proof of the preceding theorem about factorization of functions in H_1. We need first of all to note the following analogue of a result already established for functions in H_1 or in H_∞:

Theorem. *If $f \in H_2$ and $f(t)$ is not a.e. zero on $\mathbb{R}$,*

$$\boxed{\int_{-\infty}^{\infty} \frac{\log^- |f(t)|}{1+t^2}\,dt \quad < \quad \infty.}$$

Also, for

$$f(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} f(t)\,dt,$$

one has

$$\log|f(z)| \quad \leqslant \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log|f(t)|\,dt$$

when $\Im z > 0$, the integral on the right converging absolutely.

Proof. Is very similar to that of the corresponding theorem in H_1.* Here, when considering the difference

$$\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log^+|f(t+ih)|\,dt \;-\; \int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log^+|f(t)|\,dt,$$

one first observes that it is bounded in absolute value by

$$\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\big|\,|f(t+ih)| - |f(t)|\,\big|\,dt$$

and then applies Schwarz' inequality. The rest of the argument is the same as for H_1.

Corollary. *Unless $f(t) \in H_2$* vanishes *a.e., $|f(t)| > 0$ a.e. on $\mathbb{R}$.*

Definition (Beurling). A function f in H_2 which is not a.e. zero on $\mathbb{R}$ is called *outer* if, for the function $f(z)$ of the above theorem we have

$$\log|f(z)| \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log|f(t)|\,dt$$

whenever $\Im z > 0$.

* One may also appeal directly to that theorem after noting that $f^2 \in H_1$.

Theorem. *Let $f \in H_2$, not a.e. zero on $\mathbb{R}$, be outer. Then the finite linear combinations of the $e^{i\lambda t}f(t)$ with $\lambda \geqslant 0$ are $\| \quad \|_2$ dense in H_2.*

Remark. This result is due to Beurling, who also established its *converse*. The latter will not be needed in our work; it is set at the end of this article as problem 42.

Proof of theorem. In order to show that the $e^{i\lambda t}f(t)$ with $\lambda \geqslant 0$ generate H_2, it suffices to verify that if φ is any element of L_2 such that

$$\int_{-\infty}^{\infty} e^{i\lambda t} f(t)\varphi(t)\,dt \quad = \quad 0$$

for all $\lambda \geqslant 0$, then

$$\int_{-\infty}^{\infty} g(t)\varphi(t)\,dt \quad = \quad 0$$

for each $g \in H_2$. This will follow if we can show that such a φ belongs to H_2, for then the products $g\varphi$ with $g \in H_2$ will be in H_1.

Since f and $\varphi \in L_2$, $f\varphi \in L_1$, and our assumed relation makes $f\varphi$ in H_1. The function

$$F(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} f(t)\varphi(t)\,dt$$

is thus *analytic* for $\Im z > 0$.

If $\varphi(t) \equiv 0$ a.e. there is nothing to prove, so we may assume that this is not the case. By the preceding corollary, $|f(t)| > 0$ a.e.; therefore $f(t)\varphi(t)$ is not a.e. zero on $\mathbb{R}$. Hence, by an earlier result,

$$\log|F(z)| \quad \leqslant \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} \log|f(t)\varphi(t)|\,dt$$

when $\Im z > 0$, with the right-hand integral *absolutely convergent.*

At the same time, for

$$f(z) \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} f(t)\,dt$$

we have

$$\log|f(z)| \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} \log|f(t)\,dt$$

by hypothesis whenever $\Im z > 0$. The integral on the right is certainly $> -\infty$, being absolutely convergent, so $F(z)/f(z)$ is *analytic* in $\Im z > 0$.

For that ratio, the previous two relations give

$$\log\left|\frac{F(z)}{f(z)}\right| \quad \leqslant \quad \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2}\log|\varphi(t)|\,\mathrm{d}t, \qquad \Im z \;>\; 0.$$

Thence, by the inequality between arithmetic and geometric means,

$$\left|\frac{F(z)}{f(z)}\right|^2 \quad \leqslant \quad \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2}|\varphi(t)|^2\,\mathrm{d}t, \qquad \Im z \;>\; 0,$$

from which, by Fubini's theorem,

$$\int_{-\infty}^{\infty}\left|\frac{F(x+\mathrm{i}y)}{f(x+\mathrm{i}y)}\right|^2\mathrm{d}x \quad \leqslant \quad \|\varphi\|_2^2.$$

According to a previous theorem, there is hence a function $\psi\in H_2$ with

$$\frac{F(z)}{f(z)} \quad = \quad \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2}\psi(t)\,\mathrm{d}t$$

for $\Im z \;>\; 0$, and

$$\frac{F(t+\mathrm{i}y)}{f(t+\mathrm{i}y)} \quad \longrightarrow \quad \psi(t) \quad \text{a.e.}$$

as $y\longrightarrow 0$.

We have, however, by the formula for $F(z)$,

$$F(t+\mathrm{i}y) \quad \longrightarrow \quad f(t)\varphi(t) \quad \text{a.e. as } y\longrightarrow 0,$$

and, for $f(z)$,

$$f(t+\mathrm{i}y) \quad \longrightarrow \quad f(t) \quad \text{a.e. as } y\longrightarrow 0.$$

Therefore, since $|f(t)| \;>\; 0$ a.e.,

$$\varphi(t) \quad = \quad \psi(t) \quad \text{a.e.},$$

i.e., $\varphi\in H_2$, as we needed to show.

The theorem is proved.

Remark. The function f in H_2 appearing near the end of the above factorization theorem for H_1 is outer. In general, given *any* function $M(t) \;\geqslant\; 0$ such that

$$\int_{-\infty}^{\infty}\frac{\log^- M(t)}{1+t^2}\mathrm{d}t \quad < \quad \infty$$

and

$$\int_{-\infty}^{\infty} (M(t))^2\,dt \quad < \quad \infty,$$

we can construct an outer function $f \in H_2$ for which

$$|f(t)| \;=\; M(t) \qquad \text{a.e. on } \mathbb{R}.$$

To do this, one first puts

$$F(z) \;=\; \exp\left\{\frac{1}{\pi i}\int_{-\infty}^{\infty}\left(\frac{1}{t-z} - \frac{t}{t^2+1}\right)\log M(t)\,dt\right\}$$

for $\Im z > 0$; the conditions on M ensure absolute convergence of the integral figuring on the right. We have

$$\log|F(z)| \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2}\log M(t)\,dt, \qquad \Im z > 0,$$

so that, in the first place,

$$\log|F(t+iy)| \longrightarrow \log M(t) \quad \text{a.e.}$$

for $y \longrightarrow 0$. In the second place, since geometric means do not exceed arithmetic means,

$$\int_{-\infty}^{\infty} |F(x+iy)|^2\,dx \quad \leqslant \quad \int_{-\infty}^{\infty} (M(t))^2\,dt$$

for $y > 0$, by an argument like one in the above proof. There is thus an $f \in H_2$ with

$$F(z) \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2} f(t)\,dt, \qquad \Im z > 0,$$

and

$$F(t+iy) \longrightarrow f(t) \quad \text{a.e.}$$

as $y \longrightarrow 0$.

Comparing the above two limit relations we see, first of all, that

$$|f(t)| \;=\; M(t) \qquad \text{a.e.,} \quad t \in \mathbb{R}.$$

Therefore

$$\log|F(z)| \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Im z}{|z-t|^2}\log|f(t)|\,dt$$

for $\Im z > 0$. Here, our function $F(z)$ is in fact the $f(z)$ figuring in the proof of the last theorem. Hence f is *outer*.

This construction works in particular whenever $M(t) = |g(t)|$ with $g(t)$ in H_2 not a.e. zero on $\mathbb{R}$. Therefore, *any such g in H_2 coincides a.e. in modulus with an outer function in H_2.*

Problem 42

Prove the converse of the preceding result. Show, in other words, that if $f \in H_2$ is *not outer*, the $e^{i\lambda t}f(t)$ with $\lambda \geqslant 0$ *do not generate* H_2 (in norm $\| \ \|_2$). (Hint: One may as well assume that $f(t)$ is not a.e. zero on $\mathbb{R}$. Take then the *outer* function $g \in H_2$ with $|g(t)| = |f(t)|$ a.e., furnished by the preceding remark. Show first that the ratio $\omega(t) = f(t)/g(t)$ – it is of modulus 1 a.e. – belongs to H_∞. For this purpose, one may look at $f(z)/g(z)$ in $\Im z > 0$.

