Seifert van Kampen Theorem

MAT327

Let $(G, \cdot)$ be an arbitrary group. Denote its identity element as 1, and for arbitrary $x \in G$, denote its inverse as x^{-1}, and $x^0 = 1$. Let $x^n = \underbrace{x \cdot x \cdots x}_{n}$.

Definition 1. Suppose $\{G_\alpha\}$ is a collection of subgroups of G. These subgroups **generate** G if for any $x \in G$ we have

$$x = x_1 \cdot x_2 \cdots x_n$$

where $x_i \in G_{\alpha_i}$. The above product is a **word** representing x. A **reduced word** is one where no two adjacent elements in the product are in the same subgroup.

Definition 2. Let G have a collection $\{G_\alpha\}$ as subgroups. Suppose $G_{\alpha_1} \cap G_{\alpha_2} = \{1\}$ when $\alpha_1 \neq \alpha_2$. Then G is the **free product** of G_α is for any $x \in G$, there is a unique reduced word representing x:

$$G = \dot{\prod_{\alpha \in J}} G_\alpha$$

Definition 3. Let $\{G_\alpha\}_{\alpha \in J}$ be a family of groups. Suppose G is a group, and $i_\alpha : G_\alpha \to G$ are monomorphisms. If G is the free product of $\{i_\alpha(G_\alpha)\}$, then G is the **external free product**.

Definition 4. Let G be a group, and $\{a_\alpha\}$ be a family of elements of G. $\{a_\alpha\}$ generate G is every element of G can be written as a product of the $\{a_\alpha\}$. If $\{a_\alpha\}$ is finite, G is finitely generated. Suppose each a_α generates an **infinite cyclic group** (think $\mathbb{Z}$) G_α of G. If G is a free product of $\{G_\alpha\}$, then G is said to be **free**.

Example 1. *Free group with one generator.* If one element $\{a\}$ generates the free group G, then elements take the form

$$a,\ a^{-1},\ a^3,\ \varnothing$$

and G is isomorphic to $\mathbb{Z}$.

Example 2. *Free group with two generators.* Two elements $\{a, b\}$ generate the free group G. Elements take the form

$$a,\ b,\ ababa^3b,\ ab^{-4}$$

and is denoted $\mathbb{Z} * \mathbb{Z}$. In algebraic topology, this can be thought of the fundamental group at the wedge of two loops $a,\ b$, and our elements denote travelling around the loops in clockwise/counterclockwise directions!

Lemma 1. Let G be a group. Let $\{a_\alpha\}_{\alpha \in G}$ be a system of free generators of G. Given any group H, and a family $\{y_n\}$ of elements of H, there exists (!) a homomorphism $h : G \to H$ such that $h(a_\alpha) = y_\alpha$ for each α.

Definition 5. Let $\{a_\alpha\}_{\alpha \in J}$ be arbitrary indexed family, G_α denote the set of all symbols of the form $a_\alpha^n, n \in \mathbb{Z}$. We can make G_α into a group ($a^n \cdot a^m = a^{n+m}$, and a^{-n} inverse). The **external free product** of G_α is called the free group on the elements of a_α. Free groups are isomorphic if and only if the sets of free generators have the same cardinality.

Definition 6. Given G, let $\{a_\alpha\}_{\alpha \in J}$ be a family of generators for G. Let F be the free group of the elements $\{a_\alpha\}$. Take $h : F \to G$, $h(a_\alpha) = a_\alpha \in G$. h extends to a surjective homomorphism $F \to G$. Letting $N = \ker(h)$, then

$$F/N \cong G$$

and each element is called a **relation.** We may specify N by a family $\{r_\beta\}$ of all elements of F such that these elements together with their conjugates generate N. N is called the least normal subgroup of F that contains $\{r_\beta\}$. We denote the **presentation** of G as

$$\langle \{a_\alpha\} : \{r_\beta\} \rangle$$

Theorem 1 (Seifert van Kampen)**.** Let $X = U_1 \cup U_2$ where U_1, U_2 are open in X. Let $U_1 \cap U_2 \neq \varnothing$, and $U_1, U_2, U_1 \cap U_2$ are path connected. Consider the commutative diagram

$$\begin{array}{ccc} U_1 \cap U_2 & \xrightarrow{\phi_1} & U_1 \\ \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \psi_1} \\ U_2 & \xrightarrow{\psi_2} & X \end{array}$$

Let $x_0 \in U_1 \cap U_2$. We may consider the induced homomorphisms

$$\begin{array}{ccc} \pi_1(U_1 \cap U_2, x_0) & \xrightarrow{\phi_1^*} & \pi_1(U_1, x_0) \\ \downarrow{\scriptstyle \phi_2^*} & & \downarrow{\scriptstyle \psi_1^*} \\ \pi_1(U_2, x_0) & \xrightarrow{\psi_2^*} & \pi_1(X, x_0) \end{array}$$

Given the presentations

$$\pi_1(U_1 \cap U_2, x_0) = \langle S : R \rangle, \quad \pi_1(U_1, x_0) = \langle S_1 : R_1 \rangle, \quad \pi_1(U_2, x_0) = \langle S_2 : R_2 \rangle$$

The Seifert van Kampen theorem states that

$$\pi_1(X, x_0) = \langle S_1 \cup S_2 : R \cup R_2 \cup R_3 \rangle$$

This theorem is a powerful tool to compute fundamental groups of complex spaces by looking at smaller parts of the space.