Lecture Note: Data Center Power Optimisation

Case Study Inspired by Hyperscale Cloud Data Centers

1 Introduction

Modern hyperscale data centers are among the fastest growing electricity consumers globally. With the rise of artificial intelligence workloads, GPU clusters, and always-on cloud services, power optimisation has become a central design objective. Leading operators have demonstrated that optimisation must be performed jointly across:

- IT workload scheduling,
- cooling system control,
- renewable energy integration,
- energy storage and grid interaction.

This note presents a simplified optimisation formulation motivated by large-scale production data centers.

2 Data Center Power Structure

Total facility power can be decomposed as:

$$P_{\text{total}}(t) = P_{\text{IT}}(t) + P_{\text{cool}}(t) + P_{\text{aux}}(t) \tag{1}$$

A commonly used metric is Power Usage Effectiveness (PUE):

$$PUE(t) = \frac{P_{\text{total}}(t)}{P_{\text{IT}}(t)} \tag{2}$$

Hence,

$$P_{\text{total}}(t) = PUE(t) \cdot P_{\text{IT}}(t) \tag{3}$$

Typical large data centers achieve:

- PUE $\approx 1.1 - 1.3$ (highly efficient sites)

3 Cooling Power Model

Cooling power is often modeled using Coefficient of Performance (COP):

$$P_{\text{cool}}(t) = \frac{P_{\text{IT}}(t)}{COP(t)} \tag{4}$$

COP depends on:

- ambient temperature,
- cooling technology (air, liquid, immersion),
- operating mode.

4 Energy Sources

At each time step, power may come from:

- Grid electricity $P_{\text{grid}}(t)$
- Renewable generation $P_{\text{ren}}(t)$
- Battery discharge $P_{\text{bat}}(t)$

Power balance:

$$P_{\text{grid}}(t) + P_{\text{ren}}(t) + P_{\text{bat}}(t) = P_{\text{IT}}(t) + P_{\text{cool}}(t) \tag{5}$$

5 Generic Optimisation Objective

A simplified operational optimisation problem can be written as:

$$\min \sum_t \left[c_{\text{grid}}(t) P_{\text{grid}}(t) + c_{\text{bat}} P_{\text{bat}}(t) \right] \tag{6}$$

subject to:

5.1 Power Balance

$$P_{\text{grid}}(t) + P_{\text{ren}}(t) + P_{\text{bat}}(t) = PUE(t)\, P_{\text{IT}}(t) \tag{7}$$

5.2 Renewable Availability

$$0 \le P_{\text{ren}}(t) \le P_{\text{ren}}^{\max}(t) \tag{8}$$

5.3 Battery Constraints

$$E(t+1) = E(t) + \eta_c P_{\text{ch}}(t) - \frac{1}{\eta_d} P_{\text{bat}}(t) \tag{9}$$

$$0 \le E(t) \le E^{\max} \tag{10}$$

6 Sustainability Constraint

If a minimum renewable fraction α is required:

$$P_{\text{ren}}(t) \geq \alpha\, P_{\text{total}}(t) \tag{11}$$

7 Case Study Insights from Hyperscale Data Centers

Large production data centers have demonstrated several optimisation principles:

7.1 Workload-Aware Energy Optimisation

AI training jobs can be scheduled based on:

- renewable generation forecast,
- electricity price signals,
- grid carbon intensity.

7.2 Machine Learning for Cooling Control

Neural network models can:

- predict thermal behaviour,
- adjust cooling setpoints,
- reduce cooling energy consumption.

7.3 Grid Interaction

Data centers can act as flexible loads via:

- demand response,
- battery dispatch,
- workload shifting.

8 Research Directions

- Stochastic optimisation under renewable uncertainty
- Carbon-aware workload scheduling
- Joint IT + thermal + electrical co-optimisation
- Hybrid classical–quantum optimisation approaches

9 Conclusion

Power optimisation in data centers is a multi-layer control problem involving:

- electrical systems,
- thermodynamics,
- optimisation theory,
- machine learning.

Future data centers are expected to operate as grid-interactive, sustainability-aware computational infrastructure rather than passive electricity consumers.