Next observe that

$$\int_{-\infty}^{\infty} e^{i\lambda t} f(t)\overline{\omega(t)g(t)}\,dt = 0$$

for all $\lambda \geqslant 0$, so that it suffices to show that

$$\int_{-\infty}^{\infty} \varphi(t)\overline{\omega(t)g(t)}\,dt$$

cannot be zero for all $\varphi \in H_2$. Assume that *were* the case. Then

$$\int_{-\infty}^{\infty} e^{i\lambda t}\psi(t)\overline{\omega(t)g(t)}\,dt = 0,$$

i.e.,

$$\int_{-\infty}^{\infty} \overline{\omega(t)\psi(t)e^{i\lambda t}g(t)}\,dt = 0$$

for all $\lambda \geqslant 0$ and every $\psi \in H_2$.

Use now the preceding theorem (!) and another result to argue that

$$\int_{-\infty}^{\infty} \overline{\omega(t)}h(t)\,dt = 0$$

for all $h \in H_1$, making $\overline{\omega(t)}$ *also* in H_∞, together with $\omega(t)$. This means that

$$\omega(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\omega(t)\,dt$$

and $\overline{\omega(z)}$ are *both analytic* in $\Im z > 0$. Since f is *not* outer, however, $|\omega(z)| = |f(z)/g(z)| < 1$ for some such z. A contradiction is now easily obtained.)

Remark. The $\omega \in H_\infty$ figuring in the argument just indicated is called an *inner function.*

2. Statement of the problem, and simple reductions of it

Given a function $w \geqslant 0$ belonging to $L_1(\mathbb{R})$, we want to know whether there is an $\omega \geqslant 0$ defined on $\mathbb{R}$, *not a.e. zero*, such that

$$\int_{-\infty}^{\infty} |\tilde{U}(t)|^2 \omega(t)\,\mathrm{d}t \quad \leqslant \quad \int_{-\infty}^{\infty} |U(t)|^2 w(t)\,\mathrm{d}t$$

for the Hilbert transforms $\tilde{U}(t)$ (specified in some definite manner) of the functions $U(t)$ belonging to a certain class. Depending on that class, the answer is different for different specifications of $\tilde{U}(t)$.

Two particular specifications are in common use in analysis. The *first* is preferred when dealing with functions U for which only the convergence of

$$\int_{-\infty}^{\infty} \frac{|U(t)|}{1+t^2}\,\mathrm{d}t$$

is assured; in that case one takes

$$\tilde{U}(x) \quad = \quad \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{1}{x-t} + \frac{t}{t^2+1} \right) U(t)\,\mathrm{d}t.$$

The expression on the right – not really an integral – is a *Cauchy principal value*, defined for almost all real x. (At this point the reader should look again at §H.1, Chapter III and the second part of §C.1, Chapter VIII.)

A *second* definition of $\tilde{U}$ is adopted when, for $\delta > 0$, the integrals

$$\int_{|t-x| \geqslant \delta} \frac{U(t)}{x-t}\,\mathrm{d}t$$

are *already absolutely convergent*. In that case, one drops the term $t/(t^2+1)$ figuring in the previous expression and simply takes

$$\tilde{U}(x) \quad = \quad \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{U(t)}{x-t}\,\mathrm{d}t,$$

in other words, $1/\pi$ times the limit of the preceding integral for $\delta \longrightarrow 0$. This specification of $\tilde{U}$ was employed in §C.1 of Chapter VIII (see especially the scholium to that article). It is useful even in cases where the above integrals are *not absolutely convergent for* $\delta > 0$ *but merely exist as limits,* viz.,

$$\lim_{A\to\infty}\left(\int_{-A}^{x-\delta}+\int_{x+\delta}^{A}\right)\frac{U(t)}{x-t}\,dt.$$

This happens, for instance, with certain kinds of functions $U(t)$ bounded on $\mathbb{R}$ and *not* dying away to zero as $t \longrightarrow \pm\infty$. The Hilbert transforms thus obtained are the ones listed in various tables, such as those issued in the Bateman Project series.

If now our question is posed for the *first* kind of Hilbert transform, it turns out to have substance when the given class of functions U is so large as to *include all bounded ones.* In those circumstances, it is most readily treated by first making the substitution $t = \tan(\vartheta/2)$ and then working with functions $U(t)$ equal to *trigonometric polynomials in* ϑ and with certain auxiliary functions analytic in the unit disk. One finds in that way that the question has a positive answer (i.e., that a non-zero $\omega \geqslant 0$ exists) if and only if

$$\int_{-\infty}^{\infty}\frac{1}{(t^2+1)^2 w(t)}\,dt \quad < \quad \infty$$

(under the initial assumption that $w \in L_1(\mathbb{R})$); the reader will find this work set as problem 43 below, which may serve as a test of how well he or she has assimilated the procedures of the present §.

Except in problem 43, we do not consider the first kind of Hilbert transform any further. Instead, we turn to the *second* kind, taking $⨍_{-\infty}^{\infty}$ in its most general sense, as $\lim_{\delta\to\infty}\lim_{A\to\infty}(\int_{-A}^{x-\delta}+\int_{x+\delta}^{A})$. Then

$$\tilde{U}(x) \quad = \quad \frac{1}{\pi}⨍_{-\infty}^{\infty}\frac{U(t)}{x-t}\,dt$$

is defined for $U(t)$ equal to $\sin\lambda t$ and $\cos\lambda t$, and hence for finite linear combinations of such functions (the so-called *trigonometric sums*). In the following articles, we restrict our attention to trigonometric sums $U(t)$, for which the definition of $\tilde{U}$ by means of the preceding formula presents no problem. As explained at the beginning of §D, one may think crudely of the collection of trigonometric sums as 'filling out' $L_\infty(\mathbb{R})$ 'for all practical purposes'.

By elementary contour integration, one readily finds that

$$\frac{1}{\pi}⨍_{-\infty}^{\infty}\frac{e^{i\lambda t}}{x-t}\,dt \quad = \quad \begin{cases} -ie^{i\lambda x}, & \lambda > 0, \\ ie^{i\lambda x}, & \lambda < 0. \end{cases}$$

When $\lambda = 0$, the quantity on the left is *zero.* The reader should do this computation. One of the original applications made of contour integration by Cauchy, who *invented* it, was precisely the evaluation of such principal values! In terms of real valued functions, the formula just written goes as follows:

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos \lambda t}{x-t}\,dt \quad = \quad \sin \lambda t, \qquad \lambda > 0;$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin \lambda t}{x-t}\,dt \quad = \quad -\cos \lambda t, \qquad \lambda > 0.$$

From this we see already that our question (about the existence of non-zero $\omega \geqslant 0$) is *without substance* for the *present* specification of the Hilbert transform, when posed for *all trigonometric sums U. There can never be an $\omega \geqslant 0$, not a.e. zero, such that*

$$\int_{-\infty}^{\infty} |\tilde{U}(t)|^2 \omega(t)\,dt \quad \leqslant \quad \int_{-\infty}^{\infty} |U(t)|^2 w(t)\,dt$$

for all such U, when w is integrable. This follows immediately on taking

$$U(t) \;=\; \sin \lambda t, \qquad \tilde{U}(t) \;=\; -\cos \lambda t$$

in such a presumed relation and then making $\lambda \longrightarrow 0$; in that way one concludes by Fatou's lemma that

$$\int_{-\infty}^{\infty} \omega(t)\,dt \quad = \quad 0.$$

The same state of affairs prevails whenever our given class of functions U includes pure oscillations of arbitrary phase with frequencies tending to zero. For this reason, we should require the class of trigonometric sums $U(t)$ under consideration to *only contain terms involving frequencies bounded away from zero*, as we did in §D. The simplest non-trivial version of our problem thus has the following formulation:

Let $a > 0$. Under what conditions on the given $w \geqslant 0$ belonging to $L_1(\mathbb{R})$ does there exist an $\omega \geqslant 0$, not a.e. zero, such that

$$\int_{-\infty}^{\infty} |\tilde{U}(t)|^2 \omega(t)\,dt \quad \leqslant \quad \int_{-\infty}^{\infty} |U(t)|^2 w(t)\,dt$$

for all finite trigonometric sums

$$U(t) \quad = \quad \sum_{|\lambda| \geqslant a} C_\lambda e^{i\lambda t} \quad ?$$

Here, we are dealing with the *second* kind of Hilbert transform, so, for the sum $U(t)$ just written,

$$\tilde{U}(t) \quad = \quad \sum_{|\lambda| \geqslant a} (-\mathrm{i}C_\lambda \operatorname{sgn} \lambda)\mathrm{e}^{\mathrm{i}\lambda t}.$$

Such functions $U(t)$ can, of course, also be expressed thus:

$$U(t) \quad = \quad \sum_{\lambda \geqslant a} (A_\lambda \cos \lambda t + B_\lambda \sin \lambda t).$$

Then

$$\tilde{U}(t) \quad = \quad \sum_{\lambda \geqslant a} (A_\lambda \sin \lambda t - B_\lambda \cos \lambda t).$$

This manner of writing our trigonometric sums will be preferred in the following discussion; it has the advantage of making the *real-valued* sums $U(t)$ be *precisely* the ones involving *only real coefficients* A_λ *and* B_λ.

