Smooth Particle Hydrodynamics With Strength of Materials

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INTRODUCTION

An elastic-perfectly-plastic constitutive model is formulated within the Smooth Particle Hydrodynamics (SPH) framework. Many nice features of SPH such as robustness, simplicity and ease of adding new physics can therefore be extended to problems in which material strength is important, such as impacts below approximately 2 km/s. Although SPH has been applied to astrophysical problems for several years now, it is still in its infancy with regard to use in typical hydrocode production problems and its fitness for such work is yet to be determined. The last of the virtues listed is again shown to be true in that inclusion of our elasticity model required only 30 lines of FORTRAN.

SPH BASICS

Smooth Particle Hydrodynamics is unique in computational fluid dynamics in that SPH uses no grid. It was the genius of the inventors, Lucy (1977) and Gingold & Monaghan (1977) to figure out how to take a derivative (get the force on a fluid element) without using a mesh. Previously a mesh was the only known way to compute a spatial derivative using finite differences. The mathematical theory of SPH will not be discussed here. The reader is referred to Gingold & Monaghan (1977,1982), Monaghan (1982,1985) and Monaghan & Gingold (1983) for detailed treatment of the subject. We present here only some basic features as discussed by Benz (1989) that are necessary to understand the method. Consider a function f, a kernel W which has a width measured by the parameter h, and the following equation:

$$< f(\mathbf{r}) > = \int W(\mathbf{r}-\mathbf{r}', h) f(\mathbf{r}')d\mathbf{r}' \quad . \tag{1}$$

If we impose a normalization such that the integral of W is unity, then it follows that

$$< f(\mathbf{r}) > \overset{h \to 0}{\longrightarrow} f(\mathbf{r}) \quad . \tag{2}$$

Relation (1) therefore defines the kernel estimate $<f>$ of f. If W is the Dirac delta function then we have the equality $<f>=f$. Now suppose f is known only at N discrete points that are spatially distributed according to the number density distribution:

$$n(\mathbf{r}) = \sum_{j=1}^{N} \delta(\mathbf{r}-\mathbf{r}_j) \quad . \tag{3}$$

If the number density at $\mathbf{r}_j$ is written as

$$< n(\mathbf{r}_j) > = \frac{\varrho(\mathbf{r}_j)}{m_j} \quad , \tag{4}$$

thus introducing the concept of particle mass (m), the following equation can be derived:

$$< f(\mathbf{r}) > = \int f(\mathbf{r}')W(\mathbf{r}-\mathbf{r}',h)d\mathbf{r}' \doteq \sum_j f_j \; W(\mathbf{r}-\mathbf{r}_j,h)\frac{m_j}{\varrho_j} \quad . \tag{5}$$

This equation defines a procedure for transforming integral equations to particle equations and is therefore called "integral evaluation by the particle method." The choice of kernel or "smoothing function" is discussed by Monaghan and Lattanzio (1985). The W most frequently used in SPH codes is a B–spline with compact support which goes to a zero at a distance $2h$ from its peak.

FORMULATION

In SPH, the density at a point in space is computed by the sum

$$\varrho_i \doteq \sum_{j=1}^{N} m_j \; W(\mathbf{r}_i-\mathbf{r}_j,h) \quad . \tag{6}$$

Notice that this is just (5) with $f=\rho$. Every particle of mass m is smoothed in space according to W which can be regarded as its density distribution in space. The density at any point in space is then obtained by summing up the contributions from all particles at that point. Smooth Particle Hydrodynamics derives its name from this interpretation. Equation (6) requires only particle coordinates and masses to compute the density and automatically satisfies the continuity equation, provided the particle masses are constant. The problem with (6) is that edge effects appear since particles close to a boundary will appear to have a smaller density than the same particles further removed from the boundary. Monaghan (1988) suggested the problem could be fixed by solving the continuity equation,

$$\frac{\partial \varrho_i}{\partial t} + \varrho_i \frac{\partial U^a}{\partial x^a} = 0 \quad , \tag{7}$$

instead of (6). Greek superscripts have been used to indicate coordinate directions with implied summation on repeated indices. Roman subscripts will be used to label particles. Summation is not implied on repeated subscripts (the summation sign must appear explicitly). In this method a particle's density changes only as other particles move toward or away from it. Benz (1989) points out, however, that then the number density of particles is no longer equal to the fluid density divided by the particles mass, requiring modification of the usual SPH momentum and energy equations for proper normalization. If (7) is used to calculate the density the divergence of the velocity field, D, must be computed. Differentiating (6) and using the fact that the gradient of the smoothing function vanishes when integrated over all space we find

$$D_i = -\dot{\varrho}/\varrho \doteq \sum_j \frac{m_j}{\varrho_i} \frac{\partial W}{\partial x_i^a} \left(U_i^a - U_j^a \right) \quad . \tag{8}$$