10 Battery Energy Storage Model

In data center power optimisation, batteries are commonly used to store excess renewable energy and supply power during peak demand or grid constraints. The battery is modeled using a discrete-time energy balance equation together with physical capacity limits.

10.1 Battery Energy Update Equation

The evolution of stored battery energy is given by:

$$E(t+1) = E(t) + \eta_c P_{ch}(t)\Delta t - \frac{1}{\eta_d} P_{dis}(t)\Delta t \tag{12}$$

10.2 Symbol Definitions

Symbol	Meaning	Units
t	Time index	hour or time step
$E(t)$	Energy stored in battery at time t	MWh
$E(t+1)$	Energy stored at next time step	MWh
$P_{ch}(t)$	Battery charging power	MW
$P_{dis}(t)$	Battery discharging power	MW
η_c	Charging efficiency (0–1)	dimensionless
η_d	Discharging efficiency (0–1)	dimensionless
Δt	Duration of time step	hour

Table 1: Battery model variable definitions

10.3 Physical Interpretation

The battery energy at the next time step equals:

- Current stored energy,
- Plus effective energy charged into the battery,

- Minus effective energy discharged from the battery.

Charging efficiency accounts for losses during charging:

$$\text{Stored energy from charging} = \eta_c \times \text{Input energy} \tag{13}$$

Discharging efficiency accounts for losses during discharge:

$$\text{Energy removed from battery} = \frac{\text{Delivered energy}}{\eta_d} \tag{14}$$

10.4 Battery Capacity Constraints

The stored energy must remain within physical limits:

$$0 \leq E(t) \leq E^{\max} \tag{15}$$

where $E^{\max}$ is the maximum energy storage capacity of the battery (MWh).

10.5 Example Calculation

Suppose:

- $E(t) = 5$ MWh,
- $P_{ch}(t) = 2$ MW,
- $P_{dis}(t) = 1$ MW,
- $\eta_c = 0.95$,
- $\eta_d = 0.90$,
- $\Delta t = 1$ hour.

Then:

$$E(t+1) = 5 + 0.95 \times 2 - \frac{1}{0.9} \times 1 \tag{16}$$

$$= 5 + 1.9 - 1.11 \tag{17}$$

$$= 5.79 \text{ MWh} \tag{18}$$

10.6 Role in Data Center Optimisation

This battery model allows optimisation algorithms to:

- Store excess renewable energy,
- Reduce peak grid electricity consumption,
- Provide backup during grid stress events,
- Improve overall energy cost and carbon efficiency.

10.7 Alternative Single-Variable Battery Model

Some optimisation models use a single net battery power variable $P_{bat}(t)$, where positive values represent discharge and negative values represent charging. In such cases:

$$E(t+1) = E(t) + \eta_c P_{bat}^{-}(t)\Delta t - \frac{1}{\eta_d} P_{bat}^{+}(t)\Delta t \tag{19}$$

where:

$$P_{bat}^{+}(t) = \max(P_{bat}(t), 0) \tag{20}$$
$$P_{bat}^{-}(t) = \max(-P_{bat}(t), 0) \tag{21}$$

10.8 Summary

The battery dynamics equation ensures:

- Energy conservation,
- Physical feasibility,
- Time-coupling across optimisation decisions.

11 Numerical Optimisation Problems for Data Center Power Optimisation

Time is discretised into slots $t = 1, 2, \ldots$ with step length Δt (hours).

- $P_{IT}(t)$: IT power demand (MW)
- $P_{cool}(t)$: cooling power (MW)
- $P_{total}(t)$: facility total power (MW)
- $P_{grid}(t)$: grid import power (MW)
- $P_{ren}(t)$: renewable power used (MW)
- $P_{ch}(t)$, $P_{dis}(t)$: battery charge/discharge power (MW)
- $E(t)$: battery stored energy (MWh)
- η_c, η_d: charging/discharging efficiency
- $c_{grid}(t)$: grid price (currency per MWh)

Power balance (typical form):

$$P_{grid}(t) + P_{ren}(t) + P_{dis}(t) = P_{total}(t) + P_{ch}(t).$$

Battery dynamics (discrete time):

$$E(t+1) = E(t) + \eta_c P_{ch}(t)\Delta t - \frac{1}{\eta_d} P_{dis}(t)\Delta t, \qquad 0 \le E(t) \le E^{max}.$$