We see in particular that if $U(t)$ is a complex-valued sum of the above kind, $\Re U(t)$ and $\Im U(t)$ are also sums of the same form. This means that our relation

$$\int_{-\infty}^{\infty} |\tilde{U}(t)|^2 \omega(t)\,\mathrm{d}t \quad \leqslant \quad \int_{-\infty}^{\infty} |U(t)|^2 w(t)\,\mathrm{d}t$$

holds for all complex-valued U of the above form iff it holds for the real valued ones.

Given any trigonometric sum $U(t)$ (real-valued or not) of the form in question, we have

$$U(t) \ + \ \mathrm{i}\tilde{U}(t) \quad = \quad \sum_{\lambda \geqslant a} C_\lambda \mathrm{e}^{\mathrm{i}\lambda t}$$

with certain coefficients C_λ. *Conversely*, if $F(t)$ is *any* finite sum like the one on the right,

$$\Re F(t) \ = \ U(t)$$

is a sum of the form under consideration, and then

$$\tilde{U}(t) \ = \ \Im F(t).$$

These statements are immediately verified by simple calculation.

Lemma. *Given $w \geqslant 0$ in $L_1(\mathbb{R})$, let $a > 0$. The relation*

$$\int_{-\infty}^{\infty} |\tilde{U}(t)|^2 \omega(t)\,\mathrm{d}t \quad \leqslant \quad \int_{-\infty}^{\infty} |U(t)|^2 w(t)\,\mathrm{d}t$$

holds for all trigonometric sums

$$U(t) \quad = \quad \sum_{\lambda \geqslant a} (A_\lambda \cos \lambda t + B_\lambda \sin \lambda t)$$

with the function $\omega(t) \geqslant 0$ *iff*

$$\left| \int_{-\infty}^{\infty} (w(t) + \omega(t))(F(t))^2 \, \mathrm{d}t \right| \quad \leqslant \quad \int_{-\infty}^{\infty} (w(t) - \omega(t)) |F(t)|^2 \, \mathrm{d}t$$

for all finite sums

$$F(t) \quad = \quad \sum_{\lambda \geqslant a} C_\lambda \mathrm{e}^{\mathrm{i}\lambda t}.$$

Proof. As remarked above, our relation holds for trigonometric sums U of the given form iff

$$\int_{-\infty}^{\infty} (\tilde{U}(t))^2 \omega(t) \, \mathrm{d}t \quad \leqslant \quad \int_{-\infty}^{\infty} (U(t))^2 w(t) \, \mathrm{d}t$$

for all such *real-valued* U. Multiply this relation by 2 and then add to both sides of the result the quantity

$$\int_{-\infty}^{\infty} \{ (\tilde{U}(t))^2 w(t) \; - \; (U(t))^2 w(t) \; - \; (\tilde{U}(t))^2 \omega(t) \; - \; (U(t))^2 \omega(t) \} \, \mathrm{d}t.$$

We obtain the relation

$$\int_{-\infty}^{\infty} (w(t) \; + \; \omega(t)) \{ (\tilde{U}(t))^2 \; - \; (U(t))^2 \} \, \mathrm{d}t$$

$$\leqslant \quad \int_{-\infty}^{\infty} (w(t) \; - \; \omega(t)) \{ (U(t))^2 \; + \; (\tilde{U}(t))^2 \} \, \mathrm{d}t$$

which must thus be *equivalent* to our original one (see the remark immediately following this proof).

In terms of $F(t) = U(t) + \mathrm{i}\tilde{U}(t)$, the last inequality becomes

$$-\Re \int_{-\infty}^{\infty} (w(t) + \omega(t)(F(t))^2 \, \mathrm{d}t \quad \leqslant \quad \int_{-\infty}^{\infty} (w(t) - \omega(t)) |F(t)|^2 \, \mathrm{d}t,$$

so, according to the statements preceding the lemma, our original relation holds with the trigonometric sums $U(t)$ iff the present one is valid for the finite sums

$$F(t) \quad = \quad \sum_{\lambda \geqslant a} C_\lambda \mathrm{e}^{\mathrm{i}\lambda t}.$$

If, however, $F(t)$ is of this form, so is $e^{i\gamma}F(t)$ for each real constant γ. The preceding condition is thus equivalent to the requirement that

$$-\Re e^{2i\gamma}\int_{-\infty}^{\infty}(w(t)+\omega(t))(F(t))^2\,dt \quad\leqslant\quad \int_{-\infty}^{\infty}(w(t)-\omega(t))|F(t)|^2\,dt$$

for each function F and all real γ, and that happens iff

$$\left|\int_{-\infty}^{\infty}(w(t)+\omega(t))(F(t))^2\,dt\right|$$

is $\leqslant$ the integral on the right for any such F. This last condition is hence equivalent to our original one, Q.E.D.

Remark. The argument just made tacitly assumes finiteness of $\int_{-\infty}^{\infty}(\tilde{U}(t))^2\omega(t)\,dt$ and $\int_{-\infty}^{\infty}(U(t))^2\omega(t)\,dt$, as well as that of $\int_{-\infty}^{\infty}(\tilde{U}(t))^2 w(t)\,dt$ and $\int_{-\infty}^{\infty}(U(t))^2 w(t)\,dt$. About the latter two quantities, there can be no question, w being assumed integrable. Then, however, the former two must also be finite, whether we suppose the *first relation* of the lemma to hold or the *second*. Indeed, if it is the *first* one that holds,

$$\int_{-\infty}^{\infty}\{(U(t))^2+(\tilde{U}(t))^2\}\omega(t)\,dt$$
$$\leqslant\quad \int_{-\infty}^{\infty}\{(U(t))^2+(\tilde{U}(t))^2\}w(t)\,dt \quad<\quad \infty,$$

since, for $F(t) = U(t) + i\tilde{U}(t)$, $\tilde{F}(t) = -iF(t)$. And, if the *second* holds, we surely have

$$\int_{-\infty}^{\infty}(w(t)-\omega(t))\big((U(t))^2+(\tilde{U}(t))^2\big)\,dt \quad\geqslant\quad 0.$$

Theorem. *Given $w \geqslant 0$ in $L_1(\mathbb{R})$ and $a > 0$, any $\omega \geqslant 0$ such that*

$$\int_{-\infty}^{\infty}|\tilde{U}(t)|^2\omega(t)\,dt \quad\leqslant\quad \int_{-\infty}^{\infty}|U(t)|^2 w(t)\,dt$$

for all sums U of the form

$$U(t) \quad=\quad \sum_{\lambda\geqslant a}(A_\lambda\cos\lambda t + B_\lambda\sin\lambda t)$$

must satisfy

$$\omega(t) \;\leqslant\; w(t) \qquad \text{a.e. on } \mathbb{R}.$$

Proof. Such an ω must in the first place belong to $L_1(\mathbb{R})$. For, putting first

$U(t) = \sin at$, $\tilde{U}(t) = -\cos at$ in our relation, and then $U(t) = \sin 2at$, $\tilde{U}(t) = -\cos 2at$, we get

$$\int_{-\infty}^{\infty} (\cos^2 at + \cos^2 2at)\omega(t)\,dt < \infty.$$

Here,

$$\cos^2 at + \cos^2 2at = \tfrac{1}{2}(1 + \cos 2at + 2\cos^2 2at) \geqslant \tfrac{7}{16}$$

for $t \in \mathbb{R}$, so ω is integrable.

Knowing that w and ω are both integrable, we can prove the theorem by verifying that

$$\int_{-\infty}^{\infty} (w(t) - \omega(t))\varphi(t)\,dt \geqslant 0$$

for each continuous function $\varphi \geqslant 0$ of compact support.

