CLASS 1 (PHRAGMÉN-LINDELOF PRINCIPLE)

§ THE MAXIMUM PRINCIPLE

LET $f : \mathbb{D} \to \mathbb{C}$ ANALYTIC $\mathbb{D} = \{ |z| < 1 \}$ AND CONTINUOUS IN $\overline{\mathbb{D}}$, $z = x + iy \in \mathbb{C}$

$\left(\exists f'(z) \text{ OR } \partial_x f = \frac{1}{i} \partial_y f \text{ C-R EQ.} \right)$.

THM A: THEN $\max\limits_{|z| \leq 1} |f| = \max\limits_{|z|=1} |f|$

PROOF: $$|f(0)| = \left| \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{f(\xi)}{\xi} \, d\xi \right| \leq \max_{|z|=1} |f(z)|$$

$$T(z) = \frac{a - z}{1 - \bar{a} z} \qquad a \in \mathbb{D} \qquad T(0) = a \quad , \quad T : \mathbb{D} \to \mathbb{D}$$

$$T(\partial \mathbb{D}) = \partial \mathbb{D} .$$

$$|f(a)| = |f \circ T(0)| \leq \max_{|z|=1} |f \circ T(z)| = \max_{|z|=1} |f(z)| \qquad \blacksquare$$

REMARK: $\Omega \subset \mathbb{C}$ SIMPLY CONNECTED, $f : \Omega \to \mathbb{C}$ ANALYTIC AND CONTINUOUS ON $\overline{\Omega}$ THEN

$$\max_{\overline{\Omega}} |f| = \max_{\partial \Omega} |f|$$

§ MAXIMUM PRINCIPLE IN $\mathbb{C}^+ = \{ z \in \mathbb{C} : y > 0 \}$

$f(z) = e^{-iz}$ $\qquad |f(x)| = 1 \quad \forall x$ REAL

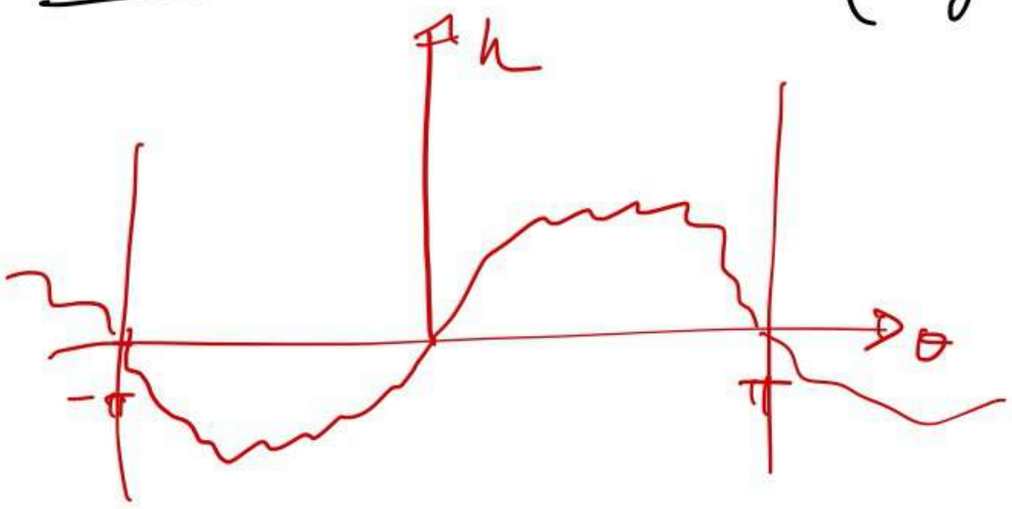
$$|f(iy)| = e^{y} \to +\infty \quad \text{AS} \quad y \to +\infty .$$

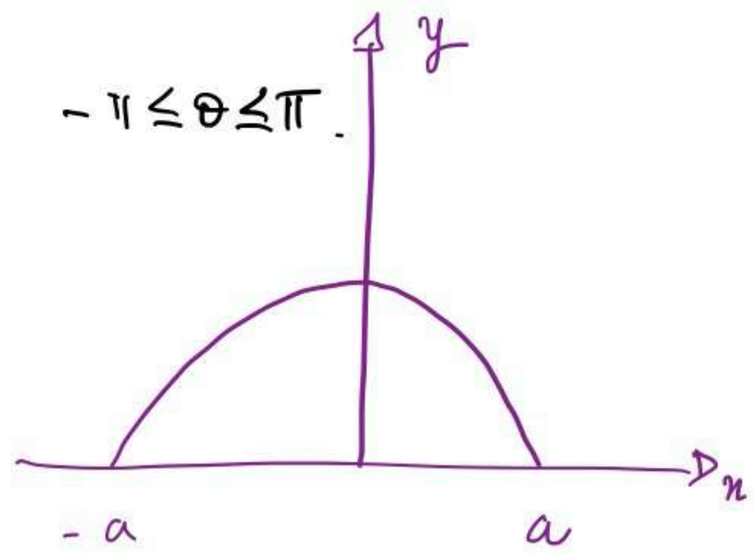
THM 1 : ASSUME $f : \mathbb{C}^+ \to \mathbb{C}$ ANALYTIC AND $|f(z)|$ is CONTINUOUS ON $\overline{\mathbb{C}^+}$. ASSUME $\left(\log^+ = \max(0, \log) \right)$

$$\underline{\lim}_{a \to \infty} \frac{1}{a} \int_0^{\pi} \log^+ |f(a e^{i\theta})| \sin\theta \, d\theta = 0 .$$

THEN IF $|f(x)| \leq 1$ $\forall x$ REAL $\Longrightarrow$ $|f(z)| \leq 1$ $\forall z \in \mathbb{C}^+$.

PROOF: LET $h(\theta) = \left(\log^+ |f(e^{i\theta})| \right) \operatorname{sign}(\theta)$ $-\pi \leq \theta \leq \pi$.





$$g(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} h(\theta) d\theta \quad , \quad \operatorname{Re} g(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{\frac{1 - |z|^2}{|e^{i\theta} - z|^2}} h(\theta) d\theta$$

$|z| < 1$

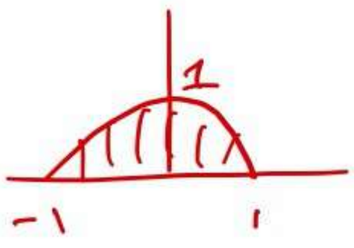
$$\boxed{P_z(\theta) \quad \text{POISSON KERNEL} \quad = \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - \theta_0)} \quad \text{IF} \quad z = r e^{i\theta_0}}$$

NOW

$$g(z) = \frac{1}{2\pi} \int_0^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} h(\theta) d\theta - \frac{1}{2\pi} \int_0^{\pi} \frac{e^{-i\theta} + z}{e^{-i\theta} - z} h(\theta) d\theta$$

$$\operatorname{Re} g(z) = \frac{1 - |z|^2}{2\pi} \int_0^{\pi} \frac{4y \, h(\theta) \sin\theta}{|e^{i\theta} - z|^2 |e^{-i\theta} - z|^2} d\theta$$

$$F(z) = \frac{f(z)}{e^{g(z)}} \quad \text{FOR} \quad |z| \leq 1 \quad \text{AND} \quad y \geq 0$$



- $|F(z)| \leq |f(z)|$ FOR $|z| \leq 1$ AND $y \geq 0$

$\operatorname{Re} g(z) = 0$ IF $y = 0$.

- $|F(x)| = |f(x)| \leq 1$ x REAL $(-1 \leq x \leq 1)$

SINCE $\quad \operatorname{Re} g(e^{i\theta}) = \log^+ |f(e^{i\theta})| \quad 0 \le \theta \le \pi$

$$|e^{g(e^{i\theta})}| \geq |f(e^{i\theta})| \implies |F(e^{i\theta})| \leq \frac{|f(e^{i\theta})|}{|f(e^{i\theta})|} = 1.$$

$$\implies |F(z)| \leq 1 \quad \text{FOR} \quad |z| < 1 \quad y > 0.$$

$$\log^+ |f(z)| \leq \frac{1-|z|^2}{2\pi} \int_0^\pi \frac{4y \log^+ |f(e^{i\theta})| \sin\theta}{|e^{i\theta} - z|^2 |e^{-i\theta} - z|^2} \, d\theta \qquad |z|<1, \; y>0.$$

THE SAME ARGUMENT WORKS FOR $f(az)$, $a>0$, REPLACE z BY w/a.

$$\log^+ |f(w)| \leq \frac{a^2 - |w|^2}{2\pi} \int_0^\pi \frac{4a \operatorname{Im} w \log^+ |f(ae^{i\theta})| \sin\theta}{|ae^{i\theta} - w|^2 |ae^{-ia} - w|^2} \, d\theta$$

$|w| < a$

$\operatorname{Im} w > 0.$

IF $\quad |w| < \varepsilon a, \quad 0 < \varepsilon < 1$

$$|ae^{i\theta} - w| \geq a(1 - |w|/a) > a(1-\varepsilon)$$

$$\log^+ |f(w)| \leq \frac{2 \operatorname{Im} w}{\pi (1-\varepsilon)^4} \frac{1}{a} \int_0^\pi \log^+ |f(ae^{i\theta})| \sin\theta \, d\theta$$

$$\log^+ |f(w)| \leq \frac{2 \operatorname{Im} w}{\pi (1-\varepsilon)^4} \cancel{\varliminf_{a \to \infty} \frac{1}{a} \int_0^\pi (\cdots) \, d\theta} \to 0$$

$$\implies \log^+ |f(w)| \leq 0 \implies |f(w)| \leq 1. \qquad \blacksquare$$

<u>THM 2</u> $\quad$ LET $h(x) \geq 1$ CONTINUOUS. ASSUME

$$\int_{-\infty}^{\infty} \frac{\log h(x)}{1+x^2} \, dx < \infty.$$

THEN THE FORMULA

$$f(z) = \exp\left[\frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{1+tz}{t-z}\,\frac{\log h(t)}{1+t^2}\,dt\right], \quad z \in \mathbb{C}^+$$

DEFINES AN ANALYTIC FUNCTION ON $\mathbb{C}^+$, CONTINUOUS ON $\overline{\mathbb{C}^+}$ AND

$|f(x)| = h(x)$, AND $|f(z)| \geq 1$ FOR $y > 0$.

<u>PROOF</u>: $\operatorname{Re} \dfrac{1+tz}{i(t-z)} = y\,\dfrac{1+t^2}{|t-z|^2}$ THEN

$$\log|f(z)| = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\log h(t)}{|t-z|^2}\,dt = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\log h(t)}{(t-x)^2+y^2}\,dt$$

$$\Rightarrow \log|f(z)| \geq 0 \Rightarrow |f(z)| \geq 1 \quad \text{IF} \quad y>0.$$

$$\lim_{\substack{x\to u\\ y\to 0}} \log|f(z)| = \log h(u)$$

$$\boxed{\frac{y}{\pi}\int_{-\infty}^{\infty} \frac{1}{(t-x)^2+y^2}\,dt = 1 \quad y>0}$$

RHS

$\|$

$$\left|\log|f(z)| - \log h(u)\right| \leq \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{|\log h(t) - \log h(u)|}{(t-x)^2+y^2}\,dt \xrightarrow[\substack{x\to u\\ y\to 0}]{?} 0$$

TAKE $\varepsilon>0$, SO $\exists\,\delta>0$ IF $|t-u|\leq\delta$ THEN

$|\log h(t) - \log h(u)| \leq \varepsilon$. ASSUME $|x-u|<\delta/2$ AND $\boxed{t-x > t-u-\frac{\delta}{2}}$ USE

$$\text{RHS} \leq \frac{y}{\pi}\int_{x-\delta}^{u+\delta} + \int_{u+\delta}^{\infty} + \int_{-\infty}^{u-\delta} (\cdots)\,dt$$

$$\leq \varepsilon + O(y) + O(y)$$

RHS $\leq 2\varepsilon$ IF y IS SMALL AND $|x-u|<\delta/2$. ∎

PROBLEMS NOV 10 (MIDNIGHT): PROBLEMS 1, 2, 3, 4, 5

CLASS 2 (POISSON REPRESENTATION)

THM 3 LET $\mu(x)$ BE A NONDECREASING FUNCTION FOR REAL x.

ASSUME $\displaystyle\int_{-\infty}^{\infty} \frac{d\mu(t)}{1+t^2} < \infty$. THEN

$$f(z) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{1+tz}{t-z} \frac{d\mu(t)}{1+t^2}$$

is ANALYTIC IN $\mathbb{C}^+$ AND $\operatorname{Re} f(z) \geq 0$ for $y>0$. IF a AND b ARE POINTS OF CONTINUITY OF μ THEN

$$\mu(b) - \mu(a) = \lim_{y \searrow 0} \int_a^b \operatorname{Re} f(x+iy)\, dx \qquad \left(\text{"}\operatorname{Re} f(x)\, dx = d\mu(x)\text{"}\right)$$

RECALL: RIEMANN-STIELTJES INTEGRAL: $\mu : [a,b] \to \mathbb{R}$ NONDECREASING

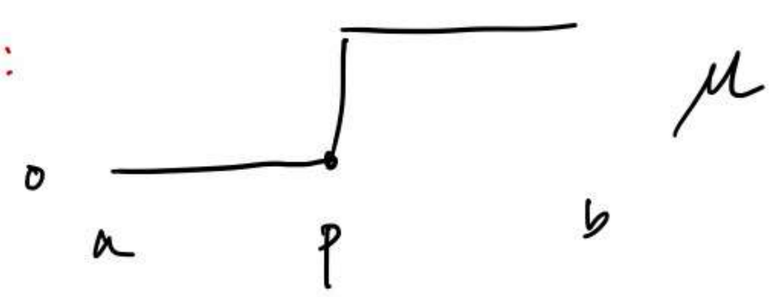
IF g IS CONTINUOUS

$$\int_a^b g(x)\, d\mu(x) = \lim_{|P| \to 0} \sum_{i=1}^{N} g(c_i)\left[\mu(x_i) - \mu(x_{i-1})\right]$$

$P = \{x_0 = a < x_1 < \dots < x_N = b\}$, $c_i \in [x_{i-1}, x_i]$.

$|P| = \min\limits_{1 \leq i \leq N} x_i - x_{i-1}$.

EXAMPLE:

- 

$$\int_a^b g(x)\, d\mu(x) = g(p)$$

$$d\mu(x) = \delta_p(x)$$

- $\mu \in C^1$ THEN $d\mu(x) = \mu'(x)\, dx$.

- IF a, b POINTS OF CONTINUITY THE $\displaystyle\int_{a^+}^{b^+} d\mu(x) = \mu(b) - \mu(a)$

Proof: $\operatorname{Re} f(z) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{d\mu(t)}{(t-x)^2+y^2} \geq 0.$

$$\boxed{\begin{aligned} &\frac{d}{dt}\,\mathrm{ATAN}(t) = \frac{1}{1+t^2} \\ &\text{OR} \\ &\frac{d}{dx}\,\mathrm{TAN}(x) = 1+\mathrm{TAN}^2(x) \end{aligned}}$$

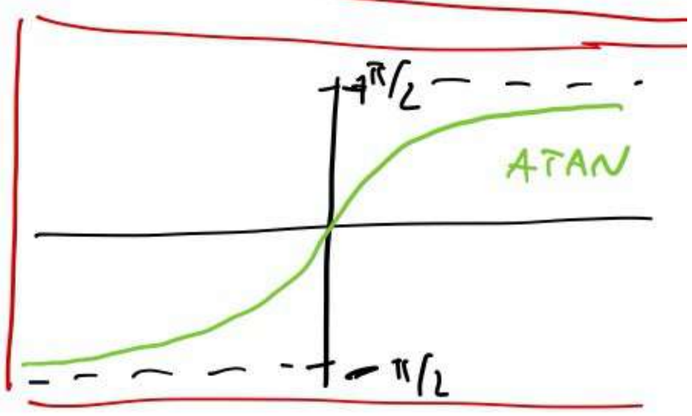
$$\lim_{y\to 0}\int_a^b \operatorname{Re} f(x+iy)\,dx \overset{?}{=} \mu(b)-\mu(a)$$

$$\int_a^b \operatorname{Re} f(x+iy)\,dx = \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\mathrm{ATAN}\,\frac{b-t}{y} + \mathrm{ATAN}\,\frac{t-a}{y}\right)d\mu(t)$$

$$\left|\int_a^b d\mu(t) - \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\mathrm{ATAN}\,\frac{b-t}{y} + \mathrm{ATAN}\,\frac{t-a}{y}\right)d\mu(t)\right|$$

$$\boxed{\int_{-\infty}^{\infty} = \int_a^b + \int_b^{\infty} + \int_{-\infty}^{a}}$$

$$\leq \frac{1}{\pi}\int_a^b\left(\pi - \mathrm{ATAN}\,\frac{b-t}{y} - \mathrm{ATAN}\,\frac{t-a}{y}\right)d\mu(t)$$



$$+\frac{1}{\pi}\left(\int_b^{\infty} + \int_{-\infty}^{a}\right)\left|\mathrm{ATAN}\,\frac{b-t}{y} + \mathrm{ATAN}\,\frac{t-a}{y}\right| d\mu(t)$$

$$= I_1 + I_2 + I_3$$

For $\varepsilon>0$ CHOSE $\delta>0$ SO THAT $|\mu(a+\delta)-\mu(a-\delta)|, |\mu(b+\delta)-\mu(b-\delta)| < \varepsilon$

- $$I_1 = \frac{1}{\pi}\left(\int_a^{a+\delta} + \int_{a+\delta}^{b-\delta} + \int_{b-\delta}^{b}\right)\left(\pi - \mathrm{ATAN}\,\frac{b-t}{y} - \mathrm{ATAN}\,\frac{t-a}{y}\right)d\mu(t)$$

$$\leq 2\varepsilon + \int_{a+\delta}^{b-\delta}\left(\pi - \mathrm{ATAN}\,\frac{b-t}{y} - \mathrm{ATAN}\,\frac{t-a}{y}\right)d\mu(t)$$

$$= 2\varepsilon + O(y)$$

$\geq \frac{\delta}{y}$

$$\boxed{\begin{aligned} &\mathrm{ATAN}(s) = \frac{\pi}{2} + O\left(\frac{1}{s}\right) \\ &s\to\infty \end{aligned}}$$

- $I_2 = \frac{1}{\pi}\left(\int_b^{b+\delta} + \int_{b+\delta}^{\infty}\right)\left|\mathrm{ATAN}\,\frac{y(b-a)}{y^2+(t-a)(t-b)}\right| d\mu(t)$

$$\boxed{\mathrm{ATAN}(X)+\mathrm{ATAN}(Y) = \mathrm{ATAN}\,\frac{X+Y}{1-XY}}$$

$$\boxed{\mathrm{ATAN}(x) = \int_0^x \frac{dt}{1+t^2} \le x}$$

$$\le \varepsilon + \frac{y(b-a)}{\pi}\int_{b+\delta}^{\infty} \frac{1}{(t-a)(t-b)}\, d\mu(t)$$

$$= \varepsilon + O(y)$$

- $I_3 \le \varepsilon + O(y)$ (SAME WAY AS IN I_2)

$$\Rightarrow I_1 + I_2 + I_3 \le 4\varepsilon + O(y) \qquad \blacksquare$$

<u>THM 4</u> (CONVERSE OF THM 3)

LET $f : \mathbb{C}^+ \to \mathbb{C}$ BE ANALYTIC, $\mathrm{Re}\, f(z) \geq 0$ $y>0$. THEN $\exists\, c, p \in \mathbb{R}$ $p \geq 0$ AND NONDECREASING $\mu(x)$ SUCH THAT

$$f(z) = i(c - pz) + \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{1+tz}{t-z}\,\frac{d\mu(t)}{1+t^2}$$

<u>HELLY'S SELECTION PRINCIPLE</u> IF $\{\mu_n\}_{n\geq 1}$ NONDECREASING FUNCTIONS ON $[a,b]$. ASSUME $\mu_n(a)$, $\mu_n(b)$ ARE BOUNDED SEQUENCES. THEN $\exists\, n_k$ AND $\mu : [a,b] \to \mathbb{R}$ NONDEC. SUCH THAT

$$\mu_{n_k}(x) \to \mu(x) \qquad \forall x \in [a,b].$$

<u>PROOF</u>: $\mu_n(x)$ IS BOUNDED FOR ANY $x \in [a,b]$ $\underline{\mu_n(a)} \le \mu_n(x) \le \underline{\mu_n(b)}$

BY USING CANTOR'S DIAGONAL PROCEDURE $\exists\, n_k$ SUCH THAT

$\mu_{n_k}(x) \longrightarrow \mu(x) \quad \forall x \in [a,b] \cap \mathbb{Q}$ THEN EXTEND μ

$$\mu(x) = \frac{1}{2}\left[\lim_{\substack{y \nearrow x \\ y \in \mathbb{Q}}} \mu(y) + \lim_{\substack{y \searrow x \\ y \in \mathbb{Q}}} \mu(y) \right] . \quad \mu : [a,b] \to \mathbb{R}$$

NONDEC. THEN IS EASY TO SEE $\mu_{n_k}(x) \longrightarrow \mu(x)$ IF x IS A POINT OF CONTINUITY OF μ. DESCONTINUITIES OF μ ON $[a,b]$, SAY S, IS COUNTABLE. THEN FURTHER TAKE A SUBSEQUENCE n_{k_j} AND REDEFINE μ ON S BY $\mu(x) = \lim \mu_{n_{k_j}}(x)$ AND SHOW THIS IS THE μ WE WANT. ∎

g CONTINUOUS

$$\int_{[a,b]} g(x)\, d\mu_n(x) \longrightarrow \int_{[a,b]} g(x)\, d\mu(x)$$

PROOF OF THM 4

PROBLEMS 2 AND 3 | IF $\operatorname{Re} f(x)$ IS CONTINUOUS THEN THM 4 IS TRUE FOR $d\mu(x) = \operatorname{Re} f(x)\, dx$

PROB. 4

$$p = \lim_{y \to \infty} \frac{\operatorname{Re} f(iy)}{y}$$

$$\operatorname{Re} f(x+iy) = py + \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\operatorname{Re} f(t)}{(t-x)^2 + y^2}\, dt$$

$$\operatorname{Re} f(z+i\varepsilon) = py + \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\operatorname{Re} f(i\varepsilon + t)}{(t-x)^2 + y^2}\, dt \qquad y > 0.$$

$$= py + \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{d\mu_\varepsilon(t)}{(t-x)^2 + y^2}$$

$$d\mu_\varepsilon(t) = \operatorname{Re} f(i\varepsilon + t)\, dt$$

$$\mu_\varepsilon(b) - \mu_\varepsilon(a) = \int_a^b dt$$

BY HELLY'S SEL. PRINCIPLE $\mu_\varepsilon(t) \xrightarrow[\varepsilon \to 0]{} \mu(t)$

$$\mu_\varepsilon(a) - \mu_\varepsilon(-a) = \int_{-a}^{a} d\mu_\varepsilon(t) \leq (1+a^2) \int_{-a}^{a} \frac{d\mu_\varepsilon(t)}{1+t^2}$$

$$\int_{-\infty}^{\infty} \frac{d\mu_\varepsilon(t)}{1+t^2} \leq \operatorname{Re} f(i + i\varepsilon)$$

$$\operatorname{Re} f(z+i\varepsilon) \geq py + \frac{y}{\pi}\int_{-a}^{a} \frac{d\mu_\varepsilon(t)}{(t-x)^2+y^2}$$

$a \to \infty$
$\varepsilon \to 0$

$$\Longrightarrow \operatorname{Re} f(z) \geq py + \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{d\mu(t)}{(t-x)^2+y^2}$$

, IN PARTICULAR $\int_{-\infty}^{\infty} \frac{d\mu(t)}{1+t^2} < \infty$

NOW DEFINE $g(z) = -ipz + \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{1+tz}{t-z}\,\frac{d\mu(t)}{1+t^2}$ SO THAT

$$\operatorname{Re} g(z) = py + \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{d\mu(t)}{(t-x)^2+y^2} \Longrightarrow \operatorname{Re} f-g \geq 0$$

SAY $h = f-g$, AND a AND b POINTS OF CONTINUITY OF μ

$$\int_a^b \operatorname{Re} h(x+i\varepsilon)\,dx = \int_a^b \operatorname{Re} f(x+i\varepsilon)dx - \int_a^b \operatorname{Re} g(x+i\varepsilon)dx$$

$$\xrightarrow{\varepsilon \to 0} \mu(b) - \mu(a) - (\mu(b) - \mu(a)) = 0$$

$$\lim_{y\to\infty} \frac{\operatorname{Re} h(iy)}{y} = 0.$$

$$\Longrightarrow \operatorname{Re} h(z+i\delta) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\operatorname{Re} h(t+i\delta)}{(t-x)^2+y^2}\,dt$$

PROB 2, 3 AND 4

$$0 \leq \frac{y}{\pi}\int_{-a}^{a} \frac{\operatorname{Re} h(t+i\delta)}{(t-x)^2+y^2}\,dt \leq \frac{1}{\pi y}\int_{-a}^{a} \operatorname{Re} h(t+i\delta)\,dt$$

$$\xrightarrow[\delta=\varepsilon\to 0]{} 0$$

$$\operatorname{Re} h(z) = \lim_{\varepsilon \to 0} \operatorname{Re} h(z+i\varepsilon) = \lim_{\varepsilon \to 0} \frac{y}{\pi} \int_a^{\infty} + \int_{-\infty}^{-a} \frac{\operatorname{Re} h(t+i\varepsilon)}{(t-x)^2+y^2}\, dt$$

$a > 0$ LARGE, POINT OF CONT OF μ.

IF $-a \leq x_1, x_2 \leq a$ AND $|t| \geq a$

$$\frac{(t-x_1)^2+y^2}{(t-x_2)^2+y^2} \leq \left(\frac{1+|x_2-x_1|}{\min(a-x_2, a+x_2)} \right)^2$$

$$\Longrightarrow \quad \operatorname{Re} h(x_2+iy) \leq \left(\frac{1+|x_2-x_1|}{\min(a-x_2, a+x_2)} \right)^2 \operatorname{Re} h(x_1+iy)$$

$$\overset{a \to \infty}{\Longrightarrow} \quad \operatorname{Re} h(x_2+iy) \leq \operatorname{Re} h(x_1+iy)$$

$$\Longrightarrow \quad \operatorname{Re} h(x+iy) = c(y)$$

THEN IS NOT HARD TO SHOW THAT $h \equiv ic$, $c \in \mathbb{R}$.

$$\Longrightarrow \quad f = g + ic \qquad \blacksquare$$

CLASS 3 ($h(\varphi)$ SPACES)

LET $\varphi : \mathbb{C}^+ \to \mathbb{C}$ ANALYTIC AND $\operatorname{Re} \varphi \geq 0$ IN $\mathbb{C}^+$. BY THM. 4

$$\operatorname{Re} \varphi(z) = py + \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{d\mu(t)}{(t-x)^2+y^2} \qquad (p \geq 0)$$

BY PROB. 5

$$\frac{\overline{\varphi(w)} + \varphi(z)}{\pi i(\overline{w} - z)} = \frac{p}{\pi} + \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{d\mu(t)}{(t-z)(t-\overline{w})}, \quad z, \overline{w} \in \mathbb{C}^+$$

DEF:

$$\mathcal{L}(\varphi) = \left\{ F : \mathbb{C}^+ \to \mathbb{C} : F \text{ ANALYTIC}, \ \exists \lambda \in \mathbb{C}, \ \exists f \in L^2(\mu) \right.$$

$$\left. \text{SO THAT } F(z) = \lambda + \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{f(t)\, d\mu(t)}{t - z} \right\}$$

$$\|F\|^2_{\mathcal{L}(\varphi)} = |\lambda|^2 \frac{\pi}{\pi} + \int_{-\infty}^{\infty} |f(t)|^2 \, d\mu(t)$$

REMARK: THIS IS WELL-DEFINED BECAUSE IF $G(z) = F(z) - \lambda$

$$\operatorname{Re} G(z) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(t)\, d\mu(t)}{(t-x)^2 + y^2} \qquad f = f_+ - f_-$$

$$v_\pm(b) - v_\pm(a) = \int_a^b f_\pm(t)\, d\mu(t)$$

$$\int_a^b f_\pm(t)\, d\mu(t) = \int_a^b dv_\pm(t) = \lim_{y \searrow 0} \int_a^b \left(\frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f_\pm(t)\, d\mu(t)}{(t-x)^2 + y^2} \right) dx$$

WHENEVER $a, b \in \operatorname{CONT}(v_+) \cap \operatorname{CONT}(v_-)$. IF

$$G(z) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\tilde{f}(t)\, d\mu(t)}{t - z}$$

$$\Rightarrow \int_a^b f_\pm(t)\, d\mu(t) = \int_a^b \tilde{f}_\pm(t)\, d\mu(t) \Rightarrow f_\pm = \tilde{f}_\pm \quad \mu\text{-a.e.}$$

NOTICE: $F(i) - \frac{1}{\pi i} \int \frac{f(t)\, d\mu(t)}{t - i} = \lambda$.

THM 5 (PART 1) $\mathcal{L}(\varphi)$ IS A HILBERT SPACE. IF

$$K_\varphi(w, z) = K_w(z) = \frac{\overline{\varphi(w)} + \varphi(z)}{\pi i (\bar{w} - z)} \quad \text{THEN}$$

$K_\omega \in \mathcal{H}(\varphi)$ $\forall \omega \in \mathbb{C}^+$ AND $F(\omega) = \langle F, K_\omega \rangle_{\mathcal{H}(\varphi)}$ $\forall F \in \mathcal{H}(\varphi)$ AND $\forall \omega \in \mathbb{C}^+$. IN PARTICULAR, THE FUNCTIONAL

$F \in \mathcal{H}(\varphi) \longmapsto F(\omega)$ is CONTINUOUS. MOREOVER,

$U : f \in L^2(\mu) \longmapsto \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t-z}\, d\mu(t) \in \mathcal{L}(\varphi)$ THEN U is ISOMETRIC AND $(\mathrm{RAN}\ U)^\perp = p \times \mathrm{CONST}$. (SO IF $p=0$ THEN U is UNITARY = ISOM. ISOMORPHISM)

PROOF: $p=0$, AND YOU FILL THE GAPS FOR $p>0$.