Problem1 (Single-hour LP with cooling + renewable share) **Data (one hour, $\Delta t = 1$ h).**

- IT load: $P_{IT} = 10$ MW
- Cooling COP: $COP = 4$
- Renewables: $0 \le P_{ren} \le 6$ MW
- Battery discharge: $0 \le P_{dis} \le 3$ MW, battery cost $c_{bat} = 20$
- Grid price: $c_{grid} = 100$
- Renewable share target: $\alpha = 0.30$

Variables: $P_{grid}, P_{ren}, P_{dis} \ge 0$.
Model. Cooling: $P_{cool} = \frac{P_{IT}}{COP}$. Total: $P_{total} = P_{IT} + P_{cool}$.

$$P_{grid} + P_{ren} + P_{dis} = P_{total}, \qquad 0 \le P_{ren} \le 6, \qquad 0 \le P_{dis} \le 3,$$

Renewable fraction:

$$P_{ren} \ge \alpha P_{total}.$$

Objective:

$$\min\ 100P_{grid} + 20P_{dis}.$$

Solution Compute cooling:

$$P_{cool} = \frac{10}{4} = 2.5 \text{ MW}, \qquad P_{total} = 10 + 2.5 = 12.5 \text{ MW}.$$

Renewable share constraint:

$$P_{ren} \ge 0.30 \times 12.5 = 3.75 \text{ MW}.$$

Since grid is most expensive, we use renewables up to their maximum, then battery, then grid:

$$P^{\star}_{ren} = 6, \quad P^{\star}_{dis} = 3, \quad P^{\star}_{grid} = 12.5 - 6 - 3 = 3.5.$$

Cost:

$$100(3.5) + 20(3) = 350 + 60 = 410.$$

Optimal solution: $(P_{ren}, P_{dis}, P_{grid}) = (6, 3, 3.5)$ with objective value 410.
Problem2 (Three-hour battery dispatch LP with time-coupling) **Data (3 hours, $\Delta t = 1$ h).**

- IT load $P_{IT}(t) = 8$ MW (constant)
- PUE $(t) = 1.25$ (constant), hence $P_{total}(t) = PUE \cdot P_{IT} = 10$ MW
- Renewable limits: $P^{max}_{ren} = [6, 2, 0]$ MW
- Grid prices: $c_{grid} = [50, 120, 200]$
- Battery parameters: $E(1) = 2$ MWh, $E^{max} = 4$ MWh, $\eta_c = 0.95$, $\eta_d = 0.90$
- Power limits: $0 \le P_{ch}(t) \le 2$, $0 \le P_{dis}(t) \le 2$

Variables (each hour t): $P_{grid}(t), P_{ren}(t), P_{ch}(t), P_{dis}(t) \geq 0$ and $E(t)$.
Model.

$$P_{grid}(t) + P_{ren}(t) + P_{dis}(t) = 10 + P_{ch}(t)$$

$$0 \leq P_{ren}(t) \leq P_{ren}^{max}(t)$$

$$E(t+1) = E(t) + 0.95\, P_{ch}(t) - \frac{1}{0.9} P_{dis}(t), \qquad 0 \leq E(t) \leq 4.$$

Objective:

$$\min \sum_{t=1}^{3} c_{grid}(t)\, P_{grid}(t).$$

Solution Intuition: Charge as much as possible in cheap hour 1; discharge as much as possible in expensive hour 3; use remaining discharge in hour 2.

Hour 1. Use all renewables (6 MW). Charge at maximum $P_{ch}(1) = 2$ MW. No discharge. Power balance:

$$P_{grid}(1) + 6 + 0 = 10 + 2 \;\Rightarrow\; P_{grid}(1) = 6.$$

Battery energy:

$$E(2) = E(1) + 0.95 \cdot 2 = 2 + 1.9 = 3.9.$$

Cost: $50 \cdot 6 = 300$.

Allocate discharge across hours 2 and 3. To discharge at maximum $P_{dis}(3) = 2$ MW in hour 3, the energy required at start of hour 3 is:

$$\frac{1}{0.9} \cdot 2 = 2.222 \text{ MWh}.$$

Thus we must keep $E(3) \geq 2.222$ after hour 2.