Fix any such φ, and pick an $\varepsilon > 0$. Choose first an L so large that $\varphi(t)$ *vanishes identically* outside $(-L, L)$ and that

$$\|\varphi\|_\infty \cdot \int_{|t| \geqslant L} (w(t) + \omega(t))\,dt < \varepsilon;$$

since w and ω are in $L_1(\mathbb{R})$, such a choice is possible. Then expand $\sqrt{(\varphi(t))}$ in a Fourier series on $[-L, L]$:

$$\sqrt{(\varphi(t))} \sim \sum_{n=-\infty}^{\infty} a_n e^{\pi i n t/L}, \quad -L \leqslant t \leqslant L.$$

According to the rudiments of harmonic analysis, the Fejér means

$$s_N(t) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) a_n e^{\pi i n t/L}$$

of this Fourier series tend *uniformly* to $\sqrt{(\varphi(t))}$ on $[-L, L]$ as $N \longrightarrow \infty$. Also,

$$\|s_N\|_\infty \leqslant \|\sqrt{(\varphi)}\|_\infty.$$

We thus have

$$\int_{-\infty}^{\infty} (w(t) - \omega(t))\varphi(t)\,dt = \int_{-L}^{L} (w(t) - \omega(t))\varphi(t)\,dt$$

$$= \lim_{N\to\infty} \int_{-L}^{L} (w(t) - \omega(t))(s_N(t))^2\,dt.$$

And, for each N,

$$\int_{-L}^{L}(w(t)-\omega(t))(s_N(t))^2\,dt \;=\; \left(\int_{-\infty}^{\infty}-\int_{|t|\geq L}\right)(w(t)-\omega(t))(s_N(t))^2\,dt$$

$$\geq\; \int_{-\infty}^{\infty}(w(t)-\omega(t))(s_N(t))^2\,dt \;-\; \|s_N\|_\infty^2\int_{|t|\geq L}(w(t)+\omega(t))\,dt$$

$$\geq\; \int_{-\infty}^{\infty}(w(t)-\omega(t))(s_N(t))^2\,dt \;-\; \varepsilon$$

by choice of L, since $\|s_N\|_\infty^2 \leq \|\varphi\|_\infty$.

Putting

$$F_N(t) \;=\; e^{iat}e^{\pi iNt/L}s_N(t),$$

we have $|F_N(t)|^2 = (s_N(t))^2$. $F_N(t)$, however, is of the form

$$\sum_{\lambda\geq a} C_\lambda e^{i\lambda t},$$

so

$$\int_{-\infty}^{\infty}(w(t)-\omega(t))(s_N(t))^2\,dt \;=\; \int_{-\infty}^{\infty}(w(t)-\omega(t))|F_N(t)|^2\,dt$$

is ≥ 0 by the lemma. Using this in the last member of the previous chain of inequalities, we see that

$$\int_{-L}^{L}(w(t)-\omega(t))(s_N(t))^2\,dt \;\geq\; -\varepsilon$$

for each N, so, by the above limit relation,

$$\int_{-\infty}^{\infty}(w(t)-\omega(t))\varphi(t)\,dt \;\geq\; -\varepsilon.$$

Squeezing ε, we see that the integral on the left is ≥ 0, which is what we needed to show to prove the theorem. Done.

Lemma. *Given $w \geq 0$ in $L_1(\mathbb{R})$, let $a > 0$. A necessary and sufficient condition that there be an $\omega \geq 0$, not a.e. zero on $\mathbb{R}$, such that*

$$\int_{-\infty}^{\infty}|\tilde{U}(t)|^2\omega(t)\,dt \;\leq\; \int_{-\infty}^{\infty}|U(t)|^2 w(t)\,dt$$

for the finite sums

$$U(t) \;=\; \sum_{\lambda\geq a}(A_\lambda\cos\lambda t + B_\lambda\sin\lambda t),$$

is that there exist a function $\rho(t)$ not a.e. zero, $0 \leqslant \rho(t) \leqslant w(t)$, with

$$\left| \int_{-\infty}^{\infty} w(t)(F(t))^2 \,dt \right| \leqslant \int_{-\infty}^{\infty} (w(t)-\rho(t))|F(t)|^2 \,dt$$

for all functions F of the form

$$F(t) = \sum_{\lambda \geqslant a} C_\lambda e^{i\lambda t}.$$

When an ω fulfilling the above condition exists, ρ may be taken equal to it. When, on the other hand, a function ρ is known, the ω equal to $\frac{1}{2}\rho$ will work.

Proof. If a function ω with the stated properties exists, we know by the previous theorem that $0 \leqslant \omega(t) \leqslant w(t)$ a.e.. Therefore, if $U(t)$ is any sum of the above form,

$$\begin{aligned} \frac{1}{2}\int_{-\infty}^{\infty} |\tilde{U}(t)|^2 \omega(t)\,dt &\leqslant \frac{1}{2}\int_{-\infty}^{\infty} |U(t)|^2 w(t)\,dt \\ &\leqslant \int_{-\infty}^{\infty} |U(t)|^2\left(w(t) - \tfrac{1}{2}\omega(t)\right)dt. \end{aligned}$$

The first condition of the previous lemma is thus fulfilled with

$$\omega_1(t) = \tfrac{1}{2}\omega(t)$$

in place of $\omega(t)$ and

$$w_1(t) = w(t) - \tfrac{1}{2}\omega(t)$$

in place of $w(t)$. Hence, by that lemma,

$$\left| \int_{-\infty}^{\infty} (w_1(t)+\omega_1(t))(F(t))^2 \,dt \right| \leqslant \int_{-\infty}^{\infty} (w_1(t)-\omega_1(t))|F(t)|^2 \,dt$$

for the functions F of the form described. This relation goes over into the asserted one on taking $\rho(t) = \omega(t)$.

If, conversely, the relation involving functions F holds for some ρ, $0 \leqslant \rho(t) \leqslant w(t)$, we certainly have

$$\begin{aligned} \left| \int_{-\infty}^{\infty} \left(w(t) + \tfrac{1}{2}\rho(t)\right)(F(t))^2 \,dt \right| &\leqslant \int_{-\infty}^{\infty} (w(t)-\rho(t))|F(t)|^2 \,dt \\ &\quad + \frac{1}{2}\int_{-\infty}^{\infty} \rho(t)|F(t)|^2 \,dt \\ &= \int_{-\infty}^{\infty} \left(w(t) - \tfrac{1}{2}\rho(t)\right)|F(t)|^2 \,dt \end{aligned}$$

for such F, so, by the previous lemma,

$$\frac{1}{2}\int_{-\infty}^{\infty} |\tilde{U}(t)|^2 \rho(t)\,\mathrm{d}t \quad \leqslant \quad \int_{-\infty}^{\infty} |U(t)|^2 w(t)\,\mathrm{d}t$$

for the sums U. Our relation for the latter thus *holds* with $\omega(t) = \rho(t)/2$, and this is not a.e. zero if $\rho(t)$ is not. Done.

Theorem. *If, for given $w \geqslant 0$ in $L_1(\mathbb{R})$ and some $a > 0$ there is any $\omega \geqslant 0$, not a.e. zero, such that*

$$\int_{-\infty}^{\infty} |\tilde{U}(t)|^2 \omega(t)\,\mathrm{d}t \quad \leqslant \quad \int_{-\infty}^{\infty} |U(t)|^2 w(t)\,\mathrm{d}t$$

for the finite sums

$$U(t) \quad = \quad \sum_{\lambda \geqslant a} (A_\lambda \cos \lambda t + B_\lambda \sin \lambda t),$$

we have

$$\int_{-\infty}^{\infty} \frac{\log^- w(t)}{1+t^2}\,\mathrm{d}t \quad < \quad \infty.$$

Remark. Of course,

$$\int_{-\infty}^{\infty} \frac{\log^+ w(t)}{1+t^2}\,\mathrm{d}t \quad < \quad \infty$$

by the inequality between arithmetic and geometric means, w being in L_1.

Proof of theorem. If an ω having the stated properties exists, there is, by the preceding lemma, a function ρ, not a.e. zero, $0 \leqslant \rho(t) \leqslant w(t)$, such that

$$\left| \int_{-\infty}^{\infty} w(t)(F(t))^2\,\mathrm{d}t \right| \quad \leqslant \quad \int_{-\infty}^{\infty} (w(t) - \rho(t))|F(t)|^2\,\mathrm{d}t$$

for the functions

$$F(t) \quad = \quad \sum_{\lambda \geqslant a} C_\lambda \mathrm{e}^{\mathrm{i}\lambda t}.$$

Suppose now that

$$\int_{-\infty}^{\infty} \frac{\log^- w(t)}{1+t^2}\,\mathrm{d}t \quad = \quad \infty;$$

then we will show that the function $\rho(t)$ figuring in the previous relation must be zero a.e., thus obtaining a contradiction. For this purpose, we use a variant of *Szegő's theorem* which, under our assumption on $\log^- w(t)$, gives us a sequence of functions $F_N(t)$, having the form just indicated, such that

$$\int_{-\infty}^{\infty} |1 - F_N(t)|^2 w(t)\,\mathrm{d}t \xrightarrow[N]{} 0.$$

The reader should refer to Chapter II and to problem 2 at the end of it. *There*, Szegő's theorem was established for the weighted L_1 norm, and problem 2 yielded functions $F_N(t)$ of the above form for which

$$\int_{-\infty}^{\infty} |1 - F_N(t)|\, w(t)\,\mathrm{d}t \xrightarrow[N]{} 0.$$

However, after making a simple modification in the argument of Chapter II, §A, which should be apparent to the reader, one obtains a proof of Szegő's theorem for weighted L_2 norms – indeed, for weighted L_p ones, where $1 \leqslant p < \infty$. There is then no difficulty in carrying out the steps of problem 2 for the weighted L_2 norm.