- $\{F_n\}$ is CAUCHY SEQ. IN $\mathcal{H}(\varphi) \Longleftrightarrow \{f_n\}$ is CAUCHY SEQ. IN $L^2(\mu)$. $\exists f$ SUCH THAT $f_n \xrightarrow[L^2(\mu)]{} f$. THEN $F = Uf$. BELONGS TO $\mathcal{H}(\varphi) \Longrightarrow F_n \to F$ IN $\mathcal{H}(\varphi)$.
- IT IS TRIVIAL TO SEE $\|Uf\|_{\mathcal{H}(\varphi)} = \|f\|_{L^2(\mu)}$ AND $\mathrm{RAN}\,U = \mathcal{H}(\varphi)$ SO $(\mathrm{RAN}\,U)^\perp = \{0\}$.
- $$K_\varphi(\omega, z) = \frac{\overline{\varphi(\omega)} + \varphi(z)}{\pi i(\overline{\omega} - z)} = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{d\mu(t)/-\pi i(t-\overline{\omega})}{(t-z)}$$

$$\Longrightarrow K_\omega = U\left(\frac{1}{-\pi i(t-\overline{\omega})}\right) \Longrightarrow K_\omega \in \mathcal{H}(\varphi)$$

$$\langle F, K_\omega \rangle = \int_{-\infty}^{\infty} f(t) \overline{\frac{1}{-\pi i(t-\overline{\omega})}}\, d\mu(t) \overset{\text{BY DEF.}}{=} F(\omega). \quad \blacksquare$$

REMARK: NOTE THAT IF $\mathcal{H}(\varphi) = \mathcal{H}(\tilde{\varphi})$ THEN $\varphi = \tilde{\varphi} + ic_0$, $c_0 \in \mathbb{R}$.

- $K_\varphi(\omega, z) = K_{\tilde{\varphi}}(\omega, z)$

$$K_\varphi(\omega, z) = \langle K_\varphi(\omega, \cdot), K_{\tilde{\varphi}}(z, \cdot) \rangle = \overline{\langle K_{\tilde{\varphi}}(z, \cdot), K_\varphi(\omega, \cdot) \rangle}$$

$$= \overline{K_{\tilde\varphi}(z,w)} = K_{\tilde\varphi}(w,z)$$

$$\varphi(z) - \tilde\varphi(z) = c$$

$$K_\varphi(w,z) - K_{\tilde\varphi}(w,z) = \frac{\bar c + c}{\pi i(\bar w - z)}$$

$$\boxed{\nu = \mu - \tilde\mu}$$

$$= \frac{p - \tilde p}{\pi} + \frac{1}{\pi^2}\int_{-\infty}^{\infty} \frac{d(\mu-\tilde\mu)(t)}{(t-z)(t-\bar w)}$$

$$\Rightarrow \bar c + c = i(\bar w - z)(p - \tilde p) - \frac{1}{\pi i}\int_{-\infty}^{\infty}\left(\frac{1}{t-\bar w} - \frac{1}{t-z}\right) d\nu(t)$$

$$\Rightarrow \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{d\nu(t)}{t-z} = A + \cancel{i(p-\tilde p)z}$$

$\left(\begin{array}{l} z = iy, \text{ TAKE} \\ \text{Re BOTH SIDES} \\ \text{DIVIDE } y \\ \Rightarrow p = \tilde p \end{array}\right)$

IT IS EASY TO SEE NOW (BY POISSON REP.) THAT $d\nu(t) \equiv \operatorname{Re} A$.

$$\Rightarrow \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{d\nu(t)}{t-z} = A + id, \quad d \in \mathbb{R}.$$

$$\Rightarrow c = i c_0, \quad c_0 \in \mathbb{R}.$$

$w \in \mathbb{C}^+$

THM 5 (PART 2) LET $T_w : F(z) \longrightarrow \dfrac{F(z) - F(w)}{z - w}$, FOR $F \in \mathcal{L}(\varphi)$. THEN T_w IS CONTINUOUS, IN FACT

$$\|T_w F\| \leq \frac{1}{\operatorname{Im} w}\|F\|.$$ AND

(i) $T_w^* = T_{\bar w}$

(ii) $\dfrac{T_\alpha - T_\beta^*}{\alpha - \bar\beta} = T_\beta^* T_\alpha$

(iii) $$U\left(\frac{f(x)}{x-\omega}\right)(z) = T_\omega F(z).$$

PROOF:

$(p=0)$ (iii) $$\frac{F(z)-F(\omega)}{z-\omega} = \frac{1}{\pi i}\int_{-\infty}^{\infty} f(t)\,d\mu(t)\left(\frac{1}{t-z}-\frac{1}{t-\omega}\right)\frac{1}{z-\omega}$$

$$= \frac{1}{\pi i}\int_{-\infty}^{\infty}\left(\frac{f(t)}{t-\omega}\right)\frac{d\mu(t)}{t-z} = U\left(\frac{f(x)}{x-\omega}\right)(z)$$

(i) $$\langle F, T_\omega G\rangle = \int_{-\infty}^{\infty} f(t)\left(\frac{g(t)}{t-\omega}\right)^{-} d\mu(t) = \langle T_{\bar\omega} F, G\rangle$$

$$\Rightarrow \quad T_\omega^* = T_{\bar\omega}.$$

(ii) $$\frac{T_\alpha - T_{\bar\beta}}{\alpha-\bar\beta} = T_{\bar\beta}T_\alpha$$ IS NOW A SIMPLE COMPUTATION.

$\underline{\alpha=\beta=\omega}$: $$\left\langle \frac{T_\omega - T_{\bar\omega}}{\omega-\bar\omega} F, G\right\rangle$$

$$= \langle T_{\bar\omega}T_\omega F, G\rangle = \langle T_\omega F, T_\omega G\rangle$$

IF $F = G$

$$\operatorname{Im}\langle T_\omega F, F\rangle = \operatorname{Im}(\omega)\,\|T_\omega F\|^2$$

$$\Rightarrow \quad \operatorname{Im}(\omega)\,\|T_\omega F\|^2 \le \|T_\omega F\|\,\|F\|$$

$$\|T_\omega F\| \le \frac{1}{\operatorname{Im}(\omega)}\|F\|.$$ ∎

THM 6 LET $\mathcal{L}$ BE A <u>HILB.</u> SPACE OF ANALYTIC FUNCTIONS $F: \mathbb{C}^+ \to \mathbb{C}$ SO THAT.

① $\exists\, \omega_0 \in \mathbb{C}^+$ $\quad F \longrightarrow F(\omega_0)$ IS CONTINUOUS

② $T_\omega F \in \mathcal{H}$ $\quad \forall F \in \mathcal{H}$ AND $\omega \in \mathbb{C}^+$.

③ $$\frac{T_\alpha - T_\beta^*}{\alpha - \overline{\beta}} = T_\beta^* T_\alpha \qquad \forall \alpha, \beta \in \mathbb{C}^+.$$

THEN $\exists\, \varphi$ SO THAT $\mathcal{H} = \mathcal{H}(\varphi)$ ISOMETRICALLY.

PROOF: BY ③

$$\left\langle \frac{T_\alpha - T_\alpha^*}{\alpha - \overline{\alpha}} F, F \right\rangle = \langle T_\alpha^* T_\alpha F, F \rangle$$

$$\Rightarrow \quad \operatorname{Im} \langle T_\alpha F, F \rangle = \operatorname{Im} \alpha \, \|T_\alpha F\|^2$$

$$\Rightarrow \quad \|T_\alpha F\| \leq \frac{1}{\operatorname{Im} \alpha} \|F\|, \quad \text{SO} \quad T_\alpha : \mathcal{H}(\varphi) \to \mathcal{H}(\varphi)$$

IS BOUNDED.

$$F(\omega) = \left[F(z) - (\omega_0 - \omega) \frac{F(z) - F(\omega)}{z - \omega} \right]_{z = \omega_0} = \left[F - (\omega_0 - \omega) T_\omega \right]_{z = \omega_0}$$

SO $F \to F(\omega)$ IS A CONTINUOUS FUNCT. FOR EVERY ω.

BY RIESZ REPRESENTATION $\exists\, K_\omega(z) = K(\omega, z)$

$$F(\omega) = \langle F, K_\omega \rangle_{\mathcal{H}}$$

WE ARE AIMING

$$K(\omega, z) = \frac{\overline{\varphi(\omega)} + \varphi(z)}{\pi i (\overline{\omega} - z)}$$

EXERCISE: SHOW $T_\alpha^* = T_{\overline{\alpha}}$ FROM ③.

$$\langle F, T_\beta K_\alpha \rangle = \langle T_\beta^* F, K_\alpha \rangle = T_\beta^* F(\alpha)$$

$$= \frac{F(\alpha) - F(\overline{\beta})}{\alpha - \overline{\beta}} = \left\langle F, \frac{K_\alpha - K_{\overline{\beta}}}{\overline{\alpha} - \beta} \right\rangle$$

$$\Rightarrow \frac{K_\alpha(z) - K_\alpha(\beta)}{z - \beta} = \frac{K_\alpha(z) - K_{\bar\beta}(z)}{\bar\alpha - \beta}. \quad (\bullet)$$

$$K(\alpha,\beta) = \overline{K(\beta,\alpha)}$$

$$K(\alpha,\beta) = \langle K_\alpha, K_\beta \rangle = \overline{\langle K_\beta, K_\alpha \rangle} = \overline{K(\beta,\alpha)} \quad (\bullet\bullet)$$

USING $(\bullet) + (\bullet\bullet)$ WE CAN DEDUCE

$$(\bullet\bullet\bullet)\ \pi i(\bar\alpha - z)K(\alpha,z) = \pi i(\bar\alpha - \beta)K(\alpha,\beta) + \pi i(\beta - z)K(\bar\beta, z)$$

SO NOW DEFINE

$$\varphi(z) = \pi i(\alpha_0 - z)K(\bar\alpha_0, z) + \underbrace{\frac{\pi i}{2}(\bar\alpha_0 - \alpha_0)\overbrace{K(\alpha_0,\alpha_0)}^{=\langle K_{\alpha_0}, K_{\alpha_0}\rangle \geq 0}}_{\geq 0}$$

USING $(\bullet\bullet\bullet)$ WE CAN SHOW

$$K(w,z) = \frac{\overline{\varphi(w)} + \varphi(z)}{\pi i(\bar w - z)}.$$

$$K(w,w) \geq 0$$

$$\hookrightarrow \quad = \frac{\operatorname{Re}\varphi(w)}{\pi \operatorname{Im}(w)} \Rightarrow \operatorname{Re}\varphi \geq 0 \text{ IN } \mathbb{C}^+.$$

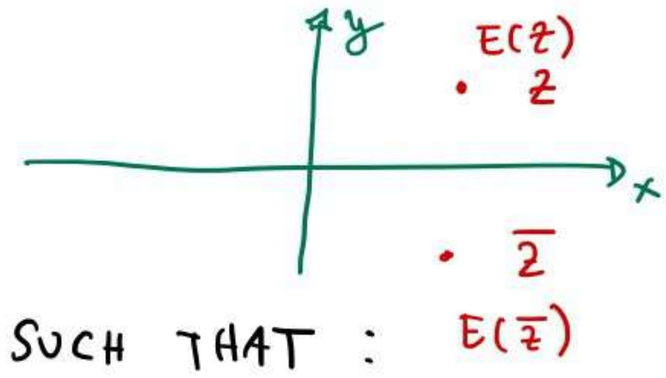
WE CAN DEFINE $\mathcal{H}(\varphi)$ AND WE NOTE THAT $\mathcal{H}(\varphi)$ AND $\mathcal{H}$ HAVE THE <u>SAME</u> KERNEL.

(i) SPAN$\{K_w : w \in \mathbb{C}^+\}$ IS DENSE IN $\mathcal{H}(\varphi)$ OR $\mathcal{H}$

(ii) $\| \ \|_{\mathcal{H}}$ AND $\| \ \|_{\mathcal{H}(\varphi)}$ COINCIDE IN SPAN$\{K_w\}$.

$$\Rightarrow \mathcal{H}(\varphi) = \mathcal{H}. \qquad \blacksquare$$

CLASS 4 (PÓLYA CLASS)



$$\mathcal{P} = \left\{ E : \mathbb{C} \to \mathbb{C} \;\middle|\; \begin{array}{l} E \text{ ENTIRE AND SUCH THAT :} \\ ① \quad E(z) \neq 0 \quad \forall z \in \mathbb{C}^+ \\ ② \quad |E(\bar{z})| \leq |E(z)| \quad \forall z \in \mathbb{C}^+ \\ ③ \quad |E(x+iy)| \text{ is NONDECREASING FOR } y>0 \end{array} \right\}$$

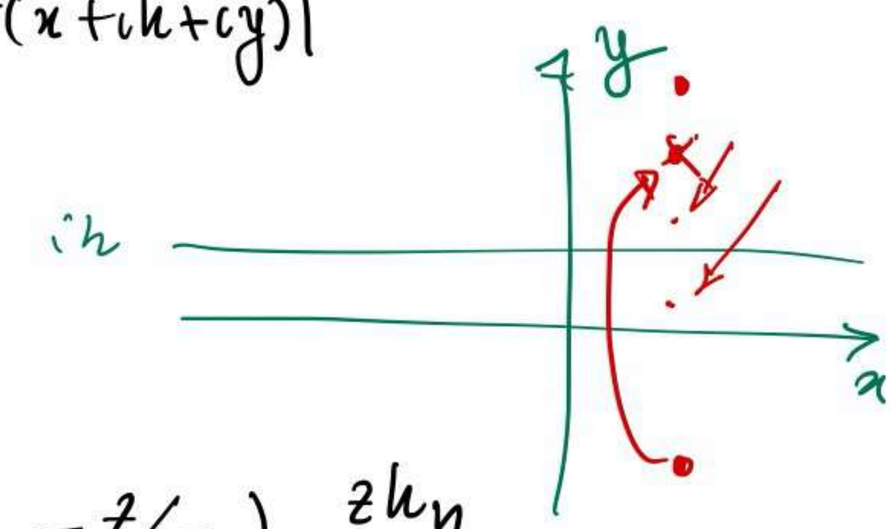
REMARK 1: $\mathcal{P}$ IS CLOSED BY PRODUCT AND UNIF. CONVERGENCE.

THAT IS $E_n \in \mathcal{P}$ AND $E_n \to E$ UNIF. IN COMPACT SETS OF $\mathbb{C}$.

THEN $E \in \mathcal{P}$. $\quad E \neq 0$.

REMARK 2 ② & ③ $\iff |E(\bar{z}+ih)| \leq |E(z+ih)| \quad \forall h \geq 0, \ \forall z \in \mathbb{C}^+$.

$\iff |E(x+ih-iy)| \leq |E(x+ih+iy)|$



THM 1: IF $E \in \mathcal{P}$ THEN

$$E(z) = E^{(r)}(0)\, \frac{z^r}{r!}\, e^{-az^2-ibz} \prod_{n=1}^{\infty} \left(1 - z/\bar{z}_n\right) e^{zh_n}$$

r = MULT. OF ZERO AT ORIGIN, $a \geq 0$, $\operatorname{Re} b \geq 0$, $\bar{z}_n = \text{zeros}(E) \setminus \{0\}$

AND $h_n = \dfrac{x_n}{x_n^2 + y_n^2}$ $\quad (z_n = x_n + iy_n,\ y_n \geq 0)$. WHERE

$$(*) \quad \sum_{n=1}^{\infty} \frac{1+y_n}{x_n^2+y_n^2} < \infty.$$

PROOF: PROB. 9: IF $(*)$ IS TRUE THEN $P_\infty(z) = \prod_{n=1}^{\infty} \left(1 - z/\bar{z}_n\right) e^{zh_n} \in \mathcal{P}$.

THAT IS, $P_n(z) = \prod_{k=1}^{n} \left(1 - z/\bar{z}_k\right) e^{zh_k} \to P_\infty(z) \in \mathcal{P}$. $(P_n \in \mathcal{P})$

LEMMA 1: IF $E \in \mathcal{P}$ AND $E(\bar{w}) = 0$. THEN $\dfrac{E(z)}{z-\bar{w}} \in \mathcal{P}$.

PROOF: WE KNOW THAT $|E(x+ih-iy)| \leq |E(x+ih+iy)|$. $F(z) = \dfrac{E(z)}{z-\bar{w}}$.

$$\left| \frac{F(\bar{z}+ih)}{F(z+ih)} \right| = \frac{|E(\bar{z}+ih)/(\bar{z}+ih-\bar{w})|}{|E(z+ih)/(z+ih-\bar{w})|} \leq \left| \frac{z+ih-\bar{w}}{\bar{z}+ih-\bar{w}} \right| = \frac{O(|z|)}{O(|z|)}$$

BUT NOTE THAT AT $z=x$ IS EQUAL TO 1.

$$\Longrightarrow \quad |F(\bar{z}+ih)| \leq |F(z+ih)| \quad \forall z \in \mathbb{C}^+.$$

LEMMA 2: IF $E \in \mathcal{P}$ AND E HAS NO ZEROS THEN

$$E(z) = e^{-az^2 - ibz + c}, \quad a \geq 0, \quad \operatorname{Re} b \geq 0.$$

PROOF: $E(z) = e^{F(z)} \Longrightarrow \operatorname{Re} \overline{F(\bar{z}+ih)} \leq \operatorname{Re} F(z+ih)$

$$G_h(z) = F(z+ih) - \overline{F(\bar{z}+ih)} \Rightarrow \operatorname{Re} G_h \geq 0 \quad \text{FOR } y \geq 0 \qquad z \in \mathbb{C}^+, \; h \geq 0.$$

$$\operatorname{Re} G_h(x) = 0 \quad \text{FOR } x \in \mathbb{R}.$$

PROB 3 $\Longrightarrow$ $G_h(z) = -ipz + ic$, $p \geq 0$.

$$\partial_z^2 G_h(z) = 0 \underset{y=0}{\Longrightarrow} F''(x+ih) = \overline{F''(x+ih)}$$

$$\Longrightarrow \operatorname{Im} F''(x+ih) \equiv 0 \Longrightarrow F''(z) = \text{REAL CONST.}$$

$$\Longrightarrow F(z) = -az^2 - ibz + c$$

$$G_h(z) = 4ahy + 2y \operatorname{Re} b \geq 0 \quad \forall y, h \geq 0.$$

$$\Longrightarrow a \geq 0, \quad \operatorname{Re} b \geq 0.$$

NOW WE CAN CONTINUE WITH THE PROOF OF THM 7.

I CAN ASSUME: $E(0) = 1$. $P_n(z) = \prod_{k=1}^{n} (1 - z/z_k) e^{z h_k} \in P$

$F_n = E/P_n \in P$. By $\boxed{\text{PROB. 9}}$ $P_n \to P_\infty \in P$

$\boxed{\text{LEMMA 1}}$

$\Rightarrow F_n \to F_\infty = E/P_\infty \in P$ BUT F_∞ HAS NO ZEROS.

$\overset{\text{LEMMA 2}}{\Longrightarrow} F_\infty = e^{-az^2 - ibz}$, $a \geq 0$ AND $\operatorname{Re} b \geq 0$.

WHY I CAN APPLY PROB 9:

$$\partial_y \log |P_n(x+iy)| = \operatorname{Re}\left(\frac{i P_n'(z)}{P_n(z)}\right) = \sum_{k=1}^{n} \frac{y + y_k}{(x-x_k)^2 + (y+y_k)^2}$$

$$E = P_n F_n \underset{\boxed{y>0}}{\Longrightarrow} \partial_y \log |E(x+iy)| = \partial_y \log |P_n(x+iy)| + \underbrace{\partial_y \log |F_n(x+iy)|}_{\geq 0}$$

$$\Rightarrow \partial_y \log |E(x+iy)| \Big|_{z=1} \geq \sum_{k=1}^{n} \frac{1 + y_k}{x_k^2 + (y_k+1)^2} \Rightarrow \sum_{k=1}^{\infty} \frac{1+y_k}{x_k^2 + y_k^2} < \infty.$$

SHOW IF $|x_k| \geq 1$ OR $y_k \geq 1 \Rightarrow x_k^2 + (y_k+1)^2 \leq 3(x_k^2 + y_k^2)$

PROBLEM 18: SHOW $\cos z \in P$ AND USE ITS FACTORIZATION TO SHOW $\pi^2/8 = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$.

SOL: $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$ $\quad \partial_y \log |\cos(z)| = \dfrac{e^{2y} - e^{-2y}}{|e^{iz} + e^{-iz}|^2} > 0$ FOR $y > 0$.

$\cos(\bar{z}) = \overline{\cos(z)}$

$\Rightarrow \cos(z) \in P$

$$\cos(z) = \prod_{n \in \mathbb{Z}} \left(1 - \frac{z}{\pi(2n+1)\frac{1}{2}}\right) e^{z/\frac{\pi}{2}(2n+1)} \; e^{-az^2 - ibz}$$

$$= \prod_{n\geq 0} \left(1 - \frac{z^2}{\frac{\pi^2}{4}(2n+1)^2}\right) e^{-az^2 - ibz}$$

$$0 = \frac{\partial}{\partial z} \log \cos z \Big|_{z=0} = \left[\sum_{n\geq 0} \frac{2z}{\left(z^2 - \frac{\pi^2}{4}(2n+1)^2\right)} - 2az - ib\right]_{z=0}$$

$$= -ib \qquad \Longrightarrow \quad b = 0.$$

$$\partial_y \log|\cos z| \geq \partial_y \log e^{-a(x^2-y^2)} = 2ay$$

$$\| $$

$$\mathrm{Re}\, -i\,\frac{\sin(z)}{\cos(z)}, \quad \text{if} \quad z = iy \quad y \to \infty. \quad \Longrightarrow \quad a = 0.$$

$$\Longrightarrow \quad \cos(z) = \prod_{\substack{n\geq 0 \\ c = \frac{\pi}{2}(2n+1)}} \left(1 - z^2/c^2\right)$$

$$\partial_z^2 \log \cos z = -\sum_{\substack{n\geq 0 \\ c = \frac{\pi}{2}(2n+1)}} \frac{2(z^2+c^2)}{(z^2-c^2)^2}$$

$$\| $$

$$-1 - \frac{\sin^2 z}{\cos z}$$

EVALUATE AT $z = 0$

$$1 = \sum_{n\geq 0} \frac{2}{\frac{\pi^2}{4}(2n+1)^2}$$

$$\Longrightarrow \quad \boxed{\sum_{n\geq 0} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}}$$

$\blacksquare$

CLASS 5 (BOUNDED TYPE)

DEF: AN ANALYTIC FUNCTION $F : \Omega \to \mathbb{C}$ IS OF BOUNDED TYPE IF $F = P/Q$ AND $P, Q : \Omega \to \mathbb{C}$ ARE ANALYTIC AND BOUNDED.

($\exists Q$ BOUNDED AND ANALYTIC SO THAT $QF = P$ IS ALSO BOUNDED) $BT(\Omega)$

$BT = BT(\mathbb{C}^+)$

PROBLEMS OF THIS WEEK: 11, 12, 13, 16, 17, 19, 24

PROB 20: $F \in BT$ IF $\operatorname{Re} F \geq 0$ ON $\mathbb{C}^+$.

PROB 21: BT IS CLOSED BY $+$ AND $\times$. ✓

PROB 22: POLYNOMIALS $\subset BT$.

SOL. OF 20: $1 + F = 1/(1/(1+F))$, $P = 1$, $Q = \frac{1}{1+F}$ $\quad |Q| \leq \frac{1}{|\operatorname{Re}(1+F)|} \leq 1$. ∎

SOL. OF 22: $-iz \in BT$? $\operatorname{Re} -iz = y \geq 0$, YES!

PROB 23: LET $z_n \in \mathbb{C}^+$, $z_n = x_n + iy_n$, $y_n > 0$ AND ASSUME

$$\sum_n \frac{y_n}{x_n^2 + y_n^2} < \infty.$$

THEN THE BLASCHKE PRODUCT

$$B(z) = \prod_n \left(\frac{1 - z/z_n}{1 - z/\overline{z_n}} \right)$$

SATISFIES:

1. CONVERGES UNIF. IN COMPACT SETS OF $\mathbb{C} \setminus \{\overline{z_n}\}_{n=1}^{\infty}$.
2. $B(z) B^*(z) = 1$ $\quad (B^*(z) = \overline{B(\bar{z})})$
3. $|B(z)| < 1$ $\quad z \in \mathbb{C}^+$.

SOLUTION: (2) IS TRIVIAL. (3) $\left| \frac{1 - z/z_n}{1 - z/\overline{z_n}} \right|^2 = \frac{(x - x_n)^2 + (y - y_n)^2}{(x - x_n)^2 + (y + y_n)^2} < 1$

FIRST WE USE $1 + x \leq e^x$ FOR $x \geq 0$.

$$1 + \left| \frac{1 - z/z_n}{1 - z/\overline{z_n}} - 1 \right| \leq \exp\left(\left| \frac{1 - z/z_n}{1 - z/\overline{z_n}} - 1 \right| \right)$$

$$= \exp\left(\frac{2}{|1/z - 1/\bar{z}_n|} \frac{y_n}{x_n^2+y_n^2} \right) \leq \exp\left(\frac{2}{\delta} \frac{y_n}{x_n^2+y_n^2} \right)$$

IF $|1/z - 1/\bar{z}_n| \geq \delta$ ($z \in K \subset \mathbb{C} \setminus \{\bar{z}_n\}_{n=1}^{\infty}$, K COMPACT).

PROB 8: $1 + |ab-1| \leq (1+|a-1|)(1+|b-1|)$ $\forall a, b \in \mathbb{C}$.

$$\boxed{1 + |a_1 \cdots a_n - 1| \leq \prod_{j=1}^{n} (1 + |a_j - 1|)}.$$

$$B_n(z) = \prod_{j=1}^{n} \left(\frac{1 - z/z_n}{1 - z/\bar{z}_n} \right)$$

IF $r > n$

$$|B_r(z) - B_n(z)| = \left| \prod_{j=1}^{n} \left(\frac{1 - z/z_j}{1 - z/\bar{z}_j} \right) \right| \left| \prod_{j=n+1}^{r} \frac{1 - z/z_j}{1 - z/\bar{z}_j} - 1 \right|$$

$$\leq |\cdots| \left\{ \prod_{j=n+1}^{r} \left(1 + \left| \frac{1 - z/z_j}{1 - z/\bar{z}_j} - 1 \right| \right) - 1 \right\}$$

$|a| \leq 1 + |a-1|$

$$\leq \prod_{j=1}^{n} \left(1 + \left| \frac{1 - z/z_j}{1 - z/\bar{z}_j} - 1 \right| \right) \left\{ \prod_{j=n+1}^{r} (\cdots) - 1 \right\}$$

$$\leq \prod_{j=1}^{n} \exp\left(\frac{2}{\delta} \frac{y_j}{x_j^2+y_j^2} \right) \left\{ \prod_{j=n+1}^{r} \exp\left(\frac{2}{\delta} \frac{y_j}{x_j^2+y_j^2} \right) - 1 \right\}$$

$$= \exp\left(\frac{2}{\delta} \sum_{j=1}^{r} \frac{y_j}{x_j^2+y_j^2} \right) - \exp\left(\frac{2}{\delta} \sum_{j=1}^{n} \frac{y_j^2}{x_j^2+y_j^2} \right)$$

$$\leq \varepsilon, \quad \text{IF } n, r \text{ ARE LARGE!}$$

∎

THM 8 LET $F \in BT$ AND SUCH THAT THE ORIGIN IS NOT A LIMIT OF ZEROS (F). THEN IF ZEROS $(F) = \{z_n = x_n + iy_n\}$ $(y_n > 0)$ WE HAVE

$$\sum_{n \geq 1} \frac{y_n}{x_n^2 + y_n^2} < \infty.$$

THEN $F = BG$ WHERE

$$B(z) = \prod \frac{1 - z/z_n}{1 - z/\bar{z}_n}$$

AND $G \in BT$ HAS NO ZEROS.

PROOF: WE CAN ASSUME $\{z_n\}$ IS INFINITE. $\exists Q$ BOUNDED SO THAT $QF = P$ IS BOUNDED. $M = \sup_{z \in \mathbb{C}^+} |P(z)| < \infty$.

$$(B_0 = 1) \qquad B_n(z) = \prod_{j=1}^{n} \frac{1 - z/z_j}{1 - z/\bar{z}_j}, \qquad G_n = F/B_n.$$

$B_n, G_n \in BT$. WE CLAIM THAT $|Q G_n| \leq M$ IN $\mathbb{C}^+$ $\forall n$.

$F = B_n G_n$, SO $|Q G_0| \leq M$ IN $\mathbb{C}^+$. ASSUME BY INDUCTION THAT $|Q G_{n-1}| \leq M$ IN $\mathbb{C}^+$.

$$|Q G_n(z)| = \left| Q(z) G_{n-1}(z) \frac{1 - z/\bar{z}_n}{1 - z/z_n} \right| \leq M \left| \frac{1 - z/\bar{z}_n}{1 - z/z_n} \right| = M \frac{O(|z|)}{O(|z|)}$$

ASSUME $z = x + iy$ AND $y = h \in (0, y_n)$.

$$\left| \frac{1 - z/\bar{z}_n}{1 - z/z_n} \right|^2 = 1 + \frac{4 h y_n}{(x - x_n)^2 + (h - y_n)^2} \leq \frac{y_n + h}{y_n - h}$$

BY THE PL MAXIMUM PRINCIPLE $\Longrightarrow$ $|Q G_n(z)| \leq M \sqrt{\frac{y_n + h}{y_n - h}}$ $\forall y \geq h$.

THEN $h \to 0 \Longrightarrow |Q G_n| \leq M$ IN $\mathbb{C}^+$.

