For the approximate reconstruction problem, including Propositions 7.8 and 7.9 and refinements of Theorems 7.10, 7.11 see Smith, Solmon and Wagner [1977], Solmon [1976] and Hamaker and Solmon [1978]. Theorem 7.11 is due to Halperin [1962], the proof in the text to Amemiya and Ando [1965]. For an account of some of those applications see e.g. Deans [1983], Natterer [1986] and Ramm and Katsevich [1996]. Applications in radio astronomy appear in Bracewell and Riddle [1967].