Hour 2. Use all renewables (2 MW). Let $P_{ch}(2) = 0$. Choose discharge $P_{dis}(2)$ so that $E(3) = 2.222$:

$$E(3) = E(2) - \frac{1}{0.9} P_{dis}(2) = 3.9 - \frac{1}{0.9} P_{dis}(2) = 2.222$$

$$\Rightarrow\; \frac{1}{0.9} P_{dis}(2) = 1.678 \Rightarrow P_{dis}(2) = 0.9 \times 1.678 = 1.5102.$$

Power balance:

$$P_{grid}(2) + 2 + 1.5102 = 10 \;\Rightarrow\; P_{grid}(2) = 6.4898.$$

Cost: $120 \cdot 6.4898 = 778.776$.

Hour 3. No renewables. No charging. Discharge at max $P_{dis}(3) = 2$. Power balance:

$$P_{grid}(3) + 0 + 2 = 10 \;\Rightarrow\; P_{grid}(3) = 8.$$

Battery energy:

$$E(4) = E(3) - \frac{1}{0.9} \cdot 2 = 2.222 - 2.222 = 0.$$

Cost: $200 \cdot 8 = 1600$.

Total cost:

$$300 + 778.776 + 1600 = 2678.776.$$

Optimal schedule (one optimal solution):

t	$P_{ren}(t)$	$P_{ch}(t)$	$P_{dis}(t)$	$P_{grid}(t)$	$E(t)$ (start)
1	6	2	0	6.0000	2.0000
2	2	0	1.5102	6.4898	3.9000
3	0	0	2.0000	8.0000	2.2220

Final energy $E(4) = 0$.

Problem3 (MILP: binary charge/discharge mode) Take Problem 2 and add binary variables to forbid simultaneous charging and discharging.

Add binaries: $z_{ch}(t), z_{dis}(t) \in \{0, 1\}$.

Add constraints:

$$P_{ch}(t) \leq 2\, z_{ch}(t), \qquad P_{dis}(t) \leq 2\, z_{dis}(t),$$

$$z_{ch}(t) + z_{dis}(t) \leq 1 \qquad \forall t.$$

Solution Problem 2's solution already satisfies the MILP logic (no hour has both $P_{ch}(t) > 0$ and $P_{dis}(t) > 0$). Hence it remains feasible and optimal under this MILP variant:

$$z_{ch}(1) = 1,\ z_{dis}(1) = 0; \qquad z_{ch}(2) = 0,\ z_{dis}(2) = 1; \qquad z_{ch}(3) = 0,\ z_{dis}(3) = 1.$$

Total cost remains 2678.776.

12 Notes

- Problem 1 illustrates a one-period LP with cooling coupling and a renewable-share constraint.
- Problem 2 illustrates time-coupling via $E(t)$ and motivates multi-period optimisation.
- Problem 3 illustrates why adding binary mode variables turns an LP into a MILP.

13 Additional Toy (AI Controller + Quantum/QUBO Scheduling)

Problem4 (AI-assisted cooling control + QUBO battery scheduling) This problem mirrors two ideas used in modern data-center power optimisation: (i) an *AI model* predicting cooling efficiency (COP) from telemetry and a control setpoint, and (ii) a *binary scheduling* subproblem that can be mapped to a QUBO for quantum optimisation.

Part A: AI-assisted cooling control (one-step decision)

Data (one hour, $\Delta t = 1$ h).

- IT load: $P_{IT} = 10$ MW
- Available renewables: $P_{ren} = 6$ MW (fixed for this hour)
- Grid price: $c_{grid} = 100$
- Two candidate cooling setpoints: $s \in \{0, 1\}$ (think: *normal* vs *more aggressive free cooling*)

- A simple learned COP predictor (toy "AI model"):

$$\widehat{COP}(s) = 3 + 1.5\,s.$$

- Optional comfort/thermal penalty for aggressive setting: penalty$(s) = \gamma s$ with $\gamma = 30$.

Task. For each setpoint s, compute cooling power $P_{cool}(s) = \frac{P_{IT}}{\widehat{COP}(s)}$, total power $P_{total}(s) = P_{IT} + P_{cool}(s)$, grid import $P_{grid}(s) = \max\{0, P_{total}(s) - P_{ren}\}$, and total cost

$$J(s) = c_{grid} P_{grid}(s) + \gamma s.$$

Choose the best s.