Once we have functions F_N satisfying the above relation, we see that

$$\int_{-\infty}^{\infty} w(t)(F_N(t))^2\,\mathrm{d}t \quad \xrightarrow[N]{} \quad \int_{-\infty}^{\infty} w(t)\,\mathrm{d}t.$$

Indeed, using Schwarz and the triangle inequality, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} w(t)\,|(F_N(t))^2 - 1|\,\mathrm{d}t \quad &= \quad \int_{-\infty}^{\infty} w(t)\,|F_N(t) - 1|\,|F_N(t) + 1|\,\mathrm{d}t \\
&\leqslant \quad \sqrt{\left(\int_{-\infty}^{\infty} w(t)|F_N(t) + 1|^2\,\mathrm{d}t \cdot \int_{-\infty}^{\infty} w(t)|F_N(t) - 1|^2\,\mathrm{d}t\right)} \\
&\leqslant \quad \left(\sqrt{\left(\int_{-\infty}^{\infty} w(t)|F_N(t) - 1|^2\,\mathrm{d}t\right)} \; + \; \sqrt{\left(4\int_{-\infty}^{\infty} w(t)\,\mathrm{d}t\right)}\right) \times \\
&\qquad \times \sqrt{\left(\int_{-\infty}^{\infty} w(t)|F_N(t) - 1|^2\,\mathrm{d}t\right)},
\end{aligned}$$

and the last expression goes to zero as $N \longrightarrow \infty$.

We also see by this computation that

$$\int_{-\infty}^{\infty} w(t)\,|F_N(t)|^2\,\mathrm{d}t \quad \xrightarrow[N]{} \quad \int_{-\infty}^{\infty} w(t)\,\mathrm{d}t,$$

and again, since $0 \leqslant \rho(t) \leqslant w(t)$, that

$$\int_{-\infty}^{\infty} \rho(t)|F_N(t)|^2\,dt \xrightarrow[N]{} \int_{-\infty}^{\infty} \rho(t)\,dt.$$

Using these relations and making $N \longrightarrow \infty$ in the inequality

$$\left|\int_{-\infty}^{\infty} w(t)(F_N(t))^2\,dt\right| \leqslant \int_{-\infty}^{\infty} (w(t) - \rho(t))|F_N(t)|^2\,dt,$$

we get

$$\int_{-\infty}^{\infty} w(t)\,dt \leqslant \int_{-\infty}^{\infty} (w(t) - \rho(t))\,dt,$$

i.e.,

$$\rho(t) = 0 \quad \text{a.e.},$$

since $\rho(t) \geqslant 0$.

We have reached our promised contradiction. This shows that the integral $\int_{-\infty}^{\infty}(\log^- w(t)/(1+t^2))\,dt$ must indeed be finite, as claimed. The theorem is proved.

3. Application of H_p space theory; use of duality

The last theorem of the preceding article shows that our problem can have a positive solution only when

$$\int_{-\infty}^{\infty} \frac{\log^- w(t)}{1+t^2}\,dt < \infty;$$

we may thus limit our further considerations to functions $w \geqslant 0$ *in* $L_1(\mathbb{R})$ *fulfilling this condition.* According to a remark at the end of article 1, there is, corresponding to any such w, an *outer function* φ in H_2 with

$$|\varphi(t)| = \sqrt{(w(t))} \qquad \text{a.e., } t \in \mathbb{R}.$$

Theorem. *Let* $w \geqslant 0$, *belonging to* $L_1(\mathbb{R})$, *satisfy the above condition on its logarithm, and let* $a > 0$. *In order that there exist an* $\omega \geqslant 0$, *not a.e. zero, such that*

$$\int_{-\infty}^{\infty} |\tilde{U}(t)|^2\omega(t)\,dt \leqslant \int_{-\infty}^{\infty} |U(t)|^2 w(t)\,dt$$

for the functions

$$U(t) \quad = \quad \sum_{\lambda \geqslant a} (A_\lambda \cos \lambda t + B_\lambda \sin \lambda t),$$

it is necessary and sufficient that there be a function $\sigma(t)$, not a.e. zero, with

$$0 \leqslant \sigma(t) \leqslant 1 \quad \text{a.e.}$$

and

$$\left| \int_{-\infty}^{\infty} \mathrm{e}^{2\mathrm{i}at} \frac{\overline{\varphi(t)}}{\varphi(t)} f(t)\,\mathrm{d}t \right| \quad \leqslant \quad \int_{-\infty}^{\infty} (1 - \sigma(t)) |f(t)|\,\mathrm{d}t$$

for all $f \in H_1$. Here, $\varphi(t)$ is any outer function in H_2 with

$$|\varphi(t)| \quad = \quad \sqrt{(w(t))} \qquad \text{a.e., } t \in \mathbb{R}.$$

If we have *a function ω for which the above relation holds, $\sigma(t)$ can be taken equal to $\omega(t)/w(t)$. If, on the other hand, a σ is furnished, $\omega(t)$ can be taken equal to $\sigma(t)w(t)/2$.*

Remark. Any two outer functions φ in H_2 with

$$|\varphi(t)| \quad = \quad \sqrt{(w(t))} \quad \text{a.e.}$$

differ by a constant factor of modulus 1.

Proof of theorem. Is based on an idea from the *Bologna Annali* paper of Helson and Szegő.

According to the second lemma of the preceding article, the existence of an ω having the properties in question is *equivalent* to that of a ρ not a.e. zero, $0 \leqslant \rho(t) \leqslant w(t)$, such that

$$\left| \int_{-\infty}^{\infty} w(t)(F(t))^2\,\mathrm{d}t \right| \quad \leqslant \quad \int_{-\infty}^{\infty} (w(t) - \rho(t)) |F(t)|^2\,\mathrm{d}t$$

for the functions

$$F(t) \quad = \quad \sum_{\lambda \geqslant a} C_\lambda \mathrm{e}^{\mathrm{i}\lambda t}.$$

This relation is in turn equivalent to the requirement that

$$\left| \int_{-\infty}^{\infty} w(t)P(t)Q(t)\,\mathrm{d}t \right| \quad \leqslant \quad \frac{1}{2} \int_{-\infty}^{\infty} (w(t) - \rho(t))(|P(t)|^2 + |Q(t)|^2)\,\mathrm{d}t$$

for all *pairs* P, Q of finite sums of the form

$$\sum_{\lambda \geqslant a} C_\lambda e^{i\lambda t}.$$

To see this, one notes in the first place that the present inequality goes over into the preceding one on taking $P = Q = F$. If, on the other hand, the preceding one always holds for our functions F, it is true both with

$$F = P + Q$$

and with

$$F = P - Q$$

whenever P and Q are two sums of the given form. Therefore,

$$\left|\int_{-\infty}^{\infty} w(t)P(t)Q(t)\,dt\right| = \frac{1}{4}\left|\int_{-\infty}^{\infty} w(t)\{(P(t)+Q(t))^2 - (P(t)-Q(t))^2\}\,dt\right|$$

$$\leqslant \frac{1}{4}\int_{-\infty}^{\infty} (w(t)-\rho(t))\{|P(t)+Q(t)|^2 + |P(t)-Q(t)|^2\}\,dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} (w(t)-\rho(t))(|P(t)|^2 + |Q(t)|^2)\,dt,$$

and we have the second of the two inequalities.

Take now an outer function $\varphi \in H_2$ such that $|\varphi(t)|^2 = w(t)$ a.e., and write

$$\sigma(t) = \rho(t)/w(t),$$

so that $0 \leqslant \sigma(t) \leqslant 1$ a.e.. Our condition on $\log w(t)$ makes

$$w(t) > 0 \quad \text{a.e.},$$

so the ratio $\sigma(t)$ will be > 0 on a set of positive measure iff $\rho(t)$ is. In terms of φ and σ, the relation involving P and Q can be rewritten

$$\left|\int_{-\infty}^{\infty} \frac{\overline{\varphi(t)}}{\varphi(t)}(\varphi(t)P(t))(\varphi(t)Q(t))\,dt\right|$$

$$\leqslant \frac{1}{2}\int_{-\infty}^{\infty} (1-\sigma(t))\{|\varphi(t)P(t)|^2 + |\varphi(t)Q(t)|^2\}\,dt.$$

Write now

$$P(t) = e^{iat}p(t), \qquad Q(t) = e^{iat}q(t),$$

so that $p(t)$ and $q(t)$ are sums of the form

$$\sum_{\lambda \geq 0} C_\lambda e^{i\lambda t}.$$

Then the last inequality becomes

$$\left| \int_{-\infty}^{\infty} e^{2iat} \frac{\overline{\varphi(t)}}{\varphi(t)} (\varphi(t)p(t))(\varphi(t)q(t))\, dt \right|$$
$$\leq \frac{1}{2} \int_{-\infty}^{\infty} (1 - \sigma(t)) \{ |\varphi(t)p(t)|^2 + |\varphi(t)q(t)|^2 \}\, dt.$$

Since $\varphi \in H_2$ is *outer*, the products $\varphi(t)p(t)$, $\varphi(t)q(t)$ are $\| \ \|_2$ dense in H_2 when p and q range through the collection of finite sums

$$\sum_{\lambda \geq 0} C_\lambda e^{i\lambda t},$$

according to the last theorem of article 1. The preceding relation is therefore *equivalent* to the condition that

$$\left| \int_{-\infty}^{\infty} e^{2iat} \frac{\overline{\varphi(t)}}{\varphi(t)} g(t)h(t)\, dt \right| \leq \frac{1}{2} \int_{-\infty}^{\infty} (1 - \sigma(t)) \{ |g(t)|^2 + |h(t)|^2 \}\, dt$$

for *all* g and h in H_2.