NOW $QF = Q B_n G_n \Longrightarrow |QF| \leq M |B_n|$

$$-\log |QF(z)| \geq -\log M + \sum_{j=1}^{n} \log \left| \frac{1 - z/\bar{z}_j}{1 - z/z_j} \right|$$

$$\underset{z = iy}{=} -\log M + \frac{1}{2} \sum_{j=1}^{n} \log \left(1 + \frac{4 y y_j}{x_j^2 + (y - y_j)^2} \right)$$

$y \leq \frac{3}{4} \delta_0 \quad \frac{4}{3} y \leq \delta_0$

$\log(1+x) \geq \frac{x}{2}$ IF $0 \leq x \leq \delta_0$.

$$-\log |QF(z)| \geq -\log M + y \sum_{j=1}^{n} \frac{y_j}{x_j^2 + y_j^2}$$

$x_j^2 + y_j^2 \geq \delta^2$ $(\delta = \frac{3}{4}\delta_0)$

IF $0 < y < \delta$

$x_j^2 + (y - y_j)^2 \leq 3(x_j^2 + y_j^2)$

$$\frac{4 y_j y}{x_j^2 + (y-y_j)^2} \geq \frac{4}{3} y \frac{y_j}{x_j^2 + y_j^2}$$

AND $\frac{y_j}{x_j^2 + y_j^2} \leq 1$.

Fix $0 < y \leq \frac{3}{4}\delta_0$ SO THAT $|QF(iy)| > 0$ AND TAKE $n \to \infty$

$$\Longrightarrow \sum_{j=1}^{\infty} \frac{y_j}{x_j^2 + y_j^2} < \infty.$$

SO $B_n \to B$, $G_n \to G$ AND $|QG| \leq M$ IN $\mathbb{C}^+$

SO $G \in BT$ AND HAS NO ZEROS. $F = B_n G_n \to BG$ ∎.

CLASS 6 (NEVALINNA'S FACTORIZATION ⊕ MEAN TYPE)

THM 9: LET $F : \mathbb{C}^+ \to \mathbb{C}$ ANALYTIC AND $F(z) \neq 0$ IN $\mathbb{C}^+ \cap \{|z| < r\}$.

THEN $F \in BT$ IFF

$$F(z) = B(z)\, e^{G_1 - G_2}$$

AND $\operatorname{Re} G_j \geq 0$ IN $\mathbb{C}^+$, $j = 1, 2$.

REMARK: $G_j(z) = i c_j - i p_j z + \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{1+tz}{t-z} \frac{d\sigma_j(t)}{1+t^2}$; $p_j, c_j \in \mathbb{R}$, $p_j \geq 0$

$$\operatorname{Re} G_j(z) = p_j y + \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{d\sigma_j(t)}{(t-x)^2 + y^2}$$

$h = p_1 - p_2$, $G = i(c_1 - c_2) + \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{1+tz}{t-z} \frac{d(\sigma_1 - \sigma_2)}{1+t^2}$

$\sigma_1 - \sigma_2 = \mu$. (*) $\int_{-\infty}^{\infty} \frac{|d\mu(t)|}{1+t^2} = \sup_P \frac{|\mu(t_k) - \mu(t_{k-1})|}{1 + \max(t_k, t_{k-1})^2}$

$$\leq \int_{-\infty}^{\infty} \frac{d\sigma_1(t)}{1+t^2} + \int_{-\infty}^{\infty} \frac{d\sigma_2(t)}{1+t^2}$$

CONVERSELY, LEMMA 3 P. 22 DB SHOWS THAT WHENEVER $\mu : \mathbb{R} \to \mathbb{R}$ SO THAT (*) IS FINITE THEN $\mu = \nu_1 - \nu_2$ $\nu_j : \mathbb{R} \to \mathbb{R}$ IS NON DEC. AND (*) IS FINITE FOR EACH ν_j.

$$(N) \quad F = B\, e^{-ihz}\, e^{G}, \qquad \operatorname{Re} G = \frac{y}{\pi} \int \frac{d\mu(t)}{(t-x)^2 + y^2}$$

AND (*) IS FINITE.

- IF $\mu \in C^1$ $\quad |d\mu(t)| = |\mu'(t)|\, dt$.

PROOF OF THM 9: BY THM. 8 $\quad B(z) = \prod_n \frac{1 - z/z_n}{1 - z/\bar{z}_n}$ IS CONV.

WHERE $\{z_n = x_n + iy_n\} = \text{ZEROS}(F)$ AND $F = B \cdot L$, $L \in BT$. AND HAS NO ZEROS.

BY PROB. 24 $\quad L = P/Q$, $\quad |P|, |Q| \leq 1$

Q NO ZEROS.

$$P = e^{-G_2}, \quad Q = e^{-G_1} \quad \text{AND} \quad \operatorname{Re} G_j \geq 0.$$

THE CONVERSE IS OBVIOUS! ∎

THM 10: LET $F \in BT$ AND ZEROS(F) DO NOT ACCUMULATE AT $z = 0$. THEN IN $F = B\, e^{-ihz}\, e^{G}$ (N) WE HAVE

$$h = \overline{\lim_{y \to \infty}} \frac{\log |F(iy)|}{y}$$

$h = \nu(F)$ MEAN TYPE OF F

$$h = \frac{2}{\pi} \lim_{r \to \infty} \frac{1}{r} \int_0^{\pi} \log |F(re^{i\theta})| \sin\theta \, d\theta$$

LEMMA 4: IF $F : \mathbb{C}^+ \to \mathbb{C}$ ANALYTIC, $|F| \leq 1$, $F \not\equiv 0$ THEN

$$H = \underline{\lim_{r \to \infty}} \frac{2}{\pi} \frac{1}{r} \int_0^{\pi} \log |F(re^{i\theta})| \sin\theta \, d\theta$$

IS FINITE $(0 \geq H > -\infty)$ AND $|F(z)\, e^{iHz}| \leq 1$ IN $\mathbb{C}^+$.

PROOF: WE ASSUME F IS CONTINUOUS IN $\mathbb{R}$. (OTHERWISE APPLY THE ε-TRICK: USE $F(z+i\varepsilon)$ INSTEAD AND TAKE $\varepsilon \searrow 0$).

LET $p>0$ $\quad F_p(z) = F(z)\, e^{-ipz}$. BY THE PROOF OF PL PRINCIPLE (THM 1) WE HAVE

$$\log |F_p(z)| \leq \frac{a^2-|z|^2}{2\pi} \int_0^\pi \frac{4ay \log^+ |F_p(ae^{i\theta})| \sin\theta}{|ae^{i\theta}-z|^2\, |ae^{-i\theta}-z|^2}\, d\theta$$

BY PROB. 25:

$$y = \frac{a^2-|z|^2}{2\pi} \int_0^\pi \frac{4a^2 y \sin^2\theta}{|\cdots|^2\, |\cdots|^2}\, d\theta \qquad |z|<a, \quad z \in \mathbb{C}^+$$

$$\to \log |F(z)| + py$$

$$\Rightarrow \log |F(z)| \leq \frac{a^2-|z|^2}{2\pi} \int_0^\pi \frac{4ay\left(\log^+ |F_p(ae^{i\theta})| - pa\sin\theta\right)\sin\theta\, d\theta}{|\cdots|^2\, |\cdots|^2}$$

TAKING $p \to 0$ WE GET

$$\log |F(z)| \leq \frac{a^2-|z|^2}{2\pi} \int_0^\pi \frac{4ay \log |F(ae^{i\theta})| \sin\theta\, d\theta}{|\cdots|^2\, |\cdots|^2}$$

$a \to \infty$

$$\log |F(z)| \leq y\, \frac{2}{\pi} \overline{\lim_{a\to\infty}} \frac{1}{a} \int_0^\pi \log |F(ae^{i\theta})| \sin\theta\, d\theta$$

$$= Hy \iff |F(z)\, e^{iHz}| \leq 1 \text{ IN } \mathbb{C}^+. \blacksquare$$

LEMMA 5: LET $F: \mathbb{C}^+ \to \mathbb{C}$ ANALYTIC, $|F| \leq 1$, $F \not\equiv 0$. IF

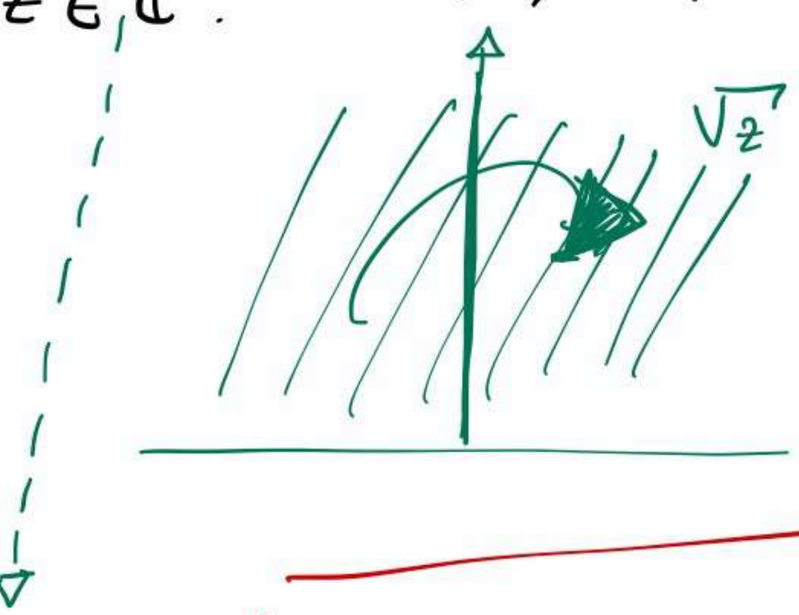
$$H = \overline{\lim_{y\to\infty}} \frac{\log |F(iy)|}{y}$$

THEN $0 \geq H > -\infty$ AND $|F(z)\, e^{iHz}| \leq 1$.

PROOF: WE CAN ASSUME F IS CONTINUOUS $\overline{\mathbb{C}^+}$ AND $H < 0$.

TAKE $H < b < 0$. THEN $|F(z)\, e^{ibz}| \leq M$, FOR $z = iy$, $y \geq 0$.

LET $G(z) = \frac{1}{M} F(\sqrt{z})\, e^{ib\sqrt{z}}$, $z \in \mathbb{C}^+$. $M \geq 1$.



G IS ANALYTIC IN $\mathbb{C}^+$, CONTINUOUS ON $\mathbb{R}$

$$|G(x)| = \left| \frac{1}{M} F(\sqrt{x})\, e^{ib\sqrt{x}} \right| \leq 1 \quad x > 0$$

IF $x > 0$

$$|G(-x)| = \left| \frac{1}{M} F(i\sqrt{x})\, e^{-b\sqrt{x}} \right| \leq 1$$

$$\boxed{\begin{aligned} &|G(re^{i\theta})| \leq \frac{1}{M} e^{-b\sqrt{r}} \\ &\log|G(re^{i\theta})| = O(\sqrt{r}) \end{aligned}}$$

$$\varliminf_{r \nearrow \infty} \frac{1}{r} \int_0^{\pi} \log^+ |G(re^{i\theta})| \sin\theta \, d\theta = 0.$$

THEN BY PL PRINCIPLE $|G| \leq 1 \Rightarrow |F(z)e^{ibz}| \leq M$ IF $z \in \mathbb{C}^+ \cap \{x > 0\}$

DO THE SAME FOR $\overline{F(-\bar{z})} = F^*(-z) \Rightarrow |F(z)e^{ibz}| \leq M$ IF $z \in \mathbb{C}^+ \cap \{x < 0\}$

$$\Rightarrow |F(z)\, e^{ibz}| \leq M \text{ IN } \mathbb{C}^+.$$

$$\overset{P.L.}{\Longrightarrow} |F(z)\, e^{ibz}| \leq 1 \text{ IN } \mathbb{C}^+.$$ THEN TAKE $b \searrow H$. ∎

LEMMA 6: IF B IS A BLASCHKE PRODUCT AND $|B(z)e^{iHz}| \leq 1$ THEN $H \geq 0$.

PROOF: $$B(z) = \prod_{k>n} \frac{1 - z/z_k}{1 - z/\bar{z}_k} \prod_{k=1}^{n} \frac{1 - z/z_k}{1 - z/\bar{z}_k} = B_n(z) \prod_{k=1}^{n} (\cdots)$$

SHOW BY INDUCTION THAT $|B_n(z)\, e^{iHz}| \le 1$ $\forall n$ (ε TRICK + P.L.) ∎

OBS: $$0 = \overline{\lim_{y\to\infty}} \frac{\log|B(iy)|}{y} = \lim_{r\to\infty} \frac{2}{\pi}\frac{1}{r}\int_0^{\pi} \log|B(re^{i\theta})|\sin\theta\, d\theta$$

FOR INSTANCE, $$H = \overline{\lim_{y\to\infty}} \frac{\log|B(iy)|}{y}.$$ LEMMA 5 $\Rightarrow |B(z)e^{iHz}| \le 1$

BUT BY LEMMA 6 $H \ge 0 \Rightarrow H = 0$.

PROOF OF THM 10: $$F(z) = B(z)\, e^{-ihz}\, e^{G}$$

$$\operatorname{Re} G = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{d\mu(t)}{(t-x)^2+y^2}$$

$$e^{G} = e^{-U_1}/e^{-U_2}$$

$\operatorname{Re} U_j \ge 0$.

$$\overline{\lim_{y\to\infty}} \frac{\log|F(iy)|}{y} = h. \quad ✓$$

$$\lim_{r\to\infty} \frac{2}{\pi}\frac{1}{r}\int_0^{\pi} \log|F(re^{i\theta})|\sin\theta\, d\theta = h + \lim_{r\to\infty}(\dots).$$

IS ZERO

$$H_j = \underline{\lim_{r\to\infty}} \frac{2}{\pi}\frac{1}{r}\int_0^{\pi} \log|e^{-U_j(re^{i\theta})}|\sin\theta\, d\theta > -\infty$$

LEMMA 4

AND $|e^{-U_j + iH_j z}| \le 1$ IN $\mathbb{C}^+$.

$$0 \ge \log|e^{-U_j + iH_j z}| = y\left(-H_j - \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{d\sigma_j(t)}{(t-x)^2+y^2}\right)$$

$y>0$

$$\Rightarrow 0 \le -H_j \le \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{d\sigma_j(t)}{(t-x)^2+y^2} \to 0 \text{ IF } y\to+\infty.$$

SO $H_j = 0$. THUS $$\lim_{r\to\infty}\frac{2}{\pi}\frac{1}{r}\int_0^{\pi}\log|e^{G(re^{i\theta})}|\sin\theta\, d\theta = 0.$$ ∎

CLASS 7 (HARDY SPACES)

RECALL: IF $F : \Omega \to \mathbb{C}$ IS ANALYTIC AND Ω SIMPLY CONNECTED DOMAIN

$$\left(\partial_x F = \partial_{iy} F \quad \boxed{\text{C.R.} - \text{EQ'S}} \quad \begin{cases} \partial_x u = \partial_y v \\ \partial_y u = -\partial_x v \end{cases} \quad F = u + iv \right)$$

THEN IS EASY TO SHOW THAT u AND v ARE HARMONIC FUNCTIONS.

$\partial_{xx} u + \partial_{yy} u = 0$, v IS CALLED THE HARMONIC CONJUGATE OF u.

CONVERSELY, IF $u : \Omega \to \mathbb{C}$ HARMONIC YOU CAN FIND v SATISFYING THE C.R. EQ'S AND $F = u + iv$ AND F WILL BE ANALYTIC.

$D = \{ z \in \mathbb{C} : |z| < 1 \}$. IF F IS ANALYTIC IN D AND $F \in C(\overline{D})$

$$F(0) = \frac{1}{2\pi i} \int_{\partial D} \frac{F(z)}{z} \, dz \implies u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta}) \, d\theta$$

THEN USING A MÖBIUS TRANSFORM $\tau(z) = \dfrac{z - z_0}{1 - \overline{z_0} z}$ FOR SOME $z_0 \in D$

IF $\tau(e^{i\theta}) = e^{i\varphi} \implies \dfrac{d\varphi}{d\theta} = \dfrac{1 - r^2}{1 - 2r\cos(\theta - \theta_0) + r^2} = P_r(\theta - \theta_0)$; $z_0 = re^{i\theta_0}$

$$\implies u(z_0) = u \circ \tau^{-1}(0) = \frac{1}{2\pi} \int_0^{2\pi} u \circ \tau^{-1}(e^{i\varphi}) \, d\varphi = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta}) P_r(\theta_0 - \theta) \, d\theta$$

$$= f * P_r(\theta_0)$$

WHERE $f(\theta) = u(e^{i\theta})$ AND $f * g(\theta_0) = \fint_0^{2\pi} f(\theta) g(\theta_0 - \theta) \, d\theta$ $\quad \boxed{\fint_0^{2\pi} = \frac{1}{2\pi} \int_0^{2\pi}}$

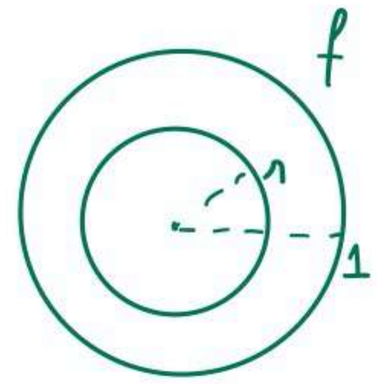
$$\implies \boxed{u(z_0) = f * P_r(\theta_0) \qquad z = re^{i\theta_0}.} \quad (i)$$

REMARK: (1) FORMULA (i) DEFINES A HARMONIC FUNCTION u WHENEVER $f \in L^1[0, 2\pi]$. $|P_r(\theta)| \leq \dfrac{1 + r}{1 - r}$ AND $P_r(\theta - \theta_0) = \operatorname{Re} \dfrac{e^{i\theta} + z_0}{e^{i\theta} - z_0}$

WHERE $z_0 = re^{i\theta_0}$, SO $P_r(\theta - \theta_0)$ IS HARM. IN z_0.

THM B

① IF $f \in L^p[0,2\pi]$ $(1 \le p < \infty)$ THEN $\| P_r * f - f \|_p \to 0$ WHEN $r \nearrow 1$.



② IF $d\mu$ IS A FINITE MEASURE ON $[0,2\pi]$ THEN

$P_r * \mu \xrightarrow{*} \mu$ WEAK-STAR TOPOLOGY, THAT IS,

$$\int_0^{2\pi} g(\theta)\, P_r * \mu(\theta)\, d\theta \longrightarrow \int_0^{2\pi} g(\theta)\, d\mu(\theta) \qquad \forall g \in C[0,2\pi].$$

③ IF $f \in L^p[0,2\pi]$ AND IS CONTINUOUS AT θ_0 THEN $P_r * f(\theta_0) \to f(\theta_0)$

④ IF $f \in C[0,2\pi]$ AND 2π-PERIODIC THEN $P_r * f \to f$ UNIFORMLY.

PROOF:
$$\left| P_r * f(\theta) - f(\theta) \right| \le \int_0^{2\pi} P_r(\varphi) \left| f(\theta-\varphi) - f(\theta) \right| d\varphi$$

$$\| P_r * f - f \|_{L^p_\theta} \le \int_0^{2\pi} P_r(\varphi) \underbrace{\| f(\theta-\varphi) - f(\theta) \|_{L^p_\theta}}_{\eta(\varphi)} d\varphi \qquad (1 \le p < \infty)$$

MINKOWSKI'S INEQ

$$\boxed{\int_0^{2\pi} P_r(\varphi)\, d\varphi = 1}$$

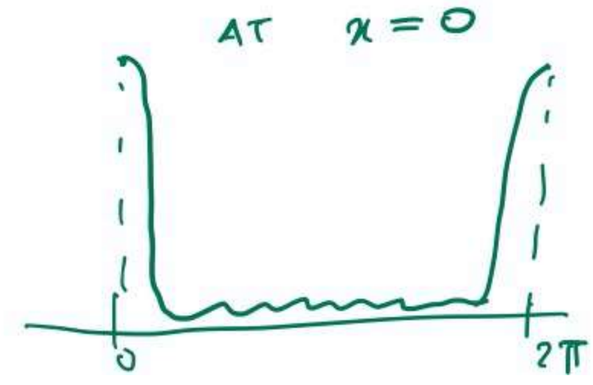
$$\| P_r * f - f \|_{L^\infty_\theta} \le \int_0^{2\pi} P_r(\varphi) \overbrace{\| f(\theta-\varphi) - f(\theta) \|_{L^\infty_\theta}}^{\eta(\varphi)} d\varphi \leftarrow \left(\begin{array}{l} \text{ASSUME} \\ f \in C[0,2\pi] \end{array} \right)$$

η CONTINUOUS AT $\varphi = 0$, BOUNDED, $\eta(0) = 0$, η IS 2π-PERIODIC, $\eta \ge 0$.

WHAT WE HAVE
$$\int_0^{2\pi} P_r(\varphi)\, \eta(\varphi)\, d\varphi \longrightarrow 0 \quad \text{AS} \quad r \nearrow 1$$

$$0 \le \int_0^{2\pi} P_r(\varphi)\, \eta(\varphi)\, d\varphi \le \int_0^{\delta} + \int_{2\pi-\delta}^{2\pi} P_r(\varphi)\, \eta(\varphi)\, d\varphi$$
$$+ M \int_{\delta}^{2\pi-\delta} P_r(\varphi)\, d\varphi$$

P_r CONCENTRATES AT $x = 0$



$$\leq \varepsilon + M \frac{1-r^2}{2rc\delta^2} \xrightarrow[r\to 1]{} \varepsilon$$

$$P_r(\varphi) = \frac{1-r^2}{1+r^2-2r\cos\varphi} \quad \text{IF} \quad \delta \leq \varphi \leq 2\pi-\delta \quad \text{THEN} \quad \cos\varphi \leq 1-c\delta^2$$

THEN $$P_r(\varphi) \leq \frac{1-r^2}{1+r^2-2r(1-c\delta^2)} = \frac{1-r^2}{(1-r)^2+2rc\delta^2} \leq \frac{1-r^2}{2rc\delta}.$$

THIS PROVES (1) AND (4). I LEAVE (2) AND (3) AS EXERCISES. ∎

THM C: LET $u: D \to \mathbb{C}$ BE HARMONIC. THEN

(1) $u = P_r * f(\theta)$ $(z = re^{i\theta})$ FOR SOME $f \in L^p[0,2\pi]$ IFF

$$\sup_{0<r<1} \left(\int_0^{2\pi} |u(re^{i\theta})|^p d\theta \right)^{1/p} < \infty. \qquad 1<p<\infty$$

(2) $u = P_r * \mu$ FOR SOME FINITE MEASURE μ IFF

$$\sup_{0<r<1} \int_0^{2\pi} |u(re^{i\theta})| d\theta < \infty.$$

(3) $u \geq 0$ IFF

$$u(z) = cy + P_y * \mu \quad ; \quad c \geq 0, \mu \geq 0 \text{ FINITE.}$$

PROOF: NECESSITY IS TRIVIAL $\|P_r * f\|_p \leq \|f\|_p$ OR $\|P_r * \mu\|_1 \leq \int_0^{2\pi} |d\mu(t)|$.

CONVERSELY, $$u(tz) = \int_0^{2\pi} u(te^{i\theta}) P_r(\theta-\theta_0) d\theta \qquad z = re^{i\theta_0}$$
$$= (u(te^{i\cdot}) * P_r)(\theta_0)$$

$\sup_{0<t<1} \|u(te^{i\theta})\|_{L^p_\theta} < \infty$. USING BANACH-ALAOGLU THM $\exists f \in L^p[0,2\pi]$ AND $t_n \nearrow 1$ SO THAT $u(t_n e^{i\theta}) \to f(\theta)$ WEAK-STAR TOPOLOGY.

$$\Longrightarrow u(z) = \int_0^{2\pi} f(\theta) P_r(\theta-\theta_0) d\theta,$$ THIS SOLVES (1).

I LEAVE ② AND ③ AS EXERCISES. ∎

REMARK: USING THE CONFORMAL MAP $w \in D \xrightarrow{\varphi} \frac{i(1-w)}{1+w} \in \mathbb{C}^+$.

$$\varphi(\partial D) = \mathbb{R} \cup \{+\infty\}, \quad \varphi(e^{i\theta}) = t \quad (\text{if } \theta \neq \pi)$$

$$\frac{1}{2\pi} P_r(\theta-\theta_0)\frac{d\theta}{dt} = \pi \frac{y}{(x-t)^2+y^2} = P_y(x-t), \quad z = \varphi(w), \quad w = r e^{i\theta_0}$$

THM. B': REPLACE IN THM B D BY $\mathbb{C}^+$, $L^p[0,2\pi]$ BY $L^p(\mathbb{R})$ AND IN ④ ASK f TO BE BOUNDED AND UNIF. CONTINUOUS. SO IF $f \in L^p(\mathbb{R})$ THEN $u(x,y) = f * P_y(x)$ IS HARMONIC AND $u(x,y) \xrightarrow{L^p} f(x)$ IF $y \searrow 0$ $(1 < p < \infty)$. IF $p=1$ THEN $u(x,y) \xrightarrow{\text{WEAK-STAR}} f(x)$ AS $y \searrow 0$, IF f IS BOUNDED AND UNIF. CONT. THEN $u(x,y) \to f(x)$ IN $L^\infty(\mathbb{R})$ AS $y \searrow 0$.

THM. C': LET u HARMONIC IN $\mathbb{C}^+$. THEN

① $u = f * P_y$ FOR $f \in L^p(\mathbb{R})$ IFF $\sup_{y>0} \|u(x,y)\|_{L^p_x} < \infty$
$(1 < p \leq \infty)$

② $u = f * \mu$ IFF $\sup_{y>0} \int |u(x,y)|\,dx < \infty$

③ $u \geq 0$ IFF $u(z) = py + P_y * \mu(x)$, $\mu \geq 0$ FINITE

HARDY SPACE: $(1 \leq p < \infty)$

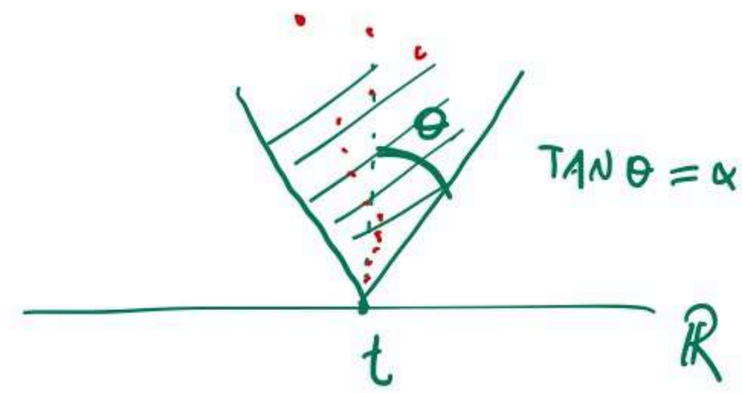
$$H^p(\mathbb{C}^+) = \left\{ f : \mathbb{C}^+ \to \mathbb{C} \;\middle|\; f \text{ ANALYTIC}, \|f\|_{H^p} = \sup_{y>0} \|f(x+iy)\|_{L^p_x} < \infty \right\}$$

$$H^p(D) = \left\{ f : D \to \mathbb{C} \;\middle|\; f \text{ ANALYTIC}, \|f\|_{H^p} = \sup_{0<r<1} \|f(re^{i\theta})\|_{L^p_\theta} < \infty \right\}$$

CLASS 8 (HARDY SPACES: PART 2)

THM D (THM 3.1) LET $1 \le p < \infty$ AND $f \in H^p(\mathbb{C}^+)$. FOR $\alpha > 0$

$$\mathcal{M}_\alpha f(t) = \sup_{z \in \Gamma_\alpha(t)} |f(z)|$$



WHERE $\Gamma_\alpha(t) = \{ z \in \mathbb{C}^+ : |x - t| < \alpha y \}$

THEN $\|\mathcal{M}_\alpha f\|_p \le C_\alpha \|f\|_{H^p}$ FOR SOME C_α. IN PARTICULAR, THE FOLLOWING LIMIT EXISTS a.e.

$$\lim_{\substack{z \to t \\ z \in \Gamma_\alpha(t)}} f(z) = f(t)$$

$f \in L^p(\mathbb{R})$, $\|f\|_{L^p} = \|f\|_{H^p}$ AND $f(x+iy) \xrightarrow[a.e.]{L^p_x} f(x)$ AS $y \searrow 0$.

THM E IF $1 \le p < \infty$, $f \in H^p$ AND $f(x) = \lim_{y \searrow 0} f(x+iy) \in L^p$

THEN

$$f(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(t)}{t - z}\, dt = \frac{y}{\pi} \int_{\mathbb{R}} \frac{f(t)}{(x-t)^2 + y^2}\, dt$$

WORKS $p = \infty$

PROOF: WE KNOW FROM THM. C' WE HAVE

$$f(z + i\varepsilon) = \frac{y}{\pi} \int_{\mathbb{R}} \frac{f(t + i\varepsilon)}{(x-t)^2 + y^2}\, dt \qquad (1 \le p \le \infty)$$

WHEN $y \searrow 0$ (BY THM D, B') WE HAVE

$$f(z) = \frac{y}{\pi} \int_{\mathbb{R}} \frac{f(t)}{(x-t)^2 + y^2}\, dt \qquad (1 \le p \le \infty).$$

$$= \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(t)}{t - z}\, dt - \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(t)}{t - \bar{z}}\, dt$$

$$= U + V$$

IF $p<\infty$ THEN U IS ANALYTIC, SO $f-U=V$ IS ANALYTIC

$\Rightarrow V = c^{te}$ AND (EXERCISE) THUS $V \equiv 0$. ∎

OBS 1: $\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(t)}{t-z}\,dt \equiv 0$ FOR $z \in \mathbb{C}^+$.

OBS 2: AN ANALOGUE OF THM'S D AND E HOLD FOR $H^p(D)$ $(1 \le p \le \infty)$

THM F: $H^p(\mathbb{C}^+)$ AND $H^p(D)$ ARE COMPLETE $(1 \le p \le \infty)$.