Part B: QUBO battery discharge scheduling (3-slot binary)

Data (3 slots).

- Prices: $c = [5, 12, 20]$ (currency per unit)
- Decision: $x_t \in \{0, 1\}$ indicates discharging one unit in slot t
- Must discharge exactly one unit total: $\sum_{t=1}^{3} x_t = 1$
- Penalty parameter: $\lambda = 50$

Task. Form the QUBO objective

$$\min_{x \in \{0,1\}^3} \sum_{t=1}^{3} c_t x_t + \lambda \left(\sum_{t=1}^{3} x_t - 1 \right)^2,$$

expand it into a quadratic polynomial, and identify the optimal schedule.

solution

Part A Solution (AI-assisted cooling control)

Evaluate both options.

Case $s = 0$.

$$\widehat{COP}(0) = 3, \quad P_{cool}(0) = \frac{10}{3} = 3.\overline{3} \text{ MW}.$$

$$P_{total}(0) = 10 + 3.\overline{3} = 13.\overline{3} \text{ MW}.$$

Renewables are fixed at 6 MW, so

$$P_{grid}(0) = 13.\overline{3} - 6 = 7.\overline{3} \text{ MW}.$$

Cost:

$$J(0) = 100 \cdot 7.\overline{3} + 30 \cdot 0 = 733.\overline{3}.$$

Case $s = 1$.

$$\widehat{COP}(1) = 3 + 1.5 = 4.5, \quad P_{cool}(1) = \frac{10}{4.5} = 2.\overline{2} \text{ MW}.$$

$$P_{total}(1) = 10 + 2.\overline{2} = 12.\overline{2} \text{ MW}.$$

$$P_{grid}(1) = 12.\overline{2} - 6 = 6.\overline{2} \text{ MW}.$$

Cost:

$$J(1) = 100 \cdot 6.\overline{2} + 30 \cdot 1 = 622.\overline{2} + 30 = 652.\overline{2}.$$

Decision. Since $J(1) < J(0)$, the optimal setpoint is:

$$s^{\star} = 1.$$

Interpretation. The "AI" model predicts a higher COP at $s = 1$, lowering cooling power and therefore grid import. The penalty γs represents a constraint/comfort trade-off (e.g., thermal risk or humidity bounds).

Part B Solution (QUBO battery scheduling)

We minimise:

$$\sum_{t=1}^{3} c_t x_t + \lambda \left(x_1 + x_2 + x_3 - 1\right)^2.$$

Expand the square:

$$(x_1 + x_2 + x_3 - 1)^2 = (x_1 + x_2 + x_3)^2 - 2(x_1 + x_2 + x_3) + 1.$$

Using $x_i^2 = x_i$ for binary variables:

$$(x_1 + x_2 + x_3)^2 = x_1 + x_2 + x_3 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3.$$

Hence,

$$(x_1 + x_2 + x_3 - 1)^2 = -(x_1 + x_2 + x_3) + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 + 1.$$

Therefore the QUBO objective (dropping the constant $\lambda \cdot 1$ since it does not affect arg min) is:

$$\min \underbrace{\sum_{t=1}^{3} c_t x_t}_{\text{linear costs}} + \lambda \left(-(x_1 + x_2 + x_3) + 2x_1x_2 + 2x_1x_3 + 2x_2x_3\right).$$

Plugging $c = [5, 12, 20]$ and $\lambda = 50$:

$$\min\ (5 - 50)x_1 + (12 - 50)x_2 + (20 - 50)x_3 + 100x_1x_2 + 100x_1x_3 + 100x_2x_3.$$

So:

$$\min\ (-45)x_1 + (-38)x_2 + (-30)x_3 + 100(x_1x_2 + x_1x_3 + x_2x_3).$$

Optimal schedule (by inspection). The penalty strongly discourages selecting more than one $x_t = 1$ because pairwise terms add $+100$. Thus the minimiser will choose exactly one discharge slot, and among single-slot choices it picks the smallest price:

$$x_1^{\star} = 1, \quad x_2^{\star} = 0, \quad x_3^{\star} = 0.$$

Interpretation. This illustrates how a constrained binary scheduling problem can be turned into an unconstrained quadratic objective (QUBO), suitable for quantum annealing or Ising formulations, while classical MILP can solve the same decision exactly.