Suppose $f \in H_1$. According to the factorization theorem from article 1 and the remark thereto, there are functions $g, h \in H_2$ such that

$$f(t) = g(t)h(t) \quad \text{a.e.}$$

and

$$|g(t)| = |h(t)| = \sqrt{(|f(t)|)} \quad \text{a.e..}$$

Substituting these relations into the previous one, we get

$$\left| \int_{-\infty}^{\infty} e^{2iat} \frac{\overline{\varphi(t)}}{\varphi(t)} f(t)\, dt \right| \leq \int_{-\infty}^{\infty} (1 - \sigma(t)) |f(t)|\, dt.$$

This inequality is thus a *consequence* of the preceding one. But it also *implies* the latter. Let, indeed, g and h be in H_2. Then gh is in H_1 by a result of article 1, so, if the present relation holds,

$$\left| \int_{-\infty}^{\infty} e^{2iat} \frac{\overline{\varphi(t)}}{\varphi(t)} g(t)h(t)\, dt \right| \leq \int_{-\infty}^{\infty} (1 - \sigma(t)) |g(t)h(t)|\, dt$$
$$\leq \frac{1}{2} \int_{-\infty}^{\infty} (1 - \sigma(t)) \{ |g(t)|^2 + |h(t)|^2 \}\, dt.$$

Our final inequality, involving functions $f \in H_1$ and the quantity $\sigma(t)$, is hence *fully equivalent* with the initial one for our functions $F(t)$, involving the quantity $\rho(t)$. Since, as we have observed, $\rho(t)$ is > 0 on a set of positive measure iff $\sigma(t)$ is, the first and main conclusion of our theorem now follows directly from the second lemma of the preceding article. Again, since $\rho(t) = \sigma(t)w(t)$, the second conclusion also follows by that lemma. We are done.

In order to proceed further, we use the duality between $L_1(\mathbb{R})$ and $L_\infty(\mathbb{R})$. When one says that the latter space is the *dual* of the former, one means that each (bounded) linear functional Ψ on L_1 corresponds to a unique $\psi \in L_\infty$ such that

$$\Psi(F) \quad = \quad \int_{-\infty}^{\infty} F(t)\psi(t)\mathrm{d}t$$

for $F \in L_1$. Here, we need the linear functionals on the *closed subspace* H_1 of L_1. These can be described according to a well known recipe from functional analysis, in the following way.

Take the (w^*) closed subspace E of L_∞ consisting of the ψ therein for which

$$\int_{-\infty}^{\infty} f(t)\psi(t)\,\mathrm{d}t \quad = \quad 0$$

whenever $f \in H_1$; the *quotient space* L_∞/E can then be identified with the dual of H_1. This is how the identification goes: to each bounded linear functional Λ on H_1 corresponds precisely one subset of L_∞ of the form $\psi_0 + E$ (called a *coset* of E) such that

$$\Lambda(f) \quad = \quad \int_{-\infty}^{\infty} f(t)\psi(t)\,\mathrm{d}t$$

whenever $f \in H_1$ for any $\psi \in \psi_0 + E$, and *only* for those ψ.

From article 1, we know that E is H_∞. *The dual of H_1 can thus be identified with the quotient space L_∞/H_∞.* We want to use this fact to investigate the criterion furnished by the last result. For this purpose, we resort to a trick, consisting of the *introduction of new norms, equivalent to the usual ones, for L_1 and L_∞.* If the inequality in the conclusion of the last theorem *holds* with any function σ, $0 \leqslant \sigma(t) \leqslant 1$, *it certainly does so when $\sigma(t)/2$ stands in place of $\sigma(t)$.* According to that theorem, however, it is the *existence* of such functions σ *different from zero on a set of positive* *measure* which is of interest to us here. We may therefore limit our search for one for which the inequality is valid to those satisfying

$$0 \leqslant \sigma(t) \leqslant 1/2 \quad \text{a.e..}$$

This restriction on our functions σ we henceforth assume.

Given such a σ, we then put

$$\|f\|_1^\sigma = \int_{-\infty}^{\infty} (1-\sigma(t))|f(t)|\,dt$$

for $f \in L_1$; $\|f\|_1^\sigma$ is a norm equivalent to the usual one on L_1, because

$$\tfrac{1}{2}\|f\|_1 \leqslant \|f\|_1^\sigma \leqslant \|f\|_1.$$

On L_∞, we use the dual norm

$$\|\psi\|_\infty^\sigma = \operatorname*{ess\,sup}_{t\in\mathbb{R}} \frac{|\psi(t)|}{1-\sigma(t)};$$

here, the $1-\sigma(t)$ goes in the denominator although we multiply by it when defining $\|\ \ \|_1^\sigma$. We have

$$\|\psi\|_\infty \leqslant \|\psi\|_\infty^\sigma \leqslant 2\|\psi\|_\infty$$

for $\psi \in L_\infty$, so $\|\ \ \|_\infty^\sigma$ and $\|\ \ \|_\infty$ are equivalent on that space.

If $\psi \in L_\infty$, we have, for the functional

$$\Psi(f) = \int_{-\infty}^{\infty} f(t)\psi(t)\,dt$$

on L_1 corresponding to it,

$$|\Psi(f)| \leqslant \|\psi\|_\infty^\sigma \|f\|_1^\sigma.$$

Moreover, the *supremum* of $|\Psi(f)|$ for the $f \in L_1$ with $\|f\|_1^\sigma \leqslant 1$ is *precisely* $\|\psi\|_\infty^\sigma$. These facts are easily verified by writing

$$f(t)\psi(t) \quad \text{as} \quad (1-\sigma(t))f(t)\cdot\frac{\psi(t)}{1-\sigma(t)}$$

in the preceding integral.

For elements of the quotient space L_∞/H_∞ – these are just the cosets $\psi_0 + H_\infty$, $\psi_0 \in L_\infty$ – we write, following standard practice,

$$\|\psi_0 + H_\infty\|_\infty^\sigma = \inf\{\|\psi_0 + h\|_\infty^\sigma :\ h \in H_\infty\}.$$

We have already observed that to each such coset corresponds a linear functional Λ on H_1 given by the formula

$$\Lambda(f) \quad = \quad \int_{-\infty}^{\infty} f(t)\psi(t)\,dt, \quad f \in H_1,$$

where ψ is *any* element of $\psi_0 + H_\infty$. By choosing the $\psi \in \psi_0 + H_\infty$ to have $\|\psi\|_\infty^\sigma$ arbitrarily close to $\|\psi_0 + H_\infty\|_\infty^\sigma$, we see that

$$|\Lambda(f)| \quad \leqslant \quad \|\psi_0 + H_\infty\|_\infty^\sigma \, \|f\|_1^\sigma, \qquad f \in H_1.$$

It is important that this inequality is *sharp*. Even more than that is true:

The infimum appearing in the above formula for $\|\psi_0 + H_\infty\|_\infty^\sigma$ is actually attained for some $h \in H_\infty$, and is equal to the supremum of $|\Lambda(f)|$ for $f \in H_1$ and $\|f\|_1^\sigma \leqslant 1$.

This statement is a straightforward consequence of results from elementary functional analysis. However, lest the reader suspect that something is being produced out of nothing here by mere juggling of notation, let us give the proof.