PROOF: I WILL SHOW ONLY FOR $H^p(D)$. $f(w) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(z)}{z-w}\,dz$, $|w|<1$

$$|f(w)| \le \|f\|_{H^p} \left\| \frac{1}{e^{i\theta}-w} \right\|_{L^p_\theta}$$

$\Rightarrow$ IF (f_n) IS A CAUCHY SEQ. IN H^p THEN (f_n) IS CAUCHY IN $L^\infty(K)$ FOR ANY COMPACT $K \subset D$. $\Rightarrow f_n \to f$ UNIF. IN COMPACT SETS OF D. IF $1 \le p < \infty$ THEN FATOU'S LEMMA

$$\|f(re^{i\theta}) - f_n(re^{i\theta})\|_{L^p_\theta} \le \varliminf_{m\to\infty} \|f_m(re^{i\theta}) - f_n(re^{i\theta})\|_{L^p_\theta}$$

$$\le \varliminf_{m\to\infty} \|f_m - f_n\|_{H^p}$$

$$\Rightarrow \|f - f_n\|_{H^p} \le \varliminf_{m\to\infty} \|f_m - f_n\|_{H^p} \to 0 \text{ AS } n\to\infty.$$

EXERCISE: DO $p=\infty$. ∎

OBS 3: $\|f(x+iy)\|_{L^p_x} \nearrow \|f(x)\|_{L^p_x}$ WHEN $y \searrow 0$. $f \in H^p(\mathbb{C}^+)$

$\|f(re^{i\theta})\|_{L^p_\theta} \nearrow \|f(e^{i\theta})\|_{L^p_\theta}$ WHEN $r \nearrow 1$ $f \in H^p(D)$

OBS 4: $H^p(D) = \overline{\mathbb{C}[z]}$. IF $f(z) = \sum_{n=0}^{\infty} a_n z^n$ THEN

$$\lim_{r\nearrow 1} \int_0^{2\pi} |f(re^{i\theta}) - f(e^{i\theta})|^p\,d\theta \to 0$$

$$\Longleftrightarrow \quad \sum_{n=0}^{\infty} a_n e^{i\theta n} r^n \xrightarrow[a.e.]{L^p} f(e^{i\theta}) \quad \text{WHEN } r \nearrow 1.$$

$$f(e^{i\theta}) \sim \sum_{n=0}^{\infty} a_n e^{i\theta n}$$

$$\forall \varepsilon > 0, \ \forall r < 1 \ \exists N_r \quad \sup_{|z| \le r} \left| \sum_{n=0}^{N} a_n z^n - f(z) \right| < \varepsilon.$$

$$\Longrightarrow \quad \overline{\lim_{r \nearrow 1}} \left(\int_0^{2\pi} \Big| \underbrace{\sum_{n=0}^{N_r} a_n r^n e^{in\theta}}_{P_r(z) = \sum_{n=0}^{N_r} r a_n z^n} - f(e^{i\theta}) \Big|^p d\theta \right)^{1/p} \le \varepsilon.$$

$$\hookrightarrow \quad \| P_r(e^{i\theta}) - f(e^{i\theta}) \|_{L^p_\theta} = \| P_r - f \|_{H^p}$$

OBS 5: • $H^p(\mathbb{C})$ ISOMETRIC TO A CLOSED SUBSPACE OF L^p
ISOMORPHIC

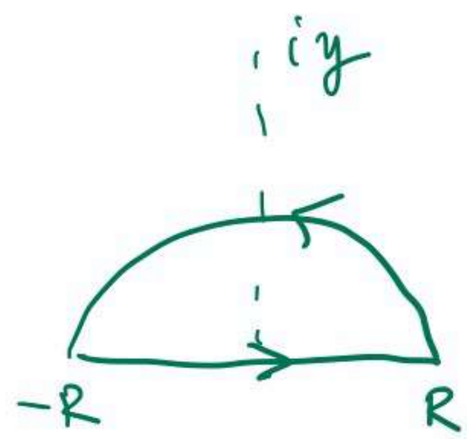
• $H^p(D)$ " TO A CLOSED SUB. OF L^p.

OBS 6: $H^1(\mathbb{C}^+) \cong \{ f \in L^1(\mathbb{R}) : \hat{f}(s) = 0 \text{ IF } s \le 0 \}$.

IF $f \in H^1(\mathbb{C}^+)$ THEN $F(z) = f(z) e^{-2\pi i s z} \in H^1(\mathbb{C}^+)$ IF $s \le 0$.

$$\hat{f}(s) = \int_{\mathbb{R}} f(t) e^{-2\pi i t s} dt = \int_{\mathbb{R}} F(t) dt$$

. JUST NEED TO SHOW $\int F(t) dt$ VANISHES FOR EVERY $f \in H^1(\mathbb{C}^+)$.



$$|f(z)| \le \|f\|_{H^1} \left\| \frac{1}{2\pi(z-t)} \right\|_{L^\infty_t(\mathbb{R})} = \frac{1}{2\pi y} \|f\|_{H^1}$$

$$0 = \int_{\partial(D \cap \mathbb{C}^+)} f(z) dz = \int_{-R}^{R} f(t) dt + \underbrace{R \int_0^{\pi} f(Re^{i\theta}) i e^{i\theta} d\theta}$$

$$\text{so} \quad \left| R \int_0^{\pi} f(Re^{i\theta}) i e^{i\theta} d\theta \right| \le R \int_0^{\pi} |f(Re^{i\theta})| d\theta \xrightarrow{?} 0.$$

SO IF $f_\varepsilon(z) = \dfrac{f(z+i\varepsilon)}{(z+i\varepsilon)^2}$ THEN $R\,|f_\varepsilon(Re^{i\theta})| \leq \dfrac{1}{R\,2\pi\varepsilon}\|f\|_{H^1}$

$$\Longrightarrow \int_{-\infty}^{\infty} f_\varepsilon(t)\,dt = 0,$$ BUT SINCE $f_\varepsilon \to f$ IN H^1

$$\Longrightarrow \int_{-\infty}^{\infty} f(t)\,dt = 0.$$ THE PART $H^1(\mathbb{C}^+) \supset \{f : \hat{f}(s) = 0 \text{ FOR } s \leq 0\}$

IS THM. 3.8 IN GARNETT'S BOOK.

OBS 7 $H^2(D) = \{ f : D \to \mathbb{C} \mid f(z) = \sum_{n=0}^{\infty} a_n z^n$ AND $\sum_{n=0}^{\infty} |a_n|^2 < \infty \}$

$f(e^{i\theta}) \sim \sum_{n=0}^{\infty} a_n e^{in\theta}$, BUT BY FOURIER SERIES $\{e^{in\theta}\}$ ARE ORTHOGONAL $\Longrightarrow \{z^n\}_n$ IS ORTHOGONAL BASIS OF $H^2(D)$

$$\langle f, g\rangle_{H^2} = \int f(e^{i\theta})\,\overline{g(e^{i\theta})}\,d\theta = \sum_{n=0}^{\infty} a_n \overline{b_n}$$

IF $f(z) = \sum_{n=0}^{\infty} a_n z^n$ AND $g(z) = \sum_{n=0}^{\infty} b_n z^n$.

$\left(H^2(D) \cong \ell^2(\mathbb{N}) \quad f(z) = \sum_{n=0}^{\infty} a_n z^n \longmapsto (a_n)_{n=0}^{\infty} \right)$.

$$f(w) = \int_{\partial D} \frac{f(z)}{2\pi i(w-z)}\,dz = \int f(z)\overline{\left(\frac{1}{2\pi i(z-\overline{w})}\right)}\,dz = \langle f, K_w\rangle$$

$K_w(z) = \dfrac{1}{2\pi i(z-\overline{w})}$ IS THE REPRODUCING KERNEL OF $H^2(D)$.

CLASS 9

THM. 12 LET $f \in BT(\mathbb{C}^+)$ AND CONTINUOUS ON $\overline{\mathbb{C}^+}$. ASSUME

- f HAS NONPOSITIVE MEAN TYPE $\left(v(f) = \overline{\lim_{y \to \infty}} \dfrac{\log |f(iy)|}{y} \leq 0 \right)$
- $\displaystyle\int_{-\infty}^{\infty} |f(t)|^p dt < \infty \qquad (1 \leq p < \infty)$

THEN
$$\begin{cases} f(z) = \dfrac{1}{2\pi i} \displaystyle\int_{-\infty}^{\infty} \dfrac{f(t)}{t-z} dt \\ 0 = \dfrac{1}{2\pi i} \displaystyle\int_{-\infty}^{\infty} \dfrac{f(t)}{t-\bar{z}} dt \end{cases} \qquad z \in \mathbb{C}^+.$$

PROOF: $\boxed{\text{PROB. 27}}$: IF $F \in BT(\mathbb{C}^+)$ AND $|F|$ is CONTINUOUS IN $\overline{\mathbb{C}^+}$,

AND $F = B\, e^{-ihz} e^{G}$, $h \in \mathbb{R}$, $\operatorname{Re} G = \dfrac{y}{\pi} \displaystyle\int_{-\infty}^{\infty} \dfrac{d\mu(t)}{(t-x)^2 + y^2}$,

THEN $\operatorname{Re} G \leq \dfrac{y}{\pi} \displaystyle\int_{-\infty}^{\infty} \dfrac{\log |F(t)|}{(t-x)^2 + y^2} dt$ EQUALITY IF $F(x) \neq 0$ $\forall x \in \mathbb{R}$.

$$\log |f(x+iy)| = \underbrace{\log |B(x+iy)|}_{\leq 0} + \underbrace{v(f) y}_{\leq 0} + \operatorname{Re} G$$
$$\leq \operatorname{Re} G \leq \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\log |f(t)|}{(t-x)^2 + y^2} dt$$

JENSEN'S INEQ: φ is CONVEX THEN $\varphi\left(\int g\right) \leq \int \varphi(g)$ (PROB. MEAS.)

φ is CONCAVE THEN $\int \varphi(g) \leq \varphi\left(\int g\right)$.

$$\boxed{|f(x+iy)| \leq \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{|f(t)|}{(t-x)^2 + y^2} dt}$$

(WORKS EVEN IF $p = \infty$)

$$\operatorname{Re} g(x+iy) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{|f(t)|\, dt}{(t-x)^2 + y^2} \Rightarrow \operatorname{Re}(g \pm f) \geq \operatorname{Re} g - |f| \geq 0$$

$$\mathrm{Re}\,(g \pm f) = c_\pm y + \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{|f(t)| \pm \mathrm{Re}\, f(t)}{(t-x)^2 + y^2}\,dt \qquad \boxed{c_\pm \geq 0}$$

$$c_\pm \underset{\substack{\text{PROB}\\ 2,3,4}}{=} \lim_{y \nearrow \infty} \frac{\mathrm{Re}\,(g \pm f)(iy)}{y} = \pm \lim_{y \nearrow \infty} \frac{\mathrm{Re}\, f(iy)}{y} \Longrightarrow c_\pm = 0.$$

$$\Longrightarrow \mathrm{Re}\, f(x+iy) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{Re}\, f(t)}{(t-x)^2 + y^2}\,dt$$

REPEAT THE ARGUMENT FOR i TIMES f AND OBTAIN

$$f(x+iy) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(t)}{(t-x)^2 + y^2}\,dt \qquad (p=\infty \text{ works})$$

$$f(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(t)}{t-z}\,dt - \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(t)}{t-\bar{z}}\,dt$$

$$\Longrightarrow c = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(t)}{t-\bar{z}}\,dt, \quad |c| \leq \|f(t)\|_{L^p_t}\, y^{-1/p}\, C_{p'}$$

$$y \to \infty \Longrightarrow c = 0. \qquad \blacksquare$$

THM 14: LET E BE ENTIRE AND HAVE NO ZEROS IN $\mathbb{C}^+$. ASSUME $|E(\bar{z})| \leq |E(z)|$ $\forall z \in \mathbb{C}^+$ AND $E(0) = 1$. THEN $E \in \mathcal{P}$ IFF

$$(*) \quad \mathrm{Re}\left(\frac{i \log E(z)}{z}\right) \geq 0 \qquad z \in \mathbb{C}^+$$

PROOF: NECESSITY: SUPPOSE $E \in \mathcal{P}$. PROB. 13 $\exists P_n \in \mathrm{Poly} \cap \mathcal{P}$ SUCH THAT $P_n \xrightarrow{\text{UNIF.}} E$, WE CAN ASSUME $P_n(0) = 1$.

$$P_n(z) = (1 - z/\bar{w}_1)\cdots(1 - z/\bar{w}_n)$$

$w_j \in \overline{\mathbb{C}^+}$. WE JUST NEED TO $(*)$ FOR EACH TERM.

$$w_1 = \rho e^{i\varphi} \qquad 0 \le \varphi \le \pi.$$

$$\frac{\partial}{\partial \rho} \operatorname{Re}\left(\frac{i \log (1 - z/\bar{w}_1)}{z} \right) = \operatorname{Re}\left(\frac{i}{\cancel{z}} \frac{\cancel{z}/\rho^2 e^{-i\varphi}}{1 - z/\rho e^{-i\varphi}} \right)$$

$$= \frac{-(\sin\varphi + y/\rho)}{|\rho e^{-i\varphi} - z|^2} \le 0.$$

SUFF. ASSUME NOW $\operatorname{Re}\left(\frac{i \log E(z)}{z} \right) \geq 0.$

$$\operatorname{Re} \frac{i \log E(z)}{z} = ay + \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{d\mu(t)}{(t-x)^2 + y^2}$$

LET $\varphi(t) = \operatorname{Re}(i \log E(t))$ $\left(e^{2i\varphi(t)} = \overline{E(t)}/E(t), \ E(t) \neq 0 \right)$

$$\int_a^b d\mu(t) = \lim_{y \searrow 0} \int_a^b \operatorname{Re}\left(\frac{i \log E(t+iy)}{t+iy} \right) dt = \int_a^b \frac{\varphi(t)}{t} dt$$

IF $E(x) \neq 0$ $\forall x \in [a,b]$.

$$\Longrightarrow d\mu(t) = \frac{\varphi(t)}{t} dt.$$

IF E HAS NO ZEROS: $\left| E^*(z)/E(z) \right| \le 1 \quad z \in \mathbb{C}^+$, SO $F = E^*/E$

HAS A NEVALINNA FACTORIZATION, SO $F = e^{-ihz} e^{-G}$ $\quad F(0) = 1.$ $(h \le 0)$

AND $\operatorname{Re} G(x+iy) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{d\mu(t)}{(t-x)^2 + y^2}$, SINCE $|F(x)| = 1$

$\forall x \in \mathbb{R}$.

$$\underset{\text{PROB. 27}}{\Longrightarrow} \quad d\mu(t) = \log |F(t)| dt \equiv 0.$$

THEN IT IS EASY TO SHOW (USING THAT $\operatorname{Re} \frac{i \log E(z)}{z} \geq 0$.)

$E(z) = e^{-az^2 - ibz}$ $\quad \operatorname{Re} b \geq 0, \ a \geq 0 \quad \Longrightarrow E \in \mathcal{P}$

IF E HAS A ZERO $\overline{\omega_1}$ $\qquad E_1(z) = \dfrac{E(z)}{1 - z/\overline{\omega_1}}$

- $$\left|\frac{E_1^*}{E_1}\right| = \left|\frac{E^*}{E}\right| \left|\frac{1 - z/\overline{\omega_1}}{1 - z/\omega_1}\right| \leq \frac{O(|z|)}{O(|z|)} \quad \text{FOR} \quad z \in \mathbb{C}^+.$$

$$\Longrightarrow \quad |E_1^*| \leq |E_1| \quad \text{IN } \mathbb{C}^+.$$

P.2.

- $$\operatorname{Re}\left(i \frac{\log E(z)}{z}\right) = ay + \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\varphi(t)/t \, dt}{(t-x)^2 + y^2}$$

$$\operatorname{Re} i \frac{\log 1 - z/\overline{\omega_1}}{z} = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\varphi(t)/t - \varphi_1(t)/t}{(t-x)^2 + y^2}\, dt \qquad \Big| \; \varphi_1 = \operatorname{Re} i \log E_1(t)$$

$$\Longrightarrow \operatorname{Re} \frac{i \log E_1(z)}{z} = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\varphi_1(t)/t}{(t-x)^2 + y^2}\, dt$$

$E_1(0) = 1$ so $\varphi_1(0) = 0$. $\qquad \dfrac{\overline{E_1(t)}}{E_1(t)} = e^{2i\varphi(t)}$

$$\Longrightarrow \varphi_1'(t) = \operatorname{Re} i \frac{E'(t)}{E(t)} = \partial_y \log |E(t+iy)|\Big|_{y=0} \geq 0$$

$$\Longrightarrow \quad \varphi_1(t)/t \geq 0. \qquad \Longrightarrow \quad \operatorname{Re}\left(\frac{i \log E_1(z)}{z}\right) \geq 0.$$

SO THEN $\quad E(z) = E_n(z)\,(1 - z/\overline{\omega_n}) \cdots (1 - z/\overline{\omega_1}) \cdot \left|e^{h_n z}\right|$

AND E_n SATISFIES THE COND. OF THE THEOREM.

$$\operatorname{Re} \frac{i \log E(z)}{z} = \operatorname{Re} \frac{i \log E_n(z)}{z} + \sum_{j=1}^{n} \operatorname{Re} \frac{i \log (1 - z/\overline{\omega_j})}{z}$$

$$\geq \sum_{j=1}^{n} \operatorname{Re} \frac{i \log (1 - z/\overline{\omega_j})}{z} \underset{z=i}{=} \sum_{j=1}^{n} \log |1 - z/\overline{\omega_j}|$$

$$\Longrightarrow \operatorname{Re} i \frac{\log E(z)}{z} \geq c \sum_{\substack{j=1 \\ |w_j| \geq R}}^{n} \frac{1+y_j}{|\overline{w_j}|^2} \qquad \operatorname{Re} i \frac{\log E_n(z)}{z} \geq 0$$

THM. 7

$$\Longrightarrow F = \prod_{j=1}^{\infty} (1 - z/\overline{w_j})\, e^{z \operatorname{Re} \frac{1}{w_j}} \in \underline{P}$$

$$E(z) = \underbrace{\lim_{n\to\infty} \left(E_n(z)\, e^{-z \sum_{j=1}^{n} \operatorname{Re} \frac{1}{w_j}} \right)}_{H} F$$

H HAS NO ZEROS, $H(0)=1$, $|H^*| \leq |H|$ IN $\mathbb{C}^+$, $\operatorname{Re} i \frac{\log H(z)}{z} \geq 0$

$$\Longrightarrow H(z) = e^{-az^2 - ibz}, \quad a \geq 0, \quad \operatorname{Re} b \geq 0. \Longrightarrow H \in \underline{P} \quad \blacksquare$$

CLASS 10 LET $f \in L^1(\mathbb{R})$ WE DEFINE THE FOURIER TRANSFORM OF f

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i \xi t} f(t)\, dt \qquad \left(\Longrightarrow \|\hat{f}\|_\infty \leq \|f\|_1 \right)$$

WE CAN SHOW $\hat{f} \in C(\mathbb{R})$ AND IT DECAYS AT INFINITY.
(R-L LEMMA)

- MULTIPLICATION FORMULA: IF $f, g \in L^1$ THEN

$$\boxed{\int \hat{f}\, g = \int f\, \hat{g}} \qquad (\text{FUBINI'S THM.})$$

- CONVOLUTION: IF $f, g \in L^1$ $\quad f * g(t) = \int_{\mathbb{R}} f(t-s) g(s)\, ds$

$\Longrightarrow f * g \in L^1$ AND (BY FUBINI'S THM) $\widehat{f*g} = \hat{f}\,\hat{g}$.

- APROXIMATION OF IDENTITY: LET $\phi_\varepsilon \in L^1$

① $\|\phi_\varepsilon\|_1 < \infty \qquad \forall \varepsilon > 0$

② $\displaystyle\int_{-\infty}^{\infty} \phi_\varepsilon(x)\,dx = 1 \qquad \forall \varepsilon > 0$

③ $\displaystyle\int_{|x| \geq X} \phi_\varepsilon(x)\,dx \longrightarrow 0$ AS $\varepsilon \to 0$.

(EXAMPLE: $P_y(x) = \dfrac{y/\pi}{t^2+y^2}$ "$y=\varepsilon$")

USUALLY $\phi_\varepsilon(x) = \frac{1}{\varepsilon}\phi(x/\varepsilon)$ FOR SOME FIXE $\phi \in L^1$ $\int \phi = 1$.

- IF $f \in L^p(\mathbb{R})$ $(1 \leq p < \infty)$ THEN $f * \phi_\varepsilon \in L^p(\mathbb{R})$ AND

$f * \phi_\varepsilon \longrightarrow f$ AS $\varepsilon \to 0$.

- $$\underline{e^{-\pi\lambda|x|^2} \xrightarrow{\wedge} \lambda^{-\frac{1}{2}} e^{-\pi|x|^2/\lambda}}$$

$$\widehat{e^{-\pi\lambda|\cdot|^2}}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i\xi x - \pi x^2\lambda}\,dx = \lambda^{-1/2} e^{-\pi\xi^2/\lambda} \int_{-\infty}^{\infty} e^{-\pi(x+i\xi/\sqrt{\lambda})^2}\,dx$$

$$= \lambda^{-1/2} e^{-\pi\xi^2/\lambda} \int_{\operatorname{Im} z = \xi/\sqrt{\lambda}} e^{-\pi z^2}\,dz = \lambda^{-\frac{1}{2}} e^{-\pi\xi^2/\lambda} \underbrace{\int_{\operatorname{Im} z = 0} e^{-\pi z^2}\,dz}_{=1}$$

- <u>PLANCHEREL IDENTITY</u>: IF $f \in L^1 \cap L^2$ THEN $\hat{f} \in L^2$ AND $\int |\hat{f}|^2 = \int |f|^2$

LET $g_\lambda(x) = \lambda^{-1/2} e^{-\pi x^2/\lambda}$ AND $f_\lambda = f * g_\lambda$

$$\widehat{f_\lambda}(\xi) = \hat{f}(\xi)\,\widehat{g_\lambda}(\xi) = \hat{f}(\xi)\, e^{-\pi\lambda\xi^2}$$

$$\int_{-\infty}^{\infty} |\widehat{f_\lambda}(\xi)|^2\,d\xi = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 e^{-2\pi\lambda\xi^2}\,d\xi \xrightarrow[\lambda \to 0]{\text{M.C.T}} \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\,d\xi$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi\lambda\xi^2} e^{-2\pi i\xi(t-s)} f(t)\overline{f(s)}\,dt\,ds\,d\xi$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\lambda}} e^{-\pi\frac{(t-s)^2}{2\lambda}} f(t)\overline{f(s)}\,dt\,ds = \int_{-\infty}^{\infty} f_{2\lambda}(s)\overline{f(s)}\,ds$$

$$\xrightarrow[\lambda\to 0]{} \int |f|^2$$

SO Λ CAN BE EXTENDED TO $L^2(\mathbb{R})$ AND PRESERVES THE NORM.

- <u>FOURIER INVERSION</u> : $\check{f}(x) = \int_{-\infty}^{\infty} e^{2\pi i\xi t} f(t)\,dt$ ONE CAN SHOW

$\Lambda V = V\Lambda = Id$. ALSO, ONE CAN SHOW IF $f \in L^1$ AND $\hat{f} \in L^1$

THEN $$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{2\pi i\xi x}\,d\xi \quad (\text{so } f \in C_0(\mathbb{R})).$$

<u>PALEY-WIENER SPACES</u> : LET $f \in L^2(\mathbb{R})$ AND BAND LIMITED, SO SAY,

$\hat{f}(\xi) = 0$ IF $|\xi| \geq b$.

$$f(x) = \int_{-b}^{b} \hat{f}(\xi)\, e^{2\pi i x\xi}\,d\xi.$$

$\Rightarrow$ f EXTENDS AS AN ENTIRE FUNCTION $f(z) = \int_{-b}^{b} \hat{f}(\xi) e^{2\pi i z\xi}\,d\xi$

ALSO $$|f(z)| \leq \int_{-b}^{b} |\hat{f}(\xi)|\, e^{2\pi|z|b}\,d\xi = C\, e^{2\pi|z|b}.$$ SO

$$\overline{\lim_{|z|\to\infty}} \frac{\log|f(z)|}{|z|} \leq 2\pi b$$

SO f HAS EXPONENTIAL TYPE AT MOST $2\pi b$. [AN ENTIRE FUNCTION OF EXP. TYPE f IS ONE THAT SATISFIES

$$\tau(f) = \overline{\lim_{|z|\to\infty}} \frac{\log|f(z)|}{|z|} < \infty .\Big]$$

THM F: LET $f \in L^2(\mathbb{R})$ AND $b>0$. THE FOLLOWING ARE EQUIV.

① $\hat{f}$ IS SUPPORTED IN $[-b,b]$

② f EXTENDS AS AN ENTIRE FUNCT. OF EXP. TYPE AT MOST $2\pi b$.

PROOF: ① $\Rightarrow$ ② ✓. LETS SHOW ② $\Rightarrow$ ①.

STEP 1: WE CAN ASSUME $f \in L^1 \cap L^2$. $\quad \dfrac{\sin 2\pi\varepsilon x}{\pi x} = \displaystyle\int_{-\varepsilon}^{\varepsilon} e^{2\pi i t x}\,dt$

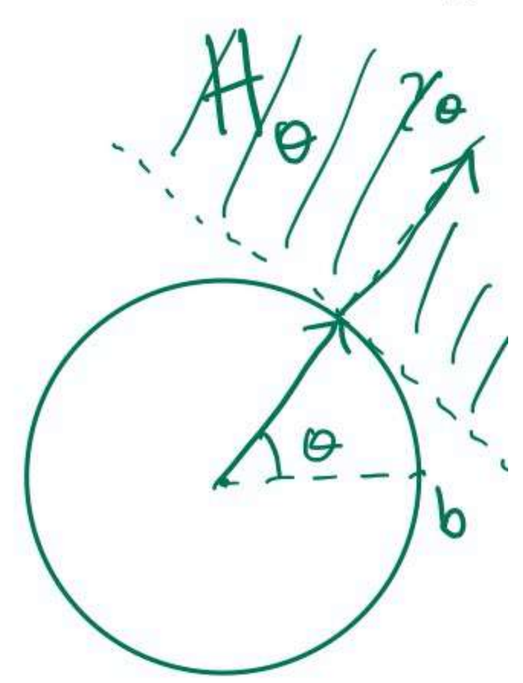
$$f_\varepsilon(x) = f(x)\,\frac{\sin 2\pi\varepsilon x}{\pi x} \in L^1 \cap L^2 \quad \text{AND} \quad \tau(f_\varepsilon) \le 2\pi(b+\varepsilon).$$

IF I SHOW $\hat{f}_\varepsilon(x) = 0$ IN $|x| > b+\varepsilon$. AND $\varepsilon \to 0$

$f_\varepsilon \to f$ IN L^2 AND SO $\hat{f}(x) = 0$ $|x| > b$.

STEP 2: LET $f \in L^1(\mathbb{R})$ AND $\tau(f) \le 2\pi b$.

$$F_\theta(z) = \int_{\gamma_\theta} f(\overline{w})\, e^{-2\pi \overline{w} z}\, d\overline{w} \qquad \gamma_\theta = \{\rho e^{i\theta} : \rho \ge 0\}$$



$$F_0(iy) = \int_0^\infty f(\rho)\, e^{-2\pi\rho i y}\, d\rho$$

$$F_\pi(iy) = -\int_0^\infty f(-\rho)\, e^{2\pi i \rho i y}\, d\rho$$

$$F_0(iy) - F_\pi(iy) = \hat{f}(y).$$

FIRST, F_θ WELL-DEFINED AND ANALYTIC IN $H_\theta = \{x\cos\theta + y\sin\theta > b\}$

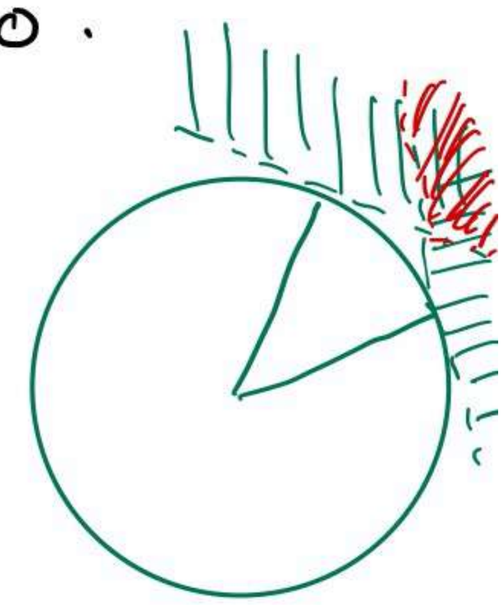
$$(*) \quad \int_{\gamma_\theta} |f(\bar\omega)\, e^{-2\pi\bar\omega z}|\,|d\bar\omega| = \int_0^\infty |f(\rho e^{-i\theta})|\, e^{-2\pi\rho(x\cos\theta + y\sin\theta)}\, d\rho$$

$\forall \delta>0\ \exists C_\delta \quad |f(\rho e^{i\theta})| \le C_\delta e^{2\pi(b+\delta)\rho} \quad \forall \rho \geq 0$. SO IF $z \in K \subset H_\theta$

THEN $x\cos\theta + y\sin\theta \geq b + \delta_0$ AND IF $\delta < \delta_0$ THEN

SO $$(*) \le C_\delta \int_0^\infty e^{2\pi(b+\delta)\rho}\, e^{-2\pi\rho(b+\delta_0)}\, d\rho$$

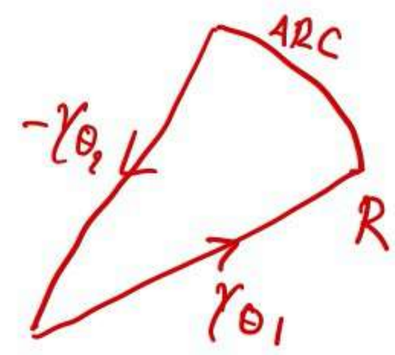
$$= C_\delta \int_0^\infty e^{-2\pi\rho(\delta_0-\delta)}\, d\rho < \infty.$$



SECONDLY, $F_{\theta_1} = F_{\theta_2}$ IN $H_{\theta_1} \cap H_{\theta_2}$

BY CAUCHY'S THM

$$0 = \int_{\gamma_{\theta_1}\cap D_R} + \int_{\substack{ARC_R \\ \theta_1\to\theta_2}} - \int_{\gamma_{\theta_2}\cap D_R} f(\bar\omega)\, e^{-2\pi\bar\omega z}\, d\bar\omega$$



$R \to \infty$

$$0 = F_{\theta_1}(z) - F_{\theta_2}(z) + \lim_{R\to\infty} \int_{\substack{ARC_R \\ \theta_1\to\theta_2}} f(\bar\omega)\, e^{-2\pi\bar\omega z}\, d\bar\omega$$

AND IT IS EASY TO SHOW THAT THE LAST LIMIT IS ZERO.