Notation (used across problems)

Time is discretised into slots $t = 1, 2, \dots, T$ with step length Δt (hours).

[leftmargin=*]

- $P_{IT}(t)$: IT power (MW)
- $P_{cool}(t)$: cooling power (MW)
- $P_{total}(t)$: total facility power (MW)
- $P_{grid}(t)$: grid import (MW)
- $P_{ren}(t)$: renewable used (MW), $0 \le P_{ren}(t) \le P_{ren}^{\max}(t)$
- $P_{ch}(t), P_{dis}(t)$: battery charge/discharge power (MW), nonnegative
- $E(t)$: battery energy (MWh)
- $\eta_c, \eta_d \in (0, 1]$: charge/discharge efficiency

Power balance (typical):

$$P_{grid}(t) + P_{ren}(t) + P_{dis}(t) = P_{IT}(t) + P_{cool}(t) + P_{ch}(t).$$

Battery dynamics:

$$E(t+1) = E(t) + \eta_c P_{ch}(t)\Delta t - \frac{1}{\eta_d} P_{dis}(t)\Delta t, \qquad 0 \le E(t) \le E^{\max}.$$

14 Problem 1: One-hour LP (cooling via COP + renewable share)

Data (one hour, $\Delta t = 1$ h).

- $P_{IT} = 12$ MW
- Cooling COP is controllable: choose $COP \in \{3, 4, 5\}$ (discrete choice)
- Renewable availability: $0 \le P_{ren} \le 7$ MW
- Battery discharge: $0 \le P_{dis} \le 4$ MW, battery cost $c_{bat} = 25$
- Grid price: $c_{grid} = 110$
- Renewable share constraint: $P_{ren} \ge \alpha P_{total}$ with $\alpha = 0.35$

Tasks.

1. For each COP choice, compute $P_{cool} = \frac{P_{IT}}{COP}$ and $P_{total} = P_{IT} + P_{cool}$.
2. For each COP, solve the single-hour LP:

   $$\min\ 110P_{grid} + 25P_{dis} \quad \text{s.t.} \quad P_{grid} + P_{ren} + P_{dis} = P_{total},\ 0 \le P_{ren} \le 7,\ 0 \le P_{dis} \le 4,\ P_{ren} \ge 0.35P_{total}.$$

3. Which COP is best overall (minimum cost)? Explain the trade-off.

15 Problem 2: Two-hour battery arbitrage (LP with time-coupling)

Data ($T = 2$, $\Delta t = 1$ h).

- $P_{total}(t) = 10$ MW for $t = 1, 2$ (assume PUE already absorbed)
- Renewable limits: $P_{ren}^{\max} = [8, 0]$ MW
- Grid prices: $c_{grid} = [40, 200]$
- Battery: $E(1) = 1$ MWh, $E^{\max} = 5$ MWh, $\eta_c = 0.95$, $\eta_d = 0.90$
- Limits: $0 \le P_{ch}(t) \le 3$, $0 \le P_{dis}(t) \le 3$

Model.

$$P_{grid}(t) + P_{ren}(t) + P_{dis}(t) = 10 + P_{ch}(t), \quad 0 \le P_{ren}(t) \le P_{ren}^{\max}(t).$$

Battery dynamics as given above.

Tasks.

1. Formulate the full LP: minimize $\sum_{t=1}^{2} c_{grid}(t) P_{grid}(t)$.
2. Solve by reasoning: how much should you charge in hour 1 and discharge in hour 2?
3. Compare with the no-battery case. What is the savings?

16 Problem 3: Demand response (peak cap) + cost minimisation (LP)

Data ($T = 3$, $\Delta t = 1$ h).

- $P_{IT}(t) = [9, 9, 9]$ MW
- PUE(t) depends on ambient temperature: $PUE(t) = [1.20, 1.35, 1.25]$
- Therefore $P_{total}(t) = PUE(t) P_{IT}(t)$
- Renewable limits: $P_{ren}^{\max} = [3, 6, 2]$ MW
- Grid prices: $c_{grid} = [60, 90, 150]$
- Demand response cap: $P_{grid}(t) \le 6$ MW for all t (contractual peak cap)
- No battery in this problem.