Denote the supremum of $|\Lambda(f)|$ for $f \in H_1$ and $\|f\|_1^\sigma \leqslant 1$ by M. *According to the Hahn–Banach theorem*, there is an extension Λ^* of the linear functional Λ *to all of* L_1, such that

$$|\Lambda^*(F)| \quad \leqslant \quad M\|F\|_1^\sigma$$

for $F \in L_1$. Corresponding to Λ^* there is, as observed earlier, a $\psi \in L_\infty$ with

$$\Lambda^*(F) \quad = \quad \int_{-\infty}^{\infty} F(t)\psi(t)\,dt, \qquad F \in L_1,$$

and, according to what was also noted above,

$$\|\psi\|_\infty^\sigma \quad = \quad \sup\{|\Lambda^*(F)|: \ F \in L_1 \text{ and } \|F\|_1^\sigma \leqslant 1\} \quad \leqslant \quad M.$$

Then, for $f \in H_1$,

$$\Lambda(f) \quad = \quad \Lambda^*(f) \quad = \quad \int_{-\infty}^{\infty} f(t)\psi(t)\,dt,$$

so $\psi \in \psi_0 + H_\infty$; there is, in other words, an $h \in H_\infty$ with $\psi = \psi_0 + h$, and

$$\|\psi_0 + h\|_\infty^\sigma \quad = \quad \|\psi\|_\infty^\sigma \quad \leqslant \quad M.$$

But, as we remarked previously,

$$|\Lambda(f)| \quad \leqslant \quad \|\psi_0 + H_\infty\|_\infty^\sigma \, \|f\|_1^\sigma, \qquad f \in H_1,$$

so $\|\psi_0 + h\|_\infty^\sigma \geqslant \|\psi_0 + H_\infty\|_\infty^\sigma \geqslant M$. Hence

$$\|\psi_0 + H_\infty\|_\infty^\sigma \quad = \quad \|\psi_0 + h\|_\infty^\sigma \quad = \quad M.$$

Once we are in possession of the above facts, it is easy to establish the following key result.

Theorem. *Let $w \geqslant 0$ in $L_1(\mathbb{R})$ and a number $a > 0$ be given. In order that there exist an $\omega \geqslant 0$, not a.e. zero, such that*

$$\int_{-\infty}^{\infty} |\tilde{U}(t)|^2 \omega(t)\,dt \quad \leqslant \quad \int_{-\infty}^{\infty} |U(t)|^2 w(t)\,dt$$

for the sums

$$U(t) \quad = \quad \sum_{\lambda \geqslant a} (A_\lambda \cos \lambda t + B_\lambda \sin \lambda t),$$

it is necessary and sufficient that, first of all,

$$\int_{-\infty}^{\infty} \frac{\log^- w(t)}{1+t^2}\,dt \quad < \quad \infty$$

and that then, if φ is any outer function in H_2 with

$$|\varphi(t)| \quad = \quad \sqrt{(w(t))} \quad \text{a.e.},$$

we have

$$\left| e^{2iat} \frac{\overline{\varphi(t)}}{\varphi(t)} \; - \; h(t) \right| \quad \leqslant \quad 1 \qquad \text{a.e., } t \in \mathbb{R},$$

for some h, not a.e. zero, *belonging to H_∞.*

A function ω equal to a constant multiple of w will satisfy our conditions iff there is an $h \in H_\infty$ for which

$$\left| e^{2iat} \frac{\overline{\varphi(t)}}{\varphi(t)} \; - \; h(t) \right| \quad \leqslant \quad \text{const.} \quad < \quad 1 \quad \text{a.e.}.$$

Proof. As we saw at the end of the last article, there can be no ω with the above properties unless

$$\int_{-\infty}^{\infty} \frac{\log^- w(t)}{1+t^2}\,dt \quad < \quad \infty.$$

Assuming, then, this condition, we take one of the outer functions φ specified in the statement, and see by the preceding theorem and discussion following it that the existence of an ω having the properties in question is *equivalent* to that of a σ, not a.e. zero,

$$0 \leqslant \sigma(t) \leqslant 1/2 \quad \text{a.e.,}$$

such that

$$\left| \int_{-\infty}^{\infty} e^{2iat} \frac{\overline{\varphi(t)}}{\varphi(t)} f(t)\,dt \right| \leqslant \|f\|_1^{\sigma}, \qquad f \in H_1.$$

According to what we observed above, however, the last relation is equivalent to the existence of an $h \in H_\infty$ for which

$$\frac{|(e^{2iat}\overline{\varphi(t)}/\varphi(t)) - h(t)|}{1 - \sigma(t)} \leqslant 1 \qquad \text{a.e., } t \in \mathbb{R}.$$

Suppose in the first place that *there is no non-zero $h \in H_\infty$ for which*

$$|(e^{2iat}\overline{\varphi(t)}/\varphi(t)) - h(t)| \leqslant 1 \qquad \text{a.e..}$$

Then, since $1/2 \leqslant 1 - \sigma(t) \leqslant 1$ a.e., no $h \in H_\infty$ other than the zero one could satisfy the previous relation. The latter must therefore reduce to

$$|e^{2iat}\overline{\varphi(t)}/\varphi(t)|/(1 - \sigma(t)) \leqslant 1 \qquad \text{a.e.,}$$

i.e., $1 - \sigma(t) \geqslant 1$ a.e., so that

$$\sigma(t) \equiv 0 \quad \text{a.e., } t \in \mathbb{R}.$$

As has just been said, this means that *there can be no non-zero* ω fulfilling our conditions, and necessity is proved.

Consider now the situation where there *is* a non-zero $h \in H_\infty$ making

$$|(e^{2iat}\overline{\varphi(t)}/\varphi(t)) - h(t)| \leqslant 1 \qquad \text{a.e..}$$

Then; since

$$(e^{2iat}\overline{\varphi(t)}/\varphi(t)) - \tfrac{1}{2}h(t) = \tfrac{1}{2}\big(\{(e^{2iat}\overline{\varphi(t)}/\varphi(t)) - h(t)\} + e^{2iat}\overline{\varphi(t)}/\varphi(t)\big),$$

the expression on the *left* also has modulus $\leqslant 1$ a.e.. It is in fact of modulus < 1 *on a set of positive measure.* Indeed, the expression in curly brackets on the right has modulus $\leqslant 1$, and the remaining right-hand term (without the factor $1/2$) has modulus equal to 1. Therefore, since the unit circle is *strictly convex*, the whole right side cannot have modulus equal to 1 *unless* the expression in curly brackets and $e^{2iat}\overline{\varphi(t)}/\varphi(t)$ are *equal*, that is, *unless* $h(t) = 0$. We are, however, assuming that $h(t) \neq 0$

on a set of positive measure; the modulus in question must hence be < 1 on such a set.

Put

$$\sigma(t) \quad = \quad \min\left(1/2, \quad 1 \; - \; |(e^{2iat}\overline{\varphi(t)}/\varphi(t)) \; - \; (h(t)/2)|\right).$$

We then have $0 \leqslant \sigma(t) \leqslant 1/2$ a.e., and $\sigma(t) > 0$ on a set of positive measure by what we have just shown. Finally,

$$\frac{|(e^{2iat}\overline{\varphi(t)}/\varphi(t)) \; - \; (h(t)/2)|}{1 \; - \; \sigma(t)} \quad \leqslant \quad 1 \qquad \text{a.e.},$$

so there must, by the above equivalency statements, be a non-zero $\omega \geqslant 0$ for which our inequality on functions U is satisfied. Sufficiency is proved.

We have still to verify the last part of our theorem. It follows, however, from the last part of the preceding one that an ω *equal to a constant multiple of* w *will work* iff, in the relation involving H_1 functions and σ, we can take σ equal to some constant c, $0 < c < 1/2$. By the above discussion, this is *equivalent* to the existence of an $h \in H_\infty$ for which

$$\frac{|(e^{2iat}\overline{\varphi(t)}/\varphi(t)) \; - \; h(t)|}{1 \; - \; c} \quad \leqslant \quad 1 \qquad \text{a.e.},$$

and we have what was needed.

The theorem is completely proved.

Establishment of the remaining results in this § is based on the criterion furnished by the one just obtained. In that way, we get, first of all, the

Theorem. *Let $w \geqslant 0$ in $L_1(\mathbb{R})$ and a number $a > 0$ be given. If there is any $\omega \geqslant 0$ at all, different from zero on a set of positive measure, such that*

$$\int_{-\infty}^{\infty} |\tilde{U}(t)|^2 \omega(t)\, dt \quad \leqslant \quad \int_{-\infty}^{\infty} |U(t)|^2 w(t)\, dt$$

for the sums

$$U(t) \quad = \quad \sum_{\lambda \geqslant a} (A_\lambda \cos \lambda t + B_\lambda \sin \lambda t),$$

there is one with

$$\int_{-\infty}^{\infty} \frac{\log^- \omega(t)}{1+t^2}\, dt \quad < \quad \infty.$$

Proof. If any ω enjoying the above properties exists, we know by the preceding result that

$$\int_{-\infty}^{\infty} \frac{\log^- w(t)}{1+t^2}\,dt \quad < \quad \infty,$$

and, taking an outer function $\varphi \in H_2$ with $\varphi(t) \;=\; \sqrt{(w(t))}$ a.e., that

$$\left| e^{2iat}\frac{\overline{\varphi(t)}}{\varphi(t)} \;-\; h(t)\right| \quad \leqslant \quad 1 \qquad \text{a.e.}$$

for some *non-zero* $h \in H_\infty$.