The equations of motion for a viscous fluid are:

$$\frac{dU^a}{dt} = -\frac{1}{\varrho} \frac{\partial \Pi^{a\beta}}{\partial x^\beta} \quad . \tag{9}$$

Symbols used refer to the density (ρ), velocity components (U), viscous stress tensor (Π), spacial coordinates (x) and the time (t). In order to cast (9) into the SPH framework we follow the procedure given by Benz (1989) and Campbell (1989). Rewrite the right hand side (rhs) of (9) as two terms one of which is in conservative form, then change the independent variable to a primed quantity, multiply by the smoothing function and integrate over all space:

$$\int_V \frac{dU^a}{dt} W d^3x' = -\int_V \frac{\partial}{\partial x'^\beta} \left[\frac{\Pi^{a\beta}}{\varrho} \right] W d^3x' - \int_V \frac{\Pi^{a\beta}}{\varrho^2} \frac{\partial \varrho}{\partial x'^\beta} W d^3x' \quad . \tag{10}$$

The first integral on the rhs is integrated by parts assuming the surface terms to vanish (Campbell discusses situations where this assumption cannot be made). The second integral is linearized by taking the expected value of the product to be the product of the expected values (this approximation is second order accurate):

$$\int_V \frac{dU^a}{dt} W d^3x' \doteq -\int_V \frac{\Pi^{a\beta}}{\varrho} \frac{\partial W}{\partial x^\beta} d^3x' - \frac{\Pi_i^{a\beta}}{\varrho_i^2} \int_V \varrho \frac{\partial W}{\partial x^\beta} d^3x' \tag{11}$$

We have also assumed a symmetric kernel in obtaining (11). These integrals, when evaluated by particle method, give the desired result.

$$\frac{dU_i^a}{dt} \doteq -\sum_j m_j \left(\frac{\Pi_i^{a\beta}}{\varrho_i^2} + \frac{\Pi_j^{a\beta}}{\varrho_j^2} \right) \frac{\partial W}{\partial x^\beta} \quad . \tag{12}$$

The stress tensor is normally defined in terms of an isotropic part which is the pressure (P) and the traceless symmetric deviatoric stress (S):

$$\Pi^{a\beta} = P\delta^{a\beta} - S^{a\beta} \quad . \tag{13}$$

For pure hydrodynamic flow with zero deviatoric stress (12) reduces to the standard SPH momentum equations (Benz, 1989). If we adopt an elastic constitutive model, then the rate of change of the stress is given by

$$\dot{S}^{a\beta} = \mu\left(\epsilon^{a\beta} - \frac{1}{3}\delta^{a\beta}\epsilon^{\gamma\gamma} \right) + S^{a\gamma}R^{\beta\gamma} + S^{\gamma\beta}R^{a\gamma} = \mu\bar{\epsilon}^{a\beta} + S^{a\gamma}R^{\beta\gamma} + S^{\gamma\beta}R^{a\gamma}, \tag{14}$$

where μ is the shear modulus, ϵ is the strain rate tensor defined by

$$\epsilon^{a\beta} = \frac{1}{2}\left[\frac{\partial U^a}{\partial x^\beta} + \frac{\partial U^\beta}{\partial x^a} \right] \quad , \tag{15}$$

$\bar{\epsilon}$ is its traceless part, and R is the rotation rate tensor defined by

$$R^{a\beta} = \frac{1}{2}\left[\frac{\partial U^a}{\partial x^\beta} - \frac{\partial U^\beta}{\partial x^a} \right] \quad . \tag{16}$$

Notice that only the off-diagonal component of R is non-zero and that it differs from the shear strain rate only in that the velocity gradients are subtracted and not added. Computing R is therefore, only trivially different from computing the shear strain. To obtain particle equations for the strain rate we proceed as before, multiplying by the smoothing function, integrating by parts, neglecting surface terms, and evaluating the remaining integrals by the particle method (5).