Tasks.

1. Compute $P_{total}(t)$ for each hour.
2. Formulate the LP:

$$\min \sum_{t=1}^{3} c_{grid}(t) P_{grid}(t) \quad \text{s.t.} \quad P_{grid}(t) + P_{ren}(t) = P_{total}(t),\ 0 \le P_{ren}(t) \le P_{ren}^{\max}(t),\ P_{grid}(t) \le 6.$$

3. Is the problem feasible? If infeasible, propose a minimal modification (e.g., allow curtailment or add backup generator) and re-formulate.

17 Problem 4: Workload shifting (convex cost + deadline)

This models moving flexible AI jobs across time to reduce cost.

Data ($T = 4$).

- Base load (must-run): $B(t) = [6, 6, 6, 6]$ MW
- Flexible job energy requirement: total $Q = 8$ MWh over 4 hours
- Decision: schedule flexible power $x(t) \geq 0$ MW so that $\sum_{t=1}^{4} x(t)\Delta t = 8$
- Per-hour grid cost is increasing convex: $\text{cost}(t) = c_t\,(B(t) + x(t))^2$, with $c = [1, 3, 2, 4]$ (arbitrary units)
- Renewable is ignored (or assume already subtracted from load).

Tasks.

1. Formulate the convex optimisation problem:

$$\min_{x(t)\geq 0} \sum_{t=1}^{4} c_t (B(t) + x(t))^2 \quad \text{s.t.} \quad \sum_{t=1}^{4} x(t) = 8.$$

2. Solve using KKT / "water-filling" reasoning: which hours get more of the flexible load?
3. Optional: add per-hour limit $x(t) \leq 4$ MW and discuss how it changes the solution.

18 Problem 5: MILP for generator on/off (unit commitment toy)

Data ($T = 3$).

- Net demand after renewables: $D = [7, 10, 9]$ MW
- Grid price: $c_{grid} = [120, 120, 120]$ (flat)
- Backup generator: if ON, can supply $0 \leq P_g(t) \leq 6$ MW
- Generator marginal cost: $c_g = 70$ per MWh
- Startup cost: $S = 200$ if turned on (per day), binary $y \in \{0, 1\}$
- If generator ON, it may run any subset of hours, but only if $y = 1$.

Tasks.

1. Formulate the MILP:

$$\min \sum_{t=1}^{3} \big(120 P_{grid}(t) + 70 P_g(t)\big) + 200y$$

subject to

$$P_{grid}(t) + P_g(t) = D(t), \quad 0 \leq P_g(t) \leq 6y, \quad y \in \{0, 1\}.$$

2. Solve by inspection: should the generator be used or not? In which hours, and why?
3. Find the break-even startup cost $S^\star$ above which the generator is never used.

19 Problem 6: QUBO toy (binary discharge selection with a coupling)

Data ($T = 4$).

- Prices: $c = [6, 10, 14, 20]$
- Decision $x_t \in \{0, 1\}$ indicates discharging 1 unit at time t
- Must pick exactly 2 slots: $\sum_{t=1}^{4} x_t = 2$
- Additional coupling penalty: avoid discharging in adjacent slots

$$\text{AdjPenalty} = \mu(x_1 x_2 + x_2 x_3 + x_3 x_4), \quad \mu = 30.$$

- Quadratic penalty weight for equality constraint: $\lambda = 50$.

Tasks.

1. Write the QUBO:

$$\min_{x \in \{0,1\}^4} \sum_{t=1}^{4} c_t x_t + \lambda \left(\sum_{t=1}^{4} x_t - 2 \right)^2 + \mu(x_1 x_2 + x_2 x_3 + x_3 x_4).$$

2. Expand the square using $x_i^2 = x_i$ and identify linear and quadratic coefficients.
3. Find the optimal schedule (try by reasoning: cheap slots but not adjacent).

Methodology to solve the problems.

- Identify which problems are LP, convex QP, MILP, and QUBO.
- Discuss modelling choices: PUE vs COP; renewables as upper bounds; storage losses.
- Ask: which constraints make the problem "hard" (integer variables, time coupling, nonlinearity)?