The proof of the sufficiency part of the last theorem shows, however, that *once we have such an h*, we can put

$$\sigma(t) \quad = \quad \min\left\{1/2, \quad 1 \;-\; |(e^{2iat}\overline{\varphi(t)}/\varphi(t)) \;-\; (h(t)/2)|\right\},$$

and then, with *this* function σ, the conditions of the *first* result in the present article will be satisfied, ensuring that

$$\omega(t) \quad = \quad \tfrac{1}{2}\sigma(t)w(t)$$

has the desired properties. It is claimed that

$$\int_{-\infty}^{\infty} \frac{\log^- \omega(t)}{1+t^2}\,dt \quad < \quad \infty$$

for *this* function ω. In view of the above condition on $\log^- w(t)$, it is enough to verify that

$$\int_{-\infty}^{\infty} \frac{\log \sigma(t)}{1+t^2}\,dt \quad > \quad -\infty$$

for our present function σ.

Write

$$\psi(t) \quad = \quad e^{-2iat}\varphi(t)/\overline{\varphi(t)};$$

then $|\psi(t)| \;=\; 1$ a.e., and the last inequality is implied by

$$\int_{-\infty}^{\infty} \frac{1}{1+t^2}\log(\,1 \;-\; |\,1 \;-\; (\psi(t)h(t)/2)|\,)\,dt \quad > \quad -\infty$$

which we proceed now to establish.

We have $|1 \;-\; \psi(t)h(t)| \;\leqslant\; 1$ a.e., so

$$1 \;-\; \tfrac{1}{2}\psi(t)h(t) \quad = \quad \tfrac{1}{2} \;+\; \tfrac{1}{2}(1 \;-\; \psi(t)h(t))$$

lies, for almost all t, in a circle of radius 1/2 about the point 1/2:

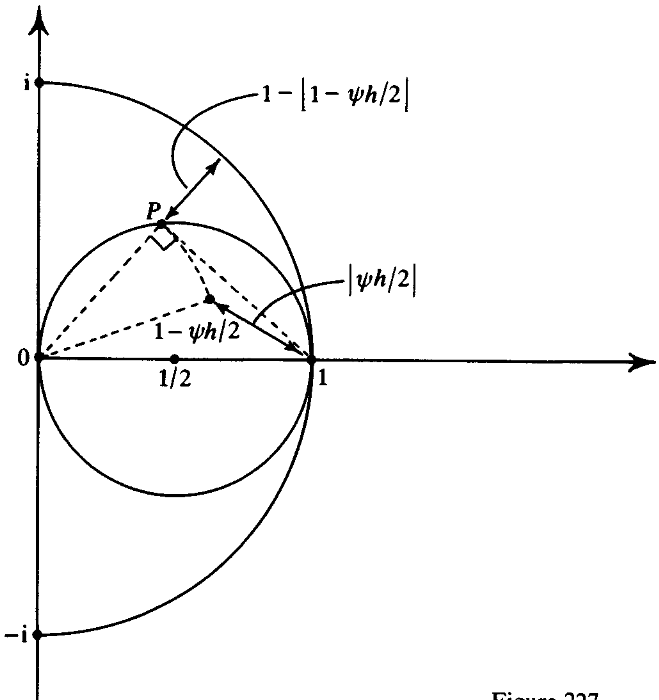

Figure 227

In this figure,

$$1 \;-\; |1 \;-\; (\psi(t)h(t)/2)| \;=\; 1 \;-\; \overline{OP},$$

where

$$\overline{OP}^2 \;\leqslant\; 1 \;-\; |\psi(t)h(t)/2|^2.$$

Therefore, for almost all t,

$$1 \;-\; |1 \;-\; (\psi(t)h(t)/2)|$$

$$\geqslant\; 1 \;-\; \sqrt{\left(1 \;-\; \frac{|\psi(t)h(t)|^2}{4}\right)} \;\geqslant\; \frac{|\psi(t)h(t)|^2}{8} \;=\; \frac{|h(t)|^2}{8}.$$

Thus,

$$\int_{-\infty}^{\infty} \frac{1}{1+t^2}\log(\,1 \;-\; |1 \;-\; (\psi(t)h(t)/2)|\,)\,\mathrm{d}t \;\geqslant\; \int_{-\infty}^{\infty} \frac{\log|(h(t))^2/8|}{1+t^2}\,\mathrm{d}t.$$

However, $h \in H_\infty$ is not a.e. zero. Therefore, *by a theorem from article* 1,

$$\int_{-\infty}^{\infty} \frac{\log|h(t)|}{1+t^2}\,\mathrm{d}t \;>\; -\infty.$$

The preceding integral is thus also $> -\infty$ and our desired relation is established. We are done.

4. Solution of our problem in terms of multipliers

We are now able to prove the

Theorem. *Let $w(t) \geqslant 0$ belonging to $L_1(\mathbb{R})$ and the number $a > 0$ be given. In order that there exist an $\omega \geqslant 0$, not a.e. zero, such that*

$$\int_{-\infty}^{\infty} |\tilde{U}(t)|^2 \omega(t)\,\mathrm{d}t \quad \leqslant \quad \int_{-\infty}^{\infty} |U(t)|^2 w(t)\,\mathrm{d}t$$

for the sums

$$U(t) \quad = \quad \sum_{\lambda \geqslant a} (A_\lambda \cos \lambda t + B_\lambda \sin \lambda t),$$

it is necessary and sufficient that there be a non-zero entire function $f(z)$ of exponential type $\leqslant a$ making

$$\int_{-\infty}^{\infty} \frac{|f(t)|^2}{(1+t^2)w(t)}\,\mathrm{d}t \quad < \quad \infty.$$

Remark. We see that in the typical situation where $w(t)$ is bounded above and *very small for large values of* $|t|$, one has, under the conditions of the theorem, an entire function f of exponential type acting as a *multiplier* (in the sense adopted at the beginning of §A) for the *large* function $1/\sqrt{((1+t^2)w(t))}$.

Proof of theorem: necessity. Is based partly on a result from §D.

Suppose there *is* an ω having the properties in question. Then, by the last theorem of the preceding article, we have one for which

$$\int_{-\infty}^{\infty} \frac{\log^- \omega(t)}{1+t^2}\,\mathrm{d}t \quad < \quad \infty.$$

Let $U(t)$ be any *real-valued* sum of the form indicated above, i.e., one for which the coefficients A_λ and B_λ are real. The function

$$F(t) \quad = \quad U(t) \;+\; \mathrm{i}\tilde{U}(t)$$

is of the form

$$\sum_{\lambda \geqslant a} C_\lambda \mathrm{e}^{\mathrm{i}\lambda t}$$

and hence belongs to H_∞ according to a simple lemma in article 1. By the first theorem of article 2,

$$\omega(t) \leqslant w(t) \quad \text{a.e.},$$

so

$$\int_{-\infty}^{\infty} |F(t)|^2 \omega(t)\,dt = \int_{-\infty}^{\infty} \{(U(t))^2 + (\tilde{U}(t))^2\}\omega(t)\,dt$$
$$\leqslant 2\int_{-\infty}^{\infty} (U(t))^2 w(t)\,dt.$$

For $\Im z > 0$, put, as in article 1,

$$F(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} F(t)\,dt;$$

then, since $F \in H_\infty$,

$$\log|F(z)| \leqslant \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2} \log|F(t)|\,dt$$

according to a result from that article. (Here, of course, $F(z)$ is just

$$\sum_{\lambda \geqslant a} C_\lambda e^{i\lambda z},$$

a function continuous up to $\mathbb{R}$, so one may, if one prefers, apply §G.2 of Chapter III directly.) The right side of the relation just written equals

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Im z \log(|F(t)|^2\omega(t))}{|z-t|^2}\,dt - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Im z \log \omega(t)}{|z-t|^2}\,dt,$$

and this, by the inequality between arthmetic and geometric means, is

$$\leqslant \frac{1}{2}\log\left\{\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}|F(t)|^2\omega(t)\,dt\right\} + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Im z}{|z-t|^2}\log^{-}\omega(t)\,dt.$$

Here, the *first* term is

$$\leqslant \frac{1}{2}\log\left\{\frac{1}{\pi\Im z}\int_{-\infty}^{\infty} |F(t)|^2\omega(t)\,dt\right\}$$

which is in turn

$$\leqslant \frac{1}{2}\log\left\{\frac{2}{\pi\Im z}\int_{-\infty}^{\infty} (U(t))^2 w(t)\,dt\right\}$$