$$\epsilon_i^{\alpha\beta} = \frac{1}{2}\int_V \left[\frac{\partial U^\alpha}{\partial x^\beta} + \frac{\partial U^\beta}{\partial x^\alpha}\right] W d^3x = \frac{1}{2}\int_V \left(U^\alpha \frac{\partial W}{\partial x^\beta} + U^\beta \frac{\partial W}{\partial x^\alpha}\right) d^3x$$

$$\doteq \frac{1}{2}\sum_j \left[U_j^\alpha \frac{\partial W}{\partial x_i^\beta} + U_j^\beta \frac{\partial W}{\partial x_i^\alpha}\right]\frac{m_j}{\varrho_j} \tag{17}$$

From (15) we see that the trace of the strain rate tensor is the divergence of the velocity, D. The trace of (17) does not give this result, nor does it contain the desired velocity differences. To get the correct trace and express ϵ in terms of velocity differences we note that because of the neglect of surface terms

$$0 = \int \frac{\partial W}{\partial x_i^\alpha} d^3x \doteq \sum_j \frac{m_j}{\varrho_j}\frac{\partial W}{\partial x_i^\alpha} . \tag{18}$$

We therefore subtract from Eq (17) the following term which is zero in our approximation.

$$\frac{1}{2} U_i^\alpha \sum_j \frac{m_j}{\varrho_j}\frac{\partial W}{\partial x_i^\beta} \quad + \quad \frac{1}{2} U_i^\beta \sum_j \frac{m_j}{\varrho_j}\frac{\partial W}{\partial x_i^\alpha} \quad . \tag{19}$$

The equation for the strain rate then becomes

$$\epsilon_i^{\alpha\beta} = \frac{1}{2}\sum_j \frac{m_j}{\varrho_j}\left(U_j^\alpha - U_i^\alpha\right)\frac{\partial W}{\partial x_i^\beta} + \frac{1}{2}\sum_j \frac{m_j}{\varrho_j}\left(U_j^\beta - U_i^\beta\right)\frac{\partial W}{\partial x_i^\alpha} . \tag{20}$$

This expression gives a trace which differs from (8) only in that the density of particle j rather than particle i appears in the denominator. The difference is of the same order as the difference between the product of the expected values and the expected value of the product. The equation for the specific internal energy (E) of a particle is written heuristically as

$$\frac{dE_i}{dt} \;=\; -\frac{1}{\varrho_i}\left[P_iD_i - S_i^{\alpha\beta}\bar{\epsilon}_i^{\alpha\beta}\right] \quad . \tag{21}$$

The work done on a particle is made up up two parts, the hydrostatic pressure times the volumetric strain and the deviator stress times the deviator strain. For pure hydrodynamic flow the stress deviators vanish and (21) reduces to the standard SPH equations for the internal energy with D determined by (8).

COMPUTATIONAL PROCEDURE

As Monaghan and Gingold (1983) point out, even though SPH does not need grid cells, a huge savings in computing time is achieved by using them as a bookkeeping device to determine which particles might interact with any given other particle. If particles are assigned to cells and identified through linked lists the calculation time is proportional to the number of particles, not the number squared. Since we use the B-spline kernel which has a $2h$ cutoff, convenient cell dimensions are $2h$ so that each particle in a cell will have neighbors only in it's own cell and the surrounding 8 cells (or in three dimensions the surrounding 26 cells). We then loop over all grid cells computing interactions between each particle in the cell and particles within the centered 3 x 3 sub-grid of cells. If a zone lies along a boundary, ghost cells are added to construct the sub-grid. Ghost particles are placed in these cells by reflection or translation of particles in the adjacent "real zone", depending on whether "reflective" or "transmissive" boundary conditions are desired. In the transmissive case, which simulates a open boundary, ghost particles have identical properties to the real particles. In the reflective case, which simulates a rigid wall, the same is true except that the sign of the normal velocity component is reversed.

For the time integration we use the leap-frog algorithm (Lattanzio et.al., 1985) with time step δt^n calculated from the configuration at t^n to advance the particles velocity and position.

$$\mathbf{U}_i^{n+1/2} = \mathbf{U}_i^{n-1/2} + 1/2(\delta t^n + \delta t^{n-1})\mathbf{f}_i \tag{22}$$

$$\mathbf{x}_i^{n+1} = \mathbf{x}_i^n + \mathbf{U}_i^{n+1/2}\delta t^n \quad . \tag{23}$$

Here $\mathbf{f}_i$ is the right hand side of (9), the total force per unit mass acting on particle i. The time step is the minimum over all particles of $\sigma h/(c+s)$, where c is the adiabatic sound speed, s the particle speed and σ a constant factor. Choosing $\sigma = 0.3$ seems adequate.