SO $\exists F$ ANALYTIC OUTSIDE D_b AND $F = F_\theta$ IN H_θ.

SO IF $|y|>b$: $\hat f(y) = F_0(iy) - F_\pi(iy) = F(iy) - F(iy) = 0$.

$\blacksquare$

- $PW(2\pi b) = \left\{ f \text{ ENTIRE FUNCTION}, \ f|_{\mathbb{R}} \in L^2, \ \hat{f}|_{\mathbb{R}} \text{ is SUPP. IN } [-b,b] \right\}$

$$= \left\{ f \text{ ENTIRE FUNC.}, \ f|_{\mathbb{R}} \in L^2, \ \tau(f) \leq 2\pi b \right\}$$

INNER PRODUCT IS JUST THE $L^2(\mathbb{R})$ - INNER PRODUCT.

$$h \in L^2[-b,b] \longmapsto \int_{-b}^{b} h(t) e^{2\pi i z t} dt \in PW(2\pi b)$$

ISOMETRIC ISOMORPHISM (UNITARY).

- KERNEL OF $PW(2\pi b)$ IS $K(\omega, z) = \dfrac{\sin 2\pi b (z - \bar{\omega})}{\pi (z - \bar{\omega})}$

$\forall \omega \in \mathbb{C}$ $\quad K_\omega(z) = K(\omega, z) \in PW(2\pi b)$ AND $\forall f \in PW(2\pi b)$

$$f(\omega) = \langle f, K_\omega \rangle = \int_{-\infty}^{\infty} f(t) \overline{K(\omega, t)} \, dt$$

$$K(\omega, z) = \int_{-b}^{b} e^{2\pi i (z - \bar{\omega}) t} dt.$$

- <u>FOURIER SERIES</u>: $h \in L^2[-b,b]$ $\quad h(t) = \sum_{n \in \mathbb{Z}} a_n e^{-\frac{\pi i n t}{b}}$

$$a_n = \frac{1}{2b} \int_{-b}^{b} h(t) e^{\pi i \frac{n}{b} t} dt.$$

BUT IF $h = \hat{f}$ OF SOME $f \in PW(2\pi b)$ THEN

$$a_n = \frac{1}{2b} \int_{-b}^{b} \hat{f}(t) e^{2\pi i \frac{n}{2b} t} dt$$

$$= \frac{1}{2b} f(n/2b)$$

$$\hat{f}(t) = \sum_{n \in \mathbb{Z}} \frac{f(\frac{n}{2b})}{2b} e^{-\pi i n t / b} \quad \text{IN } L^2[-b,b]$$

By Fourier inversion $f(z) = \int_{-b}^{b} \hat{f}(t)\, e^{2\pi i z t}\, dt$

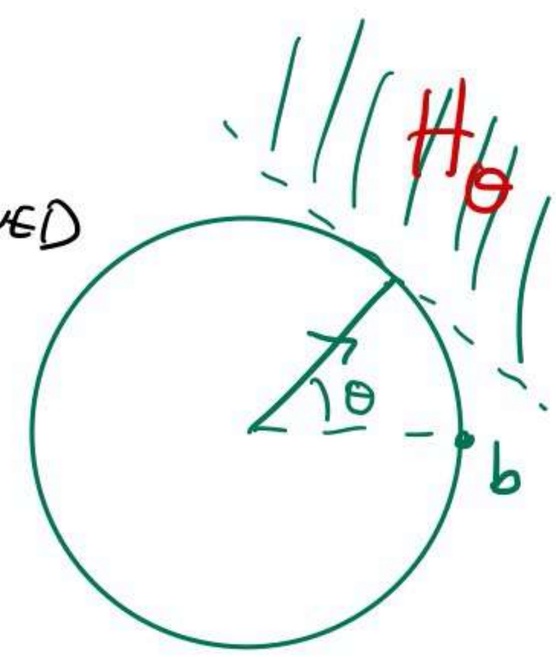
$$= \sum_n f\left(\frac{n}{2b}\right) \frac{\sin 2\pi b\,(z - n/2b)}{2\pi b\,(z - n/2b)}$$

$$= \sum_n f\left(\frac{n}{2b}\right) \frac{K\left(\frac{n}{2b}, z\right)}{K\left(\frac{n}{2b}, \frac{n}{2b}\right)}$$

$\boxed{\text{Shannon Whittaker interp. formula}}$

Proof of Thm. 4 continued:

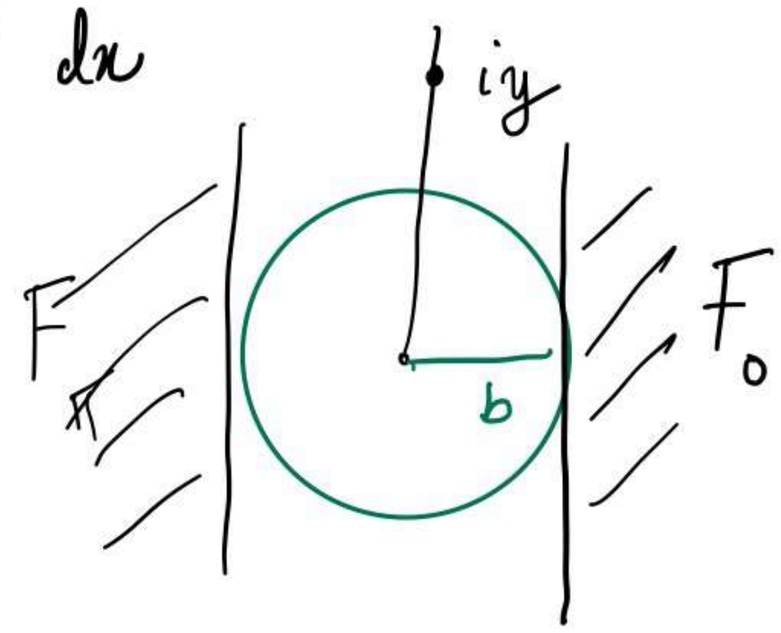
$$F_\theta(z) = \int_{\gamma_\theta} f(\bar{w})\, e^{-2\pi \bar{w} z}\, d\bar{w}$$



This analytic in H_θ. Then later we showed $\exists F : \{|z| > b\} \to \mathbb{C}$ analytic and $F = F_\theta$ in H_θ.

$$F_0(iy) - F_\pi(iy) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i y x}\, dx$$

$$= \hat{f}(y)$$



We have to use that $f \in L^1(\mathbb{R})$ at some point. Actually, F_0 is analytic for $x > 0$ and continuous for $x \geq 0$.

$$|F_0(z)| \leq \int_0^{\infty} |f(\rho)|\, e^{-2\pi \rho x}\, d\rho < \infty$$

By similar computation F_π is analytic for $x < 0$ and cont. for $x \leq 0$.

THEN $$F_0(iy) - F_\pi(iy) = \lim_{x\to 0^+} F_0(x+iy) - F_\pi(-x+iy)$$
$$= \lim_{x\to 0^+} F(x+iy) - F(-x+iy) = 0.$$
∎

THM G: LET F ENTIRE AND $1 \le p \le \infty$. THEN THE FOLL. ARE EQUIV:

① $F \in L^p(\mathbb{R})$ AND $\tau(F) = \overline{\lim_{|z|\to\infty}} \dfrac{\log|F(z)|}{|z|} \le 2\pi b$

② $F/e^{-2\pi i b z}$ AND $F^*/e^{-2\pi i b z} \in H^p(\mathbb{C}^+)$.

BY THIS RESULT WE CAN DEFINE

$$PW^p(2\pi b) = \{F \text{ ENTIRE},\ F \in L^p(\mathbb{R}),\ \tau(F) \le 2\pi b\}$$
$$= \{F \text{ ENTIRE},\ F/e^{-2\pi i b z} \text{ AND } F^*/e^{-2\pi i b z} \in H^p(\mathbb{C}^+)\}$$

PROOF: ② $\Rightarrow$ ① $(p<\infty)$

$$\frac{e^{-2\pi i b z}}{2\pi i}\int_{-\infty}^{\infty} \frac{F(t)/e^{-2\pi i b t}}{t-z}\,dt = \begin{cases} F(z), & z \in \mathbb{C}^+ \\ 0, & z \in \mathbb{C}^- \end{cases}$$

$$-\frac{e^{2\pi i b z}}{2\pi i}\int_{-\infty}^{\infty} \frac{F(t)/e^{2\pi i b t}}{t-z}\,dt = \begin{cases} F(z), & z \in \mathbb{C}^- \\ 0, & z \in \mathbb{C}^+ \end{cases}$$

ADDING

$$F(z) = \int_{-\infty}^{\infty} F(t)\,\frac{e^{-2\pi i b z}/e^{-2\pi i b t} - e^{2\pi i b z}/e^{2\pi i b t}}{2\pi i\,(t-z)}\,dt$$

$$F(z) = \int_{-\infty}^{\infty} F(t) \frac{\sin 2\pi b (t-z)}{\pi (t-z)} dt \qquad \boxed{z \in \mathbb{C}.}$$

$$|F(z)| \leq \|F\|_p \left\| \frac{\sin 2\pi b (t-iy)}{\pi(t-iy)} \right\|_{p'} \leq \|F\|_p C_{p'} e^{2\pi b|y|}$$

(!) EXERCISE

$$\frac{1}{p} + \frac{1}{p'} = 1$$

$$p' > 1.$$

$$\Longrightarrow \quad \tau(F) \leq 2\pi b.$$

<u>if $p = \infty$.</u> $\quad G(z) = \dfrac{F(z) - F(0)}{z}$ EASY TO SEE

$$G(z)/e^{-2\pi i b z}, \quad \frac{G^+(z)}{e^{-2\pi i b z}} \in H^2(\mathbb{C}^+)$$

$$\Longrightarrow \quad |G(z)| \leq C_2 e^{2\pi b|y|} \Longrightarrow |F(z)| \leq |F(0)| + C_2 |z| e^{2\pi b|z|}$$

$$\Longrightarrow \quad \tau(F) \leq 2\pi b.$$

<u>① $\Longrightarrow$ ②</u>

$$G_\varepsilon(z) = F(z) \left(\frac{\sin \pi \varepsilon z}{\pi \varepsilon z} \right)^2 \Longrightarrow \begin{cases} G_\varepsilon \in L^1(\mathbb{R}) \cap L^p(\mathbb{R}) \\ \tau(G_\varepsilon) \leq 2\pi(b+\varepsilon). \end{cases}$$

$$G_\varepsilon(z) = \int_{-\infty}^{\infty} G_\varepsilon(t) \frac{\sin 2\pi(b+\varepsilon)(z-t)}{\pi(z-t)} dt$$

(WE CAN EVEN TAKE $\varepsilon \to 0$ AND OBTAIN A FORMULA FOR F)

$$\|G_\varepsilon(x+iy)\|_{L^p_x} \leq \|G_\varepsilon\|_1 \left\| \frac{\sin 2\pi(b+\varepsilon)(z-t)}{\pi(z-t)} \right\|_{L^p_x} \qquad y > 0.$$

$$\leq \|G_\varepsilon\|_1 C e^{2\pi(b+\varepsilon)y}$$

$$\left\| \frac{G_\varepsilon(x+iy)}{e^{-2\pi i(b+\varepsilon)(x+iy)}} \right\|_{L^p_x} \leq \|G_\varepsilon\|_1 C$$

$$\Longrightarrow \quad G_\varepsilon(z)/e^{-2\pi i(b+\varepsilon)z} \in H^p(\mathbb{C}^+).$$

BUT THEN

$$\sup_{y>0} \left\| \frac{G_\varepsilon(x+iy)}{e^{-2\pi i(b+\varepsilon)(x+iy)}} \right\|_{L^p_x} = \left\| \frac{G_\varepsilon(x)}{e^{-2\pi i(b+\varepsilon)x}} \right\|_{L^p_x} \leq \|F\|_{L^p}$$

THEN TAKE $\varepsilon \to 0$ TO CONCLUDE

$$\left\| \frac{F(x+iy)}{e^{-2\pi i b(x+iy)}} \right\|_{L^p_x} \leq \|F\|_{L^p}$$

SO $F(z) / e^{-2\pi i b z} \in H^p(\mathbb{C}^+)$. THE DO THE SAME FOR F^*. ∎

APPLICATION IN ANALYTIC NUMBER THEORY:

A VERY IMPORTANT INEQUALITY IN NUMBER THEORY IS THE HILBERT-INEQ.

$$(*) \quad \left| \sum_{\substack{n,m=1 \\ n\neq m}}^{N} \frac{a_n \overline{a_m}}{(n-m)} \right| \leq \pi \sum_{n=1}^{N} |a_n|^2 \qquad (\text{SHARP!})$$

THIS INEQ. AND THE TECHNIQUES WE WILL SEE TODAY WAS USED BY SELBERG TO PROVE THE SHARP FORM OF THE LARGE SIEVE.

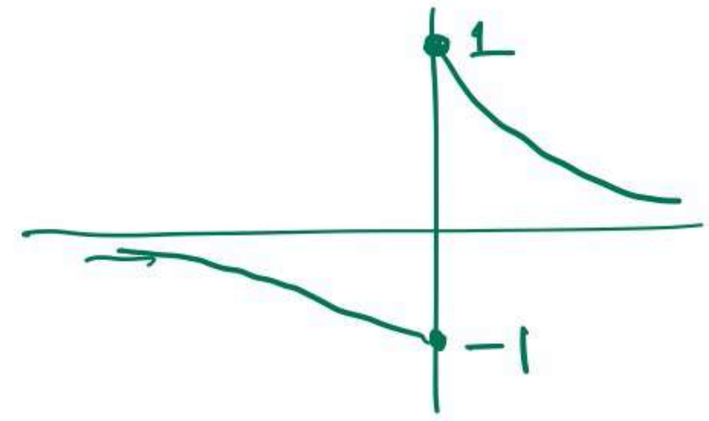
[SOME EXTREMAL FUNCTIONS IN FOURIER ANALYSIS, 1985, VAALER]

$$Q(\vec{a}) = \sum_{\substack{n,m=1 \\ n\neq m}}^{N} \frac{a_n \overline{a_m}}{\pi i(n-m)}$$

WE WANT TO SHOW THE LARGEST EIGENVALUE OF $\left[\frac{(1-\delta_{nm})}{\pi i(n-m)} \right]_{n,m=1,\dots,N}$ IN MODULI IS $\underline{1}$. FIRST NOTE THAT

$$\widehat{\operatorname{Sgn}}(\xi) = \frac{1}{\pi i \xi} \qquad (\text{IN THE DISTRIBUTION SENSE})$$

THEN DEFINE $\quad \mathrm{sgn}_\lambda(t) = e^{-\lambda|t|}\,\mathrm{sgn}(t)$



$$\widehat{\mathrm{sgn}_\lambda}(\xi) = \frac{4\pi\xi}{(\lambda^2 + 4\pi^2\xi^2)\,i} \xrightarrow{\lambda\to 0} \frac{1}{\pi i \xi}$$

$$Q_\lambda(\vec{a}) = \sum_{\substack{n,m=1\\ n\neq m}}^{N} a_n\overline{a_m}\,\frac{4\pi(n-m)}{(\lambda^2+4\pi^2(n-m)^2)\,i} \xrightarrow{\lambda\to 0} Q(\vec{a})$$

$$Q_\lambda(\vec{a}) = \sum_{n,m=1}^{N} a_n\overline{a_m}\int_{-\infty}^{\infty} e^{-\lambda|t|}\,\mathrm{sgn}(t)\,e^{-2\pi i t(n-m)}\,dt$$

$$= \int_{-\infty}^{\infty} \mathrm{sgn}_\lambda(t)\left|\sum_{n=1}^{N} a_n e^{-2\pi i t n}\right|^2 dt$$

SUPPOSE $\quad L_\lambda, U_\lambda \in PW^1(2\pi) \quad$ AND $\quad L_\lambda \le \mathrm{sgn}_\lambda \le U_\lambda$

$$Q_\lambda(\vec{a}) \ge \int_{-\infty}^{\infty} L_\lambda(t)\left|\sum_{n=1}^{N} a_n e^{-2\pi i t n}\right|^2 dt$$

$$= \sum_{n,m=1}^{N} a_n\overline{a_m}\int_{-\infty}^{\infty} L_\lambda(t)\,e^{-2\pi i t(n-m)}\,dt = \sum_{n,m=1}^{N} a_n\overline{a_m}\,\hat{L}_\lambda(n-m)$$

$$= \left(\sum_{n=1}^{N}|a_n|^2\right)\int_{-\infty}^{\infty} L_\lambda(t)\,dt$$

SIMILARLY

$$Q_\lambda(\vec{a}) \le \left(\int_{-\infty}^{\infty} U_\lambda(t)\,dt\right)\sum_{n=1}^{N}|a_n|^2$$

WE HAVE TO MAXIMIZE $\int_{-\infty}^{\infty} L_\lambda(x)\,dx$ AND MINIMIZE $\int_{-\infty}^{\infty} U_\lambda(x)\,dx$.

THIS IS IN MY THESIS.

IT TURNS OUT, THERE IS UNIQUE L_λ AND U_λ MINIMIZING THE L^1-DISTANCE TO sgn_λ $\left(-U_\lambda(-z) = L_\lambda(z)\right)$ AND

$$\widehat{L_\lambda}(0) = -1 = -\widehat{U_\lambda}(0).$$

$$\Longrightarrow \quad |Q_\lambda(\vec{a})| \leq \sum_{n=1}^{N} |a_n|^2 \quad \text{THEN } \lambda \to 0$$

WE OBTAIN $\circledast$.

(THE PEOPLE THAT FIRST USED THESE MACHINERY: SELBERG, BEURLING, MONTGOMERY)

THE TOOL TO "COOK UP" L_λ AND U_λ IS INTERPOLATION WITH DERIVATIVES.

$$f \in PW^p(2\pi) \qquad 1 \leq p < \infty.$$

$$f(z) = \sum f(n)\left(\frac{\sin \pi(z-n)}{\pi(z-n)}\right)^2 + f'(n)\,\frac{\sin^2 \pi(z-n)}{\pi^2(z-n)}$$

THEN

$$\begin{cases} U_\lambda(z) = \sum \mathrm{sgn}_\lambda(n)\left(\dfrac{\sin \pi(z-n)}{\pi(z-n)}\right)^2 + \mathrm{sgn}_\lambda'(n)\,\dfrac{\sin^2 \pi(z-n)}{\pi^2(z-n)} \\ L_\lambda(z) = -U_\lambda(-z) \end{cases}$$

CLASS 12: (BOUNDING THE ZETA FUNCTION)

- RIEMANN ZETA-FUNCTION $\qquad \zeta(s) = 1 + 2^{-s} + 3^{-s} + \cdots \qquad (\mathrm{Re}\, s > 1)$

THE ZETA-FUNCTION IS VERY IMPORTANT IN NUMBER THEORY BECAUSE OF ITS RELATION WITH PRIME NUMBERS!

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}} \qquad (\mathrm{Re}\, s > 1)$$

WE KNOW THAT $\zeta(s)$ EXTENDS AS ANALYTIC FUNCTION IN $\mathbb{C}\setminus\{1\}$ AND AT $s=1$ WE HAVE A POLE OF RESIDUE 1.

$$\xi(s) = \frac{1}{2}\, s(s-1)\, \pi^{-s/2}\, \Gamma(s/2)\, \zeta(s) \qquad \xi \text{ is ENTIRE}$$

AND $\boxed{\xi(s) = \xi(1-s)}$ (FUNCTIONAL EQ.)

- <u>RIEMANN HYPOTHESIS</u>: • BY THE PRODUCT FORMULA $\zeta(s) \neq 0$ IF $\operatorname{Re} s > 1$.
- BY THE FACT THAT $\xi(s)$ IS ENTIRE WE HAVE $\zeta(-2n) = 0$. (TRIVIAL ZEROS)
- ONE CAN SHOW $\zeta(1+it) \neq 0$ $\forall t \in \mathbb{R}$, WHICH IS EQUIV. TO THE PRIME NUMBER THEOREM $\left(\#\text{ PRIMES} \leq x \sim \frac{x}{\log x} \right.$, BECAUSE OF THE EXPLICIT FORMULA

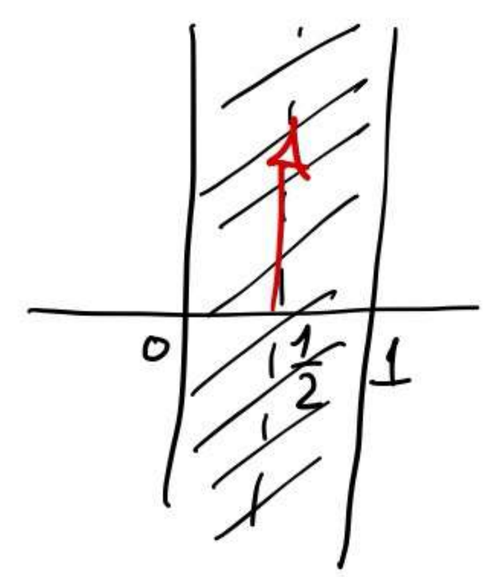
$$\pi(x)\log x \approx x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \log 2\pi$$

ρ ZEROS OF ZETA $0 \leq \operatorname{Re}\rho \leq 1$.

$$\boxed{\begin{array}{l} \text{RH} = \text{ALL ZEROS OF } \zeta(s) \text{ IN } 0 < \operatorname{Re} s < 1 \\ \text{ARE IN THE LINE } \operatorname{Re} s = \frac{1}{2}. \end{array}}$$

- <u>LITTLEWOOD 1924</u> SHOWED THAT UNDER RH

$$\left|\zeta\left(\tfrac{1}{2}+it\right)\right| \leq A\, e^{B \frac{\log t}{\log\log t}} \qquad (t \to \infty)$$



WHAT IS THE BEST B WE CAN PUT?

<u>THM H</u>: UNDER RH $\left|\zeta\left(\tfrac{1}{2}+it\right)\right| \leq A\, e^{\left(\frac{\log 2}{2} + o(1)\right) \frac{\log t}{\log\log t}}$

[BOUNDING $|\zeta(\frac{1}{2}+it)|$ ON THE RIEMANN HYPOTHESIS, 2009, CHANDEE SOUNDARARAJAN]

PROOF: STEP 1: $\xi(s) = e^{A+Bs} \prod_\rho \left(1 - s/\rho\right) e^{s/\rho}$ $\quad$ ξ HAS ORDER 1

AND AFTER SOME COMPUTATION $B = -\sum_\rho \operatorname{Re} \frac{1}{\rho}$. $\quad \rho = \frac{1}{2} + i\gamma$

$$\left| \frac{\xi(\frac{1}{2}+it)}{\xi(-\frac{3}{2}+it)} \right| = e^{2B} \prod_\rho \left| \frac{1 - \frac{1/2+it}{\rho}}{1 - \frac{-3/2+i}{\rho}} \right| e^{2\operatorname{Re}\frac{1}{\rho}}$$

$$= \prod_\gamma \left| \frac{\gamma - t}{2 + (\gamma - t)i} \right| \qquad \boxed{f(x) = \log \frac{4+x^2}{x^2}}$$

$$\log \left| \frac{\xi(\frac{1}{2}+it)}{\xi(-\frac{3}{2}+it)} \right| = -\frac{1}{2} \sum \log \frac{4 + (\gamma-t)^2}{(\gamma-t)^2} = -\frac{1}{2} \sum_\gamma f(t-\gamma)$$

$\xi(s) = \xi(1-s) \Rightarrow \xi(-\frac{3}{2}+it) = \xi(\frac{5}{2} - it)$. WRITE $\begin{cases} s = \frac{1}{2}+it \\ 2+\bar{s} = \frac{5}{2} - it \end{cases}$

$$\Rightarrow \log \left| \frac{\xi(s)}{\xi(2+\bar{s})} \right| = \log \left| \frac{s(s-1)\pi^{-s/2}\Gamma(s/2)\,\varphi(s)}{(2+\bar{s})(1+\bar{s})\pi^{-1-\bar{s}/2}\Gamma(1+\frac{\bar{s}}{2})\,\varphi(2+\bar{s})} \right| \qquad (t \to \infty)$$

$$= O(1) + O(1) + \operatorname{Re} \log \frac{\Gamma(s/2)}{\Gamma(1+\bar{s}/2)} + \log|\varphi(s)| + O(1)$$

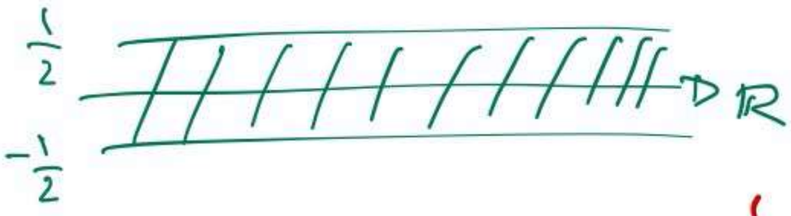
USING STIRLING'S FORMULA: $\log \Gamma(z) = \log\sqrt{2\pi} - z + (z - \frac{1}{2}) \log z + O(|z|^{-1})$

EXERCISE: $\operatorname{Re} \log \frac{\Gamma(s/2)}{\Gamma(1+\bar{s}/2)} = O(\frac{1}{t}) - \log t$

$$\Rightarrow \log \left| \frac{\xi(s)}{\xi(2+\bar{s})} \right| = O(1) + \log|\varphi(\tfrac{1}{2}+it)| - \log t$$

$$\Rightarrow \log|\varphi(\tfrac{1}{2}+it)| = \log t - \frac{1}{2} \sum_\gamma f(t-\gamma) + O(1)$$

STEP 2: (GUINAND–WEIL EXPLICIT FORMULA)



IF $h(z)$ ANALYTIC IN $|\mathrm{Im}(z)| \leq 1/2^+$ AND IT DECAYS LIKE

$$|h(z)| \leq \frac{C}{|z|^{1+\delta}} \qquad (\delta > 0,\ |z| \geq 1,\ |\mathrm{Im}\, z| \leq 1/2^+).$$

$$\sum_{\gamma} h(\gamma) = \left[h(1/2i) + h(-1/2i) - \hat{h}(0)\frac{\log \pi}{2\pi}\right]$$
$$+ \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} h(u)\, \mathrm{Re}\, \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{iu}{2}\right) du\right]$$
$$\left[-\frac{1}{2\pi}\sum_{n \geq 2} \frac{\Lambda(n)}{\sqrt{n}}\left(\hat{h}\left(\frac{\log n}{2\pi}\right) + \hat{h}\left(-\frac{\log n}{2\pi}\right)\right)\right]$$

$\Lambda(n)$ VON-MONGOLDT FUNCTION $\quad \Lambda(n) = \begin{cases} \log p, & n = p^e \\ 0, & \text{OTHERWISE} \end{cases}$ $\quad n \to \infty$.

IDEA: BOUND f FROM BELOW BY SOME ADMISSIBLE h.

LEMMA: LET $g_\Delta \in PW^1(2\pi\Delta)$, EVEN $(\Delta \geq 1)$, SUCH THAT

$$-\frac{C}{1+x^2} \leq g_\Delta(x) \leq f(x) \qquad \forall x \in \mathbb{R} \quad \forall \Delta \geq 1$$

(NOTE $f \leq \frac{C}{x^2}$ $|x| \geq 1$ AND $\int f = 4\pi$).

THEN

$$\log |\zeta(\tfrac{1}{2} + it)| \leq A + B\Delta e^{\pi\Delta} + \frac{\log t}{4\pi}\int_{-\infty}^{\infty} [f(x) - g_\Delta(x)]\, dx$$

PROOF: FIRST WE SHOW g IS ADMISSIBLE FOR GUINAND–WEIL FORMULA.

NOTE $\quad f(x) = \log \frac{4+x^2}{x^2} \underset{(x \geq 1)}{\leq} \frac{C}{1+x^2} \implies x^2 g_\Delta(x) \in PW^\infty(2\pi\Delta)$

$$\implies \frac{z^2 g_\Delta(z)}{e^{-2\pi i\Delta z}} \in H^\infty(\mathbb{C}^+) \implies \sup_{\mathrm{Im}\, z = 0} \left|\frac{z^2 g_\Delta(z)}{e^{-2\pi i\Delta z}}\right| \leq \sup_x |x^2 g_\Delta(x)| = C$$

$$\implies |g_\Delta(z)| \leq \frac{C}{|z|^2} e^{2\pi\Delta|y|}.$$

$$-\sum_{\gamma} f(t-\gamma) \leq -\sum_{\gamma} g(t-\gamma) = T_1 + T_2 + T_3$$

- $$T_1 = -\left(g_\Delta\left(t-\tfrac{1}{2i}\right) + g_\Delta\left(t+\tfrac{1}{2i}\right) - \widehat{g_\Delta}(0)\,\frac{\log \pi}{2\pi}\right)$$
$$= O(e^{\pi\Delta}) + O(1).$$

$h_\Delta(z) = g_\Delta(t-z)$

- $$|T_3| \leq \frac{1}{2\pi} \sum_{n\geq 2} \frac{\Lambda(n)}{\sqrt{n}} \left[\left|\widehat{g_\Delta}\left(\frac{\log n}{2\pi}\right)\right| + \left|\widehat{g_\Delta}\left(-\frac{\log n}{2\pi}\right)\right| \right]$$
$$\leq \frac{A}{2\pi} \sum_{\substack{n\geq 2 \\ n \leq e^{2\pi\Delta}}} \frac{\log n}{\sqrt{n}} = A'\,\Delta\, e^{\pi\Delta}$$

$$\boxed{\frac{d}{dx}\sqrt{x}(\lg x - 2) = \frac{\lg x}{\sqrt{x}}}$$

- $$T_2 = -\frac{1}{2\pi}\int_{-\infty}^{\infty} g_\Delta(v)\, \mathrm{Re}\, \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{i(t-v)}{2}\right) dv$$

STIRLING'S FORMULA

$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty} g_\Delta(v)\left[\log t + O(\log(2+|v|))\right] dv$$
$$= -\log t\, \frac{1}{2\pi}\int_{-\infty}^{\infty} g_\Delta(v)\, dv + O(1).$$

$$\boxed{\int f = 4\pi}$$

$$-\frac{1}{2}\sum f(t-\gamma) + \log(t) + O(1) \leq \begin{cases} A + B\Delta e^{\pi\Delta} - \log t\, \frac{1}{4\pi}\int_{-\infty}^{\infty} g_\Delta(v)\,dv \\ + \log(t) + O(1) \end{cases}$$

$$\log|\zeta(\tfrac{1}{2}+it)| \leq A + B\Delta e^{\pi\Delta} + \frac{\log t}{4\pi}\int_{-\infty}^{\infty}\left[f(x) - g_\Delta(x)\right] dx. \quad \blacksquare$$

STEP 3: I WANT TO MINIMIZE $\int f - g_\Delta$ AMONG $g_\Delta \in PW^1(2\pi\Delta)$ AND $g_\Delta(x) \leq f(x)$.

THM I : $(\vec{a}, \vec{b}) = (a(m), b(m))_{m \in \mathbb{Z}+\theta} \in \ell^p \times \ell^p$ $\quad \sum |a(m)|^p + |b(m)|^p < \infty$

THEN

$$F_\Delta(\vec{a}, \vec{b}; z) = \sum_{m \in \mathbb{Z}+\theta} a(m) \left(\frac{\sin \pi\Delta(z-m)}{\pi(z-m)} \right)^2 + \sum b(m) \frac{\sin^2 \pi\Delta(z-m)}{\pi^2 (z-m)}$$

BELONGS TO $PW^p(2\pi\Delta)$ AND

$$\begin{cases} F_\Delta(m) = a(m) \\ F_\Delta'(m) = b(m) \end{cases} \quad m \in \mathbb{Z}+\theta.$$

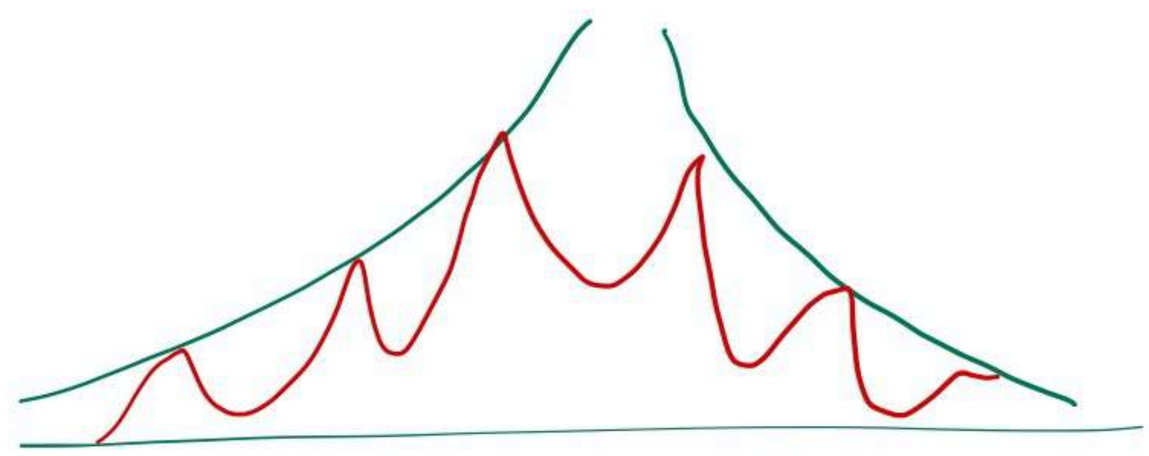
$\boxed{1 \leq p < \infty}$

IT TURNS OUT THE FUNCTION WE WANT IS

$$g_\Delta(z) = F_\Delta\left((f(m), f'(m))_{m \in \mathbb{Z}+\frac{1}{2}} ; z \right).$$

WE HAVE TO SHOW THAT IT HAS THE NEEDED PROPERTIES OF THE LEMMA ABOVE.

[SOME EXTREMAL FUNCTIONS IN FOURIER ANALYSIS II, CARNEIRO AND VAALER]



ONE CAN SHOW

$$\int_{-\infty}^{\infty} f(x) - g_\Delta(x)\, dx = \frac{2\log 2 + o(1)}{\Delta} \qquad (\Delta \to \infty).$$

$$\log |\zeta(\tfrac{1}{2}+it)| \leq A + B\Delta e^{\pi\Delta} + \frac{1}{4\pi} \log t \left(\frac{2\log 2}{\Delta} + o(1) \right)$$

$$\Delta = \frac{1}{\pi} \left(\log\log t - 3 \log\log\log t \right)$$

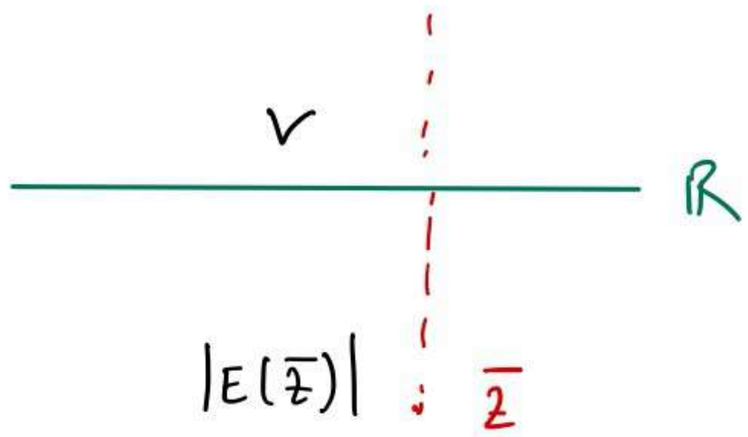
$$\Rightarrow \quad \leq \left(\frac{\log 2}{2} + o(1) \right) \frac{\log t}{\log\log t}$$

$\blacksquare$

CLASS 13 (DE BRANGES SPACES)

DEF: A FUNCTION $E : \mathbb{C} \to \mathbb{C}$ (ENTIRE) IS HERMITE-BIEHLER IF

$$|E(z)| \qquad |E^*(z)| < |E(z)| \qquad z \in \mathbb{C}^+.$$



DEF (L^p DE BRANGES SPACE, $1 \le p \le \infty$)

$$\mathcal{H}^p(E) = \Big\{ F \text{ ENTIRE}, \ \bullet\ F/E \text{ AND } F^*/E \in BT(\mathbb{C}^+),$$
$$\bullet\ \max\{ v(F/E), v(F^*/E) \} \le 0,$$
$$\bullet\ F/E \in L^p(\mathbb{R}) \Big\}$$

$$\|F\|_{\mathcal{H}^p} = \left(\int_{-\infty}^{\infty} \left| \frac{F(x)}{E(x)} \right|^p dx \right)^{1/p} \qquad p < \infty$$

$$\|F\|_{\mathcal{H}^\infty} = \sup_{x \in \mathbb{R}} |F/E(x)|$$

COMMENT: IF $x_0 \in \mathbb{R}$ IS A ZERO OF E WITH MULTIPLICITY m THEN ANY $F \in \mathcal{H}^p(E)$ HAS A ZERO AT x_0 WITH MULT. $m' \ge m$.

YET ANOTHER DEF. FOR $PW^p(2\pi b)$

$$PW^2(2\pi b) = \{ F \in L^2(\mathbb{R}), \ \operatorname{supp}(\hat{F}) \subset [-b, b] \}$$
$$= \{ F \in L^2(\mathbb{R}), \ \tau(F) \le 2\pi b \}$$
$$= \{ F/E_b, F^*/E_b \in H^2(\mathbb{C}^+) \}$$
$$\overset{!}{=} \mathcal{H}^2(E_b) \qquad \text{(TRUE ALSO FOR ANY } 1 \le p \le \infty).$$

$$\boxed{E_b(z) = e^{-2\pi i b z}}$$

KREIN THM (PROB. 37)

F IS OF EXP. TYPE AND $\iff$ $F, F^* \in BT(\mathbb{C}^+)$.

$$\int_{-\infty}^{\infty} \frac{\log^+ |F(t)|}{1+t^2}\, dt < \infty$$

IF THIS IS THE CASE $\tau(F) = \max\{\sigma(F), \sigma(F^*)\}$.

IF $F \in PW^p(2\pi b) \Rightarrow \tau(F) \leq 2\pi b$ $\Rightarrow$ $F, F^* \in BT(\mathbb{C}^+)$

$$\int_{-\infty}^{\infty} \frac{\log^+ |F(t)|}{1+t^2}\, dt \qquad \tau(F) = \max(\sigma(F), \sigma(F^*))$$

BUT SINCE $E_b \in BT(\mathbb{C}^+)$ AND $\sigma(E_b) = 2\pi b$, SO

$F/E_b,\ F^*/E_b \in BT(\mathbb{C}^+)$ AND $\sigma(F/E_b) = \sigma(F) - \sigma(E_b)$

$$\leq \tau(F) - 2\pi b$$

$$\leq 0$$

AND $\sigma(F^*/E_b) \leq 0 \Rightarrow F \in \mathcal{H}^p(E_b)$.

CONVERSELY, IF $F \in \mathcal{H}^p(E_b) \Rightarrow F/E_b,\ F^*/E_b \in BT(\mathbb{C}^+)$

$\Rightarrow F, F^* \in BT(\mathbb{C}^+) \Rightarrow F$ IS OF EXP. TYPE AND EASY TO SHOW $\tau(F) \leq 2\pi b$.

KERNEL FUNCTION: IF $F \in \mathcal{H}^p(E)$ AND $G \in \mathcal{H}^q(E)$ $\boxed{\frac{1}{p}+\frac{1}{q}=1}$

THE $$\langle F, G\rangle = \int_{-\infty}^{\infty} \frac{F(t)\,\overline{G(t)}}{|E(t)|^2}\, dt.$$

$E(z) = A(z) - iB(z)$ WHERE A, B ARE REAL ENTIRE

$A = \frac{1}{2}(E + E^*)$ AND $B = \frac{i}{2}(E - E^*)$ $\quad$ ($A(x) \in \mathbb{R}$ IF $x \in \mathbb{R}$)

$$K(\omega,z) = \frac{E(z)\,\overline{E(\omega)} - E^*(z)\,\overline{E^*(\omega)}}{2\pi i(\overline{\omega}-z)}$$

$$= \frac{B(z)\,\overline{A(\omega)} - A(z)\,\overline{B(\omega)}}{\pi(z-\overline{\omega})}$$

$$\boxed{K(\omega,z) = K_\omega(z)}$$

PW: $E_b = \cos(2\pi b z) - i\sin(2\pi b z)$

$$= \frac{\sin 2\pi b(z-\overline{\omega})}{\pi(z-\overline{\omega})}$$

THM 19: LET $E \in HB$ AND $1 \le p < \infty$. THEN $K_\omega \in \mathcal{H}^q(E)$ $(1 < q \le \infty)$ AND

$$F(\omega) = \langle F, K_\omega \rangle = \int_{-\infty}^{\infty} \frac{F(t)\,\overline{K(\omega,t)}}{|E(t)|^2}\,dt$$

$\forall F \in \mathcal{H}^p(E)$. (SO IF $p=q=2$ $K_\omega \in \mathcal{H}^2$ IS THE REPRODUCING KERNEL OF $\mathcal{H}^2$).

PROOF: NOTE $K_\omega^* = K_{\overline{\omega}}$, SO I NEED TO SHOW

$K_\omega/E \in BT(\mathbb{C}^+)$, NONPOSITIVE MEAN TYPE (FOR ALL $\omega \in \mathbb{C}$)

AND $K_\omega/E \in L^p(\mathbb{R})$.

$$\frac{K(\omega,z)}{E(z)\,\overline{E(\omega)}} = \frac{1 - \Theta(z)\,\overline{\Theta(\omega)}}{2\pi i(\overline{\omega}-z)};$$

$$\boxed{\Theta(z) = \frac{E^*(z)}{E(z)} \qquad |\Theta(z)| < 1 \text{ if } z \in \mathbb{C}^+.}$$

FROM THIS WE SEE $\dfrac{K(\omega,z)}{E(z)} \in BT(\mathbb{C}^+)$

FOR ANY $\omega \in \mathbb{C}$ AND

$$\nu\left(\frac{K_\omega}{E}\right) \le \nu(\Theta) \le 0.$$

$$\frac{K(\omega,t)}{E(t)} = \overline{E(\omega)}\,\frac{\left(1-\Theta(t)\,\overline{\Theta(\omega)}\right)}{2\pi i\,(\overline{\omega}-t)} \in L^q(\mathbb{R}) \quad \text{if } q>1.$$

NOW LET $F \in \mathcal{H}^p(E)$. THEN $F/E \in BT(\mathbb{C}^+)$ AND $\frac{F}{E} \in L^p(\mathbb{R})$.

BY THM. 12

$$E(\omega)\int_{-\infty}^{\infty} \frac{F(t)/E(t)}{2\pi i\,(t-\omega)}\,dt = \begin{cases} F(\omega), & \omega \in \mathbb{C}^+ \\ 0, & \omega \in \mathbb{C}^-. \end{cases}$$

DOING THE SAME FOR F^*/E WE GET

$$-E^*(\omega)\int_{-\infty}^{\infty} \frac{F(t)/E(t)}{2\pi i\,(t-\omega)} = \begin{cases} 0, & \omega \in \mathbb{C}^+ \\ F(\omega), & \omega \in \mathbb{C}^- \end{cases}$$

ADDING THE FORMULAS (PLUS SOME COMPUTATIONS)

$$F(\omega) = \int_{-\infty}^{\infty} \frac{F(t)\,\overline{K(\omega,t)}}{|E(t)|^2}\,dt \qquad \omega \in \mathbb{C} \qquad \blacksquare$$

REMARK:

- $K(\omega,z) = \langle K_\omega, K_z \rangle$

$$\Rightarrow |K(\omega,z)|^2 \le \|K_\omega\|^2_{\mathcal{H}^2}\,\|K_z\|^2_{\mathcal{H}^2} = K(\omega,\omega)\,K(z,z)$$

- $|K(\omega,z)| \le \|K_\omega\|_{\mathcal{H}^p}\,\|K_z\|_{\mathcal{H}^q}$

- $|F(\omega)| = |\langle F, K_\omega\rangle| \le \|F\|_{\mathcal{H}^p}\,\|K_\omega\|_{\mathcal{H}^q}$ $\boxed{F \in \mathcal{H}^p}$

$$\underset{p=2}{\le} \|F\|_{\mathcal{H}^2}\sqrt{K(\omega,\omega)}$$

THM I (BARANOV)

$$\mathcal{H}^p(E) = \{ F \text{ ENTIRE} ; F^*/E , F/E \in H^p(\mathbb{C}^+) \}$$

PROOF: $F \in \mathcal{H}^p(E)$ WE KNOW THAT (BY THE PROOF OF THM.12)

$$\frac{F(z)}{E(z)} = F/E * P_y(x) \qquad z = x+iy \quad y>0.$$

$$\sup_{y>0} \left\| \frac{F(x+iy)}{E(x+iy)} \right\|_{L^p_x} = \|F/E\|_{L^p(\mathbb{R})} = \|F\|_{\mathcal{H}^p} < \infty \quad (= \|F/E\|_{H^p})$$

SO $F/E \in H^p(\mathbb{C}^+)$. SIMILARLY, $F^*/E \in H^p(\mathbb{C}^+)$.

CONVERSELY, IF $F/E , F^*/E \in H^p(\mathbb{C}^+)$. ASSUME $p \geq 2$.

LET $g_\varepsilon(z) = \left(\frac{\sin \pi\varepsilon z}{\pi \varepsilon z}\right)^2$ AND $E_\varepsilon(z) = E(z)\, e^{-2\pi i \varepsilon z}$.

$$\frac{F g_\varepsilon}{E_\varepsilon} , \frac{F^* g_\varepsilon^*}{E_\varepsilon} \in H^2(\mathbb{C}^+)$$

$$\boxed{\frac{1}{2} = \frac{1}{p} + \frac{1}{\frac{2p}{p-2}}}$$

HÖLDER INEQ.

$$\left\| \frac{F g_\varepsilon(x+iy)}{E_\varepsilon(x+iy)} \right\|_{L^2_x} \leq \left\| F/E(x+iy) \right\|_{L^p_x} \left\| \frac{g_\varepsilon(x+iy)}{e^{-2\pi i \varepsilon (x+iy)}} \right\|_{L^{\frac{2p}{p-2}}_x}$$

$$\leq C \left\| F/E(x+iy) \right\|_{L^p_x}$$

C.F.

$$\Longrightarrow \quad F(\omega) g_\varepsilon(\omega) = \int_{-\infty}^{\infty} F(t) g_\varepsilon(t) \frac{\overline{K_\varepsilon(\omega,t)}}{|E_\varepsilon(t)|^2} dt$$

$$|F(\omega) g_\varepsilon(\omega)|^2 \leq \left\| \frac{F g_\varepsilon}{E_\varepsilon} \right\|^2_{L^2} \cdot K_\varepsilon(\omega,\omega)$$

$$\boxed{\begin{array}{l} \omega = x+iy \\ \omega \in \mathbb{C}^+ \end{array}}$$

$$\left|\frac{F(\omega)\, g_\varepsilon(\omega)}{E_\varepsilon(\omega)}\right|^2 \leq C_\varepsilon \frac{K_\varepsilon(\omega,\omega)}{|E_\varepsilon(\omega)|^2} = C_\varepsilon \frac{1-|\Theta_\varepsilon(\omega)|^2}{4\pi y}$$

$$\leq \frac{C_\varepsilon}{4\pi y}$$

USING THM. 11 (SEE ALSO THM. 20) WE CAN SHOW THAT

$\dfrac{F(\omega)\, g_\varepsilon(\omega)}{E_\varepsilon(\omega)} \in BT(\mathbb{C}^+)$ AND NON POSITIVE MEAN TYPE.

(READING EXERCISE)

$\Longrightarrow F/E \in BT(\mathbb{C}^+)$ $\quad 0 \geq \nu\left(\dfrac{F g_\varepsilon}{E\, e^{-2\pi i \varepsilon z}}\right) = \nu(F/E) + 0$

$\Longrightarrow \nu(F/E) \leq 0.$ SIMILARLY, $F^*/E \in BT(\mathbb{C}^+)$ AND

$\nu(F^*/E) \leq 0.$

IN CASE $1 \leq p < 2$ $\quad \dfrac{F^* g_\varepsilon^*}{E_\varepsilon}, \dfrac{F g_\varepsilon}{E_\varepsilon} \in \mathcal{H}^1(E)$

$$\xRightarrow{\text{C. F}} \quad F(\omega)\, g_\varepsilon(\omega) = \langle F g_\varepsilon, K_{\varepsilon,\omega} \rangle$$

$$\Longrightarrow \left|\frac{F(\omega)\, g_\varepsilon(\omega)}{E_\varepsilon(\omega)}\right| \leq \left\| \frac{F g_\varepsilon}{E_\varepsilon} \right\|_{L^1} \sup_t \left| \frac{K_\varepsilon(\omega,t)}{E_\varepsilon(t)\,\overline{E_\varepsilon(\omega)}} \right|$$

$$\sup_t \left| \frac{K(\omega,t)}{E_\varepsilon(t)\,\overline{E_\varepsilon(\omega)}} \right| = \sup_t \left| \frac{1 - \overline{\Theta(\omega)}\, \Theta(t)}{2\pi i (\overline{\omega} - t)} \right| \leq \frac{C}{y}$$

$\boxed{\omega = x + iy \quad y > 0}$

SO $\left| \dfrac{F(\omega)\, g_\varepsilon(\omega)}{E_\varepsilon(\omega)} \right| \leq C/y$, NOW THE SAME PROCEDURE

AS BEFORE SHOWS $F/E,\ F^*/E \in BT(\mathbb{C}^+)$

AND $v(F/E),\ v(F^*/E) \leq 0$. ∎

MODEL SPACE

<u>THM 21</u> : $\mathcal{H}^p(E)$ is COMPLETE!

<u>PROOF</u>: $\mathcal{H}^p(E) \cong H^p(\mathbb{C}^+) \cap \Theta H^p(\mathbb{C}^-)$

ISOMETRIC

ISOMORPHIC

$= \{ F \in H^p(\mathbb{C}^+),\ F/\Theta$ EXTENDS AS ENTIRE, $F/\Theta \in H^p(\mathbb{C}^-) \}$

$$F \longmapsto F/E$$

$$F/E \overset{?}{\in} \Theta H^p(\mathbb{C}^-) \iff F/\cancel{E} = \frac{\overset{\Theta}{E^*}}{\cancel{E}} G \quad (G \in H^p(\mathbb{C}^-))$$

$$\iff F/E^* \in H^p(\mathbb{C}^-) \iff F^*/E \in H^p(\mathbb{C}^+).$$

IT IS EASY TO SHOW THAT $H^p(\mathbb{C}^+) \cap \Theta H^p(\mathbb{C}^-)$ is A CLOSED SUBSPACE OF $H^p(\mathbb{C}^+)$. ∎

CLASS 14 (ORTHOGONAL SETS) LET $E \in HB$.

RECALL REP. KERNEL $K(w,z) = \dfrac{B(z)\overline{A(w)} - \overline{B(w)}A(z)}{\pi(z-\overline{w})}$ $\left(= K_w(z)\right)$ OF $\mathcal{H}^2(E)$

$(E = A - iB)$. LET $a, b \in \mathbb{C}$ SO THAT $K(a,b) = 0$.

$$0 = K(a,b) = \langle K_a, K_b \rangle \iff K_a \perp K_b.$$

$$K(a,a) = \langle K_a, K_a \rangle = \|K_a\|_2^2 \geq 0$$

PROB. 45: $K(a,a) > 0$ IFF $a \in \mathbb{C} \setminus$ REAL ZERO (E)

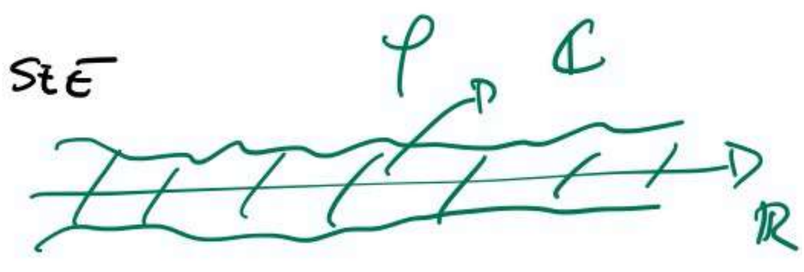
(PROB. 48)

PHASE FUNCTION: $\varphi : \mathbb{R} \to \mathbb{R}$ SO THAT $e^{i\varphi(x)} E(x) \in \mathbb{R}$ $\forall x \in \mathbb{R}$.

$$\varphi(z) = \frac{1}{2i} \log\left(\frac{E^*(z)}{E(z)}\right) \qquad \left(\varphi(z) = \int_0^z \frac{(E^*/E)'(w)}{E^*/E(w)}\, dw\right)$$

$\varphi : U \supset \mathbb{R} \to \mathbb{C}$ ANALYTIC, WHEN U IS SOME NEIB. OF $\mathbb{R}$

SO $\varphi^* = \varphi$. $(\varphi(x) \in \mathbb{R}$ FOR $x \in \mathbb{R})$ AND WE SEE



$e^{i\varphi(x)} E(x) \in \mathbb{R}$ FOR $x \in \mathbb{R}$.

$$\varphi'(z) = \frac{1}{2i}\left(\frac{\partial_z E^*}{E^*} - \frac{\partial_z E}{E}\right) \quad \text{so}$$

$$\varphi'(x) = -\operatorname{Im} E'(x)/E(x)$$

$$= \operatorname{Re} i\frac{E'(x)}{E(x)}$$

$$= -\frac{1}{2}\partial_y \log\left|\frac{E^*(x+iy)}{E(x+iy)}\right|_{y=0}$$

$$\geq 0$$

$\boxed{E \in HB}$

$$K(x,x) = \frac{E(x)\overline{E'(x)} - \overline{E(x)}E'(x)}{2\pi i}$$

$$= \frac{|E(x)|^2}{\pi}\frac{1}{2i}\left(\overline{\left(\frac{E'(x)}{E(x)}\right)} - \frac{E'(x)}{E(x)}\right)$$

$$= \frac{|E(x)|^2}{\pi}\varphi'(x)$$

$$\Rightarrow \varphi'(x) = \frac{\pi K(x,x)}{|E(x)|^2} > 0 \quad \text{IF } E(x) \neq 0.$$

CLAIM: $\varphi'(x) > 0 \quad \forall x \in \mathbb{R}$.

IF E HAS NO REAL ZEROS ✓

IF E HAS SOME REAL ZEROS $E = S E_0$ WHERE $E_0 \in HB$

NO REAL ZEROS AND S REAL ENTIRE (S = WEIRSTRASS PRODUCT OF REAL ZEROS OF E)

$\mathcal{H}^2(E_0)$ EXISTS AND WE CAN SHOW THAT

$$\mathcal{H}^2(E_0) \longrightarrow \mathcal{H}^2(E)$$
$$F \longrightarrow SF$$

ISOMETRIC ISOMORPHISM

$$E^*/E = E_0^*/E_0 \implies \varphi_E = \varphi_{E_0} \quad \text{so}$$

$$\varphi_E'(x) = \varphi_{E_0}'(x) = \frac{K_{E_0}(x,x)}{|E_0(x)|^2} > 0 \quad \left(= \frac{K_E(x,x)}{|E(x)|^2}\right)$$

OBS: IF $a, b \in \mathbb{R}$, $a \neq b$ AND $E(a)E(b) \neq 0$

$$\langle K_a, K_b \rangle = K(a,b) = \frac{E(a)\overline{E(b)} - \overline{E(a)}E(b)}{2\pi i(a-b)}$$

$$= \overline{E(a)}E(b)\left(\frac{\frac{E(a)}{\overline{E(a)}}\overline{\frac{E(b)}{E(b)}} - 1}{2\pi i(a-b)}\right)$$

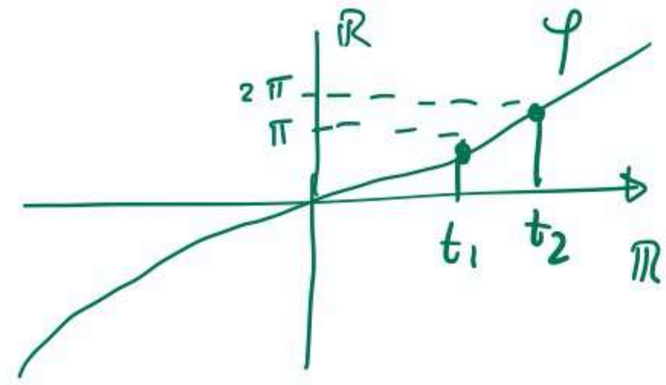
$$\boxed{\frac{\overline{E(x)}}{E(x)} = e^{2i\varphi(x)}}$$

$$= \overline{E(a)}E(b)\left[\frac{e^{2i(\varphi(b)-\varphi(a))} - 1}{2\pi i(a-b)}\right]$$

$$K_a \perp K_b \iff \varphi(b) \equiv \varphi(a) \mod \pi.$$

$\implies \{K(t_n, z)\}$ WHERE $t_n = \varphi^{-1}(\alpha + \pi n)$ $(n \in \mathbb{Z})$ is AN ORTHOGONAL SET. $(\varphi(t_n) \equiv \alpha \mod \pi)$

JUST FOR NOW ASSUME E HAS NO REAL ZEROS.



$\alpha = 0$: $\quad e^{i\varphi(x)}\left(A(x) - iB(x)\right) \in \mathbb{R} \quad \forall x \in \mathbb{R}.$

SO IF $\varphi(x) \equiv 0 \mod \pi$ IFF $B(x) = 0$.

$\alpha = \pi/2$: SO $\varphi(x) \equiv \pi/2 \mod$ IFF $A(x) = 0$.

$PW^2(\pi)$: $\quad E(z) = e^{-\pi i z} \quad \left(PW^2(\pi) = \mathcal{H}^2(E)\right)$

$= \cos \pi z - i \sin \pi z$

NODES : • ZEROS OF $\sin \pi z = \mathbb{Z}$

• ZERO' OF $\cos \pi z = \mathbb{Z} + \frac{1}{2}$

$e^{i\varphi(x)} e^{-\pi i x} \in \mathbb{R}$

$\varphi(x) = \pi x$ SO $\varphi(z) = \pi z$

$\pi x \equiv \alpha \mod \pi$

$\Rightarrow \mathbb{Z} + \alpha/\pi$

THM 22 : LET $E \in HB$ HAVE NO REAL ZEROS. $E = A - iB$ LET $\{t_n\} = \text{ZEROS}(B)$.