Concerning artificial viscosity, a good deal of work has gone into an adequate formulation for SPH (Monaghan & Gingold, 1983) and we apply those results herein without further discussion. The equations contain the dissipation in the Ω term. We have not investigated the effect of our strength model on the artificial viscosity formulation.

For clarity and convenience we list in Table 1 the complete set of SPH equations with strength written out from the tensor equations into 2D Cartesian coordinates. We use subscripts to represent differentiation of the smoothing function.

TEST PROBLEM

An iron rod (actually plate in 2D Cartesian) travelling at 200 m/s impacts a rigid surface. The rod is 2.540 cm long and 0.760 cm thick. The motion is normal to the rigid surface. We modeled the iron as an elastic-perfectly plastic material. A Grueneisen equation of state was used to calculate the pressure. Constants characterizing the iron are the density (ρ=7.85 gm/cc), yield strength, (Y_o=6.00 Kb), shear modulus (μ=800 Kb), Grueneisen parameter (γ=1.81) and Hugoniot fit (U_s=3.63+1.80U_p) where U_s and U_p are the shock and particle speeds in km/s. The left part of Figure 1 is a particle plot showing the initial positions. The right part shows the final particle positions as computed by SPH along with results obtained using the Lagrangian code **EPIC-2**. We compare SPH results against another code because these are plate-on-plate impacts for which no experimental data exists. **EPIC-2** has been compared to experiment in many other situations and is therefore good to normalize to. The SPH calculation used 4 particles per $2h$ in each coordinate direction arranged in a regular array. The smoothing length was taken to be a tenth the rod diameter (h=.076). The total number of iron particles was 1320.

Table 1

Hydrodynamics

$$\varrho_i = \sum_j m_j W_{ij} \quad or \quad \frac{d\varrho_i}{dt} = -\varrho_i D_i$$

$$\frac{dU_i}{dt} = -\sum_j \left\{ m_j\left(\frac{P_i}{\varrho_i^2} + \frac{P_j}{\varrho_j^2} + \Omega_{ij}\right)W_x - m_j\left(\frac{S_i^{xx}}{\varrho_i^2} + \frac{S_j^{xx}}{\varrho_j^2}\right)W_x - m_j\left(\frac{S_i^{xy}}{\varrho_i^2} + \frac{S_j^{xy}}{\varrho_j^2}\right)W_y \right\}$$

$$\frac{dV_i}{dt} = -\sum_j \left\{ m_j\left(\frac{P_i}{\varrho_i^2} + \frac{P_j}{\varrho_j^2} + \Omega_{ij}\right)W_y - m_j\left(\frac{S_i^{xy}}{\varrho_i^2} + \frac{S_j^{xy}}{\varrho_j^2}\right)W_x - m_j\left(\frac{S_i^{yy}}{\varrho_i^2} + \frac{S_j^{yy}}{\varrho_j^2}\right)W_y \right\}$$

$$\frac{dE_i}{dt} = -\frac{1}{\varrho_i}\left[(P_i + ½\Omega_{ij})\ D_i - S_i^{\alpha\beta}\bar{\epsilon}_i^{\alpha\beta}\right] \qquad D_i = \sum_j \frac{m_j}{\varrho_j}\left[(U_j - U_i)W_x + (V_j - V_i)W_y\right]$$

Strength — *Elastic*

$$\frac{dS_i^{xx}}{dt} = \mu\left[\sum_j \frac{m_j}{\varrho_j}(U_j - U_i)W_x - \frac{1}{3}D_i\right] + 2R_i^{xy}S_i^{xy}$$

$$\frac{dS_i^{yy}}{dt} = \mu\left[\sum_j \frac{m_j}{\varrho_j}(V_j - V_i)W_y - \frac{1}{3}D_i\right] - 2R_i^{xy}S_i^{xy}$$

$$\frac{dS_i^{zz}}{dt} = -\mu D_i/3$$

$$\frac{dS_i^{xy}}{dt} = \frac{\mu}{2}\sum_j \frac{m_j}{\varrho_j}\left[(U_j - U_i)W_y + (V_j - V_i)W_x\right] - R_i^{xy}(S_i^{xx} - S_i^{yy})$$