$(= \varphi^{-1}(\pi n))$

SO THE SET $\{K(t_n, z)\}_n$ IS ORTHOGONAL AND

$$\left(\text{SPAN}\ \{K(t_n, z)\}\right)^{\perp} \subset \left(\text{SPAN}\ B\right) \cap \mathcal{H}^2(E)$$

THUS IF $B \notin \mathcal{H}^2(E)$ THEN $\{K(t_n, z)\}$ IS AN ORTHOGANAL BASIS FOR $\mathcal{H}^2(E)$ AND

$$F(z) = \sum F(t_n) \frac{K(t_n, z)}{K(t_n, t_n)}$$

$$\|F\|_2^2 = \sum \frac{|F(t_n)|^2}{K(t_n, t_n)} = \sum \left|\frac{F(t_n)}{E(t_n)}\right|^2 \frac{\pi}{\varphi'(t_n)}$$

REMARKS: (i) $E_\alpha(z) = e^{i\alpha} E(z)$ $\qquad \mathcal{H}^2(E) = \mathcal{H}^2(E_\alpha)$

$$= A_\alpha - i B_\alpha$$

$$\Rightarrow \quad B_\alpha(z) = \frac{i}{2} e^{i\alpha} E(z) - e^{-i\alpha} E^*(z) \quad \text{so}$$

$$B_\alpha(x) = \frac{i}{2} e^{i\alpha} E(x) - e^{-i\alpha} \overline{E(x)} = \frac{i}{2} E(x) e^{i\alpha} \left(1 - e^{2i(\varphi(x)-\alpha)}\right)$$

$$\frac{B_\alpha(x)}{\overline{E(x)}} = \frac{i e^{i\alpha}}{2} \left(1 - e^{2i(\varphi(x)-\alpha)}\right)$$

THEN $\varphi(x) \equiv \alpha \mod \pi$ IFF $\left(\frac{B_\alpha}{E}\right)(x) = 0$.

THEREFORE, THM. 22 ALSO WORKS FOR $\{t_n\} = \varphi^{-1}(\alpha + \pi\mathbb{Z})$.

(ii) THM. 22 ALSO WORKS IF E HAS REAL ZEROS. USE

$$\{t_n\} = \varphi^{-1}(\alpha + \pi\mathbb{Z}) \quad \left(= \text{ZEROS}\left(B/E\right)\right)$$

AND

$$\frac{K(t_n, z)}{\overline{E(t_n)}} \quad \text{INSTEAD OF} \quad K(t_n, z)$$

PROOF: $\langle K_{t_n}, K_{t_m} \rangle = K(t_n, t_m) = \dfrac{B(t_n)\overline{A(t_m)} - B(t_m)\overline{A(t_n)}}{\pi(t_n - t_m)}$

$$= 0$$

$$\|K_{t_n}\|^2 = K(t_n, t_n) = \frac{B'(t_n) A(t_n)}{\pi} > 0.$$

SO DEFINE

$$PF(z) = \sum F(t_n) \frac{K(t_n, z)}{K(t_n, t_n)}$$

SO P IS THE PROJECTION OF F IN SPAN $\{K(t_n, z)\}$.

THEN WE HAVE TO SHOW $PF - F = \lambda B$.

SINCE $\underset{w\in\mathbb{C}}{\text{SPAN}}\{K(w,z)\}$ IS DENSE IN $\mathcal{H}^2(E)$, SO I ONLY NEED TO SHOW $\boxed{PK_w - K_w = \lambda_w B.}$

$$F_N = \sum_{j=1}^{N} a_j K_{w_j} \longrightarrow F$$

$$PF(z) - F(z) = \lim PF_N(z) - F_N(z) = \lim \left(\sum \lambda_{w_j} a_j\right) B(z)$$
$$= \lambda B(z).$$

<u>WE NEED PROB. 47.</u> LET $f : \mathbb{C}\setminus\{t_n\}_{n\in\mathbb{Z}} \xrightarrow{\text{ANALYTIC}} \mathbb{C}$ $(t_n \in \mathbb{R})$, $f^* = f$ AND $\mathrm{Re}(-if) > 0$ IN $\mathbb{C}^+$. THEN $\exists p \geq 0$ $\exists p_n \geq 0$ SO THAT

$$\frac{f(z) - \overline{f(w)}}{z - \overline{w}} = p + \sum \frac{p_n}{(t_n - z)(t_n - \overline{w})}$$

$p_n = \lim_{z\to t_n} (t_n - z) f(z)$ AND CONVERGENCE IS UNIFORM AND ABS. FOR $z, \overline{w}$ AWAY FROM $\{t_n\}$ $\left(\text{BECAUSE } \sum \frac{p_n}{1+t_n^2} < \infty\right)$.

<u>SOL</u>: BY POISSON REP.

$$-if(z) = -izp + iC + \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{1+tz}{t-z}\,\frac{d\mu(t)}{1+t^2} \qquad (d\mu \geq 0)$$

AND $\int_{-\infty}^{\infty} \frac{d\mu(t)}{1+t^2} < \infty$. $\int_a^b d\mu(t) = \lim_{y\downarrow 0} \int_a^b \mathrm{Re}\, -if(x+iy)\,dx$

$$\underset{\substack{\uparrow \\ \text{IF } t_n < a < b < t_{n+1}}}{=} \int_a^b \mathrm{Re}\, -if(x)\,dx = 0.$$

$$\Longrightarrow d\mu(t) = \sum_n \pi p_n \delta_{t_n}(t)$$

$$\Rightarrow f(z) = zp - c + \sum_n \frac{1+t_n z}{t_n - z} \frac{p_n}{1+t_n^2}$$

$$\frac{f(z) - \overline{f(w)}}{z - \overline{w}} = p + \sum_n \frac{p_n}{(t_n - z)(t_n - \overline{w})}$$

APPLYING PROB 47 TO $f = -A/B$ (*) $\boxed{\text{Re } iA/B > 0 \text{ in } \mathbb{C}^+}$

$$\frac{-\frac{A}{B}(z) + \overline{\frac{A}{B}(w)}}{z - w} = p + \sum \frac{A(t_n)/B'(t_n)}{(t_n - z)(t_n - \overline{w})}$$

MULTIPLY BY $\frac{B(z)\overline{B(w)}}{\pi}$

$$K(w,z) = \frac{p}{\pi} B(z)\overline{B(w)} + \sum_n \frac{A(t_n)}{\pi B'(t_n)} \frac{B(z)}{z - t_n} \frac{\overline{B(w)}}{\overline{w} - t_n}$$

$$= \frac{p}{\pi} B(z)\overline{B(w)} + \underbrace{\sum_n K(w,t_n) \frac{K(t_n,z)}{K(t_n,t_n)}}_{pK_w}$$

$$\Rightarrow K_w - pK_w = \lambda \overline{B(w)} B(z)$$

THIS FINISHES THE PROOF MOD. (*). TO PROVE (*)

$z \in \mathbb{C}^+$

$$0 < K(z,z) = \frac{B(z)\overline{A(z)} - \overline{B(z)}A(z)}{\pi(z - \overline{z})} = \frac{|B(z)|^2}{\pi y}\left(\frac{\left(\frac{A}{B}\right)^- - \frac{A}{B}}{2i}\right)$$

$$= \frac{|B(z)|^2}{\pi y} \text{Re } i\frac{A}{B}$$

$$\Rightarrow \text{Re } \frac{iA}{B} > 0 \text{ in } z \in \mathbb{C}^+.$$

CLASS 15 (CHARACTERIZATION OF $\mathcal{H}^2(E)$)

THM. 23 LET $\mathcal{H}^2$ BE A NON-TRIVIAL HILBERT SPACE OF ENTIRE FUNCTIONS F WITH NORM $\|F\|^2 = \langle F, F\rangle$. ASSUME THAT:

(H1) IF $F \in \mathcal{H}^2$ AND $F(\omega) = 0$, ω NONREAL, THEN
$$G(z) = F(z)\,\frac{z-\overline{\omega}}{z-\omega} \in \mathcal{H}^2 \quad \text{AND} \quad \|G\| = \|F\|.$$

(H2) THE FUNCTIONAL $F \longrightarrow F(\omega)$ IS BOUNDED.

(H3) IF $F \in \mathcal{H}^2$ THEN $F^* \in \mathcal{H}^2$ AND $\|F^*\| = \|F\|$.

THEN $\exists\, E \in HB$ SUCH THAT $\mathcal{H}^2 = \mathcal{H}^2(E)$.

PROOF: (H2) $\Longrightarrow$ $F(\omega) = \langle F, K_\omega\rangle$ FOR SOME $K_\omega(z) = K(\omega, z)$ WHICH IS THE REP. KERNEL OF $\mathcal{H}^2$. CLAIM: $K(\omega,\omega) = \|K_\omega\|^2 > 0$ IF $\omega \in \mathbb{C}\setminus\mathbb{R}$. IF $K_\omega \equiv 0$ $(K(\omega,\omega) = 0)$ THEN $F(\omega) = 0$ $\forall F \in \mathcal{H}^2$. SO BY (H1) $F(z)\,\frac{z-\overline{\omega}}{z-\omega} \in \mathcal{H}^2 \Longrightarrow F$ VANISHES AT $z = \omega$ WITH INFINITE MULTIPLICITY $\Longrightarrow F \equiv 0$. $\mathcal{H}^2 = \{0\}$. CONTRADICTION! $K(\omega,\omega) > 0$ $\forall \omega \notin \mathbb{R}$. (*)

RECALL THAT WE ARE AIMING TO

$$\boxed{K(\omega, z) \overset{?}{=} \frac{E(z)\overline{E(\omega)} - E^*(z)\overline{E^*(\omega)}}{2\pi i(\overline{\omega} - z)}}$$

NOTE THAT IF THIS IS TRUE FOR SOME ENTIRE FUNCTION E

$$0 < K(\omega,\omega) = \frac{|E(\omega)|^2 - |E^*(\omega)|^2}{4\pi y} \quad \text{IF} \quad \omega = x + iy.$$

$\Longrightarrow$ $|E(\omega)| > |E^*(\omega)|$ $\omega \in \mathbb{C}^+ \Longrightarrow E \in HB$. SO $\mathcal{H}^2(E)$ EXISTS AND $K_E = K$. SINCE

$$\left.\begin{array}{l} \operatorname{SPAN}_{w\in\mathbb{C}}\{K(w,\cdot)\} \text{ is DENSE } \mathcal{H}^2 \\ \qquad\qquad \| \\ \operatorname{SPAN}_{w\in\mathbb{C}}\{K_E(w,\cdot)\} \text{ is DENSE } \mathcal{H}^2(E) \end{array}\right\} \Rightarrow \mathcal{H}^2 \cap \mathcal{H}^2(E)$$

COINCIDE IN A DENSE SET WITH EQUAL NORM

$\Longrightarrow \mathcal{H}^2 = \mathcal{H}^2(E)$ ISOMETRICALLY. SO WE ONLY NEED TO SHOW (*).

DEFINE $L(w,z) = 2\pi i(\bar{w} - z)\, K(w,z)$ $\quad \left(= E(z)\overline{E(w)} - E^*(z)\overline{E^*(w)}\right)$

Q: <u>HOW TO WRITE E IN TERMS OF L?</u>

$$\text{(i)}\quad L(\alpha,z)L(w,\alpha) \overset{?}{=} \left(E(z)\overline{E(\alpha)} - E^*(z)\overline{E^*(\alpha)}\right)\left(E(\alpha)\overline{E(w)} - E^*(\alpha)\overline{E^*(w)}\right)$$

$$= |E(\alpha)|^2 E(z)\overline{E(w)} + |E^*(\alpha)|^2 E^*(z)\overline{E^*(w)}$$

$$- \overline{E(\alpha)}E^*(\alpha)E(z)\overline{E^*(w)} - E^*(z)\overline{E(w)}\,\overline{E^*(\alpha)}E(\alpha)$$

$$\text{(ii)}\quad L(\bar{\alpha},z)L(w,\bar{\alpha}) \overset{?}{=} |E^*(\alpha)|^2 E(z)\overline{E(w)} + |E(\alpha)|^2 E^*(z)\overline{E^*(w)}$$

$$- E^*(\alpha)\overline{E(\alpha)}E(z)\overline{E^*(w)} - E^*(z)\overline{E(w)}E(\alpha)\overline{E^*(\alpha)}$$

$$\text{(i)} - \text{(ii)} \overset{?}{=} \underbrace{\left(|E(\alpha)|^2 - |E^*(\alpha)|^2\right)}_{L(\alpha,\alpha)} \underbrace{\left(E(z)\overline{E(w)} - E^*(z)\overline{E^*(w)}\right)}_{L(w,z)}$$

$$\Longrightarrow \boxed{\frac{L(\alpha,z)L(w,\alpha)}{L(\alpha,\alpha)} + \frac{L(\bar{\alpha},z)L(w,\bar{\alpha})}{L(\bar{\alpha},\bar{\alpha})} \overset{?}{=} L(w,z)} \quad (**)$$

THIS IS PROP. 51

<u>NOW WE USE (H3).</u> $\|F^*\| = \|F\| \Longleftrightarrow \langle F,G\rangle = \overline{\langle F^*, G^*\rangle}$

$$\left\{\begin{array}{l} \|F-G\|^2 = \|F^*-G^*\|^2 \\ \|F-iG\|^2 = \|F^*+iG^*\|^2 \end{array}\right\}$$

$$\langle F, K_\omega^* \rangle = \overline{\langle F^*, K_\omega \rangle} = \overline{F^*(\omega)} = F(\bar{\omega}) = \langle F, K_{\bar{\omega}} \rangle$$

$$\Longrightarrow \quad K_\omega^* = K_{\bar{\omega}}, \text{ THAT IS, } \overline{K(\omega, \bar{z})} = K(\bar{\omega}, z).$$

BY PROB. S2 AND S3 : IF $K_\omega^* = K_{\bar{\omega}}$ AND L SATISFIES (**)

THEN $K = K_E$ FOR SOME FUNCTION $E \in HB$.

NOW IT IS ENOUGH TO PROVE (**) IS TRUE IN $\mathcal{H}^2$.

DEFINE

$$F_\alpha(\omega, z) = K(\omega, z) - \frac{K(\alpha, z)\, K(\omega, \alpha)}{K(\alpha, \alpha)} \qquad \left| \begin{array}{l} \alpha, \omega \in \mathbb{R} \\ z \in \mathbb{C}. \end{array} \right.$$

NOTE $F_\alpha(\omega, \cdot) \in \mathcal{H}^2$ AND $F(\alpha) = 0$. SO BY (H1)

$$G_\alpha(\omega, z) \overset{(= T_\alpha F_\alpha)}{=} \left(\frac{z - \bar{\alpha}}{z - \alpha} \right) F_\alpha(\omega, z) \in \mathcal{H}^2 \text{ AND } G_\alpha(\bar{\alpha}) = 0.$$

SUPPOSE $U, V \in \mathcal{H}^2$ $\quad U(\alpha) = 0, \quad V(\bar{\alpha}) = 0 \quad \left(\begin{array}{l} U = F_\alpha \\ V = G_\alpha \end{array} \right)$

I CLAIM $\langle T_\alpha U, V \rangle = \langle U, T_{\bar{\alpha}} V \rangle$ WHERE

$$T_\alpha : F(z) \longrightarrow \left(\frac{z - \bar{\alpha}}{z - \alpha} \right) F(z) \quad \text{IF} \quad F(\alpha) = 0. \qquad \left(\text{"} T_\alpha^* = T_{\bar{\alpha}} \text{"} \right)$$

$$\left.\begin{aligned} 0 &\overset{(H1)}{=} \| T_\alpha U + V \|^2 - \| U + T_{\bar{\alpha}} V \|^2 \\ &= 2\,\mathrm{Re}\left(\langle T_\alpha U, V \rangle - \langle U, T_{\bar{\alpha}} V \rangle \right) \\ 0 &= \| T_\alpha U + iV \|^2 - \| U + i T_{\bar{\alpha}} V \|^2 \\ &= 2\,\mathrm{Im}\left(\langle T_\alpha U, V \rangle - \langle U, T_{\bar{\alpha}} V \rangle \right) \end{aligned}\right\} \Longrightarrow \boxed{\begin{array}{c} \langle T_\alpha U, V \rangle \\ \| \\ \langle U, T_{\bar{\alpha}} V \rangle \end{array}}$$

$$\langle G_\alpha(w,\cdot), F_{\bar\alpha}(\beta,\cdot)\rangle = \langle T_\alpha F_\alpha(w,\cdot), F_{\bar\alpha}(\beta,\cdot)\rangle$$

$$= \langle F_\alpha(w,\cdot), T_{\bar\alpha} F_{\bar\alpha}(\beta,\cdot)\rangle$$

$$\left\langle \left(\frac{z-\bar\alpha}{z-\alpha}\right)\left[K(w,z) - \frac{K(\alpha,z)K(w,\alpha)}{K(\alpha,\alpha)}\right], K(\beta,z) - \frac{K(\bar\alpha,z)K(\beta,\bar\alpha)}{K(\bar\alpha,\bar\alpha)}\right\rangle_z$$

$$= \frac{\beta-\bar\alpha}{\beta-\alpha} K(w,\beta) - \frac{K(\alpha,\beta)K(w,\alpha)}{K(\alpha,\alpha)}$$

$$= \frac{\bar w-\bar\alpha}{\bar w-\alpha}\overline{K(\beta,w)} - \frac{\overline{K(\bar\alpha,w)}\,\overline{K(\beta,\bar\alpha)}}{\overline{K(\bar\alpha,\bar\alpha)}}$$

$$= \frac{\bar w-\bar\alpha}{\bar w-\alpha} K(w,\beta) - \frac{K(w,\bar\alpha)K(\bar\alpha,\beta)}{K(\bar\alpha,\bar\alpha)}$$

NOTE $\overline{K(w,z)}$ —
$= \overline{\langle K_w, K_z\rangle}$
$= \langle K_z, K_w\rangle$
$= K(z,w)$

EXERCISE: USING $K(w,z) = \dfrac{L(w,z)}{2\pi i(\bar w - z)}$ SHOW THIS IS EXACTLY IDENTITY (**).

$\blacksquare$

CLASS 16

THM J LET $E \in HB \cap \mathcal{P}$ AND NO REAL ZEROS. E HAS AT LEAST ONE ZERO. ASSUME $E^*(z) = E(z-i)$. ASSUME THAT THE MAP

$$T : F(z) \longrightarrow F(z+i)$$

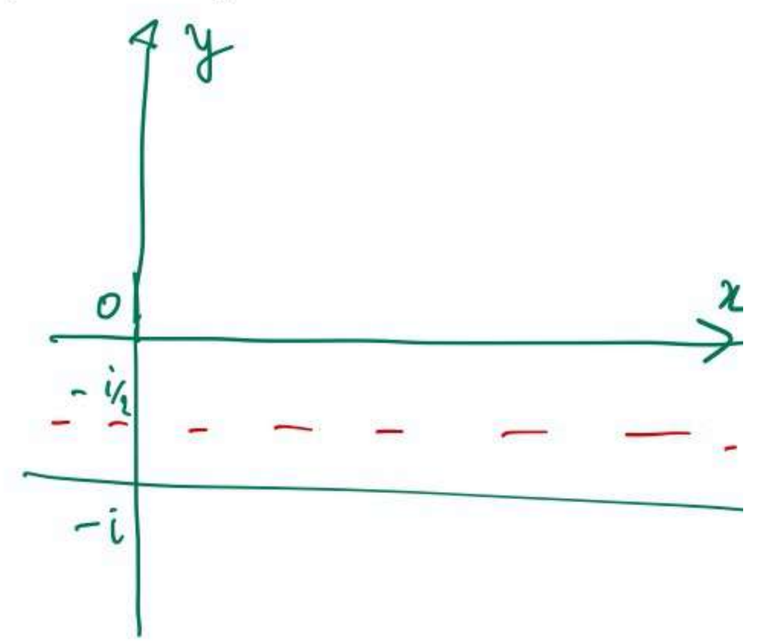
$\boxed{T + T^* \geq 0}$

IS SUCH THAT $\boxed{\mathrm{Re} \langle TF, F \rangle \geq 0}$ $\forall F \in \mathcal{H}(E)$ SUCH THAT $TF \in \mathcal{H}(E)$ $\left(D(T) = \{ F \in \mathcal{H} : F(z+i) \in \mathcal{H} \} \right)$.

THEN ALL ZEROS OF E LIE IN THE LINE

$$\mathrm{Im}\, z = -1/2 .$$

IF $\mathrm{Re} \langle TF, F \rangle > 0$ IF $F \in D(T) \setminus \{0\}$ THEN ALL ZEROS OF E ARE SIMPLE.



REMARK:

$$E^*(z) = E(z-i)$$
$$E^*(z+i) = E(z)$$

IF $E(\omega) = 0$ THEN $E^*(\omega + i) = 0$, SO

$E(\overline{\omega} - i) = 0$ SO $\mathrm{zeros}(E) \subset \{ -1 < \mathrm{Im}\, z < 0 \}$

PROOF:

$$K(\omega, z) = \frac{E(z)\,\overline{E(\omega)} - E^*(z)\,\overline{E^*(\omega)}}{2\pi i\,(\overline{\omega} - z)} \qquad \text{IF } \omega \in \mathrm{zero}(E)$$

$$K(\omega, z+i) = \frac{-E(z)\,\overline{E(\omega - i)}}{2\pi i\,(\overline{\omega} - z - i)} = -\frac{E(z)\,\overline{E(\omega+i)}}{2\pi i\,(\overline{\omega+i} - z)} \; \frac{\overline{E(\omega-i)}}{\overline{E(\omega+i)}}$$

$$= -K(\omega+i, z)\, \frac{\overline{E(\omega-i)}}{\overline{E(\omega+i)}}$$

SO

$K(\omega, z) \in D(T)$ IF $E(\omega) = 0$

NOW DEFINE FOR $F, G \in D(T)$

$$(F \mid G) = \langle F(z+i), G(z) \rangle + \langle F(z), G(z+i) \rangle$$

AND NOTE $(F|F) = 2\,\mathrm{Re}\,\langle F(z+i), F(z)\rangle \geq 0$. SO THE C-S INEQ. SAYS

$$|(F|G)|^2 \leq (F|F)(G|G)$$

NOW LET'S COMPUTE, FOR $w \in \mathrm{ZERO}(E)$,

$$(F|K(w,z)) = \langle F(z+i), K(w,z)\rangle + \langle F(z), K(w,z+i)\rangle$$
$$= F(w+i)\left(1 - \frac{E(w-i)}{E(w+i)}\right)$$

$$\left|F(w+i)\left(1 - \frac{E(w-i)}{E(w+i)}\right)\right|^2 \leq (F|F)\left(K(w,z)|K(w,z)\right)$$

$2\mathrm{Re}\,K(w,w+i)$

$$= (F|F)\left(K(w,w+i) + K(w+i,w)\right)$$

- <u>IF $K(w,w+i) \neq 0$</u>

$$K(w,w+i) = \left.\frac{-E(z)\,\overline{E(w-i)}}{2\pi i(\bar{w}-z-i)}\right|_{z=w} \quad \text{THEN} \quad \bar{w} - i = w$$

, OR EQUIV., $\mathrm{Im}\,w = -1/2$. IN THIS CASE,

$$K(w,w+i) = \frac{E'(w)\,\overline{E(w-i)}}{2\pi i} \qquad \left(\mathrm{Re}\,\frac{E'(w)\,E(w-i)}{2\pi i} \geq 0\right)$$

- <u>IF $K(w,w+i) = 0$</u> : WELL, PICK $F \in D(T)\setminus\{0\}$ AND

$$\tilde{F}(z) = \frac{F(z)}{(z-w-i)^m} \quad \text{IF} \quad F(w+i) = 0 \text{ OF ORDER } m.$$

NOTE $\tilde{F} \in \mathcal{H}(E)$ AND $\tilde{F}(z+i) \in \mathcal{H}(E)$, SO $\tilde{F} \in D(T)$

AND $\tilde{F}(\omega+i) \neq 0$. THEN USING C-S INEQ. FOR $F = \tilde{F}$

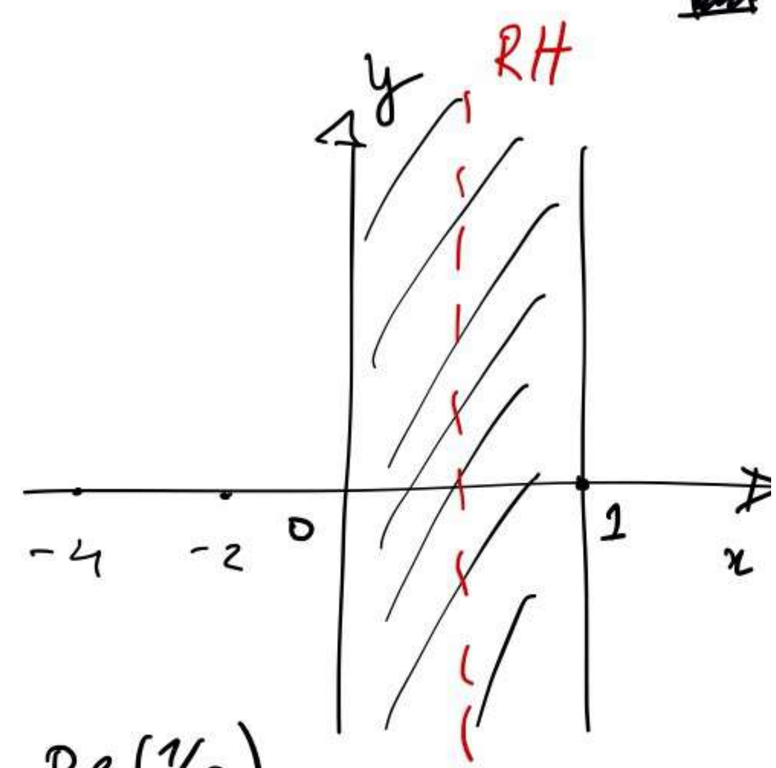
WE GET $\underset{\underset{\tilde{E}(\omega)}{\parallel}}{E(\omega - i)} = E(\omega + i) \implies \boxed{E(\omega+i) = \overline{E(\overline{\omega})}}$

SINCE $\overline{\omega}, \omega + i \in \mathbb{C}^+$ AND HAVE THE SAME REAL PART,

AND $E \in \mathcal{P}$ (PROB. 15 $E(z) = E(0)\, e^{hz}$, $h \in \mathbb{R}$). THEN

$\overline{\omega} = \omega + i \implies \operatorname{Im} \omega = -1/2$. ∎

§ CONNECTION WITH RH



- $\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$ $(\operatorname{Re} s > 1)$

$$\xi(s) = \frac{1}{2} s(s-1)\, \pi^{-s/2}\, \Gamma(s/2)\, \zeta(s)$$

$$= \xi(0)\, e^{Bs} \prod_{\rho} (1 - s/\rho)\, e^{s/\rho}, \quad B = -\sum \operatorname{Re}(1/\rho)$$

- IF χ IS A DIRICHLET CHARACTER MODULO q

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \qquad (\operatorname{Re} s > 1)$$

RECALL: $\chi : \mathbb{Z}_q^\times \longrightarrow \{|z| = 1\}$ $\quad \chi(mn) = \chi(m)\chi(n)$.

AND WE EXTEND χ TO $\mathbb{N} = \{1, 2, \dots\}$ BY RESTRICTION.

$\chi = \chi_0$ IS PRINCIPAL IF $\chi(n) = 1$ IF n IS COPRIME WITH q

χ IS INDUCED IF $\exists\, \tilde{\chi}$ MOD a, WHERE $a \mid q$, AND

$\chi = \tilde{\chi}\, \chi_0$.

χ IS PRIMITIVE IF IT IS NOT INDUCED.

- <u>EULER PRODUCT</u>: $L(s,\chi) = \prod_p \left(1 - \chi(p)\, p^{-s}\right)^{-1} \quad (\mathrm{Re}\, s > 1)$

SO $L(s,\chi) \neq 0$ IF $\mathrm{Re}\, s > 1$. ONE CAN SHOW $L(1+it,\chi) \neq 0 \quad \forall t \in \mathbb{R}$

- <u>FUNCTIONAL EQ.</u>: IF χ IS PRIMITIVE, NON-PRINCIPAL THEN

$$\xi(s,\chi) = \left(q/\pi\right)^{\frac{s+a}{2}} \Gamma\left(\frac{s+a}{2}\right) L(s,\chi) \qquad a = \begin{cases} 0, & \chi(-1)=1 \\ 1, & \chi(-1)=-1 \end{cases}$$

THEN $\xi(s,\chi) = \lambda\, \xi(1-s,\overline{\chi}) \left(= \hat{\chi}(1)\, \xi(1-s,\chi)^*\right)$

WHERE $|\lambda| = 1$ $\left(\lambda = \hat{\chi}(1)\right.$ IF $\chi(-1)=1$ OR $\left.\lambda = i\,\hat{\chi}(1)\right.$ IF $\hat{\chi}(-1)=-1.\left.\right)$

ALL ZEROS OF $\xi(s,\chi)$ ARE IN $0 < \mathrm{Re}\, s < 1$.

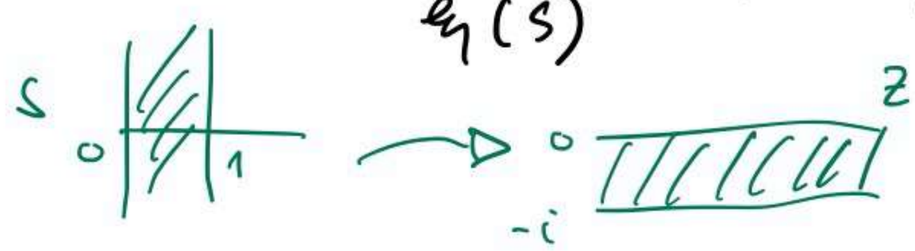
- <u>HADAMARD FACT.</u> $\xi(s,\chi) = \xi(0,\chi)\, e^{Bs} \prod_\rho \left(1 - s/\rho\right) e^{s/\rho}$

$B = -\sum \mathrm{Re}\,(1/\rho)$.