Perfeclly Plastic

$$J_i^2 = S_i^{xx}S_i^{xx} + S_i^{yy}S_i^{yy} + S_i^{zz}SS_i^{zz} + S_i^{xy}S_i^{xy} + S_i^{xx}S_i^{yy}$$

$$f = \min\left\{\sqrt{\frac{Y_o^2/3}{J_i^2}}\ ,\ 1\right\}$$

$$S_i^{xx} = f\ S_i^{xx}$$
$$S_i^{yy} = f\ S_i^{yy}$$
$$S_i^{zz} = f\ S_i^{zz}$$
$$S_i^{xy} = f\ S_i^{xy}$$

Smoothing Function

$$W = \frac{1}{\pi h^2}\begin{cases} \frac{15}{7}\left(\frac{2}{3} - \nu^2 + \frac{1}{2}\nu^3\right) & 0 < \nu < 1 \\ \frac{5}{14}(2-\nu)^3 & 1 < \nu < 2 \\ 0 & \text{otherwise} \end{cases} \qquad \nu = r/h.$$

Artificial Viscosity

$$\Omega_{ij} = \begin{cases} \dfrac{-\alpha\overline{C}_{ij}\omega + \beta\omega^2}{\overline{\varrho}_{ij}} & U_{ij}X_{ij} + V_{ij}Y_{ij} < 0 \\ 0 & \text{otherwise} \end{cases}$$

where

$$\omega = \frac{h(U_{ij}X_{ij} + V_{ij}Y_{ij})}{X_{ij}^2 + Y_{ij}^2 + \epsilon h^2}$$

$$X_{ij} = X_i - X_j \quad Y_{ij} = Y_i - Y_j$$
$$U_{ij} = U_i - U_j \quad V_{ij} = V_i - V_j$$

$$\varrho_{ij} = ½(\varrho_i - \varrho_j)$$
$$C_{ij} = ½(C_i - C_j)$$

$$\alpha = 0.5$$
$$\beta = 0.5$$
$$\epsilon = .01$$

Table 1. Equations used in the test problem.

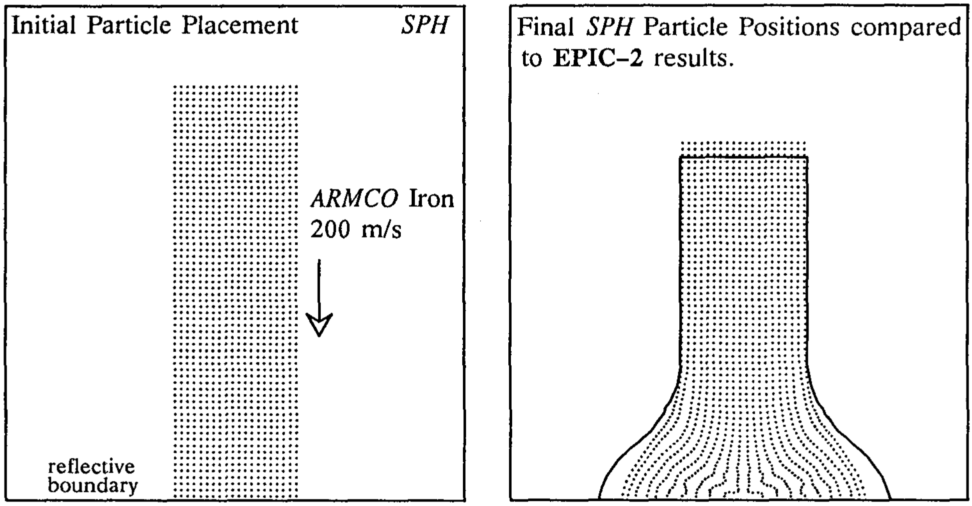

Fig. 1. Deformation of iron in plane-strain at 50 μs as computed by *SPH* (points) and **EPIC-2** (line).

DISCUSSION

We have put an elastic-perfectly-plastic material strength model into Smooth Particle Hydrodynamics. Results for the Taylor Anvil Test for iron in plane-strain show less bulging than predicted by the reliable **EPIC-2** code. We do not understand the reason for the discrepancy at this time. Doubling the number of particles does not change the final shape significantly. The tiny wiggles in the **EPIC-2** curve result from our digitization. Although these first results for SPH with strength of materials are not as accurate as we would like, we are nevertheless encouraged by the qualitative features of the calculation and anticipate improvements as the model is further exercised and examined. The robustness of the SPH method is truly remarkable. Ease of adding new physics to SPH is demonstrated in that our strength model required only 30 lines of FORTRAN.

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