<u>THM K</u> $\quad E(z) = \xi(1-iz,\chi) \in \underline{P} \cap \mathcal{H}B$

<u>PROOF</u>: $\partial_y \log|E(z)| = \mathrm{Re}\; i\,\dfrac{E'(z)}{E(z)} = \mathrm{Re}\; \dfrac{\xi'(s)}{\xi(s)} \quad (s = 1-iz)$

$\mathrm{Im}\, z > 0 \quad (\mathrm{Re}\, s > 1)$



$$\mathrm{Re}\,\frac{\xi'(s)}{\xi(s)} = B + \mathrm{Re}\left(\sum_\rho \frac{1}{s-\rho} + \frac{1}{\rho}\right) = \sum_\rho \frac{\mathrm{Re}\,(s-\rho)}{|s-\rho|^2} > 0$$

TO SHOW $E \in \mathcal{H}B$ USE HADAMARD FACTORIZATION

$$\xi(s,\chi) = \xi(0,\chi) \prod_\rho \left(1 - s/\rho\right) e^{i s\, \mathrm{Im}(1/\rho)}$$

AND THEN PROVE $E \in \mathcal{H}B$ TERM BY TERM (EXERCISE) $\blacksquare$

NOW WE COULD APPLY THE THM. TO $E(z) = \xi(1-iz, \chi)$

IF χ is PRIMITIVE AND NON-PRINCIPAL BECAUSE THE FUNCT. EQ.

FOR $\xi \Rightarrow \lambda E^*(z) = E(z-i)$. IF $\tilde{E}(z) = \lambda^{-1/2} E(z)$

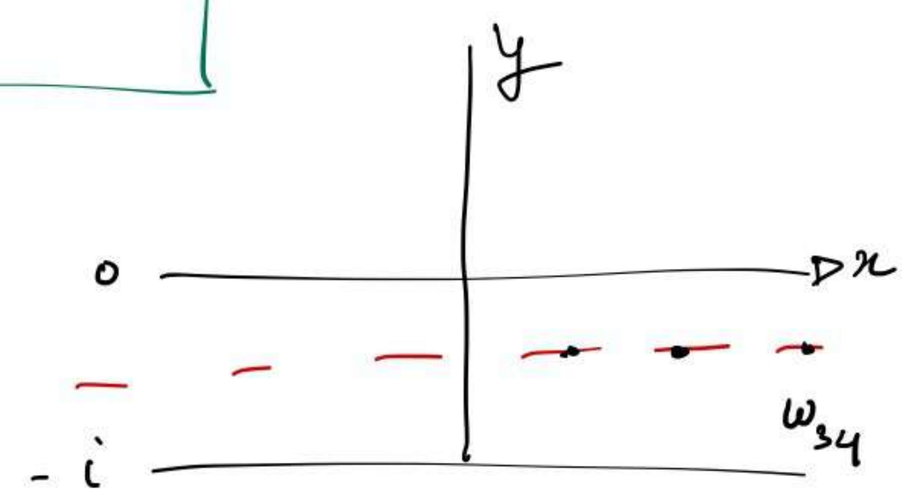
THEN $\tilde{E}^*(z) = \lambda^{1/2} E^*(z) = \lambda^{-1/2} E(z-i) = \tilde{E}(z-i)$.

OR IF $E(z) = \xi(1-iz) = \frac{1}{2} s(s-1) \pi^{-s/2} \Gamma(s/2) \zeta(s)$.

RECALL: $\mathrm{Re} \dfrac{E'(w)\,\overline{E(w-i)}}{i} \geq 0$ MUST HOLD

IF $E(w) = 0$. $\boxed{\mathrm{Re} \dfrac{E'(w)\,\overline{E(w-i)}}{i} \geq 0}$

ZETA: $w = w_{34}$



$$\mathrm{Re} \frac{E'(w_{34})\,\overline{E(w_{34}-i)}}{i} = -5.38\ldots \times 10^{-69} < 0$$

χ_4 1 0 −1 0 $\quad w =$ ONE OF THE ZEROS

$$\mathrm{Re} \frac{E'(w, \chi_4)\,\overline{E(w-i, \chi_4)}}{i} = -2.31 \times 10^{-45} < 0$$

REF: A NOTE ON SOME POSITIVITY CONDITIONS RELATED TO ZETA AND L-FUNCTIONS.

CONREY AND LI, (2000)

CLASS 17 (HOMOGENEOUS SPACES)

A HOM. DE BRANGES SPACE SATISFIES THAT $a^{1+\nu} F(az) \in \mathcal{H}(E)$ WHENEVER $F \in \mathcal{H}(E)$ $(0 < a < 1)$ AND HAS THE SAME NORM. (CHAPTER 50)

$(\Rightarrow \nu > -1$ AND E HAS NO ZEROS IN $\mathbb{R}\setminus\{0\})$ (NOTE PW^2 is HOM. $\nu = -1/2$)

§ BESSEL FUNCTIONS $(\nu > -1)$

- $$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(m+\nu)!} \left(z/2\right)^{2m+\nu} \qquad (-\pi < \arg z < \pi)$$

$$\boxed{\alpha! = \Gamma(\alpha+1)}$$

(BESSEL FUNCTION OF THE FIRST KIND OF ORDER ν)

- BESSEL DIFF. EQ: $$z^2 \frac{d}{dz^2} J + z \frac{dJ}{dz} + (z^2 - \nu^2) J = 0 \qquad (J = J_\nu)$$

$$J_{-1/2}(z) = \sqrt{\frac{2}{z}} \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{2^{2m} m!\,(m-1/2)!} = \sqrt{\frac{2}{\pi z}} \sum_{m=0}^{\infty} \frac{(-1)\, z^{2m}}{(2m)!} = \sqrt{\frac{2}{\pi z}} \cos(z)$$

$$\boxed{\begin{aligned} &2^{2m} m!\,(m-1/2)! \\ &= (2m)!\,(-1/2)! \end{aligned}}$$

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sin(z)$$

IN GENERAL: $n \geq -1$ $$J_{n+1/2}(z) = \sqrt{\frac{2}{\pi z}} \left(e^{iz} P_n(1/z) + e^{-iz} P^*(1/z) \right)$$

- INTEGRAL REP. (BETA FUNCTION: $B(x,y) = \dfrac{(x-1)!\,(y-1)!}{(x+y-1)!} = \displaystyle\int_0^1 t^{x-1}(1-t)^{y-1}\,dt$)

$$(z/2)^{-\nu} J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{2^{2m} m!\,(m+\nu)!} = \frac{1}{(-1/2)!\,(\nu-1/2)!} \sum_{m=0}^{\infty} \frac{z^{2m}(-1)^m}{(2m)!} \frac{(\nu-1/2)!\,(m-1/2)!}{(m+\nu)!}$$

$$= \frac{1}{(\cdots)} \sum_m \frac{z^{2m}(-1)^m}{(2m)!} \int_0^1 t^{m-1/2} (1-t)^{\nu-1/2}\,dt$$

$$= \frac{1}{(\cdots)} \int_0^1 \cos(z\sqrt{t})\,(1-t)^{\nu-1/2} \frac{dt}{\sqrt{t}} \qquad (s = \sqrt{t})$$

$$\Rightarrow \boxed{\left(\tfrac{z}{2}\right)^{-\nu} J_\nu(z) = \frac{1}{(-\frac{1}{2})!\,(\nu-\frac{1}{2})!}\int_{-1}^{1} \cos(sz)\,(1-s^2)^{\nu-1/2}\,ds} \qquad \boxed{\nu > -\tfrac{1}{2}}$$

THUS $\left(z/2\right)^{-\nu} J_\nu(z)$ IS REAL ENTIRE AND OF EXP. TYPE $\underline{1}$.

(AND FOR $-1<\nu<-1/2$ THERE ARE OTHER INT. REP. THAT SHOWS THE SAME)

- IT IS ALSO KNOWN THAT $\left(z/2\right)^{-\nu} J_\nu(z)$ HAS ONLY REAL ZEROS,

BESSEL ZEROS $\{j_{\nu,n}\}_{n\geq 1} \longrightarrow a_{\nu,n} = \begin{cases} j_{\nu,n}, & n\geq 1 \\ -j_{\nu,n}, & n\leq -1 \end{cases}$

THE HB FUNCTION

$$E_\nu(z) = \nu!\,\left(z/2\right)^{-\nu}\left(J_\nu(z) - i\,J_{\nu+1}(z)\right) \qquad (\nu>-1)$$

$$= A_\nu(z) - i\,B_\nu(z)$$

- A_ν is EVEN, B_ν is ODD
- $\begin{cases} A_\nu' = -B_\nu \\ B_\nu' = A_\nu - (2\nu+1)\dfrac{B_\nu}{z} \end{cases}$
- $A_\nu(z) = e^{hz}\prod_{n\geq 1}\left(1 - z^2/j_{\nu,n}\right)$
- $B_\nu(z) = e^{uz}\, z\prod_{n\geq 1}\left(1 - z^2/j_{\nu+1,n}\right)$

- $A_\nu(0) = 1$, $B_\nu(0) = 0$
- $E_\nu^* = A_\nu + iB_\nu = E_\nu(-z)$
- $E_{-1/2}(z) = e^{-iz}$
- THINK OF "$A = \cos$" "$B = \sin$"

$\Rightarrow A_\nu, B_\nu \in \mathcal{P}$ (LAGUERRE-POLYA CLASS ACTUALLY)

ZEROS OF $J_\nu, J_{\nu+1}$ INTERLACE

CLAIM: $E_\nu \in HB$. LET $z \in \mathbb{C}^+$ (AND NO REAL ZEROS)

$$\frac{|E_\nu^*(z)|^2}{|E_\nu(z)|^2} = \frac{|A_\nu|^2 + |B_\nu|^2 - 2\,\mathrm{Re}\left(i\bar{A}_\nu A_\nu'\right)}{|A_\nu|^2 + |B_\nu|^2 + 2\,\mathrm{Re}\left(i\bar{A}_\nu A_\nu'\right)} < 1$$

IFF $\quad \mathrm{Re}\,(i\,\overline{A_\nu}\,A_\nu') > 0 \quad$ IN $\quad \mathbb{C}^+$. $\quad \mathrm{Re}\; i\,\overline{A_\nu}\,A_\nu' = |A_\nu|^2\,\mathrm{Re}\; i\,\dfrac{A_\nu'}{A_\nu}$

$$= |A_\nu|^2\,\partial_y \log|A_\nu| > 0$$

SINCE $\quad A_\nu \in \underline{\mathcal{P}}$.

- ASYMPTOTICS $\quad J_\nu(z) = \sqrt{\dfrac{2}{\pi z}}\left(\cos\left(z - \dfrac{\nu\pi}{2} - \pi/4\right) + O\left(\dfrac{1}{z}\right)\right)$

FOR $\quad -\pi < \arg z < \pi$

THIS SHOW $\quad \dfrac{E_\nu(x)}{\nu!\,(x/2)^{-\nu}} = J_\nu(x) - i\,J_{\nu+1}(x) \sim \sqrt{\dfrac{2}{\pi x}}\left(e^{-i(x\cdots)} + O\left(\dfrac{1}{x}\right)\right)$

$$\Longrightarrow \quad |E_\nu(x)| \approx |x|^{-\nu-1/2} \quad \text{FOR} \quad |x| \gg 1.$$

THM L $\quad$ LET $\mathcal{H}_\nu$ BE THE SPACE OF ENTIRE FUNCTIONS F OF EXP. TYPE ≤ 1 AND SUCH THAT

$$\|F\|_\nu^2 = \int_{-\infty}^{\infty} |F(x)|^2\,|x|^{2\nu+1}\,dx < \infty \qquad (\nu > -1)$$

THEN $\mathcal{H}_\nu$ IS A DE BRANGES SPACE, $\mathcal{H}_\nu = \mathcal{H}(E_\nu)$

AND $\quad \displaystyle\int_{-\infty}^{\infty} |F(x)|^2\,|x|^{2\nu+1}\,dx = \left(2^\nu \nu!\right)^2 \frac{2}{\pi} \int_{-\infty}^{\infty} \left|\frac{F(x)}{E_\nu(x)}\right|^2 dx$

$\forall F \in \mathcal{H}_\nu = \mathcal{H}(E_\nu)$.

PROOF: $\quad F \in \mathcal{H}_\nu \Longrightarrow F \in \mathcal{H}(E_\nu)$ $\qquad \boxed{|x|^{2\nu+1} \approx |E_\nu(x)|^{-2}}$

$E_\nu = A_\nu - iB_\nu$ IS OF EXP. TYPE $= 1$ AND $\displaystyle\int_{-\infty}^{\infty} \frac{\log|E_\nu(x)|}{1+x^2}\,dx < \infty$

$\xRightarrow{\text{KREIN THM}} \quad E_\nu,\ E_\nu^* \in BT(\mathbb{C}^+)$. $\quad \underset{\underset{1}{\parallel}}{\tau(E)} = \max\left(\nu(E), \nu(E^*)\right) = \nu(E)$

ALSO, IF $F \in \mathcal{H}_\nu$ IT IS NOT HARD TO SHOW $\displaystyle\int_{-\infty}^{\infty} \frac{\log^+ |F(x)|}{1+x^2}\,dx < \infty$

KREIN THM

$\Longrightarrow F, F^* \in BT(\mathbb{C}^+)$, $\tau(F) = \max(\nu(F), \nu(F^*)) \leq 1$.

$$F/E_\nu, \quad F^*/E_\nu \in BT(\mathbb{C}^+).$$

$$\nu(F/E_\nu) = \nu(F) - \nu(E_\nu) = \nu(F) - 1 \leq 0$$

$$\nu(F^*/E_\nu) = \nu(F^*) - \nu(E_\nu) = \nu(F^*) - 1 \leq 0.$$

AND CLEARLY $\displaystyle\int |F/E_\nu|^2 < \infty$.

$\underline{F \in \mathcal{H}(E_\nu) \Longrightarrow F \in \mathcal{H}_\nu}$ (MORE OR LESS THE SAME)

$$c \int_{-\infty}^{\infty} \frac{F(x)\,\overline{G(x)}}{|E(x)|^2}\,dx \overset{(*)}{=} \int_{-\infty}^{\infty} F(x)\,\overline{G(x)}\,|x|^{2\nu+1}\,dx \qquad \forall F, G \in \mathcal{H}(E)$$

BY THE BESSEL ASYMPTOTIC $A \notin \mathcal{H}(E)$, SO $\{K(a_n, z)\}_n$ IS A ORTHOGONAL BASIS OF $\mathcal{H}(E)$, a_n ZEROS OF A.

SO I ONLY NEED TO SHOW $(*)$ FOR $F, G \in \{K(a_n, z)\}_n$.

$\underline{\text{IF } n \neq m}$

$$\text{LHS}_{(*)} = c \int_{-\infty}^{\infty} \frac{K(a_n, x)\,\overline{K(a_m, x)}}{|E(x)|^2}\,dx = c\,K(a_n, a_m) = 0$$

$$\text{RHS}_{(*)} = \frac{B(a_n)B(a_m)}{\pi^2} \int_{-\infty}^{\infty} \frac{A(x)^2}{(x-a_n)(x-a_m)}\,|x|^{2\nu+1}\,dx$$

$$= \frac{B(a_n)B(a_m)(\nu!\,2^\nu)^2}{\pi^2} \int_{-\infty}^{\infty} \frac{J_\nu(|x|)^2}{(x-a_n)(x-a_m)} |x|\, dx$$

$$= \frac{L}{a_n - a_m} \int_{-\infty}^{\infty} J_\nu(|x|)^2 \left(\frac{|x|}{x-a_n} - \frac{|x|}{x-a_m} \right) dx$$

$$= \frac{L}{a_n - a_m} \int_{0}^{\infty} J_\nu(x)^2 \left(\frac{2a_n x}{x^2 - a_n^2} - \frac{2a_m x}{x^2 - a_m^2} \right) dx$$

CONTOUR INT. ARGUMENT (WATSON, 13.53, (4))

$$\int_0^\infty J_\nu(x) \frac{x}{x^2 - r^2} dx = \frac{\pi i}{2} J_\nu(r) \left(J_\nu(r) + i Y_\nu(r) \right) \qquad (\operatorname{Im} r > 0)$$

$$\Longrightarrow \int_0^\infty J_\nu(x) \frac{x}{x^2 - j_{\nu,n}^2} = 0$$

SO RHS $(*)$ $= 0$.

IF $a_n = a_m$

$$\text{RHS } (*) = c\, K(a_n, a_n) = -c\, A'(a_n) B(a_n) = c\, B(a_n)^2$$

$$\text{LHS } (*) = \frac{L}{\tilde{a}_n - \tilde{a}_m} \int_0^\infty J_\nu(x)^2 \left(\frac{2\tilde{a}_n x}{x^2 - \tilde{a}_n^2} - \frac{2\tilde{a}_m x}{x^2 - \tilde{a}_m^2} \right) dx \Bigg|_{\varepsilon=0}$$

$\varepsilon = 0$

$\tilde{a}_n = a_n + i2\varepsilon$

$\tilde{a}_m = a_n + i\varepsilon$

$$= \frac{L}{\tilde{a}_n - \tilde{a}_m} \frac{\pi i}{2} \left(2\tilde{a}_n J_\nu(\tilde{a}_n) \left(J_\nu(\tilde{a}_n) + i Y_\nu(\tilde{a}_n) \right) - (\cdots) \right) \Bigg|_{\varepsilon=0}$$

$$= (2^{\nu}\nu!)^2 \frac{2}{\pi} B(a_n)^2 \implies \boxed{c = (2^{\nu}\nu!)^2 \frac{2}{\pi}.}$$

CLASS 18

§1. PRIME NUMBER THM. VIA TAUBERIAN THEORY

$$\pi(x) = \sum_{\substack{p \le x \\ p \text{ prime} \ge 2}} 1 \quad , \quad \psi(x) = \sum_{1 \le n \le x} \Lambda(n)$$

$$\Lambda(n) = \begin{cases} \log p, & n = p^a \\ 0, & \text{otherwise} \end{cases}$$

$$\zeta(s) = \sum_{n \ge 1} \frac{1}{n^s} \qquad (\operatorname{Re} s > 1)$$

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

P.N.T $\quad \pi(x) \sim \dfrac{x}{\log x}$, OR EQUIV., $\psi(x) \sim x$

WHY? • $\psi(x) = \displaystyle\sum_{p \le x} \left\lfloor \frac{\log x}{\log p} \right\rfloor \log p \le \log x \; \pi(x)$

$$\implies \frac{\psi(x)}{x} \overset{(1)}{\le} \frac{\log x}{x} \pi(x)$$

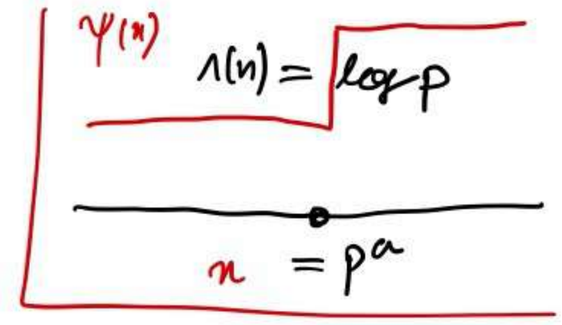
• $$\sum_{x^{\alpha} < p \le x} \log p \ge \alpha \log x \left(\pi(x) - \pi(x^{\alpha}) \right) \le \psi(x)$$

$0 < \alpha < 1$

$$\implies \frac{\psi(x)}{x} \overset{(2)}{\ge} \alpha \frac{\log x}{x} \left(\pi(x) - \pi(x^{\alpha}) \right) \ge \alpha \frac{\log x}{x} \pi(x) - \alpha \frac{\log x}{x^{1-\alpha}}$$

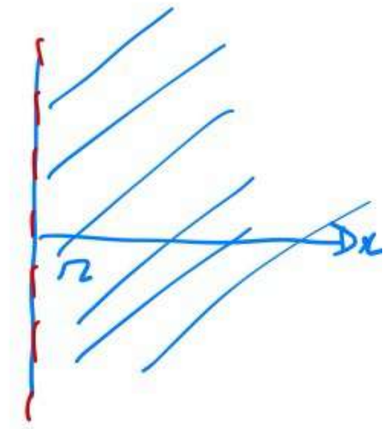
SO (1) AND (2) SHOW $\quad$ "$\displaystyle\lim_{x \to \infty} \frac{\psi(x)}{x} = \lim_{x \to \infty} \frac{x\,\pi(x)}{\log x}$".

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = \int_{1^-}^{\infty} x^{-s}\, d\psi(x) = \int_{0^-}^{\infty} e^{-st}\, d\psi(e^t)$$



THM M (WIENER-IKEHARA) LET $\alpha : [0,\infty) \to \mathbb{R}$ BE NONDECREASING AND $\alpha(0) = 0$. LET

$$F(s) = \int_{0^-}^{\infty} e^{-st}\, d\alpha(t)$$



BE THE LAPLACE TRANSF. OF $d\alpha$. ASSUME IT CONVERGES ABS. FOR $\operatorname{Re} s > r$. ASSUME THAT FOR SOME $A > 0$

$$F(s) - \frac{A}{s-r}$$

is CONTINUOUS FOR $\operatorname{Re} s \geq r$. THEN

$$\alpha(t) \sim \frac{A}{r} e^{rt} \quad (t \to \infty).$$

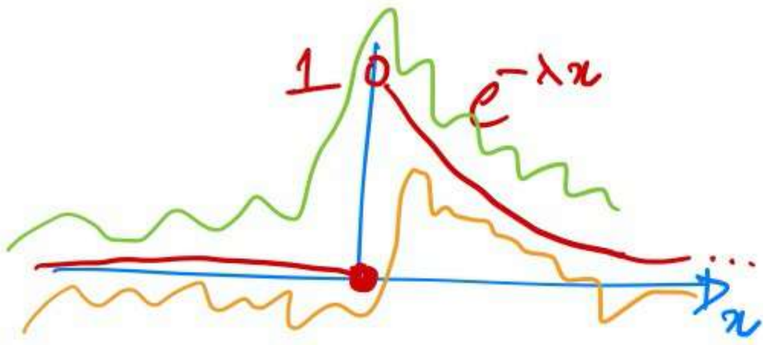
INDEED, $-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}$ is CONTINUOUS FOR $\operatorname{Re} s \geq 1$

$$\Longrightarrow \quad \psi(e^t) \sim e^t \quad \Longleftrightarrow \quad \psi(x) \sim x.$$

SO THE PNT FOLLOWS.

Q. WHAT HAPPENS WHEN WE HAVE MULTIPLE SINGULARITIES?

§ ONE-SIDED APROXIMATIONS $(\lambda > 0)$



LET $$E(\lambda, x) = \begin{cases} e^{-\lambda x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

$$m \leq E(\lambda, x) \leq M$$

I WANT TO BUILD $m, M \in PW^1(2\pi)$ $\left(\operatorname{supp} \hat{m}, \hat{M} \subset [-1,1] \right)$

AND SUCH $\hat{M}(0) = \int_{-\infty}^{\infty} M(x)\,dx$ IS MINIMAL

$\hat{m}(0) = \int_{-\infty}^{\infty} m(x)\,dx$ IS MAXIMAL

TAKE m, $$\sum_{n \in \mathbb{Z}} m(n) = \sum_{n \in \mathbb{Z}} \hat{m}(n) = \hat{m}(0)$$

$$\wedge\!\!\!\backslash$$

$$\sum_{n=1} e^{-\lambda n} = \frac{1}{e^{\lambda} - 1} \implies \hat{m}(0) \leq \frac{1}{e^{\lambda} - 1}$$

EQUALITY IS ATTAINED IFF $$\begin{cases} m(n) = 0, & n \leq 0 \\ m(n) = e^{-\lambda n}, & n \geq 1 \\ m'(n) = 0, & n \leq -1 \\ m'(n) = -\lambda e^{-\lambda n}, & n \geq 1 \end{cases}$$

INTERPOLATION WITH DERIVATIVES

$F \in PW^1(2\pi)$ THEN

$$F(z) = \sum_n F(n) \left(\frac{\sin \pi(z-n)}{\pi(z-n)} \right)^2 + F'(n) \frac{\sin^2 \pi(z-n)}{\pi^2 (z-n)}.$$

CONVERSELY, AS LONG AS $(a_n), (b_n) \in \ell^1$ THEN WE CAN PUT $F(n) = a_n$, $F'(n) = b_n$

SO $$m(z) = \sum_{n \geq 1} \left\{ e^{-\lambda n} \left(\frac{\sin \pi(z-n)}{\pi(z-n)} \right)^2 - \lambda e^{-\lambda n} \frac{\sin^2 \pi(z-n)}{\pi^2 (z-n)} \right\} + a \frac{\sin^2 \pi z}{\pi^2 z}$$

IT TURNS OUT THAT $\exists!\, a$ THAT WORKS AND $m(x) \leq E(\lambda, x)$!!!

THIS m IS OPTIMAL IN THE SENSE $\displaystyle\int_{-\infty}^{\infty} m(x)\,dx = \frac{1}{e^{\lambda}-1}$ IS MAXIMAL.

SIMILARLY, $\exists!\, M \geq E(\lambda, x)$ AND $\displaystyle\int_{-\infty}^{\infty} M(x)\,dx = \frac{e^{\lambda}}{e^{\lambda}-1}$ IS MINIMAL!

$$M(x) = m(x) + \left(\frac{\sin \pi x}{\pi x}\right)^2 !$$

THM N (A CLASS OF EXTREMAL FUNCT. FOR THE FOURIER TRANSF., GRAHAM & VAALER)

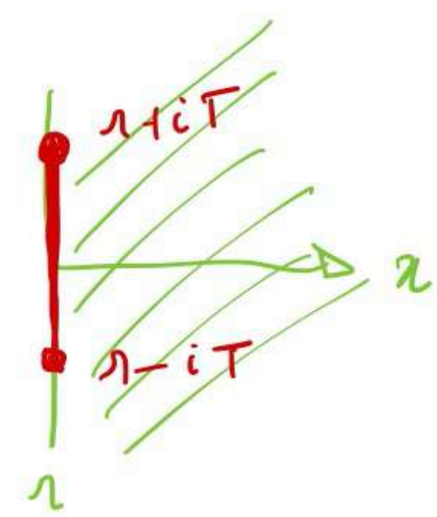
LET $\alpha : [0, \infty) \to \mathbb{R}$ BE NON-DECREASING, $\alpha(0) = 0$, AND ASSUME

$$F(s) = \int_{0^-}^{\infty} e^{-st}\, d\alpha(x)$$

CONVERGES ABS. FOR $\operatorname{Re} s > r$. ASSUME FOR SOME $A > 0$

$$F(s) - \frac{A}{s-r} \quad \text{IS CONTINUOUS}$$

IN $\{\operatorname{Re} s \geq r\} \cap \{-2\pi T < \operatorname{Im} s < 2\pi T\}$



THEN

$$o(1) + L\,\frac{A}{r} \leq \frac{\alpha(t)}{e^{rt}} \leq U\,\frac{A}{r} + o(1)$$

$$L = \frac{rT}{e^{rT}-1}, \quad U = \frac{e^{rT}\, rT}{e^{rT}-1}.$$

IN PARTICULAR, $T \to \infty$, $L, R \to 1$ AND WE RECOVER THM M.

REMARK: ONE CAN ACTUALLY ASSUME $\operatorname{Re} x_m = r$

$$F(s) - \sum_{m=-N}^{N} \frac{a_m}{s - x_m}$$

IS CONTINUOUS IN $\operatorname{Re} s \geq r$ AND $|\operatorname{Im} s| \leq 2\pi T$, AND A SIMILAR

RESULT HOLDS.

PROOF: ① WE CAN ASSUME $T=1$. $G(s) = F(Ts)$

$$= \int_{0^-}^{\infty} e^{-st}\, d\alpha(t/T)$$

② WE CAN ASSUME $A=1$.

WHAT WE WANT TO SHOW

$$\frac{1}{e^{r}-1} \le e^{-rt}\alpha(t) \le \frac{e^{r}}{e^{r}-1}$$

TAKE $\sigma > r$ AND $H(s) = F(s) - \frac{1}{s-r}$.

$$\alpha(t)\, e^{-\sigma t} = e^{-\sigma t}\int_0^t d\alpha(y) = \int_0^t e^{-\sigma(t-y)}\, e^{-\sigma y}\, d\alpha(y)$$

$$= \int_{-\infty}^{\infty} E(\sigma, t-y)\, E(\sigma, y)\, d\alpha(y) \geq \int_{-\infty}^{\infty} m_\sigma(t-y)\, E(\sigma, y)\, d\alpha(y)$$

$$= \int_{-\infty}^{\infty}\left(\int_{-1}^{1} \widehat{m}_\sigma(\xi)\, e^{2\pi i (t-y)\xi}\, d\xi\right) E(\sigma, y)\, d\alpha(y)$$

$$= \int_{-1}^{1} \widehat{m}_\sigma(\xi)\, e^{2\pi i t\xi}\, F(\sigma + 2\pi i \xi)\, d\xi$$

$$= \int_{-1}^{1} \widehat{m}_\sigma(\xi)\, e^{2\pi i t\xi}\, H(\sigma + 2\pi i \xi)\, d\xi + \int_{-1}^{1} \widehat{m}_\sigma(\xi)\, e^{2\pi i t\xi}\, \frac{1}{\sigma + 2\pi i \xi - r}\, d\xi$$

$$= (\cdots) + \int_{-1}^{1} \widehat{m}_\sigma(\xi)\, e^{2\pi i t\xi}\left(\int_0^{\infty} e^{-(\sigma + 2\pi i \xi - r)\, y}\, dy\right) d\xi$$

$$= (\cdots) + \int_0^\infty e^{-(\sigma-\lambda)y} \left(\int_{-1}^{1} \widehat{m}_\sigma(\xi)\, e^{2\pi i \xi (t-y)}\, d\xi \right) dy$$

$$= (\cdots) + \int_0^\infty e^{-(\sigma-\lambda)y}\, m_\sigma(t-y)\, dy$$

$\boxed{\sigma \to \lambda}$

$$\alpha(t)\, e^{-\lambda t} \geq \int_{-1}^{1} \widehat{m}_\lambda(\xi)\, e^{2\pi i t \xi}\, H(\lambda + 2\pi i \xi)\, d\xi$$

$$+ \int_0^\infty m_\lambda(t-y)\, dy$$

$$\alpha(t)\, e^{-\lambda t} \underset{t \to \infty}{\geq} \frac{1}{e^{\lambda}-1} + o(1)$$

THE UPPER BOUND IS VERY SIMILAR AND USES $\mathcal{M}_\lambda$ ∎